Quantum Field Theory

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The note is not intended for distribution. QFT is discussed in detail. Links below show contents of solved problems and summary of other topics. Readers who have opened this note for the first time are highly recommended to read the preface.

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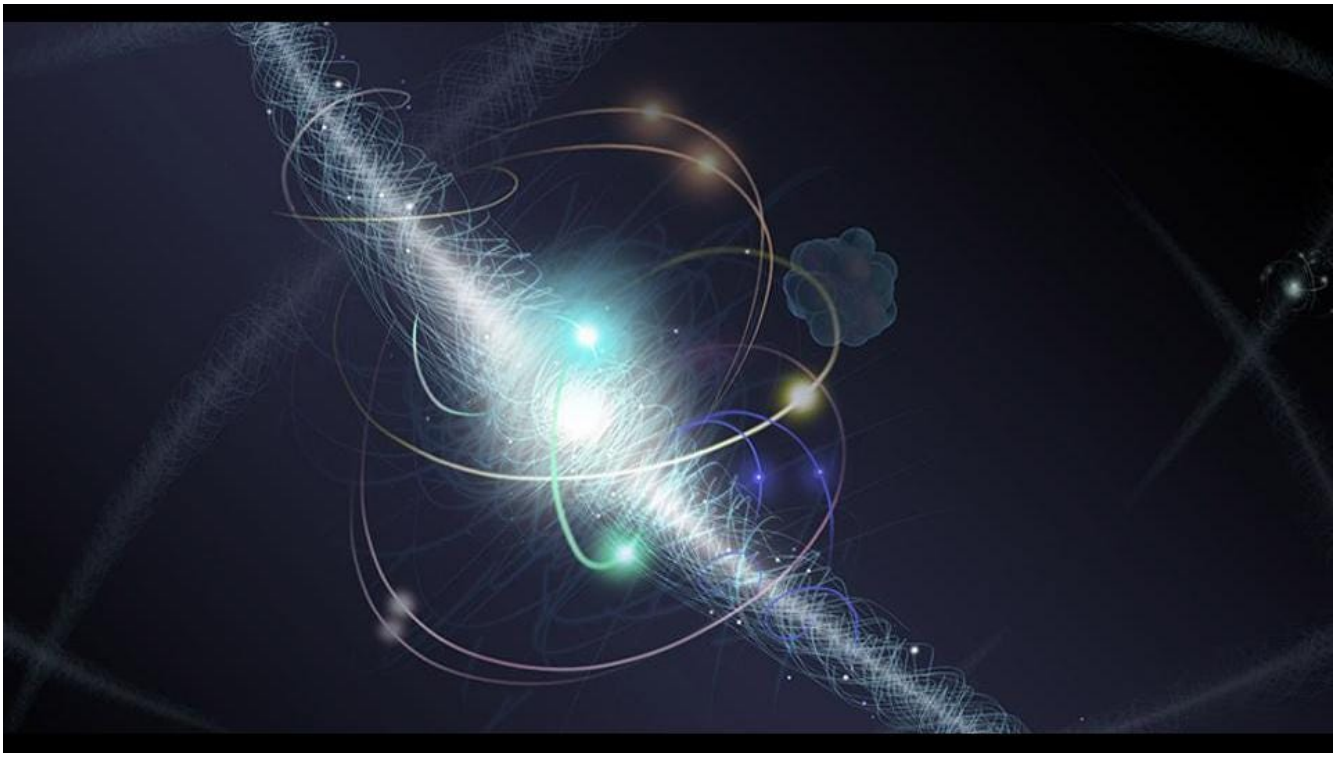
Preface, main motivation, main tips for studying



QFT Fundamentals and Free Quantum Fields

Typical Methods for Interaction of Quantum Fields

Typical QFT Processes

Other Parts and Topics of QFT

Also there are topics from Zi's book, examples of regularizations by Kazakov, good lectures by Vergeles, theory about spinors, Standard Model and SUSY by Dreiner, et al..

I used the following literature.

Let's discuss some minimum knowledge and motivation that would be good to understand for studying the subject.

Idea of the note in a nutshell

(I'll write very shortly the main idea of this approach later)

Основная мотивация

Можно понять лучше фундаментальную природу материи

(раскрою потом эту мысль так, чтобы хотелось это сделать)

Диаграммы очень много в современной физике используются, поэтому поймете формализм КТП - очень вероятно, что он поможет в какой-то теоретической физике

(раскрою потом, что на самом деле, даже если не придется заниматься КТП, уверенность в диаграммных методах - очень стоящий навык)

Лучшие идейные головоломки КТП

(напишу те, на которые ответы есть. философские интересные, но когда понимаешь, что на них нет ответов, то мотивации наоборот мало!)

m-th.1 Об суммах бесконечности и эффекте Казимира (???)

(тут пара абзацев будет, постараюсь компактнее написать, но тем не менее, в пару строчек это не выйдет)

Some instructions

Лучшие технические головоломки КТП

m-te.1 Какой на самом деле закон Кулона? (????)

(там вот поляризация вакуума еще есть, это очень прикольное КТП явление)
Some instructions

m-te.1 Чему равен магнитный момент мюона?

Почему магнитный момент на эксперименте отличен от вычисленного в квантовой механике?

(!! идейно тут укажу, что так вот к нему в квантмехе прийти, такое значение. а на эксперименте чуть другое, что делать? строчка, что так-то решится он. типичная задача КТП, результат Шивнгера. мб вопрос про электрон тут??? крч, там в теории типичное вычисление для этой задачи, тут потом укажу это!)
Some instructions

m-te.1 Как взаимодействуют фотоны?

(там через гравитацию, через ЛЛ4 один из параграфов и еще приколы в кв. оптике. укажу потом ответ на это.)
Some instructions

m-te.1 Как взаимодействуют кварки?

(вообще, идея была в том, что про госты какую-то задачу придумать, но пока не готов я еще и близко к этому.)
Some instructions

Лучшие головоломки формализма

m-f.1 Связь спина со статистикой

(про то, все-таки почему это так? напишу в теории четко!)
Some instructions

Обзор подхода к КТП

В целом, конечно же, предмет берется теми же установками и тем же подходом, как и всегда, это понимание подхода - самое важное, как и всегда.

QFT can never be learned in a hurry

If one don't sit and think about each topic in scope of basic topics of QFT, he/she will never understand QFT even on a basic level.

(I'll write better about it later)

So it is important to slow down, focus on fundamentals and go through all details. Otherwise there will be never solid understanding.

Теория - теорией, а можно решать задачи и обойти ее строной до поры до времени. Так и следует делать.

Рыться в теории тут - растягивается на годы. Это очень были бы неприятные годы, потому что хочется что-то уметь делать. Для этого можно просто сесть и начать думать про явления и задачи, много всего решено, так что можно смотреть не примеры описаний явлений.

Теорию до деталей понимать не обязательно. Кто это не понимает, тот скорее всего застрянет на КТП на долго.

Фундаментальная физика - круто, но у нее свой приоритет, не всегда он большой!

(про Понимание того, что нужно вмеру погружаться в фундаментальные вопросы, особо сильно важно в КТП, где в фундаментальных вопросах можно очень уж надолго зависнуть и ничего не сделать.)

Конечно, заниматься построением КТП, понимать поля тоже полезно, однако лучше это делать лишь небольшую часть времени, а основное время заниматься задачами и полезными методами.

И пробовать делегировать это! Пусть те, кто много времени на это потратили и кто хочет/обязан по своей работе отвечать на фундаментальные вопросы, на них и отвечают. Пусть иногда им и может казаться, что ты задаешь вопросы, на самом деле твои фундаментальные теоретические вопросы будут малой частью твоей работы, большую часть времени ты изучаешь методы и тренируешь их на задачах, которых очень много и которые очень интересные. Самому отдельно много сидеть на этом, пока не освоены методы и задачи - малополезное дело. И это речь про первые полгода.

Собственно, это же верно для любого другого предмета, но в других предметах не настолько много общей теории, только после которой идет ее применение.

Нужны лишь пара разделов КТП и пара основ для исследований и решений задач

Не нужно изучать все. Достаточно ограничится обычным, базовым уровнем. И доработать его, а потом углубляться.

Понять большинство основных разделов - дело нескольких лет, не меньше. Не нужно ставить такую цель. Куда правильнее ограничиться основами, а все остальное - только по потребности, пусть оно и важное вообще-то. Для этого у меня и есть самая 1я часть, где не только основы, но и их минимум. Вторая 1я часть - вот там другие основы указаны. Очень важное это деление, именно для того, чтобы использовать только то, что нужно, а не заниматься много чем.

А там уже когда будет движение - раз в неделю или две можно и в принципе своим пониманием заниматься, вот в таком темпе все и хорошо учится.

Part I
Typical QFT in a Nutshell

1 Main Methods and Models of QFT

1.1 Most used methods in a nutshell

1.1.1 Path integral formalism for applications in QFT

(!! очень много формул и указаний тут будет! пишу упрощенно - а далее точно!!! как и всегда со всеми формулами в 1 части.)

Z производящий функционал

(по сути его раскладываем и все про него делаем, так что он на первом месте)

Основные формулы

$$\tilde{Z}[j] := \frac{Z[j]}{Z[0]} = 1 + \sum_{k=1}^{\infty} \left(\frac{\mathrm{i}}{\hbar}\right)^k \int \mathcal{D}^{(k)}_{1\ldots k}(x_1, \ldots, x_k) \frac{1}{k!} \prod_{k'=1}^{k} j(x_{k'}) \, \mathrm{d}^4 x_{k'}$$
$$= 1 + \sum_{N=1}^{\infty} \sum_{n_k} \left(\prod_{k=1}^{\infty} \frac{1}{(n_k)!} a_k^{n_k} \right) \Bigg|_{\sum n_k = N}$$

где $a_k = \left(\frac{\mathrm{i}}{\hbar}\right)^k \int \mathcal{D}^{c(k)}_{1\ldots k}(x_1, \ldots, x_k) \frac{1}{k!} \prod_{k'=1}^{k} j(x_{k'}) \, \mathrm{d}^4 x_{k'}$, N - число произведений связных вкладов, т. е. число компонент связности в диаграмме, а n_k - количество вкладов с k-точечными связными функциями Грина, которые входят симметрично и, стало быть, допускают $(n_k)!$ перестановок, дающих одинаковые вклады, причем $\sum n_k = N$.

$$\hat{Z}[j] = \int \mathcal{D}\varphi \exp\left\{\frac{\mathrm{i}}{\hbar}\left(S_0[\varphi] + j_{x'}\varphi_{x'}\right) + \frac{\mathrm{i}}{\hbar} S_{\mathrm{int}}[\varphi]\right\} =$$
$$= \exp\left\{\frac{\mathrm{i}}{\hbar} S_{\mathrm{int}}\left(-\mathrm{i}\hbar\frac{\delta}{\delta j}\right)\right\} \int \mathcal{D}\varphi \exp\left\{\frac{\mathrm{i}}{\hbar}\left(S_0[\varphi] + j_{x'}\varphi_{x'}\right)\right\}$$

Производящий функционал от эффективного действия

$$Z_\Gamma[j] = \exp\left\{\frac{\mathrm{i}}{\hbar_\Gamma} G_\Gamma[j]\right\} = \mathcal{N} \int \mathcal{D}\varphi \exp\left\{\frac{\mathrm{i}}{\hbar_\Gamma}\left(\Gamma[\varphi] + \varphi_{x'} j_{x'}\right)\right\}$$

О разложениях по Киселеву

(тут все его методы, лекция 4, просто я мб уже это писал?)

Об обозначениях Киселева (!?!??!?!)

(чтобы в них не запутаться, отдельно напишу! что там значат его цифры и буквы! один малый раздел про это и все.)

О теореме Вика в формализме интеграла по траекториям и переходе к диаграммам

(мб в указания на теорию ниже напишу это, пока тут)
Теорема Вика:

$$\mathcal{D}^{(n)}_{1...n} = \sum_{\substack{\text{все}\\ \text{пары}}} \underbrace{\tilde{\varphi}_k \tilde{\varphi}_{k'}} \cdots \underbrace{\tilde{\varphi}_l \tilde{\varphi}_{l'}},$$

Proof:

$$\mathcal{D}^{(n)}\left(x_1, ..., x_n\right) = \left\langle \text{in} \left| \hat{\mathrm{T}}\left\{\hat{\varphi}\left(x_1\right) ... \hat{\varphi}\left(x_n\right)\right\} \right| \text{ in} \right\rangle,$$

$$Z[j] = \exp\left\{-\frac{\mathrm{i}}{\hbar}\frac{1}{2} j_x \hat{\Gamma}^{-1}_{xy} j_y\right\} = \exp\left\{\frac{\mathrm{i}}{\hbar}\frac{1}{2} j_x \mathcal{G}^{(2)}_{xy} j_y\right\} = \exp\left\{-\frac{1}{\hbar^2}\frac{1}{2} j_x \mathcal{D}^{(2)}_{xy} j_y\right\}$$

$$j(x) = -\mathrm{i}\hbar \frac{\delta}{\delta\tilde{\varphi}(x)} \mathrm{e}^{\frac{\mathrm{i}}{\hbar} j_{x'} \tilde{\varphi}_{x'}}\Big|_{\tilde{\varphi}=0} \Rightarrow F[j] = F\left[-\mathrm{i}\hbar\frac{\delta}{\delta\tilde{\varphi}}\right] \mathrm{e}^{\frac{\mathrm{i}}{\hbar} j_{x'} \tilde{\varphi}_{x'}}\Big|_{\tilde{\varphi}=0}$$

$$\mathcal{D}^{(n)}_{1...n} = (-\mathrm{i}\hbar)^n \frac{\delta^n Z[j]}{\delta j_1 ... \delta j_n}\Big|_{j=0} = \exp\left\{-\frac{\mathrm{i}\hbar}{2}\mathcal{G}^{(2)}_{xy}\frac{\delta^2}{\delta\tilde{\varphi}_x \delta\tilde{\varphi}_y}\right\} \tilde{\varphi}_1 ... \tilde{\varphi}_n\Big|_{\tilde{\varphi}=0} =$$

$$= \exp\left\{\frac{1}{2}\mathcal{D}^{(2)}_{xy}\frac{\delta^2}{\delta\tilde{\varphi}_x \delta\tilde{\varphi}_y}\right\} \tilde{\varphi}_1 ... \tilde{\varphi}_n\Big|_{\tilde{\varphi}=0}$$

$$\underbrace{\tilde{\varphi}_k \tilde{\varphi}_{k'}} := -\mathrm{i}\hbar \mathcal{G}^{(2)}_{kk'} = \mathcal{D}^{(2)}_{kk'}$$

Интеграл по путям в квантовой механике

(пока по по Пескину Шредеру, но это быстро соберется, суть тут не такая большая. в теории большой раздел про это будет.)

Континуальный интеграл для скалярного поля

континуальный интеграл для скалярного поля

$$\langle 0|\hat{\mathrm{T}}\{\hat{\Phi}(x)\hat{\Phi}^\dagger(y)\}|0\rangle = \mathrm{i}\int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-\mathrm{i}k(x-y)} \frac{1}{k^2 - m^2 + \mathrm{i}0}$$

$$\langle 0|\hat{\mathrm{T}}\{\hat{\varphi}(x)\hat{\varphi}(y)\}|0\rangle = (-\mathrm{i})^2 \frac{\delta^2 Z(j)}{\delta j(x)\delta j(y)}\Big|_{j=0} = \mathcal{D}^{(2)}(x-y)$$

$$Z(j) = \mathcal{D}^{(0)} + \mathrm{i}\mathcal{D}^{(1)}_1 j_1 + \frac{1}{2!}\mathrm{i}^2 \mathcal{D}^{(2)}_{12} j_1 j_2 + ... = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} \mathcal{D}^{(n)}_{1...n} \prod_{k=1}^{n} j_k$$

$\mathcal{D}^{(n)}_{1...n} := (-\mathrm{i})^n \frac{\delta^n Z}{\delta j_1 ... \delta j_n}\Big|_{j=0}$.

свободные поля

Для свободных полей:

$$-i\frac{\delta Z}{\delta j(x)}\bigg|_{j=0} = \mathcal{D}^{(1)}(x) = \langle 0|\hat{\varphi}(x)|0\rangle = \varphi_{\text{vac}} \equiv 0$$

а значит, значение поля в вакууме тождественно равно нулю. (??? подумаю глубже про это!!!)

Вычислим:

$$Z(j) = \mathcal{N}\int \mathcal{D}\varphi e^{\mathrm{i}\int \mathrm{d}^4x \frac{1}{2}\varphi(x)\hat{\Gamma}\varphi(x)+\mathrm{i}j_1\varphi_1}$$

где $\hat{\Gamma} := -(\partial^2 + m^2)$ с заданными вакуумными граничными условиями имеет решения уравнения $\hat{\Gamma}\varphi(x) = 0$ только $\varphi(x) = 0$. Именно это условие позволяет далее провести однозначные манипуляции с интегралом (???).

С учетом индексного обозначения:

$$Z(j) = \mathcal{N}\int \mathcal{D}\varphi e^{\mathrm{i}\left(\frac{1}{2}\varphi_1\hat{\Gamma}\varphi_1+j_1\varphi_1\right)}$$

Сперва сдвигаем $\varphi_1 = \tilde{\varphi}_1 - \hat{\Gamma}^{-1}_{12}j_2$, получим

$$Z(j) = \tilde{\mathcal{N}}\exp\left\{-\mathrm{i}\frac{1}{2}j_1\hat{\Gamma}^{-1}_{12}j_2\right\}$$

где независящий от источников вклад $\tilde{\mathcal{N}} := \mathcal{N}\int \mathcal{D}\tilde{\varphi}e^{\frac{\mathrm{i}}{2}\tilde{\varphi}_1\hat{\Gamma}\tilde{\varphi}_1}$. Введем производящий функционал

$$\hat{Z}(j) = \frac{Z(j)}{Z(0)} = \exp\left\{-\frac{\mathrm{i}}{2}j_1\hat{\Gamma}^{-1}_{12}j_2\right\}.$$

Для него

$$\mathcal{D}^{(1)}_1 = 0$$

а все высшие причинные коррелляторы, $\mathcal{D}^{(n)}_{1\ldots n}|_{n>2}$ построены из комбинаций

$$\mathcal{D}^{(2)}(x-y) = \langle 0|\hat{\mathrm{T}}\hat{\varphi}(x)\hat{\varphi}(y)|0\rangle = \mathrm{i}\hat{\Gamma}^{-1}_{xy}$$

(?? и что?? нужно тут дописать!)

Примеры полезных вычислений интегралом по траекториям

(порешаю - напишу, пока не нарешал)

Интеграл по траекториям:

$$Z(j) = \int_{\varphi_0(x)}^{\varphi_1(x)} D\varphi(x) e^{iS_0(\varphi)+i\int d_4x j(x)\varphi(x)}$$

$$\int D\varphi e^{-\frac{1}{2}\varphi\cdot K\cdot\varphi - V(\varphi)+J\cdot\varphi} = e^{-V(\delta/\delta J)}e^{\frac{1}{2}J\cdot K^{-1}\cdot J}$$

Для $V = 0$:

$$\int D\varphi e^{-\frac{1}{2}\varphi\cdot K\cdot\varphi + J\cdot\varphi} = e^{\frac{1}{2}J\cdot K^{-1}\cdot J}$$

(???) Бывает, что K не имеет обратной матрицы, например, это происходит в действии Максвелла: $S(A) = \int d^4x \left[\frac{1}{2}A_\mu \left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right) A_\nu + A_\mu J^\mu\right]$. Тут $K \sim \left(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right) \equiv Q^{\mu\nu}$, отсутствие обратной - наличие некоторых нулевых собственных значений, т.е. если при ее действии на вектор матрица зануляет этот вектор. Тензор $Q^{\mu\nu}$ зануляет любой вектор вида $\partial_\nu\Lambda(x)$: $Q^{\mu\nu}\partial_\nu\Lambda(x) = 0$. Значит, $Q^{\mu\nu}$ не имеет обратной. В этом явлении нет ничего таинственного; мы уже встречались с ним в классической физике, когда говорили о том, что смысл именно в разности потенциалов, а не их абсолютном выражении (??). На более сложном уровне мы видим, что, решая уравнение Максвелла $\partial_\mu F^{\mu\nu}$ (в виде $Q_{\mu\nu}A^\nu = J_\mu$), мы находим обратную матрицу Q^{-1} (Решение $A^\nu = (Q^{-1})^{\nu\mu}\, J_\mu$). Q^{-1} не существует! Что делать? Мы знаем, что следует наложить дополнительное условие на калибровочный потенциал A^μ, так называемую «фиксацию калибровки». (????? подумаю про это потом еще, как это спасет дело?!!)

1.1.2 Operator formalism, second quantization

Bosonic, fermionic operators

$$\left[a_r, a_s^\dagger\right]_+ = \delta_{rs}, \quad [a_r, a_s]_+ = \left[a_r^\dagger, a_s^\dagger\right]_+ = 0$$
$$(a_r)^2 = \left(a_r^\dagger\right)^2 = 0$$

$$[a_1^*, a_2] = [a_2^*, a_1] = 0, \qquad \text{скалярное поле}$$
$$\{a_1^*, a_1\} = \{a_2^*, a_2\} = 0, \quad \text{спинорное поле}$$

$\mathrm{H_B} = a^* a + a_c a_c^*$, скалярное поле
$\mathrm{H_F} = a^* a - a_c a_c^*$, спинорное поле.

$$\not\delta(\mathbf{p}) = (2\pi)^3\delta(\mathbf{p}) \text{ and } \not\delta(p) = (2\pi)^4\delta(p)$$

(тут может быть про нормировку будут формулы, для непрерывного сперва, потом для дискретногос пектра)

Formalism of confinement and bound states

(answer about how they are fitting into the basic theory)

Propagators main formulas

$$\underbrace{\phi\left(x_1\right)\phi}\left(x_2\right)\phi = \mathrm{i}\Delta_{\mathrm{F}}\left(x_1 - x_2\right)$$

$$\underbrace{\phi\left(x_1\right)\phi^{\dagger}}\left(x_2\right) = \underbrace{\phi^{\dagger}\left(x_2\right)\phi}\left(x_1\right) = \mathrm{i}\Delta_{\mathrm{F}}\left(x_1 - x_2\right)$$

$$\underbrace{\psi_{\alpha}\left(x_1\right)\bar{\psi}_{\beta}}\left(x_2\right) = -\underbrace{\bar{\psi}_{\beta}\left(x_2\right)\psi_{\alpha}}\left(x_1\right) = \mathrm{i}S_{\mathrm{F}\alpha\beta}\left(x_1 - x_2\right)$$

$$\underbrace{A^{\mu}\left(x_1\right)A^{v}}\left(x_2\right) = \mathrm{i}D_{\mathrm{F}}^{\mu v}\left(x_1 - x_2\right).$$

(?? а кстати, почему там у бозонов симметрия, а у фермионов антисимметрия в свертках???? забыл)

$$\Delta_{\mathrm{F}}(x) = \frac{1}{(2\pi)^4}\int \frac{\mathrm{d}^4 k \mathrm{e}^{-\mathrm{i}kx}}{k_0^2 - \left(\frac{\omega_{\mathrm{k}}^c}{c} - \mathrm{i}\eta\right)^2} = \frac{1}{(2\pi)^4}\int \frac{\mathrm{d}^4 k \mathrm{e}^{-\mathrm{i}kx}}{k^2 - \mu^2 + \mathrm{i}\varepsilon},$$

(пара слов про вывод)

$$S_{\mathrm{F}}(x) = \theta(t)S^{+}(x) - \theta(-t)S^{-}(x) = \left(\mathrm{i}\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} + \frac{mc}{\hbar}\right)\Delta_{\mathrm{F}}(x)$$

$$= \frac{\hbar}{(2\pi\hbar)^4}\int \mathrm{d}^4 p \mathrm{e}^{-\mathrm{i}px/\hbar}\frac{\not{p} + mc}{p^2 - m^2c^2 + \mathrm{i}\varepsilon}$$

$$D_{\mathrm{F}}^{\mu v}(x) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4 k D_{\mathrm{F}}^{\mu v}(k)\mathrm{e}^{-\mathrm{i}kx}$$

$$D_{\mathrm{F}}^{\mu v}(k) = \frac{-g^{\mu\nu}}{k^2 + \mathrm{i}\varepsilon} = \frac{1}{k^2 + \mathrm{i}\varepsilon}\sum_r \zeta_r \varepsilon_r^{\mu}(\mathbf{k})\varepsilon_r^{v}(\mathbf{k})$$

In the special frame of reference, where $\varepsilon_0^{\mu}(\mathbf{k}) = n^{\mu} \equiv (1,0,0,0), \varepsilon_r^{\mu}(\mathbf{k}) = (0, \boldsymbol{\varepsilon}_r(\mathbf{k}))\,,\ r = 1,2,3$:

$$D_{\mathrm{F}}^{\mu v}(k) = \frac{1}{k^2 + \mathrm{i}\varepsilon}\left\{\sum_{r=1}^{2}\varepsilon_r^{\mu}(\mathbf{k})\varepsilon_r^{v}(\mathbf{k}) + \frac{\left[k^{\mu} - (kn)n^{\mu}\right]\left[k^{v} - (kn)n^{v}\right]}{(kn)^2 - k^2} + (-1)n^{\mu}n^{v}\right\}$$

(???? додумаю, почему?)

(!?!?!? еще добавлю формулы для самих пропагаторов!!!)

1.1.3 Quantization of the fields

(пока идея написать обзор всего в один раздел, мб удалю его, если это невозможно.)

Canonical Quantization

Q of scalar field

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y})$$

with all other commutators equal to zero. Pointwise, these operators behave like x and p in quantum mechanics, but the position/momentum is in field space, not physical space.

$\int d\mathbf{x} e^{i\mathbf{p}\cdot\mathbf{x}} = (2\pi)^3\delta(\mathbf{p}), \quad \int d\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} = (2\pi)^3\delta(\mathbf{x})$

$$\tilde{f}(\mathbf{p}) = \int d\mathbf{x} f(\mathbf{x}) e^{-i\mathbf{p}\cdot\mathbf{x}}, \quad f(\mathbf{x}) = \int \tilde{d}\mathbf{p} \tilde{f}(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}, \quad \tilde{d}\mathbf{p} = \frac{d\mathbf{p}}{(2\pi)^3}$$

$$\not{\delta}(\mathbf{p}) := (2\pi)^3\delta(\mathbf{p}) \qquad \not{\delta}\phi(p) = (2\pi)^4\delta(p).$$

$$H = \frac{1}{2}\int d\mathbf{x} \left(\dot{\phi}^2 + (\nabla\phi)^2 + m^2\phi^2\right)$$

$\phi(\mathbf{p}) = \phi^\dagger(-\mathbf{p})$, $\pi(\mathbf{p}) = \pi^\dagger(-\mathbf{p})$, which holds because $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ are self-adjoint, giving

$$H = \frac{1}{2}\int d\mathbf{p} \left(|\pi(\mathbf{p})|^2 + \omega_p^2|\phi(\mathbf{p})|^2\right), \quad \omega_p = \sqrt{p^2 + m^2}$$

where for an operator A, $|A^2| = A^\dagger A$. This describes decoupled harmonic oscillators with frequency ω_p.

For the ordinary oscillator, $H = (1/2)\left(p^2 + \omega^2 q^2\right)$, we defined $q = \frac{a+a^\dagger}{\sqrt{2\omega}}$, $\quad p = -i\sqrt{\frac{\omega}{2}}\left(a - a^\dagger\right)$, so

$$a_\mathbf{p} = \frac{1}{2}\left(\sqrt{2\omega_p}\phi(\mathbf{p}) + i\sqrt{\frac{2}{\omega_p}}\pi(\mathbf{p})\right)$$

$$\phi(\mathbf{p}) = \frac{a_\mathbf{p} + a^\dagger_{-\mathbf{p}}}{\sqrt{2\omega_p}}$$

The calculations are a little more complicated because both $\mathbf{p}$ and $-\mathbf{p}$ terms are present. After working through the algebra (????), we find

$$\phi(\mathbf{x}) = \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}}\left(a_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}}\right), \quad \pi(\mathbf{x}) = \int d\mathbf{p}(-i)\sqrt{\frac{\omega_p}{2}}\left(a_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} - a^\dagger_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}}\right).$$

We'll use this form of the field operators almost exclusively.

Some calculation gives the commutation relations

$$\left[a_\mathbf{p}, a^\dagger_\mathbf{q}\right] = \not{\delta}(\mathbf{p} - \mathbf{q})$$

with all other commutators equal to zero. Applying these commutation relations in a slightly long but straightforward calculation gives

$$H = \int d\mathbf{p}\omega_p a^\dagger_\mathbf{p} a_\mathbf{p} + \frac{1}{2}\int d\mathbf{p}\omega_p\delta(\mathbf{0})$$

with a large divergent counterterm.

We find a divergent constant for two reasons. The $\delta(\mathbf{0})$ term is an IR divergence due to the infinite size of our space. Generally, IR divergences signal that we are asking an unphysical question. The integral of $\omega_\mathbf{p}$ also diverges in the UV because the theory must break down at some upper scale. E.g. if we take the continuum limit of a lattice with spacing a, it should break down at energy $\Lambda \sim 1/a$. We will be able to perform calculations at $E \ll \Lambda$ using the machinery of regularization and renormalization.

As an explicit example, we can instead quantize the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2 - V_0$$

where V_0 is a counterterm, and regularize by introducing a cutoff Λ. We choose $V_0(\Lambda)$ so that the vacuum energy density is a finite constant for every choice of Λ, then take the limit $\Lambda \to \infty$ without issue. The price is that our theory now no longer predicts the value of the vacuum energy density. Generically, we would expect it to be some arbitrary number between zero and M_P^4 (the cutoff for quantum gravity), and to lie somewhere in the middle of the range. But we instead observe a vacuum energy density about 10^{122} times smaller than the maximum value. This is the "old fashioned" way to explain the cosmological constant problem.

As we'll see later, the reasoning here is not quite correct, but in the modern Wilsonian framework the problem remains. Furthermore, we cannot simply ignore the vacuum energy entirely, defining it to always be zero, because we can measure differences in it via the Casimir effect.

• The spatial momentum is

$$P^i = \int d\mathbf{x}\dot{\phi}(\mathbf{x})\partial^i\phi(\mathbf{x}) = \int d\mathbf{p}p^i a^\dagger_{\mathbf{p}}a_{\mathbf{p}} + \frac{1}{2}\int d\mathbf{p}p^i\delta(\mathbf{0})$$

where the constant is zero by "symmetry". (This can also be justified more carefully by regulating the delta function in any way.) Thus in general we define the four-momentum

$$P^\mu = \int d\mathbf{p}p^\mu a^\dagger_{\mathbf{p}}a_{\mathbf{p}}, \quad p^\mu = (\omega_p, \mathbf{p})$$

The four-momentum operator satisfies:

$$\left[P^\mu, a^\dagger_{\mathbf{p}}\right] = p^\mu a^\dagger_{\mathbf{p}}, \quad \left[P^\mu, a_{\mathbf{p}}\right] = -p^\mu a_{\mathbf{p}}$$

$$|\mathbf{p}\rangle = \sqrt{2E_p}a^\dagger_{\mathbf{p}}|0\rangle$$

$$\langle\mathbf{q}|\mathbf{p}\rangle = (2E_p)\,\phi(\mathbf{p}-\mathbf{q})$$

$$1_{\text{one-particle}} = \int \frac{d\mathbf{p}}{2E_p}|\mathbf{p}\rangle\langle\mathbf{p}|$$

$$\int dp\not{\theta}\left(p^2 - m^2\right)\Theta\left(p^0\right) = \int \frac{d\mathbf{p}}{2E_p}.$$

Left-hand side is Lorentz invariant, so $d\mathbf{p}/2E_p$ is as well.

• Similarly, $(2E_p)\,\delta(\mathbf{p}-\mathbf{q})$ is Lorentz-invariant, since it gives one when integrated against the Lorentz invariant measure. Thus the normalization of the one-particle states is Lorentz invariant; when a Lorentz transformation Λ is represented as a unitary operator $U(\Lambda)$, it will satisfy $U(\Lambda)|\mathbf{p}\rangle = |\Lambda\mathbf{p}\rangle$, where $\Lambda\mathbf{p}$ is defined as the spatial part of $\Lambda\left(E_p, \mathbf{p}\right)$.

$|\mathbf{x}\rangle = \phi(\mathbf{x})|0\rangle$.

$$\langle\mathbf{p}|\mathbf{x}\rangle = e^{-i\mathbf{p}\cdot\mathbf{x}}$$

which is similar to how momentum and position eigenstates are related in quantum mechanics. Then $|\mathbf{x}\rangle$ represents a particle localized near $\mathbf{x}$.

• There are no perfectly localized states in relativistic quantum field theory. Using the definition above, $\langle \mathbf{y}|\mathbf{x}\rangle$ is not a delta function, but rather has range on the order of the Compton wavelength $1/m$, which stems from the Lorentz invariant measure factor of $1/2E_p$. Alternatively, one could omit this factor, yielding perfectly localized "Newton-Wigner" states. However, these states don't transform nicely under Lorentz transformations, and in particular, they do not remain localized states in other frames.

• Note that the quantum field has $1/\sqrt{2E_p}$ rather than the Lorentz invariant $1/2E_p$. This is because the creation operators have to be multiplied by $\sqrt{2E_p}$ to give relativistically normalized states, i.e. the expression is really $(d\mathbf{p}/2E_p)\sqrt{2E_p}a^\dagger$.

Example:

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - m^2 \phi^\dagger \phi$$

ϕ and $\phi^\dagger$ as independent fields. $\phi^\dagger$ are $\pi = \dot{\phi}^\dagger$ and $\pi^\dagger = \dot{\phi}$ giving Hamiltonian

$$\mathcal{H} = \pi^\dagger \pi + \nabla \phi^\dagger \cdot \nabla \phi + m^2 \phi^\dagger \phi$$

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = \left[\phi^\dagger(\mathbf{x}), \pi^\dagger(\mathbf{y})\right] = i\delta(\mathbf{x}-\mathbf{y})$$

other [ ] = 0.

$$\phi(\mathbf{x}) = \int \frac{d\mathbf{p}}{\sqrt{2E_p}} \left(a_\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} + b^\dagger_\mathbf{p} e^{-i\mathbf{p}\cdot\mathbf{x}}\right)$$

$$\left[a_\mathbf{p}, a^\dagger_\mathbf{q}\right] = \left[b_\mathbf{p}, b^\dagger_\mathbf{q}\right] = \not{\delta}(\mathbf{p}-\mathbf{q})$$

$$P^\mu = \int d\mathbf{p} p^\mu \left(a^\dagger_\mathbf{p} a_\mathbf{p} + b^\dagger_\mathbf{p} b_\mathbf{p}\right)$$

$$Q = i\int d\mathbf{x}\,(\pi\phi - \phi^*\pi^*) = \int d\mathbf{p}\left(b^\dagger_\mathbf{p} b_\mathbf{p} - a^\dagger_\mathbf{p} a_\mathbf{p}\right)$$

E.g.:

$$Q^i = i\int d\mathbf{x}\left(\pi_a \sigma^i_{ab} \phi_b - \phi^*_a \sigma^i_{ab} \pi^*_b\right)$$

(абзац про то, в каком смысле можно считать скалярное поле осцилляторами??? пока не очень понял, в теории это подробно опишу. наверняка многие это неверно понимают.)

Типичные тонкости с квантованием полей (???)

Что делать с вакуумной энергией при квантовании? (???)

(пока вопрос, напишу потом. очень важный для теории вопрос.)

Appearance of entanglement in quantum fields

(I'll explain, where does it appear, in 2nd 1 part there there is a special section about it)

1.1.4 Wick's theorem, origin of diagrams, averaging

(тут же ниже будет указание, как разные правила для диаграмм делать. не только фейнмановские.)

О теореме Вика в операторном формализме (!?!?!)

(допишу скоро)

О нормальном упорядочении

Мы операторы с крестами двигаем влево, ну или поля и плюсовыми (+) модами двигаем вправо (? поч - теория).

$$\underbrace{A(x_1)B}(x_2) \equiv \langle 0 |\, \mathrm{T}\left\{A\left(x_1\right) B\left(x_2\right)\right\} |\, 0\rangle$$

$$T[\psi(x)\bar{\psi}(y)] = N[\psi(x)\bar{\psi}(y)] + \overbrace{\psi(x)\bar{\psi}}(y)$$

(????? тут очень важный абзац, зачем это мы делаем??? чтобы причинность спасти??? напишу четко!)

$$N(\overbrace{\bar{\psi}_1\psi_2}\bar{\psi}_3\bar{\psi}_4) = -\overbrace{\bar{\psi}_1\bar{\psi}_3}N\left(\psi_2\bar{\psi}_4\right) = -S_F\left(x_1 - x_3\right) N\left(\psi_2\bar{\psi}_4\right).$$

(!!! додумаю это!!!!)

$$T[\psi(x)\bar{\psi}(y)] = N[\psi(x)\bar{\psi}(y)] + \overbrace{\psi(x)\bar{\psi}(y)}$$

О Т-упорядочении

$$\text{bosons}: \quad \mathrm{T}\{\phi(x)\phi(x')\} := \begin{cases} \phi(x)\phi(x'), & \text{if } t > t' \\ \phi(x')\phi(x), & \text{if } t' > t \end{cases} = \theta\left(t - t'\right)\phi(x)\phi\left(x'\right) + \theta\left(t' - t\right)\phi\left(x'\right)\phi(x).$$

$$\text{fermions}: \quad \mathrm{T}\{\phi(x)\phi(x')\} := \begin{cases} \phi(x)\phi(x'), & \text{if } t > t' \\ -\phi(x')\phi(x), & \text{if } t' > t \end{cases}$$

The idea is to make 'earlier' operators operate 'first'. (???? цвета поменяю, допишу формулу!!!)

Main formulas of expansion (!?!?!?!?)

(напишу их скоро, к ним все понимание КТП и сводится!!!)

$$S = \sum_{n=0}^{\infty} S^{(n)} \equiv \sum_{n=0}^{\infty} \frac{(-\mathrm{i})^n}{n!} \int \ldots \int \mathrm{d}^4x_1 \ldots \mathrm{d}^4x_n \ \mathrm{T}\left\{\mathcal{H}_{\mathrm{I}}\left(x_1\right) \ldots \mathcal{H}_{\mathrm{I}}\left(x_n\right)\right\}.$$

$$\text{QED}: \qquad \begin{aligned} \mathscr{H}_{\mathrm{I}}(x) &= -e\ \mathrm{N}\{\bar{\psi}(x)\boldsymbol{A}(x)\psi(x)\} \\ &= -e\ \mathrm{N}\left\{\left(\bar{\psi}^+ + \bar{\psi}^-\right)\left(\boldsymbol{A}^+ + \boldsymbol{A}^-\right)\left(\psi^+ + \psi^-\right)\right\} \end{aligned}$$

With $\psi^+\left(\bar{\psi}^-\right), \bar{\psi}^+\left(\psi^-\right)$and $A^+\left(A^-\right)$being linear in absorption (creation) operators of electrons, positrons and photons, respectively. It gives rise to 8 basic processes.

(тут еще есть разложение S матрицы до 2го порядка, там много членов от теоремы Вика, пока не пишу их, мб это не нужно)

S-matrix expansion

(четко напишу, что так-то её раскладыаем)

О S-матрице по ПШ (?!?!!)

(!!! скоро додумаю и напишу нормаьлно!!!)

$$\langle \mathbf{p}_1\mathbf{p}_2...|S|\mathbf{k}_{\mathscr{A}}\mathbf{k}_{\mathscr{B}}\rangle = \lim_{T\to\infty}\left\langle \mathbf{p}_1\mathbf{p}_2... \left| e^{-iH(2T)}\right| \mathbf{k}_{\mathscr{A}}\mathbf{k}_{\mathscr{B}}\right\rangle$$

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle.$$

На этот раз мы хотим найти соотношение вида

$$|\mathbf{k}_{\mathscr{A}}\mathbf{k}_{\mathscr{B}}\rangle \propto \lim_{T\to\infty(1-i\varepsilon)} e^{-iHT}\left|\mathbf{k}_{\mathscr{A}}\mathbf{k}_{\mathscr{B}}\right\rangle_0,$$

$$\lim_{T\to\infty(1-i\varepsilon)}\langle \left\langle \mathbf{p}_1...\mathbf{p}_n \left| e^{-iH(2T)}\right| \mathbf{p}_{\mathscr{A}}\mathbf{p}_{\mathscr{B}}\right\rangle_0 \propto$$

$$\propto \lim_{T\to\infty(1-i\varepsilon)} \left\langle \mathbf{p}_1...\mathbf{p}_n \left| T\left(\exp\left[-i\int\limits_{-T}^{T} dtH_I(t)\right]\right)\right| \mathbf{p}_{\mathscr{A}}\mathbf{p}_{\mathscr{B}}\right\rangle_0.$$

$$\langle \mathbf{p}_1...\mathbf{p}_n|iT|\mathbf{p}_{\mathscr{A}}\mathbf{p}_{\mathscr{B}}\rangle =$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \left({}_0\left\langle \mathbf{p}_1...\mathbf{p}_n \left| T\left(\exp\left[-i\int\limits_{-T}^{T} dtH_I(t)\right]\right)\right| \mathbf{p}_{\mathscr{A}}\mathbf{p}_{\mathscr{B}}\right\rangle_0\right)_{\substack{\text{амputированная}\\ \text{связная,}}}.$$

Сначала следует понять, как представить матричный элемент из (4.90) в виде суммы диаграмм Фейнмана. Вычислим явно несколько первых слагаемых в ϕ^4-теории для случая двух частиц в конечном состоянии. Первое слагаемое будет равно:

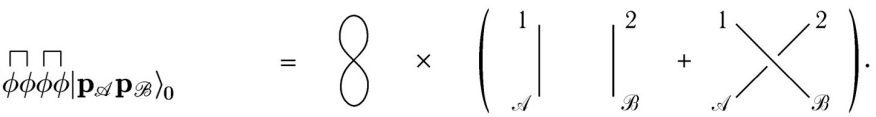
$${}_0\langle \mathbf{p}_1\mathbf{p}_2|\mathbf{p}_{\mathscr{A}}\mathbf{p}_{\mathscr{B}}\rangle_0 = \sqrt{2E_1 2E_2 2E_{\mathscr{A}} 2E_{\mathscr{B}}}\left\langle 0\left| a_1a_2a^\dagger_{\mathscr{A}}a^\dagger_{\mathscr{B}}\right| 0\right\rangle =$$
$$= 2E_{\mathscr{A}}2E_{\mathscr{B}}(2\pi)^6\left(\delta\left(\mathbf{p}_{\mathscr{A}}-\mathbf{p}_1\right)\delta\left(\mathbf{p}_{\mathscr{B}}-\mathbf{p}_2\right)+\delta\left(\mathbf{p}_{\mathscr{A}}-\mathbf{p}_2\right)\delta\left(\mathbf{p}_{\mathscr{B}}-\mathbf{p}_1\right)\right).$$

(тут две диаграммы, параллельные прямые и крест без пересечения)
Используя теорему Вика, получим следующее слагаемое в $\langle \mathbf{p}_1\mathbf{p}_2|S|\mathbf{p}_{\mathscr{A}}\mathbf{p}_{\mathscr{B}}\rangle$:

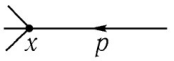
$${}_0\langle \mathbf{p}_1\mathbf{p}_2|\, T\left(-i\frac{\lambda}{4!}\int d^4x\phi_I^4(x)\right)|\mathbf{p}_{\mathscr{A}}\mathbf{p}_{\mathscr{B}}\rangle_0 =$$
$$= {}_0\langle \mathbf{p}_1\mathbf{p}_2|\, N\left(-i\frac{\lambda}{4!}\int d^4x\phi_I^4(x) + \text{ свертки }\right)|\mathbf{p}_{\mathscr{A}}\mathbf{p}_{\mathscr{B}}\rangle_0.$$

$$\phi^+{}_I(x)|\mathbf{p}\rangle_0 = \int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}a_{\mathbf{k}}e^{-ik\cdot x}\sqrt{2E_{\mathbf{p}}}a^\dagger_{\mathbf{p}}|0\rangle =$$
$$= \int\frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}e^{-ik\cdot x}\sqrt{2E_{\mathbf{p}}}(2\pi)^3\delta(\mathbf{k}-\mathbf{p})|0\rangle =$$
$$= e^{-ip\cdot x}|0\rangle.$$

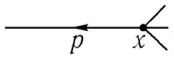
$$-i\frac{\lambda}{4!}\int d^4x\,{}_0\langle \mathbf{p_1}\mathbf{p_2}|\overbrace{\phi\phi\phi\phi}|\mathbf{p}_{\mathscr{A}}\mathbf{p}_{\mathscr{B}}\rangle_0 =$$



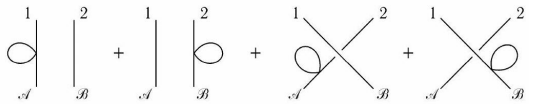
$$\overbrace{\phi_I(x)|\mathbf{p}\rangle} =$$



$$\langle\mathbf{p}|\overbrace{\phi_I(x)} =$$



$\overbrace{\phi\phi\phi\phi}$ Приводит к



(еще чуть текста добавлю!)

Например, для $\phi\phi\phi\phi$. мы должны свернуть 2 операторов ϕ с $|\mathbf{p}_{\mathscr{A}}\mathbf{p}_{\mathscr{B}}\rangle$ и другие два - с $\langle\mathbf{p}_1\mathbf{p}_2|$. Для этого существует 4! способов. Таким образом, получаем диаграмму:

$$\chi^2_{\mathscr{B}} = (4!)\cdot\left(-i\frac{\lambda}{4!}\right)\int d^4x e^{-i(p_{\mathscr{A}}+p_{\mathscr{B}}-p_1-p_2)\cdot x} =$$
$$= -i\lambda(2\pi)^4\delta^{(4)}\left(p_{\mathscr{A}}+p_{\mathscr{B}}-p_1-p_2\right).$$

Это выражение в точности имеет вид $i\mathscr{M}(2\pi)^4\delta^{(4)}\left(p_{\mathscr{A}}+p_{\mathscr{B}}-p_1-p_2\right)$ с $\mathscr{M} = -\lambda$. Прежде чем продолжить обсуждение диаграмм Фейнмана для элементов S-матрицы, применим полученный результат к сечению рассеяния. Для рассеяния в системе центра инерции можно просто подставить $|\mathscr{M}|^2 = \lambda^2$ в (4.85), в результате:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{СМ}} = \frac{\lambda^2}{64\pi^2 E^2_{\text{ци}}}.$$

Таким образом, мы вычислили наше первое сечение рассеяния в квантовой теории поля. (!!!! все, это раза 3 нужно проработать!!!)

$$\sigma_{\text{total}} = \frac{\lambda^2}{32\pi E^2_{\text{ци}}}.$$

О других разложениях S-матрицы (?!?!!)

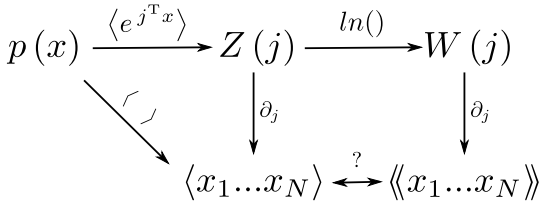
(много что может быть, важное указание)

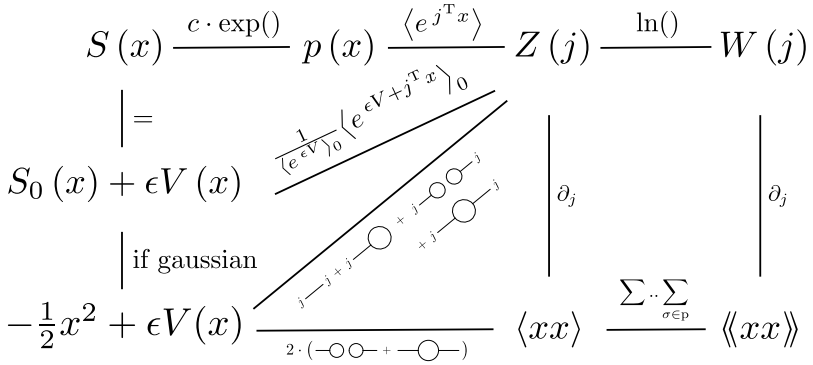
О разложении в квантовой оптике (!??!)

(важное указание, скорее всего только текстом будет, мб пара формул еще, что там важно понимать. слишком тут много особенностей, мб потом во 2й 1й напишу про это.)

О моментах, коммулянтах (????)

(см. статфиз про это, это большие, глубокие темы)





Feynman Rules for QED (in Feynman gauge)

• Each interaction vertex has two fermion legs and one photon leg, and gives the factor $-ie\gamma^\mu$. Here, the spinor indices are contracted along the fermion lines as usual, while the Lorentz indices are contracted along photon lines. Momentum is conserved at each vertex and loop momenta are integrated as usual.

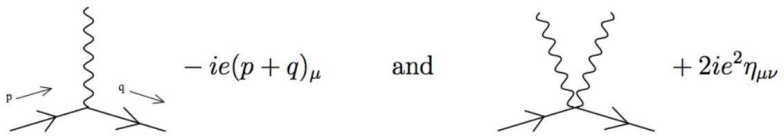
• Internal lines get the Feynman propagators,

$$\text{fermion: } \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}, \qquad \text{photon: } \frac{-i\eta^{\mu\nu}}{p^2+i\epsilon}$$

• External fermion receive the usual factors, while incoming and outgoing photons of momentum $\mathbf{p}$ receive factors $\epsilon_\mu(\mathbf{p})$ and $\epsilon^*_\mu(\mathbf{p})$, respectively. As with the fermion factors, this can be derived from the LSZ analysis for photon fields or less rigorously in the interaction picture.

• Strictly speaking, both external fermion and photon lines should receive factors of $\sqrt{Z_e}$ and $\sqrt{Z_A}$, respectively. We ignore these issues for now because we are focusing on tree-level QED.

• For charged scalars, we have the Feynman rules



The factor of 2 arises because the $A^\mu A_\mu \phi^\dagger \phi$ term has two identical particles but no compensating factor of 1/2!. Note that for the sign to be right in the first term, the momentum arrows need to be oriented along the particle flow arrows.

Note. If we had quantized in Coulomb gauge, we would have more work to do. In the presence of matter, we no longer have $A_0 = 0$. Instead we find $\nabla^2 A_0 = -ej^0$ and we may eliminate A_0 by substituting

$$A_0(\mathbf{x},t) = e\int d\mathbf{x}' \frac{j^0(\mathbf{x}',t)}{4\pi|\mathbf{x}-\mathbf{x}'|}$$

However, this makes the action nonlocal in j. Thus we have both a non-Lorentz invariant propagator and a nonlocal current-current interaction. With some effort, they can be combined into the Lorentz invariant propagator above.

Symmetries give us some useful identities.

• If we had quantized with a general ξ above, we would have found the propagator

$$-\frac{i}{p^2+i\epsilon}\left(\eta^{\mu\nu}+(\xi-1)\frac{p^\mu p^\nu}{p^2}\right)$$

We will show later that we always get the same quantum theory regardless of the value of ξ.

• For now, we can show that ξ doesn't matter at tree level. For concreteness, consider s-channel electron-positron scattering, with incoming momenta p and q. Then the photon has momentum $p+q$, and the extra term above contributes

$$\bar{v}(p)\gamma^\mu(p_\mu+q_\mu)u(q) = (\bar{v}(p)\not{p})u(q) + \bar{v}(p)(\not{q}u(q)) = (m-m)\bar{v}(p)u(q) = 0$$

where we used the on-shell spinor conditions,

$$(\not{p}-m)u(\mathbf{p}) = \bar{u}(\mathbf{p})(\not{p}-m) = (\not{p}+m)v(\mathbf{p}) = \bar{v}(\mathbf{p})(\not{p}+m) = 0$$

Similar reasoning holds for all tree-level diagrams.

• The Ward identity gives a useful result for polarization sums. Define $\mathcal{M}^\mu$ as above and let $k^\mu \propto (1,0,0,1)$. Then the Ward identity implies

$$k_\mu\mathcal{M}^\mu = 0, \quad \mathcal{M}^0 = \mathcal{M}^3$$

Now, the sum over physical polarizations is

$$\sum_{\lambda=1}^{2}\left|\epsilon_\mu^\lambda\mathcal{M}^\mu\right|^2 = \left|\mathcal{M}^1\right|^2+\left|\mathcal{M}^2\right|^2 = \left|\mathcal{M}^1\right|^2+\left|\mathcal{M}^2\right|^2+\left|\mathcal{M}^3\right|^2-\left|\mathcal{M}^0\right|^2 = -\eta_{\mu\nu}\mathcal{M}^\mu\mathcal{M}^\nu$$

by the Ward identity, so we can replace a sum over physical polarizations with a sum over all polarizations; this will be useful in some QED calculations.

• This result shows why it is possible to ignore the negative-norm states: the amplitudes to produce the unphysical states cancel out. Let P be the projector onto the space of transverse polarizations. Then our result above heuristically says

$$S^\dagger PS = S^\dagger S$$

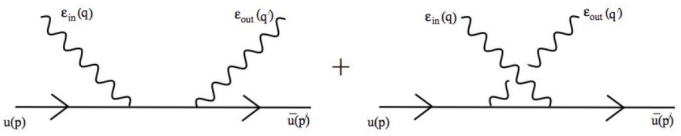
Using the unitarity of the S-matrix on the entire Fock space, we have $(PSP)^\dagger(PSP) = P$. Then the restriction of the S-matrix to the subspace of transverse polarizations is unitary, as desired.

• Note that unphysical polarizations are still produced in intermediate states, as shown by the photon propagator being proportional to $\eta_{\mu\nu}$. We cannot simply ignore these unphysical polarizations entirely, because the transverse polarizations alone are not Lorentz invariant; one can perform a Lorentz transformation on the field to convert a transverse polarization into an

unphysical one. That is, gauges that restrict to transverse polarizations only, such as Coulomb gauge, are necessarily not Lorentz invariant.

• In the case of a massive vector field, there is no gauge symmetry; instead we have three physical polarizations and a single negative-norm polarization with $\epsilon^\mu(k) \propto k^\mu$. In this case, a direct application of the Ward identity $k_\mu \mathcal{M}^\mu = 0$, which still holds, is enough to show the unphysical state decouples.

Example. Compton scattering. Two diagrams contribute to the amplitude $i\mathcal{M}$, as shown below.



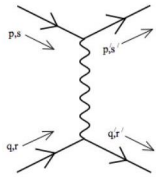
$$= i(-ie)^2 \bar{u}^{r'}(\vec{p}') \left(\frac{\gamma_\mu (\not{p} + \not{q} + m)\gamma_\nu}{(p+q)^2 - m^2} + \frac{\gamma_\nu (\not{p} - \not{q}' + m)\gamma_\mu}{(p-q')^2 - m^2} \right) u^s(\vec{p})\, \epsilon^\nu_{\text{in}}\, \epsilon^\mu_{\text{out}}$$

If we wish to sum over final polarizations and average over initial polarizations, we could use the identity derived above from the Ward identity. Alternatively, we can use this result to check the Ward identity. Setting $\epsilon_{\text{in}} = q$, we get

$$i\mathcal{M} \propto \frac{2p \cdot q}{(p+q)^2 - m^2} + \frac{2p' \cdot q}{(p'-q)^2 - m^2} = 1 - 1 = 0$$

where we used the on-shell spinor equations and $q^2 = \not{q}q = 0$. Note that the Ward identity doesn't hold for each diagram individually; it is a result for amplitudes, not diagrams.

• The same sign flip can be found in scalar QED. E.g. we have:



$$= -i\eta_{\mu\nu}(-ie)^2 \frac{(p+p')^\mu (q+q')_\nu}{(p'-p)^2} \;\to\; -i(-ie)^2 \frac{(2m)^2}{-(\vec{p}-\vec{p}')^2}$$

Then particles with the same charge repel, and again only A_0 contributes. If we flipped the charge of one of the incoming scalars, then we would get a sign flip because the momentum factors are correlated with the charge flow, giving attraction.

Note. More intuition for the sign. We consider coupling to a current j^μ, so that at lowest order the energy of interaction of two currents is

$$E \sim \int dk J^\mu(k)^* D_{\mu\nu}(k) J^\nu(k)$$

We work with a massive photon for simplicity. Naively, since each component of A_μ satisfies the Klein-Gordan equation, we should sum over all four polarizations, giving a numerator of $\eta_{\mu\nu}$. However, since we automatically have $\partial_\mu A^\mu = 0$, the polarizations satisfy $k^\mu \epsilon_\mu = 0$, or equivalently we are missing the "timelike" polarization $\epsilon^\mu \sim k^\mu$, called so because it is $(1,0,0,0)$ in the particle's rest frame. Then in a general frame, we have

$$\sum_\lambda \epsilon^{\mu *}_\lambda(k) \epsilon^\nu_\lambda(k) = -\eta^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}$$

This gives the propagator for a massive vector,

$$D_{\mu\nu}(k) = \frac{-i\eta_{\mu\nu} + ik_\mu k_\nu/m^2}{k^2 - m^2 + i\epsilon}$$

Now we take the mass to zero. For a massless photon, we can only couple to a conserved current, so that $k_\mu J^\mu(k) = 0$ and the singular second term vanishes; then the spacelike components have the usual sign and the timelike component has the opposite sign. This is why parallel currents attract in electromagnetism but like charges repel!

1.1.5 Kinematics, cross sections, decay widths, form factors

О кинематике, переменных Мандельштама, переходах в системы отсчета для распадов и столкновений

(это же в СТО по теорполу, но тут по сути и для типичных задач по КТП)
For $2 \to 2$ scattering:

$$s = (p_1 + p_2)^2 = (p_1' + p_2')^2$$
$$t = (p_1 - p_1')^2 = (p_2 - p_2')^2$$
$$u = (p_1 - p_2')^2 = (p_2 - p_1')^2$$

E.g. for the process $a + b \to c + d$ with Mandelstam variables s, t, and u and scattering amplitude $f(s, t, u)$, crossing symmetry implies that the amplitude for $\bar{c} + b \to \bar{a} + d$, in terms of its Mandelstam variables s', t', u', is $f(u', t', s')$. An equivalent way of saying this is that upon expressing (s', t', u') back in terms of (s, t, u), the amplitude is the exact same analytic function $f(s, t, u)$.

In the center of mass frame, so that the initial two particles have four-momenta

$$p_1 = (E, 0, 0, p) \text{ and } p_2 = (E, 0, 0, -p)$$

The particles then scatter at some angle θ and leave with momenta

$$p_1' = (E, 0, p\sin\theta, p\cos\theta) \text{ and } p_2' = (E, 0, -p\sin\theta, -p\cos\theta)$$

Then from the above definitions, we have that

$$s = 4E^2 \text{ and } t = -2p^2(1 - \cos\theta) \text{ and } u = -2p^2(1 + \cos\theta)$$

О сечениях рассеяния

О том, что такое вообще сечение?

$$\text{Число событий } = \sigma \ell_{\mathscr{A}} \ell_{\mathscr{B}} \int d^2x \rho_{\mathscr{A}}(x) \rho_{\mathscr{B}}(x).$$

Для постоянных плотностей пучков и для эффективной площади A сечения пучков:

$$\text{Число событий } = \frac{\sigma N_{\mathscr{A}} N_{\mathscr{B}}}{A},$$

где $N_{\mathscr{A}}$ и $N_{\mathscr{B}}$ - полное число частиц типа $\mathscr{A}$ и $\mathscr{B}$.

Более простая измеримая величина - это вероятность распада Γ нестабильной частицы $\mathscr{A}$ (которая считается покоящейся) в определенное конечное состояние (из двух или более частиц). Эта вероятность определяется как

$$\Gamma \equiv \frac{\text{Число распадов в единицу времени}}{\text{Число имеющихся частиц типа } \mathscr{A}}.$$

$$\sigma \propto \frac{1}{\left(E - E_0\right)^2 + \Gamma^2/4}$$

для массивных:

$$\frac{1}{p^2 - m^2 + im\Gamma} \approx \frac{1}{2E_{\mathbf{p}}\left(p^0 - E_{\mathbf{p}} + i\left(m/E_{\mathbf{p}}\right)\Gamma/2\right)}$$

Сечения по лекции 5 Киселева (????)

Дифференциальная вероятность процесса в единицу времени вычисляется возведением в квадрат модуля амплитуды и делением его на время наблюдения за системой, а также делением на нормировку состояния мишени, которой мы можем считать, например, античастицу, с последующим домножением на плотность состояний в конечном состоянии

$$\mathrm{d}w = \frac{|\mathcal{M}|^2}{\Delta t}\frac{1}{2E_1' V_{[3]}}\frac{\mathrm{d}^3 k_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3 k_2'}{(2\pi)^3 2E_2'},$$

откуда

$$\mathrm{d}w = \frac{\Lambda_!^2 \Lambda_2^2}{\left(s - M^2\right)^2 + M^2\Gamma^2}\frac{1}{2E_1'}(2\pi)^4 \delta\left(k_1 + k_1' - k_2 - k_2'\right)\frac{\mathrm{d}^3 k_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3 k_2'}{(2\pi)^3 2E_2'}$$

Важные формулы для сечения в случае скалярных частиц по ПШ (????)

Волновой пакет, представляющий некоторое заданное состояние $|\phi\rangle$, может быть представлен в виде:

$$|\phi\rangle = \int \frac{d^3 k}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{k}}}}\phi(\mathbf{k})|\mathbf{k}\rangle,$$

где $\phi(\mathbf{k})$ - фурье-образ пространственной волновой функции, а $|\mathbf{k}\rangle$ - одночастичное состояние с импульсом $\mathbf{k}$ в теории с взаимодействием. В свободной теории мы имели бы $|\mathbf{k}\rangle = \sqrt{2E_{\mathbf{k}}}a_{\mathbf{k}}^\dagger|0\rangle$. Множитель $\sqrt{2E_{\mathbf{k}}}$ преобразует релятивистскую нормировку $|\mathbf{k}\rangle$ в стандартную:

$$\langle\phi|\phi\rangle = 1, \quad \text{если} \quad \int \frac{d^3 k}{(2\pi)^3}|\phi(\mathbf{k})|^2 = 1$$

(??? про все это еще раз нужно подумать, теор части переписать!!!)

Вероятность, которую мы хотим вычислить, равна:

$$\mathscr{P} = \left|\left\langle \underbrace{\phi_1\phi_2...}_{\text{будущее}} \middle| \underbrace{\phi_{\mathscr{A}}\phi_{\mathscr{B}}}_{\text{прошлое}} \right\rangle\right|^2,$$

где $|\phi_{\mathscr{A}}\phi_{\mathscr{B}}\rangle$ - состояние двух волновых пакетов в далеком прошлом и $\langle\phi_1\phi_2...|$ состояние нескольких волновых пакетов в далеком будущем.

$$|\phi_{\mathscr{A}}\phi_{\mathscr{B}}\rangle_{\text{in}} = \int \frac{d^3k_{\mathscr{A}}}{(2\pi)^3}\int \frac{d^3k_{\mathscr{B}}}{(2\pi)^3}\frac{\phi_{\mathscr{A}}(\mathbf{k}_{\mathscr{A}})\,\phi_{\mathscr{B}}(\mathbf{k}_{\mathscr{B}})\,e^{-i\mathbf{b}\cdot\mathbf{k}_{\mathscr{B}}}}{\sqrt{(2E_{\mathscr{A}})\,(2E_{\mathscr{B}})}}\,|\mathbf{k}_{\mathscr{A}}\mathbf{k}_{\mathscr{B}}\rangle_{\text{in}}\ .$$

$$_{\text{out}}\,\langle\phi_1\phi_2...| = \left(\prod_f \int \frac{d^3p_f}{(2\pi)^3}\frac{\phi_f(\mathbf{p}_f)}{\sqrt{2E_f}}\right)_{\text{out}}\langle\mathbf{p}_1\mathbf{p}_2...|$$

(?????)

$$_{\text{out}}\,\langle\mathbf{p}_1\mathbf{p}_2...|\mathbf{k}_{\mathscr{A}}\mathbf{k}_{\mathscr{B}}\rangle_{\text{in}} = \lim_{T\to\infty}\langle\underbrace{\mathbf{p}_1\mathbf{p}_2...}_{T}|\underbrace{\mathbf{k}_{\mathscr{A}}\mathbf{k}_{\mathscr{B}}}_{-T}\rangle = \\ = \lim_{T\to\infty}\left\langle\mathbf{p}_1\mathbf{p}_2...\left|e^{-iH(2T)}\right|\mathbf{k}_{\mathscr{A}}\mathbf{k}_{\mathscr{B}}\right\rangle.$$

Определение S-матрицы: (??напишу!)

$$_{\text{out}}\,\langle\mathbf{p}_1\mathbf{p}_2...|\mathbf{k}_{\mathscr{A}}\mathbf{k}_{\mathscr{B}}\rangle_{\text{in}} \equiv \langle\mathbf{p}_1\mathbf{p}_2...|S|\mathbf{k}_{\mathscr{A}}\mathbf{k}_{\mathscr{B}}\rangle$$

Выделяя множитель с T-матрицей, мы определяем инвариантный матричный элемент $\mathscr{M}$:

$$\langle\mathbf{p}_1\mathbf{p}_2...|iT|\mathbf{k}_{\mathscr{A}}\mathbf{k}_{\mathscr{B}}\rangle = (2\pi)^4\delta^{(4)}\left(k_{\mathscr{A}}+k_{\mathscr{B}}-\sum p_f\right)\cdot i\mathscr{M}\left(k_{\mathscr{A}},k_{\mathscr{B}}\to p_f\right).$$

$$\mathscr{P}(\mathscr{A}\mathscr{B}\to 12...n) = \left(\prod_f\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)|\langle\mathbf{p}_1...\mathbf{p}_n|\phi_A\phi_B\rangle|^2$$

$$N = \sum_{\substack{\text{все падающие}\\ \text{частицы } i}}\mathscr{P}_i = \int d^2bn_{\mathscr{B}}\mathscr{P}(\mathbf{b}),$$

$$\sigma = \frac{N}{n_{\mathscr{B}}N_{\mathscr{A}}} = \frac{N}{n_{\mathscr{B}}\cdot 1} = \int d^2b\mathscr{P}(\mathbf{b}).$$

$$\int d\bar{k}^z_{\mathscr{A}}d\bar{k}^z_{\mathscr{B}}\delta\left(\bar{k}^z_{\mathscr{A}}+\bar{k}^z_{\mathscr{B}}-\sum p^z_f\right)\delta\left(\bar{E}_{\mathscr{A}}+\bar{E}_{\mathscr{B}}-\sum E_f\right) = \\ = \int d\bar{k}^z_{\mathscr{A}}\delta\left(\sqrt{\bar{k}^2_{\mathscr{A}}+m^2_{\mathscr{A}}}+\sqrt{\bar{k}^2_{\mathscr{B}}+m^2_{\mathscr{B}}}-\sum E_f\right)\Bigg|_{\bar{k}^z_{\mathscr{B}}=\Sigma p^z_f-\bar{k}^z} = \\ = \frac{1}{\left|\frac{\bar{k}^z_{\mathscr{A}}}{\bar{E}_{\mathscr{A}}}-\frac{\bar{k}^z_{\mathscr{B}}}{\bar{E}_{\mathscr{B}}}\right|} \equiv \frac{1}{|v_{\mathscr{A}}-v_{\mathscr{B}}|}.$$

(???? тут мб очень важная связь с рел механикой, абзац про это напишу!!! чтобы показать, что мы имеем именно обобщение рел механики!!!)

$$d\sigma = \left(\prod_f\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\frac{|\mathscr{M}(p_{\mathscr{A}},p_{\mathscr{B}}\to\{p_f\})|^2}{2E_{\mathscr{A}}2E_{\mathscr{B}}|v_{\mathscr{A}}-v_{\mathscr{B}}|}\int\frac{d^3k_{\mathscr{A}}}{(2\pi)^3}\int\frac{d^3k_{\mathscr{B}}}{(2\pi)^3}\times \\ \times|\phi_{\mathscr{A}}(\mathbf{k}_{\mathscr{A}})|^2|\phi_{\mathscr{B}}(\mathbf{k}_{\mathscr{B}})|^2(2\pi)^4\delta^{(4)}\left(k_{\mathscr{A}}+k_{\mathscr{B}}-\sum p_f\right).$$

$$d\sigma = \frac{1}{2E_{\mathscr{A}}2E_{\mathscr{B}}|v_{\mathscr{A}}-v_{\mathscr{B}}|}\left(\prod_f\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)|\mathscr{M}(p_{\mathscr{A}},p_{\mathscr{B}}\to\{p_f\})|^2(2\pi)^4\delta^{(4)}\left(p_{\mathscr{A}}+p_{\mathscr{B}}-\sum p_f\right)$$

$$\int d\Pi_n = \left(\prod_f \int \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)(2\pi)^4\delta^{(4)}\left(P - \sum p_f\right),$$

$$\frac{1}{E_{\mathscr{A}}E_{\mathscr{B}}\left|v_{\mathscr{A}} - v_{\mathscr{B}}\right|} = \frac{1}{\left|E_{\mathscr{B}}p^z_{\mathscr{A}} - E_{\mathscr{A}}p^z_{\mathscr{B}}\right|} = \frac{1}{\left|\varepsilon_{\mu x y \nu}p^\mu_{\mathscr{A}}p^\nu_{\mathscr{B}}\right|}$$

$$\int d\Pi_2 = \int \frac{dp_1 p_1^2 d\Omega}{(2\pi)^3 2E_1 2E_2}(2\pi)\delta\left(E_{\text{ци}} - E_1 - E_2\right),$$

$$\int d\Pi_2 = \int d\cos\theta \frac{1}{16\pi}\frac{2\left|\mathbf{p}_1\right|}{E_{\text{ци}}}.$$

Сечение рассеяния для двух частиц в конечном состоянии:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{СМ}} = \frac{1}{2E_{\mathscr{A}}2E_{\mathscr{B}}\left|v_{\mathscr{A}} - v_{\mathscr{B}}\right|}\frac{\left|\mathbf{p}_1\right|}{(2\pi)^2 4E_{\text{ци}}}\left|\mathscr{M}\left(p_{\mathscr{A}}, p_{\mathscr{B}} \to p_1, p_2\right)\right|^2.$$

В частном случае, когда все четыре частицы имеют одинаковые массы (включая обычно рассматриваемый предел $m \to 0$), последнее выражение сводится к формуле, приведенной в главе 1 ПШ (????):

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{СМ}} = \frac{\left|\mathscr{M}\right|^2}{64\pi^2 E^2_{\text{ци}}} \quad \text{(все четыре массы одинаковы).}$$

$$d\Gamma = \frac{1}{2m_{\mathscr{A}}}\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\left|\mathscr{M}\left(m_{\mathscr{A}} \to \{p_f\}\right)\right|^2(2\pi)^4\delta^{(4)}\left(p_{\mathscr{A}} - \sum p_f\right).$$

(пара общих абзацев о смысле)

Методы быстро оценить зависимость сечения от чего-то или его величину (!!!!!!!???)

(в космологии, кстати, такое с ходу и нужно, так что сделаю заготовку!!! мб у Мигдала Крайнова есть про это ответы?)

Final steps for the cross section

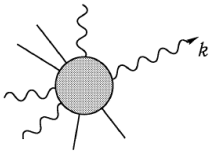
Суммирование по поляризациям фотона (?????)

(пропишу это, все-таки это часто нужно применять. не такой первый параграф, тем не менее, вначале, ибо важный.)

$\Sigma u(p)\bar{u}(p) = \not{p} + m$,

$$\sum_{\text{поляризация}} \varepsilon^*_\mu \varepsilon_\nu \longrightarrow -g_{\mu\nu}$$

Proof: (????)



$= i\mathscr{M}(k) \equiv i\mathscr{M}^\mu(k)\varepsilon^*_\mu(k).$

$$\sigma \sim \sum_{\varepsilon} \left|\varepsilon_\mu^*(k)\mathscr{M}^\mu(k)\right|^2 = \sum_{\varepsilon} \varepsilon_\mu^*\varepsilon_\nu \mathscr{M}^\mu(k)\mathscr{M}^{\nu*}(k)$$

If $k^\mu = (k,0,0,k)$, then $\varepsilon_1^\mu = (0,1,0,0)$; $\quad \varepsilon_2^\mu = (0,0,1,0)$,

$$\sum_{\varepsilon} \left|\varepsilon_\mu^*(k)\mathscr{M}^\mu(k)\right|^2 = \left|\mathscr{M}^1(k)\right|^2 + \left|\mathscr{M}^2(k)\right|^2$$

Внешние фотонные линии возникают от слагаемого с взаимодействием $\int d^4x e j^\mu A_\mu$, $j^\mu = \bar{\psi}\gamma^\mu\psi$.

$$\mathscr{M}^\mu(k) = \int d^4x e^{ik\cdot x} \left\langle f \left| j^\mu(x) \right| i \right\rangle$$

где начальное и конечное состояние включают все частицы, кроме выделенного фотона.

Из классических уравнений движения следует, что ток j^μ сохраняется: $\partial_\mu j^\mu(x) = 0$. Предполагая, что это утверждение справедливо и в квантовой теории, можно умножить (5.78) на k_μ и получить:

$$k_\mu \mathscr{M}^\mu(k) = 0$$

Амплитуда $\mathscr{M}$ обращается в нуль, если заменить вектор поляризации фотона $\varepsilon_\mu(k)$ на k_μ. Это соотношение известно как знаменитое тождество Уорда. Оно является следствием калибровочной симметрии (4.6) КЭД (????).

Полезно непосредственно проверить, что комптоновская амплитуда (5.74) удовлетворяет тождеству Уорда.

Для этого заменим $\varepsilon_\nu(k)$ на k_ν или $\varepsilon_\mu^*(k')$ на k'_μ, а затем вычислим произведения дираковских матриц.

После несложных алгебраических преобразований слагаемые, соответствующие двум разным диаграммам, сократятся, и в итоге получится нуль.

Вернемся к выводу формулы (5.75) для суммы по поляризациям.

Отметим, что если $k^\mu = (k,0,0,k)$, тождество Уорда принимает вид:

$$k\mathscr{M}^0(k) - k\mathscr{M}^3(k) = 0$$

Следовательно, $\mathscr{M}^0 = \mathscr{M}^3$ и

$$\sum_{\varepsilon} \varepsilon_\mu^*\varepsilon_\nu \mathscr{M}^\mu(k)\mathscr{M}^{\nu*}(k) = \left|\mathscr{M}^1\right|^2 + \left|\mathscr{M}^2\right|^2 = \left|\mathscr{M}^1\right|^2 + \left|\mathscr{M}^2\right|^2 + \left|\mathscr{M}^3\right|^2 - \left|\mathscr{M}^0\right|^2 =$$

$$= -g_{\mu\nu}\mathscr{M}^\mu(k)\mathscr{M}^{\nu*}(k)$$

Поэтому в вычислениях КЭД можно выбрасывать нефизические времениподобные и продольные фотоны, так как для любого процесса квадраты амплитуд рождения таких состояний сокращают друг друга. (!??!??) Отрицательная норма состояния времениподобного фотона см(???)(4.134), играет в этом сокращении существенную роль (???).

Form Factors

- Consider the scattering of an electron in the static potential due to an extended charge distribution.

-The potential at $\vec{r}$ from the centre is given by:

$$V(\vec{r}) = \int \frac{Q\rho\left(\vec{r}'\right)}{4\pi\left|\vec{r}-\vec{r}'\right|} d^3\vec{r}' \quad \text{with} \int \rho(\vec{r})\mathrm{d}^3\vec{r} = 1$$

- In first order perturbation theory the matrix element is given by:

$$\begin{aligned} M_{fi} &= \langle\psi_f|V(\vec{r})|\psi_i\rangle = \int e^{-i\vec{p}_3\cdot\vec{r}} V(\vec{r}) e^{i\vec{p}_1\cdot\vec{r}}\, \mathrm{d}^3\vec{r} \\ &= \iint e^{iq\cdot r_r} \frac{Q\rho\left(\vec{r}'\right)}{4\pi\left|\vec{r}-\vec{r}'\right|} \mathrm{d}^3\vec{r}'\mathrm{d}^3\vec{r} = \iint e^{i\dot{r}\cdot(\dot{r}'-r')} e^{iq\cdot r'} \frac{Q\rho\left(\vec{r}'\right)}{4\pi\left|\vec{r}-\vec{r}'\right|} \mathrm{d}^3\vec{r}'\mathrm{d}^3\vec{r} \end{aligned}$$

- Fix $\vec{r}'$ and integrate over $\mathrm{d}^3\vec{r}$ with substitution $\vec{R} = \vec{r} - \vec{r}'$

$$M_{fi} = \int e^{i\vec{q}\cdot\vec{R}} \frac{Q}{4\pi|\vec{R}|} \mathrm{d}^3\vec{R} \int \rho\left(\vec{r}'\right) e^{i\vec{q}\cdot\vec{r}'} \mathrm{d}^3\vec{r}' = \left(M_{fi}\right)_{\text{point}}\ F\left(\vec{q}^2\right)$$

⋆ The resulting matrix element is equivalent to the matrix element for scattering from a point source multiplied by the form factor

$$F\left(\vec{q}^2\right) = \int \rho(\vec{r}) e^{i\vec{q}\cdot\vec{r}}\, \mathrm{d}^3\vec{r}$$

1.1.6 Analysis of diagrams

(очень важный раздел, очень четко напишу!)

Классификация диаграмм (!!!)

(ниже уже написал, сюда тоже добавлю это)

Правила Фейнмана

обзор сильносвязных диаграмм

(????)

О свойствах диаграмм (!!!)

(там их много, заготовлю это!)

О тождестве Эйлера для графов (!??!?)

(тоже важное, пока не разобрался полностью.)

Связь связных и несвязных графов (!?!??)

(ну напишу потом, что еще...)

О тождестве Фарри для фермионной петли (????)

(??.)

Другие свойства диаграмм

Алгоритм вычисления диаграмм (!!!)

О факторах симметрии (!!!)

(!! проработаю уже это!!!)
These symmetry factors can all be found with the following rules:
• 1/2 for every vertex connected to itself.
• $1/k$! for k propagators connecting the same two vertices.
• 1/2 for two interchangeable vertices.

1.1.7 γ-matrices, spinors, density matrices

Contraction Identities

$$\begin{gathered}\gamma_\lambda\gamma^\lambda = 4,\\ \gamma_\lambda\gamma^\alpha\gamma^\lambda = -2\gamma^\alpha\\ \gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\lambda = 4g^{\alpha\beta},\\ \gamma_\lambda\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\lambda = -2\gamma^\gamma\gamma^\beta\gamma^\alpha\\ \gamma_i\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\gamma^\lambda = 2\left(\gamma^\delta\gamma^\alpha\gamma^\beta\gamma^\gamma + \gamma^\gamma\gamma^\beta\gamma^\alpha\gamma^\delta\right)\end{gathered} \quad .$$

For 4-vectors $A, B, ...$

$$\begin{gathered}\gamma_\lambda \not{A}\gamma^\lambda = -2\not{A} \qquad\qquad \not{A} := \gamma^\alpha A_\alpha:\\ \gamma_\lambda \not{A}\not{B}\gamma^\lambda = 4\not{A}\not{B},\\ \gamma_\lambda \not{A}\not{B}\not{C}\gamma^\lambda = -2\not{C}\not{B}\not{A}\\ \gamma_\lambda \not{A}\not{B}\not{C}\not{D}\gamma^\lambda = 2(\not{D}\not{A}\not{B}\not{C} + \not{C}\not{B}\not{A}\not{D})\end{gathered}$$

Traces of γ-matrices:

$$\begin{gathered}\mathrm{Tr}(\not{A}\not{B}) = 4(AB)\\ \mathrm{Tr}(\not{A}\not{B}\not{C}\not{D}) = 4\{(AB)(CD) - (AC)(BD) + (AD)(BC)\},\end{gathered}$$

and generally, if $A_1, A_2, ..., A_{2n}$ are four-vectors, then

$$\begin{aligned}\mathrm{Tr}\left(\not{A}_1 A_2 ... A_{2n}\right) = \{&(A_1A_2)\,\mathrm{Tr}\left(\not{A}_3 ... \not{A}_{2n}\right) - (A_1A_3)\,\mathrm{Tr}\left(A_2A_4 ... A_{2n}\right)\\ &+ ... + (A_1A_{2n})\,\mathrm{Tr}\left(A_2A_3 ... A_{2n-1}\right)\}\,.\end{aligned}$$

$$\not{A}\not{B} = AB - \mathrm{i}\sigma^{\alpha\beta}A_\alpha B_\beta = 2AB - \not{B}\not{A}, \qquad \begin{aligned}&\not{A}\not{A} = A^2;\\ &\not{A}\not{B} = -\not{B}\not{A}, \quad \text{if } AB = 0.\end{aligned}$$

(i) For any two $n \times n$ matrices U, V we have: $\mathrm{Tr}(UV) = \mathrm{Tr}(VU)$
(ii) If $\left(\gamma^\alpha\gamma^\beta\cdots\gamma^\mu\gamma^\upsilon\right)$ contains an odd number of γ-matrices: $\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\cdots\gamma^\mu\gamma^\upsilon\right) = 0$
(iii) For a product of an even number of γ-matrices:

$$\begin{gathered}\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\right) = 4g^{\alpha\beta}, \quad \mathrm{Tr}\,\sigma^{\alpha\beta} = 0\\ \mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta\right) = 4\left(g^{\alpha\beta}g^{\gamma\delta} - g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma}\right)'\end{gathered}$$

(iv) For any product of γ-matrices

$$\mathrm{Tr}\left(\gamma^{\alpha}\gamma^{\beta}...\gamma^{\mu}\gamma^{v}\right) = \mathrm{Tr}\left(\gamma^{v}\gamma^{\mu}...\gamma^{\beta}\gamma^{\alpha}\right)$$

whence

$$\mathrm{Tr}\left(\not{A}_1 A_2 ... A_{2n}\right) = \mathrm{Tr}\left(\not{A}_{2n} ... \not{A}_2 A_1'\right)$$

$$\left.\begin{array}{c} \mathrm{Tr}\,\gamma^5 = \mathrm{Tr}\left(\gamma^5\gamma^{\alpha}\right) = \mathrm{Tr}\left(\gamma^5\gamma^{\alpha}\gamma^{\beta}\right) = \mathrm{Tr}\left(\gamma^5\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\right) = 0 \\ \mathrm{Tr}\left(\gamma^5\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma^{\delta}\right) = -4\mathrm{i}\varepsilon^{\alpha\beta\gamma\delta} \end{array}\right\}.$$

Other results involving γ^5 are easily obtained from Eqs. (A.15)-(A.21), using Eqs. (A.7) and (A.8) which define γ^5 and state its main properties.

The following comments should suffice to enable the reader to derive Eqs. (A.16)(A.21). We obtain Eq. (A.16) by using $\left(\gamma^5\right)^2 = 1$ and Eq. (A.15), whence

$$\mathrm{Tr}\left(\gamma^{\alpha}...\gamma^{v}\right) = \mathrm{Tr}\left[\left(\gamma^5\right)^2\gamma^{\alpha}...\gamma^{v}\right] = \mathrm{Tr}\left(\gamma^5\gamma^{\alpha}...\gamma^{v}\gamma^5\right)$$

In the last trace, we use $\left[\gamma^{\mu},\gamma^5\right]_+ = 0$ to commute the left-hand γ^5 matrix through to the right-hand side, with the result $-\mathrm{Tr}\left(\gamma^{\alpha}...\gamma^{\nu}\right)$ for an odd number of factors in $\left(\gamma^{\alpha}...\gamma^{\nu}\right)$. Hence Eq. (A.16) follows.

Eqs. (A.17) are derived by repeated use of the anticommutation relations (A.5) of the γ matrices and of the cyclic property (A.15) of traces.

To derive Eq. (A.20a), we introduce the transposed matrices $\gamma^{\alpha\mathrm{T}}$. It follows from Eqs. (A.5) and (A.6) that the matrices $\left(-\gamma^{\alpha\mathrm{T}}\right)$ also satisfy the anticommutation relations (A.5) and the Hermiticity conditions (A.6). Hence by Pauli's fundamental theorem (A.61) there exists a unitary matrix C such that

$$C\gamma^{\alpha}C^{-1} = -\gamma^{\alpha\mathrm{T}}$$

Eq. (A.20a) then follows from

$$\begin{aligned}\mathrm{Tr}\left(\gamma^{\alpha}\gamma^{\beta}...\gamma^{v}\right) &= \mathrm{Tr}\left[\left(-C^{-1}\gamma^{\alpha\mathrm{T}}C\right)\left(-C^{-1}\gamma^{\beta\mathrm{T}}C\right)...\left(-C^{-1}\gamma^{\nu\mathrm{T}}C\right)\right] \\ &= \mathrm{Tr}\left(\gamma^{\alpha\mathrm{T}}\gamma^{\beta\mathrm{T}}\cdots\gamma^{\nu\mathrm{T}}\right) = \mathrm{Tr}\left[\left(\gamma^{\nu}\cdots\gamma^{\beta}\gamma^{\alpha}\right)^{\mathrm{T}}\right]\end{aligned}$$

Основные свойства спиноров

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\left(\gamma^0\right)^{\dagger} = \gamma^0$

$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$,

$$\mathrm{tr}\,(\text{odd N of } \gamma^\mu) = 0, \quad \mathrm{tr}\,\gamma^\mu\gamma^\nu = 4\eta^{\mu\nu}, \quad \mathrm{tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)$$

For pseudoscalar and pseudovector interactions, we will also need

$$\mathrm{tr}\left(\gamma^5\,(\text{ zero to three } \gamma^\mu)\right) = 0, \quad \mathrm{tr}\,\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = -4i\epsilon^{\mu\nu\rho\sigma}$$

• These identities can generally be proven by inserting the identity, in terms of $(\gamma^0)^2$ or $(\gamma^5)^2$, and using the anticommutation relations; the γ^5 identities can be shown by noting that the result must be a totally antisymmetric tensor.

A more general technique is to recall that $S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu{}_\nu\gamma^\nu$. The trace remains invariant, but the right-hand side transforms like a Lorentz tensor, or a pseudotensor if γ^5 appears. Thus, the right-hand side must be a Lorentz invariant (pseudo)tensor, which can only be built from $\eta^{\mu\nu}$ and $\epsilon^{\mu\nu\rho\sigma}$. This fixes the answer in all cases above, and is also useful for larger traces.

• We can express antisymmetrized products of gamma matrices as

$$\gamma^5 = -\frac{i}{24}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma, \quad \gamma^{[\nu}\gamma^\rho\gamma^{\sigma]} = i\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma^5$$

In fact, the identity, $\gamma^\mu, \gamma^{[\mu}\gamma^{\nu]}, \gamma^\mu\gamma^5$, and γ^5 form a basis for the space of 4×4 matrices; we used this when analyzing all Dirac bilinears above.

• Finally, we have

$$\gamma^\mu\gamma_\mu = 4, \quad \gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu, \quad \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho}, \quad \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$$

To show

$$\mathrm{tr}(\text{ odd num of } \gamma) = 0$$

First note that

$$\mathrm{tr}\,(\gamma^\mu) = 0$$

We'll also use two facts about the fifth gamma matrix γ^5 that says:

$$\left(\gamma^5\right)^2 = I_4, \quad \text{and} \quad \gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$$

So lets use these two facts to prove this identity for the first non-trivial case: the trace of three gamma matrices. Step one is to put in one pair of γ^5 's in front of the three original γ', and step two is to swap the γ^5 matrix back to the original position, after making use of the cyclicity of the trace.

$$\begin{aligned}
\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho) &= \mathrm{tr}\left(\gamma^5\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\right) \\
&= \mathrm{tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^5\right) \\
&= -\,\mathrm{tr}\left(\gamma^5\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\right) \text{ (using tr(ABC) = tr(BCA))} \\
&= -\,\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho)
\end{aligned}$$

This can only be fulfilled if

$$\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho) = 0$$

The extension to $2n+1$ (n integer) gamma matrices, is found by placing two gamma- 5 s after (say) the $2n$-th gamma-matrix in the trace, commuting one out to the right (giving a minus sign) and commuting the other gamma- $52n$ steps out to the left [with sign change $(-1)\hat{}2n = 1$]. Then

we use cyclic identity to get the two gamma-5s together, and hence they square to identity, leaving us with the trace equalling minus itself, i.e. 0.

Proof of $\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right) = -4i\epsilon^{\mu\nu\rho\sigma}$. For a proof of identity 7, the same trick still works unless $(\mu\nu\rho\sigma)$ is some permutation of (0123), so that all 4 gammas appear. The anticommutation rules imply that interchanging two of the indices changes the sign of the trace, so $\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right)$ must be proportional to $\epsilon^{\mu\nu\rho\sigma}$ ($\epsilon^{0123} = \eta^{0\mu}\eta^{1\nu}\eta^{2\rho}\eta^{3\sigma}\epsilon_{\mu\nu\rho\sigma} = \eta^{00}\eta^{11}\eta^{22}\eta^{33}\epsilon_{0123} = -1$). The proportionality constant is $4i$, as can be checked by plugging in $(\mu\nu\rho\sigma) = (0123)$, writing out γ^5, and remembering that the trace of the identity is 4.

Main spinor propertis

The Gordon identity,

$$\bar{u}_r\left(p'\right)\gamma^\mu u_s(p) = \bar{u}_r\left(p'\right)\left(\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}\left(p'_\nu - p_\nu\right)}{2m}\right)u_s(p)$$

where $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$. ((words about proof!!) This can be done using only the Dirac equation and the defining property of the gamma matrices.)

The completeness relation for the Γ matrices:

$$\delta_{ac}\delta_{db} = \frac{1}{4}\sum_I \tilde{\Gamma}^I_{dc}\Gamma^I_{ab}$$

((words about proof!!) it suffices to show that both sides give the same result when multiplied by a general matrix M_{cd}.)

The Fierz identity:

$$\tilde{\Gamma}^I_{ab}\Gamma^J_{cd} = \frac{1}{16}\sum_{KL}\mathrm{tr}\left[\tilde{\Gamma}^I\tilde{\Gamma}^K\Gamma^J\Gamma^L\right]\tilde{\Gamma}^L_{ad}\Gamma^K_{cb}$$

(words about proof!!)

(maybe I'll add some formulas from the articles about spinors!!!)

A set of relativistic classical fields transforms under a representation of the Lorentz group,

$$\phi_i(x) \to R(\Lambda)_{ij}\phi_j\left(\Lambda^{-1}x\right)$$

We would thus like to classify representations of $SO(1,3)$.

• As we've seen, an infinitesimal Lorentz transformation takes the form

$$\Lambda^\mu_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$$

where $\omega^{\mu\nu} = -\omega^{\nu\mu}$. We may index the six independent Lorentz transformations with two antisymmetric indices,

$$\left(J^{\rho\sigma}\right)^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}$$

• Then any infinitesimal Lorentz transformation can be written as

$$\omega^\mu_\nu = \frac{1}{2}\Omega_{\rho\sigma}\left(J^{\rho\sigma}\right)^\mu_\nu$$

and finite Lorentz transformations are given by exponentiation,

$$\Lambda^\mu_{\ \nu} = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}\,(J^{\rho\sigma})^\mu_{\ \nu}\right)$$

The factors of $1/2$ are canceled by the antisymmetry of Ω and J. Explicitly, $\Omega_{12} = -\Omega_{21} = \theta$ gives an active θ rotation about $\hat{\mathbf{z}}$, while $\Omega_{0i} = \chi_i$ gives an active boost with rapidity χ.

• Note that while the generators are antisymmetric with indices raised, they aren't with mixed indices, so the $\Lambda^\mu{}_\nu$ are not unitary. Alternatively, Lorentz transformations preserve $\eta_{\mu\nu}V^\mu V^\nu$ and not $\delta_{\mu\nu}V^\mu V^\nu$. In general, there are no finite-dimensional unitary representations of a noncompact Lie group like the Lorentz group.

• Some explicit computation yields the Lorentz algebra $\mathfrak{so}(1,3)$,

$$[J^{\rho\sigma}, J^{\tau\nu}] = \eta^{\sigma\tau}J^{\rho\nu} - \eta^{\rho\tau}J^{\sigma\nu} + \eta^{\rho\nu}J^{\sigma\tau} - \eta^{\sigma\nu}J^{\rho\tau}$$

We seek representations of this algebra. When we exponentiate them, we will generally find a projective representation of the Lorentz group, i.e. a representation of its double cover $\mathrm{Spin}(1,3) \cong SL(2,\mathbb{C})$. This is a bit strange, since classical fields don't have phase ambiguities like quantum states, but acceptable since the spinor field will not be directly measurable.

We construct the Dirac spinor representation starting from the Clifford algebra.

• The Clifford algebra is a set of four matrices γ^μ satisfying

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

That is, all of the γ^μ anticommute with each other, and they square as

$$\left(\gamma^0\right)^2 = 1, \quad \left(\gamma^i\right)^2 = -1$$

• The lowest-dimensional representation of the Clifford algebra is four-dimensional. We will use the Weyl/chiral representation,

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

We note the useful Pauli matrix identities

$$\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k, \quad \left\{\sigma^i, \sigma^j\right\} = 2\delta^{ij}, \quad \left[\sigma^i, \sigma^j\right] = 2i\epsilon^{ijk}\sigma^k$$

Other representations can be constructed by replacing γ^μ with $V\gamma^\mu V^{-1}$.

• The commutators of the γ^μ are

$$S^{\rho\sigma} = \frac{1}{4}\left[\gamma^\rho, \gamma^\sigma\right] = \left\{ \begin{array}{ll} 0 & \rho = \sigma \\ (1/2)\gamma^\rho\gamma^\sigma & \rho \neq \sigma \end{array} \right\} = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma}$$

We claim the $S^{\mu\nu}$ form a representation of the Lorentz algebra. First, note that

$$[S^{\mu\nu}, \gamma^\rho] = \frac{1}{2}\left[\gamma^\mu\gamma^\nu, \gamma^\rho\right] = \frac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho - \frac{1}{2}\gamma^\rho\gamma^\mu\gamma^\nu = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu}$$

where we used anticommutation to cancel the two terms, picking up $\gamma\eta$ terms. Next,

$$[S^{\mu\nu}, S^{\rho\sigma}] = \frac{1}{2}\left[S^{\mu\nu}, \gamma^\rho\gamma^\sigma\right] = \frac{1}{2}\left(\gamma^\mu\gamma^\sigma\eta^{\nu\rho} - \gamma^\nu\gamma^\sigma\eta^{\rho\mu} + \gamma^\rho\gamma^\mu\eta^{\nu\sigma} - \gamma^\rho\gamma^\nu\eta^{\sigma\mu}\right)$$

where we used the commutator product rule and the previous result. Finally, applying $\gamma^\mu\gamma^\nu = 2S^{\mu\nu} + \eta^{\mu\nu}$ gives the result. Note that the γ and S matrices contain complex numbers, so we have a complex representation.

Next, we exponentiate the $S^{\mu\nu}$ to find a projective representation of $SO(1,3)$.

• The $S^{\mu\nu}$ act on a Dirac spinor field, i.e. a field with four complex components $\psi^\alpha(x)$ so that

$$\psi^\alpha(x) \to S[\Lambda]^\alpha_\beta \psi^\beta\left(\Lambda^{-1}x\right), \quad S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)$$

• To see this is a projective representation, we explicitly work out $S[\Lambda]$ for rotations. For $i \neq j$,

$$S^{ij} = \frac{1}{2}\gamma^i\gamma^j = -\frac{i}{2}\epsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}$$

Defining the angle by $\Omega_{ij} = \epsilon_{ijk}\varphi^k$, we have

$$S[\Lambda] = \begin{pmatrix} e^{-i\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}/2} & 0\\ 0 & e^{-i\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}/2}\end{pmatrix}$$

which is -1 for a 2π rotation.

• Similarly, for boosts, we define $\Omega_{0i} = \chi_i$ and

$$S^{0i} = \frac{1}{2}\begin{pmatrix}-\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}, \quad S[\Lambda] = \begin{pmatrix} e^{-\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2} & 0\\ 0 & e^{\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2}\end{pmatrix}$$

We see the Dirac spinor splits into two components which transform identically under rotations but oppositely under boosts.

• Sources may differ on the signs of φ and χ. Here we have taken the active point of view, so χ refers to the amount we actively change the spinor's velocity. But χ can also label the relative velocity of a frame we transform into, which would give an extra minus sign.

• The Dirac spinor representation is not unitary. To see this at the level of the Clifford algebra, note that unitarity requires $S^{\mu\nu}$ to be anti-Hermitian, which works if the γ^μ are all Hermitian or all anti-Hermitian. But since $(\gamma^0)^2 = 1$ and $(\gamma^i)^2 = -1, \gamma^0$ can only be picked to be Hermitian and γ^i can only be picked to be anti-Hermitian. Again, the issue comes from the indefinite signature of η.

Weyl spinors

• To define the Weyl spinors in an invariant way, define

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Then one can show

$$\left\{\gamma^5, \gamma^\mu\right\} = 0, \quad \left(\gamma^5\right)^2 = 1$$

Given this definition, the matrices $(\gamma^\mu, -i\gamma^5)$ form a five-dimensional Clifford algebra. Moreover, choosing γ^0 to be Hermitian and γ^i to be anti-Hermitian as before, $\gamma^5 = \gamma^5$.

$[S_{\mu\nu}, \gamma^5] = 0$, so γ^5 is a 'Lorentz scalar'. Then we have

$$\gamma^5 = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \quad P_L = \frac{1}{2}\left(1 - \gamma^5\right), \quad P_R = \frac{1}{2}\left(1 + \gamma^5\right)$$

where P_L and P_R project on the left-handed and right-handed Weyl spinors.

the concept and properties of spinors

(добавлю многое, мб это уже написано тут)

1.2 Integrals for QFT

Integrals for QFT

(??)

Gaussian integrals

Example 1:

$$
\begin{aligned}
Z(J) &= \int\limits_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2q^2-\frac{\lambda}{4!}q^4+Jq} = \qquad (\lambda \approx 0) \\
&= \int\limits_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2q^2+Jq}\left[1-\frac{\lambda}{4!}q^4+\frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 q^8+...\right] \\
&= \left(1-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4+\frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\left(\frac{d}{dJ}\right)^8+...\right)\int\limits_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2q^2+Jq} \\
&= e^{-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4 \int\limits_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2q^2+Jq}} = \\
&= \left(\frac{2\pi}{m^2}\right)^{1/2} e^{-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4} e^{\frac{1}{2m^2}\cdot d^2}
\end{aligned}
$$

(????)

We use:

$$
\frac{1}{AB} = \int\limits_0^1 \frac{dx}{(xA+(1-x)B)^2},
$$

$$
\frac{1}{A_1...A_n} = \int\limits_0^1 dx_1...dx_n \delta\left(\sum_i x_i - 1\right)\frac{(n-1)!}{(x_1A_1+...+x_nA_n)^n}
$$

$$
\int\limits_0^1 dx_1...dx_n (n-1)!\delta\left(\sum_i x_i - 1\right) = 1
$$

After some simplification and a change of variables (???):

$$
i\Sigma_2(\not{p}) = e^2\int\limits_0^1 dx \int d\ell \frac{\gamma^\mu\left(\not{K}+m_0\right)\gamma_\mu}{\left(\ell^2-\Delta+i\epsilon\right)^2}, \qquad \begin{aligned} &\ell := k - xp, \\ &\Delta := -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2 \end{aligned}
$$

This form is useful because the denominator has a simple dependence on the integration variable. Note that the ℓ term in the numerator vanishes because it is odd in ℓ.

for integer d, we have

$$\int d\Omega_d \equiv \text{Vol}\left(S^{d-1}\right) = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

where $\Gamma(n) = (n-1)!$ and $\Gamma(z)$ has poles at $z = 0, -1, -2, ...$ This formula may be derived from $\pi^{d/2} = \left(\int dx e^{-x^2}\right)^d$ by evaluating the right-hand side in d-dimensional spherical coordinates. We then define this to hold for all $d \in \mathbb{C}$. Example:

$$\text{Vol}\left(S^1\right) = 2\pi, \quad \text{Vol}\left(S^2\right) = 4\pi, \quad \text{Vol}\left(S^3\right) = 2\pi^2$$

Therefore, the measure in d dimensions with spherical symmetry is

$$d^d p = \text{Vol}\left(S^{d-1}\right) \frac{p^{d-1} dp}{(2\pi)^d} = \frac{\left(p^2\right)^{d/2-1} d\left(p^2\right)}{(4\pi)^{d/2}\Gamma(d/2)}$$

Thus our integral generalized to d dimensions is

$$I_d = \frac{1}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty \frac{u^{d/2-1}}{(u+\Delta)^2} du = \frac{\Delta^{d/2-2}}{(4\pi)^{d/2}\Gamma(d/2)} \int_0^1 dx x^{1-d/2}(1-x)^{d/2-1}$$

where $x := \Delta/(u+\Delta)$, using the beta function, $B(\alpha,\beta) \equiv \int_0^1 dx x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

• More generally, for an arbitrary power of p in the numerator, we have

$$\int d^d p \frac{p^{2a}}{\left(p^2-\Delta\right)^b} = i(-1)^{a-b} \frac{1}{(4\pi)^{d/2}} \frac{1}{\Delta^{b-a-d/2}} \frac{\Gamma\left(a+\frac{d}{2}\right)\Gamma\left(b-a-\frac{d}{2}\right)}{\Gamma(b)\Gamma\left(\frac{d}{2}\right)}$$

by similar logic, where the sign factors come from flipping the negative-definite Euclidean signature.

In particular,

$$\begin{aligned}
\int d^d p \frac{1}{p^2-\Delta} &= \frac{-i}{(4\pi)^{d/2}} \frac{1}{\Delta^{1-d/2}} \Gamma\left(\frac{2-d}{2}\right), \\
\int d^d p \frac{1}{\left(p^2-\Delta\right)^2} &= \frac{i}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} \Gamma\left(\frac{4-d}{2}\right), \\
\int d^d p \frac{p^2}{\left(p^2-\Delta\right)^2} &= \frac{d}{2} \frac{-i}{(4\pi)^{d/2}} \frac{1}{\Delta^{1-d/2}} \Gamma\left(\frac{2-d}{2}\right), \\
\int d^d p \frac{p^2}{\left(p^2-\Delta\right)^3} &= \frac{d}{4} \frac{i}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} \Gamma\left(\frac{4-d}{2}\right).
\end{aligned}$$

We generally end up with a so-called loop factor $1/(4\pi)^2$, which suppresses the contribution of the loop.

• In Euclidean signature, we have the same result with no i or sign factors,

$$\int d^d k \frac{k^{2a}}{\left(k^2+\Delta\right)^b} = \frac{1}{(4\pi)^{d/2}} \frac{1}{\Delta^{b-a-d/2}} \frac{\Gamma\left(a+\frac{d}{2}\right)\Gamma\left(b-a-\frac{d}{2}\right)}{\Gamma(b)\Gamma\left(\frac{d}{2}\right)}$$

Alternate way - to use Schwinger parametrization:

$$\frac{1}{a^n} \equiv \frac{1}{\Gamma(n)} \int_0^\infty du u^{n-1} e^{-ua}$$

$$\int d^d k \frac{k^{2a}}{(k^2+\Delta)^b} = \frac{1}{\Gamma(b)} \int du u^{b-1} e^{-u\Delta} \int d^d k e^{-uk^2} k^{2a}$$

in Euclidean signature. After doing the angular integral, the inner integral is merely a gamma function after substitution, and so is the outer integral, giving the same result.

• Finally, we need to handle numerators, which involve spinors and the metric. We define

$$g^{\mu\nu} g_{\mu\nu} = d$$

and the Lorentz-invariant phase space as

$$d\Pi_{\text{LIPS}} = \phi^d \left(\sum p_j \right) \prod_j \frac{d^{d-1} p_j}{2E_{p_j}}$$

О гамма матрицах в $d-\varepsilon$-мерном пространстве

Formally, we suppose there are d four-dimensional gamma matrices satisfying $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, so that the trace of the identity in spinor space is still 4.

$$\gamma^\mu \gamma_\mu = d, \quad \gamma^\mu \gamma^\nu \gamma_\mu = (2-d)\gamma^\nu$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\nu\rho} + (d-4)\gamma^\nu \gamma^\rho, \quad \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu + (4-d)\gamma^\nu \gamma^\rho \gamma^\sigma$$

We also assume there exists a matrix γ_5 so that $\{\gamma_5, \gamma_\mu\} = 0$, though this runs into some subtleties with anomalies, as we'll see below.

• For the final step, we substitute $d = 4 - \epsilon$ and take the limit $\epsilon \to 0^+$, using

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon), \quad x^\epsilon = 1 + \epsilon \log x + O\left(\epsilon^2\right)$$

$\Gamma(z+1) = z\Gamma(z)$,

$$\Gamma(\epsilon - 1) = -\frac{1}{\epsilon} - 1 + \gamma + O(\epsilon), \quad \Gamma(\epsilon - 2) = \frac{1}{2\epsilon} + \frac{3}{4} - \frac{\gamma}{2} + O(\epsilon)$$

• Note that dimensional regularization is just a formal technique. It is merely a perturbative regularization scheme, and it does not provide a definition of the path integral in $d = 4$.

Note. A proof of our identity for the beta function. Let us define

$$C(\alpha, \beta) = \int_0^1 dx x^{-\alpha-1} (1-x)^{\beta-1}$$

We wish to prove that this coincides with the definition of $B(\alpha, \beta)$ in terms of the gamma function. By integration by parts, it is straightforward to see that

$$C(\alpha - 1, \beta + 1) = \frac{\beta}{\alpha - 1} C(\alpha, \beta)$$

Furthermore, by writing $x = 1 - (1 - x)$, we have

$$C(\alpha + 1, \beta) = C(\alpha, \beta) - C(\alpha, \beta + 1).$$

It is easy to see using the gamma function recursion relation $\Gamma(z+1) = z\Gamma(z)$ that the gamma function obeys the same relations. Since $C(1,1) = B(1,1)$, the above relations imply that B and C agree on integers; using their similar asymptotic behaviors suffices to show they are equal.

We now apply this technique to our divergent integral.

• Applying our results, we find

$$I_d = \frac{\Gamma(2-d/2)}{(4\pi)^{d/2}\Delta^{2-d/2}}$$

Special functions for QFT

1.3 Fields

1.3.1 Scalar Field

Основные свойства комплексного квантового скалярного поля

(I'll add here the same from field theory)

$$\left(\Box + \mu^2\right)\phi(x) = 0,$$

это же $E^2 = m^2c^4 + c^2\mathbf{p}^2$, где $\mathbf{p} = -\mathrm{i}\hbar\nabla, E = \mathrm{i}\hbar\partial/\partial t$. Ему соответствует $L = \frac{1}{2}\left(\phi_{,\alpha}\phi,^{\alpha} - \mu^2\phi^2\right)$, из которого $\pi(x) = \frac{\partial L}{\partial\dot{\phi}} = \frac{1}{c^2}\dot{\phi}(x)$.

(слова потом напишу важные)

$$\left[\phi(\mathbf{x},t), \dot{\phi}\left(\mathbf{x}',t\right)\right] = \mathrm{i}\hbar c^2\delta\left(\mathbf{x}-\mathbf{x}'\right) \qquad \left[\phi(\mathbf{x},t), \phi\left(\mathbf{x}',t\right)\right] = \left[\dot{\phi}(\mathbf{x},t), \dot{\phi}\left(\mathbf{x}',t\right)\right] = 0$$

$$\phi(x) = \phi^+(x) + \phi^-(x)$$

$$\phi^+(x) = \sum_{\mathbf{k}}\left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2} a(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx}, \qquad \phi^-(x) = \sum_{\mathbf{k}}\left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}}\right)^{1/2} a^\dagger(\mathbf{k})\mathrm{e}^{\mathrm{i}kx}.$$

(??? проверю по другим книгам, неужели везде такая запутанная нотация???? пока просто по Мандл Шаулу пишу и все.)

$$k^0 = \frac{1}{c}\omega_{\mathbf{k}} = +\left(\mu^2 + \mathbf{k}^2\right)^{1/2}$$

$$E = \hbar\omega_{\mathbf{k}} = +\left[m^2c^4 + c^2(\hbar\mathbf{k})^2\right]^{1/2}$$

$$\left[a(\mathbf{k}), a^\dagger\left(\mathbf{k}'\right)\right] = \delta_{\mathbf{k}\mathbf{k}'}$$
$$\left[a(\mathbf{k}), a\left(\mathbf{k}'\right)\right] = \left[a^\dagger(\mathbf{k}), a^\dagger\left(\mathbf{k}'\right)\right] = 0$$

$$H = \int \mathrm{d}^3\mathbf{x}\frac{1}{2}\left[\frac{1}{c^2}\dot{\phi}^2 + (\boldsymbol{\nabla}\phi)^2 + \mu^2\phi^2\right]$$

$$\mathbf{P} = -\int \mathrm{d}^3\mathbf{x}\frac{1}{c^2}\dot{\phi}\boldsymbol{\nabla}\phi.$$

или после подстановки полей через операторы:

$$H = \sum_{\mathbf{k}} \hbar\omega_{\mathbf{k}} \left(a^\dagger(\mathbf{k}) a(\mathbf{k}) + \frac{1}{2} \right),$$
$$\mathbf{P} = \sum_{\mathbf{k}} \hbar\mathbf{k} \left(a^\dagger(\mathbf{k}) a(\mathbf{k}) + \frac{1}{2} \right),$$

Вакуумное среднее можно отбросить, потому что все равно мы измеряем только разницы энергий, а не саму энергию, да и вообще классический гамильтониан можно переписать как $H = (1/2)(\omega q - ip)(\omega q + ip)$, после квантования $H = \omega a^\dagger a$ и не будет никакого вакуумного среднего. Хотя и есть явления, когда можно измерить вакуумную энергию (???)

У нас теперь пси и пси-крест.

$$\left[\phi(\mathbf{x},t), \dot{\phi}^\dagger(\mathbf{x}',t)\right] = \mathrm{i}\hbar c^2 \delta(\mathbf{x}-\mathbf{x}')$$
$$[\phi(\mathbf{x},t), \phi(\mathbf{x}',t)] = \left[\phi(\mathbf{x},t), \phi^\dagger(\mathbf{x}',t)\right] = \left[\dot{\phi}(\mathbf{x},t), \dot{\phi}(\mathbf{x}',t)\right]$$
$$= \left[\dot{\phi}(\mathbf{x},t), \dot{\phi}^\dagger(\mathbf{x}',t)\right] = \left[\phi(\mathbf{x},t), \dot{\phi}(\mathbf{x}',t)\right] = 0$$

$$\phi(x) = \phi^+(x) + \phi^-(x) = \sum_{\mathbf{k}} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}} \right)^{1/2} \left[a(\mathbf{k}) \mathrm{e}^{-\mathrm{i}kx} + b^\dagger(\mathbf{k}) \mathrm{e}^{\mathrm{i}kx} \right]$$
$$\phi^\dagger(x) = \phi^{\dagger+}(x) + \phi^{\dagger-}(x) = \sum_{\mathbf{k}} \left(\frac{\hbar c^2}{2V\omega_{\mathbf{k}}} \right)^{1/2} \left[b(\mathbf{k}) \mathrm{e}^{-\mathrm{i}kx} + a^\dagger(\mathbf{k}) \mathrm{e}^{\mathrm{i}kx} \right].$$

$$\left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] = \left[b(\mathbf{k}), b^\dagger(\mathbf{k}')\right] = \delta_{\mathbf{k}\mathbf{k}'},$$
$$[a(\mathbf{k}), a(\mathbf{k}')] = [b(\mathbf{k}), b(\mathbf{k}')] = [a(\mathbf{k}), b(\mathbf{k}')] = \left[a^\dagger(\mathbf{k}), b(\mathbf{k}')\right] = 0$$

$$Q = \frac{-\mathrm{i}q}{\hbar c^2} \int \mathrm{d}^3\mathbf{x} \mathrm{N} \left[\phi^\dagger(x)\phi(x) - \phi(x)\phi^\dagger(x) \right]$$
$$s^\alpha(x) = (c\rho(x), \mathbf{j}(x)) = \frac{-\mathrm{i}q}{\hbar} \mathrm{N} \left[\frac{\partial\phi^\dagger}{\partial x_\alpha}\phi - \frac{\partial\phi}{\partial x_\alpha}\phi^\dagger \right]$$
$$[\phi(x), \phi(y)] = \mathrm{i}\hbar c \Delta(x-y)$$
$$\Delta(x) \equiv \Delta^+(x) + \Delta^-(x) = \frac{-c}{(2\pi)^3} \int \frac{\mathrm{d}^3\mathbf{k}}{\omega_{\mathbf{k}}} \sin kx.$$

$$\left[\phi^+(x), \phi^-(y)\right] = \mathrm{i}\hbar c \Delta^+(x-y),$$

Since $[\phi^+(x), \phi^-(y)] = \frac{\hbar c^2}{2V} \sum_{\mathbf{k}\mathbf{k}'} \frac{1}{(\omega_{\mathbf{k}}\omega_{\mathbf{k}'})^{1/2}} \left[a(\mathbf{k}), a^\dagger(\mathbf{k}')\right] \mathrm{e}^{-\mathrm{i}kx + \mathrm{i}k'y} = \frac{\hbar c^2}{2(2\pi)^3} \int \frac{\mathrm{d}^3\mathbf{k}}{\omega_{\mathbf{k}}} \mathrm{e}^{-\mathrm{i}k(x-y)}$; $[\phi^-(x), \phi^+(y)] = -\mathrm{i}\hbar c \Delta^+(y-x) \equiv \mathrm{i}\hbar c \Delta^-(x-y)$.

$$\Delta(x) = \frac{-\mathrm{i}}{(2\pi)^3} \int \mathrm{d}^4k \delta\left(k^2 - \mu^2\right) \varepsilon(k_0) \mathrm{e}^{-\mathrm{i}kx}$$

where $\mathrm{d}^4k = \mathrm{d}^3\mathbf{k}\mathrm{d}k_0$, the k_0-integration is over $-\infty < k_0 < \infty$ and $\varepsilon(k_0)$ is defined by

$$\varepsilon(k_0) = \frac{k_0}{|k_0|} = \begin{cases} +1, & \text{if } k_0 > 0 \\ -1, & \text{if } k_0 < 0 \end{cases}$$

Также $\delta(k^2 - \mu^2) = \delta\left[k_0^2 - (\omega_\mathbf{k}/c)^2\right] = \frac{c}{2\omega_\mathbf{k}}\left[\delta\left(k_0 + \frac{\omega_\mathbf{k}}{c}\right) + \delta\left(k_0 - \frac{\omega_\mathbf{k}}{c}\right)\right]$ and performing the k_0-integration.

$$[\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = \mathrm{i}\hbar c\Delta(\mathbf{x} - \mathbf{y}, 0) = 0$$

which we had earlier [Eq. (3.6)]. [4] The invariance of $\Delta(x - y)$ implies that

$$[\phi(x), \phi(y)] = \mathrm{i}\hbar c\Delta(x - y) = 0, \quad \text{for } (x - y)^2 < 0,$$

(??? тут еще пара умных мыслей)

$$\Delta^{\pm}(x) = \frac{-1}{(2\pi)^4}\int\limits_{C^{\pm}} \frac{\mathrm{d}^4k\mathrm{e}^{-\mathrm{i}kx}}{k^2 - \mu^2}$$

$$\mathrm{i}\hbar c\Delta_\mathrm{F}(x - x') \equiv \langle 0 |\, \mathrm{T}\{\phi(x)\phi(x')\}|\, 0\rangle .$$

$$\Delta_\mathrm{F}(x) = \theta(t)\Delta^+(x) - \theta(-t)\Delta^-(x).$$
$$\Delta_\mathrm{F}(x) = \pm\Delta^{\pm}(x). \quad \text{if t} \gtrless 0.$$

Получили пропагатор мезона.

$$\Delta_\mathrm{F}(x) = \frac{1}{(2\pi)^4}\int\limits_{C_\mathrm{F}} \frac{\mathrm{d}^4k\mathrm{e}^{-\mathrm{i}kx}}{k^2 - \mu^2}$$

И там контур сперва снизу обходит полюс, потом сверху.

$$\begin{aligned}\Delta_\mathrm{F}(x) &= \frac{1}{(2\pi)^4}\int \frac{\mathrm{d}^4k\mathrm{e}^{-\mathrm{i}kx}}{k_0^2 - \left(\frac{\omega_\mathrm{k}}{c} - \mathrm{i}\eta\right)^2} \\ &= \frac{1}{(2\pi)^4}\int \frac{\mathrm{d}^4k\mathrm{e}^{-\mathrm{i}kx}}{k^2 - \mu^2 + \mathrm{i}\varepsilon},\end{aligned}$$

(?!??!?? я не верю теории Мандала Шава, очень абстрактная, нет примеров, нет конкретики, потом много дополнять её буду!)

Другие свойства квантового скалярного поля

О вещественным полем по Пескину Шредеру (????)

(мб добавлю в теорию выше, пока думаю, что это не такая нужная теория, хотя и многое показывающая)

Волновой пакет, представляющий некоторое заданное состояние $|\phi\rangle$, может быть представлен в виде:

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}\phi(\mathbf{k})|\mathbf{k}\rangle,$$

где $\phi(\mathbf{k})$ - фурье-образ пространственной волновой функции, а $|\mathbf{k}\rangle$ - одночастичное состояние с импульсом $\mathbf{k}$ в теории с взаимодействием. В свободной теории мы имели бы $|\mathbf{k}\rangle = \sqrt{2E_{\mathbf{k}}}a^\dagger_{\mathbf{k}}|0\rangle$. Множитель $\sqrt{2E_{\mathbf{k}}}$ преобразует релятивистскую нормировку $|\mathbf{k}\rangle$ в стандартную:

$$\langle\phi|\phi\rangle = 1, \quad \text{если} \quad \int \frac{d^3k}{(2\pi)^3}|\phi(\mathbf{k})|^2 = 1$$

(??? про все это еще раз нужно подумать, теор части переписать!!!)

1.3.2 Dirac Field

Основные свойства

$$\mathrm{i}\hbar\frac{\partial\psi(x)}{\partial t} = \left[c\boldsymbol{\alpha}\cdot(-\mathrm{i}\hbar\nabla) + \beta mc^2\right]\psi(x)$$

$$\mathrm{i}\hbar\gamma^\mu\frac{\partial\psi(x)}{\partial x^\mu} - mc\psi(x) = 0$$

$$\gamma^0 = \beta, \quad \gamma^i = \beta\alpha_i, \quad i = 1,2,3,$$

$$[\gamma^\mu,\gamma^v]_+ = 2g^{\mu v}, \qquad \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

$$\bar\psi(x) := \psi^\dagger(x)\gamma^0$$

$$\mathrm{i}\hbar\frac{\partial\bar\psi(x)}{\partial x^\mu}\gamma^\mu + mc\bar\psi(x) = 0$$

$$\mathscr{L} = c\bar\psi(x)\left[\mathrm{i}\hbar\gamma^\mu\frac{\partial}{\partial x^\mu} - mc\right]\psi(x)$$

$$\pi_\alpha(x) = \frac{\partial\mathscr{L}}{\partial\dot\psi_\alpha} = \mathrm{i}\hbar\psi^\dagger_\alpha, \quad \bar\pi_\alpha(x) = \frac{\partial\mathscr{L}}{\partial\dot{\bar\psi}_\alpha} \equiv 0$$

$$H = \int \mathrm{d}^3\mathbf{x}\bar\psi(x)\left[-\mathrm{i}\hbar c\gamma^j\frac{\partial}{\partial x^j} + mc^2\right]\psi(x)$$

$$\mathbf{P} = -\mathrm{i}\hbar\int \mathrm{d}^3\mathbf{x}\psi^\dagger(x)\boldsymbol{\nabla}\psi(x)$$

The angular momentum of the Dirac field follows similarly from Eq. (2.54). The transformation of the field under an inf initesimal Lorentz transformation, i.e. Eq. (2.47), is, in the case of the Dirac field, given by

$$\psi_\alpha(x) \to \psi'_\alpha(x') = \psi_\alpha(x) - \frac{\mathrm{i}}{4}\varepsilon_{\mu\nu}\sigma^{\mu v}_{\alpha\beta}\psi_\beta(x)$$

where summation over $\mu, v = 0, ..., 3$ and $\beta = 1, ..., 4$ is implied, and where $\sigma^{\mu\nu}_{\alpha\beta}$ is the (α,β) matrix element of the 4×4 matrix

$$\sigma^{\mu v} \equiv \frac{\mathrm{i}}{2}[\gamma^\mu,\gamma^v]$$

the angular momentum of the Dirac field:

$$\mathbf{M} = \int \mathrm{d}^3\mathbf{x}\psi^\dagger(x)[\mathbf{x}\wedge(-\mathrm{i}\hbar\boldsymbol{\nabla})]\psi(x) + \int \mathrm{d}^3\mathbf{x}\psi^\dagger(x)\left(\frac{\hbar}{2}\boldsymbol{\sigma}\right)\psi(x),$$

where the 4×4 matrices

$$\boldsymbol{\sigma} = \left(\sigma^{23}, \sigma^{31}, \sigma^{12}\right)$$

are the generalizations for the Dirac theory of the 2×2 Pauli spin matrices. We see that

$$Q = q \int \mathrm{d}^3\mathbf{x}\psi^\dagger(x)\psi(x)$$

$$s^\alpha(x) = (c\rho(x), \mathbf{j}(x)) = cq\bar{\psi}(x)\gamma^\alpha\psi(x)$$

$$\frac{\partial s^\alpha}{\partial x^\alpha} = 0$$

О конструкции майорановских и вейлевских спинорах (????)

(в теории поля подробнее)

$$\psi_{\mathrm{M}}(x) = \sum_{r\mathbf{p}} \left(\frac{mc^2}{VE_{\mathbf{p}}}\right)^{1/2} \left[c_r(\mathbf{p})u_{\mathrm{M}r}(\mathbf{p})\mathrm{e}^{-\mathrm{i}px/\hbar} + d_r^\dagger(\mathbf{p})u_{\mathrm{M}r}^*(\mathbf{p})\mathrm{e}^{\mathrm{i}px/\hbar}\right]$$

$$\psi_{\mathrm{M}}^{\dagger T}(x) = \sum_{r\mathbf{p}} \left(\frac{mc^2}{VE_{\mathbf{p}}}\right)^{1/2} \left[d_r(\mathbf{p})u_{\mathrm{M}r}(\mathbf{p})\mathrm{e}^{-\mathrm{i}px/\hbar} + c_r^\dagger(\mathbf{p})u_{\mathrm{M}r}^*(\mathbf{p})\mathrm{e}^{\mathrm{i}px/\hbar}\right]$$

$$\left[\psi_\alpha(x), \psi_\beta(y)\right]_+ = \left[\bar{\psi}_\alpha(x), \bar{\psi}_\beta(y)\right]_+ = 0$$

$$\left[\psi_\alpha^\pm(x), \bar{\psi}_\beta^\mp(y)\right]_+ = \mathrm{i}\left(\mathrm{i}\gamma^\mu\frac{\partial}{\partial x^\mu} + \frac{mc}{\hbar}\right)_{a\beta}\Delta^\pm(x-y)$$

Omitting suffixes, i.e. considered as a 4×4 matrix equation, we can write Eqs. (4.52b) as

$$\left[\psi^\pm(x), \bar{\psi}^\mp(y)\right]_+ = \mathrm{i}S^\pm(x-y); \qquad S^\pm(x) := \left(\mathrm{i}\gamma^\mu\frac{\partial}{\partial x^\mu} + \frac{mc}{\hbar}\right)\Delta^\pm(x).$$

From the last two equations

$$\left[\psi(x), \bar{\psi}(y)\right]_+ = \mathrm{i}S(x-y)$$

$$S(x) := S^+(x) + S^-(x) = \left(\mathrm{i}\gamma^\mu\frac{\partial}{\partial x^\mu} + \frac{mc}{\hbar}\right)\Delta(x) = \frac{-\hbar}{(2\pi\hbar)^4}\int\limits_{C^\pm}\mathrm{d}^4p\frac{\mathrm{e}^{-\mathrm{i}px/\hbar}}{\not{p} - mc}$$

(???)

$$\psi_{\mathrm{L}}(x) \equiv \frac{1}{2}\left(1-\gamma_5\right)\psi(x), \quad \psi_{\mathrm{R}}(x) \equiv \frac{1}{2}\left(1+\gamma_5\right)\psi(x)$$

in the limit $m = 0$.

$$\mathscr{L}(x) = \mathrm{i}\hbar c\bar{\psi}_{\mathrm{L}}(x)\gamma^\mu\partial_\mu\psi_{\mathrm{L}}(x)$$

$$\sigma^\mu := \left(1, \sigma^i\right), \qquad \bar{\sigma}^\mu := \left(1, -\sigma^i\right)$$

the Dirac Lagrangian in terms of Weyl spinors:

$$\mathcal{L} = iu_-^\dagger\sigma^\mu\partial_\mu u_- + iu_+^\dagger\bar{\sigma}^\mu\partial_\mu u_+ - m\left(u_+^\dagger u_- + u_-^\dagger u_+\right)$$

Therefore the mass term couples the Weyl spinors together. To get a theory with only one Weyl spinor, we must set it to zero. The individual Weyl spinors satisfy the Weyl equations

$$i\bar{\sigma}^\mu \partial_\mu u_+ = 0, \quad i\sigma^\mu \partial_\mu u_- = 0$$

• Suppose a classical field theory has n real degrees of freedom in configuration space at each spatial point. Then it has $2n$ degrees of freedom per point in phase space, which become $2n/2 = n$ particles upon quantization. This yields $n = 1$ for a real scalar field and $n = 2$ for a complex scalar field, as expected.

• Now, the Dirac spinor naively has 16 real degrees of freedom in phase space. However, since the Dirac equation is first-order, $\pi_\psi = i\psi^\dagger$, so the phase space is parametrized by ψ and $\psi^\dagger$. Then the phase space has real dimension 8, giving 4 particles upon quantization, i.e. a spin up/down particle/antiparticle. Similarly the Weyl spinor yields 2 particles.

• Another way of saying this is that physical fermions must have constraints in their phase space; it can be shown that we otherwise inevitably get negative-norm states.

О пространственно-временных и групповых свойствах дираковского поля

(додумаю, мб отдеьлный раздел создам)

Plane Wave Solutions (!!!!!!!)

(!!!! додумаю, как связано с другим видом пл волн???? важный абзац ниже будет)

• We consider a positive frequency plane wave $\psi(x) = u(\mathbf{p})e^{-ipx}$ where $p^0 > 0$. Then the Dirac equation becomes

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} u(\mathbf{p}) = 0$$

Since the individual components satisfy the Klein-Gordan equation, $p^2 = m^2$. Now we have

$$u(\mathbf{p}) = \begin{pmatrix} mu_1 \\ (p\cdot\bar{\sigma})u_1 \end{pmatrix}$$

for some spinor u_1. Defining $mu_1 = \sqrt{p\cdot\sigma}\xi$ and using the identity

$$(p\cdot\sigma)(p\cdot\bar{\sigma}) = p\cdot p = m^2$$

on the bottom spinor, we arrive at the general form

$$u(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix}$$

We will conventionally normalize ξ by $\xi^\dagger\xi = 1$.

• Similarly, there are negative frequency solutions. Letting $\psi(x) = v(\mathbf{p})e^{ipx}$ with $p^0 > 0$,

$$v(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\eta \\ -\sqrt{p\cdot\bar{\sigma}}\eta \end{pmatrix}$$

where we normalize $\eta^\dagger\eta = 1$. Note these classical field solutions vary in space as if they had four-momentum $-p$. However, we will fix this sign discrepancy after quantization, so that the corresponding particles indeed have four-momentum p.

• As a simple example, consider the positive frequency solution with mass m and $\mathbf{p} = 0$,

$$u(\mathbf{p}) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

To interpret ξ, recall that the angular momentum operator is $J^i = \text{diag}\,(\sigma^i/2, \sigma^i/2)$. Then ξ describes the spin, with $\xi = (1, 0)^T$ yielding spin up.

• Next, consider a boost along the x^3 direction, $p^\mu = (E, 0, 0, p)$. Then for spin up,

$$u(\mathbf{p}) = \begin{pmatrix} \sqrt{E - p^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E + p^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

In particular, in the massless case we have $u(\mathbf{p}) = (0, 0, 1, 0)^T$. Similarly, for a massless spin down field we have $u(\mathbf{p}) = (0, 1, 0, 0)^T$.

• Define the helicity as the projection of spin along the direction of momentum,

$$h = \frac{i}{2}\epsilon_{ijk} p^i S^{jk} = \frac{1}{2} \begin{pmatrix} p_i \sigma^i & 0 \\ 0 & p_i \sigma^i \end{pmatrix}$$

Then the spin up solution above has positive/right-handed helicity and the spin down solution has negative/left-handed helicity. In the massless case helicity coincides with chirality. When a mass is added, helicity is no longer Lorentz invariant, and chirality is no longer conserved. However, we're playing a dangerous game here because we're assigning helicity to classical solutions, while it is really a property of quantum states; this will lead to some extra sign flips.

• Now pick orthonormal bases ξ_s and χ_s for the positive and negative frequency solutions. Then $u^{r\dagger}(\mathbf{p})u^s(\mathbf{p}) = 2p_0\delta^{rs}$, $\bar{u}^r(\mathbf{p})u^s(\mathbf{p}) = 2m\delta^{rs}$, $v^{r\dagger}(\mathbf{p})v^s(\mathbf{p}) = 2p_0\delta^{rs}$, $\bar{v}^r(\mathbf{p})v^s(\mathbf{p}) = -2m\delta^{rs}$ by direct expansion, using the identity $(p \cdot \sigma)(p \cdot \bar{\sigma}) = m^2$. Another useful identity is

$$u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p}) = v^{r\dagger}(\mathbf{p})u^s(-\mathbf{p}) = 0$$

• Finally, for outer products we have

$$\sum_s u^s(\mathbf{p})\bar{u}^s(\mathbf{p}) = \not{p} + m, \quad \sum_s v^s(\mathbf{p})\bar{v}^s(\mathbf{p}) = \not{p} - m.$$

These combinations will appear in the numerator of the propagator.

Quantizing the Dirac Field

$\pi := \frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}} = i\bar{\psi}\gamma^0 = i\psi^\dagger$, and $\mathcal{H} := \pi\dot{\psi} - \mathcal{L} = \int d\mathbf{x}\bar{\psi}\left(-i\gamma^i\partial_i + m\right)\psi$, as we found earlier when computing $T^{\mu\nu}$ (???).

• Now, we impose the usual canonical commutators

$$\left[\psi_\alpha(\mathbf{x}), \psi^\dagger_\beta(\mathbf{y})\right] = \delta_{\alpha\beta}\delta(\mathbf{x} - \mathbf{y}),$$

all other commutators zero, and perform the mode expansion

$$\psi(\mathbf{x}) = \int \frac{d\mathbf{p}}{\sqrt{2E_p}} \left(b^s_{\mathbf{p}} u^s(\mathbf{p}) e^{i\mathbf{px}} + c^{s\dagger}_{\mathbf{p}} v^s(\mathbf{p}) e^{-i\mathbf{px}}\right)$$

Taking the conjugate gives

$$\psi^\dagger(\mathbf{x}) = \int \frac{d\mathbf{p}}{\sqrt{2E_p}} \left(b^{s\dagger}_{\mathbf{p}} u^s(\mathbf{p})^\dagger e^{-i\mathbf{px}} + c^s_{\mathbf{p}} v^s(\mathbf{p})^\dagger e^{i\mathbf{px}} \right)$$

Here, the spin indices s run from 1 to 2, the spinor indices α and β run from 1 to 4, and the summation convention is used on the spin indices.

• By a similar computation to the scalar field case, we have the commutators

$$\left[b^r_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}} \right] = \delta^{rs} \not{\delta}(\mathbf{p}-\mathbf{q}), \quad \left[c^r_{\mathbf{p}}, c^{s\dagger}_{\mathbf{q}} \right] = -\delta^{rs} \not{\delta}(\mathbf{p}-\mathbf{q}) \qquad \textit{withallotherszero;}$$

On-shell spinor condition:

$$\left(-\gamma^i p_i + m \right) u^s(\mathbf{p}) = \gamma^0 p_0 u^s(\mathbf{p}), \qquad \left(\gamma^i p_i + m \right) v^s(\mathbf{p}) = -\gamma^0 p_0 v^s(\mathbf{p})$$

so we simply have

$$\left(-i\gamma^i \partial_i + m \right) \psi = \int d\mathbf{p} \sqrt{\frac{E_p}{2}} \gamma^0 \left(b^s_{\mathbf{p}} u^s(\mathbf{p}) e^{i\mathbf{px}} - c^{s\dagger}_{\mathbf{p}} v^s(\mathbf{p}) e^{-i\mathbf{px}} \right)$$

Note there is an extra minus sign here because $e^{i\mathbf{px}}$ looks like e^{-ipx}. Applying our spinor inner product identities gives

$$H = \int d\mathbf{p} E_p \left(b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} - c^s_{\mathbf{p}} c^{s\dagger}_{\mathbf{p}} \right) = \int d\mathbf{p} E_p \left(b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} - c^{s\dagger}_{\mathbf{p}} c^s_{\mathbf{p}} + \phi(\mathbf{0}) \right).$$

• Now, note that $\left[H, c^\dagger(\mathbf{p}) \right] = E_p c^\dagger(\mathbf{p})$ as usual for a creation operator; the extra minus sign in the commutation relations cancels out the minus sign in the Hamiltonian. The problem is that the particles created by $c^\dagger$ have negative norm, as

$$\| c^{r\dagger}_{\mathbf{p}} |0\rangle \|^2 = \left\langle 0 \left| \left[c^r_{\mathbf{p}}, c^{r\dagger}_{\mathbf{p}} \right] \right| 0 \right\rangle < 0$$

which violates unitarity, as we get negative probabilities. We can fix this by interpreting $c^\dagger$ as an annihilation operator and c as a creation operator, but then c creates particles with negative energy, so the spectrum is unbounded below. Either way, the theory is sick.

Note. The problem above can be traced backwards to the fact that the Hamiltonian for a spinor field is linear in time derivatives, while that of a scalar field is quadratic, which can in turn be traced back to the Lagrangian. This generalizes to higher spins. An integer spin field can be represented using a number of vector indices, while a half-integer spin field must additionally have one spinor index. The kinetic term of the Lagrangian must contract the spinor indices by a factor of γ_μ (or more properly, for a Weyl spinor index, a factor of σ_μ), whose Lorentz index must then be contracted with a single derivative ∂_μ. The remaining Lorentz indices simply come in pairs, so half-integer spin fields have equations of motion with an odd number of derivatives.

The problem is fixed by switching to anticommutation relations: (??!!)

$$\left\{ \psi_\alpha(\mathbf{x}), \psi^\dagger_\beta(\mathbf{y}) \right\} \equiv \delta_{\alpha\beta} \delta(\mathbf{x}-\mathbf{y})$$

which are equivalent to the anticommutation relations

$$\left\{ b^r_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}} \right\} = \delta^{rs} \not{\delta}(\mathbf{p}-\mathbf{q}), \quad \left\{ c^r_{\mathbf{p}}, c^{s\dagger}_{\mathbf{q}} \right\} = \delta^{rs} \not{\delta}(\mathbf{p}-\mathbf{q})$$

with all other anticommutators zero.

• All of our manipulations above go through unchanged, except that at the last step,

$$H = \int d\mathbf{p} E_p \left(b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} - c^s_{\mathbf{p}} c^{s\dagger}_{\mathbf{p}} \right) = \int d\mathbf{p} E_p \left(b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} + c^{s\dagger}_{\mathbf{p}} c^s_{\mathbf{p}} - \phi(\mathbf{0}) \right)$$

Then both types of particles have positive energy and positive norm. The vacuum energy contribution also comes out negative, canceling part of the boson contribution.

• We can construct the Hilbert space exactly as before, acting with the creation operators on the vacuum. They both raise the energy, as

$$\left[H, b^{s\dagger}_{\mathbf{p}} \right] = E_{\mathbf{p}} b^{s\dagger}_{\mathbf{p}}, \quad \left[H, c^{s\dagger}_{\mathbf{p}} \right] = E_{\mathbf{p}} c^{s\dagger}_{\mathbf{p}}$$

The anticommutation relations imply the Pauli exclusion principle: every mode can have either zero or one particle. The anticommutation makes the multiparticle wavefunction antisymmetric, as postulated in nonrelativistic quantum mechanics.

• As an example, the conserved charge from the internal vector symmetry is

$$Q = \int d\mathbf{p} \left(b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} - c^{s\dagger}_{\mathbf{p}} c^s_{\mathbf{p}} \right)$$

indicating that the c particles have negative charge. We thus interpret them as antimatter.

• One might wonder how the c particles can have positive energy when they are the quantization of a classical mode with negative frequency. Each of these modes can have occupancy either 0 or 1, and it is arbitrary which of these two states is labeled 'unoccupied'. We thus define the vacuum to be the lowest energy state, and for the vacuum to have all modes unoccupied. In this new picture, the excitations created by $c^\dagger$ are 'holes' in the Dirac sea.

• The Dirac sea also explains why the vacuum energy comes out negative for fermionic fields. All the bosonic degrees of freedom get zero-point energy $\hbar\omega/2$, while all the negative frequency fermionic degrees of freedom get Dirac sea energy $-\hbar\omega$.

• Since $c^\dagger$ makes a hole, every quantum number is flipped, so that $c^{s\dagger}_{\mathbf{p}}$ creates a particle with momentum $\mathbf{p}$ and negative charge. It's convenient to define the basis spinors so that $c^{s\dagger}_{\mathbf{p}}$ and $b^{s\dagger}_{\mathbf{p}}$ have the same spin.

Next, we switch to Heisenberg picture, establish causality, and find the fermionic propagator. Field:

$$\psi(\mathbf{x}) = \int \frac{d\mathbf{p}}{\sqrt{2E_p}} \left(b^s_{\mathbf{p}} u^s(\mathbf{p}) e^{-ipx} + c^{s\dagger}_{\mathbf{p}} v^s(\mathbf{p}) e^{ipx} \right)$$

$$\begin{aligned} iS_{\alpha\beta}(x-y) :&= \left\{ \psi_\alpha(x), \bar{\psi}_\beta(y) \right\} = \text{suppressing spinor indices} = \\ &= \int \frac{d\mathbf{p}}{2E_p} \left(u^s(\mathbf{p}) \bar{u}^s(\mathbf{p}) e^{-ip(x-y)} + v^s(\mathbf{p}) \bar{v}^s(\mathbf{p}) e^{ip(x-y)} \right) = \\ &= \int \frac{d\mathbf{p}}{2E_p} \left((\not{p} + m) e^{-ip(x-y)} + (\not{p} - m) e^{ip(x-y)} \right) = \\ &= (i\partial_x + m) \left(D(x-y) - D(y-x) \right) \end{aligned}$$

where $D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$ is the propagator for a free scalar field. (??? проделаю потом матем!)

• Since $D(x-y)$ vanishes for spacelike separations, $S(x-y)$ and hence the anticommutator vanishes for spacelike separations. Since all observables are fermion bilinears or combinations thereof, observables commute at spacelike separations. (??????????????????)

Let's look at

$$T(\psi(x)\bar{\psi}(y)) := \begin{cases} \psi(x)\bar{\psi}(y) & x^0 \geq y^0 \\ -\bar{\psi}(y)\psi(x) & y^0 > x^0 \end{cases}$$

To see one reason the minus sign is necessary, note that at spacelike separations we have $\psi(x)\bar{\psi}(y) = -\bar{\psi}(y)\psi(x)$ by the above calculation. In such cases we can perform a boost to change the time ordering of x and y, so the two cases in the time ordering must be equal.

Feynman propagator:

$$\begin{aligned} S_F(x-y) :&= \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle \\ &= \theta\left(x^0-y^0\right)\int \frac{d\mathbf{p}}{2E_p}(\not{p}+m)e^{-ip(x-y)} - \theta\left(y^0-x^0\right)\int \frac{d\mathbf{p}}{2E_p}(\not{p}-m)e^{ip(x-y)} \\ &= \left(i\not{\partial}_x + m\right)D_F(x-y) = i\int dp \frac{\not{p}+m}{p^2-m^2+i\epsilon}e^{-ip(x-y)} \end{aligned}$$

where D_F is the Feynman propagator for a free scalar theory. If we had used commutation relations to quantize the field, we would have found an ugly, non-Lorentz invariant result.

• As expected, S_F is a Green's function for the Dirac equation, since applying $i\not{\partial}_x - m$ gives $(\not{p}-m)(\not{p}+m) = p^2 - m^2$ in the numerator. Because of this identity, the Feynman propagator is sometimes formally written as $S_F(p) = i(\not{p}-m)^{-1}$. Alternatively, we can think of the Feynman propagator as a sum of Feynman propagators for each degree of freedom; in this picture the numerator is a sum over polarizations. We will prove this more generally later.

• Normal ordering for fermionic fields is also defined with an extra minus sign for every interchange. Thus the general method for normal and time ordering is to simply commute or anticommute, neglecting all extra resulting terms. The proof of Wick's theorem goes through as before, with contractions yielding a factor of S_F.

• Contractions now come with signs, so that swapping any two ψ or $\bar{\psi}$ fields in the contraction structure yields a factor of -1. If we draw lines connecting contracted fields, the number of sign flips is the number of intersections of the lines.

• Note that the contraction of ψ with ψ, or of $\bar{\psi}$ with $\bar{\psi}$, is automatically zero. Then n-point functions of spinor fields can only be nonzero if there are an equal number of ψ and $\bar{\psi}$ fields, reflecting charge conservation.

• The proof of the LSZ reduction formula also goes through mostly as before, expect that the poles are projected out by factors of $i\not{\partial} - m$ rather than $\partial^2 + m^2$.

Other properties of Dirac field

Other general properties

$$\psi \to S[\Lambda]\psi, \quad \psi^\dagger \to \psi^\dagger S^\dagger[\Lambda]$$

Therefore, the naive guess $\psi^\dagger\psi$ does not work because the $S[\Lambda]$ are not unitary.

• For concreteness, we use a representation of the Clifford algebra where γ^0 is Hermitian and the γ^i are anti-Hermitian, such as the chiral representation.

• In any specific representation, it is straightforward to show that

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$$

which means γ^0 takes the adjoint of gamma matrices when pulled through them. (Since all representations are related by unitary transformations, it suffices to show this result in any representation - but it's not straightforward to do it without ever picking one.) Then

$$(S^{\mu\nu})^\dagger = -\gamma^0 S^{\mu\nu}\gamma^0, \quad S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1}\gamma^0$$

• Defining the Dirac adjoint $\bar{\psi} = \psi^\dagger\gamma^0$, we have

$$\bar{\psi} \to \psi^\dagger S[\Lambda]^\dagger\gamma^0 = \psi^\dagger\gamma^0 S[\Lambda]^{-1} = \bar{\psi}S[\Lambda]^{-1}$$

which implies that $\bar{\psi}\psi$ is a Lorentz scalar.

• Moreover, $\bar{\psi}\gamma^\mu\psi$ is a Lorentz vector. Since we have

$$\bar{\psi}\gamma^\mu\psi \to \bar{\psi}S[\Lambda]^{-1}\gamma^\mu S[\Lambda]\psi$$

we need to show $S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu{}_\nu\gamma^\nu$. For infinitesimal Lorentz transformations, this becomes

$$[-S^{\rho\sigma}, \gamma^\mu] = (J^{\rho\sigma})^\mu_\nu\,\gamma^\nu$$

We can directly verify this is true using computations we've already done.

• Similarly, one can show that $\bar{\psi}\gamma^\mu\gamma^\nu\psi$ transforms as a Lorentz tensor, as does any combination of more gamma matrices. Taking the symmetric part yields $\eta^{\mu\nu}\bar{\psi}\psi$, which is simply a multiple of our Lorentz scalar. The new feature is the antisymmetric part, $\bar{\psi}S^{\mu\nu}\psi$.

• We use our scalar and vector to build the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi, \quad \not{\partial} = \gamma^\mu\partial_\mu$$

Note that the factor of i is necessary to make $\mathcal{L}$ real, i.e. invariant under conjugation, to cancel out a minus sign from integration by parts. We haven't seen this issue before because our other Lagrangians were second-order, requiring two integrations by parts.

• Using the trick of treating ψ and $\bar{\psi}$ as independent fields, varying with respect to $\bar{\psi}$ gives the Dirac equation

$$(i\not{\partial} - m)\psi = 0$$

Varying with respect to ψ gives the conjugate equation

$$i\not{\partial}_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = \bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0$$

where the arrow indicates the derivative acts to the left.

• Each component of the Dirac spinor satisfies the Klein-Gordan equation, as

$$(i\gamma^\nu\partial_\nu + m)\,(i\gamma^\mu\partial_\mu - m)\,\psi = -\left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2\right)\psi = -\left(\partial^2 + m^2\right)\psi = 0$$

Thus the Dirac spinor describes four types of particles of mass m. The left-hand side contains a 'factorization' of the Klein-Gordan operator $\partial^2 + m^2$, so the Dirac equation is sometimes called the 'square root' of the Klein-Gordan equation. Here it's clear that the definition of a Clifford algebra was originally chosen just to make this work out.

Next, we decompose the Dirac spinor into Weyl spinors.

• As we've seen, in the chiral representation of the Clifford algebra, the Lorentz transformations are block-diagonal. Hence the Dirac spinor splits into two irreps, $\psi = (u_+, u_-)$, called Weyl spinors, which transform as

$$u_\pm \to e^{-i\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}/2}u_\pm, \quad u_\pm \to e^{\mp\boldsymbol{\chi}\cdot\boldsymbol{\sigma}/2}u_\pm$$

The Weyl spinors are $(1/2,0)$ and $(0,1/2)$, or left-handed and right-handed respectively, and the Dirac spinor is $(1/2,0)\oplus(0,1/2)$.

Bremsstrahlung

(про это в теорполе пока, потом сюда перепишу суть короче)

1.3.3 Vector Field

(уменьшу скорее всего, пока так пусть будет, с теорией поля свяжу еще)

О векторном поле и фотонах: main formulas

Основные свойства

(???? про это потом буду думать снова, мб что-то тут неверно написано пока будет)
Нам известны уравнения Максвелла и весь формализм классической электродинамики:

$$\partial_v F^{\mu v}(x) = \frac{1}{c}s^\mu(x)$$
$$\partial^\lambda F^{\mu v}(x) + \partial^\mu F^{v\lambda}(x) + \partial^v F^{\lambda\mu}(x) = 0.$$

Тут $F^{\mu\nu}$ is antisymmetric, поэтому заряд сохраняется $\partial_\mu s^\mu(x) = 0$.
Поле описывается лагранжианом:

$$\mathcal{L} = -\frac{1}{2}\partial_v A_\mu(x)\cdot\partial^v A^\mu(x) - \frac{1}{c}s_\mu(x)A^\mu(x).$$

Вообще, можно было бы взять и $\mathscr{L} = -\frac{1}{4}F_{\mu v}(x)F^{\mu v}(x) - \frac{1}{c}s_\mu(x)A^\mu(x)$, но от него $\pi^0 \equiv 0$, что плохо, а оба лагранжиана дают одну и ту же физику. (???? в теории про это подробно подумаю.) Мы подразумеваем тут везде Лоренцеву калибровку: $\partial_\mu A^\mu(x) = 0$, из которой следует еще условие на дополнительные калибровки: если $A'^\mu(x) = A^\mu(x) + \partial^\mu f(x)$, то должно быть $\Box f(x) = 0$.
Решение в параллелепипеде с периодическими гран условиями:

$$A^\mu(x) = A^{\mu+}(x) + A^{\mu-}(x)$$
$$A^{\mu+}(x) = \sum_{r\mathbf{k}}\left(\frac{\hbar c^2}{2\ \mathrm{V}\omega_\mathbf{k}}\right)^{1/2}\varepsilon_r^\mu(\mathbf{k})a_r(\mathbf{k})\mathrm{e}^{-\mathrm{i}kx}, \qquad A^{\mu-}(x) = \sum_{r\mathbf{k}}\left(\frac{\hbar c^2}{2\ \mathrm{V}\omega_\mathbf{k}}\right)^{1/2}\varepsilon_r^\mu(\mathbf{k})a_r^\dagger(\mathbf{k})\mathrm{e}^{\mathrm{i}kx}.$$

Тут $k^0 = \frac{1}{c}\omega_\mathbf{k} = |\mathbf{k}|$, $r \in \{0,3\}$. Вообще, a тут - вообще любые функции, при которых сумма/интеграл сходится (!??!??!?!?!?да???)
У нас сразу выходят условия на вектора поляризации:

$$\varepsilon_r(\mathbf{k})\varepsilon_s(\mathbf{k}) = \varepsilon_{r\mu}(\mathbf{k})\varepsilon_s^\mu(\mathbf{k}) = -\zeta_r\delta_{rs}, \quad r,s = 0,...,3,$$
$$\sum_r \zeta_r\varepsilon_r^\mu(\mathbf{k})\varepsilon_r^v(\mathbf{k}) = -g^{\mu\nu},$$
$$\zeta_0 = -1, \quad \zeta_1 = \zeta_2 = \zeta_3 = 1.$$

(??? в теории еще подумаю, как конкретно такие условия получаются и почему конкретно там минус????)

(???? не, ниже про квантование не могу писать, я ничего не понимаю, нужно К-Г переучивать?!!!!)

Квантуем просто заменой вектор-потенциалов операторами. (??? почитаю еще, какие особенности там с квантованием????) Явно можно увидеть, что коммутаторы:

$$[A^\mu(\mathbf{x},t), A^v(\mathbf{x}',t)] = 0, \quad \left[\dot{A}^\mu(\mathbf{x},t), \dot{A}^v(\mathbf{x}',t)\right] = 0$$
$$\left[A^\mu(\mathbf{x},t), \dot{A}^v(\mathbf{x}',t)\right] = -\mathrm{i}\hbar c^2 g^{\mu v}\delta(\mathbf{x}-\mathbf{x}')$$

Точнее, the full photon propagator. It is corrected by Dyson resummation of the 1PI contribution,

$$\mu \sim \mathrm{PP} \sim\sim v \equiv i\Pi^{\mu\nu}\left(q^2\right)$$

and $\Pi^{\mu\nu}(q^2)$ is called the self-energy of the photon or the vacuum polarization.

О фотонном пропагаторе

О пространственно-временных и групповых свойствах векторного поля

Quantization of a vector field

(I'll summarize all it here, since there are a lot of questions about it)

Features of vector field quantization

(выше уже оно проквантовано, тут другие методы и другие вопросы, этого много еще)

Gauge Symmetry, quantizing the electromagnetic field

(??? все те же вопросы про это остаются, вот тут их и отвечу)

Electromagnetism is described by $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ and satisfies the equations of motion and Bianchi identity $\partial_\mu F^{\mu\nu} = 0, \quad \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$.

These each produce four equations, i.e. two of Maxwell's equations each.

(абзац про то, в чем вообще проблема с квантованием эм поля?)

• Many of the degrees of freedom in A_μ are redundant, as we know the photon has only two polarization states. The first issue is that A_0 has no kinetic term, as the Lagrangian does not depend on $\dot{A}_0$. It is a 'non-dynamical' field, whose value is totally determined by initial conditions for A_i and $\dot{A}_i$.

• Intuitively, a kinetic term $\dot{A}_0^2$ gives the action a 'penalty' for changing A_0, and hence gives the field A_0 some 'inertia'. Without this inertia, action minimization always takes $A_0(t)$ to whatever minimizes the Lagrangian given $A_i(t)$ and $\dot{A}_i(t)$ at every time t.

• More explicitly, we can solve for A_0 (up to gauge symmetry) using its equation of motion,

$$\nabla^2 A_0 + \nabla\cdot\frac{\partial\mathbf{A}}{\partial t} = 0, \quad A_0(\mathbf{x}) = \int d\mathbf{x}' \frac{(\nabla\cdot(\partial\mathbf{A}/\partial t))(\mathbf{x}')}{4\pi|\mathbf{x}-\mathbf{x}'|}$$

Note that this equation of motion is equivalent to the constraint $\nabla\cdot\mathbf{E} = 0$. We could plug this expression for A_0 back into the Lagrangian, eliminating A_0 entirely, but this would be much messier than just keeping A_0 explicit.

• The Lagrangian also has a gauge symmetry

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$$

where we assume $\alpha(x) \to 0$ as $x \to \infty$, under which the field strength is invariant. Classically, we think of states related by a gauge transformation as the same physical state. The gauge symmetry also takes away one degree of freedom because the gauge parameter α has one value at every point, leaving two.

• Gauge symmetry places powerful constraints on the action. E.g. as we will see in detail for the non-abelian case, the Maxwell action is essentially the only possible kinetic term for a photon which is gauge invariant, not irrelevant, and not a total derivative.

• Since A_0 is non-dynamical its conjugate momentum vanishes,

$$\pi^0 = 0, \quad \pi^i = -F^{0i} = E^i$$

The Hamiltonian can be straightforwardly computed to be

$$H = \int d\mathbf{x} \frac{1}{2}E^2 + \frac{1}{2}B^2 - A_0(\nabla \cdot \mathbf{E})$$

We hence recover the familiar electromagnetic field energy, with A_0 serving as a Lagrange multiplier which enforces Gauss's law. Thus we have first-class constraints $\pi^0 = \nabla \cdot \mathbf{E} = 0$.

We will look at a few different gauges.

• Lorenz gauge is the condition $\partial_\mu A^\mu = 0$. This is always achievable since we can set $\partial^2 \alpha = -\partial_\mu A^\mu$ for any starting A^μ. This is not a complete gauge fixing, because $\partial^2 \alpha = 0$ has nontrivial solutions, but Lorenz gauge is Lorentz invariant.

• Coulomb gauge is the additional restriction $\nabla \cdot \mathbf{A} = 0$, which means A_0 is time-independent in vacuum; it is achievable by the same logic as Lorenz gauge. This is still not a complete gauge fixing, because we may still apply gauge transformations with parameters α so that

$$\ddot{\alpha} = \nabla^2 \alpha = 0.$$

Usually, we further require $A_0 = 0$, which can be achieved by performing a final gauge transformation with $\alpha = A_0 t$. This brings us to Coulomb gauge, which breaks Lorentz invariance, but makes it easy to see the two independent polarizations.

• In Lorenz gauge and hence in Coulomb gauge, the equation of motion is $\partial^2 A_\mu = 0$.

• The counting of degrees of freedom can be a little puzzling. Intuitively, there is only "one degree of freedom" in $\alpha(x)$, which is a function on spacetime. Lorenz gauge is also intuitively "one constraint", so it seems nothing should be left over. The point is that the remaining freedom is of measure zero, as a solution to $\partial^2 \alpha = 0$ can be specified by initial conditions on a hypersurface. Similarly, after fixing Coulomb gauge, we have the freedom to perform gauge transformations with $\dot{\alpha} = \nabla^2 \alpha = 0$, which again is of measure zero relative to what we started with.

• For a fixed four-vector n^μ, the axial gauge is $n^\mu A_\mu = 0$. We will not deal with axial gauge here, but it is useful in Yang-Mills theory because ghost fields are not required.

Quantization in Coulomb Gauge

This gauge is popular in atomic physics, where the breaking of Lorentz invariance is not problematic, since the matter is typically nonrelativistic anyway. However, it is especially difficult to perform renormalization with it. (????)

• Since we have set $A_0 = 0$, the classical plane wave solutions are

$$\mathbf{A}(\mathbf{x}) = \int d\mathbf{p}\boldsymbol{\xi}(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}, \quad p^2 = 0$$

where $\boldsymbol{\xi}(\mathbf{p})$ is the polarization. The constraint $\nabla \cdot \mathbf{A} = 0$ yields $\boldsymbol{\xi} \cdot \mathbf{p} = 0$.

• For each momentum, there are two independent polarization vectors $\boldsymbol{\epsilon}_r(\mathbf{p})$ satisfying

$$\boldsymbol{\epsilon}_r(\mathbf{p}) \cdot \mathbf{p} = 0, \quad \boldsymbol{\epsilon}_r(\mathbf{p}) \cdot \boldsymbol{\epsilon}_s(\mathbf{p}) = \delta_{rs}$$

Note that it is impossible to choose the polarization vectors continuously by the hairy ball theorem. (??????? в теории про это) However, this topological issue has no impact on scattering calculations.

• Naively, we would impose the canonical commutators

$$[A_i(\mathbf{x}), E_j(\mathbf{y})] = i\delta_{ij}\delta(\mathbf{x} - \mathbf{y})$$

with all other commutators zero, but we must account for the constraints. Before gauge fixing, we had the first-class constraints

$$\pi^0 = 0, \quad \nabla \cdot \mathbf{E} = 0$$

In the process of gauge fixing, we imposed the additional constraints

$$A^0 = 0, \quad \nabla \cdot \mathbf{A} = 0$$

and now $\pi^0 = A^0 = 0$ are automatically satisfied since we have eliminated those variables entirely. The remaining constraints are now second-class. They are not obeyed by the naive commutators, as, for example,

$$[\nabla \cdot \mathbf{A}(\mathbf{x}), \nabla \cdot \mathbf{E}(\mathbf{x})] = i\nabla^2\delta(\mathbf{x} - \mathbf{y}) \neq 0$$

• The fix is to use the Dirac bracket, i.e. the modified bracket

$$[A_i(\mathbf{x}), E_j(\mathbf{y})] = i\left(\delta_{ij} - \frac{\partial_i\partial_j}{\nabla^2}\right)\delta(\mathbf{x} - \mathbf{y})$$

Then in momentum space, we have

$$[A_i(\mathbf{x}), E_j(\mathbf{y})] = i\int d\mathbf{p}\left(\delta_{ij} - \frac{p_i p_j}{|\mathbf{p}|^2}\right)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}$$

so that $\partial_i A_i$ and $\partial_i E_i$ each have zero bracket with everything else.

• Formally, because our theory has constraints, we would like to define it on a subset of the full phase space. However, explicitly reducing the three degrees of freedom to two would be quite ugly. Thinking geometrically of Poisson brackets $\{f, g\}$ as the rate of change of g under the flow generated by f, the Dirac bracket amounts to adjusting the flows so that they stay on the constraint manifold, giving the same result.

• Now, the usual mode expansion gives

$$\mathbf{A}(\mathbf{x}) = \int \frac{d\mathbf{p}}{\sqrt{2|\mathbf{p}|}}\sum_{r=1}^{2}\boldsymbol{\epsilon}_r(\mathbf{p})\left(a^r_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a^{r\dagger}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right)$$

and

$$\mathbf{E}(\mathbf{x}) = \int d\mathbf{p}(-i)\sqrt{\frac{|\mathbf{p}|}{2}}\sum_{r=1}^{2}\boldsymbol{\epsilon}_r(\mathbf{p})\left(a^r_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - a^{r\dagger}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right)$$

The polarizations $\boldsymbol{\epsilon}_r(\mathbf{p})$ remain vectors, not operators, obeying the same constraints as before.

• One can show that the commutation relations above lead to the usual result

$$\left[a^r_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}}\right] = \delta^{rs}\phi(\mathbf{p}-\mathbf{q})$$

with all others zero. This can be derived using the completeness relation for the polarizations,

$$\sum_{r=1}^{2}\epsilon^i_r(\mathbf{p})\epsilon^j_r(\mathbf{p}) = \delta^{ij} - \frac{p^i p^j}{|\mathbf{p}|^2}$$

• Finally, we substitute our mode expansion back into the Hamiltonian, which gives

$$H = \int d\mathbf{p}|\mathbf{p}|\sum_{r=1}^{2}a^{r\dagger}_{\mathbf{p}}a^r_{\mathbf{p}}$$

which confirms the excitations all have energy p.

• Switching to Heisenberg picture and applying the completeness relation gives the propagator

$$D_{ij}(x-y) = \langle 0\,|TA_i(x)A_j(y)|\,0\rangle = \int dp\frac{i}{p^2+i\epsilon}\left(\delta_{ij} - \frac{p_i p_j}{|\mathbf{p}|^2}\right)e^{-ip(x-y)}$$

Unfortunately, the result is far from Lorentz invariant. Instead we'll redo the quantization procedure keeping Lorentz invariance explicit throughout.

Note. When treating the electromagnetic field, we typically either consider the quantized version, or the classical limit, which arises out of states with many photons. However, it is also valid to consider states with one photon, and see what the quantum theory says about them. For massive fields, the one-particle states would be useful in the nonrelativistic limit, where there isn't enough energy to make more particles. For photons, this doesn't apply, but considering one photon at a time is still interesting since many processes involve only one, or a few photons.

We can write a general state containing one photon as

$$|\psi\rangle = \int d\mathbf{p}\sum_{r=1}^{2}\boldsymbol{\epsilon}_r(\mathbf{p})\psi_r(\mathbf{p})a^{r\dagger}_{\mathbf{p}}|0\rangle$$

Here, we are working with a basis of one-photon states that diagonalizes momentum $\mathbf{p}$ and helicity, so $\psi_r(\mathbf{p})$ is the "wavefunction in momentum/helicity space". It is normalized in the usual way,

$$\int d\mathbf{p}\sum_{r=1}^{2}|\psi_r(\mathbf{p})|^2 = 1$$

Unlike nonrelativistic quantum mechanics, in quantum field theory it is simpler to start in momentum space. We can define a position operator by noting that, by the canonical commutation relations, we should have $\mathbf{x} = -i\nabla$ where the gradient is in momentum space. (This operator is formally called the Newton-Wigner position operator. One can show that this is the unique

candidate for $\mathbf{x}$ which transforms appropriately under the Poincare group.) There is no problem in defining a position space wavefunction, by taking the Fourier transform. Next, we can define the orbital angular momentum $\mathbf{L} = \mathbf{x} \times \mathbf{p}$, which has the same value on one-photon states as the orbital angular momentum derived by Noether's theorem for the fields. The spin $\mathbf{S}$ is already defined, taking values $\pm\hat{\mathbf{p}}$ depending on the polarization, and the helicity is $h = \hat{\mathbf{p}} \cdot \mathbf{S}$.

However, this has a serious problem, which can be seen by defining a vector-valued wavefunction

$$\psi(\mathbf{p}) = \sum_{r=1}^{2} \boldsymbol{\epsilon}_r(\mathbf{p}) \psi_r(\mathbf{p})$$

This wavefunction obeys the transverse constraint, $\mathbf{p} \cdot \boldsymbol{\psi}(\mathbf{p}) = 0$, which means the position space wavefunction obeys $\nabla \cdot \boldsymbol{\psi}(\mathbf{x}) = 0$. But in position space, the position operator just multiplies by $\mathbf{x}$, which does not preserve this property. More formally, Wigner's classification tells us that photons only occupy helicity states $\lambda = \pm 1$, with $\lambda = 0$ forbidden. But $\mathbf{x}$ does not commute with helicity, since it does not commute with $\mathbf{p}$, so it introduces a $\lambda = 0$ component. One can restrict $\mathbf{x}$ to the $\lambda = \pm 1$ subspace, but then it no longer has the right transformation properties. This problem also applies to $\mathbf{L}$ and $\mathbf{S}$. Intuitively, these rotate the field rigidly in space, and the field value itself, and rotating the two independently will turn transverse polarizations into longitudinal ones. More generally, this problem does not occur for massive particles, where one can boost to the rest frame and straightforwardly construct the operators there. For massless particles, the problem only appears for helicities of $|\lambda| \geq 1$, since otherwise there are no missing helicities.

Note that $(\mathbf{p}, h)$ is not the only choice for a complete set of commuting observables for onephoton states. E.g. if one wants to talk about angular momentum, one can diagonalize (p, J^2, J_z, h) or (p, J^2, J_z, P) where P is parity. The position space wavefunctions involve spherical Bessel functions and spherical harmonics, and are known as vector multipole fields.

The obstruction above is usually described by mathematical physicists as saying that "the position of a photon is meaningless", which is probably overenthusiastic. There clearly is no problem with confining a photon in a cavity or a beam.

Note. There are yet more subtleties even when the Newton-Wigner position operator exists. In this case, there exist states that are perfectly localized in a given reference frame, but localization is not Lorentz invariant: the very same state is not perfectly localized upon applying a boost. (However, this might not be relevant in an experimental situation where the detector defines a preferred frame.) Furthermore, the wavefunction spreads out faster than the speed of light, a result known as the Hegerfeldt paradox. The easiest way to see that causality is not actually violated is to return to the field picture. This is one of the reasons modern particle physics sources often avoid the topic of one-photon wavefunctions: they have historically led to much confusion.

1.4 QED $\mathcal{L}_{\text{int}} = e\bar{\psi}\gamma^\mu \hat{A}_\mu \psi$ main properties and phenomena

(!! только на самом деле основное, дополительные свойства - следующая 1я часть!)

1.4.1 Estimations for QED (???!!)

(после квантмеха есть идея написать это тут. По Мигдалу Крайнову и просто по опыту соберу, но позже.)

elastic scattering

$$\frac{d\sigma}{d\Omega} = \underbrace{\frac{\alpha^2}{4E_1^2 \sin^4\theta/2}}_{\text{Rutherford}} \underbrace{\frac{E_3}{E_1}}_{\text{Proton recoil}} \left(\underbrace{\cos^2\frac{\theta}{2}}_{\text{Electric/ Magnetic scattering}} - \underbrace{\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}}_{\text{Magnetic term due to spin}} \right)$$

For elastic scattering of relativistic electrons from an extended proton:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2} \right) \qquad \text{(Rosenbluth Formula)}$$

$\star$ Electron elastic scattering from protons demonstrates that the proton is an extended object with rms charge radius of ~ 0.8fm

Elastic Scattering from a Finite Size Proton $\star$ In general the finite size of the proton can be accounted for by introducing two structure functions. One related to the charge distribution in the proton, $G_E(q^2)$ and the other related to the distribution of the magnetic moment of the proton, $G_M(q^2)$ - It can be shown that equation (16) generalizes to the Rosenbluth formula with with the Lorentz Invariant quantity:

$$\tau := -\frac{q^2}{4M^2} > 0$$

- Unlike our previous discussion of form factors, here the form factors are a function of q^2 rather than $\vec{q}^2$ and cannot simply be considered in terms of the FT of the charge and magnetic moment distributions. But $q^2 = (E_1 - E_3)^2 - \vec{q}^2$ and from eq (15) obtain

$$-\vec{q}^2 = q^2 \left[1 - \left(\frac{q}{2M} \right)^2 \right]$$

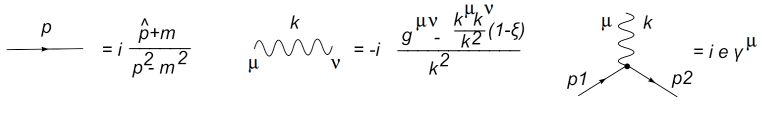
So for $\frac{q^2}{4M^2} \ll 1$ we have $q^2 \approx -\vec{q}^2$ and $G(q^2) \approx G(\vec{q}^2)$

1.4.2 1-loop QED

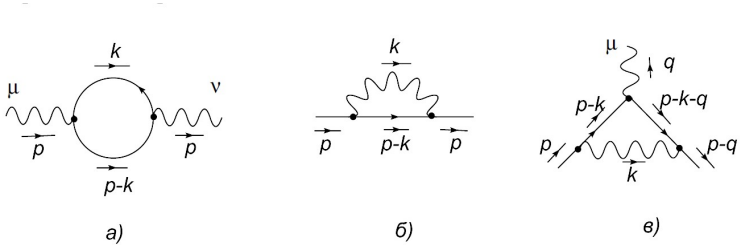
(!!! see Vergeles!)

Основные формулы и особенности

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + e\bar{\psi}\gamma^\mu A_\mu\psi - \frac{1}{2\xi}\left(\partial_\mu A_\mu\right)^2$$



Однопетлевые расходящиеся это только:



$$\partial_\nu F^{\nu\mu} = e\bar{\psi}\gamma^\mu\psi$$

Если принять калибровочное условие Лоренца $\partial_\mu A^\mu = 0$, уравнения сводятся к

$$\Box A^\mu = e\bar{\psi}\gamma^\mu\psi$$

$$\mathrm{i}\hbar\frac{\partial}{\partial t} \to \mathrm{i}\hbar\frac{\partial}{\partial t} - q\phi(x), \quad -\mathrm{i}\hbar\boldsymbol{\nabla} \to -\mathrm{i}\hbar\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}(x)$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \to D_\mu = \left[\partial_\mu + \frac{\mathrm{i}q}{\hbar c}A_\mu(x)\right].$$

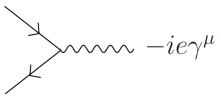
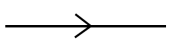
$$\psi(x) \to \psi'(x) = \psi(x)\mathrm{e}^{\mathrm{i}ef(x)/\hbar c}$$
$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)\mathrm{e}^{-\mathrm{i}ef(x)/\hbar c}$$

$$\mathscr{L}_0 \to \mathscr{L}'_0 = \mathscr{L}_0 - e\bar{\psi}(x)\gamma^\mu\psi(x)\partial_\mu f(x)$$
$$\mathscr{Y}_1 \to \mathscr{Y}'_1 = \mathscr{L}_1 + e\bar{\psi}(x)\gamma^\mu\psi(x)\partial_\mu f(x)$$

О скалярной КЭД

Правила Фейнмана в КЭД

For vertices and internal lines, we write

- Vertex:  $-ie\gamma^\mu$
- Photon Propagator: $-\dfrac{i\eta_{\mu\nu}}{p^2 + i\epsilon}$
- Fermion Propagator:  $\dfrac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$

For external lines in the diagram, we attach

- Photons: We add a polarization vector ϵ^μ_{in} /$\epsilon^\mu_{\text{out}}$ for incoming/outgoing photons. In Coulomb gauge, $\epsilon^0 = 0$ and $\vec{\epsilon}\cdot\vec{p} = 0$.

- Fermions: We add a spinor $u^r(\vec{p})/\bar{u}^r(\vec{p})$ for incoming/outgoing fermions. We add a spinor $\vec{v}^r(\vec{p})/v^r(\vec{p})$ for incoming/outgoing anti-fermions.

Массовый оператор по Вергелесу

(диаграмма тут)
аналитическое выражение которой даётся выражением

$$-i\mathcal{M}(p) = (-ie)^2 \int \frac{\mathrm{d}^{(d)}k}{(2\pi)^d} \gamma^\mu i S_c(p-k) \gamma^\nu \left(-iD^\alpha_{\mu\nu}(k)\right).$$

Проще всего вычисление проводится в калибровке Фейнмана $\alpha = 1$, в которой

$$-i\mathcal{M}(p) = -4\pi e^2 \int \frac{\mathrm{d}^{(d)}k}{(2\pi)^d} \frac{\gamma^\mu[(\not{p}-\not{k})+m]\gamma_\mu}{(p-k)^2-m^2} \frac{1}{k^2-\lambda^2}.$$

(вычисление на пару страниц, которое соберу словами в абзац)

$$\mathcal{M} = \frac{e^2}{2\pi m}(\not{p}-m)^2 \left\{ \frac{1}{2(1-\rho)} \left(1 - \frac{2-3\rho}{1-\rho}\ln\rho\right) - \right.$$
$$\left. -\frac{\not{p}+m}{m\rho} \left[\frac{1}{2(1-\rho)} \left(2-\rho+\frac{\rho^2+4\rho-4}{1-\rho}\ln\rho\right) + 1 + 2\ln\frac{\lambda}{m}\right]\right\}.$$

О поляризационном операторе по Вергелесу

(поизучаю, у него целый раздел про это)

$$\sum_\lambda \int \frac{d^3p_-}{(2\pi)^3 2p_-^0} \frac{d^3p_+}{(2\pi)^3 2p_+^0}$$

причём далее $\sum\limits_\lambda$ означает сумму лишь по поляризациям промежуточных частиц, $p_-^2 = p_+^2 = m^2$ и в сумме по промежуточным состояниям 3 -вектора $\mathbf{p}_-$и $\mathbf{p}_+$считаются независимыми. Имеем:

$$2\,\mathrm{Im}\, M_{ii} = (2\pi)^4 \sum_\lambda \int \frac{\mathrm{d}^3p_-}{(2\pi)^3 2p_-^0} \frac{\mathrm{d}^3p_+}{(2\pi)^3 2p_+^0} \delta^{(4)}\left(k-p_--p_+\right) \left|M_{\lambda i}\right|^2,$$

где амплитуда $M_{\lambda i}$ даётся частью диаграммы на рис. 24 правее крестиков:

$$iM_{\lambda i} = \bar{u}_{p_-}\left(-ie\gamma^\mu\right) u_{-p_+} \sqrt{4\pi} e_\mu.$$

Введём параметр

$$t = k^2 = (p_- + p_+)^2 = 2\left(p_-p_+ + m^2\right).$$

(???? дальше умные преобразования)

$$2e_\mu e_\nu\, \mathrm{Im}\, \mathcal{P}^{\mu\nu} = -\frac{2e^2}{3} e_\mu e_\nu \sqrt{\frac{t-4m^2}{t}} \cdot \frac{t+2m^2}{t} \left(k^2\eta^{\mu\nu} - k^\mu k^\nu\right).$$

Поэтому из (33) получаем асимптотику при больших $|t|$:

$$\mathcal{P}(t) = \begin{cases} \frac{e^2}{3\pi}\ln\frac{|t|}{m^2}, & \text{если } -t \gg 4m^2 \\ \frac{e^2}{3\pi}\left(\ln\frac{t}{m^2} - i\pi\right), & \text{если } t \gg 4m^2. \end{cases}$$

Мнимые части $\mathcal{P}(t)$, полученные при помощи (30) и (38), совпадают. Это подтверждает равенство $\ln\xi = \ln|\xi| + i\pi$ при $-1 < \xi < 0$.

Формфакторы электрона по Вергелесу

(тут диаграммы)
(допишу это, пока не до этого)

$$\delta M_{fi} = \frac{e}{m} g(0) \bar{u}_2 \left(\frac{i}{2} \varepsilon_{ijk} \sigma^{jk} \right) u_1 H^i$$

Такой же вклад в амплитуду даст добавка к гамильтониану

$$\delta \mathcal{H} = \int \mathrm{d}^{(3)} x \bar{\psi} \left[-\frac{e}{m} g(0) \left(\frac{i}{2} \varepsilon_{ijk} \sigma^{jk} \right) H^i \right] \psi \longrightarrow \int \mathrm{d}^{(3)} x \varphi^\dagger \left[-\frac{e}{m} g(0) \mathbf{sH} \right] \varphi$$

$$\mu = \frac{e\hbar}{2mc}(1 + g(0)).$$

Очевидно, что первое слагаемое в (6) не может дать вклад в аномальный магнитный момент. Действительно, при $k \to 0$ для первого слагаемого в (6) имеем $f\left(k^2\right)\left(\bar{u}_2 \gamma^\mu u_1\right) \to 2f(0)p^\mu$, где p^μ - импульс электрона.
(на этот раздел минимум часа 3 уйдет, если захочу разобраться.)

anomalous magnetic moment

$$\mu = \frac{e\hbar}{2mc} \left(1 + \frac{e^2}{2\pi\hbar c} \right)$$

(!!! это вот допишу, очень важные размышления, что вообще происходит, чтобы прийти к формуле!!!)
Ибо по Вергелесу

$$g(0) = \frac{e^2 m^2}{\pi} \int\limits_{4m^2}^{\infty} \frac{\mathrm{d}t}{t\sqrt{t\left(t - 4m^2\right)}} = \frac{e^2}{2\pi}$$

$$\mu = \frac{e\hbar}{2mc}(1 + g(0))$$

• One can show that the F_2 form factor affects the magnetic moment of the electron, by

$$\boldsymbol{\mu} = \frac{ge}{2m}\mathbf{S}, \quad g = 2 + 2F_2(0), \quad F_2(0) = \frac{\alpha}{2\pi}.$$

This is one of the most famous predictions of QED.
(кст, в теории создам раздел про вычисление этого!!!)
(!!! хотя бы одну диагрмму посчитаю по статье Волкова!!! проверю, что на самом деле могу это делать, отличная будет тренировка!)

118. Аномальный магнитный момент электрона по Ландау 4

По Зи (???)

(там мб Зи что-то годное вывел??)

По Мандл Шаулу

(???? общая идея то в чем???)

We saw in Section 9.6.1 that this correction to the magnetic moment stems from the vertex correction, Fig. 9.15(c), given by Eq. (9.48). By dimensional regularization, retaining the infrared cut-off λ, we can rewrite this equation

$$e_0^2\Lambda^\mu\left(p',p\right)=\frac{-\mathrm{i}e_0^2\mu^{4-D}}{(2\pi)^D}\int \mathrm{d}^D k\frac{N^\mu\left(p',p,k\right)}{\left(k^2-\lambda^2+\mathrm{i}\varepsilon\right)\left[\left(p'-k\right)^2-m^2+\mathrm{i}\varepsilon\right]\left[(p-k)^2-m^2+\mathrm{i}\varepsilon\right]},$$

where μ is the mass scale introduced in Section 10.3.1 and

$$N^\mu\left(p',p,k\right)=\gamma^\alpha\left(\not{p}'-\not{k}+m\right)\gamma^\mu(\not{p}-\not{k}+m)\gamma_\alpha$$

(а дальше словами пишу, потому что иначе формулы на страницу. Типа используем параметризацию Фейнмана, делаем замену переменных, дальше хрен знает что происходит, дальше разменую регуляризацию применяем (!??!?!??!?))

In order to derive the anomalous magnetic moment of the electron, we next consider the observable corrections which the vertex modification $e^2\Lambda^\mu\left(p',p\right)$ contributes to the scattering of electrons by an external static electromagnetic field. The Feynman amplitude for this is given by

$$M=\mathrm{i}e_0\bar{u}\left(\mathbf{p}'\right)e_0^2\Lambda^\mu\left(p',p\right)u(\mathbf{p})A_{\mathrm{e}\mu}\left(\mathbf{q}=\mathbf{p}'-\mathbf{p}\right).$$

The most general form for this amplitude, allowing for Lorentz invariance, the Lorentz gauge condition

$$q_\mu A_{\mathrm{e}}^\mu(\mathbf{q})=0$$

and the Gordon identity is

$$M=\mathrm{i}e_0\bar{u}\left(\mathbf{p}'\right)\left[\gamma^\mu F_1\left(q^2\right)+\frac{\mathrm{i}}{2m}\sigma^{\mu v}q_\nu F_2\left(q^2\right)\right]u(\mathbf{p})A_{\mathrm{e}\mu}(\mathbf{q})$$

where F_1 and F_2 are arbitrary functions of q^2. In order to calculate the magnetic moment, we

Получаем

$$F_2(0)=\frac{\alpha}{\pi}\int\limits_0^1 \mathrm{d}y\int\limits_0^{1-y}\mathrm{d}z\frac{1-y-z}{y+z}=\frac{\alpha}{2\pi}$$

which is the second-order correction to the magnetic moment quoted earlier, Eq. (9.72).

Lamb shift

(все указания про него в этом параграфе, должно хватить. порешаю, мб дело часов 3х.)

Радиационные поправки к закону Кулона по Вергелесу

(скоро продумаю)

(??? интеграл 48 Вергелеса какой??)

В частности:

а) $mr\gg 1$.

Интеграл (48) легко оценивается при помощи подстановки $\zeta = 1 + \xi$:

$$\varphi(r) = \frac{e_1}{r}\left[1 + \frac{e^2}{4\sqrt{\pi}} \cdot \frac{e^{-2mr}}{(mr)^{3/2}}\right], \qquad r \gg \frac{1}{m}.$$

б) $mr \ll 1$. Опуская вычисления, приводим ответ (см. ЛЛ.4, §114):

$$\varphi(r) = \frac{e_1}{r}\left[1 + \frac{2e^2}{3\pi}\left(\ln\frac{1}{mr} - C - \frac{5}{6}\right)\right], \qquad r \ll \frac{1}{m},$$

$C \approx 0,577$.

Идея в том, что....

$$\varphi_0(r) \equiv A_0^{(e)}(r) = \frac{e_1}{r}, \quad \mathbf{A}^{(e)} = 0.$$

$$\varphi_0(\mathbf{k}) = \frac{4\pi e_1}{\mathbf{k}^2}.$$

С учётом квантовых поправок:

$$\varphi = A_0^{(e)} + \mathcal{D}_{0\mu}\frac{\mathcal{P}^{\mu\nu}}{4\pi}A_\nu^{(e)}.$$

Получим первую поправку. Для этого поляризационный оператор возьмём в (32) и заменим точный фотонный пропагатор на свободный. С учётом того что $k^0 = 0$ получаем:

$$\varphi(\mathbf{k}) = \varphi_0(\mathbf{k}) + \mathcal{P}\left(-\mathbf{k}^2\right)\varphi_0(\mathbf{k}).$$

Отсюда видно, что возмущение потенциала есть

$$\delta\varphi(\mathbf{k}) = \frac{4\pi e_1}{\mathbf{k}^2}\mathcal{P}\left(-\mathbf{k}^2\right) = \frac{4\pi e_1}{y^2}\mathcal{P}\left(-y^2\right) \equiv \delta\varphi\left(-y^2\right)$$

(дальше преобразование интеграла, которое додумаю потом)

1.4.3 Coupling of Light to Matter

Theory

• While naively we can write down any Lorentz-invariant interaction term involving A_μ, we must be careful to maintain decoupling of the negative-norm and zero-norm states. By the Feynman rules, a scattering amplitude involving an external photon of momentum k and polarization $\zeta_\mu(k)$ has the form

$$\mathcal{M} = \zeta_\mu(k)\mathcal{M}^\mu$$

Note that by the Feynman rules, $\mathcal{M}^\mu$ depends on k but not on ζ.

The Ward identity:

$$k_\mu\mathcal{M}^\mu. = 0$$

It ensures the amplitude to produce a photon with $\zeta_\mu(k) \propto k_\mu$ is zero, which means the zero-norm states decouple as desired. We'll show later that the Ward identity implies that all unphysical polarizations decouple. (?????)

• Suppose the coupling to matter takes the form $A_\mu j^\mu$. The classical equation of motion is

$$\partial_\mu F^{\mu\nu} = j^\nu$$

which implies that $\partial_\mu j^\mu = 0$, so j^μ must be a conserved current. Heuristically, at the quantum level, since j^μ creates photons we have

$$M^\mu(k) \sim \int dx e^{ikx} \langle f | j^\mu(x) | i \rangle$$

so the Ward identity follows from conservation of the current j^μ.

• More generally, it can be shown that j^μ must be the conserved current arising from a global $U(1)$ symmetry of the action. To maintain gauge invariance, this symmetry must be gauged.

We will justify these statements more carefully later; now we turn to some examples. Note that different sources will differ on the sign of e below.

• First, consider coupling to Dirac fermions. We have a global $U(1)$ symmetry from phase rotation $\psi(x) \to e^{-ie\alpha}\psi(x)$, where we have introduced a dimensionless coupling constant e, which yields the conserved current $j^\mu = e\bar{\psi}\gamma^\mu\psi$.

• The interaction term can be written neatly with the covariant derivative,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi, \qquad D_\mu\psi := \partial_\mu\psi + ieA_\mu\psi$$

We gauge the global symmetry to

$$A_\mu \to A_\mu + \partial_\mu\alpha, \quad \psi \to e^{-ie\alpha}\psi$$

where α is a general function of x. Then the action is gauge-invariant because

$$D_\mu\psi \to \partial_\mu\left(e^{-ie\alpha}\psi\right) + ie\left(A_\mu + \partial_\mu\alpha\right)\left(e^{-ie\alpha}\psi\right) = e^{-ie\alpha}D_\mu\psi$$

• Given the coupling $A_\mu j^\mu$, the component j^0 can be interpreted as electric charge. Tracing back to our earlier results, this shows that particles and antiparticles have charge $\pm e$. One can also show that charge conjugation turns e to $-e$, as expected.

• Note that a gauge transformation with constant parameter is just the same as a global transformation, which does not take us to the same physical state; this must be true because we can infer a conservation law from the global symmetry. Only gauge transformations where the parameter goes to zero at infinity are 'true' gauge transformations.

• Next, we turn to scalar fields. A real scalar field has no suitable conserved current, so we focus on a complex scalar field φ, which has a symmetry $\varphi \to e^{-ie\alpha}\varphi$ and conserved current

$$j^\mu = ie\left(\varphi^\dagger\partial^\mu\varphi - \left(\partial^\mu\varphi^\dagger\right)\varphi\right)$$

However, the naive coupling $j^\mu A_\mu$ doesn't work, because j^μ depends on $\partial^\mu\varphi$. Then if we add the term $j^\mu A_\mu$ to the Lagrangian, the conserved quantity associated with the $U(1)$ global symmetry changes, so j^μ is no longer conserved!

• It's possible to fix this by manually adding terms, but a simpler, reliable method is 'minimal coupling', i.e. replacing partial derivatives with covariant derivatives as above. We have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left(D_\mu\phi\right)^\dagger D^\mu\phi - m^2\phi^\dagger\phi \qquad D_\mu\phi := \left(\partial_\mu + ieA_\mu\right)\phi,$$

$$\mathcal{L}_{\text{int}} = -\left(ie\left(\phi^\dagger\partial^\mu\phi - \left(\partial^\mu\phi^\dagger\right)\phi\right)A_\mu - e^2A_\mu A^\mu\phi^\dagger\phi\right) \equiv -j'^\mu A_\mu$$

$$\left(D_\mu D^\mu + m^2\right)\phi = 0.$$

$$j^{\prime\mu}_{U(1)\text{gl}} = ie\left(\phi^\dagger D^\mu \phi - \left(D^\mu \phi\right)^\dagger \phi\right)$$

But this final result would have been rather difficult to guess without the benefit of the covariant derivative. Also note that to lowest order in e, the coupling is $-j^\mu A_\mu$. This is a universal result for gauge theories.

1.4.4 Electron and Photon Self-Energy

(?? я хз, или тут об этом или дальше, в разделе про явления и рад. поправки???)

Electron SE

The bare propagator is the tree-level propagator:

$$S_F(x-y) = \int dp e^{-ip(x-y)} \frac{i\left(\not{p} + m_0\right)}{p^2 - m_0^2 + i\epsilon} = \int dp e^{-ip(x-y)} \frac{i}{\not{p} - m_0 + i\epsilon}$$

where m_0 - bare fermion mass, the latter notation is just suggestive, motivated by 'factoring the difference of squares'.

• We define the electron self-energy $\Sigma(\not{p})$ by the sum of 1PI diagrams,

$$A - 1\text{PI} - B \equiv -i\Sigma(p)_{AB}$$

By Dyson resummation, the full propagator is

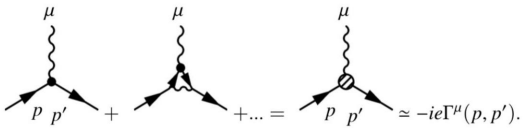
$$\int dp e^{-ip(x-y)} \frac{i}{\not{p} - m_0 - \Sigma(\not{p}) + i\epsilon}$$

and hence we call $\Sigma(\not{p})$ the self-energy of the electron. Note the self-energy has two spinor indices which contract with the spinor indices of the incoming and outgoing electron.

The full photon propagator is corrected by Dyson resummation of the 1PI contribution,

and $\Pi^{\mu\nu}\left(q^2\right)$ is called the self-energy of the photon or the vacuum polarization.

• Finally, the interaction vertex receives radiative corrections, and summing all these corrections gives the effective vertex,



We only count amputated diagrams, which in this case coincide with 1PI diagrams.

• Generally, we will find UV divergences from virtual particles with arbitrarily high momenta; they must be tamed by regularization. There are also IR divergences for loops with photons; these cancel against IR divergences from the radiation of soft photons.

Note. One strange feature of Dyson resummation is that it involves summing over diagrams of all orders, even though we only calculate the 1PI diagrams (and most other things) to a fixed order. For instance, if we calculate the sum of 1PI diagrams up to $O(e^2)$, then the Dyson sum includes additional contributions of $O(e^4)$ and higher. But this extra accuracy seems pointless, because there are also 1PI diagrams of $O(e^4)$ we're neglecting!

The real point is that in processes where the intermediate propagator is nearly on-shell, the higherorder terms in the Dyson sum become important. To see this, note that $\Sigma \sim m_0$ on dimensional grounds, so each term in the Dyson series is suppressed relative to the last by a factor

$$\Sigma\left(p^2\right) D_F^0\left(p^2\right) \sim \frac{e^2 m_0}{\not{p} - m_0} \sim \frac{e^2 m_0^2}{p^2 - m_0^2}$$

This ratio can be large when $p^2 - m_0^2$ is small enough, in which case the entire Dyson series has to be summed to get an accurate answer, even at $O(e^2)$. That is, in some situations we must sum over diagrams of all orders merely to ensure accuracy at a fixed order.

We need to perform Dyson resummation for the photon propagator for a similar reason. For processes with low-energy photons, the full Dyson series needs to be summed to get an accurate result, and the result is the "running coupling" where the effective strength of the electromagnetic interaction varies with energy, as described below. A more obscure example of this phenomenon occurs when two nonrelativistic charged particles scatter. The propagator for photon exchange in the t-channel can be very large, since $1/t \sim 1/v^2$. In this case, all the "ladder" diagrams involving n photon exchanges are comparable in size, and must be summed to get an accurate result. The result is a complex structure with resonant peaks corresponding to the bound states of the particles, which matches what one finds using the nonrelativistic Schrodinger equation. These kinds of calculations are not covered in standard introductory textbooks, but can be found in Quantum Electrodynamics by Berestetskii, Lifshitz, and Pitaevskii.

Calculation of the one-loop electron self-energy.

The one-loop contribution is due to emission and absorption of a virtual photon:

$$-i\Sigma_2(\not{p}) = (-ie)^2 \int dk \frac{\gamma^\mu \left(i\left(\not{K} + m_0\right)\right) \gamma_\mu (-i)}{\left(k^2 - m_0^2 + i\epsilon\right)\left((p-k)^2 - \mu^2 + i\epsilon\right)}$$

Feynman rules, Feynman gauge, μ - small photon mass.

(!!! проверю этот вывод, добавлю тут тоже диаграмму!)

$$i\Sigma_2(\not{p}) = e^2 \int_0^1 dx \int d\ell \frac{\gamma^\mu \left(\not{K} + m_0\right) \gamma_\mu}{\left(\ell^2 - \Delta + i\epsilon\right)^2}, \qquad \begin{array}{l} \ell := k - xp, \\ \Delta := -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2 \end{array}$$

$d\ell^0$ integral above can be viewed as a contour integral, with poles in the second and fourth quadrants. We can rotate the contour to go along the positive imaginary axis.

$\ell^0 = i\ell_E^0$,

$$\int \frac{d\ell}{\left(\ell^2 - \Delta + i\epsilon\right)^m} = i(-1)^m \int \frac{d\ell_E}{\left(\ell_E^2 + \Delta - i\epsilon\right)^m}, \quad \ell_E^2 = \left(\ell_E^0\right)^2 + \boldsymbol{\ell} \cdot \boldsymbol{\ell}$$

where the sign flip occurs because the metric for (ℓ_E^0, ℓ) is negative definite. Finally, we drop the $i\epsilon$ and regard the integral as an integral in real Euclidean space $\mathbb{R}^4$.

• In our case, we would like to evaluate the UV-divergent integral

$$I_4 = \int \frac{d\ell_E}{\left(\ell_E^2 + \Delta\right)^2}$$

The simplest method is a hard UV cutoff at $r = \Lambda$, which gives $I_4 \sim \log\left(\Lambda^2/\Delta\right)$, which is a log-divergence. However, this method breaks gauge invariance, since a gauge transformation can introduce arbitrarily high Fourier modes.

We instead regulate the integral with dimensional regularization (DR), using $d = 4 - \epsilon$ dimensions. For the angular part, note that We now set $d = 4 - \epsilon$ and taking the limit $\epsilon \to 0$. The dimensions of I_d depend on d, so we introduce a compensating mass scale $\widetilde{M}$, giving

$$I_4 = \lim_{\epsilon\to 0} \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} + \log\frac{M^2}{\Delta} + O(\epsilon)\right), \quad M^2 = 4\pi e^{-\gamma}\widetilde{M}^2$$

Simplifying the numerator with spinor identities gives

$$\Sigma_2(\not{p}) = \lim_{\epsilon\to 0} \frac{\alpha}{2\pi}\int_0^1 dx\,((2-\epsilon/2)m_0 - (1-\epsilon/2)x\not{p})\left(\frac{2}{\epsilon} + \log\frac{M^2}{\Delta}\right)$$

The same result can be carried out in Pauli-Villars regularization, where we subtract off a loop diagram containing a fictitious massive particle of mass Λ in place of the photon. This is a better approach than the hard-cutoff, and introducing an arbitrary mass scale M gives

$$\Sigma_2(\not{p}) = \lim_{\Lambda\to\infty} \frac{\alpha}{2\pi}\int_0^1 dx\,(2m_0 - x\not{p})\left(\log\frac{x\Lambda^2}{M^2} + \log\frac{M^2}{\Delta}\right)$$

The momentum-dependent terms are the same, while the log-divergence matches with that seen with the hard cutoff, and translates to a $1/\epsilon$ pole in DR.

Spectral representation

To understand the consequences of the self-energy, we recall the spectral representation.

• Applying the spectral representation, we have

$$\frac{i}{\not{p} - m_0 - \Sigma(\not{p})} = \frac{iZ_2}{\not{p} - m} + \text{ terms analytic at } m$$

The physical mass m is the location of the lowest-lying pole, and hence satisfies

$$\not{p} - m_0 - \Sigma(\not{p})\big|_{\not{p}=m} = 0$$

We can extract the residue Z_2 by differentiating with respect to $\not{p}$ here, giving

$$Z_2^{-1} = 1 - \left.\frac{d\Sigma(\not{p})}{d\not{p}}\right|_{\not{p}=m}$$

In both equations above, the equality $\not{p} = m$ looks rather strange, though it's straightforward to use. It is a valid shortcut for a more correct procedure, as justified here.

• Since we are working to lowest order in α, the mass shift is

$$\delta m = m - m_0 = \Sigma_2 \left(\not{p} = m_0\right) + O\left(\alpha^2\right)$$

where we evaluated Σ_2 at m_0 rather than m since the resulting error is second-order.

• We need to interpret the fact that the correction δm is divergent. The parameter m is physical, while m_0 can never be measured directly. Hence we interpret our mass renormalization as a definition of m_0 in terms of m in terms of the regularization parameters (e.g. ϵ and M). As the regularization is removed, m_0 diverges, but this is acceptable since it is not physical. The cost of this interpretation is that m is now an input rather than an output of the theory.

Note. In classical electrodynamics, the UV divergence of the mass is even worse, as

$$\delta m \sim \int d\mathbf{x} E^2 \sim \alpha \int \frac{r^2 dr}{r^4} \sim \alpha\Lambda$$

The reason the divergence is less strong in quantum electrodynamics is that when $m_0 = 0$, the electron separates into two uncoupled Weyl spinors, which must remain massless. Then assuming δm is analytic in m_0, the leading term is linear in m_0, and for that term the dependence on Λ must be logarithmic; indeed, we have $\delta m \sim m_0 \log\left(\Lambda^2/m_0^2\right)$.

Photon Self-Energy

By Lorentz invariance, its tensorial structure can only depend on $\eta^{\mu\nu}$ and $q^\mu q^\nu$. By the Ward identity, the photon self-energy is also transverse, $q^\mu \Pi_{\mu\nu}(q) = 0$. (Note that this does not imply the photon propagator is transverse; instead, this depends on the gauge.) Therefore, the relative magnitude of the two terms is fixed, as

$$i\Pi^{\mu\nu}(q) = \left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) f\left(q^2\right)$$

Heuristically, $i\Pi^{\mu\nu}(q)$ is regular at $q^2 = 0$ because poles can only come from single-particle massless intermediate states. (This is a bit too quick; there are subtleties here, but they don't apply to QED in four dimensions.)

• Therefore, we can pull out a q^2 factor from $f\left(q^2\right)$ to give

$$i\Pi^{\mu\nu}(q) = \left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right)\Pi\left(q^2\right)$$

with $\Pi\left(q^2\right)$ also regular at $q^2 = 0$. Note that when we talk about regularity, we always imagine a UV cutoff, since $\Pi^{\mu\nu}(q)$ is UV divergent.

• As a result, when we apply Dyson resummation, only the transverse part of the propagator is affected. In general R_ξ gauge, the exact propagator is

$$\begin{aligned}\Delta_{\mu\nu}(q) &= \Delta^0_{\mu\nu}(q) + \Delta^0_{\mu\rho}(q)\Pi^{\rho\sigma}(q)\Delta^0_{\sigma\nu}(q) + \ldots \\ &= \frac{-i}{q^2\left(1 - \Pi\left(q^2\right)\right)}\left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \underbrace{\xi\frac{-i}{q^2}\frac{q_\mu q_\nu}{q^2}}_{\approx 0 \text{ Landau gauge} \xi = 0}\end{aligned}$$

We see the photon remains massless, since $\Pi\left(q^2\right)$ is regular at $q = 0$, and

$$Z_3 = \frac{1}{1 - \Pi(0)}$$

• Explicitly, one can show that at one-loop, under DR,

$$\Pi_2\left(q^2\right) = -\frac{2\alpha}{\pi}\lim_{\epsilon\to 0}\int_0^1 dx x(1-x)\left(\frac{2}{\epsilon}+\log\frac{M^2}{\Delta}\right), \quad \Delta = m_0^2 - x(1-x)q^2$$

If we had instead used a hard cutoff, we would have found a correction to $\Pi_2^{\mu\nu}(q)$ proportional to $\eta^{\mu\nu}$, violating the Ward identity. Path integral formulation tells us that such anomalous symmetries occur because the path integral measure is not invariant under the classical symmetry.

Running coupling

The photon field renormalization leads to a running coupling.

• Consider electron-electron scattering by an intermediate photon of momentum q. Then the amplitude is proportional to

$$\frac{\left(-ie_0\right)^2}{1-\Pi\left(q^2\right)}$$

this motivates to define the effective coupling

$$e\left(q^2\right) = \frac{e_0}{\sqrt{1-\Pi\left(q^2\right)}}$$

Note that there are other processes that contribute, e.g. by the exchange of two photons. However, to 'lowest order' this is a good estimate of the total scattering amplitude.

• As with the electron mass, $\Pi\left(q^2\right)$ is divergent, so we define the physical/renormalized coupling

$$e = \frac{e_0}{\sqrt{1-\Pi(0)}} = e_0\sqrt{Z_3}$$

As before, we then flip this around and regard it as an expression for the divergent bare quantity e_0 in terms of the physical quantity e. Therefore, the effective coupling and physical coupling are related at one-loop order by

$$e^2\left(q^2\right) = \frac{1-\Pi_2(0)}{1-\Pi_2\left(q^2\right)}e^2 + O\left(\alpha^2\right) = \frac{e^2}{1-\widehat{\Pi}_2\left(q^2\right)} + O\left(\alpha^2\right)$$

$$\widehat{\Pi}_2\left(q^2\right) := \Pi_2\left(q^2\right) - \Pi_2(0) = -\frac{2\alpha}{\pi}\int_0^1 dx x(1-x)\log\frac{m^2}{m^2 - x(1-x)q^2}$$

where we replaced m_0 with m, which is accurate up to $O\left(\alpha^2\right)$ terms. Physically, this result tells us that the interaction between electrons gets stronger at high energies; note that the unphysical renormalization scale M has dropped out.

• The e that appears in Coulomb's law, which corresponds to the typical quoted value $\alpha \approx 1/137$, applies at very low energies. At energies $q^2 \sim m_W^2$, we instead have $\alpha \approx 1/128$.

Note. Heuristically, the vertex renormalization is really what renormalizes the charge e, if we think of e as the coefficient of the vertex factor. But as we'll see below, this effect is exactly canceled by the renormalization of the electron field Z_2. The effect we've computed above is really a renormalization of the electromagnetic field, but since its effects always appear in tandem with an electron-field interaction, it can be thought of as renormalizing the charge.

Interpreting the running coupling.

In the nonrelativistic limit, the potential is

$$V(\mathbf{q}) = -\frac{e^2}{|\mathbf{q}|^2\left(1-\hat{\Pi}_2\left(-|\mathbf{q}|^2\right)\right)}$$

and the real-space potential is found by Fourier transform. Taking the inverse Fourier transform is tricky, since for high $|\mathbf{q}|$ the $\hat{\Pi}_2$ factor provides a branch cut; doing this yields the Uehling potential.

High and low limit

Alternatively, for low $|\mathbf{q}|$ we can simply expand $\hat{\Pi}_2$ in a Taylor series. The zeroth order term slightly modifies e, while the $|\mathbf{q}|^2$ term contributes a constant to $V(\mathbf{q})$ and hence a delta function to the potential. This slightly lowers the energy of the $2s$ state in hydrogen relative to the $2p$ state, producing the Lamb shift.

In the opposite limit, we consider a high-energy scattering where the photon is in the t-channel or u-channel, so $-q^2 \gg m^2$. Then

$$\alpha_{\text{eff}}\left(q^2\right) = \frac{e^2\left(q^2\right)}{4\pi} \approx \alpha\left(1-\frac{\alpha}{3\pi}\log\frac{-q^2}{\exp(5/3)m^2}\right)^{-1}$$

by straightforward calculation. This diverges for very high energies at the Landau pole of QED.

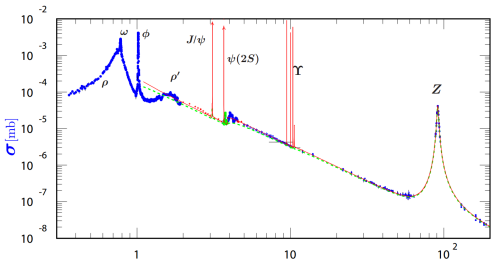
1.5 Main QFT processes and experiments

(по большей части тут КЭД. просто нет такого прямо на него ограничения, ибо часто какая-то частица вне КЭД появялется.)

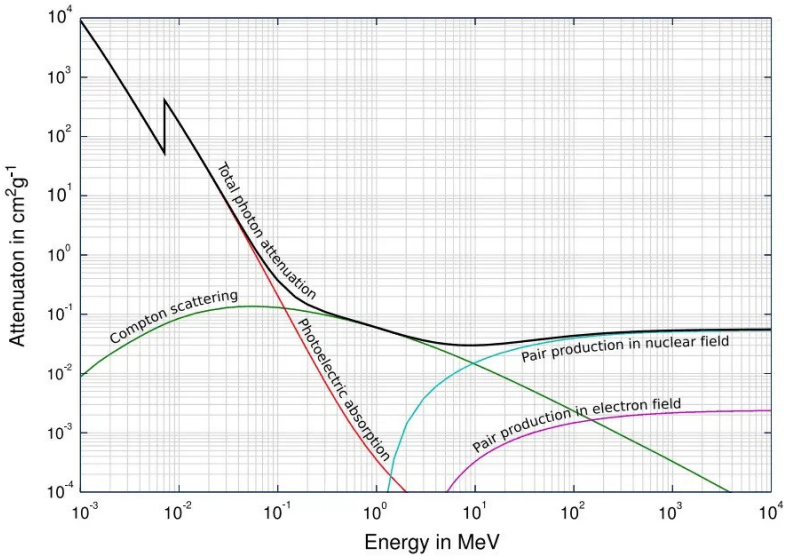
1.5.1 Overview of QFT processes and their comparison (???)

(!!! допишу обзор процессов, которые вообще мы решать будем!!! пока нужно додумать их, что вообще там происходит и какие графики вообще имеются! допишу абзацы с пояснениями каждого графика и словами указаниями про вывод!)

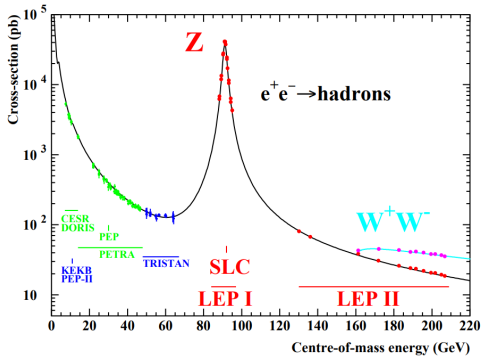
Электрон-позитронное рассеяние имеет сечение:



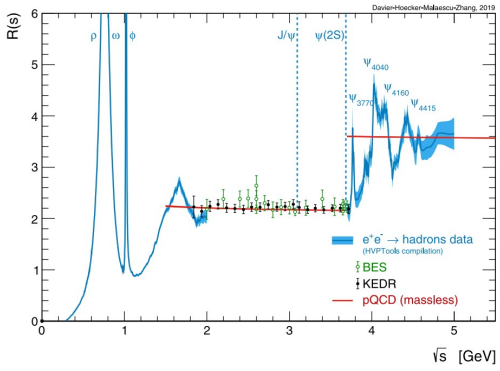
total photon attenuation:

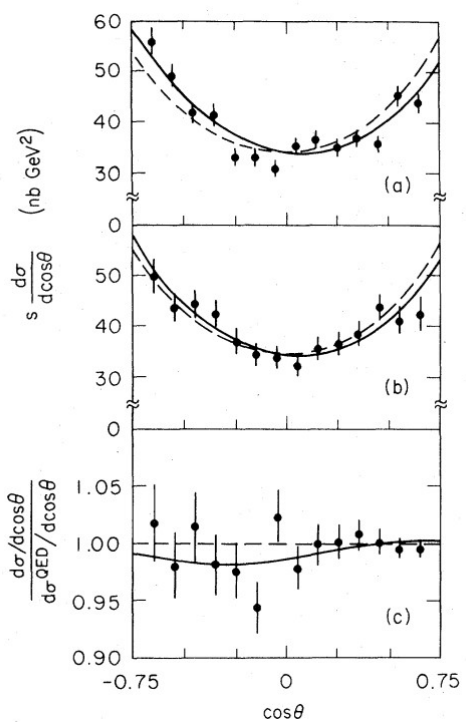


e+e-hadrons



el-el-scatt



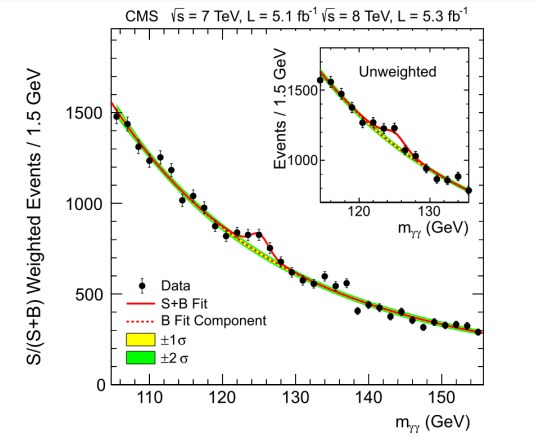


The differential cross sections for (a) Reaction (2) and (b) Reaction (3), normalized to the QED prediction. (c) The ratio of the differential cross section for Reaction (1) to the $O(\alpha^3)$ QED prediction. The experimental cross section in (c) is normalized to the electroweak cross section taken from a simultaneous fit to all three reactions. In (a) - (c) the dashed line corresponds to the $O(\alpha^3)$ QED cross section and the solid line to the fitted cross section.

$$e^+e^- \to e^+e^- \qquad (1)$$
$$e^+e^- \to \mu^+\mu^- \qquad (2)$$
$$e^+e^- \to \tau^+\tau^- \qquad (3)$$

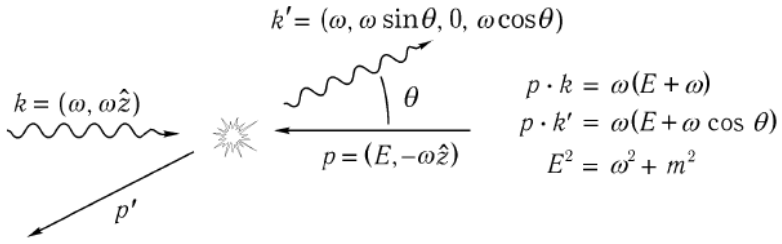
at a center-of-mass energy $\sqrt{s}$ of 29GeV. (by M. E. Levi, C. A. Blocker, et. al.)

higgs



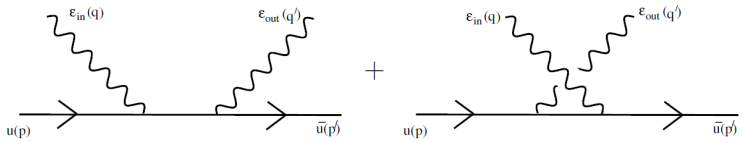
1.5.2 Typical scatterings, decays in QED

Compton Scattering $\gamma e^- \to \gamma e^-$



$$i\mathscr{M} = -ie^2\varepsilon_\mu^*(k')\,\varepsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu k\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu \not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u(p)$$

где векторы поляризации начального и конечного фотонов $\varepsilon_\nu(k)$ и $\varepsilon_\mu^*(k')$, а также были произведены некоторые преобразования (??? потом допишу).



$$= i(-ie)^2\bar{u}^{r'}(\vec{p}')\left(\frac{\gamma_\mu(\not{p}+\not{q}+m)\gamma_\nu}{(p+q)^2-m^2} + \frac{\gamma_\nu(\not{p}-\not{q}'+m)\gamma_\mu}{(p-q')^2-m^2}\right)u^s(\vec{p})\,\epsilon_{\text{in}}^\nu\,\epsilon_{\text{out}}^\mu$$

(тут связь с простейшим описанием из квантфиза (??))

еще раз прочитаю ЛЛ-86. Рассеяние фотона электроном, там точно ничего нового???

Сохранение 4-импульса при рассеянии фотона свободным электроном (эффект Комптона) выражается равенством

$$p + k = p' + k'$$

где p и k – 4-импульсы электрона и фотона до столкновения, а p' и k' - их 4-импульсы после столкновения.

$$s = (p+k)^2 = (p'+k')^2 = m^2 + 2pk = m^2 + 2p'k',$$
$$t = (p-p')^2 = (k'-k)^2 = 2\left(m^2 - pp'\right) = -2kk',$$
$$u = (p-k')^2 = (p'-k)^2 = m^2 - 2pk' = m^2 - 2p'k,$$
$$s + t + u = 2m^2.$$

Рассматриваемый процесс изображается двумя диаграммами Фейнмана (74, 14), и его амплитуда

$$M_{fl} = -4\pi e^2 e_\mu'^* e_v\left(\bar{u}'Q^{\mu v}u\right),$$

$$Q^{\mu\nu} = \frac{1}{s-m^2}\gamma^\mu(\gamma p + \gamma k + m)\gamma^\nu + \frac{1}{u-m^2}\gamma^\nu\left(\gamma p - \gamma k' + m\right)\gamma^\mu.$$

$$d\sigma = \frac{\pi e^4}{4}\frac{dt}{\left(s-m^2\right)^2}\,\mathrm{Sp}\left\{(\gamma p'+m)\,Q^{\lambda\mu}(\gamma p+m)\bar{Q}_{\lambda\mu}\right\}$$

$$g(s,u) = \frac{2m^2}{(s-m^2)\,(u-m^2)}\left\{4m^2+\left(s-m^2\right)+\left(u-m^2\right)\right\}.$$

$$d\sigma = 8\pi r_e^2\frac{m^2dt}{\left(s-m^2\right)^2}\left\{\left(\frac{m^2}{s-m^2}+\frac{m^2}{u-m^2}\right)^2+\left(\frac{m^2}{s-m^2}+\frac{m^2}{u-m^2}\right)-\frac{1}{4}\left(\frac{s-m^2}{u-m^2}+\frac{u-m^2}{s-m^2}\right)\right\}.$$

где $r_e = e^2/m$.

(вывод потом понимать буду)

$$d\sigma = \frac{r_e^2}{2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'}+\frac{\omega'}{\omega}-\sin^2\vartheta\right)do'$$

Наибольший член в сечении $(86,6)$

$$d\sigma \approx 8\pi r_e^2\frac{m^2dt}{4\,(s-m^2)\,(m^2-u)},$$

откуда

$$d\sigma = \frac{r_B^2}{2}\frac{do'}{1+(\pi-\theta)^2\omega^2/m^2}.$$

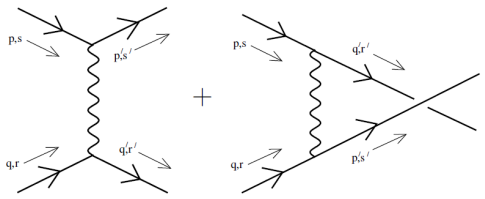
Сечение $d\sigma/do' \sim r_e^2$ в узком конусе $\pi-\theta \leqslant m/\omega$, а вне его уменьшается по порядку величины в ω^2/m^2 раз.

(write that in nonrel limit we get the Thompson cross section, ...)

Electron-electron $e^- + e^- \to e^- + e^-$ scattering (Moller scattering)

Main results

(решу еще раз, потом суть выпишу)



$$= -i(-ie)^2\left(\frac{[\bar{u}^{s'}(\vec{p}')\gamma^\mu u^s(\vec{p})]\,[\bar{u}^{r'}(\vec{q}')\gamma_\mu u^r(\vec{q})]}{(p'-p)^2}-\frac{[\bar{u}^{s'}(\vec{p}')\gamma^\mu u^r(\vec{q})]\,[\bar{u}^{r'}(\vec{q}')\gamma_\mu u^s(\vec{p})]}{(p-q')^2}\right)$$

$$S^{(2)}\left(2e^-\to 2e^-\right) = \frac{-e^2}{2!}\int \mathrm{d}^4x_1\;\mathrm{d}^4x_2\;\mathrm{N}\left[\left(\bar{\psi}^-\gamma^\alpha\psi^+\right)_{x_1}\left(\bar{\psi}^-\gamma^\beta\psi^+\right)_{x_2}\right]\mathrm{i}D_{\mathrm{F}\alpha\beta}\left(x_1-x_2\right).$$

$$|i\rangle = c^\dagger(2)c^\dagger(1)|0\rangle \qquad \to \qquad |f\rangle = c^\dagger\left(2'\right)c^\dagger\left(1'\right)|0\rangle$$

In order to obtain explicit expressions for these two contributions, let

$$\psi_j^+(x) = c(j) f_j(x), \quad \bar{\psi}_j^-(x) = c^\dagger(j) g_j(x)$$

be the parts of the operators $\psi^+(x)$ and $\bar{\psi}^-(x)$ proportional to $\mathrm{c}(j)$ and $\mathrm{c}^\dagger(j)$ respectively. $j = 1, 2, 1', 2'$ labels the electron states involved in the process. The part of the S-matrix operator (7.14) which effects the transition $|i\rangle \to |f\rangle$, Eq. (7.15), is then given by

$$S^{(2)}_{(e^-(1)+e^-(2)\to e^-(1')+e^-(2'))} = S_\mathrm{a} + S_\mathrm{b}$$

$$S_\mathrm{a} = -e^2 \int \mathrm{d}^4x_1\ \mathrm{d}^4x_2\ \mathrm{N}\left[\left(\bar{\psi}^-_{1'}\gamma^\alpha\psi^+_1\right)_{x_1}\left(\bar{\psi}^-_{2'}\gamma^\beta\psi^+_2\right)_{x_2}\right] \mathrm{i}D_{\mathrm{F}\alpha\beta}\left(x_1 - x_2\right)$$

$$S_\mathrm{b} = -e^2 \int \mathrm{d}^4x_1\ \mathrm{d}^4x_2\ \mathrm{N}\left[\left(\bar{\psi}^-_{2'}\gamma^\alpha\psi^+_1\right)_{x_1}\left(\bar{\psi}^-_{1'}\gamma^\beta\psi^+_2\right)_{x_2}\right] \mathrm{i}D_{\mathrm{F}\alpha\beta}\left(x_1 - x_2\right).$$

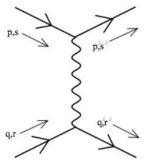
$$\begin{aligned}
p_1 \cdot p_2 = p_1' \cdot p_2' &\approx 2E^2, \\
p_1 \cdot p_2' = p_1' \cdot p_2 &\approx 2E^2 \cos^2(\theta/2), \\
p_1 \cdot p_1' = p_2' \cdot p_2 &\approx 2E^2 \sin^2(\theta/2), \\
\left(p_1' - p_1\right)^2 &\approx -2p_1 \cdot p_1' = -4E^2 \sin^2(\theta/2), \\
\left(p_2' - p_1\right)^2 &\approx -2p_1 \cdot p_2' = -4E^2 \cos^2(\theta/2).
\end{aligned}$$

$$\begin{aligned}
\frac{d\bar{\sigma}}{d\Omega} &= \frac{e^4m^4}{8(2\pi)^2} \int \frac{E^2 dE}{E^4\beta} \left|\mathcal{M}_{fi}\right|^2 \delta\left(E_1' + E_2' - E_1 - E_2\right) \\
&= \frac{\alpha^2}{8E^2}\left[\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{2}{\sin^2(\theta/2)\cos^2(\theta/2)} + \frac{1+\sin^4(\theta/2)}{\cos^4(\theta/2)}\right].
\end{aligned}$$

(а дальше я не понял, у Мандл Шаула какой-то текст, а итоговый ответ я так и не вижу, так что потом почитаю это.)

Recovering the Coulomb potential.

• First, consider electron-electron scattering, with the following amplitude.



$$= -i(-ie)^2 \frac{[\bar{u}(\vec{p}')\gamma^\mu u(\vec{p})]\,[\bar{u}(\vec{q}')\gamma_\mu u(\vec{q})]}{(p'-p)^2}$$

• In the nonrelativistic limit, the spinors become $u(p) \to \sqrt{m}(\xi, \xi)^T$. Then the only term in the spinor contractions that contributes is $\mu = 0$ (which means the interaction is mediated by a timelike photon) giving a numerator of $(2m)^2$, canceling the relativistic normalization.

• Working in the center-of-mass frame, the process is elastic, so the denominator gives a factor of $-(\Delta\mathbf{p})^2$ where $\Delta\mathbf{p}$ is the momentum transfer. The nonrelativistic amplitude for scattering is $\mathcal{M} \sim -i\widetilde{U}(\Delta\mathbf{p})$ by the Born approximation, giving

$$U(\mathbf{p}) = \frac{e^2}{p^2}$$

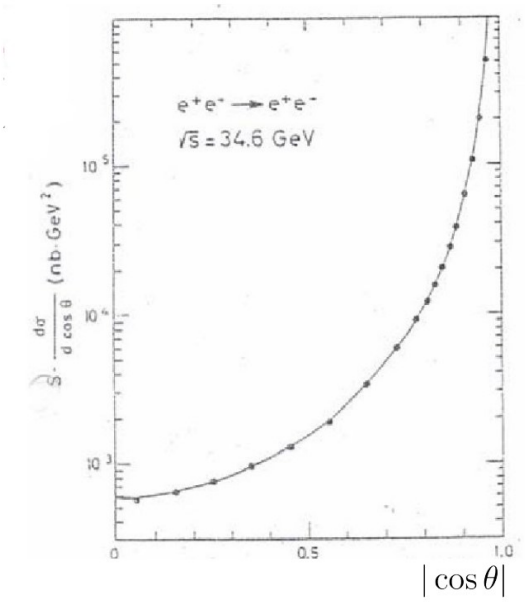
which is a repulsive Coulomb potential.

Electron-Positron $e^- + e^+ \to e^- + e^+$ Scattering (Bhabha Scattering)

Main results

Perfect agreement with QED, first order computation:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha}{2s}\left(\frac{3+\cos^2\theta}{1-\cos\theta}\right)^2$$

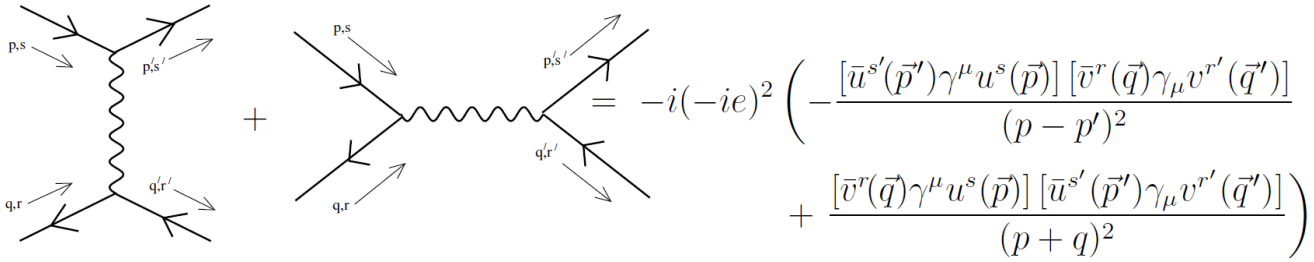


divergent for $\cos\theta \to 1$
Additional „heavy photon“:

$$\frac{1}{q^2} \to \frac{1}{q^2}\left(1 \mp \frac{q^2}{q^2-\Lambda_\pm^2}\right)$$

$\Rightarrow$ Can be described by form factor:

$$F\left(q^2\right) = 1 \mp \frac{q^2}{q^2-\Lambda_\pm^2}$$



(!! если другие продукты реакции - это другое)

О рассеянии электронов и позитронов нв электроне по Ландау 4 (???)

(!!! убеждусь, чтобы тут выше было четкое указание на методы, которые в этом параграфе выписываю!!! пока вроде четких указаний нет)

$$s = (p_1 + p_2)^2 = 2\left(m^2 + p_1 p_2\right),$$
$$t = (p_1 - p_1')^2 = 2\left(m^2 - p_1 p_1'\right),$$
$$u = (p_1 - p_2')^2 = 2\left(m^2 - p_1 p_2'\right),$$
$$s + t + u = 4m^2.$$

Рассматриваемый процесс изображается двумя диаграммами Фейнмана $(73, 13 - 14)$, и его амплитуда

$$M_{fl} = 4\pi e^2 \left\{ \frac{1}{t} \left(\bar{u}_2' \gamma^\mu u_2\right) \left(\bar{u}_1' \gamma_\mu u_1\right) - \frac{1}{u} \left(\bar{u}_1' \gamma^v u_2\right) \left(\bar{u}_2' \gamma_v u_1\right) \right\}.$$

Согласно указанным в ∅65 правилам для состояний начальных и конечных частиц, описывающихся полярнзационными мат рицами плотности $\rho_1, \rho_1', ...$, заменяем

$$|M_{fi}|^2 \to 16\pi^2 e^4 \{ \frac{1}{t^2} \operatorname{Sp}\left(\rho_2' \gamma^\mu \rho_2 \gamma^v\right) \operatorname{Sp}\left(\rho_1' \gamma_\mu \rho_1 \gamma_v\right) +$$
$$+ \frac{1}{u^2} \operatorname{Sp}\left(\rho_1' \gamma^\mu \rho_2 \gamma^v\right) \operatorname{Sp}\left(\rho_2' \gamma_\mu \rho_1 \gamma_v\right) -$$
$$- \frac{1}{tu} \operatorname{Sp}\left(\rho_2' \gamma^\mu \rho_2 \gamma^v \rho_1' \gamma_\mu \rho_1 \gamma_v\right) - \frac{1}{tu} \operatorname{Sp}\left(\rho_1' \gamma^\mu \rho_2 \gamma^v \rho_2' \gamma_\mu \rho_1 \gamma_v\right) \Big\}.$$

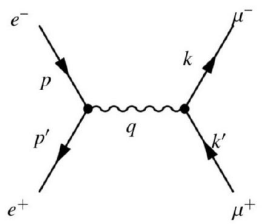
(??? результат????)

Attraction of electron and positron

• Next, we consider electron-positron scattering. In this case, we pick up a sign flip from fermionic statistics, as in Yukawa theory. In Yukawa theory, there was an second sign flip since $\bar{v}v \to -2m$, but here we instead have $\bar{v}\gamma^0 v \to 2m$. Then all the logic above goes through as before, and the result is an attractive Coulomb potential.

• Thus, we have shown that while forces mediated by spin-0 particles are universally attractive, forces mediated by spin-1 particles are not. One might say the sign flip really comes from the negative norm of the timelike photon.

Electron to muon scattering $e^+e^- \to \mu^+\mu^-$



We set the initial spin polarizations to s and s' and the final polarizations to r and r'.

$$i\mathcal{M}\left(s, s', r, r'\right) = \frac{ie^2}{q^2} \left(\bar{v}_{s'}\left(p'\right) \gamma^\mu u_s(p)\right) \left(\bar{u}_r(k) \gamma_\mu v_{r'}\left(k'\right)\right), \quad q = p + p' = k + k'$$

Often, the initial state is unpolarized, and the detector cannot measure the polarization, then

$$|\mathcal{M}|^2_{\text{tot}} = \frac{1}{2}\sum_s \frac{1}{2}\sum_{s',r,r'} |\mathcal{M}(s,s',r,r')|^2$$

Using the identities $(\gamma^0)^\dagger = \gamma^0$ and $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$, we have

$$(\bar{v}_{s'}\gamma^\mu u_s)^* = \bar{u}_s\gamma^\mu v_{s'}$$

A similar rule holds in general: the gamma matrices just reverse in order, and there's an additional sign flip for every factor of γ^5 since $\gamma^0\gamma^5\gamma^0 = -\gamma^5 = (-\gamma^5)^\dagger$. For example, $(\bar{v}_{s'}\gamma^\mu\gamma^\nu\gamma^\rho u_s)^* = \bar{u}_s\gamma^\rho\gamma^\nu\gamma^\mu v_{s'}$.

In our current case, the total squared amplitude has two factors,

$$|\mathcal{M}|^2_{\text{tot}} = \frac{1}{4}\frac{e^4}{q^4}\sum_{ss'rr'} (\bar{u}_s\gamma^\mu v_{s'}\bar{v}_{s'}\gamma^\nu u_s)(\bar{v}_{r'}\gamma_\mu u_r\bar{u}_r\gamma_\nu v_{r'})$$

where we suppress momentum arguments. After the completeness relations $\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m_e$, $\sum_s v_s(p)\bar{v}_s(p) = \not{p} - m_\mu$

$$|\mathcal{M}|^2_{\text{tot}} = \frac{1}{4}\frac{e^4}{q^4}\operatorname{tr}\left((\not{p}+m_e)\gamma^\mu(\not{p}''-m_e)\gamma^\nu\right)\operatorname{tr}\left((\not{k}''-m_\mu)\gamma_\mu(\not{k}+m_\mu)\gamma_\nu\right)$$

More generally, we will have multiple diagrams that contribute, so the squared amplitude will have cross-terms.

• Sometimes it's useful to rearrange things using the Clifford algebra, $\not{p}\not{q} = 2p\cdot q - \not{q}\not{p}$.

Note that the γ^5 identities use the convention $\epsilon^{0123} = 1$, $\epsilon_{0123} = -1$. (??? они такие???)

This is the opposite of the convention in general relativity, where the volume form naturally has lowered indices, so $\epsilon_{0123} = \sqrt{|g|} > 0$.

• Applying these identities to our result and setting m_e to zero gives

$$|\mathcal{M}|^2_{\text{tot}} = \frac{8e^4}{q^4}\left((p\cdot k)(p'\cdot k') + (p'\cdot k)(p\cdot k') + (p\cdot p')m_\mu^2\right)$$

Working in the center of mass frame, let the initial energies of the particles be E, and let the outgoing muon come out at an angle θ from the incoming electron. Then

$$|\mathcal{M}|^2_{\text{tot}} = e^4\left(1 + m_\mu^2/E^2 + \left(1 - m_\mu^2/E^2\right)\cos^2\theta\right).$$

• Finally, we may substitute this into the $2\to 2$ scattering cross section and integrate for

$$\sigma = \frac{4\pi\alpha^2}{3E_{\text{cm}}}\sqrt{1+m_\mu/E}\left(1+m_\mu^2/2E^2\right), \quad \alpha = \frac{e^2}{4\pi}$$

Here, the factor in parentheses is determined by the dynamics of QED, while everything else is determined by the "kinematics", i.e. by dimensional analysis and the volume of available phase space. E.g. for $E_{\text{cm}} \gg m_\mu$ we must have $\sigma \propto 1/E_{\text{cm}}^2$ by dimensional analysis.

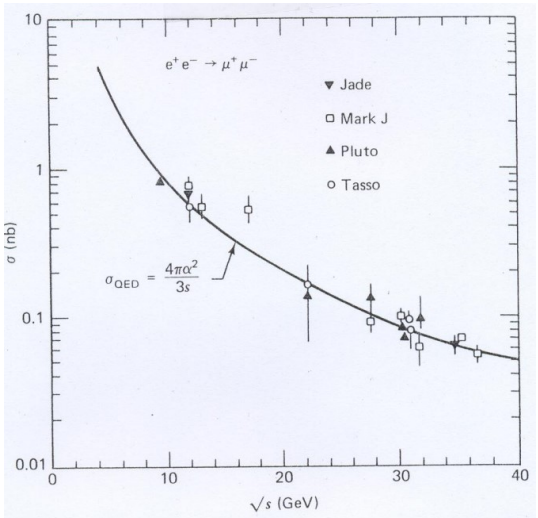
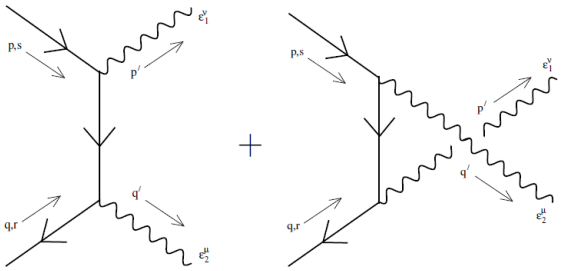


Fig. 6.6 The total cross section for $e^-e^+ \to \mu^-\mu^+$ measured at PETRA versus the center-of-mass energy.

Electron positron annihilation to photons $e + e^+ \to \gamma + \gamma$



$$= i(-ie)^2\,\bar{v}^r(\vec{q})\left(\frac{\gamma_\mu(\not{p} - \not{p}' + m)\gamma_\nu}{(p-p')^2 - m^2} + \frac{\gamma_\nu(\not{p} - \not{q}' + m)\gamma_\mu}{(p-q')^2 - m^2}\right) u^s(\vec{p})\epsilon_1^\nu(\vec{p}')\epsilon_2^\mu(\vec{q}')$$

(тоже дорешаю потом, мб в теор части или задачах?)

Scattering by an External Field

(посмотрю ЛЛ4, этого должно и хватить)

О распаде мюона, (??)

(???)

Electron-positron to hadrons scattering

$$\sigma^{qq}(s) = \sigma^{qq}_{Q\in D}(s)[1 + \underbrace{\frac{\alpha_s(s)}{\pi} + 1.411\cdot\frac{\alpha_s(s)^2}{\pi^2} + ...}_{\sim 7\%}]$$

$$R_{\text{had}} = \frac{\sigma(ee \to \text{ hadrons })}{\sigma(ee \to \mu\mu)} = 3\cdot\sum_i Q_i^2$$

$\sqrt{s}$	Quarks	$R_{\text{nad}} = 3 \cdot \sum_1^1 Q_i^2$
$<\sim$ 3GeV	uds	$3.6/9 = 2.00$
$<\sim$ 10GeV	udsc	$3 \cdot 10/9 = 3.33$
$<\sim$ 350GeV	udscb	$3 \cdot 11/9 = 3.67$
$>\sim$ 350GeV	udscbt	$3.15/9 = 5.00$

Data lies systematically higher than the prediction from Quark Parton Model (QPM) $\rightarrow$ gluon bremsstrhl.

$\star$ In the centre-of-mass frame the $e^+e^- \rightarrow \mu^+\mu^-$differential cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1 + \cos^2\theta\right)$$

NOTE: neglected masses of the muons, i.e. assumed $E \gg m_\mu$

In QED only certain combinations of LEFT- and RIGHT-HANDED CHIRAL states give non-zero matrix elements

$\star$ CHIRAL states defined by chiral projection operators

$$P_R = \frac{1}{2}\left(1 + \gamma^5\right); \quad P_L = \frac{1}{2}\left(1 - \gamma^5\right)$$

$\star$ In limit $E \gg m$ the chiral eigenstates correspond to the HELICITY eigenstates and only certain HELICITY combinations give non-zero matrix elements

1.5.3 Typical motion in external field

(От Ландау выгрузки тут, пару часов почитаю, пропишу - и будет сделано!)

32. Уравнение Дирака для электрона во внешнем поле

33. Разложение по степеням 1/c

34. Тонкая структура уровней атома водорода

35. Движение в центрально-симметричном поле

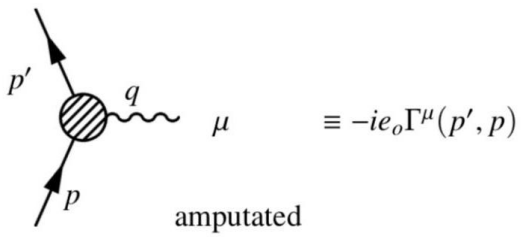
36. Движение в кулоновом поле

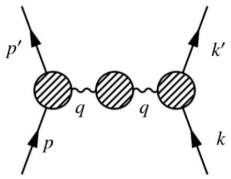
2 Most Useful Special Methods

(???)

2.1 Typical renormalizations

Typical Vertex Renormalization in QED





$$=\sim \bar{u}\left(p'\right)\left(-ie_0 Z_2 \Gamma^{\mu}(p',p)\right) u(p) \frac{-i\left(\eta_{\mu\nu}-q^{\mu}q^{\nu}/q^2\right)}{q^2\left(1-\Pi(q^2)\right)} \bar{u}\left(k'\right)\left(-ie_0 Z_2 \Gamma^{\nu}(k',k)\right) u(k)$$

The most general form possible is

$$\Gamma^{\mu} \equiv A\gamma^{\mu} + B\left(p'^{\mu}+p^{\mu}\right) + C\left(p'^{\mu}-p^{\mu}\right)$$

Since the Γ^{μ} is sandwiched between on-shell spinors, we can replace all factors of $\not{p}$ and $\not{p}''$ with m. Then the coefficients can only depend on q^2. The Ward identity requires the amplitude to vanish when Γ^{μ} is contracted with q_{μ}, setting $C=0$.

By the Gordan identity $\bar{u}\left(p'\right)\gamma^{\mu}u(p) = \bar{u}\left(p'\right)\left(\frac{p'^{\mu}+p^{\mu}}{2m} + \frac{iS^{\mu\nu}q_{\nu}}{2m}\right)u(p), \quad S^{\mu\nu} = \frac{i}{4}\left[\gamma^{\mu},\gamma^{\nu}\right]$ we arrive at the conventional form (???? why??)

$$\Gamma^{\mu}\left(p',p\right) = \gamma^{\mu}F_1\left(q^2\right) + i\frac{S^{\mu\nu}q_{\nu}}{2m_0}F_2\left(q^2\right)$$

where $F_1\left(q^2\right)$ and $F_2\left(q^2\right)$ are called form factors.

• The first form factor δF_1 is both UV and IR divergent. At one loop with PV regularization,

$$\delta F_1\left(q^2\right) = \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\left(\log\frac{z\Lambda^2}{\Delta} + \frac{(1-x)(1-y)}{\Delta}q^2 + \left(1-4z+z^2\right)m_0^2\right)$$

$$\delta F_2\left(q^2\right) = \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2m_0^2 z(1-z)}{\Delta} \qquad \text{(finite)},$$

$\Delta := -xyq^2 + (1-z)^2m_0^2 + \mu^2 z$, and μ is a photon mass that regulates the IR divergence.

• In the limit of small momentum transfer $q\to 0$, only F_1 contributes, so we define the vertex renormalization

$$\lim_{q\to 0}\Gamma^{\mu}(p+q,p) = Z_1^{-1}\gamma^{\mu}$$

The Ward-Takahashi identity can be used to show that $Z_1 = Z_2$, so

$$\lim_{q\to 0} Z_2\Gamma^{\mu}(p+q,p) = \gamma^{\mu}$$

and at each order in perturbation theory,

$$\delta F_1(0) = -\delta Z_2$$

• Comparing this with our scattering amplitude, the physical charge at $q=0$ is $e_0\left(Z_2/Z_1\right)\sqrt{Z_3} = e_0\sqrt{Z_3}$ as found earlier. As a physical application, this means that electrons and muons have the same bare charge if and only if they have the same physical charge.

At this point one could make a metaphysical objection: what does it mean for a "fundamental constant of nature" to be scale-dependent? Isn't the amount of charge on the electron fixed? The point is that there is no such thing as a direct measurement of the charge itself; instead we only measure how strongly the electron interacts in various situations. We've found that the effective interaction strength changes depending on the scale μ of the process, and there's nothing strange, physically or metaphysically, about that. The only reason one might think it was strange is that the dependence happens to be very weak at low μ, where most experiments historically took place. But once we frame it this way, $e(0)$ stops appearing particularly fundamental at all; the truly fundamental constants are defined at high scales.

Scalar field theory example of renormalization by Schwartz (????)

(доучу тоже, наверное, содержательный там пример)

Renormalized Perturbation Theory, loop corrections

• For a systematic treatment of UV divergent diagrams, it suffices to consider amputated, 1PI UV divergent diagrams; all other UV divergent diagrams can be built from them.

It is possible to classify the divergent 1PI diagrams by power counting. At one-loop order we found, in PV,

$$\begin{aligned}
\text{(photon)} \qquad \Pi_2\left(q^2\right) &= c_0^{(1)} \log \frac{\Lambda}{M} && + \text{finite}, \\
\text{(photon)} \qquad -i\Sigma_2(\not{p}) &= a_0^{(1)} m_0 \log \frac{\Lambda}{M} + a_1^{(1)} \not{p} \log \frac{\Lambda}{M} && + \text{finite}, \\
\text{(photon)} \qquad -i\Gamma_2^{\mu}\left(p', p\right) &= b_0^{(1)} \gamma^{\mu} \log \frac{\Lambda}{M} && + \text{finite},
\end{aligned}$$

M is the renormalization scale, Λ is the cutoff.

• Later, we will show that this structure persists to all orders in perturbation theory. Then the UV divergences are completely described by the four divergent constants a_0, a_1, b_0, and c_0, where e.g. $a_0^{(1)}$ is the $O(\alpha)$ term of a_0.

• In bare perturbation theory, we write the Lagrangian in terms of bare parameters m_0 and e_0 and impose a finite cutoff Λ, or in DR, a nonzero ϵ. We then compute the physical electron mass m, the physical coupling e, and the field renormalizations Z_2 and Z_3 in terms of m_0, e_0, and Λ. The renormalization step is to regard m_0 and e_0 as functions of m, e, and Λ, and vary them as $\Lambda \to \infty$ so that m and e remain fixed.

• Finally, we perform the usual perturbation theory, e.g. expanding in powers of e_0 with a cutoff Λ. If the preceding step has been performed correctly, then when the answers are written in terms of m and e, all divergences will cancel out of scattering amplitudes.

• In practice, we only compute $m\left(m_0, e_0, \Lambda\right)$ and $e\left(m_0, e_0, \Lambda\right)$ to some loop order in e_0. Then we must compute the scattering amplitudes to the same order in e_0.

• It might be disturbing that we are expanding in the formally divergent parameter e_0. In fact, this is acceptable because e_0 diverges only logarithmically, and the theory of QED has a reasonable cutoff so that $e_0(\Lambda)$ is still small. However, we will reformulate perturbation theory without e_0 below.

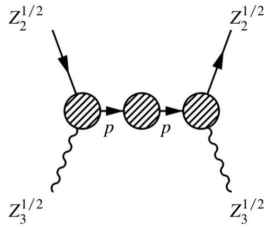
• In general, a theory is renormalizable if there are only a finite number of UV divergent resummed amputated 1PI diagrams. If there are m fields and n such diagrams, then m divergent

constants can be absorbed into field renormalizations, while $n-m$ are absorbed into bare couplings. The $n-m$ bare couplings must then be specified by measurements of physical couplings.

• Philosophically, the UV divergences signal that our theory must break down at some high scale, where some unknown physics takes over. The process of renormalization parametrizes our ignorance about this new physics into $n-m$ observables that must be measured. A renormalizable theory is one where this process only requires a finite number of experimental inputs. A truly fundamental theory must be UV finite, as string theory is believed to be.

• Finally, recall that our regularization schemes introduce an arbitrary mass scale M. The results don't depend on M, but higher-order corrections for a process with energy scale E contain factors of $\log E/M$. Then the perturbation theory converges fastest when $M \sim E$.

Example: contribution to the amplitude for Compton scattering.



$$M = \left(Z_3^{1/2} i e_0 \Gamma^\mu Z_2\right) \frac{1}{Z_2} \frac{i}{\not p - m_0 - \Sigma(\not p)} \left(Z_3^{1/2} i e_0 \Gamma_\mu Z_2\right)$$

$e = e_0 Z_3^{1/2}$ is the finite physical charge. The factor $\Gamma^\mu Z_2$ is also finite, since

$$\Gamma^\mu(p+q,p) = \gamma^\mu F_1\left(q^2\right) + \text{ finite}\,, \quad F_1\left(q^2\right) = 1 + \delta F_1(0) + \text{ finite}$$

so the divergent part of $\Gamma^\mu Z_2$ is $\delta Z_2 + \delta F_1(0) = 0$ at each order in perturbation theory. Finally, the propagator is

$$\frac{1}{Z_2} \frac{i}{\not p - m_0 - \Sigma(\not p)} = \frac{i}{\not p - m} + \frac{1}{Z_2}(\text{ finite terms analytic in } \not p = m)$$

The extra terms are finite because they come from the second and higher derivatives of $\Sigma(\not p)$, which are finite, as the divergences in $\Sigma(\not p)$ are proportional to m_0 and $\not p$.

We can also use renormalized perturbation theory, which works explicitly with physical parameters. In this formulation, the action depends on the cutoff/renormalization parameters just as before, but we split it into a finite, cutoff-independent "renormalized" piece and an infinite, cutoff-dependent "counterterm" piece.

$$\mathcal{L} = -\frac{1}{4}F^2 + \bar\psi\left(i\not\partial - m_0\right)\psi - e_0\bar\psi\gamma^\mu\psi A_\mu.$$

$$A^\mu = Z_3^{1/2} A_r^\mu, \quad \psi = Z_2^{1/2}\psi_r$$

$$\mathcal{L} = -\frac{1}{4}Z_3 F_r^2 + Z_2\bar\psi_r\left(i\not\partial - m_0\right)\psi_r - Z_2 Z_3^{1/2} e_0\bar\psi_r\gamma^\mu\psi_r A_\mu^r$$

This is simply a change of variables; the Lagrangian itself is unchanged.

The Feynman rules for the renormalized fields have no Z factors, the fermion has mass $m = Z_2 m_0$, and the coupling is $e' = Z_2 Z_3^{1/2} e_0$. Applying these rules yields the same results as the original Feynman rules.

• Next, we split the (bare) Lagrangian into the renormalized Lagrangian and the counterterms. We further define the parameters e and m by

$$e' = Z_1 e, \qquad Z_2 m_0 = m + \delta_m \qquad Z_i = 1 + \delta_i$$

and suppressing the r subscript,

$$\mathcal{L} \equiv \mathcal{L}^{(1)} + \mathcal{L}^{(2)} = \left(-\frac{1}{4}F^2 + \bar{\psi}(i\not{\partial} - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu\right) + \left(-\frac{1}{4}\delta_3 F^2 + \bar{\psi}\left(i\delta_2\not{\partial} - \delta_m\right)\psi - e\delta_1\bar{\psi}\gamma^\mu\psi A^\mu\right).$$

The perturbation parameter is now e. The four counterterms $\delta_1, \delta_2, \delta_3$, and δ_m are are defined order by order in $\alpha = e^2/4\pi$. The new arbitrary parameters e and m will be chosen to be more physically meaningful than e_0 and m_0.

• The Feynman rules for $\mathcal{L}^{(1)}$ have no Z factors, mass m, and coupling e. They are UV divergent in the exact same way as before, but with m_0 and e_0 replaced with m and e. The terms in $\mathcal{L}^{(2)}$ lead to additional diagrams, such as

The dependence on the bare parameters is entirely captured by the counterterms.

• The one-loop electron self-energy takes the form

$$\begin{aligned}
&-i\Sigma_r^{(1)}(\not{p}) = -i\Sigma_2^{(1)}(\not{p}) + i\left(\not{p}\delta_2^{(1)} - \delta_m^{(1)}\right) && (\Sigma_2^{(1)} \text{ is due to } \mathcal{L}_r^{(1)})\\
&i\Pi_r^{(1)\mu\nu}(q) = i\left(q^2\eta^{\mu\nu} - q^\mu q^\nu\right)\left(\Pi_2^{(1)}\left(q^2\right) - \delta_3^{(1)}\right). && (\text{Landau gauge})\\
&-ie\Gamma_r^{(1)\mu}(p+q,p) = -ie\Gamma^{(1)\mu}(p+q,p) - ie\gamma^\mu\delta_1^{(1)}
\end{aligned}$$

• Next, we determine the counterterms, which depend on the cutoff, by renormalization conditions. In the on-shell scheme, we let m be the physical mass of the electron and e be the physical charge measured at $q = 0$. By construction, we also require the propagators to have no Z-factors at the physical mass poles. Then we have four conditions,

$$\Sigma_r(\not{p})\big|_{\not{p}=m} = 0, \quad \frac{d}{d\not{p}}\Sigma_r(\not{p})\bigg|_{\not{p}=m} = 0, \quad \Pi_r\left(q^2\right)\big|_{q^2=0} = 0, \quad -ie\Gamma_r^\mu(p,p) = -ie\gamma^\mu$$

which fix the values of the counterterms order by order in α in terms of e, m, and the cutoff. We have removed all dependence on e_0 and m_0, and when the regularization is removed, all amplitudes are finite functions of e and m.

• At two-loop order, we have two-loop diagrams involving renormalized fields and second-order contributions to the counterterms such as $\delta_1^{(2)}$. In addition, one-loop diagrams containing first-order counterterms like $\delta_1^{(1)}$ also contribute.

• We assumed that the counterterms above were sufficient to absorb all UV divergences. As before, we can do this with finitely many counterterms only for a renormalizable theory.

Note. There's a fair deal of historical confusion about renormalization. Many sources phrase the procedure above in terms of "adding counterterms" to "cancel divergences". This is not a good way of thinking about it, because adding terms to a Lagrangian changes the theory. We are simply working with a single Lagrangian the entire time, whose terms may be formally divergent but whose physical predictions are always finite.

However, this naive picture is redeemed in Wilsonian renormalization. As we will see later, the bare Lagrangian is the fundamental Lagrangian, while the renormalized Lagrangian can be thought of as the effective Lagrangian after integrating out high-energy modes. The parameters in the effective Lagrangian are closely related to physically measurable ones, because loop integrals are cut off at a low scale and hence have little effect. The counterterm consists of the terms added to the fundamental Lagrangian as we perform Wilsonian RG flow down to the low scale.

Note. There are a wide variety of ways to set the counterterms, called subtraction schemes. E.g. in minimal subtraction, we use DR regularization and let the counterterms have zero finite part, so they're only series in $1/\epsilon$. This is more computationally efficient, and the standard in modern calculations. Alternatively, in the off-shell scheme, we do the same thing as the on-shell scheme, but let m above be an arbitrary parameter rather than the physical mass. Note that the bare Lagrangian doesn't depend on the subtraction scheme; it only affects the split into $\mathcal{L}_r^{(1)} + \mathcal{L}_r^{(2)}$.

Whenever a subtraction scheme contains a mass parameter, such as the μ parameter in DR, we have a continuum RG flow by changing the parameter. In the on-shell scheme, there is no continuum RG, though we have the Wilsonian RG as always.

Note. The point of these other subtraction schemes is to improve perturbation theory. Generically, a loop integral will give logarithmic factors like $\log(s/\Lambda^2)$, so

$$\mathcal{M}(s) \sim \lambda + \lambda^2 \log \frac{s}{\Lambda^2} + ...$$

We may define the renormalized coupling to be the physical coupling at some arbitrary scale s_0,

$$\lambda_R = \mathcal{M}(s_0)$$

$$\mathcal{M}(s) \sim \lambda_R + \lambda_R^2 \log \frac{s}{s_0} + ...$$

Then if s is very far from s_0, the perturbation series doesn't converge quickly. Thus it's very useful to be able to adjust s_0 as needed. The continuum RG gives a differential equation for $\mathcal{M}(s_0)$ as s_0 is varied, essentially resumming the perturbation series.

2.2 QFT for condensed matter: main methods

(I'll write soon)

2.3 QFT for statistical field theory: main methods

(I'll write soon)

Part II
Specific Quantum Field Theories in a Nutshell

3 Details of Fundamental QFT Methods and Properties in a Nutshell

3.1 General important properties of QFT

3.1.1 Nature of microworld and what do we mean by QFT?

(очень важный раздел для глубокого профессионального понимания КТП. В первые несколько лет изучения можно им вообще не заниматься, потому что все равно для задач мы используем формулы, для которых не так важен их фундаментальный смысл.)

Что такое квантовое поле?

(один из не многих вопросов теории, который тут я и обсуждаю, этот ответ нужно иметь готовым!)

Квантовое поле - это возмущение к большому классическому полю или само большое классическое поле - это большое количество маленьких квантовых полей?

(пока насколько я понял, четко на этот вопрос никто не отвечает, иногда считают одним образом, иногда другим.)

Что вообще из себя представляет материя на микро уровне и почему?

(сперва коротко выводы)
(добавлю какие-то графики, из которых сделано заключение, что все из кварков и т.п. Свяжу ключевыми формулами графики с выводами.)

О связи спина со статистикой (!??!?!??!?)

(!??!? а хрен его знает, я так и не понял эти доказательства, что за бред вообще????? еще раз посмотрю доказательства, в теории подробно их обсужу, потом сюда напишу лучшее)

Почему считают, что элементарных (составных) частиц так много?

(важный теор вопрос, пока не так сильно готов к этому, нужно нормально ответить, а не для галочки, как любой может)

Об экспериментальных фактах в основе КТП (??!!?)

(полезное указание.)

Что по итогу с расходимостями с физической и математической точки зрения?

(это большой вопрос, в курсах вуза проходятся методы в духе "делайте - и будет работать" и все. Может, потом подумаю про это, это важный, большой и фундаментальный вопрос)

О философии и идеи КТП (???)

(хз)

Restoring quantum mechanics from QFT

(I'll write later and detailed discussion of it will be in theory)

Аналогия между квантовой теорией поля и статистической механикой интегралы - 0 размерность, квантмех: 1, КТП: 3+1 (????)

Что за фигня? Но лучше написать, потом когда-то пойму.

Связь квантового поля с волновой функцией (??)

(пока остался вопрос.)

3.1.2 Constants of QFT: α, e, m_e, μ, $\hbar$

(more details about all of them, experiments and calculations)

On fine structure constant (????)

(впишу про нее что узнал очень коротко)

$$\alpha\left(q^2\right)=\frac{\alpha\left(\mu^2\right)}{1-\frac{\alpha(\mu^2)}{3\pi}\log\frac{q^2}{\mu^2}}$$

On magnetic moment (????)

Exact results of a magnetic moment

Note. When one speaks of the physical charge of a particle, one almost always means the charge $e(\mu)$ evaluated at zero momentum, $e(0)^2\approx 1/137$. This is the quantity relevant to all experiments besides those in high energy physics.

$$a_e=0.00115965218059(13)$$

См статью Волкова от 2023, и там куча ссылок.

$$a_e=a_e(\text{QED})+a_e(\text{ hadronic })+a_e(\text{ electroweak }),$$

$$a_e(\text{QED})=\sum_{n\geq 1}\left(\frac{\alpha}{\pi}\right)^n a_e^{2n},$$

$$a_e^{2n}=A_1^{(2n)}+A_2^{(2n)}\left(m_e/m_\mu\right)+A_2^{(2n)}\left(m_e/m_\tau\right)+A_3^{(2n)}\left(m_e/m_\mu,m_e/m_\tau\right),$$

where m_e, m_μ, m_τ are the masses of the electron, muon, and tau-lepton, respectively, α is the fine-structure constant; a similar expression is used for the muon AMM a_μ.

3.1.3 Vacuum, Casimir effect

(!!! доделаю раздел про скалярное поле!!!)

Что такое вакуум? (!?!??!?)

Это пустота.

Которая еще и имеет энергию.

А еще она может себя расталкивать.

(важное про понимание указание, потом додумаю. Большой вопрос, пока просто помню факты и модели)

Что такое энергия вакуума и почему ее иногда можно не рассматривать? (!?!??!?)

For a free scalar field $H = \int \frac{d^3k}{(2\pi)^3}\omega_k \left(a_k^\dagger a_k + \frac{1}{2}\right)$, where $\omega_k = |\vec{k}|$ (??? Шварц что ли безмассовые поля рассматривает???). So the contribution to the vacuum energy of the photon zero modes is zero-point energy:

$$E = \langle 0|H|0\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{\omega_k}{2} = \frac{1}{4\pi^2}\int k^3 dk = \infty,$$

Or from

$$\langle 0|H|0\rangle = \frac{1}{2}\int d^Dx \left\langle 0 \left| \pi^2 + (\vec{\nabla}\varphi)^2 + m^2\varphi^2 \right| 0 \right\rangle,$$

тут третий член:

$$\langle 0|\varphi(\vec{x},t)\varphi(\vec{x},t)|0\rangle = \langle 0|\varphi(\overrightarrow{0},0),\varphi(\overrightarrow{0},0)|0\rangle = \lim_{\vec{x},t\to\overrightarrow{0},0}\langle 0|\varphi(\vec{x},t)\varphi(\overrightarrow{0},0)|0\rangle =$$

$$= \lim_{\vec{x},t\to\overrightarrow{0},0}\int \frac{d^Dk}{(2\pi)^D 2\omega_k} e^{-i\left(\omega_k t - \vec{k}\cdot\vec{x}\right)} = \int \frac{d^Dk}{(2\pi)^D 2\omega_k}.$$

Первое равенство следует из трансляционной инвариантности, которая означает, что мы можем заменить $\int d^Dx$ в $\langle 0|H|0\rangle$ на объем пространства V. Вычисление двух других членов производится во многом аналогично: например, два множителя $\vec{\nabla}$ в $(\vec{\nabla}\varphi)^2$ просто дают множитель $\vec{k}^2$. Таким образом

$$\langle 0|H|0\rangle = V\int \frac{d^Dk}{(2\pi)^D 2\omega_k}\left[\frac{1}{2}\left(\omega_k^2 + \vec{k}^2 + m^2\right)\right] = V\int \frac{d^Dk}{(2\pi)^D}\frac{1}{2}\hbar\omega_k,$$

где мы восстановили $\hbar$.

Following our procedure for the harmonic oscillator, let's define the vacuum $|0\rangle$ by insisting that it is annihilated by all $a_{\vec{p}}$,

$$a_{\vec{p}}|0\rangle = 0 \quad \forall \vec{p}$$

With this definition, the energy E_0 of the ground state comes from the second term in (2.22),

$$H|0\rangle \equiv E_0|0\rangle = \left[\int d^3p \frac{1}{2}\omega_{\vec{p}}\delta^{(3)}(0)\right]|0\rangle = \infty|0\rangle$$

The subject of quantum field theory is rife with infinities. Each tells us something important, usually that we're doing something wrong, or asking the wrong question. Let's take some time to explore where this infinity comes from and how we should deal with it.

In fact there are two different ∞ 's lurking in the expression (2.24). The first arises because space is infinitely large. (Infinities of this type are often referred to as infra-red divergences although in this case the ∞ is so simple that it barely deserves this name). To extract out this infinity, let's consider putting the theory in a box with sides of length L. We impose periodic boundary conditions on the field. Then, taking the limit where $L \to \infty$, we get

$$(2\pi)^3\delta^{(3)}(0) = \lim_{L\to\infty} \int_{-L/2}^{L/2} d^3x \quad e^{i\vec{x}\cdot\vec{p}}\Big|_{\vec{p}=0} = \lim_{L\to\infty} \int_{-L/2}^{L/2} d^3x = V$$

where V is the volume of the box. So the $\delta(0)$ divergence arises because we're computing the total energy, rather than the energy density $\mathcal{E}_0$. To find $\mathcal{E}_0$ we can simply divide by the volume,

$$\mathcal{E}_0 = \frac{E_0}{V} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\omega_{\vec{p}}$$

which is still infinite. We recognize it as the sum of ground state energies for each harmonic oscillator. But $\mathcal{E}_0 \to \infty$ due to the $|\vec{p}| \to \infty$ limit of the integral. This is a high frequency - or short distance - infinity known as an ultra-violet divergence.

(??? додумаю, почему именно для единицы объема все считаем тут??? как-то не думал, что это так, допишу в указания про свойства полей это!)

(?? тут энергию вакуума для других полей тоже допишу)

Так мы всегда делаем, мол энергия отсчитывается от вакуума.

(абзац про то, почему ее нельзя измерить? никак вообще что ли?)

(важное про понимание указание, потом додумаю. пока так вот просто типичная фраза, конечно, она ничего не отвечает.)

Об эффекте Казимира (!??!)

Суть эффекта Казимира

For simple 1D box of size r with scalar field, $H = \int \frac{d^3k}{(2\pi)^3}\omega_k \left(a_k^\dagger a_k + \frac{1}{2}\right)$.

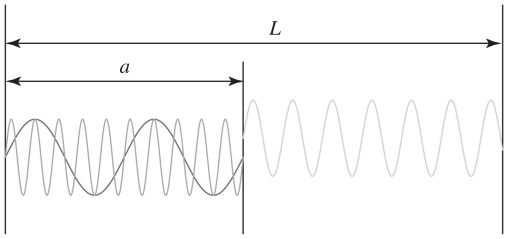
$$E(r) = \langle 0|H|0\rangle = \sum_n \frac{\omega_n}{2}, \qquad \omega_n = \frac{\pi}{r}n \qquad F = -\frac{dE}{da}$$

$$F(a) = -\frac{\pi\hbar c}{24a^2}.$$

Разные ренормировки для вычисления силы Казимира

$$
\begin{aligned}
E(r) &= \tfrac{1}{2}\sum_n \omega_n \theta\left(\pi\Lambda - \omega_n\right) && \text{(hard cutoff)},\\
E(r) &= \tfrac{1}{2}\sum_n \omega_n e^{-\frac{\omega_n}{\pi\Lambda}} && \text{(heat kernel)},\\
E(r) &= \tfrac{1}{2}\sum_n \omega_n e^{-\left(\frac{\omega_n}{\pi\Lambda}\right)^2} && \text{(Gaussian)},\\
E(r) &= \tfrac{1}{2}\sum_n \omega_n \left(\tfrac{\omega_n}{\mu}\right)^{-s} && (\zeta\text{-function}).\\
\tfrac{\pi}{2r}\sum_{n=1}^{\infty} n \quad &\to \quad -\tfrac{\pi}{24r} && 1+2+3+\cdots = -\tfrac{1}{12}
\end{aligned}
$$

All four of these regulators agree.



Casimir force by just summation:

$$E_{\text{tot}}\left(a\right) = E(a) + E(L-a) = \left(\frac{1}{a} + \frac{1}{L-a}\right)\frac{\pi}{2}\sum_{n=1}^{\infty} n = \infty$$

$$F(a) = -\frac{dE_{\text{tot}}}{da} = \left(\frac{1}{a^2} - \frac{1}{(L-a)^2}\right)\frac{\pi}{2}\sum_{n=1}^{\infty} n \overset{L\to\infty}{\longrightarrow} \frac{\pi}{2}\frac{1}{a^2}(1+2+3+\cdots) = \infty.$$

Casimir force by Hard cutoff. Very high frequency can't interact with atoms. We put in a high-frequency cutoff Λ so that $\omega < \pi\Lambda$, Λ in $\frac{1}{\text{atomic size}}$, or some other natural scale that limits the high-frequency light.

$$n_{\max}(r) = \Lambda r.$$

$$E(r) = \frac{1}{r}\frac{\pi}{2}\sum_{n=1}^{n_{\max}} n = \frac{\pi}{2r}\frac{n_{\max}\left(n_{\max}+1\right)}{2} = \frac{\pi}{4r}(\Lambda r)(\Lambda r+1) = \frac{\pi}{4}\left(\Lambda^2 r + \Lambda\right).$$

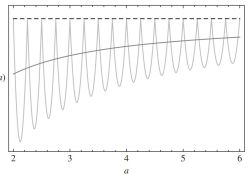
$$E_{\text{tot}} = E(L-a) + E(a) = \frac{\pi}{4}\left(\Lambda^2 L + 2\Lambda\right)$$

$$F(a) = -\frac{dE_{\text{tot}}}{da} = 0.$$

But $n_{\max}$ can only change by discrete amounts:

$$n_{\max}(r) = \lfloor \Lambda r \rfloor,$$

$$E(r) = \frac{\pi}{4r}\lfloor \Lambda r \rfloor(\lfloor \Lambda r \rfloor + 1)$$

$$E_{\text{tot}}(a) = E(L-a) + E(a) = \quad \text{  }$$

$$E(a) = \frac{\pi}{4}\left[\Lambda^2 a + \Lambda - 2\Lambda x - \frac{x(1-x)}{a}\right] \qquad x := \Lambda a - \lfloor \Lambda a \rfloor \in [0,1),$$

We will also take ΛL to be an integer, which is allowed because L was some arbitrary fixed size that does not change when we move the wall at a. Then $\lfloor \Lambda L - \Lambda a \rfloor = \Lambda L - \lceil \Lambda a \rceil$. For simplicity, let us also assume Λa is not an integer, which lets us use $\lceil \Lambda a \rceil = \lfloor \Lambda a \rfloor + 1$. Then,

$$\begin{aligned} E(L-a) &= \frac{\pi}{4}\left[\frac{(\Lambda L - \lceil \Lambda a \rceil)(\Lambda L - \lceil \Lambda a \rceil + 1)}{L-a}\right] \\ &= \frac{\pi}{4}\left[\Lambda^2(L-a) - \Lambda + 2\Lambda x - \frac{x(1-x)}{L-a}\right] \end{aligned}$$

and

$$E_{\text{tot}}(a) = E(L-a) + E(a) = \frac{\pi}{4}\left[\Lambda^2 L - \frac{x(1-x)}{a} - \frac{x(1-x)}{L-a}\right].$$

The $\Lambda^2 L$ piece is the extrinsic energy of the whole system, which does not contribute to the force, and a part that oscillates as x goes between 0 and 1. Keeping only terms up to order L^0, the total energy is

$$E_{\text{tot}}(a) = \frac{\pi}{4} L\Lambda^2 - \frac{\pi}{4a} x(1-x).$$

The $\frac{\pi}{4} L\Lambda^2$ term is the extrinsic energy, which does not contribute to the force, and a part that oscillates as $0 \le x < 1$.

Since $x = \Lambda a - \lfloor \Lambda a \rfloor$, as $\Lambda \to \infty$ at fixed a, there are more and more oscillations. In the continuum limit ($\Lambda \to \infty$), the plate will only experience the average force. Thus, we can average x between 0 and 1, using $\int x(1-x) = \frac{1}{6}$. So,

$$E_{\text{tot}}(a) \approx \frac{\pi}{4} L\Lambda^2 - \frac{\pi}{24a}.$$

$$F(a) = -\frac{dE_{\text{tot}}}{da} = -\frac{\pi}{24a^2} \longrightarrow -\frac{\pi \hbar c}{24a^2}.$$

Casimir force by Heat-kernel regularization. What if there is some penetration depth of the modes into the walls, with high-frequency modes getting further. This means that the contribution of high-frequency modes to the relevant energy sum is exponentially suppressed (??? why??)

$$E(r) = \frac{1}{2}\sum_n \omega_n e^{-\omega_n/(\pi\Lambda)}. \qquad \omega_n = \frac{\pi}{r} n$$

$$E(r) = \frac{1}{r}\frac{\pi}{2}\sum_{n=1}^{\infty} n e^{-n/(\Lambda r)} = \frac{1}{r}\frac{\pi}{2}\sum_{n=1}^{\infty} n e^{-\varepsilon n}, \quad \varepsilon = \frac{1}{\Lambda r} \ll 1.$$

$$\sum_{n=1}^{\infty} n e^{-\varepsilon n} = -\partial_\varepsilon \sum_{n=1}^{\infty} e^{-\varepsilon n} = -\partial_\varepsilon \frac{1}{1-e^{-\varepsilon}} = \frac{e^{-\varepsilon}}{(1-e^{-\varepsilon})^2} = \frac{1}{\varepsilon^2} - \frac{1}{12} + \frac{\varepsilon^2}{240} + \cdots$$

$$E(r) = \frac{1}{r}\frac{\pi}{2}\left[\Lambda^2 r^2 - \frac{1}{12} + \frac{1}{240 r^2\Lambda^2}\cdots\right] = \frac{\pi}{2} r\Lambda^2 - \frac{\pi}{24 r} + \cdots,$$

$$F(a) = -\frac{d}{da}[E(L-a)+E(a)] = -\frac{d}{da}\left[\frac{\pi}{2}L\Lambda^2 - \frac{\pi}{24}\left(\frac{1}{L-a}+\frac{1}{a}\right)+\cdots\right] = \frac{\pi}{24}\left(\frac{1}{(L-a)^2}-\frac{1}{a^2}\right)+\cdots$$

$$F(a) \overset{L\to\infty}{\longrightarrow} -\frac{\pi\hbar c}{24 a^2},$$

which is the same thing we found with the floor-function cutoff. Note, however, that the extrinsic energy term was $\frac{\pi}{4}L\Lambda^2$ in the previous case and is $\frac{\pi}{2}L\Lambda^2$ in this case.

Casimir force by ζ-function regulator:

$$E(r) = \frac{1}{2}\sum_n \omega_n \left(\frac{\omega_n}{\mu}\right)^{-s}\hbar$$

where we take $s \to 0$ instead of $\omega_{\max} \to \infty$ and have added an arbitrary scale μ to keep the dimensions correct. μ does not have to be large - it should drop out for any μ. Let us work out the ζ-function case. Substituting in for ω_n we get

$$E(r) = \frac{1}{2}\left(\frac{\pi}{r}\right)^{1-s}\mu^s\sum_n n^{1-s} = \frac{1}{r}\frac{\pi}{2}\zeta(s-1) = -\frac{\pi}{24 r}+\cdots, \qquad \sum n^{1-s} = \zeta(s-1) = -\tfrac{1}{12} - 0.165 s + \cdots$$

Casimir force by Regulator-independent derivation (Casimir 1948)

$$E(a) := \frac{\pi}{2}\sum_n \frac{n}{a} f\left(\frac{n}{a\Lambda}\right),$$

$f(x)$ is some function whose properties we will determine shortly.

$$E(L-a) = \frac{\pi}{2}(L-a)\Lambda^2\sum_n \frac{n}{(L-a)^2\Lambda^2} f\left(\frac{n}{(L-a)\Lambda}\right) \quad \overset{L\to\infty}{\longrightarrow}$$

$$\to \frac{\pi}{2}L\Lambda^2\int x dx f(x) - \frac{\pi}{2}a\Lambda^2\int x dx f(x) \qquad\qquad x := \frac{n}{(L-a)\Lambda}$$

The first integral is just the extrinsic vacuum energy, with energy density

$$\rho = \frac{\pi}{2}\Lambda^2\int x dx f(x).$$

The second integral simplifies with the change of variables $x = \frac{n}{a\Lambda}$. Adding the discrete sum, for the a side, with the continuum limit of the $L-a$ side, gives

$$E_{\text{tot}} = E(a)+E(L-a) = \rho L + \frac{\pi}{2a}\left[\sum_n n f\left(\frac{n}{a\Lambda}\right) - \int n dn f\left(\frac{n}{a\Lambda}\right)\right].$$

This contains the difference between an infinite sum and an infinite integral. Such a difference is given by the Euler-Maclaurin series:

$$\sum_{n=1}^{N} F(n) - \int_0^N F(n)dn = \frac{F(0)+F(N)}{2} + \frac{F'(N)-F'(0)}{12} + \cdots + B_j\frac{F^{(j-1)}(N)-F^{(j-1)}(0)}{j!}+\cdots,$$

where $F^{(j)}(N) = \frac{d^j F(N)}{dN^j}$ and B_j are the Bernoulli numbers ($B_2 = \frac{1}{6}$ and B_j for odd $j > 1$ they vanish). In our case, $F(n) = nf\left(\frac{n}{a\Lambda}\right)$. So, assuming that $f(x)$ dies sufficiently fast,

$$\lim_{x\to\infty} xf^{(j)}(x) = 0$$

then

$$E_{\text{tot}} = \rho L - \frac{\pi f(0)}{24a} - \frac{B_4}{4!}\frac{3\pi}{2a^3\Lambda^2}f''(0) + \cdots .$$

E.g.if $f(x) = e^{-x}$, then

$$E_{\text{tot}} = \frac{\pi}{2}\Lambda^2 L - \frac{\pi}{24a} + \mathcal{O}\left(\frac{1}{a^2\Lambda}\right),$$

which gives the correct Casimir force.

In fact, it is now clear that any regulator will give this force as long as

$$\lim_{x\to\infty} xf^{(j)}(x) = 0 \quad \text{and} \quad f(0) = 1.$$

You can see that all four of the regulators in Eq. (15.38) satisfy these requirements. The first requirement, that $f(x)$ die fast enough at high energy, means that UV modes go right through the box. It is this requirement that makes the force finite. The second requirement, that $f(0) = 1$, means that the regulator does not affect the spectrum in the IR. On physical grounds, only modes of size $\frac{1}{a}$ can reach both walls of the box to transmit the force, thus our deformation should not affect those modes.

Об экспериментах и применениях эффекта Казимира

Casimir force, first predicted in 1948 and observed 10 years later (по Зи). (по Шварцу - в 1997, Lamoreaux, S. K. 1997. Demonstration of the Casimir force in the 0.6 to 6 μm range. Phys. Rev. Lett., 78, 5–8. Кто там в итоге впервые это увидел?)

(как проверили это?)

(про то, где он на самом деле используется? проверю все это, потом напишу)

О другом про эффект Казимира

(все это в теории чуть решено, а также в отдельной записи, пока не до этих тем)

(короткий по сути абзац, а что идейно делать, если у нас другая геометрия?)

(короткий по сути абзац, а что идейно делать, если у нас температура ненулевая?)

(короткий по сути абзац, а что идейно делать, если одна из пластин движется?)

О космологической постоянной

(see cosmology for now)

О взаимодействии тел, сил Ван дер Ваальса, методах ЛЛ9

(пока мб в конд средах напишу, тоже не особо до этого мне сейчас.)

О других важных вакуумных эффектах и их применениях

(какие-то основные темы из Гриба Мамая выпишу, остальное - другая 1я часть. слишком тут много всего.)

3.1.4 On Causality, Locality, (etc)

(другие мелкие общие вопросы тоже сюда поставлю.)

Heisenberg Picture, causality (????)

To restore some of the Lorentz invariance, we switch to Heisenberg picture, where $A^H(t) = e^{iH^S t}A^S e^{-iH^S t}$, where we have assumed all Schrodinger operators are time-independent. In particular, the Hamiltonian satisfies $H^S = H^H$, and the time evolution is

$$\frac{d}{dt}A^H(t) = i\left[H, A^H(t)\right]$$

• Switching to Heisenberg picture preserves relations between operators as long as all operators are evaluated at the same time. Then the commutators become equal time commutators,

$$\left[q_i^H(t), p_j^H(t)\right] = i\delta_{ij}$$

• For fields, we will drop the indices. A Schrodinger field will have argument $\mathbf{x}$, while a Heisenberg field will have argument $x = (t, \mathbf{x})$. The equal time commutators are

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}), \quad [\phi(t,\mathbf{x}), \phi(t,\mathbf{y})] = [\pi(t,\mathbf{x}), \pi(t,\mathbf{y})] = 0$$

• The Heisenberg equation of motion for the fields is

$$\dot{\phi}(t,\mathbf{x}) = i[H, \phi(t,\mathbf{x})]$$

Now, H doesn't evolve in time, so we choose to evaluate it at time t, where

$$\mathcal{H}(t,\mathbf{x}) = \frac{1}{2}\pi^2(t,\mathbf{x}) + \frac{1}{2}(\nabla\phi(t,\mathbf{x}))^2 + \frac{1}{2}m^2\phi(t,\mathbf{x})^2$$

so we can apply the equal time commutation relations. We thus find

$$\dot{\phi}(t,\mathbf{x}) = \pi(t,\mathbf{x}), \quad \dot{\pi}(t,\mathbf{x}) = \left(\nabla^2 - m^2\right)\phi(t,\mathbf{x})$$

which correspond to the classical Hamilton's equations. The first equation gives an easy way of recovering the mode expansion for π. The two together show the field ϕ obeys the Klein-Gordon equation.

• The Heisenberg equations also 'covariantize', i.e. we can show that

$$\partial^\mu\phi(x) = i\left[P^\mu, \phi(x)\right], \quad \phi(x+a) = e^{iaP}\phi(x)e^{-iaP}$$

• Finally, we compute the mode expansion in Heisenberg picture; we choose to keep the creation and annihilation operators the same. We get factors like $e^{iHt}a_{\mathbf{p}}e^{-iHt}$ in the field expansion; to handle them note that

$$Ha_{\mathbf{p}} = a_{\mathbf{p}}\left(H - \omega_p\right), \quad f(H)a_{\mathbf{p}} = a_{\mathbf{p}}f\left(H - \omega_p\right)$$

Commuting the creation and annihilation operators in this way gives

$$\phi(x) = \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}}\left(a_{\mathbf{p}}e^{-ipx} + a_{\mathbf{p}}^\dagger e^{ipx}\right)$$

$$a_{\mathbf{p}} = \frac{i}{\sqrt{2\omega_p}} \int d\mathbf{x} e^{iqx} \overleftrightarrow{\partial_0} \phi(x) \qquad f \overleftrightarrow{\partial} g = f\partial g - g\partial f$$

Finally, we return to the question of causality.

• Using the Heisenberg picture, we can think of measurements as taking place at different times; conceptually $O^H(t)$ represents a measurement of O at time t.

• To understand this, suppose that $O_1(t_1)$ and $O_2(t_2)$ commute. Then they have a common eigenbasis $|\psi_i\rangle$ where the $|\psi_i\rangle$ are nonevolving Heisenberg states. In Schrodinger picture,

$$|\psi_i(t_j)\rangle \text{ is an eigenvector of } O_j.$$

Thus, if O_1 is measured at t_1, there is no statistical effect on a measurement of O_2 at time t_2.

• Therefore, we say operators $O_1(x)$ and $O_2(y)$ on spacetime are local if

$$[O_1(x), O_2(y)] = 0 \text{ when } (x-y)^2 < 0.$$

This enforces causality: a measurement of O_1 cannot affect O_2 superluminally.

• Since the elementary observables are the fields ϕ, we compute

$$\Delta(x-y) = [\phi(x), \phi(y)] = \int \frac{d\mathbf{p}}{2\omega_p} \left(e^{-ip(x-y)} - e^{-ip(y-x)}\right)$$

Now, the integration measure and integrand are both Lorentz invariant, so if the separation is spacelike we can switch to a frame where $x^0 = y^0$, giving factor $\left(e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right)$, which integrates to zero as desired.

• More generally, the same calculation goes through for commutators of $\partial_\mu\phi$, and all local operators $O(x)$ can be built out of $\phi(x)$ and $\partial_\mu\phi(x)$. Thus quantum fields maintain causality.

• For interacting fields, we cannot use the free mode expansion; the commutator turns out to be an operator rather than a number. Working casually, we could show it also vanishes at spacelike separation, but formally, we would take this as one of our axioms.

Next, we consider propagators and Green's functions.

• Define $|x\rangle = \phi^H(x)|0\rangle$. To interpret $|x\rangle$, switch back to Schrodinger picture for

$$|x\rangle = U(-t)\phi(\mathbf{x})U(t)|0\rangle = U(-t)\phi(\mathbf{x})|0\rangle = U(-t)|\mathbf{x}\rangle$$

Since $|\mathbf{x}\rangle$ is roughly a state with a particle at $\mathbf{x}$, $|x\rangle$ is a state that has a particle at $\mathbf{x}$ after evolution for a time t, so $|x\rangle$ has a particle at x. This particle exists for all times, both before and after t, but is only well-localized at time t.

$$D(x-y) := \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d\mathbf{p}}{2\omega_p} e^{-ip(x-y)}$$

(amplitude for a particle to go from y to x).

While the propagator is nonzero for spacelike separations, the commutator $D(x-y) - D(y-x)$ is, because the amplitudes for propagation from x to y and y to x cancel out! Note the interplay between the field intuition for ϕ (in the commutator) and the particle intuition for ϕ (in the propagator).

• When we generalize this to the complex scalar field, we see that the amplitude for a particle to go from x to y is canceled by the amplitude for an antiparticle to go from y to x. (For the real scalar field, the particles are their own antiparticles.) This is generally the reason that a multi-particle picture is necessary to preserve causality in relativistic quantum mechanics.

$$D_F(x-y) := \langle 0|T\phi(x)\phi(y)|0\rangle$$

where time ordering means to order fields "later on left", so that earlier fields are applied to the state first. Then

$$\begin{aligned} D_F(x-y) &= \theta\left(x^0-y^0\right) D(x-y) + \theta\left(y^0-x^0\right) D(y-x) \\ &= \int \frac{d\mathbf{p}}{2\omega_p} e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \left(\theta\left(x^0-y^0\right) e^{-i\omega_p\left(x^0-y^0\right)} + \theta\left(y^0-x^0\right) e^{i\omega_p\left(x^0-y^0\right)}\right) \\ &= D_F(x-y) = \int dp \frac{i}{p^2-m^2+i\epsilon} e^{-ip(x-y)} \end{aligned}$$

$$\left(\partial_x^2 + m^2\right) D_F(x-y) = -i\delta(x-y)$$

3.1.5 QED: Other conceptual properties

Low Energy Models from QED

(очень важно для понимания коротко вывести многие физические модели из КТП, пойму - напишу)

Lattice QED

(there are a lot of articles, later I'll write, how exactly it is connected with QFT)

3.1.6 Understanding gravity in QFT formalism

Guessing the propagator for gravity.

In general, the numerator of a propagator is simply the sum of projections onto polarizations. Define the spin 1 numerator as $-G_{\mu\nu}(k)$, we have

$$\sum_a \epsilon_\mu^{(a)}(k)\epsilon_\nu^{(a)}(k) = -G_{\mu\nu}(k)$$

as can be checked by explicit computation in the rest frame. Now for spin 2, there are five matrix polarizations satisfying

$$k^\mu \epsilon_{\mu\nu}^{(a)} = 0, \quad \eta^{\mu\nu}\epsilon_{\mu\nu}^{(a)} = 0$$

which project away the spin 1 and spin 0 components. Applying these constraints to the most general candidate built out of $G_{\mu\nu}$ and k_μ gives

$$D_{\mu\nu,\lambda\sigma}(k) = i\frac{G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda} - (2/3)G_{\mu\nu}G_{\lambda\sigma}}{k^2-m^2+i\epsilon}$$

where the overall constant can be found by evaluating in the rest frame. Gravity couples to the stress-energy tensor, and its conservation means we can replace $G_{\mu\nu}$ with $\eta_{\mu\nu}$. It turns out that in proper GR, the massless limit converts the 2/3 to a 1.

The sign of the gravitational force is determined by the sign of $D_{00,00}$, which is indeed positive. (Another way of understanding this is that the sign flip for spin 1 occurs twice; universal attraction

holds for even spins.) Pressure is also attractive by the same amount as energy density, since $D_{ii,ii} = D_{00,00}$, as we know from the notes on General Relativity.

3.2 Other properties of basic methods 1

(like what is in 1 part, but details about it)

3.2.1 Kinematics, cross sections, decay widths, form factors: details

Приложение A from Vergeles Выражение вероятностей процессов через амплитуды

Рассмотрим вопрос о выражении сечений рассеяния и вероятностей распадов частиц через инвариантные амплитуды рассеяния в трёхмерном пространстве.

Сначала проясним вопрос о нормировке r-частичных состояний. Согласно (39 Л2)

$$\langle k|k\rangle = 2\omega_k(2\pi)^3\delta^{(3)}(0) \tag{A1}$$

Так как объём пространства представляется как

$$V = \int \mathrm{d}^{(3)}x e^{i(\mathbf{p}-\mathbf{k})\mathbf{x}}\Big|_{\mathbf{p}=\mathbf{k}} = (2\pi)^3\delta^{(3)}(0)$$

то (A1) означает, что принятая нами лоренц-инвариантная нормировка состояний (38 Л2) такова, что в единице объёма содержится

$$n_{\mathbf{k}} = 2\omega_k = 2\sqrt{m^2+\mathbf{k}^2} \tag{A2}$$

частиц.

Построим r-частичное нормированное состояние. Пусть G - малая область в импульсном пространстве частиц вблизи значений импульсов $(\mathbf{p}_1, ..., \mathbf{p}_r)$, так что объём G равен:

$$G = \int\limits_G \mathrm{d}^{(3)}p_1 ... \mathrm{d}^{(3)}p_r = \prod_{j=1}^{r}\Delta^{(3)}p_j \tag{A3}$$

Состояние $\left(\omega_j \equiv \sqrt{m_j^2+\mathbf{p}_j^2}\right)$

$$|\tilde{f}\rangle = \prod_{j=1}^{r}\int\limits_G \frac{\mathrm{d}^{(3)}p_j}{[(2\pi)^3 2\omega_j]}\,|\mathbf{p}_1, ..., \mathbf{p}_r\rangle$$

имеет норму

$$\langle\tilde{f}|\tilde{f}\rangle = \prod_{j=1}^{r}\int\limits_G \frac{\mathrm{d}^{(3)}p_j}{[(2\pi)^3 2\omega_j]}\int\limits_G \frac{\mathrm{d}^{(3)}p'_j}{\left[(2\pi)^3 2\omega'_j\right]}\,\langle\mathbf{p}'_1, ..., \mathbf{p}'_r|\mathbf{p}_1, ..., \mathbf{p}_r\rangle =$$

$$= \prod_{j=1}^{r}\int\limits_G \frac{\mathrm{d}^{(3)}p_j}{[(2\pi)^3 2\omega_j]}\int\limits_G \mathrm{d}^{(3)}p'_j\delta^{(3)}\left(\mathbf{p}_j-\mathbf{p}'_j\right) = \prod_{j=1}^{r}\frac{\Delta^{(3)}p_j}{[(2\pi)^3 2\omega_j]} \tag{A4}$$

При получении (А4) была использована формула (45 Л2), а также тот факт, что импульсы $\mathbf{p}_j$ и $\mathbf{p}'_j$ почти совпадают при всех j, в то время как импульсы $\mathbf{p}_j$ и $\mathbf{p}'_i$ при $i \neq j$ принадлежат непересекающимся областям Δp_j и Δp_i. Следует заметить, что (А4) есть лоренц-инвариант. Таким образом состояние

$$|f\rangle = \left[\prod_{j=1}^{r} \frac{\Delta^{(3)} p_j}{[(2\pi)^3 2\omega_j]}\right]^{-1/2} |\tilde{f}\rangle \tag{A5}$$

нормировано, причём это свойство лоренц-инвариантно.

Пусть $|i\rangle$ - n-частичное состояние (43 Л2) и $|f\rangle$ - состояние (А5). Тогда для недиагональных матричных элементов (см. (2 ЛЗ)-(4 ЛЗ))

$$\langle f|S|i\rangle = \langle f|iT|i\rangle = \left[\prod_{j=1}^{r} \frac{\Delta^{(3)} p_j}{[(2\pi)^3 2\omega_j]}\right]^{-1/2} \prod_{j=1}^{r} \int\limits_G \frac{\mathrm{d}^{(3)} p_j}{[(2\pi)^3 2\omega_j]} \langle \mathbf{p}_1, ..., \mathbf{p}_r|iT|\mathbf{k}_1, ..., \mathbf{k}_n\rangle \tag{A6}$$

Согласно вышесказанному выражение (А6) лоренц-инвариантно. Перепишем (А6) при помощи амплитуды M_{fi} (см. (4 ЛЗ) и вычислим вероятность перехода из состояния $|i\rangle$ в состояние $|f\rangle$. Так как $|f\rangle$ - нормированное состояние, то согласно общим правилам квантовой механики эта вероятность

$$|\langle f|S|i\rangle|^2 = \prod_{j=1}^{r} \int\limits_G \frac{\mathrm{d}^{(3)} p_j}{[(2\pi)^3 2\omega_j]} \left[(2\pi)^4 \delta^{(4)} (P_f - P_i)\right]^2 |M_{fi}|^2 \tag{A7}$$

При получении последней формулы была учтена малость объёма G, а также равенство

$$\left[\prod_{j=1}^{r} \frac{\Delta^{(3)} p_j}{[(2\pi)^3 2\omega_j]}\right]^{-1} \prod_{j=1}^{r} \int\limits_G \frac{\mathrm{d}^{(3)} p_j}{[(2\pi)^3 2\omega_j]} = 1$$

В(А7)

$$P_i = k_1 + ... + k_n, \quad P_f = p_1 + ... + p_r$$

Положим

$$\left[(2\pi)^4 \delta^{(4)} (P_f - P_i)\right]^2 = VT(2\pi)^4 \delta^{(4)} (P_f - P_i)$$

где V - объём пространства, а T-полный промежуток времени, в течение которого рассматривается процесс. Таким образом, деля выражение (А7) на 4 -объём VT, мы получаем вероятность перехода в единице пространства в единицу времени:

$$\mathrm{d}w_{fi} = (2\pi)^4 \delta^{(4)} (P_f - P_i)\big] |M_{fi}|^2 \prod_{j=1}^{r} \frac{\mathrm{d}^{(3)} p_j}{[(2\pi)^3 2\omega_j]} \tag{A8}$$

В (А8) опущен знак интеграла по импульсам конечных частиц. Поэтому вероятность перехода (А8) является дифференциальной вероятностью перехода в бесконечно малый элемент импульсного пространства конечных состояний.

Ещё раз подчеркнём, что выражение (A8) лоренц-инвариантно и в случае применимости ТВ амплитуда M_{fi} вычисляется при помощи диаграммной техники Фейнмана согласно вышеизложенным правилам.

Теперь рассмотрим частные случаи

1. Процессы распада.

В этом процессе в начальном состоянии имеется одна частица с массой M. Чтобы получить вероятность распада одной частицы, следует разделить (A8) на плотность начальных частиц $n_{\mathbf{k}} = 2\sqrt{M^2 + \mathrm{k}^2} = 2\omega_k$. Таким образом вероятность распада частицы в единицу времени

$$\mathrm{d}\Gamma = \frac{(2\pi)^4}{2\sqrt{M^2 + \mathbf{k}^2}} \delta^{(4)}\left(P_f - k\right) |M_{fi}|^2 \prod_{j=1}^{r} \frac{\mathrm{d}^{(3)} p_j}{[(2\pi)^3 2\omega_j]} \tag{A9}$$

2. Взаимное рассеяние двух частиц

В системе отсчёта, в которой одна из рассеивающихся частиц покоится, вероятность (A8) представляется в виде

$$\mathrm{d}w_{fi} \equiv \mathrm{d}\sigma \cdot v_{\mathrm{OTH}} n_1 n_2 \tag{A10}$$

где n_1 и n_2 - плотности частиц сортов 1 и 2, соответственно, $v_{\text{отн}}$ - относительная скорость частиц и $\mathrm{d}\sigma$ - лоренц-инвариантное дифференциальное сечение рассеяния. Из определения (A10) видно, что сечение имеет размерность площади, если n_1 и n_2 в (A10) нормируются так же как и начальные состояния при вычислении амплитуды рассеяния, т.е. согласно (A2).

Для проведения лоренц-инвариантных вычислений при нахождении сечений введём инвариантную величину, которую можно назвать инвариантной плотностью потока:

$$j = \sqrt{\left(k_1 k_2\right)^2 - m_1^2 m_2^2} \tag{A11}$$

В системе отсчёта, в которой частица 2 покоится

$$j = \frac{m_1 m_2 v_{\mathrm{OTH}}}{\sqrt{1 - v_{\mathrm{OTH}}^2}} = \frac{1}{4} v_{\mathrm{OTH}} n_1 n_2 \tag{A12}$$

где

$$n_1 = \frac{2m_1}{\sqrt{1 - v_{\mathrm{OTH}}^2}}, \quad n_2 = 2m_2$$

Если же налетающая частица 1 является безмассовой, то выражение для j при помощи правой части (A12) остаётся справедливым, если положить $v_{\mathrm{O}} = 1$. Таким образом лоренц-инвариантное дифференциальное сечение рассеяния представляется как

$$\mathrm{d}\sigma = \frac{\mathrm{d}w_{fi}}{4j} \tag{A13}$$

Легко проверить следующую формулу:

$$2j = \sqrt{\left[s - (m_1 + m_2)^2\right]\left[s - (m_1 - m_2)^2\right]}, \quad s = (k_1 + k_2)^2 \tag{A14}$$

В системе центра инерции частиц

$$k_1 = (\omega_1, \mathbf{k}), \quad k_2 = (\omega_2, -\mathbf{k})$$

и плотность потока

$$j = |\mathbf{k}|(\omega_1 + \omega_2) \tag{A15}$$

Рассмотрим подробнее формулу (A13) в системе центра инерции в том случае, когда в конечном состоянии имеется также две частицы с массами m_1' и m_2'. Имеем:

$$p_1 = (\omega_1', \mathbf{p}), \quad p_2 = (\omega_1', -\mathbf{p}), \quad \omega_1'^2 - m_1'^2 = \omega_2'^2 - m_2'^2 = |\mathbf{p}|^2 \tag{A16}$$

Отсюда

$$\omega_1' \mathrm{d}\omega_1' = \omega_2' \mathrm{d}\omega_2' = |\mathbf{p}|\mathrm{d}|\mathbf{p}| \longrightarrow \mathrm{d}(\omega_1' + \omega_2') = (\omega_1' + \omega_2')\frac{|\mathbf{p}|\mathrm{d}|\mathbf{p}|}{\omega_1'\omega_2'} \tag{A17}$$

При помощи обозначений

$$\omega = \omega_1 + \omega_2, \quad \omega' = \omega_1' + \omega_2'$$

формулы (A15) и (A17) переписываются как

$$j = |\mathbf{k}|\omega, \quad |\mathbf{p}|\mathrm{d}|\mathbf{p}| = \frac{\omega_1'\omega_2'}{\omega'}\mathrm{d}\omega' \tag{A18}$$

что позволяет представить сечение (A13) в виде

$$\mathrm{d}\sigma = \frac{|M_{fi}|^2 |\mathbf{p}|\mathrm{d}o'}{64\pi^2 |\mathbf{k}|\omega^2} \tag{A19}$$

Здесь d o' - элемент телесного угла, в котором заключён импульс рассеянной частицы:

$$\mathrm{d}o' = \sin\theta \mathrm{d}\theta \mathrm{d}\phi \tag{A20}$$

так что θ есть угол между векторами $\mathbf{k}$ и $\mathbf{p}$, а угол ϕ является азимутальным углом вектора $\mathbf{p}$ относительно некого фиксированного ортогонального репера, третья ось которого совпадает с $\mathbf{k}$. Таким образом

$$t = (k_1 - p_1)^2 = m_1^2 + m_1'^2 - 2\omega\omega' + 2|\mathbf{k}||\mathbf{p}|\cos\theta$$

Поэтому при фиксированной энергии ω

$$\mathrm{d}t = 2|\mathbf{k}||\mathbf{p}|\mathrm{d}\cos\theta, \quad \mathrm{d}o' = \mathrm{d}\phi\mathrm{d}(-\cos\theta) = \frac{\mathrm{d}\phi\mathrm{d}(-t)}{2|\mathbf{k}||\mathbf{p}|}$$

так что

$$\mathrm{d}\sigma = \frac{|M_{fi}|^2\ \mathrm{d}t}{64\pi|\mathbf{k}|^2\omega^2} \cdot \frac{\mathrm{d}\phi}{2\pi} = \frac{|M_{fi}|^2\ \mathrm{d}t}{64\pi j^2} \cdot \frac{\mathrm{d}\phi}{2\pi} \tag{A21}$$

Так как в физически интересных случаях амплитуда не зависит от угла ϕ, то, интегрируя (A21) по этому углу, находим:

$$\mathrm{d}\sigma = \frac{|M_{fi}|^2 \ \mathrm{d}t}{64\pi j^2} \tag{A22}$$

Поскольку в последнем равенстве инвариантное дифференциальное сечение представлено через инвариантные же величины, то этим равенством можно пользоваться в любой системе отсчёта.

ll4-39. Система волновых функций непрерывного спектра для рассеяния в кулоновом поле

3.2.2 Poincare group for QFT

(это же в теорполе, только тут в контексте КТП и короче пояснения)

Об СТО и группе Пуанкаре для КТП

(уже почти готов написать, но пока об этом в теории поля. потом это же сюда и выпишу, мб с особенностями для КТП)

3.2.3 Path integral in QFT: details of the formalism

Построение интеграла по траекториям разбиением экранами

(типичный вывод, другого и не знаю.)

Path Integrals in Zero Dimensions

Основная физическая идея интеграла по траекториям (???)

(додумаю - напишу, пока не так въехал)

Introduction

$$\mathcal{Z} = \int\limits_{\mathcal{C}} \mathcal{D}\phi \exp\left(-\frac{S[\phi]}{\hbar}\right)$$

$$\left\langle \prod_i \mathcal{O}_i[\phi] \right\rangle = \frac{1}{\mathcal{Z}} \int\limits_{C} \mathcal{D}\phi \exp\left(-\frac{S[\phi]}{\hbar}\right) \prod_i \mathcal{O}_i[\phi]$$

• Often we are interested in local operators, such as $\phi^3(x)\partial^\mu\phi(x)\partial_\mu\phi(x)$ or $e^{\phi(x)}$, but we can also consider nonlocal operators. E.g. the Wilson loop $W_\Gamma[A] = \operatorname{tr} P\exp(-\oint A)$ depends on the value of the gauge field A along the loop Γ. (?? P экспонента это???) Again, correlation functions of such operators don't depend on the fields themselves, only on the theory.

• Formal differentiation of the partition function allows us to recover correlation functions of operators in the action. E.g. if the action has a term

$$\mathcal{O} = \lambda \int_M d^d x \phi^4(x)$$

then differentiating with respect to λ gives

$$\frac{\partial \mathcal{Z}}{\partial \lambda} = -\frac{1}{\hbar} \int_{\mathcal{C}} \mathcal{D}\phi e^{-S[\phi]/\hbar} \int_M d^d x \phi^4 = -\frac{\mathcal{Z}\langle \mathcal{O} \rangle}{\hbar}$$

Thus, correlation functions tell us about the response of the partition function to changes in the parameters of the theory.

• Similarly, suppose we add a source term to the action,

$$S[\phi] \to S[\phi] + \int_M d^d x J_i(x) \mathcal{O}_i(x)$$

Then the sources $J_i(x)$ are fields, and $\mathcal{Z}$ is a functional of it; varying with respect to it gives correlation functions of local operators,

$$-\hbar \frac{\delta \mathcal{Z}\left[J_i\right]}{\delta J_i\left(x_i\right)} = \int_{\mathcal{C}} \mathcal{D}\phi e^{-\left(S[\phi] + \int J_j \mathcal{O}_j\right)/\hbar} \mathcal{O}_i\left(x_i\right) = \mathcal{Z}\left\langle \mathcal{O}_i\left(x_i\right) \right\rangle$$

...

$$\langle \mathcal{O}_1(x_1) ... \mathcal{O}_n(x_n) \rangle = \frac{(-\hbar)^n}{\mathcal{Z}} \frac{\delta^n \mathcal{Z}[J]}{\delta J_1(x_1) ... \delta J_n(x_n)} \bigg|_{J=0}$$

E.g.:

$$\langle \phi(x_1) ... \phi(x_n) \rangle = \frac{(-\hbar)^n}{\mathcal{Z}} \frac{\delta^n \mathcal{Z}[J]}{\delta J(x_1) ... \delta J(x_n)} \bigg|_{J=0}$$

E.g.:

$$\delta S[\phi] = \frac{1}{2} \int_M d^d x \sqrt{g} T_{\mu\nu}(x) \delta g^{\mu\nu}(x)$$

$$-\frac{2\hbar}{\sqrt{g(x)}} \frac{\delta \log \mathcal{Z}}{\delta g^{\mu\nu}(x)} = \langle T_{\mu\nu}(x) \rangle \qquad \text{at } \delta g = 0$$

• Suppose our manifold M has boundaries, $\partial M = \cup_i B_i$. Then to specify the path integral, we must choose boundary conditions on the B_i. For each B_i, the set of boundary conditions forms a Hilbert space $\mathcal{H}_i$, so the path integral defines a map $\mathcal{Z} : \otimes_i \mathcal{H}_i \to \mathbb{C}$ by

$$\mathcal{Z}\left[B_i\right] = \int_{\phi|_{B_i} = \varphi_i} \mathcal{D}\phi e^{-S[\phi]/\hbar}$$

We won't specify an inner product on the $\mathcal{H}_i$, because it's unknown how to define it in general.

• An important special case is $M = N \times I$ where I is an interval of length T with respect to the metric g on M. Then the path integral gives a map

$$U(T) : \mathcal{H} \to \mathcal{H}, \quad \langle \varphi_1 | U(T) | \varphi_0 \rangle = \int_{\phi|_{N\times\{0\}}=\varphi_0}^{\phi|_{N\times\{T\}}=\varphi_1} \mathcal{D}\phi e^{-S[\phi]/\hbar}$$

This is simply the time evolution operator. However, it is difficult to show that this map is unitary; that's one of the advantages of the canonical picture.

• Now we take a detour in classical field theory, i.e. considering only the action and not integrating over the fields. The variation of the action takes the form

$$\delta S[\phi] = (\text{ bulk eom })\delta\phi + \sum_i \int_{B_i} n_i^\mu \frac{\delta \mathcal{L}}{\delta\left(\partial^\mu \phi\right)} \delta\phi \sqrt{g} d^{d-1}x$$

$$\pi_i := \sqrt{g} n_i^\mu \frac{\delta \mathcal{L}}{\delta\left(\partial^\mu \phi\right)}$$

E.g. when ∂M is a constant time slice of flat spacetime, then $\pi(t) = \delta\mathcal{L}/\delta\dot{\phi}(t)$.

• Now, the variation δ maps functions on $\mathcal{H}$ to one-forms, since $\delta S[\phi]$ is a number, so we may extend its definition so it is an exterior derivative on differential forms. Then

$$0 = \delta^2 S[\phi]\big|_{\text{eom}} = \int_{N\times\{T\}} \delta\pi \wedge \delta\phi d^{d-1}x - \int_{N\times\{0\}} \delta\pi \wedge \delta\phi d^{d-1}x$$

This implies that there is a conserved quantity

$$\Omega = \int_N \delta\pi \wedge \delta\phi d^{d-1}$$

which is a symplectic form on the space of fields. The symplectic form defines a Poisson bracket, and its conservation implies Liouville's theorem. This is the structure that we aim to quantize in canonical quantization, replacing brackets with commutators.

• E.g. for the canonical quantization on $(\mathbb{R}^4, \eta)$ of the free real scalar field,

$$\phi(x) = \int \frac{d\mathbf{p}}{\sqrt{2E}} \left(e^{ipx} a(\mathbf{p}) + e^{-ipx} a^\dagger(\mathbf{p})\right), \quad [\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y})$$

The commutation relations are only canonical at equal times, since the symplectic form is defined on timeslices. We defined $\phi(x)$ on all of spacetime, rather than timeslices, but this isn't a big difference since the values of ϕ and π on one timeslice determine it everywhere, so boundary values and full solutions are equivalent.

• Next, we look at the state space. A general state on the boundary Hilbert space is

$$|\Psi\rangle = \int_{\mathcal{C}[B]} [d\varphi] |\varphi\rangle\langle\varphi|\Psi\rangle, \quad \Psi[\varphi] = \langle\varphi|\Psi\rangle$$

where we are vague about the inner product and measure. That is, a general state is a superposition of $|\varphi\rangle$ states, each of which correspond to a function on N, representing a definite field configuration.

• In canonical quantization, we used wavefunctions that were polynomials, with monomials

$$\int\limits_{N\otimes n} d\mathbf{x}_1 ... d\mathbf{x}_n \psi\left(\mathbf{x}_1, ..., \mathbf{x}_n\right) \phi\left(\mathbf{x}_1\right) ... \phi\left(\mathbf{x}_n\right)$$

interpreted as n-particle states. E.g. the monomial $\phi(\mathbf{x})$ corresponds to the state $\widehat{\phi}(\mathbf{x})|0\rangle$, i.e. a single particle at $\mathbf{x}$. This state is a superposition of the states $|\varphi\rangle$, weighting states with larger $\phi(\mathbf{x})$ more.

• Assuming our particles are bosons, n-particle states live in the Fock space

$$\mathcal{H} = \mathcal{C} \oplus V \oplus \mathrm{Sym}^2 V \oplus \mathrm{Sym}^3 V \oplus ...$$

where V is the one-particle Hilbert space. Working in the Fock basis, which is not always available, is analogous to expanding a solution to the quantum harmonic oscillator in terms of Hermite polynomials (times Gaussian weights). The general problem is analogous to working in the position basis in quantum mechanics, which is mathematically much trickier.

• If M is non-compact, there may be a region that is asymptotically far away. Then to define the path integral, we must specify asymptotic values of the fields. The simplest choice is $\phi \to 0$, but another important choice is $\phi \to \phi_i$ in the asymptotic past and $\phi \to \phi_f$ in the asymptotic future, in which case the path integral gives the scattering amplitude $\langle \phi_f | \phi_i \rangle$. These are related to correlation functions by the LSZ reduction formula.

Zero Dimensional Field Theory

• If our spacetime M is zero-dimensional and connected, it must be a single point. There is no metric, and the Lorentz group is trivial, so all of its representations are; all fields are scalars. In the path integral, fields are just maps from the single point to $\mathbb{R}$, so $\mathcal{C} \cong \mathbb{R}$.

• The path integral measure $\mathcal{D}\phi$ becomes the usual measure $d\phi$ on $\mathbb{R}$ and the partition function $\mathcal{Z} = \int\limits_{\mathbb{R}} d\phi e^{-S(\phi)/\hbar}$ is an ordinary real integral.

• The action can't have derivatives, so there are no kinetic terms. We'll usually choose it to be a polynomial, such as $S(\phi) = m^2\phi^2/2 + g\phi^4/4!$, where the highest term must have even degree and positive coefficient for the partition function to converge. We can think of the coupling constants as coordinates on an infinite-dimensional 'space of theories'.

• The partition function is a function of these coupling constants, $\mathcal{Z} = \mathcal{Z}\left(m^2, g, ...\right)$. We denote the free partition function $\mathcal{Z}_0$ to be the partition function where the action is quadratic.

$$\langle f \rangle := \frac{1}{\mathcal{Z}} \int\limits_{\mathbb{R}} d\phi f(\phi) e^{-S(\phi)/\hbar}$$

The only restriction is that f cannot grow so fast the integral diverges; in practice we restrict f to be polynomial. Thinking of $e^{-S(\phi)/\hbar}$ as a probability distribution, $\langle f \rangle$ is simply the expectation value of f.

• Again, we can recover correlation functions from the partition function. For example,

$$\frac{1}{p!} \langle \phi^p \rangle = -\frac{\hbar}{\mathcal{Z}} \frac{\partial}{\partial g_p} \mathcal{Z}\left(m^2, \lambda_i\right)$$

where the action has terms $g_p\phi^p/p!$.

For n fields ϕ^a with action $S(\phi) = \frac{1}{2}M(\phi,\phi) = \frac{1}{2}M_{ab}\phi^a\phi^b$, where M_{ab} is real, positive-definite and symmetric the partition function $\mathcal{Z}_0 = \int\limits_{\mathbb{R}^n} d^n\phi e^{-M(\phi,\phi)/2\hbar}$ is a Gaussian integral.

• We may diagonalize M using an orthogonal transformation, which leaves the measure invariant. Then applying the standard Gaussian integral,

$$\int\limits_{\mathbb{R}} d\chi e^{-m\chi^2/2\hbar} = \sqrt{\frac{2\pi\hbar}{m}}, \quad \mathcal{Z}_0 = \frac{(2\pi\hbar)^{n/2}}{\sqrt{\det M}}$$

• More generally, we can consider a linear source term,

$$S(\phi) = \frac{1}{2}M(\phi,\phi) + J\cdot\phi = \frac{1}{2}M(\tilde{\phi},\tilde{\phi}) - \frac{1}{2}M^{-1}(J,J), \quad \tilde{\phi} \equiv \phi + M^{-1}(J,\cdot)$$

where the shift from ϕ to $\tilde{\phi}$ removes the linear term and leaves the measure invariant. Then

$$\mathcal{Z}(J) = \exp\left(\frac{M^{-1}(J,J)}{2\hbar}\right)\mathcal{Z}_0$$

• Now we consider the correlation function $\langle P(\phi)\rangle$ where P is a polynomial in the n fields. By linearity, it suffices to let P be a monomial. We will treat the case where P is a product of linear factors $\ell(\phi) = \ell\cdot\phi$, so we compute

$$\langle \ell_1(\phi)...\ell_p(\phi)\rangle = \frac{1}{\mathcal{Z}_0}\int\limits_{\mathbb{R}^n} d^n\phi e^{-M(\phi,\phi)/2\hbar}\prod_{i=1}^{p}\ell_i(\phi)$$

$p = 2k$,

$$\begin{aligned}
\langle \ell_1(\phi)...\ell_{2k}(\phi)\rangle &= \frac{1}{\mathcal{Z}_0}\int\limits_{\mathbb{R}^n} d^n\phi\prod_{i=1}^{2k}\ell_i(\phi)e^{-M(\phi,\phi)/2\hbar - J(\phi)/\hbar}\Bigg|_{J=0} \\
&= \frac{(-\hbar)^{2k}}{\mathcal{Z}_0}\int\limits_{\mathbb{R}^n} d^n\phi\prod_{i=1}^{2k}\ell_i\cdot\partial_J e^{-M(\phi,\phi)/2\hbar - J(\phi)/\hbar}\Bigg|_{J=0} \\
&= \prod_{i=1}^{2k}\ell_i\cdot\partial_J\frac{(-\hbar)^{2k}}{\mathcal{Z}_0}\int\limits_{\mathbb{R}^n} d^n\phi e^{-M(\phi,\phi)/2\hbar - J(\phi)/\hbar}\Bigg|_{J=0} \\
&= \hbar^{2k}\prod_{i=1}^{2k}\ell_i\cdot\partial_J\left(e^{M^{-1}(J,J)/2\hbar}\right)\Bigg|_{J=0}
\end{aligned}$$

Every factor of ∂_J pops out a factor of $M^{-1}(\ell_i, J)$. Each of these factors must be differentiated with respect to J, to leave a factor of $M^{-1}(\ell_i,\ell_j)$, since we set $J=0$ at the end.

• E.g. for a two-point function

$$\langle \ell_1(\phi)\ell_2(\phi)\rangle = M^{-1}(\ell_1,\ell_2)$$

and more generally

$$\langle \ell_1(\phi)...\ell_{2k}(\phi)\rangle = \hbar^k\sum_{\text{pairs }(a_i,b_i)}\prod_{i=1}^{k}M^{-1}(\ell_{a_i},\ell_{b_i})$$

where, for example, the four-point function has three terms. We call M^{-1} the (free) propagator; it is simply the inverse of the kinetic term. When all the ℓ_i are the same,

$$\left\langle (\ell \cdot \phi)^{2k} \right\rangle = \frac{(2k)!}{2^k k!} \left(\hbar M^{-1}(\ell, \ell) \right)^k$$

The terms can be represented with free field Feynman diagrams with $2k$ external points.

• The result we've just proven is called Wick's theorem. In the context of probability theory, it's a result about moments of the Gaussian distribution called Isserlis' theorem.

• For fermionic variables that anticommute, $\theta^a \theta^b = -\theta^b \theta^a$, the starting point is

$$\int d^n\theta d^n\bar{\theta} \exp(M(\bar{\theta}, \theta) + \bar{\eta} \cdot \theta + \bar{\theta} \cdot \eta) = e^{M^{-1}(\bar{\eta}, \eta)} \det M$$

and we can use this to derive an analogue of Wick's theorem.

Perturbation Theory

• In general, we can't evaluate the path integral exactly for an interacting theory, even in zero dimensions. We might hope to Taylor expand in the small parameter $\hbar$. However, the radius of convergence is zero, because the integral diverges in the region $\mathrm{Re}(\hbar) < 0$.

• However, it is possible to obtain an asymptotic expansion. Suppose $S(\phi)$ has a global minimum at ϕ_0. Then the method of stationary phase gives

$$\int\limits_{\mathbb{R}^n} d^n\phi f(\phi) e^{-S(\phi)/\hbar} \sim (2\pi\hbar)^{n/2} \frac{f(\phi_0)\, e^{-S(\phi_0)/\hbar}}{\sqrt{\det\left(\partial_a \partial_b S(\phi_0)\right)}} \left(1 + \hbar A_1 + \hbar^2 A_2 + \ldots\right)$$

as $\hbar \to 0^+$, where the first term is the 'semiclassical' term. This is an asymptotic series.

• In general, an asymptotic series $\sum\limits_n a_n \hbar^n$ for $I(\hbar)$ means that $\lim_{\hbar \to 0^+} \frac{1}{\hbar^N} \left| I(\hbar) - \sum\limits_{n=0}^{N} a_n \hbar^n \right| = 0$, so the truncated series is arbitrarily good for fixed N and sufficiently small $\hbar$. But we cannot make the series arbitrarily good for fixed $\hbar$. Instead, for sufficiently high N, the error goes back up; the entire series diverges.

• Consider a single scalar field with action $S(\phi) = m^2\phi^2/2 + \lambda\phi^4/4!$ where we need $\lambda > 0$ for convergence and $m^2 > 0$ to get a unique minimum at $\phi_0 = 0$. The leading term is

$$(2\pi\hbar)^{1/2} \frac{e^{-S(\phi_0)}}{\sqrt{S''(\phi_0)}} = \frac{\sqrt{2\pi\hbar}}{m}$$

Going further, we can construct an asymptotic series by expanding the action,

$$\mathcal{Z}\left(m^2, \lambda\right) = \int\limits_{\mathbb{R}} d\phi e^{-\left(m^2\phi^2/2 + \lambda\phi^4/4!\right)/\hbar} = \int\limits_{\mathbb{R}} d\phi e^{-m^2\phi^2/2\hbar} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-\lambda}{4!\hbar}\right)^n \phi^{4n}.$$

It is illegitimate to exchange the sum and integral because the integral does not converge when λ is negative. However, we can cut off the sum and then exchange the finite sum, for

$$\mathcal{Z}\left(m^2, \lambda\right) \sim \sum_{n=0}^{N} \frac{1}{n!} \left(\frac{-\lambda}{4!\hbar}\right)^n \int\limits_{\mathbb{R}} d\phi e^{-m^2\phi^2/2} \phi^{4n} = \frac{\sqrt{2\hbar}}{m} \sum_{n=0}^{N} \frac{1}{n!} \left(\frac{-\hbar\lambda}{3! m^4}\right)^n \Gamma(2n + 1/2)$$

where we recognized the integral representation of the gamma function, and the $\sim$ denotes asymptotic equality, i.e. equality in the limit $\hbar \to 0^+$.

• Expanding the gamma function, we have the asymptotic series

$$\mathcal{Z}\left(m^2,\lambda\right) \sim \frac{\sqrt{2\pi\hbar}}{m}\sum_{n=0}^{N}\left(\frac{-\hbar\lambda}{m^4}\right)^n \frac{(4n)!}{(4!)^n n! 4^n (2n)!} = \frac{\sqrt{2\pi\hbar}}{m}\left(1-\frac{\hbar\lambda}{8m^4}+\frac{35}{384}\frac{\hbar^2\lambda^2}{m^8}-\ldots\right)$$

Note that up to the overall factor of $\sqrt{2\pi\hbar}$, the series only depends on the combination $\hbar\lambda$. Thus our series can either be thought of as a semiclassical expansion in $\hbar$, or a weak coupling expansion in λ, and this is generally true for graphs with a fixed number of external lines by some graph theory.

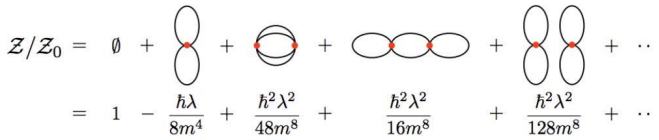
• The series is asymptotic, because $n! \sim e^{n\log n}$, so the numerical factor goes as $e^{n\log n}$. Then eventually it overwhelms the exponential suppression, and the series has zero radius of convergence. However, it is possible to recover $\mathcal{Z}$ from this asymptotic series using Borel summation.

• The theory is well-defined if $m^2 < 0$, but then our asymptotic series is invalid, because we end up integrating a divergent Gaussian. The issue is that we expanded about a maximum instead of a minimum. In physical terminology, we have expanded about an 'unstable vacuum' and found tachyons.

Asymptotic series via Feynman diagrams.

• There are two Feynman rules: a propagator is $\hbar/m^2$ and a four-point vertex is $-\lambda/\hbar$. Note that various factors of i are different since we are in Euclidean signature. Here, since the path integral's integrand is proportional to e^{-S}, every vertex comes with a factor of -1.

• The partition function is the sum of graphs with zero external points. For instance,



The hard part is explaining the numerical factors, which we do below.

• We let a graph be a Feynman diagram where every vertex and edge has a distinguishable label, and let D_n be the set of graphs with n vertices. Adding the labels introduces overcounting; formally the group $G_n = (S_4)^n \rtimes S_n$ performs arbitrary relabelings, and we have overcounted by a factor of $|G_n|$. Then

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} \sim \sum_{n=0}^{\infty}\left(\frac{-\hbar\lambda}{m^4}\right)^n \frac{|D_n|}{|G_n|}, \qquad \mathcal{Z}_0 = \mathcal{Z}\left(m^2,0\right)$$

$$|G_n| = (4!)^n n!,$$

$$|D_n| = (4n-1)(4n-3)\ldots(3)(1) = \frac{(4n)!}{4^n(2n)!}$$

We can find factors $|D_n|\,/\,|G_n|$ by symmetries of the diagram. Note that G_n acts on D_n, partitioning it into orbits O_n. Each orbit $\Gamma \in O_n$ is a 'topological graph', i.e. a graph without labellings. Then the orbit-stabilizer theorem states that

$$\frac{|D_n|}{|G_n|} = \sum_{\Gamma\in O_n}\frac{1}{|\operatorname{Aut}\Gamma|}$$

where Aut Γ is the stabilizer of any element of Γ and $|\mathrm{Aut}\Gamma|$ is also called the symmetry factor. E.g.for the lowest order term, we can flip within each loop and interchange the loops, giving a symmetry factor of 8.

• Finally, we can then express our sum as a sum over orbits,

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = \sum_{\Gamma} \frac{\hbar^{e(\Gamma)-v(\Gamma)}}{|\operatorname{Aut}\Gamma|} \frac{(-\lambda)^{v(\Gamma)}}{(m^2)^{e(\Gamma)}}$$

This recovers the usual Feynman diagram expansion.

• If we define the free energy or Wilsonian effective action by $\mathcal{W} := -\hbar \log \mathcal{Z}$,

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} \equiv e^{(-\mathcal{W}+\mathcal{W}_0)/\hbar}$$

By the standard combinatoric argument, $\mathcal{W}$ contains only connected diagrams; note that the empty diagram is not connected. Note that because of the sign, $\mathcal{W}$ is really the opposite of the sum of connected diagrams.

Let's see it explicitly. Let $F(\Gamma)$ be the contribution of a Feynman graph without the symmetry factor. We may expand any graph as the product of disconnected pieces,

$$\left|\mathrm{Aut}\left(\Gamma_1^{n_1}...\Gamma_k^{n_k}\right)\right| = \prod_{j=1}^{k} \left(n_j!\right) \left|\mathrm{Aut}\left(\Gamma_j\right)\right|^{n_j}, \qquad F\left(\prod_j \Gamma_j^{n_j}\right) = \prod_j F\left(\Gamma_j\right)^{n_j},$$

therefore we have

$$\frac{\mathcal{Z}}{\mathcal{Z}_0} = \sum_{\Gamma} \frac{F(\Gamma)}{|\operatorname{Aut}(\Gamma)|} = \sum_{\{n_j\}} \frac{F\left(\prod_j \Gamma_j^{n_j}\right)}{\left|\mathrm{Aut}\left(\prod_j \Gamma_j^{n_j}\right)\right|} = \sum_{\{n_j\}} \prod_j \frac{1}{n_j!} \left(\frac{F\left(\Gamma_j\right)}{\left|\mathrm{Aut}\left(\Gamma_j\right)\right|}\right)^{n_j} = \exp\left(\sum_j \frac{F\left(\Gamma_j\right)}{\left|\mathrm{Aut}\left(\Gamma_j\right)\right|}\right)$$

Therefore, $\mathcal{W}$ is the opposite of the sum of connected vacuum graphs,

$$-\mathcal{W} + \mathcal{W}_0 = \hbar \sum_{\text{connected } \Gamma} \frac{F(\Gamma)}{|\operatorname{Aut}(\Gamma)|}$$

In a general quantum field theory, $\mathcal{Z}$ sums over vacuum graphs (measures the "vacuum to vacuum' transition amplitude $e^{-iEt/\hbar}$ where E is the vacuum energy), $\mathcal{W}$ directly measures the vacuum energy.

Note. Counting powers of $\hbar$. Propagators are proportional to $\hbar$ and vertices are inversely proportional to $\hbar$, so a graph contributes $\hbar^{\ell(\Gamma)}$ where $\ell(\Gamma) = e(\Gamma) - v(\Gamma) + 1$ in the expansion for $\mathcal{W}$. For a connected graph, this is the number of loops by Euler's theorem, as loops are just faces of a planar graph, and the 'outside face' doesn't count.

To count powers of $\hbar$ in detail, let there be v_a vertices with a fields. Then for vacuum graphs,

$$2e = \sum_a a v_a$$

where we suppress the Γ argument, so

$$\ell = 1 + e - \sum_a v_a = 1 + \sum_a \left(\frac{a}{2} - 1\right) v_a$$

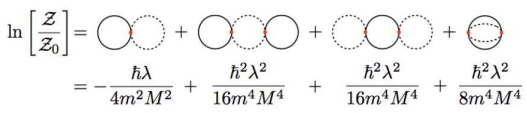
The $\hbar^0$ term is from classical field theory, and corresponds to tree-level diagrams; there are none here because we're focusing on vacuum diagrams. The $\hbar^1$ term comes from one-loop diagrams, and we see above that such diagrams only have quadratic vertices. This makes sense, because the semiclassical contribution comes from integrating over fluctuations in the Gaussian (quadratic) ensemble, and hence only depends on the quadratic part of the action. Finally, all vacuum graphs with an interaction vertex are two-loop or higher. Note that while loops correspond to powers of $\hbar$ in general, the result may be shifted by a constant by field insertions, external states, and external sources.

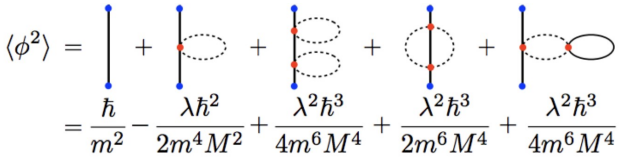
Effective Actions

Zero-dimensional field theory also provides simple examples of effective theories. Consider two real scalar fields with

$$S(\phi,\chi) := \frac{m^2}{2}\phi^2 + \frac{M^2}{2}\chi^2 + \frac{\lambda}{4}\phi^2 chi^2$$

Then we have propagators $\hbar/m^2$ and $\hbar/M^2$, and a vertex $-\lambda/\hbar$.





• Now suppose that we only want to evaluate correlation functions of ϕ. E.g. the mass M of the χ may be very high, so we cannot produce it in an experiment. Then it is useful to "integrate out" the χ, defining the effective action

$$S_{\text{eff}}(\phi) \equiv -\hbar \log\left(\int_{\mathbb{R}} d\chi e^{-S(\phi,\chi)/\hbar}\right)$$

Then correlation functions of ϕ can be evaluated with the weight $e^{-S_{\text{eff}}(\phi)/\hbar}$. Here the action is quadratic in χ, so we can explicitly compute

$$\int_{\mathbb{R}} d\chi e^{-S(\phi,\chi)/\hbar} = e^{-m^2\phi^2/2\hbar}\sqrt{\frac{2\pi\hbar}{M^2+\lambda\phi^2/2}}$$

However, in general we must use perturbation theory. Here we have, exactly,

$$S_{\text{eff}}(\phi) = \frac{m^2}{2}\phi^2 + \frac{\hbar}{2}\log\left(1+\frac{\lambda}{2M^2}\phi^2\right) + \frac{\hbar}{2}\log\frac{M^2}{2\pi\hbar}$$

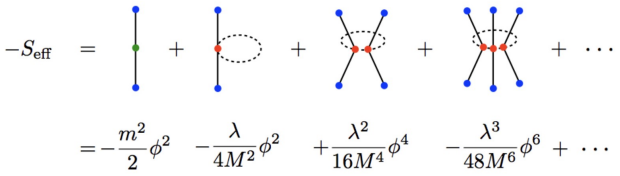
We ignore the final term, which is a constant, but its appearance is a manifestation of the cosmological constant problem.

$$S_{\text{eff}}(\phi) \approx \frac{m_{\text{eff}}^2}{2}\phi^2 + \frac{\lambda_4}{4!}\phi^4 + \frac{\lambda_6}{6!}\phi^6 + ...,$$

$$m_{\text{eff}}^2 = m^2 + \frac{\hbar\lambda}{2M^2}$$

$$\lambda_4 = -\frac{3\hbar}{2}\frac{\lambda^2}{M^4}, \qquad \lambda_6 = 15\hbar\frac{\lambda^3}{M^6}, \qquad \lambda_{2k} = (-1)^{k+1}\hbar\frac{(2k)!}{2^{k+1}k}\frac{\lambda^k}{M^{2k}}$$

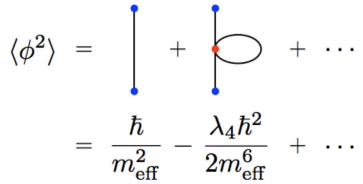
Diagrammatically, it is a polynomial in ϕ, where the contribution to the ϕ^n term comes from diagrams with n external ϕ fields, only connected diagrams are used since we take the logarithm, and only χ propagators appear since the ϕ is not dynamical:



Terms that depend only on ϕ can be simply pulled out of the integral; here we have represented the original mass term using a diagram with a special green vertex. The solid lines aren't ϕ propagators; they're for visual convenience, to separate the ϕ fields from their vertices.

• In this theory, all contributions to $S_{\text{eff}}(\phi)$ have one χ loop, so the corrections to the couplings are $O(\hbar)$. In general, the corrections will each themselves be asymptotic series in $\hbar$. We can imagine the process of integrating out χ as 'zooming out' until we can't see the loop.

• Using the effective action, it is much easier to compute $\langle\phi^2\rangle$ to $O(\lambda^2)$,



In general, we think of the world as a series of nested effective field theories. As a result, every field theory has an infinite amount of terms, i.e. all terms allowed by symmetry, and every computation is actually using an effective action. But at higher energy scale only renormalizable terms in actions are significant.

Note. We've talked about two kinds of effective action, $\mathcal{W}$ and $S_{\text{eff.}}$. These are essentially the same; consider adding an external current, so the action becomes $S[\phi, J]$. Then $\mathcal{W}[J]$ is the effective action for J after integrating out ϕ. Since we've done everything with $J = 0$, only vacuum diagrams appear in the expansion for $\mathcal{W}$, but generally there would be external J points just like in S_{eff}. Both $\mathcal{W}$ and S_{eff} are examples of Wilsonian effective actions.

1PI effective action

• The effective action $\mathcal{W}[J]$ is the effective action for J once the field ϕ has been integrated out; it is analogous to the Helmholtz free energy $F(H)$ for a magnet. However, for an isolated system there is no clear external source.

• In thermodynamics, we deal with this by switching to the Gibbs free energy $G(M)$. By minimizing $G(M)$, we may find the equilibrium magnetization $M = \langle s(\mathbf{x})\rangle$. We can perform a simple minimization because $G(M)$ already accounts for thermal effects, unlike $H(s(\mathbf{x}))$.

• Similarly, in quantum field theory we Legendre transform from the effective action $\mathcal{W}(J)$ to the 1PI effective action $\Gamma(\Phi)$, where the 'classical field' Φ is the VEV in the presence of a source,

$$\Phi = \langle\phi\rangle_J$$

We will see that minimizing $\Gamma(\Phi)$ can tell us about phase transitions, just like $G(M)$.

$$\frac{\partial\mathcal{W}}{\partial J} = -\frac{\hbar}{\mathcal{Z}(J)}\frac{\partial}{\partial J}\int d\phi e^{-(S+J\phi)/\hbar} = \frac{1}{\mathcal{Z}(J)}\int d\phi e^{-(S+J\phi)/\hbar}\phi = \langle\phi\rangle_J$$

$$\Phi = \partial\mathcal{W}/\partial J$$

Quantum effective action $\Gamma(\Phi)$:

$$\Gamma(\Phi) := \mathcal{W}(J) - \Phi J$$

$$J = -\frac{\partial\Gamma}{\partial\Phi}$$

• To understand the meaning of $\Gamma(\Phi)$, note that when $J = 0$, possible values for Φ correspond to extrema of Γ. Then in higher dimensions, the action S yields the classical equations of motion $\delta S/\delta\phi = 0$ while Γ yields the 'quantum' equations of motion for $\delta\Gamma/\delta\Phi = 0$ which account for quantum effects. In zero dimensions these are simply algebraic equations, but in higher dimensions we can have nontrivial spatial dependence, yielding solitons.

• Next, we define a quantum theory by a path integral of Φ, where we let $\Gamma(\Phi)$ play the role of the classical action.

$$e^{-W_\Gamma(J)/g} := \int d\Phi e^{-(\Gamma(\Phi)+J\Phi)/g}$$

$$W_\Gamma(J) = \sum_{\ell=0}^{\infty} g^\ell \mathcal{W}_\Gamma^{(\ell)}(J),$$

$W_\Gamma^{(\ell)}(J)$ is the sum of all ℓ-loop connected Feynman graphs built from $\Gamma(\Phi)$.

In the limit $g \to 0$ only the tree-level connected Feynman graphs contribute;

$$\lim_{g\to 0}\mathcal{W}_\Gamma(J) = W_\Gamma^{(0)}(J) \equiv \Gamma(\Phi) + J\Phi|_{J=-\frac{\partial\Gamma}{\partial\Phi}} = \mathcal{W}(J)$$

Therefore, the connected graphs built from the classical action $S(\phi) + J\phi$ are equal to the tree graphs built from the effective action $\Gamma(\Phi) + J\Phi$, another sense in which Γ includes quantum corrections to S.

• Diagrammatically, every connected diagram is a tree whose vertices are one-particle irreducible (1PI) graphs. Therefore, we can interpret $\Gamma(\Phi)$ as the sum of 1PI graphs built from $S(\phi)$, where a 1PI diagram with n external ϕ fields contributes to the Φ^n term/vertex in $\Gamma(\Phi)$.

• Explicitly, suppose we have fields ϕ^a and sources J_a. Then

$$-\hbar\frac{\partial^2\mathcal{W}}{\partial J_a\partial J_b} = -\hbar\frac{\partial}{\partial J_a}\left(\frac{1}{\mathcal{Z}(J)}\int d^n\phi e^{-(S(\phi^a)+J_a\phi^a)/\hbar}\phi^b\right) = \left\langle\phi^a\phi^b\right\rangle_J - \left\langle\phi^a\right\rangle_J\left\langle\phi^b\right\rangle_J \equiv \left\langle\phi^a\phi^b\right\rangle_J^{\text{conn}}$$

The first term contains both connected and disconnected diagrams, but the disconnected ones are canceled out by the second term, leaving the connected correlation function, also called the exact propagator. It is the quadratic term in $\mathcal{W}[J]$, as expected.

$$\left\langle \phi^a \phi^b \right\rangle_J^{\text{conn}} \equiv -\hbar \frac{\partial^2 \mathcal{W}}{\partial J_a \partial J_b} = -\hbar \frac{\partial \Phi^b}{\partial J_a} = -\hbar \left(\frac{\partial J_a}{\partial \Phi_b} \right)^{-1} = \hbar \left(\frac{\partial^2 \Gamma}{\partial \Phi^b \partial \Phi^a} \right)^{-1}$$

That is, the exact propagator is the inverse of the quadratic term of the 1PI effective action.

• To connect this to diagrams, note that we have the diagrammatic recurrence relation

$$\text{connected} \sim \text{connected} \times \text{1PI} \times \text{connected}$$

which is indeed compatible with what we found.

• For more than two external fields, we can continue differentiating $\mathcal{W}$ with respect to J. The details are messy, but the key identity is

$$\frac{\partial}{\partial \Phi} = \frac{\partial J}{\partial \Phi} \frac{\partial}{\partial J} = \left(\left\langle \phi^2 \right\rangle^{\text{conn}} \right)^{-1} \frac{\partial}{\partial J}$$

That is, adding another external Φ leg to Γ is the same as adding another external J to $\mathcal{W}$ and amputating the resulting leg; this is exactly how 1PI diagrams are produced.

• In general dimension, Γ is extensive and dividing it by the spacetime volume gives the effective potential $V_{\text{eff}}(\Phi)$. As in thermodynamics, the effective potential must be convex; taking the convex hull is the same as the Maxwell construction for the var der Waals gas.

Note. There are various sign flips above, so $\mathcal{W}[J]$ and $\Gamma[\Phi]$ are not really the sum of connected and 1PI diagrams, but rather the sum up to a sign. Explicitly, we have

$$W[J] = \sum_n \frac{(-1)^{n-1}}{n!} J^n \left\langle \phi^n \right\rangle^{\text{conn}}, \quad \Gamma[\Phi] = -\sum_n \frac{1}{n!} \Phi^n \left\langle \phi^n \right\rangle^{\text{1PI}}$$

as can be shown using the identities above. The only exception is the case $n = 2$ for the 1PI effective action. Note that for $n > 2$, the n^{th} order terms in the action contribute directly to $\langle \phi^n \rangle^{\text{1PI}}$, but the quadratic terms don't contribute directly to Π. This is because we treated the quadratic part of the action as 'free' and hence differently from the 'interacting' part when setting per perturbation theory. By summing a geometric series, we hace

$$\left\langle \phi^2 \right\rangle^{\text{conn}} \left(p^2\right) = \frac{1}{p^2 - m_0^2 - \Pi\left(p^2\right)}, \qquad \Pi\left(p^2\right) = \sum \text{1PI diagrams}$$

which implies that the quadratic term of the 1PI effective action is

$$\left\langle \phi^2 \right\rangle^{\text{conn}} \left(p^2\right)^{-1} = p^2 - m^2 - \Pi\left(p^2\right)$$

with an extra inverse bare propagator.

Example. In the case of no external sources, $J = 0$, we have

$$W[0] = \Gamma[\Phi]|_{\partial\Gamma/\partial\Phi = 0}.$$

Thus, by plugging in the equilibrium value of Φ, we get a relationship between the set of connected vacuum bubbles and the set of 1PI diagrams. E.g. in the simple case $\Phi = 0$ we find

$$\sum \text{ connected vacuum bubbles } = \sum \text{1PI vacuum bubbles}$$

which holds because all non-1PI connected vacuum bubbles have tadpoles, which vanish when $\Phi = 0$.

Analogy with statistical mechanics

Consider a magnet with spin field $s(x)$ and energy $E(s)$. Then the partition function is

$$\mathcal{Z} = \int ds e^{-\beta(E(s)-Hs)}$$

where H is an external field. We define the Helmholtz free energy by

$$F(H) = -\frac{1}{\beta}\log \mathcal{Z}$$

where we integrated out the spin field, leaving a function of the external field. We leave the dependence on T implicit, because in quantum field theory its analogue $\hbar$ can't be varied.

• We define the magnetization $M = \langle s\rangle = -\partial F/\partial H$. This is the spin seen on a macroscopic scale, accounting for thermal corrections. We define the Gibbs free energy

$$G(M) = F(H) + MH, \quad \frac{\partial G}{\partial M} = H$$

by Legendre transform so it is directly a function of M.

• To compute the equilibrium value of M, we simply find the minima of G. That is, we don't have to do a path integral over configurations of M, so G 'accounts for thermal corrections'. In particular, it's easy to tell where phase transitions occur.

• E.g.in the ferromagnetic phase transition, the energy is minimized by having the spins aligned, but thermal effects favor disorder; we can tell which wins by looking at $G(T, M)$. In quantum field theory, an analogue is the Coleman-Weinberg potential: a ϕ^4 theory can exhibit spontaneous symmetry breaking because the effective potential has a nonzero minimum.

Fermionic Fields

• A fermionic field is an anticommuting field. In zero dimensions, there is no notion of spin, but in $d = 3 + 1$ fermionic fields have half-integer spin by the spin-statistics theorem.

• The Grassmann algebra is an algebra over the complex numbers, generated by n elements θ^a that anticommute, $\theta^a\theta^b = -\theta^b\theta^a$. Note that scalar multiplication still commutes, $\theta^a\alpha = \alpha\theta^a$ for $\alpha \in \mathbb{C}$. Also, products of an even number of Grassmann numbers commute with everything, so they are 'bosonic'.

• Every Grassmann number squares to zero, so a general element of the algebra is

$$F(\theta) = \alpha + \rho_a\theta^a + \frac{1}{2!}g_{a_1a_2}\theta^{a_1}\theta^{a_2} + ... + \frac{1}{n!}h_{a_1...a_n}\theta^{a_1}...\theta^{a_n}$$

where we can take the coefficients to be totally antisymmetric.

• Next, we define differentiation and integration. The derivative is defined by

$$\frac{\partial\theta^a}{\partial\theta^b} = \delta^a_b, \quad \frac{\partial\alpha}{\partial\theta^a} = 0, \quad \alpha \in \mathbb{C}$$

so it anticommutes with multiplication by a Grassmann number,

$$\frac{\partial}{\partial\theta^a}\theta^b + \theta^b\frac{\partial}{\partial\theta^a} = \delta^b_a$$

where the derivatives act on everything to the right, i.e. on a test function. Then the Grassmann derivative obeys a product rule with an extra minus sign. (Note the spatial derivative of a product of Grassmann fields obeys the product rule, with no extra signs.)

• To define the integral, we consider the case of one Grassmann number, so the most general function is $f(\theta) = \alpha + \rho\theta$. We demand the integral be linear, so

$$\int d\theta(\alpha + \rho\theta) = \alpha \int d\theta + \rho \int d\theta\theta$$

We also demand invariance under a shift of the integration variable by η. For complex η,

$$\int d\theta f(\theta) = \int d\theta f(\theta + \eta) = (\alpha + \rho\eta) \int d\theta + \rho \int d\theta\theta$$

• Using our choice of normalization for the measure, we conclude

$$\int d\theta 1 = 0, \quad \int d\theta\theta = 1$$

More generally, η can be Grassmann valued, which shows that the $d\theta$ integral of anything not linear in θ vanishes; in particular, Grassmann differentiation and integration are equal!

• Using the definition of integration, we find

$$\int d\theta \frac{\partial}{\partial\theta} F(\theta) = 0$$

since the derivative removes the term linear in θ. This allows us to integrate with parts, though there's an extra sign flip.

• If we have n Grassmann variables θ^a, we define $d^n\theta = d\theta^n ... d\theta^1$, so

$$\int d^n\theta\theta^1 ... \theta^n = 1$$

with all lower-order terms integrating to zero, and in general,

$$\int d^n\theta\theta^{a_1} ... \theta^{a_n} = \epsilon^{a_1 ... a_n}$$

• Next, consider a change of variables $\theta'^a = N^a_b \theta^b$. Then by linearity,

$$\int d^n\theta\theta^{a_1} ... \theta'^{a_n} = N^{a_1}_{b_1} ... N^{a_n}_{b_n} \int d^n\theta\theta^{b_1} ... \theta^{b_n} = N^{a_1}_{b_1} ... N^{a_n}_{b_n} \epsilon^{b_1 ... b_n} = (\det N)\epsilon^{a_1 ... a_n}$$

which shows that

$$d^n\theta = (\det N) d^n\theta'$$

This is the exact opposite of the transformation for bosons, e.g. $d\theta = 2d(2\theta)$.

$$\int d\eta \delta(\eta - \theta) f(\eta) = f(\theta) \qquad \text{Grassmann Dirac delta-f.}$$

By considering the general form $f(\eta)$ could take, we see that $\delta(\eta - \theta) = \eta - \theta$.

Note. Concretely, the θ^a can be thought of as one-forms, and the product can be thought of as the wedge product. The Grassmann integral $\int d^n\theta$ can be thought of as integration of a differential form over $\mathbb{R}^n$, where only the integral of a top-dimensional form is nonzero.

A simple calculation in a fermionic free field theory.

• To get a nontrivial theory, we need two fermionic fields, θ^1 and θ^2. The action must be bosonic, so it must be even in the fields, so we must have

$$S(\theta) = \frac{1}{2} A \theta^1 \theta^2$$

$$\mathcal{Z}_0 = \int d^2\theta e^{-S(\theta)/\hbar} \overset{\text{Taylor}}{=} \int d^2\theta \left(1 - \frac{A}{2\hbar}\theta^1\theta^2\right) = -\frac{A}{2\hbar}$$

• More generally, consider $2m$ fermionic fields θ^a with

$$S(\theta) = \frac{1}{2} A_{ab}\theta^a\theta^b$$

where A is $2m \times 2m$ antisymmetric matrix.

$$\mathcal{Z}_0 = \int d^{2m}\theta e^{-A(\theta,\theta)/2\hbar} = \left(-\frac{1}{2\hbar}\right)^m \frac{1}{m!} \int d^{2m}\theta A_{a_1a_2}...A_{a_{2m-1}a_{2m}}\theta^{a_1}...\theta^{a_{2m}}$$

where we used the fact that only the m^{th} order term can contribute.

$$\text{Pfaff } A := \frac{1}{2^m m!}\epsilon^{a_1...a_{2m}} A_{a_1a_2}...A_{a_{2m-1}a_{2m}}$$

$$(\text{Pfaff } A)^2 = \det A$$

$$\mathcal{Z}_0 = \pm\sqrt{\frac{\det A}{\hbar^n}}$$

while for free bosons we had the $\hbar$'s in the numerator and the determinant in the denominator.

• We can also consider sources,

$$S(\theta,\eta) = \frac{1}{2} A_{ab}\theta^a\theta^b + \eta_a\theta^b$$

where the source η must be fermionic.

$$S(\theta,\eta) = \frac{1}{2}\left(\theta^a + \eta_c\left(A^{-1}\right)^{ca}\right) A_{ab}\left(\theta^b + \eta_d\left(A^{-1}\right)^{db}\right) + \frac{1}{2}\eta_a\left(A^{-1}\right)^{ab}\eta_b.$$

Here the two cross-terms are equal because A is antisymmetric and θ and η anticommute.

$$\mathcal{Z}_0(\eta) = \exp\left(-\frac{1}{2\hbar}A^{-1}(\eta,\eta)\right)\mathcal{Z}_0(0)$$

$$\left\langle\theta^a\theta^b\right\rangle = \frac{\hbar^2}{\mathcal{Z}_0(0)}\frac{\partial^2\mathcal{Z}_0(\eta)}{\partial\eta_a\partial\eta_b}\bigg|_{\eta=0} = \hbar\left(A^{-1}\right)^{ab}$$

(???? derive!) where we use Taylor expansion and the fact that derivatives anticommute with Grassmann numbers. This is just the same as the bosonic result: the free propagator is always the inverse of the kinetic term.

• The fact that functions of a finite number of Grassmann variables can always be represented as polynomials means that in $d = 0$, all fermionic path integrals can be evaluated exactly. In higher

dimensions, we use Grassmann-valued fields which contain infinitely many independent Grassmann variables, and turn to perturbation theory with Feynman diagrams.

Note. Strictly speaking, all fermionic fields in relativistic quantum field theory must be Grassmann numbers, even when we are performing canonical quantization. E.g. a Majorana mass term cannot be written down in a classical Lagrangian unless the spinors are anticommuting variables; otherwise the term vanishes identically.

One might worry our classical analysis of the Dirac Lagrangian missed some sign flips. The results are unchanged as long as we define $\pi = \partial\mathcal{L}/\partial\dot{\psi}$ as a derivative to the right of $\mathcal{L}$ acting to the left, and $\mathcal{H} = \pi\dot{\psi} - \mathcal{L}$. Practically speaking, just as there are ambiguities when going from a classical to a quantum theory (which are, e.g. fixed by normal ordering and other prescriptions), there are ambiguities when going from a classical to a semiclassical theory, where the spinors become Grassmann valued. As with normal ordering, we pick the conventions so that the results work.

How supersymmetry makes a path integral much easier to evaluate.

• Consider a theory with one bosonic field ϕ and two fermionic fields ψ_1 and ψ_2, normalizing the measure so that

$$\mathcal{Z} = \int \frac{d\phi d\psi_1 d\psi_2}{\sqrt{2\pi}} e^{-S(\phi,\psi_i)}$$

$$S(\phi,\psi_1,\psi_2) := \frac{1}{2}(\partial h)^2 - \psi_1\psi_2\partial^2 h$$

where $h(\phi)$ is a real-valued polynomial and ∂h is its derivative with respect to ϕ. This action has the most general possible fermionic term, so this just restricts the purely bosonic piece.

0D SUSY:

$$\delta\phi = \epsilon_1\psi_1 + \epsilon_2\psi_2, \qquad \delta\psi_1 = \epsilon_2\partial h, \qquad \delta\psi_2 = -\epsilon_1\partial h \qquad (\epsilon_i \text{ - fermionic})$$

$$\delta S = \partial h\partial^2 h\,(\epsilon_1\psi_1 + \epsilon_2\psi_2) - (\epsilon_2\partial h)\,\psi_2\partial^2 h - \psi_1\,(-\epsilon_1\partial h)\,\partial^2 h = 0$$

S is invariant; one can show the measure is invariant as well. We also have $\delta_1^2 = \delta_2^2 = [\delta_1,\delta_2] = 0$.

$$\forall\mathcal{O}(\phi,\psi_i) \qquad \langle\delta\mathcal{O}\rangle = \frac{1}{\mathcal{Z}_0}\int d\phi d^2\psi e^{-S}\delta\mathcal{O} = \frac{1}{\mathcal{Z}_0}\int d\phi d^2\psi\,\delta\left(e^{-S}\mathcal{O}\right)$$

Now, the quantity in parentheses is at most linear in the ψ_i, so the terms that come from varying ψ_i don't depend on ψ_i and hence integrate to zero. The terms that come from varying ϕ are total derivatives in ϕ. Then $\langle\delta\mathcal{O}\rangle = 0$.

• Now suppose we choose $\mathcal{O} = (\partial g)\psi_1$ for some $g(\phi)$. Setting $\epsilon_1 = -\epsilon_2 = \epsilon$, we have

$$0 = \langle\delta\mathcal{O}\rangle = \epsilon\left\langle\partial g\partial g - \partial^2 g\psi_1\psi_2\right\rangle$$

But this is also the first-order change of the action under the deformation $h \to h + g$. That is, we can replace h in the action with any $h + g$ as long as we don't change the behavior at infinity. In particular, if $g = h$, then we can scale h up. Then the path integral only depends on regions right around extrema of ϕ, a phenomenon known as localization.

• We can explicitly find the partition function. Near an extremum ϕ_*, we have

$$h(\phi) = h(\phi_*) + \frac{c_*}{2}(\phi - \phi_*)^2, \quad S(\phi, \psi_i) \approx \frac{c_*^2}{2}(\phi - \phi_*)^2 + c_*\psi_1\psi_2$$

The contribution to the partition function is

$$\frac{1}{\sqrt{2\pi}} \int d\phi d^2\psi e^{-c_*^2(\phi-\phi_*)^2/2}(1 - c_*\psi_1\psi_2) = \frac{c_*}{\sqrt{2\pi}} \int d\phi e^{-c_*^2(\phi-\phi_*)^2/2} = \frac{c_*}{\sqrt{c_*^2}} = \text{sign}(c_*)$$

• Then the full partition function is

$$\mathcal{Z}[h] = \sum_{\text{extrema } \phi_*} \text{sign}\left(\partial^2 h\big|_{\phi_*}\right)$$

I.e. if h has odd degree, then the partition function is 0, and if h has even degree, the partition function is ± 1.

• Localization is also useful for computing correlation functions of operators $\mathcal{O}_i$. We are interested in supersymmetry-invariant operators, $\delta\mathcal{O} = 0$. On the other hand, if $\mathcal{O} = \delta\mathcal{O}'$ then the correlator is automatically zero. Thus the nontrivial observables in the theory correspond to the cohomology of δ. That is, one can use supersymmetric quantum field theories to compute the cohomologies of interesting spaces.

Quantum Mechanics (1D QFT)

• We recall that we usually consider fields of the form $x : M \to N$ where (N, g) is a Riemannian manifold. In the one-dimensional case, M is either the circle S^1, parametrized by $t \in [0, T)$ with endpoints identified, or the interval I, parametrized by $t \in [0, T]$.

• For every time $t, x(t)$ is a point in N. We let $x^a(t)$ denote its coordinates; specifically this is the pullback to M by x of coordinates on a patch of N. A standard choice of action is

$$S[\phi] = \int_M \frac{1}{2} g_{ab}(x)\dot{x}^a\dot{x}^b + V(x)dt$$

$g_{ab}(x(t))$ is the pullback to M of the Riemannian metric on N, and $V : N \to \mathbb{R}$ is a 'potential' and $V(x(t))$ is its pullback. We've implicitly chosen the metric on M to be the Euclidean metric $\delta_{tt} = 1$.

• Directly varying this action gives the Euler-Lagrange equation

$$\frac{d^2x^a}{dt^2} + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = g^{ab}(x)\frac{dV}{dx^b}, \qquad \text{(geodesic eq.)}$$

$\Gamma^a_{bc} := \frac{1}{2}g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})$. Note that there is no minus sign on the right-hand side because of the Wick rotation $t \to it$. In Euclidean signature, $\mathbf{F} = -m\mathbf{a}$ because a picks up a factor of i^2 in the Wick rotation (????).

• We interpret the target space N as being the space where we live, and $x(t)$ as the trajectory of a quantum particle; we call either M or $x(M) \subset N$ the worldline of the particle. However, note that we usually interpret M as being the space where we live, and N as an abstract space of fields, e.g. a Grassmann algebra for fermion fields. Even within quantum mechanics, with two particles N is no longer space, but a configuration space with twice the dimension.

Path integral picture from standard quantum mechanics.

$$H = \frac{1}{2}\Delta + V, \qquad \Delta := \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^a}\left(\sqrt{g}g^{ab}\frac{\partial}{\partial x^b}\right)$$

$$K_T\left(y_0, y_1\right) := \left\langle y_1 \left| e^{-HT} \right| y_0 \right\rangle = \langle y_1, T | y_0, 0 \rangle$$

$$\frac{\partial}{\partial t}K_t(x,y) + HK_t(x,y) = 0$$

the Schrodinger equation in Euclidean signature, with $\hbar = 1$.

E.g.in $N \cong \mathbb{R}^n$, we have $g_{ab} = \delta_{ab}$, $V = 0$,

$$K_t(x,y) = \frac{1}{(2\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{2t}\right)$$

For a general metric:

$$\lim_{\Delta t \to 0} K_{\Delta t}(x,y) \sim \frac{1}{(2\pi\Delta t)^{n/2}} a(x) \exp\left(-\frac{d(x,y)^2}{2\Delta t}\right)$$

where $d(x,y)$ is the geodesic distance from x to y and $a(x)$ is a polynomial in the Riemann curvature tensor.

The heat kernel can be rewritten by inserting copies of the identity:

$$\left\langle y_1 \left| e^{-HT} \right| y_0 \right\rangle := \lim_{N\to\infty} \left(\frac{1}{2\pi\Delta t}\right)^{nN/2} \int \prod_{i=1}^{N-1} d^n x_i a\left(x_i\right) \exp\left[-\frac{\Delta t}{2}\left(\frac{d\left(x_{i+1}, x_i\right)}{\Delta t}\right)^2\right]$$

$$\mathcal{D}x := \lim_{N\to\infty} \left(\frac{1}{2\pi\Delta t}\right)^{nN/2} \int \prod_{i=1}^{N-1} d^n x_i a\left(x_i\right)$$

If the paths are differentiable, then the sum over $\left(d\left(x_{i+1}, x_i\right)/\Delta t\right)^2$ converges to the integral over $g_{ab}\dot{x}^a\dot{x}^b$ and we recover the path integral expression

$$K_T\left(y_0, y_1\right) = \int\limits_{C_T[y_0,y_1]} \mathcal{D}x e^{-S}$$

where S is the action we wrote down earlier with $V = 0$, and $C_T\left[y_0, y_1\right]$ specifies boundary conditions $x(0) = y_0$ and $x(T) = y_1$.

Computation of correlation functions.

• A local operator is one which depends on the field (in this case, $\mathbf{x}$) at only one point of the worldline. E.g. any function $\mathcal{O} : N \to \mathbb{R}$ corresponds to a local operator $\hat{\mathcal{O}}$. Let $|y,t\rangle$ be the Heisenberg state that will be peaked at y at time t. Then in Heisenberg picture,

$$\left\langle y_1, T | \hat{\mathcal{O}}(t) | y_0, 0 \right\rangle = \left\langle y_1, 0 \left| e^{-H(T-t)}\hat{\mathcal{O}}(0)e^{-Ht} \right| y_0, 0 \right\rangle = \left\langle y_1 \left| e^{-H(T-t)}\hat{\mathcal{O}}e^{-Ht} \right| y_0 \right\rangle$$

where the final expression is in Schrodinger picture.

• Inserting a complete set of states, we find this is equal to

$$\int d^n x \mathcal{O}(x) K_{T-t}(y_1, x) K_t(x, y_0) = \int_{C_T[y_1,y_0]} \mathcal{D}x e^{-S} \mathcal{O}(x(t))$$

Therefore, in general correlation functions are computed from the path integral by (??????)

$$\left\langle y_1, T \left| \hat{\mathcal{O}}_n(t_n) ... \hat{\mathcal{O}}_1(t_1) \right| y_0, 0 \right\rangle = \int_{C_T[y_0,y_1]} \mathcal{D}x e^{-S} \prod_{i=1}^{n} \mathcal{O}_i(x(t_i))$$

Note that we don't need to divide by a partition function here, because the normalization is implicitly in the definition of the path integral measure.

• Note that when we insert complete sets of states, we find a path of the form $(y_0, 0) \to (x_1, t_1) \to ... \to (x_n, t_n) \to (y_1, T)$ and integrate out the internal points to get an arbitrary path $(y_0, 0) \to (y_1, T)$. This is only possible if the points are in time order, as otherwise we'd end up with an integral over paths that go forward and backward in time. Then

$$\left\langle y_1, T \left| \mathcal{T} \prod_{i=1}^{n} \hat{\mathcal{O}}_i(t_i) \right| y_0, 0 \right\rangle = \int_{C_T[y_0,y_1]} \mathcal{D}x e^{-S} \prod_{i=1}^{n} \mathcal{O}_i(x(t_i))$$

This is fitting, as we know the $\mathcal{O}_i$ commute, but the $\hat{\mathcal{O}}_i$ do not, and time ordering makes everything inside it effectively commute. Time-ordered products appear ubiquitously when dealing with the path integral, and we will often drop the time-ordering symbol.

• Note that if there are no derivative terms, positions at arbitrarily close times 'decouple', and the path integral splits into individual independent integrals for each time. Then all correlation functions split as

$$\left\langle \hat{\mathcal{O}}_1(t_1) \hat{\mathcal{O}}_2(t_2) \right\rangle = \left\langle \hat{\mathcal{O}}_1(t_1) \right\rangle \left\langle \hat{\mathcal{O}}_2(t_2) \right\rangle$$

so there are no nontrivial correlations. This corresponds to the statement that fields without kinetic terms are nondynamical and do not propagate.

• More generally, we can consider operators that depend on derivatives of $\mathbf{x}$. In particular, the canonical momentum for our action is

$$p_a = \frac{\partial L}{\partial \dot{x}^a} = g_{ab} \dot{x}^b,$$

so we could replace the function $\mathcal{O}(x^a, \dot{x}^a)$ with the operator $\hat{\mathcal{O}} = \mathcal{O}\left(\hat{x}^a, g^{ab}(\hat{x})\hat{p}_b\right)$. However, this is also puzzling because the latter depends on an ordering prescription while the former does not (??? why???). One can simply define $\hat{\mathcal{O}}$ to be normal ordered, but we would like to understand where this comes from in the path integral.

Ordering problems and the continuum limit.

Note. Such ordering problems also occur in canonical quantization. Here, given the Poisson bracket structure $\{f, g\} = h$ we would like to define operators satisfying $[\hat{f}, \hat{g}] = i\hbar\hat{h}$ which act irreducibly on the Hilbert space. However, the Groenewald-van Hove theorem states that this is

impossible in general; in fact, it's even impossible for polynomials in the positions and momenta, once we go beyond quadratics. The idea of quantizing a classical system in general remains ambiguous.

• For a free particle in one dimension, let $0 < t_- < t < t_+ < T$. Then

$$\int\limits_{C_T[y_0,y_1]} \mathcal{D}x e^{-S} x(t)\dot{x}\left(t_-\right) = \left\langle y_1 \left| e^{-H(T-t_+)} \hat{x} e^{-H(t_+-t)} \hat{p} e^{-Ht} \right| y_0 \right\rangle$$

$$\int\limits_{C_T[y_0,y_1]} \mathcal{D}x e^{-S} x(t)\dot{x}\left(t_+\right) = \left\langle y_1 \left| e^{-H(T-t)} \hat{x} e^{-H(t-t_-)} \hat{p} e^{-Ht} \right| y_0 \right\rangle$$

Taking the limits $t_+ \to t$ from above and $t_- \to t$ from below, the difference of the right-hand sides is nonzero, as

$$\left\langle y_1 \left| e^{-H(T-t)} [\hat{x}, \hat{p}] e^{-Ht} \right| y_0 \right\rangle = \left\langle y_1 \left| e^{-HT} \right| y_0 \right\rangle$$

• To take the same limits on the left-hand side, we explicitly restore the discretization Δt. Then the furthest we can take the limit is

$$x(t)\dot{x}\left(t_-\right) - x(t)\dot{x}\left(t_+\right) \to x_t \frac{x_t - x_{t-\Delta t}}{\Delta t} - x_t \frac{x_{t+\Delta t} - x_t}{\Delta t}$$

This is a discretized second derivative times $x_t \Delta t$, which would vanish in the limit $\Delta t \to 0$ if our paths were smooth. However, we integrate over paths that are not even differentiable.

• More explicitly, the part of the path integral that depends on x_t is

$$\int dx_t K_{\Delta t}\left(x_{t+\Delta t}, x_t\right) \left(x_t \frac{x_t - x_{t-\Delta t}}{\Delta t} - x_t \frac{x_{t+\Delta t} - x_t}{\Delta t} \right) K_{\Delta t}\left(x_t, x_{t-\Delta t}\right)$$

but using the known form of the heat kernel, this is equal to

$$-\int dx_t x_t \frac{\partial}{\partial x_t} \left(K_{\Delta t}\left(x_{t+\Delta t}, x_t\right) K_{\Delta t}\left(x_t, x_{t-\Delta t}\right) \right) = K_{2\Delta t}\left(x_{t+\Delta t}, x_{t-\Delta t}\right)$$

by integration by parts. Performing the other integrals gives $\left\langle y_1 \left| e^{-HT} \right| y_0 \right\rangle$ as desired. Similarly, the operator ordering in the Hamiltonian is also determined by the discretization procedure.

• To understand the result more quantitatively, note that

$$K_t(x, y) = \frac{1}{2\pi t} e^{-(x-y)^2/2t}$$

has $\langle (x-y)^2 \rangle = t$, typical for a diffusion process. Then a typical derivative is actually $O(\sqrt{\Delta t})$ instead of $O(1)$, allowing it to contribute in ways it naively should not. E.g. changing a forward derivative to a backward derivative within a single timeslice yields a change of

$$x_{t+\Delta t} \frac{x_{t+\Delta t} - x_t}{\Delta t} - x_t \frac{x_{t+\Delta t} - x_t}{\Delta t} = \Delta t \left(\frac{x_{t+\Delta t} - x_t}{\Delta t} \right)^2 = O(1)$$

Hence the operator ordering is hidden in the discretization procedure.

The issue of the path integral measure.

• Naively, we may take the limit of infinitely many position-space integration measures to yield the path integral measure $\mathcal{D}x$. Alternatively, one might want to do this in Fourier space, taking the limit of arbitrarily high frequencies; in this picture it is clear we only need a countable infinity of integrations. However, both these limits do not exist.

• We say $d\mu$ is a Lebesgue measure on $\mathbb{R}^D$ if it assigns a strictly positive volume

$$\mathrm{vol}(U) \equiv \int_U d\mu$$

to every non-empty open set U, is translationally invariant, and if for every $x \in \mathbb{R}^D$ there is at least one open neighborhood U_x of x with finite volume. A standard choice is $d\mu = d^D x$.

• There is no non-trivial Lebesgue measure on a vector space with countably infinite dimension. Let $C(L)$ denote an open hypercube of side length L. Dividing it gives

$$\mathrm{vol}(C(L)) \geq 2^D \,\mathrm{vol}(C(L/2))$$

Then if $D \to \infty$, the only way for the measure to remain finite is for $\mathrm{vol}(C(L/2))$ to go to zero, so the measure of any hypercube must be zero. Since the dimension is countably infinite any open set can be covered with countably many cubes, so the measure is identically zero.

• For the path integral in one dimension, there is a nontrivial measure,

$$d\mu = \lim_{N\to\infty} \left(\frac{1}{2\pi\Delta t}\right)^{nN/2} \int \prod_{i=1}^{N-1} d^n x_i \exp\left(-\frac{\Delta t}{2}\left(\frac{x_{t_{i+1}} - x_{t_i}}{\Delta t}\right)^2\right)$$

This is called the Wiener measure, and it necessarily involves the $\mathcal{D}x$ and $e^{-S[x]}$ factor together. It evades the above theorem because it is not translationally invariant, i.e. shifting one of the x_i changes the measure.

• In higher dimensions, the naive generalization of the Wiener measure does not generally exist, but it is believed that quantum field theories that are asymptotically free, such as Yang-Mills, do have a continuum limit. However, a continuum limit almost certainly doesn't exist for general relativity or quantum electrodynamics, and probably doesn't exist for the Standard Model. This is acceptable in practice because we can just treat them as effective theories.

(вообще, люди курсы по теории меры проходят, чтобы разобраться с этим, но мне этот вопрос сейчас не интересен.)

О функциональном квантовании векторных полей

О функциональном квантовании спинорных полей

Симметрии в функциональном формализме

On holomorphic path integral (!!!!)

(!!! напишу скоро про это!!! типичная тема, просто что-то опустил ее)

3.2.4 Diagrams in QFT: other information about basics

Examples of diagrams for practice

(I'll collect some of them to have a reference just in case)

Other general questions about diagrams

требования к теории для применения к ней диаграммных методов

(????)

О разных диаграммных техниках

(?? why there are different diagram techniques??)

3.2.5 Spinors: other properties

Спиновые матрицы плотности (from Vergeles)

Спиновые матрицы плотности необходимы при вычислении сечений и вероятностей распадов, т.к. в процессе таких вычислений приходится делать замены выражений, билинейных относительно спиновых волновых функций начальных и конечных частиц, согласно правилу

$$e_{\mathbf{k}\mu}e^*_{\mathbf{k}\nu} \longrightarrow \rho_{\mu\nu}(k), \quad u_{\pm ka}\bar{u}_{\pm kb} \longrightarrow \rho^{(\pm)}_{ab}(k) \tag{Б1}$$

где $\rho_{\mu\nu}(k)$ - спиновая матрица плотности фотона, $\rho^{(+)}_{ab}(k)\left(\rho^{(-)}_{ab}(k)\right)$ - спиновая матрица плотности электрона (позитрона), причём импульс всех этих частиц равен $\mathbf{k}$.

Фотонные спиновые матрицы плотности.

$$\rho_{ij\text{unpol}} = \frac{1}{2}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right), \quad \rho_{0i} = \rho_{i0} = \rho_{00} = 0 \tag{Б2}$$

$$\rho^\mu_\mu = -1 \tag{Б3}$$

$$\rho_{\mu\nu}(k) \longrightarrow \rho_{\mu\nu}(k) + (\chi_\mu k_\nu + \chi_\nu k_\mu) \tag{Б4}$$

$$\chi^\mu = \left(-\frac{1}{4\omega}, \frac{\mathbf{k}}{4\omega^2}\right) \qquad \Rightarrow \qquad \rho_{\mu\nu} = -\eta_{\mu\nu}/2 \tag{Б5}$$

В общем случае матрица плотности представляется в виде

$$\rho_{\mu\nu} = \rho^*_{\nu\mu} = \sum_{\alpha,\beta=1}^{2} \rho^{\alpha\beta} e_{\alpha\mu} e^*_{\beta\nu} \tag{Б6}$$

$$\rho^{\alpha\beta} = \rho^{\beta\alpha *} \tag{Б7}$$

В случае неполяризованного фотона

$$\rho^{\alpha\beta} = (1/2)\delta^{\alpha\beta} \tag{Б8}$$

$$\det \rho^{\alpha\beta} = 1/4 \tag{Б9}$$

В случае полностью поляризованного фотона:

$$\rho^{\alpha\beta} = v^{\alpha} v^{\beta *} \tag{Б10}$$

$$\det \rho^{\alpha\beta} = 0 \tag{Б11}$$

В общем случае:

$$\sum_{\alpha=1}^{2} \rho^{\alpha\alpha} = 1 \tag{Б12}$$

$$0 \leq \det \rho^{\alpha\beta} \leq 1/4 \tag{Б13}$$

$$\rho = \frac{1}{2}(1 + \boldsymbol{\xi}\boldsymbol{\sigma}), \quad \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \tag{Б14}$$

$$\det \rho = \frac{1}{4}\left(1 - \xi_1^2 - \xi_2^2 - \xi_3^2\right) \tag{Б15}$$

$$0 \leq \xi_1^2 + \xi_2^2 + \xi_3^2 \equiv \boldsymbol{\xi}^2 \leq 1 \tag{Б16}$$

Поэтому в неполяризованном случае $\xi^2 = 0$, а в полностью поляризованном случае $\xi^2 = 1$.

Спинорные матрицы плотности.

$$\operatorname{tr} \rho^{(\pm)}(k) = \pm 2m \tag{Б19}$$

$$(\not{k} \mp m)\rho^{(\pm)}(k) = \rho^{(\pm)}(k)(\not{k} \mp m) = 0 \tag{Б20}$$

$$\langle \mathbf{s} \rangle = \frac{1}{2\omega_k} \operatorname{tr}\left(\rho^{(\pm)}\gamma^0 \boldsymbol{\Sigma}\right) = \frac{1}{4\omega_k} \operatorname{tr}\left(\rho^{(\pm)}\gamma^5\gamma\right) \tag{Б21}$$

Матрица плотности для неполяризованного состояния получается при помощи $\frac{1}{2}\sum\limits_{s=1,2} u_{\pm ks}\bar{u}_{\pm ks} = k \pm \pm$

$$\rho^{(\pm)}(k) = (1/2)(\not{k} \pm m) \tag{Б22}$$

В общем случае

$$\rho^{(\pm)}(k) = (1/2)(\not{k} \pm m)\left(1 - \gamma^5 \not{a}\right) \tag{Б23}$$

$a^\mu: \quad k_\mu a^\mu = 0 \quad \longleftrightarrow \quad \not{a}\not{k} + \not{k}\not{a} = 0$

$$\pm a^\mu = \frac{1}{2m} \operatorname{tr}\left(\rho^{(\pm)}\gamma^5\gamma^\mu\right) \tag{b25}$$

В системе покоя частицы

$$a^\mu = (0, \pm 2\langle \mathbf{s} \rangle) \tag{Б26}$$

$0 \leq \langle \mathbf{s} \rangle^2 \leq 1/4 \quad \Rightarrow -1 \leq a^2 \leq 0$

Также в сечение рассеяния следует внести дополнительный множитель 2^r, где r - суммарное число электронов, позитронов и фотонов в конечном состоянии. Этот множитель означает учёт обоих состояний спина каждой конечной частицы. (??)

3.2.6 Vortex function

(не знаю, что это вообще. пока пусть повесит)

Формулы по Ландау

(ЛЛ4 пар 106, но вообще много где это должно быть написано тоже.)

$$K^{\mu}_{ik}(x_1, x_2, x_3) = \left\langle 0 \left| \, \mathrm{T} A^{\mu}(x_1) \, \psi_i(x_2) \, \bar{\psi}_k(x_3) \right| 0 \right\rangle$$

с одним 4-векторным и двумя биспинорными индексами; в силу однородности пространства-времени она зависит лишь от разностей аргументов x_1, x_2, x_3. Выраженная через операторы в представлении взаимодействия, функция K имеет вид

$$K^{\mu}_{ik}(x_1, x_2, x_3) = \frac{\left\langle 0 \left| \, \mathrm{T} A^{\mu}_{\mathrm{int}}(x_1) \, \psi^{\mathrm{int}}_i(x_2) \, \bar{\psi}^{\mathrm{int}}_k(x_3) \, s \right| 0 \right\rangle}{\langle 0|S|0\rangle}.$$

Переход к импульсному представлению осуществляется формулой

$$(2\pi)^4 \delta^{(4)}(p_1 + k - p_2) \, K^{\mu}_{ik}(p_2, p_1; k) = \\ = \iiint K^{\mu}_{ik}(x_1, x_2, x_3) \, e^{-ikx_1 + ip_2x_2 - ip_1x_3} d^4x_1 d^4x_2 d^4x_3.$$

В диаграммной технике функциям K^{μ}_{ik} соответствуют блоки (треххвостки) вида

(тут много картинок, скорее всего выгружу ЛЛ4, потом уже скопирую их)

$$\int K_{ik,lm}(x_1, x_2; x_3, x_4) \, e^{i(p_3x_1 + p_4x_2 - p_3x_3 - p_2x_4)} d^4x_1 d^4x_2 d^4x_3 d^4x_4 = \\ = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4) \, K_{ik,lm}(p_3, p_4; p_1, p_2),$$

причем

$$K_{lk,lm}(p_3, p_4; p_1, p_2) = \\ = (2\pi)^4 \delta^{(4)}(p_1 - p_3) \, \mathscr{G}_{il}(p_1) \, \mathscr{G}_{km}(p_2) - (2\pi)^4 \delta^{(4)}(p_2 - p_3) \, \mathscr{G}_{im}(p_1) \, \mathscr{G}_{kl}(p_2) + \\ + \mathscr{G}_{ln}(p_3) \, \mathscr{G}_{kr}(p_4) \left[-i\Gamma_{nr,st}(p_3, p_4; p_1, p_2) \right] \mathscr{G}_{sl}(p_1) \, \mathscr{G}_{tm}(p_2).$$

3.2.7 Schwinger-Dyson equation

вкратце формулы и суть

Суть в том, что это квантовое уравнение Эйлера-Лагранжа
(потом допишу, в процессе размышления)
(тут место явного сравнения ЭЛ и ШД)

пример: уравнение Ш-Д для потенциала φ^4

(см википедию английскую, скоро додумаю)

3.2.8 Dyson equation

Формулы (????!!!)

(ЛЛ4 пока что)

$$\mathscr{H}(p) = G^{-1}(p) - \mathscr{G}^{-1}(p) = \\ = -ie^2 \int \gamma^\nu \mathscr{G}(p+k) \cdot \Gamma^\mu(p+k, p; k) \cdot \mathscr{D}_{\mu\nu}(k) \frac{d^4k}{(2\pi)^4}.$$

3.2.9 Bethe-Salpeter equation

(никто не понимает, как это работает и как этим пользоваться, отдельные часа 15 нужно только на эту тему. Так что пока вдали, научусь - мб в 1ю переставлю. Пока не увидел, что на самом деле требуется где-то понимание этого уравнения, хотя, конечно, запросто можно это найти)

Разберу скоро, это мб от Киселева (?????)

Основным методом решения задач со взаимодействием, бесспорно, является теория возмущений, однако это далеко не единственный метод. Существуют, так называемые, непертурбативные методы и один из них ведет к уравнению Бете - Солпитера. Рассматривается система двух связанных фермионов. В свободной теории, как известно, для одночастичной волновой функции ψ_a (где a - спинорный индекс) пропагатор определяется следующим образом:

$$\psi(x_2) = -i \int d\sigma(x_1) S_F(x_2, x_1) \not{n}(x_1) \psi(x_1)$$

Тут используется запись с использованием «перечёркнутых матриц», $n(x_1)$ - 4-х вектор внешней нормали. Интегрирование ведется по поверхности объёма, включающего в себя событие x_2, S_F. - фейнмановский пропагатор. В случае невзаимодействующих частиц он определяется как решение следующего уравнения [2].

$$(i\nabla' - m_0) S_F = \delta^4(x' - x)$$

Аналогично пропагатору для одночастичной волновой функции, можно определить пропагатор для двучастичной волновой функции следующим выражением:

$$\psi_{ab}(x_3, x_4) = \int d\sigma(x_1) d\sigma(x_2) S^{ab}(x_3, x_1 x_4, x_2) \not{x}(x_1) \not{x}(x_2) \psi_{ab}(x_1, x_2)$$

Здесь ψ_{ab} - спинор, обладающий двумя спинорными индексами a, b. В случае невзаимодействующих частиц, двучастичная волновая функция распадается в произведение одночастичных, а пропагатор в произведение пропагаторов:

$$S^{0ab}(x_3, x_4; x_1, x_2) = iS_F^a(x_3, x_1) S_F^b(x_4, x_2)$$

Однако это самый тривиальный случай. Теперь же «включим» электромагнитное взаимодействие между двумя частицами. Если бы мы следовали идеологии теории возмущений, то получили бы, следуя Фейнману, S^{ab} представляется в виде:

$$S^{ab}(x_3, x_4; x_1, x_2) = iS_F^a(x_3, x_1) S_F^b(x_4, x_2) + \Sigma$$

Под Σ понимается сумма всевозможных диаграмм, получаемых из теории возмущения. Основная идея, приводящая к уравнению заключается в том, что всю

сумму диаграмм мы обозначаем, как некоторое ядро K. Мы будем называть диаграмму приводимой, если после удаления двух фермионных линий она становится несвязной. Тогда K можно представить в виде суммы двух вкладов: вклада приводимых диаграмм и вклада неприводимых диаграмм $\bar{K}$. Можно показать [3], что выражение для $S^{ab}(x_3,x_4;x_1,x_2)$ может быть переписано как:

$$S^{ab}(x_3,x_4;x_1,x_2) = iS_F^a(x_3,x_1)\,S_F^b(x_4,x_2) + \int d^4x_5 d^4x_6 d^4x_7 d^4x_8 \cdot iS_F^a(x_3,x_5)\,iS_F^b(x_4,x_6)\,\bar{K}^{ab}(x_5,x_6;x_7,x_8)\,S^{a}$$

Подставляя это выражение в (2) получаем уравнение Бете - Солпитера:

$$\psi_{ab}(x_1,x_2) = \varphi_{ab}(x_1,x_2) + \int d^4x_3 d^4x_4 d^4x_5 d^4x_6 \cdot iS_F^a(x_1,x_5)\,iS_F^b(x_2,x_6)\,\bar{K}^{ab}(x_5,x_6;x_3,x_4)\,\psi_{ab}(x_3,x_4)$$

В этом выражении φ_{ab} - свободная двучастичная волновая функция, то есть волновая функция в отсутствии взаимодействия между частицами. Таким образом, получили интегральное уравнение Фредгольма II рода.

Основные формулы уравнения Бете-Солпитера

(вики)

Подействуем теперь на уравнение Бете-Солпитера операторами $(i\nabla_1 - m_a), (i\nabla_2 - m_b)$, в силу (1) получим следующее выражение:

$$(i\nabla_1 - m_a)(i\nabla_2 - m_b)\,\psi_{ab}(x_1,x_2) = -\int d^4x_3 d^4x_4 \bar{K}^{ab}(x_1,x_2;x_3,x_4)\,\psi_{ab}(x_3,x_4)$$

Соответственно вместо интегрального уравнения типа Фредгольма, мы получаем интегро-дифференциальное уравнение на двухчастичную волновую функцию $\psi_{ab}(x_1,x_2)$. Ещё одной возможной формой записи уравнения Бете-Солпитера, является запись в импульсном пространстве, а именно, определим преобразование Фурье двухчастичной волновой функции $\psi_{ab}(x_1,x_2)$ следующим образом:

$$\chi_{ab}(p_1,p_2) = \frac{1}{(2\pi)^4}\int d^4x_1 d^4x_2 \cdot e^{i(p_1x_1+p_2x_2)}\psi_{ab}(x_1,x_2)$$

Фурье преобразования самого уравнения Бете-Солпитера запишется следующим образом:

$$\frac{1}{(2\pi)^4}\int d^4x_1 d^4x_2 e^{i(p_1x_1+p_2x_2)}(i\nabla_1 - m_a)(i\nabla_2 - m_b)\,\psi_{ab}(x_1,x_2) = -\frac{1}{(2\pi)^4}\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{i(p_1x_1+p_2x_2)}\bar{K}^{ab}$$

В левой части можно перенести градиенты на экспоненту при помощи интегрирования по частям. Также добавим в правую часть две дельта-функции. Получим:

$$\frac{1}{(2\pi)^4}\int d^4x_1 d^4x_2 \left[(i\nabla_1 - m_a)(i\nabla_2 - m_b)\,e^{i(p_1x_1+p_2x_2)}\right]\psi_{ab}(x_1,x_2) =$$
$$= -\frac{1}{(2\pi)^4}\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 d^4x_3' d^4x_4' e^{i(p_1x_1+p_2x_2)}\delta^4(x_3'-x_3)\,\delta^4(x_4'-x_4)\,\bar{K}^{ab}(x_1,x_2;x_3',x_4')\,\psi_{ab}(x_3,x_4)$$

Используя импульсное представление дельта-фрункций со штрихованными переменными мы можем переписать ядро $\bar{K}^{ab}$ в импульсном представлении, а именно:

$$\bar{K}^{ab}\left(p_1,p_2;p_3,p_4\right)=\frac{1}{(2\pi)^8}\int d^4x_1d^4x_2d^4x_3d^4x_4e^{i(p_1x_1+p_2x_2-p_3x_3-p_4x_4)}\bar{K}^{ab}\left(x_1,x_2;x_3,x_4\right)$$

Используя это, мы получаем уравнение Бете-Солпитера в импульсной форме:

$$\left(\not{p}_1-m_a\right)\left(\not{p}_2-m_b\right)\chi_{ab}\left(p_1,p_2\right)=-\int d^4p_1'd^4p_2'\bar{K}^{ab}\left(p_1,p_2;p_1',p_2'\right)\chi_{ab}\left(p_1',p_2'\right)$$

О применениях уравнения Бете-Солпитера (!!!!)

(додумаю - займусь этим.)

3.2.10 Properties of amplitudes

In this section, we consider some of the properties of amplitudes, i.e. the quantities $\mathcal{M}(i\to f)$ simply related to S-matrix elements.

• Amplitudes are constrained by Poincare invariance. Note that

$$\langle f,\ \text{out}\ |i,\ \text{in}\ \rangle=\left\langle f,\ \text{out}\ \left|U^\dagger(\Lambda,a)U(\Lambda,a)\right|i,\ \text{in}\ \right\rangle$$

where $U(\Lambda,a)$ is our representation of the Poincare group. We know how it must act on the in and out states, since these are just combinations of independent single particles, and using this gives us a relation between two amplitudes.

• Specifically, let us label the incoming particles with momenta p_i, little group indices σ_i, and discrete labels n_i for the particle type, and likewise with primes for the outgoing particles. Then

$$S_{\{p_i,\sigma_i,n_i\},\left\{p_i',\sigma_i',n_i'\right\}}=\exp\left(ia_\mu\Lambda^\mu_\nu\left(\Sigma p_i'-\Sigma p_i\right)^\nu\right)\cdot$$
$$\cdot\left(\sum_{\{\bar{\sigma}_i\}}\prod_i D^{(j_i)}_{\bar{\sigma}_i\sigma_i}\left(W\left(\Lambda,p_i\right)\right)\right)\left(\sum_{\left\{\bar{\sigma}_i'\right\}}\prod_i D^{\left(j_i'*\right)}_{\bar{\sigma}_i'\sigma_i'}\left(W\left(\Lambda,p_i'\right)\right)\right)S_{\{\Lambda p_i,\bar{\sigma}_i,n_i\},\left\{\Lambda p_i',\bar{\sigma}_i',n_i'\right\}}$$

In other words, we just pick up the expected transformations for the momenta and little group labels. Here, we've omitted a convention-dependent normalization factor.

• We pick up additional constraints from internal symmetries, which are defined to be those that change the particle type label n_i, without changing the momenta.

• Scattering amplitudes are analytic in the momenta, so we can analytically continue them "offshell". At the perturbative level, we can define this analytic continuation by simply demanding that off-shell amplitudes can be computed using the same Feynman rules as on-shell ones.

• A simple reason this is useful is that diagrammatically, off-shell amplitudes could appear as internal parts of larger, on-shell amplitudes.

• Another reason is that they appear when one has an external source,

$$\mathcal{L}=\mathcal{L}_0+\mathcal{L}_{\text{int}}\ -J(x)\phi(x)$$

In this case, one has a Feynman rule where a particle can appear out of the source, with a factor $iJ(x)$. The diagrams that contribute to the vacuum-to-vacuum transition amplitude $Z[J]$, at n^{th} order in J, have n copies of this vertex.

• Taking the Fourier transform, since $\tilde{J}(k)$ can have support off-shell, the resulting amplitude is just an n-particle off-shell scattering amplitude times factors of $i\tilde{J}(k_i)$. This is important because, as we'll see later, $Z[J]$ is what the path integral naturally computes.

• There is one more direct interpretation of off-shell amplitudes. We recall that the S-matrix in the adiabatic formalism is $S = U_I(\infty, -\infty)$. With the external source,

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} \ + J(x)\phi(x)$$

Now, suppose we work in interaction picture by treating $\mathcal{H}_0 + \mathcal{H}_{\text{int}}$ as the "free" Hamiltonian. In this case, the interaction picture fields are what we usually call the Heisenberg fields, and the vacuum-to-vacuum amplitude is

$$Z[J] = \langle \Omega | S | \Omega \rangle = \left\langle \Omega \left| T \exp\left(-i \int dx J(x) \phi_H(x) \right) \right| \Omega \right\rangle$$

By Taylor expanding the right-hand side, we see that n-particle off-shell amplitudes are just the Fourier transforms of the VEV of the time-ordered product of n Heisenberg fields.

• This point is important because we usually take for granted that symmetries work for off-shell amplitudes or amplitudes with external currents just as they do for on-shell amplitudes, which is not strictly justified. If we define off-shell amplitudes in terms of Heisenberg fields, then we can establish this rigorously. E.g. this ensures that momentum is conserved in off-shell amplitudes, beyond perturbation theory, where it is clear from the Feynman diagram expansion.

Another useful consequence of analytic continuation is crossing symmetry.

• Crossing symmetry is the statement that

$$\mathcal{M}(\phi(p) + ... \to ...) = \mathcal{M}(... \to ... + \tilde{\phi}(-p))$$

where $\phi(p)$ represents a particle with four-momentum p, and $\tilde{\phi}(-p)$ represents an antiparticle with momentum $-p$. Note that it is impossible for both sides above to be on-shell, so this statement must be understood in terms of analytic continuations.

• For scalars, crossing symmetry can easily be shown perturbatively with the Feynman rules, as for every diagram with ϕ incoming there is a corresponding diagram with $\tilde{\phi}$ outgoing. For spinors and vectors, one picks up factors for the external polarizations, which can lead to extra sign flips; however, these can be removed if one uses appropriate conventions for the off-shell polarization vectors.

• Crossing symmetry can also be shown nonperturbatively using the LSZ reduction formula. This gives an analytic formula for S-matrix elements which only distinguishes between in and out particles by the sign of the energy in the Fourier transform. Analytically continuing a positive energy to a negative one thus yields a crossed S-matrix element.

• Also note that applying crossing symmetry to switch all incoming particles to outgoing ones, and vice versa, just yields the S-matrix element related by CPT.

3.2.11 The Optical Theorem

Об оптической теореме в КТП (???)

$$\left\langle \mathbf{p}_1 \mathbf{p}_2 \left| T^\dagger T \right| \mathbf{k}_1 \mathbf{k}_2 \right\rangle = \sum_n \left(\prod_{i=1}^n \int \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E_i} \right) \left\langle \mathbf{p}_1 \mathbf{p}_2 \left| T^\dagger \right| \{\mathbf{q}_i\} \right\rangle \left\langle \{\mathbf{q}_i\} \left| T \right| \mathbf{k}_1 \mathbf{k}_2 \right\rangle$$

Оптическая теорема. Мнимая часть амплитуды рассеяния вперед возникает за счет вкладов всех возможных многочастичных промежуточных состояний.

Theory

The unitarity of the S-matrix leads to useful constraints.

• We recall that we have defined

$$S = 1 + iT, \quad T = \mathcal{M}\phi\left(\Sigma p_{\text{in}} - \Sigma p_{\text{out}}\right)$$

The unitarity condition $S^\dagger S = 1$ gives

$$-i\left(T - T^\dagger\right) = T^\dagger T$$

• To make this more explicit, we take matrix elements of both sides, suppressing "in" and "out" labels, and introduce a resolution of the identity on the right-hand side. Canceling a delta function from both sides, and considering a $2 \to 2$ scattering for concreteness, we have

$$\begin{aligned} &- i\left(\mathcal{M}\left(k_1 k_2 \to p_1 p_2\right) - \mathcal{M}^*\left(p_1 p_2 \to k_1 k_2\right)\right) \\ &= \sum_n \left(\prod_{i=1}^n \int \frac{d\mathbf{q}_i}{2E_i}\right) \mathcal{M}^*\left(p_1 p_2 \to \{q_i\}\right) \mathcal{M}\left(k_1 k_2 \to \{q_i\}\right) \not{\delta}\left(k_1 + k_2 - \sum_i q_i\right). \end{aligned}$$

In a more abbreviated notation, this is

$$2\,\mathrm{Im}\,\mathcal{M}(a \to b) = \sum_f \int d\Pi_f \mathcal{M}^*(b \to f)\mathcal{M}(a \to f)$$

This is the optical theorem.

• The optical theorem is most useful when a and b are the same state. For a one-particle state, the right-hand side is just the decay rate, up to some constants, so

$$\mathrm{Im}\,\mathcal{M}(a \to a) = m_A \sum_f \Gamma(a \to f) = m_a \Gamma_a$$

It's not clear if the left-hand side makes sense for an unstable particle, because such particles can't appear in asymptotic states. One needs to do more work to resolve this, but we'll just ignore it. (resolve)

• For a two-particle state, the right-hand side gives the total scattering cross section, up to some constants, and we have

$$\mathrm{Im}\,\mathcal{M}(a \to a) = 2E_{\text{cm}} p_{\text{cm}} \sigma_{\text{tot}}(a \to \text{ anything })$$

where p_{cm} is the momentum of either particle in the center of mass frame.

• These results have precisely the same interpretation as in the notes on Undergraduate Physics. The imaginary part of the forward scattering amplitude is precisely the amount by which the amplitude for remaining in the same state is depleted, so it is sensible that this is related to the probability to end up in a different state.

• The optical theorem is especially useful when we work perturbatively, because it relates contributions from different orders in perturbation theory. E.g. we will see that the leading contribution to $\mathrm{Im}\,\mathcal{M}(a \to a)$ is at loop level, while the right-hand sides have leading contributions

at tree level. Thus, even if we didn't know the full theory, unitarity can still be used to constrain loop level amplitudes given tree level ones. This will be formalized below by "cutting rules".

Next, we give an explicit example of the optical theorem. We focus on a $2 \to 2$ scattering and consider $\mathcal{M}$ as an analytic function of s, defined by the Feynman rules.

• First, it's worth noting that typically, we expect the imaginary part of $\mathcal{M}$, for real s, to vanish. This is a bit confusing because the Feynman rules have many factors of i, but we can trace them as follows.

• Vertices always come with a factor of i, e.g. because they come from expanding $e^{iH_{\text{int}}\,t}$, along with a momentum conserving delta function.

• Edge (i.e. Feynman propagators) always come with a factor of i in the numerator, along with an integral over four-momentum dp.

• Each seemingly real integral over four-momentum is secretly a purely imaginary number, because the integral over p^0 can be thought of as a contour integral, which yields $2\pi i$ times the residues of relevant poles. This might sound unfamiliar but it isn't strange, because this is precisely the reverse of the logic that let us introduce four-momentum integrals in the first place.

• Thus, each edge secretly comes with an extra factor of i, as does every vertex (since delta functions remove these four-momentum integrals).

Since edges and vertices each come with two factors of i, the amplitude is real. Alternatively, if we use the delta functions to do as many four-momentum integrals as possible, then the number of remaining integrals is the number of loops. Then the total number of factors of i is the number of vertices, plus edges, plus faces (loops) mod 2, which is zero by Euler's formula.

• The idea that a loop gives a factor of i is also consistent with our previous treatment of renormalization. All of our renormalization schemes (dimensional regularization, Pauli-Villars, or a Wilsonian cutoff) begin by Wick rotating the integral over p^0, giving a factor of i.

• Another way of thinking about this is that, as a distribution,

$$\operatorname{Im} \frac{1}{p^2 - m^2 \pm i\epsilon} = \mp \pi \delta\left(p^2 - m^2\right)$$

In a typical loop integral, where only one propagator can be on-shell at once, this yields an imaginary number, making the amplitude overall real. But there is a chance for the amplitude to become imaginary if multiple propagators go on-shell at once. And if the intermediate states in a diagram are on-shell, then we can "cut" the diagram in half, giving diagrams contributing to the right-hand side of the optical theorem.

• Now we treat this explicitly. First, we note that at low s, we are "below threshold" to produce on-shell intermediate states, so we have

$$\mathcal{M}(s) = \left(\mathcal{M}\left(s^*\right)\right)^*$$

Both sides are analytic, so by analytic continuation we have, for all real s,

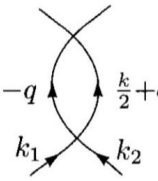
$$\operatorname{Re} \mathcal{M}(s + i\epsilon) = \operatorname{Re} \mathcal{M}(s - i\epsilon), \quad \operatorname{Im} \mathcal{M}(s + i\epsilon) = -\operatorname{Im} \mathcal{M}(s - i\epsilon)$$

At threshold, $\mathcal{M}(s)$ picks up a branch cut along the real axis, across which the imaginary part is discontinuous. The physically measured quantity corresponds to $\mathcal{M}(s + i\epsilon)$, which is related to the discontinuity across the cut by

$$\operatorname{Disc} \mathcal{M}(s) = 2i \operatorname{Im} \mathcal{M}(s + i\epsilon)$$

• As an example, we consider ϕ^4 theory. For $2 \to 2$ scattering, at loop level we have s, t, and u-channel diagrams. The latter two don't give rise to branch cuts, so we focus on the s-channel.

$$i(\delta\mathcal{M}) = \quad \frac{\lambda^2}{2}\int \frac{dq}{((k/2-q)^2-m^2+i\epsilon)\,((k/2+q)^2-m^2+i\epsilon)},$$



where $k = k_1 + k_2$.

Now we consider the (contour) integral over q^0. In the center of mass frame, $k = (k^0, \mathbf{0})$, and defining $E_\mathbf{q} = \sqrt{|\mathbf{q}|^2 + m^2}$ as usual, the integral has poles at

$$q^0 = \frac{1}{2}k^0 \pm (E_\mathbf{q} - i\epsilon)\,, \quad q^0 = -\frac{1}{2}k^0 \pm (E_\mathbf{q} - i\epsilon)$$

To perform the integral, we close the contour downward, picking up the residues of the poles in the lower half-plane. Only the pole at $q^0 = -(1/2)k^0 + E_\mathbf{q}$ will contribute to the discontinuity, and accounting for it is equivalent to replacing

$$\frac{1}{(k/2+q)^2-m^2+i\epsilon} \to -i\phi\left((k/2+q)^2-m^2\right)$$

in the original integral.

• Straightforwardly performing the integral gives

$$i(\delta\mathcal{M}) = -i\frac{\lambda^2}{2}\int \frac{d\mathbf{q}}{2E_\mathbf{q}}\frac{1}{(k^0-E_\mathbf{q})^2-E_\mathbf{q}^2} = -i\frac{\lambda^2}{2}\frac{1}{2\pi^2}\int_m^\infty dE_\mathbf{q}E_\mathbf{q}|\mathbf{q}|\frac{1}{2E_\mathbf{q}}\frac{1}{k^0\,(k^0-2E_\mathbf{q})}$$

The integrand has a pole at $E_\mathbf{q} = k^0/2$, which thus produces a branch cut if the integral goes over it, $k^0 > 2m$. To find the discontinuity across the branch cut, note that

$$\frac{1}{k^0-2E_\mathbf{q}\pm i\epsilon} = P\frac{1}{k^0-2E_\mathbf{q}} \mp i\pi\delta\left(k^0-2E_\mathbf{q}\right)$$

so we may replace this factor with a delta function. This is equivalent to replacing

$$\frac{1}{(k/2-q)^2-m^2+i\epsilon} \to -i\phi\left((k/2-q)^2-m^2\right)$$

in the original integral. (Of course, the original integral is also divergent and must be regularized. We don't need to talk about a specific regulator here, however, because any reasonable regulator must preserve unitarity.)

• Now we return to the original integral, rewriting the loop integral as

$$\int dq = \int dp_1 \int dp_2 \phi\,(p_1+p_2-k)$$

If we make the two replacements above, we simply put the momenta p_i on shell, and integrating the p_i^0 against the delta functions produces the standard factors for Lorentz invariant phase space, giving

$$\text{Disc}\,\mathcal{M}(k) = 2i\,\text{Im}\,\mathcal{M}(k) = \frac{i}{2}\int \frac{d\mathbf{p}_1}{2E_1}\frac{d\mathbf{p}_2}{2E_2}\lambda^2 \not{\phi}\left(p_1 + p_2 - k\right)$$

This is precisely the prediction of the optical theorem, where the $1/2$ in front is the phase space factor for identical final particles, and the λ^2 is the squared matrix element.

• This logic above can be generalized to yield "Cutkosky cutting rules" to compute the discontinuity of any diagram:

• Cut through the diagram, so that it falls apart into two pieces, separating the initial and final particles.

• For each cut, replace $1/\left(p^2 - m^2 + i\epsilon\right) \to -i\not{\phi}\left(p^2 - m^2\right)$.

• Sum over all possible cuts.

This can be used to prove the optical theorem to all orders in perturbation theory.

• Another way to arrive at this is to note that the Feynman propagator and retarded propagator differ only in that the retarded propagator has both poles in the upper half plane, so

$$D_F(p) = D_R(p) + \frac{\pi}{E_{\mathbf{p}}}\delta\left(p^0 - E_{\mathbf{p}}\right)$$

If we apply this formula to all propagators in a loop, then the resulting term consisting of a product of only retarded propagators has zero contour integral, by closing the contour in the lower half plane. By applying the formula in reverse, we recover terms with Feynman propagators, but each term has at least one delta function, thereby reducing a loop amplitude to a tree amplitude.

• We could also treat the example by just evaluating the integral explicitly in, e.g. dimensional regularization, in which case the discontinuity appears through a logarithm.

• Now consider applying the same logic to particles with spin. The right-hand side of the optical theorem will involve spin sums, e.g. for decay into two fermions $\psi\left(p_1\right)\bar{\psi}\left(p_2\right)$ we get $\text{tr}\left(\left(\not{p}_1 + m\right)\left(\not{p}_2 - m\right)\right)$, as shown in the previous section. The optical theorem hence requires that the numerator of a propagator must involve a sum over physical spin states.

• For Dirac fermions this is clearly true, as $\not{p} + m$ indeed appears in the numerator. The result is also clearly true for massive vectors. But for massless vectors, the spin sum is over the two physical polarizations, while the propagator numerator is ξ-dependent! However, as we've seen, we can replace the spin sum with $\eta^{\mu\nu}$ by the Ward identity, and we can freely choose $\xi = 1$ by gauge invariance to make the propagator match.

Next, we apply the optical theorem to unstable particles.

• Recall that we wrote the exact propagator for a scalar particle as

$$D_F\left(p^2\right) = \frac{i}{p^2 - m_0^2 - M^2\left(p^2\right)}$$

where $-iM^2\left(p^2\right)$ was the sum of all 1PI diagrams. On the other hand, these same diagrams can be used to compute $1 \to 1$ forward scattering, and the LSZ reduction formula gives

$$\mathcal{M}(p \to p) = -ZM^2\left(p^2\right)$$

• Suppose that the scalar particle is stable. Then by the optical theorem, $\mathcal{M}(p \to p)$ is real, so the self-energy is real, which implies that the pole of the propagator is simply shifted along the real axis, as we've seen before.

• If the particle can decay, then $M^2\left(p^2\right)$ acquires an imaginary part, shifting the pole off the real axis. In this case, the particle is not an asymptotic state, so we can't assign it an unambiguous

mass; equivalently, we have some freedom over what to call the mass. A reasonable choice for a slowly decaying particle is to let m be the "real pole mass",

$$m^2 - m_0^2 - \operatorname{Re} M^2\left(m^2\right) = 0$$

Then for $p^2 \approx m^2$, the exact propagator is

$$D_F\left(p^2\right) \sim \frac{iZ}{p^2 - m^2 - iZ \operatorname{Im} M^2\left(p^2\right)}$$

For particles such as the bottom quark, which immediately hadronize, the standard choice is to instead use the $\overline{\text{MS}}$ mass.

• If the propagator appears in the s-channel of a Feynman diagram, then the cross section obeys

$$\sigma(s) \propto \left|\frac{1}{s - m^2 - iZ \operatorname{Im} M^2(s)}\right|^2 = \frac{1}{\left(s - m^2\right)^2 + \left(Z \operatorname{Im} M^2(s)\right)^2}$$

which gives a Breit-Wigner resonance peak.

The optical theorem thus tells us that the width of this peak is related to the particle's decay rate,

$$\Gamma = -\frac{Z}{m} \operatorname{Im} M^2\left(m^2\right).$$

Above we have treated $\operatorname{Im} M^2\left(p^2\right)$ as a constant near the peak, so this holds in the case of narrow resonances, $\Gamma \ll m$. For broad resonances, there can be deviations from the Breit-Wigner shape.

• Suppose we are searching for a weakly coupled particle produced in the s-channel of a scattering process. There is naively a paradox here. If we consider just the first half of the Feynman diagram, involving the on-shell production of the new particle, then there is only one vertex, so the cross section scales as g^2. But if we consider the full Feynman diagram, which includes the particle's subsequent decay, then there are two vertices and the cross section scales as g^4.

• The resolution is that in the full diagram, we should perform the integral $\int \sigma(s) ds$ to get the full rate, and the sharp peak of the Breit-Wigner distribution gives a factor of $1/\Gamma \propto 1/\mathrm{g}^2$. Therefore, the two calculations give equivalent results. This enhancement was used by LEP when it ran at the Z-pole.

• The above story does not hold if the kinematics are such that the weakly coupled particle is always far off-shell; in that case the diagram cannot be cut, and the rate scales as g^4. Also, there is a limit to how long in g one can probe, because for sufficiently low g the particle becomes so long-lived that it does not decay before exiting the detector.

• If we trace through the definitions above, the shifted pole in s is below the real axis, while for stable particles, the physical values of s were $i\epsilon$ above the real axis. So if we place the multiparticle branch cut along the real axis, we reach this pole by analytically continuing downward through it, so the pole is really on the "second sheet".

Unitarity bounds

Finally, we discuss unitarity bounds.

• Consider a $2 \to 2$ elastic scattering in the CM frame. The total cross section is

$$\sigma_{\text{el}} = \frac{1}{32\pi E_{\text{cm}}^2} \int d\cos\theta |\mathcal{M}(\theta)|^2$$

where θ is the final angle between the particles. In quantum field theory, the partial wave expansion is simply the decomposition

$$\mathcal{M}(\theta) = 16\pi \sum_{j=0}^{\infty} a_j (2j+1) P_j(\cos\theta)$$

- By logic similar to the nonrelativistic version of the partial wave expansion,

$$\sigma_{\text{el}} = \frac{16\pi}{E_{\text{cm}}^2} \sum_{j=0}^{\infty} (2j+1) \left|a_j\right|^2$$

- The optical theorem gives us the relation

$$\text{Im}\, \mathcal{M}(\theta = 0) = 2E_{\text{cm}} p_{\text{cm}} \sigma_{\text{tot}} \geq 2E_{\text{cm}} p_{\text{cm}} \sigma_{\text{el}}$$

where σ_{tot} is the cross section for scattering into any final state. This yields

$$\sum_{j=0}^{\infty} (2j+1)\, \text{Im}\, a_j \geq \frac{2p_{\text{cm}}}{E_{\text{cm}}} \sum_{j=0}^{\infty} (2j+1) \left|a_j\right|^2$$

In fact, the sum on j can be dropped, by considering scattering of angular momentum eigenstates rather than plane waves.

- For simplicity, consider the ultrarelativistic limit, where $p_{\text{cm}} \approx E_{\text{cm}}/2$. Then

$$\text{Im}\, a_j \geq \left|a_j\right|^2$$

which is the interior of a circle of unit diameter in the complex plane, centered at $i/2$, so

$$|a_j| \leq 1, \quad 0 \leq \text{Im}\, a_j \leq 1, \quad |\text{Re}\, a_j| \leq \frac{1}{2}$$

- The coefficients a_j can be computed order by order in perturbation theory. If the partial wave unitarity bound is violated at some order, we say the theory violates "perturbative unitarity". This doesn't mean that the theory violates unitarity; rather, it means that the results of perturbation theory cannot be trusted, and we must switch to a new description that can be.

- This was one of the main motivations for searching for the Higgs boson. If one calculates the amplitude for $W^+W^- \to W^+W^-$in the Standard Model without including the diagrams with the Higgs boson, the amplitude scales as $\mathcal{M} \sim g^2 E^2/m_W^2$, leading to a breakdown of perturbative unitarity above a few hundred GeV. However, adding the diagrams involving the Higgs precisely cancels this growth.

- Another useful fact is the Froissart bound: the total cross section cannot grow faster than $\log^2(E_{\text{cm}})$ at high energies.

- Finally, recall that the spectral representation writes the exact time-ordered two-point function as a combination of bare Feynman propagators with positive spectral weight. This implies that propagators cannot fall off faster than $1/p^2$ at large p^2.

- E.g. the theory

$$\mathcal{L} \supset -\frac{1}{2}\phi \left(\partial^2 + \frac{\partial^4}{\Lambda^2} + m^2\right) \phi$$

has a propagator that falls off as $1/p^4$ at large p^2. Thus, it is tempting to use this to construct a UV finite quantum field theory, but the point above forbids this. As we'll see later, terms like this do appear in effective theories, but this is acceptable because this behavior only kicks in as we approach the cutoff, at which point infinitely many other terms become important and the naive calculation of the propagator falls apart.

3.3 Other properties of basic methods 2

3.3.1 Soft Theorems

Soft limit and Lorentz invariance can establish strong constraints on how massless particles can interact.

• Consider an arbitrarily complicated amplitude with an incoming particle of momentum p_i, $\mathcal{M}_0(p_i)$. Now suppose a photon of momentum q is emitted off that particle's incoming line, giving the amplitude $\mathcal{M}_i(p_i, q)$.

• For concreteness, we suppose the particle is a scalar with charge e. Using the Feynman rules of scalar QED, and assuming the photon polarization is real to avoid having to write ϵ_μ^* everywhere,

$$\mathcal{M}_i(p_i, q) = (ie)\frac{i\left(p_i^\mu + (p_i^\mu - q^\mu)\right)}{(p_i - q)^2 - m^2}\epsilon_\mu \mathcal{M}_0(p_i - q)$$

Since external particles are on-shell, we have $p_i^2 = m^2$ and $q^2 = 0$. Since the photon polarizations are transverse, $q^\mu \epsilon_\mu = 0$. Thus, we have

$$\mathcal{M}_i(p_i, q) = e\frac{p_i \cdot \epsilon}{p_i \cdot q}\mathcal{M}_0(p_i - q)$$

• Next, we take the soft limit of small q, which strictly speaking means $|q \cdot p_i| \ll |p_j \cdot p_k|$ for all external momenta p_j and p_k. Then we have

$$\mathcal{M}_i(p_i, q) \approx \left(e\frac{p_i \cdot \epsilon}{p_i \cdot q}\right)\mathcal{M}_0(p_i)$$

• Now, consider the total scattering amplitude $\mathcal{M}(q)$ involving all of the original particles in $\mathcal{M}_0$, and an extra outgoing soft photon with momentum q. To compute this, we need to sum over all possible attachment points for the photon. However, the "soft factor" that we pick up is large only when the photon attaches to an on-shell particle. For generic amplitudes, this can only happen on external lines or in loops, but in loops the on-shell part is softened by an integration over momenta, so the contributions of external lines dominate.

• Repeating the reasoning above, we find that when we attach the photon on incoming particles, we get a factor of the charge eQ_i, while attaching the photon on outgoing particles gives the same result with the opposite sign. Thus,

$$\mathcal{M}(q) \approx e\left(\sum_{i \text{ incoming}} Q_i\frac{p_i \cdot \epsilon}{p_i \cdot q} - \sum_{i \text{ outgoing}} Q_i\frac{p_i \cdot \epsilon}{p_i \cdot q}\right)\mathcal{M}_0$$

• Now, the amplitude $\mathcal{M}(q)$ must be Lorentz invariant. Under a Lorentz transformation, the amplitude can only pick up a trivial phase factor due to the little group action on the photon. But we also know that in general, a Lorentz transformation shifts the polarization vector, $\epsilon_\mu \to \epsilon_\mu + q_\mu$, which changes the amplitude by

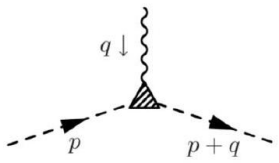
$$e\left(\sum_{i \text{ incoming}} Q_i - \sum_{i \text{ outgoing}} Q_i\right)\mathcal{M}_0$$

Demanding this be zero implies conservation of the charge the photon couples to.

• Note that we have already shown that a photon must couple to a conserved current using the canonical formalism, but this derivation is much more minimal, requiring only Lorentz invariance of the S-matrix and the soft limit.

Next, we generalize this derivation.

• First, it's not clear that the logic above holds for any type of external particle. Let's suppose that the particle is a scalar, but the interaction vertex is arbitrary,



$$= -ie\Gamma_\mu\left(p,q\right).$$

• In general, we can write the vertex in terms of form factors,

$$\Gamma^\mu = 2p^\mu F\left(p^2,q^2,p\cdot q\right) + q^\mu G\left(p^2,q^2,p\cdot q\right)$$

The form factor G is irrelevant because $q^\mu\epsilon_\mu = 0$.

• Since $p^2 = m^2$ and $q^2 = 0$, the dependence of F can be written by dimensional analysis as

$$\Gamma^\mu = 2p^\mu F\left(\frac{p\cdot q}{m^2}\right)$$

In the soft limit, the vertex simplifies further to $2p^\mu F(0)$. Thus everything goes through as before, with $F(0)$ being the definition of the conserved charge, as we've already seen in QED.

• For a minimally coupled Dirac spinor, we would instead pick up the factor

$$\bar{u}^s(\mathbf{p})\gamma^\mu(\not{p}+m) = \sum_{s'}\bar{u}^s(\mathbf{p})\gamma^\mu u^{s'}(\mathbf{p})\bar{u}^s(\mathbf{p})$$

in the numerator. However, we end up getting precisely the same result, by the identity

$$\bar{u}^s(\mathbf{p})\gamma^\mu u^{s'}(\mathbf{p}) = 2p^\mu\delta^{ss'}$$

The analysis proceeds similarly for a non-minimally coupled Dirac spinor.

• Next, we consider a massless spin 2 field. Such a field has two polarizations $\epsilon^i_{\mu\nu}$, satisfying

$$q^\mu\epsilon_{\mu\nu} = 0, \quad \eta^{\mu\nu}\epsilon_{\mu\nu} = 0$$

Under a little group transformation, we have

$$\epsilon_{\mu\nu} \to \epsilon_{\mu\nu} + \Lambda_\mu q_\nu + q_\mu\Lambda_\nu + \Lambda q_\mu q_\nu$$

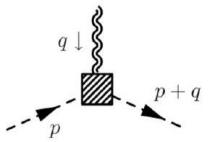
where Λ_μ and Λ depend on the little group transformation in a way we do not write out here.

• Therefore, the Ward identity must be that for any amplitude involving such an external particle, $\mathcal{M} = \epsilon_{\mu\nu}\mathcal{M}^{\mu\nu}$, we must have

$$q_\mu\mathcal{M}^{\mu\nu} = q_\nu\mathcal{M}^{\mu\nu} = 0$$

so that the amplitude is properly Lorentz invariant.

• Now we can repeat our above argument. By similar logic, an arbitrary interaction vertex is



$$= -i\Gamma^{\mu\nu}(p,q) = -2ip^\mu p^\nu \tilde{F}\left(\frac{p\cdot q}{m^2}\right),$$

where the index structure must be $p^\mu p^\nu$, because we cannot use factors of q^μ as the polarizations are transverse, and we cannot use factors of $\eta^{\mu\nu}$ as the polarizations are traceless.

• Therefore, in the soft limit,

$$\mathcal{M}(q) \approx \left(\sum_{i \text{ incoming}} \tilde{F}_i(0) \frac{p_i^\mu \epsilon_{\mu\nu} p_i^\nu}{p_i \cdot q} - \sum_{i \text{ outgoing}} \tilde{F}_i(0) \frac{p_i^\mu \epsilon_{\mu\nu} p_i^\nu}{p_i \cdot q} \right) \mathcal{M}_0$$

By Lorentz invariance, this must vanish if we substitute $\epsilon_{\mu\nu} \to q_\mu \Lambda_\nu$, so

$$\mathcal{M}_0 \Lambda_\nu \left(\sum_{i \text{ incoming}} \tilde{F}_i(0) p_i^\nu - \sum_{i \text{ outgoing}} \tilde{F}_i(0) p_i^\nu \right) = 0$$

However, we already know that the sum of incoming and outgoing momenta must be equal, by momentum conservation. So the quantity above can only vanish in general if the $\tilde{F}_i(0)$ are all equal, which means that massless spin 2 particles must couple with universal strength. This is indeed the case for the only massless spin 2 particle we know of, the graviton.

• For higher spin, the soft limit becomes even more restrictive. For massless spin 3,

$$\sum_{i \text{ incoming}} \tilde{F}_i(0) p_i^\mu p_i^\nu = \sum_{i \text{ outgoing}} \tilde{F}_i(0) p_i^\mu p_i^\nu$$

but when combined with momentum conservation, this only has solutions for general momenta if the $\tilde{F}_i(0)$ are all zero. As a result, higher spin massless particles cannot mediate long-range forces, though they can mediate interactions with a faster falloff.

• By applying this reasoning to two soft particles, one can get further constraints. E.g. it is possible to show that Yang-Mills theory is essentially the unique theory of interacting massless spin 1 particles (e.g. the quartic gauge boson interaction that appears in Yang-Mills is required by the soft limit). Similar reasoning can be used to constrain the graviton self-coupling, in a way that matches what appears in the Einstein-Hilbert action.

3.3.2 Spectral Representation

We now carefully treat interacting scalar fields.

• We consider an interacting Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}\,, \quad \mathcal{L}_{\text{int}} = -V_{\text{int}}$$

where V_{int} is a power series in ϕ starting at cubic order.

• We still have Poincare symmetry, i.e. we still have unitary operators realizing these symmetries. Thus we can still talk, for example, about the momentum of a state. We also still have the equal-time commutation relations.

• We assume the full Hamiltonian has a unique vacuum $|\Omega\rangle$ with zero four-momentum,

$$P^\mu |\Omega\rangle = 0$$

Hence the vacuum is translation invariant, $e^{ixP}|\Omega\rangle = |\Omega\rangle$.

• Since we still have translation invariance, the Hamiltonian and 3-momentum commute, so we can choose a basis of states with definite four-momentum,

$$H\,|\lambda_{\mathbf{p}}\rangle = E_p(\lambda)\,|\lambda_{\mathbf{p}}\rangle\,, \qquad \mathbf{P}\,|\lambda_{\mathbf{p}}\rangle = \mathbf{p}\,|\lambda_{\mathbf{p}}\rangle$$

One can show that, by applying a Lorentz boost, we get a family of states $|\lambda_{\mathbf{p}}\rangle$ for every λ, whose energies are $E_p = (\mathbf{p}^2 + m_\lambda^2)^{1/2}$.

• The index λ labels irreps of the Poincare group in the interacting Hilbert space. In a weakly interacting theory, the sum includes "dressed" versions of the particles in the free theory, bound states of those particles, and unbound multiparticle states. In the latter case, λ has a continuum of values, as necessary to specify the relative motion of the particles.

• In a strongly interacting theory, there is no sharp distinction between dressed particles and bound states. The only meaningful distinction we can make is between multiparticle states and others, because the continuous values of λ for multiparticle states will have analytic consequences.

• Since the equation of motion is now nonlinear, the mode decomposition of ϕ is no longer useful. Even at the classical level, such a decomposition doesn't work because the interaction mixes the modes. At the quantum level, we could formally expand $\phi(x)$ in terms of ladder operators as in the free field case, but as we'll shortly see, these operators wouldn't have a simple interpretation in terms of creating and annihilating particles.

• The completeness relation, with relativistic normalization, is

$$1 = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d\mathbf{p}}{2E_p(\lambda)} |\lambda_{\mathbf{p}}\rangle\langle\lambda_{\mathbf{p}}|$$

• Translation operators and Lorentz boost act on the field as

$$\phi(x) = e^{ixP}\phi(0)e^{-ixP}, \quad \phi(x) = U^{-1}(\Lambda)\phi\left(x'\right)U(\Lambda)$$

where $U(\Lambda)$ carries out the Lorentz transformation $x' = \Lambda x$. To check the signs, we can take matrix elements on both sides and act on the states instead.

With this setup, we now compute the Feynman propagator.

• Consider $\langle\Omega|\phi(x)|\Omega\rangle$. Using the translational invariance of the vacuum, we find that $\langle\Omega|\phi(x)|\Omega\rangle = \langle\Omega|\phi(0)|\Omega\rangle$ for all x. We can set the constant to zero by shifting the field, as this preserves the canonical commutators.

• Therefore, inserting the identity in a correlation function,

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{d\mathbf{p}}{2E_p(\lambda)} \langle\Omega|\phi(x)|\lambda_{\mathbf{p}}\rangle \langle\lambda_{\mathbf{p}}|\phi(y)|\Omega\rangle$$

where the vacuum term vanishes. Using the fact that the $|\lambda_{\mathbf{p}}\rangle$ have definite momentum,

$$\langle\Omega|\phi(x)|\lambda_{\mathbf{p}}\rangle = \langle\Omega|\phi(0)|\lambda_{\mathbf{p}}\rangle\, e^{-ipx}$$

Next, applying a Lorentz boost such that $|\lambda_{\mathbf{p}}\rangle = U^{-1}\,|\lambda_0\rangle$,

$$\langle\Omega|\phi(0)|\lambda_{\mathbf{p}}\rangle = \left\langle\Omega\left|U^{-1}U\phi(0)U^{-1}U\right|\lambda_{\mathbf{p}}\right\rangle e^{-ipx} = \langle\Omega|\phi(0)|\lambda_0\rangle$$

where we used the Lorentz invariance of the zero vector and the vacuum.

• Putting these results together, we have

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{d\mathbf{p}}{2E_p(\lambda)} e^{-ip(x-y)} \left|\langle\Omega|\phi(0)|\lambda_0\rangle\right|^2$$

Doing the exact same manipulations for the time ordered correlator gives

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int dp \frac{i}{p^2 - m_\lambda^2 + i\epsilon} e^{-ip(x-y)} \left|\langle\Omega|\phi(0)|\lambda_0\rangle\right|^2$$

We parametrize the result by mass:

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty d\left(M^2\right) \rho\left(M^2\right) D_F^0\left(x-y, M^2\right);$$

$$D_F^0\left(x-y, M^2\right) := \int dp \frac{i}{p^2 - M^2 + i\epsilon} e^{-ip(x-y)},$$

$$\rho\left(M^2\right) := \sum_\lambda \not{\delta}\left(M^2 - m_\lambda^2\right) \left|\langle\Omega|\phi(0)|\lambda_0\rangle\right|^2$$

$\rho\left(M^2\right)$ is called the spectral function, and by its definition, it is real and positive.

• Typically the spectral function will have poles for one-particle states and bound states. Multiparticle states have continuously "smeared out" poles, i.e. branch cuts. Separating out the one-particle contribution,

$$\rho\left(M^2\right) = \not{\delta}\left(M^2 - m^2\right) Z + \text{ higher}, \quad Z = \left|\langle\Omega|\phi(0)|1_0\rangle\right|^2$$

so that

$$\int dx e^{ip(x-y)} \langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{m^2_{\text{bound}}}^\infty d\left(M^2\right) \rho\left(M^2\right) \frac{i}{p^2 - M^2 + i\epsilon}.$$

The quantity Z is called the wavefunction renormalization, where $Z = 1$ when the theory is free. We see that in an interacting theory, $\phi(x)$ creates not only a one-particle state, but a superposition of many kinds of states. However, as long as $Z \neq 0$, it has at least some amplitude to create the single-particle states that we need to calculate scattering amplitudes.

• Note that this derivation was very general, and never required an expansion into creation and annihilation operators; it works for any field operator, e.g. also for composite operators. However, one can show that for bare fields (i.e. fields that satisfy $[\phi_0(t,\mathbf{x}), \partial_t\phi_0(t,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y})$), the spectral function has unit integral, and hence $Z \leq 1$.

• There is also a similar spectral representation for the Dirac field,

$$\int dx e^{ip(x-y)} \langle\Omega|T\psi(x)\bar{\psi}(y)|\Omega\rangle = \frac{iZ_2 \sum_s u^s(p)\bar{u}^s(p)}{p^2 - m^2 + i\epsilon} + ..., \quad \sqrt{Z_2} u^s(p) = \langle\Omega|\psi(0)|p,s\rangle$$

To derive it, we also need to keep track of how the Lorentz boosts above affect the field.

• Interactions could also render one-particle states unstable, e.g. by allowing them to decay to lighter particles. Such states are not energy eigenstates at all. They instead appear in the Fourier transform of the two-point function as poles off the real axis, as we'll discuss later. Another possible subtlety is that in a theory with massless particles, the branch cut extends down to $M^2 = 0$.

Note. A cheap RG analysis. Consider interactions of the form $\lambda_n \phi^n$. Since $[\mathcal{L}] = 4$ and $[\phi] = 1$, we have $[\lambda_n] = 4 - n$. We consider a few examples.

• For the cubic interaction, $[\lambda_3] = 1$, so the effect of the interaction is described by the dimensionless parameter λ_3/E, where E is the energy scale of the process. (This is clearest in the path integral picture, where λ_3/E estimates the action contribution.) Then the interaction is strong at low energies, so it is called relevant.

• For the quartic interaction, $[\lambda_4] = 0$, and we say it is marginal.

• For the quintic interaction, $[\lambda_5] = -1$, so its effect is described by $\lambda_5 E$, which is weak at low energies, so it is called irrelevant.

• Now suppose we have a fundamental theory defined at some high scale Λ. Then the dimensionless coupling constant for ϕ^5 would be $g_5 = \Lambda\lambda_5$, where by naturalness we expect $g_5 = O(1)$. Then the effect of the interaction at scale E is $g_5(E/\Lambda)$.

• If Λ is taken to be the GUT scale or the Planck scale, then E/Λ is extremely small, so it is an excellent approximation to ignore irrelevant terms.

• But there is a subtlety: consider the dimension 2 mass term. If the above argument worked, then we would expect it to scale as $\Lambda^2\phi^2$, which implies a mass of order Λ. But clearly, there exist plenty of particles with masses well below the Planck scale. This violation of naive dimensional analysis can result from approximate symmetries, tuning, or nontrivial dynamics.

Note that this analysis depends on the spatial dimension. In six dimensions, $[\mathcal{L}] = 6, [\phi] = 2$, and it is λ_6 that is marginal.

3.3.3 The LSZ Reduction Formula

Обсудим формулу Лемана-Симанчика-Циммермана (ЛСЦ)
(мб выше потом вставлю, пока тут.)

Основные формулы

$$= \frac{i}{p^2 - m_0^2} + \frac{i}{p^2 - m_0^2}\left(-iM^2\right)\frac{i}{p^2 - m_0^2} + \cdots =$$
$$= \frac{i}{p^2 - m_0^2 - M^2\left(p^2\right)}.$$

Иллюстративные примеры (?!?)

Theory

• We are still working in Heisenberg picture, so all states are "fixed in time". Note that a better way of thinking about Heisenberg picture is that all the states "extend through time", so they may be described by how they look at any given time.

• An in state $|i, \text{ in } \rangle$ is a state which looks like several widely separated, incoming particles as $t \to -\infty$. Similarly, an out state $|f, \text{ out } \rangle$ is a state which has widely separated, outgoing particles as $t \to \infty$. Here, i and f are shorthand for a specification like "two particles, of momenta p^μ and q^μ". The asymptotic vacua are $|$ vac, in $\rangle$ and $|$ vac, out $\rangle$.

• The S matrix maps out states to in states,

$$|i, \text{ in } \rangle = S|i, \text{ out } \rangle$$

In most reasonable physical theories, we have

$$| \text{ vac, in } \rangle = | \text{ vac, out } \rangle = |\Omega\rangle, \quad S|\Omega\rangle = |\Omega\rangle, \quad S^\dagger = S^{-1}$$

We are interested in calculating the transition amplitudes

$$\langle f, \text{ out } | i, \text{ in } \rangle = \langle f, \text{ in } | S | i, \text{ in } \rangle.$$

Hence these transition amplitudes are also called S-matrix elements. Note that all of these states live in the full interacting Hilbert space.

• To formally construct the in states, we define an "in field" ϕ_{in} with the following properties.

• The in field is a free Klein-Gordan field $\phi(x)$ which creates in states from $|vac, \text{in}\rangle$ as $t \to -\infty$. Hence we have the mode expansion

$$\phi_{\text{in}}(x) = \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} \left(a_{\text{in},\mathbf{p}} e^{-ipx} + a^\dagger_{\text{in},\mathbf{p}} e^{ipx} \right)$$

• Since the in states have mass m, where m is the single-particle mass in the full theory, the in field must have mass m.

• The in field approaches the interacting field as $t \to -\infty$ in the sense that

$$\langle \alpha | \phi(x) | \beta \rangle \to \sqrt{Z} \langle \alpha | \phi_{\text{in}}(x) | \beta \rangle$$

for all states as $t \to -\infty$. The constant $\sqrt{Z}$ is determined by setting $|\beta\rangle = |\Omega\rangle$ and $|\alpha\rangle = |1_{\mathbf{p}}\rangle$ and using the spectral form.

• The out field ϕ_{out} has analogous properties as $t \to \infty$. The two are related by

$$\phi_{\text{in}}(x) = S \phi_{\text{out}}(x) S^{-1}.$$

We now apply our setup to derive the LSZ reduction formula.

• For concreteness, consider the S-matrix element $\langle p_1, p_2, \text{ out } | q_1, q_2, \text{ in } \rangle$. We have

$$\begin{aligned}
\langle p_1, p_2, \text{ out } | q_1, q_2, \text{ in } \rangle &= \sqrt{2\omega_{q_1}} \left\langle p_1, p_2, \text{ out } \left| a^\dagger_{\text{in},\mathbf{q}_1} \right| q_2, \text{ in } \right\rangle \\
&= \frac{1}{i} \int d\mathbf{x} e^{-iq_1 x} \overleftrightarrow{\partial_0} \langle p_1, p_2, \text{ out } | \phi_{\text{in}}(x) | q_2, \text{ in } \rangle \Big|_{x^0 = t} \\
&= \lim_{t \to -\infty} Z^{-1/2} \frac{1}{i} \int d\mathbf{x} e^{-iq_1 x} \overleftrightarrow{\partial_0} \langle p_1, p_2, \text{ out } | \phi(x) | q_2, \text{ in } \rangle
\end{aligned}$$

where we used the expression for $a^\dagger_{\text{in}}$ in terms of ϕ_{in}, which holds for arbitrary t, then took $t \to -\infty$ to match with the interacting field.

• The next step is to act with $\phi(x)$ on the left to get rid of one of the out particles. This is only possible if we have an out field, which requires a limit $t \to +\infty$, so we use

$$\lim_{t \to -\infty} f(t) = \lim_{t \to +\infty} f(t) - \int dt \partial_0 f(t)$$

The first term will give us terms like $\langle p_2, \text{ out } | q_2, \text{ in } \rangle \delta(\mathbf{p}_1 - \mathbf{q}_1)$ with another term with p_1 and p_2 swapped. These represent "disconnected" contributions to the S-matrix, i.e. one where the first particle doesn't scatter at all. We don't care about them here because it can be recursively computed from S-matrix elements with fewer particles.

• The connected term yields an integral over spacetime, and directly carrying out the derivatives and integrating by parts yields

$$iZ^{-1/2}\int dx_1 e^{-iq_1x_1}\left(\Box_1+m^2\right)\langle p_1,p_2,\text{ out }|\phi(x_1)|\,q_2,\text{ in }\rangle$$

There is a sticky point here, which is that the integration by parts is not legal, because e^{-iq_1x} doesn't go to zero at infinity. This occurs because momentum eigenstates unphysically cover all of space. To define our in and out states properly, we should have constructed finite wavepackets and worked with them from the start, but we sweep this under the rug to avoid the complication.

• Next, we would like to repeat this process for all of the other incoming and outgoing particles. The only snag is that when we flip $t\to-\infty$ to $t\to+\infty$ with the identity above, the new fields may be in the "wrong place" and can't be commuted past the others. The fix is to replace the correlation functions with time-ordered correlation functions, which automatically put the in and out fields on the right side.

• Carrying out this procedure and generalizing to $n+r$ particles,

$$\langle p_1,...,p_n,\text{ out }|q_1,...,q_4,\text{ in }\rangle=\sum\text{ disconnected }+$$
$$+\left(iZ^{-1/2}\right)^{n+r}\int dy_1...dx_1...\prod_k e^{ip_ky_k}\left(\Box_{y_k}+m^2\right)\prod_\ell e^{-iq_\ell x_\ell}\left(\Box_{x_\ell}+m^2\right)\langle\Omega\,|T\phi\,(y_1)\,...\phi(x_1)...|\,\Omega\rangle\,.$$

This is the LSZ reduction formula, relating S-matrix elements to time ordered correlation functions. Note that all momenta here are on-shell, since they are the momenta of asymptotic particles, so we can't treat the scattering of unstable particles.

• In momentum space, the connected part becomes

$$\left(iZ^{-1/2}\right)^{n+r}\prod_k\left(-p_k^2+m^2\right)\prod_\ell\left(-q_\ell^2+m^2\right)\langle\Omega\,|T\phi\,(p_1)\,...\phi\,(q_1)\,...|\,\Omega\rangle$$

For the S-matrix element to be nonsingular, the time-ordered correlators must have a corresponding pole structure, with a Feynman propagator factor giving poles for on-shell particles.

• Another useful rewriting is

$$\prod_k\int dy_k e^{ip_ky_k}\prod_\ell\int dx_\ell e^{-iq_lx_l}\langle\Omega|T\phi(y_1)...\phi(x_1)...|\Omega\rangle=$$
$$=\prod_k\frac{i\sqrt{Z}}{p_k^2-m^2}\prod_\ell\frac{i\sqrt{Z}}{q_\ell^2-m^2}\left\langle p_1,...|S|q_1,...\right\rangle\Big|_{\text{connected}}\,.$$

Note. In summary, in free field theory, the fields always create or destroy a particle near a point. Then position-space correlation functions describe amplitudes for particle propagation between those points, and LSZ states that momentum-space correlation functions are essentially equal to S-matrix elements after removing the poles.

In an interacting field theory, $\phi(x)$ still creates some kind of excitation near the point x (as can be seen by its transformation properties) but it is generally a complicated combination of single-particle and multiparticle states, as shown in the spectral representation. The point of the LSZ reduction formula is that we can project out the part we want (i.e. above, single-particle states of mass m) by looking at the residues of the appropriate poles, up to a $\sqrt{Z}$ correction factor.

More generally, the LSZ reduction theorem means that any field can be used to compute S-matrix elements of any kind of particle, as long as that field has nonzero overlap with the particle. E.g.we can compute S-matrix elements using the canonical momentum $\pi(x)$, because it has some amplitude to make single-particle states. For a complex scalar field, we can compute S-matrix elements for bound particle-antiparticle pairs using the field $\psi^\dagger(x)\psi(x)$.

3.3.4 Time-Ordered Correlators

We compute time-ordered correlators in the interaction picture.

• In the interaction picture, we split

$$H = H_0 + H_{\text{int}}$$

and evolve the states with H_{int} and the operators with H_0. In particular, this means the interaction picture field

$$\phi_I(x) = e^{iH_0 t}\phi(0,\mathbf{x})e^{-iH_0 t}$$

act exactly like the free Heisenberg fields which we introduced earlier. They satisfy the KleinGordan equation with the bare mass m_0, not the physical mass m. Moreover, using the free mode expansion to define $a_{I,\mathbf{p}}$, the interacting annihilation operator annihilates the free vacuum $|0\rangle$, not the interacting vacuum $|\Omega\rangle$.

• At general times, the Heisenberg picture operators are related by

$$\phi(t,\mathbf{x}) = U^\dagger(t)\phi_I(t,\mathbf{x})U(t), \qquad U(t) = e^{iH_0 t}e^{-iHt}.$$

Taking matrix elements of both sides shows that states evolve in interaction picture via $U(t)$.

$$i\frac{\partial}{\partial t}U(t) = H_I(t)U(t), \qquad H_I(t) = e^{iH_0 t}H_{\text{int}}e^{-iH_0 t}$$

$$U(t) = T\exp\left(-i\int_0^t dt' H_I(t')\right)$$

$$= 1 - i\int_0^t dt_1 H_I(t_1) + (-i)^2\int_0^t\int_0^t dt_1 dt_2 T H_I(t_1)H_I(t_2) + \ldots \qquad \text{(Dyson's equation)}$$

$$U(t,t_0) = T\exp\left(-i\int_{t_0}^t dt' H_I(t')\right)$$

and the time ordering ensures $U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$.

• Next, we relate the interacting and free vacuum. Let $H|n\rangle = E_n|n\rangle$. Then

$$e^{-iHt}|0\rangle = \sum_n e^{iE_n t}|n\rangle\langle n|0\rangle$$

by inserting the energy basis. The vacuum state will have some nonzero energy E_Ω (as we already used up our vacuum energy counterterm to set $E_0 = 0$), but all other states will have higher energy. Thus, sending time to infinity with a small damping,

$$|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)}\frac{e^{-iHt}|0\rangle}{e^{-iE_\Omega t}\langle\Omega|0\rangle}$$

Note that we must assume $\langle\Omega|0\rangle$ is nonzero, which should be true if H_{int} is 'weak'. This trick also fails if the theory has massless particles, which yield states of arbitrarily low energy.

• Next, note that

$$e^{-iHt}|0\rangle = e^{-iHt}e^{iH_0t}|0\rangle = U(-t)^\dagger|0\rangle = U(t)|0\rangle = U(0,-t)|0\rangle$$

where we used $H_0|0\rangle = 0$. Plugging this into a correlation function,

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \left(|\langle 0|\Omega\rangle|^2 e^{-2iE_\Omega t}\right)^{-1} \cdot \\ \cdot \left\langle 0 \left| U(t,0)U\left(x^0,0\right)^\dagger \phi_I(x) U\left(x^0,0\right) U\left(y^0,0\right)^\dagger \phi_I(y) U\left(y^0,0\right) U(0,-t) \right| 0 \right\rangle$$

where we simply switched to interaction picture.

• Using the multiplication rule for $U(t,t')$, we have

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \left(|\langle 0|\Omega\rangle|^2 e^{-2iE_\Omega t}\right)^{-1} \cdot \left\langle 0 \left| U\left(t,x^0\right)\phi_I(x) U\left(x^0,y^0\right)\phi_I(y) U\left(y^0,-t\right) \right| 0 \right\rangle$$

We see that as long as the fields were time ordered, everything inside the expectation value is automatically timed ordered!

• To deal with the prefactor, note that

$$1 = \langle\Omega|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\langle 0|U(t,-t)|0\rangle}{|\langle 0|\Omega\rangle|^2 e^{-2iE_\Omega t}}$$

Combining these results together and generalizing to n fields, we find

$$\langle\Omega\left|T\phi(x_1)...\phi(x_n)\right|\Omega\rangle = \lim_{t\to\infty(1-i\epsilon)} \frac{\langle 0\left|T\phi_I(x_1)...\phi_I(x_n)U(t,-t)\right|0\rangle}{\langle 0|U(t,-t)|0\rangle}$$

We use Wick's theorem to compute these interaction picture correlators. Since we'll be working exclusively with interaction picture fields from now on, we drop the subscript.

• We decompose the interaction picture field into positive and negative free modes,

$$\phi(x) = \phi^+(x) + \phi^-(x) = \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} a_{\mathbf{p}} e^{-ipx} + \int \frac{d\mathbf{p}}{\sqrt{2\omega_p}} a^\dagger_{\mathbf{p}} e^{ipx}$$

The plus and minus fields are easier to deal with because $\phi^+(x)|0\rangle = \langle 0|\phi^-(x) = 0$.

• Given an operator O define as a string of creation and annihilation operators, its normal ordering $:O:$ is the same string but with all the annihilation operators moved to the right. As long as O does not have a c-number piece cI,

$$\langle 0|:O:|0\rangle = 0.$$

• Note that normal ordering is not a map on operators, but a map on strings of creation and annihilation operators. E.g. $aa^\dagger - a^\dagger a = 1$ is an operator equation, but normal ordering both sides gives $0 = 1$. (??? и что делать с этим???)

• Now, for the product of two fields, we have

$$\phi(x)\phi(y) = \phi^-(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^+(y)\phi^+(y) + \phi^+(x)\phi^-(y).$$

The first three terms are in normal order, but the last is not. Normal ordering both sides (which is legal because we haven't done anything nontrivial like applying commutation relations) gives

$$\phi(x)\phi(y) =: \phi(x)\phi(y) : + \left[\phi^+(x), \phi^-(y)\right]$$

• Applying this to a time-ordered pair of fields, we have

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : +\theta\left(x^0 - y^0\right)\left[\phi^+(x), \phi^-(y)\right] + \theta\left(y^0 - x^0\right)\left[\phi^+(y), \phi^-(x)\right]$$

The last two terms are c-numbers by the commutation relations. By taking the VEV of both sides, we see they must sum to the VEV of $T\phi(x)\phi(y)$, i.e. the (bare) Feynman propagator, so

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : +D_F^0(x-y).$$

• More generally, for a string of n fields, we get 2^n terms in total, $(n!)^2$ of which are already normal ordered. For the other terms, we have to do some number of commutations to reach normal ordering, each of which produces a two-point function.

• The full result can be written in terms of contractions, where contracting two fields removes them from the normal ordering and produces a factor of their two-point function. Wick's theorem states that

$$T\left(\phi_1 ... \phi_n\right) =: \phi_1 ... \phi_n : + \text{ all possible contractions}$$

which includes terms with any number of contractions; here we write $\phi_i = \phi\left(x_i\right)$.

• The sum includes terms which are not fully contracted, such as $D_F^0\left(x_1 - x_2\right) : \phi_3\phi_4 :$, which vanish when we take the vev. The fully contracted terms all come from the term that begins anti-normal ordered. We thus conclude

$$\left\langle 0\left|T\phi_1 ... \phi_{2n}\right|0\right\rangle = D_F^0\left(x_1 - x_2\right)\cdots D_F^0\left(x_{2n-1}, x_{2n}\right) + \text{ all other full contractions}$$

while the time-ordered correlator of an odd number of fields vanishes.

3.3.5 Ward-Takahashi Identity

(не нужно будет - во 2ю 1ю переставлю!!!)

Теория

$$\phi(x): \qquad \frac{\delta S}{\delta\phi(x)} = -\left(\partial^2 + m^2\right)\phi(x) = 0.$$

$$\left(\partial_x^2 + m^2\right) i\left\langle 0\left|T\phi(x)\phi(x_1)\right|0\right\rangle = \delta\left(x - x_1\right),$$

where the delta function comes from differentiating a step function, and we use the equal-time commutation relations (???). Домножая это уравнение на ϕ_i и подставляя уравнение движения через действие, we have the Schwinger-Dyson equation

$$\left\langle 0\left|T\frac{\delta S}{\delta\phi(x)}\phi(x_1)...\phi(x_n)\right|0\right\rangle = i\sum_{j=1}^{n}\left\langle 0\left|T\phi(x_1)...\hat{\phi}\left(x_j\right)\delta\left(x - x_j\right)...\phi(x_n)\right|0\right\rangle,$$

hat indicates a missing argument; it is the analog of the classical 'equations of motion' for correlation functions. (??? я вот не понял, там мы то ли еще обкладки берем, то ли изначально как-то в обоих частях они появляются? в теории додумаю!)

The Schwinger-Dyson equation says that correlation functions in classical field theory obey the classical equations of motion, while correlation functions in quantum field theory are corrected by 'contact terms' which correspond to the emission and absorption of virtual particles (????); accordingly, classical field theory Feynman diagrams have no loops.

• Off-shell, the Noether current associated with the symmetry $\delta\phi(x)$ satisfies $\partial_\mu j^\mu = -\frac{\delta S}{\delta\phi(x)}\delta\phi(x)$, so the Schwinger-Dyson equations become the Ward-Takahashi identity:

$$\partial_\mu \langle 0 | T j^\mu(x)\phi(x_1)...\phi(x_n) | 0\rangle + i\sum_{j=1}^{n} \left\langle 0 \left| T\phi(x_1)...\hat{\phi}(x_j)\,\delta\phi(x)\delta(x-x_j)...\phi(x_n)\right| 0\right\rangle = 0$$

it is the analog of classical current conservation for correlation functions. They are generalized in Yang-Mills to the Slavnov-Taylor identities. (??? formulas???)

• We have only justified these results for a free theory, but the same results hold for an interacting theory, where $\delta S/\delta\phi(x)$ simply gets more complicated; we will justify this with the path integral later. One caveat is that correlation functions generally involve divergent loop integrals that must be regulated. If we cannot find a regulator that respects the classical symmetry, then the symmetry is anomalous and does not hold at the quantum level.

Now we see how these results imply the Ward identity.

• By the LSZ reduction formula, a scattering amplitude involving an external photon $|k^\mu, \xi^\mu(k)\rangle$ and n other external particles has the form

$$\langle f|i\rangle \sim \xi^\mu(k)\int dx e^{-ikx}\partial_x^2 ... \langle 0|TA_\mu(x)...|0\rangle$$

• In Feynman gauge, the classical equation of motion for A^μ is $\partial^2 A^\mu = j^\mu$. Then we have

$$\partial_x^2 \langle 0|TA_\mu(x)...|0\rangle = \langle 0|Tj_\mu(x)...|0\rangle + \text{ contact terms}$$

where the contact terms contract our photon with another external photon. This represents a disconnected contribution and hence doesn't contribute to the connected S-matrix.

• Now set $\xi^\mu = k^\mu$. Integrating by parts turns the momentum into a derivative, giving

$$\partial^\mu \langle 0|Tj_\mu(x)...|0\rangle = \text{ contact terms}$$

by the Ward-Takahashi identity. Since j^μ is the conserved current associated with the $U(1)$ symmetry of the electron field, we get a contact term for each external electron. None of these terms contribute to the S-matrix because in each term, the momentum of one external leg will be displaced by $\pm k$ from where it should be; this proves the Ward identity.

• Alternatively, backing up a step and removing the LSZ pole cancellation factors, we have derived an identity between off-shell momentum-space correlation functions, where a correlation function involving an external photon is equal to a sum of correlation functions without it, where the external electron momenta are shifted by $\pm k$. This is the form of the Ward-Takahashi identity presented in Peskin, where it is proven diagrammatically.

Note. More generally, any amplitude $\mathcal{M}^{\mu\nu\cdots}\epsilon_\mu^{(1)}\epsilon_\nu^{(2)}...$ will vanish if any nonzero number of the $\epsilon^{(i)}$ are made longitudinal. However, in non-abelian gauge theory the equations of motion are

more complicated, and the above argument is not applicable. It turns out that the amplitude still vanishes if any one of the $\epsilon^{(i)}$ are made longitudinal, but the rest must be transverse.

This leads to some subtleties. For processes involving off-shell gluons, we cannot naively replace sums over physical polarizations with $\eta_{\mu\nu}$ as we did for QED. However, the extra contribution due to including the extra unphysical polarizations can be cancelled by ghosts. E.g. in the amplitude for $q\bar{q} \to gg$, the extra unphysical contribution can be cancelled by adding the amplitude for $q\bar{q} \to c\bar{c}$. This latter amplitude contributes negatively to the cross section, which is possible because the ghosts have negative norm.

We now explicitly link current conservation and gauge invariance.

• Write the action as $S = S_A[A] + S_{\text{int}}[A, \phi] + S_\phi[\phi]$.

In retrospect, we see that our derivation of the Ward identity only requires S to have a global continuous symmetry with conserved current j^μ that couples to A^μ as

$$\frac{\delta S_{\text{int}}[A, \phi]}{\delta A_\mu} = -j^\mu$$

This is the first step in the derivation, and conservation of j^μ is the second.

• We claim that if A^μ couples to a conserved current in a massless vector theory, the action S must be gauge invariant. To see this, consider the global continuous symmetry $\phi(x) \to \phi(x) + \epsilon\delta\phi(x)$, which leaves S invariant. If we promote ϵ to $\epsilon(x)$, then

$$\delta S = \int dx \tilde{j}^\mu \partial_\mu \epsilon(x) = -\int dx \left(\partial_\mu \tilde{j}^\mu\right) \epsilon(x)$$

where the first step follows by Lorentz invariance, and because δS is zero when $\epsilon(x)$ is constant. Comparing this to our earlier expression shows that $\tilde{j}^\mu = j^\mu$.

• Next, using our original hypothesis we have

$$\int dx \left(\frac{\delta S_{\text{int}}}{\delta A_\mu} + j^\mu\right) \partial_\mu \epsilon(x) = 0$$

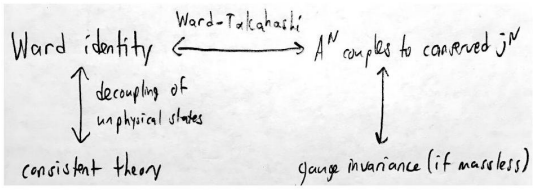
As we've just seen, the second term is the change in the action due to the local transformation $\phi(x) \to \phi(x) + \epsilon(x)\delta\phi(x)$. The first term is the change in the action due to the local transformation $A_\mu \to A_\mu + \partial_\mu \epsilon(x)$, where we used the fact that $S_A[A]$ is gauge invariant. Thus we have shown that the full action has a gauge symmetry.

• The converse of this statement follows from running the steps in reverse. If there is a gauge symmetry, taking ϵ constant gives a global symmetry with conserved current j^μ, and the variation of the action under the gauge transformation is

$$\int dx \left(\frac{\delta S_{\text{int}}}{\delta A_\mu} + j^\mu\right) \partial_\mu \epsilon(x) = 0$$

as before. Since $\partial_\mu \epsilon(x)$ is arbitrary, the term in brackets is zero.

• In the case of a massive vector theory, there is no gauge invariance, but the Ward identity still holds if A^μ couples to a conserved current j^μ. A map of equivalences is given below.



All of these statements are so closely related that the Ward identity is sometimes called a statement of 'current conservation' or 'gauge invariance'.

3.3.6 Feynman Diagrams

We can represent the terms in Wick's theorem diagrammatically.

• For $\langle 0\,|T\phi_1...\phi_n|\,0\rangle$, draw n points, then draw all possible diagrams made by connecting disjoint pairs of points. For an edge from i to j, write down the factor $D_F^0\left(x_i - x_j\right)$.

• Interactions can be expanded order by order, e.g. for a ϕ^4 interaction,

$$\left\langle 0\left|T\phi(x)\phi(y)e^{-i(\lambda/4!)\int dz\phi^4(z)}\right|0\right\rangle = \langle 0|T\phi(x)\phi(y)|0\rangle + \left\langle 0\left|T\phi(x)\phi(y)\int dz\left(\frac{-i\lambda}{4!}\right)\phi^4(z)\right|0\right\rangle + ...$$

The $1/n!$ factor in the Taylor series is always canceled by the permutation symmetry of the vertices. Naively, the $1/4!$ factors are canceled because we have $4\,!$ contractions possible for each factor of $\phi^4(z)$, but this is not true for symmetric diagrams; the overall result is that we must divide by the symmetry factor of the diagram.

• This gives the position-space Feynman rules for $\langle 0\,|T\phi(x_1)...\phi(x_n)U(\infty,-\infty)|\,0\rangle$.

• Draw a point for every 'external' point x_i. These are the original fields in the correlator.

• Draw some internal vertices z_i to account for the interaction. For every vertex, multiply by $-i\lambda$ and integrate over z_i.

• Draw directed lines between points until every external point is connected to one line, and every internal vertex is connected to four.

• For a line from x to y, write down the Feynman propagator

$$D_F^0(x-y) = \int dp \frac{i}{p^2 - m_0^2 + i\epsilon} e^{-ip(x-y)}$$

The orientation of the line doesn't matter, since $D_F^0(x-y) = D_F^0(y-x)$.

• Divide by the symmetry factor of the diagram and sum over all diagrams.

• Note that we can contract fields at a point to other fields at the same point, giving factors of $D_F^0(0)$ which diverges. This is a loop diagram, which must be addressed by renormalization.

• To simplify this, we can perform the position integrals, leaving behind only momentum integrals. If lines with momentum p_1 and p_2 point into a vertex z and lines with momentum p_3 and p_4 point out, the position integral is

$$\int dz e^{ip_1 z} e^{ip_2 z} e^{-ip_3 z} e^{-ip_4 z} = \not{\delta}\left(p_1 + p_2 - p_3 - p_4\right)$$

Therefore, we have the momentum-space Feynman rules shown below.

• Give each line a directed momentum p_i and write down the factor

$$\tilde{D}_F^0(p) = \frac{i}{p^2 - m_0^2 + i\epsilon}$$

• For each vertex, multiply by

$$(-i\lambda)\phi\left(\Sigma_{\text{in}}\, p_i - \Sigma_{\text{out}}\, p_j\right)$$

• For each external point, multiply by e^{-ipx} if momentum p points out of the point, and e^{ipx} if it points in.

• Integrate dp over each internal momentum.

• For each connected component that is a vacuum bubble, multiply by the volume of spacetime VT.

• To explain the last rule, note that for vacuum bubbles, not all of the position integrals can be performed. We are left with integrals parametrizing the location of each bubble,s

$$\int dz = \not{\delta}(0) = VT$$

where VT is the volume of the spacetime. This is generally what $\delta(0)$ means when it appears.

Working with disconnected diagrams.

• The nomenclature is a bit confusing. We say a diagram is connected if every vertex is connected, in the graph theory sense, to an external point, i.e. there are no vacuum bubbles. We say a diagram is fully connected if it has one connected component. Neither condition implies the other. E.g. a diagram with only a vacuum bubble is fully connected but not connected.

• Every diagram can be factorized into the product of a connected diagram and a diagram with no external points (vacuum bubbles):

$$\sum \text{ all diagrams } = \left(\sum \text{ connected diagrams }\right)\left(\sum \text{ vacuum bubbles }\right).$$

• By applying our rules above to the case of zero external points, we have

$$\lim_{t\to\infty(1-i\epsilon)} \langle 0|U(t,-t)|0\rangle = \sum \text{ vacuum bubbles } \equiv Z$$

which tells us that the correlator of Heisenberg fields is

$$\langle \Omega \,|T\phi_1...\phi_n|\, \Omega\rangle = \sum \text{ connected diagrams with } n \text{ external points.}$$

We call the sum of the vacuum bubbles Z the partition function.

• The partition function has further structure. Let $\{V_i\}$ be the set of fully connected vacuum bubbles. Then a vacuum bubble with n_i copies of V_i comes with the additional symmetry factor $\prod_i 1/n_i!$, so

$$Z = \exp\left(\sum \text{ fully-connected vacuum bubbles }\right)$$

• On the other hand, by our previous work we have

$$\lim_{t\to\infty(1-i\epsilon)} \langle 0|U(t,-t)|0\rangle = \lim_{t\to\infty(1-i\epsilon)} |\langle\Omega|0\rangle|^2 e^{-iE_\Omega T}$$

Taking the logarithm, the $|\langle\Omega|0\rangle|^2$ term is finite and vanishes in the limit, giving

$$E_\Omega = \lim_{t\to\infty(1-i\epsilon)} \frac{i}{T}\log Z$$

Thus the sum of fully-connected vacuum bubbles gives the vacuum energy.

• We can divide by V to get the vacuum energy density. On the right-hand side, this combines with T to give the volume of the spacetime, which cancels with the factor of $\phi(0)$ which comes

with every fully-connected vacuum bubble. This removes the IR divergence, though we still have a UV divergence, which must be handled by renormalization.

Next, we introduce 1PI diagrams to compute the propagator.

• In perturbation theory, the exact propagator can be expanded as,

$$D_F(x-y) = \langle\Omega|T\phi(x)\phi(y)|\Omega\rangle = \sum \text{ connected diagrams with 2 external points.}$$

To organize the perturbation series, define a one-particle irreducible (1PI) diagram to be one which cannot be separated into two separate nontrivial diagrams by cutting a single line.

• We define

$$-iM^2\left(p^2\right) = \sum \text{ all non-trivial 1PI diagrams}$$

where the 1PI diagrams have incoming and outgoing momentum p^2, and do not include factors for the incoming/outgoing propagators or external points.

• We define the Fourier transform

$$D_F(x-y) = \int dp e^{-ip(x-y)} D_F\left(p^2\right)$$

Then $D_F\left(p^2\right)$ is computed with the same Feynman rules as $D_F(x-y)$, but there are no factors for external points.

• Therefore by Dyson resummation:

$$D_F\left(p^2\right) = D_F^0\left(p^2\right) + D_F^0\left(p^2\right)\left(-iM^2\left(p^2\right)\right)D_F^0\left(p^2\right) + \ldots = \frac{i}{p^2-m_0^2+i\epsilon}\frac{1}{1-\frac{M^2(p^2)}{p^2-m_0^2+i\epsilon}}$$

$$D_F\left(p^2\right) = \frac{i}{p^2-\left(m_0^2+M^2\left(p^2\right)\right)+i\epsilon}.$$

$M^2\left(p^2\right)$ is the self-energy of the particle.

• Let m^2 be the first analytic pole of $D_F\left(p^2\right)$. Then we may write

$$D_F\left(p^2\right) = \frac{iZ}{p^2-m^2+i\epsilon} + \text{ terms regular at } m^2$$

Comparing this with the spectral representation, Z is the wavefunction renormalization and m is the physical mass. Thus we have found a way to compute Z and m perturbatively.

3.3.7 Scattering Amplitudes

We now use LSZ to relate our correlators back to S-matrix elements.

• We return to our statement of LSZ, with $n+r$ total incoming and outgoing particles,

$$\prod_k \int dy_k e^{ip_k y_k} \prod_\ell \int dx_\ell e^{-iq_l x_l} \langle\Omega|T\phi\left(y_1\right)\ldots\phi(x_1)\ldots|\Omega\rangle$$
$$= \prod_k \frac{i\sqrt{Z}}{p_k^2-m^2} \prod_\ell \frac{i\sqrt{Z}}{q_\ell^2-m^2} \left\langle p_1,\ldots|S|q_1,\ldots\right\rangle\Big|_{\text{connected}}.$$

Above, all momenta are on-shell. The S-matrix element cannot cancel any of the poles, because then it would be proportional to $p_k^2 - m^2$ or $q_\ell^2 - m^2$, and hence vanish entirely. Thus contributions to $S|_{\text{connected}}$ come from correlation functions with $n + r$ poles at m^2.

• The only diagrams which produce such poles are fully connected diagrams. E.g. if one particle was in its own connected component, it would only contribute one factor of $iZ/\left(p^2 - m^2\right)$ from its propagator rather than the two it should. This lines up with our intuition that $S|_{\text{connected}}$ gives scattering amplitudes where all particles participate together.

• Every fully connected diagram can be decomposed as $n + r$ exact propagators attaching to an "amputated" diagram. Specifically, to amputate a leg, we start at some external point and cut off as much as possible so that the diagram splits into two components, one of which contains only that external point. Note that all 1PI diagrams are amputated but not vice versa.

• As we've seen above, each resummed propagator contributes $iZ/\left(p^2 - m^2 + i\epsilon\right)$, plus a regular piece which is not relevant to our calculation; these cancel with the factors on the right-hand side of the LSZ reduction formula. Taking the Fourier transform has the sole effect of removing the exponential factors for the external points.

• Therefore, the Feynman rules for $\langle p_1, ..., p_n|S|q_1, ..., q_r\rangle|_{\text{connected}}$ are

• Add internal points and lines as specified earlier, giving every line a directed momentum.

• For each internal line of momentum p, multiply by

$$\frac{i}{p^2 - m_0^2 + i\epsilon}$$

• For each vertex, multiply by

$$(-i\lambda)\phi\left(\Sigma_{\text{in}}\, p_i - \Sigma_{\text{out}}\, p_j\right)$$

• Integrate dp over each internal momentum and divide by the symmetry factor.

• Sum over fully-connected, amputated diagrams and multiply by $\sqrt{Z}^{n+r}$.

Formally, $\sqrt{Z} = 1 + O(\lambda)$, so we can set $Z = 1$ for a lowest-order computation. We will always get an overall factor of $\phi\left(q_1 + ... + q_r - p_1 + ... - p_n\right)$.

• For the physical computations which follow, we will almost never be interested in vacuum bubbles, i.e. all diagrams will be connected. Hence we will rename "fully connected" to "connected" for the rest of these notes; note that we've already been using this nomenclature to refer to the connected S-matrix.

We will now introduce a naive version of perturbation theory which is easier to handle at tree level.

• Working in Schrodinger picture, we suppose that $H = H_0 + f(t)H_{\text{int}}$, where $f(t)$ is chosen so the interaction adiabatically turns on and off. This makes calculations much easier, though we lose generality. E.g. if H_0 is just taken to be the free part of H, then it is impossible to describe the scattering of bound states, which fall apart when the interaction is turned off; however, it is possible if H_0 is chosen appropriately.

• We define an "in" state $\left|i^{\text{in}}\left(t\right)\right\rangle$ to be a state which approaches a free state for $t \to -\infty$,

$$\left|i^{\text{in}}(-\infty)\right\rangle = |i(-\infty)\rangle$$

where $\left|i^{\text{in}}\right\rangle$ evolves by H and $|i\rangle$ evolves by H_0, and i stands for a state specification like 'two one-particle wavepackets with momenta $\mathbf{p}$ and $\mathbf{q}$ approaching each other'.

• We define the S-matrix by

$$\langle f(0)|S|i(0)\rangle = \left\langle f^{\text{out}}\,(0)|i^{\text{in}}\,(0)\right\rangle .$$

Thus, when the particles in the free state $|i(0)\rangle$ are about to meet and pass through each other, S takes them to a superposition of other free particle states in order to reproduce the dynamics of an interaction. This definition is equivalent to our earlier one, though everything is evaluated at time $t = 0$ on the left rather than at $t \to -\infty$.

• Letting U and U_0 be the full and free time evolution operators,

$$\begin{aligned}\left\langle f^{\text{out}}\,(0)|i^{\text{in}}\,(0)\right\rangle &= \left\langle f^{\text{out}}\,(\infty)|U(\infty,-\infty)|i^{\text{in}}\,(-\infty)\right\rangle \\ &= \langle f(\infty)|U(\infty,-\infty)|i(-\infty)\rangle \\ &= \langle f(0)\,|U_0(\infty,0)U(\infty,-\infty)U_0(-\infty,0)|\,i(0)\rangle \\ &= \left\langle f(0)\,\middle|U_I(\infty,0)U_I^\dagger(-\infty,0)\middle|\,i(0)\right\rangle \\ &= \langle f(0)\,|U_I(\infty,-\infty)|\,i(0)\rangle\end{aligned}$$

which shows that $S = U_I(\infty,-\infty)$. In other words, the point of the adiabatic turn-off is that it allows us to get a simple formal expression for the S-matrix itself. We can also show that $[S, H_0] = 0$, so scattering does not change the free particle energy.

Calculations can also be done with the adiabatic approach introduced above.

Example: scalar Yukawa theory

$$\mathcal{L} = |\partial_\mu\psi|^2 + \frac{1}{2}\left(\partial^\mu\phi\right)^2 - M^2|\psi|^2 - \frac{1}{2}m^2\phi^2 - g\psi^\dagger\psi\phi$$

$$S = U_I(\infty,-\infty) = T\exp\left(-ig\int dx\psi^\dagger\psi\phi\right)$$

where the fields above are interaction picture fields.

• We would like to calculate the amplitudes $\langle f(0)|S|i(0)\rangle$. For convenience, we construct the states in interaction picture, i.e. we view $|i(0)\rangle_S = |i(0)\rangle_I$, so that particles in the initial state are also created by interaction picture operators. Since the interaction picture states don't change at all (as the states are free), we drop the $t = 0$ argument.

• Concretely, we call the real particles mesons and the complex particles nucleons, with

$$\phi \sim a + a^\dagger, \quad \psi \sim b + c^\dagger, \quad \psi^\dagger \sim b^\dagger + c$$

where $b^\dagger$ creates a nucleon and $c^\dagger$ creates an antinucleon. Then for meson decay,

$$|i\rangle = \sqrt{2E_{\mathbf{p}}}a^\dagger_{\mathbf{p}}|0\rangle, \quad |f\rangle = \sqrt{4E_{\mathbf{q}_1}E_{\mathbf{q}_2}}b^\dagger_{\mathbf{q}_1}c^\dagger_{\mathbf{q}_2}|0\rangle.$$

• To lowest order in g, we have

$$\langle f|S-1|i\rangle = -ig\left\langle f\left|\int dx\psi^\dagger(x)\psi(x)\phi(x)\right|i\right\rangle$$

This can be simplified using the commutation relations for free field creation and annihilation operators; only one combination of operators gives a nonzero contribution. We find

$$\langle f|S-1|i\rangle = -ig\phi\left(q_1+q_2-p\right)$$

in accordance with the Feynman rules above. To save space, we define

$$\langle f|S-1|i\rangle = \langle f|iT|i\rangle = i\mathcal{M}\phi(\cdots)$$

since the delta function always appears, so that $i\mathcal{M} = -ig$.

• For more complicated processes, we use Wick's theorem, but slightly differently. Since the initial and final states aren't vacuum, we don't care about full contractions; instead we want enough fields left over to turn the initial and final states to vacuum. In this formalism, contractions correspond to only internal lines in diagrams.

Example: nucleon-nucleon scattering, $\psi\psi \to \psi\psi$

$$|i\rangle = |p_1, p_2\rangle = \sqrt{4E_{\mathbf{p}_1}E_{\mathbf{p}_2}}b^\dagger_{\mathbf{p}_1}b^\dagger_{\mathbf{p}_2}|0\rangle, \quad |f\rangle = |q_1, q_2\rangle = \sqrt{4E_{\mathbf{q}_1}E_{\mathbf{q}_2}}b^\dagger_{\mathbf{q}_1}b^\dagger_{\mathbf{q}_2}|0\rangle$$

and the $O(g^2)$ contribution to the S-matrix element is

$$\frac{(-ig)^2}{2}\left\langle f\left|\int dx_1 dx_2 T\left(\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2)\right)\right|i\right\rangle$$

Then the only contraction which contributes is

$$T\left(\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2)\right) \supset :\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2): D_F^0(x_1-x_2)$$

Since all the annihilation operators are moved to the right, we have

$$\left\langle f\left|:\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2):\right|i\right\rangle = \left\langle f\left|\psi^\dagger(x_1)\psi^\dagger(x_2)\right|0\right\rangle\langle 0|\psi(x_1)\psi(x_2)|i\rangle$$

Expanding the fields into creation and annihilation operators, each piece has two terms, giving

$$\frac{(-ig)^2}{2}\int dx_1 dx_2\left(e^{i(q_1x_1+q_2x_2)}+e^{i(q_1x_2+q_2x_1)}\right)\left(e^{-i(p_1x_1+p_2x_2)}+e^{-i(p_1x_2+p_2x_1)}\right)\int\frac{dk i e^{ik(x_1-x_2)}}{k^2-m^2+i\epsilon}$$

Doing the position integrals gives momentum-conserving delta functions, giving

$$i\mathcal{M} = (-ig)^2\left(\frac{i}{(p_1-q_1)^2-m^2}+\frac{i}{(p_1-q_2)^2-m^2}\right)$$

We dropped the $i\epsilon$ because the denominator is never zero; to see this, work in the center of mass frame where $|\mathbf{p}_1| = |\mathbf{q}_1|$. Then the timelike component of $p_1 - q_1$ is zero, so $(p_1-q_1)^2 < 0$. There is also an $O(g^0)$ contribution when $|i\rangle = |f\rangle$, which is not included in $S-1$.

We can generalize this calculation into Feynman rules.

• The Feynman rules for computing $i\mathcal{M}$, valid at tree level, are the following.

• Add internal vertices and lines as specified earlier, giving every line a directed momentum, and conserving momentum at every vertex.

• For each internal line of momentum p and mass m, multiply by $i/(p^2-m^2+i\epsilon)$.

• For every vertex, multiply by $-ig$.

• Divide by the symmetry factor, and sum over tree-level diagrams, excluding the totally disconnected diagram.

For complex scalar fields, we draw arrows on the lines to indicate the flow of particle number, but define the direction of the momentum so that p^0 is positive for external legs. There is no fixed convention for internal lines; there we might as well align the two.

- As another example, for $\psi\bar{\psi} \to \psi\bar{\psi}$, we have

$$i\mathcal{M} = (-ig)^2 \left(\frac{i}{(p_1 - q_1)^2 - m^2} + \frac{i}{(p_1 + p_2)^2 - m^2 + i\epsilon} \right)$$

where the $i\epsilon$ cannot be dropped in the second term, as it diverges when the incoming nucleons form an on-shell meson. As we'll describe in more detail later, this appears as a resonance peak in the scattering cross section.

Note. The calculational formalism we have built up in this chapter is very different from the material one learns in ordinary quantum mechanics. It's important to keep in mind that quantum field theory is in principle a special case of quantum mechanics, so the same techniques still work. E.g. one can calculate cross sections using the standard techniques of time-independent perturbation theory, in a procedure particle physicists call "old-fashioned perturbation theory", but which atomic and optical physicists still commonly use. E.g. one gets expressions like

$$\frac{dP}{dt} \sim |\mathcal{M}|^2, \quad \mathcal{M} \sim \sum_k \frac{\langle f | H_i | k \rangle \langle k | H_i | i \rangle}{E_i - E_k}$$

for a process that occurs at second order. Old-fashioned perturbation theory doesn't require learning complex, abstract new tools such as the LSZ reduction theory. However, it relies on the Hamiltonian and hence breaks Lorentz invariance. In particular, single Feynman diagrams break into multiple diagrams in this formalism, one for each possible time ordering of the interaction vertices, making calculations more complicated. (By combining pairs of such diagrams, one combines the energy denominators $1/(E_i - E_k)$ into Lorentz invariant Feynman propagators.)

Note. Above, we have seen that the connected part of the S-matrix contains a single overall momentum conserving delta function, so the connected amplitude $\mathcal{M}$ contains no delta functions. In our treatment, this followed automatically from the structure of the perturbative expansion, in terms of Feynman diagrams.

More generally, this result can be understood as a requirement of the cluster decomposition principle, the notion that distant experiments should give uncorrelated results. To see this, first note that the connected S-matrix can be defined independently of the dynamics; it is simply the part of the S-matrix, for n-particle scattering, that cannot be constructed from combining S-matrix elements for $(n-1)$-particle scattering and lower. Thus, the connected S-matrix tells us about the part of the scattering that involves all n particles together. Now, the S-matrix is unchanged under translating all of the particles, which is equivalent in momentum space to requiring an overall momentum conserving delta function. If there were further delta functions in the connected amplitude, it would imply that the connected S-matrix was invariant under translating a subset of the particles, i.e. that these particles would continue to have an effect on the others if they were translated arbitrarily far away. Hence this is forbidden by the cluster decomposition principle.

Following Weinberg, the fact that this result follows automatically from a local field description is one of the key motivations for introducing quantum fields at all, rather than focusing on just the particle states.

3.3.8 Physical Observables

We now link our scattering amplitudes to physical quantities. We begin with the potential energy $U(r)$ of two nucleons.

• At the classical level, a delta-function source for the meson field yields

$$\phi(\mathbf{x}) = \frac{e^{-mr}}{4\pi r}$$

where m is the meson mass. (?????)

We would like to think of nucleons as such delta-function sources, so that this gives the potential energy of two nucleons.

• To see this picture at the quantum level, we consider the process $\psi\psi \to \psi\psi$ at tree level. We work in the center of mass frame, so $\mathbf{p} = \mathbf{p}_1 = -\mathbf{p}_2$ and $\mathbf{q} = \mathbf{q}_1 = -\mathbf{q}_2$. Then

$$i\mathcal{M} = ig^2 \left(\frac{1}{(\mathbf{p}-\mathbf{q})^2 + m^2} + \frac{1}{(\mathbf{p}+\mathbf{q})^2 - m^2} \right)$$

On the other hand, we know from nonrelativistic quantum mechanics that the amplitude for this scattering is $\widetilde{U}(\mathbf{p}-\mathbf{q}) + \widetilde{U}(\mathbf{p}+\mathbf{q})$ in the first Born approximation, where we took into account the fact that the ψ particles are identical. Then

$$\widetilde{U}(\mathbf{p}-\mathbf{q}) = \frac{-\lambda^2}{(\mathbf{p}-\mathbf{q})^2 + m^2}, \quad U(\mathbf{r}) = -\frac{\lambda^2}{4\pi r} e^{-mr}$$

where $\lambda = g/2M$ and the extra factors of $2M$ are from the relativistic normalization. To get the result for $U(\mathbf{r})$, we have to work in spherical coordinates and perform a contour integral.

• The exact same logic holds for $\psi\bar{\psi} \to \psi\bar{\psi}$. In this case, the s-channel doesn't contribute because it vanishes in the nonrelativistic limit, as long as $M \gg m$, but the t-channel contributes in the same way, giving the exact same potential. Therefore, the force between a nucleon and antinucleon is also attractive. In general, forces mediated by particles of even spin are universally attractive.

• Similarly, in ϕ^4 theory with $H_{\text{int}} = \lambda\phi^4/4!$ the tree-level amplitude is

$$i\mathcal{M} = -i\lambda, \quad \widetilde{U}(\mathbf{p}) \propto \lambda, \quad U(\mathbf{r}) \propto \delta(\mathbf{r})$$

That is, the potential in ϕ^4 theory has zero range.

• In general, the Born approximation is a useful way of relating the scattering amplitudes we know how to compute to "position-space" quantities such as potentials. E.g. we can use it to relate the magnetic moment of the electron to its scattering amplitude off a fixed background vector potential, which can be computed diagrammatically.

• For $2 \to 2$ scattering, the Mandelstam variables

$$s = (p_1 + p_2)^2, \quad t = (p_1 - q_1)^2, \quad u = (p_1 - q_2)^2$$

appear frequently. We say that $\psi\psi \to \psi\psi$ has t-channel and u-channel diagrams at $O(g^2)$, while $\psi\bar{\psi} \to \psi\bar{\psi}$ has s-channel and t-channel diagrams. If the particles have masses m_i,

$$s + t + u = \sum_i m_i^2$$

Suppose two identical particles scatter by an angle θ in the center-of-mass frame. Then

$$s = 4E_{\text{cm}}^2, \quad t = -2p^2(1 - \cos\theta), \quad u = -2p^2(1 + \cos\theta)$$

In this frame, s determines the energy of each particle, while t and u give directional information.

Next, we turn to the calculation of cross sections and decay rates.

• Let V be the spatial volume and T be the total time separating the asymptotic past and future. Then $\phi(0) = VT$ and $\phi(\mathbf{0}) = V$, as we've seen earlier. We'll need these results because we will encounter squared delta functions, which arise from working in an infinite space.

• By definition, the probability to transition from state $|i\rangle$ (with particles of momentum p_i) to $|f\rangle$ (with particles of momentum q_j) is

$$P = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}$$

$$\langle f|f\rangle = \prod_j 2E_{\mathbf{q}_i}\phi(\mathbf{0}) = \prod_j 2E_{\mathbf{q}_j}V, \quad \langle i|i\rangle = \prod_i 2E_{\mathbf{p}_i}V$$

$$|\langle f|S|i\rangle|^2 = |\mathcal{M}_{fi}|^2\,\phi\left(p_I - p_F\right)^2 = |\mathcal{M}_{fi}|^2\,\phi\left(p_I - p_F\right)VT$$

$$\frac{1}{V}\sum_{\mathbf{p}} \to \int d\mathbf{p}, \quad V\delta_{\mathbf{p},\mathbf{q}} \to \phi(\mathbf{p}-\mathbf{q})$$

$$P = \left(\prod_i \frac{1}{2E_{\mathbf{p}_i}V}\right)VT\int |\mathcal{M}_{fi}|^2\,d\Pi, \qquad d\Pi := \not{\phi}\left(p_F - p_I\right)\prod_j \frac{d\mathbf{q}_j}{2E_{\mathbf{q}_j}}$$

If some of the particles in the final state are identical, the Lorentz invariant phase space $d\Pi$ should contain $1/N!$ factors to avoid overcounting.

• First, consider the decay rate of a particle with mass m. Taking a rate cancels the T. The result is not Lorentz invariant, because of the factor of $1/E_{\mathbf{p}}$, but this is correct because it accounts for time dilation. Usually we will work in the rest frame, where $E_{\mathbf{p}} = m$, giving

$$\Gamma = \frac{1}{2m}\int |\mathcal{M}_{fi}|^2\,d\Pi$$

If there are multiple decay channels, we must also sum over types of final states.

• Next, consider scattering of two particles with energies E_1 and E_2. Then we wish to calculate the differential cross section

$$d\sigma = \frac{\text{differential probability}}{\text{unit time} \times \text{unit flux}} = \frac{1}{4E_1E_2V}\frac{1}{F}\,|\mathcal{M}_{fi}|^2\,d\Pi.$$

The flux F is the number of incoming particles per area per unit time.

• Working in the center of mass frame, the probability per volume is $1/V$ for each initial particle, and the relative velocity is $|\mathbf{v}_1 - \mathbf{v}_2|$, giving

$$d\sigma = \frac{1}{4E_1E_2\,|\mathbf{v}_1 - \mathbf{v}_2|}\,|\mathcal{M}_{fi}|^2\,d\Pi$$

• The denominator is Lorentz-invariant under boosts along $\mathbf{v}_1 - \mathbf{v}_2$, but not others, since the cross section can length contract. To see this, note that for velocities parallel to $\hat{\mathbf{z}}$,

$$E_1E_2\,|\mathbf{v}_1 - \mathbf{v}_2| = |E_2\mathbf{p}_1 - E_1\mathbf{p}_2| = |\epsilon_{\mu xy\nu}p_1^\mu p_2^\nu|$$

Thus, the inverse of this quantity has precisely the Lorentz transformation quantities of the xy component of an antisymmetric rank 2 tensor, which is precisely the geometrical representation of an area element in the xy plane.

• Some other sources define the cross section in a slightly different, Lorentz invariant way,

$$d\sigma = \frac{1}{4v_{\text{rel}}\left(p_1 \cdot p_2\right)} \left|\mathcal{M}_{fi}\right|^2 d\Pi$$

$$v_{\text{rel}} = \sqrt{1 - \frac{m_1^2 m_2^2}{\left(p_1 \cdot p_2\right)^2}}$$

This matches the definition above in the center of mass frame, so it will make no difference for these notes. On the other hand, in situations where the particles may not have opposite momenta (such as in a thermal average) one has to be careful about which definition is used.

• For scattering with two particles in the final state, we are interested in the quantity $d\sigma/d\Omega$, which can be found by writing $d\Pi = (d\Pi/d\Omega)d\Omega$. One can show, in the CM frame,

$$d\Pi = \frac{|\mathbf{q}_1|}{(2\pi)^2\left(4E_{\text{cm}}\right)} d\Omega$$

where $\mathbf{q}_1$ is the final momentum, which gives

$$\frac{d\sigma}{d\Omega} = \frac{1}{4E_1E_2\left|\mathbf{v}_1 - \mathbf{v}_2\right|} \frac{|\mathbf{q}_1|}{(2\pi)^2\left(4E_{\text{cm}}\right)} \left|\mathcal{M}_{fi}\right|^2$$

• For all equal masses, this reduces to

$$\frac{d\sigma}{d\Omega} = \frac{\left|\mathcal{M}_{fi}\right|^2}{64\pi^2 E_{\text{cm}}^2}$$

E.g. we see that s-wave scattering is isotropic, and $\sigma \propto 1/s$ for ϕ^4 theory. This is a general phenomenon for scattering off pointlike sources.

• The traditional unit for a cross section is the typical cross sectional area of a uranium nucleus,

$$1 \text{ barn } = 10^{-28} \text{ m}^2 = 100\text{fm}^2$$

which corresponds to a scatterer whose radius is a few femtometers. For context, the total cross section for proton-proton scattering is of order 100mb, and the instantaneous luminosity of the LHC is about $10\text{nb}^{-1}\ \text{s}^{-1}$. The total integrated luminosity of the LHC so far is about $200\text{fb}^{-1} = 0.2\text{ab}^{-1}$, while the future HL-LHC will accumulate 3ab^{-1}.

Final state with 3 particles

In the CM frame, the momenta $\mathbf{q}_i$ lie in a common plane, called the event plane.

$$x_i := \frac{2E_i}{E_{\text{cm}}}, \qquad x_1 + x_2 + x_3 = 2$$

In these variables, a laborious calculation shows that the phase space takes a very simple form,

$$d\Pi = \frac{E_{\text{cm}}^2}{128\pi^3} dx_1 dx_2,$$

the right-hand side is implicitly supported only over the kinematically allowed region.

• The result is also simple in terms of the invariant masses of pairs:

$$m_{12}^2 = (p_1 + p_2)^2, \quad m_{23}^2 = (p_2 + p_3)^2$$

$$d\Pi = \frac{1}{128\pi^3 E_{\text{cm}}^2} dm_{12}^2 dm_{23}^2$$

• Again, there are implicit and messy theta functions on the right-hand side to mark the kinematically allowed region. Thus, the utility of this parametrization is really on the experimental side. Plotting events in the (m_{12}^2, m_{23}^2) plane yields a Dalitz plot, and features of the plot can be attributed to the scattering amplitude itself, and hence indicate physically interesting effects.

(??? когда-то буду продумывать, см. литературу в теории поля про это, открывал, но не моя это специализация, не до этого было)

3.3.9 1-loop QED: other properties and approaches

(по ПШ мб суть напишу. пока тут, мб потом увеличу раздел.)

Излучение мягких фотонов по ПШ

Электронная вершинная функция: формальная структура по ПШ

Электрон-фотонная вершина: оценка выражения по ПШ

Электрон-фотонная вершина: инфракрасная расходимость по ПШ

Суммирование и интерпретация инфракрасных расходимостей по ПШ

3.4 Renormalization: other well-known methods

3.4.1 Physical Interpretation

We now give some insight into the meaning of renormalization using simple toy models.

• We consider the general problem of fitting a quantum mechanical model to data. We try a candidate class of Hamiltonians $H(\theta)$ with parameters θ, which output predictions $v(\theta)$, such as energy eigenvalues. We then compute the predictions to data to infer θ.

• An important point is that the parameters θ need not have a direct physical meaning. E.g. different models of quarks will have significantly different quark masses in θ, and this is acceptable because quark masses are not physically measurable.

• Parameters that are model-independent and directly measurable by experiment, such as the physical electron mass, are called renormalized or dressed parameters, in contrast to bare parameters. The purpose of renormalization is to reparametrize $H(\theta)$ as H_{ren} (θ_{ren}) so that ideally the θ_{ren} are directly measurable, or at least can be easily inferred from the data.

• As a trivial example, consider a two-state system with

$$H(g, \Lambda) = H_0 + gV_\Lambda, \quad H_0 = \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix}, \quad V_\Lambda = \begin{pmatrix} -1 & \Lambda \\ \Lambda & 0 \end{pmatrix}$$

where Λ plays a role like a UV cutoff. The energy eigenvalues are

$$E_\pm = \frac{1}{2}\left(\omega - g \pm \sqrt{(\omega + g)^2 + 4g^2\Lambda^2}\right)$$

If Λ is a very large number, the outputs $E_\pm$are extremely sensitive to changes in g, indicating a poor choice of parametrization.

• To avoid this problem, we could reparametrize in terms of $E_\pm$, but these quantities are somewhat complicated. A simpler option is to use $g_{\text{ren}} = g\Lambda$, giving the renormalized Hamiltonian

$$H_{\text{ren}}\left(g_{\text{ren}}, \Lambda\right) = \begin{pmatrix} -g_{\text{ren}}\Lambda^{-1} & g_{\text{ren}} \\ g_{\text{ren}} & \omega \end{pmatrix}$$

In this case, the energy levels are less sensitive to g_{ren} for large Λ. The reparametrization is so nice that we can even take the limit $\Lambda \to \infty$ and find a finite result.

• Situations of this form typically arise when there is a naturally large parameter, such as a volume or energy cutoff. Note that taking $\Lambda \to \infty$ is not logically necessary; in the Wilsonian picture we never do this because realistic theories will have some finite cutoff. The point is that after a successful renormalization we have removed the extreme sensitivity to Λ, so we could take $\Lambda \to \infty$ if we wanted to; either way it shouldn't affect the experimental outputs.

Next, we turn to perturbation theory.

• Usually, for practical reasons, we expand perturbatively about a reference Hamiltonian $H_0(\mu)$, where $\mu = \mu(\theta)$. E.g. the decomposition $H = H_0 + gV_\Lambda$ expands about $H_0(\omega)$. The energy levels are

$$E_- = -g - \frac{g^2\Lambda^2}{\omega} + O\left(g^3\right), \quad E_+ = \omega + \frac{g^2\Lambda^2}{\omega} + O\left(g^3\right)$$

• We see that g is a bad expansion parameter because for large Λ, we require $g \ll \Lambda^{-1}$ instead of the typical $g \ll 1$. Alternatively, it is impossible to expand in g at all in the limit $\Lambda \to \infty$.

• On the other hand, after renormalization and taking the limit $\Lambda \to \infty$, we have

$$E_- = -\frac{g_{\text{ren}}^2}{\omega} + O\left(g_{\text{ren}}^4\right), \quad E_1 = \omega + \frac{g_{\text{ren}}^2}{\omega} + O\left(g_{\text{ren}}^4\right)$$

which is a perfectly well-behaved series. Thus the quality of the perturbation series depends on the parametrization θ.

• As another example of this phenomenon, consider the Taylor series of the logarithm,

$$\log(1+g) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} g^k$$

This only converges for $|g| < 1$. But if we expand in terms of $z = g/(g+2)$, we have

$$\log(1+g) = \log\left(\frac{1+z}{1-z}\right) = \sum_{k=0}^{\infty} \frac{2}{2k+1} z^{2k+1}$$

which converges for $|z| < 1$ and hence for all $\operatorname{Re} g > -1$, much faster than the original series. The singularity at $g = -1$ remains at $z = -1$, but all $\operatorname{Re} g > -1$ are 'sucked into' the region $|z| < 1$.

• The quality of the perturbation series also depends on the choice $H_0(\mu)$ as well as the specific value of μ. Since the total Hamiltonian is always the same, these choices do not affect the exact results, but they can affect the accuracy of a truncated series expansion. One might worry that this reduces predictivity; thus a good choice of μ is one which minimizes the sensitivity of the results to changes in μ.

• As an example, consider finding $x(t)$ for a particle in the potential $V(x) = ax^2 + \epsilon x^4$ at a large time $t = T$, so that T is the large parameter. Then $x(T)$ depends very sensitively upon the bare parameters a and b. Now suppose we attempt a perturbative expansion. Taking $H_0 = p^2/2m$ is clearly a very poor choice; a more reasonable choice is $H_0 = p^2/2m + ax^2$. However, expanding in ϵ gives a bad perturbation series; the terms in the series depend on T and diverges for $T \to \infty$.

• The solution is to work in terms of a renormalized frequency. We set

$$H_0 = \frac{p^2}{2m} + cx^2, \quad c = a + \sum_n c_n \epsilon^n$$

so that H_0 describes a harmonic oscillator with the same period as the anharmonic oscillator. Then if we perform a first-order expansion in ϵ, we can set the 'counterterm' c_1 so that the first-order correction is small for all T. Similarly to perform a second-order expansion we would have to also determine c_2.

• The sensitivity to T has been reduced; changing variables from (a, ϵ) to (c, ϵ'), where c is the renormalized frequency and ϵ describes the amount of anharmonicity. The result $x(T)$ depends sensitively on ϵ, but not on ϵ'.

О нерелятивистском пределе (!?!?!?)

(важный переход туда, укажу тут это! мб в другие переходы тоже тут напишу, нпр в квантовую оптику, пока сам это не понял.)

3.4.2 Power Counting

In this section, we explore the systematics of perturbative renormalization. We begin by estimating the degree of divergence of a Feynman diagram.

• We consider a scalar field theory with a ϕ^n interaction. Consider a connected diagram with V such vertices, I internal propagators, E external lines, and L loops. Euler's formula gives

$$L = I - V + 1$$

while counting the number of edges in two different ways gives

$$nV = 2I + E$$

• The momentum integral will contain Ld^dk integrals and I factors of $1/k^2$, so we define the superficial degree of divergence

$$D = dL - 2I$$

We estimate $\mathcal{M} \sim \Lambda^D$, so diagrams with $D > 0$ are superficially divergent, diagrams with $D = 0$ are superficially log-divergent, and diagrams with $D < 0$ are superficially finite. Note that for particles with half-integer spin, propagators would instead contribute as $1/k$.

• Superficially divergent diagrams can actually be finite due to a symmetry. Moreover, superficially finite diagrams may diverge: the superficial estimate assumes that all of the loop momenta are of the same order Λ, but divergences can come from regions where only some of the loop momenta are large. One can show, rather tediously, that all such divergences come from superficially divergent subdiagrams.

• Rewriting D in terms of V and E, we have

$$D = d - \left(d - n\frac{d-2}{2}\right)V - \frac{d-2}{2}E$$

Note that the quantity in brackets is simply the mass dimension of the ϕ^n coupling, while the coefficient of E is simply the mass dimension of the field. This generalizes straightforwardly to multiply types of vertices and fields.

• We see in general that more external lines decreases D, while more internal vertices increases or decreases D depending on the dimension of V. We say that a theory is (power-counting)

• renormalizable if the number of superficially divergent amplitudes is finite, but superficial divergences appear at every order in perturbation theory,

• superrenormalizable if the number of superficially divergent diagrams is finite,

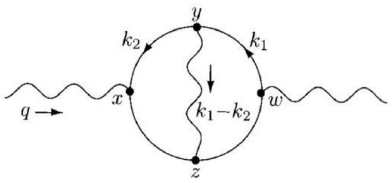
• nonrenormalizable if the number of superficially divergent amplitudes is infinite.

In general, the couplings in a renormalizable or superrenormalizable theory all have nonnegative mass dimensions, while in a superrenormalizable theory the dimensions are all strictly positive. A single coupling with negative mass dimension renders a theory nonrenormalizable.

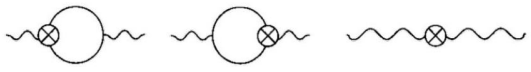
• Ignoring the issue of divergent subdiagrams, if a theory is (power-counting) renormalizable, then at every order in perturbation theory only a finite number of divergent diagrams appear, parametrized by a finite number of divergent constants.

• The BPHZ theorem ensures that divergent subdiagrams cannot change these conclusions; the divergences there are simply canceled by "counterterm subdiagrams", requiring no new counterterms to be introduced. The proof is by giving an explicit algorithm to perform this cancellation, and it is nontrivial because of the possibility of "overlapping divergences", where two divergent subdiagrams overlap, and hence their loop integrals can't be treated separately.

Note. A glimpse of higher-order renormalization. Consider the following two-loop diagram in QED.



This diagram contains two overlapping divergences. It is canceled by the counterterm diagrams shown below.



When k_2 is large, the points x, y, and z are "close together", so the divergence is canceled by the first counterterm diagram. Similarly, the second counterterm diagrams accounts for the divergence where k_1 is large, while the third accounts for the divergence where both k_1 and k_2 are large. In DR, such a divergence appears as a double pole in ϵ.

As an example, we apply power counting to QED.

• In QED, the interaction term is $e\bar{\psi}A\psi$ and the superficial degree of divergence of an amplitude with E_γ external photons and E_e external electrons is

$$D = 4 - E_\gamma - \frac{3}{2}E_e$$

since $[e] = 0$. The theory is hence renormalizable.

• The amplitudes with $D \geq 0$, that don't vanish automatically by Lorentz invariance, are:

• The vacuum energy with $D = 4$.

• The photon tadpole with $D = 3$, and the three-photon amplitude with $D = 1$.

• The four-photon amplitude with $D = 0$.

• The photon propagator with $D = 2$, the electron propagator with $D = 1$, and the renormalized vertex with $D = 0$.

Only the amplitudes in the final group are important, as we'll justify below.

• The vacuum energy is divergent, and we can absorb it with a vacuum energy counterterm. Since it has no observable consequences, we typically ignore it entirely by normal ordering.

• The one-photon and three-photon amplitudes all vanish by symmetry. QED is invariant under charge conjugation, $j^\mu \to -j^\mu$, which requires $A^\mu \to -A^\mu$. Therefore a correlation function of an odd number of photons vanishes, a result known as Furry's theorem. Diagrammatically, each diagram cancels against another with all of the electron loops reversed.

• The four-photon amplitude is finite by the Ward identity. To see this, let the amplitude be $A^{\mu\nu\rho\sigma}$. By the Ward identity, $k_\mu A^{\mu\nu\rho\sigma} = 0$, which implies that the amplitude must be proportional to something like $g^{\mu\nu}k^\sigma - g^{\mu\sigma}k^\nu$.

• To show this more explicitly, the divergent part of one diagram is

$$A^{\mu\nu\rho\sigma} \sim \int dk \frac{\mathrm{tr}\left(k\gamma^\mu \not{k}\gamma^\nu \not{k}\gamma^\rho \not{k}\gamma^\sigma\right)}{k^8}$$

and there are five more diagrams corresponding to the different ways to attach the photons to the electron loop. Using spherical symmetry, we simplify the integral to

$$A^{\mu\nu\rho\sigma} \sim \int dk \frac{\mathrm{tr}\left(\gamma^{\alpha_1}\gamma^\mu\gamma^{\alpha_2}\gamma^\nu\gamma^{\alpha_3}\gamma^\rho\gamma^{\alpha_4}\gamma^\sigma\right)}{k^4}\left(\eta_{\alpha_1\alpha_2}\eta_{\alpha_3\alpha_4} + \text{ perms. }\right)$$

which can be simplified using the standard trace identities. Finally, symmetrizing over $\mu\nu\rho\sigma$ to account for the six diagrams yields zero.

• Pulling this factor out leaves an amplitude with $D = -1$, which is hence finite. One might naively think this means the amplitude is proportional to $1/\Lambda$ and hence vanishes, but the four-photon vertex is nonzero; it mediates light-by-light scattering. The point is that D only gives us the UV dependence of the amplitude on Λ, while the finite contribution comes from the IR part of the loop integral.

• Finally, we arrive at the propagators. In our one-loop analysis of QED we found they were only logarithmically divergent, and this is indeed guaranteed by symmetries as we see below. For the degree of divergence, it suffices to look at the self energies.

• The electron self energy takes the form

$$A_0 + A_1 \not{p} + A_2 p^2 + ...$$

and has mass dimension 1. Then the only place a linear divergence can appear is in the A_0 coefficient. Now, A_0 is an analytic function in the electron mass,

$$A_0 \sim \Lambda + m_e \log(...) + \frac{m_e^2}{\Lambda}\log(...) + ...$$

When $m_e = 0$ the theory has an axial symmetry, by opposite rotations of the left and righthanded Weyl spinors; then a mass term cannot be generated by renormalization. Thus A_0 has no term independent of m_e and the divergence is at most logarithmic in Λ.

• The photon self energy has mass dimension 2 and takes the form

$$\Pi^{\mu\nu} \sim B_0\eta_{\mu\nu} + B_2 q_\mu q_\nu + B_2'\eta_{\mu\nu}q^2 + ...$$

where quadratic divergences can come from B_0. But we saw that the Ward identities required

$$\Pi^{\mu\nu} = \left(\eta^{\mu\nu}q^2 - q^\mu q^\nu\right)\Pi\left(q^2\right)$$

where $\Pi\left(q^2\right)$ is regular in q. Then we must have $B_0 = 0$.

• These results ensure that the divergences of QED can be absorbed by renormalizing terms already present in the Lagrangian. E.g. if B_0 had been nonzero, we would have had a quadratic divergence independent of q in $\Pi^{\mu\nu}$. This could only be absorbed by adding a term $A_\mu A^\mu$ to the Lagrangian, which would break gauge invariance. Similarly, if the four-photon amplitude had been divergent, we would have had to add the term $\left(A_\mu A^\mu\right)^2$. Of course, these terms would appear, and be necessary, in QED with a massive "photon".

• The more general lesson here is that, in order to absorb all divergences, a renormalizable theory must contain all renormalizable terms that are allowed by symmetry, and no more.

Note. The logic above requires that the divergent part of an amplitude be written as a polynomial in external momenta, with divergent coefficients. To see this in general, note that we may simply differentiate any superficially divergent loop integral $D+1$ times with respect to external momenta to get a convergent integral (ignoring the issue of divergent subdiagrams), which has a finite result. Integrating $D+1$ times produces a finite piece plus the advertised divergent polynomial. The finite piece contains all the analytically nontrivial dependence on the external momenta, such as the logarithms.

3.4.3 Renormalization of ϕ^4 Theory

We perform renormalization in ϕ^4 theory with a hard cutoff. We will consider the renormalization of the propagator and the ϕ^4 coupling. Equivalently, we are computing the quadratic and quartic terms in the 1PI effective action.

• We consider the action

$$S_{\Lambda_0}[\phi] = \int dx \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

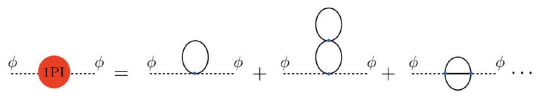
where we are working in four dimensions and Euclidean signature. We work perturbatively, which means we are near the Gaussian fixed point, so λ is marginal.

• The exact propagator/connected two-point function in momentum space is

$$\Delta\left(k^2\right) = \int dx e^{ikx}\langle\phi(x)\phi(0)\rangle_c = \frac{1}{k^2 + m^2 - \Pi\left(k^2\right)}$$

which $\Pi\left(k^2\right)$ is the sum of 1PI diagrams, by the usual geometric series argument.

• The terms in $\Pi\left(k^2\right)$ up to two loops are shown below.



Here, a dotted line denotes an amputated propagator, i.e. they are not associated with $1/\left(p^2+m^2\right)$ factors. At one loop, the contribution is simply

$$\Pi\left(k^2\right) \supset -\frac{\lambda}{2}\int\limits_{|p|\leq\Lambda_0}\frac{dp}{p^2+m^2} = -\lambda\frac{\mathrm{Vol}\left(S^3\right)}{2(2\pi)^4}\int\limits_0^{\Lambda_0}\frac{p^3dp}{p^2+m^2} = -\frac{\lambda}{2}\frac{m^2}{16\pi^2}\int\limits_0^{\Lambda_0^2/m^2}\frac{udu}{1+u}$$

Here, we used $\mathrm{Vol}\left(S^3\right)=2\pi^2$ and let $u=p^2/m^2$. We see the appearance of the generic $1/16\pi^2$ "loop factor" that appears in four dimensions. Explicitly performing the integral gives

$$\Pi\left(k^2\right) \supset -\frac{\lambda}{32\pi^2}\left(\Lambda_0^2 - m^2\log\left(1+\frac{\Lambda_0^2}{m^2}\right)\right)$$

which has a quadratic and a logarithmic divergence.

• To "cancel" these divergences, we "add counterterms" to the action, subject to the same caveats as before. That is, we work with the action

$$S_{\Lambda_0}[\phi] + \hbar S^{\mathrm{CT}}\left[\phi,\Lambda_0\right], \quad S^{\mathrm{CT}}\left[\phi,\Lambda_0\right] = \int dx\frac{1}{2}\delta Z(\partial\phi)^2 + \frac{1}{2}\delta m^2\phi^2 + \frac{1}{4!}\delta\lambda\phi^4$$

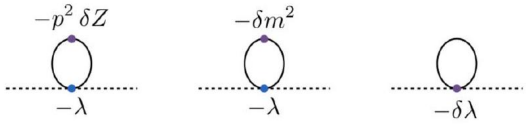
Note that this is rather different from what we did earlier, where we started from the low-energy effective action we wanted and added counterterms to get the high-energy action. At this point, the terms S_{Λ_0} and S^{CT} have no physical meaning individually.

• At $O(\hbar)$, the counterterms contribute two diagrams,

$$\Pi^{1\text{ loop}}\left(k^2\right) = -k^2\delta Z - \delta m^2 - \frac{\lambda}{32\pi^2}\left(\Lambda_0^2 - m^2\log\left(1+\frac{\Lambda_0^2}{m^2}\right)\right)$$

and we must choose δZ and δm^2 so the result is finite.

• At two loops, the corresponding counterterm diagrams have one loop, as shown.



• Since the counterterms guarantee we get finite results, we may take the continuum limit $\Lambda_0 \to \infty$ while fixing our action to match experiment. This is done by requiring the exact propagator have a pole at $p^2=-m_{\mathrm{phys}}^2$ with residue one, where m_{phys} is the physical mass, which is measured by 'weighing' a stable particle, or more generally by looking for peaks in cross sections for processes involving the exchange of the particle.

• Quantitatively, this means we require

$$\Pi\left(-m_{\mathrm{phys}}^2\right) = m^2 - m_{\mathrm{phys}}^2, \quad \left.\frac{\partial\Pi}{\partial k^2}\right|_{k^2=-m_{\mathrm{phys}}^2} = 0$$

Note that these are only constraints on the total action; we are still free to pick how it is split into S_{Λ_0} and S^{CT}. In the on-shell scheme, we choose S_{Λ_0} so that the parameter m is equal to the physical mass, so

$$\Pi\left(-m^2\right) = 0, \quad \left.\frac{\partial \Pi}{\partial k^2}\right|_{k^2=-m^2} = 0$$

Hence, in the on-shell scheme S_{Λ_0} does have a physical meaning, being roughly equivalent to the low-energy effective action encountered in Wilsonian renormalization. However, in other schemes S_{Λ_0} has no direct interpretation.

• In our case, at one loop we have

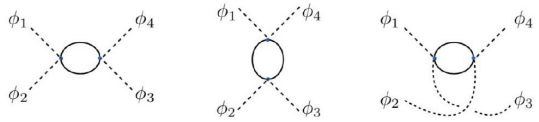
$$\delta Z = 0, \quad \delta m^2 = -\frac{\lambda}{32\pi^2}\left(\Lambda_0^2 - m^2 \log\left(1 + \frac{\Lambda_0^2}{m^2}\right)\right)$$

so that the one loop correction exactly vanishes. This is rather special; the reason it happened is that the loop is a tadpole, which doesn't depend on the external momenta at all. The tadpole only renormalizes the mass, so its effects may be entirely canceled by a counterterm. One can reformulate perturbation theory so that all tadpoles are set to zero from the start, reflecting the fact that they don't affect the physics.

• More generally, a loop will generate an infinite series of terms, such as $\phi^2, (\partial\phi)^2$, and all higher derivatives, though the higher derivative terms will be suppressed by powers of Λ_0 rather than divergent. The on-shell scheme only fixes Π and its derivative at a single point, so loop effects can be seen away from this point, e.g. in virtual particle exchange.

Next, we turn to the renormalization of the quartic coupling, which we take to mean the quartic part of the 1PI effective action.

• The one-loop corrections come from the graphs below.



Letting the external momenta k_i all flow into the diagram, the diagrams sum to

$$\frac{\lambda^2}{2}\int^{\Lambda_0} \frac{dp}{p^2+m^2}\left(\frac{1}{(p+k_1+k_2)^2+m^2} + \frac{1}{(p+k_1+k_4)^2+m^2} + \frac{1}{(p+k_1+k_3)^2+m^2}\right).$$

Since the external momenta also flow through the loop, we generate an infinite series of derivative terms, as anticipated above.

• The contribution to the pure quartic coupling λ comes from the momentum-independent part,

$$\frac{3\lambda^2}{2}\int^{\Lambda_0} \frac{dp}{(p^2+m^2)^2} = \frac{3\lambda^2}{32\pi^2}\left(\log\left(1+\frac{\Lambda_0^2}{m^2}\right) - \frac{\Lambda_0^2}{\Lambda_0^2+m^2}\right)$$

which is logarithmically divergent. One possible choice for the counterterm is

$$\delta\lambda = \frac{3\lambda^2}{32\pi^2}\left(\log\frac{\Lambda_0^2}{m^2} - 1\right)$$

The quartic term in the 1PI effective action is the negative of the sum of 1PI diagrams, so

$$\lambda_{\text{eff}} = \lambda - \frac{3\hbar\lambda^2}{32\pi^2}\left(\log\left(1+\frac{m^2}{\Lambda_0^2}\right) + \frac{m^2}{m^2+\Lambda_0^2}\right)$$

This choice ensures that in the continuum limit, $\lambda_{\text{eff}} = \lambda$, in the spirit of the on-shell scheme.

• The contributions for terms with more derivatives come in a power series in k^2/Λ_0^2, and are hence finite, vanishing in the continuum limit $\Lambda_0 \to \infty$.

• The integrals here are essentially the same as in Wilsonian renormalization. There, we integrated loops over momenta in $[\Lambda, \Lambda_0]$, getting a renormalized coupling constant. Here, we are integrate over momenta in $[0, \Lambda_0]$ in essentially the same expressions. Hence in the limit $\Lambda_0 \to \infty$ the functional dependence is the same. This mean we know, e.g. that a marginal coupling gets logarithmically divergent corrections.

• While the above is true at one loop, matters are complicated at higher loops due to divergent subdiagrams, and we need the BPHZ theorem to ensure the divergences don't get worse than expected. But the Wilsonian intuition makes it clear that the BPHZ procedure must work.

• We now do a full calculation to see the derivative terms explicitly:

$$\frac{1}{(p+k_{12})^2+m^2}\frac{1}{p^2+m^2} \equiv \int_0^1 \frac{dx}{\left[x\left((p+k_{12})^2+m^2\right)+(1-x)\left(p^2+m^2\right)\right]^2}$$

so completing the square gives the loop integral

$$\frac{\lambda^2}{2}\int_0^1 dx \int \frac{d\ell}{\left(\ell^2+M^2\right)^2}, \quad \ell = p - xk_{12}, \quad M^2 = m^2 + x(1-x)k_{12}^2$$

Note that integrating over the region $|p| \leq \Lambda_0$ is the same as integrating over the region $|\ell| \leq \Lambda_0$ up to error terms of order $|k_{12}|/\Lambda$, which vanish in the continuum limit $\Lambda_0 \to \infty$. For simplicity, we discard these terms now.

• Working in spherical coordinates, we get a factor of $\text{Vol}\left(S^3\right)$ for

$$\pi^2 \int_0^1 dx \int_0^{\Lambda_0^2} \frac{\ell^2 d\left(\ell^2\right)}{\left(\ell^2+M^2\right)^2} = \pi^2 \int_0^1 dx \log\frac{\Lambda_0^2+M^2}{M^2} + \frac{M^2}{\Lambda_0^2+M^2} - 1$$

Keeping only terms that survive in the continuum limit, our three loop diagrams yield

$$\frac{\lambda^2}{32\pi^2}\int_0^1 dx \log\frac{\Lambda_0^2}{m^2-x(1-x)s} + \log\frac{\Lambda_0^2}{m^2-x(1-x)t} + \log\frac{\Lambda_0^2}{m^2-x(1-x)u} - 3$$

$$s := -\left(k_1+k_2\right)^2, \qquad t := -\left(k_1+k_4\right)^2, \qquad u := -\left(k_1+k_3\right)^2$$

• Using the same choice of counterterm as in our rough calculation above, the quartic term in the effective action is

$$\mathcal{A}\left(k_i\right) = \lambda + \frac{\hbar\lambda^2}{32\pi^2}\int_0^1 dx \log\left(1-\frac{x(1-x)s}{m^2}\right) + \log\left(1-\frac{x(1-x)t}{m^2}\right) + \log\left(1-\frac{x(1-x)u}{m^2}\right)$$

Since this expression is in momentum space, expanding to get powers of the Mandelstam variables yields derivative terms. As expected, these terms correspond to no UV divergences, since they are irrelevant. But they are nonzero, because they are generated by the ϕ^4 coupling.

• At this point, we can compute everything else using just tree-level diagrams. E.g. the amplitude for ϕ^4 scattering only contains one diagram, the quartic vertex, which is $-\delta^4\Gamma/\delta\phi^4 = \mathcal{A}(k_i)$. Similarly for ϕ^6 scattering we have

so we only need the ϕ^2, ϕ^4, and ϕ^6 terms in $\Gamma[\phi]$.

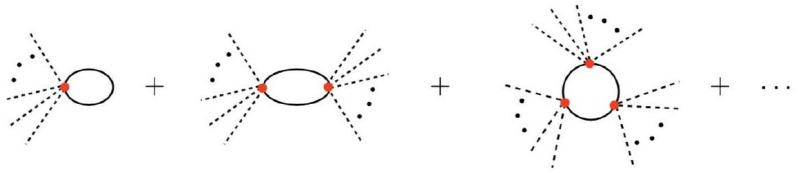
Note. A subtlety with getting a "pure coupling" is that with a hard cutoff, the effective action in momentum space is not analytic! Concretely, if the coupling is $f(k)$, then RG modifies it by an analytic function times step functions in the external momenta. This was not a problem here, but in general, computing $f(0)$ by taking zero external momenta will not give the same result as computing $f(k)$ for nonzero k, expanding in a Taylor series in k, and taking the constant term. The proliferation of step functions is one reason the hard cutoff is not practical.

Another fact we know from the Wilsonian picture is that irrelevant couplings in the high-energy theory should not affect anything at low energies. We can see this explicitly by including irrelevant couplings from the start.

• In this case, we take the action

$$S_{\Lambda_0}[\phi] = \int dx \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + V(\phi), \quad V(\phi) = \sum_{k\geq 2} g_{2k}\Lambda_0^{4-2k}\phi^{2k}$$

We would like to compute the ϕ^{2m} coupling in the 1PI effective action. At one loop, all such diagrams have the form below.



This is merely the $O(\hbar)$ contribution to the effective action we've seen in the zero-dimensional case; it can also be found by expanding the functional determinant $\det\left(-\nabla^2 + m^2 + V''(\phi)\right)^{-1/2}$.

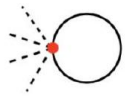
• We already know that the derivative terms are suppressed by powers of k/Λ_0, so we focus on the renormalization of the ϕ^{2m} coupling. A one-loop graph with e propagators contributes

$$\int dp \prod_{j=1}^{e} \frac{1}{(p+K_j)^2 + m^2} \sim \int \frac{dp}{p^{2e}} \sim \begin{cases} \Lambda_0^2 & e = 1 \\ \log\Lambda_0 & e = 2 \\ \text{finite but nonzero} & e > 2 \end{cases}$$

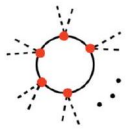
For each vertex ϕ^{2k+2}, we also pick up a factor of $\Lambda_0^{2(1-k)}$.

• Therefore, the only divergent one-loop diagrams are the mass and quartic vertex diagrams,

which we've already considered. Almost all other one-loop diagrams vanish in the continuum limit, but we find finite contributions from



which has no new effect, since we are already setting the ϕ^4 coupling with a counterterm, and



where all of the vertices are ϕ^4. That is, in the continuum limit the effective strength of the irrelevant interactions is totally fixed by the $\lambda\phi^4$ coupling, while the original high-energy irrelevant couplings play no role at all, just as in the Wilsonian picture.

Note. This result is very different from what we got by naive power counting, where we concluded that a single irrelevant coupling made every amplitude diverge. The difference is that there, we fixed the value of the dimensionful coupling. This corresponds to thinking about an effective field theory (e.g. the dimensionful masses should be fixed to match experiment), and indeed one cannot take the continuum limit of an effective field theory, and one should require infinitely many counterterms.

By contrast, here we've been thinking about S_{Λ_0}, the action of a fundamental theory. Then we've been fixing the dimensionless couplings, since they should all be presumably $O(1)$, and there is no issue with irrelevant couplings.

Finally, we repeat our calculations using DR and $\overline{\text{MS}}$.

• First, we consider the mass renormalization. We define the dimensionless coupling $g(\mu)$ by

$$\lambda = \mu^{4-d} g(\mu)$$

where μ is an arbitrary mass scale. Then the one-loop correction to $\Pi\left(k^2\right)$ is

$$\Pi\left(k^2\right) \supset -\frac{1}{2} g(\mu)\mu^{4-d} \int \frac{dp}{p^2+m^2} = -\frac{g(\mu)m^2}{2(4\pi)^{d/2}} \left(\frac{\mu}{m}\right)^{4-d} \Gamma\left(1-\frac{d}{2}\right)$$

by our usual DR formulas, with some extra signs due to the Euclidean signature.

• Setting $d = 4-\epsilon$ and expanding as usual, we find

$$\frac{g(\mu)m^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log\left(\frac{4\pi\mu^2}{m^2}\right) - 1\right) + O(\epsilon)$$

as $\epsilon \to 0^+$. In MS we choose the counterterm

$$\delta m^2 = -\frac{g(\mu)m^2}{16\pi^2\epsilon}$$

to remove the divergence only, while in $\overline{\text{MS}}$ we choose

$$\delta m^2 = -\frac{g(\mu)m^2}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + \log 4\pi\right), \quad \Pi_{1\text{ loop}}\left(k^2\right) = \frac{g(\mu)m^2}{32\pi^2}\left(\log\frac{\mu^2}{m^2} - 1\right)$$

The one-loop mass correction now depends on μ, but this is perfectly acceptable, as μ is part of the specification of the renormalization scheme. If we sum the corrections to all orders, μ must

drop out, since we should just arrive back at the pole mass. Note that the pole mass is no longer equal to m, as we're not using the on-shell scheme.

• Next, we turn to the quartic coupling, where the 1PI diagrams are

$$\frac{g^2\mu^{4-d}}{2}\int\frac{d^dp}{(p^2+m^2)\left((p+k_1+k_2)^2+m^2\right)}+2\text{ other channels.}$$

We set the external momenta k_i to zero to focus on the pure quartic coupling, giving

$$\frac{3g^2}{2(4\pi)^{d/2}}\left(\frac{\mu}{m}\right)^{4-d}\Gamma\left(2-\frac{d}{2}\right)=\frac{3g^2}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma+\log\frac{4\pi\mu^2}{m^2}\right)+O(\epsilon)$$

Then in the $\overline{\text{MS}}$ scheme we have

$$\delta\lambda=\frac{3g^2}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma+\log 4\pi\right),\quad g_{\text{eff}}(\mu)=g(\mu)-\frac{3\hbar g^2(\mu)}{32\pi^2}\log\frac{\mu^2}{m^2}+O\left(\hbar^2\right)$$

where we restored $\hbar$ to emphasize the one-loop effect is quantum.

• If we work to all orders, the dependence of $g_{\text{eff}}(\mu)$ on μ must drop out, because it's just the ϕ^4 coefficient of the effective action. Therefore its derivative is zero,

$$\frac{dg_{\text{eff}}}{d\log\mu}=0$$

This is a "continuum RG" equation, not to be confused with a Wilsonian RG equation.

• Differentiating our result and keeping only the lowest order terms, we have

$$\beta(g)=\frac{dg}{d\log\mu}=\frac{3\hbar g^2}{16\pi^2}+O\left(\hbar^2\right)$$

where the classical term, independent of $\hbar$, vanishes since the ϕ^4 coupling is marginal. Therefore, at one loop the coupling is marginally irrelevant.

• Separating and integrating, we find

$$\frac{1}{g\left(\mu'\right)}=\frac{1}{g(\mu)}+\frac{3\hbar}{16\pi^2}\log\frac{\mu}{\mu'},\quad g(\mu)=\frac{16\pi^2}{3\hbar}\frac{1}{\log\left(\Lambda_{\phi^4}/\mu\right)}$$

at least at one-loop order. The fact that a dimensionless initial condition can be exchanged for a dimensionful scale, Λ_{ϕ^4}, where the coupling diverges, is called dimensional transmutation.

Note. Interpreting the running coupling $g(\mu)$. The amplitude to all orders can't depend on μ, but changing μ modifies each of the terms. Generically, we have a perturbation series in $g(\mu)\log\left(\mu^2/E^2\right)$ where E is the energy scale of the process. Thus we can significantly improve the convergence of the series if we take $\mu\approx E$. In particular, when $\mu\approx E$ the sum of the series g_{eff} is well-approximated by its first term $g(\mu)$. Thus $g(\mu)$ estimates the strength of interactions at energy scale μ.

Previously, we used the on-shell scheme, where the coupling e was fixed. Using this fixed coupling we computed the one-loop amplitude for $e^+e^-\to e^+e^-$by exchange of a photon with momentum q, and found it depended on q^2, yielding a running coupling. While the particulars were different, in both cases the running coupling referred to the fact that the vertices in the 1PI effective action are momentum-dependent.

Note. The energy scale μ is not to be interpreted as a cutoff, but it does have a relationship. Note that DR integral measures take the form $\int d^{4-\epsilon}p\mu^\epsilon$. For positive ϵ, we have more momentum

modes at energies less than μ, and less at energies greater than μ. DR removes a greater and greater fraction of the momentum modes as their energy increases, acting like a smooth cutoff.

Of course, there's more to DR than just that, because DR is defined in more subtle way that preserves symmetries, but this can lead to some useful intuition. Note that the density of momentum modes starts to be significantly changed when $(p/\mu)^\epsilon \sim 1 + \epsilon \log(p/\mu)$ is significantly greater than one, which corresponds to $\epsilon \log(p/\mu) \sim 1$. Treating this momentum scale as a UV cutoff, $p \sim \Lambda$, we have $1/\epsilon \sim \log(\Lambda/\mu)$. This is the reason $1/\epsilon$ poles in DR correspond to logarithmic divergences in cutoff regularization.

3.4.4 Renormalization of Quantum Electrodynamics

We work in Euclidean signature.

$$S[A,\psi] := \int d^d x \frac{1}{4e^2} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(\not{D} + m)\psi, \quad \not{D} = \not{\partial} + i\not{A}$$

Under these conventions we have

$$[e] = \frac{4-d}{2}, \quad [A] = 1$$

in general dimensions, so the kinetic term is irrelevant when $d < 4$. Then the only relevant and marginal terms are topological ones, such as the Chern-Simons term, and this normalization is convenient for highlighting this fact: irrespective of normalization, the topological terms dominate the kinetic ones.

• However, when we do practical calculations for QED in $d = 4$, we would like the photon field to be canonically normalized. This is done by multiplying it by e^{-1}, giving

$$S[A,\psi] = \int d^d x \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(\not{D} + m)\psi + ie\bar{\psi} A \psi, \quad \not{D} = \not{\partial} + ie\not{A}$$

so that the photon field now has the usual dimension $[A] = (d-2)/2$ and the kinetic term is marginal in all dimensions. There is no contradiction: while being marginal or relevant depends on the normalization, the Chern-Simons term is more relevant in all normalizations. We'll use this convention below and suppress the d dependence.

• The Clifford algebra is modified since the metric is, giving

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad (\gamma^\mu)^\dagger = -\gamma^\mu, \quad \operatorname{tr} \gamma^\rho \gamma^\sigma = 4\delta^{\rho\sigma}$$

so that γ^0 is no longer distinguished. Then the boost generators

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

are Hermitian, reflecting the fact that $SO(d)$ is compact, and the action is real, and

$$\operatorname{tr}(\gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma) = 4(\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\nu}\delta^{\rho\sigma} + \delta^{\mu\sigma}\delta^{\nu\rho})$$

• To find the classical photon propagator, we expand the kinetic term in momentum space, for

$$\begin{aligned}\frac{1}{4}\int dx F^{\mu\nu}F_{\mu\nu} &= \frac{1}{4}\int dk(-i)\left(k^\mu \tilde{A}^\nu(-k) - k^\nu \tilde{A}^\mu(-k)\right) i\left(k_\mu \tilde{A}_\nu(k) - k_\nu \tilde{A}_\mu(k)\right) \\ &= \frac{1}{2}\int dk k^2 \left(\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\tilde{A}_\mu(-k)\tilde{A}_\nu(k).\end{aligned}$$

The propagator is the inverse of the kinetic term, and one possible inverse is

$$\Delta^0_{\mu\nu}(k) = \frac{1}{k^2}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

The other possibilities are indexed by the ξ parameter in R_ξ gauge. Here we have chosen Landau/Lorenz gauge, $\partial^\mu A_\mu(x) = 0$ or equivalently $k^\mu \Delta^0_{\mu\nu}(k) = 0$.

• Next, we define the exact photon propagator

$$\Delta_{\mu\nu}(k) = \int dx e^{ikx} \langle A_\mu(x) A_\nu(0)\rangle$$

and by the usual argument, denoting the sum of 1PI diagrams as Π^ρ_σ,

$$\Delta_{\mu\nu}(k) = \Delta^0_{\mu\nu}(k) + \Delta^0_{\mu\rho}(k)\Pi^\rho_\sigma(k)\Delta^{0\sigma}{}_\nu + ...$$

• Using the Ward identity, we can show that

$$\Pi_{\mu\nu}(k) = k^2\pi\left(k^2\right)P_{\mu\nu}, \quad P_{\mu\nu} = \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)$$

The quantity in brackets is just an idempotent projection operator, $k^\mu P_{\mu\nu} = 0$, so the series sums to

$$\Delta_{\mu\nu}(k) = \frac{\Delta^0_{\mu\nu}(k)}{1 - \pi\left(k^2\right)}$$

• Since $\Delta_{\mu\nu}(k)$ is the inverse of the quadratic term in the 1PI effective action S_{eff} ,

$$S^{(2)}_{\text{eff}}[\tilde{A}] = \frac{1}{2}\int dk \left(1 - \pi\left(k^2\right)\right) k^2 \left(\delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\tilde{A}_\mu(-k)\tilde{A}_\nu(k)$$

Then in position space, we get a wavefunction renormalization of $1-\pi(0)$, along with an infinite series of derivative interactions suppressed by powers of the electron mass.

Note. Showing that the photon self-energy is transverse. By diagrammatics, we have

$$\langle j^\mu j^\nu\rangle(k) = \Pi^{\mu\nu}(k) + (\Pi\Delta\Pi)^{\mu\nu}(k)$$

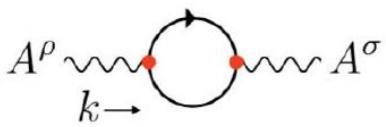
where $j^\mu = \bar{\psi}\gamma^\mu\psi$. Then the Ward identity ensures

$$k_\mu(\Pi + \Pi\Delta\Pi)^{\mu\nu} = 0$$

Expanding order by order and using the fact that Π starts at order e gives $k_\mu\Pi^{\mu\nu} = 0$, as claimed. Note that in both this proof and our earlier proof, we've had to work perturbatively. Indeed, the statement does not hold in general; it fails in the case of spontaneous symmetry breaking.

Now we turn to the one-loop contribution to $\Pi^{\rho\sigma}(k)$.

• We use dimensional regularization, defining the dimensionless coupling g by $e^2 = \mu^{4-d}g^2(\mu)$. The only diagram that contributes is



where there is no symmetry factor, since the electron and positron are distinct.

• Applying the Feynman rules, we have

$$\Pi^{\rho\sigma}_{1\text{ loop}}(k) = -\mu^{4-d}(ig)^2 \int dp \,\mathrm{tr}\left(\frac{1}{i\not{p}+m}\gamma^\rho \frac{1}{i(\not{p}-\not{k})+m}\gamma^\sigma\right)$$

where we picked up a minus sign from the fermion loop. As in the canonical formalism, we can understand this minus sign by going back to the level of contractions. When we apply Wick's theorem, we will always need to perform an odd number of Grassmann anticommutations to get the spinors all 'in order'. Equivalently, the path integral for fermions gives $\det(\not{D}+m)$ in the numerator rather than the denominator, and taking a logarithm gives the sign.

• Simplifying, the loop integral becomes

$$\int dp \frac{\mathrm{tr}\left((-i\not{p}+m)\gamma^\rho(-i(\not{p}-\not{k})+m)\gamma^\sigma\right)}{(p^2+m^2)\left((p-k)^2+m^2\right))}$$

We begin by using the Feynman trick to complete the square, turning the denominator into

$$\int_0^1 \frac{dx}{\left((p^2+m^2)(1-x)+\left((p-k)^2+m^2\right)x\right)^2} = \int_0^1 \frac{dx}{\left((p-kx)^2+m^2+k^2x(1-x)\right)^2}.$$

We then shift $p \to p+kx$, noting that this also changes the numerator.

• Next, we apply the trace identities, giving

$$\Pi^{\rho\sigma}_{1\text{ loop}}(k) = 4\mu^{4-d}g^2 \int dp \int_0^1 dx \frac{\text{numerator}}{\left(p^2+\Delta\right)^2}, \quad \Delta = m^2+k^2x(1-x)$$

$$\text{numerator} = -(p+kx)^\rho(p-k(1-x))^\sigma + (p+kx)\cdot(p-k(1-x))\delta^{\rho\sigma} - (p+kx)^\sigma(p-k(1-x))^\rho + m^2\delta^{\rho\sigma}.$$

• We can dramatically simplify the numerator using symmetry. All terms odd in p must vanish, while since the integral is isotropic we may replace

$$p^\mu p^\nu \to \frac{1}{d}\delta^{\mu\nu}p^2$$

We finally use two of our DR results to find

$$\Pi^{\rho\sigma}_{1\text{ loop}} = \left(k^2\delta^{\rho\sigma}-k^\rho k^\sigma\right)\pi_{1\text{ loop}}\left(k^2\right)$$

as required by the Ward identity, where

$$\pi_{1\text{ loop}}\left(k^2\right) = -\frac{8g^2(\mu)\Gamma(2-d/2)}{(4\pi)^{d/2}}\int_0^1 dx x(1-x)\left(\frac{\mu^2}{\Delta}\right)^{2-d/2}$$

Finally, setting $d = 4-\epsilon$ yields

$$\pi_{1\text{ loop}}\left(k^2\right) = -\frac{g^2(\mu)}{2\pi^2}\int_0^1 dx x(1-x)\left(\frac{2}{\epsilon}-\gamma+\log\frac{4\pi\mu^2}{\Delta}\right)$$

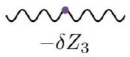
Next, we set the counterterms.

• As argued earlier, we require three counterterms for QED, which are

$$S^{\mathrm{CT}}[A,\psi]=\int dx\delta Z_3\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\delta Z_2\bar{\psi}\not{D}\psi+\delta m\bar{\psi}\psi$$

Here we've used $Z_1=Z_2$ to combine the electron field strength renormalization and the vertex renormalization into one term. This guarantees the counterterm Lagrangian is gauge invariant, as it must be if our regulator preserved gauge invariance.

• In $\overline{\mathrm{MS}}$ the δZ_3 counterterm contributes



$$\delta Z_3=-\frac{g^2(\mu)}{12\pi^2}\left(\frac{2}{\epsilon}-\gamma+\log 4\pi\right)$$

$$\pi_{1\text{ loop}}\left(k^2\right)=\frac{g^2(\mu)}{2\pi^2}\int_0^1 dxx(1-x)\log\frac{m^2+x(1-x)k^2}{\mu^2}$$

• The logarithm produced here is typical for loop corrections. Moreover, it creates a branch cut for the photon propagator in the region $m^2+x(1-x)k^2\leq 0$. This is inaccessible for real Euclidean momenta, but in Lorentz signature with $k_0=iE$ we have

$$x(1-x)\left(E^2-\mathbf{k}^2\right)\geq m^2$$

so the branch cut starts at $E=2m$, the threshold for creating a real electron-positron pair.

• Now, reverting to our original normalization of the photon field, where the coefficient of the kinetic term is $1/e^2$, we find

$$\frac{1}{g_{\mathrm{eff}}^2}=\frac{1-\pi(0)}{g^2(\mu)}=\frac{1}{g^2(\mu)}+\frac{\hbar}{12\pi^2}\log\frac{\mu^2}{m^2}$$

where we explicitly restore factors of $\hbar$. Note that there is no contribution from the vertex renormalization because, as we've shown, the covariant derivative $D_\mu=\partial_\mu+ieA_\mu$ is renormalized as a 'single piece' (i.e. $Z_1=Z_2$) and hence is absorbed by a field renormalization of ψ alone, without changing e.

• Setting $\partial g_{\mathrm{eff}}/\partial\log\mu$ to zero gives

$$\beta(g)=\frac{\hbar g^3(\mu)}{12\pi^2}+O\left(\hbar^2\right),\quad g^2(\mu)=\frac{6\pi^2}{\hbar}\frac{1}{\log\left(\Lambda_{\mathrm{QED}}/\mu\right)},\quad \Lambda_{\mathrm{QED}}\approx 10^{286}\mathrm{GeV}$$

where we set $\alpha\left(m_e\right)=g^2\left(m_e\right)/4\pi\approx 1/137$. That is, the QED coupling is marginally irrelevant and there is a Landau pole.

Note. How are our results here compatible with our earlier analysis of QED? In this section, we think in terms of the 1PI effective action. For simplicity, we set $m=0$. Then

$$\mathcal{L}=\frac{1}{4}F_0^2+\bar{\psi}_0\not{\partial}\psi_0+ie_0\bar{\psi}_0A_0\psi_0$$

This is gauge invariant because the interactions are built from the covariant derivative $D=\partial+ie_0A_0$. The effective Lagrangian in Fourier space takes the form

$$\mathcal{L}^{\text{eff}} \supset \frac{1}{4} f_3\left(q^2\right)\left|F_0^q\right|^2+\frac{1}{Z_2} f_2\left(q^2\right) \bar{\psi}_0^q \not{\partial} \psi_0^q+i e_0 f_1(q, p) \bar{\psi}_0^{q+p} A_0^q \psi_0^p$$

Here, we're using an abbreviated notation where superscripts stand for momenta; we get this momentum dependence because the effective action has an infinite series of derivative terms. The parameters Z_1, Z_2, and Z_3 are defined by

$$\frac{1}{Z_3}=f_3\left(q^2\right)\big|_{q^2=0}, \quad \frac{1}{Z_2}=f_2\left(q^2\right)\big|_{q^2=m_e^2}, \quad \frac{1}{Z_1}=f_1(q, p)|_{q^2=0}$$

where m_e is the pole mass of the electron, in this case zero. Therefore, at low momenta, we have

$$\mathcal{L}^{\text{eff}} \supset \frac{1}{4} \frac{1}{Z_3} F_0^2+\frac{1}{Z_2} \bar{\psi}_0 \not{\partial} \psi_0+\frac{i e_0}{Z_1} \bar{\psi}_0 A_0 \psi_0$$

We are thus motivated to define the renormalized fields and renormalized/physical coupling

$$A=\frac{A_0}{\sqrt{Z_3}}, \quad \psi=\frac{\psi_0}{\sqrt{Z_2}}, \quad e=\frac{Z_2}{Z_1} \sqrt{Z_3} e_0$$

which gives

$$\mathcal{L}=\frac{1}{4} Z_3 F^2+Z_2 \bar{\psi} \not{\partial} \psi+i Z_1 e \bar{\psi} A \psi, \quad \mathcal{L}^{\text{eff}} \supset \frac{1}{4} F^2+\bar{\psi} \not{\phi} \psi+i e \bar{\psi} A \psi$$

Therefore, e is a direct measure of the all-orders interaction strength at low momentum. In terms of these rescaled fields and couplings, the covariant derivative in $\mathcal{L}$ is $D=\partial+i\left(Z_1 / Z_2\right) e A$. On the other hand, $\mathcal{L}^{\text{eff}}$ is built from $\partial+i e A$. Hence conceptually the result $Z_1=Z_2$ tells us that the effective action is built from covariant derivatives in the same way as the original action.

We are now working solely in terms of physical variables, which is the essential point of renormalization. The last step is to split $\mathcal{L}$ into a finite and "counterterm" part. In renormalized perturbation theory, we set the finite part of $\mathcal{L}$ equal to $\mathcal{L}^{\text{eff}}$ evaluated at on-shell momenta, i.e. low momenta here. The running coupling arises because the coefficient of $\bar{\psi} A \psi$ in $\mathcal{L}^{\text{eff}}$ depends on momentum.

In $\overline{\text{MS}}$, we instead set the counterterm to have a simple form; then the parameters in the finite part of $\mathcal{L}$ are not physical parameters; instead we have an ' $\overline{\text{MS}}$ mass' and ' $\overline{\text{MS}}$ coupling' $e(\mu)$. But as argued earlier, $e(\mu)$ is physically meaningful because it approximates the coefficient of $\bar{\psi} A \psi$ in $\mathcal{L}^{\text{eff}}$ at momentum scale μ.

Note. The $\overline{\text{MS}}$ scheme is a bit strange, since it does not have "decoupling". The coupling continues to run even when we are far below the mass of the electron, even though one would intuitively think the electron should have no effect at these energies. As another example, applying $\overline{\text{MS}}$ to the $SU(5)$ GUT, all three gauge couplings are equal at all energies, since $\overline{\text{MS}}$ knows nothing about the symmetry breaking. This is perfectly consistent, as the full perturbation series in principle sums to the same thing, but not physically transparent.

An alternative scheme that makes the decoupling explicit is to demand that the quantum corrections vanish at an arbitrary scale $k^2=\mu^2$, which is the MOM/momentum space subtraction scheme. That is, we set

$$\delta Z_3=-\frac{\hbar g^2(\mu)}{2 \pi^2} \int_0^1 d x x(1-x)\left(\frac{2}{\epsilon}-\gamma+\log \frac{4 \pi \mu^2}{m^2+x(1-x) \mu^2}\right)$$

so the total one-loop contribution is

$$\pi\left(k^2\right) = \frac{\hbar g^2(\mu)}{2\pi^2}\int_0^1 dx x(1-x)\log\left(\frac{m^2+x(1-x)k^2}{m^2+x(1-x)\mu^2}\right) + O\left(\hbar^2\right)$$

Then the beta function is

$$\beta(\mu) = \frac{g^3(\mu)}{2\pi^2}\int_0^1 dx \frac{x^2(1-x)^2\mu^2}{m^2+x(1-x)\mu^2}$$

which indeed goes to zero when $\mu \ll m$, freezing $\alpha \approx 1/137$. This is an example of the AppelquistCarazzone decoupling theorem. In general, a mass-dependent (or "physical") scheme is one where the beta function depends explicitly on the renormalization scale μ, and hence explicitly on the particle mass m. Decoupling only holds for mass-dependent schemes, which include this scheme and a hard cutoff, where Λ plays the role of the renormalization scale.

The disadvantage of mass-dependent schemes is that perturbation theory is harder; we've seen that for a hard cutoff the loops are not small. Thus in practice, a modified version of $\overline{\text{MS}}$ is used, where the decoupling is put in by hand. We consider two theories, one with an electron, valid for $\mu > m_e$, and one with no electron, valid for $\mu < m_e$. We perform renormalization in both theories with $\overline{\text{MS}}$ or another mass-independent scheme and match physical quantities at $\mu = m_e$, such as S-matrix elements. At leading order, this means that $g(\mu)$ is continuous at $\mu = m_e$, and stops running below m_e. However, at higher orders, or if we choose $\mu \neq m_e$, the coupling jumps discontinuously at the matching. This is a "threshold correction", and must be computed carefully to get, e.g. gauge coupling unification to work out correctly.

This method is called "continuum EFT" in contrast to "Wilsonian EFT", since we never have a momentum cutoff. The idea is so common today that it is often used implicitly. E.g. the QCD beta function is written in terms of n_f, the "number of light quarks". But in $\overline{\text{MS}}$ all of the quarks contribute in the same way to the beta function, regardless of mass. The standard $g(\mu)$ for QCD is actually using successive matchings, integrating out each heavy quark as we pass its mass.

Note. The first two terms of the beta function, corresponding to the one-loop and two-loop contributions, are independent of the subtraction scheme, provided that we maintain canonical normalization. Changing the normalization clearly adds terms to the beta function even at tree level; in this case that corresponds to setting $Z = 1$ at the physical mass pole. We did this for $\overline{\text{MS}}$ but not for our other scheme, which instead sets $Z = 1$ at μ.

Note. In $\overline{\text{MS}}$, the mass is not the same as the pole mass, so what is its physical meaning? One can think of the mass term as like any other renormalized coupling. E.g. the up quark mass provides an extra two-point vertex in QCD scattering problems. The $\overline{\text{MS}}$ mass at scale μ approximates the overall all-order effect of this interaction term at tree level. (Note that there is nothing preventing $\overline{\text{MS}}$ mass terms from being negative!) The PDG gives most quark masses as $\overline{\text{MS}}$ masses (the universal choice for modern theory calculations), since there is no way to define a pole mass for them. But its top quark mass is a "Pythia" mass, meaning it's simply what one gets by fitting to Pythia, which has no direct link to any theoretical calculation. When speaking of the mass of any fundamental particle, we need to specify the scheme in which it's defined.

Finally, we prove that $Z_1 = Z_2$ using the Ward-Takahashi identity.

• Starting from the global $U(1)$ symmetry of QED,

$$\psi \to \psi' = e^{i\alpha\psi}, \quad j^\mu = i\bar{\psi}\gamma^\mu\psi$$

we may apply the Ward-Takahashi identity to find

$$\partial_\mu \left\langle j^\mu(x)\psi(x_1)\bar{\psi}(x_2)\right\rangle = -i\delta\left(x-x_1\right)\left\langle \psi(x_1)\bar{\psi}(x_2)\right\rangle + i\delta\left(x-x_2\right)\left\langle \psi(x_1)\bar{\psi}(x_2)\right\rangle .$$

Note that the presence of gauge fixing terms makes no difference here, because we're only considering global symmetry transformations. However, we will see an alternate derivation below that uses the local symmetry.

• We define the exact electron propagator by Fourier transforming the two-point function,

$$\left\langle \psi\left(k_1\right)\bar{\psi}\left(-k_2\right)\right\rangle \equiv \int dx_1 dx_2 e^{ik_1x_2}e^{-ik_2x_2}\left\langle \psi(x_1)\bar{\psi}(x_2)\right\rangle = \not{\delta}\left(k_1-k_2\right)S(k)$$

where we defined

$$S(k) = \int dx e^{ikx}\langle\psi(x)\bar{\psi}(0)\rangle = \frac{1}{i\not{k}+m-\Sigma(\not{k})}$$

and $\Sigma(\not{k})$ is the sum of 1 PI diagrams.

• We define the exact vertex by

$$\left\langle j_\mu(k)\psi\left(k_1\right)\bar{\psi}\left(-k_2\right)\right\rangle \equiv \phi\left(p+k_1-k_2\right)S\left(k_1\right)\Gamma_\mu\left(k_1,k_2\right)S\left(k_2\right)$$

To understand this expression, consider computing the left-hand side diagrammatically. Here, it can be useful conceptually to draw a 'phantom' photon attached to x, though no such photon field actually appears in the correlator. The lowest-order term is γ_μ, so indeed

$$\Gamma_\mu\left(k_1,k_2\right) = \gamma_\mu + \text{ quantum corrections.}$$

At one-loop order, either the ψ and $\bar{\psi}$ can emit and reabsorb a photon. This has nothing to do with the vertex, and is accordingly absorbed by the $S\left(k_1\right)$ and $S\left(k_2\right)$ factors. Recalling that 1PI and amputated diagrams are equivalent for three external fields, we end up computing the sum of 1PI diagrams with an external photon and two external fermions.

• Finally, taking the Fourier transform of the Ward-Takahashi identity,

$$\left(k_1-k_2\right)_\mu S\left(k_1\right)\Gamma^\mu\left(k_1,k_2\right)S\left(k_2\right) = iS\left(k_1\right) - iS\left(k_2\right)$$

which is equivalent to

$$\left(k_1-k_2\right)\Gamma^\mu\left(k_1,k_2\right) = iS^{-1}\left(k_2\right) - iS^{-1}\left(k_1\right)$$

This result relates the vertex to electron kinetic term in the 1PI effective action.

• To extract the desired result, differentiate with respect to k_1 and then take $k_1, k_2 \to k$ for

$$\Gamma_\mu(k,k) = -i\frac{\partial}{\partial k^\mu}S^{-1}(k) = \gamma_\mu + i\frac{\partial}{\partial k^\mu}\Sigma(\not{k})$$

Thus, at lowest order in k, the effective action looks like

$$S^{\text{eff}} \supset \bar{\psi}\left(\gamma_\mu + i\frac{\partial}{\partial k^\mu}\Sigma(\not{k})\right)D\psi$$

so the covariant derivative $D\psi$ is renormalized as a single piece. Expanding our result in a series in k shows that interactions must come in the form $\bar{\psi}D^p\psi$.

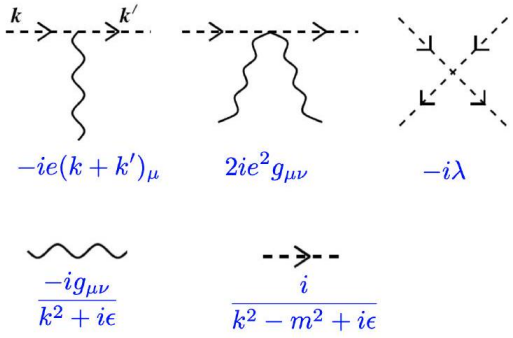
3.4.5 Scalar QED

As a final example, we'll consider the one-loop renormalization of scalar QED in $(+---)$ signature with $\overline{\text{MS}}$. Note that switching from MS to $\overline{\text{MS}}$ is equivalent to using $\mu^2_{\text{MS}} = \mu^2_{\text{MS}} 4\pi e^{-\gamma}$ throughout, so if we define $\mu = \mu_{\overline{\text{MS}}}$, we won't need to add extra "macaroni and pie" to the counterterms. We will use a new technique, to be explained below, that can compute the beta functions from only the divergent parts of the counterterms, hence displaying the simplicity of the $\overline{\text{MS}}$ scheme.

The scalar QED Lagrangian is

$$\mathcal{L} = |\partial_\mu \phi|^2 - m^2|\phi|^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{\lambda}{4}|\phi|^4 - ieA_\mu\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right) + e^2 A_\mu A^\mu \phi^*\phi.$$

Here we must include a ϕ^4 interaction because it is consistent with the symmetries, or equivalently because it may be generated by the other interaction terms. The Feynman rules are:



$\epsilon^\mu_{\lambda_i}(k), \epsilon^{\mu*}_{\lambda_i}(k)$ for incoming and outgoing photons respectively

However, for this analysis, we will work in Landau gauge, so the photon propagator comes with a factor of $P_{\mu\nu} = \eta_{\mu\nu} - q_\mu q_\nu/q^2$ rather than just $\eta_{\mu\nu}$. As we'll see, this sets many diagrams to zero, shortening the analysis.

• Relabeling all the fields above as bare fields with 0 subscripts, and defining physical fields as

$$\phi = \frac{\phi_0}{\sqrt{Z_2}}, \quad A = \frac{A_0}{\sqrt{Z_3}}, \quad e = \frac{Z_2}{Z_1}\sqrt{Z_3}e_0$$

we have

$$\mathcal{L} = Z_2\,|\partial_\mu\phi|^2 - Z_m m^2|\phi|^2 - \frac{Z_3}{4}F^{\mu\nu}F_{\mu\nu} - Z_\lambda\frac{\lambda}{4}|\phi|^4 - iZ_1 eA_\mu\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right) + Z_4 e^2 A_\mu A^\mu\phi^*\phi$$

Here we have $Z_4 = Z_1^2/Z_2$, and we should have $Z_1 = Z_2$. In the calculations below we will not assume these results, but rather directly compute the Z_i at one-loop order in DR and $\overline{\text{MS}}$ to demonstrate they hold.

• Defining $Z_i = 1 + \delta_i$, it is convenient to split the Lagrangian as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{\text{ct}}$$

where

$$\mathcal{L}_0 = |\partial_\mu\phi|^2 - m^2|\phi|^2 - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \quad \mathcal{L}_{\text{ct}} = \delta_2\,|\partial_\mu\phi|^2 - \delta_m m^2|\phi|^2 - \frac{\delta_3}{4}F^{\mu\nu}F_{\mu\nu}$$

and $\mathcal{L}_1$ contains the interactions. Below we will work to order $O\left(e^2\right)$ and $O(\lambda)$ in the Z_i. First, we consider the photon propagator.

• As we've seen before, the exact photon propagator remains transverse in Landau gauge, as

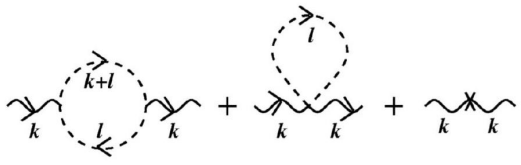
$$\Delta_{\mu\nu}(k) = \frac{\Delta^0_{\mu\nu}(k)}{1-\pi\left(k^2\right)}$$

where $\pi\left(k^2\right)$ is determined by the sum of 1PI diagrams,

$$\Pi_{\mu\nu}\left(k^2\right) = k^2\pi\left(k^2\right)P_{\mu\nu}$$

which is transverse by the Ward identity.

• At order e^2, we have the contributing diagrams:



• Since $Z_i = 1 + O\left(e^2\right)$, we can replace Z_1 and Z_4 in the first two diagrams with just 1, giving

$$i\Pi^{\mu\nu}\left(k^2\right) = (-ie)^2 i^2 \int d\ell \frac{(2\ell+k)^\mu (2\ell+k)^\nu}{(\ell+k)^2-m^2)\left(\ell^2-m^2\right)} + (2i)e^2\eta^{\mu\nu} i\int \frac{d\ell}{\ell^2-m^2} - i\left(Z_3-1\right)\left(k^2\eta^{\mu\nu}-k^\mu k^\nu\right).$$

The first two terms can be combined into

$$e^2\int d\ell \frac{N^{\mu\nu}}{\left((\ell+k)^2-m^2\right)\left(\ell^2-m^2\right)}, \quad N^{\mu\nu} = (2\ell+k)^\mu(2\ell+k)^\nu - 2\left((\ell+k)^2-m^2\right)\eta^{\mu\nu}.$$

• Applying Feynman parameters as usual, this term becomes

$$e^2\int_0^1 dx\int dq \frac{N^{\mu\nu}}{\left(q^2-\Delta\right)^2}, \quad q = \ell + xk, \quad \Delta = -x(1-x)k^2+m^2$$

Replacing ℓ with q in $N^{\mu\nu}$, it simplifies to

$$N^{\mu\nu} \sim 4q^\mu q^\nu + (1-2x)^2 k^\mu k^\nu - 2\left(q^2+(1-x)^2k^2-m^2\right)\eta^{\mu\nu}$$

where we removed terms odd in q as they integrate to zero. We replace $q^\mu q^\nu \to q^2\eta^{\mu\nu}/D$ by isotropy, leaving two standard DR integrals.

• Performing these integrals with the standard formulas and then performing the x integral gives

$$i\Pi^{\mu\nu}\left(k^2\right) = ie^2\mu^\epsilon\left(k^\mu k^\nu - k^2\eta^{\mu\nu}\right)\left(Z_3 - 1 + \frac{1}{3}\frac{1}{(4\pi)^{d/2}}\Gamma\left(\frac{4-d}{2}\right) + \text{ finite }\right)$$

where $\epsilon = 4-d$, with the transverse structure required by the Ward identity. The finite parts are momentum-dependent and can be used to compute the running coupling, as in QED.

• In this section, we'll work with $\overline{\text{MS}}$, and hence can ignore the finite parts. Then we have

$$Z_3 = 1 - \frac{e^2}{24\pi^2}\frac{1}{\epsilon}$$

at one loop. We will use this later to recover the running coupling by another way.

Note. In QED, we used the Ward-Takahashi identity associated with the global $U(1)$ symmetry to derive $Z_1 = Z_2$. For variety, we can do the same here using a somewhat different setup, essentially using the Schwinger-Dyson equation corresponding to a $U(1)$ gauge transformation. In this case, we must account for the gauge fixing term, which in R_ξ gauge is

$$\mathcal{L} \supset \mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2$$

The bare photon propagator is now

$$\Delta^0_{\mu\nu}(k) = \frac{1}{k^2}\left(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) = \frac{1}{k^2}\left(P_{\mu\nu}(k) + \xi\frac{k_\mu k_\nu}{k^2}\right)$$

Now, the sum of 1PI diagrams remains transverse, so the additional non-transverse piece is unaffected when we sum over 1PI diagrams, giving the exact propagator

$$\Delta_{\mu\nu}(k) = \frac{P_{\mu\nu}}{k^2\left(1-\pi\left(k^2\right)\right)} + \xi\frac{k_\mu k_\nu}{k^2}$$

The strategy we will use to derive our relations is to start with the generating functional with currents. Schematically, consider some local symmetry $\phi \to \phi' = \phi + \epsilon(x)\delta\phi$ under which the path integral measure is invariant. Then we have

$$Z[J] = \int \mathcal{D}\phi e^{i(S[\phi]+\phi J)} = \int \mathcal{D}\phi' e^{i(S[\phi']+\phi' J)} = \int \mathcal{D}\phi e^{i(S[\phi]+\phi J)}\left(1 + i\left(\delta S + \int dx J\delta\phi\right)\right)$$

Therefore, we must have

$$\int \mathcal{D}\phi e^{i(S[\phi]+\phi J)}\left(\delta S + \int dx J\delta\phi\right) = 0$$

Integrating by parts to get an overall factor of $\epsilon(x)$, we can use the fact that $\epsilon(x)$ is arbitrary to peel it off and remove the dx integral, leaving an correlation function identity. Since we are working with the generating functional, we can differentiate with respect to J to yield identities for higher-point functions.

In this context, we will use the gauge transformation

$$\delta\phi(x) = ie\alpha(x)\phi(x), \quad \delta A_\mu(x) = \partial_\mu\alpha(x)$$

In this case δS is solely due to the gauge fixing term $\mathcal{L}_{\text{gf}}$,

$$\delta S = -\int dx \frac{1}{\xi}\alpha(x)\partial^2\partial_\mu A^\mu(x)$$

We also get three terms from the variations of the currents, so we have, suppressing x arguments,

$$\int \mathcal{D}\phi\mathcal{D}\phi^*\mathcal{D}A_\mu e^{i(S[\phi,A_\mu]+\phi J^*+\phi^* J+J^\mu A_\mu)}\int dx\left(-\frac{1}{\xi}\alpha\partial^2\partial_\mu A^\mu\right) + J^*\delta\phi + J\delta\phi^* - iJ^\mu\delta A_\mu = 0.$$

Integrating by parts to get a factor of $\alpha(x)$ and using the fact that $\alpha(x)$ is arbitrary, we have

$$\left\langle -\frac{1}{\xi}\partial^2\partial_\mu A^\mu + ie\phi J^* - ie\phi^* J + i\partial_\mu J^\mu \right\rangle = 0$$

Dividing by $Z\left[J^\mu, J^*, J\right]$ puts this in terms of the connected generating functional $W\left[J^\mu, J^*, J\right]$,

$$\frac{1}{\xi}\partial^2\partial_\mu \frac{\delta W}{\delta J_\mu(x)} - ieJ^*(x)\frac{\delta W}{\delta J^*(x)} + ieJ(x)\frac{\delta W}{\delta J(x)} + \partial_\mu J^\mu(x) = 0$$

Acting with $(\delta/\delta J^*(y))\,(\delta/\delta J(z))$ on both sides and setting the currents to zero,

$$\frac{1}{\xi}\partial^2\partial_\mu \left\langle A_\mu(x)\phi(y)\phi(z)\right\rangle_c + ie\delta(y-x)\left\langle \phi(x)\phi^*(z)\right\rangle_c - ie\delta(z-x)\left\langle \phi^*(x)\phi(y)\right\rangle_c = 0$$

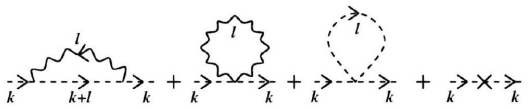
The last two terms are exact scalar propagators, while the first term is the product of the exact photon propagator, two exact scalar propagators, and the exact three-point vertex. However, upon a Fourier transform the derivatives on the photon propagator yield

$$\frac{1}{\xi}k^2 k^\mu \Delta_{\mu\nu}(k) = k^\mu$$

This reduces the identity we have here to the Ward-Takahashi identity we saw earlier for QED, and the proof continues in the same way.

Next, we consider the scalar propagator, where Landau gauge will be somewhat useful.

• The scalar propagator $i\Pi_\phi\left(k^2\right)$ has five contributions:



Here, the final diagram contains both the δ_2 and δ_m counterterms.

• The second diagram is a "scaleless" integral, and using the DR formulas gives an answer of zero. But this is too quick: the DR formulas are not actually valid at all, because there does not exist any value of d for which a scaleless integral converges (i.e. it is always either UV or IR divergent), so there are no values of d to analytically continue from!

• A glib way to justify the result is to say "the DR answer has to depend on μ, but there are no other mass scales in the integral, so what else could it be?" A bit more seriously, the integral can be IR regulated, in which case the DR formulas legitimately give

$$i\Pi_\phi\left(k^2\right) \supset \left(2Z_4 ie^2\eta^{\mu\nu}\right)(-i)\int d\ell \frac{P_{\mu\nu}}{\ell^2 - m_\gamma^2}$$

The resulting integral is proportional to m_γ^2/ϵ in DR, which vanishes in the limit $m_\gamma \to 0$.

• The first diagram contributes

$$i\Pi_\phi\left(k^2\right) \supset \left(-iZ_1 e\right)^2(-i)i\int d\ell \frac{P_{\mu\nu}(\ell)(\ell+2k)^\mu(\ell+2k)^\nu}{\ell^2\left((\ell+k)^2 - m^2\right)}$$

in Landau gauge, and since $\ell_\mu P^{\mu\nu} = 0$ we have

$$i\Pi_\phi\left(k^2\right) \supset -e^2\int d\ell \frac{4\left(\ell^2 k^2 - (k\cdot\ell)^2\right)}{\ell^4\left((\ell+k)^2 - m^2\right)}$$

where we again set $Z_1 = 1$ since we only want the $O(e^2)$ terms.

• The denominator can be simplified using Feynman parameters,

$$\frac{1}{\ell^4\left((\ell+k)^2-m^2\right)} = 2\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{dx_1 dx_2}{\left(\left(1-x_1-x_2\right)\ell^2 + x_2\ell^2 + x_1\left((\ell+k)^2-m^2\right)\right)^3}.$$

The x_2 dependence drops out, so evaluating the x_2 integral and renaming x_1 to x,

$$\frac{1}{\ell^4\left((\ell+k)^2-m^2\right)} = 2\int \frac{(1-x)dx}{\left(\ell^2+2x\ell\cdot k + xk^2 - xm^2\right)^3}.$$

Using the usual shift $q = \ell + xk$ we have

$$i\Pi_\phi\left(k^2\right) \supset -8e^2\int_0^1 dx \int dq \frac{q^2k^2-(q\cdot k)^2}{\left(q^2-\Delta\right)^3}, \quad \Delta = -x(1-x)k^2+m^2.$$

Applying isotropy and a standard DR formula we find

$$i\Pi_\phi\left(k^2\right) \supset -i\frac{3e^2}{8\pi^2}\frac{1}{\epsilon}k^2$$

• Next, for the scalar loop contribution, we have

$$i\Pi_\phi\left(k^2\right) \supset (-i\lambda)(i)\int \frac{d\ell}{\ell^2-m^2} = \frac{i\lambda}{8\pi^2}\frac{1}{\epsilon}m^2$$

by a standard DR formula.

• For comparison, the counterterms contribute

$$i\Pi_\phi\left(k^2\right) \supset i\delta_2 k^2 - i\delta_m m^2$$

from which we conclude

$$Z_2 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\epsilon}, \quad Z_m = 1 + \frac{\lambda}{8\pi^2}\frac{1}{\epsilon}$$

Note. A bit more about scaleless integrals in DR. The argument above is sketchy, because it requires a separate IR regulator; in principle DR can handle both UV and IR divergences by itself. Within pure DR, the justification for setting the integral to zero is the 't Hooft-Veltman conjecture, which simply states that "no inconsistencies" arise from doing this.

A bit more precisely, a proper treatment of DR would begin with a list of postulates that DR integrals must obey, just as we begin with postulates for Grassmann integration. Changing notation for clarity, these postulates are linearity, shift invariance, and the scaling

$$\int d^d\mathbf{p} f(\lambda\mathbf{p}) = \lambda^{-d}\int d^d\mathbf{p} f(\mathbf{p})$$

It can be shown (e.g. in Collins) that these postulates essentially uniquely determine the DR prescription. The result is equivalent to how we've presented DR earlier: take the measure to be $d^d\mathbf{p} = p^{d-1}\Omega_d dp$, do the integral for the d where it is defined, and analytically continue in d.

Now, using linearity, we can split the scaleless integral as

$$I = \int \frac{d\ell}{\ell^2} = \int d\ell \frac{\ell^2}{\ell^2\left(\ell^2 - m^2\right)} - \int d\ell \frac{m^2}{\ell^2\left(\ell^2 - m^2\right)}$$

where m is an arbitrary scale, thereby converting it into two ordinary DR integrals. Both integrals are convergent in an open region of the complex plane, so they may be analytically continued as usual, and the standard DR formulas apply as $d \to 4$. We find that as long as the ϵ 's and μ 's used in each are the same, the two results cancel, yielding the desired result. Further arguments can be used to show that, due to the DR postulates, $\int d\ell \left(\ell^2\right)^\alpha = 0$ for any α.

Of course, in this view the 't Hooft-Veltman conjecture simply reduces to the conjecture that these postulates do not lead to inconsistencies. (This is by no means guaranteed, as inconsistencies can arise when we try to account for spinors and the associated γ^μ.) The "proof" for our purposes is simply experience. However, it turns out that it will eventually lead to trouble, in the form of "renormalons", i.e. poor convergence at large orders in perturbation theory, which are connected to the fact that the QFT perturbation series is asymptotic.

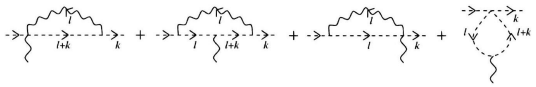
Note. There are more subtleties for integrals which are simultaneously UV and IR divergent, like

$$I = \int \frac{d\ell}{\ell^4}$$

Using the same trick as above, we can split this into a UV divergent and an IR divergent integral, which are each evaluated in DR with $(\epsilon_{\text{UV}}, \mu_{\text{UV}})$ and $(\epsilon_{\text{IR}}, \mu_{\text{IR}})$ respectively. The integrals converge for $\epsilon_{\text{UV}} > 0$ and $\epsilon_{\text{IR}} < 0$ respectively, and the integral is zero when we take $\epsilon_{\text{UV}} = \epsilon_{\text{IR}}$. The subtlety here is that this integral cannot simply be ignored, because the counterterm, being a UV quantity, only can cancel the $1/\epsilon_{\text{UV}}$ pole; something else is required to cancel the $1/\epsilon_{\text{IR}}$ pole. That is, the UV and IR divergences are conceptually distinct. (think more about this)

Now we consider the vertices, where clever choices for the external momenta will greatly simplify the calculation in Landau gauge.

• First, we consider the one-loop corrections to the three-point vertex V_3^μ.



These are renormalized by Z_1. Since we only care about the divergence structure, we are free to choose any external momenta we want, so we choose the incoming ϕ to have zero momentum and the outgoing ϕ to have k. Then in the second and third diagrams, the left vertex yields a factor of ℓ_μ which vanishes when contracted with the $P^{\mu\nu}(\ell)$ from the photon propagator in Landau gauge.

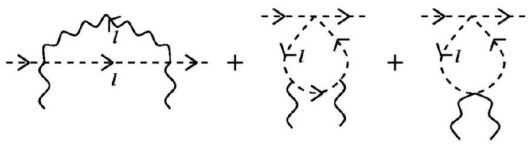
• The remaining terms take the form

$$\begin{aligned} iV_3^\mu(k,0) = ieZ_1 k^\mu + (iZ_1 e)\left(-2iZ_4 e^2 \eta^{\mu\nu}\right)(-i)^2 \int d\ell \frac{P_{\nu\rho}(\ell)(\ell+2k)^\rho}{\ell^2\left((\ell+k)^2+m^2\right)} \\ + (-iZ_\lambda \lambda)(iZ_1 e)(-i)^2 \int d\ell \frac{(2\ell+k)^\mu}{\left(\ell^2+m^2\right)\left((\ell+k)^2+m^2\right)} \end{aligned}$$

Upon Feynman parametrization, shifting the loop momentum, and integrating over x, the final term is proportional to ℓ^μ and hence vanishes by symmetry. Evaluating the remaining integral by the usual methods gives

$$Z_1 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\epsilon}$$

• Next, consider the four-point, scalar-scalar-photon-photon vertex. In this case, a convenient choice is to set all external momenta to zero, so that only five diagrams survive.



For each of the first two diagrams, there is another diagram which swaps how the photon lines attach. Then we have

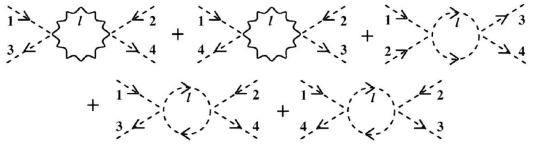
$$\begin{aligned} iV_4^{\mu\nu}(0,0,0) = -2iZ_4e^2\eta^{\mu\nu} &+ \left(-2iZ_4e^2\right)^2(-i)^2\int d\ell \frac{P^{\mu\nu}}{\ell^2\left(\ell^2+m^2\right)} + (\mu \leftrightarrow \nu) \\ &+ (iZ_1e)^2(-iZ_\lambda\lambda)(-i)^3\int d\ell \frac{(2\ell)^\mu(2\ell)^\nu}{\left(\ell^2+m^2\right)^3} + (\mu \leftrightarrow \nu) \\ &+ (-iZ_\lambda\lambda)\left(-2iZ_4e^2\eta^{\mu\nu}\right)(-i)^2\int d\ell \frac{1}{\left(\ell^2+m^2\right)^2}. \end{aligned}$$

All of these may be evaluated in the usual ways to find

$$Z_4 = 1 + \frac{3e^2}{8\pi^2}\frac{1}{\epsilon}$$

We've now shown $Z_1 = Z_2 = Z_4$, as expected and required by general considerations above.

• Finally, consider the four-scalar vertex. Again we may set all external momenta to zero, giving the remaining diagrams below.



Note that the first three have a symmetry factor of 2. The first two and the last three all have the same structure, giving

$$\begin{aligned} iV_4(0,0,0) = -iZ_\lambda\lambda &+ \left(\frac{1}{2}+\frac{1}{2}\right)\left(-2iZ_4e^2\right)^2(-i)^2\int d\ell \frac{P^{\mu\nu}(\ell)P_{\mu\nu}(\ell)}{\left(\ell^2+m_\gamma^2\right)^2} \\ &+ \left(\frac{1}{2}+1+1\right)(-iZ_\lambda\lambda)^2(-i)^2\int d\ell \frac{1}{\left(\ell^2+m^2\right)^2} \end{aligned}$$

which may be evaluated in the usual ways to find

$$Z_\lambda = 1 + \left(\frac{3e^4}{2\pi^2\lambda} + \frac{5\lambda}{16\pi^2}\right)\frac{1}{\epsilon}$$

We now consider the running couplings.

• One of the best features of $\overline{\text{MS}}$ is that we can read off the beta functions directly from the Z_i, without ever having to compute the momentum-dependence of the vertices. We note that

$$e_0 = Z_3^{-1/2}\mu^{\epsilon/2}e, \quad \lambda_0 = Z_2^{-2}Z_\lambda\mu^\epsilon\lambda$$

The Z_i may be expanded in a series in the couplings e^2 and λ, and in $1/\epsilon$.
• The key result is that the e_0 and λ_0 are independent of μ. Now note that

$$\log e_0 = \log\left(Z_3^{-1/2}\right) + \log e + \frac{\epsilon}{2}\log\mu + ... = \frac{e^2}{48\pi^2}\frac{1}{\epsilon} + \log e + \frac{\epsilon}{2}\log\mu + ...$$

Differentiating with respect to $\log\mu$ on both sides and letting $\beta_e = de/d\log\mu$, we have

$$0 = \frac{\epsilon e}{2} + \beta + \frac{e^2}{24\pi^2}\frac{1}{\epsilon}\beta$$

Solving for β and taking the limit $e \to 0$ (crucially, before the limit $\epsilon \to 0$), we get

$$\beta = -\frac{\epsilon e}{2}\left(1 + \frac{e^2}{24\pi^2}\frac{1}{\epsilon}\right)^{-1} + ... = -\frac{\epsilon e}{2}\left(1 - \frac{e^2}{24\pi^2}\frac{1}{\epsilon}\right) + ... = \frac{e^3}{48\pi^2} + ...$$

which is the beta function in scalar QED.
• A slightly more complicated calculation along the same lines gives

$$\beta_\lambda = \frac{1}{16\pi^2}\left(5\lambda^2 - 6\lambda e^2 + 24e^4\right) + ...$$

for the quartic coupling.
• For reference, the analogous factors for QED are

$$Z_1 = Z_2 = 1 - \frac{e^2}{8\pi^2\epsilon}, \quad Z_m = 1 - \frac{\delta_m}{m} = 1 - \frac{e^2}{2\pi^2\epsilon}, \quad Z_3 = 1 - \frac{e^2}{6\pi^2\epsilon}$$

where Z_3 immediately gives the QED beta function

$$\beta = \frac{e^3}{12\pi^2}$$

by the same argument as in scalar QED.

3.4.6 Basics of Regularization

Анализ и индекс расходимости диаграмм

Для произвольной диаграммы введём следующие обозначения:
L_e - число внешних электронных линий;
L_γ - число внешних фотонных линий;
N_e - число внутренних электронных линий;
N_γ - число внутренних фотонных линий;
n - число вершин;
l - число петель.
Тогда индекс расходимости диаграммы

$$\omega = dl - N_e - 2N_\gamma.$$

Для числа петель имеем:

$$l = N_e + N_\gamma - n + 1,$$

так как каждому пропагатору соответствует интегрирование (по своему d-импульсу), но каждая верщина содержит δ-функцию по входящим импульсам, однако одна δ-функция даёт общий закон сохранения. Для числа вершин имеем:

$$n = L_\gamma + 2N_\gamma = \frac{1}{2}\left(L_e + 2N_e\right).$$

(??? как его применять - тут)

О размерной регуляризации

Метрика: $g_4^{\mu\nu} \to g_{4-2\varepsilon}^{\mu\nu}$.
$g^{\mu\nu}g_{\mu\nu} = \delta_\mu^\mu = D = 4 - 2\varepsilon$.
Мера: $d^4q \to (\mu^2)^\varepsilon d^{4-2\varepsilon}q$, где μ есть параметр размерной регуляризации размерности массы.

$$\gamma^\mu\gamma_\mu = D = 4 - 2\epsilon; \quad \operatorname{Tr}\gamma^\mu\gamma^\nu = g^{\mu\nu}\operatorname{Tr}1 = g^{\mu\nu}\begin{cases} 2^{[D/2]} \\ 4 \end{cases}.$$

$$\operatorname{Tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \operatorname{Tr}1\left[g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\nu\rho}g^{\mu\sigma}\right],$$
$$\gamma^\mu\gamma^\nu\gamma^\mu = -\gamma^\mu\gamma^\mu\gamma^\nu + 2g^{\mu\nu}\gamma^\mu = -(4-2\epsilon)\gamma^\nu + 2\gamma^\nu = -(2-2\epsilon)\gamma^\nu, \quad \text{и т.д.}$$

$$\frac{1}{p^2 - m^2} \to \frac{1}{p^2 - m^2}.$$

$$\int \frac{d^4p}{p^2}e^{ipx} \sim \frac{1}{x^2} \Rightarrow \int \frac{d^{4-2\epsilon}p}{p^2}e^{ipx} \sim \frac{1}{[x^2]^{1-\epsilon}}.$$

$$\int \frac{d^Dk}{\left[k^2 + M^2\right]^2} = \frac{\Omega_D}{2}\int\limits_0^\infty \frac{(k^2)^{D/2-1}\,dk^2}{\left[k^2 + M^2\right]^2}, \quad M^2 \equiv p^2x(1-x) + m^2$$

$$I\left(p^2\right) = \frac{i}{(2\pi)^D}\frac{\Omega_D}{2}\int\limits_0^1 dx\frac{\Gamma(D/2)\Gamma(2-D/2)}{\left[p^2x(1-x) + m^2\right]^{2-D/2}} =$$

$$= \frac{i(-\pi)^{2-\varepsilon}}{(2\pi)^{4-2\varepsilon}}\Gamma(\varepsilon)\int\limits_0^1 \frac{dx\,(\mu^2)^\varepsilon}{\left[p^2x(1-x) - m^2\right]\varepsilon} =$$

$$= \frac{i}{16\pi^2}\Gamma(1+\varepsilon)\left(\frac{1}{\varepsilon} - \int\limits_0^1 dx\log\left[\frac{p^2x(1-x) - m^2}{-\mu^2}\right] + \log(4\pi)\right)$$

$$\int \frac{d^Dp}{\left[p^2 - 2kp + m^2\right]^\alpha} = i\frac{\Gamma(\alpha - D/2)}{\Gamma(\alpha)}\frac{(-\pi)^{D/2}}{\left[m^2 - k^2\right]^{\alpha - D/2}}$$

или дифференцируя

$$\int \frac{d^{4-2\epsilon}p}{[p^2-2kp+m^2]^2} = i\frac{\Gamma(\epsilon)}{\Gamma(2)}\frac{(-\pi)^{2-\epsilon}}{[m^2-k^2]^\epsilon}, \quad \Gamma(\epsilon)\sim\frac{1}{\epsilon}\to\infty$$
$$\int \frac{d^{4-2\epsilon}pp_\mu}{[p^2-2kp+m^2]^2} = i\frac{\Gamma(\epsilon)}{\Gamma(2)}\frac{(-\pi)^{2-\epsilon}k_\mu}{[m^2-k^2]^\epsilon}$$
$$\int \frac{d^{4-2\epsilon}pp_\mu p_\nu}{[p^2-2kp+m^2]^2} = i(-\pi)^{2-\epsilon}\left[\frac{\Gamma(\epsilon)}{\Gamma(2)}\frac{k_\mu k_\nu}{[m^2-k^2]^\epsilon}+\frac{g^{\mu\nu}}{2}\frac{\Gamma(\epsilon-1)}{\Gamma(2)}\frac{1}{[m^2-k^2]^{\epsilon-1}}\right]$$

$$\int \frac{d^Dk}{(k^2)^\alpha} = 0, \quad \forall\alpha.$$

На самом деле здесь происходит сокращение ультрафиолетовой и инфракрасной расходимостей, которые обе имеют вид полюса по $1/\varepsilon$.

О регуляризации обрезанием на верхнем пределе

(спишу у Казакова, пока не очень готов к этому)

О регуляризации Паули-Вилларса

$$\frac{1}{p^2-m^2}\to\frac{1}{p^2-m^2}-\frac{1}{p^2-M^2},$$

где $M\to\infty$ - масса полей Паули-Вилларса.

Регуляризация Паули-Вилларса является лоренц и калибровочная инвариантность (в абелевом случае), но она весьма сложна при вычислениях, поскольку приходится вычислять диаграммы с массами, в то время как безмассовые интегралы заметно проще.

Иногда используют разновидности регуляризации Паули - Вилларса, когда замену (2.9) производят не для каждого пропагатора, а для петли в целом. Такой способ регуляризации называется регуляризацией по циклам. Этот приём применяется в абелевых калибровочных теориях для петель образованных полями материи. При этом удаётся сохранить калибровочную инвариантность. Однако в неабелевых теориях мы сталкиваемся с трудностью в связи с наличием петель образованных калибровочными полями, которым нельзя придать массу, не нарушая калибровочную инвариантность. Эту проблему часто решают путём введения дополнительной регуляризации для векторных частиц, например, с помощью высших производных.

Основные критические соображения об адекватности регуляризаций (???????)

(??? пока просто их прошел, критически особо не думал про них. Может, каждый способ - бред, не знаю пока.)

Основное о переходе к физически осмысленным значениям (?????)

(очень важный вопрос, как же конкретно к ним приходить, тут раздел про это.)

Анализ возможности регуляризовать теорию

Иллюстрация

$$\begin{aligned}
&\mathcal{L}_{\varphi^3} = -\lambda\varphi^3 && \Rightarrow\ [\lambda] = 1, && \text{СуперРен.}\\
&\mathcal{L}_{\varphi^4} = -\lambda\varphi^4 && \Rightarrow\ [\lambda] = 0, && \text{Рен.}\\
&\mathcal{L}_{QED} = e\overline{\psi}\gamma^\mu A_\mu\psi && \Rightarrow\ [e] = 0, && \text{Рен.}\\
&\mathcal{L}_{gauge} = -\tfrac{1}{4}F^2_{\mu\nu} = -\tfrac{1}{4}\left[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu\right]^2 && \Rightarrow\ [g] = 0, && \text{Рен.}\\
&\mathcal{L}_{Yukawa} = y\overline{\psi}\varphi\psi && \Rightarrow\ [y] = 0. && \text{Рен.}
\end{aligned}$$

Итак, все эти модели перенормируемы.

$$\begin{aligned}
&\mathcal{L} = -h\varphi^6 && \Rightarrow\ [h] = -2, && \text{НеРен.}\\
&\mathcal{L} = G\overline{\psi}\psi\overline{\psi}\psi && \Rightarrow\ [G] = -2 && \text{НеРен.}\\
&\mathcal{L} = \kappa\overline{\psi}\partial_\mu V_\mu\psi && \Rightarrow\ [\kappa] = -1 && \text{НеРен.}\\
&\mathcal{L} = \gamma\overline{\psi}\partial_\mu\varphi\gamma^\mu\psi && \Rightarrow\ [\gamma] = -1. && \text{НеРен.}
\end{aligned}$$

3.4.7 Systematics of Renormalization after Peskin & Schroeder

Приглашение к теме: ультрафиолетовое обрезание и критические флуктуации

Подсчет ультрафиолетовых расходимостей

Перенормируемая теория возмущений

Перенормировка квантовой электродинамики

Перенормировка в высших порядках

Пример диаграммы с двумя петлями

3.4.8 Renormalization and Symmetry after Peskin & Schroeder

Спонтанное нарушение симметрии

Перенормировка и симметрия: явный пример

Эффективное действие

Вычисление эффективного действия

Эффективное действие как производящий функционал

Перенормировка и симметрии: общий анализ

3.4.9 On the Renormalization Group (???)

(по Пескину Шредеру структуру заготовил, много где это есть!)

Вильсоновский подход к теории перенормировок

уравнение Каллана-Симанчика

Эволюция констант связи

Перенормировка локальных операторов

Эволюция массовых параметров

3.4.10 symptotic Freedom and the Renormalization Group

(?? в чем ее суть - пока так все еще хз.)

Об асимптотической свободе (?????)

(вот тут спишу основные формулы.)

Асимптотическая свобода по Владимирову

обзор

Изобразим историю борьбы с расходимостями в виде схемы:

$$F(k,g,m) \xrightarrow{\infty} F_{\text{reg}}(k,M,g,m) \overset{M\to\infty}{\longrightarrow} F_{\text{ren}}(k,\mu,g,m) \xrightarrow{\text{РГ}} F_{\text{phys}}(k,\Lambda,m)$$

Здесь k условно обозначает все импульсные аргументы, μ можно понимать как $\sqrt{\mu^2}$????, и т.п.

Смысл этой диаграммы такой.

Изначально можно было ожидать, что некая интересующая нас функция F будет зависеть от внешних импульсов k, константы связи g и массы m.

Но вмешались бесконечности, и нам пришлось ввести регуляризацию (M).

Добавив в лагранжиан сингулярные контрчлены, мы смогли перейти к пределу $M \to \infty$ и получить перенормированную функцию F_{ren}, свободную от бесконечностей.

(заработав, правда, при этом новый параметр μ; такого рода ренормировочный параметр с неизбежностью появляется при любой регуляризации).

Далее мы освободимся и от μ, ибо на самом деле F_{ren} окажется функцией от некоторой комбинации $\Lambda(\mu, g)$, вобравшей в себя всю зависимость от μ и g.

Тем самым, эффективное число аргументов функции F вернется к прежнему значению, только вместо безразмерного заряда g будет фигурировать размерная величина Λ.

Этот оставшийся шаг основан на идеях ренормгруппы и тесно связан с вопросом об однозначности определения функции F_{ren} в процессе перенормировки.

С одной стороны, сингулярные компенсирующие вклады (контрчлены) произвольно менять нельзя, иначе не сократятся расходимости.

Но с другой - никто не мешает, наряду с бесконечными, добавлять к лагранжиану и конечные поправки, тем более что такие действия вполне укладываются в рамки произвола, изначально заложенного в определение T-экспоненты. ?????

С физической же точки зрения, добавление контрчленов (конечных и даже бесконечных) может быть оправдано тем, что истинные значения g, m и других возможных параметров лагранжиана все равно неизвестны, и выясняются только по результатам сравнения предсказаний теории с экспериментом.

?????

Эффективный заряд и β -функция

В основе ренормгруппы лежит следующий фундаментальный принцип ренорм-инвариантности, эквивалентный утверждению о перенормируемости теории: добавление любых конечных (несингулярных) контрчленов в точности компенсируется конечным изменением масс и зарядов.

Под точной компенсацией имеется в виду неизменность любых физических величин, вычисляемых в данной теории (нефизические величины типа n -хвосток могут при этом приобретать дополнительный множитель).

??? пример: изменение значения ренормировочного параметра

Рассмотрим очевидный (хотя и не самый общий) пример введения конечных контрчленов - изменение значения ренормировочного параметра $\mu \to \mu'$.

Отвечающее ему конечное изменение заряда

$$g \to g' = \bar{g}\left(\frac{\mu'}{\mu}, g\right) \quad \Longrightarrow \quad f\left(\frac{k}{\mu}, g\right) = f\left(\frac{k}{\mu'}, g'\right) = f\left(\frac{k}{\mu'}, \bar{g}\left(\frac{\mu'}{\mu}, g\right)\right)$$

подбирается так, чтобы физика (в лице безразмерной функции f) оставалась неизменной.

Функция $\bar{g}\left(\mu'/\mu, g\right)$ - эффективный заряд или бегущая константа связи (running coupling), задает преобразование заряда $g \to g'$ в явном виде.

(Для простоты изложения мы забываем про массы; в общем случае надо еще вводить эффективную массу $\bar{m}$).

Исходно, термин ренормгруппа относился именно к этой однопараметрической группе преобразований $\mu \to \mu', g \to g'$.

По сути, речь идет о функциональной зависимости $g(\mu)$, налагаемой требованием физической инвариантности и реализуемой функцией $\bar{g}$.

Очевидное условие согласования для $\bar{g}$,

$$g'' = \bar{g}\left(\frac{\mu''}{\mu}, g\right) = \bar{g}\left(\frac{\mu''}{\mu'}, g'\right) = \bar{g}\left(\frac{\mu''}{\mu'}, \bar{g}\left(\frac{\mu'}{\mu}, g\right)\right),$$

выражает тот факт, что изменения $\mu \to \mu' \to \mu''$ можно проделать и за один шаг, и за два.

β -функция

$$\beta(g) \doteq \left.\frac{dg}{d\ln\mu}\right|_{\text{физ} = \text{const}} = \left.\frac{dg'}{d\ln\mu'}\right|_{g'=g} = \left.\frac{\partial}{\partial \ln t}\bar{g}(t,g)\right|_{t=1} = \left.\frac{\partial}{\partial t}\bar{g}(t,g)\right|_{t=1}$$

β -функция - скорость изменения заряда g в ответ на изменение μ, иначе говоря -производная функции $\bar{g}$:

С ее помощью ренормгрупповую зависимость $g(\mu)$ можно представить в виде

$$d\ln\mu = \frac{dg}{\beta(g)} \equiv d\psi(g) \quad \Longleftrightarrow \quad d\Lambda = 0, \quad \psi(g) = \int^{g} \frac{du}{\beta(u)}, \quad \Lambda(\mu, g) = \mu e^{-\psi(g)}$$

Точкам на плоскости $\{\mu, g\}$, связанным преобразованиями ренормгруппы, соответствует одна и та же физика, и одно и то же значение $\Lambda(\mu, g)$.

Это и означает, что наблюдаемые величины реально зависят от Λ, а не от μ и g по отдельности.

вычисление β -функций

Несколько слов о методике практического вычисления β -функций.

Если вкратце, то считать нужно (так)???????

Как видно из (256), ренормгрупповую зависимость $g(\mu)$ можно задавать не только в форме (262), но и условием $g_B = \text{const}$.

Значит, определение β -функции можно переписать и так:

$$\beta(g) = \left.\frac{dg}{d\ln\mu}\right|_{g_B=\text{ const}} = -\frac{\partial g_B/\partial \ln\mu}{\partial g_B/\partial g}.$$

Например, в M -регуляризации сингулярный заряд есть формальный ряд вида

$$g_B\left(\ln\frac{M}{\mu}, g\right) = g + a(g)\ln\frac{M}{\mu} + b(g)\ln^2\frac{M}{\mu} + \mathcal{O}\left(\ln^3\frac{M}{\mu}\right)$$

поэтому

$$\beta(g) = \frac{a(g) + 2b(g)\ln(M/\mu) + ...}{1 + a'(g)\ln(M/\mu) + ...} = a(g) + [2b(g) - a(g)a'(g)]\ln\frac{M}{\mu} + \mathcal{O}\left(\ln^2\frac{M}{\mu}\right)$$

??????

Все сингулярности в выражении для $\beta(g)$ обязаны сократиться, так что из правой части мы извлекаем не только ответ для β -функции, но и явные формулы, выражающие через нее функции от коэффициентов???? при любых старших степенях $\ln(M/\mu)$:

$$\beta(g) = a(g), \quad b(g) = \frac{1}{2}a(g)a'(g) = \frac{1}{2}\beta(g)\beta'(g), ...$$

ε-регуляризация

Аналогичное вычисление в ε -регуляризации имеет ту особенность, что в определении (263) надо еще взять предел $\varepsilon \to 0$:

$$g_B = \mu^\varepsilon\left(g + \frac{a(g)}{\varepsilon} + \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)\right), \quad \beta(g) = -\lim_{\varepsilon\to 0}\frac{\partial g_B/\partial\ln\mu}{\partial g_B/\partial g} = \left(g\frac{\partial}{\partial g} - 1\right)a(g)$$

Несмотря на некоторое различие правил нахождения β -функции в этих двух подходах, общий принцип ясен: $\beta(g)$ определяется коэффициентной функцией при самой младшей сингулярности, а коэффициенты при всех старших через нее выражаются.

Уравнения ренормгруппы

Перепишем условия

$$g \to g' = \bar{g}\left(\frac{\mu'}{\mu}, g\right) \quad \Longrightarrow \quad f\left(\frac{k}{\mu}, g\right) = f\left(\frac{k}{\mu'}, g'\right) = f\left(\frac{k}{\mu'}, \bar{g}\left(\frac{\mu'}{\mu}, g\right)\right)$$

и

$$g'' = \bar{g}\left(\frac{\mu''}{\mu}, g\right) = \bar{g}\left(\frac{\mu''}{\mu'}, g'\right) = \bar{g}\left(\frac{\mu''}{\mu'}, \bar{g}\left(\frac{\mu'}{\mu}, g\right)\right),$$

в более удобном виде

$$f(xt,g)=f(x,\bar g(t,g)),\quad \bar g(xt,g)=\bar g(x,\bar g(t,g)),\quad \bar g(1,g)=g$$

Дифференцируя второе из этих равенств по x в точке $x=1$, и привлекая определение β -функции $\beta(g)\doteq\frac{dg}{d\ln\mu}\Big|_{\text{физ}=\text{const}}$, получим ренормгрупповое уравнение для $\bar g(t,g)$:

$$t\frac{\partial}{\partial t}\bar g(t,g)=\beta(\bar g(t,g))$$

Выразить t -зависимость функции $\bar g(t,g)$, считая g просто параметром, можно и в виде следующих соотношений, эквивалентных уравнению ренормгруппы:

$$d\ln t=\frac{d\bar g(t,g)}{\beta(\bar g(t,g))}=d\psi(\bar g(t,g)),\quad \ln t=\psi(\bar g(t,g))-\psi(g)=\int\limits_g^{\bar g(t,g)}\frac{du}{\beta(u)}$$

Аналогично, дифференцируя два первых равенства в условии согласования для $\bar g$ по t при $t=1$, а потом переобозначая x через t, приходим к альтернативной форме уравнений ренормгруппы:

$$\left(t\frac{\partial}{\partial t}-\beta(g)\frac{\partial}{\partial g}\right)\bar g(t,g)=0,\quad \left(t\frac{\partial}{\partial t}-\beta(g)\frac{\partial}{\partial g}\right)f(t,g)=0$$

Ясно, что с учетом начального условия $\bar g(1,g)=g$ эффективный заряд $\bar g(t,g)$ однозначно восстанавливается по функции $\beta(g)$ с помощью любого из уравнений реномгруппы, в этом смысле они эквивалентны.

восстановление всего ряда по коэффициентам

Далее, подставляя формальное разложение $\bar g(t,g)$ по степеням $\ln t$ в первое равенство уравнения реномгруппы, получаем

$$\bar g(t,g)=g+\beta(g)\ln t+b(g)\ln^2 t+...,\quad b(g)=\frac{1}{2}\beta(g)\beta'(g),\quad ...$$

и приходим к следующему выводу: зная коэффициент при первой степени $\ln t$, т.е. функцию $\beta(g)$, мы знаем и весь ряд (271).

Другими словами, ренормгруппа восстанавливает все логарифмические вклады по самому младшему из них.

Особенно важно, что при этом мы иногда имеем возможность улучшать свойства рядов теории возмущений.

Пусть t в $f(xt,g)=f(x,\bar g(t,g))$ имеет смысл большого по величине импульсного аргумента (точнее, большого множителя, равномерно увеличивающего все импульсы, ибо неоднородные асимптотики часто выходят за рамки возможностей ренормгруппы).

??? асимптотическая свобода

Мы видим, что асимптотическая зависимость f от импульса эффективно перешла в заряд, оправдывая название функции $\bar{g}$. ?????

Если же $\bar{g}(t,g)$ при больших t еще и стремится к нулю, то действительно можно говорить об улучшении теории возмущений: вместо исходного заряда g (малым параметром, возможно, не являющегося) мы получаем право использовать асимптотически малую величину $\bar{g}(t,g)$ в качестве нового параметра разложения.

?????

Такая ситуация носит название асимптотической свободы и осуществляется при отрицательном знаке однопетлевого вклада в β -функцию: $\beta(g) = ag^2 + \mathcal{O}(g^3)$, $a < 0$.

логарифмическое убывание заряда

Ограничиваясь этим приближением для $\beta(g)$, получим из ренормгруппового уравнения

$$t\frac{\partial}{\partial t}\bar{g}(t,g) = a\bar{g}^2(t,g) \quad \Longrightarrow \quad \bar{g}(t,g) = \frac{g}{1-ag\ln t} \sim \frac{1}{|a|\ln t}$$

то есть эффективный заряд в этом случае логарифмически убывает при больших t.

По сути, единственным примером асимптотически свободной теории служит КХД, в которой $a \sim \left(-11 + \frac{2}{3}n_f\right)$, где n_f -число сортов кварков.

По современным представлениям $n_f = 6$, благодаря чему a оказывается отрицательным.

Как было отмечено выше, у n -хвосток и им подобных объектов (обозначим их Γ) закон ренормгруппового преобразования отличается от (267) на фактор Z_Γ:

$$\Gamma(xt,g) = Z_\Gamma^{-1}(t,g)\Gamma(x,\bar{g}(t,g)), \quad Z_\Gamma(xt,g) = Z_\Gamma(t,g)Z_\Gamma(x,\bar{g}(t,g)), Z_\Gamma(1,g) = 1$$

Дифференцированием получаем отсюда уравнения ренормгруппы для Z_Γ и Γ:

$$t\frac{\partial}{\partial t}\ln Z_\Gamma(t,g) = \gamma_\Gamma(\bar{g}(t,g)), \quad \left(t\frac{\partial}{\partial t} - \beta(g)\frac{\partial}{\partial g} + \gamma_\Gamma(g)\right)\Gamma(t,g) = 0$$

Введенную здесь функцию

$$\gamma_\Gamma(g) \doteq \frac{\partial}{\partial t}\ln Z_\Gamma(t,g)\Big|_{t=1}$$

называют аномальной размерностьто.

Ее, как и $\beta(g)$, вычисляют по теории возмущений.

Решения уравнений (274) можно, с помощью (269), записать в виде

$$Z_\Gamma(t,g) = \exp\int_1^t \frac{du}{u}\gamma_\Gamma(\bar{g}(u,g)) = \exp\int_g^{\bar{g}(t,g)} \frac{du\gamma_\Gamma(u)}{\beta(u)}$$

$$\Gamma(xt,g) = \Gamma(x,\bar{g}(t,g))\exp\left[-\int_g^{\bar{g}(t,g)} \frac{du\gamma_\Gamma(u)}{\beta(u)}\right]$$

чтобы затем разлагать (в случае асимптотической свободы) по степеням $\bar{g}(t,g)$.

Схемная зависимость

Нам осталось объяснить, как осуществляется переход к аргументу Λ.

Совместим это с изучением проблемы зависимости ренормгрупповых величин от схемы перенормировки.

То, что различные варианты процедуры перенормировки (например, основанные на разных регуляризациях) приводят к различным выражениям для β -функции и многих других объектов, не удивительно.

Однако физические величины не должны, по идее, зависеть от этого.

Снова переписав ключевое соотношение (259), на этот раз в форме

$$f\left(\frac{k}{\mu}, g\right) = f\left(1, \bar{g}\left(\frac{k}{\mu}, g\right)\right) \doteq \Phi\left(\ln\frac{k}{\Lambda}\right)$$

можно ожидать, что схемная зависимость, если она есть в функциях f и $\bar{g}$, должна каким-то образом сократиться.

При этом выяснится, что функция Φ от перенормировочной схемы практически не зависит.

Согласно принципу ренорм-инвариантности, разные способы перенормировки могут отличаться лишь на конечные контрчлены.

Если условиться в обеих схемах обозначать ренормировочный параметр через μ, то всё различие схем должно свестись к переопределению заряда $g \to \tilde{g}(g)$ (массы мы не рассматриваем).

Соответственно, для физических (f) и нефизических (Γ) объектов формулы перехода имеют вид

$$f\left(\frac{k}{\mu}, g\right) = \tilde{f}\left(\frac{k}{\mu}, \tilde{g}(g)\right), \quad \Gamma\left(\frac{k}{\mu}, g\right) = q(g)\tilde{\Gamma}\left(\frac{k}{\mu}, \tilde{g}(g)\right)$$

Далее, прямыми следствиями определений (262) являются равенства

$$\frac{dg}{\beta(g)} = d\psi(g) = d\ln\mu = d\tilde{\psi}(\tilde{g}) = \frac{d\tilde{g}}{\tilde{\beta}(\tilde{g})}$$

которые говорят о том, что весь схемный произвол в $\psi(g)$ сводится к аддитивной, а в Λ -к мультипликативной константе (не зависящей от g):

$$\tilde{\psi}(\tilde{g}(g)) = \psi(g) + \text{ const}, \quad \tilde{\Lambda}(\mu, \tilde{g}(g)) = \mu\exp(-\tilde{\psi}(\tilde{g}(g))) = \Lambda(\mu, g)\cdot \text{ const}.$$

Впрочем, при внимательном рассмотрении формул (262) становится ясно, что именно такие допуски были предусмотрены уже в определениях этих функций.

Приведем также вытекающее из (280) правило пересчета β -функции в другую схему, а заодно и аналогичную формулу для аномальной размерности:

$$\tilde{\beta}(\tilde{g}(g)) = \beta(g)\frac{d\tilde{g}(g)}{dg}, \quad \tilde{\gamma}_\Gamma(\tilde{g}(g)) = \gamma_\Gamma(g) - \beta(g)\frac{d\ln q(g)}{dg}$$

Применяя формулу для β в стандартной ситуации

$$\beta(g) = ag^2 + bg^3 + cg^4 + ..., \quad \tilde{\beta}(g) = \tilde{a}g^2 + \tilde{b}g^3 + \tilde{c}g^4 + ..., \quad \tilde{g}(g) = g + Ag^2 + ...$$

наблюдаем любопытный эффект: два первых коэффициента β -функции не зависят от схемы, зависимость начинается только с третьего, $\tilde{a} = a, \tilde{b} = b, \tilde{c} \neq c$.

Чтобы найти связь между $\bar{g}$ и Λ, а также явный вид Φ, объединим (269) и (278) :

$$\psi\left(\bar{g}\left(\frac{k}{\mu}, g\right)\right) = \psi(g) + \ln\frac{k}{\mu} = \ln\frac{k}{\Lambda} \doteq L, \quad \bar{g} = \psi^{-1}(L), \quad f(1, \bar{g}) = \Phi(L)$$

для перехода к (более физическому) аргументу L надо обратить функцию ψ.

После этого, подстановка результата в f даст Φ.

Применим (279) для выяснения схемной зависимости функции Φ :

$$\Phi(\psi(x)) = f(1, x) = \tilde{f}(1, \tilde{g}(x)) = \tilde{\Phi}(\tilde{\psi}(\tilde{g}(x))) = \tilde{\Phi}(\psi(x) + \text{ const })$$

Она, как видим, исчерпывается возможностью сдвига аргумента на константу, что как раз соответствует мультипликативному произволу в Λ или аддитивному в L.

Явное нахождение $\psi(g)$ потребует разложения подынтегрального выражения из определения (262) в формальный ряд.

Типичный вид ответа таков:

$$\beta(g) = ag^2 + bg^3 + \mathcal{O}\left(g^4\right) \quad \Longrightarrow \quad \psi(g) = -\frac{1}{ag} - \frac{b}{a^2}\ln g + \text{ const } + \mathcal{O}(g)$$

Обратная функция ψ^{-1}, нужная нам для замены $\bar{g}$ на L, тоже ищется в виде формального разложения.

Окончательные результаты обычно выглядят примерно так:

$$\bar{g} = \psi^{-1}(L) = -\tfrac{1}{aL} - \tfrac{b}{a^3}\tfrac{\ln L}{L^2} + \tfrac{c}{L^2} + \mathcal{O}\left(\tfrac{\ln^2 L}{L^3}\right),$$
$$f\left(\tfrac{k}{\mu}, g\right) = f(1, \bar{g}) = \Phi(L) = 1 + \tfrac{A}{L} + \tfrac{B\ln L}{L^2} + \tfrac{C}{L^2} + \mathcal{O}\left(\tfrac{\ln^2 L}{L^3}\right) = \sum_{mn} A_{mn}\tfrac{\ln^m L}{L^n}$$

Преимущество разложений по $1/L$ вида (287) над рядами улучшенной теории возмущений по степеням $\bar{g}$ заключается в их практической независимости от схемы перенормировки.

Точнее говоря, наборы коэффициентов A_{mn} в двух разных схемах могут поначалу различаться, но одна-единственная операция - сдвиг L на надлежащую константу - эту разницу полностью устраняет.

Вильсоновский подход к теории перенормировок по ПШ (?????)

Асимптотическая свобода по Полякову (????????)

(???? тут жесть какая-то, отложил пока что, потом мб посмотрю, как основное будет сделано.)

Главные киральные поля

Модель п-поля

Неабелевы калибровочные поля при D = 4

3.5 Free quantum fields: other properties

3.5.1 Dirac field: other properties

(пока тут, всякие не такие актуальные свойства, их скорее всего много)

Электронный пропагатор

По Мандл Шалуу

$$\left\langle 0 \left| \ \mathrm{T}\left\{\psi(\mathrm{x})\bar{\psi}\left(x'\right)\right\}\right| 0\right\rangle = \mathrm{i}S_{\mathrm{F}}\left(x-x'\right)$$

$$S_{\mathrm{F}}(x) := \theta(t)S^{+}(x) - \theta(-t)S^{-}(x) = \left(\mathrm{i}\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} + \frac{mc}{\hbar}\right)\Delta_{\mathrm{F}}(x)$$

(??? тут еще текст какой-то, который я вообще не понимаю)

По Ландау (????)

(лл пар 105 хз, что-то очень далеко этот параграф, мб тут и не такое нужное?)
Подобно фотонному, точный электронный пропагатор определяется формулой

$$\mathscr{G}_{ik}\left(x-x'\right) = -i\left\langle 0 \left| \ \mathrm{T}\psi_i(x)\bar{\psi}_k\left(x'\right)\right| 0\right\rangle$$

(i, k - - биспинорные индексы), отличающейся от определения $(75,1)$ пропагатора свободных частиц

$$G_{ik}\left(x-x'\right) = -i\left\langle 0 \left| \ \mathrm{T}\psi_i^{\mathrm{int}}\ (x)\bar{\psi}_k^{\mathrm{int}}\ \left(x'\right)\right| 0\right\rangle$$

заменой ψ-операторов в представлении взаимодействия гейзенберговскими.

Те же рассуждения, что и при выводе $(103,7)$, позволяют преобразовать $\mathscr{G}_{ik}$ к виду

$$\mathscr{G}_{ik}\left(x-x'\right) = -i\frac{\left\langle 0 \left| \ \mathrm{T}\psi_i^{\mathrm{int}}\ (x)\bar{\psi}_k^{\mathrm{int}}\ \left(x'\right)s\right| 0\right\rangle}{\langle 0|S|0\rangle}.$$

(тут картинки типа поляризации вакуума, их сумма, вставлю мб.)

Путем суммирования, в точности аналогичного выводу $(103,13)$, получим

$$\mathscr{G}(p) = G(p) + G(p)\mathscr{M}(p)\mathscr{G}(p)$$

(биспинорные индексы опускаем), или, для обратных матриц,

$$\mathscr{G}^{-1}(p) = G^{-1}(p) - \mathscr{A}(p) = \gamma p - m - \mathscr{M}(p).$$

В результате преобразования $(102,8)$ пропагаторы $\mathscr{D}(103,1)$ и $\mathscr{G}(105,1)$ переходят в

$$\mathscr{D}_{\mu\nu} \to i\left\langle 0 \left| \ \mathrm{T}\left[A_{\mu}(x) - \partial_{\mu}\chi(x)\right]\left[A_{v}\left(x'\right) - \partial'_{v}\chi\left(x'\right)\right]\right| 0\right\rangle,$$

$$\mathscr{G}_{ik} \to -i\left\langle 0 \left| \ \mathrm{T}\psi_i(x)e^{ie\chi(x)}e^{-ie\chi(x')}\bar{\psi}_k\left(x'\right)\right| 0\right\rangle.$$

$$\left\langle 0 \left| \ \mathrm{T}e^{ie\delta\chi(x)}e^{-ie\delta\chi(x')}\right| 0\right\rangle \approx -\frac{1}{2}e^2\left\langle 0 \left|\delta\chi^2(x) + \delta\chi^2\left(x'\right) - 2\ \mathrm{T}\delta\chi(x)\delta\chi\left(x'\right)\right| 0\right\rangle.$$

С учетом определения $(105,12)$ находим, таким образом, следующий закон преобразования электронного пропагатора:

$$\mathscr{G} \to \mathscr{G} + \delta\mathscr{G}, \qquad \delta\mathscr{G} = ie^{2\mathscr{G}}\left(x-x'\right)\left[d^{(l)}(0) - d^{(l)}\left(x-x'\right)\right].$$

В импульсном представлении [1]):

$$\delta\mathscr{G}(p) = ie^2\int d^{(l)}(q)[\mathscr{G}(p) - \mathscr{G}(p-q)]\frac{d^4q}{(2\pi)^4}.$$

При этом функция $d^{(l)}(q)$ связана с изменением функции $\mathscr{D}^{(l)}$ согласно

$$\delta\mathscr{D}^{(l)}(q) == q^2 d^{(l)}(q)$$

Symmetries and Conserved Quantities

• Conjugating a rotation by parity leaves it invariant, but conjugating a boost with parity flips its sign. Thus we want parity to exchange the Weyl spinors,

$$P : \psi_{\pm}(\mathbf{x}, t) \mapsto \psi_{\mp}(-\mathbf{x}, t).$$

Thus, in the chiral representation,

$$P : \psi(\mathbf{x}, t) \mapsto \gamma^0 \psi(-\mathbf{x}, t)$$

The Dirac equation is parity invariant, since if $\psi(\mathbf{x}, t)$ satisfies it, so does $\gamma^0\psi(-\mathbf{x}, t)$.

• Note that $\bar{\psi}\psi$ transforms as a scalar under parity because $(\gamma^0)^2 = 1$. However, for the vector $\bar{\psi}\gamma^\mu\psi$, the spacelike parts flip sign,

$$\bar{\psi}\gamma^i\psi \to \bar{\psi}\gamma^0\gamma^i\gamma^0\psi = -\bar{\psi}\gamma^i\psi$$

which implies that $\bar{\psi}\gamma^\mu\psi$ transforms as a vector under parity.

• Similarly, $\bar{\psi}S^{\mu\nu}\psi$ transforms as a tensor under parity. We also have $\bar{\psi}\gamma^5\psi$, which transforms as a pseudoscalar, and $\bar{\psi}\gamma^5\gamma^\mu\psi$, which transforms as a pseudovector; both of these pick up an extra sign flip due to the γ^5.

• As we've seen above, the Dirac spinor is a certain representation of the Lorentz group and the set of spinor bilinears is that representation's tensor product with itself,

$$((1/2,0)+(0,1/2)) \times ((1/2,0)+(0,1/2)) = (0,0)+(1/2,1/2)+(1,0)+(0,1)+(1/2,1/2)+(0,0)$$

which decompose as $1 + 4 + 3 + 3 + 4 + 1$. The two 3's are packaged together into the rank 2 antisymmetric tensor and correspond to self-dual and anti-self-dual tensors.

• The pseudoscalar and axial vector terms treat the $u_{\pm}$asymmetrically since they are not parity invariant. In general, any theory that treats the $u_{\pm}$asymmetrically is called a chiral theory (vs. a vector-like theory). However, chiral theories can be parity invariant; for example, the term $\phi\bar{\psi}\gamma^5\psi$ respects parity if ϕ is a pseudoscalar.

Note. Sometimes people ask: why don't the gamma matrices transform under Lorentz transformations, like everything else with a four-vector index does? The representation theory above shows the right way to think about this. A gamma matrix is a set of Clebsch-Gordan coefficients which extracts the Lorentz vector representation from the tensor product of two Dirac spinor representations. These coefficients don't change under Lorentz transformations, for exactly the same reason that ordinary Clebsch-Gordan coefficients for spin don't. E.g. for spin we have $1 \subset 1/2 \times 1/2$, and the state $|\ell, m_z\rangle = |1, 0\rangle$ is always of the form $(|1/2, 1/2\rangle|1/2, -1/2\rangle + |1/2, -1/2\rangle|1/2, 1/2\rangle)/\sqrt{2}$. This doesn't depend on how you orient the z-axis, as long as you use the same z-axis to define the m_z states in the 1 and 1/2 representations, which of course we always do.

We now define charge conjugation symmetry and Majorana fermions.

• Naively, we would like to define charge conjugation of a classical field as just the complex conjugate, as is suitable for a complex scalar field. But charge conjugation should commute with Lorentz transformations, so the naive option doesn't work since $S[\Lambda]$ is not real.

• Instead, we define

$$\psi^{(c)} = C\psi^*$$

where C is a 4×4 unitary matrix satisfying

$$C^\dagger \gamma^\mu C = (-\gamma^\mu)^*$$

in a general representation of the Clifford algebra where $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$.

• To check the transformation properties, we have

$$\psi^{(c)} \to CS[\Lambda]^*\psi^* = S[\Lambda]C\psi^* = S[\Lambda]\psi^{(c)}$$

as desired. The Dirac equation is also invariant. Complex conjugating it gives

$$(-i\not{\partial}* - m)\,\psi^* = 0$$

Multiplying on the left by C and pulling it through shows that $\psi^{(c)}$ obeys the Dirac equation.

• A Majorana spinor obeys the reality condition $\psi^{(c)} = \psi$. This constraint relates the two Weyl spinors. Majorana spinors can have a mass, but they cannot have charges.

• Finally, we can write C explicitly given a fixed basis for the Clifford algebra. In the chiral basis,

$$C = i\gamma^2$$

because only γ^2 is imaginary. In terms of the Weyl spinors, this implies $u_- = -i\sigma^2 u_+^*$.

• Alternatively, in the Majorana basis,

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$$

Then all of the gamma matrices are pure imaginary, so the $S[\Lambda]$ are real and $C = 1$.

Note. This definition of the C matrix differs from the C matrix appearing in the quantum charge conjugation operation $\widehat{C}$. This is because this C arises from the classical charge conjugation on fields, which in this case acts like $\widehat{C}$ with an additional parity factor γ_0. This is explained further in the notes on the Standard Model.

Note. We check the Lagrangian is invariant under charge conjugation. Consider the mass term,

$$\bar{\psi}\psi = \psi^\dagger\gamma^0\psi \to \psi^T C^\dagger \gamma^0 C\psi^* = -\psi^T \left(\gamma^0\right)^* \psi^*$$

This quantity is, naively, equal to its own transpose. However, we must account for the fact that spinors are inherently anticommuting, and hence the classical fields here are really Grassmann numbers; taking the transpose flips the sign because of the anticommutation. Then we get

$$\bar{\psi}\psi \to \psi^\dagger {\gamma^0}^\dagger \psi = \bar{\psi}\psi$$

as desired. Demanding the invariance of the $\bar{\psi}\not{D}\psi$ term forces

$$A_\mu \to -A_\mu$$

under charge conjugation, which intuitively is because $\partial^2 A_\mu = j_\mu$ and $j_\mu \to -j_\mu$. More generally, for a non-abelian gauge theory, the gauge interaction term $\bar{\psi}_i \mathcal{A}_{ij}\psi_j$ is invariant if

$$A^\mu_{ij} \to -A^\mu_{ji}$$

This is sensible, as it's just the usual rule for conjugating a Lie algebra representation.

Finally, we consider the continuous symmetries and conserved currents of the Dirac action.

• We begin with translational symmetry, treating ψ and $\bar{\psi}$ as independent as usual. Since the Lagrangian does not depend on $\partial\bar{\psi}$, we have

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\mathcal{L}$$

We can further simplify this using the equations of motion, which state that $\mathcal{L}$ vanishes on shell. The total energy is then

$$E = \int d\mathbf{x}T^{00} = \int d\mathbf{x}i\bar{\psi}\gamma^0\dot{\psi} = \int d\mathbf{x}\psi^\dagger\gamma^0\left(-i\gamma^i\partial_i + m\right)\psi$$

• An infinitesimal Lorentz transformation gives

$$\delta\psi^\alpha = -\omega^\mu_\nu x^\nu\partial_\mu\psi^\alpha + \frac{1}{2}\Omega_{\rho\sigma}\left(S^{\rho\sigma}\right)^\alpha_\beta\psi^\beta$$

Expanding out the definitions yields $\omega^{\mu\nu} = \Omega^{\mu\nu}$. Now, the first term contributes the same thing we saw for the scalar field, while the second term contributes a "spin" term,

$$(\mathcal{J}^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} - i\bar{\psi}\gamma^\mu S^{\rho\sigma}\psi$$

Upon quantization the latter will yield a spin of $\pm 1/2$ for each particle. A similar spin/orbit decomposition appears for general fields with spin.

• The Dirac action has an internal phase symmetry, $\psi \to e^{-i\alpha}\psi$ which yields

$$j^\mu_V = \bar{\psi}\gamma^\mu\psi \qquad \text{(vector current)}$$

$$Q = \int d\mathbf{x}\bar{\psi}\gamma^0\psi = \int d\mathbf{x}\psi^\dagger\psi$$

which we will see can be interpreted as the total particle number.

• When $m = 0$, the Weyl spinors decouple, so there is an additional independent symmetry given by rotating the Weyl spinors in opposite directions,

$$\psi \to e^{i\alpha\gamma^5}\psi, \quad \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma^5}$$

This gives the 'axial current',

$$j^\mu_A = \bar{\psi}\gamma^\mu\gamma^5\psi$$

which is an axial vector. Later, we will see that the axial symmetry is anomalous. In QED with massless fermions, the Lagrangian has axial symmetry, but the current is not conserved in the quantum theory.

3.5.2 Electromagnetic field: other properties

(пока тут, всякие не такие актуальные свойства, их скорее всего много)

Quantization of a vector field: other ideas

Gupta-Bleuler Quantization

Alternatively, we may impose the constraint at the level of the Hilbert space.

• We instead consider the new, "gauge-fixed" Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2$$

where ξ is a number. Naively, we have changed the theory, and this will be justified below by showing the S-matrix is independent of ξ. But it's more correct to say that the non-gauge-fixed theory isn't defined as a quantum theory at all, and we're really checking that the specific way we gauge fix doesn't matter. We'll derive this again in a more satisfying way with the path integral, where the extra term falls out naturally by restricting the integral to gauge-inequivalent configurations.

• To motivate this choice, note that if ξ were a field, the equations of motion would be

$$\partial^2 A^\mu - (1 - 1/\xi)\partial^\mu\partial^\nu A_\nu = 0, \quad \partial_\mu A^\mu = 0$$

so ξ acts as a Lagrange multiplier that enforces Lorenz gauge.

• However, we don't want the extra auxiliary field ξ, so we instead let ξ be a number. A natural choice is $\xi = 1$, where the equation of motion becomes $\partial^2 A^\mu = 0$ as it is in Lorenz gauge. However, note that the Lorenz gauge condition is not enforced, i.e. we don't have $\partial_\mu A^\mu = 0$.

• Next, we perform canonical quantization. The new term ensures all fields are dynamical,

$$\pi^0 = -\partial_\mu A^\mu, \quad \pi^i = \partial^i A^0 - \dot{A}^i$$

Below, it will be more convenient to integrate by parts, yielding the new canonical momenta,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu, \quad \pi^\mu = -\dot{A}^\mu$$

Next, we impose the Lorentz-invariant canonical commutation relations

$$[A_\mu(\mathbf{x}), \pi_\nu(\mathbf{y})] = i\eta_{\mu\nu}\delta(\mathbf{x} - \mathbf{y})$$

with all other commutators zero; note the extra minus sign, because of the minus sign in π^μ. We haven't imposed any constraints, so there are no issues with Dirac brackets.

Next, we perform the usual mode expansion, giving

$$A_\mu(x) = \int \frac{d\mathbf{p}}{\sqrt{2|\mathbf{p}|}} \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda(\mathbf{p}) \left(a_{\mathbf{p}}^\lambda e^{-ipx} + a_{\mathbf{p}}^{\lambda\dagger} e^{ipx}\right)$$

in Heisenberg picture, and

$$\pi_\mu(x) = \int d\mathbf{p}\sqrt{\frac{|\mathbf{p}|}{2}}(+i) \sum_{\lambda=0}^{3} \epsilon_\mu^\lambda(\mathbf{p}) \left(a_{\mathbf{p}}^\lambda e^{-ipx} - a_{\mathbf{p}}^{\lambda\dagger} e^{ipx}\right)$$

where the canonical momentum has a $+i$ rather than a $-i$ because $\pi^\mu = -\dot{A}^\mu$.

• Consider a photon of momentum p^μ and let $n^\mu = (1, 0, 0, 0)$. Then we define

$-\epsilon^0_\mu = n_\mu$ is the timelike polarization

• ϵ^1_μ and ϵ^2_μ are the transverse polarizations, obeying

$$\epsilon^i \cdot n = \epsilon^i \cdot p = 0, \quad \epsilon^i \cdot \epsilon^j = -\delta^{ij}$$

$-\epsilon^3_\mu$ is the longitudinal polarization, defined to be orthogonal to all the others,

$$\epsilon^3_\mu \propto p_\mu - n_\mu (p \cdot n)$$

and normalized to have norm -1.

E.g. when $p^\mu = (E, 0, 0, E)$, these are simply the standard basis for $\mathbb{R}^4$. Using these definitions, we have the orthogonality and completeness relations

$$\epsilon^\lambda \cdot \epsilon^{\lambda'} = \eta^{\lambda\lambda'}, \quad \epsilon^\lambda_\mu \epsilon^{\lambda'}_\nu \eta_{\lambda\lambda'} = \eta_{\mu\nu}.$$

• The resulting commutation relations for the creation/annihilation operators are

$$\left[a^\lambda_{\mathbf{p}}, a^{\lambda'\dagger}_{\mathbf{q}}\right] = -\eta^{\lambda\lambda'} \not{\delta}(\mathbf{p} - \mathbf{q}).$$

Then there is a minus sign for the timelike polarizations, which have negative norm, as we saw for the incorrectly quantized spinors. That is, our naive canonical quantization procedure hasn't given us a Hilbert space at all, because the inner product is not positive definite.

Next, we remove the unphysical degrees of freedom by imposing the Lorenz gauge constraint on the naive Fock space.

• The simplest idea is to set $\partial_\mu A^\mu = 0$ as an operator equation in Heisenberg picture, but this is unacceptable as it sets $\pi^0 = 0$. We can also attempt to identify a set of physical states by

$$\partial_\mu A^\mu |\Psi\rangle = 0$$

However, this is also too restrictive since not even the vacuum is a physical state.

• Finally, consider writing $A_\mu = A^+_\mu + A^-_\mu$where A^+_μhas positive frequency, and hence contains only annihilation operators. Then we define physical states by the Gupta-Bleuler condition

$$\partial^\mu A^+_\mu |\Psi\rangle = 0$$

so that the vacuum is indeed physical. This is equivalent to

$$\langle \Psi' | \partial_\mu A^\mu | \Psi \rangle = 0$$

That is, we demand the operator $\partial_\mu A^\mu$ is zero when restricted to the physical states.

• Explicitly, we define the one-photon states by

$$|\mathbf{p}, \xi\rangle = \sqrt{2|\mathbf{p}|}\alpha_\lambda a^\dagger_\lambda(\mathbf{p})|0\rangle, \quad \xi^\mu = \sum_{\lambda,\lambda'} \alpha_\lambda \eta_{\lambda\lambda'} \epsilon^{\mu\lambda'}$$

where ξ is a polarization vector. Then we have

$$\langle \mathbf{q}, \xi' | \mathbf{p}, \xi \rangle = -2|\mathbf{p}| \phi(\mathbf{p} - \mathbf{q}) (\xi' \cdot \xi).$$

• Using the strange convention above, the Gupta-Bleuler condition is simply

$$p^\mu \xi_\mu = 0$$

This eliminates the negative-norm timelike photons, but we still have the zero-norm states

$$|\mathbf{p}, \xi_S\rangle \propto \left(a_0^\dagger(\mathbf{p}) - a_3^\dagger(\mathbf{p})\right)|0\rangle$$

which are combinations of timelike and longitudinal photons, with $p^\mu \propto \xi^\mu$. We need a relative minus sign between the terms above, even though $p^\mu \propto (1,0,0,1)$, because of the minus sign in the commutation relations for timelike photons.

• More generally, a basis for our Fock space contains elements of the form $|\Psi\rangle = |\psi\rangle|\phi\rangle$ where $|\psi\rangle$ contains transverse photons and $|\phi\rangle$ contains timelike and longitudinal photons. Then the allowed $|\phi\rangle$ states satisfy the Gupta-Bleuler condition

$$\left(a_{\mathbf{p}}^3 - a_{\mathbf{p}}^0\right)|\phi\rangle = 0$$

so that the allowed $|\phi\rangle$ are of the form $|\phi_n\rangle = \left(a_{\mathbf{p}}^{0\dagger} - a_{\mathbf{p}}^{3\,\dagger}\right)^n |0\rangle$, which have zero norm for $n > 0$. The presence of these states reflects the fact that Lorenz gauge is not a complete gauge fixing.

• We claim the zero-norm states decouple, in the sense that they have zero expectation value for all gauge-invariant observables. E.g. the Hamiltonian is

$$H = \int d\mathbf{p}|\mathbf{p}| \left(\sum_{i=1}^{3} a_{\mathbf{p}}^{i\dagger} a_{\mathbf{p}}^i - a_{\mathbf{p}}^{0\dagger} a_{\mathbf{p}}^0\right)$$

However, the Gupta-Bleuler condition implies $\langle\phi\left|a_{\mathbf{p}}^{3\dagger}a_{\mathbf{p}}^3 - a_{\mathbf{p}}^{0\dagger}a_{\mathbf{p}}^0\right|\phi\rangle = 0$, so the $|\phi_n\rangle$ have zero energy. Thus, for a free theory, one can think of the states $|\psi\rangle\,|\phi_n\rangle$ as all being 'gauge equivalent' to $|\psi\rangle\,|\phi_0\rangle$, yielding a Hilbert space where all states have positive norm.

• On the other hand, zero-norm states do affect the expectation value of A_μ, as we have

$$\langle\phi_n\,|A_\mu(x)|\,\phi_n\rangle = \partial_\mu\alpha, \quad \partial^\mu\partial_\mu\alpha = 0$$

This is acceptable since A_μ is not gauge-invariant. Indeed, this is the exact analogue of the remaining classical gauge freedom $A_\mu \to A_\mu + \partial_\mu\alpha$ with $\partial^\mu\partial_\mu\alpha = 0$.

• We cannot ignore the zero-norm states entirely. In an interacting theory, they appear as intermediate states in scattering processes, though we will see that the Ward identity guarantees they decouple from initial and final states. They are also required to maintain Lorentz invariance, because a boost can turn a transverse polarization into a partly longitudinal one.

Finally, we have the Lorentz-invariant propagator

$$\langle 0\,|TA_\mu(x)A_\nu(y)|\,0\rangle = \int dp \frac{i}{p^2 + i\epsilon}\left(-\eta_{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)e^{-ip(x-y)}$$

where above we worked with $\xi = 1$, the numerator comes from the sum of all four polarizations, and the minus sign ensures the right sign for the spacelike polarizations. The choice $\xi = 1$ is called Feynman gauge, while $\xi \to 0$ is called Landau gauge. Landau gauge is the quantum version of Lorenz gauge, as the Lorenz gauge condition is imposed exactly in this limit.

О квантовании векторного поля методом Фаддева-Попова

(додумаю еще это)

Other properties of a vector field

Калибровочная инвариантность по ЛЛ4

Момент и четность фотона по ЛЛ4

Сферические волны фотонов по ЛЛ4

Полярнзация фотона по ЛЛ4

Система двух фотонов по ЛЛ4

What if we wished to introduce a mass term for the photon field?

(мб потом больший раздел про это сделаю, мб это важный вопрос. мб в самом начале, где обсуждаю вообще суть КТП. пока не до этих вопросов)

The Proca action is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu$$

The second term is not gauge invariant, so the theory does not have a gauge symmetry. The equation of motion is

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu = 0$$

so that, as long as the mass term is nonzero, we automatically have $\partial_\mu A^\mu = 0$. Plugging this back into the equation of motion yields the Klein-Gordan equation,

$$\left(\partial^2 + m^2\right) A^\mu = 0$$

Hence we have three massive degrees of freedom. The quantization of this theory is much more straightforward than the quantization of QED. We have only the second-class constraints $\partial_\mu A^\mu = 0$ and $\pi^0 = 0$, and Dirac brackets get the job done.

Stuckelberg trick

In fact, it is possible to obtain the Proca action from a gauge invariant action by gauge fixing; The Stuckelberg action is

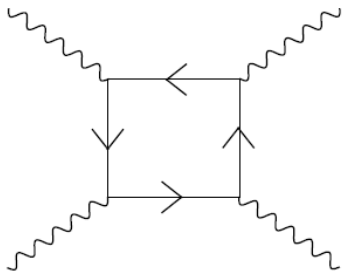
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(mA_\mu - \partial_\mu\phi\right)^2$$

The action has the gauge symmetry

$$\delta A_\mu = \partial_\mu \epsilon, \quad \delta\phi = m\epsilon$$

We recover the Proca action by gauge fixing $\phi = 0$. The benefit of starting from the Stuckelberg action is that the gauge symmetry constrains the divergences that may appear in perturbation theory, allowing one to prove the theory is renormalizable. Since the ϕ field is eaten, the Stuckelberg theory can be thought of as the abelian Higgs model in the limit where the Higgs is very heavy, and decouples. The Stuckelberg trick doesn't work for a non-abelian gauge theory, at least in this form, as in that case δA_μ is more complicated.

О взаимодействии фотонов (????)



3.5.3 Spin-3/2 field

3.5.4 Spin-2 field

3.5.5 Other about free quantum fields

(встречу - напишу, пока не до них)

3.6 Other models of interaction of fields

3.6.1 ϕ^3 theory

$$V = \frac{\Lambda}{3!}\varphi^3, \text{ где } \quad [\Lambda] = [m]$$

Основные формулы и особенности

О другом

Об операторном формализме по Киселеву для ϕ^3

$$\mathcal{D}^{(3)}_{123} = \left\langle 0 \left| \hat{\mathrm{T}} \left\{ \hat{\varphi}_1 \hat{\varphi}_2 \hat{\varphi}_3 \right\} \right| 0 \right\rangle \approx$$
$$\approx \exp\left\{ -\frac{\mathrm{i}\hbar}{2} \mathcal{G}^{(2)}_{xy} \frac{\delta^2}{\delta\tilde{\varphi}_x \delta\tilde{\varphi}_y} \right\} \tilde{\varphi}_1 \tilde{\varphi}_2 \tilde{\varphi}_3 \left(-\frac{\mathrm{i}}{\hbar} \right) \int \mathrm{d}^4 x_4 \frac{\Lambda}{3!} \tilde{\varphi}^3 (x_4) \Big|_{\tilde{\varphi}=0} .$$

Остается третья степень, в системе единиц $\hbar = c = 1$ имеем $\mathcal{G}^{(2)}_{xy} = \mathrm{i}\mathcal{D}^{(2)}_{xy}$ (??), так что

$$\mathcal{D}^{(3)}_{123} \approx -\,\mathrm{i}\frac{\Lambda}{3!} \left(\frac{1}{2} \int \mathrm{d}^4 x \; \mathrm{d}^4 y \mathcal{D}^{(2)}_{xy} \frac{\delta^2}{\delta\tilde{\varphi}_x \delta\tilde{\varphi}_y} \right)^3 \frac{1}{3!} \tilde{\varphi}_1 \tilde{\varphi}_2 \tilde{\varphi}_3 \int \mathrm{d}^4 x_4 \tilde{\varphi}^3 (x_4) \Big|_{\tilde{\varphi}=0}$$

(???? полчаса про это подумаю) Связный вклад $\mathcal{D}^c_{123}$ получается, если каждое их полей $\tilde{\varphi}_{1,2,3}$ спарить с полями $\tilde{\varphi}_4$, причем, в силу симметричности варьирования по $\tilde{\varphi}_x$ и $\tilde{\varphi}_y$ фактор $1/2$ компенсируется перестановкой $x \leftrightarrow y$, а комбинаторика по выборке $\tilde{\varphi}_4$ из $\tilde{\varphi}^3_4$ компенсирует фактор $1/3$! и комбинаторика выбора вариации из $\tilde{\varphi}_1 \tilde{\varphi}_2 \tilde{\varphi}_3$ также компенсирует фактор $1/3$!, так что

$$\mathcal{D}^c_{123} \approx -\mathrm{i}\Lambda \int \mathrm{d}^4 x_4 \mathcal{D}^{(2)}_{14} \mathcal{D}^{(2)}_{24} \mathcal{D}^{(2)}_{34}$$

3.6.2 ϕ^4 theory

Основные формулы и особенности

Правила Фейнмана для $-\frac{\lambda}{4!}\phi^4$.

Вычислим

$$Z(J) = \int D\varphi e^{i\int d^4x\left\{\frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right] - \frac{\lambda}{4!}\varphi^4 + J\varphi\right\}}$$

(??????)

О другом

3.6.3 Yukawa theory $\mathcal{L}_{\text{Yukawa}}(\phi,\psi) = -g\bar{\psi}\phi\psi$

Main formulas

$$U(\vec{r}) = \frac{-\lambda^2}{4\pi r}e^{-mr}$$

$$H = H_{\text{Дирак}} + H_{\text{Клейн-Гордон}} + \int d^3x g\bar{\psi}\psi\phi.$$

Главный вклад дает слагаемое H_I^2 в S-матрице:

$${}_0\left\langle \mathbf{p}', \mathbf{k}' \middle| T\left(\frac{1}{2!}(-ig)\int d^4x\bar{\psi}_I\psi_I\phi_I \quad (-ig)\int d^4y\bar{\psi}_I\psi_I\phi_I\right)\middle|\mathbf{p},\mathbf{k}\right\rangle_0.$$

$$\psi_I(x)|\mathbf{p},s\rangle = \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}'}}}\sum_{s'} a^{s'}_{\mathbf{p}'}u^{s'}(p')\,e^{-ip'\cdot x}\sqrt{2E_{\mathbf{p}}}a^{s\dagger}_{\mathbf{p}}|0\rangle = e^{-ip\cdot x}u^s(p)|0\rangle.$$

Пример:

$$=$$

$$(-ig)^2\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m_\phi^2}(2\pi)^4\delta(p'-p-q)\,(2\pi)^4\delta(k'-k+q)\,\bar{u}(p')\,u(p)\bar{u}(k')\,u(k)$$

(Мы опустили множитель 1/2!, потому что имеется второе, идентичное слагаемое, которое происходит от перестановки местами x и y в (4.115)). Используя любую дельта-функцию для выполнения интегрирования, находим, что это выражение имеет вид $i\mathscr{M}(2\pi)^4\delta(\Sigma p)$, где

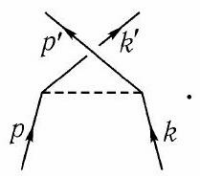
$$i\mathscr{M} = \frac{-ig^2}{q^2-m_\phi^2}\bar{u}(p')\,u(p)\bar{u}(k')\,u(k).$$

При такой записи нельзя забывать накладывать связь $p - p' = q = k' - k$.

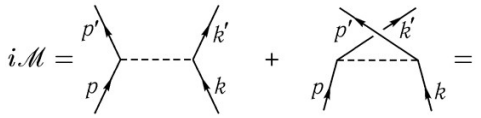
Наконец, обсудим вопрос о фермионных знаках «минус». Вернемся к примеру процесса фермион-фермионного рассеяния. Мы принимаем следующее соглашение о знаках для начальных и конечных состояний:

$$|\mathbf{p},\mathbf{k}\rangle \sim a^{\dagger}_{\mathbf{p}}a^{\dagger}_{\mathbf{k}}|0\rangle, \quad \langle\mathbf{p}',\mathbf{k}'| \sim \langle 0|a_{\mathbf{k}'}a_{\mathbf{p}'},$$

так что $(|p,k\rangle)^{\dagger} = \langle p,k|$. Тогда свертка



Наконец-то ниже важный итог наинисшего порядка!!!



$$= (-i\,g^2)\left(\bar{u}\,(p')\,u\,(p)\frac{1}{(p'-p)^2-m_\phi^2}\,\bar{u}\,(k')\,u\,(k) - \bar{u}\,(p')\,u\,(k)\frac{1}{(p'-k)^2-m_\phi^2}\,\bar{u}\,(k')\,u\,(p)\right).$$

Собирая вместе вклады в эту амплитуду рассеяния, находим:

$$i\mathscr{M} = \frac{ig^2}{|\mathbf{p}'-\mathbf{p}|^2+m_\phi^2}2m\delta^{ss'}2m\delta^{rr'}.$$

Это выражение надо сравнить с борновским приближением для амплитуды рассеяния в нерелятивистской квантовой механике, записанной с помощью потенциала $V(\mathbf{x})$:

$$\langle p'|iT|p\rangle = -i\tilde{V}(\mathbf{q})(2\pi)\delta\left(E_{\mathbf{p}'}-E_{\mathbf{p}}\right), \quad (\mathbf{q}=\mathbf{p}'-\mathbf{p}).$$

Очевидно, для взаимодействия Юкавы

$$\widetilde{V}(\mathbf{q}) = \frac{-g^2}{|\mathbf{q}|^2+m_\phi^2}$$

После обращения фурье и небольшого интегрирования, получаем

$$V(r) = -\frac{g^2}{4\pi}\frac{1}{r}e^{-m_\phi r}.$$

Потенциал взаимодействия!!!

Feynman Rules Illustration

(пока тут, мб тут что-то, что в другую часть вставляется лучше)

The same results can also be found by the same rigorous route followed for the scalar field, starting with the LSZ reduction theorem (??? сделаю это!!!).

For Yukawa theory with

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\not{\partial} - m)\psi - \lambda\phi\bar{\psi}\psi$$

where ϕ is a scalar field. We think of the ψ particles as nucleons and the ϕ particles as mesons (pions).

• We now carefully work out $\psi\psi \to \psi\psi$ scattering, with initial and final states

$$|i\rangle = \sqrt{4E_pE_q}\,b^{s\dagger}_{\mathbf{p}}b^{r\dagger}_{\mathbf{q}}|0\rangle, \quad |f\rangle = \sqrt{4E_{p'}E_{q'}}\,b^{s'\dagger}_{\mathbf{p}'}b^{r'\dagger}_{\mathbf{q}'}|0\rangle.$$

Taking the adjoint to get $\langle f|$ reverses the order of the operators (!!!!!!); we were careless about this before because the operators commuted.

The $O(\lambda^2)$ contribution to $\langle f|S-1|i\rangle$ is

$$\frac{(-i\lambda)^2}{2}\int dx_1dx_2T\left(\bar{\psi}(x_1)\psi(x_1)\phi(x_1)\bar{\psi}(x_2)\psi(x_2)\phi(x_2)\right)$$

where the two ϕ fields must be contracted, just as in scalar Yukawa theory.

• Next, we need to be careful about how the spinor indices are contracted. While we've kept this implicit, there is a contraction between $\bar{\psi}$ and ψ in the interaction term, so explicitly

$$\begin{aligned}&:\bar{\psi}(x_1)\psi(x_1)\bar{\psi}(x_2)\psi(x_2): b^{s\dagger}_{\mathbf{p}}b^{r\dagger}_{\mathbf{q}}|0\rangle = \\ &\quad = -\int\frac{d\mathbf{k}_1d\mathbf{k}_2}{\sqrt{4E_{k_1}E_{k_2}}}\left(\bar{\psi}(x_1)\cdot u^m(\mathbf{k}_1)\right)\left(\bar{\psi}(x_2)\cdot u^n(\mathbf{k}_2)\right)e^{-i(k_1x_2+k_2x_2)}b^m_{\mathbf{k}_1}b^n_{\mathbf{k}_2}b^{s\dagger}_{\mathbf{p}}b^{r\dagger}_{\mathbf{q}}|0\rangle\end{aligned}$$

where the dot indicates spinor contraction, and we picked up a sign from moving $\psi(x_1)$ past $\bar{\psi}(x_2)$. Applying the anticommutation relations gives two terms which differ by a sign.

• Finally, we need to take care of the final state. Each of the two terms above yields two possible contraction structures, canceling the $1/2$ in front of the amplitude. We are left with the position integrations, which yield delta functions, finally giving

$$\mathcal{M} = (-i\lambda)^2\left(\frac{\left(\bar{u}^{s'}(\mathbf{p}')\cdot u^s(\mathbf{p})\right)\left(\bar{u}^{r'}(\mathbf{q}')\cdot u^r(\mathbf{q})\right)}{(p'-p)^2-\mu^2+i\epsilon} - \frac{\left(\bar{u}^{s'}(\mathbf{p}')\cdot u^r(\mathbf{q})\right)\left(\bar{u}^{r'}(\mathbf{q}')\cdot u^s(\mathbf{p})\right)}{(q'-p)^2-\mu^2+i\epsilon}\right)$$

• We can summarize our results with the following Feynman rules.

• For every incoming fermion with momentum p and spin r, write down $u^r(\mathbf{p})$. For outgoing fermions, write $\bar{u}^r(\mathbf{p})$.

• For incoming and outgoing antifermions write $\bar{v}^r(\mathbf{p})$ and $v^r(\mathbf{p})$, respectively,

• Every vertex gives a factor of $-i\lambda$.

• Every internal line gets a Feynman propagator,

$$\text{scalar: } \frac{i}{p^2-\mu^2+i\epsilon}, \qquad \text{fermion: } \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}.$$

• Conserve momentum at every vertex, and integrate dp over loop momenta.

• For every fermion line, draw an arrow indicating the flow of particle number, and contract spinor indices along these lines. This is independent of the momentum flow arrows, though the two can always be aligned for internal lines.

An easy way to remember the external fermion factors is to note that the matrix multiplications, read left to right, always go opposite the particle flow lines.

• There are also extra minus signs in various places.

• Interchange of identical particles gives a factor of -1. E.g. the t-channel and u-channel diagrams have opposite signs for $\psi\psi \to \psi\psi$, as we saw above.

• Every closed fermion loop has a factor of -1. This is because we will always need an odd number of swaps to 'untangle' all of the contractions. We write 'tr' to indicate the spinor indices are contracted in a loop.

• Unfortunately, there are further signs that are more subtle, for which these two rules don't suffice. The most reliable way is to simply return to the level of contractions and count the anticommutations necessary to form one. An equivalent diagrammatic method is to redraw the diagram so that all fermion lines start on the left (in a standard order) and end on the right; the sign is the sign of the permutation of the lines on the right.

• Finally, we can find the Yukawa potential in the nonrelativistic limit. For $\psi\psi \to \psi\psi$ scattering, note that the spinors become

$$u(\mathbf{p}) \to \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}, \quad v(\mathbf{p}) \to \sqrt{m}\begin{pmatrix} \xi \\ -\xi \end{pmatrix}$$

so the spin-dependence in the terms is just $\delta^{ss'}\delta^{rr'}$ and $\delta^{rs'}\delta^{sr'}$, so the interaction conserves spin and is spin-independent. Besides that, the amplitude is the same as in Yukawa theory, so the potential is attractive.

• Next, consider the process $\psi\bar{\psi} \to \psi\bar{\psi}$. There is a sign flip from fermionic statistics as shown above, but another sign flip from $\bar{v}^{s'} \cdot v^{s} = -2m\delta^{ss'}$, which means nucleons and antinucleons still attract. This is because forces mediated by spin 0 particles are universally attractive.

• More realistically, the pion is a pseudoscalar, so we should considering the coupling

$$\mathcal{L}_{\text{Yuk}} = -\lambda\phi\bar{\psi}\gamma^5\psi$$

which obeys parity. Then the interaction vertex becomes $-i\lambda\gamma^5$. In this case, the calculation becomes significantly more complicated, since taking the naive nonrelativistic limit above just yields zero. The result is a spin-dependent potential which goes as $1/r^3$ for massless pions. For details, see the paper New Macroscopic Forces?

О рассеянии частиц на античастицах по Юквае (?????)

(дальше у ПШ про это)

3.6.4 Standard Model main properties, algebra

(страниц 10 будет самого основного, подробнее - см. пока отдеьлную запись, но может и во 2ю 1ю часть добавлю)

Основные формулы и особенности

(тут абзац про части стандартной модели, типа это КХД и лептоны и они не сильно друг с другом связаны. все общее важное!)

О лептонном и хиггсовском секторе по лекции 13 Киселеву

(когда-то за часа 3 пойму и напишу про это тут что нужно, чисто введение)

$(V-A)$-структура слабого взаимодействия

связь лептнного тока с массивным заряженным W-бозоном и дублет левых лептонов,

группа локальной калибровочной симметрии $U(1)\times SU(2)_L$ и необходимость введения слабых нейтральных токов,

дублет скалярных полей Хиггса и нормировка гиперзаряда,

спонтанное нарушение электрослабой симметрии,

масса W-бозона,

угол Вайнберга,

связь оператора электрического заряда с гиперзарядом и слабым изоспином,

масса Z-бозона,

вершина взаимодействия Z-бозона с нейтральными токами,

вакуумное среднее поля Хиггса и постоянная Ферми в эффективной теории 4-фермионного взаимодействия при низких энергиях,

юкавский сектор и массы заряженных фермионов.

О другом в стандартной модели

3.6.5 Other toy illustrative theories

$\varphi_M\Phi^\dagger\Phi$ theory

Ширина распада (?!)

$$\mathrm{d}\Gamma = \frac{|\mathfrak{M}|^2}{2M} d\mathfrak{p}_f$$

где $\mathrm{d}\mathfrak{p}_f = (2\pi)^4\delta\left(p-\sum\limits_{n=1}^{n_f} k_{(n)}\right)\prod_{n=1}^{n_f}\frac{\mathrm{d}^3k_{(n)}}{2E_{(n)}}$.

Дифференциальное сечением рассеяния частиц с 4-импульсами p и $\tilde{p}$:

$$\mathrm{d}\sigma = \frac{|\mathfrak{M}|^2}{4\sqrt{(p\cdot\tilde{p})^2 - m^2\tilde{m}^2}} d\mathfrak{p}_f$$

$$\mathrm{d}\mathfrak{p}_f := (2\pi)^4\delta\left(p+\tilde{p}-\sum_{n=1}^{n_f} k_{(n)}\right)\prod_{n=1}^{n_f}\frac{\mathrm{d}^3 k_{(n)}}{2E_{(n)}}.$$

Эти правила относятся к случаю, когда одноточечная функция Грина поля равна нулю (??? поч????). Однако, ненулевое вакуумное значение поля $v = \langle 0|\hat{\varphi}(x)|0\rangle$ отвечает просто случаю внешнего поля с нулевым импульсом. Такую внешнюю линию обычно изображают с символом $\otimes$ на конце (???? или такого нет????). При этом вершины взаимодействия внешнего вакуумного поля легко определить, если в исходном лагранжиане произвести замену поля, скажем, $\varphi \mapsto v+\varphi$, так что новое поле уже не имеет вакуумного среднего.

У нас всегда распадаются частицы по Брейту-Вигнеру и там всегда есть параметр - ширина распада.

Ширина для $-\mathcal{L}_{\text{int}} = \Lambda\varphi_M\Phi^\dagger\Phi$ по Киселеву

(мб добавлю формулы Киселева, пока думаю, по ПШ хватит теории.)

$$\begin{aligned}
\mathrm{i}\mathcal{M}^c_{(3)} &= -\mathrm{i}\Lambda\int \mathrm{d}^4x_4\varphi^{\text{cl}}_M(x_4)\,\Phi^\dagger_{\text{cl}}(x_4)\,\Phi_{\text{cl}}(x_4)\\
&= -\mathrm{i}\Lambda(2\pi)^4\delta(p-k_1-k_2)\\
\left|\mathcal{M}^c_{(3)}\right|^2 &= \Lambda^2(2\pi)^4\delta(p-k_1-k_2)\left\{(2\pi)^4\delta(k')\right\}\big|_{k'\to 0}
\end{aligned}$$

Стандартным приемом найдем «сингулярное значение»

$$\left\{(2\pi)^4\delta(k')\right\}\big|_{k'\to 0} = \int\limits_{V_{[4]}} \mathrm{d}^4x \mathrm{e}^{\mathrm{i}k'x}\Big|_{k'\to 0} = V_{[4]} = \Delta t V_{[3]},$$

(тут мб абзац будет про эти бесконечности, что просто нормируем на них и все.)
(словами напишу переходы простые)

$$\Gamma = \frac{\Lambda^2}{2M}\int\delta\left([p-k_2]^2\right)\frac{\mathrm{d}^3k_2}{(2\pi)^2 2E_2}.$$

$$\frac{\mathrm{d}^3k_2}{(2\pi)^2 2E_2} = \frac{4\pi}{4\pi^2}\quad\frac{|\boldsymbol{k}_2|^2\,\mathrm{d}\,|\boldsymbol{k}_2|}{2E_2} = \frac{|\boldsymbol{k}_2|\,\mathrm{d}E_2}{2\pi}.$$

В итоге, релятивистски инвариантный фазовый объем

$$\int\delta\left([p-k_2]^2\right)\frac{\mathrm{d}^3k_2}{(2\pi)^2 2E_2} = \frac{1}{4\pi}\frac{|\boldsymbol{k}_2|}{M} = \frac{1}{8\pi}.$$

Ширина найдена

$$\Gamma = \frac{\Lambda^2}{16\pi M},\quad [\Gamma] = [m].$$

(!!! подумаю про это, очень содержательная формула!)

В теории возмущений необходимо, чтобы

$$\Gamma \ll M \quad\Rightarrow\quad \Lambda \ll M.$$

3.7 Useful background for QFT

3.7.1 Other mathematics for QFT

Commutators

$$[A, BC] = [A, B]C + B[A, C]$$

$$[A, BCD] = [A, B]CD + B[A, C]D + BC[A, D]$$

$$[A, BCDE] = [A, B]CDE + B[A, C]DE + BC[A, D]E + BCD[A, E]$$

$$[AB, C] = A[B, C] + [A, C]B$$

$$[ABC, D] = AB[C, D] + A[B, D]C + [A, D]BC$$

$$[ABCD, E] = ABC[D, E] + AB[C, E]D + A[B, E]CD + [A, E]BCD$$

$$\begin{aligned}[AB, CD] &= A[B, C]D + [A, C]BD + CA[B, D] + C[A, D]B = \\ &= A[B, C]D + AC[B, D] + [A, C]DB + C[A, D]B\end{aligned}$$

$$[[A, C], [B, D]] = [[[A, B], C], D] + [[[B, C], D], A] + [[[C, D], A], B] + [[[D, A], B], C]$$

$$e^A e^B e^{-A} e^{-B} = \exp\left([A, B] + \frac{1}{2!}[A + B, [A, B]] + \frac{1}{3!}\left(\frac{1}{2}[A, [B, [B, A]]] + [A + B, [A + B, [A, B]]]\right) + \cdots\right)$$

$$\left[A^N, B^M\right] = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} A^n B^m [A, B] A^{N-n-1} B^{M-m-1}$$

$$[AB, C]_\pm = A[B, C]_- + [A, C]_\pm B$$

$$[AB, CD]_\pm = A[B, C]_- D + AC[B, D]_- + [A, C]_- DB + C[A, D]_\pm B$$

$$[[A, B], [C, D]] = \left[[[B, C]_+, A]_+, D\right] - \left[[[B, D]_+, A]_+, C\right] + \left[[[A, D]_+, B]_+, C\right] - \left[[[A, C]_+, B]_+, D\right]$$

$$[A, [B, C]_\pm] + [B, [C, A]_\pm] + [C, [A, B]_\pm] = 0$$

$$[A, BC]_\pm = [A, B]_- C + B[A, C]_\pm = [A, B]_\pm C \mp B[A, C]_-$$

$$[A, BC] = [A, B]_\pm C \mp B[A, C]_\pm$$

Tensors for QFT

Levi-Chivita symbol $\varepsilon^{\alpha\beta\gamma\delta}$:

$$\varepsilon^{\alpha\beta\mu v}\varepsilon_{\alpha\beta\sigma\tau} = -2\left(g^{\mu}_{\sigma}g^{v}_{\tau} - g^{\mu}_{\tau}g^{v}_{\sigma}\right)$$
$$\varepsilon^{\alpha\beta\gamma v}\varepsilon_{\alpha\beta\gamma\tau} = -6g^{v}_{\tau}$$
$$\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\alpha\beta\gamma\delta} = -24$$

(мб в др место вставлю)

3.7.2 Quantum Mechanics for QFT

(перегруппирую некоторые отсюда темы потом, вообще поставлю или под начало, где смысл КТП, или под конец, где всякее общее не актуальные соображения)

Attempts at nonrelativistic quantum mechanics

(I'll write in compact form later, or maybe not write it at all. Maybe 1/2 or 1 page will be about it.)

Schrodinger equation from quantum field theory

The Klein-Gordan equation for complex (for probability) scalar field $(\partial^2 + m^2)\,\phi = 0$ and setting $\phi = e^{-imt}\chi/\sqrt{2m}$ yields

$$i\dot{\chi} = \frac{\ddot{\chi} - \nabla^2\chi}{2m}$$

$\ddot{\chi}/m \ll \dot{\chi}$,

$$\mathcal{L} = i\chi^*\dot{\chi} - \frac{1}{2m}\nabla\chi^*\cdot\nabla\chi$$

$$\pi = i\chi^*, \quad \mathcal{H} = \frac{1}{2m}\nabla\chi^*\cdot\nabla\chi, \quad \left[\chi(\mathbf{x}), \chi^\dagger(\mathbf{y})\right] = \delta(\mathbf{x}-\mathbf{y})$$

$$\chi(\mathbf{x}) = \int d\mathbf{p} a_{\mathbf{p}} e^{ipx}, \quad \left[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}\right] = \not{\delta}(\mathbf{p}-\mathbf{q})$$

$$H = \int d\mathbf{p}\frac{p^2}{2m}a^\dagger_{\mathbf{p}}a_{\mathbf{p}}, \quad \mathbf{P} = \int d\mathbf{p}\mathbf{p}a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$$

which are simply the usual definitions in nonrelativistic quantum field theory.
$|\mathbf{x}\rangle = \chi^\dagger(\mathbf{x})|0\rangle$

$$\mathbf{X} = \int d\mathbf{x}\mathbf{x}\chi^\dagger(\mathbf{x})\chi(\mathbf{x}), \quad \mathbf{X}|\mathbf{x}\rangle = \mathbf{x}|\mathbf{x}\rangle$$

$$|\psi\rangle = \int d\mathbf{x}\psi(\mathbf{x})|x\rangle$$

$$\mathbf{X}|\psi\rangle = \int d\mathbf{x}\mathbf{x}\psi|\mathbf{x}\rangle, \quad \mathbf{P}|\psi\rangle = \int d\mathbf{x}(-i\nabla\psi)|\mathbf{x}\rangle, \quad H|\psi\rangle = \int d\mathbf{x}\left(-\frac{1}{2m}\nabla^2\psi\right)|\mathbf{x}\rangle$$

Основные формулы гармонического осциллятора и его применения в КТП

О когерентных состояниях

О другом про гармонический осциллятор

3.7.3 Classical field theory for QFT

(это же в теорполе, только тут в контексте КТП и короче пояснения)
(Нетер, ТЭИ, параметры, все это тут укажу, чтобы не листать в запись по теории поля)

О Лагранжевом формализме

(суть Мандела Шаула, начала)

4 Other QFT processes

4.1 Overview of other processes

(тоже обзор всяких рассеяний, распадов, т.п., с содежательным графиком, формулой, сутью описания. Не КЭД, КЭД был в 1й части.)

4.1.1 Other QED processes

38. Рассеяние в ультрарелятавистском случае

42. Рассеяние нейтронов в электрическом поле

Уравнение Брейта по Ландау 4 (???)

Позитроний по Ландау 4 (???)

Взаимодействие атомов на далеких расстояниях по Ландау 4 (???)

$$U(r) = \frac{i}{2} \int\limits_{-\infty}^{\infty} \omega^4 \alpha_1(\omega) \alpha_2(\omega) \left[D_{ik}(\omega, \mathbf{r}) \right]^2 \frac{d\omega}{2\pi}$$

Что следует из какого-то замысловатого усреднения S-матрицы (????) Тут

$$D_{ik}(\omega, \mathbf{r}) = -\left(\delta_{ik} + \frac{1}{\omega^2} \frac{\partial^2}{\partial x_i \partial x_k} \right) D(\omega, \mathbf{r})$$

где еще можно чуть преобразовать это. (?? теория пар 76 тут используется, я хз)

В (?? каких-то одних упрощениях) имеем формулу Лондона

$$U(r) = -\frac{2}{3r^6} \sum_{n,n'} \frac{\left|\mathbf{d}_{0n}\right|^2 \left|\mathbf{d}_{0n'}\right|^2}{\hbar \left(\omega_{n0} + \omega_{n'0} \right)}$$

В (??? других??) имеем формулу Казимира-Полдера (1948):

$$U(r) = -\frac{23}{4\pi} \frac{\hbar c \alpha_1(0) \alpha_2(0)}{r^7}$$

(??? пока не понял суть, потом добавлю важные слова сюда!)

4.2 Special Phenomena of Electrons and Photons

4.2.1 Electron Scattering, Ionization Losses, Positronium, Interaction of Atoms

37. Рассеяние в центрально-симметричном поле

40. Электрон в поле плоской электромагнитной волны

41. Движение спина во внешнем поле

О рассеянии электрона во внешнем поле по Ландау 4 (???)

Ионизационные потери быстрых частиц по Ландау 4 (???)

80. Рассеяние электрона во внешнем поле

81. Рассеяние электронов и позитронов на электроне

82. Ионизационные потери быстрых частиц

83. Уравнение Брейта

84. Позитроний

85. Взаимодействие атомов на далеких расстояниях

$$S_{fi} = -i \int \psi_1'^* (\mathbf{r}_1)\, \psi_2'^* (\mathbf{r}_2) U(r) \psi_1 (\mathbf{r}_1)\, \psi_2 (\mathbf{r}_2)\, d^3x_1 d^3x_2 \times \\ \times \int \exp\left\{-i\left(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2'\right) t\right\} dt.$$

$$S_{fi} = \int \psi_1'^* \psi_2'^* \langle S(r)\rangle \psi_1 \psi_2 d^3x_1 d^3x_2$$

Сравнив с $(85, 2)$, мы увидим, что если получить выражение для $\langle S(r)\rangle$ в виде

$$\langle S(r)\rangle = -itU(r),$$

$$\hat{S}^{(4)} = \frac{(-i)^4}{4l} \int dt_1 \ldots \int dt_4 \cdot \mathrm{T}\left\{\hat{V}(t_1)\, \hat{V}(t_2)\, \hat{V}(t_3)\, \hat{V}(t_4)\right\}$$

$$D_{ik}(t, \mathbf{r}) = \int D_{ik}(\omega, \mathbf{r}) e^{-i\omega t} \frac{d\omega}{2\pi}$$

$$D_{lk}^E(t, \mathbf{r}) = -i \int \omega^2 D_{ik}(\omega, \mathbf{r}) e^{-i\omega t} \frac{d\omega}{2\pi}$$

$$\alpha_{ik}(t_1 - t_2) = i \langle \mathrm{T}(d_i(t_1)\, d_k(t_2))\rangle$$

$$\alpha_{ik}(\omega) = \sum_n \left\{ \frac{(d_i)_{0n} (d_k)_{n0}}{\omega_{n0} - \omega - i0} + \frac{(d_k)_{0n} (d_i)_{n0}}{\omega_{n0} + \omega - i0} \right\}$$

Если основное состояние является S-состоянием, то этот тензор сводится к скаляру: $\alpha_{ik} = \alpha\delta_{ik}$, где

$$\dot{\alpha(\omega)} = \frac{1}{3} \sum_n |\mathbf{d}_{0n}|^2 \left(\frac{1}{\omega_{n0} - \omega - i0} + \frac{1}{\omega_{n0} + \omega - i0} \right).$$

$$\langle S(r)\rangle = -itU(r),$$
$$U(r) = \frac{i}{2}\int\limits_{-\infty}^{\infty} \omega^4 \alpha_1(\omega)\alpha_2(\omega)\left[D_{ik}(\omega,\mathbf{r})\right]^2 \frac{d\omega}{2\pi}$$

В предельном же случае больших расстояний, $r \gg \lambda_0$, в интеграле существенны значения $\omega \leqslant c/r \ll \omega_0$; при $\omega \geqslant \omega_0$ интеграл погашается быстро осциллирующим множителем $\exp(2i\omega r)$. Поэтому можно заменить поляризуемости $\alpha_1(\omega)$ и $\alpha_2(\omega)$ их статическими значениями $\alpha_1(0)$ и $\alpha_2(0)$. После этого интегрирование производится элементарно (прнчем для обеспечения сходимости следует заменить в экспоненте $r \to r + i0$). В результате окончательно находим (в обычных единицах):

$$U(r) = -\frac{23}{4\pi}\frac{\hbar c \alpha_1(0)\alpha_2(0)}{r^7}$$

(H.B.G. Casimir, D. Polder, 1948) [1]).

4.2.2 $\gamma e \to \gamma e,\ ee \to \gamma\gamma,\ ee^+ \to \gamma\gamma\gamma$

87. Рассеяние фотона электроном. Поляризационные эффекты

Общие формулы

$$e^{(1)} = \frac{N}{\sqrt{-N^2}}, \quad e^{(2)} = \frac{P}{\sqrt{-P^2}},$$

где

$$P^\lambda = \left(p^\lambda + p'^\lambda\right) - K^\lambda \frac{pK + p'K}{K^2}, \quad N^\lambda = e^{\lambda\mu\nu\rho}P_\mu q_v K_\rho$$
$$K^\lambda = k^\lambda + k'^\lambda, \quad q^\lambda = k'^\lambda - k^\lambda = p^\lambda - p'^\lambda.$$

Непосредственным проецированием тензора $Q_{\mu\nu}$ находим

$$Q_0 = \frac{1}{2}Q^{\mu\nu}\left(e_\mu^{(1)}e_v^{(1)} + e_u^{(2)}e_v^{(2)}\right)$$

и т. д. Для вычисления удобно сначала выразить $Q_{\mu v}$ через взаимно ортогональные 4-векторы P, N, q, K:

$$Q^{\mu\nu} = \gamma^\mu \frac{\gamma P/2 + m}{s - m^2}\gamma^\nu + \gamma^\nu \frac{\gamma P/2 + m}{u - m^2}\gamma^\mu - \frac{1}{t}\left[\gamma^\mu(\gamma K)\gamma^\nu - \gamma^\nu(\gamma K)\gamma^\mu\right].$$

$$\bar u'\left(\gamma p + \gamma p'\right)u = 2m\bar u' u,$$
$$\bar u'\gamma^5(\gamma q)u = \bar u'\left[\gamma^5(\gamma p) + (\gamma p')\,\gamma^5\right]u = 2m\bar u'\gamma^5 u$$

то можно заменить в $Q^{\mu\nu}$ слагаемые

$$\gamma p + \gamma p' \to 2m, \quad \gamma^5(\gamma q) \to 2m\gamma^5$$

Поэтому матрицы плотности начального и конечного фотонов, отнесенные к 4-ортам $e^{(1)}, e^{(2)}$, будут

$$\rho^{(\gamma)} = \tfrac{1}{2}(1 + \xi\sigma), \qquad \xi = (\xi_1, \xi_2, \xi_3)\,;$$
$$\rho^{(\gamma)'} = \tfrac{1}{2}\left(1 + \xi'\sigma\right), \quad \xi' = (-\xi_1', -\xi_2', \xi_3')\,.$$

Теперь тензорный след

$$\rho^{(\nu')}_{\lambda\mu} Q^{\mu\nu} \rho^{(\nu)}_{\nu\rho} \tilde{\tilde{Q}}^{\rho\lambda}$$

Рассеяние на неполяризованных электронах

$$\rho^{(e)} = \frac{1}{2}(\gamma p + m), \quad \rho^{(e)'} = \frac{1}{2}(\gamma p' + m)$$

и удвоить результат, который должен быть подставлен вместо $|M_{fi}|^2$ в формулу для сечения $(64, 22)$

$$d\sigma = \frac{1}{32\pi^2} \frac{dt d\varphi}{(s - m^2)^2} |M_{fi}|^2$$

В лабораторной системе формула $(87, 11)$ принимает вид

$$d\sigma = \frac{r_e^2}{4} \left(\frac{\omega'}{\omega}\right)^2 do' \{F_0 + F_3(\xi_3 + \xi_3') + F_{11}\xi_1\xi_1' + F_{22}\xi_2\xi_2' + F_{33}\xi_3\xi_3'\}$$
$$do' = \sin\theta d\theta d\varphi, \quad (87, 12$$

где

$$F_0 = \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\vartheta, \quad F_3 = \sin^2\vartheta$$
$$F_{11} = 2\cos\vartheta, \quad F_{22} = \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega}\right)\cos\vartheta, \quad F_{33} = 1 + \cos^2\vartheta$$

(U. Fano, 1949).

Согласно изложенным в §65 правилам величины ξ_i'' равны отношениям коэффициентов при $\xi_i'\dot{\kappa}$ члену, не содержащему ξ':

$$\xi_1^{(f)} = \frac{F_{11}}{F}\xi_1, \quad \xi_2^{(f)} = \frac{F_{22}}{F}\xi_2, \quad \xi_3^{(f)} = \frac{F_3 + F_{33}\xi_3}{F}.$$

В частности, при рассеянии неполяризованного фотона

$$\xi_1^{(f)} = \xi_2^{(f)} = 0, \quad \xi_3^{(f)} = \frac{\sin^2\theta}{\omega/\omega' + \omega'/\omega - \sin^2\theta}$$

$$d\sigma = \frac{r_e^2}{4}\left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2 + 4\cos^2\Theta\right) do',$$

Рассеяние на поляризованных электронах

$$d\sigma = \frac{1}{2} d\sigma(\xi, \xi') +$$
$$+ \frac{r_e^2}{8}\left(\frac{\omega'}{\omega}\right)^2 do' \{\mathbf{f}\xi\xi_2 + \mathbf{f}'\xi_2' + \mathbf{g}\xi'\xi_2 + \mathbf{g}'\xi'\xi_2' + G_{ik}\xi_i\xi_k' + ...\}.$$

фотона поляризованным электроном

$$d\sigma(\xi, \xi) = d\sigma(\xi) + \frac{1}{2} r_e^2 \left(\frac{\omega'}{\omega}\right)^2 \xi_2 \mathrm{f}\xi do'$$

отличается от $d\sigma(\xi)$ только при наличии у фотона круговой поляризации, а у электронов-отличной от нуля проекции среднего спина на плоскость рассеяния. По той же причине электрон отдачи поляризуется только в случае, если фотон обладает круговой поляризацией; вектор же возникающей поляризации электрона лежит при этом в плоскости рассеяния:

$$\xi^{(f)} = \frac{1}{F}\xi_2 \mathbf{g}.$$

Соотношения симметрии

Качественные свойства поляризационных эффектов при рассеянии фотонов на электронах следуют уже из общих требований симметрин.

Параметр ξ_2 циркулярной поляризации - псевдоскаляр (см. §8). Поэтому в силу требования P-инвариантности члены вида $\sim \xi_2$ (или $\propto \xi_2'$) в сечении рассеяния могли бы возникнуть лишь как произведение ξ_2 на какой-либо псевдоскаляр, составленный из имеющихся в нашем распоряжении векторов $\mathbf{k}$ и $\mathbf{k}'$[1]). Но из двух полярных векторов нельзя составить псевдоскаляр. Отсюда и следует, что указанных членов в сечении не может быть.

Параметры линейной поляризации ξ_1 и ξ_3 связаны с компонентами двумерного (в плоскости, перпендикулярной k) симметричного тензора

$$S_{\alpha\beta} = \frac{1}{2}\left(\rho^{(\nu)}_{\alpha\beta} + \rho^{(\nu)}_{\beta\alpha}\right) = \frac{1}{2}\begin{pmatrix} 1+\xi_3 & \xi_1 \\ \xi_1 & 1-\xi_3 \end{pmatrix}.$$

88. Двухфотонная аннигиляция электронной пары

Они отличаются от диаграмм рассеяния фотона на электроне заменой

$$p \to p_-, \quad p' \to -p_+, \quad k \to -k_1, \quad k' \to k_2.$$

Оба процесса - два перекрестных канала одной и той же (обобщенной) реакции. После замены $(88,2)$ кинематические инварианты $(86,2)$ приобретают следующий смысл:

$$\begin{aligned} s &= (p_- - k_1)^2, \\ t &= (p_- + p_+)^2 = (k_1 + k_2)^2, \\ u &= (p_- - k_2)^2. \end{aligned}$$

Если рассеяние фотона было s-каналом, то аннигиляция есть t-канал.

Физическая область аниигиляционного канала есть область II на рис. 7 (см. с. 300). При заданном t (заданной знергии в системе центра инерцин) интервал изменения s определяется уравнением границы $su = m^4$. Вместе с соотношением $s + t + u = 2m^2$ это дает $-\frac{t}{2} - \frac{1}{2}\sqrt{t\,(t-4m^2)} \leqslant s - m^2 \leqslant -\frac{t}{2} + \frac{1}{2}\sqrt{t\,(t-4m^2)}$. Интегрирование выражения $(88,4)$ элементарно; результат надо еше разделить на 2, учитывая тождественность двух конечных частиц (фотонов). Таким образом, получим

$$\sigma = \frac{\pi r_e^2}{2\tau^2(\tau-1)}\left[\left(\tau^2+\tau-\frac{1}{2}\right)\ln\frac{\sqrt{\tau}+\sqrt{\tau-1}}{\sqrt{\tau}-\sqrt{\tau-1}} - (\tau+1)\sqrt{\tau(\tau-1)}\right],$$

$(88,6)$ где $\tau = t/4m^2$ (P. A. M. Dirac, 1930).

В нерелятивистском пределе ($\tau \to 1$) находим отсюда

$$\sigma = \frac{\pi r_e^2}{2\sqrt{\tau - 1}} \text{ (н. р.)}.$$

В системе центра инерции электрон, позитрон и оба фотона имеют одинаковые энергии $\varepsilon = \omega$. Инварианты:

$$m^2 - s = 2\varepsilon(\varepsilon - |\mathbf{p}|\cos\theta), \quad m^2 - u = 2\varepsilon(\varepsilon + |\mathbf{p}|\cos\theta),$$
$$t = 4\varepsilon^2,$$

где θ - угол между импульсами электрона и одного из фотонов. Подставив $(88, 12)$ в $(88, 4)$, получим угловое распределение аннигиляционных фотонов

$$d\sigma = \frac{r_e^2 m^2}{4\varepsilon|\mathrm{p}|}\left[\frac{\varepsilon^2 + \mathrm{p}^2\left(1 + \sin^2\theta\right)}{\varepsilon^2 - \mathrm{p}^2\cos^2\theta} - \frac{2\mathrm{p}^4\sin^4\theta}{\left(\varepsilon^2 - \mathrm{p}^2\cos^2\theta\right)^2}\right] do.$$

В ультрарелятивистском случае оно имеет симметричные максимумы в направлениях $\theta = 0$ и $\theta = \pi$. Вблизи $\theta = 0$

$$d\sigma \approx \frac{r_e^2 m^2 do}{2\varepsilon^2\left(\theta^2 + m^2/\varepsilon^2\right)} \text{ (у. р.)}.$$

Подставив $(88, 12)$ в $(88, 4)$, получй угловое распределение аннигиляционных фотонов

$$d\sigma = \frac{r_e^2 m^2}{4\varepsilon|\mathrm{p}|}\left[\frac{\varepsilon^2 + \mathrm{p}^2\left(1 + \sin^2\theta\right)}{\varepsilon^2 - \mathrm{p}^2\cos^2\theta} - \frac{2\mathrm{p}^4\sin^4\theta}{\left(\varepsilon^2 - \mathrm{p}^2\cos^2\theta\right)^2}\right] do.$$

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$$d\sigma \approx \frac{r_e^2 m^2 do}{2\varepsilon^2\left(\theta^2 + m^2/\varepsilon^2\right)} \text{ (у. р.)}.$$

Полное сечение получается из $(88, 6)$:

$$\sigma = \pi r_e^2 \frac{1 - v^2}{4v}\left[\frac{3 - v^4}{v}\ln\frac{1 + v}{1 - v} - 2\left(2 - v^2\right)\right],$$

где $v = |\mathbf{p}|/\varepsilon = \sqrt{\varepsilon^2 - m^2}/\varepsilon$ - скорость сталкивающихся частин.

89. Аннигиляция позитрония

В нерелятивистском пределе согласно $(88, 11)$[1])

$$\overrightarrow{\sigma}_{2\gamma} = \pi\left(\frac{e^2}{mc^2}\right)^2 \frac{c}{v},$$

где v-относительная скорость частиц. Мы получим вероятность аннигиляции $\overline{\boldsymbol{\omega}}_{2\gamma}$, умножив $\bar{\sigma}_{2\gamma}$ на плотность потока, равную $v|\psi(0)|^2$. Здесь $\psi(r)$ - нормированная на 1 волновая функция основного состояния позитрония:

$$\psi(r) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}, \quad a = \frac{2\hbar^2}{me^2}$$

(боровский радиус позитрония a в два раза больше радиуса атома водорода из-за вдвое меньшей приведенной массы). Эта

Аналогичным образом найдем, что вероятность распада ортопозитрония связана с усредненным по спинам сечением трехфотонной аннигиляции свободной пары соотношением

$$w_1 = \frac{4}{3}\bar{w}_{3\gamma} = \frac{4}{3}|\psi(0)|^2 \left(v\bar{\sigma}_{3\gamma}\right)_{0\to 0}$$

(3/4 - относительный статистический вес состояния со спином 1). Забегая вперед, укажем, что

$$\bar{\sigma}_{3\nu} = \frac{4\left(\pi^2-9\right)c}{3v}\alpha\left(\frac{e^2}{mc^2}\right)^2.$$

Поэтому продолжительность жизни ортопозитрония

$$\tau_1 = \frac{9\pi}{2\left(\pi^2-9\right)}\frac{\hbar}{mc^2\alpha^5} = 1,4\cdot 10^{-7}\text{с}.$$

Неравенство $\Gamma_1 \ll |E_{\text{осн}}|$ в этом случае, разумеется, выполняется еще в большей степени, чем для парапозитрония.

Вычислим сечение трехфотонной аннигиляции свободной пары (A. Ore, J. L. Powell, 1949).

Соответствующую амплитуду запишем в виде

$$M_{fi} = (4\pi)^{s/2} e_\lambda^{(3)^*} e_\mu^{(2)^*} e_v^{(1^{**})} \bar{u}\left(-p_+\right) Q^{\lambda\mu v} u\left(p_-\right),$$

где

$$Q^{\lambda\mu\nu} = \sum_{\text{пер}} \gamma^\lambda G\left(k_3 - p_+\right)\gamma^\mu G\left(p_- - k_1\right)\gamma^\nu,$$

причем сумма берется по всем перестановкам номеров фотонов $1,2,3$ вместе с одновременными такими же перестановками соответствующих тензорных индексов $\lambda\mu v$.

Квадрат модуля амплитуды, усредненный по поляризациям электрона и позитрона и просуммированный по поляризациям фотонов:

$$\frac{1}{4}\sum_{\text{поляр}} |M_{fi}|^2 = (4\pi)^3 \,\text{Sp}\left\{\rho_+ Q^{\lambda\mu\nu}\rho_- \bar{Q}_{\lambda\mu\nu}\right\},$$

где

$$\rho_- = \frac{1}{2}\left(\gamma p_- + m\right), \quad \rho_+ = \frac{1}{2}\left(\gamma p_+ - m\right).$$

В результате все же довольно длинного вычисления пояучается

$$\frac{1}{4}\sum_{\text{поляр}} |M_{fi}|^2 = (4\pi)^3 e^6 \cdot 16\left[\left(\frac{m-\omega_1}{\omega_2\omega_3}\right)^2 + \left(\frac{m-\omega_2}{\omega_1\omega_3}\right)^2 + \left(\frac{m-\omega_3}{\omega_1\omega_2}\right)^2\right].$$

Подставив это выражение в $(89,8)$, найдем дифференциальное сечение трехфотонной аннигиляции:

$$d\tilde{\sigma}_{3\gamma} = \frac{e^6}{\pi^2 m^2 v}\left[\left(\frac{m-\omega_1}{\omega_2\omega_3}\right)^2 + \left(\frac{m-\omega_2}{\omega_1\omega_3}\right)^2 + \left(\frac{m-\omega_3}{\omega_1\omega_2}\right)^2\right]\times$$
$$\times\,\delta\left(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3\right)\delta\left(\omega_1+\omega_2+\omega_3-2m\right)\frac{d^3k_1 d^3k_2 d^3k_3}{\omega_1\omega_2\omega_3}.$$

Здесь надо еще исключить δ-функции. Первая нз них устраняется интегрированием по d^3k_3, после чего заменяем остальные дифференциалы:

$$d^3k_1 d^3k_2 \to 4\pi\omega_1^2 d\omega_1 \cdot 2\pi\omega_2^2 d\left(\cos\theta_{12}\right) d\omega_2$$

где θ_{12} - угол между $\mathbf{k}_1$ и $\mathbf{k}_2$; подразумевается, что уже произведено интегрирование по направлениям $\mathbf{k}_1$ и азимуту $\mathbf{k}_2$ относительно k_1. Дифференцируя равенство

$$\omega_3 = \sqrt{\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2\cos\theta_{12}},$$

находим

$$d\cos\theta_{12} = \frac{\omega_3}{\omega_1\omega_2} d\omega_3$$

Полнсе сечение аннигиляции получается интегрированием $(89,14)$ по обеим частотам:

$$\bar{\sigma}_{3y} = \frac{4\varepsilon^6}{3vm^2} 3 \int_0^m \int_{m-\omega_1}^m \frac{\left(\omega_1+\omega_2-m\right)^2}{\omega_1^2\omega_2^2} d\omega_1 d\omega_2.$$

Стояший здесь двойной интеграл равен $\left(\pi^2-9\right)/3$, и мы приходим к приведенной выше формуле $(89,6)$.

4.2.3 Photons in Magnetic Field, in Nucleus, Pairs in the Ultrarelativistic Case

90. Магнитотормозное излучение

квантовая отдача при испускании фотона. Последняя определяется отношением $\hbar\omega/\varepsilon$, и условие применимости классической теории требует его малостн. В этой связи удобно ввести параметр

$$\chi = \frac{H}{H_0}\frac{|p|}{m} \approx \frac{H\varepsilon}{H_0 m} \approx \frac{\hbar\omega_0}{\varepsilon}\left(\frac{\varepsilon}{m}\right)^3,$$

где $H_0 = m^2/|e|\hbar\,(= m^2c^3/|e|\hbar) = 4,4\cdot 10^{13}\Gamma$ с. В классической области $\chi \sim \hbar\omega/\varepsilon \ll 1$. В случае $\chi \geqslant 1$ энергия излученного фотона $\hbar\omega \sim \varepsilon$, причем при $\chi \gg 1$ (как мы увидим в дальнейшем) существенная область спектра простирается до частот, при которых энергия электрона после испускания

$$\varepsilon' \sim m\frac{H_0}{H} \ll \varepsilon.$$

Для того чтобы электрон оставался ультрарелятивистским, поле должно удовлетворять условию

$$\frac{H}{H_0} \ll 1$$

символическом виде

$$\psi = (2\hat{H})^{-1/2} u(\hat{p}) \exp\left(-\frac{i}{\hbar}\hat{H}t\right)\varphi(\mathbf{r}),$$

где $\varphi(\mathbf{r}) \sim \exp(iS/\hbar)$ - квазиклассические волновые функцин бесспиновой частицы ($S(\mathbf{r})$ - ее классическое действие); $u(\hat{p})$ - операторный биспинор

$$u(\hat{p}) = \begin{pmatrix} (\hat{H}+m)^{1/2} w \\ (\hat{H}+m)^{-1/2}(\sigma\hat{p})\omega \end{pmatrix},$$

получающийся из биспинорной амплитуды плоской волны $u(p)$ $(23,9)$ заменой $\mathbf{p}$ и ε операторами [2])

$$\hat{\mathbf{p}} = \hat{\mathbf{p}} - e\mathbf{A} = -i\hbar\nabla - e\mathbf{A}, \quad \hat{H} = \left(\hat{\mathbf{p}}^2 + m^2\right)^{1/2},$$

$$V_{fi}(t) = e\sqrt{\frac{2\pi}{\hbar\omega}}\langle f|Q(t)|i\rangle e^{i\omega t},$$

где $\hat{Q}(t)$ обозначает гейзенберговский оператор

$$\widehat{Q}(t) = \frac{u_f^+(\hat{p})}{(2\hat{H})^{1/2}}\,(\alpha \mathrm{e}^*)\,e^{-i\hat{\mathrm{k}}(t)}\frac{u_i(\hat{p})}{(2\hat{H})^{1/2}},$$

$$dI = \frac{e^2}{4\pi^2}d^3k\int e^{-i\omega\tau}\left\langle i\left|Q^+\left(t+\frac{\tau}{2}\right)Q\left(t-\frac{\tau}{2}\right)\right|i\right\rangle d\tau.$$

$$\hat{\mathbf{p}}\exp(-i\mathbf{k}\hat{\mathbf{r}}) = \exp(-i\mathbf{k}\hat{\mathbf{r}})(\hat{\mathbf{p}} - \hbar\mathbf{k})$$
$$H(\hat{\mathbf{p}})\exp(-i\mathbf{k}\hat{\mathbf{r}}) = \exp(-i\mathbf{k}\hat{\mathbf{r}})H(\hat{\mathbf{p}} - \hbar\mathbf{k})$$

сдвига в импульсном пространстве. С помощью $(90,11)$. выносим в $(90,8)$ оператор $\exp(-i\mathrm{k}\hat{\mathbf{r}}(t))$ налево и записываем $\widehat{Q}(t)$ в виде

$$\hat{Q}(t) = \exp[-i\mathbf{k}\hat{\mathbf{r}}(t)]\hat{R}(t), \quad \hat{R}(t) = \frac{u_f^+\left(\hat{p}'\right)}{\left(2\widehat{H}'\right)^{1/2}}\,(\alpha \mathrm{e}^*)\,\frac{u_i(\hat{p})}{(2\widehat{H})^{1/2}},$$

где $\hat{H}' = \hat{H} - \hbar\omega, \hat{\mathbf{p}}' = \hat{\mathbf{p}} - \hbar\mathbf{k}$. Теперь

$$\hat{Q}_2^+\hat{Q}_1 = \hat{R}_2\exp\left(i\mathbf{k}\hat{\mathbf{r}}_2\right)\exp\left(-i\mathbf{k}\hat{\mathbf{r}}_1\right)\hat{R}_1$$

$$L(\tau) = \exp\left\{i\frac{\boldsymbol{\varepsilon}}{\boldsymbol{\varepsilon}'}\left(\mathbf{k}\mathbf{r}_2 - \mathbf{k}\mathbf{r}_1 - \omega\tau\right)\right\}$$

До сих пор мы не использовали конкретного вида траектории электрона. Выразим теперь $\boldsymbol{r}_2 - \mathbf{r}_1$ в $(90,18)$ через $\mathbf{p}_1$ с помощью уравнения движения электрона в плоскости, перпендикулярной полю H (см. II, §21):

$$\mathbf{r}_2 - \mathbf{r}_1 = \frac{\mathbf{p}_1}{e\ddot{H}}\sin\frac{eH\tau}{\varepsilon} + \frac{[\mathbf{p}_1\mathbf{H}]}{eH^2}\left(1 - \cos\frac{eH\tau}{\varepsilon}\right).$$

Разлагая по степеням τ, имеем отсюда

$$\mathbf{k}\left(\mathbf{r}_2 - \mathbf{r}_1\right) - \omega\tau \approx \omega\tau\left\{\left(\mathbf{v}_1\mathbf{n} - 1\right) + \tau\frac{e\mathbf{n}\left[\mathbf{p}_1\mathbf{H}\right]}{2\varepsilon^2} - \tau^2\frac{e^2H^2}{6\varepsilon^2}\right\}$$

Таким образом, окончательно имеем

$$\exp(-i\omega t)\left\langle i\left|Q_2^+Q_1\right|i\right\rangle = R_2^*R_1L(\tau)$$
$$R_2^*R_1 = \mathrm{Sp}\,\frac{1+\xi_i\boldsymbol{\sigma}}{2}\left(\left(\mathrm{A}_2 - i\left[\mathbf{B}_2\boldsymbol{\sigma}\right]\right)\mathrm{e}\right)\frac{1+\xi_i\boldsymbol{\sigma}}{2}\left(\left(\mathbf{A}_1 + i\left[\mathbf{B}_1\boldsymbol{\sigma}\right]\right)\mathrm{e}^*\right).$$

Множители $(1+\xi\sigma)/2$ - двухрядные поляризационные матрицы плотности начального и конечного электронов.

$$\frac{dI}{d\omega}=\frac{ie^2\omega}{2\pi}\int\limits_{-\infty}^{\infty}\left(\frac{m^2}{\varepsilon^2\tau}+\frac{\varepsilon^2+\varepsilon'^2}{4\varepsilon\varepsilon'}\omega_0^2\tau\right)\exp\left\{-\frac{i\omega\tau\varepsilon}{\boldsymbol{\varepsilon}'}\left(1-v+\frac{\tau^2}{24}\omega_0^2\right)\right\}d\tau,$$

причем контур интегрирования выбирается указанным выше способом. Используя интегральное представление, функции Эйри Φ (см. III § b), нетрудно показать, что первый член сводится к интегралу от функции Эйри, а второй-к производной от нее. Окончательно находим

$$\frac{dI}{d\omega}=-\frac{e^2m^2\omega}{\sqrt{\pi}\varepsilon^2}\left\{\int\limits_x^{\infty}\Phi(\xi)d\xi+\left(\frac{2}{x}+\frac{\hbar\omega}{\varepsilon}\chi x^{1/2}\right)\tilde{\Phi}'(x)\right\},$$

$$x=\left(\frac{\hbar\omega}{\varepsilon'\chi}\right)^{2/3}=\frac{m\iota^2}{\varepsilon^2}\left(\frac{\varepsilon\omega}{\varepsilon'\omega_0}\right)^{2/s}$$

(А. И. Никишов, В. И. Ритус, 1967).

Магнитотормозное излучение ириводит к возникновению поляризации движущихся электронов (А. А. Соколов, И. М. Тернов, 1963). Для рассмотрения этого вопроса надо найти всроятность радиационного перехода с обращением направления спина. Положнв в $(90,21)\xi_i=-\xi_f\equiv\xi,|\zeta|=1$, получим

$$R_2^*R_1=(\mathbf{B}_1\mathbf{B}_2)-(\mathrm{e}^*\mathbf{B}_1)(\mathbf{e}_2)-(\mathbf{e}^*[\mathbf{B}_1\zeta])(\mathbf{e}[\mathbf{B}_2\zeta])-i(\zeta\mathrm{e}^*)(\mathbf{e}[\mathbf{B}_1\mathbf{B}_2]).$$

окончательно полную вероятность радиационного перехода с обращением спина:

$$w=\frac{5\sqrt{3}\alpha}{16}\frac{\hbar^2}{m^2}\left(\frac{\varepsilon}{m}\right)^5\omega_0^3\left(1-\frac{2}{9}\zeta_\parallel^2-\frac{8\sqrt{3}}{15}\frac{e}{|e|}\zeta_\perp\right),$$

где $\zeta_\parallel=\zeta\mathbf{v},\zeta_\perp=\zeta\mathbf{H}/H$. Эта формула пригодна как для электронов $(e<0)$, так и для позитронов $(e>0)$.

Вероятность $(90,30)$ не зависит от знака продольной поляризации $\zeta_\parallel$, но зависит от знака $\zeta_\perp$.

91. Образование пар фотоном в магнитном поле

Образование электрон-позитронной пары фотоном в магнитном поле и магнитотормозное излучение - два перекрестных канала одной и той же реакции. Поэтому амплитуда M_{fi} процесса образования пары получается из амплитуды тормозного излучения просто путем замены

$$\varepsilon,p\to-\varepsilon_+,-p_+;\quad\varepsilon',p'\to\varepsilon_-,p_-;\quad\omega,k\to-\omega,-k\quad(91,1)$$

(здесь ε_-,p_-и ε_+,p_+−энергии и импульсы электрона и пози трона в паре; $\boldsymbol{\varepsilon},\mathbf{p}$ и $\boldsymbol{\varepsilon}',\mathbf{p}'$ - начальные и конечные энергии и импульсы электрона при тормозном излучении). В терминах углов и абсолютных значений преобразование импульсов есть

$$|\mathbf{p}|\to|\mathbf{p}_+|,|\mathbf{p}'|\to|\mathbf{p}_-|,\theta\to\pi-\theta_+,\theta'\to\theta_-,\varphi\to\varphi-\pi,$$

где $\theta_\pm$−углы между $\mathbf{p}_\pm$и $\mathbf{k},\varphi$-угол между плоскостями $\mathbf{k},\mathbf{p}_+$и $\mathbf{k},\mathbf{p}_-$.

В случае же образования пары фотоном сечение выражается через амплитуду согласно

$$d\sigma = |M_{fi}|^2 \frac{1}{8\omega\varepsilon_-\varepsilon_+}\delta\left(\omega - \varepsilon_+ - \varepsilon_-\right)\frac{d^3p_+ d^3p_-}{(2\pi)^5}$$

или, после исключения δ-функции:

$$d\sigma = |M_{fi}|^2 \frac{|\mathrm{p}_+|\,|\mathbf{p}_-|}{8(2\pi)^5\omega} do_+ do_- d\varepsilon_+.$$

$$dw = \frac{m^2e^2}{\sqrt{\pi}}\frac{d\varepsilon_+}{\omega^2}\left\{\int_x^\infty \Phi(\xi)d\xi + \left(\frac{2}{x} - xx^{1/2}\right)\Phi'(x)\right\}.$$

где теперь

$$x = \left(\frac{m^3\omega}{|e|H{\varepsilon_+}^2-}\right)^{2/3}, \quad x = \frac{|e|H\omega}{m^3}\left(= \frac{\hbar^2|e|H\omega}{m^3c^5}\right).$$

Полная вероятность рождения пары в единицу времени получается интегрированием $(91,8)$ по ε_+(причем, ввиду очевидной симметрии по отношению к ε_+и $\varepsilon_- = \omega - \varepsilon_+$, достаточно интегрировать от 0 до $\omega/2$ и затем удвонть результат). Производя замену переменной интегрирования ε_+на x и интегрируя первый член в $(91,8)$ дважды по частям, получаем

$$w = -\frac{|e|^3H}{mx\sqrt{\pi}}\int_{(4/x)^{2/6}}^\infty \frac{2\left(x^{3/2}+1/x\right)\Phi'(x)}{x^{1/4}\left(x^{3/2}-4/x\right)^{1/2}}dx$$

(А. И. Никищов, В. И. Ритус, 1967).

92. Тормозное излучение электрона на ядре. Нерелятивистский случай

93. Тормозное излучение электрона на ядре. Релятивистский случай

94. Образование пар фотоном в поле ядра

95. Точная теория рождения пар в ультрарелятивистском случае

96. Точная теория тормозного излучения в ультрарелятивистском случае

4.2.4 Special Effects with Photons

97. Тормозное излучение электрона на электроне в ультрарелятивистском случае

98. Излучение мягких фотонов при столкновениях

99. Метод эквивалентных фотонов

100. Образование пар при столкновениях частиц

101. Излучение фотона электроном в поле интенсивной электромагнитной волны

(мб в другой разедл перенесу, пока тут)

4.2.5 Radiative Corrections to the Mass Operator, Scattering in External Field

119. Вычисление массового оператора по Ландау 4

120. Испускание мягких фотонов с ненулевой массой по Ландау 4

121. Рассеяние электрона во внешнем поле во втором борновском приближении по Ландау 4

122. Радиационные поправки к рассеянию электрона во внешнем поле по Ландау 4

4.2.6 Radiative Level Shifts, Bound States

123. Радиационное смещение атомных уровней по Ландау 4

124. Радиационное смещение уровней мезоатомов по Ландау 4

125. Релятивистское уравнение для связанных состояний по Ландау 4

(тут нужно много диаграмм, так что потом выгружу ЛЛ4, их и вставлю.)

$$\Gamma_{ik,lm} = -e^2\gamma^\mu_{il}\gamma^v_{km}D_{\mu\nu}\left(p_- - p'_-\right) + e^2\gamma^\mu_{im}\gamma^v_{kl}D_{\mu\nu}\left(p_- + p_+\right)$$

Будем рассматривать все величины в системе «центра инерции». Поскольку, однако, 4-импульсы внешних концов диаграмм не предполагаются физическими (т. е. $p^2 \neq m^2$), то хотя в этой системе $\mathbf{p}_+ = -p_-$, но $\varepsilon_+ \neq \varepsilon_-$. Таким образом, 4-импульсы концОВ

$$\begin{gathered} p_- = (\varepsilon_-, \mathbf{p}), \quad p_+ = (\varepsilon_+, -\mathbf{p}), \\ p'_- = \left(\varepsilon'_-, \mathbf{p}'\right), \quad p'_+ = \left(\varepsilon'_+, -\mathbf{p}'\right), \\ \varepsilon_- + \varepsilon_+ = \varepsilon'_- + \varepsilon'_+. \end{gathered}$$

Энергия связи электрона и позитрона в позитронии $\sim m\alpha^2$. Поэтому в интересующей нас окрестности полюсов амплитуды рассеяния

$$\begin{gathered} |\mathbf{p}| \sim |\mathbf{p}'| \sim m\alpha \ll m, \\ |\varepsilon_- - m| \sim |\boldsymbol{\varepsilon}_+ - m| \sim \mathbf{p}^2/m \sim m\alpha^2, \ldots \end{gathered}$$

Вклад в вершинную часть от диаграммы $(125, 4, a)$

$$\Gamma^{(4a)}_{ik,lm} = -ie^4 \int \left(\gamma^\lambda G(q)\gamma^\mu\right)_{il} \left(\gamma^v G\left(q - p_- - p_+\right)\gamma^0\right)_{km} \times \\ \times D_{\lambda\rho}\left(q - p'_-\right) D_{\mu v}\left(p_- - q\right) \tfrac{d^4q}{(2\pi)^4}.$$

$$\Gamma^{(4a)} \sim e^4 \int \frac{d^3q}{\cdot \left(q - p'_-\right)^2 \left(p_- - q\right)^2 \left(2m - \varepsilon_- - \varepsilon_+ + q^2/m\right)}$$

и отсюда, с учетом $(125, 6)$, оценку

$$\Gamma^{(4\alpha)} \sim \alpha^2 \frac{(m\alpha)^3}{(m\alpha)^4 m\alpha^2} = \frac{1}{m^2\alpha}.$$

4.2.7 $\gamma\gamma \to (?)$, Photon Splitting

126. Двойное дисперсионное соотношение по Ландау 4

(диаграммы и умные рассуждения)

Каждую из инвариантных амплитуд (которые мы обозначим здесь той же буквой M) можно представить интегралом вида

$$M = \int \frac{iBd^4q}{\left[q^2-m^2\right]\left[(q-k_4)^2-m^2\right]\left[(q-k_1-k_2)^2-m^2\right]\left[(q-k_2)^2-m^2\right]},$$

где B - некоторая функция всех 4-импульсов; множители в знаменателе происходят от пропагаторов четырех виртуальных электронов.

Как и для всякой диаграммы вида

(???? тут два кружка с входящими 4-мя линиями и соединенные двумя линиями??)

$A_{1s}(s,t)$ вычисляется по правилу $(115,9)$ заменой в интеграле $(126,4)$ соответствующих полюсных множителей δ-функциями: $2iA_{1s}(s,t) = (2\pi i)^2 \int \frac{iB\delta\left(q^2-m^2\right)\delta\left\lfloor(q-k_1-k_2)^2-m^2\right\rfloor}{\left[(q-k_4)^2-m^2\right]\left[(q-k_2)^2-m^2\right]}d^4q$, причем интегрирование производится по половине q-пространства, в которой $q^0>0$.

$$M(s,0) = \frac{s}{\pi}\int \frac{f(s')\,ds'}{(s'-s)\,s'}$$

с однократным дисперсионным соотношением $(126,5)$, написанным «с вычитанием», отвечающим условию $(126,20)$:

$$M(s,t) = \frac{s}{\pi}\int \frac{A_{1s}(s',t)}{(s'-s)\,s'}ds'.$$

Таким образом, окончательное двойное дисперсионное соотношение «с вычитанием»:

$$M(s,t) = \frac{st}{\pi^2}\int\int \frac{A_2(s',t')}{(s'-s)(t'-t)\,s't'}ds'dt' + \\ + \frac{s}{\pi}\int \frac{A_{1s}(s',0)}{(s'-s)\,s'}ds' + \frac{t}{\pi}\int \frac{A_{1t}(0,t')}{(t'-t)\,t'}dt'.$$

127. Рассеяние фотона на фотоне по Ландау 4

$$M_{fi} = 2\left(M^{(a)} + M^{(\sigma)} + M^{(\theta)}\right),$$

где $M^{(a)}, M^{(\sigma)}, M^{(\theta)}$ - вклады диаграмм $a),\,),\,)$. Согласно $(64,19)$ сечение рассеяния

$$d\sigma = \frac{1}{64\pi^2}\left|M_{fi}\right|^2\frac{do'}{(2\omega)^2},$$

где do' - элемент телесных углов для направления $\mathbf{k}'$ в системе центра инерции. Угол рассеяния в этой системе обозначим θ. Инвариантные амплитудь Выделив поляризационные множители четырех фотонов, представим M_{fi} в виде

$$M_{fi} = e_1^{\lambda}e_2^{\mu}e_3^{\upsilon^*}e_4^{\rho^*}M_{\lambda\mu\nu\rho};$$

$$M_{\lambda_1\lambda_2\lambda_:\lambda_4}(s,t,u) = M_{\lambda_1\lambda_4\lambda_1\lambda_2}(s,t,u).$$

Наконец, еще одно соотношение является следствием инвариантности амплитуды M_{fi} относительно перестановки двух начальных или двух конечных фотонов. Если произвести сразу обе перестановки $(k_1 \leftrightarrow k_2, k_3 \leftrightarrow k_4)$, то переменные s,t,u ие изменятся, а перестановка в поляризационных индексах приведет к соотношению

$$M_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,t,u) = M_{\lambda_2\lambda_1\lambda_1\lambda_3}(s,t,u).$$

Легко убедиться, что в силу свойств симметрии $(127,10-12)$ цисло независимых инвариантных амплитуд сводится к пяти; в качестве них можно, например, выбрать

$$M_{++++}, M_{++--}, M_{+++-}, M_{+-++}, M_{+++-}$$

(индексы «+", «-" означают спиральности +1 и -1). Если подставить в $(127,3)$ вместо M_{fi} одну из амплитуд $M_{\lambda_1\lambda_2\lambda_5\lambda_4}$, то мы получим сечение рассеяния с заданными поляризациями начальных и конечных фотонов. Сечение же, просуммированное по конечным и усредненное по начальным поляризациям, получится заменой

$$|M_{fi}|^2 \to 1/4\left\{2\left|M_{++++}\right|^2 + 2\left|M_{++--}\right|^2 + 2\left|M_{+-+-}\right|^2 + \right.$$
$$\left. + 2\left|M_{+-++}\right|^2 + 8\left|M_{+++-}\right|^2\right\}.$$

Вычисление амплитуд Интеграл $M_{fi}^{(a)}$, отвечающий диаграмме $(127,1,a)$, имеет вид $(126,4)$, причем

$$B^{(a)} = \frac{e^4}{\pi^2}\,\mathrm{Sp}\left\{(\gamma e_1)(\gamma q - \gamma k_2 + m)(\gamma e_2)(\gamma q + m)\times\right.$$
$$\left.\times(\gamma e_4^*)(\gamma q - \gamma k_4 + m)(\gamma e_3^*)(\gamma q - \gamma k_1 - \gamma k_2 + m)\right\}.$$

(тут большие формулы, нет смысла писать их)

128. Когерентное рассеяние фотона в поле ядра по Ландау 4

Другими (наряду с рассеянием фотона на фотоне) нелинейными эффектами, описывающимися квадратными диаграммами вида $(127,1)$, являются распад одного фотона во внешнем поле на два фотона (и обратный процесс «слияния» двух фотонов в один) и рассеяние фотона во внешнем поле. Первому процессу отвечают диаграммы, в которых один из четырех внешних фотонных концов заменен линией внешнего поля. Второму же процессу отвечают диаграммы с двумя внешними линиями реальных и двумя - виртуальных фотонов.

К последней категории относится, в частности, когерентное (упругое) рассеяние фотона в постоянном электрическом поле неподвижного ядра. В общем случае вычисления приводят к очень громоздким формулам (содержащим кратные квадратуры))[1]. Мы ограничимся здесь лишь некоторыми оценками. В силу требований калибровочной инвариантности амплитуда рассеяния при $\omega \to 0$ должна содержать произведения компонент 4-импульса начального (k) и конечного (k') фотонов (подобно тому как разложение амплитуды рассеяния фотона на фотоне начинается с четверных произведений компонент 4-импульсов всех фотонов). Другими словами, амплитуда рассеяния фотона

малой частоты пропорциональна ω^2. Учитывая также, что эта амплитуда содержит внешнее поле (поле ядра с зарядом Ze) во втором порядке, заключаем, что сечение рассеяния

$$d\sigma \sim Z^4\alpha^4 r_e^2 \left(\frac{\omega}{m}\right)^4 do, \quad \omega \ll m.$$

Зависимость от частоты находится, разумеется, в соответствии с общнми заключениями §59.

Коэффициент в (128, 1) нельзя вычислить с помощью функции Лагранжа однородного электромагнитного поля (как это можно было сделать для рассеяния света на свете). Причина заключается в том, что в данном процессе существенны расстояния от ядра $r \sim 1/m$, на которых поле ядра нельзя рассматривать как однородное. Приведем результат точного расчета:

$$d\sigma_{++} = d\sigma_{--} = 1,004 \cdot 10^{-3}(Z\alpha)^4 r_e^2 \left(\frac{\omega}{m}\right)^4 \cos^4\frac{\theta}{2} do,$$
$$d\sigma_{+-} = d\sigma_{-+} = 3,81 \cdot 10^{-4}(Z\alpha)^4 r_e^2 \left(\frac{\omega}{m}\right)^4 \sin^4\frac{\theta}{2} do.$$

130. Расщепление фотона в магнитном поле по Ландау 4

131. Вычисление интегралов по четырехмерным областям по Ландау 4

4.2.8 Hadrons, Scattering of Electrons on Them, Transformations

138. Электромагнитные формфакторы адронов по Ландау 4

139. Рассеяние электронов адронами по Ландау 4

140. Низкоэнергетическая теорема для тормозного излучения по Ландау 4

В §98 был исследован процесс испускания фотона при столкновении частиц в пределе, когда частота фотона стремится k нулю. Оказалюсь, что амплитуда процесса обратно пропорциональна ω и простым образом выражается через амплитуду того же столкновения без испускания мягкого фотона (об этой носледней мы будем снова говорить условно как об амплитуде щем по ю приближении будет

$$M_{fi} = M_{fi}^{(-1)} + M_{fi}^{(0)},$$

где к главному члену (ω^{-1}) добавляется не зависящий от ω ($\infty\omega^0$) лоправочный член. Мы увидим, что и этот поправочный (как и главный) член может быть выражен через $M_{fi}^{(y)}$, причем независимо от деталей электромагнитной структуры адрона. Это утверждение называют низкоэнергетической теоремой для тормозного излучения (F. E. Low, 1958).

141. Низкоэнергетическая теорема для рассеяния фотона на адроне по Ландау 4

142. Мультипольные моменты адронов по Ландау 4

143. Неупругое рассеяние электронов адронами по Ландау 4

144. Превращение электрон-позитронной пары в адроны по Ландау 4

4.3 Hypothetical processes

4.3.1 Examples from the Standard Model and its SUSY extension by Dreiner et.al.

(see their wonderful article for now. I'll add it with mathpix later here, because there are diagrams that are not working by code!)

(I make it subsection, not subsubsection, because it is important to have an overview of examples. No subsubsections are planned here!)

5 Other Well-Known Methods of QFT

5.1 On Other Properties of QFT (?)

(много еще может быть применений кстати.)

5.1.1 On typical spontaneous symmetry breaking, Higgs boson (???)

(тоже потом напишу сюда, типичная тема. в ПШ и у Вергелеса про это написано, дело 2 часов.)

Спонтанное нарушение симметрии в классической теории

Теорема Голдстоуна

Эффективное действие

Эффект Хиггса в абелевой теории по Киселеву

Пусть

$$S = \int \mathrm{d}^4x \left\{ (\nabla_\mu \Phi)^* \nabla_\mu \Phi - \frac{\lambda}{4}\left(\Phi^*\Phi - \frac{v^2}{2}\right)^2 - \frac{1}{4}F_{\mu\nu}^2 \right\},$$

где $\nabla_\mu = \partial_\mu + \mathrm{i}e\mathcal{A}_\mu(x)$. Калибровочную свободу зафиксируем с помощью $\mathfrak{Im}\Phi \equiv 0$.

$$\Phi = \frac{1}{\sqrt{2}}\{v + \chi(x)\}.$$

$$(\nabla_\mu \Phi)^* \nabla^\mu \Phi = \frac{1}{2}\left\{(\partial_\mu \chi)^2 + e^2(v+\chi(x))^2 \mathcal{A}_\mu^2\right\}.$$

$$S_0 = \int \mathrm{d}^4x \left\{ \frac{1}{2} \left(\partial_\mu \chi\right)^2 - \frac{1}{2} m_\chi^2 \chi^2 - \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} m_{\mathcal{A}}^2 \mathcal{A}_\mu^2 \right\}, \qquad \begin{aligned} m_{\mathcal{A}}^2 &= e^2 v^2, \\ m_\chi^2 &= \frac{1}{2} \lambda v^2, \end{aligned}$$

$$S_{\text{int}} = \int \mathrm{d}^4x \left\{ -\frac{\lambda}{4} \left(v\chi^3 + \frac{\chi^4}{4} \right) + \frac{e^2}{2} \left(2v\chi + \chi^2 \right) \mathcal{A}_\mu^2 \right\}$$

$$\frac{1}{2} \mathcal{A}^\mu \hat{\Gamma}_{\mu\nu} \mathcal{A}^\nu, \quad \hat{\Gamma}_{\mu\nu} = -g_{\mu\nu} \left(p^2 - m_A^2 \right) + p_\mu p_\nu.$$

$$\mathcal{D}_{\mu\nu}(p) = \mathrm{i} \Gamma_{\mu\nu}^{-1},$$

$$\mathcal{D}_{\mu\nu}(p) = \frac{\mathrm{i}}{p^2 - m_A^2 + \mathrm{i}0} \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_A^2} \right).$$

$$\mathcal{L}_\varphi = \frac{1}{2} \left(\partial_\mu \varphi\right)^2 - \lambda_\varphi \varphi^2 \Phi^* \Phi,$$

$\lambda_\varphi > 0$. Тогда при спонтанном нарушении симметрии это поле приобретает массу за счет взаимодействия с вакуумным средним бозона Хиггса:

$$m_\varphi^2 = \lambda_\psi v^2$$

(!!! как сделаю основные темы - додумаю этот типичный тоже эффект!!!)

5.1.2 Wilsonian Renormalization, effective action, RG flow, effective theories (!!!!!)

(окажется, что нужно пару разделов нужно - сделаю)

Effective Actions

In QFT we no longer have to sit at a minimum of the action, quantum fluctuations appear which involve degrees of freedom at arbitrarily high energies, e.g. in loop integrals. The content of renormalization is that it is still possible to calculate at low energies without knowing what those high-energy degrees of freedom are. (???? отслежу потом, как конкретно то, что действие не минимальное проявляется???)

• We consider a scalar field theory with action

$$S_{\Lambda_0}[\varphi] = \int d^dx \frac{1}{2} \partial^\mu \varphi \partial_\mu \varphi + \sum_i \Lambda_0^{d-d_i} g_{i0} \mathcal{O}_i(x)$$

The subscript Λ_0 refers to the UV cutoff, which must be included to define the path integral. The $\mathcal{O}_i(x)$ are arbitrary local operators with dimensions $d_i > 0$, where the dimensions are found with ordinary dimensional analysis, and the g_{i0} are defined so they are dimensionless.

• The partition function is

$$\mathcal{Z}_{\Lambda_0}\left(g_{i0}\right) = \int\limits_{C^\infty(M)_{\leq \Lambda_0}} \mathcal{D}\varphi e^{-S_{\Lambda_0}[\varphi]/\hbar}$$

where $C^\infty(M)_{\leq\Lambda_0}$ is the space of smooth functions on M with momenta bounded by Λ_0. There are also IR divergences, which we can regulate by putting the system in a box.

• Since $C^\infty(M)_{\leq\Lambda_0}$ is a vector space, we may split the modes as

$$\varphi(x) = \phi(x) + \chi(x), \quad \phi \in C^\infty(M)_{\leq\Lambda}, \quad \chi \in C^\infty(M)_{(\Lambda,\Lambda_0]}, \quad \mathcal{D}\varphi = \mathcal{D}\phi\mathcal{D}\chi$$

Integrating over the high-energy modes χ gives the effective action

$$S_\Lambda^{\text{eff}}[\phi] = -\hbar \log \int\limits_{C^\infty(M)_{(\Lambda,\Lambda_0]}} \mathcal{D}\chi \exp\left(-S_{\Lambda_0}[\phi+\chi]/\hbar\right)$$

We can iterate this process, yielding a semigroup called the renormalization group. Note that the effective action $\mathcal{W}[J]$ is just a Wilsonian effective action with $\Lambda = 0$.

• Setting $\hbar = 1$, we may separate out the kinetic terms of the action as

$$S_{\Lambda_0}[\phi+\chi] = S^0[\phi] + S^0[\chi] + S^{\text{int}}_{\Lambda_0}[\phi,\chi]$$

where there are no $\phi\chi$ mixing terms by momentum conservation. Therefore the interaction terms, which include mass terms, obey

$$S_\Lambda^{\text{int}}[\phi] = -\log \int\limits_{C^\infty(M)_{(\Lambda,\Lambda_0]}} \mathcal{D}\chi \exp\left(-S^0[\chi] - S^{\text{int}}_{\Lambda_0}[\phi,\chi]\right)$$

which yields effective couplings $g_i(\Lambda)$. Note that if we expand the right-hand side in powers of $S^{\text{int}}_{\Lambda_0}$, the even-power terms contribute with an extra minus sign, and the logarithm ensures we only sum over connected diagrams.

• By definition, we must have

$$\mathcal{Z}_\Lambda\left(g_i(\Lambda)\right) = \mathcal{Z}_{\Lambda_0}\left(g_{i0}\right)$$

because both sides are the same integral; the left-hand side depends on Λ through both the cutoff and the couplings. We can also include external currents to have partition functions $\mathcal{Z}[J]$. Then we have $\mathcal{Z}_\Lambda[J] = \mathcal{Z}_{\Lambda_0}[J]$ as long as J has no Fourier components in $(\Lambda, \Lambda_0]$.

• Therefore, the total derivative with respect to $\log\Lambda$ is zero,

$$\frac{d\mathcal{Z}_\Lambda(g)}{d\log\Lambda} = \left(\left.\frac{\partial}{\partial\log\Lambda}\right|_{g_i} + \frac{\partial g_i(\Lambda)}{\partial\log\Lambda}\left.\frac{\partial}{\partial g_i}\right|_\Lambda\right)\mathcal{Z}_\Lambda(g) = 0$$

This is an example of an RG equation, which we'll generically call a Callan-Symanzik equation.

• It will be useful to adjust our definitions to maintain canonical normalization of the kinetic term. With a generic initial action, the effective action will have the form

$$S_\Lambda^{\text{eff}}[\phi] = \int d^dx \frac{Z_\Lambda}{2}\partial^\mu\phi\partial_\mu\phi + \sum_i \Lambda^{d-d_i} Z_\Lambda^{n_i/2} g_i(\Lambda)\mathcal{O}_i(x)$$

where n_i is the number of factors of ϕ in $\mathcal{O}_i(x)$, and we have redefined $g_i(\Lambda)$ to pull out factors of Z_Λ. We then define the renormalized field

$$\varphi = Z_\Lambda^{1/2}\phi$$

so that $S_\Lambda^{\text{eff}}[\varphi]$ has a canonically normalized kinetic term and dimensionless couplings $g_i(\Lambda)$.

• For convenience, we define the beta function

$$\beta_i\left(g_j(\Lambda)\right) = \frac{\partial g_i}{\partial \log \Lambda} = \left(d_i - d\right) g_i(\Lambda) + b_i^q\left(g_j\right)$$

where the first term is just from the variation of the explicit power of Λ, and the second represents the inherently quantum effect of integrating out high-energy modes. This term generically depends on all of the couplings; for example a ϕ^6 vertex can renormalize a ϕ^4 vertex by contracting two of the legs. It also depends on the field renormalization.

• We define the anomalous dimension γ_ϕ of the field ϕ by

$$\gamma_\phi = -\frac{1}{2}\frac{\partial \log Z_\Lambda}{\partial \log \Lambda}$$

This is essentially what the beta function for the kinetic term would be, if we weren't fixing its normalization, and it appears in the RG flow of correlation functions. More generally, for multiple fields we would have a matrix of wavefunction renormalization factors, since the fields may mix as modes are integrated out. In practice, this matrix can be computed perturbatively, as we saw in the notes on Statistical Field Theory.

We now use our setup to compute correlation functions.

• We define the n-point correlator

$$\langle \phi(x_1)...\phi(x_n)\rangle = \frac{1}{\mathcal{Z}} \int\limits_{C^\infty(M)_{\leq\Lambda}} \mathcal{D}\phi e^{-S_\Lambda^{\text{eff}}} \phi(x_1)...\phi_n(x_n)$$

In terms of the canonically normalized field, this is

$$\langle \phi(x_1)...\phi(x_n)\rangle = Z_\Lambda^{-n/2}\langle \varphi(x_1)...\varphi(x_n)\rangle \equiv Z_\Lambda^{-n/2}\Gamma_\Lambda^{(n)}\left(x_1,...,x_n; g_i(\Lambda)\right)$$

where all other factors cancel out due to division by $\mathcal{Z}$.

• As long as the ϕ insertions involve modes with energy less than Λ, we can compute the ϕ correlator in either the original theory or the effective theory. Therefore we have

$$Z_{s\Lambda}^{-n/2}\Gamma_{s\Lambda}^{(n)}\left(x_1,...,x_n; g_i(s\Lambda)\right) = Z_\Lambda^{-n/2}\Gamma_\Lambda^{(n)}\left(x_1,...,x_n; g_i(\Lambda)\right)$$

for $s < 1$. Taking the differential gives the Callan-Symanzik equation

$$\frac{d}{d\log\Lambda}\Gamma_\Lambda^{(n)}\left(x_1,...,x_n; g_i(\Lambda)\right) = \left(\frac{\partial}{\partial\log\Lambda} + \beta_i\frac{\partial}{\partial g_i} + n\gamma_\phi\right)\Gamma_\Lambda^{(n)}\left(x_1,...,x_n; g_i(\Lambda)\right) = 0$$

• It is also useful to consider an "autonomous" RG transformation, where the cutoff Λ remains the same. To do this, we perform the change of variables

$$x' = sx, \quad \phi'\left(x'\right) = s^{(2-d)/2}\phi(x)$$

where the new field ϕ is chosen so that the kinetic term is invariant,

$$\int d^dx(\partial\phi(x))^2 = \int d^dx\left(\partial\phi'(x)\right)^2 = \int d^dx'\left(\partial'\phi'\left(x'\right)\right)^2$$

If $\phi(x)$ has UV cutoff $k = s\Lambda$, then $\phi'(x')$ has UV cutoff $k' = \Lambda$ where k' is formally the Fourier conjugate to x'. Note that all of the dimensionless couplings are left invariant by this transformation. (Equivalently, everything dimensionful transforms with its engineering dimension.)

• At this point, the physical UV cutoff scale has not actually been changed; we have merely performed a change of coordinates. We next perform an active rescaling of the metric so that physical distances shrink by a factor of s. Then the new primed metric coordinates match the old unprimed metric coordinates.

• The composition of these steps (integrating out, changing to primed coordinates, actively rescaling the metric, renaming primed coordinates to unprimed) leaves Λ and the metric invariant, but changes the dimensionless couplings from $g_i(\Lambda)$ to $g_i(s\Lambda)$. It is hence an autonomous RG transformation.

• It can be a bit puzzling to physically interpret the rescaling. In the nonrelativistic context, one might imagine pointing a camera at a physical sample. Then scaling the metric corresponds to "zooming out" the camera. In high energy physics, we would not include the scaling at all: we should not physically rescale the UV cutoff because it represents the scale of new physics. The calculational issues that would arise in the absence of rescaling are moot, because we do not use Wilsonian RG in practice for high energy physics anyway; we instead use "continuum RG". However, for the remainder of this section we will include the rescaling.

• Integrating out by a factor of s and then rescaling by a factor of s gives

$$\begin{aligned}\Gamma_{\Lambda}^{(n)}\left(x_1, ..., x_n; g_i(\Lambda)\right) &= \left(\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2} \Gamma_{s\Lambda}^{(n)}\left(x_1, ..., x_n; g_i(s\Lambda)\right) \\ &= \left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2} \Gamma_{\Lambda}^{(n)}\left(sx_1, ..., sx_n; g_i(s\Lambda)\right)\end{aligned}$$

Finally, replacing x_i with x_i/s gives

$$\Gamma_{\Lambda}^{(n)}\left(x_1/s, ..., x_n/s; g_i(\Lambda)\right) = \left(s^{2-d}\frac{Z_\Lambda}{Z_{s\Lambda}}\right)^{n/2} \Gamma_{\Lambda}^{(n)}\left(x_1, ..., x_n; g_i(s\Lambda)\right)$$

For $s \to 0$, this says we can study the long-distance behavior of our theory by consider correlation functions at fixed separation, but using the couplings in the low-energy effective theory, which is intuitive.

• Taking the differential, every factor of the field scales as if the field had mass dimension

$$\Delta_\phi = (d-2)/2 + \gamma_\phi$$

so the anomalous dimension is the difference between the scaling dimension and the naive classical dimension. Note that we pick up a minus sign here since mass dimension is inverse to length dimension.

RG Flow

• A critical point is a point where all of the beta functions vanish, so the couplings are scale independent. A simple example is the Gaussian critical point, where $g_i^* = 0$. At a nontrivial fixed point, quantum effects exactly cancel the classical scaling; this usually requires strong coupling, since the classical terms are always $O(1)$.

• Now, the anomalous dimension γ_ϕ is only a function of the couplings, so at a critical point it is scale-invariant as well, $\gamma_\phi\left(g_i^*\right) = \gamma_\phi^*$. Then we have

$$\frac{\partial \Gamma_\Lambda^{(2)}(x,y)}{\partial \log \Lambda} = -2\gamma_\phi^* \Gamma_\Lambda^{(2)}(x,y)$$

so that $\Gamma^{(2)} \propto \Lambda^{-2\gamma_\phi^*}$. By Lorentz invariance, correlation functions only depend on $|x-y|$. By classical dimensional analysis,

$$\langle \phi(x)\phi(y)\rangle = \Lambda^{d-2} G\left(\Lambda|x-y|, g_i^*\right)$$

Combining these results together, we have

$$\Gamma_\Lambda^{(2)}\left(x,y;g_i^*\right) = \frac{\Lambda^{d-2}}{\Lambda^{2\Delta_\phi}} \frac{c\left(g_i^*\right)}{|x-y|^{2\Delta_\phi}}$$

This power-law behavior of correlation functions is typical at critical points; this is just a special case of our more general result above.

• Performing an RG step at a critical point leaves the partition function invariant, since it leaves the couplings invariant. On the other hand, an RG step is composed of (1) integrating out degrees of freedom, (2) changing to primed coordinates, and (3) rescaling the metric. The first two never change the partition function by definition. Hence at a critical point, the partition function is invariant under rescaling the metric alone, $g^{\mu\nu} \to e^{2\Omega} g^{\mu\nu}$. Now we have

$$0 = \delta g^{\mu\nu}(x) \frac{\delta \log \mathcal{Z}}{\delta g^{\mu\nu}(x)} = -\delta g^{\mu\nu}(x) \left\langle \frac{\delta S}{\delta g^{\mu\nu}(x)} \right\rangle \propto \delta g^{\mu\nu}(x) \langle T_{\mu\nu}(x)\rangle$$

so at an RG critical point, the spacetime integral of $\langle T^\mu{}_\mu \rangle$ is zero.

• All known examples of Lorentz invariant, unitary QFTs that are scale invariant are actually invariant under conformal transformations $g^{\mu\nu} \to e^{2\Omega(x)} g^{\mu\nu}$. It is believed this is true in general; it has been proven in $d = 2$ and is an open question in high dimensions. Assuming this holds, at RG critical points $\left\langle T_\mu^\mu(x)\right\rangle = 0$.

We now linearize about a critical point.

• Near a critical point, at $g_i^* + \delta g_i$, the beta functions are

$$\beta_i = B_{ij}\delta g_j + O\left(\delta g^2\right)$$

Let σ_i be an eigenvector of B_{ij} and let its eigenvalue be $\Delta_i - d$, so the coupling scales as

$$\sigma_i(\Lambda) = \left(\frac{\Lambda}{\Lambda_0}\right)^{\Delta_i - d} \sigma_i\left(\Lambda_0\right)$$

The definition of Δ_i is slightly different from that of Δ_ϕ because we are taking the couplings to be dimensionless but the field to be dimensionful.

• Classically, we would have $\Delta_i = d_i$, but more generally σ_i has an anomalous dimension

$$\gamma_i = \Delta_i - d_i$$

which is defined analogously to the anomalous dimension for ϕ. For the Gaussian fixed point, all anomalous dimensions are zero, as we'll show below.

• It is a bit of an approximation to assign σ_i a classical dimension, because in general the σ_i will be complicated linear combinations of all the operators. However, when the corrections are 'weak' the off-diagonal elements of B_{ij} are small compared to the classical diagonal terms, and the eigenvectors are close to the classical ones.

• Now imagine starting near a critical point and turning on the coupling to any operator with $\Delta_i > d$. Then the coupling becomes smaller as the scale Λ is lowered, so we say the corresponding operator is irrelevant, as it does not affect long-distance physics; we just flow back to the critical point.

• More generally, the critical surface $\mathcal{C}$ is the set of points that flow back to the critical point under RG flow; then the irrelevant operators provide coordinates for $\mathcal{C}$ in the neighborhood of the critical point.

• Couplings with $\Delta_i < d$ instead grow as the scale is lowered, and are called relevant; RG flow instead drives us away from the critical surface. This flow may eventually terminate at a different critical point, or in exotic cases perform a limit cycle. Since each new field or derivative increases Δ_i classically, there are finitely many relevant operators for fixed d.

• In the classical approximation, and the convention where couplings have dimensions, irrelevant operators have couplings with negative mass dimension, and relevant operators have couplings with positive mass dimension. Then it is clear that a mass term is relevant.

• A marginal coupling is an RG eigenvector with $\Delta_i = d$, or equivalently about the Gaussian fixed point, $d_i = d$. In this case, we need to expand to second order about the fixed point, yielding a weak, logarithmic dependence on Λ. Depending on the sign of the next term, the coupling is marginally irrelevant or marginally relevant. Note that the kinetic term is exactly marginal by definition.

Note. Consider calculating the beta function to first order about the Gaussian fixed point. This is equivalent to considering diagrams with only one vertex, so renormalization only goes one way. For example, ϕ^6 can renormalize ϕ^4 but not vice versa. The matrix B_{ij} is thus upper-triangular with the classical values on the diagonal, so all anomalous dimensions are zero. The only effect is that the eigenvectors are slightly tilted.

We must go to second order to figure out what classically marginal operators do. Note that in most practical computations in particle physics, we are implicitly expanding about the Gaussian fixed point. Also note that while the dimensionless couplings change, the dimensionful couplings are approximately constant.

Note. Given a term $\lambda_i \mathcal{O}_i$ in a weakly coupled field theory, its contribution to the action for a field configuration of energy $E \ll \Lambda$ is $\lambda_i E^{\Delta_i - d}$ by dimensional analysis. We don't use Λ here because the field configuration can't know what Λ is. To show this more quantitatively, consider a field configuration with lengthscale $L \sim 1/k$ and dimensionless amplitude $\hat{\phi} = \phi/k$. Then

$$\int dx(\partial\phi)^2 \sim \hat{\phi}^2, \quad \int dx m^2 \phi^2 \sim \frac{m^2}{k^2}\hat{\phi}^2, \quad \int dx(\partial\phi)^p \phi^q \sim k^{2p+q-4}\hat{\phi}^{p+q}$$

Assuming the kinetic term dominates, when we perform the path integral, configurations with $\hat{\phi} \sim 1$ dominate. Then the contribution of a term depends on k just as our dimensional analysis suggests.

Then a dimensionless coupling g_i has effect

$$\Delta S_i \sim g_i \left(\frac{E}{\Lambda}\right)^{\Delta_i - d}$$

where the kinetic term has $\Delta S \sim 1$. This measures the effect of the term on observable quantities derived from the action, such as cross sections. The tree-level RG evolution of g_i simply states that ΔS_i is independent of Λ. Because loops are sensitive to Λ, anomalous dimensions appear at loop-level.

Relevant terms such as masses are more important at $E \ll \Lambda$, while all irrelevant terms become important as E approaches Λ. For marginal terms, which are the majority of couplings we consider, the value of g_i itself is a good estimate of its effect, so perturbation theory really does break down when g_i is $O(1)$.

Example. For a scalar field $d = 4$ with $\mathbb{Z}_2$ symmetry near the gaussian fixed point, we have

$$\text{relevant: } 1, \phi^2, \quad \text{marginal: } \phi^4, (\partial\phi)^2$$

and everything else is irrelevant. The physical mass is independent of the cutoff, as one would expect. In $d = 3$, ϕ^4 becomes relevant while ϕ^6 becomes marginal. In $d = 2$, the field is dimensionless, so

$$\text{relevant: } 1, \phi^2, \phi^4, ..., \quad \text{marginal: } (\partial\phi)^2, \phi^2(\partial\phi)^2, ...$$

That is, we get an infinite series of relevant terms, which is quite rare.

Note. The origin of universality. The theory space is generically infinite-dimensional. Now consider the finite-dimensional set of theories obtained by starting from an RG fixed point, turning on a relevant operator, and performing RG flow, creating a 'renormalized trajectory'. By the discussion above, every initial condition will approach a renormalized trajectory, so theories in the IR are described by a finite number of parameters.

If we continue RG flowing into the deep IR, we often end up at a trivial fixed point, like the "infinite temperature" fixed point in statistical field theory, since we integrate out all massive particles. Alternatively, we may end up at a nontrivial fixed point/CFT, such as the Wilson-Fisher fixed point, where the particles are massless; renormalization exactly cancels the tree-level mass.

We now consider a converse question, more relevant to particle physics: is it possible to fix a low-energy theory at scale Λ while sending the cutoff Λ_0 to infinity?

• The simplest case is when S_{Λ_0} is on the critical surface $\mathcal{C}$. Then taking $\Lambda_0 \to \infty$ gives S_Λ^{eff} exactly at the critical point. Since $\mathcal{C}$ has finite codimension, we have to tune a finite number of parameters to do this, e.g. the temperature in a statistical field theory.

• Theories such as QCD and Yang-Mills are not CFTs, but instead have relevant and marginally relevant terms in their actions. In this case, to fix the low-energy theory we must modify the high-energy theory as Λ_0 varies, which is done perturbatively by adding counterterms. For example, we could parametrize the high-energy theory as

$$S_{\Lambda_0}[\varphi] = S_\Lambda^{\text{eff}}[\varphi] + \hbar S_{\text{CT}}[\varphi, \Lambda_0]$$

so that the counterterms vanish when $\Lambda = \Lambda_0$. The counterterms are tuned so that the limit

$$e^{-S_\Lambda^{\text{eff}}[\phi]/\hbar} = \lim_{\Lambda_0 \to \infty} \int_{C^\infty(M)_{(\Lambda,\Lambda_0]}} \mathcal{D}\chi \exp\left(-\frac{S_\Lambda^{\text{eff}}[\phi+\chi]}{\hbar} - S_{\text{CT}}[\phi+\chi, \Lambda_0]\right)$$

exists. We separate out the counterterms explicitly to organize perturbation theory. In perturbation theory, we work order by order in $\hbar$, canceling the (regularized) divergences produced by the bare action S_Λ^{eff} by counterterms. The extra factor of $\hbar$ is because the one-loop divergences are canceled by tree-level counterterm diagrams, and so on.

• We need our low-energy theory to lie on a renormalized trajectory. Then the counterterms are chosen so that, as Λ_0 is increased, S_{Λ_0} gets closer and closer to the critical point along the renormalized trajectory. Then there's one counterterm for each relevant operator, as we've seen perturbatively, and the continuum limit is the critical point.

• The existence of relevant operators leads to a fine-tuning problem. Suppose we have a very high, fixed cutoff $\Lambda_0 = \Lambda_{\text{Planck}}$. Then if the dimensionless coupling at scale Λ is not extremely large, the dimensionless coupling at scale Λ_0 must be extremely small. This is 'unnatural', because we expect a fundamental theory, where the couplings can be calculated, to have $O(1)$ dimensionless couplings. Note there's no problem with having very large or very small dimensionless couplings in an effective theory.

• Marginally relevant operators aren't nearly as problematic as $\log(\Lambda_{\text{Planck}}/1\text{TeV}) \approx 40$ isn't huge, where 1TeV is about the minimum possible cutoff Λ for the Standard Model.

• In the case of a mass term, the only way to avoid unnaturalness would be to have $m \sim \Lambda_0$, i.e. we need particles so heavy we can't even produce them at Λ in the first place.

• More generally, we could include finite coefficients for irrelevant operators in S_{Λ_0}. This affects the required tuning for the relevant operators, since the two will renormalize each other. However, the effects of the irrelevant operators will disappear exactly in the limit $\Lambda_0 \to \infty$. The point is that we only need one fine tuning for each relevant parameter.

• Note that generically, we should always think of every operator as being present. Even the points on a renormalized trajectory don't only have the relevant couplings turned on; instead these relevant couplings immediately turn on all others, so in general every coupling is nonzero, though the irrelevant operators remain small because they decay away as they're produced.

• As a result, the true coefficient of an irrelevant operator in the IR isn't nearly as small as naive scaling suggests. The real meaning of "irrelevant" is that the value of the coupling in the UV is irrelevant to the value in the IR, since the quick initial decay washes it out; irrelevant operators are still present in S_Λ^{eff} even when $\Lambda_0 \to \infty$, but with small coefficients.

Now suppose that our low-energy effective theory contains an irrelevant operator with a large coefficient, which is often required to match experimental results.

• In this case, there is no hope of our theory lying on a renormalized trajectory. Since there are infinitely many irrelevant operators, which generically all renormalize each other, we now need an infinite number of counterterms and an infinite number of renormalization conditions.

• Moreover, the counterterm coefficients apparently diverge in the limit $\Lambda_0 \to \infty$, so we cannot take the continuum limit. Instead, we say the theory comes with a cutoff $\Lambda_{\text{cutoff.}}$.

• Specifically, Λ_{cutoff} is the scale where the dimensionless irrelevant couplings are $O(1)$. Now, the theory is perfectly predictive at energies $E \ll \Lambda_{\text{cutoff}}$, because the effects of more irrelevant terms are suppressed by more powers of $E/\Lambda_{\text{cutoff}}$, so we can truncate the Lagrangian after a few terms. On the other hand, perturbation theory breaks down as E approaches $\Lambda_{\text{cutoff.}}$. All of the terms become important at once, and we cannot determine all of the couplings with finite data, so the theory is not predictive.

• Generally, what must happen is that new physics takes over. This means that we switch to a new set of degrees of freedom (i.e. new fields, or even non-fields, such as in the Ising model) where the theory is either renormalizable or effective with a higher cutoff, at which point we can truncate the Lagrangian and do calculations again.

• Another possibility is a Landau pole, where a coupling diverges at finite energy, such as the marginally irrelevant coupling in QED. In practice this just means that new physics must take over before then. If it doesn't, then the theory is 'quantum trivial', meaning that the only way for it to have a continuum limit is for the coupling to be exactly zero.

• The usual exposition of the Landau pole uses a perturbatively computed beta function. This isn't valid since perturbation theory breaks down; instead we must establish a Landau pole nonperturbatively, through lattice simulations. Simulations of ϕ^4 theory indicate a Landau pole indeed exists, so the theory is quantum trivial. On the other hand, simulations of QED indicate

that we don't hit the Landau pole; instead chiral symmetry breaking occurs.

• The Standard Model is also suspected to be quantum trivial, through the marginally irrelevant quartic Higgs interaction. Of course, this isn't the main reason we suspect the Standard Model to be incomplete, or even a minor reason. A more compelling reason is the hierarchy problem: the Higgs mass is relevant, so as the usual logic goes, the Standard Model must be extended at the TeV scale to preserve naturalness. A critical evaluation of this argument is given here.

• A final possibility is that we simply flow to some unknown UV fixed point. This is the hypothesis of the asymptotically safe approach to quantum gravity. In this case, new physics need not take over, and our description can be valid up to arbitrarily high energies.

• All of these issues also occur in reverse for relevant couplings, such as in QCD, where the coupling would naively hit a Landau pole at $\Lambda_{\text{QCD}} \sim 1\text{GeV}$. In this case, confinement and chiral symmetry breaking occur; we must describe the dynamics in terms of hadrons rather than quarks and gluons.

Note. Relating our results to the old picture of perturbative renormalization.

• In the old picture, the counterterms are divergent order by order in $\hbar$. Here, they are perfectly finite. This is simply because limits don't commute with sums; for instance,

$$\lim_{x\to\infty} e^{-x} = 0$$

but all the individual terms in the Taylor series for e^{-x} diverge as $x \to \infty$. If we always maintain a finite cutoff and do the path integral exactly, there are never any divergences.

• Before, we thought of the regularization scale Λ_0 as an unphysical scale that had to be sent to infinity at the end of the calculation. Now, we think of it as a physical cutoff, beyond which might lie new physics. Taking Λ_0 to infinity is not essential, and not physical.

• Before, in renormalized perturbation theory, we thought of a bare Lagrangian as composed of a renormalized Lagrangian plus counterterms. Now, we identify the bare Lagrangian with the fundamental theory, at scale Λ_0, and the renormalized Lagrangian with the effective theory, at scale Λ.

• Before, we thought of irrelevant (nonrenormalizable) operators as dangerous and relevant (superrenormalizable) operators as benign. Now, we see that turning on an irrelevant operator changes nothing in the IR, while relevant operators present fine-tuning problems.

• Before, we identified a renormalizable theory as one that required only a finite number of counterterms to absorb the infinities. Now, we think of a renormalizable theory as one that admits a sensible continuum limit.

Note. Conventions differ between SFT and high energy QFT, for physical reasons. In SFT, we typically know the UV theory and want to find the IR behavior, so we keep Λ the same by rescaling; the only thing that matters about the atomic cutoff is that it's very far away. In high energy physics, we know the IR behavior and are trying to find the UV behavior, so we keep the variation of Λ explicit; this is what we've done above.

There is also another convention, which is whether to work with dimensionless couplings by pulling out factors of Λ, or to work with dimensionful couplings. For concreteness, consider $\mathcal{L} \supset m^2\phi^2$. In a free theory, integrating out UV degrees of freedom keeps the dimensionful coupling m^2 the same. However, one could also consider the dimensionless coupling $g = m^2/\Lambda^2$ (common in QFT), or stick with dimensionful couplings but keep Λ the same by rescaling (common in SFT). These are essentially the same idea phrased in different ways. In both cases, the coupling will grow.

Philosophically, in SFT, the slogan is to "find something relevant". This is because we're interested in macroscopic behavior, and the atomic scale is so small that all effects of irrelevant operators are completely undetectable. The fact that relevant operators require fine tuning is acceptable because there is fine tuning in the lab, e.g. by tuning the temperature to a phase transition. In QFT, we already know the low-energy physics, and irrelevant operators are useful as a window into higher-energy physics. Relevant operators are a larger problem because there is nothing outside the universe that does the tuning.

Note. Before, we've argued that contributions to the action from various operators go as $g_i(E/\Lambda)^{\Delta_i - d}$. However, naive power counting can break down because of loops. For instance, in scalar field theory with a hard cutoff, a ϕ^6 loop renormalizes the ϕ^4 coupling at one-loop as

$$\delta g_4 \sim \frac{g_6}{\Lambda^2} \int^{\Lambda} \frac{dk}{k^2 - m^2} \sim g_6$$

which is too large. This occurs because only the external legs are fixed at energy E, while particles in the loop can go up to Λ. But if we regulate with a mass-independent scheme such as DR, we get a result compatible with power counting,

$$\delta g_4 \sim \frac{g_6 \mu^{2\epsilon}}{\Lambda^2} \int \frac{d^{4-\epsilon}k}{k^2 - m^2} \sim \frac{g_6}{\epsilon}\frac{m^2}{\Lambda^2} + g_6 \frac{m^2}{\Lambda^2} \log \frac{m^2}{\mu^2} = O\left(m^2/\Lambda^2\right)$$

That is, DR removes the Λ^2 power divergence entirely; intuitively DR "only sees logarithmic divergences", which show up as $1/\epsilon$ terms that are subtracted out. Thus, if we don't use something like DR, it is difficult to self-consistently truncate the Lagrangian in an EFT. It is still possible, but one has to carefully keep track of counterterms, which can remove the contributions that violate naive power counting. (In this sense, it's the same as using a regulator that violates a symmetry: it's perfectly legal, but requires counterterms to be added with care.) For this reason, $\overline{\text{MS}}$ is used almost universally in practical EFT calculations.

Note. It is tempting to say that if a dimension- 6 operator is measured with coefficient $1/\Lambda^2$, then new physics appears at scale Λ. This is deceptive, because operators come with UV couplings. What we really can conclude is that if nothing UV completes the theory, then the theory becomes strongly coupled at scale Λ. It is perfectly possible for new particles to appear at a lower scale. Indeed, there are many very small couplings in the SM, such as the electron Yukawa coupling, to which this reasoning applies.

Calculating RG Evolution

Next, we present an explicit, though somewhat impractical method for computing RG evolution.

• Concretely, we would like to evaluate

$$S^{\text{int}}_{\Lambda}[\phi] = -\log \int_{C^\infty(M)_{(\Lambda,\Lambda_0]}} \mathcal{D}\chi \exp\left(-S^0[\chi] - S^{\text{int}}_{\Lambda_0}[\phi + \chi]\right)$$

but this is hard in general, as the right-hand side is a general interacting path integral. The general approach is to expand $e^{-S^{\text{int}}_{\Lambda_0}[\phi+\chi]}$ as a series in χ,

$e^{-S^{\text{int}}_{\Lambda_0}[\phi+\chi]} = e^{-S^{\text{int}}_{\Lambda_0}[\phi]} + \int dx \chi(x) \frac{\delta}{\delta\phi(x)} e^{-S^{\text{int}}_{\Lambda_0}[\phi]} + \frac{1}{2}\int dx dy \chi(x)\chi(y) \frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)} e^{-S^{\text{int}}_{\Lambda_0}[\phi]} + \ldots$

then perform the χ path integral. Note that we're Taylor expanding in position space, since eventually we want an action in position space.

• When we perform the χ path integral, we get factors of the position space propagator,

$$D_\Lambda(x,y) = \int\limits_{\Lambda<|p|\leq\Lambda_0} dp \frac{e^{ip(x-y)}}{p^2+m^2}$$

Now the trick is to consider an infinitesimal RG step, $\Lambda = \Lambda_0 - \delta\Lambda$, so that

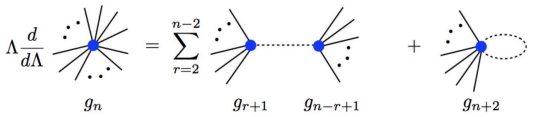
$$D_\Lambda(x,y) = \frac{1}{(2\pi)^d}\frac{\Lambda^{d-1}\delta\Lambda}{\Lambda^2+m^2}\int\limits_{S^{d-1}} d\Omega e^{i\Lambda\hat{p}(x-y)}$$

This is an enormous improvement, because we only need keep track of terms linear in $\delta\Lambda$, which means we only want diagrams with a single χ propagator.

• Therefore, only the zeroth and second order terms in the Taylor expansion matter, and an explicit calculation yields Polchinski's equation,

$$-\frac{\partial S_\Lambda^{\text{int}}[\phi]}{\partial \log\Lambda} = \int dxdy \frac{\delta S_\Lambda^{\text{int}}}{\delta\phi(x)} D_\Lambda(x,y) \frac{\delta S_\Lambda^{\text{int}}}{\delta\phi(y)} - D_\Lambda(x,y)\frac{\delta^2 S_\Lambda^{\text{int}}}{\delta\phi(x)\delta\phi(y)}$$

More concretely, let g_n be the coefficient of ϕ^n. Then schematically we have



which corresponds with the intuition of zooming out, shrinking χ propagators to points.

• Note that since the action is local, we must have $x = y$ in the second term to get a nonzero result. On the other hand, we can have $x \neq y$ in the first term, leading to non-local contributions, as the χ propagator falls off as $e^{-\sqrt{\Lambda^2+m^2}r}/r^{d-3}$. As usual, we simply Taylor expand to get a series of local terms with derivatives.

• Note that the variation of the action is

$$\frac{\delta S}{\delta\phi(x)} \equiv \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)} + ...$$

so Polchinski's equation indeed accounts for the derivative terms.

• It's convenient to rewrite Polchinski's equation as

$$\frac{\partial}{\partial t}e^{-S^{\text{int}}[\phi]} = -\int dxdy D_\Lambda(x,y)\frac{\delta^2}{\delta\phi(x)\delta\phi(y)}e^{-S^{\text{int}}[\phi]}, \quad t = \log\Lambda$$

which has the general form of a heat equation, with the 'Laplacian'

$$\Delta = \int dxdy D_\Lambda(x,y)\frac{\delta^2}{\delta\phi(x)\delta\phi(y)}$$

The eigenfunctions of the Laplacian grow or decay exponentially under RG flow; they are simply the *RG* eigenvectors we've seen earlier.

Next, we use the local potential approximation to explicitly compute RG evolution.

• We work in $d > 2$ near the Gaussian fixed point. Then besides the kinetic term, all operators involving derivatives are irrelevant, so we restrict attention to actions of the form

$$S_\Lambda^{\text{eff}}[\phi] = \int d^dx \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + V(\phi), \quad V(\phi) = \sum_k \Lambda^{d-k(d-2)}\frac{g_{2k}}{(2k)!}\phi^{2k}$$

where for simplicity we've imposed $\mathbb{Z}_2$ symmetry. Neglecting the derivative interactions is called the local potential approximation, and works for slowly varying fields.

• Next, we set up our approximation somewhat differently. Before, we expanded $e^{-S_\Lambda^{\text{int}}[\phi+\chi]}$, yielding a Gaussian path integral for χ. Now we instead write

$$S_\Lambda^{\text{eff}}[\phi+\chi] = S_\Lambda^{\text{eff}} + \int dx \frac{1}{2}(\partial\chi)^2 + \frac{1}{2}\chi^2 V''(\phi) + ...$$

so that we maintain a nontrivial action for χ.

• We now consider an infinitesimal RG step $\delta\Lambda$ and work in momentum space. Then the momentum-space propagators are not small, but every loop picks up a factor of $\delta\Lambda$, so we only consider one-loop graphs. Note that there are no tree level graphs at all since the action is even in χ, though we could also relax this assumption.

• We claim that one-loop graphs can only contain the χ^2 vertex. To prove this rigorously, let there be v_i copies of the χ^i vertex. Then Euler's identity gives

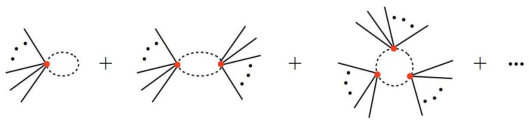
$$e - \sum_i v_i = \ell - 1$$

where ℓ is the number of loops. Since there are no external lines,

$$2e = \sum_i iv_i, \quad \ell = 1 + \sum_i \frac{i-2}{2}v_i$$

Therefore only v_2 can be nonzero, and we recover a quadratic action for χ, which is tractable.

• The types of diagrams that contribute are shown below.



Now, suppressing the position-dependence of ϕ, the action for χ becomes

$$\begin{aligned} S^{(2)}[\chi,\phi] &= \int\limits_{\Lambda-\delta\Lambda<|p|\leq\Lambda} dp\tilde{\chi}(p)\left(\frac{1}{2}p^2 + \frac{1}{2}V''(\phi)\right)\tilde{\chi}(-p) \\ &= \frac{1}{2}\frac{\Lambda^{d-1}\delta\Lambda}{(2\pi)^d}\left(\Lambda^2 + V''(\phi)\right)\int\limits_{S^{d-1}} d\Omega\tilde{\chi}(\Lambda\hat{p})\tilde{\chi}(-\Lambda\hat{p}) \end{aligned}$$

• Performing the Gaussian integration, we find

$$e^{-\delta\Lambda S^{\text{eff}}[\phi]} = \int \mathcal{D}\chi e^{-S^{(2)}[\chi,\phi]} = C\left(\frac{\pi}{\Lambda^2 + V''(\phi)}\right)^{N/2}$$

where N is the number of momentum modes. To regularize this, we put the theory in a box of side length L and use periodic boundary conditions, giving

$$N = \frac{\text{vol}\left(S^{d-1}\right)}{(2\pi)^d}\Lambda^{d-1}\delta\Lambda L^d$$

We can thus rewrite our result as

$$\delta_\Lambda S^{\text{eff}}[\phi] = a\Lambda^{d-1}\delta\Lambda \int dx \log\left(\Lambda^2 + V''(\phi(x))\right)$$

where we restored the position-dependence of ϕ, turning a factor of L^d into an integral over x.

• Finally, expanding the logarithm gives an infinite series of corrections,

$$\frac{dg_{2k}}{d\log\Lambda} = (k(d-2)-2)g_{2k} - a\Lambda^{k(d-2)}\frac{\partial^{2k}}{\partial\phi^{2k}}\log\left(\Lambda^2 + V''(\phi)\right)\Big|_{\phi=0}$$

where the first term is from the classical scaling. For example, the first two terms are

$$\frac{dg_2}{d\log\Lambda} = -2g_2 - \frac{ag_4}{1+g_2}, \quad \frac{dg_4}{d\log\Lambda} = (d-4)g_4 - \frac{ag_6}{a+g_2} + \frac{3ag_4^2}{(1+g_2)^2}$$

For example, we see that in $d = 4$, g_4 is marginally irrelevant.

• Looking back, we've traded one kind of complexity for another. In the local potential approximation, the beta functions are exact at one loop, but can contain arbitrarily high powers of the couplings. In Polchinski's equation, the beta functions are also exact at one loop, and are linear in the couplings, but we need to account for all derivative terms. One can think of the local potential approximation as solving for the derivative term couplings in terms of the non-derivative ones, and plugging them back into the beta functions to eliminate them.

Using our result, we can explicitly look at some simple RG flows.

• First, we consider the Gaussian fixed point. Linearizing the beta function about it gives

$$\beta_{2k} = (k(d-2)-d)g_{2k} - ag_{2k+2}$$

Thus as anticipated, there are no anomalous dimensions since the matrix B in $\beta_i = B_{ij}g_j$ is upper-triangular, though the eigenvectors are mixed a bit.

• In $d = 4$, we see that the mass term is relevant, while ϕ^6 and higher couplings are irrelevant, as expected. To understand g_4, we expand to quadratic order for

$$\beta_4 = \frac{3}{16\pi^2}g_4^2$$

so that the quartic interaction is marginally irrelevant; explicitly,

$$g_4(\Lambda) = \frac{16\pi^2}{3\log\left(\Lambda_0/\Lambda\right)}$$

where Λ_0 is an integration constant, we need $\Lambda_0 > \Lambda$ for $g_4 > 0$, ensuring stability, and the theory has a Landau pole at $\Lambda_0 = \Lambda$.

• The fact that a dimensionless coupling implies an energy scale is known as dimensional transmutation. We've already seen this for irrelevant couplings, which give a natural cutoff for the effective field theory; marginal couplings have the scale appear in a logarithm. There's nothing puzzling about this, since all theories come with a natural energy scale, the cutoff Λ.

• We can find another critical point in $d = 4 - \epsilon$, the Wilson-Fisher fixed point, where

$$g_2 = -\frac{1}{6}\epsilon + O\left(\epsilon^2\right), \quad g_4 = \frac{1}{3a}\epsilon + O\left(\epsilon^2\right), \quad g_{2k} = O\left(\epsilon^2\right) \text{ for } k > 2$$

Staying near $d = 4$, where g_4 is marginal, allows us to find a nontrivial fixed point without requiring strong coupling.

• Linearizing about the fixed point gives

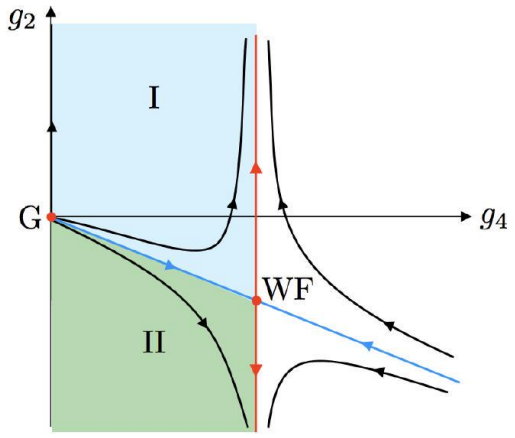
$$\frac{d}{d\log\Lambda}\begin{pmatrix}\delta g_2\\ \delta g_4\end{pmatrix} = \begin{pmatrix}\epsilon/3-2 & -a(1+\epsilon/6)\\ 0 & \epsilon\end{pmatrix}\begin{pmatrix}\delta g_2\\ \delta g_4\end{pmatrix}$$

which has eigenvalues $\epsilon/3 - 2$ and ϵ, so only one direction is relevant. Explicitly, we have

$$a = \frac{1}{(4\pi)^{d/2}}\frac{1}{\Gamma(d/2)}\bigg|_{d=4-\epsilon} = \frac{1}{16\pi^2} + \frac{\epsilon}{32\pi^2}(1-\gamma+\log 4\pi) + O\left(\epsilon^2\right)$$

More sophisticated techniques can be used to show that the Wilson-Fisher fixed point indeed exists in $d = 3$ and $d = 2$.

• The RG flows in three dimensions are shown below.



Theories in region I flow to an interacting massive theory in the IR, and have the Gaussian theory as the continuum limit. Theories in region II are similar, but exhibit spontaneous symmetry breaking since $g_2 < 0$. Theories on the red line have the Wilson-Fisher fixed point as their continuum limit, while all others have no continuum limit at all.

• The situation in two dimensions in more complicated, as all of the operators ϕ^{2n} are marginal. It turns out there are infinitely many fixed points, where the n^{th} fixed point can be reached from the Gaussian fixed point by turning on $\phi^{2(n+1)}$. The n^{th} fixed point has n relevant operators, which are essentially $\phi^2, \phi^4, ..., \phi^{2n}$. This result can be derived by conformal field theory.

Effective Field Theories

In this section, we give an array of examples of effective field theories, illustrating the general way of thinking about theories in the Wilsonian picture.

Example. In some cases, we can pick up extra symmetries just from the requirement of renormalizability; such symmetries are called "accidental" or "emergent". A familiar example is the emergence of rotational symmetry for a lattice spin system. As another example, consider electromagnetism with several fermions, where the most general renormalizable Lagrangian is

$$\mathcal{L} = \frac{1}{4e^2} Z_3 F^{\mu\nu} F_{\mu\nu} + (Z_L)_{ij}\, \bar{\psi}_{L_j} \not{D} \psi_{L_i} + (Z_R)_{ij}\, \bar{\psi}_{R_j} \not{D} \psi_{R_i} + M_{ij} \bar{\psi}_{L_i} \psi_{R_j} + \bar{M}_{ij} \bar{\psi}_{R_i} \psi_{L_j}.$$

Here, $Z_{L/R/3}$ are general wavefunction renormalizations. To have a real Lagrangian, we require $Z_{L/R}$ to be Hermitian. Note that we have no Majorana masses since the fields are charged.

We can dramatically simplify the Lagrangian by defining

$$\psi_L = S_L \psi'_L, \quad \psi_R = S_R \psi'_R$$

so the Lagrangian for the primed fields has

$$Z'_L = S_L^\dagger Z_L S_L, \quad Z'_R = S_R^\dagger Z_R S_R, \quad M' = S_L^\dagger M S_R$$

Then it is possible to choose S_L so that Z'_L is the identity, and similarly choose S_R so that Z'_R is the identity, giving the usual kinetic terms. We can still redefine $S_{L/R}$ by unitaries, which we can use to make M diagonal. Dropping the primes,

$$\mathcal{L} = \frac{1}{4e^2} Z_3 F^{\mu\nu} F_{\mu\nu} + \sum_i \bar{\psi}_i \left(\not{D} + m_i \right) \psi_i$$

Then the number of ψ_i particles is conserved, but this is an accidental symmetry; it could be broken by a nonrenormalizable term such as $Y_{ijkl} \bar{\psi}_i \gamma^\mu \psi_j \bar{\psi}_k \gamma_\mu \psi_\ell$. In the Standard Model, this is the origin of baryon and lepton number symmetry.

Note. More generally, it is believed that there are no continuous global symmetries in a quantum theory of gravity. Then from this perspective, all continuous symmetries we've found in particle physics are accidental! One piece of evidence in favor of this picture is Hawking radiation. The formation and evaporation of a black hole does not conserve baryon or lepton number, but it does conserve gauge quantum numbers, such as electric charge, since they may be measured by flux integrals at infinity. In string theory, it is a folk theorem that all exact symmetries, even discrete symmetries, are gauge symmetries.

Example. The Schrodinger field has kinetic term $\nabla^2 \psi / 2m$. Since the coefficient $1/2m$ has negative mass dimension, the operator is irrelevant. Hence upon renormalization, the Hamiltonian should have all terms allowed by symmetry, giving

$$H \sim \frac{p^2}{2m} \left(1 + a_1 \frac{p^2}{m^2} + a_2 \frac{p^4}{m^4} + ... \right) + V(r)$$

where the factors of m are put in by dimensional analysis. Note that here we are thinking of the renormalized theory which has no cutoff, $\Lambda \to \infty$, so the only mass scale is m.

This theory is perfectly predictive at energies where $p^2 \ll m^2$, despite the infinitely many unknown parameters a_i. But when $p^2 \sim m^2$, perturbation theory breaks down and all of the a_i are important. The Schrodinger field can be UV-completed to either the Klein-Gordan or Dirac field, both of which are renormalizable.

Example. The classic example of an effective field theory is the 4-Fermi theory for β decay,

$$\mathcal{L} \sim G_F \bar{\psi}_p \psi_n \bar{\psi}_e \psi_\nu$$

where we suppress gamma matrices and spinor indices. Similar operators can be used to describe the decay of the muon; this is used to determine

$$G_F = \left(\frac{1}{292.9\text{GeV}}\right)^2$$

which indicates the energy where the theory breaks down; higher-dimension terms contribute in a series in $p^2 G_F$. The specific values of these terms may be found by UV completing the 4-Fermi theory to the electroweak theory.

Sticking with only the dimension 6 term, the 4-Fermi theory is quite predictive. It can compute relations between decay rates, and angular distributions of decays. One can also include loops to get genuinely quantum predictions, which appear in cross sections as logarithmic dependence on the external momenta.

Example. Light scattering; or, why the sky is blue. We consider Rayleigh scattering of visible light off a nitrogen molecule, where the wavelength of the light is much larger than the size of the atom. We assume the photon energy is too weak to excite the molecule, so we can ignore its internal degrees of freedom and model it as a complex scalar field ϕ. The molecule remains at rest since it is heavy, so we break Lorentz invariance, allowing us to use the four-velocity $u^\mu = (1, 0, 0, 0)$ in our

Lagrangian. Finally, the nitrogen molecule is electrically neutral, so the minimal coupling vanishes, $D_\mu \phi = \partial_\mu \phi$.

Given this setup, the kinetic term is $\pi^\dagger i v^\mu \partial_\mu \phi$, which shows that ϕ has mass dimension $3/2$. Here, we can use relativistic dimensional analysis because the photon is relativistic, and the nitrogen is static. The effective interactions are

$$S^{\text{int}}\left[A, \phi\right] = \int dx \frac{g_1}{8\Lambda^3}\phi^2 F^2 + \frac{g_2}{8\Lambda^3}\phi^2 (u \cdot F)^2 + ...$$

where the g_i are dimensionless and the higher terms are suppressed by more powers of Λ. Now consider the tree-level contribution to $\phi + \gamma \to \phi + \gamma$ scattering. Since the field F contains a derivative, the amplitude is proportional to ω^2 and hence the cross section is proportional to ω^4,

$$\sigma \sim g_2^2 \frac{\omega^4}{\Lambda^6}$$

Thus we get the frequency dependence of Rayleigh scattering almost for free! Note that when the scatterer is comparable in size to the wavelength, our effective field theory breaks down and we enter the regime of Mie scattering. It also breaks down once the light can excite the atom, which creates a Lorentzian scattering peak.

Example. Light propagation in an insulator. In vacuum, the unique gauge, Lorentz, and parity invariant action we can write for the photon is

$$S[A] = \frac{1}{4}\int dx \epsilon_0 E^2 - \frac{1}{\mu_0} B^2$$

Now we suppose the light propagates in a medium, whose degrees of freedom are modeled with additional fields Φ. The Euler-Lagrange equations become

$$\partial^\mu F_{\mu\nu} = \mu_0 J_\nu, \quad J_\mu(x) = \frac{\delta S^{\text{int}}}{\delta A^\nu(x)}$$

For example, in a conductor, we might ignore higher energy bands for the electrons, so Λ would be the band gap. Then in an insulator, Λ has the same interpretation, but since the Fermi surface lies in a band gap there are no relevant degrees of freedom at all! Then it would appear that S^{int}

must contain only irrelevant terms. But the insulator also picks out a preferred frame of reference, so we can write down non-Lorentz invariant terms,

$$S_\Lambda^{\text{int}}\left[A,\Phi\right]=\int dx\frac{1}{2}\left(\chi_e E^2-\chi_m B^2+...\right)$$

where the dimensionless couplings $\chi_e(\Lambda)$ and $\chi_m(\Lambda)$ are the susceptibilities. Then light can travel at a different speed, and this is the leading effect an insulator has. More generally, a crystalline insulator breaks rotational symmetry, so we can have

$$S_\Lambda^{\text{int}}[A,\Phi]=\int dx\frac{1}{2}\left(\left(\chi_e\right)_{ij}E_iE_j-\left(\chi_m\right)_{ij}B_iB_j+...\right)$$

which leads to birefringence. At higher orders, we will find nonlinear terms such as E^4/Λ^2, yielding nonlinear optical effects.

Note. Quantum gravity may also be treated as an effective field theory. The action is

$$S[g]=\int dx\sqrt{-g}\left(\lambda+\frac{R}{16\pi G}\right)$$

where the dimensions are

$$[\lambda]=4,\quad [g_{\mu\nu}]=0,\quad [R]=[\partial\partial g]=2,\quad [G]=-2$$

In particular, one can see the metric is dimensionless because in flat spacetime its elements are simply δ. Perturbing about the flat metric,

$$g=\delta+\sqrt{G}h$$

where h is scaled to have canonically normalized kinetic terms. Higher-order terms in h come with positive powers of G and hence negative powers of M_{pl}, so they are all irrelevant. Their form can be obtained from taking the most general diffeomorphism invariant action,

$$S_\Lambda^{\text{eff}}[g]=\int dx\sqrt{-g}\left(c_0\Lambda^4+c_1\Lambda^2R+c_2R^2+c_3R^{\mu\nu}R_{\mu\nu}+c_4R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}+...\right)$$

where the c_i are dimensionless, and ordinary GR only uses c_0 and c_1. In the UV, either new degrees of freedom appear, such as strings, or the theory flows to a fixed point, a proposal known as asymptotic safety. (Note that "asymptotic freedom" as used above is just the case where this fixed point is the Gaussian fixed point.)

Of course, even though the interaction of gravity with matter is "irrelevant", it is very relevant to everyday life! This is because the gravitational interactions between the particles in macroscopic objects add up coherently, a fact which is not captured in our simple analysis above.

Note. Multiple kinds of renormalizability. So far we've been talking about Wilsonian renormalizability, which, near the Gaussian fixed point, simply corresponds to having no couplings with negative mass dimension. However, historically renormalizability was defined by demanding that all the divergences in a theory could be absorbed by redefinitions of the couplings already in the Lagrangian, which in turn must have a finite number of terms; we will call this counterterm renormalizability. In our picture, this means the Lagrangian must include all renormalizable terms consistent with symmetries; for instance, massless ϕ^4 theory is not permitted. (Thus counterterm renormalizability is obliquely related to naturalness.) The main point is that it directly demands that all predictions up to arbitrary energy can be made given a finite number of parameters.

By contrast, in an effective field theory one requires more and more parameters to get an accurate prediction as one goes to higher energies. In practice, one fixes a desired precision, and then computes up to the order in $1/\Lambda$ required (or more generally, to the required order in the power counting parameter), neglecting all effects associated with higher-order terms. Then it is useful to say an EFT is renormalizable if it is renormalizable in the historic sense, at each order of the power counting expansion. (That is, divergences can appear associated with higher-order terms, but they can be simply neglected.) For example, ϕ^6 theory is renormalizable at this sense, even though it induces a ϕ^8 coupling, because the ϕ^8 coupling is order $1/\Lambda^4$, while the ϕ^6 coupling is only order $1/\Lambda^2$. EFTs are discussed further in the notes on the Standard Model.

Note. Counterterm renormalizability can be much more difficult to show than Wilsonian renormalizability, though the two are related.

• Consider the linear sigma model, a case with spontaneously broken global symmetry. Naively, there are more divergences than there are free parameters, so it looks like the theory is not renormalizable. But the divergences are still constrained by the broken $SO(n)$ symmetry, which is still present but nonlinearly realized, so "secret symmetry yields secret renormalizability".

• Consider a gauge theory. The gauge symmetry constrains the terms in the Lagrangian, but it does not manifestly constrain the divergences, because we need to gauge fix before quantization; that is, the divergences never "see" the gauge symmetry. Thus renormalizability is nontrivial, but can be proven using Ward identities. Using BRST symmetry, one can also prove Yang-Mills is indeed renormalizable.

• Now consider Yang-Mills with an explicit mass term $m^2 A^\mu A_\mu$. This makes the analysis of renormalizability even more difficult because the propagator no longer falls off at high momenta, so superficial degrees of divergence get worse. However, it turns out that it is renormalizable if and only if the gauge group is abelian.

• The Standard Model is more complicated, as it is a gauge theory with spontaneous symmetry breaking. 't Hooft proved its renormalizability in 1971 and we can perform this proof explicitly by working in R_ξ gauge, introduced in the notes on the Standard Model.

These nontrivial facts should not be confused with Wilsonian renormalizability, which is easy to see in all these cases. Sometimes, counterterm and Wilsonian renormalizability are called perturbative and nonperturbative renormalizability. However, Wilsonian renormalizability does not imply counterterm renormalizability, e.g. in the case of quantum gravity if it were asymptotically safe. Here perturbative renormalizability fails because it expands about the Gaussian fixed point, while the nonperturbatively the theory is determined by a different fixed point in the UV.

5.1.3 Corrections to Polarization Operator and to Coulomb Law, Form Factors

113. Вычисление поляризационного оператора по Ландау 4

114. Радиационные поправки к закону Кулона по Ландау 4

115. Вычисление мнимой части поляризационного оператора по интегралу Фейнмана по Ландау 4

При прямом вычислении по диаграмме (петля на диаграмме $(113, 1)$) поляризационный оператор в первом приближении теории возмущений давался бы интегралом

$$\frac{i\mathscr{F}^{\mu\nu}}{4\pi} \to -e^2 \int \mathrm{Sp}\,\gamma^\mu G(p)\gamma^\nu G(p-k)\frac{d^4p}{(2\pi)^4}.$$

Однако этот интеграл, взятый по всему четырехмерному p-пространству, квадратично расходится и для получения конечного результата должен быть регуляризован по описанным в 112 правилам.

Мы не будем воспроизводить здесь полностью такой вывод, но покажем, каким образом можно вычислить по интегралу $(115,1)$ мнимую часть поляризационного оператора (которая была определена нами в \$113 с помощью условия унитарности); этот вывод содержит в себе ряд поучительных моментов.

Мнимая часть интеграла $(115,1)$ не содержит расходимости и не требует поэтому регуляризации. Для скалярной функции $\operatorname{Im}\mathscr{P} = 1/3\operatorname{Im}\mathscr{P}^{\mu}_{\mu}$ имеем

$$\operatorname{Im}\mathscr{P} = \operatorname{Im}\left\{ i\frac{4\pi e^2}{3(2\pi)^4}\int \frac{\operatorname{Sp}\gamma^{\mu}(\gamma p+m)\gamma_{\mu}\left(\gamma p+\gamma^{k}+m\right)}{(p^2-m^2+i0)\left[(p-k)^2-m^2+i0\right]}d^4p \right\}.$$

$$\operatorname{Im}\mathscr{P}\left(k^2\right) = \operatorname{Im}\int d^3p \int\limits_{-\infty}^{\infty} dp_0 \frac{i\varphi\left(p_0,\mathbf{p}\right)}{\left(p_0^2-\varepsilon^2+i0\right)\left[\left(p_0-k_0\right)^2-\varepsilon^2+i0\right]}$$

$$\varphi\left(p_0,\mathbf{p}\right) = \frac{2e^2}{3\pi^3}\left(m^2+\varepsilon^2+p_0k_0-p_0^2\right).$$

Подынтегральное выражение имеет четыре полюса по переменной p_0: а) $p_0=\varepsilon-i0,\ a')\, p_0=-\varepsilon+i0$, b) $p_0=k_0-\varepsilon+i0,\ b')\, p_0=k_0+\varepsilon-i0$.

После этого находим окончательно

$$\Delta\mathscr{P}\left(k^2\right) = i(2\pi i)^2\int\limits_{p_0>0} d^4p\cdot\varphi(p)\delta\left(p^2-m^2\right)\delta\left[(p-k)^2-m^2\right].$$

Ввиду наличия δ-функций интегрирование производится факти чески лишь в области пересечения гиперповерхностей

$$p^2=m^2,\quad (p-k)^2=m^2.$$

Поскольку в этой области все 4-векторы p времениподобны, то условие интегрирования по $p_0>0$ имеет инвариантный характер (верхняя полость конуса $p^2=m^2$).

Сравним (115,7) с исходной формулой $(115,2)$. Мы видим, что скачок функции $\mathscr{P}(t)$ на разрезе в плоскости t можно получить, если в исходном фейнмановском интеграле произвести замену

$$\frac{1}{p^2-m^2+i0}\to -2\pi i\delta\left(p^2-m^2\right)$$

в пропагаторах, отвечающих пересеченным на диаграмме $(113,1)$. линиям петли (S. Mandelstam, 1958, R. Cutkosky, 1960).

116. Электромагнитные формфакторы электрона по Ландау 4

117. Вычисление формфакторов электрона по Ландау 4

129. Радиационные поправки к уравнениям электромагнитного поля по Ландау 4

(поставил выше, потому что это скорее теория)

5.1.4 Asymptotic Formulas of QED, High Energies

(!! скоро допройду!)

132. Асимптотическое поведение фотонного пропагатора при больших импульсах

133. Связь между «затравочным» и истинным зарядами

134. Асимптотическое поведение амплитуд рассеяния при высоких энергиях

135. Выделение дважды логарифмических членов в вершинном операторе

(большой на самом деле параграф такой)

Поправки вида $(\alpha L)^n (L$ - большой логарифм) могут стать существенными, как уже было отмечено в конце §133, лишь при фантастически высоких энергиях и потому имеют только теоретическое значение. Но в амплитудах реальных процессов рассеяния возникают также и гораздо бОльшие поправки - вида $(\alpha L^2)^n$. Такие члены, содержащие по квадрату логарифма на каждую степень α, называют дважды логарифмическими.

Характерным параметром разложения в дважды логарифмических поправках является величина

$$\frac{a}{\pi} \ln^2 \frac{\varepsilon^2}{m^2},$$

где ε-фигурирующие в задаче энергии (скажем, суммарная энергия сталкивающихся частиц в системе их центра инерции). Условие применимости теории возмущений требует малости этой величнны; оно нарушается при энергиях

$$\varepsilon \sim m \exp\left(\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\right) \sim 3 \cdot 10^4 m$$

Случай виртуальных электронных линий Пусть импульсы p_1, p_2 отвечают виртуальным электронам, причем

$$\left|p_1^2\right|, \left|p_2^2\right| \gg m^2$$

Мы увидим, что основной областью интегрирования, приводящей к дважды логарифмическому выражению, является в этом случае область, определяемая неравенствами $0 < \rho \ll |tu|, |tv|; \left|\frac{p_1^2}{t}\right| \ll |v| \ll 1; \left|\frac{p_2^2}{t}\right| \ll |u| \ll 1$. Соответственно этому в знаменателе подынтегрального выражения в $(135,9)$ можно пренебречь m^2, p_1^2, p_2^2, f^2 по сравнению с $(p_1 f)$ или $(p_2 f)$, так что

$$I_1 = \int \frac{d^4 f}{2\,(p_2 f) \cdot 2\,(p_1 f)\,(f^2 + i0)}.$$

Случай физических электронных концов Пусть теперь импульсы p_1, p_2 отвечают реальным электронам, так что

$$p_1^2 = p_2^2 = m^2.$$

В этом случае существенна область интегрирования

$$0 < \rho \ll |tu|, |tv|; \quad 0 < |v|, |u| \ll 1.$$

136. Дважды логарифмическая асимптотика вершинного оператора

137. Дважды логарифмическая асимптотика амплитуды рассеяния электрона на мюоне

5.1.5 Entanglement in Quantum Fields

(!????????!?!?!?!?!?!?!!!!! an important topic in current physics)

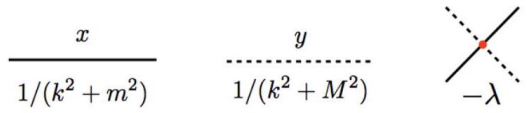
5.2 On Special Path Integral Methods

5.2.1 Effective Quantum Mechanics

- First, we consider a circular worldline with two fields x and y, with action

$$S[x,y]=\int_{S^1}\frac{1}{2}\dot{x}^2+\frac{1}{2}\dot{y}^2+V(x,y)dt,\qquad V(x,y):=\frac{1}{2}\left(m^2x^2+M^2y^2\right)+\frac{\lambda}{4}x^2y^2$$

As a quantum mechanical system, it's two coupled harmonic oscillators in periodic time T. In quantum field theory language, it's a theory of two interacting fields with masses m and M.



where k is the worldline momentum, which is quantized in units $2\pi/T$.

- Alternatively, we can integrate out the y field, giving

$$\int\mathcal{D}y\exp\left(-\frac{1}{2}\int_0^T dty\left(-\frac{d^2}{dt^2}+M^2+\frac{\lambda}{2}x^2\right)y\right)\sim\det\left(-\frac{d^2}{dt^2}+M^2+\frac{\lambda}{2}x^2\right)^{-1/2}$$

(we integrated by parts, boundary terms cancel by periodicity, and performed a formal Gaussian integral.)

$$S_{\text{eff}}[x]=\int_0^T dt\left(\frac{1}{2}\dot{x}^2+\frac{m^2}{2}x^2\right)+\frac{1}{2}\log\det\left(-\frac{d^2}{dt^2}+M^2+\frac{\lambda}{2}x^2\right)$$

$\log\det(AB)=\log\det A+\log\det B$, so $\log\det(AB)=\operatorname{tr}\log A+\operatorname{tr}\log B$.

$$G(t,t'):\qquad\left(\frac{d^2}{dt^2}-M^2\right)G(t,t')=\delta(t-t')$$

$$G(t,t')=\frac{1}{2M}\sum_{n\in\mathbb{Z}}e^{-M|t-t'+nT|}$$

With this:

$$\operatorname{tr}\log\left(-\frac{d^2}{dt^2}+M^2+\frac{\lambda}{2}x^2\right)=\operatorname{tr}\log\left(-\frac{d^2}{dt^2}+M^2\right)+\operatorname{tr}\log\left(1-\lambda\left(\frac{d^2}{dt^2}-M^2\right)^{-1}\frac{x^2}{2}\right)$$

The first term is a divergent constant, while the next term can be expanded in a series. The inverse is simply the Green's function, and the trace is over time, giving

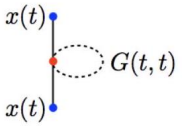
$$-\frac{\lambda}{2}\int_{S^1} dtG(t,t)x^2(t) - \frac{\lambda^2}{8}\int_{S^1\times S^1} dtdt'G\left(t',t\right)x^2(t)G\left(t,t'\right)x^2\left(t'\right) + ...$$

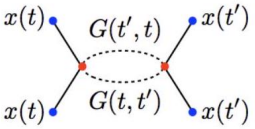
The general term here is

$$-\sum_{n=1}^{\infty}\frac{\lambda^n}{2^n n}\int_{(S^1)^n} dt_1...dt_nG\left(t_n,t_1\right)x^2\left(t_1\right)G\left(t_1,t_2\right)x^2\left(t_2\right)...G\left(t_{n-1},t_n\right)x^2\left(t_n\right)$$

and we generate an infinite series of new interactions as usual.

• The striking new feature is that the new interactions are non-local in time. To understand this, we consider the Feynman diagrams for the first two terms,





Since the y field is dynamical, it has its own propagator which allows it to move around on the worldline, so integrating it out gives a nonlocal interaction.

• To estimate the size of the nonlocal interactions, we Taylor expand in $t-t'$ for

$$\int dtdt'G\left(t,t'\right)^2x^2(t)x^2\left(t'\right) = \int dtdt'G\left(t,t'\right)^2x^t(t)\left(x^2(t)+2x(t)\dot{x}(t)\left(t-t'\right)+...\right).$$

Note that $G\left(t,t'\right)$ only depends on $t-t'$ through the combination $M\left(t-t'\right)$. Then we can integrate over $t-t'$, giving a power of $1/M$ for every time derivative. The terms look like

$$\int dt\frac{x^4(t)}{M} + \frac{\text{two derivatives}}{M^3} + \frac{\text{four derivatives}}{M^5} + ...$$

where we've suppressed numerical factors, and all terms are quartic in x. Therefore, as long as the derivatives are small in units of M^{-1}, we can truncate the series and get a local action.

• Note that we have a double expansion in λ and M. The former is relatively innocuous, but the latter breaks down for energies on the scale of M, where all terms are important; if we continue to try to use only a few terms, we'll find violations of unitarity, signaling that perturbation theory is breaking down. To fix it in a tractable way, we must 'un-integrate out' the y field.

• This same scenario was important in the discovery of the W boson. Fermi's theory of weak decay was an effective theory that had integrated out the W boson, but which led to a violation of unitarity at high energies.

Note. Perturbative unitarity violation is a typical feature of nonrenormalizable theories. For example, consider the innocuous case of massless ϕ^6 theory in $d=4$, with interaction term $\lambda\phi^6/M^2$. The coupling constant gives a mass scale M where perturbation theory must break down. Specifically, the lowest-order contribution to $2\phi\to 4\phi$ scattering is

$$\mathcal{M}\sim\frac{\lambda}{M^2},\quad |\mathcal{M}|^2\sim\frac{\lambda^2}{M^4},\quad \sigma\sim\frac{\lambda^2}{M^4}p^2$$

where we included the p^2 factor by dimensional analysis. Then for $p \gtrsim M$ the cross section violates unitarity bounds. The full nonperturbative result could still be unitary, but in reality the most common case is that some new physics takes over. This situation looks a bit different from the infinite tower of terms found above, but it's really the same, since these terms will be generated by renormalization.

The worldline approach to perturbative quantum field theory.

In this approach, we think about the particles of a quantum field theory in terms of ordinary quantum mechanics.

• The simplest way to describe a relativistic particle is to consider maps $x : [0, T] \to \mathbb{R}^n$ with

$$S[x] = \int_0^T dt \sqrt{G_{ab}(x)\dot{x}^a\dot{x}^b}$$

where G is the metric on $\mathbb{R}^n$ and t is interpreted as a parameter; this is just the standard path length action, and it is invariant under diffeomorphisms of the worldline.

• We claim that this is equivalent to the action

$$S[g, x] = \int_M dt\sqrt{g}\left(\frac{1}{2}G_{ab}(x)g^{tt}(t)\dot{x}^a\dot{x}^b + \frac{1}{2}V(x)\right)$$

where we vary with respect to both the worldline metric g on M and the fields x. Without the square root, this naively looks like the nonrelativistic action. The difference is that now we're dealing with a one-dimensional quantum gravity theory, since g is varied too.

• The metric g is specified by a single function $\sqrt{g} = |e|$ with $g^{tt} = e^{-2}$. The Riemann tensor vanishes, so the Einstein-Hilbert term vanishes. The Einstein equation is simply

$$T_{tt} = \frac{2}{\sqrt{g}}\frac{\delta(\sqrt{g}\mathcal{L})}{\delta g^{tt}} \propto \frac{1}{|e|}\left(G_{ab}(x)\dot{x}^a\dot{x}^b - e^2(t)V(x)\right) = 0$$

which yields

$$g_{tt}(t) = e^2(t) = \frac{1}{V(x)}G_{ab}(x)\dot{x}^a\dot{x}^b$$

We see the metric is non-dynamical; it is simply determined by a constraint.

• Plugging this into our action $S[g, x]$ in the case $V(x) = V_0$ gives

$$S[x] = \sqrt{V_0}\int_M dt\sqrt{G_{ab}(x)\dot{x}^a\dot{x}^b}$$

which is the geometrically natural action we encountered earlier.

• To see this another way, the momentum conjugate to the field x^a is

$$p_a = \frac{\partial\mathcal{L}}{\partial\dot{x}^a} = \frac{1}{|e|}G_{ab}(x)\dot{x}^b$$

so the Einstein equation gives the constraint

$$G^{ab}(x)p_a p_b + V(x) = 0$$

which becomes the Klein-Gordan equation, where $m^2 = V_0$. The Hamiltonian is

$$H = p_a \dot{x}^a - \mathcal{L} = p^2 + m^2$$

which vanishes on shell; this occurs generically for reparametrization-invariant theories. (typos here?)

• To see this a third way, we can perform the path integral explicitly in the case where G is the Minkowski metric. In one dimension, diffeomorphisms of the worldline, which are gauge transformations, let us set $g_{tt} = 1$. Then integrating over all possible metrics is equivalent to integrating over T, which is called a Schwinger parameter. In formal language, we are integrating over the moduli space of Riemannian metrics.

• With fixed T, the propagator is

$$\left\langle y \left| e^{-HT} \right| x \right\rangle = \int dpdq \langle y|p\rangle \left\langle p \left| e^{-HT} \right| q \right\rangle \langle q|x\rangle = \int dp e^{ip(x-y)} e^{-T\left(p^2+m^2\right)}$$

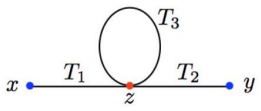
Integrating over T, we indeed find the propagator for a free scalar field in the target space,

$$\int_0^\infty dT \int_{C_I[x,y]} \mathcal{D}x e^{-S} = \int_0^\infty dT \left\langle y \left| e^{-HT} \right| x \right\rangle = \int dp \frac{e^{ip(x-y)}}{p^2+m^2}$$

• We would like to use one-dimensional quantum gravity to describe interacting scalar fields. We can do this by summing over possible topologies of M, replacing the worldline I with a worldgraph Γ. More elaborate setups allow us to include spin.

• Specifically, to compute an r-point correlation function, we sum over all graphs with r endpoints, where, e.g. for ϕ^4 theory all vertices have four edges. Each edge e has a Schwinger parameter T_e integrated over $(0, \infty)$. Each vertex has a position, which is also integrated over, producing momentum-conserving delta functions, and an appropriate factor involving the coupling constant. As usual, we also divide by the symmetry factor.

• Explicitly, in ϕ^4 theory one contribution to the propagator is



$$=$$

$$-\frac{\lambda}{4}\int_0^\infty dT_1 \int_{C_{T_1}[x,z]} \mathcal{D}x e^{-S} \int_0^\infty dT_2 \int_{C_{T_2}[y,z]} \mathcal{D}x e^{-S} \int dz \int_0^\infty dT_3 \int_{C_{T_3}[z,z]} \mathcal{D}x e^{-S}$$

$$=$$

$$-\frac{\lambda}{4}\int dz dp dq d\ell \frac{e^{ip(x-z)}}{p^2+m^2}\frac{e^{iq(y-z)}}{q^2+m^2}\frac{e^{i\ell(z-z)}}{\ell^2+m^2} = -\frac{\lambda}{4}\int dp d\ell \frac{e^{ip(x-y)}}{\left(p^2+m^2\right)^2\left(\ell^2+m^2\right)}$$

• The formalism we've used here is called the worldline approach to quantum field theory, a predecessor to the modern path integral formulation using fields. (The quantum gravity interpretation came much later.) It is explicitly perturbative, since it directly deals with particles and Feynman diagrams.

• In $d = 2$ and $d = 3$, we have a Riemann tensor but no Weyl tensor, so gravitational degrees of freedom do not propagate and the path integral for quantum gravity can be performed. The standard approach to string theory is just the case $d = 2$, where the graph becomes a Riemann surface, and we sum over the topologies of this surface. This approach is thus a perturbative picture of "string field theory", a mysterious subject about which little is known.

5.2.2 Quantum Statistical Mechanics

Our results above are also useful for thermal/statistical field theory (SFT).

Analogies between SFT and QFT

• Consider a QFT with fields $\phi : M \to N$ where $\dim M = D$, so

$$Z_{\text{QFT}} = \int \mathcal{D}\phi e^{iS/\hbar}$$

Wick rotation flips the sign the relative signs of the potential and kinetic terms, giving

$$Z_{\text{QFT}} = \int \mathcal{D}\phi e^{-S_E/\hbar}, \quad \beta = \frac{1}{\hbar}$$

That is, we have

QFT in D spacetime dimensions $\sim$ classical SFT in D spatial dimensions in the sense that their partition functions are simply related. For example, for $D = 1$ we have quantum mechanics $\sim$ classical SFT in 1 spatial dimension.

We've been making this correspondence implicitly above.

• A different correspondence starts with a quantum SFT defined on D spatial dimensions. Partition functions in SFT have the form of a path integral, but with a periodic imaginary "time",

$$Z_{\text{SFT}} = \operatorname{tr} e^{-\beta H} = \sum_n \left\langle n \left| e^{-\beta H} \right| n \right\rangle = \sum_n \int\limits_{C_t[n,n]} \mathcal{D}\phi e^{-S_E/\hbar} = \int\limits_{\mathcal{C}[M\times S^1]} \mathcal{D}\phi e^{-S_E/\hbar}$$

Here S_E is the Euclidean action in $D + 1$ spatial dimensions. Hence we have

quantum SFT in D spatial dimensions $\sim$ classical SFT in $D + 1$ spatial dimensions

where one dimension in the classical SFT has period $t = \hbar\beta$.

• To compute an amplitude in such a theory, integrals $\int dk$ are replaced with sums over discrete frequencies in the periodic time direction. In the high temperature limit $\beta \to 0$, only the zero frequency matters, so we recover classical SFT in D dimensions. Hence the high temperature limit of a quantum SFT in D dimensions is a classical SFT in D dimensions, as expected.

• As we saw in the notes on Undergraduate Physics, amplitudes in quantum mechanics are related to matrix elements of the thermal density matrix $\rho = e^{-\beta H}$ in quantum statistical mechanics, if the time evolution is taken to be imaginary. The diagonal elements of ρ are of course the occupancies, but the off-diagonal elements are harder to interpret.

• Conceptually, the reason that Z_{QFT} doesn't have boundary conditions but Z_{SFT} does is because the time direction in a QFT is infinite, and the $i\epsilon$ damping automatically projects out the vacuum at temporal infinity. By contrast, Z_{SFT} doesn't have a dynamical time at all.

How to perform computations in quantum statistical mechanics.

• In quantum statistical mechanics, our goal is to compute thermal expectation values. Quantum statistical mechanics takes place in zero spatial dimensions and periodic time, so

$$\left\langle \prod_i \hat{O}_i \right\rangle = \frac{\int\limits_{\mathcal{C}[S^1]} \mathcal{D}\phi \prod_i \hat{O}_i e^{-S_E/\hbar}}{\int\limits_{\mathcal{C}[S^1]} \mathcal{D}\phi e^{-S_E/\hbar}}$$

• More explicitly, in the case of a one-dimensional harmonic oscillator,

$$\langle x^n \rangle = \frac{\int dx x^n \int\limits_{C_t[x,x]} \mathcal{D}\phi e^{-S_E/\hbar}}{\int dx \int\limits_{C_t[x,x]} \mathcal{D}\phi e^{-S_E/\hbar}}$$

$$t = \hbar\beta,$$

$$-\frac{1}{\hbar}S_E[x(\tau)] = -\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + \frac{k}{2}x^2 = -\frac{1}{2}\int_0^{\hbar\beta} d\tau x(\tau)\hat{D}x(\tau), \quad \hat{D} = -\frac{m}{\hbar}\frac{d^2}{d\tau^2} + \frac{\kappa}{\hbar}$$

Here we pulled the x^n out of the functional integral, because it doesn't depend on τ. However, we can formally consider τ-dependent operators, which behave just as in quantum mechanics. In particular, we get τ-ordered correlators on the left-hand side.

• As before, we may define a generating functional

$$Z[J] = \int \mathcal{D}x(\tau) \exp\left(\int_0^{\hbar\beta} d\tau - \frac{1}{2}x(\tau)\hat{D}x(\tau) + J(\tau)x(\tau)\right)$$

Then $\langle x^n \rangle$ may be obtained by differentiating with respect to $J(0)$ and then setting $J(\tau) = 0$.

• By the same logic as in the zero-dimensional theory, for a free theory $Z[J]$ may be written in terms of the imaginary time Green's function,

$$Z[J] = Z[0]\exp\left(\frac{1}{2}\int d\tau d\tau' J(\tau)G(\tau-\tau')J(\tau')\right), \quad \hat{D}_\tau G(\tau-\tau') = \delta(\tau-\tau'), \quad G(0) = G(\beta\hbar).$$

• Since the Green's function takes place in periodic time, we take a Fourier series,

$$G(\tau) = \frac{1}{\sqrt{\beta\hbar}}\sum_n g_n e^{i\omega_n\tau}, \quad \omega_n = \frac{2\pi n}{\beta\hbar}, \quad n \in \mathbb{Z}$$

where the ω_n are the Matsubara frequencies. Plugging this into the defining equation,

$$G(\tau) = \frac{1}{\beta\kappa}\sum_n \frac{\omega^2}{\omega^2+\omega_n^2}e^{i\omega_n\tau}, \quad \omega = \sqrt{\frac{\kappa}{m}}$$

This is equivalent to

$$G(\tau) = \frac{\hbar\omega}{2\kappa}\left(\frac{e^{\omega|\tau|}}{e^{\hbar\beta\omega}-1} + \frac{e^{-\omega|\tau|}}{1-e^{-\hbar\beta\omega}}\right)$$

as can be shown by contour integrating the latter.

Now we give some explicit examples of computations.

• Correlators may be expanded by Wick's theorem. For example, we have

$$\left\langle \hat{x}^{2n+1} \right\rangle = 0, \quad \left\langle \hat{x}^2 \right\rangle = G(0).$$

To extend to an arbitrary imaginary time, we define the "Heisenberg" operators

$$\bar{x}(\tau) = e^{H\tau/\hbar} \hat{x} e^{-H\tau/\hbar}.$$

One must take some care, since $\bar{x}(\tau)$ is not Hermitian. Now we have

$$\langle T\bar{x}(\tau_1)\,\bar{x}(\tau_2)\rangle = G(\tau_1 - \tau_2)$$

while higher correlators are also decomposed by Wick's theorem.

• Using the form of the Green's function, we have

$$\left\langle \hat{x}^2 \right\rangle = \frac{\hbar\omega}{\kappa}\left(\frac{1}{e^{-\beta\hbar\omega}-1} + \frac{1}{2}\right)$$

Using $\langle H\rangle = 2\langle V\rangle = \kappa G(0)$ and $\hat{H} = \hbar\omega(\hat{n} + 1/2)$, we find the Bose-Einstein distribution.

• We can compute the position probability distribution by using our knowledge of the moments. Alternatively, we can perform the computation in one step, as the probability distribution is

$$\langle \delta(\hat{x} - x_0)\rangle \equiv \int dk \left\langle e^{ik(\hat{x}-x_0)} \right\rangle = \int dk e^{-ikx_0} \frac{Z[ik\delta(\tau)]}{Z[0]} = \int dk e^{-ikx_0} e^{-k^2 G(0)/2}$$

where we used the explicit form of $Z[J]$. Thus, we conclude

$$\langle \delta(\hat{x} - x_0)\rangle = \frac{e^{-x_0^2/2G(0)}}{\sqrt{2\pi G(0)}}$$

In the low temperature limit this gives the ground state probability distribution, while in the high temperature limit this gives the Boltzmann distribution.

• For the harmonic oscillator, the computations done above can also be done in the "Heisenberg" picture. Starting with the usual creation and annihilation operators, define

$$\bar{a}(\tau) = e^{H\tau/\hbar} a e^{-H\tau/\hbar}, \quad \bar{a}^\dagger = e^{H\tau/\hbar} a^\dagger e^{-H\tau/\hbar}$$

Again we must be careful, as $\bar{a}^\dagger(\tau) \neq (\bar{a}(\tau))^\dagger$. Integrating the Heisenberg equation of motion,

$$\bar{a}(\tau) = e^{-\omega\tau} a, \quad \bar{a}^\dagger(\tau) = e^{\omega\tau} a^\dagger$$

• Now we may straightforwardly compute imaginary time-dependent thermal averages,

$$\left\langle \bar{a}(\tau)\bar{a}^\dagger(0) \right\rangle = \frac{1}{Z(\beta)} \operatorname{tr} e^{-\beta H} \bar{a}(\tau)\bar{a}^\dagger(0) = \frac{e^{-\omega\tau}}{1 - e^{-\beta\hbar\omega}}$$

Using the relationship between $\hat{x}$ and $\hat{a}^\dagger$ and $\hat{a}$, which continues to hold with τ-dependence, we may compute $\langle \bar{x}(\tau)\bar{x}(0)\rangle$, which agrees with the Green's function computed above.

• We can set up perturbation theory with Feynman diagrams just as we did earlier. For example, for the anharmonic oscillator with a $\lambda\hat{x}^4/4!$ term, we have

$$Z_\lambda[J] = \int \mathcal{D}x(\tau) \exp\left(-\frac{S_E}{\hbar} + \int_0^{\hbar\beta} d\tau Jx - \frac{\lambda}{4!\hbar}x^4\right) = \exp\left(-\frac{\lambda}{4!\hbar}\int_0^{\hbar\beta} d\tau \frac{\delta^4}{\delta J(\tau)^4}\right) Z_0[J]$$

where S_E is the Euclidean action for the harmonic oscillator. We can then expand the exponential in a series, using our simple expression for $Z_0[J]$, and simplify each term by Wick's theorem. Each set of contractions yields a Feynman diagram.

- Hence we have the following Feynman rules for computing the partition function $Z_\lambda[0]$.
- Draw n internal points at times τ_i.
- For every point, multiply by $-\lambda/\hbar$ and integrate over τ_i.
- Contract the $4n$ edges pairwise. For each edge, write $G(\tau_i - \tau_j)$.
- Multiply by $Z_0[0]$ and sum over all contractions.

Summing over contractions, we get equivalence classes of diagrams, where we must divide by the symmetry factor as usual. We can evaluate correlators similarly, though now there are external points.

5.2.3 Quantum Fields

Finally, we generalize to quantum fields, returning to Lorentzian signature, and tie up some loose ends from the previous sections.

- For a scalar field, we may define field eigenstates by

$$\hat{\phi}(x)|\phi(x)\rangle = \phi(x)|\phi(x)\rangle$$

and similarly field momentum eigenstates $|\pi(x)\rangle$, and use them to construct a path integral as before using the formal completeness relation

$$\int \mathcal{D}\phi(\mathbf{x})|\phi(\mathbf{x})\rangle \left\langle \phi(\mathbf{x}) \right| = \int \mathcal{D}\pi(\mathbf{x}) \left| \pi(\mathbf{x}) \right\rangle \langle \pi(\mathbf{x})| = 1$$

The path integral then yields the transition amplitude between two states $|\phi_a\rangle$ and $|\phi_b\rangle$.

- In relativistic quantum field theory, we are usually interested in vacuum expectation values, since they appear in the LSZ reduction formula. If we let the vacuum be $|\Omega\rangle$, then we can isolate it from arbitrary boundary conditions, as long as they have some overlap with the vacuum state, by taking the time to infinity in a slightly imaginary direction. The result is

$$\left\langle \Omega \left| \mathcal{T} \prod_{i=1}^{n} \hat{\mathcal{O}}_i(x_i) \right| \Omega \right\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi(x) e^{iS} \prod_{i=1}^{n} \mathcal{O}_i(x_i)}{\int \mathcal{D}\phi(x) e^{iS}}$$

where the paths on the right-hand side run from $-T$ to T, and the denominator cancels out the phase and overlap factors, so that $\langle\Omega|\Omega\rangle = 1$. We will keep this implicit below.

- Taking the time to infinity this way effectively provides an infinitesimal damping which makes the path integral converge. It is always necessary in the definition of the path integral, even in Lorentzian signature, and it is what makes the transition to Euclidean signature natural. It is equivalent to add an $i\epsilon$ term in the action like

$$\mathcal{L} \supset \left(m^2 - i\epsilon\right)\phi^2$$

This manifestly yields the expected $i\epsilon$ in the Feynman propagator. In all cases, whenever we write m^2, we really mean $m^2 - i\epsilon$.

• A general state of the quantum field is a superposition of the field eigenstates, so it is a wavefunctional of the classical field configurations. For example, the vacuum wavefunctional can be computed from the path integral as

$$\langle \phi(x)|\Omega\rangle = \lim_{t_i \to -\infty(1-i\epsilon)} \int \mathcal{D}\phi(\mathbf{x}) e^{iS}$$

where the final boundary condition is $\phi(x)$ and the initial boundary condition is arbitrary.

• The partition function can be interpreted as the vacuum to vacuum amplitude,

$$\left\langle \Omega \left| e^{-iHT} \right| \Omega \right\rangle = \int\limits_{\text{time } T} \mathcal{D}\phi(x) e^{iS} = \mathcal{Z}[0] = \text{ sum of vacuum diagrams}$$

which implies that it is equal to e^{-iET} where E is the vacuum energy, which is generally divergent. Thus dividing by $\mathcal{Z}[0]$ to compute expectation values is equivalent to renormalizing the vacuum energy to zero. Then $\mathcal{W}[0]$ is simply the vacuum energy times T.

Since Dirac fields are complex, we'll also have to deal with complex Grassmann numbers.

• We can construct complex Grassmann numbers from real Grassmann numbers, e.g.

$$\theta = \frac{1}{\sqrt{2}}\left(\eta_1 + i\eta_2\right), \quad \theta^* = \frac{1}{\sqrt{2}}\left(\eta_1 - i\eta_2\right)$$

Using the change of variables formula derived earlier,

$$\int d\theta^* d\theta \theta\theta^* = (-i)(i) \int d\eta_2 d\eta_1 \eta_1 \eta_2 = 1$$

For multiple Grassmann variables we define $d^n\theta d^n\theta^* = d\theta_n d\theta_n^* ... d\theta_1 d\theta_1^*$. Note that $d\theta_i d\theta_i^*$ is Grassmann-even.

• By convention, we take complex conjugation to reverse the order of products, $(\theta\eta)^* = \eta^*\theta^*$.

• A complex Gaussian integral takes the form

$$\int d^n\theta d^n\theta^* e^{-\theta_i^* M_{ij} \theta_j} = \int d^n\theta d^n\theta^* e^{-\sum\limits_i \theta_i^* m_i \theta_i} = \prod_i m_i = \det M$$

where the m_i are the eigenvalues of M. Note that the derivation of this formula does not require θ^* to be the conjugate of θ, and that the final result has no square root.

• The partition function with anticommuting sources χ and χ^* is

$$Z(\chi) = \int d^n\theta d^n\theta^* e^{-\theta_i^* M_{ij} \theta_j + \chi_i^* \theta_i + \theta_i^* \chi_i}$$

and performing the shift $\theta \to \theta + M^{-1}\chi$ yields

$$Z(\chi) = (\det M) e^{\chi_i^* \left(M^{-1}\right)_{ij} \chi_j}$$

Note that M must be antisymmetric. When n is odd, M has a zero eigenvalue and hence M^{-1} does not exist, so the above formula does not apply.

• In the case of a Dirac field, there is an independent Grassmann algebra at every point in spacetime, and we write

$$\psi(x) = \sum_i \psi_i \phi_i(x)$$

where the $\phi_i(x)$ are a basis of four-component spinors, and the ψ_i form a basis of the Grassmann algebra at x. In particular, this is how one should think of relativistic spinors even on the classical level; otherwise, the anticommutativity must be put in 'by hand' upon quantization.

• As an example, in $d = 4$, every interaction of the form $(\bar{\psi}\psi)^5$ is trivial, because there are only eight independent Grassmann variables at every point. This is physically sensible, because interactions are contact interactions, and we cannot put five Dirac fermions at the same point.

Next, we explicitly construct the fermionic path integral using coherent states.

• We consider a single fermionic degree of freedom, e.g. a single mode that could be occupied by one kind of fermion. We construct the path integral using coherent states,

$$\hat{\psi}|\psi\rangle = \psi|\psi\rangle$$

where ψ is a complex Grassmann number and $\hat{\psi}$ is an annihilation operator. The operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ obey

$$\left\{\hat{\psi}, \hat{\psi}^\dagger\right\} = 1$$

with all other anticommutators zero. Note that we are allowing 'scalar multiplication' with Grassmann numbers in our Hilbert space, though our final answers will be ordinary numbers.

• We postulate a unique ground state $|0\rangle$ so that $\hat{\psi}|0\rangle = 0$ and define $\hat{\psi}^\dagger|0\rangle = |1\rangle$. Then there are no other states. The desired coherent state is

$$|\psi\rangle = |0\rangle - \psi|1\rangle.$$

To show this very explicitly, note that

$$\hat{\psi}|\psi\rangle = -\hat{\psi}\psi|1\rangle = \psi\hat{\psi}|1\rangle = \psi\hat{\psi}\hat{\psi}^\dagger|0\rangle = \psi|0\rangle = \psi|\psi\rangle$$

It is easy to make a mistake, as everything can have Grassmann parity. The matrix elements of $\hat{\psi}$ are Grassmann numbers, and the state $|\psi\rangle$ itself has Grassmann coefficients.

• To make the analogy clearer, note that

$$|\psi\rangle = e^{-\psi\hat{\psi}^\dagger}|0\rangle$$

In the bosonic case, coherent states are defined as

$$a|\mu\rangle = \mu|\mu\rangle, \quad |\mu\rangle = e^{\mu a^\dagger}|0\rangle.$$

The latter expression is identical up to a sign.

• The complex conjugate state $\langle\psi|$ is defined by

$$\langle\psi|\hat{\psi}^\dagger = \langle\psi|\psi^*, \quad \langle\psi| = \langle 0| - \langle 1|\psi^* = \langle 0|e^{-\hat{\psi}\psi^*}$$

The inner product of two coherent states is

$$\langle\psi|\psi'\rangle = \langle 0|0\rangle + \langle 1\,|\psi^*\psi'|\,1\rangle = 1 + \psi^*\psi' = e^{\psi^*\psi'}.$$

• Finally, the completeness relation is

$$\int d\psi^* d\psi |\psi\rangle e^{-\psi^*\psi} \langle\psi| = 1$$

This result holds because the left-hand side is equal to $|0\rangle\langle 0| + |1\rangle\langle 1|$.

We now use this setup to compute transition amplitudes.

• We compute the transition amplitude

$$\left\langle \psi_f \left| e^{-i\hat{H}\left(t_f - t_i\right)} \right| \psi_i \right\rangle, \quad \hat{H} = \hat{\psi}^\dagger M \hat{\psi}$$

for a quadratic Hamiltonian. As in the bosonic path integral, we split the time interval by inserting copies of the identity.

• Each copy comes with a factor of

$$\int d\psi_j^* d\psi_j \left\langle \psi_{j+1} \left| e^{-i\hat{H}\delta t} \right| \psi_j \right\rangle e^{-\psi_j^*\psi_j} \left\langle \psi_j \right|$$

Expanding the exponential, we find

$$\left\langle \psi_{j+1} \left| e^{-i\hat{H}\delta t} \right| \psi_j \right\rangle = \left(1 - i\psi_{j+1}^* M \psi_j \delta t\right) \langle \psi_{j+1} | \psi_j \rangle = e^{\psi_{j+1}^*\psi_j} e^{-i\psi_{j+1}^* M \psi_j \delta t}.$$

Then some of the phase factors telescope, and we find

$$\begin{aligned} \langle \psi_f, t_f | \psi_i, t_i \rangle &= \lim_{N\to\infty} \int \prod_{j=1}^{N} d\psi_j^* d\psi_j \exp\left(\sum_{j=0}^{N} -\psi_{j+1}^* \left(\psi_{j+1} - \psi_j\right) - i\delta t H\left(\psi_{j+1}^*, \psi_j\right) \right) \\ &= \int \mathcal{D}\psi^* \mathcal{D}\psi e^{i\int_{t_i}^{t_f} dt \left(\psi^* i\frac{\partial\psi}{\partial t} - H\right)} \end{aligned}$$

and in the usual case the quantity in parentheses is the Lagrangian.

• Now, the Dirac field is a relativistic field, so the field operator necessarily contains both creation and annihilation operators. Hence the interpretation is somewhat different from above, but the same manipulations work. In this case we integrate $\mathcal{D}\bar{\psi}\mathcal{D}\psi$ and define the generating functional

$$Z[\eta, \bar{\eta}] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS[\psi,\bar{\psi}] + i\bar{\eta}\psi + i\bar{\psi}\eta}$$

and by the same manipulations in the bosonic case, we have

$$\left\langle \Omega \left| T\psi(x_1)\bar{\psi}(x_2) \right| \Omega \right\rangle = \frac{1}{Z[0,0]} \frac{\delta}{i\delta\bar{\eta}(x_1)} \frac{-\delta}{i\delta\eta(x_2)} Z[\eta, \bar{\eta}] \Big|_{\eta=\bar{\eta}=0}$$

where we pick up an extra minus sign by anticommutation.

• By our earlier result for a Gaussian integral with linear sources, the Feynman propagator is again the inverse of the kinetic term $-i(i\not{\partial} - m)$, recovering the usual result in Fourier space.

Note. As mentioned earlier, spinors should be Grassmann-valued in classical field theory, leading to tricky signs. We defined the complex conjugate so that $(\theta_1\theta_2)^* = \theta_2^*\theta_1^*$, and define the transpose of a single Grassmann number to do nothing. Then for spinor fields ψ_i,

$$\left(\bar{\psi}_1\psi_2\right)^T = -\psi_2^T\psi_1$$

because the terms in the sum have the order of the Grassmann variables flipped. But more nicely,

$$\left(\bar{\psi}_1\psi_2\right)^\dagger = -\left(\psi_2^T\bar{\psi}_1^T\right)^* = \psi_2^\dagger\bar{\psi}_1^\dagger$$

so the adjoint simply applies to everything in flipped order.

5.2.4 Symmetries of the Path Integral

In this section, we derive the Ward-Takahashi identities. They are exact, nonperturbative relations between correlation functions that result from symmetries of the path integral. We begin by reviewing Noether's theorem.

• We work in curved spacetime using the language of differential forms, restating every result in coordinates. Consider the infinitesimal transformation

$$\delta_\epsilon\phi(x) = \epsilon f\left(\phi, \partial_\mu\phi\right)$$

The transformation is local if f depends only on the fields and their derivatives at x, in which case it is generated by the vector

$$V_f = \int\limits_M d^dx\sqrt{g}f(\phi,\partial\phi)\frac{\delta}{\delta\phi(x)}$$

acting on the space of fields.

• This transformation is a symmetry if it leaves the action invariant. Now, we introduce the so-called "Noether trick", promoting ϵ to be spacetime-dependent. Since we have a symmetry for constant ϵ, the variation of the action should depend only on derivatives of ϵ, so we can always find some j so that

$$\delta_\epsilon S[\phi] = -\int\limits_M *j\wedge d\epsilon = -\int\limits_M d^dx\sqrt{g}g^{\mu\nu}j_\mu(x)\partial_\nu\epsilon(x)$$

One can check j is just the conserved current found in the usual proof of Noether's theorem.

• When the equations of motion hold, the action is invariant under any infinitesimal change in the fields, $\delta_\epsilon S[\phi] = 0$. Integrating by parts and choosing $\epsilon(x)$ to have compact support,

$$d*j = \partial_\mu\left(\sqrt{g}g^{\mu\nu}j_\nu\right) = 0$$

• We define the charge Q on a hypersurface N of codimension one by

$$Q[N] = \int\limits_N *j = \int\limits_N d^{d-1}x\sqrt{g_N}j_\mu n^\mu$$

where g_N is the metric pulled back to N and n^μ is a unit normal.

• Now consider two such hypersurfaces N_0 and N_1 bounding a region M'. Then

$$Q\left[N_1\right] - Q\left[N_0\right] = \int\limits_{\partial M'} *j = \int\limits_{M'} d*j = 0$$

That is, the charge depends on N only through its homology class. In the simple case where the N_i are constant time-slices of Minkowski space, this means Q is conserved in time.

Example. For the complex scalar field we have

$$S[\phi] = \int_M d\bar{\phi} \wedge *d\phi + *V\left(|\phi|^2\right)$$

where we write conjugation with a bar to avoid confusion with the Hodge star. Then by direct computation, the current is $j = i(\phi d\bar{\phi} - \bar{\phi} d\phi)$ and for a time-slice,

$$Q[N] = i \int_N *(\phi d\bar{\phi} - \bar{\phi} d\phi) = i \int d\mathbf{x} \phi \partial_0 \bar{\phi} - \bar{\phi} \partial_0 \phi$$

The canonical momenta are

$$\pi = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi\right)} = \partial_0 \bar{\phi}, \quad \bar{\pi} = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \bar{\phi}\right)} = \partial_0 \phi$$

so the charge is

$$Q = i \int d\mathbf{x} (\phi \pi - \bar{\phi} \bar{\pi})$$

which indeed generates the transformations by Poisson brackets.

Note. Dynamical symmetries are symmetries of the equations of motion that are not symmetries of the Lagrangian; they are associated with integrable systems. For example, for a free particle, the Lagrangian has an $O(n)$ symmetry, while the equation of motion has a $\mathrm{GL}_n(\mathbb{R})$ symmetry because it only requires the particle to move uniformly on a line.

Next, we turn to symmetries in the quantum theory.

• In the quantum theory, the local field transformation $\phi \to \phi'(\phi)$ must leave the product of the path integral weight and measure invariant to be a symmetry,

$$\mathcal{D}\phi e^{-S[\phi]} = \mathcal{D}\phi' e^{-S[\phi']}$$

This is necessary and sufficient, but since we have more experience with symmetries of the action from classical field theory, we often use those and hope the measure is invariant as well.

• For example, suppose $S[\phi]$ only depends on the derivatives of ϕ. Then classically we have the shift symmetry $\phi(x) \to \phi(x) + \phi_0$. It would appear that the measure would obviously be invariant, but the measure is not defined without regularization, and the regularized measure might not be invariant.

• If $M = T^d$, we might expand $\phi(x)$ in a Fourier series, then cut off the sum. Since the constant translation only affects the lowest Fourier mode, and the regularized measure integrates over all values of this coefficient, the measure is indeed invariant.

• As another example, consider rotational symmetry in Euclidean space. The action is $SO(d)$ invariant if it is a scalar under rotation. We can regularize the path integral by integrating over all Fourier modes where the $SO(d)$ invariant quantity $p^\mu p_\mu$ is less than some cutoff; this is slightly more subtle for $SO(d-1,1)$ in Minkowski space. Alternatively, we can regularize by replacing space with a lattice, but this breaks $SO(d)$ down to the lattice's point group.

• Sometimes we may run into tradeoffs, where only some symmetries can be preserved, and we may intentionally choose which to lose. A worse situation is when there doesn't exist any regulator that preserves a symmetry, in which case the symmetry is said to be anomalous; it is said to be incompatible with quantum mechanics itself.

• For example, QED in four dimensions with a massless fermion is conformally invariant, but there is a conformal anomaly; indeed, the beta functions are nonzero in DR, with a hard cutoff, and on a lattice. It's possible to prove that a symmetry is anomalous by considering the geometry and topology of the space of fields, using sophisticated mathematics such as the Atiyah-Singer index theorem.

• The Standard Model contains a number of anomalies, such as the global symmetry of baryon number. In general, it's acceptable for global symmetries to be anomalous, but not gauge symmetries, since we lose the Ward identity and hence unitarity. The hypercharges in the Standard Model are fixed by requiring gauge anomalies to vanish.

• Consider operators whose only variation under a symmetry transformation is through the transformation of the field, e.g. scalar operators for rotations. Then $\mathcal{O}(\phi) \to \mathcal{O}(\phi')$, and on a compact manifold M we have

$$\int \mathcal{D}\phi e^{-S[\phi]} \prod_i \mathcal{O}_i(\phi(x_i)) = \int \mathcal{D}\phi' e^{-S[\phi']} \prod_i \mathcal{O}_i(\phi'(x_i)) = \int \mathcal{D}\phi e^{-S[\phi]} \prod_i \mathcal{O}_i(\phi'(x_i))$$

where we simply renamed the dummy variable in the first step, then defined ϕ' in terms of ϕ and used the definition of a symmetry in the second.

• Therefore, we conclude (leaving the time ordering implicit)

$$\langle \mathcal{O}_1(\phi(x_1)) ... \mathcal{O}_n(\phi(x_n)) \rangle = \langle \mathcal{O}_1(\phi'(x_1)) ... \mathcal{O}_n(\phi'(x_n)) \rangle$$

In general, we will call any identity between correlation functions derived from a symmetry a Ward or Ward-Takahashi identity. This is an example of a 'global' Ward-Takahashi identity.

Example. Consider the $U(1)$ symmetry of a complex scalar field,

$$\phi \to \phi' = e^{i\alpha}\phi, \quad \bar{\phi} \to \bar{\phi}' = e^{-i\alpha}\bar{\phi}.$$

The path integral measure is invariant under this symmetry as long as we integrate over as many modes of $\bar{\phi}$ as we do of ϕ. Therefore for $\mathcal{O}_i = \phi^{r_i}\bar{\phi}^{s_i}$, we have

$$\langle \mathcal{O}_1(x_1) ... \mathcal{O}_n(x_n) \rangle = e^{i\alpha \sum_i (r_i - s_i)} \langle \mathcal{O}_1(x_1) ... \mathcal{O}_n(x_n) \rangle$$

Then correlators vanish unless they contain the same number of ϕ and $\bar{\phi}$ fields.

Example. If the action and path integral measure are translationally invariant, and the operators $\mathcal{O}_i$ depend on x only through their dependence on $\phi(x)$, then

$$\langle \mathcal{O}_1(x_1) ... \mathcal{O}_n(x_n) \rangle = \langle \mathcal{O}_1(x_1 - a) ... \mathcal{O}_n(x_n - a) \rangle$$

so correlators only depend on position differences. Similarly, Lorentz invariance ensures that correlators can only depend on the invariant interval between the insertion points.

Note. Formally, the path integral measure picks up a Jacobian factor,

$$\mathcal{D}\phi' = \det\left(\frac{\delta\phi'(x)}{\delta\phi(y)}\right)\mathcal{D}\phi$$

In the case of the transformation described above, we have

$$\frac{\delta}{\delta\phi(y)}(\phi(x) + \epsilon f(x)) = \delta(x - y) + \epsilon\frac{\partial f(\phi(x))}{\partial\phi(y)}$$

and hence the Jacobian is

$$\det\left(\frac{\delta\phi'(x)}{\delta\phi(y)}\right) = 1 + \mathrm{tr}\left(\epsilon\frac{\delta f(\phi(x))}{\delta\phi(y)}\right)$$

Then classical symmetries linear in the fields should remain symmetries in the quantum theory, because the argument of the determinant is field-independent, so the Jacobian is just a constant that cancels out. However, this formal argument is deceptive, because we are working with an undefined, "ideal" path integral measure. Instead we must check invariance directly.

These results roughly correspond to charge conservation, but we can derive more powerful identities that correspond to current conservation, usually called the Ward-Takahashi identities. To warm up, we derive the Schwinger-Dyson and Ward-Takahashi identities in standard notation.

• We consider the partition function of a generic theory $Z[J]$ and a transformation

$$\phi(x) \to \phi'(x) = \phi(x) + \epsilon\Delta\phi(x)$$

not necessarily a symmetry, which leaves the measure invariant. Then infinitesimally

$$Z[J] = \int \mathcal{D}\phi e^{i(S[\phi]+\phi\cdot J)} = \int \mathcal{D}\phi' e^{i(S[\phi']+\phi'\cdot J)} = \int \mathcal{D}\phi e^{i(S[\phi]+\phi\cdot J)}\left(1 + i\left(\frac{\delta S}{\delta\phi} + J\right)\epsilon\Delta\phi\right)$$

which shows that

$$\int \mathcal{D}\phi e^{i(S[\phi]+\phi\cdot J)} \int dy \left(\frac{\delta S}{\delta\phi(y)} + J(y)\right)\Delta\phi(y) = 0.$$

• Next, we consider the specific transformation that shifts the field at one point,

$$\Delta\phi(y) = \delta(y - x)$$

which should leave a reasonable measure invariant; this shift-invariance was the motivation behind our definition of Grassmann integration. Then we find

$$\int \mathcal{D}\phi \left(\frac{\delta S}{\delta\phi(x)} + J(x)\right) e^{i(S[\phi]+J\cdot\phi)} = 0$$

which states that the classical equations of motion hold in vacuum expectation.

• To get relationships between correlation functions, we act by n field derivatives $\delta/i\delta J(x_i)$. This yields one term with n factors of ϕ, and n terms with $n-1$ factors of ϕ, where one of the derivatives hits the $J(x)$ factor. Setting $J = 0$ gives

$$\left\langle \frac{\delta S}{\delta\phi(x)} \prod_{i=1}^{n} \phi(x_i) \right\rangle = i \sum_{i=1}^{n} \left\langle \phi(x_1)...\widehat{\phi}(x_i)\,\delta(x - x_i)\,...\phi(x_n) \right\rangle$$

where the hat denotes a missing argument. This is the Schwinger-Dyson equation, as we derived earlier. Letting X stand for all the fields, it can be concisely written as

$$\left\langle \frac{\delta S}{\delta\phi(x)} X \right\rangle = i \left\langle \frac{\delta X}{\delta\phi(x)} \right\rangle$$

Indeed, here X can be any function of the fields.

• The classical equivalent of a time-ordered correlator is just a product of numbers,

$$\langle \phi(x_1)...\phi(x_n)\rangle \to \phi(x_1)...\phi(x_n)$$

so in the classical limit, the left-hand side is zero as $\delta S/\delta\phi(x) = 0$. Thus the contact terms on the right encode the difference between classical products and quantum correlation functions.

• As an example, consider a free scalar field. Then the Schwinger-Dyson equation for $n = 1$ reduces to the statement that the Feynman propagator is a Green's function for the Klein-Gordan operator; the contact term supplies the required delta function. For an interacting theory, we would instead find a relationship between the two-point correlator and higher correlators, which we can use to organize a perturbative expansion.

• To derive the Ward-Takahashi identities, we do the same procedure, but for a classical global symmetry associated with a field change $\delta\phi$. Here we have

$$\Delta\phi(y) = \delta(y-x)\delta\phi(x), \quad \frac{\delta S}{\delta\phi(x)}\delta\phi(x) = -\partial_\mu j^\mu(x)$$

Assuming this transformation leaves the measure invariant as well, we have

$$\int \mathcal{D}\phi e^{i(S[\phi]+\phi\cdot J)}\left(-\partial_\mu j^\mu(x) + J(x)\delta\phi(x)\right) = 0$$

Taking derivatives as before gives the Ward-Takahashi identity

$$\partial_\mu \left\langle j^\mu(x)\prod_{i=1}^n \phi(x_i)\right\rangle = -i\sum_{i=1}^n \left\langle \phi(x_1)...\widehat{\phi}(x_i)\,\delta\phi(x)\delta(x-x_i)...\phi(x_n)\right\rangle$$

where the derivative acts on x. In the case $n = 0$, we see the current j^μ is conserved in expectation, $\partial_\mu\langle j^\mu\rangle = 0$.

Note. It is conventional in theoretical physics to write the Schwinger-Dyson equation as, e.g.

$$\frac{\delta S}{\delta\phi(x)}\phi(y) = i\delta(x-y)$$

and call this an "operator equation", even though the two sides are not equal as operators on the Hilbert space. The sociological reason behind this is that statements about operators are difficult to make, since they are fraught with short-distance singularities. In the mathematical physics community, these difficulties are addressed in a formalism which replaces operators with operator-valued distributions. In the theoretical physics community, these issues are avoided by simply redefining the phrase "operator equation" to mean something weaker. The operator equation $\mathcal{O}_1 = \mathcal{O}_2$ means that

$$\langle \mathcal{O}_1 X\rangle = \langle \mathcal{O}_2 X\rangle$$

for any product of operators in X involving fields evaluated at times different from those of $\mathcal{O}_1$ and $\mathcal{O}_2$. (If the times matched, we would pick up extra contact terms.)

Ward-Takahashi in curved spacetime, using symmetries of the path integral

• We consider an infinitesimal symmetry of the path integral, $\phi \to \phi' = \phi + \epsilon\delta\phi$, where ϵ is constant. Then if we allow ϵ to vary in space, falling off at infinity to avoid boundary terms,

$$\mathcal{Z} = \int \mathcal{D}\phi' e^{-S[\phi']} = \int \mathcal{D}\phi e^{-S[\phi]}\left(1 - \int_M *j\wedge d\epsilon\right)$$

as we saw classically, but the variation of the measure may also contribute to j. Then

$$0 = -\int_M *\langle j(x)\rangle \wedge d\epsilon = \int_M \epsilon(x) d * \langle j(x)\rangle$$

which implies that $j^\mu(x)$ is a conserved current in expectation, $\partial_\mu \langle j^\mu(x)\rangle = 0$.

• Now, as before, we introduce local operators which transform as

$$\mathcal{O} \to \mathcal{O}' = \mathcal{O} + \epsilon\delta\mathcal{O}$$

Accounting for both the change in the action and measure, and the operators, we find

$$\int_M \epsilon(x) \wedge d * \left\langle j(x) \prod_{i=1}^n \mathcal{O}_i(x_i) \right\rangle = -\sum_{i=1}^n \left\langle \epsilon(x_i)\, \delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j) \right\rangle$$

Note that the exterior derivative d acts on x, and the correlators are time-ordered as usual.

• To finish up, we would like to strip off the parameter $\epsilon(x)$. Note that

$$\epsilon(x_i)\, \delta\mathcal{O}_i(x) = \int_M *\delta(x - x_i)\, \epsilon(x) \delta\mathcal{O}_i(x_i) = \int_M \delta(x - x_i)\, \epsilon(x) \delta\mathcal{O}_i(x_i) \sqrt{g} dx$$

Having expressed both sides as integrals, we simply have

$$d * \left\langle j(x) \prod_{i=1}^n \mathcal{O}_i(x_i) \right\rangle = - * \sum_{i=1}^n \delta(x - x_i) \left\langle \delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j) \right\rangle$$

This is the Ward-Takahashi identity in a curved spacetime. In more pedestrian notation, in flat spacetime, this reduces to

$$\partial_\mu \left\langle j^\mu(x) \prod_{i=1}^n \mathcal{O}_i(x_i) \right\rangle = -\sum_{i=1}^n \delta(x - x_i) \left\langle \delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j) \right\rangle$$

• Next, we integrate over x to investigate charge conservation. Consider integrating over some region $M' \subset M$ with $\delta M' = N_1 - N_0$, as we studied classically, where M' contains all the points x_i. Then

$$\left\langle Q[N_1] \prod_i \mathcal{O}_i(x_i) \right\rangle - \left\langle Q[N_0] \prod_i \mathcal{O}_i(x_i) \right\rangle = -\sum_{i=1}^n \left\langle \delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j) \right\rangle$$

If M is compact without boundary and $M' = M$, the left-hand side vanishes, and

$$\sum_{i=1}^n \left\langle \delta\mathcal{O}_i(x_i) \prod_{j\neq i} \mathcal{O}_j(x_j) \right\rangle = 0$$

which is just the infinitesimal form of the global Ward-Takahashi identity under the symmetry generated by Q.

• If M has a boundary, we pick up extra terms. For example, we saw earlier that correlation functions of ϕ vanish if they have nonzero $U(1)$ charge. In this more general context, they vanish unless their charge is equal to the difference of the initial and final $U(1)$ charges.

• A more subtle example is when M is non-compact. In this case we must impose boundary conditions at infinity, e.g. that the field approaches a constant that is a minimum of the effective potential. This can result in spontaneous symmetry breaking.

Note. We may use the Ward-Takahashi identity to show that the conserved charge generates the symmetry transformation. We integrate over a spacetime region bounded by $t = t_-$and $t = t_+$, where the time interval contains x_1^0 but none of the other x_i^0. Then

$$\langle Q\left(t_+\right)\mathcal{O}_1(x_1)Y\rangle - \langle Q\left(t_-\right)\mathcal{O}_1(x_1)Y\rangle = -\langle \delta\mathcal{O}_1(x_1)Y\rangle\,, \quad Y = \prod_{i=2}^{n}\mathcal{O}_i\left(x_i\right)$$

Taking the limits $t_- \to x_1^0$ and $t_+ \to x_1^0$, the left-hand side becomes a commutator because the correlators are time-ordered. Since Y is arbitrary, we conclude

$$[Q, \mathcal{O}] = -\delta\mathcal{O}$$

as an operator equation in the Hilbert space.

Relation of quantum symmetries and symmetries of the 1PI effective action.

• Under the infinitesimal symmetry $\phi \to \phi' = \phi + \epsilon f(\phi, \partial\phi)$ we have

$$\begin{aligned}
\mathcal{Z}[J] &= \int \mathcal{D}\phi' \exp\left(-\frac{1}{\hbar}\left(S\left[\phi'\right] + \int_M dx J(x)\phi'(x)\right)\right) \\
&= \int \mathcal{D}\phi \exp\left(-\frac{1}{\hbar}\left(S[\phi] + \int_M dx J(x)\phi'(x)\right)\right) \\
&= \mathcal{Z}[J]\left(1 - \frac{\epsilon}{\hbar}\int_M dx J(x)\langle f(\phi, \partial\phi)\rangle_J\right)
\end{aligned}$$

where the expectation value is taken in the presence of the source J. Then we have

$$\int_M dx J(x)\langle f(\phi, \partial\phi)\rangle_J = 0$$

• Next, we evaluate the current at

$$J_\Phi(y) = -\frac{\delta\Gamma[\Phi]}{\delta\Phi(y)}, \quad \langle\phi\rangle_{J_\Phi} = \Phi$$

to give

$$\int_M dx \frac{\delta\Gamma[\Phi]}{\delta\Phi(x)}\langle f(\phi, \partial\phi)\rangle_{J_\Phi} = 0$$

Therefore, the effective action is invariant under the transformation

$$\Phi \to \Phi' = \Phi + \epsilon\langle f(\phi, \partial\phi)\rangle_{J_\Phi}$$

which involves expectation values of the transformations, which can be rather complicated. This result is called a Slavnov-Taylor identity.

• When the symmetry is linear, i.e. f is linear in the fields, we have

$$\langle f(\phi, \partial\phi)\rangle_{J_\Phi} = f(\Phi, \partial\Phi)$$

so the effective action is invariant under the transformation

$$\Phi \to \Phi' = \Phi + \epsilon f(\Phi, \partial\Phi)$$

Therefore, all symmetries of the classical action linear in the fields, under which the path integral measure is invariant, become symmetries of the quantum 1PI action. For example, this result ensures that a ϕ^3 coupling cannot be generated in ϕ^4 theory by $\mathbb{Z}_2$ symmetry, that $O(n)$ symmetry is preserved in the $O(n)$ model, and that Lorentz invariance is preserved.

6 Experiments and methods for experiments

6.0.1 Ideas for experimental determining parameters of fields

(связь с экспериментом, вообще, тоже очень важный вопрос)

Как определяли массу лептонов?

(важный теор вопрос, пока не готов к этому)

6.0.2 Properties of fundamental particles

(не уверен пока, что тут, но по идее суть физики элементарных части, с позиции экспериментов и итоговых свойств. в 1й части же просто модели не нужно будет - удалю)

Overview of particles

(общие свойства и сравнения)

Particle	Mass
Neutrinos	$\sim 10^{-2}$eV
Electron	0.5MeV
Muon	100MeV
Pions	140MeV
Proton, Neutron	1GeV
Tau	2GeV
W,Z Bosons	$80 - 90$GeV
Higgs Boson	125GeV

Properties of electron and photon

(разные физические свойства соберу, мб полезно будет. не полезно - удалю раздел.)

Properties of typical nuclei

(по идее тоже много можно важных графиков вставить, изучу мб)

Which particles can also be?

(что гипотетическое и исследуется - то тут)

On other particles

On dark photon (?)

In 2023 there are many experiments searching for the dark photon, a hypothetical particle whose motivation is described in the notes on Cosmology. The dark photon acts like the regular photon, but it has a mass $m_{A'}$ and couplings ϵe to fermions, where ϵ is the kinetic mixing parameter. Accelerator experiments attempt to produce the dark photon and detect its decay. The dominant production channel depends on the experiment, such as whether it uses lepton or hadron beams, but some important production channels are:

• Electron-positron annihilation, $e^+e^- \to \gamma A'$. At high energies, the rate for this is ϵ^2 smaller than the rate for $e^+e^- \to \gamma\gamma$.

• A' Bremsstrahlung. Most processes can also produce an A' by attaching it to a fermion line.

• Meson decay. For example, pions can decay as $\pi^0 \to \gamma\gamma$, though we won't be able to calculate this rate for some time. However, it is simply related to the rate for $\pi^0 \to \gamma A'$.

The A' can then decay, e.g. through $A' \to e^+e^-$, which is trivial to calculate given what we already know, or through $A' \to \pi^+\pi^-$, which is somewhat more tricky. The way we detect this decay depends on the lifetime of the A', which generally scales as $\tau \sim 1/\Gamma \sim 1/\alpha\epsilon^2 m_{A'}$. The dark photon will travel a distance $\gamma c\tau$, but the opening angle of the decay products is $\sim 1/\gamma$, so the quantity that determines experimental resolution turns out to be

$$\ell \sim c\tau \sim \left(10^{-6}\text{ m}\right)\left(\frac{10^{-3}}{\epsilon}\right)^2 \frac{10\text{MeV}}{m_{A'}}$$

For $\ell \lesssim 10\mu$m, the decay is "prompt", i.e. it happens at the same point as production, up to the detector resolution. In this case, individual dark photon events look like background events, but they can be found by computing the invariant mass of the decay products and looking for a statistically significant bump at $m_{A'}$. For larger ℓ, we enter the "displaced" regime, where the decay products can be measured to be coming from a point away from the interaction point. Here the signal rate tends to be lower, but measuring a displaced decay dramatically cuts down on backgrounds.

7 Special Phenomena and Theoretical Extensions of QFT

(тут же разные калибровочные важные теории!)

7.1 Non-Abelian Gauge Theory in a nutshell

7.1.1 Classical Yang-Mills, quantization, Faddeev-Popov method

(см. также лекции Киселева, там в конце про это есть.)

Classical Yang-Mills

• We consider a Lie group H with Lie algebra $\mathfrak{h}$, so every $h \in H$ can be written as

$$h = e^{ig\alpha}, \quad \alpha = \alpha_a T^a, \quad T^a \in \mathfrak{h}.$$

Here the generators T^a satisfy

$$\left[T^a, T^b\right] = i f_c^{ab} T^c$$

Note that conventions differ here, as one can flip the sign of g, α, or A_μ. In our current convention, also used by Peskin and Schroeder, the covariant derivative comes with a minus sign, $D \sim \partial - igA$, and gauge transformations come with plus signs.

• The Lie algebra contains a symmetric bilinear form called the Killing form,

$$\kappa^{ab} \equiv T^a \circ T^b$$

which is invariant under the adjoint action,

$$hT^a h^{-1} \circ hT^b h^{-1} = T^a \circ T^b.$$

Note that when we focus on the Lie algebra alone, we usually take the infinitesimal version of this equation by setting $h \approx 1 - ig\alpha$, for

$$[T^a, T^c] \circ T^b + T^a \circ \left[T^c, T^b\right] = 0$$

as defined earlier.

• For an abstract Lie algebra, the Killing form is defined as we've seen earlier,

$$\kappa^{ab} = \operatorname{tr}\left[T^a, \left[T^b, \cdot\right]\right]$$

On the other hand, for a matrix Lie group where the generators are traceless we have

$$T^a \circ T^b = \operatorname{tr} T^a T^b$$

We will normalize generators in the fundamental representation as

$$T^a \circ T^b = \frac{1}{2}\delta^{ab}$$

and think of all generators below as being in the fundamental representation. There's no reason to do this at this point, but when we introduce physical matter, it will be in the fundamental representation, so we'll only need one set of generators.

• Under a gauge transformation $U(x) = e^{ig\alpha(x)}$ the gauge potential/connection transforms as

$$A_\mu \to U A_\mu U^{-1} - \frac{i}{g}\left(\partial_\mu U\right) U^{-1}$$

where infinitesimally we have

$$A_\mu \to A_\mu + D_\mu \alpha, \quad D_\mu \alpha \equiv \partial_\mu \alpha - ig\left[A_\mu, \alpha\right]$$

where we have defined the covariant derivative acting on a $\mathfrak{h}$-valued field. Note that A_μ doesn't transform in a representation of H, but the difference of two A_μ 's transforms in the adjoint.

• It will be useful to write the above in components,

$$D_\mu \alpha^a = \partial_\mu \alpha^a + g A^b_\mu \alpha^c f^{abc}.$$

The same applies to any other quantity transforming in the adjoint.
• We define the curvature/field strength as

$$F_{\mu\nu} = \frac{i}{g}\left[D_\mu, D_\nu\right] = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right]$$

as can be shown by acting with both sides on a "test object".
• The field strength transforms in the adjoint representation, so that

$$F_{\mu\nu} \to U F_{\mu\nu} U^{-1} = F_{\mu\nu} + ig\left[\alpha, F_{\mu\nu}\right] + O\left(\alpha^2\right)$$

Since the covariant derivative obeys the Jacobi identity, we have the Bianchi identity,

$$D_\mu F_{\alpha\beta} + D_\beta F_{\mu\alpha} + D_\alpha F_{\beta\mu} = 0$$

• The pure Yang-Mills Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\operatorname{tr} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$$

Note that this automatically contains terms cubic and quartic in A.
• To derive the equation of motion, it is useful to note that

$$\delta F_{\mu\nu} = D_\mu \delta A_\nu - D_\nu \delta A_\mu$$

Then the variation of the action is proportional to

$$\operatorname{tr}\left(F_{\mu\nu}\left(\partial^\mu \delta A^\nu + ig\left[A^\mu, \delta A^\nu\right]\right)\right) = -\operatorname{tr}\left(\delta A^\nu D^\mu F_{\mu\nu}\right)$$

using the fact that δA^μ is in the adjoint, the cyclic property of the trace, and integration by parts. Since δA^μ is arbitrary, and the Killing form is nondegenerate, we have the equation of motion

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} - ig\left[A_\mu, F^{\mu\nu}\right] = 0.$$

• Matter fields in a representation R will be introduced through the Lagrangian

$$\mathcal{L} \supset \bar{\psi}(i\not{D} - m)\psi, \quad (i\not{D} - m)\psi = 0, \quad \psi \to e^{ig\alpha}\psi = \left(1 + ig\alpha^a T^a_R\right)\psi + O\left(\alpha^2\right)$$

The covariant derivative acts on ψ in whatever representation it transforms in,

$$D_\mu \psi_i = \partial_\mu \psi_i - ig A^a_\mu \left(T^a_R\right)_{ij} \psi_j$$

which leads to the interaction term

$$\mathcal{L} \supset g\bar{\psi}_i \gamma^\mu A^a_\mu \left(T^a_R\right)_{ij} \psi_j$$

• The matter provides a current for the gauge fields, so that the equation of motion is

$$D_\mu F^{\mu\nu} = J^\nu, \quad J^{\mu a} = -g\bar{\psi}\gamma^\mu T^a_R \psi$$

By the transformation properties of $D_\mu F^{\mu\nu}$, we see that J^ν transforms in the adjoint. Acting with D_ν on both sides and using symmetry, we have $D_\mu J^\mu = 0$, so the current is "covariantly conserved" rather than conserved; this can also be shown with the equation of motion for ψ.

• In fact, there is no gauge-invariant conserved current in this theory. This result is a special case of the Weinberg-Witten theorem, which states that any theory with a global non-abelian symmetry under which massless spin 1 particles are charged does not admit a gauge-invariant conserved current.

• The second part of the Weinberg-Witten theorem implies that any theory with a conserved, Lorentz-covariant energy-momentum tensor cannot contain a massless particle of spin 2. This rules out scenarios where the graviton is a composite particle and space is fundamentally flat.

Note. The reason the classical Yang-Mills equations are not relevant to everyday life is best seen in the "theoretical" normalization, where the Yang-Mills action has a prefactor $1/g^2\hbar$. Then the classical limit is the weak coupling limit. However, in everyday life we're looking at the low energy limit, and since Yang-Mills is asymptotically free, this is precisely where the coupling is strong!

Note. The statements made above are straightforward to prove by expanding in components, but one can show them using differential forms, in a clean index-free notation. We define

$$A' = -igA$$

and hereafter work with A', dropping the prime. We regard A as as one-form and suppress explicit wedge products. Then the transformation of A is

$$A \to UAU^{-1} - (dU)U^{-1} = UAU^{-1} + UdU^{-1}$$

Taking the exterior derivative of both sides,

$$dA \to (dU)AU^{-1} + U(dA)U^{-1} - UAdU^{-1} + (dU)\left(dU^{-1}\right)$$

where we picked up a minus sign from anticommuting d and A. Similarly, we have

$$A^2 \to UA^2U^{-1} + UA\left(dU^{-1}\right) + U\left(dU^{-1}\right)UAU^{-1} + U\left(dU^{-1}\right)U\left(dU^{-1}\right)$$

This may be simplified by noting that $0 = d\left(UU^{-1}\right) = U\left(dU^{-1}\right) + (dU)U^{-1}$, giving

$$A^2 \to UA^2U^{-1} + UA\left(dU^{-1}\right) - (dU)AU^{-1} - (dU)\left(dU^{-1}\right)$$

Adding these expressions, we see that

$$F \to UFU^{-1}, \quad F = dA + A^2$$

In this notation, the covariant derivative is

$$D = d + A, \quad F = D^2$$

Furthermore, the Bianchi identity is simply $DF = 0$. To prove this, note that the A in D acts on F by commutator, so

$$DF = dF + AF - FA = d\left(A^2\right) + AdA + A^3 - (dA)A - A^3 = 0$$

where we expanded $d\left(A^2\right) = (dA)A - AdA$. In this notation, the usual kinetic term is $\mathrm{tr}(F \star F)$, and the theta term is $\mathrm{tr}\, F^2$. Expanding the theta term, we have

$$\operatorname{tr} F^2 = \operatorname{tr}\left((dA)(dA) + A^2 dA + (dA)A^2 + A^4\right)$$

Due to the trace, the final term vanishes while the second and third term are in fact equal. This is easiest to see by expanding explicitly in components. For instance,

$$\operatorname{tr} A^4 = A^a_\mu A^b_\nu A^c_\rho A^d_\sigma \operatorname{tr}\left(T^a T^b T^c T^d\right) dx^\mu dx^\nu dx^\rho dx^\sigma$$

which vanishes because the trace is symmetric under cyclic permutations while the wedge product on the right is antisymmetric. In fact, we have

$$\operatorname{tr} F^2 = d \operatorname{tr}\left(AdA + \frac{2}{3}A^3\right)$$

where the quantity differentiated is the Chern-Simons current. This result will play a role in the theory of anomalies below.

Faddeev-Popov Quantization

First, we discuss some of the challenges associated with quantizing the theory.

• In canonical quantization, we again find that

$$\Pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$$

as in QED, so it is a non-dynamical field which enforces the constraint

$$D_i F^{0i} = 0$$

which is the non-abelian generalization of Gauss's law.

• As in QED, we can enforce the constraints using Dirac brackets, as we did in Coulomb gauge. Alternatively, we can quantize naively and then restrict our Hilbert space to a 'physical' Hilbert space, as we did in the Gupta-Bleuler method. However, these methods are more involved for non-abelian gauge theory because the constraints are more complicated.

• Again as in QED, we still have a gauge symmetry. The gauge symmetry must always be fixed to define the quantum theory in the first place; for instance, with a gauge symmetry the propagator is not defined since the kinetic term is not invertible.

• Unlike for QED, we will first quantize using the path integral, which is much easier. To handle first-class constraints/gauge redundancies, we restrict the path integral to integrate over only one point in each gauge orbit. Note that in Yang-Mills, there are no second-class constraints, so we don't have to worry about them.

Next, we formally introduce the Faddeev-Popov method.

• Let $\mathcal{A}$ be the space of all gauge fields $A_\mu(x)$, and let H be the space of gauge transformations. The physically inequivalent gauge field configurations live in the quotient space $\mathcal{A}/H$, where we identify

$$A_\mu \sim A^h_\mu, \quad A^h_\mu = h A_\mu h^{-1} + \frac{i}{g}\left(\partial_\mu h\right) h^{-1}$$

Thus we want to define the path integral partition function as

$$\mathcal{Z} = \int\limits_{\mathcal{A}/H} d\mu[A] e^{iS[A]}$$

but this is computationally inconvenient, since $\mathcal{A}/H$ is complicated. In the Faddeev-Popov method, we cleverly insert the identity to equate this path integral with another path integral over all of $\mathcal{A}$.

• Specifically, note that the naive path integral measure $\mathcal{D}A$ over $\mathcal{A}$ factors as

$$\int\limits_{\mathcal{A}} \mathcal{D}A = \int\limits_{H} d\mu[h] \int\limits_{\mathcal{A}/H} d\mu[A]$$

Here, we will take $\mu[h]$ to be the Haar measure over H, which is shift-invariant,

$$\mu[g(x)h(x)] = \mu[h(x)]$$

• Now consider a Lie-algebra valued gauge fixing function F, such as $F(A) = \partial_\mu A^\mu$, so that $F(A) = 0$ once on every gauge orbit, say $F\left(A^{h_0}\right) = 0$. Then we have

$$1 = \int\limits_{H} d\mu[H] \delta\left[F\left(A^h\right)\right] M\left(A^h\right), \quad M\left(A^h\right) = \det \frac{\partial F\left(A^h\right)}{\partial h}$$

which holds in analogy with the delta function identity

$$1 = \int dx \delta[f(x)] \left|\frac{\partial f}{\partial x}\right|$$

• Inserting the identity, we have

$$\int\limits_{\mathcal{A}} \mathcal{D}A = \int\limits_{H} d\mu[h] \int\limits_{\mathcal{A}} \mathcal{D}A \delta\left[F\left(A^h\right)\right] M\left(A^h\right)$$

Next, we make the nontrivial assumption that we can find a measure $\mathcal{D}A$ so that

$$\mathcal{D}A = \mathcal{D}A^h$$

which means the gauge symmetry is not anomalous. Upon relabeling, we thus have

$$\int\limits_{\mathcal{A}} \mathcal{D}A = \left(\int\limits_{H} d\mu[h]\right) \int\limits_{\mathcal{A}} \mathcal{D}A \delta[F(A)] M(A)$$

• Equating this to our other expression for $\int\limits_{\mathcal{A}} \mathcal{D}A$ and canceling a factor of the volume of H, we find that the partition function is

$$\mathcal{Z} = \int\limits_{\mathcal{A}} \mathcal{D}A \delta[F(A)] M(A) e^{iS[A]}$$

where we used the fact that the action was gauge-invariant.

• Next, we find a more explicit expression for M. Using the shift-invariance of the Haar measure we can bring the delta function peak to the identity,

$$1 = \int\limits_H d\mu[h]\delta\left[F\left(\tilde{A}^h\right)\right] M\left(\tilde{A}^h\right), \quad \tilde{A} = A^{h_0}$$

so that the peak is at $h(x) = e$. Expanding $h = e^{i\alpha}$, we get

$$1 = \int\limits_{\mathfrak{h}} d\alpha\delta\left[F\left(\tilde{A}^\alpha\right)\right] M\left(\tilde{A}^\alpha\right), \quad M\left(\tilde{A}^\alpha\right) = \det\frac{\partial F\left(\tilde{A}^\alpha\right)}{\partial\alpha}$$

for a suitably normalized Lie algebra measure $d\alpha$. Using the chain rule,

$$\frac{\partial F\left(\tilde{A}^\alpha\right)}{\partial\alpha} = \frac{\partial F\left(\tilde{A}^\alpha\right)}{\partial\tilde{A}^\alpha}\frac{\partial\tilde{A}^\alpha}{\partial\alpha} = \frac{\partial F\left(\tilde{A}^\alpha\right)}{\partial\tilde{A}^\alpha_\mu}D_\mu$$

• Finally, using the gauge invariance of $\mathcal{D}A$ and $S[A]$, we have the partition function

$$\mathcal{Z} = \int\limits_{\mathcal{A}} \mathcal{D}A\delta[F(A)]\det\Delta_{\mathrm{FP}}e^{iS[A]}, \quad \Delta_{\mathrm{FP}} = \frac{\partial F(A)}{\partial A_\mu}D_\mu$$

Note that the covariant D_μ acts on the Lie algebra, in the adjoint representation. Similarly, we may define expectation values of gauge invariant operators.

Note. In the absence of matter, the measure is, formally, automatically invariant under gauge transformations. Infinitesimally we have

$$A'^a_\mu = A^a_\mu + \partial_\mu\alpha^a + igf^{ade}A^d_\mu\alpha^e.$$

The Jacobian factor is

$$\det\frac{\partial A'^a_\mu}{\partial A^b_\nu} = \det\left(\delta^\mu_\nu\delta^a_b + ig\delta^\mu_\nu f^{abc}\alpha^c\right)$$

where the determinant is over both color and Lorentz indices. Using $\det(1+A) = 1 + \mathrm{tr}(A) + O\left(A^2\right)$,

$$\det\frac{\partial A'^a_\mu}{\partial A^b_\nu} = 1 + igf^{aac}\alpha^c = 1$$

by the antisymmetry of the structure constants, so we pick up no Jacobian at all.

Note. In more mathematical language, we begin with a principal G-bundle $P \to M$ and pick a 'base-point' connection A_0. Then any connection can be written as

$$A = A_0 + \delta A, \quad \delta A \in \mathcal{A}$$

The space $\mathcal{A}$ is an affine space, i.e. a vector space without an origin, and there is a natural inner product on it, obtained by integration over the manifold along with contraction by the Killing form, giving a natural path integral measure.

We want to integrate over connections on the bundle, but note that $\mathcal{A}$ counts a connection and the very same connection with a different local trivialization as distinct. Hence we want to integrate over $\mathcal{A}/\mathcal{G}$ where $\mathcal{G}$ is the space of all gauge transformations; note this is much larger than

G. The space $\mathcal{A}/\mathcal{G}$ is much more complicated than $\mathcal{A}$, and it is unclear how to define a measure on it.

The purpose of the Faddeev-Popov procedure is to write the desired path integral over $\mathcal{A}/\mathcal{G}$ in terms of a path integral over all of $\mathcal{A}$ but with a delta function, which is easier to handle. Many sources describe this procedure essentially in reverse, starting with a naive path integral over all of $\mathcal{A}$ and then "factoring out the volume of $\mathcal{G}$ ".

Note. Gauge fixing is more complicated in the non-abelian case. It's difficult to show that Lorenz gauge can even be attained. A worse problem is the Gribov ambiguity: generically gauge orbits intersect the gauge slice more than once. To see this, note that $\mathcal{A}$ is a principal bundle over $\mathcal{A}/\mathcal{G}$ with structure group $\mathcal{G}$, and a gauge slice is equivalent to a global section. Thus we require the bundle to be trivial,

$$\mathcal{A} \cong \mathcal{B} \times \mathcal{G}$$

Since $\mathcal{A}$ is a vector space, it has trivial homotopy groups. However, it can be shown that $\mathcal{G}$ doesn't, giving the result. We won't worry about the 'Gribov copies' since we'll only work perturbatively.

At this point, we have a path integral with a $\mathcal{D}A$ measure, but it contains inconvenient extra factors. We now perform a few more tricks to absorb these terms into the action.

• First, we rewrite the delta function as an action contribution by introducing a new field, called the Nakanishi-Lautrup field. The simplest way to do this is

$$\delta[F(A)] = \int \mathcal{D}B^a(x) e^{i\int dx B^a(x)F^a(A)}$$

Since $F(A)$ is Lie-algebra valued, $B(x) = B^a(x)T^a$ is as well and hence transforms in the adjoint; it is a bosonic scalar. The auxiliary field just acts like a Lagrange multiplier. Note that B^aF^a is not gauge-invariant and we shouldn't expect it to be, because we are using the delta function to do gauge fixing.

• Next, we can simplify the determinant using the general formula

$$\int d\theta^* d\theta e^{\theta_i^* M_{ij}\theta_j} = \det M$$

Since the Faddeev-Popov matrix acts on the Lie algebra, we have

$$\det \Delta_{\mathrm{FP}} = \int \mathcal{D}\bar{c}\mathcal{D}c e^{-i\int dx \bar{c}^a(x)(\Delta_{\mathrm{FP}}c(x))^a}$$

where c and $\bar{c}$ are $\mathfrak{h}$-valued Grassmann fields, called Faddeev-Popov ghosts and anti-ghosts, which transform in the adjoint. Note that we have added a factor of $-i$ to Δ_{FP}. This is allowed since it merely multiplies $Z[J]$ by a phase and will ensure a canonical kinetic term for the ghosts in the gauge we'll use below.

• The ghosts are fermionic scalars, violating the spin-statistics theorem; this is allowed because the theorem assumes a positive-definite norm, but the ghost states do not have one. Their role is to cancel the unphysical polarizations of the Yang-Mills field: if the A_μ field has 4 polarizations, the ghosts heuristically have -2 because of their fermionic statistics, which cause them to contribute oppositely in loops.

• Now, the partition function is

$$\mathcal{Z} = \int \mathcal{D}[A, B, \bar{c}, c] e^{iS[A,B,\bar{c},c]}, \quad S = \int dx \left(-\frac{1}{4} \left(F^a_{\mu\nu} \right)^2 + B^a F^a(A) + \bar{c}^a \left(\Delta_{\text{FP}} c \right)^a \right)$$

which is the desired form. However, we can make it even more convenient.

• Our gauge fixing condition was $F(A) = 0$, but we could also have required $F(A(x)) = f(x)$ for any $\mathfrak{h}$-valued function $f(x)$. We may also integrate over f with a Gaussian weight,

$$\delta[F(A)] \to \int \mathcal{D}f \delta[F(A) - f] e^{-\frac{i}{2\xi} \int dx f^a(x) f^a(x)}$$

The point of this manipulation is that we do not need an auxiliary field; we may simply perform the integral over f, using the delta function, to get a contribution to the action quadratic in F,

$$\mathcal{Z} = \int \mathcal{D}[A, \bar{c}, c] e^{iS[A,\bar{c},c]}, \quad S = \int dx \left(-\frac{1}{4} \left(F^a_{\mu\nu} \right)^2 - \frac{1}{2\xi} F^a F^a + \bar{c}^a \left(\Delta_{\text{FP}} c \right)^a \right)$$

• Alternatively, we can express the delta function using the auxiliary field as before, then integrate over f. This gives

$$\mathcal{Z} = \int \mathcal{D}[A, B, \bar{c}, c] e^{iS[A,B,\bar{c},c]}, \quad S = \int dx \left(-\frac{1}{4} \left(F^a_{\mu\nu} \right)^2 + B^a F^a + \frac{\xi}{2} B^a B^a + \bar{c}^a \left(\Delta_{\text{FP}} c \right)^a \right)$$

Both of these results are commonly used, and integrating out B in the latter gives the former.

• Specializing to $F(A) = \partial_\mu A^\mu$, we have

$$\Delta_{\text{FP}} = \partial^\mu D_\mu$$

so the former action becomes

$$\mathcal{Z} = \int \mathcal{D}[A, \bar{c}, c] e^{iS[A,\bar{c},c]}, \quad S = \int dx \left(-\frac{1}{4} \left(F^a_{\mu\nu} \right)^2 - \frac{1}{2\xi} F^a F^a + \bar{c}^a \left(-\partial^\mu D_\mu c \right)^a \right)$$

This is known as R_ξ gauge.

• We now revisit QED. In R_ξ gauge, we recover the gauge-fixed QED action we had postulated earlier. The covariant derivative D_μ in the adjoint representation is simply ∂_μ, giving a determinant factor of $\det(-\partial^2)$. This is independent of the gauge field, so we factor it out as a constant; this is why we didn't see ghosts in QED. However, we can encounter ghosts in QED if we use other gauges.

• The choice $\xi = 0$ is called Landau gauge, while $\xi = 1$ is called Feynman gauge; we often use it by default. In the path integral, it's conceptually clear the results do not depend on ξ, since ξ just parametrizes how we do the gauge fixing.

Example. The axial gauge, $F(A) = n^\mu A_\mu$. In this case we have

$$\Delta_{\text{FP}} = \frac{\partial F(A)}{\partial A_\mu} D_\mu = n^\mu D_\mu$$

The ghosts automatically decouple, because when we use the delta function, the dependence on Δ_{FP} on A_μ via $n^\mu A_\mu$ vanishes. Then $\det \Delta_{\text{FP}}$ is simply a constant which can be ignored. The downside is that the gauge explicitly breaks Lorentz invariance.

Alternatively, we can perform the same integration over f as above, giving

$$\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 - \frac{1}{2\xi}\left(n^\mu A_\mu\right)^2 - \bar{c}n^\mu D_\mu c$$

In this case it is straightforward, if a bit tedious, to invert the quadratic part of the Lagrangian to find the propagator for A_μ, which gives

$$i\Delta^{\mu\nu}_{ab}(k) = \frac{i\delta_{ab}}{k^2 + i\epsilon}\left(\eta^{\mu\nu} - \frac{k^\mu n^\nu + n^\mu k^\nu}{k\cdot n} - \frac{\xi k^2 - n^2}{(k\cdot n)^2}k^\mu k^\nu\right)$$

Note that the interaction vertex between the ghosts and gauge boson contains the factor $n_\mu A^\mu$, and hence is proportional to

$$n_\mu \Delta^{\mu\nu}_{ab}(k) = -\frac{\xi}{k\cdot n}\delta_{ab}k^\nu$$

Hence in the limit $\xi \to 0$ where the gauge condition is exactly imposed, all diagrams where a ghost attaches to a gluon vanish, so the ghosts decouple as we'd expect. A useful special case of this gauge is light cone gauge, the limit $\xi \to 0$ with $n^2 = 0$, giving

$$i\Delta^{\mu\nu}_{ab}(k) = \frac{i\delta_{ab}}{k^2 + i\epsilon}\left(\eta^{\mu\nu} - \frac{k^\mu n^\nu + n^\mu k^\nu}{k\cdot n}\right)$$

Canonical Quantization of Yang-Mills theory

• Motivated by the path integral treatment above, we take the gauge-fixed Lagrangian

$$\mathcal{L} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 - \partial^\mu B^a A^a_\mu + \frac{\xi}{2}B^a B^a + \partial^\mu \bar{c}^a D_\mu c^a$$

where the derivatives only act on the field immediately to their right.

• The field momenta are

$$\Pi^{a\mu}_A = \frac{\partial\mathcal{L}}{\partial\dot{A}^a_\mu} = F^{a\mu 0}, \quad \Pi^a_c = \frac{\partial\mathcal{L}}{\partial\dot{c}^a} = -\dot{\bar{c}}^a, \quad \Pi^a_B = \frac{\partial\mathcal{L}}{\partial\dot{B}^a} = -A^a_0, \quad \Pi^a_{\bar{c}} = \frac{\partial\mathcal{L}}{\partial\dot{\bar{c}}^a} = \dot{c}^a - gf^{abc}A^b_0 c^c$$

The conjugate momentum of A^0 still vanishes, but this is just because we have the auxiliary field B. We can eliminate B by plugging in its equations of motion.

• Next, we perform canonical quantization, imposing the usual commutation relations

$$[A_\mu(t,\mathbf{x}), \Pi_{\nu A}(t,\mathbf{y})] = i\eta_{\mu\nu}\delta(\mathbf{x}-\mathbf{y})$$

Then by direct computation, we find for spacelike j and k,

$$\left[A^a_j(t,\mathbf{x}), \dot{A}^b_k(t,\mathbf{y})\right] = i\delta^{ab}\delta_{kj}\delta(\mathbf{x}-\mathbf{y})$$

Meanwhile, we have

$$\left[A^a_0(t,\mathbf{x}), B^b(t,\mathbf{y})\right] = i\delta^{ab}\delta(\mathbf{x}-\mathbf{y})$$

• Since the ghosts were Grassmann variables, we impose the anti-commutation relations

$$\left\{c^a(t,\mathbf{x}), \Pi^b_c(t,\mathbf{y})\right\} = \left\{\bar{c}^a(t,\mathbf{x}), \Pi^b_{\bar{c}}(t,\mathbf{y})\right\} = i\delta^{ab}\delta(\mathbf{x}-\mathbf{y})$$

We also define

$$c^{a\dagger} = c^a, \quad (\bar{c}^a)^\dagger = -\bar{c}^a$$

which ensures that $\mathcal{L} = \mathcal{L}^\dagger$, yielding a unitary S-matrix.

• At this point we would usually construct the Fock space, but Yang-Mills is an intrinsically interacting theory. Thus the spectrum will contain complicated bound states, e.g. hadrons and glueballs. On the other hand, to compute S-matrix elements it suffices to define 'in' and 'out' states via the LSZ reduction formula. These asymptotic states are free fields, which are fully renormalized but have gauge coupling $g = 0$.

• Therefore, we instead consider the free Lagrangian

$$\mathcal{L}_0 = -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 - \partial^\mu B^a A^a_\mu + \frac{\xi}{2} B^a B^a + \partial^\mu \bar{c}^a \partial_\mu c^a$$

We set $\xi = 1$ and integrate out B^a using the equation of motion $B^a = -\partial^\mu A^a_\mu$, giving

$$\left[A^a_\mu(t,\mathbf{x}), \dot{A}^b_\nu(t,\mathbf{y})\right] = -i\delta^{ab}\eta_{\mu\nu}\delta(\mathbf{x}-\mathbf{y})$$

just as we had in QED.

• The gauge field and ghost field have free mode expansions,

$$A^a_\mu(x) = \int \frac{d\mathbf{k}}{\sqrt{2E_k}} \sum_\lambda \epsilon^a_\mu(\mathbf{k},\lambda)\left(a^a_\lambda(\mathbf{k})e^{-ikx} + a^{a\dagger}_\lambda(\mathbf{k})e^{ikx}\right), \quad \epsilon(\mathbf{k},\lambda)\cdot\epsilon\left(\mathbf{k},\lambda'\right) = \eta_{\lambda\lambda'}$$

and

$$c^a(x) = \int \frac{d\mathbf{p}}{\sqrt{2E_p}} c^a(\mathbf{p})e^{-ipx} + c^{a\dagger}(\mathbf{p})e^{ipx}$$

with the commutation relations

$$\left[a^{a\dagger}_\lambda(\mathbf{k}), a^b_{\lambda'}\left(\mathbf{k}'\right)\right] = \delta^{ab}\eta_{\lambda\lambda'}\phi\left(\mathbf{k}-\mathbf{k}'\right), \quad \left\{c^a(\mathbf{p}), \bar{c}^{b\dagger}\left(\mathbf{p}'\right)\right\} \propto \left\{\bar{c}^a(\mathbf{p}), c^{b\dagger}\left(\mathbf{p}'\right)\right\} \propto \delta^{ab}\delta\left(\mathbf{p}-\mathbf{p}'\right)$$

Thus we have negative norm and zero norm gluon states, as well as zero norm ghost states.

• The physical Hilbert space H_{phys} must have a positive-definite norm. In addition, the S-matrix must be unitary when restricted to H_{phys} , i.e. the unphysical states must decouple. This is much more difficult to guarantee, but it follows if we define H_{phys} using a symmetry of the full interacting theory, as in this case the S-matrix respects this symmetry by assumption.

• We cannot use gauge symmetry for this purpose, because we had to fix the gauge to quantize at all. In QED, we used the $U(1)$ global symmetry, which gave the Ward-Takahashi identities. But in a non-abelian gauge theory, this is complicated by the ghost and auxiliary fields. Instead, we will use the more subtle BRST symmetry.

SU(3) **калибровки по Мандлу Шаулу**

We next generalize the SU(3) transformations (11.22) from global to local phase transformations. The development will be very similar to that for QED in Section 11.1. In order to retain

invariance under local phase transformations, we shall have to introduce gauge fields, and this will automatically generate the interactions.

The first step is to replace the global SU(3) transformations (11.21) by the local phase transformations

$$\left.\begin{array}{l}\Psi f(x) \to \Psi^f(x) = \exp\left[\mathrm{i}g_s\lambda_j\omega_j(x)/2\right]\Psi^f(x) \\ \bar{\Psi}^f(x) \to \bar{\Psi}^{f\prime}(x) = \bar{\Psi}^f(x)\exp\left[-\mathrm{i}g_s\lambda_j\omega_j(x)/2\right]\end{array}\right\}.$$

Here $\omega_j(x)(j = 1, 2, ..., 8)$ are arbitrary real differentiable functions and g_s is a real constant which will later be identified with a coupling constant.

The free-field Lagrangian density (11.20) is not invariant under the transformation (11.26), but transforms as

$$\mathscr{L}_0 \to \mathscr{L}_0' = \mathscr{L}_0 - \frac{1}{2}g_s\bar{\Psi}^f(x)\lambda_j\not{\partial}\omega_j(x)\Psi^f(x)$$

for infinitesimal $\omega_j(x)$. As we shall see, an invariant Lagrangian density can be obtained if, in analogy with Eqs. (11.7) and (11.8) in QED, we replace $\partial^\mu\Psi^f(x)$ in Eq. (11.20) by the covariant derivative

$$D^\mu\Psi f(x) = \left[\partial^\mu + \mathrm{i}g_s\lambda_j A_j^\mu(x)/2\right]\Psi^f(x)$$

so that $\mathscr{L}_0$ goes over to

$$\mathscr{L}_q(x) = \bar{\Psi}^f(x)\left[\mathrm{i}\not{D} - m_f\right]\Psi^f(x) = \mathscr{L}_0 + \mathscr{L}_\mathrm{I}$$

where

$$\mathscr{L}_\mathrm{I} = -\frac{1}{2}g_s\bar{\Psi}^f(x)\gamma_\mu\lambda_j\Psi^f(x)A_j^\mu(x)$$

7.1.2 BRST Symmetry

BRST symmetry is a fermionic global symmetry of the interacting, gauge-fixed Yang-Mills Lagrangian, which roughly corresponds to gauge symmetry when applied to the gauge field alone. It is theoretically useful because it allows us to recover some of the consequences of gauge symmetry, even though the Lagrangian is gauge-fixed.

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} + \frac{\xi}{2}B^a B^a + B^a\partial_\mu A_\mu^a + \bar{c}^a\left(-\partial^\mu D_\mu c\right)^a$$

We define a Grassmann odd operator S:

$$SA_\mu = -D_\mu c = -\left(\partial_\mu c + ig\left[A_\mu, c\right]\right), \quad Sc = \frac{i}{2}gf^{abc}c^b c^c t^a \equiv \frac{g}{2}[c,c], \quad S\bar{c} = -B, \quad SB = 0$$

$Sc \neq 0$, since c is Grassmann odd. Sc is not just the conjugate of $S\bar{c}$. We regard c and $\bar{c}$ as independent real Grassmann fields, not conjugates of each other.

• Note that S obeys a graded Leibniz rule: if c is Grassmann odd,

$$S(cA) = (Sc)A - c(SA)$$

We now show that $S^2\Phi = 0 \forall \Phi \in \{A, c, \bar{c}, B\}$. This is obvious for B and $\bar{c}$, and we have

$$S^2 c \propto t^a f^{abc}f^{bde}c^c c^d c^e \propto t^a\left(\sum_{\mathrm{cde}}^{\mathrm{cyclic}} f^{abc}f^{bde}\right)c^c c^d c^e = 0$$

by the Jacobi identity. The proof that $S^2 A_\mu = 0$ is similar, but more complicated.

• Now, for a product of any two fields, we have

$$S^2(\Phi_1\Phi_2) = S(S\Phi_1\Phi_2 \pm \Phi_1 S\Phi_2) = \mp S\Phi_1 S\Phi_2 \pm S\Phi_1 S\Phi_2 = 0$$

where we used $S^2\Phi_i = 0$ and the fact that S flips the Grassmann parity. Similar logic applies for any product of fields, which implies S is nilpotent, $S^2 = 0$.

From direct expansion:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - S\left(\bar{c}^a\partial^\mu A^a_\mu + \frac{\xi}{2}\bar{c}^a B^a\right)$$

$$S\mathcal{L} = 0,$$

where the first term vanishes by gauge invariance and the second by nilpotence.

We define the BRST symmetry by

$$\delta_\epsilon\Phi = \epsilon S\Phi$$

where ϵ is a constant independent Grassmann number, so $\delta_\epsilon\mathcal{L} = 0$. The BRST transformation preserves the Grassmann parity, and obeys the Leibniz rule without grading.

$$J^\mu = \sum_I \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_I)}\delta_\epsilon\Phi_I, \qquad \partial_\mu J^\mu = 0, \qquad Q = \int d\mathbf{x}J^0, \qquad \dot{Q} = 0$$

• Upon canonical quantization, one can show that we find a BRST charge operator $\hat{Q}^\dagger = \hat{Q}$ which generates the BRST transformation, by

$$[\hat{Q}, \hat{X}] = iS\hat{X}$$

where the bracket is a commutator or anticommutator depending on whether $\hat{X}$ is bosonic or fermionic. The conservation of $\hat{Q}$ is expressed as $[\hat{H}, \hat{Q}] = 0$, and the nilpotency of the BRST transformation implies

$$\hat{Q}^2 = 0$$

• The continuous symmetries of the Yang-Mills Lagrangian are Lorentz invariance, global gauge invariance, BRST invariance, ghost number (a $U(1)$ assigning $+1$ charge to c^a and -1 to $\bar{c}^a$), and anti-ghost translation invariance $\bar{c} \to \bar{c} + \chi$. In fact, the Lagrangian already contains all terms consistent with these symmetries whose coefficients have nonnegative mass dimension. Upon renormalization, BRST symmetry will ensure that the separate appearances of g renormalize in the same way, providing identities analogous to $Z_1 = Z_2$ in QED.

• There are some minor variations we can consider. We can easily include matter fields, which, like the gauge field, transform as under a gauge transformation with gauge parameter $\alpha = -\epsilon c$,

$$\delta\psi_i = -ig\epsilon c^a t^a_{ij}\psi_j$$

It is straightforward to show that $S^2\psi_i$ as well.

• We could also integrate out the B field, in which case the only change is

$$S\bar{c} = \frac{1}{\xi}F^a = \frac{1}{\xi}\partial_\mu A^\mu$$

in R_ξ gauge. However, to show that $S^2\bar{c} = 0$, we have to use the equation of motion for the ghost field. This is a general phenomenon: if we eliminate auxiliary fields, then often symmetries that held off-shell will only hold on-shell, reducing their power.

We now use the BRST operator to define a cohomology and physical Hilbert space.

• Since $\hat{Q}^2 = 0$, it defines a cohomology.

• An element $|\psi\rangle \in \ker \hat{Q}$ is called $\hat{Q}$-closed.

• An element $|\psi\rangle \in \operatorname{im} \hat{Q}$ is called $\hat{Q}$-exact. All exact states are closed.

• For $|\psi\rangle$ exact, note that

$$\langle\psi|\psi\rangle = \left\langle \chi \left| \hat{Q}^\dagger \hat{Q} \right| \chi \right\rangle = \left\langle \chi \left| \hat{Q}^2 \right| \chi \right\rangle = 0$$

Then all exact states are null. More generally, all exact states are orthogonal to all closed states, and any two closed states differing by an exact state have the same norm. Physically, an exact state is gauge-equivalent to vacuum.

• We define the $\hat{Q}$-cohomology by

$$\operatorname{cohom}(\hat{Q}) = \frac{\ker(\hat{Q})}{\operatorname{im}(\hat{Q})}$$

Then there is a well-defined inner product on $\operatorname{cohom}(\hat{Q})$.

• Consider the variation of the time evolution operator

$$\hat{U} = \exp\left(i \int \Pi \dot{\phi} - \mathcal{L} dt \right), \quad \mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - S\psi$$

under a change in the gauge fixing condition $F(A) = 0$. Only S depends on F, so

$$\delta_F \langle\alpha|\hat{U}|\beta\rangle = \langle\alpha\,|i\delta_F S\psi|\,\beta\rangle = \langle\alpha\,|iS\delta_F\psi|\,\beta\rangle = \left\langle \alpha \left| \hat{Q}\delta_F\psi - \delta_F\psi\hat{Q} \right| \beta \right\rangle$$

For this to vanish for all $\delta_F\psi$, we require $\hat{Q}|\beta\rangle = 0$. Since the time-evolution of physical states should not depend on the gauge choice, the physical Hilbert space must be BRST closed.

• Within this space, the BRST exact states have zero overlap with all other states and hence can never be measured. Thus we identify $H_{\text{phys}} = \operatorname{cohom}(\hat{Q})$.

• We can explicitly find the physical Hilbert space for the free in/out states. The states are

$$\left|A^a_\mu(\mathbf{k})\right\rangle = \sum_\lambda \epsilon^a_\mu(\mathbf{k},\lambda) a^{a\dagger}_\lambda(\mathbf{k})|0\rangle, \quad |c^a(\mathbf{k})\rangle = c^{a\dagger}(\mathbf{k})|0\rangle, \quad |\bar{c}^a(\mathbf{k})\rangle = \bar{c}^{a\dagger}(\mathbf{k})|0\rangle$$

and we don't have separate B states since $B \sim \partial A$. The BRST transformation is

$$SA_\mu = -\partial_\mu c, \quad Sc = 0, \quad S\bar{c} = -B = -\partial A, \quad SB = 0$$

where we work in Feynman gauge.

• Therefore, the BRST charge acts as

$$\hat{Q}\,|A^{\mu a}(\mathbf{k})\rangle = \alpha k^\mu\,|c^a(\mathbf{k})\rangle\,, \quad \hat{Q}\,|c^a(\mathbf{k})\rangle = 0, \quad \hat{Q}\,|\bar{c}^a(\mathbf{k})\rangle = \beta k^\mu \left|A^a_\mu(\mathbf{k})\right\rangle$$

where $\alpha, \beta \neq 0$. Since $\hat{Q}^2 = 0$, we have $k^2 = 0$. We now drop the a index.

• The states that are BRST closed must have no $\bar{c}$ excitations, and moreover

$$|\xi\rangle = \xi_\mu\,|A^\mu(\mathbf{k})\rangle$$

is only closed if $\xi^\mu k_\mu = 0$, removing one of the unphysical polarizations.

• Note that c excitations are BRST exact, as is $k^\mu |A_\mu(\mathbf{k})\rangle$. Therefore, the physical one-particle Hilbert space consists of states

$$|\psi\rangle = \xi^\mu |A_\mu(\mathbf{k})\rangle , \quad k^2 = 0, \quad \xi^\mu k_\mu = 0$$

where $\xi^\mu \sim \xi^\mu + k^\mu$. These are indeed the two degrees of freedom we want.

• Next, we check unitarity of the S-matrix. Since $\mathcal{L}^\dagger = \mathcal{L}$, the S-matrix is unitary on the entire Fock space. Letting $|\gamma\rangle$ be a basis,

$$\sum_\gamma \left\langle \alpha \left| S^\dagger \right| \gamma \right\rangle \langle \gamma | S | \beta \rangle = \langle \alpha | \beta \rangle$$

The nontrivial thing to check is that it is unitary on H_{phys} ,

$$\sum_{\chi_T} \left\langle \phi_T \left| S^\dagger \right| \chi_T \right\rangle \langle \chi_T | S | \psi_T \rangle = \langle \phi_T | \psi_T \rangle , \quad |\phi_T\rangle , |\psi_T\rangle \in H_{\text{phys}}$$

where $|\chi_T\rangle$ is a basis for H_{phys} .

• Since $\hat{Q}$ commutes with $\hat{H}$, it commutes with S, so

$$\hat{Q} S |\psi_T\rangle = S \hat{Q} |\psi_T\rangle = 0$$

Therefore, S maps $\ker \hat{Q}$ to itself. However, it can and does produce BRST exact states. These don't spoil unitarity because such states are orthogonal to BRST closed states. Explicitly,

$$\langle \phi_T | \psi_T \rangle = \sum_\gamma \left\langle \phi_T \left| S^\dagger \right| \gamma \right\rangle \langle \gamma | S | \psi_T \rangle = \sum_{\chi_T} \left\langle \phi_T \left| S^\dagger \right| \chi_T \right\rangle \langle \chi_T | S | \psi_T \rangle$$

as desired. Here, γ ranges over physical states, BRST exact states, and states that are not BRST closed, and the latter two don't contribute by the arguments above.

• In the special case of QED in Lorenz gauge, the ghosts automatically decouple, and the constraints above reduce to the Gupta-Bleuler condition $\partial A^+ |\psi\rangle = 0$. The difference in the non-abelian case is that such a constraint is not preserved by time evolution, as we can create BRST exact states.

• Accordingly, the Ward identity $k_\mu \mathcal{M}^\mu = 0$ does not hold in the non-abelian case, and this is related to the nonexistence of a conserved current. Instead, we use the BRST current to construct Slavnov-Taylor identities, which can be used to prove decoupling just like the Ward identities. The Slavnov-Taylor identities also ensure the gluon remains massless, as the Ward identities did for the photon.

• To see the role of the ghosts, note that the optical theorem relates a loop amplitude to a production amplitude squared, where the latter sums over physical external particles. Only two gauge boson polarizations are physical, but all four run around the loop. To maintain the optical theorem and hence unitarity, the ghosts also run around the loop with an extra factor of -1, canceling the unphysical polarizations. In the abelian case, the ghosts aren't necessary because the unphysical polarizations contribute nothing to loop amplitudes.

7.1.3 Perturbative Renormalization, asymptotic freedom

Next, we show the non-abelian gauge theories are asymptotically free. We continue to follow the Peskin and Schroeder conventions.

• By the same reasoning as in QED, the gauge boson propagator is

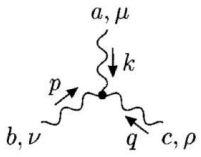
$$D_F^{\mu\nu ab}(p) = \frac{-i\delta^{ab}}{p^2 + i\epsilon}\left(\eta^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)$$

The propagator is diagonal in color space. Incoming gauge bosons get $\epsilon_\mu^a(\mathbf{k}, \lambda)$ and outgoing gauge bosons get $\epsilon_\mu^{*a}(\mathbf{k}, \lambda)$.

• Next, in our convention the gauge boson interaction terms are

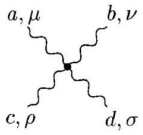
$$\mathcal{L} \supset g f^{abc}\eta^{\nu\rho}\partial^\mu A_\nu^a A_\mu^b A_\rho^c - \frac{g^2}{4} f^{abc} f^{ade}\eta^{\mu\rho}\eta^{\nu\sigma} A_\mu^b A_\nu^c A_\rho^d A_\sigma^e$$

For the cubic Feynman rule, note that every interaction comes with an automatic factor of i, while the derivative contributes $-ip^\mu$ for incoming momentum p. There are six distinct contractions, yielding:



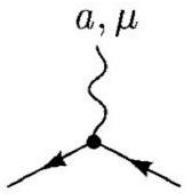
$$= gf^{abc}\big[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\big]$$

• As for the quartic interaction, there are 4 ! ways to perform the contractions. It turns out that they come in identical groups of 4, canceling the 1/4 factor and giving six terms:



$$= -ig^2\big[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\big]$$

• Next, when we include matter fields in a representation t^a, they have the interaction



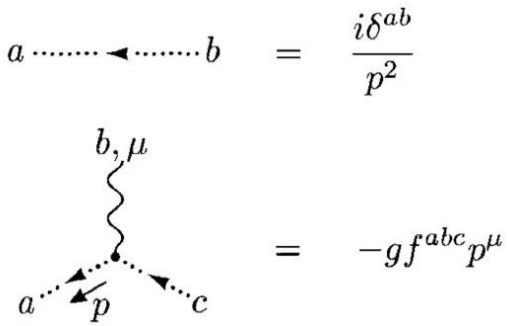
$$= ig\gamma^\mu t^a$$

where again our convention's sign is flipped. The propagator is the same as usual, with an extra δ_{ij} in flavor space.

• Finally, we consider the ghost fields, where

$$\mathcal{L} \supset \bar{c}^a\left(-\partial^\mu D_\mu c\right)^a = \bar{c}^a\left(-\partial^2\delta^{ab}\right)c^b + g f^{abc}\bar{c}^a\partial^\mu\left(A_\mu^b c^c\right)$$

The resulting Feynman rules are shown below.



The propagator is just the usual one, but it has a direction because the ghosts are fermionic; also note that a ghost loop contributes a factor of -1. To get the ghost vertex, we integrate by parts to put the derivative on $\bar{c}^a$. Also note that here, p is outgoing rather than incoming.

Next, we briefly discuss the "group theory" factors that appear in amplitudes.

• In QED, we saw that the spinor-related parts of amplitudes could be constructed by following every fermion line, then writing the entries right-to-left. Similarly, in QCD, we get color factors by following the quark lines, with a generator t^a every time a gluon attaches. The ordering is the same: reading matrix indices along the fermion line will give them ordered right-to-left.

• We also get color factors from gluons, since they themselves are colored. These yield products of structure factors, which are essentially matrix products in the adjoint representation, since $(t_A^a)^{bc} = if^{abc}$.

• We define the Dynkin index and quadratic Casimir as

$$\mathrm{tr}\left(t_R^a t_R^b\right) = T(R)\delta^{ab}, \quad t_R^a t_R^a = C(R)$$

with implicit summation over repeated indices and an identity in the second result. Conventionally, we normalize the generators so that

$$T(F) = \frac{1}{2}$$

which implies that

$$C(F) = \frac{N^2-1}{2N} = \frac{4}{3}, \quad T(A) = C(A) = N = 3$$

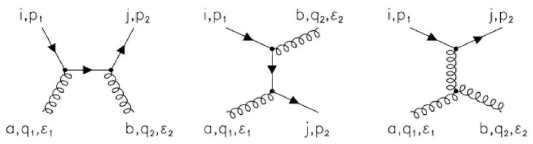
Here F stands for the fundamental representation.

• We also often get factors of the dimensions of these representations,

$$d(F) = N = 3, \quad d(A) = N^2 - 1 = 8$$

Further rules for computing these constants are given in the lecture notes on Group Theory, but these will suffice for our purposes.

Example. The process $qg \to qg$ at tree level. There are three diagrams:



which are in the s-channel, u-channel, and t-channel respectively. We'll focus on the color factors, since the kinematics are quite complicated. Setting the quark mass to zero for simplicity, we have

$$\mathcal{M}_s = g^2 \bar{u}_{s'}(p_2) \not{\epsilon}_2^* \frac{i(\not{p}_1 + \not{p}_2)}{s} \not{\epsilon}_1 u_s(p_1) t^b_{jk} t^a_{ki}$$

and

$$\mathcal{M}_u = g^2 \bar{u}_{s'}(p_2) \not{\epsilon}_1 \frac{i(\not{p}_1 - \not{q}_2)}{u} \not{\epsilon}_2^* u_s(p_1) t^a_{jk} t^b_{ki}$$

and

$$\mathcal{M}_t = g^2 \bar{u}_{s'}(p_2) \gamma^\mu u_s(p_1) \left(\frac{-i}{t}\right) f^{abc} t^c_{ji} \epsilon_2^{*\rho} \epsilon_1^\nu \left(\eta^{\nu\rho}(q_1+q_2)^\mu + \eta^{\rho\mu}(q_1-2q_2)^\nu + \eta^{\mu\nu}(q_2-2q_1)^\rho\right)$$

where repeated color indices are not summed. Upon squaring, and summing over final spins and averaging over initial spins, we get complicated expressions involving traces of up to eight gamma matrices, which can be simplified using the Clifford algebra. Ignoring this, we get, for example,

$$|\mathcal{M}_s|^2 \sim \left(t^b t^a\right)_{ji} \left(t^b t^a\right)^*_{ji} = \left(t^b t^a\right)_{ji} \left(t^b t^a\right)^\dagger_{ij} = \left(t^b t^a\right)_{ji} \left(t^a t^b\right)_{ij}$$

Summing over final colors and averaging over initial colors, and keeping only the color factor,

$$|\mathcal{M}_s|^2 \sim \frac{1}{3}\frac{1}{8} \operatorname{tr}\left(t^b t^a t^a t^b\right) = \frac{1}{24} d(F) C(F)^2 = \frac{2}{9}$$

As another example, we have

$$|\mathcal{M}_t|^2 \sim \frac{1}{24} f^{abc} f^{abd} \operatorname{tr}\left(t^c t^c\right) = \frac{1}{24} T(F) f^{abc} f^{abc} = \frac{1}{24} T(F) C(A) d(A) = \frac{1}{2}$$

As a final example,

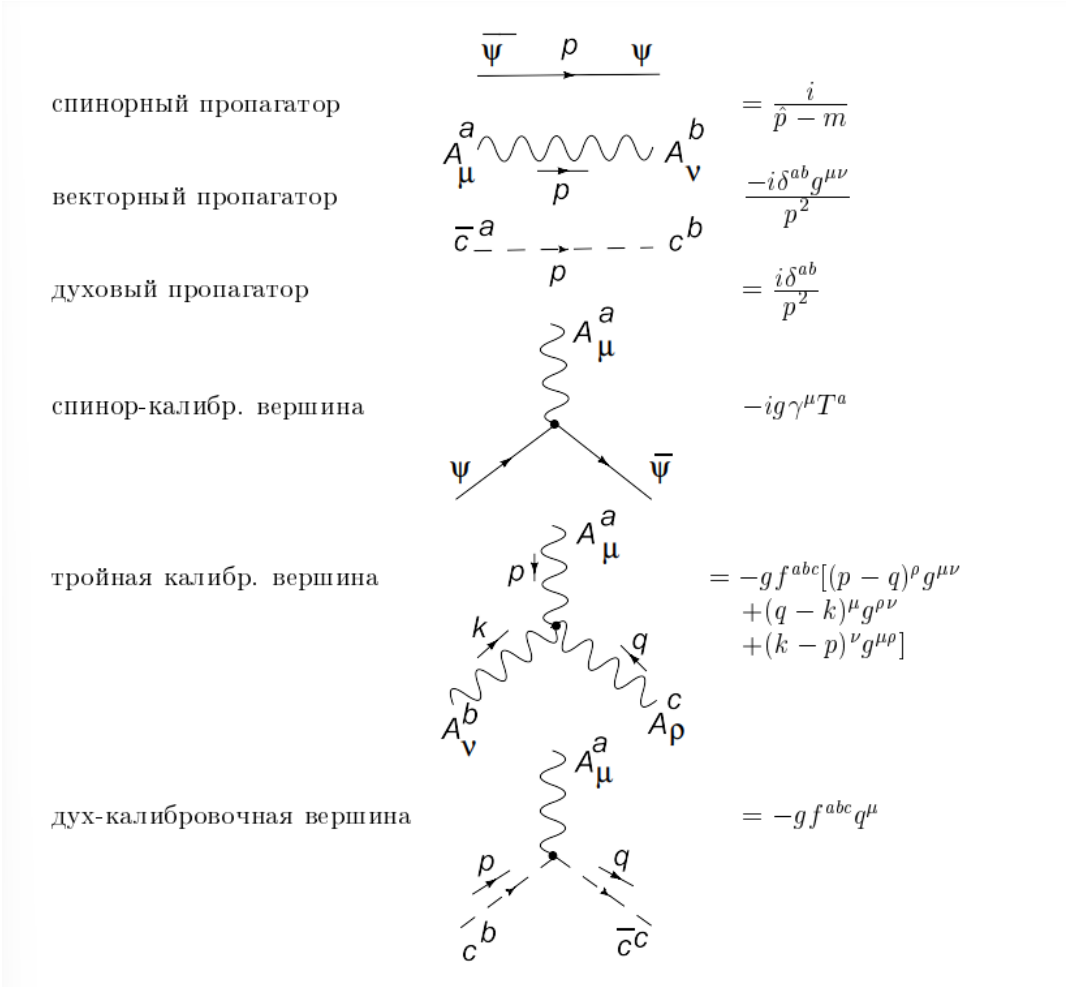
$$\mathcal{M}_t \mathcal{M}_s^* \sim \frac{1}{24} f^{abc} \operatorname{tr}\left(t^a t^b t^c\right) = \frac{1}{48} f^{abc} \operatorname{tr}\left(t^a \left[t^b, t^c\right]\right) = \frac{i}{48} f^{abc} f^{bcd} \operatorname{tr}\left(t^a t^d\right) = \frac{i}{48} C(A) d(A) T(F) = \frac{i}{4}$$

Now we proceed to computing the beta function of Yang-Mills.

7.1.4 QCD: main laws and ideas, quarks, confinement

Основные формулы и особенности КХД

$$\mathcal{L}_{\text{QCCD}} = -\frac{1}{4}\left(F^a_{\mu\nu}\right)^2 + \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi + g\bar{\psi}\gamma^\mu A^a_\mu T^a \psi - \frac{1}{2\xi}\left(\partial_\mu A^a_\mu\right)^2 + \partial_\mu \bar{c}^a \partial_\mu c^2 + g f^{abc} \partial_\mu \bar{c}^a A^b_\mu c^c$$



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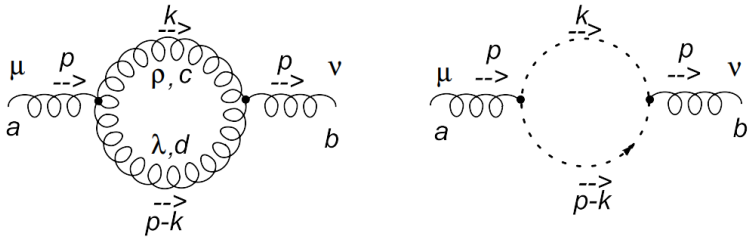


О другом

Симметрия БРСТ (On BRS symmetry)

Однопетлевые расходимости в неабелевой калибровочной теории

Асимптотическая свобода: метод фонового поля

Асимптотическая свобода: качественное объяснение

От кварков к КХД

Quantum chromodynamics is the theory defined by the Lagrangian density

$$\mathscr{L} = \bar{\Psi}^f(x)\left[\not{D} - m_f\right]\Psi^f(x) - \frac{1}{4}G_{i\mu v}(x)G_i^{\mu v}(x)$$

obtained by combining Eqs. (11.28) and (11.34) for the quark and gluon terms, respectively. It incorporates the minimal interactions needed to make the free quark Lagrangian (11.20) gauge invariant. As in QED, it is easy to add more complicated gauge-invariant terms to the Lagrangian, for example,

$$\mathrm{i}'_s\bar{\Psi}^f(x)\Psi^f(x)G_{i\mu v}(x)G_i^{\mu v}(x).$$

However, such terms are excluded by the criterion of renormalizability, as we shall show in Section 11.3.

In contrast to QED, QCD describes a strong interaction and lowest-order perturbation theory is not, in general, a good approximation. We will defer further discussion of this until later.

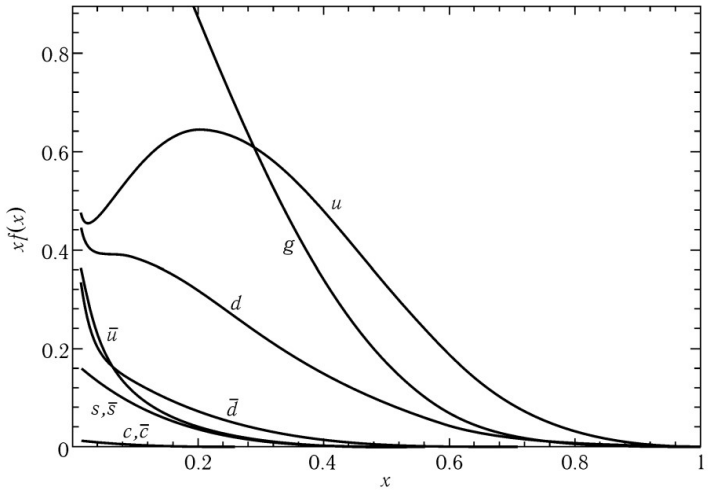
For the moment, we will summarize some important physical properties of QCD which can be inferred directly from the Lagrangian density (11.35). We start by considering the gluons. On substituting Eq. (11.33) in Eq. (11.34) one obtains

$$\begin{aligned}\mathscr{L}_G = &-\frac{1}{4}F_{i\mu v}(x)F_i^{\mu v}(x) + g_s f_{ijk}A_{i\mu}(x)A_{jv}(x)\partial^\mu A_k^v(x)\\ &-\frac{1}{4}g_s^2 f_{ijk}f_{ilm}A_j^\mu(x)A_k^v(x)A_{l\mu}(x)A_{mv}(x)\end{aligned}$$

for the purely gluonic contribution to the Lagrangian density.

О протоне (uud)

(у ПШ есть формулы про это, я пока тупой.)



$$\int_0^1 dx x \left[f_u(x) + f_d(x) + f_{\bar{u}}(x) + f_{\bar{d}}(x) + f_g(x) \right] = 1$$

Функции распределения кварков и антикварков, извлеченные из данных по глубоконеупругим рассеяниям, дают только половину значения этого интеграла. Остальные импульсы должны уносить глюоны.

(??? абзац про то, откуда вообще $SO(3)$???? хз)

are three-dimensional analogs of the Pauli matrices (A.62). [3] The operators $\hat{F}_i (i = 1, 2, ..., 8)$ are called colour charges and satisfy commutation relations of the form

$$\left[\hat{F}_i, \hat{F}_j \right] = \mathrm{i} f_{ijk} \hat{F}_k$$

where a sum over repeated colour indices $k = 1, 2, ..., 8$ is understood here and in what follows. The structure constants f_{ijk} are numbers which are easily determined using Eq. (11.11). They are totally antisymmetric

$$f_{ijk} = -f_{jik} = f_{jki} = -f_{kji} = f_{kij} = -f_{ikj}$$

(потом выгружу, пока лень)

О конфайнменте (!??!?!?)

(важный вопрос, напишу потом)

Качественно, обзорно и упрощенно о процессах КХД (!?!??!?!?)

(то, что на 3 курсе нам рассказывали, важная заготовка!!!! пока ничего не помню.)

7.1.5 Partons, Deep Inelastic Scattering, Drell–Yan Process

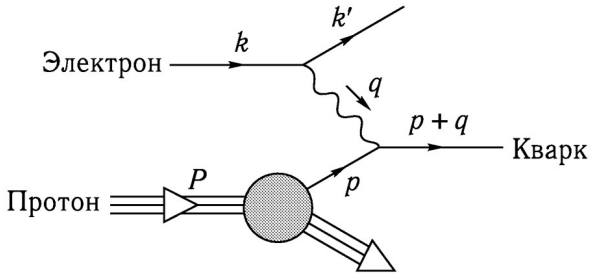
Аннигиляция e^+e^- в адроны

Об испускании глюонов (??)

$$\frac{d\sigma_{(e^+e^-\to q\bar{q}g)}}{dx_1dx_2} = \sigma_0 \cdot \left(3\sum_f Q_f^2\right) \cdot \frac{2\alpha_s}{3\pi}\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}$$

Глубоконеупругое рассеяние

О глубоконеупругом рассеянии электрона (рассеянии электрона на протоне)



$f_f(\xi)$ - партонная функция распределения:

$$\left(\begin{array}{c}\text{вероятность найти партон } f\\ \text{с продольной долей } \xi\end{array}\right) = f_f(\xi)d\xi.$$

и вообще тут партонная модель. (????)

$$\frac{d\sigma}{d\widehat{t}}\left(e^-q\to e^-q\right) = \frac{2\pi\alpha^2Q_f^2}{\widehat{s}^2}\left[\frac{\widehat{s}^2+\widehat{u}^2}{\widehat{t}^2}\right]$$

Кинематика (????)

$$\widehat{s}+\widehat{t}+\widehat{u}=0.$$
$$\widehat{t}=-Q^2$$
$$\widehat{s}=2p\cdot k=2\xi P\cdot k=\xi s.$$

В итоге:

$$\frac{d\sigma}{dQ^2} = \int\limits_0^1 d\xi\sum_f f_f(\xi)Q_f^2\frac{2\pi\alpha^2}{Q^4}\left[1+\left(1-\frac{Q^2}{\xi s}\right)^2\right]\theta\left(\xi s-Q^2\right)$$

(вводим новые параметры про которые долго писать) через них

$$\frac{d^2\sigma}{dxdy}\left(e^-p\to e^-X\right) = \left(\sum_f xf_f(x)Q_f^2\right)\frac{2\pi\alpha^2s}{Q^4}\left[1+(1-y)^2\right]$$

тут имеем соотношение Каллана-Гросса (??????)

(в ПШ много глубоких мыслей)

О глубоконеупругом рассеянии нейтрино (рассеянии нейтрино на протоне)

(?????? тут слабое взаимодействие, для него все еще нет заготовок, плохо)

$$\Delta\mathscr{L} = \frac{g^2}{2}\frac{1}{m_W^2}\left[\bar{\ell}\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\nu\right]\left[\bar{u}\gamma_\mu\left(\frac{1-\gamma^5}{2}\right)d\right] + \text{ э.с.}$$

где ℓ, ν, d, u - фермионные поля, связанные с заряженным лептоном, нейтрино, и u-кварками, а g - константа слабого взаимодействия. Множитель $1/m_W^2$ возникв ет из-за пропагатора W-бозона, рассматриваемого в пределе $q^2 \ll m_W^2$.

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}$$

$$\frac{d\sigma}{d\widehat{t}}\left(\nu d \to \mu^- u\right) = \frac{\pi g^4}{2(4\pi)^2\widehat{s}^2}\left[\frac{\widehat{s}^2}{m_W^4}\right] = \frac{G_F^2}{\pi}.$$

(проверю ее потом)

$$\frac{d\sigma}{d\widehat{t}}\left(\bar{\nu}u \to \mu^+ d\right) = \frac{\pi g^4}{2(4\pi)^2\widehat{s}^2}\left[\frac{\widehat{u}^2}{m_W^4}\right] = \frac{G_F^2}{\pi}(1-y)^2$$

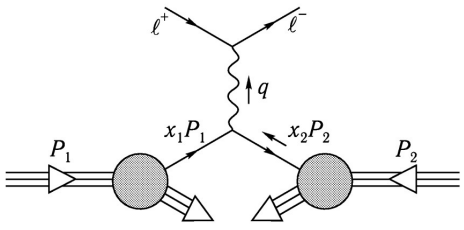
$$\frac{d^2\sigma}{dxdy}\left(\nu p \to \mu^- X\right) = \frac{G_F^2 s}{\pi}\left[x f_d(x) + x f_{\bar{u}}(x)\cdot(1-y)^2\right],$$

$$\frac{d^2\sigma}{dxdy}\left(\bar{\nu} p \to \mu^+ X\right) = \frac{G_F^2 s}{\pi}\left[x f_u(x)\cdot(1-y)^2 + x f_{\bar{d}}(x)\right].$$

Процессы жесткого рассеяния при столкновениях адронов

$$\sigma\left(p\left(P_1\right) + p\left(P_2\right) \to Y + X\right) = \int\limits_0^1 dx_1 \int\limits_0^1 dx_2 \sum_f f_f\left(x_1\right) f_{\bar{f}}\left(x_2\right)\cdot\sigma\left(q_f\left(x_1 P\right) + \bar{q}_f\left(x_2 P\right) \to Y\right),$$

Процесс Дрелла-Яна: $pp \to \ell^+\ell^-$+что-то еще. (!??!?!?!?!??)



(based on 17.4 (?????))

$$\sigma\left(q_f\bar{q}_f \to \ell^+\ell^-\right) = \frac{1}{3}Q_f^2\cdot\frac{4\pi\alpha^2}{3\widehat{s}}.$$

$$\frac{d^2\sigma}{dM^2dY}\left(pp \to e^+e^- + X\right) = \sum_f x_1 f_f\left(x_1\right) x_2 f_{\bar{f}}\left(x_2\right) \cdot \frac{1}{3} Q_f^2 \cdot \frac{4\pi\alpha^2}{3M^4}$$

$$q = x_1 P_1 + x_2 P_2 = \left(\left(x_1 + x_2\right) E, 0, 0, \left(x_1 - x_2\right) E\right)$$

$$M^2 = x_1 x_2 s$$

$$x_1 = \frac{M}{\sqrt{s}} e^Y, \quad x_2 = \frac{M}{\sqrt{s}} e^{-Y}.$$

$$\frac{d^3\sigma}{dx_1 dx_2 d\widehat{t}}(pp \to 3 + 4 + X) = f_1\left(x_1\right) f_2\left(x_2\right) \frac{d\sigma}{d\widehat{t}}(1 + 2 \to 3 + 4).$$

О другом про жесткое рассеяние по ПШ (?????)

(просто сил на это нет, мне основы ктп бы изучить сперва!!! тут еще много всего, но уже сюда хз, что буду вписывать, актуального вряд ли много будет.)

Эволюция партонов

Измерения α_s

7.1.6 On Operator Products and Effective Vertices

Перенормировка массы кварка

КХД перенормировка слабых взаимодействий

Операторное разложение

Операторный анализ e^+e^--аннигиляции

Операторный анализ глубоконеупругого рассеяния

7.1.7 On the Anomaly of Perturbation Theory

Аксиальный ток в двух измерениях

Аксиальный ток в четырех измерениях

Голдстоуновские бозоны и киральные симметрии в КХД

Киральные аномалии и киральные калибровочные теории

Аномальное нарушение масштабной инвариантности

Квантовые аномалии по Вергелесу

(когда-то засяду, дело мб 1 дня)

Диаграммный метод вычисления аксиальной аномалии по Вергелесу

Неинвариантность функциональной меры как источник аномалии по Вергелесу

7.1.8 On Gauge Theories with Spontaneous Symmetry Breaking

Механизм Хиггса

Теория слабых взаимодействий Глэшоу-Вайнберга-Салама

Симметрии теории кварков и лептонов

7.1.9 On Quantization of Spontaneously Broken Gauge Theories

R_ξ калибровки

Теорема эквивалентности голдстоуновских бозонов

Однопетлевые поправки к калибровочной теории слабого взаимодействия

7.2 Radiation Theory in a Nutshell

(это же упрощенно в теорполе, квантмехе, оптике. I expect that it is useful, thus I created a whole subsection for it. maybe I'll decrease it later)

7.2.1 Radiation of the Simplest Atoms and Nuclei, Photoelectric Effect

(это же будет в квантмехе)

49. Излучение атомов. Электрический тип

50. Излучение атомов. Магнитный тип

51. Излучение атомов. Эффекты Зеемана и Штарка

52. Излучение атомов. Атом водорода

53. Излучение двухатомных молекул. Электронные спектры

54. Излучение двухатомных молекул. Колебательный и вращательный спектры

55. Излучение ядер

О фотоэффекте

(тут абзац с основными формулами, до которых дойду потом, пока не до них.)

Фотоэффект по Киселеву (!??!?)

(да тут простые формулы, просто собрать в задачу то, к чему уже полно заготовок нужно и все)

$$|\mathrm{i}\rangle = |\boldsymbol{k}_\mathrm{i}, \lambda_\mathrm{i}\rangle\, |\psi_{1\ \mathrm{s}}\rangle$$

$$\langle \boldsymbol{r}|\psi_{1\ \mathrm{s}}\rangle = \frac{1}{\sqrt{\pi a^3}}\mathrm{e}^{-r/a}.$$

$$\hat{V}(t) = -\boldsymbol{d} \cdot \hat{\mathcal{E}} = -e\boldsymbol{r} \cdot \hat{\mathcal{E}}(\boldsymbol{r}, t),$$

$$|\mathfrak{f}\rangle = |\,\mathrm{vac}\rangle\, |\boldsymbol{k}_\mathfrak{f}\rangle\,.$$

$$\mathrm{d}w_{\mathfrak{f}\mathfrak{i}} = \frac{2\pi}{\hbar^2}\delta\left(\omega_\mathrm{i} + \omega_\mathrm{at} - \omega_\mathfrak{f}\right) |V_{\mathrm{f}\mathrm{i}}(0)|^2 \;\mathrm{d}\nu_\mathfrak{f},$$

$$\mathrm{d}\nu_\mathrm{f} = \frac{\mathrm{d}^3 k_\mathrm{f}}{(2\pi)^3},$$

$$\frac{E_\mathrm{at}}{\hbar\omega_\mathrm{i}} \ll 1 \Rightarrow \frac{\hbar^2}{2ma^2}\left(\frac{\hbar^2 k_\mathrm{f}^2}{2m}\right)^{-1} = \frac{1}{\left(k_\mathfrak{f} a\right)^2} \ll 1.$$

(???? а дальше словами нужно писать, тут прямо очень много преобразований, формулами на страницу минимум выйдет.)

$$\frac{1}{2}\sum_{\lambda_\mathrm{i}} |V_{\mathrm{f}\mathrm{i}}(0)|^2 = 2^9\pi^2\frac{\hbar^5 c}{m^2}\alpha_\mathrm{em}\frac{a}{\left(k_\mathrm{f} a\right)^6}\left(1 - \cos^2\theta\right).$$

В итоге, сечение фотоэффекта на основном состоянии атома водорода выражается через боровский радиус, постоянную тонкой структуры и отношение энергии ионизации к энергии фотона и равно

$$\sigma = \frac{8}{3}\pi a^2 \alpha_\mathrm{em}\left(\frac{4E_\mathrm{at}}{\hbar\omega_\mathrm{i}}\right)^{7/2}$$

(??? отлично, а на не атоме водорода мы абсолютно ничего не можем сказать, потому что мы и близко к этому не готовы.)

56. Фотоэффект. Нерелятивистский случай

57. Фотоэффект. Релятивистский случай

58. Фоторасщепление дейтрона

7.2.2 Dipole and Multipole Electromagnetic Radiation

(??? maybe the same will be in the field theory, I'll think later. or maybe here it will be deeper)
(по идее это углубления теорпола и квантмеха про это)

43. Оператор электромагнитного взаимодействия

44. Испускание и поглощение

О времени жизни состояний (!??!)

(типичная задача квантмеха)

45. Дипольное излучение

46. Электрическое мультипольное излучение

47. Магнитное мультипольное излучение

48. Угловое распределение и поляризация излучения

7.3 Solitons, kinks, monopoles, vortices, instantons

(также у Зи хорошо написано, поизучаю потом мб, в этой записи это уже написано.)

7.3.1 Kinks

Introduction

• Solitons are stable, spatially localized smooth solutions of classical PDEs. We will focus on their description in classical field theory. Often, we will find that they "connect distinct vacua", giving them topological stability. The solitons have a topological charge that makes them distinct from the vacuum.

• To have soliton solutions, we need nonlinear equations, which can be achieved by nontrivial interactions between fields, or by self-interaction, as in ϕ^4 theory.

• Solitons can be quantized, at which point we can interpret them as particles. For example, solitons carry energy and momentum. By Lorentz invariance, a stationary soliton with energy E_0 can be boosted to yield moving solitons, obeying

$$E^2 = p^2 + m^2$$

where the mass of the soliton is $m = E_0$. Solitons can also interact with each other like particles.

• However, solitons cannot be seen by perturbing about a vacuum state. Instead, we fix a soliton solution and treat it as a background for quantization. This is difficult, as we typically can't write down exact soliton solutions; we won't consider this subject too closely.

• Another feature that distinguishes solitons is that ordinary quantum particles have masses proportional to $\hbar$, since $E = \hbar\omega$, while the masses of solitons are independent of $\hbar$.

• We will focus on a few types of solitons, all in relativistic field theory.

• In one spatial dimension, we have kink solutions.

• In two spatial dimension, we have vortices. We will also investigate solitons in nonlinear σ models.

• In three spatial dimensions, we have Skyrmions.

Overview of Skyrmions, domain walls, solitons in cosmology

Nonrelativistic solitons include domain walls in ferromagnets and two-dimensional "baby" Skyrmions in exotic magnets. Solitons also appear in cosmology, where we can have domain walls, cosmic strings, and monopoles.

• Skyrmions are solitons in an effective field theory of interacting pions, representing the (fermionic) nucleons. They were pioneered by Skyrme, who had the strong philosophical opinion that all fermionic fields had to be emergent in this way. Though the skyrmion approach is not particularly accurate in real QCD, it was famously shown by Witten to be a good description in the large N_c limit.

• Finally, we comment on quantization. Typically, for a field whose particles have mass m and coupling g, solitons classically have energy and size

$$E \sim \frac{m}{g}, \quad \ell \sim \frac{1}{m}$$

in natural units; this will hold for our examples below.

• Then the Compton wavelength of a soliton is

$$\lambda \sim \frac{1}{E} \sim g\ell$$

Therefore, we don't expect quantization to significantly affect the soliton when the coupling is weak; we simply find perturbative corrections to E. The fact that E diverges as $g \to 0$ also indicates the solitons cannot be seen in perturbation theory.

• We can also have some insight in the strong coupling regime. For example, the sine-Gordan theory and massive Thirring model are dual, with strong coupling in one mapping to weak coupling in the other, and solitons exchanging with elementary particles. Hence there is no intrinsic difference between an elementary particle and a soliton; all that matters is which viewpoint is more convenient for calculation in a particular regime.

Next, we introduce the basic kink solution.

• We work in $1+1$ dimensions with metric $\mathrm{diag}(1,-1)$. The action is

$$S = \int d^2x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) \right)$$

The equation of motion is

$$\partial_\mu \partial^\mu \phi + \frac{dU}{d\phi} = 0$$

which is called a nonlinear Klein-Gordan equation.

• For explicit calculations, it will be useful to work 'nonrelativistically'. We define $\dot{\phi} = \partial\phi/\partial t$ and $\phi' = \partial\phi/\partial x$ with

$$L = T - V, \quad T = \int \frac{1}{2}\dot{\phi}^2 dx, \quad V = \int \frac{1}{2}\phi^2 + U(\phi) dx$$

• We are interested in static solutions, which obey $\phi'' = dU/d\phi$. We choose the potential

$$U(\phi) = \frac{1}{2}\left(1 - \phi^2\right)^2$$

so two static solutions to the field equations are the vacua $\phi(x) = \pm 1$. There are soliton solutions that connect these vacua.

• There are plenty of other choices, such as

$$U(\phi) = 1 - \cos\phi$$

which is called the sine-Gordan theory since the field equation contains $\sin\phi$. This theory is integrable, which means that we can find explicit exact solutions involving interacting solitons.

• Sticking with our original choice, the field equation is

$$\phi'' = -2\left(1-\phi^2\right)\phi$$

This is a nonlinear second order differential equation, so instead we think in terms of energy. By Noether's theorem, we have a conserved energy

$$E = \int \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\phi^2 + U(\phi)dx$$

which is simply V for a static solution. For static solutions, the action is proportional to V and is also extremized, which means that E is minimized.

• We shift $U(\phi)$ so that it is nonnegative, and its minimum value is zero. Then finite energy solutions have fixed boundary conditions $U(\pm\infty)$. Now, we define the function W by

$$U(\phi) = \frac{1}{2}\left(\frac{dW}{d\phi}\right)^2$$

Now the energy is quadratic, so we may complete the square for

$$\begin{aligned} E &= \frac{1}{2}\int \phi'^2 + \left(\frac{dW}{d\phi}\right)^2 dx \\ &= \frac{1}{2}\int \left(\phi' \mp \frac{dW}{d\phi}\right)^2 dx \pm \int \frac{dW}{d\phi}\frac{d\phi}{dx}dx \\ &= \frac{1}{2}\int \left(\phi' \mp \frac{dW}{d\phi}\right)^2 dx \pm (W(\phi(\infty)) - W(\phi(-\infty))). \end{aligned}$$

• Since the second term is fixed, the energy is minimized if the first term vanishes, so

$$\phi' = \pm\frac{dW}{d\phi}$$

while the energy of the solution is

$$E = \pm(W(\phi(\infty)) - W(-\infty))$$

where we take the positive solution. Moreover, any soliton will have at least this energy; this is an example of a so-called Bogomolny bound.

• Note that the equation $\phi'' = dU/d\phi$ for static solutions is equivalent to that of a particle of unit mass and position $x(t)$ in the potential $-U$. This is just like the situation for instantons, and provides some intuition.

• In the case of the ϕ^4 kink, we have

$$W = \phi - \frac{1}{3}\phi^3$$

Fixing the boundary conditions $\phi(\infty) = 1$ and $\phi(-\infty) = -1$, we have $E = \pm 4/3$. Since the energy is positive, we take the plus sign, so $M = 4/3$ and

$$\phi' = \frac{dW}{d\phi} = 1 - \phi^2, \quad \phi(x) = \tanh(x-a)$$

where a is a constant of integration, the position of the soliton. We thus call a a modulus (or collective coordinate) of the solution, and the set of possible moduli is the moduli space; in this case it is $\mathbb{R}$.

• We can also have moving kink solutions, by simply performing a Lorentz transformation,

$$\phi(x,t) = \tanh\gamma(x-vt)$$

It is useful to focus on slowly moving solitons. Then

$$\phi(x,t) \approx \tanh(x-a(t)), \quad a(t) = vt$$

which looks like a static solution with a time-dependent modulus.

• We can write the action in terms of the time-varying modulus in this 'adiabatic' approximation. We have $\dot{\phi} = -\dot{a}\phi'$, so

$$T = \int \frac{1}{2}\dot{\phi}^2 dx = \frac{1}{2}\dot{a}^2 \int \phi'^2 dx = \frac{1}{2}M\dot{a}^2$$

where the final result can be simply written down using Lorentz invariance or computed by an ugly integral. Alternatively, the Bogomolny equation shows that the two terms in the E integral above contribute equally, which means that the integral of ϕ^2 is just M.

• Meanwhile, V doesn't depend on $\dot{a}$, so it is simply $4/3$, and we have the particle Lagrangian

$$L = \frac{1}{2}M\dot{a}^2 - \frac{4}{3}$$

This is an 'effective field theory' style approach; we are restricting our variational problem in field space to a 'valley' given by static solutions, parametrized by $a(t)$.

• The soliton Lagrangian has the equation of motion

$$M\ddot{a} = 0, \quad a(t) = vt + a_0$$

which is exactly what we found above. Geometrically, we can think of the equation of motion as the geodesic equation on the moduli space $\mathbb{R}$, where M specifies the metric. This is trivial here, but generalizes to higher-dimensional systems and multi-soliton dynamics.

• In general, zero modes arise whenever a soliton breaks a symmetry in the Lagrangian; in this case the soliton breaks translational symmetry. Later we'll see more complex examples where a soliton breaks an internal symmetry, i.e. it carries a conserved charge.

Dynamics of Kinks

Next, we consider interactions of kinks.

• In our current theory, there aren't multi-kink solutions. However, we can ask how a solution consisting of a kink and anti-kink behaves.

• First, it's useful to look at conserved quantities. We have

$$T^\mu_\nu = \partial^\mu\phi\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}$$

The energy is the integral of T^0_0, giving the same result we found earlier. The momentum is

$$P = -\int T^0_1 dx = -\int \dot{\phi}\phi' dx$$

For our moving kink solution $\phi = \tanh(x - a(t))$ we have, by the same tricks as above,

$$P = M\dot{a}$$

which is what we would expect for a particle of mass M.

• Next, we would like to compute the interaction force between a widely separated kink and antikink, as only this regime is analytically tractable. Let the kink and antikink be located at $\pm a$, and let b be a point in between them far away from each of them; we'll show the result doesn't depend sensitively on b. We define the momentum of the kink as

$$P = -\int_{-\infty}^{b} \dot{\phi}\phi' dx$$

By conservation of energy-momentum we have

$$\partial_t T_1^0 + \partial_x T_1^1 = 0$$

• The force on the kink is defined as

$$F = \dot{P} = -\int_{-\infty}^{b} \partial_t T_1^0 dx = \int_{-\infty}^{b} \partial_x T_1^1 dx = T_1^1(b) = \left(-\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 + U\right)_{x=b}$$

• To make further progress, we need to explicitly write down ϕ. An approximate solution is

$$\phi(x) = \tanh(x + a) - \tanh(x - a) - 1 \equiv \phi_1 + \phi_2 - 1$$

as long as $a \gg 1$. We will assume that the kinks are initially at rest, $\dot{\phi} = 0$, though they will begin moving as they exert a force on each other.

• Next, at the point $b, \phi_2 - 1$ and hence ϕ_2' are small, so we expand to first order in them,

$$F = \left(-\frac{1}{2}\phi_1^2 + U(\phi_1) - \phi_1'\phi_2' + (\phi_2 - 1)\frac{dU}{d\phi}(\phi_1)\right)_{x=b}$$

The first two terms cancel out by the Bogomolny equation; physically they must cancel because a kink cannot exert a force on itself.

• Next, the field equation gives $(dU/d\phi)(\phi_1) = \phi_1''$, so we have

$$F = \left(-\phi_1'\phi_2' + (\phi_2 - 1)\phi_1''\right)_{x=b}$$

This is as far as we can go without using the explicit solution.

• To keep the calculations manageable we use the asymptotic form of tanh,

$$\phi_1(x) \sim 1 - 2e^{-2(x+a)}, \quad \phi_2 \sim 1 - 2e^{2(x-a)}$$

which is valid when x is far from both $\pm a$. The factors of 2 here are called the 'amplitude of the tail'. Plugging this in and simplifying,

$$F = 32e^{-4a}$$

where the value of b drops out.

• To understand this solution better, we restore parameters to find

$$F = 2m^2 A^2 e^{-ms}$$

where $s = 2a$ is the separation, A is the amplitude of the tail, and m is the mass of the quanta in the theory. But this looks just like the result from tree-level exchange of quanta of mass m. Physically, applying perturbation theory makes sense because the interaction happens at b, where the field is close to vacuum.

• Evidently, the kink and antikink attract, eventually colliding. We have to resort to numerics to find what happens. Generally, if the kinks are moving slowly, they annihilate into 'meson radiation'. If they are moving very quickly, they can bounce off each other.

Next, we turn to kinks in sine-Gordan theory.

• In this case, the potential has minima at multiples of 2π,

$$U(\phi) = 1 - \cos\phi, \quad \frac{dW}{d\phi} = 2\sin\frac{\phi}{2}$$

The Bogomolny equation is

$$\frac{d\phi}{dx} = 2\sin\frac{\phi}{2}, \quad \phi(x) = 4\tan^{-1}\left(e^{x-a}\right)$$

which describes a kink that interpolates between 0 and 2π, with $M = 8$.

• However, there is no solution to the Bogomolny equations which interpolates between 0 and 4π. The reason is that sine-Gordan kinks repel each other (since we know a kink and antikink attract), no matter how far apart they are. Thus the energy can be continually lowered by bringing the kinks further apart; there are no static multi-kink solutions.

• We can describe dynamical multi-kink solutions. One explicit example is

$$\phi(x,t) = 4\tan^{-1}\frac{v\sinh\gamma x}{\cosh\gamma vt}$$

Physically, the kinks move towards each other until $t = 0$, then bounce off each other, so evidently they repel. There is also a "breather" solution consisting of a kink and antikink bound together and oscillating.

• We might expect that a system of many kinks and antikinks, all with different velocities, will behave in a complicated way. Numerical simulations indicate that the result is simple; there is no energy loss due to "radiation". This is because the sine-Gordan theory is integrable.

• Since everything is periodic modulo 2π, we can choose to physically identify $\phi \sim \phi + 2\pi$. Then we have $\phi : \mathbb{R} \to S^1$ and the boundary condition is $\phi(x) = 0$ at infinity.

• For a topological approach to classifying solitons, we can compactify $\mathbb{R}$ to get $\phi : S^1 \to S^1$, and solitons are classified by their winding number, i.e. their homotopy class in $\pi_1(S^1)$. We call this winding number the topological charge Q.

• We can also take a "physics" style approach. Define the topological current

$$j^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi = \frac{1}{2\pi}\left(\partial_x\phi, -\partial_t\phi\right), \quad \epsilon^{01} = 1$$

This current is conserved by the symmetry of mixed partial derivatives. This is remarkable because it is completely independent of the field equations, making its conservation topological rather than dynamical.

• The conserved charge associated with the current is

$$Q = \int j^0 dx = \frac{1}{2\pi} \int \partial_x \phi dx$$

which is simply the topological charge Q as defined earlier.

• Finally, we can define Q geometrically. The target space S^1 has a normalized volume form

$$\int_{S^1} \omega = 1$$

where one example is $\omega = d\phi/2\pi$.

• Given a mapping $\phi : \mathbb{R} \to S^1$ we can pullback the volume form,

$$\phi^* \omega = \frac{1}{2\pi} \frac{d\phi}{dx} dx$$

and we define the degree of the map to be

$$Q = \int_{\mathbb{R}} \phi^* \omega = \frac{1}{2\pi} \int \frac{d\phi}{dx} dx$$

which agrees with the previous expressions.

Next, we briefly discuss quantization of the kink.

• At the simplest level, we can use the moduli space approximation,

$$L = \frac{1}{2} M \dot{a}^2, \quad H = \frac{P^2}{2M}$$

and hence upon canonical quantization we get the ordinary Schrodinger equation for a free particle of mass M. Stationary states take the plane wave form

$$\psi(a) = e^{ika}, \quad P = \hbar k, \quad E = \frac{\hbar^2 k^2}{2M}$$

Note that the wavefunction is on the moduli space.

• More properly, we would quantize the field about a kink solution; this is similar to quantizing a field in a curved spacetime, in that we just generalize the usual plane-wave mode expansion. For example, one can form wavepackets far from the kink that don't feel it at all, while there are also modes that are "bound to the kink", and look like the kink's shape wobbling.

• The zero-point energies of these modes provide an infinite renormalization of the kink mass. This is acceptable, since we get a finite result when we subtract it against the zero-point energy of the vacuum solution. The remainder is a finite shift to the kink mass, which is indeed small when the coupling is weak.

• This computation will be complicated by the presence of zero-frequency modes. These correspond to moduli, and we can't ignore them; they remain important even when the coupling goes all the way to zero. Accounting for the moduli alone and ignoring all other terms in the Lagrangian is exactly what we did above.

• Another way to see the weak coupling requirement is to say that the soliton is essentially unchanged by quantization if there is a length scale L, much smaller than the kink size, where the size of the quantum fluctuations is small; this is the same requirement to treat the electromagnetic field classically. One can show explicitly this is equivalent to weak coupling.

• In the case of strong coupling, the soliton typically survives in the quantum theory, but it doesn't behave anything like the classical soliton.

7.3.2 Vortices, Monopoles, Instantons

Vortices in a complex scalar field in $2+1$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^* \left(\partial^\mu \phi\right) - \frac{\lambda}{4}\left(|\phi|^2 - \frac{\mu^2}{\lambda}\right)^2,$$

the potential is minimized for $|\phi| = v = \sqrt{\mu^2/\lambda}$.

• We expect solitons on topological grounds. Letting $\phi = \rho e^{i\alpha}$ and defining the winding number

$$N = \frac{1}{2\pi}\int_C d\ell \cdot \nabla\alpha$$

we find that N takes integer values. On the other hand, it appears that N must be zero if ϕ is non-singular since we can shrink the loop to zero. Thus for N to be nonzero, there must be at least one point where ϕ vanishes. The solutions with nonzero $N = n$ are topologically stable with vorticity $n \in \mathbb{Z}$.

• To make further progress we must solve the field equations

$$\nabla^2\phi - \lambda\left(|\phi|^2 - v^2\right)\phi = 0$$

We will try the rotationally symmetric ansatz $\phi(\mathbf{x}) = f(r)e^{i\theta}$. Also, if we demand symmetry by a reflection about the x-axis followed by complex conjugation, $f(r)$ must be real.

• There is a general reason the ansatz will work. Naively, taking an ansatz makes the equations of motion overdetermined, so we generically have no solutions at all. But suppose the action is invariant under a group of transformations G, and the ansatz has the most general possible form invariant under G. A variation of the ansatz can be decomposed into a part that is G-invariant and a part that is not, where the second part averages to zero upon integration. Thus the only nontrivial equations of motion come from G-invariant variations, so the ansatz has enough parameters to generically yield a solution. A similar principle in quantum mechanics is that energy eigenstates have the symmetries of the Hamiltonian.

• However, we do need to look at variations that are not G-invariant to determine whether G invariant solutions are stable, since physically there will be external influences that slightly break the symmetry.

• Now, plugging into the equations of motion gives a single equation for the single function f,

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{f}{r^2} + \lambda\left(v^2 - f^2\right)f = 0$$

so we indeed get a solution, by the general reasoning above, with $f(0) = 0$ and $f(\infty) = v$.

• However, the solution has infinite energy, because

$$E \supset \int d^2x \frac{1}{2}|\phi|^2(\nabla\theta)^2 \sim \int \frac{dr}{r}$$

which diverges logarithmically. Such configurations could be physically relevant, i.e. we would have a finite energy per vortex if we had a finite density of vortices, but a single vortex is not physically meaningful.

It's possible to understand this result using general scaling reasoning.

• The energy of a finite-energy, static solution $\bar{\phi}$ can be written as $E = I_K + I_V$ where the gradient energy/"kinetic energy" I_K is bilinear in the first derivatives of ϕ and the potential energy I_V is an integral of $V(\phi)$.

• Now consider the scaled field $\phi(\mathbf{x}) = \bar{\phi}(\lambda\mathbf{x})$. In D spatial dimensions,

$$E(\lambda) = \lambda^{2-D} I_K + \lambda^{-D} I_V$$

Since $\bar{\phi}$ is a static solution, it locally minimizes the energy, so $dE/d\lambda = 0$ at $\lambda = 1$, giving

$$0 = (D-2) I_K + D I_V$$

This places strong restrictions on solitons.

• For $D = 1$, we get $I_K = I_V$, which is a generalization of a result we found earlier.

• For $D = 2$, we require $I_V = 0$. This rules out non-singular solitons in our model above, since ϕ must vanish at some point, so we cannot have $|\phi| = v$ everywhere. Solitons can exist in more complicated models, such as the $O(3)$ sigma model. However, then $E(\lambda) = \lambda^{2-D} I_K$ is independent of λ, so the soliton is neutrally stable against expansion and contraction.

• For $D \geq 3$, we require $I_K = I_V = 0$, so the only solution is the constant vacuum solution.

This result is a version of Derrick's theorem, and forbids simple solitons in $D > 1$.

• Solitons can evade Derrick's theorem in several ways. They can have infinite energy (as for our vortices), they can necessarily vary in time (which we won't consider), or they can be unstable or neutrally stable against compression, as we saw for $D = 2$. In this last case, however, this means we don't actually have stable solitons at all: when the soliton size shrinks close to the lattice cutoff, the topological obstruction fades away and the soliton can vanish entirely.

• Thus, in order to have stable finite-energy static solitons, we must add more structure to the theory. One example is an abelian Higgs model, where the energy is now

$$E = I_F + I_K + I_V, \quad I_F = \frac{1}{2}\int d^D x \, \mathrm{tr}\, F_{ij}^2, \quad I_K = \frac{1}{2}\int d^D x \, (D_j\phi)^* \, (D_j\phi)$$

The new term I_F transforms differently under scaling.

• We now consider the scaled fields

$$\phi(x) = \bar{\phi}(\lambda x), \quad A(x) = \lambda \bar{A}(\lambda x)$$

where the factor of λ is necessary to maintain the functional form of I_K. Then

$$E(\lambda) = \lambda^{4-D} I_F + \lambda^{2-D} I_K + \lambda^{-D} I_V$$

which is stationary at $\lambda = 1$ if

$$(D-4) I_F + (D-2) I_K + D I_V = 0$$

which allows soliton solutions in $D = 2$ and $D = 3$.

• Note that in pure Yang-Mills, we have only I_F, in which case a neutrally stable soliton exists only when $D = 4$. These solitons are just Yang-Mills instantons in spacetime dimension $d = 4$, after a Wick rotation to $D = 4$. The neutral stability indicates that instantons have any size.

We will next investigate vortices in the abelian Higgs model in $2 + 1$ dimensions.

• We have a complex scalar field ϕ and a $U(1)$ gauge potential a_μ with

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(D_\mu\phi\right)^*\left(D^\mu\phi\right) - \frac{\lambda}{4}\left(|\phi|^2 - \frac{\mu^2}{\lambda}\right)^2, \quad D_\mu\phi = \partial_\mu\phi + ieA_\mu\phi$$

The gauge transformations take the form

$$\phi(x) \to e^{ie\Lambda(x)}\phi(x), \quad A_\mu(x) \to A_\mu(x) - \partial_\mu\Lambda(x)$$

Under such a gauge transformation, the winding number transforms as

$$N \to N + \frac{e}{2\pi}\int_C d\boldsymbol{\ell}\cdot\nabla\Lambda = N$$

as long as Λ is nonsingular and single-valued, so our topological argument still works.

• Taking spatial components with $\phi = \rho e^{i\alpha}$, we find

$$\mathbf{D}\phi = e^{i\alpha}(\nabla\rho + i\rho(\nabla\alpha - e\mathbf{A}))$$

which means the gradient energy is zero at large distances if

$$\mathbf{A} = \frac{1}{e}\nabla\alpha$$

Hence the energy does not diverge as it did above.

• Evaluating the winding number by a loop at large r, for a solution with vorticity n,

$$n = \frac{1}{2\pi}\int_C d\boldsymbol{\ell}\cdot\nabla\alpha = \frac{e}{2\pi}\int_C d\boldsymbol{\ell}\cdot\mathbf{A} = \frac{e}{2\pi}\int d^2x B, \quad \Phi = \frac{2\pi n}{e}$$

so we find a quantized magnetic flux. Since the field is pure gauge at large r, the flux is localized at small r.

• We now consider a static vortex solution with $n = 1$. We work in the gauge $A_0 = 0$ and take the ansatz

$$\phi(\mathbf{x}) = ve^{i\theta}f(evr), \quad A_j(\mathbf{x}) = \epsilon_{jk}\hat{x}^k\frac{a(evr)}{er}$$

where f and a are real. This is the most general ansatz with the same symmetries we used for the global vortex case. The boundary conditions are

$$f(0) = a(0) = 0, \quad f(\infty) = a(\infty) = 1$$

which ensure the vortex is nonsingular at the origin, and has finite energy.

• There is no closed-form analytic solution, but the form of the solution is intuitive. When $|\phi| \neq 0$, the gauge field acquires a mass, so it is energetically costly to have a magnetic field. Hence it is localized near the origin, where $|\phi|$ is small. Since the magnetic energy depends on B^2, it favors a larger vortex core, but the potential for ϕ favors a smaller vortex core. The relative strength of these two effects is determined by the ratio λ/e^2.

• Also note that the same vortex solution can be written in other gauges, though it will no longer appear rotationally symmetric.

Next, we consider multiple vortices.

• We may also consider the interaction of vortices. Two vortices will interact through both the ϕ field and the A_μ field. Since the former is spin zero, it mediates a universally attractive

force, while the latter makes like-charged vortices repel. Hence at large distances, the interaction is determined by the field with the slower falloff; the vortices attract if and only if $m_\phi < m_A$, or equivalently $\lambda/e^2 < 1/2$.

• It's harder to say what will happen at smaller distances, since the problem becomes nonlinear, but numerical simulations indicate that the sign of the force only depends on whether m_ϕ/m_A is greater or less than one. When they are equal, vortices don't interact at all, and there are static solutions with multiple vortices.

• Otherwise, such static solutions don't exist, but we can still find a static solution consisting of a single vortex with $n > 1$. However, when the force between vortices is repulsive, such a vortex is not energetically stable.

• This result is important for superconductivity, which is described by the same theory, but in three dimensions; our vortices are replaced with vortex lines. Consider forcing a magnetic flux through a superconductor by applying a field; it will then be energetically favorable to form vortices.

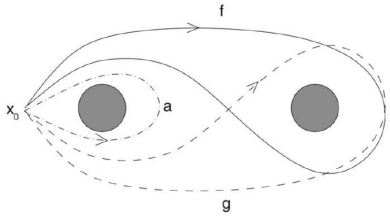
• In a type I superconductor, the superconductivity is lost at a relatively low external field, while a type II superconductor can persist up to a much higher field in a mixed state where the flux is confined to separated vortex lines. Physically, type I superconductors have $\lambda/e^2 < 1/2$, so the vortex lines attract, combining into extended regions where the superconductor breaks down. In a type II superconductor they repel, forming a lattice.

Vortices and Homotopy

• We consider a theory with symmetry group G broken down to H, so the vacuum manifold is $\mathcal{M} = G/H$. In two spatial dimensions, we consider a loop at spatial infinity (or at least well-separated from the vortex cores), which yields a loop in field space. Then naively, if this loop is not homotopic to the trivial loop, it must contain a soliton.

• However, there is a subtlety in this standard argument. When we consider homotopy of loops, we deform the loop while fixing a base point. However, when we consider the stability of solitons, we allow arbitrary variations of the field; there is no point on the loop where the field value is fixed. We are considering loops up to "free homotopy".

• As a specific example, consider the setup shown below.



The loops f and g are not homotopic, but they are related by $g \sim afa^{-1}$. Hence they are freely homotopic. More generally, the set of loops up to free homotopy is the set of conjugacy classes of $\pi_1(\mathcal{M})$.

• Note that the set of conjugacy classes is not a group! Given multiple vortices, there are multiple paths that go around all of them, whose corresponding loops lie in different conjugacy classes.

• We must also account for gauge invariance. One can show that the freedom due to gauge transformations keeps loops within the same free homotopy class, so we don't have to further modify our conclusions.

• Below, we will focus on the abelian case, where these subtleties do not appear, as all conjugacy classes have one element. In this case, the quantum numbers of vortices simply add.

Example. In the models above, the symmetry group was $U(1)$ and completely broken, so $\mathcal{M} = U(1)$, and $\pi_1(\mathcal{M})$ indexes the vortices.

Example. Consider $SO(n)$ broken to $SO(n-1)$, which occurs when a scalar field transforming in the vector representation of $SO(n)$ acquires a vev. Then $\mathcal{M} = S^{n-1}$ and for $N \geq 3, \pi_1(\mathcal{M}) = 0$, so there are no topologically stable vortices.

Example. Consider an $SO(3)$ gauge theory with two scalar fields transforming in the vector representation, with the potential

$$V(\boldsymbol{\phi}, \boldsymbol{\chi}) = \frac{\lambda_\phi}{4}\left(\phi^2 - v_\phi^2\right)^2 + \frac{\lambda_\chi}{4}\left(\chi^2 - v_\chi^2\right)^2 + g(\phi \cdot \chi)^2$$

If g is negative, the vevs are parallel, and the symmetry is broken to $U(1)$. If g is positive they must be orthogonal, so the $SO(3)$ symmetry is completely broken, so $\mathcal{M} = SO(3)$ and $\pi_1(\mathcal{M}) = \mathbb{Z}_2$. Hence we have $\mathbb{Z}_2$ vortices, where a combination of two is topologically trivial. A simple ansatz is

$$\phi = v_\phi(0,0,1), \quad \chi = v_\chi f(r)(\cos\theta, \sin\theta, 0), \quad \mathbf{A}_j = \epsilon_{jk}\hat{x}^k \frac{a(r)}{r}(0,0,1)$$

where the latter two are just an embedding of the $U(1)$ gauged vortex.

Note. Consider the case $v_\phi \gg v_\chi$ above. Then if we integrate out the ϕ field, we are left with a $U(1)$ gauge theory, and a combination of two vortices is topologically stable, even though it isn't in the original $SO(3)$ gauge theory. This is due to energetics. The deformation that takes two $\mathbb{Z}_2$ vortices to the untwisted configuration must involve a non-constant ϕ field at intermediate stages, but if $v_\phi \gg v_\chi$ this has a very large energy cost. The lesson here is that topological stability is not absolute; it can be 'effective' like everything else in field theory.

Note. There is some freedom in the gauge groups. In the previous example, we could have taken gauge group $SU(2)$, in which case a $\mathbb{Z}_2$ symmetry would remain when $g > 0$. We get the same vacuum manifold, as $\mathcal{M} = SU(2)/\mathbb{Z}_2 = SO(3)$.

Example. The electroweak sector of the Standard Model has

$$G = SU(2) \times U(1), \quad H = U(1), \quad \mathcal{M} = S^3$$

because the Higgs VEV can take any value with the same norm. Since $\pi_1\left(S^3\right) = 0$, there are no topological vortices. On the other hand, consider a theory with a local $U(1)$ symmetry but only a global $SU(2)$ symmetry. Then a vortex configuration with

$$\phi = \begin{pmatrix} 0 \\ f(r)e^{i\theta} \end{pmatrix}$$

is indeed topologically stable, as unwinding it would cost an infinite amount of $SU(2)$ gradient energy. Such a vortex is called a semilocal vortex. As before, we must have $f(0) = 0$ to avoid a singularity.

However, the semilocal vortex has a different source of instability. Recall that we can use a local $SU(2)$ symmetry to set the upper component of ϕ to zero everywhere, without loss of generality. With a global $SU(2)$ symmetry, we must consider the more general possibility

$$\phi = \begin{pmatrix} g(r) \\ f(r)e^{i\theta} \end{pmatrix}$$

The scalar potential is minimized if $f^2 + g^2 = v^2$, so a vortex may be unstable to the formation of larger and larger regions where $f(r) = 0$, as long as $g(r) = v$. By the topological arguments, the vorticity must still remain at spatial infinity, but it will become unobservable at any finite radius. This instability occurs when the $U(1)$ gauge coupling is small compared to the scalar self-coupling.

Example. Alice strings. Consider an $SO(3)$ gauge theory with a Higgs field in the 5, i.e. an traceless symmetric 3×3 matrix transforming as

$$\phi \to R\phi R^T$$

Also suppose that $V(\phi)$ is minimized when two of its eigenvalues are equal, so

$$\phi = R\,\mathrm{diag}(a, a, -2a)R^T = a\left(1 - 3\mathbf{e}\mathbf{e}^T\right), \quad \mathbf{e} = R\mathbf{e}_0$$

Then the vacuum manifold is $S^2/\mathbb{Z}_2$ with fundamental group $\mathbb{Z}_2$. To find the unbroken gauge group, note that the vacuum remains invariant under rotations about the e axis, but it is also invariant under a π rotation about the $\mathbf{e}'$ axis, where $\mathbf{e}'$ is an arbitrary vector perpendicular to $\mathbf{e}$. These operations do not commute; instead the group is

$$U(1) \rtimes \mathbb{Z}_2 = \mathrm{Pin}(2)$$

Of course, these results are consistent with the general principle $\mathcal{M} = G/H$.

Now consider a vortex solution described by $\mathbf{e}(\theta) = R(\theta)\mathbf{e}_0$ at large distances. If $R(\theta)$ traverses a nontrivial path through $S^2/\mathbb{Z}_2$, then the vortex is topologically stable. If it is present, it is not possible to define the charge of a particle under the unbroken $U(1)$ subgroup unambiguously, because it is generated by rotations about $\mathbf{e}$, and following $\mathbf{e}$ around a circle flips the sign. In order to remove the ambiguity, one may perform a gauge transformation so that $\mathbf{e}$ is constant everywhere, except for a "branch cut" running out to infinity, across which it flips. Then we may say the sign of the charge flips upon crossing the branch cut. In three dimensions, the vortices become strings, and the branch cuts become surfaces. The location of the surface is not gauge invariant, but its existence is, leading to the term "Alice strings".

Now consider two charges and an Alice string that closes on itself to make a circular loop. We consider two particles with the same charge starting next to each other; this is a physical statement, as it means that, e.g. they cannot annihilate. If we transport one of them in a circle that goes through the loop, it will come back with the opposite charge, and hence the two particles can annihilate; hence this ambiguity in the charge has a real physical effect. We can define the total charge of the system of the two charges and string at all times by a flux integral at infinity, and this must be conserved, so charge must have been transferred to the string. But since the surface associated with the string is arbitrary, it is meaningless to ask when the transfer occurred; hence we have a system where the total charge is defined but cannot be unambiguously localized!

Example. A non-abelian fundamental group. Consider the same theory as above, with a potential that is minimized when the eigenvalues of ϕ are all distinct,

$$\phi = R\,\mathrm{diag}\,(a_1, a_2, a_3)\,R^T, \quad a_1 \neq a_2 \neq a_3$$

This vacuum is invariant under only the identity and rotations by π about the x, y, and z axes, which form the Klein four group. To find $\pi(\mathcal{M})$, it is more convenient to use the universal cover $G = SU(2)$. The unbroken group H has eight elements, $\{\pm I, \pm\sigma_x, \pm\sigma_y, \pm\sigma_z\}$ forming the quaternion group. Since G is simply connected,

$$\pi_1(G/H) = \pi_0(H)$$

so the fundamental group is non-abelian.

Review of topological defects in general, in the SM and GUT theories.

• For a theory with vacuum manifold $\mathcal{M} = G/H$, the topological defects are classified by $\pi_0(\mathcal{M})$ (domain walls), $\pi_1(\mathcal{M})$ (vortices/strings), $\pi_2(\mathcal{M})$ (monopoles), and $\pi_3(\mathcal{M})$ (textures).

• The behavior of these objects changes dramatically if there is gauge symmetry, as in the SM. We call these defects with gauge symmetry "local", in contrast to "global" textures.

• Only local strings and monopoles are interesting as cosmological defects, because global strings and monopoles have infinite energy. However, collections of global strings or monopoles can have finite energy and play a role in other contexts, such as the Kosterlitz-Thouless transition.

• The global/local distinction is not relevant for domain walls; one could add a discrete gauge symmetry, but this doesn't do much because it doesn't produce a gauge field. Local textures are a bit trivial because the field lies in the vacuum manifold at all points, so the energy vanishes. Textures are instead interpreted as other vacuum states. (right? anything else?)

• In the SM, the relevant groups are $G = SU(2)_L \times U(1)_Y$ and $H = U(1)_A$, and $\mathcal{M} \cong S^3$. (This is easier to see by considering the possible states of the Higgs doublet, which takes values in $\mathbb{C}^2$.) As a result, the only possible defects are local textures, which are not interesting.

• Using results from the notes on Geometry and Topology, for the $SU(5)$ GUT, G is connected and simply connected and $H = SU(3)_C \times U(1)_A$, so

$$\pi_1(G/H) = \pi_0(H) = 0, \quad \pi_2(G/H) = \pi_1(H) = \mathbb{Z}$$

This yields monopoles, but not other topological defects. But for more complex GUTs, symmetry breaking generally occurs in multiple stages, and generically produces cosmic strings.

• Cosmologically, there are strong constraints on domain walls and (local) monopoles coming from high-scale physics; both generically contain far too much energy and must be inflated away if they appear. Cosmic (local) strings are acceptable, but CMB measurements show they cannot play a dominant role in structure formation. Global textures were once hypothesized to play a role in structure formation, but this has also been ruled out.

Instantons in Quantum Mechanics

An instanton is a classical solution of the equations of motion with finite, non-zero action. They can be used to describe transitions between vacuum states in a semiclassical approximation.

• First, consider ϕ^4 theory,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - g^2\phi^4$$

In the classical case, we are free to scale the Lagrangian. Letting $\phi' = g\phi$, we have

$$\mathcal{L} = \frac{1}{g^2}\left(\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}m^2\phi^2 - \phi'^4\right)$$

which means that g is not a physical parameter.

• In the quantum case, the scale of the Lagrangian is set by $\hbar$, so g is important; the physically relevant quantity is

$$\frac{\mathcal{L}}{\hbar} = \frac{1}{g^2\hbar}\left(\frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' + ...\right)$$

Then the relevant dimensionless parameter is $g^2\hbar$, and an expansion in $\hbar$ is the same as an expansion in g, as we've seen before.

• Our approximations will be semiclassical and hence require weak coupling. However, they will extract nonperturbative information essentially by the WKB approximation, which says that the probability of barrier penetration for a particle of unit mass is

$$|T(E)| = \exp\left(-\frac{1}{\hbar}\int_{x_1}^{x_2} dx\sqrt{2(V-E)}\right)(1+O(\hbar))$$

which is zero to all orders in $\hbar$.

• We can't imagine barrier penetration as a classical process, because the velocity would be imaginary in the classically forbidden region. But this is exactly what we would expect if we worked in Euclidean (imaginary) time. This motivates us to consider Euclidean solutions.

• Specifically, Euclidean solutions are precisely those which maximize the WKB probability of barrier penetration. To see this, consider the system

$$L = \int \frac{1}{2}\dot{q}^2 - V(\mathbf{q})$$

and consider a tunneling path $\mathbf{q}(s)$ between minima, where $ds^2 = d\mathbf{q}^2, \mathbf{q}(0) = \mathbf{q}_0, \mathbf{q}(s_f) = \mathbf{q}_f$. The tunneling amplitude is approximately $e^{-B/2}$ where

$$B = 2\int_0^{s_f} ds\sqrt{2(V(\mathbf{q}(s)) - E)}$$

• In the case where the motion is between maxima, and hence classically allowed, Jacobi's principle tells us these paths correspond to stationary points of the action. By similar reasoning, for tunneling between minima, the paths minimize the Euclidean action

$$S_E = \int_{\tau_0}^{\tau_f} d\tau\left(\frac{1}{2}\left(\frac{d\mathbf{q}}{d\tau}\right)^2 + V(\mathbf{q})\right)$$

which has the potential negated.

• The equation of motion is

$$\frac{d^2q_i}{d\tau^2} = \frac{\partial V}{\partial q_i}$$

and the particle is at rest at the start and end of the path $\overline{\mathbf{q}}(\tau)$, so

$$\frac{1}{2}\left(\frac{d\overline{\mathbf{q}}}{d\tau}\right)^2 = V(\overline{\mathbf{q}}) - E, \quad E = V(\overline{\mathbf{q}}_0)$$

• Explicitly, the Euclidean action of the path is

$$S_E[\overline{\mathbf{q}}] = \int_{\tau_0}^{\tau_f} d\tau 2\left(V(\overline{\mathbf{q}}) - V\left(\mathbf{q}_0\right)\right) + \int_{\tau_0}^{\tau_f} d\tau V\left(\mathbf{q}_0\right)$$

and the first term is simply $B/2$ upon a change of variables, so

$$\frac{B}{2} = S_E(\overline{\mathbf{q}}) - S_E\left(\mathbf{q}_0\right)$$

where $\mathbf{q}_0$ is the trivial path $\mathbf{q}_0(\tau) = \mathbf{q}_0$.

• Such a Euclidean solution is called an 'instanton' because it is centered around some point of Euclidean time. However, this is not physically related to real time; at this point it's simply some parameter related to s. Originally, instantons were called pseudoparticles.

• We may also consider tunneling out of a metastable minimum. In this case, the WKB amplitude goes across the potential barrier, but the Euclidean solution goes out and 'bounces' back in, doubling the action. Hence we have

$$B = S_E(\overline{\mathbf{q}}) - S_E\left(\mathbf{q}_0\right)$$

Physically, the amplitude for a particle in the minimum $\mathbf{q}_0$ to stay there falls exponentially in time, with the exponent being proportional to $e^{-B/2}$. Alternatively, one can say the state peaked in $\mathbf{q}_0$ has complex energy. This could be made more precise by starting with a system where $\mathbf{q}_0$ is a true minimum and analytically continuing to the system considered here.

• It is clear here that there are many possible escape paths, and we've just found the most probable one. The other reasonably likely escape paths are simply perturbed about the most probable one, and their effect on B can be found perturbatively.

Next, we'll turn to the path integral, which is better at calculating the subleading corrections.

• Setting $m = \hbar = 1$ and restricting to one dimension,

$$\left\langle x_f \left| e^{-HT} \right| x_i \right\rangle = \int_{x(-T/2)=x_i}^{x(T/2)=x_f} \mathcal{D}x e^{-S_E}, \quad H = \frac{p^2}{2} + V(x)$$

where S_E is the Euclidean action as above. Expanding in energy eigenstates, $H|n\rangle = E_n|n\rangle$,

$$\left\langle x_f \left| e^{-HT} \right| x_i \right\rangle = \sum_n e^{-E_n T} \left\langle x_f | n \right\rangle \left\langle n | x_i \right\rangle$$

so for large T, we may compute the energy and wavefunction of the lowest energy states by evaluating the path integral.

• To define the measure, we choose a set of orthonormal functions $x_n(\tau)$ vanishing at the boundaries and choose $\bar{x}$ obeying the boundary conditions; this is equivalent to the usual $\mathcal{D}x$ measure up to a constant Jacobian. Then we define the measure by

$$x(\tau) = \bar{x}(\tau) + \sum_n c_n x_n(\tau), \quad \mathcal{D}x \equiv N \prod_n \frac{dc_n}{\sqrt{2\pi}}$$

where N is a divergent normalization factor.

• In the semiclassical limit, we expand around stationary points of S. Suppose $\bar{x}$ is the only one; it obeys the classical equations of motion in the potential $-V(x)$. We let the basis functions $x_n(t)$ be eigenfunctions of the second variational derivative of S at $\bar{x}$,

$$-\frac{d^2x_n}{dt^2} + V''(\bar{x})x_n = \lambda_n x_n$$

Expanding about the stationary point, the path integral is a Gaussian, so

$$\left\langle x_f \left| e^{-HT} \right| x_i \right\rangle = Ne^{-S(\bar{x})} \prod_n \lambda_n^{-1/2}(1 + O(\hbar)) \approx Ne^{-S(\bar{x})} \det\left(-\partial_\tau^2 + V''(\bar{x})\right)^{-1/2}$$

Higher order terms here come from the higher-order variation of the action with respect to c_n.

• This result is a generalization of the usual functional determinants seen in field theory. In that case, we usually only explicitly perform path integrals when the action is quadratic, in which case the functional determinant $\det \delta^2 S|_{\bar{x}}$ is independent of $\bar{x}$ and diagonalized by plane waves.

Example. A trivial example. Let $x_i = x_f = 0$ and consider a potential with a minimum at $x = 0$ and $V(0) = 0$. Then the only solution for $\bar{x}$ is the constant $\bar{x} = 0$, where $S = 0$, so

$$\left\langle 0 \left| e^{-HT} \right| 0 \right\rangle = N \det\left(-\partial_t^2 + \omega^2\right)^{-1/2} (1 + O(\hbar)), \quad \omega = V''(0)$$

To evaluate the functional determinant, note the eigenvalues are

$$\lambda_n = \frac{\pi n^2}{T^2} + \omega^2$$

so we have

$$N \det\left(-\partial_\tau^2 + \omega^2\right)^{-1/2} = N \prod_n \left(\frac{\pi^2}{n^2} T^2\right)^{-1/2} \prod_n \left(1 + \frac{\omega^2 T^2}{\pi^2 n^2}\right)^{1/2}$$

The constant N is fixed by matching with the result for the free particle, $\omega = 0$, giving

$$N \det\left(-\partial_\tau^2 + \omega^2\right)^{-1/2} = \frac{1}{\sqrt{2\pi T}} \prod_n \left(1 + \frac{\omega^2 T^2}{\pi^2 n^2}\right)^{1/2} = \frac{1}{\sqrt{2\pi T}} \left(\frac{\sinh(\omega T)}{\omega T}\right)^{-1/2}$$

Taking the limit $T \to \infty$ and neglecting exponentially small terms, and restoring $\hbar$,

$$N \det\left(-\partial_\tau^2 + \omega^2\right)^{-1/2} = \sqrt{\frac{\omega}{\pi\hbar}} e^{-\omega T/2}$$

which implies that the ground state energy is $\hbar\omega/2 + O(\hbar^2)$, and

$$|\langle x = 0 | n = 0 \rangle|^2 = \sqrt{\frac{\omega}{\pi\hbar}}(1 + O(\hbar))$$

These are indeed the correct semiclassical results.

Next, we consider the extended example of the double well.

• Consider an even potential with minima at $\pm a$ with $V(a) = 0$ and $V''(a) = \omega^2$. We will compute $\left\langle a \left| e^{-HT} \right| a \right\rangle$ and $\left\langle a \left| e^{-HT} \right| -a \right\rangle$ semiclassically. As before, we have classical solutions that stay motionless at $\pm a$, but we also have the 'instanton' solution that moves from $-a$ to a, and the 'anti-instanton' that moves from a to $-a$.

• Since we'll take T to infinity anyway, we take this limit now. Then the instanton solutions have zero energy, and

$$\frac{dx}{d\tau} = \sqrt{2V}, \quad \tau = \tau_1 + \int_0^x \frac{dx'}{\sqrt{2V}}$$

Such a solution is called an instanton centered at τ_1. They are like solitons, but localized in Euclidean time rather than space. The action of an instanton is

$$S_0 = \int dt \frac{1}{2}(dx/dt)^2 + V = \int dt (dx/dt)^2 = \int_{-a}^{a} dx \sqrt{2V}$$

• For large times, when x approaches a, the equation of motion is

$$\frac{dx}{dt} = \omega(a - x)$$

which is exponential decay, so instantons have size on the order of $1/\omega$. Then in the $T \to \infty$ limit, we must account for multi-instanton solutions, where the instanton separation is $O(T)$, since they are approximate solutions to the equation of motion.

• Consider a solution with n widely separated instantons, centered at $t_1, ..., t_n$, with

$$T/2 > t_1 > ... > t_N > -T/2$$

The action of this solution is nS_0. The functional determinant is

$$N \det\left(-\partial_t^2 + V''(\bar{x})\right)^{-1/2} = \sqrt{\frac{\omega}{\pi}} e^{-\omega T/2} K^n$$

where the factor K accounts for how each instanton modifies the functional determinant. We may multiply by K^n because the effect of each instanton on the spectrum of fluctuations is localized about each one, and they are widely separated.

• Integrating over the instanton locations gives a factor of $T^n/n!$, so we have

$$\left\langle a \left| e^{-HT} \right| a \right\rangle = \sqrt{\frac{\omega}{\pi}} e^{-\omega T/2} \sum_{n \text{ even}} \frac{\left(K e^{-S_0} T\right)^n}{n!}$$

and carrying out the sums gives

$$\left\langle \pm a \left| e^{-HT} \right| a \right\rangle = \sqrt{\frac{\omega}{\pi}} e^{-\omega T/2} \frac{\exp\left(K e^{-S_0} T\right) \mp \exp\left(-K e^{-S_0} T\right)}{2}.$$

• Thus, we have two low-lying energy eigenstates, with energies

$$E_\pm = \frac{1}{2}\omega \pm K e^{-S_0}$$

These are the two expected energy eigenstates, consisting of even and odd combinations of the ground states of the two wells; the degeneracy is broken by barrier tunneling. Unlike the WKB approximation, the path integral allows us to calculate the preexponential factor K, and if we worked to more accuracy, we could find higher-order corrections.

• Above, we have assumed we were working with an 'instanton gas', i.e. that the instantons were too widely separated to affect each other. To check self-consistency, note that the largest term in the series $\sum_n x^n/n!$ comes when $x \sim n$, so

$$n \sim KTe^{-S_0}$$

If the instanton has characteristic size $\delta\tau$, we require

$$(\delta\tau)Ke^{-S_0} \ll 1$$

On dimensional grounds $K \sim 1/\delta\tau$, so we just need $S_0 \gg \hbar$, which we've already assumed.

• To compute the value of K more carefully, formally we would have

$$K = \left| \frac{\det\left(-\partial_\tau^2 + \omega^2\right)}{\det\left(-\partial_\tau^2 + V''(\bar{x})\right)} \right|^{1/2}$$

However, this is not well-defined because the functional determinant in the denominator has a zero eigenvalue. This zero mode results because the instanton breaks time translation invariance. Explicitly, it is $x_1 = S_0^{-1/2} d\bar{x}/dt$.

• Making the zero mode into a collective coordinate and evaluating its contribution explicitly,

$$K = \left(\frac{S_0}{2\pi}\right)^{1/2} \left| \frac{\det\left(-\partial_\tau^2 + \omega^2\right)}{\det'\left(-\partial_\tau^2 + V''(\bar{x})\right)} \right|^{1/2}$$

where the prime denotes exclusion of the zero mode. The remaining determinants can be computed using standard methods.

Example. Consider a periodic potential with minima at the integers. By similar reasoning,

$$\left\langle j_+ \left| e^{-HT} \right| j_- \right\rangle = \left(\frac{\omega}{\pi}\right)^{1/2} e^{-\omega T/2} \sum_{n,\bar{n}} \frac{1}{n!\bar{n}!} \left(Ke^{-S_0}T\right)^{n+\bar{n}} \delta_{n-\bar{n}-j_++j_-}$$

where n is the number of instantons and n' is the number of anti-instantons. Using the identity $\delta_{ab} = \int_0^{2\pi} d\theta \frac{e^{i\theta(a-b)}}{2\pi}$ we have

$$\left\langle j_+ \left| e^{-HT} \right| j_- \right\rangle = \left(\frac{\omega}{\pi}\right)^{1/2} e^{-\omega T/2} \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i(j_- - j_+)\theta} \exp\left(2KT\cos\theta e^{-S_0}\right)$$

Then the Hamiltonian is diagonalized by states of the form $|\theta\rangle$ where

$$\langle\theta|j\rangle = e^{ij\theta}, \quad E(\theta) = \frac{1}{2}\omega + 2Ke^{-S_0}\cos\theta$$

These $|\theta\rangle$ states are analogous to the " θ-vacua" we will encounter in gauge theories.

Example. Decay from a metastable vacuum. We consider a local minimum at $x = a$ with $V(a) = 0$ and wish to compute the rate of decay. Naively, the instanton solutions $\bar{x}(\tau)$ are bounces, and we can sum over any number of bounces, giving the answer

$$\left\langle a \left| e^{-HT} \right| b \right\rangle = \sqrt{\frac{\omega}{\pi}} e^{-\omega T/2} \sum_n \frac{\left(Ke^{-S_0}T\right)^n}{n!} = \sqrt{\frac{\omega}{\pi}} e^{-\omega T/2} \exp\left(Ke^{-S_0}T\right)$$

where as before,

$$K = \left(\frac{S_0}{2\pi}\right)^{1/2} \left| \frac{\det\left(-\partial_\tau^2 + \omega^2\right)}{\det'\left(-\partial_\tau^2 + V''(\bar{x})\right)} \right|^{1/2}$$

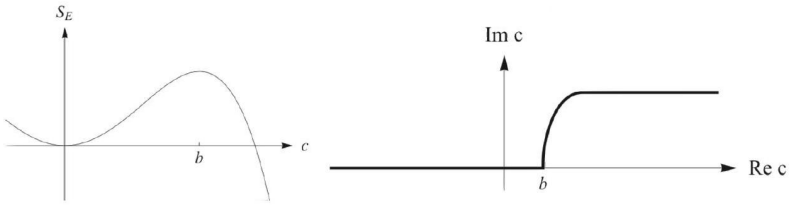
We expect the amplitude to decay exponentially, or equivalently for the state in the well to have a complex energy. Hence taking the logarithm, at large times we have

$$E_0 = \frac{1}{2}\omega - Ke^{-S_0}$$

We get the desired imaginary part because there is a mode with a negative eigenvalue, which renders K imaginary. To see this, note that the equation for the eigenfunctions x_n has the form of the Schrodinger equation, and hence the mode with lowest eigenvalue should have no nodes. However, since the instanton solution bounces, the zero mode has one node, and hence there is one lower mode. This implies that the instanton solution is merely a saddle, not a minimum, of the potential.

To understand this, suppose the 'bounce' occurs at $x = b$, and consider the family of solutions that bounce at $x = c$. Then $S_E(c)$ has a local minimum at $c = 0$, corresponding to the trivial solution, and an extremum at $c = b$ corresponding to the instanton; hence it must be a maximum.

The reason that the bounce can be a local minimum of the barrier penetration integral B but not a local minimum of S_E is that they have different boundary conditions. In our earlier calculations, both were restricted over paths that went from one minimum to the other. But in this case, we're just going from one minimum back to itself, and B has the additional constraint that the paths must bounce at $x = c$. (However, note that we must always consider minima of B, since otherwise our approximation will give a completely wrong result.)



To integrate over this mode, we must essentially perform the integral

$$J = \int_{-\infty}^{\infty} \frac{dc}{\sqrt{2\pi}} e^{-S(c)}$$

This integral seems to diverge. To get a sensible result, we deform the contour as shown above, which gives a finite result. In the semiclassical limit, the only contribution to the integral comes from the vertical region near $c = b$, where we may apply the steepest descent approximation to get

$$\operatorname{Im} J = \frac{1}{2} e^{-S(b)} \left|S''(b)\right|^{-1/2}$$

where the factor of $1/2$ is because we only have half of the Gaussian peak; this is the factor of $1/2$ we anticipated on intuitive grounds earlier. Hence

$$K = \frac{i}{2}\left(\frac{S_0}{2\pi}\right)^{1/2} \left|\frac{\det\left(-\partial_\tau^2 + \omega^2\right)}{\det'\left(-\partial_\tau^2 + V''(\bar{x})\right)}\right|^{1/2}$$

and the decay rate is

$$\Gamma = -2\operatorname{Im} E_0 = \left(\frac{S_0}{2\pi}\right) \left|\frac{\det\left(-\partial_\tau^2 + \omega^2\right)}{\det'\left(-\partial_\tau^2 + V''(\bar{x})\right)}\right|^{1/2} e^{-S_0}$$

Note. The ideas above generalize directly to field theory. For the instantons to have finite B, the deviation of the fields from their initial value must be localized in space as well as time, resulting in additional zero modes; integrating over them gives a factor of the volume of the space. There will also be additional UV divergences that must be renormalized.

It's subtle why tunneling should be permitted in field theory at all. Naively, there are infinitely many degrees of freedom that all have to tunnel at once, giving an infinite suppression; this is exactly why spontaneous symmetry breaking is possible in quantum field theory but not in quantum mechanics. The first caveat is that in $d = 1+1$, the tunneling can occur by the formation and separation of domain walls, which incurs a finite cost; this corresponds to the Mermin-Wagner theorem. The second is gauge symmetry, which allows instantons to tunnel between vacua while being locally gauge-equivalent to zero on almost all of space, avoiding an infinite energy cost.

Note. The above formalism, based on analytic continuation to a Euclidean potential, is almost universally used in quantum field theory, but physically opaque. The instanton trajectories we sum over in Euclidean time bear no resemblance to the tunneling events occurring in physical time. (As they say, "you can't eat an instanton".) For a more physical approach based on the Minkowski path integral, see this paper and this paper.

Yang-Mills Vacua

We will work primarily in $A_0 = 0$ gauge, and begin with some classical subtleties.

• We use conventions where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie\left[A_\mu, A_\nu\right], \quad D_\mu = \partial_\mu - ieA_\mu$$

and normalize generators so that $\operatorname{tr}\left(T^a T^b\right) = \delta^{ab}/2$.

• In the gauge $A_0 = 0$, the Lagrangian is

$$\mathcal{L} = \operatorname{tr} \dot{A}_j^2 - \frac{1}{2}\operatorname{tr} F_{ij}^2, \quad F_{0i} = \dot{A}_i$$

The dynamical variables are the A_i, and the conjugate momenta are

$$\Pi^j = F^{j0}.$$

Note that we are working in the non-abelian case, but we suppress group indices.

• The classical equations of motion are

$$D_\mu F^{\mu j} = 0.$$

By comparison, if we had not fixed the gauge, it would have been $D_\mu F^{\mu\nu} = 0$. Hence we are missing the Gauss's law constraint

$$C(\mathbf{x}) = D_\mu F^{\mu 0}(\mathbf{x}) = D_j \Pi^j(\mathbf{x}) = 0$$

However, the equations of motion do imply $\dot{C}(\mathbf{x}) = 0$.

• Physically, we have the freedom to make time-independent gauge transformations. If we view them as genuine symmetries, then they yield conserved charges, which correspond to the conservation of $C(\mathbf{x})$. Specifically, for a gauge transformation generated by $\Lambda(\mathbf{x})$ which vanishes at spatial infinity, the Noether charge is

$$C_\Lambda = 2\int d\mathbf{x} \operatorname{tr} D_j \Lambda(\mathbf{x}) \Pi^j(\mathbf{x}) = -2\int d\mathbf{x} \operatorname{tr} \Lambda(\mathbf{x}) D_j \Pi^j(\mathbf{x}) = -2\int d\mathbf{x} \operatorname{tr} \Lambda(\mathbf{x}) C(\mathbf{x})$$

By Noether's theorem, C_Λ is conserved for any $\Lambda(\mathbf{x})$, so $C(\mathbf{x})$ is. Hence we can just restrict to configurations where $C(\mathbf{x}) = 0$.

• The Lagrangian is in the standard $T - V$ form with quadratic kinetic energy, so going to Euclidean signature is straightforward, with

$$\mathcal{L}_E = \operatorname{tr}(\partial_4 A_j)^2 + \frac{1}{2}\operatorname{tr} F_{ij}^2, \quad x_4 = \tau$$

Conventionally, we take $\epsilon^{0123} = \epsilon^{123} = \epsilon^{1234} = 1$.

• Finally, we may include a field $A_4(x)$ which acts as a Lagrange multiplier that enforces Gauss's Law, in which case the Euclidean Lagrangian becomes

$$\mathcal{L}_E = \frac{1}{2}\operatorname{tr} F_{rs}^2, \quad r, s \in \{1, 2, 3, 4\}$$

along with the boundary conditions $F_{j4} = 0$ on the temporal boundaries. This procedure, of first eliminating A_0, then rotating to Euclidean spacetime, then introducing A_4, is a bit longer than the naive approach but avoids some subtleties.

In $A_0 = 0$ gauge, there are topologically distinct vacua.

• Vacuum configurations have $F_{\mu\nu} = 0$, which implies in our gauge that

$$A_j = \frac{i}{g} G^{-1} \partial_j G$$

where $G(\mathbf{x})$ is an element of the gauge group, which we take to be $SU(2)$.

• We assume that G tends to a constant G_∞ as $r \to \infty$, to be justified later, giving a map $S^3 \to SU(2)$. The homotopy classes of such maps are classified by $\pi_3(SU(2)) = \mathbb{Z}$.

• To see why homotopy classes are relevant, we need to distinguish between 'large' and 'small' gauge transformations. Gauge transformations map $G(\mathbf{x})$ to $\Lambda(\mathbf{x})G(\mathbf{x})$. A small gauge transformation is one that can be built from a series of infinitesimal gauge transformations, with gauge functions $\Lambda(\mathbf{x})$ that vanish at spatial infinity. Large gauge transformations instead connect vacua in different homotopy classes.

• Physically, small gauge transformations must be 'do nothing' transformations, but it is ambiguous whether large gauge transformations are; this is a choice that affects the theory. For our purposes, it won't matter; we'll choose them to not be 'do nothing' transformations for clarity.

• We can explicitly compute the homotopy class by the integral

$$N[G] = \frac{1}{24\pi^2}\epsilon^{ijk}\int d\mathbf{x} \operatorname{tr} G^{-1}\partial_i G G^{-1}\partial_j G G^{-1}\partial_k G$$

which is normalized so that if G is taken to be in the fundamental representation of $SU(2)$, with $T_a = \sigma_a/2$, then $N[G]$ is the usual winding number. In the case of gauge group $SU(2)$ only, it is also the Brouwer degree of the mapping G.

• To see this, note that near a point where G is equal to G_0,

$$G(\mathbf{x}) = G_0 \exp(i\sigma_a \Lambda_a(\mathbf{x})) \approx G_0 (I + i\sigma_a \Lambda_a(\mathbf{x}))$$

Plugging this in, we find the spatial integral of

$$-\frac{i}{24\pi^2}\epsilon^{ijk}\operatorname{tr}(\sigma_a\sigma_b\sigma_c)\,\partial_i\Lambda_a\partial_j\Lambda_b\partial_k\Lambda_c = \frac{1}{12\pi^2}\epsilon^{ijk}\epsilon^{abc}\partial_i\Lambda_a\partial_j\Lambda_b\partial_k\Lambda_c = \frac{1}{2\pi^2}\epsilon^{ijk}\partial_i\Lambda_1\partial_j\Lambda_2\partial_k\Lambda_3$$

However, this is precisely the Jacobian factor for the transformation from the spatial coordinates x_j to the group coordinates Λ_a, and $2\pi^2$ is simply the area of S^3.

• An example of a configuration with unit winding number is

$$g_1(x) = \exp\left(i\hat{r}_a\sigma_a f(r)\right)$$

where $f(r)$ is any monotonic function with $f(0) = -\pi$ and $f(\infty) = 0$.

• Next, we have the useful identity

$$N\left[G_1G_2\right] = N\left[G_1\right] + N\left[G_2\right]$$

Here the error term is

$$\frac{1}{8\pi^2}\epsilon^{ijk}\int d\mathbf{x}\,\mathrm{tr}\left(G_1^{-1}\partial_i G_1\partial_j G_2 G_2^{-2}\partial_k G_2 G_2^{-1} + \partial_i G_2 G_2^{-1}G_1^{-1}\partial_j G_1 G_1^{-1}\partial_k G_1\right)$$

and it can be shown to vanish using the product rule, the antisymmetry of ϵ^{ijk}, and the fact that the G_i tend to a constant at infinity.

• Also note that for an infinitesimal gauge transformation, the winding number is the integral of

$$\frac{1}{2\pi^2}\epsilon^{ijk}\partial_i\Lambda_1\partial_j\Lambda_2\partial_k\Lambda_3 = \frac{1}{2\pi^2}\epsilon^{ijk}\partial_i\left(\Lambda_1\partial_j\Lambda_2\partial_k\Lambda_3\right)$$

which is a total derivative, so the winding number is zero. Hence $N[G]$ is invariant under continuous deformations/small gauge transformations.

• Next, we define the current

$$j_A^\mu = \frac{g^2}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left(A_\nu\partial_\rho A_\sigma - \frac{2ig}{3}A_\nu A_\rho A_\sigma\right) = \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left(A_\nu F_{\rho\sigma} + \frac{2ig}{3}A_\nu A_\rho A_\sigma\right).$$

The current is not gauge invariant, but it has a gauge-invariant divergence,

$$\partial_\mu j_A^\mu = \frac{g^2}{16\pi^2}\,\mathrm{tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu}, \quad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

To verify this equation, note that the term quartic in A_μ on the right-hand side drops out by contraction with the Levi-Civita symbol.

• The charge associated with this current is

$$Q_A = \int d\mathbf{x} j_A^0 = \frac{g^2}{16\pi^2}\int d\mathbf{x}\epsilon^{ijk}\,\mathrm{tr}\left(A_iF_{jk} + \frac{2ig}{3}A_iA_jA_k\right)$$

Therefore, for a vacuum configuration we have

$$Q_A = N[G]$$

Next, we consider tunneling between these vacua.

• Consider a sequence of finite energy configurations $A_\mu(\mathbf{x}, t)$ interpolating between two vacuum configurations with winding numbers N_1 and N_2. We have

$$\int dx\partial_\mu j_A^\mu = \int dx\left(\partial_0 j_A^0 + \partial_i j_A^i\right)$$

which can be reduced to integrals over the bounding surface, conventionally depicted as a cylinder. The first term yields $N_2 - N_1$, while the second is a surface integral over $r = \infty$, whose only contribution in $A_0 = 0$ gauge is from the $A_j F_{0k}$ term. However, for finite energy configurations F_{0k} must fall off faster than $1/r^{3/2}$, so A_j must fall off faster than $1/r^{1/2}$, and hence this term vanishes in the limit $r \to \infty$. Thus,

$$\Delta N = \frac{g^2}{16\pi^2} \int dx \, \mathrm{tr} \, F_{\mu\nu} \tilde{F}^{\mu\nu}$$

• Note that our derivation above is agnostic over whether t is real or Euclidean time. In the context of Euclidean time, the most important vacuum tunneling solutions will be instantons. They connect vacua with N differing by 1, and hence obey

$$\int dx \, \mathrm{tr} \, F_{\mu\nu} \tilde{F}^{\mu\nu} = \frac{16\pi^2}{g^2}$$

• In the quantum theory, the state can be regarded as a wavefunctional of classical field configurations. The C_Λ generate small gauge transformations, and the Gauss's law constraint means $C_\Lambda = 0$. Hence the wavefunctional must be constant on classical vacua related by small gauge transformations, in the same way that a wavefunction with $L_z = 0$ is independent of θ. From this point on we ignore these states.

• Next, we have vacua $|n\rangle$ related by large gauge transformations, indexed by an integer n. These are not energy eigenstates, as tunneling between the vacua can occur.

• We define an operator T by

$$T|n\rangle = |n+1\rangle$$

and note that by gauge invariant, T commutes with the Hamiltonian. Then the energy eigenstates diagonalize T. They are the θ-vacua,

$$|\theta\rangle = \sum_n e^{-in\theta} |n\rangle$$

The θ-vacua cannot evolve into each other, and this property extends to Fock spaces built upon these vacua. Hence we could simply regard θ as a constant of nature.

• When we perform the path integral, we must sum over gauge field configurations with all possible winding numbers. The net effect is that living in $|\theta\rangle$ is equivalent to ignoring such configurations, at the cost of adding the term

$$\Delta \mathcal{L} = \frac{\theta g^2}{16\pi^2} \, \mathrm{tr} \, F_{\mu\nu} \tilde{F}^{\mu\nu}$$

to the Lagrangian. This is a total derivative, but has nonperturbative effects.

Note. The conclusions above depend on the gauge we're choosing. For example, in the axial gauge

$$A_3 = 0, \quad A_2(x, y, 0, t) = 0, \quad A_1(x, 0, 0, t) = 0, \quad A_0(0, 0, 0, t) = 0$$

the gauge fixing is complete; there is a one-to-one relationship between $F_{\mu\nu}$ and A_μ and hence one classical vacuum, where $A_\mu = 0$. Intuitively, this is because the first condition leaves only z-independent gauge transformations; the second condition fixes z and leaves only z-independent and y-independent gauge transformations, and so on, leaving only global gauge transformations.

In this gauge, the winding number of the vacuum is always zero. Instanton configurations still exist, connecting this vacuum to itself. However, in the surface integral of $n_\mu j^\mu_A$ considered above, only the sides of the cylinder contribute rather than the caps; the 'charge' created by an instanton flows out to spatial infinity. In this picture, the θ term is not a consequence of the θ-vacuum, but simply a parameter in the Lagrangian. (This viewpoint also holds in $A_0 = 0$ gauge, so the theta term would be the sum of a bare, Lagrangian term and the effective term from living in a θ vacuum.) The fundamental difference between this gauge and the previous one is that this one counts the large gauge transformations as 'do nothing' transformations. However, the θ term has the same consequences in both pictures.

A simple analogy is given by the Lagrangian

$$L = \frac{1}{2}B\dot{\alpha}^2 - K(1 - \cos\alpha)$$

For $\alpha \in \mathbb{R}$, it describes a particle of mass B moving in a periodic potential, and the energy eigenstates are the analogues of the θ-vacua. However, we can also regard α as an angle, in which case the Lagrangian describes a pendulum with moment of inertia B. The analogue of the θ term is

$$\Delta L = \frac{\theta}{2\pi}\dot{\alpha}$$

and appears in either case.

Yang-Mills Instantons

Next, we take a closer look at the instanton solutions. First, we take a detour into topology.

• We compactify Euclidean spacetime to S^4 and consider G-bundles over S^4. We may cover S^4 with two patches, which overlap on the "equator" S^3. Here the gauge fields are related by

$$A_p^{\mathbf{II}} = U A_p^{\mathbf{I}} U^{-1} - \frac{i}{g}\left(\partial_p U\right) U^{-1}$$

We would like to classify all topologically distinct bundles.

• First, we consider the simpler situation of a $U(1)$-bundle over S^2, where

$$A_p^{\mathbf{II}} - A_p^{\mathbf{I}} = -\partial_p \Lambda$$

on the overlap region. We define

$$I_1 = \frac{1}{4\pi}\int d^2x \epsilon^{pq} F_{pq}$$

where the x^p are coordinates on S^2. This quantity is manifestly gauge invariant.

• Next, we claim I_1 is a topological invariant, i.e. it does not change under a smooth deformation

$$A_p^{\mathbf{I}} \to A_p^{\mathbf{I}} + \delta A_p^{\mathbf{I}}, \quad A_p^{\mathbf{II}} \to A_p^{\mathbf{II}} + \delta A_p^{\mathbf{II}}$$

In particular, it is continuous from one patch to the next, i.e. on the overlap region

$$\delta A_p^{\mathbf{II}} = \delta A_p^{\mathbf{I}}$$

Now the integrand is a total derivative, and

$$\delta I_1 = \frac{1}{4\pi}\int d^2x\epsilon^{pq}\delta F_{pq} = \frac{1}{2\pi}\int d^2x\partial_p\left(\epsilon^{pq}\delta A_q\right) = 0$$

since S^2 has no boundary.

• Similarly, the value of I_1 itself is

$$I_1 = \frac{1}{2\pi}\int d^2x\partial_p\left(\epsilon^{pq}A_q\right) \equiv \int d^2x\partial_p j^p$$

However, it is incorrect to simply set this to zero, because the current varies from patch to patch; it is not gauge invariant. Instead, we split it into integrals over the two hemispheres, yielding two integrals over the equator Σ,

$$I_1 = \frac{1}{2\pi}\int\limits_{\Sigma} d\ell_p\epsilon^{pq}\left(A_q^{\mathbf{I}} - A_q^{\mathbf{II}}\right) = \frac{1}{2\pi}\int\limits_{\Sigma} d\ell_p\epsilon^{pq}\partial_q\Lambda$$

which is simply the winding number of the map $\Lambda : \Sigma \to U(1)$. Hence we have related I_1 to the winding number, a topological invariant of the fibre bundle. The integrand of I_1 is called the first Chern class/form and I_1 itself is called the first Chern number; the Chern class/form is only locally the derivative of the Chern-Simons 1-form.

• In four dimensions, we have the second Chern number

$$I_2 = \frac{g^2}{32\pi^2}\int d^4x\epsilon^{pqrs}\,\mathrm{tr}\,F_{pq}F_{rs}$$

Again, this is a topological invariant, but now we have

$$\delta A_p^{\mathbf{II}} = U\delta A_p^{\mathbf{I}}U^{-1}$$

Under a deformation we have

$$\delta I_2 = \frac{g^2}{16\pi^2}\int d^4\epsilon^{pqrs}\,\mathrm{tr}\,F_{pq}\delta F_{rs} = \frac{g^2}{8\pi^2}\int d^4x\epsilon^{pqrs}\,\mathrm{tr}\,F_{pq}D_r\delta A_s$$

where the covariant derivative is taken with respect to the unperturbed potential. Now

$$\epsilon^{pqrs}\,\mathrm{tr}\,F_{pq}D_r\delta A_s = \epsilon^{pqrs}\partial_r\left(\mathrm{tr}\,F_{pq}\delta A_s\right) - \epsilon^{pqrs}\,\mathrm{tr}\left(D_rF_{pq}\right)\delta A_s$$

and the second term vanishes by the Bianchi identity. The first term is gauge invariant, so by the same argument as before $\delta I_2 = 0$ since S^4 has no boundary.

• Next, to relate I_2 to the topology of the bundle, we have shown above that

$$\frac{g^2}{32\pi^2}\epsilon^{pqrs}\,\mathrm{tr}\,F_{pq}F_{rs} = \partial_p j_A^p$$

The current j_A^p is the Chern-Simons 3-form, and is not gauge invariant. Hence by the same argument as above, we have

$$I_2 = \int\limits_{\Sigma} dS_p\left(j_A^{\mathbf{I}p} - j_A^{\mathbf{II}p}\right)$$

where Σ is the 'equator' with topology S^3.

• An explicit computation shows that

$$j_A^{\mathbf{I}p} - j_A^{\mathbf{II}p} = \frac{1}{24\pi^2}\epsilon^{pqrs}\,\mathrm{tr}\,U^{-1}\partial_q U U^{-1}\partial_r U U^{-1}\partial_s U - \frac{ig}{8\pi^2}\epsilon^{pqrs}\partial_q\left(\mathrm{tr}\,U^{-1}\partial_q U A_s\right)$$

The second term, while not gauge invariant, is non-singular everywhere on S^3. That is, it can be computed using only the gauge field on a single patch. Then since S^3 has no boundary, the integral of this term vanishes, leaving

$$I_2 = \frac{1}{24\pi^2}\epsilon^{pqrs}\int_\Sigma dS_p\,\mathrm{tr}\,U^{-1}\partial_q U U^{-1}\partial_r U U^{-1}\partial_s U$$

As shown above, for $G = SU(2)$ is is simply the winding number that counts the number of times $SU(2)$ is covered as one goes over Σ. Hence the bundles are classified by $\pi_3\left(S^3\right)$.

• More generally, they are classified by $\pi_3(G)$. Bott's theorem states that any continuous mapping from S^3 into a simple Lie group G can be continuously deformed into a mapping into an $SU(2)$ subgroup of G. Hence our analysis holds unchanged for more general gauge groups. For a general compact Lie group, the $U(1)$ factors don't matter, while each other factor gives an independent winding number.

Finally, we return to the tricky business of justifying the boundary conditions.

• Mathematically, a gauge transformation is a fiber-preserving automorphism $P \to P$, and a large gauge transformation is one which is not connected to the identity. In particular, all gauge transformations preserve transition functions and hence the instanton number. They simply are basis changes for the fibers, which change the description of the connection.

• The transition functions between two patches are sometimes called gauge transformations, but they are completely different objects; they characterize the structure of the bundle itself and are fixed prior to the introduction of the connection.

• Now, the vacua and instantons were classified in different ways. The vacua are invariant under spatial gauge transformations; whether or not they are invariant under large spatial gauge transformations depends on the convention. The instantons, counted by the Chern number, are invariant under arbitrary continuous deformations of the gauge field, which include gauge transformations as a subset, as well as under large gauge transformations because the Chern form is gauge invariant. While instantons correspond to nontrivial fiber bundles over S^4, vacua are all trivial fiber bundles over S^3.

• In the case of instantons, one can shrink one of the patches to a very small region, which makes the transition functions singular. Naively, without the machinery of bundles, this looks like a singularity in the gauge field; one example is the Dirac string for the magnetic monopole.

• As we'll see below, we can write down instanton gauge fields on $\mathbb{R}^4$, but they will always be singular in at least one point. If this is the point at infinity, the instanton can described as the 'winding of the gauge field at long distances', allowing a sensible description without bundles. Alternatively, in the bundle language, the winding number of instanton is encoded in the transition function between this gauge field and a trivial gauge field in a topologically trivial patch containing the point at infinity.

• A more common approach in practice, which we will use below, is to take the gauge field to be constant at $x \to \infty$ but singular at $x = 0$. This is equally valid, and in the bundle language the winding would be encoded in the transition function between the gauge field in an infinitesimal patch containing the origin, and a patch containing everything else.

• The compactification to S^4 is a mathematical tool rather than a physical requirement. The physical content of an instanton is that it is a local minimum of the Euclidean action. Compactification of S^4 is only necessary to get topologically nontrivial fibre bundles, where the instanton

number is additionally invariant under all continuous deformations, but this requirement has nothing to do with the physical business of computing tunneling rates: since instantons have finite sizes, modifying boundary conditions at infinity doesn't affect the Euclidean action.

• Note that there are two definitions of 'topological' going on here. Both the Chern and ChernSimons forms are topological terms in the sense that they don't depend on the metric, but only the Chern form is invariant under arbitrary deformations of the gauge field. These two types of terms yield topological information about the spacetime manifold and bundles over it, respectively. In the special case where the bundle is the tangent bundle, we can use it to get topological information about the manifold; the Gauss-Bonnet theorem is one example.

• Instantons are sometimes mistakenly called "vacua". The confusion arises from the fact that one may take an instanton background and perturb around it to compute subleading corrections, just like one does for a vacuum.

• It is simple to justify the condition $A_\mu(\mathbf{x}) \to \text{const}$ in the physical gauge, since this is required by the uniqueness of the vacuum $A_\mu(\mathbf{x}) = 0$.

• The situation is more subtle in $A_0 = 0$ gauge. If we do not take this condition, the quantity $N[G]$ will not be an integer, intuitively because there is a singular winding 'near the point at infinity'. On the other hand, the difference of $N[G]$ for two vacua will always be an integer, because it's simply the instanton number. Hence by performing a large gauge transformation we can always set $N[G]$ to an integer, and instantons will preserve this. Hence we can really just take $A_\mu(\mathbf{x})$ to be constant without loss of generality.

We now find explicit instanton solutions on $\mathbb{R}^4$ using the 't Hooft symbols.

• The trick is to use the relation between $\mathfrak{so}(4)$ and $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Define the matrices

$$e_p = \begin{cases} i\tau_p & p = 1,2,3 \\ I & p = 4 \end{cases}, \quad e_p^\dagger = \begin{cases} -i\tau_p & p = 1,2,3 \\ I & p = 4 \end{cases}$$

where the τ_p are the usual Pauli matrices. Then for an $SO(4)$ vector V_p, define

$$V = e_p^\dagger V_p = \begin{pmatrix} V_4 - iV_3 & -V_2 - iV_1 \\ V_2 - iV_1 & V_4 + iV_3 \end{pmatrix}, \quad \det V = V_p V_p$$

Then V is a unitary matrix times a multiple of the identity. We map the other way by

$$V_p = \frac{1}{2} \operatorname{tr} V e_p$$

• Rotations of V_p can be implemented on V by

$$V \to U_L^{-1} V U_R, \quad U_L, U_R \in SU(2)$$

where we need U_L/U_R to be unitary to preserve the form of V and special unitary to preserve the determinant. Since the transformation $U_L = U_R = -I$ does nothing, we conclude

$$SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2.$$

• To compute the effect of an infinitesimal transformation, let

$$U_L = e^{-i\boldsymbol{\omega}_L \cdot \boldsymbol{\tau}}, \quad U_R = e^{-i\boldsymbol{\omega}_R \cdot \boldsymbol{\tau}}.$$

Then infinitesimally we have

$$\delta V = i\left(\boldsymbol{\omega}_L \cdot \boldsymbol{\tau}\right) V - iV\left(\boldsymbol{\omega}_R \cdot \boldsymbol{\tau}\right)$$

• By some messy but direct calculation, we can show this implies

$$\delta V_p = \left(\omega_L^j \bar{\eta}_{pq}^j + \omega_R^j \eta_{pq}^j\right) V_q$$

and the numbers η_{pq}^j are best written in terms of the matrices

$$\eta_{pq} = \eta_{pq}^j \tau_j = -i\left(e_p e_q^\dagger - \delta_{pq} I\right), \quad \bar{\eta}_{pq} = \bar{\eta}_{pq}^j \tau_j = -i\left(e_p^\dagger e_q - \delta_{pq} I\right)$$

• These matrices are antisymmetric, with the nonzero η_{pq}^j and $\bar{\eta}_{pq}^j$ fixed by

$$\eta_{ij}^k = \bar{\eta}_{ij}^k = \epsilon_{ijk}, \quad \eta_{i4}^k = -\bar{\eta}_{i4}^k = \delta_{ik}$$

This implies that η_{pq} is self-dual and $\bar{\eta}_{pq}$ is anti-self-dual,

$$\eta_{pq} = \frac{1}{2}\epsilon_{pqrs}\eta_{rs}, \quad \bar{\eta}_{pq} = -\frac{1}{2}\epsilon_{pqrs}\bar{\eta}_{rs}$$

Above, we have seen that $\bar{\eta}_{pq}$ determines the effect of U_L while η_{pq} determines the effect of U_R. This is because an antisymmetric rank two $SO(4)$ tensor transforms as $(1,0) \oplus (0,1)$ under $SU(2) \times SU(2)$. These two terms each transform trivially under one of the $SU(2)$ factors, and are self-dual and anti-self-dual.

• Finally, we will use the identities

$$\eta_{pq}\eta_{qr} = -2i\eta_{pr} - 3\delta_{pr} I, \quad \bar{\eta}_{pq}\bar{\eta}_{qr} = -2i\bar{\eta}_{pr} - 3\delta_{pr} I$$

Next, we explicitly construct the unit instanton. It is convenient to proceed without gauge fixing.

• We can bound the action by noting that $F_{rs}^2 = \tilde{F}_{rs}^2$, so

$$\int d^4x \frac{1}{2} \operatorname{tr} F_{rs}^2 = \int d^4x \frac{1}{4} \operatorname{tr}\left(F_{rs}^2 + \tilde{F}_{rs}^2\right) = \int d^4x \frac{1}{4} \operatorname{tr}\left(F_{rs} - \tilde{F}_{rs}\right)^2 + \int d^4x \frac{1}{2} \operatorname{tr} F_{rs}\tilde{F}_{rs} \geq \frac{8\pi^2}{g^2} k$$

where k is the instanton number. The unit instanton has $k = 1$ and locally minimizes the action, which means its action is $8\pi^2/g^2$ and the field strength is self-dual. In the case of $k < 0$, we may flip a sign above to conclude that anti-instantons have anti-self-dual field strength.

• The result above is a Bogomolny bound, and it is useful because imposing self-duality requires solving only a first-order differential equation, while naively we would have to solve a secondorder differential equation.

• We take the ansatz

$$A_p = \eta_{pq} x_q f\left(x^2\right)$$

with the corresponding field strength

$$F_{pq} = 2\eta_{qp} f + 2\left(\eta_{qr} x_r x_p - \eta_{pr} x_r x_q\right) f' - ig\left[\eta_{pr}\eta_{qs}\right] x_r x_s f^2, \quad f' = \frac{df}{d\left(x^2\right)}$$

Requiring self-duality imposes constraints on $f\left(x^2\right)$.

• By rotational invariance, it suffices to examine the fields along the positive x_4 axis, where

$$F_{ij} = -2\eta_{ij} f - ig\left[\eta_{i4}, \eta_{j4}\right] x^2 f^2, \quad F_{k4} = -2\eta_{k4} f - 2\eta_{k4} x^2 f'$$

and the self-equality condition $F_{ij} = \epsilon_{ijk} F_{k4}$ reduces to

$$f' = -gf^2, \quad f = \frac{1}{g}\frac{1}{x^2 + \lambda^2}$$

• Therefore, the unit instanton solution, also called the BPST instanton, is

$$A_p = \frac{1}{g}\frac{\eta_{pq} x_q}{x^2 + \lambda^2}, \quad A_p^j = \frac{2}{g}\frac{\eta_{pq}^j x_q}{x^2 + \lambda^2}$$

The parameter λ gives the characteristic size of the instanton; all values of λ are allowed by the scale invariance of classical Yang-Mills. The anti-self-dual anti-instanton has the same form, with η_{pq} replaced with $\bar{\eta}_{pq}$. The field strength is

$$F_{pq} = -\frac{2}{g}\frac{\eta_{pq}\lambda^2}{\left(x^2 + \lambda^2\right)^2}$$

Note that this falls off as $1/x^4$ even though A_p falls off as $1/x$. This is because the leading large-distance component of A_p is pure gauge.

• Alternatively, by taking a gauge transformations, we have

$$A_p' = \frac{1}{g}\frac{\bar{\eta}_{pq} x_q \lambda^2}{x^2\left(x^2 + \lambda^2\right)}$$

In terms of bundles over S^4, the original gauge field is singular at the point at infinity, while this gauge field is singular at the origin. Such a singularity is necessary, since the fiber bundle is nontrivial.

• Yet another way to write the unit instanton is to let

$$G = \frac{x_4 + i\mathbf{x}\cdot\boldsymbol{\tau}}{|x|}, \quad A_p = \frac{x^2}{x^2 + \lambda^2}\left(-\frac{i}{g}\right)\left(\partial_p G\right) G^{-1}$$

This works because G wraps around $SU(2)$ once in an S^3 slice at constant $|x|$. Multiple instantons can be found by taking powers of G. Finally, one can explicitly transform this to $A_0 = 0$ gauge, giving explicit tunneling between vacua with different winding.

Physical Consequences

Finally, we investigate the physical consequence of instantons. First we connect them to anomalies.

• Consider a theory of N_f massless fermions. The $U(1)_A$ symmetry is anomalous by

$$\partial_\mu j_5^\mu = \frac{N_f g^2}{8\pi^2}\,\mathrm{tr}\, F_{\mu\nu}\tilde{F}^{\mu\nu}$$

We can form a divergenceless current

$$J_5^\mu = j_5^\mu - 2N_f j_A^\mu$$

where j_A^μ is as defined above; this current is not gauge invariant.

• In the $A_0 = 0$ gauge, the instanton corresponds to tunneling between vacua of different winding number; moreover this current vanishes at spatial infinity. Hence we have a conserved charge

$$Q_5 = n_R - n_L - 2N_f n$$

where n is the winding number of the gauge field. Hence a change in winding number must be accompanied by a change in fermion chirality,

$$\Delta\left(n_R - n_L\right) = 2N_f \Delta n$$

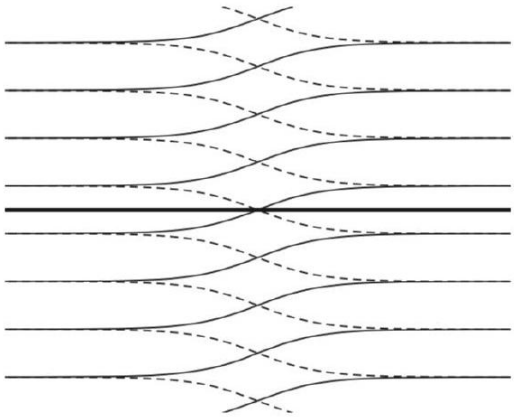
• To understand exactly how this happens, we consider the Dirac equation. The Hamiltonian for a single particle is

$$H = -i\alpha^j D_j, \quad \alpha^j = \gamma^0\gamma^j = \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}$$

in a particular representation of the γ^μ matrices. The four-component fermions naturally split into a pair of two-component Weyl fermions, where

$$H_R = -i\sigma^j D_j, \quad H_L = i\sigma^j D_j = -H_R$$

• Now we work in the Dirac sea picture. Both H_L and H_R have both positive and negative energy eigenstates, and we suppose all the negative energy states are filled. During the tunneling process, A_μ is not in a vacuum state, so the fermion spectrum changes; however it must be the same in both the initial and final states. However, the spectrum can 'flow' as shown below.



Here the heavy line is the zero of energy. If the change is slow, the adiabatic theorem holds, so that the initial Dirac sea turns into a Dirac sea plus a right-chiral particle and a left-chiral negative energy hole, corresponding to a right-chiral positive energy antiparticle.

• Even if the adiabatic theorem does not hold, the Hamiltonian only couples fermion fields of the same chirality, so any transitions between states do not affect the change in net chirality. Similarly, the initial state need not be the vacuum state.

• The fact that the instanton number is generally equal to the number of energy levels that cross zero can be proven with an index theorem, which we won't go into here.

• Now consider the physical gauge. In this case, the vacua have been gauge transformed to all have the same winding number. However the effect of the instanton of course cannot be gauged

away; the spectral flow occurs as before. Then the charge Q_5 is not conserved, which appears paradoxical. The resolution is that, as we've seen already, the current j^μ_A does not vanish at spatial infinity in this gauge; instead it yields a finite surface integral.

• The chirality-violating processes associated with instantons can be represented by a nonlocal effective Lagrangian density,

$$\mathcal{L}_{\text{eff}} = Ce^{-8\pi^2/g^2(\lambda)} \prod_{s=1}^{N_f} \left(\bar{\psi}^s_R \omega\right) \left(\bar{\omega} \psi^s_L\right)$$

where we must integrate over the positions of the fermion fields, ω is a fixed Dirac spinor transforming under the fundamental representation of the gauge group which depends on the gauge orientation of the instanton, and C is obtained from the one-loop corrections to the instanton. The Hermitian conjugate of this term accounts for the anti-instanton, and one must also integrate over instanton positions and scales. One may also integrate over gauge orientations of ω to get a gauge-invariant result.

• To solve the $U(1)$ problem of QCD, note that if we define

$$\mathcal{M}_{rs} = \left(\bar{q}_{Rr} \omega\right) \left(\bar{\omega} q_{Ls}\right)$$

then the anticommutativity of the fermion fields implies

$$\prod_{s=1}^{N_f} \left(\bar{q}_{Rs} \omega\right) \left(\bar{\omega} q_{Ls}\right) = \frac{1}{N_f!} \det \mathcal{M}$$

The chiral transformations of the quark fields are

$$q_L \to U_L q_L, \quad q_R \to U_R q_R, \quad \mathcal{M} \to U_R^\dagger \mathcal{M} U_L$$

under which $\det \mathcal{M}$ is not invariant for a $U(1)_A$ transformation. Hence $U(1)_A$ is explicitly broken when one accounts for instanton effects, and so is not a spontaneously broken symmetry.

• Computing the coefficient C amounts to evaluating functional determinants as we did for quantum mechanical tunneling, to find how the action varies as the instanton solution is deformed. We must also integrate over collective coordinates. The troublesome coordinate is the instanton size; we find a divergence due to large instantons, where the gauge coupling is strong. Hence unless we have a natural cutoff on the instanton size, it is difficult to estimate C. We only know that in QCD, instanton effects are significant.

Note. However, the $U(1)_A$ problem is a bit more confusing when one thinks in terms of θ-vacua instead of Lagrangian terms. The charge associated with $U(1)_A$ generates θ translations. Hence it appears the $U(1)_A$ symmetry is spontaneously broken by the choice of a θ vacuum, and hence should yield a PNBG.

Formally, any explicitly broken symmetry can be thought of as a spontaneously broken symmetry by enlarging the Hilbert space (i.e. treating the symmetry breaking parameter as a spurion), but it will not correspond to a physical excitation. The key property that distinguishes $U(1)_A$ breaking from the genuine spontaneous rotational symmetry breaking of a ferromagnet is that the magnetization $\mathbf{M}(x)$ is a local field which can have local excitations, while θ is a global property of the state, so it doesn't make sense to promote it to $\theta(x)$. We dwell on this point because often the introduction of the axion is oversimplified to saying that "axions result from promoting θ to a field". As shown in the notes on the Standard Model, one has to work a bit harder than that!

Note. A very similar phenomenon occurs in the simpler context of QED with a massless fermion in $d = 2$, where we can do the calculations simply and explicitly. In this case, the particles are either right-movers or left-movers, with the numbers n_R and n_L separately conserved. The axial charge, proportional to $n_R - n_L$, is violated by the anomaly,

$$\Delta\left(n_R - n_L\right) = \int d^2x \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu}$$

To relate this to spectral flow, suppose we adiabatically turn on a constant A^1 field, and for simplicity compactify space to length L. The Hamiltonian for a Weyl field is

$$H = \int dx\psi^\dagger\left(-i\alpha^1 D_a\right)\psi, \quad \alpha = \gamma^0\gamma^1 = \gamma^5$$

Splitting this into right-moving and left-moving components,

$$H = \int dx\left(-i\psi_+^\dagger\left(\partial_1 - ieA^1\right)\psi_+ + i\psi_-^\dagger\left(\partial_1 - ieA^1\right)\psi_-\right)$$

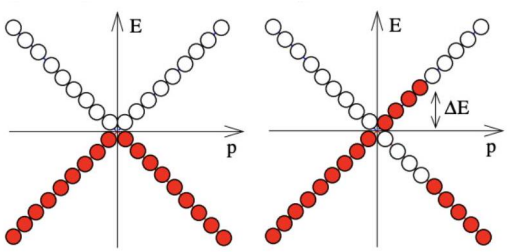
The eigenfunctions of the covariant derivative the form $e^{ik_n x}$ with $k = 2\pi n/L$, which implies the corresponding single-particle eigenstates of H have the energies

$$E_n^+ = k_n - eA^1, \quad E_n = -\left(k_n - eA^1\right)$$

Suppose we shift A^1 by $\Delta A^1 = 2\pi/eL$. Then spectral flow occurs in exactly the same manner as we saw above for $d = 4$, changing $n_R - n_L$ by -2. Correspondingly, we have

$$\int d^2x \frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} = \int dtdx\frac{e}{\pi}\partial_0 A_1 = -2$$

as expected. We can understand the spectral flow in a very concrete way: turning on the A_1 field corresponds to giving an equal impulse $e\Delta A^1$ to each particle. This affects the Dirac sea as follows:



The impulse creates right-moving particles and left-moving antiparticles, violating axial charge conservation.

This derivation didn't need instantons: we just postulated a background field configuration, nonzero over all space. Instantons are necessary to get a nontrivial result if we also require the fields to vanish quickly at infinity, i.e. they are the only contributions that matter in the infinite volume limit, where constant field configurations like this one would have infinite action.

Note. As with all anomaly effects, the result depends on the UV regularization. This is because the spectral flow affects energy levels all the way to positive and negative infinity. For example, if we had a hard cutoff, then spectral flow would never change the charges; in the above example, for example, the creation of low-energy right-moving particles would be compensated by the creation

of high-energy right-moving antiparticles. But we cannot just use this regulator because, as we've seen before, it would ultimately lead to an anomaly in the vector current, rendering the theory inconsistent.

Next, we investigate baryon number violation.

• In the $SU(2) \times U(1)$ electroweak theory there are also instantons. The Higgs acquires its usual VEV at infinity but vanishes at the center of the instanton. The Higgs VEV hence breaks the classical scale invariance, by favoring a smaller instanton size. As a result the integration over instanton size is finite, peaking around $\lambda \sim 1/\langle\phi\rangle$.

• Each electroweak instanton leads to a violation of baryon and lepton number by

$$\Delta B = \Delta L = 3$$

where the factor of 3 is from the three generations. The effective Lagrangian is nonlocal at the electroweak scale and hence, at longer distances, one can approximate the terms as local. For example, an instanton could mediate the scattering process $p + n \to \bar{p} + e^+ + e^+ + \bar{\nu}_e$.

• The rate of the associated processes is suppressed by

$$\left(e^{-8\pi^2/g^2}\right)^2 = e^{-16\pi^2 \sin^2\theta_W/e^2} \sim 10^{-161}$$

where we evaluated e at the Z mass, since the typical instanton size is set by the electroweak scale. The universe contains about 10^{78} protons and has an age of about 10^{10} years, or 10^{40} times a typical strong interaction time of 10^{-23} seconds. Hence baryon number violation by instantons appears to be completely negligible.

• However, one can also travel between different vacua by thermal fluctuations, rather than quantum tunneling. For every path connecting two vacua in spacetime, there is a point of highest energy; the minimum of all such maxima determines the thermal fluctuation rate. Such a solution is called a sphaleron, and it heuristically looks like the "midpoint" of the unit instanton solution. In particular, it has half-integer winding number. Sphalerons shouldn't be confused with instantons: instantons are spacetime configurations that are minima of the Euclidean action, while sphalerons are spatial configurations that are saddle points of the energy.

• Since a sphaleron is a critical point of the potential, it corresponds to a static solution (in $A_0 = 0$ gauge) of the field equations. Since it is a saddle point, it is unstable, leading to its name; sphaleron is etymologically related to "slippery".

• The size of the sphaleron is set by the Higgs vev, leading to the estimate

$$E_{\text{sph}} \sim \frac{4\pi v}{g} \sim 10\text{GeV}$$

where g is the $SU(2)$ gauge coupling. We then estimate $\Gamma \sim e^{-E_{\text{sph}}/T}$, though a more accurate calculation would give $\Gamma \sim e^{-F_{\text{sph}}/T}$ where the free energy F_{sph} accounts for finite temperature corrections to the effective potential, which change the expectation value of the Higgs field.

• More quantitatively, the rate per spacetime volume goes as

$$\Gamma \sim \begin{cases} T^4 e^{-F_{\text{sph}}/T} & T \lesssim F_{\text{sph}} \\ T^4 \alpha_w^4 & T \gtrsim F_{\text{sph}} \end{cases}$$

where the T^4 is by dimensional analysis, and the casework occurs because electroweak symmetry is unbroken for $T \gtrsim F_{\text{sph}}$.

• In any case, it is clear that at temperatures near or above that of the electroweak phase transition, baryon number violation can be significant. Depending on the model, this can be an obstacle to baryogenesis, diluting existing baryon number asymmetry, or the reason for baryogenesis itself.

Next, we discuss the QCD θ term.

• As we've motivated above, the Yang-Mills Lagrangian contains the θ term

$$\Delta\mathcal{L} = \frac{\theta g^2}{16\pi^2}\,\mathrm{tr}\,F_{\mu\nu}\tilde{F}^{\mu\nu} = \frac{\theta g^2}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\,F_{\mu\nu}F_{\rho\sigma}$$

Such a term does not contribute classically but can in quantum mechanics.

• As a simple analogue, consider a particle on a ring, at angular coordinate α, and

$$L = \frac{1}{2}B\dot{\alpha}^2 + \frac{\theta}{2\pi}\dot{\alpha}$$

The conjugate momentum to α is

$$J = B\dot{\alpha} + \frac{\theta}{2\pi}$$

is quantized and takes on integer values, so for $J = n$,

$$E_n = \frac{1}{2B}\left(n - \frac{\theta}{2\pi}\right)^2$$

Hence a total derivative term affects the spectrum, though θ only matters up to multiples of 2π. For example, when $\theta = 0$ or $\theta = \pi$, the energy levels are doubly degenerate; in both cases this is a consequence of time reversal symmetry.

• Now add a potential energy $K(1 - \cos\alpha)$ where $K \gg 1/B$, so we may think of the particle as a rigid pendulum. In this case, the θ dependence only comes from tunneling processes, so the θ-dependence of the energy levels is exponentially suppressed.

• As for the physical consequences of the θ term, we cannot consider the θ-dependence of the vacuum energy, since θ is fixed. (However, such dependence is important when there is an axion.) The θ term also cannot explain observed CP violating processes, such as kaon decay, because neutron EDM requirements mean θ must be extremely small. In fact, no effects of the θ term have ever been observed. This is the strong CP problem.

Finally, we describe the Witten effect.

• The Witten effect links the θ term to the electric charge of magnetic monopoles. Here we must distinguish between two senses of the electric charge: the quantity that is coupled to electric fields which can be measured by their Coulomb tail, and the conserved Noether charge Q_n associated with phase rotations.

• Now consider a global $U(1)$ transformation composed with a gauge transformation,

$$\delta\mathbf{A}_\mu = \frac{1}{e}\mathbf{D}_\mu(\phi/|\phi|), \quad \delta\phi = 0$$

in a theory with gauge coupling e where $SU(2)$ is broken to $U(1)$ by a triplet, and the bold refers to $SU(2)$ indices.

• The Noether charge is

$$Q_n = \int d\mathbf{x}\delta\mathbf{A}_j \cdot \mathbf{\Pi}^j$$

where the conjugate momentum $\mathbf{\Pi}^j$ is

$$\mathbf{\Pi}_j = \frac{\partial \mathcal{L}}{\partial \left(\partial_0 \mathbf{A}_j\right)} = \mathbf{F}^{j0}$$

which implies that when $\theta = 0$,

$$eQ_n = \int d\mathbf{x} \mathbf{D}_j \hat{\boldsymbol{\phi}} \cdot \mathbf{F}^{j0} = \int d\mathbf{x} \left(\partial_j \left(\hat{\boldsymbol{\phi}} \cdot \mathbf{F}^{j0} \right) - \hat{\boldsymbol{\phi}} \cdot \mathbf{D}_j \mathbf{F}^{j0} \right)$$

The field equation $\mathbf{D}_j \mathbf{F}^{j0} = \boldsymbol{\phi} \times \mathbf{D}^0 \boldsymbol{\phi}$ shows the second term vanishes, so

$$eQ_n = \int d^2 S_j \hat{\boldsymbol{\phi}} \cdot \mathbf{F}^{j0} = Q_E$$

where electric charge is defined by the unbroken $U(1))$ subgroup. Since the Noether charge Q_n is conjugate to a periodic value, it is an integer, so the charge is quantized.

• Adding the θ term changes the conjugate momenta to

$$\mathbf{\Pi}^j = \mathbf{F}^{j0} - \frac{\theta e^2}{16\pi^2} \epsilon^{jkl} \mathbf{F}_{kl}$$

and repeating the steps above, using the Bianchi identity $\epsilon^{jkl} \mathbf{D}_j \mathbf{F}_{kl} = 0$ gives

$$eQ_n = \int d^2 S_j \hat{\boldsymbol{\phi}} \cdot \mathbf{F}^{j0} - \frac{\theta e^2}{16\pi^2} \epsilon^{jkl} \hat{\boldsymbol{\phi}} \cdot \mathbf{F}_{kl} = Q_E - \frac{e\theta}{2\pi} \left(\frac{eQ_M}{4\pi} \right)$$

• However, the Noether charge remains quantized to integer values. Thus, a monopole with magnetic charge $Q_M = 4\pi/e$ must have electric charge

$$Q_E = ne + \frac{e\theta}{2\pi}$$

for some integer n. Note that the result is periodic in θ as required.

• This can be understood by looking at the monopole from a distance, where it looks like a point charge. The Lagrangian is that of ordinary electromagnetism with the additional term

$$\Delta \mathcal{L} = -\frac{\theta e^2}{8\pi^2} \mathbf{E} \cdot \mathbf{B}$$

so that the abelian Gauss's law becomes

$$\nabla \cdot \mathbf{E} - \frac{\theta e^2}{8\pi^2} \nabla \cdot \mathbf{B} = \rho$$

where ρ is a purely electric source, which vanishes for a monopole. Hence magnetic charge must be accompanied by electric charge.

• This doesn't contradict the fact that the θ term does not affect the equations of motion, because for trivial gauge bundles $\nabla \cdot \mathbf{B}$ vanishes identically. It can only be nonzero if the bundle is nontrivial, in which case the θ term, which is still a total derivative, cannot be converted to a vanishing surface integral at infinity; instead we get a contribution at the overlap of the patches.

• Note that the Witten effect is totally independent of instantons; it only relies on there being a θ term in the Lagrangian. This term is thought of as "caused" by instantons for historical reasons.

7.3.3 Anomalies: Pion decay, triangle diagrams, features

(см. также последние лекции Вергелеса)

Pion Decay

We begin with an overview of anomalies.

• A anomaly is a symmetry of a classical theory that is not present in the quantum theory. Specifically, anomalies mean the Ward-Takahashi identities will no longer hold. They arise because the theory cannot be regularized without breaking the symmetry, or equivalently because the path integral measure cannot be chosen to be invariant.

• Anomalies can apply to either gauge or global symmetries. A gauge anomaly is dangerous because it destroys the Ward identities, and hence prevents the decoupling of unphysical states, leading to violation of unitarity. Thus we must arrange our theories so that all gauge anomalies cancel. Such a theory is called 'anomaly free'.

• Global anomalies are not dangerous, and are instead ubiquitous. For example, the $U(1)_B$ baryon number symmetry is anomalous, and this allows for nonconservation of baryon number, which is required for baryogenesis.

• We will see that anomalies are infrared effects, resulting from massless particles in the spectrum. This leads to the idea of anomaly matching, which relates the spectrum of massless particles above and below a phase transition. In renormalizable gauge theories, anomalies arise exclusively from chiral fermions.

• More specifically, consider a theory with gauge group G and global symmetry group $\tilde{G}$, with currents j^μ and $\tilde{j}^\mu$. It will turn out that anomalies can be computed by considering correlators of three currents, each of which may be from G or $\tilde{G}$, at one-loop order with a massless particle in the loop.

• For reasons that will become clear later, we will write, e.g. the anomaly resulting from three j^μ 's as $\mathrm{tr}(GGG)$. As a result, there are four important cases.

– $\mathrm{tr}(GGG)$ corresponds to an anomalous gauge symmetry, which is dangerous for the reasons explained above. The Ward-Takahashi identities correspond roughly to $\partial_\mu j^\mu \sim F\tilde{F}$.

– $\mathrm{tr}(GG\tilde{G})$ and $\mathrm{tr}(G\tilde{G}\tilde{G})$ may correspond to either an anomalous gauge symmetry or an anomalous global symmetry, depending on how we regularize the theory; for consistency we always choose the latter. This results in a non-conserved global current, $\partial_\mu \tilde{j}^\mu \sim F\tilde{F}$. For historical reasons, this is called a chiral anomaly.

• Note that many anomalies of this mixed form automatically vanish, because $\mathrm{tr}\,(T_a) = 0$ for any generator of a semi-simple Lie group.

– $\mathrm{tr}(\tilde{G}\tilde{G}\tilde{G})$ corresponds to a 't Hooft anomaly. These do not cause nonconservation of the current, $\partial_\mu \tilde{j}^\mu \sim 0$, but they are an obstruction to gauging $\tilde{G}$. It is these anomalies that are used in anomaly matching.

• There are also linear/mixed/gravitational anomalies of the form $\mathrm{tr}\,(G\,\mathrm{grav}^2)$ or $\mathrm{tr}\left(\tilde{G}\,\mathrm{grav}^2\right)$ where two of the external legs are gravitons. These cause nonconservation of a current in the presence of spacetime curvature. This can be understood in terms of the previous anomalies by working with a vierbein, in which case coupling to gravity is like coupling to an $SO(3,1)$ gauge field.

• Since $F\tilde{F}$ is a total derivative term, one might think that an anomalous global symmetry still has a conserved charge. However, this is not true if we account for instantons, the subject of the

next chapter, which can occur in non-abelian gauge theories. For gravitational anomalies, there are also corresponding gravitational instantons.

• Finally, quantum field theories generically break scale invariance. In this case the anomaly is called the trace anomaly, since scale invariance makes the energy-momentum tensor traceless, and it is proportional to the β-function. Conformal field theories are trace anomaly free.

• Historically, anomalies were first encountered in the computation of the decay rate for $\pi^0 \to \gamma\gamma$, where the chiral anomaly has an effect. In this context, it is also called an ABJ anomaly or the axial anomaly.

• The anomalies we have described above are sometimes called "perturbative" anomalies, in the sense that they can be computed using perturbative techniques. There are also more subtle anomalies, such as Witten's global $SU(2)$ anomaly (not to be confused with the anomalies of global symmetries we've considered above), which states that an $SU(2)$ gauge theory with an odd number of fermions in the fundamental representation is inconsistent.

First, we consider the decay $\pi^0 \to \gamma\gamma$ in the context of QCD with the up and down quark.

• An overview of the symmetries of the theory is given in the notes on the Standard Model. As a quick summary, the symmetry group is

$$\tilde{G} = U(1)_L \times U(1)_R \times SU(2)_L \times SU(2)_R$$

This is spontaneously broken to

$$\tilde{G} = U(1)_V \times SU(2)_V$$

by the formation of the quark condensate. The $U(1)_V$ symmetry is the total quark number, while the symmetry $U(1)_{\text{EM}} \subseteq SU(2)_V$ is gauged. The spontaneously broken symmetries $U(1)_A$ and $SU(2)_A$ correspond to the η' and the pions, respectively.

• However, we should also account for the quark masses. Equal masses for the up and down quark break the symmetry group explicitly to

$$\tilde{G} = U(1)_V \times SU(2)_V$$

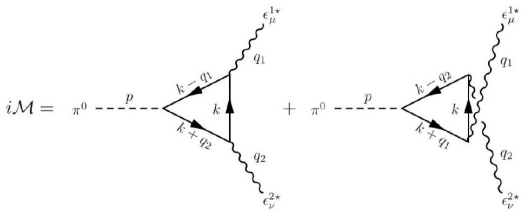
independent of the quark condensate. In this context $SU(2)_V$ is known as isospin. It is broken by the up-down quark mass difference, which we neglect.

• Now, suppose the π^0 couples to the proton ψ by the interaction

$$\mathcal{L}_{\text{int}} = i\lambda\pi\bar{\psi}\gamma^5\psi$$

and it decays to photons by a proton loop, where the proton has mass m.

• The two contributing diagrams are shown below.



The diagrams may be computed in the usual way. The amplitude is superficially divergent, of the form $\int d^4k/k^3$, but the combination of the two terms turns out to be finite, with

$$\mathcal{M} = \frac{\lambda e^2}{4\pi^2 m}\epsilon^{\mu\nu\rho\sigma}\epsilon_\mu^{1*}\epsilon_\nu^{2*}q_1^\rho q_2^\sigma, \quad \Gamma = \frac{\alpha^2}{64\pi^3}\frac{\lambda^2 m_\pi^3}{m^2}$$

• To make the result more quantitative, we need to find the value of λ. In the context of chiral perturbation theory, this coupling is related at tree level to the nucleon mass by $\lambda = m/f_\pi$, so

$$\Gamma = \frac{\alpha^2}{64\pi^3}\frac{m_\pi^3}{F_\pi^2}$$

• Alternatively, the computation can be done by current algebra. The key fact is that the neutral pion is a Goldstone boson associated with the spontaneous breaking of $SU(2)_A$. The three pions satisfy

$$\left\langle \Omega \left| j_\mu^{5a}(x) \right| \pi^a(p) \right\rangle = ie^{ipx}F_\pi p_\mu, \quad j_\mu^{5a} = \bar{\Psi}\tau^a\gamma_\mu\gamma^5\Psi$$

where Ψ can be taken to be either (p, n) or (u, d). The neutral pion corresponds to $a = 3$.

• Now, if we had used $\Psi = (u, d)$ instead, there would be contributions from both up quark loops and down quark loops, of which there are N colors, giving an overall factor of

$$N\left((2/3)^2 - (1/3)^2\right) = \frac{N}{3}$$

in the amplitude, where the minus sign comes from the negative isospin of the down quark, and we get q^2 factors from the two photon vertices. Hence the pion decay rate was an early test that $N = 3$ in QCD.

• One might complain that higher-order effects in QCD should be significant. The above result is only accurate because the chiral anomaly is exact at one loop, as we'll see below.

• The fact that the spinor ψ in the loop has a mass means that it violates $U(1)_A$ axial symmetry,

$$j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi, \quad \partial_\mu j^{\mu 5} = 2im\bar{\psi}\gamma^5\psi$$

This basic fact holds whether we think of the loop as containing nucleons or quarks.

• Now, the calculation we did above essentially says that

$$\left\langle A \left| \bar{\psi}\gamma^5\psi \right| A \right\rangle = i\frac{e^2}{32\pi^2}\frac{1}{m}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

where $F_{\mu\nu}$ is a background electromagnetic field. But then we have

$$\left\langle A \left| \partial_\mu j^{\mu 5} \right| A \right\rangle = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

which means that the axial current is not conserved in a background electromagnetic field, even when we take the mass m to zero! Hence our simple calculation shows us something strange is going on with the axial symmetry - an anomaly is present.

• The reason the chiral anomaly is important here is that, if one reasoned from chiral symmetry alone, the decay rate should be much smaller. Naively we could have a term in the effective Lagrangian of the form

$$\mathcal{L}_{\text{eff}} = \frac{e^2}{8\pi^2 f_\pi}\pi^0\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

where the f_π is by dimensional analysis, and the extra $1/4\pi^2$ is because we are dealing with a loop effect. This yields the same result that we found above.

• This is surprising since the pion is a Goldstone boson of $SU(2)_A$, and hence should not have a non-derivative coupling. We know that $SU(2)_A$ is not an exact symmetry, so such a coupling may arise due to the breaking of $SU(2)_A$ by the quark masses. But it must be proportional to $m_u + m_d \propto m_\pi^2$, and hence smaller by a factor of m_π^2/f_π^2, giving a result that is too small.

• The resolution comes from the chiral anomaly, which in this case shows that there is a violation of $SU(2)_A$ symmetry independent of the quark masses. We know that under an $SU(2)_A$ transformation with $a = 3$, the π^0 field shifts, $\delta\pi^0 = \theta F_\pi$. When the quarks are integrated out,

$$\mathcal{L}_{\text{eff}} \supset -\frac{e^2}{16\pi^2 f_\pi}\pi^0 \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

almost exactly as expected by the naive argument. This is the vertex in the effective theory that replaces the triangle diagrams.

• In the language we used above, the π^0 decay rate is due to an $SU(2)_A U(1)_{\text{EM}}^2$ anomaly. There is also a $U(1)_A U(1)_{\text{EM}}^2$ anomaly. However, what distinguishes $U(1)_A$ is that it is anomalous even in pure QCD, by a $U(1)_A SU(3)_C^2$ anomaly. This has a large effect on the mass of the η' because QCD is strongly coupled, resolving the $U(1)_A$ problem.

Triangle Diagrams

Let's compute the anomaly directly, starting with a massless Dirac fermion. In this case we have the global symmetry $U(1)_A \times U(1)_V$.

• We consider the correlation function $\langle j^{\alpha 5}(x) j^\mu(y) j^\nu(z)\rangle$. Under an axial transformation we have $\delta j^\mu = 0$, so the Ward-Takahashi identity is simply

$$\partial_\alpha \left\langle j^{\alpha 5}(x) j^\mu(y) j^\nu(z)\right\rangle = 0$$

This correlator is closely related to the pion decay amplitude, but without the external lines or the coupling constants. • Defining

$$iM_5^{\alpha\mu\nu}(p, q_1, q_2)\not{\delta}(p - q_1 - q_2) := \int dx dy dz e^{ipx} e^{iq_1 y} e^{iq_2 z} \left\langle j^{\alpha 5}(x) j^\mu(y) j^\nu(z)\right\rangle .$$

Feynman rules at one loop give:

$$M_5^{\alpha\mu\nu} = \int dk \,\mathrm{tr}\left(\gamma^\mu \frac{1}{\not{k}} \gamma^\nu \frac{1}{\not{k} + q/2} \gamma^\alpha \gamma^5 \frac{i}{\not{k} - q/1} + \begin{pmatrix} \mu \leftrightarrow \nu \\ 1 \leftrightarrow 2 \end{pmatrix}\right)$$

• Using standard identities, we may show

$$p_\alpha M_5^{\alpha\mu\nu} = 4i\epsilon^{\mu\nu\rho\sigma} \int dk \left(\frac{k^\rho q_2^\sigma}{k^2 (k + q_2)^2} + \frac{k^\rho q_1^\sigma}{k^2 (k - q_1)^2}\right) + \begin{pmatrix} \mu \leftrightarrow \nu \\ 1 \leftrightarrow 2 \end{pmatrix} .$$

By Lorentz invariance the integrals must be proportional to $q_1^\rho q_1^\sigma$ and $q_2^\rho q_2^\sigma$ respectively, and hence vanish when contracted with the Levi-Civita symbol. So naively it looks like the WardTakahashi identity is obeyed.

• Similarly, by contracting with q_μ^1, we find

$$q_\mu^1 M_5^{\alpha\mu\nu} = -4i\epsilon^{\mu\nu\rho\sigma}\int dk\left(\frac{(k-q_1)^\rho(k+q_2)^\sigma}{(k-q_1)^2(k+q_2)^2}-\frac{(k-q_2)^\rho(k+q_1)^\sigma}{(k-q_2)^2(k+q_1)^2}\right)$$

If we shift $k\to k+q_1$ in the first term and $k\to k+q_2$ in the second, the integral vanishes identically, so naively it looks like the vector current is conserved as well.

• However, this calculations are deceptive. If an integral is regularized, the integration variable can always be shifted, but we haven't done any regularization here. In fact, our standard regulators don't work, because DR has trouble with γ^5 and PV explicitly breaks chiral symmetry by introducing a massive fermion. One can still make sense of linearly divergent integrals, but their value changes upon a shift.

• To see this, consider the integral

$$\Delta(a)=\int dx f(x+a)-f(x)$$

where the integration bounds are $\pm\infty$, and $f(x)$ goes to a constant at infinity. Naively the integral is zero by shifting the first term, but by Taylor expanding we have

$$\Delta(a)=\int dx a f'(x)+\text{ higher derivatives }=a(f(\infty)-f(-\infty))$$

where the higher derivatives don't contribute since f goes to a constant at infinity. This result holds regardless of whether or how we regulate $f(x)$ at infinity.

• By similar reasoning, one can show that for the linearly divergent integral

$$\Delta^\alpha(a^\mu)=\int dk\left(F^\alpha(k+a)-F^\alpha(k)\right),\quad \lim_{k_E\to\infty}F^\alpha(k_E)=A\frac{k_E^\alpha}{k_E^4}$$

where k_E is a Euclidean momentum, we have

$$\Delta^\alpha(a^\mu)=\frac{i}{32\pi^2}Aa^\alpha$$

where the factor of i comes from Wick rotation.

• Returning to the vector current calculation above, the quadratic divergences cancel, while for the linear divergence gives

$$q_\mu^1 M_5^{\alpha\mu\nu}=\frac{1}{4\pi^2}\epsilon^{\alpha\nu\rho\sigma}q_1^\rho q_2^\sigma$$

Hence it appears the vector current is not conserved. The resolution is that the definitions of the loop momenta k are independent between the two contributing diagrams. In other words, by choosing the k 's we did above, we had implicitly fixed a regularization scheme for the correlator.

• We may choose the vector current to be conserved by shifting $k\to k+q_1$ in the first diagram, which fixes $k\to k+q_2$ in the second by Bose symmetry. Then the linear divergences cancel exactly, but one can show that

$$p_\alpha M_5^{\alpha\mu\nu}=\frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\sigma}q_1^\rho q_2^\sigma$$

Taking the most general possible shift $k\to k+b_1q_1+b_2q_2$ it can be shown that it is impossible to preserve both the vector and axial symmetry; we choose to preserve the former.

• To compare with what we found earlier, we can relate correlation functions to matrix elements by LSZ reduction. The form of the LSZ reduction formula for photons is

$$\langle f|i\rangle = i\epsilon^\mu \int dx e^{-ikx} \left(-\partial^2\right) ... \langle A_\mu(x)...\rangle$$

By the Schwinger-Dyson equation associated with the classical equation of motion $-\partial^2 A_\mu = j_\mu$,

$$\langle f|i\rangle = i\epsilon^\mu \int dx e^{-ikx} ... \langle j_\mu(x)...\rangle + \text{ contact terms}$$

where the latter do not contribute to S-matrix elements and hence don't matter here.

• Applying the LSZ reduction formula, we have

$$\left\langle q_1, q_2 \left| j^{\alpha 5}(p) \right| 0 \right\rangle = (ig)^2 \epsilon_\mu \epsilon'_\nu \left(iM^{\alpha\mu\nu}\left(p, q_1, q_2\right) \not{\delta}\left(p - q_1 - q_2\right)\right).$$

Contracting both sides with p_α, we have

$$\left\langle q_1, q_2 \left| \partial_\alpha j^{\alpha 5}(x) \right| 0 \right\rangle = -\frac{g^2}{2\pi^2} \epsilon^{\mu\nu\rho\sigma} q_{1\rho} q_{2\sigma} \epsilon_\mu \epsilon'_\nu e^{-i(q_1+q_2)z} + O\left(g^4\right)$$

where the higher-order terms vanish by the one-loop exactness of the anomaly. This is consistent with the operator equation

$$\partial_\mu j^{\mu 5} = -\frac{g^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$$

Note. We haven't proven the above operator equation, only that a certain matrix element of both sides matches. Later, we will use the path integral to give a more general proof, which establishes that the two sides are equal when placed inside arbitrary time-ordered correlation functions.

The reason we hesitate to work with the actual operators is because they are difficult to define: quantum fields are operator-valued distributions and hence two fields at the same point can't be multiplied, but $j^{\mu 5}$ contains products of this form. In this case, we can "regularize" by point-splitting, separating the two ψ fields in $j^{\mu 5}$ and connecting them with a Wilson line,

$$j^{\mu 5} = \lim_{\epsilon \to 0} \bar{\psi}(x + \epsilon/2) \gamma^\mu \gamma^5 \exp\left(-ie \int_{x-\epsilon/2}^{x+\epsilon/2} dz^\mu A_\mu(z) \right) \psi(x - \epsilon/2)$$

Taking the limit carefully proves the desired operator equation. This somewhat old-fashioned approach is given in detail in chapter 19 of Peskin and Schroeder. In modern high-energy theory, many simply circumvent this issue by working exclusively with the path integral. In this language, an "operator" is just defined as any expression that can go under a path integral, and an "operator equation" is defined to be any statement that holds underneath a path integral up to contact terms.

Note. As stated above, dimensional regularization has problems with γ^5. So far, we've mostly treated dimensional regularization as a series of ad hoc rules. That is, we assume that the trace obeys the usual properties (linearity, cyclicity), and further assume that

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \{\gamma_5, \gamma^\mu\} = 0, \quad \gamma_5^2 = 1$$

These properties can be used to derive others; for instance,

$$d \operatorname{tr} \gamma_5 = \operatorname{tr} (\gamma_5 \gamma^\mu \gamma_\mu) = -\operatorname{tr} (\gamma_5 \gamma_\mu \gamma^\mu) = -d \operatorname{tr} \gamma_5$$

which implies that $\operatorname{tr} \gamma_5 = 0$ for all $d \neq 0$. If $\operatorname{tr} \gamma_5$ is to be meromorphic in d, then $\operatorname{tr} \gamma_5 = 0$ for all d.

If γ^5 really could satisfy all these properties simultaneously, the axial anomaly would vanish. However, other results would come out incorrect. For instance, it can be shown that

$$(4 - d) \operatorname{tr} (\gamma_5 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) = 0$$

which this trace to be zero for all d, including $d = 4$, making many amplitudes vanish identically, and giving completely wrong physical results.

To avoid problems of this sort, we need to give a more constructive definition of dimensional regularization. For example, Collins defines integrals in non-integer dimensions axiomatically in chapter 4. One consistent way to define γ_5 is to take $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ in all dimensions. This formally means that γ^5 anticommutes with 4 of the gamma matrices, and commutes with $d - 4$ of them, breaking Lorentz invariance. Practically, this means that to evaluate loop integrals, we must split the loop momentum as $\ell = \ell_\parallel + \ell_\perp$, where the two pieces have 4 and $d - 4$ components, respectively, and treat the latter piece like a typical dimensional regularization integral in $d - 4$ dimensions. This is used to derive the axial anomaly in chapter 19 of Peskin and Schroeder.

There are other methods for defining γ_5, which are described in article by F. Jegerlehner. For instance, one can proceed naively if one only is interested in amplitudes with an even number of γ_5 's. This is used in chapter 21 of Peskin and Schroeder to handle one-loop computations in the Standard Model.

Comments about results

• We see that we may choose which symmetry to preserve; we could also choose to preserve neither. If we give the Dirac fermion a charge, we must choose the vector symmetry to be preserved to avoid a gauge anomaly. If we give the fermion a mass, then axial symmetry is automatically broken and we may regulate with PV, so none of the subtleties above apply.

• The reason we considered this correlator was that taking more currents would yield divergences weaker than a linear one. Also, we needed one or three axial currents, because we need an odd number of γ^5 matrices (otherwise they can cancel out and we can use DR) and at least one vector current to check vector current conservation. However, we could also derive the axial anomaly by considering a correlator of three axial currents.

• In general, anomalies in four dimensions are studied with triangle diagrams, while anomalies in $2n$ dimensions are studied with $(n + 1)$-gons. Moreover, we don't need to look at higher-loop diagrams since the chiral anomaly is one-loop exact. (why not 2 currents?)

• In general, for QED with any number of Dirac fermions, the gauge anomaly can be made to vanish by similar manipulations. However, for a single left-chiral Weyl fermion, this is not the case. The photon couples to the current

$$j_L^\mu = \bar{\psi} \gamma^\mu P_L \psi$$

The factors of P_L can be manipulated to show that

$$\langle j_L^\alpha j_L^\mu j_L^\nu \rangle = \frac{1}{2} \left(\langle j^\alpha j^\mu j^\nu \rangle - \langle j_5^\alpha j^\mu j^\nu \rangle \right)$$

The former vanishes when contracted with any of the momenta, as we may use DR there. Now we can have $p_\alpha \langle j_5^\alpha j^\mu j^\nu \rangle = 0$ or $q_{1\mu} \langle J_5^\alpha j^\mu j^\nu \rangle = 0$, but not both. Then the Ward-Takahashi identity for j_L^μ cannot be satisfied, and QED with a single Weyl fermion is inconsistent.

• Now suppose we have a left-chiral and right-chiral Weyl fermion with different charges. In this case the gauge boson A_μ couples to

$$j_{LR}^\mu = Q_L \bar{\psi} \gamma^\mu P_L \psi + Q_R \bar{\psi} \gamma^\mu P_R \psi$$

Since the mass is zero, there is no mixing between left-chiral and right-chiral fermions, so the contributions just add, giving

$$\langle j_{LR}^\alpha j_{LR}^\mu j_{LR}^\nu \rangle = Q_L^3 \langle j_L^\alpha j_L^\mu j_L^\nu \rangle + Q_R^3 \langle j_R^\alpha j_R^\mu j_R^\nu \rangle \supset \frac{1}{2} \left(Q_R^3 - Q_L^3 \right) M_5^{\alpha\mu\nu}$$

Hence the theory is consistent if $Q_L = Q_R$.

• Another way to understand the relative minus sign is to convert the right-chiral Weyl fermions to left-chiral Weyl fermions, which flips their charge; then the total anomaly is just proportional to $\sum_i Q_i^3$. In the case of a non-chiral gauge theory, every Q_i is paired with another of the opposite sign, which guarantees the sum is zero.

• In general, we only need to worry about vector symmetries being anomalous if the gauge theory is chiral. And chiral fermions in four dimensions must be massless, so we only need the lowenergy spectrum; the chiral anomaly is an infrared effect. This allows us to check gauge anomaly cancellation in the Standard Model below. (However, we shouldn't forget that axial symmetries can be anomalous even in non-chiral theories, as we saw in π^0 decay.)

• By generalizing the computation of the triangle diagram to a non-abelian gauge symmetry, for a massless Dirac fermion in a representation R, we have

$$\partial_\mu j^{\mu 5} = -\frac{g^2}{16\pi^2} T(R) \epsilon^{\mu\nu\rho\sigma} \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a \right) \left(\partial_\rho A_\sigma^a - \partial_\sigma A_\rho^a \right) + O\left(g^3\right)$$

which suggests the result

$$\partial_\mu j^{\mu 5} = -\frac{g^2}{16\pi^2} T(R) \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a = -\frac{g^2}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} \operatorname{tr} F_{\mu\nu} F_{\rho\sigma}$$

by gauge invariance; it can properly be derived by computing square and pentagon diagrams.

• By the LSZ logic above, this anomaly will also affect Ward-Takahashi identities for correlation functions of the form $\langle j^5 jjj \rangle$ and $\langle j^5 jjjj \rangle$, since the term $F\tilde{F}$ can also annihilate three or four gauge bosons.

• Massive fermions don't contribute to the chiral anomaly, because there we can simply use PV regularization. Hence the chiral anomaly is an infrared effect. Note that the (historically prior) calculation we presented above can obscure this, since it makes it look like anomalies are due to loop divergences.

Anomalies from the Path Integral

Next, we see how anomalies can arise from the path integral.

• We consider the quantity

$$Z(A) = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} e^{iS(A)}, \quad S(A) = \int dx \bar{\Psi} (i \not{D}) \Psi, \quad D_\mu = \partial_\mu - igA_\mu$$

for a fixed background gauge field A, under which the Dirac fermion Ψ transforms in a representation R. Formally, we have

$$Z(A) = \det(i\not{D})$$

but this expression must be regularized.

• Now consider a spacetime-dependent axial $U(1)$ transformation,

$$\Psi'(x) = e^{-i\alpha(x)\gamma_5}\psi(x), \quad \bar{\Psi}'(x) = \bar{\psi}(x)e^{-i\alpha(x)\gamma_5}.$$

If the measure were invariant, we would have

$$Z(A) = \int \mathcal{D}\Psi'\mathcal{D}\overline{\Psi'} \exp\left(i\int dx\overline{\Psi'(i\not{D})\Psi'}\right) = \int \mathcal{D}\Psi\mathcal{D}\bar{\Psi} \exp\left(i\int dx\bar{\Psi}(i\not{D})\Psi + j_A^\mu(x)\partial_\mu\alpha(x)\right)$$

where we used invariance of the measure in the second step. Integrating by parts and setting this equal to $Z(A)$, we have $\langle\partial_\mu j_A^\mu\rangle = 0$. If we had other operators present, we would have this result up to contact terms, which is precisely the Ward-Takahashi identity.

• Now we consider the Jacobian of the transformation more closely. We have

$$J(x,y) = \delta(x-y)e^{-i\alpha(x)\gamma_5}, \quad \mathcal{D}\Psi'\mathcal{D}\overline{\Psi'} = (\det J)^{-2}\mathcal{D}\Psi\mathcal{D}\bar{\Psi}$$

where the negative power is due to Grassmann variables, and

$$(\det J)^{-2} = \exp(-2\,\mathrm{tr}\log J) = \exp\left(2i\int dx\alpha(x)\,\mathrm{tr}\,(\delta(x-x)\gamma_5)\right)$$

where the trace is over spin and group indices.

• Again, this result is meaningless without regularization, as the delta function is infinite and the trace vanishes. Since we need to regularize $Z(A)$ in any case, we should use $\not{D}$, so we replace

$$\delta(x-y) \to e^{-\left(i\not{D}_x\right)^2/M^2}\delta(x-y) = \int dk e^{-\left(i\not{D}_x\right)^2/M^2}e^{ik(x-y)}$$

for a mass M that will be taken to infinity. This replaces the delta function with a Gaussian in a gauge-invariant way, as we'll see below. Other cutoff functions would also work.

• If one is unsatisfied, one can also define the path integral more explicitly, using the same method we use for instantons below. That is, we expand Ψ in a basis of suitably orthonormalized eigenfunctions of $\not{D}$,

$$\psi(x) = \sum_n a_n\phi_n(x), \quad i\not{D}\phi_n = \lambda_n\phi_n$$

and define the measure by $\mathcal{D}\Psi = \prod_n da_n$, with a similar definition for $\mathcal{D}\bar{\Psi}$. One can then derive the Jacobian above more carefully, getting

$$\mathrm{tr}\,(\delta(x-x)\gamma_5) \to \sum_n \bar{\phi}_n(x)\gamma^5\phi_n(x)$$

which is equally divergent without regularization, but less manifestly nonsensical. One can then regulate the sum by adding a factor of $e^{-\lambda_n^2/M^2}$, then switch to a plane wave basis, recovering the expression above.

• Next, moving the derivative through the exponential with the identity

$$f(\partial)e^{ikx} = e^{ikx}f(\partial + ik)$$

we have

$$\delta(x-y) \to \int dk e^{ik(x-y)} e^{-(i\not{D}-\not{k})^2/M^2}$$

Expanding the exponential, we have

$$-(i\not{D}-\not{k})^2 = -\not{k}^2 + i\{\not{k},\not{D}\} + \not{D}^2 = -k^2 + 2ik\cdot D + D^2 + \frac{1}{2}\left[\gamma^\mu,\gamma^\nu\right] D_\mu D_\nu$$

where we used standard gamma matrix identities and $\gamma^\mu\gamma^\nu = \left(\{\gamma^\mu,\gamma^\nu\} + [\gamma^\mu,\gamma^\nu]\right)/2$. Finally, we replace $D_\mu D_\nu$ with $[D_\mu, D_\nu]/2 = igF_{\mu\nu}/2$.

• Finally, rescaling k by M, we have

$$\delta(x-y) \to M^4 \int dk e^{iMk(x-y)} e^{-k^2} e^{2ik\cdot D/M + D^2/M^2 + igF_{\mu\nu}[\gamma^\mu,\gamma^\nu]/4M^2}$$

Therefore, we have

$$\operatorname{tr}\delta(x-x)\gamma_5 \to M^4 \int dk e^{-k^2} \operatorname{tr} e^{2ik\cdot D/M + D^2/M^2 + igF_{\mu\nu}[\gamma^\mu,\gamma^\nu]/4M^2}\gamma_5$$

• We now expand the exponential in powers of $1/M$, where only terms up to M^{-4} survive the $M\to\infty$ limit. The only way to get a nonzero trace is to have four gamma matrices, so we get
$\operatorname{tr}\delta(x-x)\gamma_5 \to \int dk e^{-k^2} \frac{1}{2}\left(\frac{ig}{4M^2}\right)^2 \operatorname{tr} F_{\mu\nu}F_{\rho\sigma}\left[\gamma^\mu,\gamma^\nu\right]\left[\gamma^\rho,\gamma^\sigma\right]\gamma_5 = \frac{ig^2}{2}\epsilon^{\mu\nu\rho\sigma}\operatorname{tr}\left(F_{\mu\nu}F_{\rho\sigma}\right)\int dk e^{-k^2}$
using standard identities, where the remaining trace is over the group. The Gaussian integral is

$$\int dk e^{-k^2} = \frac{i}{(4\pi)^2}$$

where the factor of i is from Wick rotation, and the denominator is the usual loop factor.

• Putting everything together, we have

$$(\det J)^{-2} = \exp\left(-\frac{ig^2}{16\pi^2}\int dx\alpha(x)\epsilon^{\mu\nu\rho\sigma}\operatorname{tr} F_{\mu\nu}F_{\rho\sigma}\right)$$

Thus, the invariance of $Z(A)$ implies that

$$\left\langle \partial_\mu j_A^\mu + \frac{g^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\operatorname{tr} F_{\mu\nu}F_{\rho\sigma}\right\rangle = 0$$

with the same holding up to contact terms in correlators, as desired.

Now we reflect on the result.

• Taking $\alpha(x)$ to be constant, we have additionally shown that a chiral redefinition of a fermion field can induce extra terms in the Lagrangian, which is important for understanding the strong CP problem.

• The Adler-Bardeen theorem states that the anomaly is one-loop exact. It can be proven by a careful study of Feynman diagrams, or by noting that we never expanded in g above. (right?)

• By direct computation, we see the divergence of the current is a total derivative,

$$\epsilon^{\mu\nu\rho\sigma} \operatorname{tr} F_{\mu\nu} F_{\rho\sigma} = \partial_\mu \left(4\epsilon^{\mu\nu\rho\sigma} \operatorname{tr} \left(A_\nu \partial_\rho A_\sigma - \frac{2}{3} i g A_\nu A_\rho A_\sigma \right) \right)$$

where the quantity in brackets is called a Chern-Simons current. In particular, given an anomalous symmetry, we can construct an alternative current that is conserved. However, note that this current is not gauge-invariant.

• Note that the total value of the axial charge cannot change for topologically trivial configurations. Such changes must be mediated via instantons, but instantons don't exist for $U(1)$ gauge fields, so the axial charge is conserved even if the current locally isn't.

• A better proof for the one-loop exactness of the anomaly uses topology. Imagine higher-order corrections to the anomaly. The most general answer has the form

$$\partial_\mu j_A^\mu = f(g) g^2 \epsilon^{\mu\nu\rho\sigma} \operatorname{tr} F_{\mu\nu} F_{\rho\sigma}$$

by dimensional analysis and parity invariance. Integrating over spacetime,

$$\Delta Q_A = f(g) \int dx g^2 \epsilon^{\mu\nu\rho\sigma} \operatorname{tr} F_{\mu\nu} F_{\rho\sigma}$$

Assuming the gauge group is non-abelian, both the integral and the left-hand side are integers, so $f(g)$ cannot depend on g, and hence is one-loop exact. (This proof doesn't work in the abelian case because both sides vanish identically.)

• A similar derivation can be used for any even dimension $d = 2n$, giving

$$\partial_\mu j^{\mu 5} = (-1)^{n+1} \frac{2g^n}{n!(4\pi)^n} \epsilon^{\mu_1 \cdots \mu_{2n}} F_{\mu_1\mu_2} \cdots F_{\mu_{2n-1}\mu_{2n}}$$

Note that in $d = 2$, anomalies only arise from abelian gauge groups, as otherwise $\operatorname{tr} F_{\mu\nu} = 0$.

Note. The meaning of an anomaly for a global symmetry can be described directly in terms of the path integral as follows. Suppose j^μ is the current for a global symmetry. We can formally couple the symmetry to a background gauge field, which in the simplest case can be done by adding $j^\mu A_\mu$ to the Lagrangian for a background field A_μ, giving a partition function $Z[A]$. The symmetry can be gauged if we add in the Yang-Mills action for A_μ and integrate $Z[A]$ over gauge-inequivalent A_μ. This cannot be done if the global symmetry is anomalous, which occurs precisely when $Z[A]$ is not gauge-invariant.

Anomalies in the Standard Model

Next, we show gauge anomaly cancellation in the SM, introducing non-abelian anomalies.

• In the SM, we have the three currents j_μ^C, j_μ^L and j_μ^Y, which couple to $SU(3)_C, SU(2)_L$ and $U(1)_Y$ respectively. (For brevity, we'll drop the subscripts.) Gauge anomaly cancellation requires

$$\partial_\mu \left\langle j_\mu^i j_\nu^j j_\rho^k \right\rangle = 0$$

for all combinations of i, j, and k.

• In the case where all of the currents are j_μ^Y, we have the $U(1)^3$ anomaly

$$\partial_\mu j_Y^\mu = \left(\sum_{\text{left}} Y_i^3 - \sum_{\text{right}} Y_i^3 \right) \frac{g^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} B_{\rho\sigma}$$

where $B_{\mu\nu}$ is the $U(1)$ field strength and g is the associated coupling.

• For the anomaly to vanish, we need

$$0 = \left(2Y_L^3 - Y_e^3 - Y_\nu^3\right) + 3\left(2Y_Q^3 - Y_u^3 - Y_d^3\right)$$

where the 3 accounts for color charge, the 2 accounts for the two members in a doublet, and we've added a right-handed neutrino for later.

• Plugging in the numbers

$$Y_L = -\frac{1}{2}, \quad Y_e = -1, \quad Y_\nu = 0, \quad Y_Q = \frac{1}{6}, \quad Y_u = \frac{2}{3}, \quad Y_d = -\frac{1}{3}$$

we find the anomaly vanishes for each generation, but not for the quarks or leptons individually.

• Next, consider triangle diagrams with three of the same non-abelian current. Here a fermion ψ in a representation R contributes to the current as

$$j_\mu^a = \bar{\psi}_i \left(T_R^a\right)_{ij} \gamma^\mu T_j$$

Therefore, the two diagrams pick up factors of $\operatorname{tr}\left(T_R^a T_R^b T_R^c\right)$ and $\operatorname{tr}\left(T_R^a T_R^c T_R^b\right)$ respectively.

• By substituting $T_R^a T_R^b = \left(\left[T_R^a, T_R^b\right] + \left\{T_R^a, T_R^b\right\}\right)/2$, we have

$$\operatorname{tr}\left(T_R^a T_R^b T_R^c\right) = \frac{i}{2} T_R f^{abc} + \frac{1}{4} d_R^{abc}, \quad d_R^{abc} = 2\operatorname{tr}\left(T_R^a \left\{T_R^b, T_R^c\right\}\right)$$

The quantity d_R^{abc} is a totally symmetric rank 3 tensor. In the case of $SU(n)$, there is only one such tensor, so we have

$$d_{abc}^R = A(R) d_{abc}$$

where d_{abc} is defined with the generators in the fundamental representation, and $A(R)$ is called the anomaly coefficient of the representation R.

• The term proportional to f^{abc} contributes through the difference of the diagrams and is UV divergent. However, it does not contribute to the anomaly; instead it merely renormalizes the three gauge boson vertex.

• The term proportional to d_{abc}^R contributes to the anomaly by

$$\partial_\mu j^{\mu a}(x) = \left(\sum_{\text{left}} A\left(R_i\right) - \sum_{\text{right}} A\left(R_i\right)\right) \frac{g^2}{128\pi^2} d^{abc} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^b F_{\rho\sigma}^b$$

where $F_{\mu\nu}$ is the associated gauge field strength and g is the coupling. The overall factor is fixed by $U(1)$, where we have $T^a = 1$ and hence $d^{abc} = 4$.

• Since $T_{\bar{R}}^a = -\left(T_R^a\right)^T$, the anomaly coefficient obeys

$$A(R) = -A(\bar{R})$$

Since the representations of $SU(2)$ are all pseudoreal, there is no $SU(2)^3$ anomaly because the anomaly coefficients all vanish. Moreover, there is no $SU(3)^3$ anomaly because QCD is non-chiral. The anomaly coefficient obeys the useful identities

$$A\left(R_1 \oplus R_2\right) = A\left(R_1\right) + A\left(R_2\right), \quad A\left(R_1 \otimes R_2\right) = A\left(R_1\right) d\left(R_2\right) + d\left(R_1\right) A\left(R_2\right)$$

so we can compute anomaly coefficients by building up from the fundamental.

Now we consider the mixed anomalies involving different currents. In this case, the trace is taken over matrices which are tensor products of the matrices associated with each group factor.

• An $SU(n)U(1)^2$ anomaly would be proportional to $\operatorname{tr}(T_R^a\{1,1\}) \propto \operatorname{tr} T_R^a = 0$, so they automatically vanish. Similarly, any anomaly with exactly one factor of $SU(2)$ or $SU(3)$ vanishes. Therefore, the only cases to check are $SU(3)^2U(1)$ and $SU(2)^2U(1)$.

• One has to be careful when applying this point: not all $SU(n)U(1)^2$ anomalies vanish, only those where $SU(n)\times U(1)$ is a symmetry. For example, the $SU(2)_A U(1)^2$ anomaly does not vanish, and this is precisely what accounts for the decay $\pi^0 \to \gamma\gamma$.

• The $SU(3)^2U(1)$ anomaly only receives contributions from quarks, of the form

$$Y \operatorname{tr}\left(\left\{T^a, T^b\right\}\right) \propto \delta^{ab}$$

which means the overall contribution is

$$\delta^{ab}\left(6Y_Q - 3Y_u - 3Y_d\right) = 0$$

where the right-handed quarks contribute negatively as usual.

• The $SU(2)^2U(1)$ anomaly receives contributions from left-handed fields only, giving

$$\delta^{ab}\left(2Y_L + 6Y_Q\right) = 0$$

• The final constraint is from the gravitational anomaly, which comes from diagrams with two external gravitons and one external gauge boson. We find

$$\partial^\mu j_\mu^a \propto \operatorname{tr}(T_R^a)\,\epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\alpha\beta} R_{\rho\sigma}^{\alpha\beta}$$

Since the generators of $SU(n)$ are traceless, the only constraint is from $\text{grav}^2\, U(1)$, which gives

$$\sum_{\text{left}} Y_i - \sum_{\text{right}} Y_i = 0$$

• In summary, we have four constraints on six hypercharges. There are two two-parameter families of allowed hypercharges, one of which is ruled out by demanding $Y_Q \neq 0$. The other is

$$Y_L = -\frac{a}{2} - b, \quad Y_e = -a - b, \quad Y_\nu = -b, \quad Y_Q = \frac{a}{6} + \frac{b}{3}, \quad Y_u = \frac{2a}{3} + \frac{b}{3}, \quad Y_d = -\frac{a}{3} + \frac{b}{3}$$

Note that we may also swap Y_d and Y_u, and Y_e and Y_ν, since the constraints don't distinguish between them; there is also an arbitrary hypercharge normalization we've neglected. The coupling a corresponds to hypercharge as it is in the SM, while b corresponds to $B - L$.

• No matter what the values of a and b are, we must have $Y_L + 3Y_Q = 0$ exactly, indicating that the electron and proton have exactly opposite electric charges. However, for general $b \neq 0$, the neutron is not electrically neutral, and charge is not quantized.

• If there is no right-handed neutrino, or if it is Majorana, then we must have $Y_\nu = 0$, implying $b = 0$ and therefore fixing the SM hypercharges up to scaling. On the other hand, if the opposite is true, then it is possible to extend the Standard Model by gauging both $U(1)_Y$ as in the usual Standard Model, and $U(1)_{B-L}$.

For more about anomaly coefficients, see the notes on Group Theory. Next, we consider global anomalies in the Standard Model.

• Recall the $U(1)_A$ axial symmetry considered earlier. We cannot compute an anomaly from a $U(1)_A^3$ triangle diagram, because there is nothing $\partial_\mu j_A^\mu$ can be equal to. Instead the anomaly is due to $U(1)_A U(1)_V^2$, since the latter is associated with the electromagnetic gauge group.

• By similar reasoning, all anomalies for a global symmetry G come from GH^2 triangle diagrams, where H is one of the gauge groups of the SM. We must use H^2 since the trace of a single factor is zero.

• Two important global symmetries are baryon number and lepton number. Now, we have anomalies due to $SU(2)^2U(1)_B$ and $SU(2)^2U(1)_L$, giving

$$\partial_\mu j^{\mu B} = \partial_\mu j^{\mu L} \supset n_g \frac{g^2}{32\pi^2} \epsilon^{\mu\nu\rho\sigma} W^a_{\mu\nu} W^a_{\rho\sigma}$$

where $n_g = 3$ is the number of generations. But if there is a sterile neutrino, $U(1)_{B-L}$ is nonanomalous, as we saw earlier, so it can be gauged. Such gauge bosons are common in grand unified theories and cause baryon number violation. Note that in neutral atoms, such a gauge boson would couple to neutron number. If gauged $U(1)_{B-L}$ is unbroken, it would need an exceptionally small coupling. If it is broken, the gauge boson mass is at least 3.5TeV.

• The right-hand side is a total derivative, which means that baryon and lepton number are conserved perturbatively. However, there are $SU(2)$ instantons for which the right-hand side integrates to a nonzero value over spacetime; these mediate violation of baryon and lepton number. The rate for this process is extremely small, but at high temperatures, it can be much higher because the transitions are caused by thermal fluctuations rather than quantum effects. In this case, we say the process is mediated by sphalerons.

• There are also anomalies due to $U(1)_Y^2 U(1)_B$ and $U(1)_Y^2 U(1)_L$. However, since there are no $U(1)$ instantons, such anomalies cannot lead to global nonconservation of baryon or lepton number, so they are less important.

• Next, we consider theta terms. The theta term of $U(1)_Y$ has no physical effects, because there are no $U(1)$ instantons. The theta term of $SU(2)$ may be rotated away by the $SU(2)^2U(1)_B$ anomaly. There is also an $SU(2)^2U(1)_L$ anomaly, which matches the $U(1)_B$ anomaly so that $U(1)_{B-L}$ is nonanomalous.

• However, the theta term of $SU(3)$ cannot be removed. The reason that we could easily remove the theta term of $SU(2)$ was because of the chiral structure of the weak force; rotating all quarks contributes to the anomaly because $SU(2)$ only couples to left-handed quarks.

• For $SU(3)$, we may try $SU(3)^2U(1)_A$ since $U(1)_A$ is chiral, but $U(1)_A$ is not a symmetry of the SM because of the quark masses. Instead, such an axial transformation changes the phases of the quark Yukawa couplings, and the invariant quantity is

$$\bar{\theta} = \theta - \arg\det M$$

where M is the quark mass matrix. The fact that $\bar{\theta}$ is observed to be zero within experimental error is the strong CP problem, as explained further in the notes on the Standard Model.

Note. A bit more about baryon and lepton number violation. In order to see what an $SU(2)_L$ instanton does, we note there are $12 SU(2)_L$ doublets in the SM (3 families of left-handed leptons, and 3×3 families and colors of left-handed quarks). The number of corresponding particles changes by 1 in each, so for a unit instanton we have

$$\Delta L_e = \Delta L_\mu = \Delta L_\tau = 1, \quad \Delta B = 3$$

Of course such a process must still conserve energy, electric charge, and color charge. Also note that instantons can't make isolated protons decay, because baryon number is violated in multiples of 3.

Finally, we describe the technique of anomaly matching.

• As an example, consider pure QCD with three massless quark flavors, where the global symmetry is $G = SU(3)_L \times SU(3)_R \times U(1)_V$. There are no $SU(3)^2 G$ anomalies, which ensures G remains a symmetry in a background gauge field, though there are G^3 anomalies.

• We now imagine gauging the symmetry G with an arbitrarily weak coupling. The G^3 anomalies now render the theory inconsistent, but they can be removed by adding spectator fermions. They are only coupled to the existing particles through the gauge field, and can be effectively decoupled by taking the gauge coupling arbitrarily small.

• Now consider the low-energy theory where quarks are confined. Since anomalies are infrared effects, gauge anomaly cancellation must still hold. This must be due to massless particles in the spectrum, which could be either Goldstone bosons or massless hadrons.

• Suppose the latter occurs, so $SU(3)_L \times SU(3)_R$ is not spontaneously broken, and consider the $SU(3)_L^3$ anomalies. In the deconfined phase, the left-handed quarks each contribute A (fund) = 1, and there are three colors, so the spectator fermions contribute an anomaly coefficient of -3.

• Now, for the anomalies to be canceled in the confined phase, color singlet fermions constructed from quarks must provide a total anomaly coefficient of 3. We have

$$3 \times 3 \times 3 = 10 + 8 + 8 + 1$$

and the 8 cannot contribute since it is a real representation. To evaluate $A(10)$, note

$$A(6) = A(3 \times 3) = A(\overline{3}) = 3A(3) + 3A(3) - A(\overline{3}) = 7$$

and

$$A(10) = A(6 \times 3) - A(8) = 3A(6) + 6A(3) - A(8) = 27.$$

Hence a 10 of baryons can only contribute in multiples of 27. Thus, for consistency, pure QCD with three flavors must have spontaneous chiral symmetry breaking!

• In real QCD, chiral symmetry breaking indeed occurs, and the anomalies are transferred over to the Goldstone bosons. For example, anomaly matching allows us to conclude there must be a $U(1)_{\pi^0} U(1)_V^2$ anomaly, where $U(1)_{\pi^0} \subseteq SU(2)_A$ is associated with the π^0. Indeed, this was our original motivating example.

7.4 Hypothetical particles and their phenomena in a nutshell

(тоже всякое можно предполагать, вот сюда и напишу коротко осоновные предположения)

On dark photon

Intro to dark photon by Zhou (??)

Currently, there are many experiments searching for the dark photon, a hypothetical particle whose motivation is described in the notes on Cosmology. The dark photon acts like the regular photon, but it has a mass $m_{A'}$ and couplings ϵe to fermions, where ϵ is the kinetic mixing parameter. Accelerator experiments attempt to produce the dark photon and detect its decay. The dominant

production channel depends on the experiment, such as whether it uses lepton or hadron beams, but some important production channels are:

- Electron-positron annihilation, $e^+e^- \to \gamma A'$. At high energies, the rate for this is ϵ^2 smaller than the rate for $e^+e^- \to \gamma\gamma$.

- A' Bremsstrahlung. Most processes can also produce an A' by attaching it to a fermion line.

- Meson decay. For example, pions can decay as $\pi^0 \to \gamma\gamma$, though we won't be able to calculate this rate for some time. However, it is simply related to the rate for $\pi^0 \to \gamma A'$.

The A' can then decay, e.g. through $A' \to e^+e^-$, which is trivial to calculate given what we already know, or through $A' \to \pi^+\pi^-$, which is somewhat more tricky. The way we detect this decay depends on the lifetime of the A', which generally scales as $\tau \sim 1/\Gamma \sim 1/\alpha\epsilon^2 m_{A'}$. The dark photon will travel a distance $\gamma c\tau$, but the opening angle of the decay products is $\sim 1/\gamma$, so the quantity that determines experimental resolution turns out to be

$$\ell \sim c\tau \sim \left(10^{-6}\text{ m}\right)\left(\frac{10^{-3}}{\epsilon}\right)^2 \frac{10\text{MeV}}{m_{A'}}.$$

For $\ell \lesssim 10\mu$m, the decay is "prompt", i.e. it happens at the same point as production, up to the detector resolution. In this case, individual dark photon events look like background events, but they can be found by computing the invariant mass of the decay products and looking for a statistically significant bump at $m_{A'}$. For larger ℓ, we enter the "displaced" regime, where the decay products can be measured to be coming from a point away from the interaction point. Here the signal rate tends to be lower, but measuring a displaced decay dramatically cuts down on backgrounds. Finally, if ℓ is larger than the detector, we enter the "long-lived" regime, where the dark photon flies out undetected. These can be detected by looking for "missing energy/momentum", but this procedure introduces its own backgrounds and uncertainties. Alternatively, one can search for the dark photon's decay inside a shielded secondary detector, far downstream of the main detector. These events would be very rare, but backgrounds are very low. Furthermore, in this case the dark photon doesn't need to be traced back to the event that produced it, so we can enhance the luminosity by using a "beam dump", crashing a beam into a wall rather than colliding it with another beam. Every one of these options is being considered by several completed, ongoing and proposed experiments around the world, including BaBar, Belle II, APEX, MAMI, MESA, DarkLight, HPS, DarkQuest, NA48, and NA42 (for prompt and displaced decays) and GAZELLE, CODEXb, FASER, MATHUSLA, E137, E141, E774, NA64, CHARM, NuCal, and NA62 (for long-lived particles).

7.5 Gauge symmetries, theories, lattices

(тут будет конспект Полякова и чего-то еще, мб Тонга)

7.5.1 Strong Coupling Approximation

Модель Изинга

Непрерывные глобальные симметрии

Калибровочные симметрии

7.5.2 On non-renormalizable theories

(У Шварца почитаю, пока не думал про это особо, ну есть такие и есть)

Summary of non-renormalizable theories

We have looked at four important non-renormalizable theories:

- The Schrödinger equation is perturbative for $E < m_e$. Its UV completion is the Dirac equation and QED, which is perturbative up to its Landau pole, $E \sim 10^{286}$GeV.

- The Fermi theory of weak interactions is perturbative for $E < G_F^{-1/2} \sim 300$GeV. Its UV completion is the electroweak theory with massive vector bosons W and Z. The electroweak theory is also non-renormalizable. Its UV completion contains a Higgs boson.

- The Chiral Lagrangian is the low-energy theory of pions. It is perturbative and very predictive for $E < 4\pi F_\pi \sim 1200$MeV. Its UV completion is QCD. QCD is predictive at high energies. The fields in QCD, quarks and gluons, are related to pions and other hadrons (quark and gluon bound states) in a complicated, non-perturbative way. Thus, to study hadrons in QCD, we need non-perturbative methods, such as the lattice. In contrast, at low energy the Chiral Lagrangian is perturbative and therefore more useful than QCD for answering certain questions.

- General relativity is the low-energy theory of gravity. It is perturbative for $E < M_{\rm Pl} \sim 10^{19}$GeV. It is extremely predictive at low energies, including predictive quantum corrections. One possible UV completion is string theory. Gravitational physics at distances larger than the Planck length, 10^{-33} cm, is independent of the UV completion, which explains why string theory (as a quantum theory of gravity) is so hard to test.

These four examples correspond to the four forces of nature: the electromagnetic force, the weak force, the strong force, and gravity. Notice that the UV completions are all qualitatively very different. In some cases, certainly for many physical applications, the non-renormalizable theory is more useful than the renormalizable one. Renormalizable just means there are a finite number of counterterms; it does not mean that you can calculate every observable perturbatively.

7.5.3 Instantons in Abelian Systems (after Polyakov)

(??? по Полякову это ведь???)

Инстантоны в квантовой механике и в модели Изинга

Инстантоны в модели с глобальной O(2)-симметрией

Компактная КЭД O(2)-калибровочная модель

7.6 Specific properties of QFT (???)

(тут спецификa, которой без большой потребности не планирую заниматься)

7.6.1 On QFT in Real Time (???)

(это современная тема, которой заниматься имеет смысл только если сделаны уже хооро известные науке темы.)

7.6.2 On Integrable Quantum Fields

(Есть такой курс для магистратуры теоретиков, мне пока это не интересно. Вообще, почитаю потом хот бы Лашкевича про это когда-то)

7.6.3 On Other Number Fields, Special Algebra for QFT

(что если у нас кватернионы описывают квантовые поля? или p-аддические числа? или еще какая-то алгебраическая модификация КТП? Если все эти большие вопросы буду изучать, то тут буду указывать выводы.)

7.7 Condensed matter models

(see note on cond matter for now)

7.7.1 Critical Exponents and Scalar Field Theory (after Peskin & Schroeder)

(см. пока запись по статфизу.)

Теория критических показателей

Критическое поведение в четырех измерениях

Нелинейная сигма-модель

Потенциал Коулмена-Вайнберга

(ПШ, конец 2 части)

7.7.2 Typical Phase Transitions in Matter (!?)

(см. пока запись по конд. средам.)

Пайерлсовская неустойчивость

(По Зи)

7.7.3 Quantum Phase Transitions in Matter (!?)

(see note on statphys for now)

7.7.4 Superconductivity [see note on SC]

(см. пока запись по конд. средам.)

О диаграммной технике для сверхпроводников

О явления со сверхпроводниками методами КТП

7.7.5 Disorder in Matter

(см. пока запись по конд. средам.)

7.8 Other Modifications of QFT in a Nutshell (?)

(нужно будет какая-то - укажу особенности)

7.8.1 QFT in curved space-time

(???)

7.8.2 The Euler-Heisenberg Lagrangian

Finally, we sketch the derivation of the Euler-Heisenberg Lagrangian, which results from integrating out the electron.

• We expect we should be able to integrate out the electron because it appears quadratically in the QED action,

$$S[A,\psi] = \int dx \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(\not{D} + m)\psi$$

Since the electron is fermionic, we get a functional determinant in the numerator,

$$\Gamma_{\text{eff}}[A] = \frac{1}{4}\int dx F^{\mu\nu} F_{\mu\nu} - \hbar \log\det(\not{D} + m)$$

Here, the determinants are in both spinor space and functional space.

• To understand the functional determinant better, note that we can expand

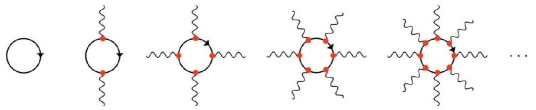
$$\log\det(\not{D} + m) = \log\det(\not{\partial} + m) + \operatorname{tr}\log\left(1 + ie(\not{\partial} + m)^{-1}\not{A}\right)$$

where we used $\log\det M = \operatorname{tr}\log M$. The first term comes from an electron loop vacuum bubble. The other term gives an infinite series in powers of A,

$$\operatorname{tr}\log\left(1 + ie(\not{\partial} + m)^{-1}\mathcal{A}\right) = -\sum_{n=1}^{\infty}\frac{(-ie)^n}{n}\int\prod_{i=1}^{n} dx_i \operatorname{tr}\left(S\left(x_n, s_1\right)\mathcal{A}(x_1)...S\left(x_{n-1}, x_n\right)\mathcal{A}(x_n)\right)$$

where the trace on the right is just a Dirac trace, and the functional trace is handled by the integration. All terms odd in A vanish by Furry's theorem, and $S\left(x_i, x_{i+1}\right)$ is the Dirac propagator, i.e. the inverse of the kinetic term.

• Thus the diagrams that contribute are



so the effective action has interaction terms for any even number of photons. To make the calculation tractable, we will assume the electromagnetic field is constant while taking the trace, which is equivalent to neglecting derivative terms; from an effective field theory standpoint this is valid if we consider light with frequency much less than the electron mass.

• As a result, the first diagram contributes only to vacuum energy, the second renormalizes the photon field, and all the others produce new interactions, which mediate light-by-light scattering. Thus the Euler-Heisenberg Lagrangian is useful when considering high-intensity, low-frequency light.

• We now turn to the explicit evaluation of $\log\det(\not{D} + m)$.

Since the trace of an odd number of gamma matrices vanishes, we must have

$$\operatorname{tr}\log(\not{D} + m) = \operatorname{tr}\log(-\not{D} + m)$$

which gives

$$\operatorname{tr}\log(\not{D}+m)=\frac{1}{2}(\log\det(\not{D}+m)+\log\det(-\not{D}+m))=\frac{1}{2}\log\det\left(-\not{D}^2+m^2\right).$$

• Next we can simplify $\not{D}^2$ as

$$\not{D}^2=\gamma^\mu\gamma^\nu D_\mu D_\nu=\left(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}+\frac{1}{2}[\gamma^\mu,\gamma^\nu]\right)D_\mu D_\nu=D^\mu D_\mu-eS^{\mu\nu}F_{\mu\nu}$$

The same manipulations yield the electron's magnetic moment in the Dirac equation.

• Therefore, we want to evaluate the functional

$$\begin{aligned}\frac{1}{2}\operatorname{tr}\log\left(-D^2+eS^{\mu\nu}F_{\mu\nu}+m^2\right)&\\ =\frac{1}{2}\operatorname{tr}\log\left(-\left((\partial+ieA)^2+m^2\right)\begin{pmatrix}1&0\\0&1\end{pmatrix}+e\begin{pmatrix}(\mathbf{B}+i\mathbf{E})\cdot\boldsymbol{\sigma}&0\\0&(\mathbf{B}-i\mathbf{E})\cdot\boldsymbol{\sigma}\end{pmatrix}\right)&\end{aligned}$$

where the $\boldsymbol{\sigma}$ are the usual Pauli matrices and the electromagnetic field is constant.

• Suppressing the spinor indices temporarily, we can perform the functional trace in position space, giving

$$\Gamma_{\text{eff}}[A]=\int dx\frac{1}{4}F^{\mu\nu}F_{\mu\nu}+\frac{1}{2}\langle x|\log\left(-D^2+eS^{\mu\nu}F_{\mu\nu}+m^2|x\right\rangle$$

To remove the logarithm, we use the asymptotic relation

$$\lim_{s_0\to0^+}\int_{s_0}^{\infty}e^{-sX}\frac{ds}{s}=-\log(s_0X)+\text{ finite}$$

to find

$$\frac{1}{2}\left\langle x\left|\log\left(-D^2+eS^{\mu\nu}F_{\mu\nu}+m^2\right)\right|x\right\rangle=\lim_{s_0\to0^+}\frac{1}{2}\int_{s_0}^{\infty}\frac{ds}{s}e^{-sm^2/2}\left\langle x\left|e^{-s\left(-D^2+eS^{\mu\nu}F_{\mu\nu}\right)}\right|x\right\rangle.$$

Thus the problem has been reduced to nonrelativistic quantum mechanics; we essentially have to find the energy eigenstates of the Pauli equation in position space. For example, in the case of a constant magnetic field, the solutions are Landau levels.

• A complicated calculation yields the effective Lagrangian

$$\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\lim_{s_0\to0^+}\frac{e^2}{64\pi^2}\int_{s_0}^{\infty}\frac{ds}{s}e^{-sm^2}\frac{\operatorname{Re}\cosh(esX)}{\operatorname{Im}\cosh(esX)}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma},\quad X=\sqrt{\mathbf{E}^2-\mathbf{B}^2+\mathbf{E}\cdot\mathbf{B}}.$$

Finally, we need to handle the UV divergences, which occur for small s. Expanding the integrand as a series in es, we find

$$\frac{\operatorname{Re}\cosh(esX)}{\operatorname{Im}\cosh(esX)}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \frac{4}{e^2s^2} + \frac{2}{3}F^{\mu\nu}F_{\mu\nu} - \frac{e^2s^2}{45}\left((F^{\mu\nu}F_{\mu\nu})^2 + \frac{7}{4}\left(\widetilde{F}^{\mu\nu}F_{\mu\nu}\right)^2\right) + O\left(e^4\right)$$

and the first two terms are a vacuum energy and photon field renormalization, which we remove by minimal subtraction, and all other terms are finite. This yields the Euler-Heisenberg Lagrangian. It is fully nonperturbative in e, since the real expansion parameter is E/m_e.

Note. One might wonder what the most general terms in the series above look like. In particular, is it true that every term is built out of $F^2 \sim E^2 - B^2$ and $F\tilde{F} \sim \mathbf{E}\cdot\mathbf{B}$? One slick way to prove this is to consider the complex vector $\mathbf{E} + i\mathbf{B}$. Boosts correspond to imaginary rotations of this vector, so rotations and boosts together cover all possible rotations. But the only invariant of a vector with respect to rotation is its square, and

$$(\mathbf{E} + i\mathbf{B})\cdot(\mathbf{E} + i\mathbf{B}) = \left(E^2 - B^2\right) + 2i(\mathbf{E}\cdot\mathbf{B}).$$

Note that this proof only works in $d = 4$.

7.8.3 Supersymmetry (???)

(Вайнберг целую книгу написал, я хотя бы раздел один создам с указанием)

7.8.4 String Theory

(see note on string theory first)

7.9 Specific applications of QFT in a nutshell

7.9.1 On methods of QFT for quantum optics

(see note on q. optics first)

7.9.2 Waveguide QFT: basics ans useful methods

(see note on wQED, it is a very specific field of science)

7.9.3 Circuit QFT: basics ans useful methods

(see something in some note on spin models, a specific topic. I'll write maybe later main ideas, models will be discussed in another note)

7.9.4 QFT for finances

(?????)

Part III
QFT Fundamentals and Free Quantum Fields

Обсудим подробно свободные квантовые поля.

8 Formalisms and a general view of QFT

8.1 Fundamental Experiments for QFT

(a brief discussion of the experimental results to support why the formalism is such. details will be at the latter part of the note)

8.2 Philosophy of QFT: Main Ideas (??)

(отдельно прорабатываю это, типичная первая тема. эти все соображения дальше и будут применяться.)

8.2.1 The Idea of the Quantum-Field Approach (????)

Обсудим подробно, зачем конкретно нам вообще КТП?

Почему мы описываем микромир полями?

Для чего мы должны изучать квантование полей? (ПШ,????)

8.2.2 When in Physics "all is field" and when it is not?

На макро уровне это, конечно, бред, однако на микро уровне это начинает иметь смысл. Вот и напишу, чтобы лишний раз про это не думать.

8.2.3 Examples for Introducing the Essence of QFT (??)

(такое очень полезно иметь под рукой)

Рождение пар e^+e^- как введение в КТП по Пескину Шредеру

Главная цель части I этой книги заключается в развитии основного вычислительного метода квантовой теории поля - формализма диаграмм Фейнмана. Затем мы применим этот формализм для вычислений в квантовой электродинамике - квантовой теории электронов и фотонов.

Квантовая электродинамика (КЭД) - вероятно, наилучшая из имеющихся фундаментальных теорий. Она формулируется как набор простых уравнений (Максвелла и Дирака), вид которых определяется, по существу, релятивистской инвариантностью. Квантово-механические решения этих уравнений дают детальные предсказания

электромагнитных явлений в масштабах от макроскопических расстояний вплоть до расстояний, в несколько сотен раз меньших размера протона.

Фейнмановские диаграммы обеспечивают соответствующую элегантную вычислительную схему для этой элегантной теории. Достаточно представить процесс, который может происходить между электронами и фотонами, нарисовать его диаграмму, а затем с помощью диаграммы записать математическое выражение для квантово-механической амплитуды этого процесса.

В этой части книги мы развиваем как теорию КЭД, таю и метод фейнмановских диаграмм, исходя из основных принципов квантовой механики и теории относительности. В конце концов мы приходим к состоянию, которое позволит рассчитывать наблюдаемые величины, представляющие большой интерес при изучении элементарных частиц. Однако чтобы достичь цели и развить простой вычислительный метод, нам, к сожалению, придется сначала серьезно погрузиться в формализм теории. Три главы, следующие за этой, почти полностью формальны, и при их изучении читатель в какой-то момент может задаться вопросом, а куда мы, собственно, движемся? Мы хотели бы хоть частично заранее ответить на этот вопрос, обсудив в этой главе физику особенно простого процесса КЭД, многие свойства которого следуют непосредственно из физической интуиции. Конечно, такой интуитивный приземленный подход содержит много прорех. В гл. 5 мы вернемся к рассмотрению этого процесса, вооруженные всей мощью формализма фейнмановских диаграмм. Спускаясь вниз с горных высот, мы увидим, как ликвидируются все трудности интуитивного подхода.

Простейшая ситуация

Так как большинство экспериментов в физике частиц включает рассеяние, наиболее часто вычисляемыми в квантовой теории поля величинами являются сечения рассеяния. Мы хотим вычислить сечение простейшего, из возможных, процесса КЭД: рождение пары более тяжелых лептонов (например, мюонов) при аннигиляции электрона с его античастицей - позитроном. Существование античастиц, на самом деле, предсказывает квантовая теория поля, и мы будем обсуждать это предсказание в гл. 2

и 3. Сейчас примем его на веру.

Опыт по измерению вероятности такой аннигиляции заключается в наблюдении столкновения пучка электронов с пучком позитронов. Измеряемой при этом величиной является сечение реакции $e^+e^- \to \mu^+\mu^-$как функция энергии в системе центра инерции и относительного угла θ между направлениями полета начальных электронов и конечных мюонов. Процесс показан на рис. 1.1. Для простоты будем работать в системе центра инерции (с. ц. и.), где импульсы частиц удовлетворяют условию $\mathbf{p}' = -\mathbf{p}$ и $\mathbf{k}' = -\mathbf{k}$. Предположим также, что энергия пучков E много больше как массы электрона, так и массы мюона, так что $|\mathbf{p}| = |\mathbf{p}'| = |\mathbf{k}| = |\mathbf{k}'| = E \equiv E_{\text{ци}}/2$. (Мы используем полужирный шрифт для обозначения 3-векторов и обычный курсивный шрифт для обозначения 4-векторов.)

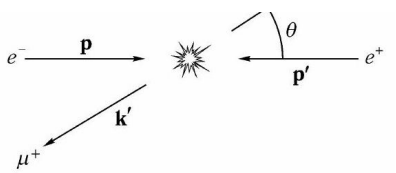


Рис. 1.1. Реакция аннигиляции $e^+e^- \to \mu^+\mu^-$, показанная в системе центра инерции.

Так каю и электрон, и мюон имеют спин 1/2, то нужно задать ориентации их спинов. Удобно выбрать ось, определяющую квантование спина каждой частицы, в направлении ее движения. Тогда спин каждой частицы может быть направлен либо параллельно, либо антипараллельно этой оси. На практике пучки электронов и позитронов часто не поляризованы, а мюонные детекторы не регистрируют поляризацию мюонов. Поэтому нам следует усреднить сечение по ориентациям спина электрона и позитрона и просуммировать его по ориентациям спина мюонов. Для любой заданной ориентации спинов всех частиц принято записывать дифференциальное сечение этого процесса с вылетом μ^+в заданный элемент телесного угла $d\Omega$ как

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{\text{ци}}^2}|\mathscr{M}|^2$$

Множитель $E_{\text{ци}}^2$ обеспечивает правильную размерность сечения, так как в выбранной системе единиц (энергия $)^{-2} \sim$ длина $)^{2}$[2]. Следовательно, величина $\mathscr{M}$ безразмерна. Она представляет квантово-механическую амплитуду рассматриваемого процесса (аналогичную амплитуде f в нерелятивистской квантовой механике). Главный вопрос заключается в том, как вычислить эту амплитуду, исходя из фундаментальных принципов теории. Другие множители в (1.1) являются исключительно результатом соглашения. Формула (1.1) представляет, на самом деле, частный случай более общей формулы, справедливой для рассеяния в с. ц. и., когда конечное состояние содержит две безмассовые частицы. Общую формулу нельзя получить на основании размерных соображений, и мы выведем ее в разделе 4.5.

Теперь наступает время плохих и хороших новостей.

Плохая новость заключается в том, что даже для простейшего из процессов КЭД точное выражение для $\mathscr{M}$ неизвестно. Это не должно удивлять, так как даже в не- релятивистской квантовой механике задача рассеяния редко имеет точное решение. Лучшее, что можно сделать, это получить формальное выражение для $\mathscr{M}$ в виде ряда теории возмущений по константе электромагнитного взаимодействия и вычислить несколько первых членов этого ряда.

Хорошая новость состоит в том, что Фейнман разработал красивый способ упорядочить и сделать наглядным ряд теории возмущений, получивший название метода фейнмановских диаграмм. Грубо говоря, диаграммы изображают потоки электронов и фотонов в процессе рассеяния. В нашем частном случае слагаемое низшего порядка в ряде теории возмущений можно представить единственной диаграммой рис. 1.2. В этой диаграмме имеются элементы трех типов: внешние линии (представляющие четыре начальные и конечные частицы), внутренние линии (представляющие «виртуальные» частицы; в нашем случае - один виртуальный фотон) и вершины. Принято изображать фермионы сплошными линиями, а фотоны - волнистыми. Стрелки на сплошных линиях указывают не направление импульса, а направление потока отрицательного заряда. Каждой внешней линии приписывается определенный 4-вектор импульса, как показано на рисунке. Импульс q внутренней линии на этой диаграмме определяется законом сохранения импульса в каждой из вершин: $q = p + p' = k + k'$. Кроме того, каждому внешнему фермиону следует сопоставить определенное спиновое состояние (либо «вверх»,

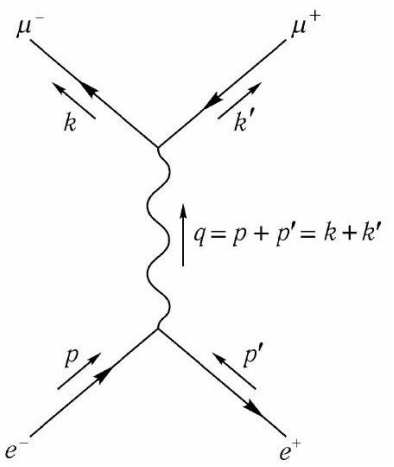


Рис. 1.2. Фейнмановская диаграмма низшего порядка для сечения рассеяния $e^+e^- \to \mu^+\mu^-$. В этом порядке единственным возможным промежуточным состоянием является фотон (γ). либо "Вниз»).

Согласно фейнмановским правилам каждой диаграмме соответствует совершенно определенный вклад в $\mathscr{M}$. Правила приписывают каждому элементу диаграммы компактный алгебраический множитель, так что произведение этих множителей дает значение соответствующего слагаемого в ряде теории возмущений. Однако приведение результирующего выражения для $\mathscr{M}$ к пригодному для использования виду может оказаться нетривиальной задачей. В последующих главах мы разработаем полезную технологию подобных вычислений. Но поскольку сейчас мы ею не владеем, воспользуемся для нахождения ответа для нашей конкретной задачи не точными фейнмановскими правилами, а некоторыми эвристическими аргументами.

Напомним, что в квантово-механической теории возмущений амплитуда перехода в первом порядке может быть записана в виде

$$\langle \text{ конечное состояние } |H_I| \text{ начальное состояние } \rangle ,$$

где H_I - часть гамильтониана, отвечающая за «взаимодействие». В нашем случае начальным является состояние $|e^+e^-\rangle$, а конечным - состояние $\langle\mu^+\mu^-|$. Но гамильтониан взаимодействия связывает электроны с мюонами не напрямую, а только через электромагнитное поле (т. е. фотоны). Поэтому выражение амплитуды первого порядка (1.2) обращается в нуль, и мы должны перейти к рассмотрению вклада второго порядка

$$\mathscr{M} \sim \left\langle \mu^+\mu^- \left|H_I\right| \gamma \right\rangle^{\mu} \left\langle \gamma \left|H_I\right| e^+e^- \right\rangle_{\mu} .$$

В этом и заключается эвристический способ записи $\mathscr{M}$ по диаграмме рис. 1.2. Внешним электронным линиям соответствует множитель $|e^+e^-\rangle$, внешним мюонным линиям - множитель $\langle\mu^+\mu^-|$. Вершинам соответствуют H_I, а внутренней фотонной линии - оператор $|\gamma\rangle\langle\gamma|$. Мы добавили векторные индексы μ, так как фотон является векторной частицей с четырьмя компонентами. Имеются четыре возможных промежуточных состояния, по одному на каждую компоненту. Согласно правилам теории возмущений следует просуммировать по промежуточным состояниям. Заметим, что поскольку сумма в (1.3) имеет вид скалярного произведения 4-векторов, амплитуда $\mathscr{M}$ является лоренц-инвариантным скаляром, если только каждый множитель в (1.3) есть 4-вектор.

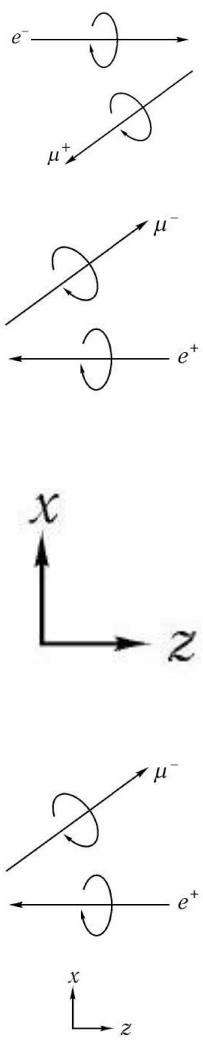


Рис. 1.3. Один возможный набор спиновых ориентаций. Электрон и отрицательный мюон «правые», тогда как позитрон и положительный мюон - «левые».

Попробуем угадать форму вектора $\langle \gamma \left| H_I \right| e^+ e^- \rangle_\mu$. Так как H_I связывает электроны с фотонами с константой e (заряд электрона), матричный элемент должен быть пропорционален e. Рассмотрим конкретный набор начальных и конечных спиновых ориентаций, показанных на рис. 1.3. Спины электрона и мюона параллельны своим направлениям движения: говорят, что эти частицы «правые». Аналогично, античастицы «левые». Спины электрона и позитрона складываются в единицу углового момента в направлении $+z$. Так как H_I должен сохранять угловой момент, фотон, с которым связаны эти частицы, должен обладать правильным вектором поляризации, отвечающим тому же угловому моменту: $\varepsilon^\mu = (0, 1, i, 0)$. Следовательно,

$$\left\langle \gamma \left| H_I \right| e^+ e^- \right\rangle_\mu \propto e(0, 1, i, 0)$$

Аналогично, мюонный матричный элемент должен иметь поляризацию, соответствующую единице углового момента вдоль направления импульса отрицательно заряженного мюона. Чтобы получить правильный вектор, повернем (1.4) на угол θ в плоскости xz:

$$\left\langle \gamma \left| H^I \right| \mu^+ \mu^- \right\rangle^\mu \propto e(0, \cos\theta, i, -\sin\theta)$$

Для вычисления амплитуды $\mathscr{M}$ комплексно сопрягаем этот вектор и скалярно умножаем его на (1.4). Тогда для заданной ориентации спинов получим

$$\mathscr{M}(RL \to RL) = -e^2(1+\cos\theta)$$

Конечно, таким способом невозможно установить общий множитель, но по случайности оказывается, что в (1.6) он правилен благодаря тем соглашениям, которые приняты в (1.1). Заметим, что при $\theta = 180^\circ$ амплитуда обращается в нуль. Этого следовало ожидать: состояние с угловым моментом в направления $+z$ не переходит в состояние с угловым моментом в направлении $-z$. Рассмотрим теперь случай, когда и электрон, и позитрон правополяризованы. Их полный спиновый момент равен нулю. В этом случае нужны более тонкие рассуждения. Можно ожидать, что мы получим продольно поляризованный фотон, причем появится коэффициент Клебша-Гордана $\frac{1}{\sqrt{2}}$, по аналогии со случаем сложения трехмерных угловых моментов: $|\uparrow\downarrow\rangle = (1/\sqrt{2})(|j=1, m=0\rangle + |j=0, m=0\rangle)$. Но на самом деле мы складываем угловые моменты в рамках четырехмерной группы Лоренца, поэтому следует принимать во внимание не только спин (свойства преобразования состояний при вращениях), но и преобразования состояний при бустах. Оказывается (мы покажем это в гл. 3), что коэффициент Клебша-Гордана, связывающий 4-вектор с состоянием $\left|e_R^- e_R^+\right\rangle$безмассовых фермионов, равен нулю. (Суть в том, что такое состояние есть суперпозиция скаляра и антисимметричного тензора.) Следовательно, амплитуда $M(RR \to RL)$ равна нулю, как и одиннадцать других амплитуд, в которых либо начальное, либо конечное состояние имеет полный нулевой угловой момент.

Остающиеся ненулевые амплитуды можно найти тем же способом, как и первую. Имеем:

$$\mathscr{M}(RL \to LR) = e^2(1-\cos\theta)$$
$$\mathscr{M}(LR \to RL) = e^2(1-\cos\theta)$$
$$\mathscr{M}(LR \to LR) = e^2(1+\cos\theta)$$

Подставляя эти выражения в (1.1), усредняя по четырем начальным и суммируя по четырем конечным спиновым ориентациям, находим

$$\frac{d\Sigma}{d\Omega} = \frac{\alpha^2}{4E_{\text{ци}}^2}\left(1+\cos^2\theta\right),$$

где $\alpha = e^2/4\pi \simeq 1/137$. Интегрируя по углам θ и ϕ, получаем полное сечение

$$\sigma_{\text{полн}} = \frac{4\pi\alpha^2}{3E_{\text{ци}}^2}.$$

Результаты (1.8) и (1.9) согласуются с экспериментом с точностью порядка 10%. Почти все расхождение обязано вкладу следующего слагаемого ряда теории возмущений, отвечающего диаграммам рис. 1.4. Качественные особенности полученных выражений - угловая зависимость и резкое уменьшение сечения с энергией - явно подтверждаются данными эксперимента. (Подробнее свойства полученных результатов обсуждаются в разделе 5.1.)

Уточнения и вопросы

Угловое распределение в реакции $e^+e^- \to \mu^+\mu^-$, предсказываемое квантовой электродинамикой, было получено с помощью аргументов, связанных с сохранением

углового момента, без обращения к формализму, лежащему в основании теории. Однако мы существенно использовали упрощения, связанные с пределом высоких энергий и выбором системы центра инерции. Если отказаться от любого из упрощающих предположений, то проделанный нами анализ разрушится. Как же нужно проводить вычисления в КЭД в общем случае? Чтобы ответить на этот вопрос, следует вернуться к фейнмановским правилам.

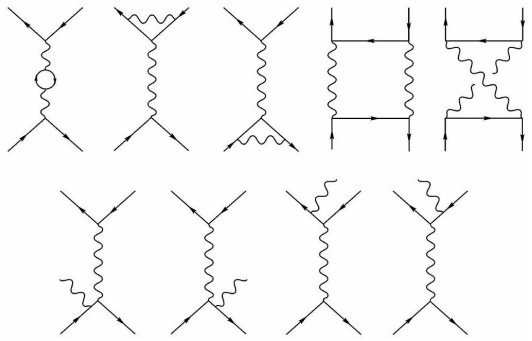


Рис. 1.4. Фейнмановские диаграммы, которые дают вклад α^3 в сечение рассеяния $e^+e^- \to \mu^+\mu^-$.

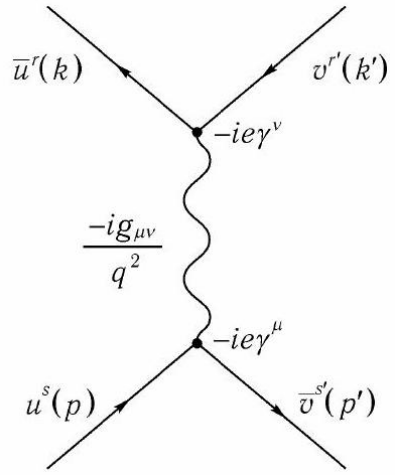


Рис. 1.5. Диаграмма из рис. 1.2, где указаны выражения, соответствующие вершинам, внутренним и внешним линиям.

Выше упоминалось, что эти правила предписывают нам нарисовать диаграмму (диаграммы) рассматриваемого процесса и сопоставить каждому элементу диаграммы определенный алгебраический множитель. На рис. 1.5 приведена диаграмма нашей реакции $e^+e^- \to \mu^+\mu^-$, где показаны предписываемые правилами множители.

Для внутренней фотонной линии пишем множитель $-ig_{\mu\nu}/q^2$, где $g_{\mu\nu}$ - обычный метрический тензор в пространстве Минковского, а q – 4-импульс виртуального фотона. Этот множитель соответствует оператору $|\gamma\rangle\langle\gamma|$ в нашем эвристическом выражении (1.3).

Каждой вершине сопоставляется множитель $-ie\gamma^\mu$, что соответствует H_I в (1.3). Величины γ^μ представляют набор постоянных 4×4-матриц. Именно они осуществляют «сложение угловых моментов», связывая состояния двух частиц со спином $\frac{1}{2}$ с векторной частицей.

Внешним линиям соответствуют выражения для четырехкомпонентных столбцов-спиноров u, v, или строк-спиноров $\bar{u}, \bar{v}$. Они, по существу, являются волновыми функциями начальных и конечных частиц в импульсном пространстве и соответствуют $|e^+e^-\rangle$и $\langle\mu^+\mu^-|$в (1.3). Индексы s, s', r и r' отмечают одно из двух возможных спиновых состояний.

Теперь можно выписать выражение для $\mathscr{M}$, считывая множители непосредственно с диаграммы:

$$\mathscr{M} = \bar{v}^{s'}(p')\,(-ie\gamma^\mu)\,u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^r(k)\,(-ie\gamma^\nu)\,v^{r'}(k') =$$
$$= \frac{ie^2}{q^2}\left(\bar{v}^{s'}(p')\,\gamma^\mu u^s(p)\right)\left(\bar{u}^r(k)\gamma^\nu v^{r'}(k')\right)$$

Полезно детально сравнить это выражение с (1.3). Чтобы получить сечение (1.8) из (1.10), можно было бы вернуться к аргументам, связанным с угловым моментом, дополнив их некоторыми конкретными сведениями о γ-матрицах и дираковских спинорах. Подобные вычисления будут проведены в разделе 5.2. Однако существует множество полезных трюков, используемых при манипуляциях с выражениями типа (1.10), особенно в тех случаях, когда нас интересует вычисление сечения для неполяризованных частиц. Используя эту фейнмановскую "технологию следов» (название происходит от того, что приходится вычислять следы от произведения γ-матриц), даже не обязательно знать явный вид этих матриц и дираковских спиноров. Вычисления становятся почти бездумными, и ответ (1.8) получается после алгебраических преобразований, занимающих меньше страницы. Однако поскольку фейнмановские правила и технология следов столь мощны, то можно отказаться и от некоторых упрощающих предположений. В завершение этого раздела обсудим те ситуации, при которых вычисления станут более трудными.

Простейшее ограничение, от которого можно отказаться, - это равенство нулю массы мюона. Если энергия пучков не намного больше массы мюона, то все наши предсказания будут зависеть от отношения $\frac{m_\mu}{E_{\text{ци}}}$. (Поскольку электрон примерно в 200 раз легче мюона, его можно рассматривать безмассовым во всех случаях, когда энергия пучка достаточна для рождения мюонов.) Используя фейнмановскую технологию следов, чрезвычайно просто учесть в вычислениях массу мюона. Объем необходимых алгебраических вычислений возрастает процентов на 50; кроме того, немного меняется выражение (1.1) для связи амплитуды и сечения, но ответ оправдывает затраченные усилия. Мы проделаем эти вычисления в разделе 5.1.

Столь же просто рассмотреть вычисления в другой системе отсчета. Единственное место, которое подвергнется модификации - это связь (1.1) между амплитудой и сечением. Впрочем, можно просто совершить преобразование Лоренца в формуле в с. ц. и., перейдя к другой системе отсчета.

Когда известны спиновые состояния начальных и/или конечных частиц и мы хотим учесть массу мюона, то вычисления становятся более громоздкими, хотя, в принципе, остаются несложными. Технология следов обобщается на такой случай, но часто легче вычислить выражение (1.10) непосредственно, используя выражения для спиноров u и v.

Далее, можно рассчитать сечения разных процессов.

Процесс $e^+e^- \to e^+e^-$, называемый рассеянием Баба, более сложен, так как существует вторая диаграмма (см. рис. 1.6). Сначала следует сложить амплитуды, отвечающие этим диаграммам, а затем возвести результат в квадрат.

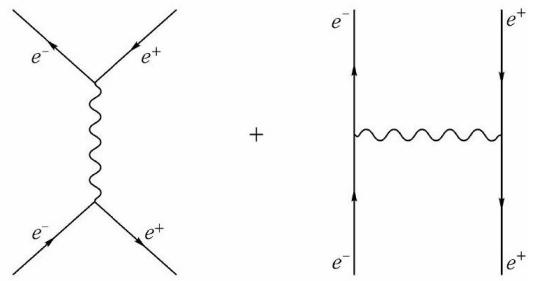


Рис. 1.6. Две диаграммы низшего порядка для рассеяния Баба $e^+e^- \to e^+e^-$.

Другие процессы содержат фотоны в начальном и/или конечном состояниях. Классическим примером является комптоновское рассеяние, две диаграммы низшего порядка для которого показаны на рис. 1.7. Фейнмановские правила для внешних фотонных и внутренних электронных линий не сложнее тех, которые уже рассмотрены. Детальный анализ комптоновского рассеяния проведен в разделе 5.5.

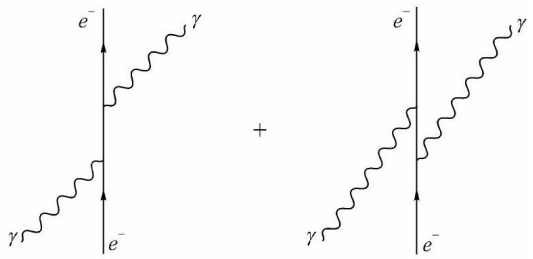


Рис. 1.7. Две диаграммы низшего порядка для комптоновского рассеяния.

Наконец, можно вычислить вклады слагаемых более высокого порядка в ряде теории возмущений. Благодаря Фейнману, по крайней мере, нетрудно нарисовать необходимые диаграммы. На рис. 1.4 были показаны те диаграммы, которые дают вклад в следующий порядок теории возмущений для сечения процесса $e^+e^- \to \mu^+\mu^-$. Весьма примечательно, что алгоритм, приписывающий алгебраические множители отдельным элементам диаграммы, верен для всех вкладов более высоких порядков и позволяет вычислить их непосредственно, хотя такие расчеты очень громоздки. Расчет полного вклада всех девяти диаграмм - серьезная работа на уровне научной статьи.

В этой книге, начиная с гл. 6, мы достаточно подробно проанализируем ту физику, которая возникает при учете фейнмановских диаграмм более высокого порядка типа, показанных на рис. 1.4. Мы увидим, что последние четыре из них, содержащие дополнительный фотон в конечном состоянии, необходимо учитывать потому, что ни один детектор не способен регистрировать фотоны необычайно малых энергий. Поэтому конечное состояние с такими фотонами неотличимо от интересующего нас состояния, где есть только мюонная пара.

Другие пять диаграмм на рис. 1.4 содержат не один виртуальный фотон, а промежуточные состояния нескольких виртуальных частиц. В каждой из этих диаграмм имеется хотя бы одна виртуальная частица, импульс которой не определяется законом сохранения импульсов в вершинах. Поскольку теория возмущений требует взять сумму по всем возможным промежуточным состояниям, мы должны проинтегрировать по всем возможным значениям этого импульса. Однако на этом этапе возникает новая трудность. В первых трех диаграммах при наивном подходе интегралы по импульсу в петле оказываются бесконечными. В конце части I мы сформулируем рецепт решения этой проблемы с тем,

чтобы получить конечные результаты. Однако вопрос о физической природе таких расходимостей просто не решается. Это будет главной темой части II книги.

Мы обсуждали фейнмановские правила как алгоритм для проведения вычислений. В последующих главах мы проиллюстрируем мощь этого метода. Но после того как мы рассмотрим все больше применений диаграмм, мы убедимся, что они начнут жить своей собственной жизнью. Диаграммы укажут на неожиданные связи между различными физическими процессами и позволят сформулировать интуитивные аргументы, которые потом можно будет проверить с помощью вычислений. Мы надеемся, что наша книга поможет читателю научиться этому методу и найти ему новые многообещающие применения.

Квантовые поля в иллюстративных примерах

Поле одного кванта

Поле двух квантов (!?!??!?!?!?!??)

(завис вопрос, подумаю потом.)

Поле одного кванта спинорного поля

(пропишу позже скоро.)

8.2.4 What is a Quantum Field (??????)

Идея квантовых полей (???)

Основная формула квантового поля (????)

(тут решение ур К-Г и Максвелла и рассуждение по поводу того, в чем их смысл? очень важный параграф.)

Если поля фундаментальны, то что делать с ядрами атомов? (????)

(приехали. пока не знаю внятного ответа на этот вопрос)

На каких масштабах квантовая механика неверна и необходима КТП? (????)

Можно ли на нано масштабах использовать квантовую механику?

Ее внаглую используют и все, и ничего не поделать.

Примеры, когда квантовая механика неверна и необходима КТП? (????)

Что такое квант поля

(подробно про это)

Что такое квазичастица? (????)

(подробно про это)

8.2.5 What Does It Mean to “Quantize” a Field (??????)

(четко с примерами напишу)

8.2.6 Causality (?!??!?)

(подробно об этом!! мб даже еще один раздел про это создам!)

Почему нельзя проквантовать релятивистские частицы так же, как мы квантуем нерелятивистские? (ПШ, ????)

Ответ можно дать несколькими способами. Вероятно, лучше всего выписать одночастичное релятивистское волновое уравнение (например, уравнений Клейна-Гордона или Дирака) и убедиться, что оно приводит к состояниям с отрицательной энергией и другим несообразностям. Так как это рассмотрение обычно проводится в конце продвинутого курса нерелятивистской квантовой механики, мы не будем здесь его повторять. Тем не менее нетрудно понять, почему не работает одночастичный подход. Мы не имеем права предполагать, что какой-либо релятивистский процесс можно объяснить на языке отдельной частицы, так как соотношение Эйнштейна $E = mc^2$ допускает образование пар частица-античастица.

Даже в случае, когда для рождения пары не хватает энергии, многочастичные состояния возникают, например, как промежуточные состояния во втором порядке теории возмущений.

В соответствии с соотношением неопределенностей $\Delta E \cdot \Delta t = \hbar$ можно считать, что эти состояния существуют только в течение очень короткого времени и в более высоких порядках теории возмущений возможно рождение произвольного числа таких «виртуальных» частиц.

Менее очевидно, что необходимость многочастичной теории вытекает из требования причинности.

Рассмотрим амплитуду распространения свободной частицы из точки $\mathbf{x}_0$ в точку $\mathbf{x}$:

$$U(t) = \left\langle \mathbf{x} \left| e^{-iHt} \right| \mathbf{x}_0 \right\rangle$$

В нерелятивистской квантовой механике имеем $E = \mathbf{p}^2/2m$, так что

$$\begin{aligned} U(t) = \left\langle \mathbf{x} \left| e^{-i\left(\mathbf{p}^2/2m\right)t} \right| \mathbf{x}_0 \right\rangle = \int \frac{d^3p}{(2\pi)^3} \left\langle \mathbf{x} \left| e^{-i\left(\mathbf{p}^2/2m\right)t} \right| \mathbf{p} \right\rangle \langle \mathbf{p} | \mathbf{x}_0 \rangle = \\ = \frac{1}{(2\pi)^3} \int d^3p e^{-i\left(\mathbf{p}^2/2m\right)t} \cdot e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)} = \left(\frac{m}{2\pi i t}\right)^{3/2} e^{im(\mathbf{x}-\mathbf{x}_0)^2/2t} \end{aligned}$$

Это выражение отлично от нуля для всех x и t, указывая на то, что частица может распространяться между любыми двумя точками за произвольно короткое время. В релятивистской теории это говорило бы о нарушении причинности. Можно ожидать, что положение исправит использование релятивистского соотношения $E = \sqrt{p^2 + m^2}$, но оказывается, что это не так. По аналогии с нерелятивистским случаем, находим:

$$\begin{aligned} U(t) = \left\langle \mathbf{x} \left| e^{-it\sqrt{\mathbf{p}^2+m^2}} \right| \mathbf{x}_0 \right\rangle = \frac{1}{(2\pi)^3} \int d^3p e^{-it\sqrt{\mathbf{p}^2+m^2}} \cdot e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_0)} = \\ = \frac{1}{2\pi^2 \left|\mathbf{x}-\mathbf{x}_0\right|} \int_0^\infty dp p \sin\left(p\left|\mathbf{x}-\mathbf{x}_0\right|\right) e^{-it\sqrt{p^2+m^2}}. \end{aligned}$$

Этот интеграл можно вычислить явно через функции Бесселя*. Мы ограничимся рассмотрением асимптотического поведения этого интеграла при $x^2 \gg t^2$ (достаточно далеко от светового конуса), используя метод стационарной фазы. Фаза $px - t\sqrt{p^2+m^2}$ имеет стационарную точку при $p = imx/\sqrt{x^2-t^2}$. Можно свободно замкнуть контур интегрирования в верхней полуплоскости так, чтобы он прошел через эту точку. Подставляя данное значение p, находим, что с точностью до рациональной функции от x и t,

$$U(t) \sim e^{-m\sqrt{x^2-t^2}}.$$

Таким образом, вне светового конуса амплитуда распространения хотя и мала, но отлична от нуля, и причинность по-прежнему нарушается.

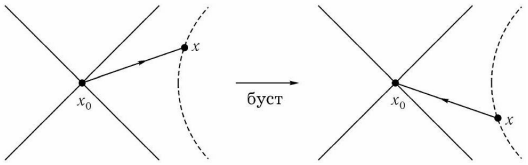


Рис. 2.1. Распространение из точки x_0 в точку x в одной системе отсчета выглядит как распространение из точки x в точку x_0 в другой системе отсчета.

Квантовая теория поля разрешает проблему причинности чудесным способом, который мы обсудим в разделе 2.4. Мы увидим, что в многочастичной теории поля распространение частицы на пространственноподобное расстояние не отличается от распространения античастицы в противоположном направлении (рис. 2.1). Когда мы спрашиваем, может ли наблюдение, сделанное в точке x_0 повлиять на наблюдение, сделанное в точке x, мы находим, что амплитуды распространения для частицы и античастицы в точности сокращаются, так что причинность не нарушается.

*См. Градштейн и Рыжик (1980), #3.914. Квантовая теория поля естественным образом способна описывать не только многочастичные состояния, но и переходы между состояниями с разным числом частиц. Она решает проблему причинности путем введения античастиц и затем позволяет объяснить связь между спином и статистикой. Но самое главное то, что квантовая теория поля дает в руки исследователей инструменты, необходимые для расчета бесчисленного количества сечений процессов рассеяния. Экспериментальное подтверждение этих предсказаний, часто имеющих беспрецедентную точность, и есть главная причина изучения квантовой теории поля.

Допустим, мы не знаем никаких экспериментов, из каких соображений вообще мы начнем именно в квантами поля работать?

(это проблема классической теории поля, потому что там так нас обучают, что нет вообще этих соображений. вот, продумаю, напишу.)

8.2.7 Does the Idea of Quantizing Fields Lead to Contradictions?

(подробно про это)

Допустим мы проквантовали электромагнитное поле вблизи заряда, которые его создает. Далеко вдали от него неужели наши детекторы квантов будут случайным образом щелкать: долетел ли квант до них или не долетел? (????)

(давний вопрос, пока не думал про это, но мб ответ в том, что кванты все-таки квазичастицы, которые не реальные, а реально именно поле или волна. И дальше уже она может проявлять свои квантовые свойства.)

8.2.8 Connection with Quantum Mechanics and Wave Functions (???)

(пропишу потом, как они связаны?)

8.3 Philosophy of QFT: Other Ideas (??)

8.3.1 Locality

(discuss, what it is? how could it be otherwise???)

8.3.2 How Were Particles Created?

Обсудим в двух словах, как создаются частицы

Идейно, почему инфляция рождает частицы

(???)

8.3.3 What is Time and Space?

(this is answered in field theory, but it is important to also write about it here)

8.4 What is Really a Vacuum?

(intro to casimir effect, article by В. Л. Гинзбург, В. П. Фролов. a very deep question)

8.4.1 Why is Something in the Quantum World Renormalized?

(general intro to renormalizations)

8.4.2 How Rigorous should be Mathematics for QFT?

(?? big question)

8.4.3 Зи.III.1 Cutting Off Our Ignorance

Следите за руками

Первых исследователей квантовой теории поля приводили в сильнейшее затруднение расходящиеся интегралы, которые часто встречались по ходу расчетов, так что в 1930-1940-е гг. ученые много времени тратили на борьбу с бесконечностями.

Многочисленные неудачи, следующие за, казалось бы, уже сделанными открытиями, все больше подталкивали к тому, чтобы отказаться от квантовой теории поля. Наконец ученые разработали так называемую процедуру перенормировки, которая позволила решить проблему бесконечных интегралов и получить конечные физические результаты. Однако, еще долгие годы, вплоть до конца 1960-х и начала 1970-х гг., многие физики скептически относились к теории перенормировки, считая ее простой ловкостью рук. Шутили, что в квантовой теории поля бесконечность приравнивается к нулю или что под ковром в кабинете физиков запрятано много бесконечностей.

Наконец благодаря усилиям Кена Вильсона и многих других ученых в начале 1970-х гг. появилось новое понимание квантовой теории поля. Постепенно теоретики пришли к выводу, что в квантовой теории поля вовсе не существует проблемы расходимости. Сейчас мы рассматриваем квантовую теорию поля как эффективную низкоэнергетическую теорию. Я кратко затрону этот вопрос здесь и подробнее в главе VIII.3.

Теория поля рушится

Мы должны увидеть саму бесконечность, прежде чем говорить о том, как работать с бесконечностями. В главе I.7 мы уже сталкивались с одним из них. Вспомним, что поправка к амплитуде мезон-мезонного рассеяния порядка λ^2 (выражение I.7.23) расходится. При $K \equiv k_1 + k_2$ имеем:

$$\mathcal{M} = \frac{1}{2}(-i\lambda)^2 i^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\varepsilon} \frac{1}{(K-k)^2 - m^2 + i\varepsilon}$$

Как говорилось в главе I.7, даже не проводя никаких расчетов, мы можем увидеть проблему, которая ставила в тупик пионеров квантовой теории поля. Для больших значений k подынтегральное выражение равно $1/k^4$, т. е. интеграл будет логарифмически расходиться как $\int [d^4k/(2\pi)^4]\,(1/k^4)$. (Обычный интеграл $\int^{\infty} dr r^n$ линейно расходится при $n = 0$, квадратично расходится при $n = 1$ и т. д., а интеграл $\int^{\infty} \frac{dr}{r}$ расходится логарифмически.) Поскольку эта расходимость связана с большими значениями k, ее называют ультрафиолетовой расходимостью.

Чтобы узнать, как работать с возникшей бесконечностью, мы должны четко представлять разницу между концептуально отличными между собой сюжетами, названными по историческим причинам пугающими терминами «регуляризация» и «перенормировка».

Параметризация неведения

Предположим, вместо искусственной φ^4-теории мы изучаем квантовую электродинамику. Было бы чрезвычайно неразумно утверждать, что теория взаимодействия электрона с фотоном работает до произвольно высокой энергии. По крайней мере с увеличением энергии появятся другие частицы и окажется, что электродинамика станет просто частью большей электрослабой теории. Для увеличения энергии необходимо как минимум участие других частиц, так что электродинамика является лишь частью более обширной теории электрослабых взаимодействий. Действительно, с современной точки зрения при переходе к все более высоким энергиям вся «конструкция» квантовой теории поля с необходимостью оказывается приближением к неизвестной пока нам теории. По мнению некоторых физиков, такой теорией является теория струн.

Современный взгляд состоит в том, что сегодня квантовая теория поля должна рассматриваться как эффективная низкоэнергетическая теория, справедливой до некоторого масштаба энергии (или, в лоренцинвариантной теории, до некоторого масштаба импульса) Λ. Представьте себе, что мы живем во Вселенной, описываемой игрушечной φ^4-теорией. По мере того как физики в этой Вселенной будут изучать физику все более высоких энергий, они придут к выводу, что их Вселенная является матрацем, состоящим из материальных точек и пружин, а масштаб Λ примерно равен обратному размеру решетки.

Когда я преподаю квантовую теорию поля, то, чтобы подчеркнуть свои слова, пишу на доске: «Не стыдитесь своего неведения». Любая физическая теория должна иметь свою область применимости, за пределами которой мы ничего не знаем о соответствующей физике. В противном случае не был бы возможен прогресс знаний в облаcти физики. Хорошо, что Фейнман, Швингер, Томонага и другие исследователи квантовой электродинамики не имели представления о кварках, например.

Подчеркну, что масштаб Λ следует рассматривать не как математический, а как физический элемент, параметризующий порог нашего неведения [1]. Действительно, в физически осмысленных квантовых теориях поля всегда должна появляться неявно заданная величина Λ. Если кто-то вздумает «впариты» вам теорию поля, которая якобы справедлива для произвольно высоких энергий, вам следует проверить, не торговал ли раньше этот человек подержанными автомобилями. (Когда я писал эти строки, мой коллега - редактор Phsical Review Letters - рассказал, что во время каникул в средней школе он подрабатывал сборщиком мусора, и шутя добавил, что такая практика хорошо подготовила его к нынешней профессии.)

[1] В главе I.8 мы видели яркий тому пример. Определив проводящую пластинку как поверхность, на которой перестает существовать тангенциальное поле. мы не учли физику электрона, движение которого противодействует любому наложенному полю. При крайне высоких частотах электроны не могут двигаться достаточно быстро, значит, здесь действуют другие законы физики, согласно которым высокочастотные моды не видят пластинки. При расчете силы Казимира мы параметризуем свое незнание величиной $a \sim \Lambda^{-1}$.

Таким образом, вычисляя интеграл $\int d^4k/(2\pi)^4$ в (1), мы должны интегрировать лишь до значения Λ, называемого параметром обрезания. Мы буквально обрезаем интегрирование по импульсу (рис. III.1.1)2. Говорят, что тем самым «регулиризовали» интеграл.

В этой книге я стремлюсь подчеркнуть скорее концептуальные, нежели вычислительные аспекты, поэтому я не стану сейчас явно вычислять интеграл, но скажу, что он равен $2iC\log(\Lambda^2/K^2)$, где C - некоторая численная константа, которую вы можете вычислить сами (см. приложение 1 к этой главе). Пусть для простоты $m^2 \ll K^2$, тогда величиной m^2 в подынтегральном выражении можно пренебречь. Удобно ввести кинематические переменные $s \equiv K^2 = (k_1+k_2)^2, t \equiv (k_1-k_3)^2$ и $u \equiv (k_1-k_4)^2$. (Записав явным образом параметры k_j в системе центра масс, увидим, что переменные s,t и u соответствуют довольно обычным величинам: энергии центра масс и углу рассеяния). С учетом всего вышеизложенного амплитуда мезон-мезонного рассеяния будет равна:

$$\mathcal{M} = -i\lambda + iC\lambda^2\left[\log\left(\frac{\Lambda^2}{s}\right)+\log\left(\frac{\Lambda^2}{t}\right)+\log\left(\frac{\Lambda^2}{u}\right)\right]+O\left(\lambda^3\right).$$

Это довольно легко понять. После регуляризации мы говорим о величинах, зависящих от обрезания, а не о расходящихся величинах. Параметры расходимости будем называть параметрами, зависимыми от обрезания; $\mathcal{M}$ зависит от обрезания логарифмически.

Что мы в действительности измеряем

Давайте теперь после регуляризации перейдем к перенормировке. Этот термин неудачен, ведь он предполагает, что мы что-то нормируем повторно, когда на самом деле это не так.

Ключевым моментом здесь является то, как бы мы описали экспериментатору процесс измерения мезон-мезонного рассеяния. Мы говорим ей (или ему, если вы настаиваете), что нам необходимо задать обрезание Λ, а он не проявляет ни капли беспокойства; тот факт, что любая заданная теория имеет конечную область применимости, для экспериментатора очевиден.

Мы, вероятно, скажем ему, как рассеяние будет зависеть от энергии в системе центра масс и угла рассеяния. Затем продемонстрируем ей выражение (2). Она укажет на параметр λ и воскликнет: «Что это такое?»

Мы ответим: «Константа связи», на что она воскликнет: «Какая такая константа связи, если это всего-навсего греческая буква!»

Смущенный студент, пусть будет Конфузио, слышавший разговор, в этом случае ответил бы: «В чем дело? Я изучал физику много лет, преподаватели познакомили нас со множеством уравнений, в которых фигурируют латинские и греческие буквы, к примеру, с законом Гука $F = -kx$, и никто не утверждал при этом, что k - всего лишь латинская буква».

Сообразительный экспериментатор скажет: «Но это потому, что я смогу пойти и измерить значение k, если вы дадите мне пружину. В этом все дело! Пускай мистер Заумный Теоретик скажет мне, как я могу измерить эту вашу λ.

Да, вот это действительно хороший экспериментатор! Теперь нам следует более тщательно продумать, что в действительности означает константа связи. Вспомним про константу связи в квантовой электродинамике, α. Ну, это коэффициент при $1/r$ в законе Кулона. Отлично, месье Кулон измерил его с помощью металлических шаров или чего-то вроде этого. Но современный экспериментатор тоже мог бы измерить α, рассеивая электрон при той или иной энергии и под тем или иным углом на протоне. Итак, мы объясним все это нашему другу-экспериментатору.

Он, кивая, согласится: «Да, конечно. Не так давно мои коллеги измеряли константу связи в мезон-мезонном взаимодействии путем рассеяния одного мезона на другом при той или иной энергии и том или ином угле рассеяния. Эти параметры соответствуют вашим переменным s, t и u и имеют значения s_0, t_0 и u_0. Но что общего между той константой связи, которую измеряли мои коллеги (назовем ее λ_P, где индекс P означает «физический»)[3]), с вашей теоретической λ, которая, насколько мне известно, является всего лишь греческой буквой в том, что вы называете лагранжианом!»

Конфузио скажет: «Эй, если вы беспокоитесь по поводу малого лямбда, тогда я начну беспокоиться по поводу большого лямбда. Откуда я знаю, насколько велика область применимости?»

СЭ заметит: «Вы не настолько глупы, как кажетесь! Мистер Заумный Теоретик, если я решу воспользоваться формулой (2), какое конкретно зна-

[3] От англ. «physical». - Прим. пер. чение Λ я должен в нее подставить? Зависит ли выбор исключительно от вашего настроения, мистер Заумный Теоретик? Если вы проснетесь с оптимистическим настроем, примените ли вы значение 2Λ, вместо Λ? А если вас бросит подружка, воспользуетесь ли вы значением $\frac{1}{2}\Lambda$?»

Мы усмехнемся: «Ха! Мы можем ответить на ваш вопрос. Взгляните на формулу (2): считается, что $\mathcal{M}$ является настоящей амплитудой рассеяния и не зависит от Λ. Если кто-то захочет изменить значение Λ, можно просто сдвинуть λ таким образом, чтобы величина $\mathcal{M}$

не изменилась. Пара строчек выкладок, и мы можем точно сказать, каким должно быть значение $d\lambda/d\Lambda$ (см. упражнение III.1.3)».

Экспериментатор: «Хорошо. Значит, λ есть скрытая функция от Λ. Ваше обозначение ничего не стоит».

Мы согласимся: «Это правда. Неудачное обозначение вводило в заблуждение не одно поколение физиков».

Экспериментатор: «Я так и не услышал от вас, каким образом параметр λ_P, который измеряли мои коллеги, связан с вашим λ.

Мы скажем: «Это просто. Посмотрите на уравнение (2), которое мы еще раз приведем здесь для наглядности:

$$\mathcal{M} = -i\lambda + iC\lambda^2 \left[\log\left(\frac{\Lambda^2}{s}\right) + \log\left(\frac{\Lambda^2}{t}\right) + \log\left(\frac{\Lambda^2}{u}\right)\right] + O\left(\lambda^3\right)$$

Согласно нашей теории, параметр λ_P равен:

$$-i\lambda_P = -i\lambda + iC\lambda^2 \left[\log\left(\frac{\Lambda^2}{s_0}\right) + \log\left(\frac{\Lambda^2}{t_0}\right) + \log\left(\frac{\Lambda^2}{u_0}\right)\right] + O\left(\lambda^3\right).$$

Обозначим сумму логарифмов в квадратных скобках в выражениях (3) и (4) как L и L_0 соответственно. Перепишем данные выражения в более компактной форме:

$$\mathcal{M} = -i\lambda + iC\lambda^2 L + O\left(\lambda^3\right)$$

и

$$-i\lambda_P = -i\lambda + iC\lambda^2 L_0 + O\left(\lambda^3\right)$$

Вот так связаны параметры λ_P и λ.

Экспериментатор скажет: «Если вы выразите амплитуду рассеяния через физическую константу связи λ_P, то это будет полезно для меня, но никак не для λ. Я понимаю смысл λ_P, но не могу понять смысла λ.

На это мы ответим: «Хорошо, нам ничего не стоит исключить из формулы параметр λ, вместо λ_P. Большое дело! Разрешив уравнение (6) относительно λ, получим:

$$-i\lambda = -i\lambda_P - iC\lambda^2 L_0 + O\left(\lambda^3\right) = -i\lambda_P - iC\lambda_P^2 L_0 + O\left(\lambda_P^3\right).$$

Второе равенство верно с указанной точностью. Теперь подставим его в (5):

$$\mathcal{M} = -i\lambda + iC\lambda^2 L + O\left(\lambda^3\right) = -i\lambda_P - iC\lambda_P^2 L_0 + iC\lambda_P^2 L + O\left(\lambda_P^3\right)$$

Проверьте пожалуйста, что все выкладки верны с указанной точностью».

«Чудо»

И вдруг! Вот оно, чудо перенормировки!

Теперь в амплитуду рассеяния $\mathcal{M}$ входит комбинация $L - L_0 = [\log(s_0/s) + \log(t_0/t) + \log(u_0/u)]$. Другими словами, амплитуда рассеяния сводится к виду:

$$\mathcal{M} = -i\lambda_P + iC\lambda_P^2 \left[\log\left(\frac{s_0}{s}\right) + \log\left(\frac{t_0}{t}\right) + \log\left(\frac{u_0}{u}\right)\right] + O\left(\lambda_P^3\right)$$

С триумфальным чувством мы объявляем нашему другу-экспериментатору, что в случае представления амплитуды рассеяния через зависимость от физической константы связи λ_P, как ей и хотелось, обрезание Λ полностью исчезает!

Ответ всегда должен быть выражен через физически измеримые величины

Урок в том, что физические величины всегда следует выражать не через «фиктивные» теоретические параметры, такие как λ, а через физически измеримые параметры, такие как λ_P.

Кстати, в литературе параметр λ_P часто обозначают как λ_R и по историческим причинам называют «перенормированной константой связи». Я считаю, что физический смысл «перенормировки» становится более понятным, если употреблять термин «физическая константа связи» и, следовательно, подстрочный индекс «р». К тому же, у нас никогда не было «нормированной константы связи».

Неожиданно Конфузио появляется опять; мы про него почти забыли: «Вы начали работу с выражением для $\mathcal{M}$, в котором фигурировали две нефизические величины λ и Λ, а в конечном итоге их «нефизическая природа» нейтрализовала друг друга». Экспериментатор соглашается: «Да, это и отличает хороших теоретиков от плохих. Хорошие теоретики всегда допускают четное число ошибок в знаках, тогда как плохие - всегда делают нечетное число».

Интегрирование лишь по медленным модам

В формулировке интеграла по траекториям амплитуда рассеяния $\mathcal{M}$ получается вычислением интеграла (глава I.7):

$$\int D\varphi\varphi\left(x_1\right)\varphi\left(x_2\right)\varphi\left(x_3\right)\varphi\left(x_4\right)e^{i\int d^dx\left\{\frac{1}{2}\left[(\partial\varphi)^2-m^2\varphi^2\right]+\frac{\lambda}{4!}\varphi^4\right\}}.$$

Используемая здесь регуляризация соответствует, грубо говоря, тому, что мы интегрируем при вычислении интеграла $\int D\varphi$, лишь по тем модам поля $\varphi(x)$, фурье-преобразование которых равно нулю при $k \gtrsim \Lambda$. Иначе говоря, поли, соответствующие внутренним линиям на фейнмановских диаграммах (рис. I.7.10), не могут флуктуировать со слишком большой энергией. Позже мы вернемся к этой формулировке интеграла по траекториям, когда будем рассматривать ренорм-группу.

Альтернативные пути

Возможно, следует упомянуть некоторые альтернативные способы регуляризации фейнмановских диаграмм, каждый из которых имеет свои преимущества и недостатки, что делает их удобными в одних случаях, а в других - нет. Регуляризация, о которой мы говорили в настоящей главе, известна под названием регуляризации Паули-Вилларса. Ее преимуществом считается прозрачность с точки зрения физики. Другой часто используемой регуляризацией является размерная регуляризация. Предположим, наши расчеты производятся в d-мерном пространстве-времени. После приведения интеграла Фейнмана к удобной для нас форме, мы прибегаем к методу аналитического продолжения по d и в конце концов кладем $d = 4$. Зависимость различных интегралов от обрезания характеризуется теперь полюсами ($d \to 4$). Аналогично тому, как после представления амплитуды рассеяния в виде зависимости от физической константы связи λ_P исчезает масштаб обрезания λ, в размерной регуляризации амплитуда рассеяния, выраженная в виде

зависимости от λ_P, не содержит полюсов. Во многих случаях применение размерной регуляризации оказывается предпочтительнее, как я покажу в следующих главах, хотя, по сравнению с регуляризаци- ей Паули-Вилларса, она носит более абстрактный и формальный характер. Когда дело доходит до регуляризации, делайте выбор по своему вкусу.

Поскольку основное внимание в книге уделяется концептуальным, а не вычислительным моментам квантовой теории поля, я не буду описывать другие схемы регуляризации, но вкратце покажу, как работают регуляризация Паули-Вилларса и размерная регуляризация в двух приложениях к этой главе.

Приложение 1. Регуляризация Паули-Вилларса

Важным моментом настоящей главы является концептуальное утверждение, согласно которому выражение физических амплитуд в терминах физических констант связи приводит к исчезновению зависимости от обрезания. Фактическое вычисление интеграла Фейнмана не имеет для нас значения. Однако на всякий случай я продемонстрирую вам процесс его вычисления, ведь в будущем это может вам понадобиться.

Начнем со сходящегося интеграла

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{\left(k^2 - c^2 + i\varepsilon\right)^3} = \frac{-i}{32\pi^2 c^2}$$

Зависимость от c^2 следует из анализа размерностей. Общий множитель вычисляется в приложении D.

Применяя тождество (D.15)

$$\frac{1}{xy} = \int_0^1 d\alpha \frac{1}{[\alpha x + (1-\alpha)y]^2}$$

к (1), получаем

$$\mathcal{M} = \frac{1}{2}(-i\lambda)^2 i^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha \frac{1}{D^2}$$

где

$D = \alpha(K-k)^2 + (1-\alpha)k^2 - m^2 + i\varepsilon = (k - \alpha K)^2 + \alpha(1-\alpha)K^2 - m^2 + i\varepsilon.$

Сдвинем переменную интегрирования $k \to k + \alpha K$ и получим интеграл $\int [d^4k/(2\pi)^4] \left[1/\left(k^2 - c^2 + i\varepsilon\right)^2\right]$, где $c^2 = m^2 - \alpha(1-\alpha)K^2$. Паули и Вилларс предложили заменять его интегралом

$$\int \frac{d^4k}{(2\pi)^4} \left[\frac{1}{\left(k^2 - c^2 + i\varepsilon\right)^2} - \frac{1}{\left(k^2 - \Lambda^2 + i\varepsilon\right)^2}\right].$$

Если k намного меньше Λ, дополнительный второй член в подынтегральной функции имеет порядок Λ^{-4} и, следовательно, ничтожно малое значение, по сравнению со значением первого члена, поскольку априори считается, что Λ намного больше c. Если k существенно превышает Λ, оба члена взаимно сокращаются и подынтегральное выражение быстро стремится к нулю с ростом значения k, что эффективно обрезает интеграл.

Дифференцируя интеграл (12) по c^2 и используя (10), мы получаем, что интеграл (12) равен $(i/16\pi^2)\log(\Lambda^2/c^2)$. Таким образом, интеграл

$$\int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{\left(k^2 - c^2 + i\varepsilon\right)^2} = \frac{i}{16\pi^2} \log\left(\frac{\Lambda^2}{c^2}\right)$$

действительно логарифмически зависит от масштаба обрезания, как и утверждалось в тексте.

Итак, получаем

$$\mathcal{M} = \frac{i\lambda^2}{32\pi^2} \int_0^1 d\alpha \log\left(\frac{\Lambda^2}{m^2 - \alpha(1-\alpha)K^2 - i\varepsilon}\right)$$

Приложение 2. Размерная регуляризация

В основе размерной регуляризации лежит очень простая идея. Когда мы приходим к интегралу $I = \int [d^4k/(2\pi)^4] \left[1/\left(k^2 - c^2 + i\varepsilon\right)^2\right]$, мы делаем поворот в евклидово пространство и обобщаем его до d измерений (см. приложение D):

$$I(d) = i \int \frac{d_E^d k}{(2\pi)^d} \frac{1}{\left(k^2 + c^2\right)^2} = i \left[\frac{2\pi^{d/2}}{\Gamma(d/2)}\right] \frac{1}{(2\pi)^d} \int_0^\infty dk k^{d-1} \frac{1}{\left(k^2 + c^2\right)^2}$$

Как я уже говорил, мне бы не хотелось пускаться в длинные вычисления, но приходится делать то, что делать необходимо. Заменим переменную интегрирования $k^2 + c^2 = c^2/x$ и получим выражение:

$$\int_0^\infty dk k^{d-1} \frac{1}{\left(k^2 + c^2\right)^2} = \frac{1}{2} c^{d-4} \int_0^1 dx (1-x)^{d/2-1} x^{1-d/2}$$

которое оказывается интегральным представлением бета-функции. После некоторых манипуляций получаем:

$$i \int \frac{d_E^d k}{(2\pi)^d} \frac{1}{\left(k^2 + c^2\right)^2} = i \frac{1}{(4\pi)^{d/2}} \Gamma\left(\frac{4-d}{2}\right) c^{d-4}$$

При $d \to 4$ правая часть уравнения равна:

$$i \frac{1}{(4\pi)^2} \left[\frac{2}{4-d} - \log c^2 + \log(4\pi) - \gamma + O(d-4)\right],$$

где $\gamma = 0,577....$.сть постоянная Эйлера-Маскерони.

Сравнивая полученное выражение с (13), видим, что в регуляризации Паули-Вилларса логарифм $\log \Lambda^2$ эффективно заменяется полюсом $2/(4-d)$. Как говорилось в тексте, представление физических величин в виде зависимости от физических констант связи приводит к сокращению всех полюсов.

Зи-у.

III.1.1. Проделайте выкладки, приводящие к (9), не пользуясь текстом.

Зи-у.

III.1.2. Поработайте с выражением (1) как с аналитической функцией от K^2. Покажите, что она имеет разрез от $4m^2$ до бесконечности.

[Указание: если вы не можете получить этот ответ прямо из (1), посмотрите на (14).]

Зи-у.

III.1.3. Замените Λ на $e^{\varepsilon}\Lambda$. Покажите, что, чтобы в указанном порядке $\mathcal{M}$ не изменился, λ изменится на $\delta\lambda = 6\varepsilon C\lambda^2 + O\left(\lambda^3\right)$, так что:

$$\Lambda\frac{d\lambda}{d\Lambda} = 6C\lambda^2 + O\left(\lambda^3\right)$$

8.4.4 Can a Diagram Be Something More?

(string theory tells that yes, it can be, there are surfaces. but why? I'll write this later here, it is a conceptually important question?)

8.4.5 What Is Life?

(I'll write the best answer I can think about it here)

8.4.6 O, Actually, QFT Is in Condensed Matter

(write that for some reason, the same concepts are in absolutely different part of physics. a good intro to it)

Why QFT Is Applicable in Condensed Matter?

(????? I can say something now, but I want a really good answer about it)

8.4.7 Gravity is Not a Quantum Field, Right?

(intro to discussion about quantization of gravity. just main ideas, details will be in the end of the note)

8.5 Poincare Group for QFT

(тут суть, подробно в классической теории поля. тут глубокие мысли, потом подумаю про них)

8.5.1 Mathematical Details of Relativistic Invariance

(see field theory about it for now)

8.5.2 Field Equations from Poincare Group

(see field theory about it for now)

8.6 Field Formalism for QFT

(все это в теории поля я подробно писал, так что тут очень коротко и в контексте ктп)

8.6.1 Lagrange Field Formalism

Фундаментальная величина классической механики - действие S, являющееся интегралом по времени от лагранжиана L.

В локальной теории поля лагранжиан может быть записан как пространственный интеграл от плотности лагранжиана, обозначаемой $\mathscr{L}$, которая является функцией одного или более полей $\phi(x)$ и их производных $\partial_\mu \phi$. Таким образом,

$$S = \int L dt = \int \mathscr{L}\left(\phi, \partial_\mu \phi\right) d^4 x$$

Так как это книга по теории поля, далее мы называем $\mathscr{L}$ просто лагранжианом.

Принцип наименьшего действия утверждает, что когда система эволюционирует от одной заданной конфигурации к другой за промежуток времени от t_1 до t_2, то ее «путь» в конфигурационном пространстве таков, что действие S имеет экстремум (обычно минимум). Мы можем записать это условие в виде:

$$\begin{aligned} 0 = \delta S &= \int d^4 x \left\{ \frac{\partial \mathscr{L}}{\partial \phi} \delta\phi + \frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi\right)} \delta\left(\partial_\mu \phi\right) \right\} = \\ &= \int d^4 x \left\{ \frac{\partial \mathscr{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi\right)} \right) \delta\phi + \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi\right)} \delta\phi \right) \right\}. \end{aligned}$$

Последнее слагаемое можно превратить в поверхностный интеграл по границе четырехмерной пространственно-временной области интегрирования. Так как начальная и конечная полевые конфигурации считаются заданными, то $\delta\phi$ равно нулю в начальный и конечный моменты времени на этой области. Если мы также ограничимся рассмотрением вариаций $\delta\phi$, исчезающих на пространственной границе области, то поверхностный интеграл равен нулю. Вынося $\delta\phi$ как множитель из первых двух слагаемых и замечая, что интеграл должен исчезать для любой вариации $\delta\phi$, делаем вывод, что множитель при $\delta\phi$ должен исчезать во всех точках. Таким образом, мы приходим к уравнениям движения Эйлера-Лагранжа для поля:

$$\partial_\mu \left(\frac{\partial \mathscr{L}}{\partial\left(\partial_\mu \phi\right)} \right) - \frac{\partial \mathscr{L}}{\partial \phi} = 0$$

Если лагранжиан содержит более одного поля, то имеется по одному такому уравнению для каждого из них.

8.6.2 Conservation Laws for Quantum Fields (after Mandl & Shaw)

Теория

It follows from the Heisenberg equation of motion of an operator $O(t)$

$$\mathrm{i}\hbar \frac{\mathrm{d}O(t)}{\mathrm{d}t} = [O(t), H]$$

(we are not considering operators with explicit time dependence) that O is a constant of the motion provided

$$[O, H] = 0.$$

Constants of the motion generally stem from invariance properties of systems under groups of transformations, e.g. translational and rotational invariance lead to conservation of linear and angular momentum, respectively. Such transformations lead to equivalent descriptions of the system; for example, referred to two frames of reference related by a Lorentz transformation. Quantum-mechanically, two such descriptions must be related by a unitary transformation U under which states and operators transform according to

$$|\Psi\rangle \to |\Psi'\rangle = U|\Psi\rangle, \quad O \to O' = UOU^\dagger$$

The unitarity of the transformation ensures two things. Firstly, operator equations are covariant, i.e. have the same form whether expressed in terms of the original or the transformed operators. In particular, this will be true of the commutation relations of the fields and of the equations of motion, e.g. Maxwell's equations will be covariant with respect to Lorentz transformations. Secondly, under a unitary transformation, amplitudes, and hence observable predictions, are invariant.

If one deals with continuous transformations, the unitary operator U can be written

$$U = \mathrm{e}^{\mathrm{i}\alpha T}$$

where $T = T^\dagger$ and α is a real continuously variable parameter. For $\alpha = 0, U$ goes over into the unit operator. For an infinitesimal transformation

$$U \approx 1 + \mathrm{i}\delta\alpha T$$

and Eq. (2.33) becomes

$$O' = O + \delta O = (1 + \mathrm{i}\delta\alpha T)O(1 - \mathrm{i}\delta\alpha T)$$

i.e.

$$\delta O = \mathrm{i}\delta\alpha[T, O].$$

If the theory is invariant under this transformation, the Hamiltonian H will be invariant, $\delta H = 0$, and taking $O = H$ in Eq. (2.35) we obtain $[T, H] = 0$, i.e. T is a constant of the motion. For a field theory derived from a Lagrangian density $\mathscr{L}$, one can construct conserved quantities from the invariance of $\mathscr{L}$ under symmetry transformations. We shall show that for such a theory, the invariance of $\mathscr{L}$ leads to equations of the form

$$\frac{\delta f^\alpha}{\delta x^\alpha} = 0$$

where the f^α are functions of the field operators and their derivatives. If we define

$$F^\alpha(t) = \int \mathrm{d}^3\mathbf{x} f^\alpha(\mathbf{x}, t),$$

where integration is over all space, then the continuity equation (2.36) gives

$$\frac{1}{c}\frac{\mathrm{d}F^0(t)}{\mathrm{d}t} = -\int \mathrm{d}^3\mathbf{x} \sum_{j=1}^{3} \frac{\partial}{\partial x^j} f^j(\mathbf{x}, t) = 0$$

where the last step follows by transforming the integral into a surface integral by means of Gauss's divergence theorem and assuming (as always) that the fields, and hence the f^j, tend to zero sufficiently fast at infinity. [3] Hence

$$F^0 = \int \mathrm{d}^3\mathbf{x} f^0(\mathbf{x}, t)$$

is a conserved quantity. With $T = F^0$, the corresponding unitary operator is then given by Eq. (2.34).

The interpretation of the four-vector f^α follows from Eqs. (2.36)-(138). f^0/c and f^j are the three-dimensional volume and current densities of the conserved quantity F^0/c. Eq. (2.36a), applied to a finite three-dimensional volume V bounded by a surface S, then states that the rate of decrease of F^0/c within V equals the current of F^0/c flowing out through S. Correspondingly, the four-vector $f^\alpha(x)$, satisf ying the conservation equation (2.36), is called a conserved current. (Strictly speaking, one should call it a four-current

density.) The result, that the invariance of the Lagrangian density $\mathscr{L}$ under a continuous one-parameter set of transformations implies a conserved quantity, is known as Noether's theorem. We apply these ideas to the transformation

$$\phi_r(x) \to \phi'_r(x) = \phi_r(x) + \delta\phi_r(x)$$

of the fields. The change induced in $\mathscr{L}$ is given by

$$\delta\mathscr{L} = \frac{\partial\mathscr{L}}{\partial\phi_r}\delta\phi_r + \frac{\partial\mathscr{L}}{\partial\phi_{r,\alpha}}\delta\phi_{r,\alpha} = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial\mathscr{L}}{\partial\phi_{r,\alpha}}\delta\phi_r\right),$$

where the last step follows since $\phi_r(x)$ satisfies the field equations (2.16), and summations over repeated indices r and α are implied as previously. If $\mathscr{L}$ is invariant under the transformation (2.39) so that $\delta\mathscr{L} = 0$, the last equation reduces to the continuity equation (2.36) with

$$f^\alpha = \frac{\partial L}{\partial\phi_{r,\alpha}}\delta\phi_r$$

and the constant of the motion, from Eqs. (2.38) and (2.22), is

$$F^0 = c\int \mathrm{d}^3\mathbf{x}\pi_r(x)\delta\phi_r(x).$$

An important particular case of the above arises for complex fields ϕ_r, i.e. nonHermitian operators in the quantized theory. ϕ_r and $\phi_r^\dagger$ are then treated as independent fields, as discussed earlier. Suppose $\mathscr{L}$ is invariant under the transformation

$$\left.\begin{array}{l}\phi_r \to \phi'_r = \mathrm{e}^{\mathrm{i}\varepsilon}\phi_r \approx (1+\mathrm{i}\varepsilon)\phi_r \\ \phi_r^\dagger \to \phi_r^{\dagger\prime} = \mathrm{e}^{-\mathrm{i}\varepsilon}\phi_r^\dagger \approx (1-\mathrm{i}\varepsilon)\phi_r^\dagger\end{array}\right\}$$

where ε is a real parameter, and the right-hand expressions result for very small ε. From Eqs. (2.41)

$$\delta\phi_r = \mathrm{i}\varepsilon\phi_r, \quad \delta\phi_r^\dagger = -\mathrm{i}\varepsilon\phi_r^\dagger,$$

and Eq. (2.40) becomes

$$F^0 = \mathrm{i}\varepsilon c\int \mathrm{d}^3\mathbf{x}\left[\pi_r(x)\phi_r(x) - \pi_r^\dagger(x)\phi_r^\dagger(x)\right].$$

Since F^0 multiplied by any constant is also conserved, we shall, instead of F^0, consider

$$Q = -\frac{\mathrm{i}q}{\hbar}\int \mathrm{d}^3\mathbf{x}\left[\pi_r(x)\phi_r(x) - \pi_r^\dagger(x)\phi_r^\dagger(x)\right]$$

where q is a constant to be determined later. The reason for this change is that $\pm q$ will turn out to be the electric charges of the particles represented by the fields.

We evaluate the commutator $[Q, \phi_r(x)]$. Since ϕ_r and $\phi_r^\dagger$ are independent fields, ϕ_r commutes with all fields except π_r [see Eq. (2.31)]. Hence taking $(x')^0 = x^0 = ct$,

$$[Q, \phi_r(x)] = -\frac{\mathrm{i}q}{\hbar}\int \mathrm{d}^3\mathbf{x}' \left[\pi_s\left(x'\right), \phi_r(x)\right]\phi_s\left(x'\right)$$

and using the commutation relations (2.31) one obtains

$$[Q, \phi_r(x)] = -q\phi_r(x)$$

From this relation one easily verifies that if $|Q'\rangle$ is an eigenstate of Q with the eigenvalue Q', then $\phi_r(x)\,|Q'\rangle$ is also an eigenstate of Q belonging to the eigenvalue $(Q' - q)$, and correspondingly $\phi_r^\dagger(x)$ | $(Q'\rangle$ belongs to $(Q' + q)$. In the next chapter we shall see that, consistent with these results, ϕ_r and $\phi_r^\dagger$ are linear in creation and absorption operators, with ϕ_r absorbing particles of charge $(+q)$ or creating particles of charge $(-q)$, while $\phi_r^\dagger$ absorbs particles of charge $(-q)$ or creates particles of charge $(+q)$. Hence, we interpret the operator Q, Eq. (2.42), as the charge operator. We have therefore shown that charge is conserved $(\mathrm{d}Q/\mathrm{d}t = 0, [Q, H] = 0)$, provided the Lagrangian density $\mathscr{L}$ is invariant with respect to the transformation (2.41), which is known as a global phase transformation [corresponding to the fact that the phase ε in Eq. (2.41) is independent of x] or as a gauge transformation of the first kind. We see from Eq. (2.42) that we require complex, i.e. nonHermitian, fields to represent particles with charge. Real, i.e. Hermitian, fields represent uncharged particles. Interpreting Eq. (2.42) as an operator involves the usual ambiguity as to the order of factors. We shall have to choose these so that for the vacuum state | $(0\rangle$, in which no particles are present, Q | $(0) = 0$. We shall return to this point in the next chapter.

The unitary transformation corresponding to the phase transformation (2.41) can, from Eq. (2.34), be written

$$U = \mathrm{e}^{\mathrm{i}\alpha Q}.$$

Hence for infinitesimally small α, we obtain from Eq. (2.33)

$$\begin{aligned}\phi_r' &= \mathrm{e}^{\mathrm{i}\alpha Q}\phi_r\mathrm{e}^{-\mathrm{i}\alpha Q}\\ &= \phi_r + \mathrm{i}\alpha\,[Q, \phi_r] = (1 - \mathrm{i}\alpha q)\phi_r\end{aligned}$$

where the last line follows from Eq. (2.43). Comparing Eqs. (2.45) and (2.41), we see that they are consistent if we take $\varepsilon = -\alpha\mathrm{q}$.

Although we have talked of electric charge, with which one is familiar, this analysis applies equally to other types of charge, such as hypercharge.

Conservation of energy and momentum and of angular momentum follows from the invariance of the Lagrangian density $\mathscr{L}$ under translations and rotations. Since these transformations form a continuous group we need only consider infinitesimal transformations. Any finite transformation can be built up through repeated infinitesimal transformations. In four dimensions these are given by

$$x_\alpha \to x'_\alpha \equiv x_\alpha + \delta x_\alpha = x_\alpha + \varepsilon_{\alpha\beta}x^\beta + \delta_\alpha$$

where δ_α is an infinitesimal displacement and $\varepsilon_{\alpha\beta}$ is an infinitesimal antisymmetric tensor, $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$, to ensure the invariance of $x_\alpha x^\alpha$ under homogeneous Lorentz transformations ($\delta_\alpha = 0$).

The transformation (2.46) will induce a transformation in the fields which we assume to be

$$\phi_r(x) \to \phi'_r(x') = \phi_r(x) + \frac{1}{2}\varepsilon_{\alpha\beta} S^{\alpha\beta}_{rs} \phi_s(x).$$

In this section, summation over repeated indices r, s, labelling fields, as well as over Lorentz indices α, β, is implied. Here x and x' label the same point in space-time referred to the two frames of reference, and ϕ_r and ϕ'_r are the field components referred to these two coordinate systems. The coefficients $S^{\alpha\beta}_{rs}$ in Eq. (2.47) are antisymmetric in α and β, like $\varepsilon_{\alpha\beta}$, and are determined by the transformation properties of the fields. For example, for the vector potential $A_\alpha(x)$, Eq. (2.47) reduces to the transformation law of a vector. Invariance under the transformations (2.46) and (2.47) means that the Lagrangian density expressed in terms of the new coordinates and fields has the same functional form as when expressed in the original coordinates and fields:

$$\mathscr{L}\left(\phi_r(x), \phi_{r,\alpha}(x)\right) = \mathscr{L}\left(\phi'_r(x'), \phi'_{r,\alpha}(x')\right).$$

(Here $\phi_{r,\alpha}(x') \equiv \partial\phi'_r(x')/\partial x'^\alpha$.) From Eq. (2.48) the covariance of the field equations, etc. follows; i.e. they will have the same form expressed in terms of either the original or the transformed coordinates and fields.

The conservation laws follow by expressing the right-hand side of Eq. (2.48) in terms of the original coordinates and fields by means of Eqs. (2.46) and (2.47). We shall first state and discuss these results, postponing their derivation to the end of this section. For a translation (i.e. $\varepsilon_{\alpha\beta} = 0$), one obtains the four continuity equations

$$\frac{\partial \mathscr{T}^{\alpha\beta}}{\partial x^\alpha} = 0,$$

where

$$\mathscr{J}^{-\alpha\beta} \equiv \frac{\partial \mathscr{L}}{\partial \phi_{r,\alpha}} \frac{\partial \phi_r}{\partial x_\beta} - \mathscr{L} g^{\alpha\beta},$$

and the four conserved quantities are

$$cP^\alpha \equiv \int \mathrm{d}^3\mathbf{x} \mathscr{T}^{-\alpha\alpha} = \int \mathrm{d}^3\mathbf{x} \left\{ c\pi_r(x) \frac{\partial \phi_r(x)}{\partial x_\alpha} - \mathscr{L} g^{0\alpha} \right\}.$$

P^α is the energy-momentum four-vector, with

$$\left.\begin{aligned} cP^0 &= \int \mathrm{d}^3\mathbf{x} \left\{ \pi_r(x)\dot{\phi}_r(x) - \mathscr{L}\left(\phi_r, \phi_{r,\alpha}\right) \right\} \\ &= \int \mathrm{d}^3\mathbf{x} \mathscr{H} = H \end{aligned}\right\}$$

being the Hamiltonian, Eqs. (2.24) and (2.25), and

$$P^j = \int \mathrm{d}^3\mathbf{x} \pi_r(x) \frac{\partial \phi_r(x)}{\partial x_j}$$

being the momentum components of the fields. This interpretation will be confirmed when we come to express these operators in the number representation. Correspondingly $\mathscr{F}^{\alpha\beta}$ is called

the energy-momentum tensor. For a rotation (i.e. $\delta_\alpha = 0$) Eqs. (2.46)-(2.48) give the continuity equations

$$\frac{\partial \|^{\alpha\beta\gamma}}{\partial x^\alpha} = 0,$$

where

$$\|^{\alpha\beta\gamma} \equiv \frac{\partial,}{\partial \phi_{r,\alpha}} S_{rs}^{\beta\gamma} \phi_s(x) + \left[x^\beta, -\alpha\gamma - x^\gamma, -\alpha\beta\right],$$

and the six conserved quantities (note that $\|^{\alpha\beta\gamma} = -\mathbb{1}^{\alpha\gamma\beta}$) are

$$\begin{aligned} cM^{\alpha\beta} &= \int \mathrm{d}^3\mathbf{x} \|^{0\alpha\beta} \\ &= \int \mathrm{d}^3\mathbf{x} \left\{ \left[x^\alpha \mathscr{F}^{-0\beta} - x^\beta \mathcal{J}^{0\alpha} \right] + c\pi_r(x) S_{rs}^{\alpha\beta} \phi_s(x) \right\}. \end{aligned}$$

For two space-like indices ($i, j = 1, 2, 3$), M^{ij} is the angular momentum operator of the field (M^{12} being the z-component, etc.). Remembering that $\mathscr{F}^{oi}/c$ is the momentum density of the field [see Eqs. (2.51)], we interpret the term in square brackets in Eq. (2.54)

as the orbital angular momentum and the last term as the intrinsic spin angular momentum.

We return to the derivation of the continuity equations (2.49) and (2.52).[4] The variation of a function ϕ_r(x) with the argument unchanged was defined in Eq. (2.39) as

$$\delta\phi_r(x) \equiv \phi'_r(x) - \phi_r(x).$$

In addition, we now define the variation

$$\delta_{\mathrm{T}}\phi(x) \equiv \phi'_r(x\prime) - \phi_r(x).$$

which results from changes of both the form and the argument of the function. We can then write

$$\begin{aligned} \delta_{\mathrm{T}}\phi_r(x) &= \left[\phi'_r\left(x'\right) - \phi_r\left(x'\right)\right] + \left[\phi_r\left(x'\right) - \phi_r(x)\right] \\ &= \delta\phi_r\left(x'\right) + \frac{\partial \phi_r}{\partial x_\beta} \delta x_\beta, \end{aligned}$$

where δx_β is given by Eq. (2.46). To first order in small quantities this can be written

$$\delta_{\mathrm{T}}\phi_r(x) = \delta\phi_r(x) + \frac{\partial \phi_r}{\partial x_\beta} \delta x_\beta.$$

We can similarly write Eq. (2.48) as

$$\begin{aligned} 0 &= \mathscr{L}\left(\phi'_r\left(x'\right), \phi_{r,\alpha}\left(x'\right)\right) - \mathscr{L}\left(\phi_r(x), \phi_{r,\alpha}(x)\right) \\ &= \delta y + \frac{\partial \mathscr{L}}{\partial x^\alpha} \delta x^\alpha \end{aligned}$$

For $\delta\mathscr{L}$ we obtain, since $\phi_r(x)$ satisfies the field equations (2.16),

$$\begin{aligned} \delta\mathscr{L} &= \frac{\partial \mathscr{L}}{\partial \phi_r} \delta\phi_r + \frac{\partial L}{\partial \phi_{r,\alpha}} \delta\phi_{r,\alpha} \\ &= \frac{\partial}{\partial x^\alpha} \left\{ \frac{\partial y}{\partial \phi_{r,\alpha}} \delta\phi_r \right\} = \frac{\partial}{\partial x^\alpha} \left\{ \frac{\partial y}{\partial \phi_{r,\alpha}} \left[\delta_{\mathrm{T}}\phi_r - \frac{\partial \phi_r}{\partial x_\beta} \delta x_\beta \right] \right\}. \end{aligned}$$

We combine Eqs. (2.58) and (2.59) to obtain the continuity equations

$$\frac{\partial f^\alpha}{\partial x^\alpha} = 0,$$

where

$$f^\alpha \equiv \frac{\partial \mathscr{L}}{\partial \phi_{r,\alpha}} \delta_{\mathrm{T}} \phi_r - \mathcal{T}^{\alpha\beta} \delta x_\beta$$

with $\mathcal{I}^{\alpha\beta}$ given by Eq. (2.50). We first consider translations, i.e. $\varepsilon_{\alpha\beta} = 0$, so that from Eqs. (2.46) and (2.47) $\delta x_\beta = \delta_\beta$ and $\delta_{\mathrm{T}} \phi_r = 0$. Eq. (2.61) reduces to $f^\alpha = -\mathcal{J}^{\alpha\beta} \delta x_\beta$, and, since the four displacements δ_β are independent of each other, Eq. (2.60) reduces to the four continuity equations (2.49) for energy and momentum conservation.

Finally, we consider rotations, i.e. $\delta_\alpha = 0$. From Eqs. (2.46) and (2.47) for δx_β and $\delta_{\mathrm{T}} \phi_r$, and the antisymmetry of $\varepsilon_{\alpha\beta}$, Eq. (2.61) becomes

$$f^\alpha = \frac{1}{2} \varepsilon_{\beta\gamma} \| ^{\alpha\beta\gamma}$$

where $\| ^{\alpha\beta\gamma}$ is the tensor defined in Eq. (2.53). Since the rotations $\varepsilon_{\beta\gamma}$ are independent of each other, Eq. (2.60) reduces to the continuity equations (2.52).

8.6.3 Idea of Symmetries (after Zi)

Симметрия, преобразование и инвариантность

Важность симметрии для современной физики трудно переоценить 1. Когда физический закон не меняется при некотором преобразовании, говорят, что он обладает симметрией.

Я уже использовал лоренц-инвариантность для ограничения формы действия. Конечно, лоренц-инвариантность является симметрией пространства-времени, но поля могут также преобразовываться в так называемом внутреннем пространстве. На самом деле мы уже видели тому пример. В главе I.3 я бегло упомянул, что в теории скалярного поля мы могли бы потребовать инвариантность действия относительно преобразования $\varphi \to -\varphi$ и таким образом исключить из действия члены нечетной степени по φ, например, φ^3.

Если член φ^3 присутствует, два мезона могут рассеяться и перейти в три мезона, как показано для примера на рис. I.9.1.

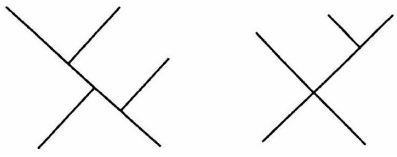


Рис. I.9.1

Но в отсутствие этого члена такой процесс запрещен, и вы легко можете убедиться в этом. Нельзя нарисовать диаграмму Фейнмана с нечетным числом внешних линий. (Подумайте о модификации интеграла из нашей младенческой задачи

[1] A. Zee, Fearful Symmetry. в главе I.7 до вида $\int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2q^2 - gq^3 - \lambda q^4 + Jq}$.) Таким образом, простая симметрия относительно отражений $\varphi \to -\varphi$ подразумевает, что в любом процессе рассеяния число мезонов сохраняется по модулю 2.

После того как мы рассмотрели случай одного скалярного поля, обратимся к теории с двумя скалярными полями φ_1 и φ_2, удовлетворяющими симметрии относительно отражений $\varphi_a \to -\varphi_a (a = 1, 2)$:

$$\mathcal{L} = \frac{1}{2}(\partial\varphi_1)^2 - \frac{1}{2}m_1^2\varphi_1^2 - \frac{\lambda_1}{4}\varphi_1^4 + \frac{1}{2}(\partial\varphi_2)^2 - \frac{1}{2}m_2^2\varphi_2^2 - \frac{\lambda_2}{4}\varphi_2^4 - \frac{\rho}{2}\varphi_2^2\varphi_2^2$$

Имеем две скалярные частицы, назовем их 1 и 2, с массами m_1 и m_2. В низшем порядке они могут рассеиваться следующим образом: $1+1 \to 1+ +1, 2+2 \to 2+2, 1+2 \to 1+2, 1+1 \to 2+2$ и $2+2 \to 1+1$ (убедитесь в этом самостоятельно). Если пять параметров $m_1, m_2, \lambda_1, \lambda_2$ и ρ абсолютно произвольны, между частицами нет никакой взаимосвязи.

Точка зрения, которую среди прочих активно поддерживал Эйнштейн и согласно которой фундаментальные законы должны быть упорядоченными и простыми, а не произвольными и сложными, является почти символом веры среди физиков-теоретиков.

Такая упорядоченность находит свое отражение в симметрии действия.

Предположим, что $m_1 = m_2$ и $\lambda_1 = \lambda_2$. Тогда две частицы будут иметь одинаковые массы и одинаково взаимодействовать как с собой, так и друг с другом. Лагранжиан $\mathcal{L}$ оказывается инвариантным относительно симметрии перестановки $\varphi_1 \leftrightarrow \varphi_2$.

Предположим, далее, что мы налагаем условие $\rho = \lambda_1 = \lambda_2$, так что лагранжиан становится равным

$$\mathcal{L} = \frac{1}{2}\left[(\partial\varphi_1)^2 + (\partial\varphi_2)^2\right] - \frac{1}{2}m^2\left(\varphi_1^2 + \varphi_2^2\right) - \frac{\lambda}{4!}\left(\varphi_1^2 + \varphi_2^2\right)^2$$

и инвариантным относительно двумерных вращений

$\{\varphi_1(x) \to \cos\theta\varphi_1(x) + \ +\sin\theta\varphi_2(x), \varphi_2(x) \to -\sin\theta\varphi_1(x) + \cos\theta\varphi_2(x)\}$

с произвольным углом θ. Говорят, что теория обладает «внутренней» $SO(2)$-симметрией, внутренней в том смысле, что преобразование не имеет ничего общего с пространством-временем. В отличие от симметрии перестановки $\varphi_1 \leftrightarrow \varphi_2$, это преобразование зависит от непрерывного параметра θ, а соответствующая симметрия называется непрерывной.

Из этого простого примера мы видим, что существуют иерархии симметрий.

Непрерывные симметрии

Если мы будем достаточно долго смотреть на уравнения движения $(\partial^2 + m^2)\varphi_a + \lambda\vec{\varphi}^2\varphi_a = 0$, то увидим, что если определить $J^\mu \equiv i(\varphi_1\partial^\mu\varphi_2 - \ -\varphi_2\partial^\mu\varphi_1)$, то $\partial_\mu J^\mu = i(\varphi_1\partial^2\varphi_2 - \varphi_2\partial^2\varphi_1) = 0$, то есть J^μ является сохраняющимся током. Соответствующий заряд $Q = \int d^D x J^0$, подобно электрическому заряду, сохраняется.

Обратимся к истории. Когда Гейзенберг заметил, что масса только что открытого нейтрона почти совпадает с массой протона, он высказал предположение, что если бы электромагнитное взаимодействие можно было бы каким-то образом выключить, то существовала бы внутренняя симметрия, переводящая протон в нейтрон.

Внутренняя симметрия точно так же накладывает ограничения на форму теории, как и лоренц-инвариантность. В качестве обобщения нашего простого примера предположим, что мы хотим построить теорию поля, содержащую N скалярных полей $\varphi_a, a = 1, ..., N$, так, чтобы она была инвариантна относительно преобразований $\varphi_a \to R_{ab}\varphi_b$ (с суммированием по повторяющимся индексам), где матрица R есть элемент группы вращений $SO(N)$ (см. краткое изложение теории групп в приложении В). Поля φ_a преобразуются как вектор $\vec{\varphi} = (\varphi_1, ..., \varphi_N)$. Мы можем составить только один инвариант, а именно скалярное произведение

$\vec{\varphi}\cdot\vec{\varphi} = \varphi_a\varphi_b = \vec{\varphi}^2$ (как обычно, по повторяющимся индексам, если не оговорено особо, производится суммирование). Лагранжиан, таким образом, ограничивается формой

$$\mathcal{L} = \frac{1}{2}\left[(\partial\vec{\varphi})^2 - m^2\vec{\varphi}^2\right] - \frac{\lambda}{4}\left(\vec{\varphi}^2\right)^2.$$

Правила Фейнмана приведены на рис. 1.9.2. Когда мы рисуем диаграммы Фейнмана, каждая линия, помимо импульса, несет на себе внутренний индекс.

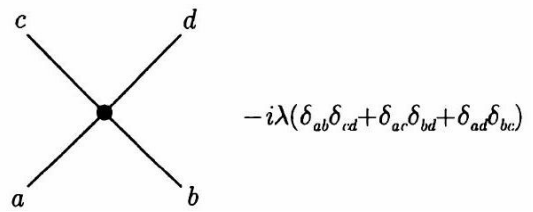


Рис. I.9.2

Симметрия проявляется в физических амплитудах. Представим, например, что мы вычисляем пропагатор $iD_{ab}(x) = \int D\vec{\varphi}e^{iS}\varphi_a(x)\varphi_b(0)$. Предпо- ложим, что мера $D\vec{\varphi}$ инвариантна относительно $SO(N)$. Рассуждая о том, как $D_{ab}(x)$ преобразуется под действием группы симметрии $SO(N)$, вы легко придете к выводу (см. упражнение I.9.2), что он должен быть пропорционален δ_{ab}. Можно проверить справедливость полученного вывода, нарисовав несколько диаграмм Фейнмана или рассмотрев обыкновенный интеграл $\int d\vec{q}e^{-S(q)}q_aq_b$. Независимо от того, насколько сложна нарисованная вами диаграмма (например, рис. I.9.3), вы всегда получите этот множитель δ_{ab}. Аналогично амплитуды рассеяния также должны обладать симметрией.

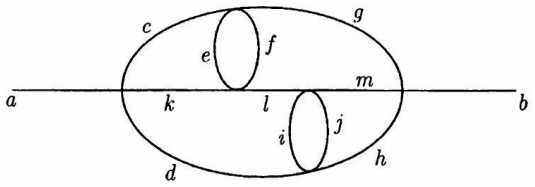


Рис. I.9.3

В отсутствие $SO(N)$-симметрии, в (3) могли бы присутствовать и другие члены (например, $\varphi_a\varphi_b\varphi_c\varphi_d$ с произвольно выбранными a, b, c и d).

Можно записать $R = e^{\theta\cdot T}$, где $\theta\cdot T = \sum_A \theta^A T^A$ есть вещественная антисимметричная матрица. Группа $SO(N)$ имеет $N(N-1)/2$ генераторов, обозначаемых как T^A. [Рассмотрите более знакомый вам случай $SO(3)$.] При инфинитезимальном преобразовании (с суммированием по повторяющимся индексам) $\varphi_a \to R_{ab}\varphi_b \simeq \left(1+\theta^A T^A\right)_{ab}\varphi_b$, то есть, другими словами, имеем бесконечно малое изменение $\delta\varphi_a = \theta^A {T^A}_{ab}\varphi_b$.

Теорема Нётер

Мы подошли к одному из самых принципиальных открытий теоретической физики - теореме Нётер, которая гласит, что каждому генератору группы непрерывной симметрии соответствует сохраняющийся ток. Появление сохраняющегося тока в (2) не было случайным.

Как это часто бывает с особо важными теоремами, доказательство теоремы Нётер удивительно просто. Обозначим поля нашей теории как φ_a. Поскольку симметрия является

непрерывной, можно рассмотреть бесконечно малое изменение $\delta\varphi_a$. Так как $\mathcal{L}$ при этом не меняется, получаем

$$0 = \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\varphi_a}\delta\varphi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\delta\partial_\mu\varphi_a = \frac{\delta\mathcal{L}}{\delta\varphi_a}\delta\varphi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\partial_\mu\delta\varphi_a$$

На этом месте мы придем в тупик, если не воспользоваться уравнениями движения $\delta\mathcal{L}/\delta\varphi_a = \partial_\mu\left(\delta\mathcal{L}/\delta\partial_\mu\varphi_a\right)$ и не объединить два члена, получив

$$0 = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\delta\varphi_a\right)$$

Если определить

$$J^\mu \equiv \frac{\delta\mathcal{L}}{\delta\partial_\mu\varphi_a}\delta\varphi_a$$

то (5) будет означать, что $\partial_\mu J^\mu = 0$. Мы нашли сохраняющийся ток! [Из наших действий очевидно, что в (6) производится суммирование по повторяющемуся индексу a.

Теперь немедленно проиллюстрируем это на примере простой теории скалярного поля (3). Подставляя $\delta\varphi_a = \theta^A\left(T^A\right)_{ab}\varphi_b$ в (6) и замечая, что θ^A является произвольным, получаем $N(N-1)/2$ сохраняющихся токов $J^A_\mu = \partial_\mu\varphi_a\left(T^A\right)_{ab}\varphi_b$, по одному на каждый генератор группы симметрии $SO(N)$

В частном случае $N = 2$ можно определить комплексное поле $\varphi \equiv (\varphi_1 + i\varphi_2)/\sqrt{2}$. Лагранжиан (3) перепишем в виде

$$\mathcal{L} = \partial\varphi^\dagger\partial\varphi - m^2\varphi^\dagger\varphi - \lambda\left(\varphi^\dagger\varphi\right)^2$$

Видим, что он инвариантен относительно $\varphi \to e^{i\theta}\varphi$ и $\varphi^\dagger \to e^{-i\theta}\varphi^\dagger$. Из (6) находим $J_\mu = i\left(\varphi^\dagger\partial_\mu\varphi - \partial_\mu\varphi^\dagger\varphi\right)$ - это ток, использованный нами в упражнении I.8.4. Математически это объясняется изоморфизмом групп $SO(2)$ и $U(1)$ (см. приложение B).

Пример скалярного поля, преобразующегося как вектор под действием группы $SO(N)$, я рассмотрел из педагогических соображений. Очевидно, что все рассмотренное выше остается справедливым и для случая произвольной группы G с полями φ, преобразующимися в условиях произвольного представления $\mathcal{R}$ группы G. Сохраняющиеся токи также будут определяться выражениями $J^A_\mu = \partial_\mu\varphi_a\left(T^A\right)_{ab}\varphi_b$ с генераторами T^A, вычисленными в представлении $\mathcal{R}$. Например, если φ преобразуется как пятимерное представление группы $SO(3)$, то T^A является 5×5-матрицей.

Для того чтобы физика была инвариантна относительно группы преобразований, нужно лишь, чтобы инвариантным было действие. При этом плотность лагранжиана может измениться на полную производную: $\delta\mathcal{L} = \partial_\mu K^\mu$ при условии, что соответствующий граничный член можно опустить. Тогда из (5) будет следовать, что для вывода формулы для сохраняющегося тока необходимо будет лишь модифицировать (6) до вида $J^\mu \equiv \delta\mathcal{L}/\delta\partial_\mu\varphi_a\delta\varphi_a - K^\mu$. Как мы увидим в главе VIII.4, многие суперсимметричные теории поля принадлежат к этому типу.

Заряд как генераторы

Используя канонический формализм из главы I.8, мы можем получить изящное выражение для заряда, связанного с сохраняющимся током

$$Q \equiv \int d^3x J^0 = \int d^3x \frac{\delta \mathcal{L}}{\delta \partial_0 \varphi_a} \delta \varphi_a.$$

Если узнать в $\delta\mathcal{L}/\delta\partial_0\varphi_a$ канонический импульс, сопряженный полю φ_a, получим

$$i\,[Q, \varphi_a] = \delta\varphi_a.$$

Оператор заряда генерирует соответствующее преобразование полей. Рассмотренный нами частный случай для комплексного поля φ в $SO(2) \simeq U(1)$-теории является важным; здесь $[Q, \varphi] = \varphi$ и $e^{i\theta Q}\varphi e^{-i\theta Q} = e^{i\theta}\varphi$.

Упражнения

I.9.1. Некоторые авторы предпочитают более детальную формулировку теоремы Нётер. Предположим, что действие не меняется под действием инфинитезимального преобразования $\delta\varphi_a(x) = \theta^A V_a^A$ [здесь θ^A - параметры, маркируемые индексом A, а V_a^A - функция от полей φ_b и, возможно, их первых производных по x]. Следует подчеркнуть, что, когда мы говорим о неизменности действия S, мы не используем уравнения движения. В конце концов, уравнения движения Эйлера - Лагранжа следуют из условия $\delta S = 0$ для произвольного изменения $\delta\varphi_a$, удовлетворяющего определенным граничным условиям. Рассмотренный нами пример теории скалярного поля прекрасно иллюстрирует этот момент, который в некоторых книгах может приводить читателей к замешательству: $\delta S = 0$ просто потому, что S построено с помощью скалярного произведения $O(N)$ векторов.

А теперь проделаем нечто, что на первый взгляд выглядит несколько странным. Рассмотрим написанное выше инфинитезимальное изменение, но с параметрами θ^A, зависящими от x. Другими словами, рассмотрим $\delta\varphi_a(x) = \theta^A(x)V_a^A$. В этом случае никак не объяснить факт сокращения δS, но, с другой стороны, мы знаем, что раз δS сводится к нулю при постоянном θ^A, то выражение для δS должно иметь вид $\delta S = \int d^4x J^\mu(x)\partial_\mu\theta^A(x)$. На практике это позволяет нам легко получить ток $J^\mu(x)$; он представляет собой всего лишь коэффициент при $\partial_\mu\theta^A(x)$ в δS.

Покажите, как все это работает для лагранжиана (3). I.9.2. Покажите, что $D_{ab}(x)$ должно быть пропорционально δ_{ab}, как утверждается в тексте.

I.9.3. Напишите лагранжиан $SO(3)$-инвариантной теории для скалярного (относительно преобразований Лоренца) поля φ, преобразующегося по пятимерному представлению, до членов четвертого порядка по полю. [Указание: удобно представить φ симметричной 3×3-матрицей с нулевым следом.]

I.9.4. Добавьте к лагранжиану из упражнения I.9.3 скалярное (относительно преобразований Лоренца) поле η, преобразующееся как вектор под действием $SO(3)$, сохраняя при этом $SO(3)$-инвариантность. Получите нётеровские токи этой теории. Используя уравнения движения, проверьте сохранение токов.

8.6.4 Hamiltonian Field Formalism

Лагранжева формулировка теории поля особенно подходит для релятивистской динамики, потому что все выражения в ней явно лоренц-инвариантны. Тем не менее, в первой части этой книги мы будем использовать гамильтонову формулировку, так как с ее помощью легче осуществить переход к квантовой механике.

Напомним, что в дискретной системе для каждой динамической переменной q можно определить канонически сопряженный импульс $p \equiv \partial L/\partial\dot{q}($ где $\dot{q} = \partial q/\partial t)$. Тогда гамильтониан $H \equiv \sum p\dot{q} - L$. Обобщение на непрерывную систему легче всего понять, если представлять точки пространства $\mathbf{x}$ расположенными дискретно. Мы можем определить

$$p(\mathbf{x}) \equiv \frac{\partial L}{\dot{\phi}(\mathbf{x})} = \frac{\partial}{\partial\dot{\phi}(\mathbf{x})}\int \mathscr{L}(\phi(\mathbf{y}),\dot{\phi}(\mathbf{y}))d^3y \sim \frac{\partial}{\partial\dot{\phi}(\mathbf{x})}\sum_{\mathbf{y}} \mathscr{L}(\phi(\mathbf{y}),\dot{\phi}(\mathbf{y}))d^3y = \pi(\mathbf{x})d^3x,$$

где

$$\pi(\mathbf{x}) \equiv \frac{\partial\mathscr{L}}{\partial\dot{\phi}(\mathbf{x})}$$

называется плотностью импульса, сопряженного к $\phi(\mathbf{x})$. Таким образом, гамильтониан может быть записан в виде

$$H = \sum_{\mathbf{x}} p(\mathbf{x})\dot{\phi}(\mathbf{x}) - L$$

Переходя к континууму, имеем

$$H = \int d^3x[\pi(\mathbf{x})\dot{\phi}(\mathbf{x}) - \mathscr{L}] \equiv \int d^3x\mathscr{H}$$

В конце этого раздела, используя другой метод, мы заново получим это же выражение для плотности гамильтониана $\mathscr{H}$.

В качестве простого примера рассмотрим теорию одного поля $\phi(x)$, лагранжиан которого равен

$$\mathscr{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2.$$

Пока будем считать ϕ вещественным полем. В разделе 2.3 величина m будет интерпретирована как масса, но пока будем рассматривать ее как некий параметр. Исходя из этого лагранжиана, с помощью обычной процедуры получаем уравнение движения:

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = 0 \quad \text{или} \quad \left(\partial^\mu\partial_\mu + m^2\right)\phi = 0$$

которое является известным уравнением Клейна-Гордона. (В этом контексте это классическое полевое уравнение, аналогичное уравнениям Максвелла, а не квантово-механическое волновое уравнение.) Замечая, что канонически сопряженная к $\phi(x)$ плотность импульса есть $\pi(x) = \dot{\phi}(x)$, можно также построить гамильтониан

$$H = \int d^3x\mathscr{H} = \int d^3x\left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]$$

Три слагаемых в правой части можно рассматривать, соответственно, как энергию «перемещения» во времени, энергию «сдвига» в пространстве и энергию, связанную с наличием самого поля. Мы исследуем этот гамильтониан более детально в разделах 2.3 и 2.4.

8.6.5 Noether Theorem: Basics

Обсудим теперь связь между симметриями и законами сохранения в классической теории поля, устанавливаемую теоремой Нетер. В этой теореме рассматриваются непрерывные преобразования полей ϕ, которые в инфинитезимальной форме могут быть записаны в виде

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi(x),$$

где α - инфинитезимальный параметр и $\Delta\phi$ - некоторая деформация конфигурации поля. Мы называем это преобразование симметрией, если оно оставляет инвариантными уравнения движения. Это заведомо имеет место, если действие инвариантно по отношению к преобразованию (2.9). В более общем случае можно допустить, чтобы добавка к действию имела вид поверхностного слагаемого, так как наличие такого слагаемого не влияет на вывод уравнений движения Эйлера-Лагранжа (2.3). Поэтому лагранжиан должен быть инвариантен относительно (2.9) с точностью до 4-дивергенции от некоторого вектора $\mathscr{J}^\mu$:

$$\mathscr{L}(x) \to \mathscr{L}(x) + \alpha\partial_\mu \mathscr{J}^\mu(x),$$

Сравним ожидаемое выражение для $\Delta\mathscr{L}$ с результатом, полученным при вариации полей:

$$\begin{aligned}\alpha\Delta\mathscr{L} &= \frac{\partial\mathscr{L}}{\partial\phi}(\alpha\Delta\phi) + \left(\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi\right)}\right)\partial_\mu(\alpha\Delta\phi) = \\ &= \alpha\partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi\right)}\Delta\phi\right) + \alpha\left[\frac{\partial\mathscr{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi\right)}\right)\right]\Delta\phi\end{aligned}$$

В силу уравнения Эйлера-Лагранжа (2.3) второе слагаемое обращается в нуль. Полагая левую часть этого уравнения равной $\alpha\partial_\mu\mathscr{J}^\mu$, находим:

$$\partial_\mu j^\mu(x) = 0, \quad \text{для \%} \quad \%j^\mu(x) = \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi\right)}\Delta\phi - \mathscr{J}^\mu.$$

(Если симметрия включает больше одного поля, то первое слагаемое в этом выражении для $j^\mu(x)$ должно быть заменено суммой таких слагаемых по одному для каждого поля.) Этот результат означает, что ток $j^\mu(x)$ сохраняется. Каждой непрерывной симметрии $\mathscr{L}$ соответствует такой закон сохранения.

Закон сохранения может быть также сформулирован как сохранение заряда

$$Q \equiv \int\limits_{\text{все пространство}} j^0 d^3x$$

во времени. Заметим, однако, что формулировка теории поля в терминах локальной плотности лагранжиана непосредственно приводит к локальной форме закона сохранения (2.12).

Самый простой пример такого закона возникает в теории с лагранжианом, содержавшим лишь кинетическое слагаемое: $\mathscr{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2$. Преобразование $\phi \to \phi + \alpha$, где α - константа, оставляет $\mathscr{L}$ неизменным, и мы заключаем, что ток $j^\mu = \partial^\mu\phi$ сохраняется. В качестве менее тривиального примера рассмотрим лагранжиан

$$\mathscr{L} = |\partial_\mu\phi|^2 - m^2|\phi|^2$$

где теперь ϕ - комплексное поле. Можно легко показать, что уравнение движения для этого лагранжиана вновь является уравнением Клейна-Гордона (2.7). Этот лагранжиан инвариантен относительно преобразования $\phi \to e^{i\alpha}\phi$; для бесконечно малого преобразования имеем:

$$\alpha\Delta\phi = i\alpha\phi; \quad \alpha\Delta\phi^* = -i\alpha\phi^*$$

(Мы рассматриваем ϕ и ϕ^* как независимые поля. Альтернативно можно было бы работать с вещественной и мнимой частями поля ϕ.) Теперь несложно показать, что сохраняющийся нетеровский ток равен

$$j^\mu = i\left[\left(\partial^\mu\phi^*\right)\phi - \phi^*\left(\partial^\mu\phi\right)\right].$$

(Общий множитель был выбран произвольно.) Используя уравнение Клейна-Гордона, можно непосредственно проверить, что дивергенция этого тока равна нулю. Ниже мы добавим к этому лагранжиану слагаемые, которые связывают ϕ с электромагнитным полем. Тогда j^μ можно будет рассматривать как плотность электромагнитного тока комплексного скалярного поля, а пространственный интеграл j^0 - как электрический заряд.

Теорему Нетер можно также применить к пространственно-временным преобразованиям типа трансляций и вращений. Бесконечно малая трансляция

$$x^\mu \to x^\mu - a^\mu$$

может быть альтернативно описана как преобразование поля:

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu\partial_\mu\phi(x).$$

Лагранжиан также является скаляром, поэтому он должен преобразовываться аналогично:

$$\mathscr{L} \to \mathscr{L} + a^\mu\partial_\mu\mathscr{L} = \mathscr{L} + a^\nu\partial_\mu\left(\delta^\mu{}_\nu\mathscr{L}\right).$$

Сравнивая это уравнение с (2.10), видим, что теперь имеется ненулевой ток $\mathscr{J}^\mu$. Учитывая это, можно применить теорему Нетер, чтобы получить четыре отдельно сохраняющихся тока

$$T^\mu_\nu \equiv \frac{\partial\mathscr{L}}{\partial\left(\partial_\mu\phi\right)}\partial_\nu\phi - \mathscr{L}\delta^\mu_\nu$$

Это - в точности тензор энергии-натяжения, называемый также тензором энергиц-импульса поля ϕ. Сохраняющийся заряд, связанный с временными трансляциями, есть гамильтониан

$$H = \int T^{00}d^3x = \int \mathscr{H}d^3x$$

Вычисляя эту величину для поля Клейна-Гордона, можно еще раз получить результат (2.8). Связанные с пространственными трансляциями сохраняющиеся заряды имеЮт вид:

$$P^i = \int T^{0i}d^3x = -\int \pi\partial_i\phi d^3x$$

Мы интерпретируем их как (физический) импульс, переносимый полем (не путать с каноническим импульсом).

8.6.6 Noether Theorem: Introduction to Consequences

(there are a lot of them in QFT)

8.7 So, What Is Exactly a Quantum Field?

8.7.1 What Do We Mean By "Operator Quantum Field"?

(write this in detail)

8.7.2 The Transition from Classical to Quantum Fields

(отдельно прорабатываю это, типичная первая тема.)

8.7.3 Is a Particle a One Field or Several?

(???? It is a fundamental question. I'll write some answer which I have)

8.7.4 Single-quantum Field

(??????????????)

общие соображения о локальных полях

Свободные поля эквивалентны частицам, поэтому их можно описывать в виде операторов рождения и уничтожения элементарных квантов в фоковском пространстве.

Однако поле является первичной сущностью, а его кванты производной от поля сущностью, потому что не всякое квантованное поле представимо как набор квантов.

Например, при наличии потенциала скалярного поля с минимумом потенциала, который достигается при не равном нулю поле, это вакуумное поле вовсе не соответствует каким-либо квантам, оно является самостоятельной сущностью.

Другой пример - поле, которое в силу внешних условий зависит только от времени, также вполне самостоятельная физическая сущность, более общая, чем кванты.

(что?)

условия, про которых квант поля это частица

Мы сформулируем условия, при которых понятие кванта локального поля сводится к точечной частице, которая обладает всеми измеримыми параметрами поля в форме дискретных порций этих параметров, таких как энергия, импульс, заряд и проч.

Мы будем рассуждать в терминах свободного поля, а значит, все наши построения останутся справедливыми в областях физической системы, где становится обоснованным приближение пренебрежимо малого взаимодействия полей.

Эти области называются асимптотически свободными.

Тогда в асимптотически свободных областях физической системы можно ввести понятие фоковского пространства и вакуума - состояния с минимальной энергией без квантов.

итоговые методы

(сперва простая теория)
Вводится причинный коррелятор, который также обозначается как

$$\mathcal{D}^{(2)}(x-y) \equiv \langle 0|\hat{\mathrm{T}}\{\hat{\varphi}(x)\hat{\varphi}(y)\}|0\rangle$$

поле с одним квантом

Рассмотрим релятивистское поле, когда оператор гамильтониана поля после нормального упорядочивания равен

$$\hat{H} = \int \frac{\mathrm{d}^3k}{(2\pi)^3 2\hbar\omega(\boldsymbol{k})}\hbar\omega(\boldsymbol{k})\hat{\mathfrak{a}}^\dagger(\boldsymbol{k})\hat{\mathfrak{a}}(\boldsymbol{k})$$

Изучим характеристики состояния с одним квантом

$$|k\rangle = \hat{\mathfrak{a}}^\dagger(k)|0\rangle.$$

Действие гамильтониана поля на состояние с одним квантом дает

$$\hat{H}|k\rangle = \int \frac{\mathrm{d}^3k'}{(2\pi)^3 2\hbar\omega(k)}\hbar\omega(k)\hat{\mathfrak{a}}^\dagger\left(k'\right)\hat{\mathfrak{a}}\left(k'\right)\hat{\mathfrak{a}}^\dagger(k)|0\rangle$$

преобразуем это выражение. Перестановка операторов рождения и уничтожения приводит к

$$\hat{\mathfrak{a}}\left(\boldsymbol{k}'\right)\hat{\mathfrak{a}}^\dagger(\boldsymbol{k}) = \hat{\mathfrak{a}}^\dagger(\boldsymbol{k})\hat{\mathfrak{a}}\left(\boldsymbol{k}'\right) + (2\pi)^3\delta\left(\boldsymbol{k}-\boldsymbol{k}'\right)2\hbar\omega(\boldsymbol{k}),$$

поэтому интеграл преобразуется:

$$\hat{H}|k\rangle = \hbar\omega(k)|k\rangle + \int \frac{\mathrm{d}^3k'}{(2\pi)^3 2\hbar\omega(k)}\hbar\omega(k)\hat{\mathfrak{a}}^\dagger\left(k'\right)\hat{a}^\dagger(k)\hat{a}\left(k'\right)|0\rangle = \hbar\omega(k)|k\rangle,$$

поскольку действие опрератора уничтожения кванта на вакуум дает нуль $\hat{\mathfrak{a}}\left(k'\right)|0\rangle = 0$.

Получается, что состояние с одним квантом $|k\rangle$ - это собственное состояние гамильтониана поля с квантом энергии $\hbar\omega(k)$.

Конкретно, в чем выражается локальность?

(пропишу, важный вопрос!)

импульс поля с одним квантом

Изучим поле с одним квантом.
Посмотрим на импульс поля
Импульс поля, который задается через плотность импульса

$$p^\alpha = \int \mathrm{d}^3x T_0^\alpha = -\frac{1}{c}\int \mathrm{d}^3x \nabla_\alpha\varphi\partial_t\varphi$$

на уравнениях движения имеет вид (??? мы что ли подставили сюда решение?)

$$\begin{aligned} p = \tfrac{1}{c}\int \tfrac{c^2d^3k}{4\omega^2(2\pi)^3}\big\{&k\omega e^{-2\mathrm{i}\omega t}\alpha(\omega,k)\alpha(\omega,-k) - \\ &-k\omega e^{2\mathrm{i}\omega t}\alpha^*(\omega,k)\alpha^*(\omega,-k) + \\ &+k\omega\left[\alpha(\omega,k)\alpha^*(\omega,k)+\alpha^*(\omega,k)\alpha(\omega,k)\right]\big\} \end{aligned}$$

Преобразуем. В этом выражении две первых строчки справа от знака равенства тождественно равны нулю,

поскольку, глядя на явный вид подынтегральных функций,

при замене переменной интегрирования $k \mapsto -k$ интегралы переходят сами в себя со знаком минус! (???) В итоге импульс поля на уравнениях движения

$$p = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2\hbar\omega(k)} \hbar k |\alpha(\omega, k)|^2$$

после квантования и применения операции нормального упорядочивания переходит в оператор

$$\hat{p} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2\hbar\omega(k)} \hbar k \hat{\mathrm{a}}^\dagger(k) \hat{\mathrm{a}}(k)$$

Вспомним, что оператор энергии поля имеет с точностью до константы такой же вид, так верно:

$$\hat{p}|k\rangle = \hbar k |k\rangle$$

а значит, состояние с одним квантом $|k\rangle$ является собственным для оператора импульса поля,

и этот квант несет импульс поля $\hbar k$.

квант - точечная частица

Покажем теперь, что состояние с одним квантом поля $|k\rangle$ - это точечная частица.

Делается это за счет (?????)

Для этого сначала рассмотрим матричный элемент

$$\langle 0|\hat{\varphi}(x)|k\rangle$$

Физический смысл этого матричного элемента следующий: на состояние поля с одним квантом импульса $\hbar k$ и энергии $\hbar\omega(k)$ в момент времени t в точке x подействовал оператор квантованного поля $\hat{\varphi}(x)$, а в конечном состоянии мы рассматриваем состояние вакуума, а значит, рассматриваем ситуацию, когда данный квант поля исчез в точке x!

С учетом выражения для квантованного поля (2.46),

$$\hat{\varphi}(x) = \int \frac{c d^3 k}{2\omega(2\pi)^3} \left\{ e^{\mathrm{i}kx} \hat{\mathfrak{a}}^\dagger(\boldsymbol{k}) + e^{-\mathrm{i}kx} \hat{\mathfrak{a}}(\boldsymbol{k}) \right\}$$

мысленно повторим упражнения с перестановкой операторов рождения и уничтожения, после чего легко найдем, что искомый матричный элемент равен

$$\langle 0|\hat{\varphi}(x)|\boldsymbol{k}\rangle = e^{-\mathrm{i}kx} \hbar c$$

Таким образом, амплитуда перехода состояния поля с одним квантом в вакуум при действии локального поля отлична от нуля. Более того, с точностью до фактора $\hbar c$ амплитуда (3.2) есть просто положительно частотная мода поля. Подчеркнем, что амплитуда исчезновения частицы с заданным импульсом в

начальном состоянии отвечает именно положительно частотной моде. Ясно, что аналогичный матричный элемент

$$\langle \boldsymbol{k}|\hat{\varphi}(x)|0\rangle = (\langle 0|\hat{\varphi}(x)|\boldsymbol{k}\rangle)^\dagger = e^{\mathrm{i}kx} \hbar c$$

отвечает появлению частицы в конечном состоянии из вакуума, и он описывается, подчеркнем, отрицательно частотной модой.

Логическая трактовка этого матричного элемента возможна только,

если полагать, что сам квант является точечным объектом, т. е. частицей!

квант локального поля является точечной частицей, которая несет вполне определенную порцию энергии и импульса поля.

Если у поля, в добавок к энергии и импульсу, есть другие характеристики, например, заряд, то, как мы покажем, частица также будет обладать этими характеристиками, в частности, квантом заряда.

причинный коррелятор

Вывод (??) можно сделать еще более убедительным, если рассмотреть матричный элемент

$$\langle 0|\hat{\varphi}(x)\hat{\varphi}(y)|0\rangle \text{ при } x_0 > y_0$$

Физический смысл квантованного поля нам известен: квантованное скалярное определенными значениями энергии и импульса.

Действие этого поля на вакуум дает

$$\hat{\varphi}(y)|0\rangle = \int \frac{cd^3k}{2\omega(2\pi)^3} e^{\mathrm{i}ky}\hat{\mathfrak{a}}^\dagger(\boldsymbol{k})|0\rangle = \int \frac{cd^3k}{2\omega(2\pi)^3} e^{\mathrm{i}ky}|\boldsymbol{k}\rangle$$

Как локальное поле действует на состояние одним квантом мы уже выяснили, так что

$$\langle 0|\hat{\varphi}(x)\hat{\varphi}(y)|0\rangle = \hbar c \int \frac{cd^3k}{2\omega(2\pi)^3} e^{-\mathrm{i}k(x-y)} \quad \text{при } x_0 > y_0$$

Физический смысл этого матричного элемента следующий: в момент времени $t_0 = y_0/c$ в точке y действие локального поля породило суперпозицию квантов поля с определенными квантами энергии и импульса, каждый из этих квантов затем при $t_0' = x_0/c > t_0$ исчез в точке x, так что резонно считать, что эти кванты локального поля - точечные частицы.

Построенный матричный элемент с упорядочиванием аргументов по времени можно записать в более общем виде, если воспользоваться T-произведением, которое для бозонных полей определяется как

$$\hat{\mathrm{T}}\{\hat{\varphi}(x)\hat{\varphi}(y)\} = \vartheta(x_0 - y_0)\,\hat{\varphi}(x)\hat{\varphi}(y) + \vartheta(y_0 - x_0)\,\hat{\varphi}(y)\hat{\varphi}(x)$$

где бозонная природа отвечает знаку плюс при перестановке полей.

Тогда

$$\langle 0|\hat{\mathrm{T}}\{\hat{\varphi}(x)\hat{\varphi}(y)\}|0\rangle = \vartheta(x_0 - y_0)\,\hbar c \int \frac{cd^3k}{2\omega(2\pi)^3} e^{-\mathrm{i}k(x-y)} + \vartheta(y_0 - x_0)\,\hbar c \int \frac{cd^3k}{2\omega(2\pi)^3} e^{-\mathrm{i}k(y-x)}$$

Отметим, что при собственных ортохорнных преобразованиях группы Лоренца данное выражение является инвариантным: мера интегрирования явно инвариантна, а аргумент экспоненты - скаляр.

Этот матричный элемент можно переписать более компактно, если воспользоваться интегрированием в плоскости комплексной переменной k_0.

В самом деле, интегрирование вдоль вещественной оси выражения

$$\frac{1}{2\omega/c}\left(\frac{1}{k_0-\omega/c+\mathrm{i}0}-\frac{1}{k_0+\omega/c-\mathrm{i}0}\right)e^{-\mathrm{i}k_0(x_0-y_0)}$$

может быть проведено с помощью теоремы Коши для вычетов.

(????)

При $x_0>y_0$ к контуру интегрирования можно добавить полуокружность на бесконечности с $\operatorname{Im}k_0\to-\infty$, поскольку вклад такой полуокружности подавлен экспонентой.

Поэтому внутри замкнутого контура, ориентированного по часовой стрелке, останется только один полюс при $k_0=\omega/c-\mathrm{i}0$, и в итоге получится вклад

$$-2\pi\mathrm{i}\frac{1}{2\omega/c}e^{-\mathrm{i}\omega(x_0-y_0)/c}$$

где знак минус, конечно, относится к обходу замкнутого контура по часовой стрелке.

И аналогично, при $x_0<y_0$ к контуру интегрирования можно добавить полуокружность на бесконечности с $\operatorname{Im}k_0\to+\infty$, так что внутри замкнутого контура, ориентированного против часовой стрелке, останется только один полюс при $k_0=-\omega/c+\mathrm{i}0$, так получится вклад

$$2\pi\mathrm{i}\frac{1}{2\omega/c}(-1)e^{+\mathrm{i}\omega(x_0-y_0)/c}$$

Значит, матричный элемент можно записать теперь в единообразном виде

$$\langle 0|\hat{\mathrm{T}}\{\hat\varphi(x)\hat\varphi(y)\}|0\rangle=\mathrm{i}\hbar c\int\frac{\mathrm{d}^4k}{(2\pi)^4}e^{-\mathrm{i}k(x-y)}\frac{1}{2\omega/c}\left(\frac{1}{k_0-\omega/c+\mathrm{i}0}-\frac{1}{k_0+\omega/c-\mathrm{i}0}\right)$$

а элементарное преобразование с дробями при учете $\left(\frac{\omega}{c}-\mathrm{i}0\right)^2=\boldsymbol{k}^2+m^2c^2/\hbar^2-\mathrm{i}0$ дает

$$\langle 0|\hat{\mathrm{T}}\{\hat\varphi(x)\hat\varphi(y)\}|0\rangle=\mathrm{i}\hbar c\int\frac{\mathrm{d}^4k}{(2\pi)^4}e^{-\mathrm{i}k(x-y)}\frac{1}{k^2-m^2c^2/\hbar^2+\mathrm{i}0}$$

Подчеркнем, что в силу однородности пространства-времени Минковского и того факта, что вакуум сохраняет эту однородность,

двухточечный коррелятор $\langle 0|\hat{\mathrm{T}}\{\hat\varphi(x)\hat\varphi(y)\}|0\rangle$ является инвариантным относительно трансляций,

и поэтому он зависит лишь от разности координат $x-y$, что конечно связано с сохранением импульса: кванты поля, рожденные с импульсом $\hbar k$ в точке x,

распространяются с тем же импульсом до точки y и уничтожаются в этой точке оператором $\hat{\mathfrak{a}}(k)$ с тем же импульсом.

Причинный коррелятор (3.4) мы также будем обозначать как

$$\langle 0|\hat{\mathrm{T}}\{\hat\varphi(x)\hat\varphi(y)\}|0\rangle\equiv\mathcal{D}^{(2)}(x-y)$$

В естественной системе единиц имеем

$$\langle 0|\hat{\mathrm{T}}\{\hat\varphi(x)\hat\varphi(y)\}|0\rangle=\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}e^{-\mathrm{i}k(x-y)}\frac{1}{k^2-m^2+\mathrm{i}0}$$

сразу видно, что масса и импульс имеют одну размерность, так что в этой формуле применена система единиц $\hbar=c=1$.

система единиц

Для полноты представлений об этой системе единиц укажем в явном виде, чему равны размерности действия и т.п. в единицах массы:

$$[S] = [1], \quad \left[\mathrm{d}^4x\right] = \frac{1}{[m]^4}, \quad [\mathcal{L}] = [m]^4, \quad [\varphi] = [m], \quad [\partial_\mu] = [m]$$

или, как говорят, действие безразмерно, лагранжиан имеет размерность 4, скалярное поле - размерность 1 и проч.

Для дальнейшего изложения и вычислений нам достаточно будет знать значение, как говорят, причинного двухточечного коррелятора (3.4) именно в импульсном представлении.

Как мы увидим, с точностью до постоянных речь идет также о пропагаторе скалярного поля, т.е. функции распространения поля или частиц поля, которую также называют фейнмановским пропагатором или причинной функцией Грина.

В координатах пространствавремени Минковского этот коррелятор после стандартных преобразований переменных интегрирования сводится к вкладу от дельта-функции Дирака на световом конусе, который не зависит от массы, и вкладам от спецфункций Бесселя, Неймана и Ганкеля, которые также зависят от интервала $(x-y)^2$ и обращаются в нуль при $m \to 0$.

Вместе с тем, необходимо знать, что вне светового конуса коррелятор экспоненциально затухает при увеличении пространственно подобного интервала, а внутри светового конуса коррелятор имеет осциллирующее поведение.

согласие с СТО

Как такое поведение согласуется с принципом причинности в специальной теории относительности, который гласит, что события могут быть причинно связаны, только если их разделяют времениподобные интералы?

Наша интерпретация коррелятора, как амплитуды рождения частиц в одной точке с последующим исчезновением этих частиц в другой точке, казалось бы, должна приводить к заключению, что эта амплитуда отлична от нуля только внутри светового конуса будущего и обращаться в нуль вне светового конуса.

Это утверждение было бы совершенно корректным для классических частиц, у которых траектории не имеют неопределенностей даже при точно заданных значениях импульса.

У квантовых частиц, как мы знаем, в силу соотношения неопределенностей координата-импульс, траектории для частиц с заданными импульсами с необходимостью должны флуктуировать и могут проникать в область, которая выглядит запрещенной для классических частиц.

Именно поэтому коррелятор экспоненциально мал при пространственно подобных интервалах.

Однако это объяснение о характере поведения причинного двухточечного коррелятора не снимает с повестки вопрос о принципе причинности в специальной теории относительности для квантованных полей.

У нас есть необходимый аппарат для ответа на этот вопрос в квантовой теории: два эрмитово самосопряженных оператора, которые описывают наблюдаемые величины, могут быть независимо измерены, только если их коммутатор обращается в нуль.

Следовательно, операторы наблюдаемых локальных полей являются независимыми в области координат, разделенных пространственно подобным интервалом, только если коммутатор полей в этой области тождественно обращается в нуль.

функция Паули-Йордона $D(x-y)$

В случае свободного вещественного скалярного поля коммутатор является функцией, так что можно рассмотреть матричный элемент этого коммутатора по вакууму

$$\langle 0|[\hat{\varphi}(x),\hat{\varphi}(y)]|0\rangle = -\mathrm{i}D(x-y)$$

где $D(x-y)$ называют функцией Паули-Йордана.

При расчете этой функции у нас уже нет нужды следить за временными компонентами координат, как мы это делали при рассмотрении T- произведения полей (для определенности при интерпретации амплитуд).

Это значит, что зависимость в виде функций Хевисайда от временных аргументов не возникнет, но, при этом, останется в силе утверждение, что функция является скаляром, и в этом случае она зависит только от интервала $(x-y)^2$.

Тогда расчеты можно существенно упростить и, скажем, при пространственно подобных интервалах $(x-y)^2<0$ проводить рассмотрение в системе отсчета, где $x_0-y_0=0$, а $(x-y)^2\neq 0$. Подстановка этих условий в формулы для амплитуд, которые мы получили выше, дает

$$\langle 0|[\hat{\varphi}(x),\hat{\varphi}(y)]|0\rangle = \hbar c\int\frac{cd^3k}{2\omega(2\pi)^3}e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})} - \hbar c\int\frac{cd^3k}{2\omega(2\pi)^3}e^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{y}-\boldsymbol{x})}\equiv 0$$

поскольку оба интеграла совпадают, как легко увидеть, если в одном из них произвести замену переменной интегрирования $k\mapsto -k$.

функция Паули-Йордана действительно обращается в нуль вне светового конуса, поэтому квантование поля согласовано с принципом причинности в специальной теории относительности.

Эта же функция возникает и при квантовании полей с не равным нулю спином.

Бозонные и фермионные поля

(?? так в чем их отличие?? переформулирую от киселева теорию!)

При этом стоит отметить, что наше рассмотрение опиралось на построение для локальных полей, амплитуды которых для положительно и отрицательно частотных мод в классическом случае являются комплексными числами. Это справедливо для бозонных полей.

В случае с фермионными полями, для которых амплитуды положительно и отрицательно частотных мод являются грассмановыми переменными, необходимо сделать специальное замечание.

Именно, умножение грассмановой амплитуды θ_λ, где λ нумерует моду, на грассманово число χ_λ дает комплексное число $\chi_\lambda\theta_\lambda$ в смысле его перестановочности с другими грассмановыми числами.

А произведение соответствующего фермионного оператора $\hat{\theta}_\lambda$ на грассманово число χ_λ дает бозонный оператор $\chi_\lambda\hat{\theta}_\lambda$ в голоморфном представлении, для которого наши стандартные рассуждения имеют силу.

Так что вычисление коммутатора для двух таких бозонньх операторов сводится в вычислению антикоммутатора для фермионных операторов, поскольку грассмановы числа антиперестановочны.

Для иллюстрации и упрощения записи опустим символы, нумерующие моды, и запишем равенство для коммутатора составных операторов уже бозонного типа, соблюдая правила перестановок грассмановых чисел и операторов в голоморфном представлении

$$\begin{aligned}\left[\chi^{(1)}\hat{\theta}^{(1)}, \chi^{(2)}\hat{\theta}^{(2)}\right] =& \chi^{(1)}\hat{\theta}^{(1)}\chi^{(2)}\hat{\theta}^{(2)} - \chi^{(2)}\hat{\theta}^{(2)}\chi^{(1)}\hat{\theta}^{(1)} = \\ =& -\chi^{(1)}\chi^{(2)}\left(\hat{\theta}^{(1)}\hat{\theta}^{(2)} + \hat{\theta}^{(2)}\hat{\theta}^{(1)}\right) = \\ =& -\chi^{(1)}\chi^{(2)}\left[\hat{\theta}^{(1)}, \hat{\theta}^{(2)}\right]\end{aligned}$$

То есть требование равенства нулю коммутатора локальных бозонных полей, разделенных пространственно подобным интервалом, приводит к требованию равенства нулю антикоммутатора локальных фермионных полей, разделенных пространственно подобным интервалом.

Повторимся, что рассматриваемые нами далее примеры для спинорных и векторных полей в явном виде убедят нас, что для этих полей та же функция Паули-Йордана будет обеспечивать выполнение релятивистского принципа причинности. (?? где??)

Пример выполнения и невыполнения принципа причиннсти (??)

(?? чет не вижу пока)

8.7.5 Two- and Three-Quantum Fields (???)

(?? однокв дир поле??? двуквантовое поле??? все про это напишу!)

8.7.6 What Is Exactly a Quantum Field?

(мб во введение потом перенесу, а мб и тут оставлю, посмотрим.)

волновая функция поля и ее связь с функцией поля

подумаем, как ее ввести, какой точно ее смысл?
мб даже приведу примеры ее нахождения.
(?)

эквивалентность лагранжева и гамильтонова формализма

я говорю, что они эквивалентны, но это еще не проверено.
тут обсуждение, всегда ли это так?

Может ли быть скорость полей больше, чем скорость света, раз у нас в оптике фазовая скорость бывает более скорости света?

Типа чтобы групповая скорость была меньше. Говорят в оптике, что это и нормально.
(?? в ктп почему такого нету?)

Примеры квантованных полей

(соберу, такой вопрос может возникнуть)

отличие квантовых полей от классических

можно сказать, что отличие в том, что в классическом случае мы работаем с функцией поля,
а в квантовом - с операторами,
тем не менее, это не то, что хочется услышать.
сравним тут их природу нормально.

граница квантовых полей

я правильно понимаю, что реальное поле спадает к границе?

смысл перехода от частицы к полю

говорят, какая-то частица ударяется с другой.
где тут ктп?

Что фундаментальнее, свободное поле из группы Пунанкаре или квантмех, где всё квантуертся?

я пока не знаю, но это и отражается в последовательности глав в этой записи. мы же склеиваем квантмех и теорию поля, неплохо бы знать, что более фундаментальное? и почему вообще такие два соврешенно разных подхода должны так вот склеиваться совершенно сложно?

принципиальные моменты происходящего КТП

что такое родить частицу?

подействовать оператором рождения.
на а что это на самом деле значит?
обсудим тут подробно.

что такое вообще квантовое поле?

тут подробно ответим, ибо короткие ответы ничего не говорят.

Иллюстрация работы в КТП “на пальцах”

Вывод диаграмм Фейнамана из основ формализма (?????)

(пока я так и не понимаю, откуда конкретно бурутся диаграммы Фейнмана???? чет так и не увидел, лучше нужно пройти эти части еще.)

8.7.7 Wave-particle Duality

(see QM about more details, here will be a very similar discussion)

что значит, что частица эквивалентна волне?

Как поле дирака, в то же время электрон в классической электростатике?

никогда нормально не понимал.

один оператор может рождать несколько частиц

8.7.8 What is Not a Quantum Field?

(discuss some wrong statements)

8.7.9 What is QFT, and What Did We Think It Is? (Weinberg, 1996)

(a good article, which I'll read in free time)

Quantum field theory was originally thought to be simply the quantum theory of fields. That is, when quantum mechanics was developed physicists already knew about various classical fields, notably the electromagnetic field, so what else would they do but quantize the electromagnetic field in the same way that they quantized the theory of single particles? In 1926, in one of the very first papers on quantum mechanics,[1] Born, Heisenberg and Jordan presented the quantum theory of the electromagnetic field. For simplicity they left out the polarization of the photon, and took spacetime to have one space and one time dimension, but that didn't affect the main results. (*Response to comment from audience*: Yes, they were really doing string theory, so in this sense string theory is earlier than quantum field theory.) Born et al. gave a formula for the electromagnetic field as a Fourier transform and used the canonical commutation relations to identify the coefficients in this Fourier transform as operators that destroy and create photons, so that when quantized this field theory became a theory of photons. Photons, of course, had been around (though not under that name) since Einstein's work on the photoelectric effect two decades earlier, but this paper showed that photons are an inevitable consequence of quantum mechanics as applied to electromagnetism.

The quantum theory of particles like electrons was being developed at the same time, and made relativistic by Dirac[2] in 1928–1930. For quite a long time many physicists thought that the world consisted of both fields and particles: the electron is a particle, described by a relativistically invariant version of the Schrödinger wave equation, and the electromagnetic field is a field, even though it also behaves like particles. Dirac I think never really changed his mind about this, and I believe that this was Feynman's understanding when he first developed the path integral and worked out his rules for calculating in quantum electrodynamics. When I first learned about the path-integral formalism, it was in terms of electron trajectories (as it is also presented in the book by Feynman and Hibbs[3]). I already thought that wasn't the best way to look at electrons, so this gave me an distaste for the path integral formalism, which although unreasonable lasted until I learned of 't Hooft's work[4] in 1971. I feel it's all right to mention autobiographical details like that as long as the story shows how the speaker was wrong.

In fact, it was quite soon after the Born–Heisenberg–Jordan paper of 1926 that the idea came along that in fact one could use quantum field theory for everything, not just for electromagnetism. This was the work of many theorists during the period 1928–1934, including Jordan, Wigner, Heisenberg, Pauli, Weisskopf, Furry, and Oppenheimer. Although this is often talked about as second quantization, I would like to urge that this description should be banned from physics, because a quantum field is not a quantized wave function. Certainly the Maxwell field is not

the wave function of the photon, and for reasons that Dirac himself pointed out, the Klein–Gordon fields that we use for pions and Higgs bosons could not be the wave functions of the bosons. In its mature form, the idea of quantum field theory is that quantum fields are the basic ingredients of the universe, and particles are just bundles of energy and momentum of the fields. In a relativistic theory the wave function is a functional of these fields, not a function of particle coordinates. Quantum field theory hence led to a more unified view of nature than the old dualistic interpretation in terms of both fields and particles.

There is an irony in this. (I'll point out several ironies as I go along — this whole subject is filled with delicious ironies.) It is that although the battle is over, and the old dualism that treated photons in an entirely different way from electrons is I think safely dead and will never return, some calculations are actually easier in the old particle framework. When Euler, Heisenberg and Kockel[5] in the mid-thirties calculated the effective action (often called the Euler–Heisenberg action) of a constant external electromagnetic field, they calculated to all orders in the field, although their result is usually presented only to fourth order. This calculation would probably have been impossible with the old fashioned perturbation theory techniques of the time, if they had not done it by first solving the Dirac equation in a constant external electromagnetic field and using those Dirac wave functions to figure out the effective action. These techniques of using particle trajectories rather than field histories in calculations have been revived in recent years. Under the stimulus of string theory, Bern and Kosower,[6] in particular, have developed a useful formalism for doing calculations by following particle world lines rather than by thinking of fields evolving in time. Although this approach was stimulated by string theory, it has been reformulated entirely within the scope of ordinary quantum field theory, and simply represents a more efficient way of doing certain calculations.

One of the key elements in the triumph of quantum field theory was the development of renormalization theory. I'm sure this has been discussed often here, and so I won't dwell on it. The version of renormalization theory that had been developed in the late 1940s remained somewhat in the shade for a long time for two reasons: (1) for the weak interactions it did not seem possible to develop a renormalizable theory, and (2) for the strong interactions it was easy to write down renormalizable theories, but since perturbation theory was inapplicable it did not seem that there was anything that could be done with these theories. Finally all these problems were resolved through the development of the standard model, which was triumphantly verified by experiments during the mid-1970s, and today the weak, electromagnetic and strong interactions are happily all described by a renormalizable quantum field theory. If you had asked me in the mid-1970s about the shape of future fundamental physical theories, I would have guessed that they would take the form of better, more all-embracing, less arbitrary, renormalizable quantum field theories. I gave a talk at the Harvard Science Center at around this time, called "The Renaissance of Quantum Field Theory," which shows you the mood I was in.

There were two things that especially attracted me to the ideas of renormalization and quantum field theory. One of them was that the requirement that a physical theory be renormalizable is a precise and rational criterion of simplicity. In a sense, this requirement had been used long before the advent of renormalization theory. When Dirac wrote down the Dirac equation in 1928 he could have added an extra 'Pauli' term[7] which would have given the electron an arbitrary anomalous magnetic moment. Dirac could (and perhaps did) say 'I won't add this term because it's ugly and complicated and there's no need for it.' I think that in physics this approach generally makes good strategies but bad rationales. It's often a good strategy to study simple theories before you study complicated theories because it's easier to see how they work, but the purpose of physics is to find out why nature is the way it is, and simplicity by itself is I think never the answer. But

renormalizability was a condition of simplicity which was being imposed for what seemed after Dyson's 1949 papers[8] like a rational reason, and it explained not only why the electron has the magnetic moment it has, but also (together with gauge symmetries) all the detailed features of the standard model of weak, electromagnetic, and strong, interactions, aside from some numerical parameters.

The other thing I liked about quantum field theory during this period of tremendous optimism was that it offered a clear answer to the ancient question of what we mean by an elementary particle: it is simply a particle whose field appears in the Lagrangian. It doesn't matter if it's stable, unstable, heavy, light — if its field appears in the Lagrangian then it's elementary, otherwise it's composite.[***]

Now my point of view has changed. It has changed partly because of my experience in teaching quantum field theory. When you teach any branch of physics you must motivate the formalism — it isn't any good just to present the formalism and say that it agrees with experiment — you have to explain to the students why this the way the world is. After all, this is our aim in physics, not just to describe nature, but to explain nature. In the course of teaching quantum field theory, I developed a rationale for it, which very briefly is that it is the only way of satisfying the principles of Lorentz invariance plus quantum mechanics plus one other principle.

Let me run through this argument very rapidly. The first point is to start with Wigner's definition of physical multi-particle states as representations of the inhomogeneous Lorentz group.[9] You then define annihilation and creation operators $a(\vec{p},\sigma,n)$ and $a^\dagger(\vec{p},\sigma,n)$ that act on these states (where $\vec{p}$ is the three-momentum, σ is the spin z-component, and n is a species label). There's no physics in introducing such operators, for it is easy to see that any operator whatever can be expressed as a functional of them. The existence of a Hamiltonian follows from time-translation invariance, and much of physics is described by the S-matrix, which is given by the well known Feynman–Dyson series of integrals over time of time-ordered products of the interaction Hamiltonian $H_I(t)$ in the interaction picture;

$$\begin{aligned} S &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \cdots \int_{-\infty}^{\infty} dt_n \\ &\quad \times T\{H_I(t_1)H_I(t_2)\cdots H_I(t_n)\} \,. \end{aligned} \tag{8.1}$$

This should all be familiar. The other principle that has to be added is the cluster decomposition principle, which requires that distant experiments give uncorrelated results.[10] In order to have cluster decomposition, the Hamiltonian is written not just as any functional of creation and annihilation operators, but as a power series in these operators with coefficients that (aside from a *single* momentum-conservation delta function) are sufficiently smooth functions of the momenta carried by the operators. This condition is satisfied for an interaction Hamiltonian of the form

$$H_I(t) = \int d^3x \; \mathcal{H}(\vec{x},t) \tag{8.2}$$

where $\mathcal{H}(x)$ is a power series (usually a polynomial) with terms that are local in annihilation fields, which are Fourier transforms of the annihilation operators:

$$\psi_\ell^{(+)}(x) = \int d^3p \; \sum_{\sigma,n} e^{ip\cdot x} u_\ell(\vec{p},\sigma,n)\, a(\vec{p},\sigma,n) \tag{8.3}$$

[***] We should not really give quantum field theory too much credit for clarifying the distinction between elementary and composite particles, because some quantum field theories exhibit the phenomenon of bosonization: At least in two dimensions there are theories of elementary scalars that are equivalent to theories with elementary fermions.

together of course with their adjoints, the creation fields.

So far this all applies to nonrelativistic as well as relativistic theories.[†] Now if you also want Lorentz invariance, then you have to face the fact that the time-ordering in the Feynman–Dyson series (1) for the S-matrix doesn't look very Lorentz invariant. The obvious way to make the S-matrix Lorentz invariant is to take the interaction Hamiltonian density $\mathcal{H}(x)$ to be a scalar, and also to require that these Hamiltonian densities commute at spacelike separations

$$[\mathcal{H}(x), \mathcal{H}(y)] = 0 \quad \text{for spacelike} \quad x - y \ , \tag{8.4}$$

in order to exploit the fact that time ordering *is* Lorentz invariant when the separation between spacetime points is timelike. In order to satisfy the requirement that the Hamiltonian density commute with itself at spacelike separations, it is constructed out of fields which satisfy the same requirement. These are given by sums of fields that annihilate particles plus fields that create the corresponding antiparticles

$$\begin{aligned} \psi_\ell(x) \ &= \ \sum_{\sigma,n} \int d^3p \left[e^{ip\cdot x}\, u_\ell(\vec{p},\sigma,n)\, a(\vec{p},\sigma,n) \right. \\ &\qquad \left. + e^{-ip\cdot x}\, v_\ell(\vec{p},\sigma,n)\, a^\dagger(\vec{p},\sigma,\bar{n}) \right] , \end{aligned} \tag{8.5}$$

where $\bar{n}$ denotes the antiparticle of the particle of species n. For a field ψ_ℓ that transforms according to an irreducible representation of the homogeneous Lorentz group, the form of the coefficients u_ℓ and v_ℓ is completely determined (up to a single over-all constant factor) by the Lorentz transformation properties of the fields and one-particle states, and by the condition that the fields commute at spacelike separations. Thus the whole formalism of fields, particles, and antiparticles seems to be an inevitable consequence of Lorentz invariance, quantum mechanics, and cluster decomposition, without any ancillary assumptions about locality or causality.

This discussion has been extremely sketchy, and is subject to all sorts of qualifications. One of them is that for massless particles, the range of possible theories is slightly larger than I have indicated here. For example, in quantum electrodynamics, in a physical gauge like Coulomb gauge, the Hamiltonian is not of the form (2) — there is an additional term, the Coulomb potential, which is bilocal and serves to cancel a non-covariant term in the propagator. But relativistically invariant quantum theories always (with some qualifications I'll come to later) do turn out to be quantum field theories, more or less as I have described them here.

One can go further, and ask why we should formulate our quantum field theories in terms of Lagrangians. Well, of course creation and annihilation operators by themselves yield pairs of canonically conjugate variables; from the as and $a^\dagger$s, it is easy to construct qs and ps. The time-dependence of these operators is dictated in terms of the Hamiltonian, the generator of time translations, so the Hamiltonian formalism is trivially always with us. But why the Lagrangian formalism? Why do we enumerate possible theories by giving their Lagrangians rather than by writing down Hamiltonians? I think the reason for this is that it is only in the Lagrangian formalism (or more generally the action formalism) that symmetries imply the existence of Lie algebras of suitable quantum operators, and you need these Lie algebras to make sensible quantum theories. In

[†]By the way, the reason that quantum field theory is useful even in nonrelativistic statistical mechanics, where there is often a selection rule that makes the actual creation or annihilation of particles impossible, is that in statistical mechanics you have to impose a cluster decomposition principle, and quantum field theory is the natural way to do so.

particular, the S-matrix will be Lorentz invariant if there is a set of 10 sufficiently smooth operators satisfying the commutation relations of the inhomogeneous Lorentz group. It's not trivial to write down a Hamiltonian that will give you a Lorentz invariant S-matrix — it's not so easy to think of the Coulomb potential just on the basis of Lorentz invariance — but if you start with a Lorentz invariant Lagrangian density then because of Noether's theorem the Lorentz invariance of the S-matrix is automatic.

Finally, what is the motivation for the special gauge invariant Lagrangians that we use in the standard model and general relativity? One possible answer is that quantum theories of mass zero, spin one particles violate Lorentz invariance unless the fields are coupled in a gauge invariant way, while quantum theories of mass zero, spin two particles violate Lorentz invariance unless the fields are coupled in a way that satisfies the equivalence principle.

This has been an outline of the way I've been teaching quantum field theory these many years. Recently I've put this all together into a book,[11] now being sold for a negligible price. The bottom line is that quantum mechanics plus Lorentz invariance plus cluster decomposition implies quantum field theory. But there are caveats that have to be attached to this, and I can see David Gross in the front row anxious to take me by the throat over various gaps in what I have said, so I had better list these caveats quickly to save myself.

First of all, the argument I have presented is obviously based on perturbation theory. Second, even in perturbation theory, I haven't stated a clear theorem, much less proved one. As I mentioned there are complications when you have things like mass zero, spin one particles for example; in this case you don't really have a fully Lorentz invariant Hamiltonian density, or even one that is completely local. Because of these complications, I don't know how even to state a general theorem, let alone prove it, even in perturbation theory. But I don't think that these are insuperable obstacles.

A much more serious objection to this not-yet-formulated theorem is that there's already a counter example to it: string theory. When you first learn string theory it seems in an almost miraculous way to give Lorentz invariant, unitary S-matrix elements without being a field theory in the sense that I've been using it. (Of course it is a field theory in a different sense — it's a two dimensional conformally invariant field theory, but not a quantum field theory in four spacetime dimensions.) So before even being formulated precisely, this theorem suffers from at least one counter example.

Another fundamental problem is that the S-matrix isn't everything. Spacetime could be radically curved, not just have little ripples on it. Also, at finite temperature there's no S-matrix because particles cannot get out to infinite distances from a collision without bumping into things. Also, it seems quite possible that at very short distances the description of events in four-dimensional flat spacetime becomes inappropriate.

Now, all of these caveats really work only against the idea that the final theory of nature is a quantum field theory. They leave open the view, which is in fact the point of view of my book, that although you can not argue that relativity plus quantum mechanics plus cluster decomposition necessarily leads only to quantum field theory, it is very likely that any quantum theory that at sufficiently low energy and large distances looks Lorentz invariant and satisfies the cluster decomposition principle will also at sufficiently low energy *look* like a quantum field theory. Picking up a phrase from Arthur Wightman, I'll call this a folk theorem. At any rate, this folk theorem is satisfied by string theory, and we don't know of any counterexamples.

This leads us to the idea of effective field theories. When you use quantum field theory to study low-energy phenomena, then according to the folk theorem you're not really making any assumption that could be wrong, unless of course Lorentz invariance or quantum mechanics or

cluster decomposition is wrong, provided you don't say specifically what the Lagrangian is. As long as you let it be the most general possible Lagrangian consistent with the symmetries of the theory, you're simply writing down the most general theory you could possibly write down. This point of view has been used in the last fifteen years or so to justify the use of effective field theories, not just in the tree approximation where they had been used for some time earlier, but also including loop diagrams. Effective field theory was first used in this way to calculate processes involving soft π mesons,[12] that is, π mesons with energy less than about $2\pi F_\pi \approx 1200$ MeV. The use of effective quantum field theories has been extended more recently to nuclear physics,[13] where although nucleons are not soft they never get far from their mass shell, and for that reason can be also treated by similar methods as the soft pions. Nuclear physicists have adopted this point of view, and I gather that they are happy about using this new language because it allows one to show in a fairly convincing way that what they've been doing all along (using two-body potentials only, including one-pion exchange and a hard core) is the correct first step in a consistent approximation scheme. The effective field theory approach has been applied more recently to superconductivity. Shankar, I believe, in a contribution to this conference is talking about this. The present educated view of the standard model, and of general relativity,[14] is again that these are the leading terms in effective field theories.

The essential point in using an effective field theory is you're not allowed to make any assumption of simplicity about the Lagrangian. Certainly you're not allowed to assume renormalizability. Such assumptions might be appropriate if you were dealing with a fundamental theory, but not for an effective field theory, where you must include all possible terms that are consistent with the symmetry. The thing that makes this procedure useful is that although the more complicated terms are not excluded because they're non-renormalizable, their effect is suppressed by factors of the ratio of the energy to some fundamental energy scale of the theory. Of course, as you go to higher and higher energies, you have more and more of these suppressed terms that you have to worry about.

On this basis, I don't see any reason why anyone today would take Einstein's general theory of relativity seriously as the foundation of a quantum theory of gravitation, if by Einstein's theory is meant the theory with a Lagrangian density given by just the term $\sqrt{g}R/16\pi G$. It seems to me there's no reason in the world to suppose that the Lagrangian does not contain all the higher terms with more factors of the curvature and/or more derivatives, all of which are suppressed by inverse powers of the Planck mass, and of course don't show up at any energy far below the Planck mass, much less in astronomy or particle physics. Why would anyone suppose that these higher terms are absent?

Likewise, since now we know that without new fields there's no way that the renormalizable terms in the standard model could violate baryon conservation or lepton conservation, we now understand in a rational way why baryon number and lepton number are as well conserved as they are, without having to assume that they are exactly conserved.[††] Unless someone has some *a priori* reason for exact baryon and lepton conservation of which I haven't heard, I would bet very strong odds that baryon number and lepton number conservation are in fact violated by suppressed non-renormalizable corrections to the standard model.

These effective field theories are non-renormalizable in the old Dyson power-counting sense. That is, although to achieve a given accuracy at any given energy, you need only take account of a finite number of terms in the action, as you increase the accuracy or the energy you need to include more and more terms, and so you have to know more and more. On the other hand, effective field

[††]The extra fields required by low-energy supersymmetry may invalidate this argument.

theories still must be renormalizable theories in what I call the modern sense: the symmetries that govern the action also have to govern the infinities, for otherwise there will be infinities that can't be eliminated by absorbing them into counter terms to the parameters in the action. This requirement is automatically satisfied for unbroken global symmetries, such as Lorentz invariance and isotopic spin invariance and so on. Where it's not trivial is for gauge symmetries. We generally deal with gauge theories by choosing a gauge before quantizing the theory, which of course breaks the gauge invariance, so it's not obvious how gauge invariance constrains the infinities. (There is a symmetry called BRST invariance[15] that survives gauge fixing, but that's non-linearly realized, and non-linearly realized symmetries of the action are not symmetries of the Feynman amplitudes.) This raises a question, whether gauge theories that are not renormalizable in the power counting sense are renormalizable in the modern sense. The theorem that says that infinities are governed by the same gauge symmetries as the terms in the Lagrangian was originally proved back in the old days by 't Hooft and Veltman[16] and Lee and Zinn-Justin[17] only for theories that are renormalizable in the old power-counting sense, but this theorem has only recently been extended to theories of the Yang–Mills[18] or Einstein type with arbitrary numbers of complicated interactions that are not renormalizable in the power-counting sense.[‡] You'll be reassured to know that these theories are renormalizable in the modern sense, but there's no proof that this will be true of all quantum field theories with local symmetries.

I promised you a few ironies today. The second one takes me back to the early 1960s when S-matrix theory was very popular at Berkeley and elsewhere. The hope of S-matrix theory was that, by using the principles of unitarity, analyticity, Lorentz invariance and other symmetries, it would be possible to calculate the S-matrix, and you would never have to think about a quantum field. In a way, this hope reflected a kind of positivistic puritanism: we can't measure the field of a pion or a nucleon, so we shouldn't talk about it, while we do measure S-matrix elements, so this is what we should stick to as ingredients of our theories. But more important than any philosophical hang-ups was the fact that quantum field theory didn't seem to be going anywhere in accounting for the strong and weak interactions.

One problem with the S-matrix program was in formulating what is meant by the analyticity of the S-matrix. What precisely are the analytic properties of a multi-particle S-matrix element? I don't think anyone ever knew. I certainly didn't know, so even though I was at Berkeley I never got too enthusiastic about the details of this program, although I thought it was a lovely idea in principle. Eventually the S-matrix program had to retreat, as described by Kaiser in a contribution to this conference, to a sort of mix of field theory and S-matrix theory. Feynman rules were used to find the singularities in the S-matrix, and then they were thrown away, and the analytic structure of the S-matrix with these singularities, together with unitarity and Lorentz invariance, was used to do calculations.

Unfortunately to use these assumptions it was necessary to make uncontrolled approximations, such as the strip approximation, whose mention will bring tears to the eyes of those of us who are old enough to remember it. By the mid-1960's it was clear that S-matrix theory had failed in dealing with the one problem it had tried hardest to solve, that of pion–pion scattering. The strip approximation rested on the assumption that double dispersion relations are dominated by regions of the Mandelstam diagram near the fringes of the physical region, which would only make sense if π–π scattering is strong at low energy, and these calculations predicted that π–π scattering is indeed strong at low energy, which was at least consistent, but it was then discovered that π–π

[‡]I refer here to work of myself and Joaquim Gomis,[19] relying on recent theorems about the cohomology of the Batalin–Vilkovisky operator by Barnich, Brandt, and Henneaux.[20] Earlier work along these lines but with different motivation was done by Voronov, Tyutin, and Lavrov;[21] Anselmi;[22] and Harada, Kugo, and Yamawaki.[23]

scattering is *not* strong at low energy. Current algebra came along at just that time, and was used to predict not only that low energy π-π scattering is not strong, but also successfully predicted the values of the π–π scattering lengths.[24] From a practical point of view, this was the greatest defeat of S-matrix theory. The irony here is that the S-matrix philosophy is not that far from the modern philosophy of effective field theories, that what you should do is just write down the most general S-matrix that satisfies basic principles. But the practical way to implement S-matrix theory is to use an effective quantum field theory — instead of deriving analyticity properties from Feynman diagrams, we use the Feynman diagrams themselves. So here's another answer to the question of what quantum field theory is: it is S-matrix theory, made practical.

By the way, I think that the emphasis in S-matrix theory on analyticity as a fundamental principle was misguided, not only because no one could ever state the detailed analyticity properties of general S-matrix elements, but also because Lorentz invariance requires causality (because as I argued earlier otherwise you're not going to get a Lorentz invariant S-matrix), and in quantum field theory causality allows you to derive analyticity properties. So I would include Lorentz invariance, quantum mechanics and cluster decomposition as fundamental principles, but not analyticity.

As I have said, quantum field theories provide an expansion in powers of the energy of a process divided by some characteristic energy; for soft pions this characteristic energy is about a GeV; for superconductivity it's the Debye frequency or temperature; for the standard model it's 10^{15} to 10^{16} GeV; and for gravitation it's about 10^{18} GeV. Any effective field theory loses its predictive power when the energy of the processes in question approaches the characteristic energy. So what happen to the effective field theories of electroweak, strong, and gravitational interactions at energies of order 10^{15}–10^{18} GeV? I know of only two plausible alternatives.

One possibility is that the theory remains a quantum field theory, but one in which the finite or infinite number of renormalized couplings do not run off to infinity with increasing energy, but hit a fixed point of the renormalization group equations. One way that can happen is provided by asymptotic freedom in a renormalizable theory,[25] where the fixed point is at zero coupling, but it's possible to have more general fixed points with infinite numbers of non-zero nonrenormalizable couplings. Now, we don't know how to calculate these non-zero fixed points very well, but one thing we know with fair certainty is that the trajectories that run into a fixed point in the ultraviolet limit form a finite dimensional subspace of the infinite dimensional space of all coupling constants. (If anyone wants to know how we know that, I'll explain this later.) That means that the condition, that the trajectories hit a fixed point, is just as restrictive in a nice way as renormalizability used to be: It reduces the number of free coupling parameters to a finite number. We don't yet know how to do calculations for fixed points that are not near zero coupling. Some time ago I proposed[26] that these calculations could be done in the theory of gravitation by working in $2+\epsilon$ dimensions and expanding in powers of $\epsilon = 2$, in analogy with the way that Wilson and Fisher[27] had calculated critical exponents by working in $4-\epsilon$ dimensions and expanding in powers of $\epsilon = 1$, but this program doesn't seem to be working very well.

The other possibility, which I have to admit is *a priori* more likely, is that at very high energy we will run into really new physics, not describable in terms of a quantum field theory. I think that by far the most likely possibility is that this will be something like a string theory.

Before I leave the renormalization group, I did want to say another word about it because there's going to be an interesting discussion on this subject here tomorrow morning, and for reasons I've already explained I can't be here. I've read a lot of argument about the Wilson approach[28] vs. the Gell-Mann–Low approach,[29] which seems to me to call for reconciliation. There have been two fundamental insights in the development of the renormalization group. One, due to Gell-Mann and Low, is that logarithms of energy that violate naive scaling and invalidate perturbation theory arise

because of the way that renormalized coupling constants are defined, and that these logarithms can be avoided by renormalizing at a sliding energy scale. The second fundamental insight, due to Wilson, is that it's very important in dealing with phenomena at a certain energy scale to integrate out the physics at much higher energy scales. It seems to me these are the same insight, because when you adopt the Gell-Mann–Low approach and define a renormalized coupling at a sliding scale and use renormalization theory to eliminate the infinities rather than an explicit cutoff, you are in effect integrating out the higher energy degrees of freedom — the integrals converge because after renormalization the integrand begins to fall off rapidly at the energy scale at which the coupling constant is defined. (This is true whether or not the theory is renormalizable in the power-counting sense.) So in other words instead of a sharp cutoff *a la* Wilson, you have a soft cutoff, but it's a cutoff nonetheless and it serves the same purpose of integrating out the short distance degrees of freedom. There are practical differences between the Gell-Mann–Low and Wilson approaches, and there are some problems for which one is better and other problems for which the other is better. In statistical mechanics it isn't important to maintain Lorentz invariance, so you might as well have a cutoff. In quantum field theories, Lorentz invariance is necessary, so it's nice to renormalize *a la* Gell-Mann–Low. On the other hand, in supersymmetry theories there are some non-renormalization theorems that are simpler if you use a Wilsonian cutoff than a Gell-Mann–Low cutoff.[30] These are all practical differences, which we have to take into account, but I don't find any fundamental philosophical difference between these two approaches.

On the plane coming here I read a comment by Michael Redhead, in a paper submitted to this conference: 'To subscribe to the new effective field theory programme is to give up on this endeavor' [the endeavor of finding really fundamental laws of nature], 'and retreat to a position that is somehow less intellectually exciting.' It seems to me that this is analogous to saying that to balance your checkbook is to give up dreams of wealth and have a life that is intrinsically less exciting. In a sense that's true, but nevertheless it's still something that you had better do every once in a while. I think that in regarding the standard model and general relativity as effective field theories we're simply balancing our checkbook and realizing that we perhaps didn't know as much as we thought we did, but this is the way the world is and now we're going to go on the next step and try to find an ultraviolet fixed point, or (much more likely) find entirely new physics. I have said that I thought that this new physics takes the form of string theory, but of course, we don't know if that's the final answer. Nielsen and Oleson[31] showed long ago that relativistic quantum field theories can have string-like solutions. It's conceivable, although I admit not entirely likely, that something like modern string theory arises from a quantum field theory. And that would be the final irony.

8.7.10 Connection of Particles and Fields 1

От поля к частице

В предыдущей главе для свободной теории мы получили выражение

$$W(J) = -\frac{1}{2}\iint d^4x d^4y J(x) D(x-y) J(y)$$

которое мы сейчас запишем через преобразование Фурье $J(k) \equiv \int d^4x e^{-ikx} J(x)$

$$W(J) = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J(k)^* \frac{1}{k^2 - m^2 + i\varepsilon} J(k).$$

(Заметим, что $J(k)^* = J(-k)$ для вещественного $J(x)$.)

Мы можем как угодно прыгать на матраце вверх-вниз. Иными словами, мы можем выбирать любой $J(x)$ и, используя эту свободу выбора, получать большое количество физической информации.

Рассмотрим $J(x) = J_1(x) + J_2(x)$, где $J_1(x)$ и $J_2(x)$ сосредоточены в двух локальных областях 1 и 2 пространства-времени (рис. I.4.1). Тогда $W(J)$ содержит четыре члена вида $J_1^* J_1$, $J_2^* J_2$, $J_1^* J_2$ и $J_2^* J_1$. Рассмотрим два последних членах, один из которых имеет следующую форму:

$$W(J) = -\frac{1}{2}\int \frac{d^4k}{(\pi)^4} J_2(k)^* \frac{1}{k^2 - m^2 + i\varepsilon} J_1(k).$$

Мы видим, что $W(J)$ является большим только тогда, когда фурье-компоненты $J_1(x)$ и $J_2(x)$ значительно перекрываются и когда в области перекрытия в импульсном пространстве член $k^2 - m^2$ близок к нулю. В точке $k^2 = m^2$ существует пик «резонансного типа», при этом выполняется соотношение энергии и импульса для частицы массой m. (Мы будем пользоваться языком, принятым в теории относительности, и писать «импульсное пространство», вместо «пространство энергии-импульса», переходя на нерелятивистский язык только тогда, когда того требует контекст, как, например, в случае словосочетания «соотношение энергии-импульса».)

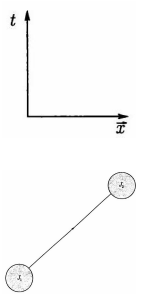


Рис. I.4.1

Итак, мы интерпретируем физические свойства, содержащиеся в нашей простой теории поля, следующим образом: источник в области 1 пространства-времени создает «возмущение поля», которое позднее поглощается стоком в области 2 пространства-времени. Экспериментаторы предпочитают называть это возмущением поля частицей с массой m. Наше предположение, основанное на уравнении движения, о том, что теория содержит частицу массой m, нашло свое подтверждение.

Немного жаргона: при $k^2 = m^2$ говорят, что k находится на массовой поверхности. Заметьте, однако, что в (3) мы интегрировали по всем k, включая k, удаленные от массовой оболочки. Для произвольного k удобно говорить о «виртуальной частице» с импульсом k, распространяющейся от источника к стоку.

От частицы к силе

Продолжим исследовать другие возможности для $J(x)$ (которые мы обобщенно будем называть источниками), например, $J(x) = J_1(x) + J_2(x)$, где $J_a(x) = \delta^{(3)}(\vec{x} - \vec{x}_a)$. Иначе говоря, $J(x)$ есть сумма источников, яв- ляющихся стационарными бесконечно острыми пиками, расположенными в $\vec{x}_1$ и $\vec{x}_2$ в пространстве. (Если вам нужна большая математическая точность, чем предлагается здесь, замените дельта-функцию функциями с

пиками в $\vec{x}_a$. Правда, этим вы лишь загромоздите формулы, не добившись ничего существенного.) Образно выражаясь, мы описываем два массивных сгустка, расположенных на матраце в $\vec{x}_1$ и являющихся абсолютно неподвижными [в $J(x)$ нет временной зависимости].

Что квантовые флуктуации поля φ, то есть вибрации матраца, делают с двумя массами, находящимися на матраце? Если вы считаете, что между этим массами существует взаимное притяжение, то вы совершенно правы.

Как и прежде, $W(J)$ содержит четыре члена. Мы пренебрежем членом «самодействия» $J_1 J_1$, так как он присутствовал бы в W вне зависимости от присутствия J_2. Нас интересует взаимодействие между двумя «массивными сгустками», представленными посредством J_1 и J_2. Аналогично мы пренебрегаем членом $J_2 J_2$.

Подставляя в (1) и вычисляя интеграл по d^3x и d^3y, мы немедленно получаем

$$W(J) = -\iint dx^0 dy^0 \int \frac{dk^0}{2\pi} e^{ik^0(x-y)^0} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}} \cdot (\vec{x}_1 - \vec{x}_2)}{k^2 - m^2 + i\varepsilon}$$

(Множитель 2 обусловлен двумя членами $J_2 J_1$ и $J_1 J_2$.) Интегрируя по y^0, получаем дельта-функцию, которая обнуляет k^0 (так что, если говорить на жаргонном языке, k определенно не находится на массовой поверхности). Итак, мы остаемся с

$$W(J) = \left(\int dx^0 \right) \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}}{\vec{k}^2 + m^2}$$

Обратите внимание, что бесконечно малое $i\varepsilon$ можно опустить, поскольку знаменатель $\vec{k}^2 + m^2$ всегда положителен.

Множитель $\left(\int dx^0 \right)$ может вселить в нас ужас и тревогу: интеграл по времени кажется бесконечным. Не бойтесь! Вспомним, что в формализме интеграла по траекториям $Z = \mathcal{C} e^{iW(J)}$ представляет собой $\left\langle 0 \left| e^{-iHT} \right| 0 \right\rangle = e^{-iET}$, где E - энергия, обусловленная двумя источниками, воздействующими друг на друга. Множитель $\left(\int dx^0 \right)$ дает в точности временной интервал T. Все в порядке. Подставляя $iW = -iET$, из (5) получаем

$$E = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}_1 - \vec{x}_2)}}{\vec{k}^2 + m^2}$$

Эта энергия отрицательна! Наличие двух дельта-функциональных источников в $\vec{x}_1$ и $\vec{x}_2$ уменьшило энергию. Иными словами, два источника притягивают друг друга в силу их взаимодействия с полем φ. Мы добились первого физического результата в квантовой теории поля!

Величина E отождествляется с потенциальной энергией между двумя статическими источниками. Даже без вычисления интеграла видно, что с увеличением расстояния $\vec{x}_1 - \vec{x}_2$ между источниками осциллирующая экспонента обрезает интеграл. Характерное расстояние является обратным характерному значению k, равному m. Итак, можно ожидать, что взаимодействие между двумя источниками быстро убывает до нуля на расстоянии $1/m$.

Радиус действия силы притяжения, порождаемой полем φ, определяется обратной массой m частицы, описываемой полем. Это понятно?

Интеграл вычисляется в приложении к этой главе, в итоге получаем

$$E = -\frac{1}{4\pi r} e^{-mr}$$

Результат вполне ожидаемый: потенциал убывает экспоненциально на масштабе расстояний $1/m$. Очевидно, $dE/dr > 0$: две расположенные на матраце массы могут уменьшить энергию, приближаясь друг к другу.

Вывод, к которому мы пришли, явился одним из наиболее выдающихся результатов в физике двадцатого века. Юкава предположил, что взаимодействие между нуклонами в атомном ядре обусловлено взаимодействием с полем, подобным вышеописанному полю φ. Известный радиус действия ядерных сил позволил ему не только предсказать существование частицы, ассоциированной с этим полем и называемой ныне π-мезоном [1] или пионом, но также определить ее массу. Как вам, вероятно, известно, в итоге было установлено, что пион наделен свойствами, в значительной степени совпадающими с предсказанными Юкавой.

Происхождение силы

Тот факт, что обмен частицей порождает силу, явился одним из наиболее значительных концептуальных достижений в физике. Сейчас мы ассоциируем частицу с каждой из известных нам сил: например, фотон с электромагнитной силой, а гравитон с гравитационной силой; первое утверждеиис однозначно подтверждается экспериментально, второе - пока никто не

1r) тимология этого слова довольно интересна. (A. Zee. Fearful Symmetry: читайте стр. 169 II 335, чтобы узнать, наряду с другими вещами, о возражении французов и связи между мезопом и иллюзией.) подтвердил опытным путем, но вряд ли кто-то сомневается в его истинности. Фотон и гравитон мы рассмотрим в следующей главе, но уже сейчас мы может ответить на вопрос, который очень любят задавать сообразительные школьники: почему гравитационная сила Ньютона и электрическая сила Кулона подчиняются закону $1/r^2$.

Из (7) видно, что если масса m промежуточной частицы равна нулю, возникающая сила будет подчиняться закону $1/r^2$. Прослеживая обратно наш вывод, видим, что это следует из того факта, что плотность лагранжиана для простейшей теории поля содержит пространственно-временную производную ∂ второй степени (так как любой член, включающий одну производную, не является лоренц-инвариантным). В самом деле, степенная зависимость потенциала следует из анализа размерностей: $\int d^3k \left(e^{i\vec{k}\cdot\vec{x}}/k^2\right) \sim 1/r$

8.7.11 Connection of Particles and Fields 2

(two sections, since it is a conceptually important question, so there will be a lot to write)

Связанные против несвязанных

В заключение сделаем несколько формальных замечаний, которые будут важны для нас только в главе I.7. Во-первых, обратите внимание, что мы могли нарисовать небольшой рисунок (рис. I.4.2) для иллюстрации подынтегрального выражения $J(x)D(x - -y)J(y)$ в $W(J)$: возмущение распространяется из точки x в точку y (или наоборот).

Рис. I.4.2

В действительности это начало фейнмановских диаграмм! Во-вторых, вспомним, что

$$Z(J) = Z(J=0)\sum_{n=0}^{\infty}\frac{[(iW(J))^n]}{n!}$$

Например, член с $n=2$ в $Z(J)/Z(J=0)$ определяется выражением

$$\frac{1}{2!}\left(-\frac{i}{2}\right)^2 \iiint\int d^4x_1 d^4x_2 d^4x_3 d^4x_4 D\left(x_1-x_2\right)\times$$
$$\times D\left(x_3-x_4\right) J\left(x_1\right) J\left(x_2\right) J\left(x_3\right) J\left(x_4\right).$$

Подынтегральное выражение графически представлено на рисунке I.4.3. Этот процесс называется несвязанным: распространение из x_1 в x_2 и из x_3 в x_4 происходит независимо. Мы вернемся к вопросу о разнице между связанными и несвязанными диаграммами в главе I.7.

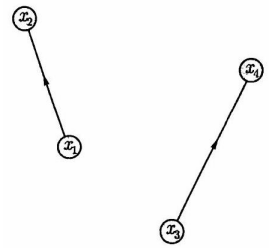


Рис. I.4.3

Приложение

С учетом того, что $\vec{x} \equiv (\vec{x}_1 - \vec{x}_2)$ и $u \equiv \cos\theta$, где θ - угол между $\vec{k}$ и $\vec{x}$, вычисляем интеграл в (6) в сферических координатах (с $k = |\vec{k}|$ и $r = |\vec{x}|$) и получаем

$$I \equiv \frac{1}{(2\pi)^2}\int_0^{\infty} dk k^2 \int_{-1}^{+1} du \frac{e^{ikru}}{k^2+m^2} = \frac{2i}{(2\pi)^2 ir}\int_0^{\infty} dk k \frac{\sin kr}{k^2+m^2}$$

Так как подынтегральное выражение четно, можно расширить область интегрирования и записать

$$\frac{1}{2}\int_{-\infty}^{\infty} dk k \frac{\sin kr}{k^2+m^2} = \frac{1}{2i}\int_{-\infty}^{\infty} dk k \frac{1}{k^2+m^2} e^{ikr}$$

Поскольку r положительно, можно замкнуть контур в верхней полуплоскости, чтобы включить полюс в $+im$, получая тем самым $(1/2i)(2\pi i)(im/2im)e^{-mr} = (\pi/2)e^{-mr}$. Итак, $I = (1/4\pi r)e^{-mr}$.

Упражнение

I.4.1. Вычислите аналог закона обратных квадратов в $(2+1)$-мерной Вселенной и в более общем случае в $(D+1)$-мерной Вселенной.

8.8 World of Quantum Fields

8.8.1 Concept of Mass

(??? discussion what is mass?)

8.8.2 Forces in Scope of QFT

(discuss, what is it in general???)

Обход понятия силы в квантовой теории поля (!??!?)

Чем мы заменяем понятие “сила”?

8.8.3 Deriving Coulomb’s Law from QFT

What is a force at all?

(this argument that it is a field, so on... (???? is it a really good argument??))

Почему одноименные заряды отталкиваются

Мы предположили, что квантовая теория поля дает естественное объяснение как гравитационной силе Ньютона, так и электрической силе Кулона. Между подобными объектами ньютоновская сила является притягивающей, а кулоновская - отталкивающей. Является ли квантовая теория поля «достаточно умной», чтобы объяснить этот наблюдаемый факт - один из основополагающих в нашем понимании физической Вселенной? Делайте ставки!

Вначале мы рассмотрим квантовую теорию электромагнитного поля, известную как квантовая электродинамика, или сокращенно КЭД. Чтобы на этом этапе избежать сложностей, связанных с калибровочной инвариантностью (подробнее о ней позже), рассмотрим теорию массивного мезона со спином 1, или векторного мезона. В конце концов, все, что нам известно экспериментально, - это верхняя граница массы фотона, которая, хотя и маленькая, не является нулем с математической точки зрения. Можно встать на прагматическую позицию: проводить вычисления с массой фотона m, в конце положить $m = 0$ и, если результат окажется разумным, считать, что все ОК[1].

Вспомним максвелловский лагранжиан для электромагнетизма $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, где $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ с векторным потенциалом $A_\mu(x)$. Причину появления важного общего знака «минус» в лагранжиане можно понять, если посмотреть на коэффициент при $(\partial_0 A_i)^2$, который должен быть положительным, как и коэффициент при $(\partial_0\varphi)^2$ в лагранжиане для

скалярного поля. Это просто означает, что вариация по времени приводит к положительному изменению действия.

'Когда. будучи студентом, я слушал курс квантовой теории поля Сиднея Коулмена. он трактовал КЭД именно таким образом. чтобы избегать обсуждения калибровочной инвариантности. Сейчас наделим фотон небольшой массой, изменив лагранжиан на $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2A_\mu A^\mu + A_\mu J^\mu$. (Массовый член мы записали по аналогии с массовым членом $m^2\varphi^2$ в лагранжиане скалярного поля; впоследствии станет очевидным, что это на самом деле масса фотона.) Я также добавил источник $J^\mu(x)$, который в этом случае более известен как ток. Будем предполагать, что ток сохраняется, то есть $\partial_\mu J^\mu = 0$.

Итак, вы знаете, что теория поля нашего векторного мезона определяется интегралом по траекториям $Z = \int DAe^{iS(A)} \equiv e^{iW(J)}$ с действием

$$S(A) = \int d^4x\mathcal{L} = \int d^4x \left\{A_\mu \left[\left(\partial^2 + m^2\right) g^{\mu\nu} - \partial^\mu\partial^\nu\right] A_\nu + A_\mu J^\mu\right\}$$

Ко второму равенству приходим путем интегрирования по частям [сравните с (I.3.15)].

Надо просто применить уравнение (I.3.16). Для этого всего лишь нужно обратить дифференциальный оператор в квадратных скобках; другими словами, необходимо решить уравнение

$$\left[\left(\partial^2 + m^2\right) g^{\mu\nu} - \partial^\mu\partial^\nu\right] D_{\nu\lambda}(x) = \delta^\mu_\lambda\delta^{(4)}(x)$$

Как и раньше [сравните с (I.3.17)], переходим в импульсное пространство, определяя

$$D_{\nu\lambda}(x) = \int \frac{d^4k}{(2\pi)^4} D_{\nu\lambda}(k)e^{ikx}$$

Подставим последнее выражение в (2) и получим $\left[-\left(k^2 - m^2\right) g^{\mu\nu} + \ + k^\mu k^\nu\right] D_{\nu\lambda}(k) = \delta^\mu_\lambda$, в итоге найдем

$$D_{\nu\lambda}(k) = \frac{-g_{\nu\lambda} + k_\nu k_\lambda/m^2}{k^2 - m^2}$$

Это и есть пропагатор фотона или, точнее, массивного векторного мезона. Таким образом

$$W(J) = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J^\mu(k)^* \frac{-g_{\mu\nu} + k_\mu k_\nu/m^2}{k^2 - m^2 + i\varepsilon} J^\nu(k)$$

Так как закон сохранения тока $\partial_\mu J^\mu(x) = 0$, сформулированный для импульсного пространства, имеет вид $k_\mu J^\mu(k) = 0$, мы можем опустить член $k_\mu k_\nu$ в фотонном пропагаторе. Эффективное действие при этом упрощается до

$$W(J) = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J^\mu(k)^* \frac{1}{k^2 - m^2 + i\varepsilon} J_\mu(k).$$

Никаких вычислений больше не требуется для получения фундаментального результата. Просто сравним его с (I.4.2). Теория поля привела к дополнительному знаку «минус». Потенциальная энергия между двумя сгустками с плотностью заряда $J^0(x)$ положительна. Электромагнитная сила между одноименными зарядами является отталкивающей.

В силу закону сохранения тока мы можем теперь спокойно положить массу фотона m равной 0. [Обратите внимание, что мы не могли этого сделать в (3).] В самом деле,

обратившись к (I.4.7) мы видим, что потенциальная энергия между одноименными зарядами равна

$$E = \frac{1}{4\pi r} e^{-mr} \to \frac{1}{4\pi r}.$$

Чтобы включить положительные и отрицательные заряды, можно просто записать $J_p^\mu - J_n^\mu$. Мы видим, что сгусток с плотностью заряда J_p^0 притягивается к сгустку с плотностью заряда J_n^0.

Идем в обход Максвелла

Покончив с электромагнетизмом за две минуты, перейду теперь к гравитации. Обратимся к массивному мезонному полю со спином 2. При анализе массивного мезонного поля со спином 1 я двигался кратчайшим путем. Предположил, что вы знакомы с максвелловским лагранжианом, просто добавил к нему массовый член и начал вычисления. Однако я сомневаюсь, что вы столь же хорошо знакомы с соответствующим лагранжианом для массивного поля со спином 2. (Так называемый линеаризованный эйнштейновский лагранжиан, речь о котором пойдет в следующей главе.) Поэтому сейчас я прибегну к другой стратегии.

Давайте, рассуждая физически, вместе получим пропагатор массивного поля со спином 2. Для начала рассмотрим массивное поле со спином 1. Фактически мы начнем с нечто еще более простого: с пропагатора $D(k) = 1/(k^2 - m^2)$ массивного поля со спином 0. Он соответствует ситуации, когда амплитуда распространения возмущения со спином 0 устремляется к бесконечности, когда возмущение - почти реальная частица. Вычет в полюсе определяется свойствами частицы. Пропагатор для поля со спином $1 D_{\nu\lambda}$ имеет пару лоренцевых индексов, и мы действительно можем получить его форму из выражения (3):

$$D_{\nu\lambda}(k) = \frac{-G_{\nu\lambda}}{k^2 - m^2}$$

где для дальнейшего удобства введено обозначение

$$G_{\nu\lambda}(k) \equiv g_{\nu\lambda} - \frac{k_\nu k_\lambda}{m^2}$$

Теперь попробуем объяснить физику, скрывающуюся за $G_{\nu\lambda}$. Я надеюсь, вы помните понятие поляризации из курса электромагнетизма. Массивная частица со спином 1 имеет три степени поляризации по той простой причине, что в собственной системе покоя вектор ее спина может быть направлен по трем разным направлениям. Эти три вектора поляризации $\varepsilon_\mu^{(a)}$ являются просто тремя единичными векторами, направленными вдоль осей x, y и z соответственно, $(a = 1, 2, 3) : \varepsilon_\mu^{(1)} = (0, 1, 0, 0), \varepsilon_\mu^{(2)} = (0, 0, 1, 0), \varepsilon_\mu^{(3)} = (0, 0, 0, 1)$. В системе покоя $k^\mu = (m, 0, 0, 0)$, и поэтому

$$k^\mu \varepsilon_\mu^{(a)} = 0$$

Поскольку мы пришли к лоренц-инвариантному уравнению, оно также верно для движущейся частицы со спином 1. Действительно, вместе с подходящим условием нормировки оно фиксирует три вектора поляризации $\varepsilon_\mu^{(a)}(k)$ для частицы с импульсом k.

Амплитуда рождения источником частицы с импульсом k и поляризацией a пропорциональна $\varepsilon_\lambda^{(a)}(k)$, а амплитуда поглощения ее в стоке пропорциональна $\varepsilon_\nu^{(a)}(k)$.

Чтобы получить амплитуду распространения частицы от источника до стока, необходимо перемножить эти амплитуды и затем просуммировать по трем возможным поляризациям. Становится понятным смысл вычета в полюсе пропагатора частицы со спином $1 D_{\nu\lambda}$: он представляет собой $\sum_a \varepsilon_\nu^{(a)}(k)\varepsilon_\lambda^{(a)}(k)$. Чтобы вычислить эту величину, заметим, что в силу лоренц-инвариантности она может выражаться лишь линейной комбинацией $g_{\nu\lambda}$ и $k_\nu k_\lambda$. Условие $k^\mu \varepsilon_\mu^{(a)} = 0$ фиксирует эту комбинацию как $g_{\nu\lambda} - k_\nu k_\lambda / m^2$. Вычислив левую часть выражения для k, находящегося в состоянии покоя, при, скажем, $\nu = \lambda = 1$, мы придем к общему знаку -1, который имеет принципиальное значение. Таким образом:

$$\sum_a \varepsilon_\nu^{(a)}(k)\varepsilon_\lambda^{(a)}(k) = -G_{\nu\lambda}(k) \equiv -\left(g_{\nu\lambda} - \frac{k_\nu k_\lambda}{m^2}\right)$$

Итак, мы построили пропагатор частицы со спином 1, двигаясь в обход Максвелла. А теперь перейдем к спину 2 ! Попробуем аналогичным образом пойти в обход Эйнштейна.

Все механические силы имеют только электромагнитную природу

(????????????? why????)

Появление силы трения из КТП (???)

(вряд ли скоро отвечу на этот вопрос)

Other about derivation of Coulomb's Law from QFT

How can we understand other interactions if we don't have force in our equations?

(?? is there a generalization of a force??? for QCD, for week interactions??)

Is force a fundamental concept?

(?? I'll think later??)

8.8.4 Deriving Newton's Law from QFT (??)

Идем в обход Эйнштейна

Массивная частица со спином 2 имеет $5 (2 \cdot 2 + 1 = 5$, помните?) степеней поляризации, характеризуемых пятью тензорами поляризации $\varepsilon_{\mu\nu}^{(a)}$ $(a = 1, 2, ..., 5)$, симметричными относительно индексов μ и ν и удовлетворяющими

$$k^\mu \varepsilon_{\mu\nu}^{(a)} = 0$$

а также условию нулевого следа

$$g^{\mu\nu} \varepsilon_{\mu\nu}^{(a)} = 0$$

Проверим выполнимость условий. Симметричный лоренцевский тензор имеет $4 \cdot 5/2 = 10$ компонент. Четыре условия (11) и одно условие (12) сокращают число компонент до

$10-4-1=5$, а нам только этого и надо. (И еще немного жаргона: помните ли вы, как построить неприводимое представление группы? Если нет, читайте приложение B.) Мы фиксируем нормировку $\varepsilon_{\mu\nu}$, полагая, что положительная величина $\sum\limits_a \varepsilon_{12}^{(a)}(k)\varepsilon_{12}^{(a)}(k)=1$.

Итак, по аналогии со случаем частицы спина 1, определим $\sum\limits_a \varepsilon_{\mu\nu}^{(a)}(k)\varepsilon_{\lambda\sigma}^{(a)}(k)$. Для этого необходимо выразить этот объект через $g_{\mu\nu}$ и k_μ, или, что эквивалентно, через $G_{\mu\nu}$ и k_μ. Объект должен представлять собой линейную комбинацию $G_{\mu\nu}G_{\lambda\sigma}, G_{\mu\nu}k_\lambda k_\sigma$, и т. д. Используя (11) и (12) (упр. I.5.1), вы легко придете к выражению

$$\sum_a \varepsilon_{\mu\nu}^{(a)}(k)\varepsilon_{\lambda\sigma}^{(a)}(k)=(G_{\mu\lambda}G_{\nu\sigma}+G_{\mu\sigma}G_{\nu\lambda})-\frac{2}{3}G_{\mu\nu}G_{\lambda\sigma}$$

Общий знак и коэффициент пропорциональности определяются из вычисления обеих частей данного равенства при, например, $\mu=\lambda=1$ и $\nu=\sigma=2$. Таким образом, мы нашли пропагатор массивной частицы со спином 2

$$D_{\mu\nu,\lambda\sigma}(k)=\frac{(G_{\mu\lambda}G_{\nu\sigma}+G_{\mu\sigma}G_{\nu\lambda})-\frac{2}{3}G_{\mu\nu}G_{\lambda\sigma}}{k^2-m^2}$$

Почему мь падаем?

Мы уже в состоянии объяснить одну из фундаментальных тайн Вселенной: почему массы притягиваются.

Вспомним из курсов электродинамики и специальной теории относительности, что плотность энергии или массы является частью тензора энергии импульса $T^{\mu\nu}$. Для наших целей вам фактически нужно помнить лишь то, что это симметричный тензор и что его компонента T^{00} является плотностью энергии.

Для взаимодействия с тензором энергии импульса необходимо тензорное поле $\varphi_{\mu\nu}$, симметричное по своим индексам. Другими словами, лагранжиан Вселенной должен содержать член вида $\varphi_{\mu\nu}T^{\mu\nu}$. Фактически это и означает, что гравитон - частица, ответственная за гравитацию, имеет спин 2, точно так же, как фотон - частица, ответственная за электромагнетизм и взаимодействующая с током J^μ, - имеет спин 1. В теории Эйнштейна, которую мы будем обсуждать позднее, $\varphi_{\mu\nu}$ является, конечно же, частью метрического тензора.

Подобно тому как мы наделяли фотон маленькой массой для того, чтобы избежать обсуждения калибровочной инвариантности, мы предположим, что гравитон имеет малую массу, уклоняясь тем самым от обсуждения общей координатной инвариантности [2]. Ага, мы только что получили пропагатор массивной частицы со спином 2. Посмотрим, как он работает. По аналогии с выражением (4)

$$W(J)=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}J^\mu(k)^*\frac{-g_{\mu\nu}+k_\mu k_\nu/m^2}{k^2-m^2+i\varepsilon}J^\nu(k)$$

описывающим взаимодействие двух электромагнитных токов, взаимодействие между двумя сгустками энергии импульса описывается формулой

$$W(T)=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}T^{\mu\nu}(k)^*\frac{(G_{\mu\lambda}+G_{\mu\sigma}G_{\nu\lambda})-\frac{2}{3}G_{\mu\nu}G_{\lambda\sigma}}{k^2-m^2+i\varepsilon}T^{\lambda\sigma}(k).$$

[2] В настоящий момент я прошу вас пренебречь нюансами и просто предположить, что для осмысления гравитации стоит положить $m \to 0$. Подробнее я остановлюсь на теории гравитации Эйнштейна в главе VIII.1. Закон сохранения энергии-импульса $\partial_\mu T^{\mu\nu}(x) = 0$ и, следовательно, $k_\mu T^{\mu\nu}(k) = 0$ позволяет заменить $G_{\mu\nu}$ в (16) на $g_{\mu\nu}$.

Наступает момент истины. Взглянем на взаимодействие двух сгустков плотности энергии T^{00}. Из (16) следует

$$W(T) = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} T^{00}(k)^* \frac{1+1-\frac{2}{3}}{k^2 - m^2 + i\varepsilon} T^{00}(k).$$

Сравнивая с (5) и учитывая хорошо известный факт, что $1 + 1 - 2/3 > 0$, мы видим, что в то время как одноименные заряды отталкиваются, массы притягиваются. Аплодисменты, пожалуйста!

Вселенная

Трудно переоценить важность (не говоря о красоте) того, что мы только что узнали: обмен частицей со спином 0 порождает притягивающую силу, со спином 1 - отталкивающую силу, а со спином 2 - притягивающую силу. Упомянутые силы реализуются в сильном взаимодействии адронов, электромагнитном взаимодействии и гравитационном взаимодействии соответственно. Универсальное гравитационное притяжение порождает нестабильности, приводящие к образованию структур в ранней Вселенной 3. Плотные области становятся еще плотнее. Притягивающая ядерная сила, обусловленная частицей со спином 0, в конечном итоге зажигает звезды. Более того, притягивающая сила между протонами и нейтронами, порождаемая обменом частицей со спином 0, способна преодолеть отталкивающую электрическую силу между протонами, порождаемую обменом частицей со спином 1, что приводит к образованию огромного разнообразия ядер, без которого мир определенно был бы довольно скучным.

Отталкивание между одноименными и, следовательно, притяжение между противоположными зарядами, вызываемое частицей со спином 1, приводит к образованию электрически нейтральных атомов.

Наш мир является результатом хрупкого равновесия между спинами 0, 1 и 2.

В нашем кратком обзоре Вселенной мы не упомянули слабое взаимодействие. В действительности слабое взаимодействие играет решающую роль в поддержании постоянной скорости горения звезд, подобных нашему Солнцу.

[3] О гравитационной неустойчивости и образовании структур во Вселенной читайте в книге A. Zee Einstein's Universe (ранее известной под названием An Old Man's Toy').

Степени свободы

Пора немножко остыть: с логической и математической точек зрения физика частицы с массой $m \neq 0$ может отличаться от физики частицы с массой $m = 0$. В самом деле, как известно из классической электродинамики, электромагнитная волна имеет 2 поляризации, то есть 2 степени свободы. Для массивной частицы со спином 1 мы можем перейти в ее собственную систему покоя, после чего группа вращений даст нам $2 \cdot 1 + 1 = 3$ степени свободы. Ключевым с точки зрения физики здесь является то, что безмассовый фотон нельзя привести в его собственную систему покоя. Математически группа вращений $SO(3)$ вырождается в группу вращений $SO(2)$, т. е. группу двумерных вращений вокруг направления импульса фотона.

В главе II.7 мы увидим, что в пределе нулевой массы продольная степень свободы массивного мезона со спином 1 отщепляется. Поэтому рассмотренное здесь взаимодействие между зарядами (6) следует признать корректным. Однако в главе VIII.1 вы узнаете, что в случае с гравитацией коэффициент $\frac{2}{3}$ в (17) заменяется на 1 в эйнштейновской теории. К счастью, знак взаимодействия в (17) не меняется. Приглушите немного аплодисменты.

Приложение

Давайте представим, что мы никогда не слышали о лагранжиане Максвелла. Нам надо написать релятивистский лагранжиан для массивного мезонного поля со спином 1. Общими усилиями мы придем к теории Максвелла. Спин 1 означает, что поле преобразуется как вектор под действием трехмерной группы вращений. Простейшим лоренцевским объектом, содержащим трехмерный вектор, является, очевидно, четырехмерный вектор. Поэтому начнем с векторного поля $A_\mu(x)$.

То, что векторное поле имеет массу m, означает, что оно удовлетворяет полевому уравнению

$$\left(\partial^2 + m^2\right) A_\mu = 0$$

Частица со спином 1 имеет три степени свободы [на научном языке: представление j группы вращений имеет размерность $(2j+1)$; здесь $j=1$. С другой стороны, поле $A_\mu(x)$ содержит 4 компоненты. Таким образом, необходимо наложить условие, которое сократило бы число степеней свободы с 4 до 3. Единственной лоренц-ковариантной возможностью (линейной по A_μ) является условие

$$\partial_\mu A^\mu = 0$$

Полезно проанализировать выражения (18) и (19), которые для импульсного пространства имеют вид $(k^2 - m^2)\, A_\mu(k) = 0$ и $k_\mu A^\mu(k) = 0$. Первое уравнение говорит о том, что $k^2 = m^2$, а из второго следует, что при переходе в собственную систему покоя, где $k^\mu = (m, \overrightarrow{0}), A^0$ сводится к 0, при этом остаются три ненулевые компоненты A^i с $i = 1, 2, 3$.

Обратите особое внимание на то, что (18) и (19) можно объединить в одно уравнение

$$\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right) A_\nu + m^2 A^\mu = 0.$$

Проверим, что (20) содержит как (18), так и (19). Подействуем с помощью ∂_μ на (20). Получим $m^2\partial_\mu A^\mu = 0$, что предполагает $\partial_\mu A^\mu = 0$. (На этом этапе критично, что $m \neq 0$, то есть мы не рассматриваем фотон с массой, строго равной 0.) Таким образом, получаем (19); используя (19) в (20), мы приходим к (18).

Теперь можно построить лагранжиан, умножая левую часть (20) на $\frac{1}{2}A_\mu$ (где $\frac{1}{2}$ есть результат соглашения, а общий знак «+» обусловлен физикой, а именно условием положительности кинетической энергии); то есть

$$\mathcal{L} = \frac{1}{2} A_\mu \left[\left(\partial^2 + m^2\right) g^{\mu\nu} - \partial^m\partial^\nu\right] A_\nu$$

Интегрируя по частям, приходим к версии лагранжиана Максвелла с ненулевой массой. В пределе $m \to 0$ получаем вариант Максвелла.

Несколько слов о терминологии. Некоторые настаивают на том, чтобы называть полем только $F_{\mu\nu}$, а A_μ - потенциалом. Но в соответствии с общепринятым обозначением мы не

будем делать этих тонких различий. Для нас любая динамическая функция в пространстве-времени является полем.

Упражнения

I.5.1. Используя свойства симметрии, запишите наиболее общую форму для $\sum_a \varepsilon^{(a)}_{\mu\nu}(k)\varepsilon^{(a)}_{\lambda\sigma}(k)$. Например, она должна быть инвариантна относительно перестановки $\{\mu\nu \leftrightarrow \lambda\sigma\}$. Вы должны получить нечто типа

$$AG_{\mu\nu}G_{\lambda\sigma} + B\left(G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda}\right) + C\left(G_{\mu\nu}k_\lambda k_\sigma + k_\mu k_\nu G_{\lambda\sigma}\right) + \\ + D\left(k_\mu k_\lambda G_{\nu\sigma} + k_\mu k_\sigma G_{\nu\lambda} + k_\nu k_\sigma G_{\mu\lambda} + k_\nu k_\lambda G_{\mu\sigma}\right) + Ek_\mu k_\nu k_\lambda k_\sigma$$

с неизвестными $A, ..., E$. Примените $k^\mu \sum_a \varepsilon^{(a)}_{\mu\nu}\varepsilon^{(a)}_{\lambda\sigma} = 0$ и выясните, что это условие означает для констант. Действуя таким образом, получите (13).

8.8.5 Зи.I.6 Inverse-Square Law and Floating 3-Brane

Почему обратный квадрат?

При первом знакомстве с физикой не задавались ли вы вопросом, почему сила действует по закону обратных квадратов, а, скажем, не по закону обратных кубов? Сейчас у вас есть исчерпывающий ответ. Когда две частицы обмениваются безмассовой частицей, потенциальная энергия между ними равна

$$V(r) \propto \int d^3k e^{i\vec{k}\cdot\vec{x}} \frac{1}{\vec{k}^2} \propto \frac{1}{r}$$

Спин обмениваемой частицы определяет общий знак, но $1/r$, как я отмечал ранее, следует из анализа размерностей. На самом деле $V(r)$ - преобразование Фурье пропагатора. Член $\vec{k}^2$ в пропагаторе обусловлен членом $(\partial_i\varphi \cdot \partial_i\varphi)$ в выражении для действия, где φ обозначает поле, ассоциированное с обмениваемой безмассовой частицей, а форма $(\partial_i\varphi \cdot \partial_i\varphi)$ объясняется инвариантностью относительно вращений. Поэтому в (1) не может фигурировать ни $\vec{k}$, ни $\vec{k}^3; \vec{k}^2$ есть простейшая возможность. То есть можно сказать, что закон обратных квадратов в некотором смысле проистекает из инвариантности относительно врашений!

С физической точки зрения закон обратных квадратов восходит к модели потока Фарадея. Рассмотрим сферу радиуса r, окружающую заряд. Электрический поток на единицу площади, протекающий сквозь сферу, изменяется по закону $1/4\pi r^2$. Этот геометрический факт отражается множителем d^3k в (1).

Мир бран

Удивительно, но тех минимальных знаний по квантовой теории поля, которые я вам успел преподать, достаточно, чтобы вывести вас на передовой край современных исследований, современных, разумеется, на момент написания книги. В теории струн наш $(3+1)$-мерный мир может быть вложен в большую Вселенную, подобно тому как $(2+1)$-мерный лист бумаги вкладывается в наш привычный $(3+1)$-мерный мир. Считается, что мы живем на 3-бране.

Предположим, существуют n дополнительных измерений с координатами $x^4, x^5, ..., x^{n+3}$. Пусть характеристическими масштабами, ассоциированными с дополнительными

координатами, будет R. Я не могу вдаваться в подробности, касающиеся различных сценариев, описывающих суть R. Я также не могу объяснить, почему мы привязаны к 3-бране. Однако могу сказать, что гравитация по своей сути связана со структурой пространства-времени и поэтому пронизывает всю $(n+3+1)$-мерную Вселенную.

Хорошо, какому же закону подчиняется гравитационная сила между двумя частицами? Конечно не закону гравитации, известному еще вашим дедам: используя преобразование Фурье

$$V(r) \propto \int d^{3+n}k e^{i\vec{k}\cdot\vec{x}} \frac{1}{\vec{k}^2} \propto \frac{1}{r^{1+n}},$$

мы получаем закон $1/r^{1+n}$.

Не противоречит ли он наблюдениям?

На самом деле нет, поскольку закон Ньютона остается справедливым для $r \gg R$. В этом пределе дополнительные координаты эффективно зануляются, по сравнению с рассматриваемым нами характеристическим масштабом длины r. Поток не может распространиться далеко в направлении n дополнительных координат. Представьте себе поток, вынужденно распространяющийся лишь по трем известным нам направлениям подобно тому, как электромагнитное поле в волноводе вынуждено распространяться только вдоль трубы. Мы эффективно возвращаемся в $(3+1)$-мерное пространство-время, а $V(r)$ возвращается к зависимости $1/r$.

Новый закон гравитации (2) справедлив только в противоположном пределе $r \ll R$. Эвристически, когда R много больше расстояния между двумя частицами, поток не знает о том, что дополнительные координаты конечны, и считает, что он живет в $(n+3+1)$-мерной Вселенной.

В силу слабости гравитации, закон Ньютона не был проверен с достаточной точностью на лабораторных масштабах расстояний, поэтому теоретикам есть чем спекулировать: R легко может существенно превышать масштаб элементарных частиц, но все же быть много меньше масштаба явлений повседневной жизни. Невероятно, но Вселенная может иметь «большие дополнительные измерения»! (Слово «большие» означает большие в масштабах физики элементарных частиц.)

Масса Планка

Обратимся к расчетам и определим массу Планка $M_{\rm Pl}$, записав закон Ньютона в более удобной форме $v(r) = G_N m_1 m_2 (1/r) = (m_1 m_2 / M_{\rm Pl}^2)\,(1/r)$. Численно $M_{\rm PI} \simeq 10^{19}$ ГэВ. Очевидно, что такое огромное значение отражает слабость гравитации.

В фундаментальных единицах, в которых $\hbar$ и c берутся равными единице, гравитация определяет собственный масштаб массы или энергии, который много больше любого экспериментально исследованного масштаба. Действительно, одна из фундаментальных загадок современной физики элементарных частиц как раз и заключается в том, почему этот массовый масштаб столь велик, по сравнению со всеми остальными известными нам масштабами. Когда придет время, я вернусь к этой так называемой проблеме иерархии. А сейчас зададимся вопросом, может ли эта новая модель гравитации - новая на рубеже последнего столетия - облегчить решение проблемы иерархии за счет уменьшения собственного гравитационного массового масштаба?

Пусть $M_{{\rm Pl}(n+3+1)}$ обозначает характеристический масштаб массы для гравитации в $(n+3+1)$-мерной Вселенной, так что гравитационный потенциал между двумя объектами с массами

m_1 и m_2, находящимися на расстоянии $r \ll R$ друг от друга, равен

$$V(r) = \frac{m_1 m_2}{\left[M_{\mathrm{Pl}(n+3+1)}\right]^{2+n}} \frac{1}{r^{1+n}}$$

Заметим, что зависимость от $M_{\mathrm{Pl}(n+3+1)}$ непосредственно следует из анализа размерностей: две степени нужны, чтобы сократить $m_1 m_2$, и n степеней - чтобы согласовать с n дополнительными степенями $1/r$. Для случая $r \gg R$, как уже говорилось ранее, геометрическая протяженность гравитационного потока ограничивается значением R так, что потенциал становится равным

$$V(r) = \frac{m_1 m_2}{\left[M_{\mathrm{Pl}(n+3+1)}\right]^{2+n}} \frac{1}{R^n} \frac{1}{r}$$

Сравнивая с экспериментально наблюдаемым законом $V(r) = (m_1 m_2 / M_{\mathrm{Pl}}^2) \times \times (1/r)$, получаем

$$M_{\mathrm{Pl}(n+3+1)}^2 = \frac{M_{\mathrm{Pl}}^2}{\left[M_{\mathrm{Pl}(n+3+1)} R\right]^n}$$

Если величина $M_{\mathrm{Pl}(n+3+1)} R$ может быть достаточно большой, то появляется интригующая возможность того, что фундаментальный масштаб гравитации $M_{\mathrm{P}(n+3+1)}$ много меньше, чем мы всегда думали.

Таким образом, величина R ограничена, с одной стороны, нашим желанием уменьшить фундаментальный гравитационный масштаб и, с другой стороны, результатами экспериментов.

Упражнения

I.6.1. Подставив числа, покажите, что случай $n = 1$ уже исключен. Для справки см. S. Nussinov and R. Schrock, Phys. Rev. D59: 105002, 1999.

8.8.6 Зи.I. Canonical Quantization and Vacuum Perturbation

Взаимодополняющие формализмы

Я рассматриваю формализм интеграла по траекториям как наиболее быстрый путь построения квантовой теории поля. Но я также должен обсудить канонический формализм не только в силу того, что исторически именно этим путем развивалась квантовая теория поля, но и потому, что он сохраняет актуальность и поныне. Интересно заметить, что канонический формализм и формализм интеграла по траекториям часто являются взаимодополняющими в том смысле, что результаты, которые трудно получить в одном, легко получаются в другом.

При обсуждении канонического формализма я буду довольно краток, отчасти из-за того, что он является стандартным подходом и потому детально излагается в большинстве книг по квантовой теории поля.

Гейзенберг и Дирак

Начнем с краткого обзора гейзенберговского подхода к квантовой механике. Для заданного классического лагранжиана одной частицы $L = \frac{1}{2}\dot{q}^2$ - - $V(q)$ (мы положили массу равной 1) канонический импульс определяется как $p \equiv \delta L/\delta\dot{q} = \dot{q}$. После этого, гамильтониан системы записывается в виде $H = p\dot{q} - L = p^2/2 + V(q)$. Гейзенберг заменяет положение $q(t)$ и импульс $p(t)$ на операторы, накладывая каноническое коммутационное соотношение

$$[p, q] = -i$$

Операторы эволюционируют во времени по закону

$$\frac{dp}{dt} = i[H, p] = -V'(q)$$

к

$$\frac{dq}{dt} = i[H, q] = p$$

Другими словами, операторы, составленные из p и q, эволюционируют как $O(t) = e^{iHt}O(0)e^{-iHt}$. В формуле (1) p и q рассматриваются в один и тот же момент времени. Объединяя (2) и (3), мы получаем операторное уравнение движения $\ddot{q} = -V'(q)$.

Следуя Дираку, рассмотрим в некоторый момент времени оператор $a \equiv (1/\sqrt{2\omega})(\omega q + ip)$ с некоторым параметром ω. Из (1) получаем

$$\left[a, a^\dagger\right] = 1$$

Оператор $a(t)$ эволюционирует по закону

$$\frac{da}{dt} = i\left[H, \frac{1}{\sqrt{2\omega}}(\omega q + ip)\right] = -i\sqrt{\frac{\omega}{2}}\left(ip + \frac{1}{\omega}V'(q)\right)$$

который можно переписать в терминах a и $a^\dagger$. Основное состояние $|0\rangle$ определяется как состояние, такое что $a|0\rangle = 0$.

В частном случае $V'(q) = \omega^2 q$ получаем особенно простой результат $\frac{da}{dt} = -i\omega a$. Он представляет собой гармонический осциллятор $L = \frac{1}{2}\dot{q}^2$ - $-\frac{1}{2}\omega^2 q^2$ и $H = \omega\left(a^\dagger a + \frac{1}{2}\right)$

Обобщение на случай многих частиц следует немедленно. Начиная с

$$L = \sum_a \frac{1}{2}\dot{q}_a^2 - V\left(q_1, q_2, ..., q_N\right)$$

получаем $p_a = \delta L/\delta\dot{q}_a$ и

$$[p_a(t), q_b(t)] = -i\delta_{ab}$$

Обобщение до теории поля следует почти немедленно. Фактически мы просто используем нашу таблицу подстановок (I.3.6) и видим, что в D-мерном пространстве L обобщается до

$$L = \int d^D x \left\{\frac{1}{2}\left(\dot{\varphi}^2 - (\vec{\nabla}\varphi)^2 - m^2\varphi^2\right) - u(\varphi)\right\}$$

где ангармоничный член (член взаимодействия в квантовой теории поля) обозначается через $u(\varphi)$. Плотность канонического импульса, сопряженного полю $\varphi(\vec{x}, t)$, при этом равна

$$\pi(\vec{x},t)=\frac{\delta L}{\delta\dot{\varphi}(\vec{x},t)}=\partial_0\varphi(\vec{x},t)$$

так что каноническое коммутационное соотношение для равных моментов времени теперь имеет вид

$$[\pi(\vec{x},t),\varphi(\vec{x}',t)]=[\partial_0\varphi(\vec{x},t),\varphi(\vec{x}',t)]=-i\delta^{(D)}(\vec{x}-\vec{x}')$$

(и, конечно, $[\pi(\vec{x},t),\pi(\vec{x}',t)]=0$ и $[\varphi(\vec{x},t),\varphi(\vec{x}',t)]=0$). Заметьте, что δ_{ab}, из (5) становится в (8) в $\delta^{(D)}(\vec{x}-\vec{x}')$. Проверьте, что (8) имеет правильную размерность. Проворачивая рукоятку канонического формализма, находим гамильтониан

$$\begin{aligned}H&=\int d^Dx\left[\pi(\vec{x},t)\partial_0\varphi(\vec{x},t)-\mathcal{L}\right]\\&=\\&=\int d^Dx\left\{\frac{1}{2}\left[\pi^2+(\vec{\nabla}\varphi)^2+m^2\varphi^2\right]+u(\varphi)\right\}\end{aligned}$$

Для случая $u=0$, соответствующего гармоническому осциллятору, приходим к теории свободного скалярного поля и можем продвинуться существенно дальше. Полевое уравнение имеет вид

$$\left(\partial^2+m^2\right)\varphi=0$$

Раскладывая в интеграл Фурье, получаем

$$\varphi(\vec{x},t)=\int\frac{d^Dk}{\sqrt{(2\pi)^D2\omega_k}}\left[a(\vec{k})e^{-i\left(\omega_kt-\vec{k}\cdot\vec{x}\right)}+a^\dagger(\vec{k})e^{i\left(\omega_kt-\vec{k}\cdot\vec{x}\right)}\right]$$

где $\omega_k=+\sqrt{\vec{k}^2+m^2}$, так что удовлетворяется полевое уравнение (10). Странный на вид множитель $(2\omega_k)^{-1/2}$ выбирается с тем, чтобы каноническое соотношение

$$\left[a(\vec{k}),a^\dagger\left(\vec{k}'\right)\right]=\delta^{(D)}\left(\vec{k}-\vec{k}'\right)$$

для операторов рождения и уничтожения приводило к каноническому коммутированию $[\partial_0\varphi(\vec{x},t),\varphi(\vec{x}',t)]=-i\delta^{(D)}(\vec{x}-\vec{x}')$ в (8). Лучше вам это проверить, хотя необходимость в множителе $(2\omega_k)^{-1/2}$ можно увидеть еще и потому, что в $\partial_0\varphi$ множитель ω_k спускается из показателя экспоненты.

Как и в квантовой механике, вакуумное, или основное, состояние $|0\rangle$ определяется условием $a(\vec{k})|0\rangle=0$ для всех $\vec{k}$. Вычислим $\langle0|\varphi(\vec{x},t)\varphi(\overrightarrow{0},0)|0\rangle$ для $t>0$. Из четырех членов $a^\dagger a^\dagger,a^\dagger a,aa^\dagger$ и aa, фигурирующих в произведении двух полей, остается вычислить только член $aa^\dagger$. Таким образом, используя (12), получаем $\int\left[d^Dk/(2\pi)^D2\omega_k\right]e^{-i\left(\omega_kt-\vec{k}\cdot\vec{x}\right)}$. Другими словами, если мы определим упорядоченное по времени произведение

$$T[\varphi(x)\varphi(y)]=\theta\left(x^0-y^0\right)\varphi(x)\varphi(y)+\theta\left(y^0-x^0\right)\varphi(y)\varphi(x),$$

то

$$\langle0|T[\varphi(\vec{x},t),\varphi(\overrightarrow{0},0)]|0\rangle=\int\frac{d^Dk}{(2\pi)^D2\omega_k}\left[\theta(t)e^{-i\left(\omega_kt-\vec{k}\cdot\vec{x}\right)}+\theta(-t)e^{+i\left(\omega_kt-\vec{k}\cdot\vec{x}\right)}\right].$$

Вернемся назад к (I.3.23). Находим, что $\langle 0|T[\varphi(x)\varphi(0)]|0\rangle = iD(x)$ есть пропагатор частицы, распространяющейся из 0 в x, полученный нами в формализме интеграла по траекториям. Полученный результат также подтверждает обоснованность i прескрипции в (I.3.22). Физический смысл состоит в том, что мы всегда рождаем, перед тем как уничтожить, а не наоборот. Это одна из форм причинности.

Обратите внимание на то, что условие эрмитовости позволяет выразить второй член в квадратных скобках в (11) через первый член. Если бы $\varphi \neq \varphi^\dagger$ (как вы увидите в главе I.9), то два члена должны были быть разными. Этот важный случай рассматривается в упражнении I.8.3, которое следует выполнить, так как его результаты понадобятся нам в дальнейшем.

Никто не совершенен

В каноническом формализме время рассматривается не так, как пространство, в силу чего у читателя может возникнуть вопрос о лоренц-инвариантности получившейся теории. В стандартном подходе, используемом во многих учебниках, начиная с этого момента мы должны были бы использовать гамильтониан для описания динамики, развивая теорию возмущений по взаимодействию $u(\varphi)$. После ряда формальных шагов мы получили бы правила Фейнмана, которые явно задавали бы лоренц-инвариантную теорию.

Был период, когда люди считали, что квантовая теория поля определяется набором правил Фейнмана, которые дают конкретный алгоритм расчета наблюдаемых величин, таких как сечения рассеяния. В своем крайнем проявлении такая точка зрения рассматривала поля лишь как математические объекты, помогающие нам добраться до правил Фейнмана, и которые после этого надо просто выбросить.

Начиная с 1970-х годов, такая точка зрения оказалась несостоятельной, так как стало понятно, что квантовая теория поля представляет собой нечто большее, чем просто набор диаграмм Фейнмана. Теория поля содержит в себе непертурбативные эффекты, которые по определению не ухватываются диаграммами Фейнмана. Многие из этих эффектов, которые мы рассмотрим в свое время, наиболее легко формулируются в формализме интеграла по траекториям.

Как я отмечал, канонический формализм и формализм интеграла по траекториям часто дополняют друг друга, поэтому я воздержусь от дискуссии на тему, который из них является более эффективным. В своей книге я принимаю прагматическую точку зрения и использую тот из них, который наилучшим образом подходит для обсуждаемой проблемы.

Тем не менее позвольте мне отметить некоторые особенности обоих формализмов, вызывающие особые затруднения. В каноническом формализме поля являются квантовыми операторами, содержащими бесконечное число степеней свободы. Были времена, когда ученые обсуждали такие тонкие вопросы, как, например, определение произведения полей. С другой стороны, в формализме интеграла по траекториям многие грехи можно замести под ковер, в качестве которого выступает мера интегрирования (см. главу IV.7).

Приложение

Вас может озадачить тот факт, что пропагатор в каноническом формализме должен определяться с упорядочиванием по времени, тогда как в формализме интеграла по траекториям в нем нет необходимости. Для разрешения этой кажущейся загадки достаточно посмотреть на квантовую механику.

Пусть $A\left[q\left(t_1\right)\right]$ - функция от q, определенная в момент времени t_1. Что представляет собой интеграл по траекториям $\int Dq(t)A\left[q\left(t_1\right)\right]e^{i\int_0^T dtL(\dot{q},q)}$ на операторном языке? Вернемся назад к (I.2.3) и вставим $A\left[q\left(t_1\right)\right]$ в подходящий множитель $\left\langle q_{j+1}\left|e^{-iH\delta t}\right|q_j\right\rangle$. Слово «подходящий» означает, что момент времени t_1 располагается между моментами времени $j\delta t$ и $(j+1)\delta t$ для этого j. В $\left\langle q_{j+1}\left|e^{-iH\delta t}A\left[q\left(t_1\right)\right]\right|q_j\right\rangle$ мы заменяем c-число $A\left[q\left(t_1\right)\right]$ оператором $A\left[\widehat{q}\left(t_1\right)\right]$. Таким образом, интеграл представляет собой выражение $\left\langle q_F\right|e^{-iH(T-t_1)}\times\ \times A\left[\widehat{q}\left(t_1\right)\right]e^{-iHt_1}\left|q_I\right\rangle$

Теперь мы готовы ответить на более сложный вопрос: что представляет собой интеграл по траекториям

$$\int Dq(t)A\left[q\left(t_1\right)\right]B\left[q\left(t_2\right)\right]e^{i\int_0^T dtL(\dot{q},q)}$$

на операторном языке? Здесь $B\left[q\left(t_2\right)\right]$ означает некоторую другую функцию от q, определенную в момент времени t_2. Мы также подставляем $B\left[q\left(t_2\right)\right]$ в соответствующий множитель в (I.2.3) и заменяем $B\left[q\left(t_2\right)\right]$ на $B\left[\widehat{q}\left(t_2\right)\right]$. Видим, что нам необходимо следить за тем, какой из двух моментов времени t_1 и t_2 наступает раньше. Если сначала наступает t_2, то оператор $A\left[\widehat{q}\left(t_1\right)\right]$ должен стоять слева от оператора $B\left[\widehat{q}\left(t_2\right)\right]$, и наоборот, если сначала наступает t_1. Итак, если t_2 раньше t_1, мы получаем следующую последовательность:

$$e^{-iH(T-t_1)}A\left[\widehat{q}\left(t_1\right)\right]e^{-iH(t_1-t_2)}B\left[\widehat{q}\left(t_2\right)\right]e^{-iHt_2}$$

Таким образом, определяется упорядоченное по времени произведение

$$T\left\{A\left[\widehat{q}\left(t_1\right)\right]B\left[\widehat{q}\left(t_2\right)\right]\right\}\equiv\theta\left(t_1-t_2\right)A\left[\widehat{q}\left(t_1\right)\right]B\left[\widehat{q}\left(t_2\right)\right]+\theta\left(t_2-t_1\right)B\left[\widehat{q}\left(t_2\right)\right]A\left[\widehat{q}\left(t_1\right)\right].$$

Имеем

$$\left\langle q_F\left|T\left\{A\left[\widehat{q}\left(t_1\right)\right]B\left[\widehat{q}\left(t_2\right)\right]e^{-iHT}\right\}\right|q_I\right\rangle=\int Dq(t)A\left[q\left(t_1\right)\right]B\left(q\left(t_2\right)\right]e^{i\int_0^T dtL(\dot{q},q)}$$

Упорядочение по времени не появляется в правой части, но играет важную роль в левой.

Зи-I.8.1.

Некоторые авторы предпочитают определять $a(\vec{k})$ и $a^\dagger(\vec{k})$ без квадратного корня. Покажите, что для произвольной функции f

$$\int d^4k\delta\left(k^2-m^2\right)\theta\left(k^0\right)f\left(k^0,\vec{k}\right)=\int\frac{d^3k}{2\omega_k}f\left(\omega_k,\vec{k}\right)$$

Обоснуйте, что $d^3k/2\omega_k$ есть лоренц-инвариантная мера. [Указание: преобразования Лоренца не изменяют знак k^0.] Значит, определенные этими авторами операторы рождения и уничтожения являются лоренц-ковариантными. Получите их коммутационные соотношения.

Зи-I.8.2.

Вычислите $\left\langle\vec{k}'|H|\vec{k}\right\rangle$, где $|\vec{k}\rangle=a^\dagger(\vec{k})|0\rangle$.

Зи-I.8.3.

Мы рассмотрели эрмитово (или вещественное, как говорят при несколько вольном употреблении терминологии) скалярное поле. Рассмотрим теперь комплексное скалярное поле, определяемое лагранжианом $\mathcal{L} = \partial\varphi^\dagger\partial\varphi$ - $-m^2\varphi^\dagger\varphi$. Считая φ и $\varphi^\dagger$ независимыми переменными, получите уравнения движения Эйлера -Лагранжа. Неэрмитовость φ означает, что (11) должно быть заменено на

$$\varphi(\vec{x},t) = \int \frac{d^Dk}{\sqrt{(2\pi)^D 2\omega_k}} \left[a(\vec{k})e^{-i\left(\omega_k t - \vec{k}\cdot\vec{x}\right)} + b^\dagger(\vec{k})e^{i\left(\omega_k t - \vec{k}\cdot\vec{x}\right)}\right.$$

Докажите, что канонические коммутационные соотношения предполагают, что $\left(a, a^\dagger\right)$ и $\left(b, b^\dagger\right)$ формируют два независимых множества операторов рождения и уничтожения. Вычислите $\left\langle 0 \left| T\left[\varphi(x)\varphi^\dagger(0)\right] \right| 0 \right\rangle$. [Указание: ∂_φ сопряжено с $\varphi^\dagger$, а не с φ.]

Зи-I.8.4.

Используя уравнения движения, проверьте, что ток $J_\mu = i\left(\varphi^\dagger\partial_\mu\varphi - -\partial_\mu\varphi^\dagger\varphi\right)$ сохраняется, а соответствующий заряд равен $Q = \int d^Dx J_0(x) = \int d^Dk \left[a^\dagger(\vec{k})a(\vec{k}) - b^\dagger(\vec{k})b(\vec{k})\right]$. Мы видим, что частица, порожденная $a^\dagger$ (назовем ее «частица»), и частица, порожденная $b^\dagger$ (назовем ее «античастица»), несут противоположные заряды. Таким образом, $\varphi^\dagger$ рождает частицу и уничтожает античастицу, то есть создает единицу заряда. Поле φ делает обратное. Покажите, что $[Q, \varphi(x)] = -\varphi(x)$.

Зи-I.8.5.

Для поля A покажите, что при $q^0 > 0$ имеет место выражение

$$\mathrm{Im}\left\{ i\int d^4x e^{iqx}\langle 0|T[A(x)A(0)]|0\rangle \right\} = \frac{1}{2}\int d^4x e^{iqx}\langle 0|[A(x), A(0)]|0\rangle$$

Для этого вставьте $1 = \sum\limits_n |n\rangle\langle n|$($|n\rangle$ - полный набор состояний) между $A(x)$ и $A(0)$ в левой части равенства. [Указание: используйте интегральные представления $\theta(t) = -i\int(d\omega/2\pi)e^{i\omega t}/(\omega - i\varepsilon)$ и $\theta(-t) = = i\int(d\omega/2\pi)e^{i\omega t}/(\omega + i\varepsilon)$.]

Зи-I.8.6.

Продемонстрируйте свое умение вычислять интегралы на примере расчета силы Казимира в $(3+1)$-мерном пространстве-времени. В помощь см. M. Kardar and R. Golestanian, Rev. Mod. Phys. 71: 1233, (1999); J. Feinberg, A. Mann, and M. Revzen, Ann. Phys. 288: 103, (2001).

8.9 Concept of Operators and Second Quantization

(intro to all this)

8.9.1 Origin of Operators

(just general idea)

8.9.2 Concept of Second Quantization

(just general idea)

8.10 Concept of Path Integral

(intro to all this)

8.10.1 Idea of Path Integral

(just general idea)

Что конкретно предполагается, чтобы работала конструкция интеграла по траекториям?

Предполагается:
Непрерывность времени
изотропность пространства
Наличие потенциалов.
(?????????????)

8.10.2 What is Exactly Meant by Path Integral?

(just general idea)

8.10.3 Quantization Using of Path Integral

(just general idea)

8.10.4 Where Path Integral is Also?

(overview, why this idea is used in gravitational lensing, finances, poluymers, so on????)

8.11 Conceptual Properties of Emptiness

Обсудим подробно вакуум.
(мб статью Гинзбурга сюда вставлю, пока она вдали. (????))

8.11.1 Vacuum Energy

Энергия вакуума

В качестве содержательного упражнения вычислим для теории свободного скалярного поля среднее

$$\langle 0|H|0\rangle = \frac{1}{2}\int d^D x \left\langle 0 \left| \pi^2 + (\vec{\nabla}\varphi)^2 + m^2\varphi^2 \right| 0 \right\rangle$$

которое мы условно назовем «энергией вакуума». Все, что нам нужно, это объединить вместе (7), (11) и (12). Сфокусируемся на третьем члене в $\langle 0|H|0\rangle$, который мы уже фактически вычислили, так как

$$\langle 0|\varphi(\vec{x},t)\varphi(\vec{x},t)|0\rangle = \langle 0|\varphi(\overrightarrow{0},0),\varphi(\overrightarrow{0},0)|0\rangle = \lim_{\vec{x},t\to 0,0}\langle 0|\varphi(\vec{x},t)\varphi(\overrightarrow{0},0)|0\rangle =$$

$$= \lim_{\vec{x},t\to 0,0}\int \frac{d^Dk}{(2\pi)^D 2\omega_k} e^{-i\left(\omega_k t-\vec{k}\cdot\vec{x}\right)} = \int \frac{d^Dk}{(2\pi)^D 2\omega_k}.$$

Первое равенство следует из трансляционной инвариантности, которая означает, что мы можем заменить $\int d^Dx$ в $\langle 0|H|0\rangle$ на объем пространства V. Вычисление двух других членов производится во многом аналогично: например, два множителя $\vec{\nabla}$ в $(\vec{\nabla}\varphi)^2$ просто дают множитель $\vec{k}^2$. Таким образом

$$\langle 0|H|0\rangle = V\int \frac{d^Dk}{(2\pi)^D 2\omega_k}\left[\frac{1}{2}\left(\omega_k^2+\vec{k}^2+m^2\right)\right] = V\int \frac{d^Dk}{(2\pi)^D}\frac{1}{2}\hbar\omega_k,$$

где мы восстановили $\hbar$.

Полученный результат должен вызвать у вас чувство удовлетворения и тревоги одновременно: удовлетворения потому, что мы узнаем в нем энергию нулевых колебаний гармонического осциллятора, проинтегрированную по всем импульсным модам, а тревоги из-за того, что интеграл по $\vec{k}$ явно расходится. Нам, однако, не следует тревожиться: энергия любой физической конфигурации, например масса частицы, должна измеряться относительно «энергии вакуума». Нас интересует разница между энергией мира с частицей и энергией мира без частицы. Другими словами, мы могли бы просто определить правильный гамильтониан как $H-\langle 0|H|0\rangle$. К некоторым из обсуждавшихся здесь вопросов мы вернемся в главах II.5, III.1 и VIII.2.

8.11.2 Idea of Casimir Effect

Эффект Казимира

А что случится, если мы возмутим вакуум?

Разумеется, не только скалярное поле вносит вклад в энергию вакуума. Электромагнитное поле, например, также испытывает квантовые флуктуации и добавляет к плотности энергии вакуума ε что-то типа $2\int[d^3k/(2\pi)^3]\,\frac{1}{2}\hbar\omega_k$. В 1948 году Казимир сделал блестящее предположение, что вызванное нами возмущение вакуума может привести к сдвигу $\Delta\varepsilon$. Хотя ε является экспериментально ненаблюдаемой величиной, сдвиг $\Delta\varepsilon$ должен поддаваться измерению, поскольку мы можем контролировать процесс возмущения нами вакуума. В частности, Казимир рассматривал ситуацию со внесением в вакуум двух параллельных «идеально» проводящих пластин (которые формально имеют бесконечную протяженность и нулевую толщину). Изменение $\Delta\varepsilon$ в зависимости от расстояния d между пластинами приводило к возникновению силы между ними, известной как сила Казимира.

Назовем направление, перпендикулярное пластинам, осью x. В силу граничных условий, которым должно удовлетворять электромагнитное поле на пластинах, волновой вектор может принимать лишь значения $(\pi n/d, k_y, k_z)$ с целым n. Таким образом, энергия между пластинами в расчете на единицу площади изменяется на величину

$$\frac{1}{2}\sum_n \int \frac{dk_y dk_z}{(2\pi)^2}\sqrt{\left(\frac{\pi n}{d}\right)^2 + k_y^2 + k_z^2}.$$

Для вычисления силы мы изменяем d, но при этом должны учитывать, как изменяется плотность энергии снаружи двух пластин. Существует, однако, хитрый прием, позволяющий этого избежать: надо вставить третью пластину, как показано на рис. I.8.1! Расстояние L между двумя внешними пластинами можно выбрать произвольно большим.

В соответствии с общей атмосферой книги (и моей философией), я стараюсь избегать, насколько это возможно, утомительных вычислений, поэтому предлагаю ввести два упрощения: (1) провести вычисления для безмассового скалярного, а не электромагнитного поля, чтобы не беспокоиться о поляризации и тому подобном, и (2) перейти в $(1+1)$-мерное пространство-время, чтобы не интегрировать по k_y и k_z. Как вы увидите, вычисления окажутся очень поучительными, они позволят ощутить прелесть выделения конечных физических результатов из явно расходящихся выражений. Это называется перенормировкой, мы рассмотрим ее в главах III. 1-3.

При такой формальной постановке энергия равна $E = f(d) + f(L-d)$, где

$$f(d) = \frac{\pi}{2d}\sum_{n=1}^{\infty} n$$

поскольку моды задаются выражением $\sin(n\pi x/d)(n = 1, ..., \infty)$, а соответствующая энергия равна $\omega_n = n\pi/d$.

Ой! А что нам делать с $\sum\limits_{n=1}^{\infty} n$? Никто из древних греков, начиная с Зенона, нам не подскажет.

Им следовало бы нам ответить, что мы решаем физическую, а не математическую задачу. Физические пластины не могут препятствовать прохождению волн произвольно большой частоты. Чтобы учесть это крайне важное физическое свойство, нам следует ввести множитель $e^{-a\omega_n}$, с тем чтобы исключить вклад мод с $\omega_n \gg a^{-1}$: этим модам не видны пластины. Таким образом, получаем

$$f(d) = \frac{\pi}{2d}\sum_{n=1}^{\infty} n e^{-an\pi/d} = -\frac{1}{2}\frac{\partial}{\partial a}\sum_{n=1}^{\infty} e^{-an\pi/d} = -\frac{1}{2}\frac{\partial}{\partial a}\frac{1}{1-e^{-a\pi/d}} = \frac{\pi}{2d}\frac{e^{a\pi/d}}{\left(e^{a\pi/d}-1\right)^2}$$

Так как мы хотим, чтобы a^{-1} было большим, рассмотрим предел малого a, когда

$$f(d) = \frac{d}{2\pi a^2} - \frac{\pi}{24d} + O\left(a^2\right).$$

Заметим, что $f(d)$ устремляется к бесконечности при $a \to 0$, как и должно быть, иначе мы бы возвратились к (17). Но сила между двумя проводящими пластинами не должна устремляться к бесконечности. Экспериментаторы это бы точно заметили!

Итак, сила дается выражением

$$\frac{\partial E}{\partial d} = f'(d) - f'(L-d) = \left(\frac{1}{2\pi a^2} + \frac{\pi}{24d^2} + ...\right) - (d \to L-d) \underset{a\to 0}{\longrightarrow}$$
$$\underset{a\to 0}{\longrightarrow} \frac{\pi}{24}\left(\frac{1}{d^2} - \frac{1}{(L-d)^2}\right)_L \widetilde{\gg} d\frac{\pi\hbar c}{24d^2}, \quad (20)$$

взятым с обратным знаком. Расходимость $1/a^2$ сократилась.

Видите, наша любимая физика приводит к разумному результату. Мы восстановили $\hbar$, чтобы подчеркнуть квантовую природу силы. Так как $\partial E/\partial d > 0$, сила Казимира между двумя пластинами является притягивающей. Чтобы измерить эту крошечную силу, потребовались довольно сложные эксперименты.

Об экспериментальных подтверждениях (???)

Другое про эффект Казимира

О модификациях и эффекте Казимира (????)

8.11.3 Emptiness in Cosmology: Main Ideas

(this is the most important topic there, there is a dark matter, so on, so it is written in cosmology, here I'll make an intro to that important question)

8.12 Details of Quantization

8.12.1 Quantization of Non-plane Waves: Main Ideas

(Указания для полноты, такое редко нужно будет, тем не менее, важная тема для понимания.)

8.12.2 Quantization on a Curved Spacetime: Main Ideas

(Указания для полноты, такое редко нужно будет, тем не менее, важная тема для понимания.)

What Is a Quantum Field If There Is No Plane Wave?

(???? actually, there are no plane waves, so this is a fundamental question)

8.12.3 Other Ideas about Quantization of Fields

(read Maslov's book and write his ideas)
(and also from QM, there are many things known about it, but why they are not in QFT?)

8.12.4 Other Ideas about Quantization

Квантование вселенной через квазиклассику

(???? is it a physical idea at all???)

8.12.5 What about that Illustrative Picture of "Quants-bullets"?

(??? is it valid or not???)

8.13 Deeper Understanding of General QFT Concepts

(reminder: there is a section below "Philosophy of QFT: Other Ideas ", where I discuss some general questions, here I assume that there are no general questions and deal with some particular)

8.13.1 Newtonian Mechanics as a Particular Case of QFT

(!??!??!?!??!)

(очень важный раздел, где очень многое выведу, что уже известно нам в клас. физике, но из общих формул, которые тут развиваем.)

переход из КТП в классическую механику

можно заметить, что легко видеть классический предел...

как мы помним, переход в классику происходит изменением масштаба, то есть $\hbar \to 0$, так что в фазе

$$\delta S = 0, \quad \delta x = \delta y = 0$$

$$\mathcal{G}_{\text{ret}}\left(x, y, t > 0\right) \approx \mathcal{A} e^{\frac{1}{\hbar} S_0}$$

и тут еще пару нюансов про него можно сказать

8.13.2 Field Theory as a Particular Case of QFT

(!??!??!?!??!)

переход из КТП в теорию поля (???)

8.13.3 Quantum Mechanics as a Particular Case of QFT

(!??!??!?!??!)

переход из КТП в квантовую механику (??)

8.13.4 Haag's theorem

While working on the mathematical physics of an interacting, relativistic, quantum field theory, Rudolf Haag developed an argument[1] against the existence of the interaction picture, a result now commonly known as Haag's theorem. Haag's original proof relied on the specific form of then-common field theories, but subsequently generalized by a number of authors, notably Dick Hall and Arthur Wightman, who concluded that no single, universal Hilbert space representation can describe both free and interacting fields.[2] A generalization due to Michael C. Reed and Barry Simon shows that applies to free neutral scalar fields of different masses,[3] which implies that the interaction picture is always inconsistent, even in the case of a free field.

(?????? I have no idea how it works)

9 Free Quantum Fields

(for a detailed discussion without focus on quantum properties see my note on field theory)

9.1 Scalar Field (bosons)

9.1.1 The Klein-Gordon field as a set of harmonic oscillators

Теория

Начнем обсуждение квантовой теории поля с довольно формального рассмотрения простейшего типа поля - вещественного поля Клейна-Гордона. Идея состоит в том, чтобы начать с классической теории поля (теории классического скалярного поля, описываемого лагранжианом (2.6)) и затем «проквантовать» ее, то есть реинтерпретировать динамические переменные как операторы, которые подчиняются каноническим коммутационным соотношениям*. Далее мы «решаем» теорию, находя собственные значения и собственные состояния гамильтониана, в качестве аналогии используя гармонический осциллятор.

Классическая теория вещественного поля Клейна-Гордона была кратко (но достаточно) обсуждена в предыдущем разделе; соответствующие выражения даны в формулах (2.6), (2.7) и (2.8). При квантовании теории мы следуем тем же путем, что и для любой другой динамической системы: полагаем ϕ и π операторами и накладываем на них подходящие коммутационные соотношения. Напомним, что для дискретной системы одной или более частиц коммутационные соотношения имеют вид:

$$[q_i, p_j] = i\delta_{ij}$$
$$[q_i, q_j] = [p_i, p_j] = 0.$$

Обобщение для непрерывной системы вполне естественно: так как $\pi(\mathbf{x})$ является плотностью импульса, то вместо символа Кронекера мы берем дельта-функцию Дирака:

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})$$
$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0.$$

(До сих пор мы используем представление Шредингера, в котором ϕ и π не зависят от времени. В следующем разделе мы перейдем к представлению Гейзенберга и эти «одновременные»коммутационные соотношения будут по-прежнему верны при условии, что оба оператора рассматриваются в один и тот же момент времени.)

Так как гамильтониан является функцией ϕ и π, то он также становится оператором. Следующая задача состоит в том, чтобы, зная гамильтониан, найти его спектр. Так как нет очевидного способа сделать это, попробуем записать уравнение КлейнаГордона в фурье-пространстве. Если разложить классическое поле Клейна-Гордона

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t)$$

(где $\phi^*(\mathbf{p}) = \phi(-\mathbf{p})$, так что $\phi(\mathbf{x})$ вещественно), то уравнение (2.7) принимает вид:

$$\left[\frac{\partial^2}{\partial t^2} + \left(|\mathbf{p}|^2 + m^2\right)\right] \phi(\mathbf{p}, t) = 0.$$

Это - уравнение движения для простого гармонического осциллятора с частотой

$$\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}.$$

- Эта процедура иногда называется вторичным квантованием, чтобы отличать возникающее в результате уравнение Клейна - Гордона (в котором ϕ является оператором) от старого одночастичного уравнения Клейна - Гордона (в котором ϕ было волновой функцией). В этой книге мы никогда не обращаемся ко второй точке зрения; мы стартуем с классического уравнения (в котором ϕ является классическим полем) и квантуем его только один раз. Мы знаем, как найти спектр простого гармонического осциллятора (ПГО). Коротко напомним, как это делается. Запишем гамильтониан в виде:

$$H_{\text{Пго}} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\phi^2.$$

Чтобы найти собственные значения $H_{\text{П}}$, запишем ϕ и p через лестничные операторы:

$$\phi = \frac{1}{\sqrt{2\omega}}\left(a + a^\dagger\right); \quad p = -i\sqrt{\frac{\omega}{2}}\left(a - a^\dagger\right).$$

Каноническое коммутационное соотношение $[\phi, p] = i$ эквивалентно соотношению

$$\left[a, a^\dagger\right] = 1.$$

Гамильтониан можно теперь переписать в виде:

$$H_{\text{Пго}} = \omega\left(a^\dagger a + \frac{1}{2}\right).$$

Состояние $|0\rangle$, такое, что $a|0\rangle = 0$, является собственным состоянием H с собственным значением $\frac{1}{2}\omega$, равным энергии нулевых колебаний. Далее, с помощью коммутаторов

$$\left[H_{\text{Пго}}\,, a^\dagger\right] = \omega a^\dagger, \quad \left[H_{\text{Пго}}\,, a\right] = -\omega a$$

легко проверить, что состояния

$$|n\rangle \equiv \left(a^\dagger\right)^n |0\rangle$$

являются собственными состояниями $H_{\text{ПГО}}$ с собственными значениями $\left(n+\frac{1}{2}\right)\omega$. Этими состояниями исчерпывается спектр.

Спектр гамильтониана Клейна-Гордона можем найти, используя тот же прием, но теперь каждая мода Фурье поля рассматривается как независимый осциллятор со своими собственными a и $a^\dagger$. По аналогии с (2.23) мы пишем:

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right)$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right)$$

Обратные выражения для $a_{\mathbf{p}}$ и $a_{\mathbf{p}}^{\dagger}$ через ϕ и π легко выводятся, но они редко используются. Для дальнейших вычислений полезно переписать (2.25) и (2.26) следующим образом:

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger}\right) e^{i\mathbf{p}\cdot\mathbf{x}}$$
$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} - a_{-\mathbf{p}}^{\dagger}\right) e^{i\mathbf{p}\cdot\mathbf{x}}.$$

Коммутационное соотношение (2.24) принимает вид:

$$\left[a_{\mathbf{p}}, a_{\mathbf{p}'}^{\dagger}\right] = (2\pi)^3 \delta^{(3)}\left(\mathbf{p} - \mathbf{p}'\right),$$

при помощи которого вычисляется коммутатор ϕ и π:

$$[\phi(\mathbf{x}), \pi\left(\mathbf{x}'\right)] = \int \frac{d^3p d^3p'}{(2\pi)^6} \frac{-i}{2} \sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}} \left(\left[a_{-\mathbf{p}}^{\dagger}, a_{\mathbf{p}'}\right] - \left[a_{\mathbf{p}}, a_{-\mathbf{p}'}^{\dagger}\right]\right) e^{i(\mathbf{p}\cdot\mathbf{x} + \mathbf{p}'\cdot\mathbf{x}')} =$$
$$= i\delta^{(3)}\left(\mathbf{x} - \mathbf{x}'\right).$$

Этот коммутатор имеет правильный вид. (Если вычисления такого типа незнакомы читателю, то мы рекомендуем тщательно их проделать; после небольшой практики они становятся совершенно простыми, но имеют фундаментальное значение для формализма, рассматриваемого в следующих двух главах.)

Теперь можно выразить гамильтониан через повышающие и понижающие операторы. Начиная с выражения (2.8) через ϕ и π, получаем:

$$H = \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6} e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \left\{ -\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{4} \left(a_{\mathbf{p}} - a_{-\mathbf{p}}^{\dagger}\right) \left(a_{\mathbf{p}'} - a_{-\mathbf{p}'}^{\dagger}\right) + \right.$$
$$\left. + \frac{-\mathbf{p}\cdot\mathbf{p}' + m^2}{4\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}} \left(a_{\mathbf{p}} + a_{-\mathbf{p}}^{\dagger}\right) \left(a_{\mathbf{p}'} + a_{-\mathbf{p}'}^{\dagger}\right) \right\} =$$
$$= \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \left(a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} + \frac{1}{2}\left[a_{\mathbf{p}}, a_{\mathbf{p}}^{\dagger}\right]\right)$$

Второе слагаемое пропорционально бесконечному с-числу $\delta(0)$. Это есть просто сумма по всем модам энергий нулевых колебаний $\omega_{\mathbf{p}}/2$. Наличие этого слагаемого следовало ожидать, но оно вызывает беспокойство. К счастью, этот бесконечный сдвиг энергии не может быть обнаружен экспериментально, так как эксперименты измеряют только разность между данной энергией и энергией основного состояния H. Поэтому мы игнорируем это бесконечное постоянное слагаемое во всех вычислениях. Возможно, что этот сдвиг энергии основного состояния может создавать проблему на более глубоком уровне теории; мы обсудим этот вопрос в Эпилоге.

Используя выражение для гамильтониана через $a_{\mathbf{p}}$ и $a_{\mathbf{p}}^{\dagger}$, нетрудно найти коммутаторы:

$$\left[H, a_{\mathbf{p}}^{\dagger}\right] = \omega_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}; \quad [H, a_{\mathbf{p}}] = -\omega_{\mathbf{p}} a_{\mathbf{p}}$$

Теперь можно выписать спектр теории так же, как и для гармонического осциллятора. Состояние $|0\rangle$, такое, что $a_{\mathbf{p}}|0\rangle = 0$ для всех $\mathbf{p}$, является основным состоянием, или вакуумом, причем его энергия $E = 0$ после отбрасывания бесконечной константы в (2.31). Все другие собственные состояния энергии могут быть построены путем действия операторами рождения

на $|0\rangle$. Состояние $a^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{q}}\cdots|0\rangle$ является собственным состоянием гамильтониана H с энергией $\omega_{\mathbf{p}}+\omega_{\mathbf{q}}+\cdots$. Эти состояния исчерпывают спектр.

Найдя спектр гамильтониана, попробуем дать интерпретацию его собственных состояний. При помощи (2.19) и вычислений, подобных (2.31), можно записать оператор полного импульса

$$\mathbf{P}=-\int d^3x\pi(\mathbf{x})\nabla\phi(\mathbf{x})=\int\frac{d^3p}{(2\pi)^3}\mathbf{p}a^\dagger_{\mathbf{p}}a_{\mathbf{p}}$$

Таким образом, оператор $a^\dagger_{\mathbf{p}}$ рождает состояние с импульсом $\mathbf{p}$ и энергией $\omega_{\mathbf{p}}=\sqrt{|\mathbf{p}|^2+m^2}$. Аналогично, состояние $a^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{q}}\cdots|0\rangle$ имеет импульс $\mathbf{p}+\mathbf{q}+....$ Естественно называть эти возбуждения частицами, так как они являются дискретными объектами, Удовлетворяющими правильному релятивистскому соотношению между импульсом и энергией. (Под частицей мы не подразумеваем нечто такое, что должно быть локализовано в пространстве; $a^\dagger_{\mathbf{p}}$ рождает частицы как собственные состояния импульса). С этого момента мы обозначаем $\omega_{\mathbf{p}}$ через $E_{\mathbf{p}}$ (или просто E), так как это действительно есть энергия частицы. Обратим внимание, что энергия всегда положительна: $E_{\mathbf{p}}=+\sqrt{|\mathbf{p}|^2+m^2}$

Обсуждаемый формализм позволяет также определить статистику наших частиц. Рассмотрим двухчастичное состояние $a^\dagger_{\mathbf{p}}a^\dagger_{\mathbf{q}}|0\rangle$. Так как $a^\dagger_{\mathbf{p}}$ и $a^\dagger_{\mathbf{q}}$ коммутируют, то это состояние тождественно состоянию $a^\dagger_{\mathbf{q}}a^\dagger_{\mathbf{p}}|0\rangle$, в котором две частицы переставлены местами. Более того, одна мода $\mathbf{p}$ может содержать произвольное количество частиц (точно так же, как простой гармонический осциллятор может быть возбужден до произвольно высокого уровня). Отсюда можно заключить, что частицы Клейна-Гордона подчиняются статистике Бозе-Эйнштейна.

Вакуумное состояние естественно считать нормированным условием $\langle0|0\rangle=1$. Одночастичные состояния $|\mathbf{p}\rangle\propto a^\dagger_{\mathbf{p}}|0\rangle$ также достаточно часто встречаются в теории, поэтому необходимо принять соглашение об их нормировке. Простейшая нормировка $\langle\mathbf{p}|\mathbf{q}\rangle=(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})$ (используемая во многих книгах) не является лоренц-инвариантной, что можно показать, рассматривая действие буста в направлении оси z. В результате такого буста имеем $p'_3=\gamma\left(p_3+\beta E\right),E'=\gamma\left(E+\beta p_3\right)$. Используя тождество для дельта-функции

$$\delta\left(f(x)-f\left(x_0\right)\right)=\frac{1}{\left|f'\left(x_0\right)\right|}\delta\left(x-x_0\right),$$

можно записать:

$$\begin{aligned}\delta^{(3)}(\mathbf{p}-\mathbf{q})&=\delta^{(3)}\left(\mathbf{p}'-\mathbf{q}'\right)\cdot\frac{dp'_3}{dp_3}=\\&=\delta^{(3)}\left(\mathbf{p}'-\mathbf{q}'\right)\gamma\left(1+\beta\frac{dE}{dp_3}\right)=\\&=\delta^{(3)}\left(\mathbf{p}'-\mathbf{q}'\right)\frac{\gamma}{E}\left(E+\beta p_3\right)=\\&=\delta^{(3)}\left(\mathbf{p}'-\mathbf{q}'\right)\frac{E'}{E}\end{aligned}$$

Проблема состоит в том, что объемы не инвариантны по отношению к бустам; ящик объемом V в собственной системе покоя имеет объем V/γ в движущейся системе в силу лоренцевского сокращения. Но из приведенного вычисления видно, что величина $E_{\mathbf{p}}\delta^{(3)}(\mathbf{p}-\mathbf{q})$ лоренц-инвариантна. Поэтому мы определяем

$$|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}}a^\dagger_{\mathbf{p}}|0\rangle,$$

так что

$$\langle\mathbf{p}|\mathbf{q}\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})$$

(Множитель 2 не является необходимым, но он удобен из-за множителя 2 в (2.25).)

В гильбертовом пространстве квантовых состояний преобразование Лоренца Λ действует как унитарный оператор $U(\Lambda)$. Из условия нормировки (2.35) вытекает, что

$$U(\Lambda)|\mathbf{p}\rangle = |\Lambda\mathbf{p}\rangle.$$

Если рассматривать это преобразование действующим на оператор $a^\dagger_{\mathbf{p}}$, то можно записать

$$U(\Lambda)a^\dagger_{\mathbf{p}}U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\mathbf{p}}}{E_{\mathbf{p}}}}a^\dagger_{\Lambda\mathbf{p}}$$

При выбранной нормировке следует в других местах разделить некоторые выражения на $2E_{\mathbf{p}}$. Например, соотношение полноты для одночастичных состояний примет вид:

$$(\mathbf{1})_{\text{одночастичные}} = \int \frac{d^3p}{(2\pi)^3}|\mathbf{p}\rangle\frac{1}{2E_{\mathbf{p}}}\langle\mathbf{p}|,$$

где оператор слева тождествен единичному на подпространстве одночастичных состояний и равен нулю на остальной части гильбертова пространства состояний. Интегралы такого вида будут встречаться достаточно часто; действительно, интеграл

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} = \int \frac{d^4p}{(2\pi)^4}(2\pi)\delta\left(p^2-m^2\right)\Big|_{p^0>0}$$

является лоренц-инвариантным интегралом по пространству 3-импульсов, в том смысле, что если $f(p)$ лоренц-инвариантна, то и выражение $\int \frac{d^3pf(p)}{(2E_{\mathbf{p}})}$ лоренц-инвариантно. Предполагается, что интегрирование идет по ветви $p^0 > 0$ гиперболоида $p^2 = m^2$ в пространстве 4-импульсов (рис. 2.2).

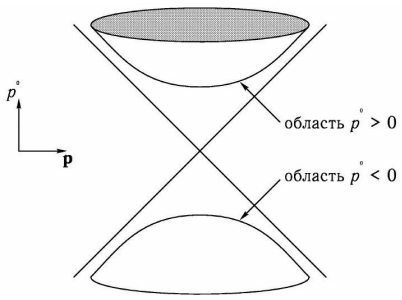


Рис. 2.2. Лоренц-инвариантный интеграл по пространству 3-импульсов берется по верхней ветви гиперболоида $p^2 = m^2$.

В заключение рассмотрим интерпретацию состояния $\phi(\mathbf{x})|0\rangle$. Из разложения (2.25) видно, что

$$\phi(\mathbf{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-i\mathbf{p}\cdot\mathbf{x}}|\mathbf{p}\rangle$$

является линейной суперпозицией одночастичных состояний, имеющих хорошо определенный импульс. Если бы не множитель $1/2E_{\mathbf{p}}$, то это выражение совпало бы с известным нерелятивистским выражением для собственного состояния оператора координаты $|\mathbf{x}\rangle$; на самом деле дополнительный множитель почти постоянен для малых (нерелятивистских) $\mathbf{p}$. Поэтому мы придерживаемся такой же интерпретации и считаем, что оператор $\phi(\mathbf{x})$, действующий на вакуум, рождает частицу в точке x. Эта интерпретация подтверждается следующими вычислениями:

$$\langle 0|\phi(\mathbf{x})|\mathbf{p}\rangle = \left\langle 0 \left| \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}'}}} \left(a_{\mathbf{p}'} e^{i\mathbf{p}'\cdot\mathbf{x}} + a^\dagger_{\mathbf{p}'} e^{-i\mathbf{p}'\cdot\mathbf{x}} \right) \sqrt{2E_{\mathbf{p}}} a^\dagger_{\mathbf{p}} \right| 0 \right\rangle = e^{i\mathbf{p}\cdot\mathbf{x}}$$

Мы можем интерпретировать этот результат как координатное представление одночастичной волновой функции состояния $|\mathbf{p}\rangle$, точно так же, как в нерелятивистской квантовой механике $\langle \mathbf{x}|\mathbf{p}\rangle \propto e^{i\mathbf{p}\cdot\mathbf{x}}$ является волновой функцией состояния $|\mathbf{p}\rangle$ в координатном представлении.

свойства настоящего скалярного поля

(здесь по итогу останутся только свойства, пока что это не так)

Квантованное поле в релятивистской нормировке имеет вид оператора

$$\hat{\varphi}(x) = \int \frac{c d^3k}{2\omega(2\pi)^3} \left\{ e^{\mathrm{i}kx} \hat{\mathfrak{a}}^\dagger(k) + e^{-\mathrm{i}kx} \hat{\mathfrak{a}}(k) \right\}$$

Подчеркнем, что классическое поле не имеет смысла амплитуды вероятности,

т. е. волновой функции.

Оно является средним значением оператора квантованного поля по когерентным состояниям.

Согласно принципу соответствия, при большом количестве квантов $|\alpha(k)|^2 \gg 1$ флуктуациями их количества можно пренебречь, и мы имеем наблюдаемое классическое поле.

По построению гамильтониан квантованного поля получается из гамильтониана классического поля подстановкой квантованного поля и взятием нормального символа:

$$\hat{\mathcal{H}} = \frac{1}{2} \int \mathrm{d}^3\boldsymbol{x} : \left\{ \frac{1}{c^2} (\dot{\hat{\varphi}}(x))^2 + (\boldsymbol{\nabla}\hat{\varphi}(x))^2 + \frac{m^2c^2}{\hbar^2} \hat{\varphi}^2(x) \right\}$$

определение объектов

Комплексное скалярное поле с массой m определяется как суперпозиция двух вещественных полей φ_1, φ_2 с той же массой,

$$\Phi(x) = \frac{1}{\sqrt{2}} \left\{ \varphi_1(x) + \mathrm{i}\varphi_2(x) \right\}$$

так что сумму действий для вещественных компонент можно представить в виде

$$S = \frac{1}{c\hbar^2} \int \mathrm{d}^4x \Phi^\dagger(x) \left(p^2 - m^2c^2 \right) \Phi(x)$$

основные свойства

(соберем тут их)

Выделение в комплексном поле пары динамических вещественных полей допускает введение произвольной глобальной комплексной фазы u:

$$\Phi_u(x) = e^{-\mathrm{i}ue/(\hbar c)}\Phi(x)$$

что, как говорят, определяет калибровочные преобразования поля, оставляющие инвариантным действие.

Мы ввели в определение фазы фактор $\frac{e}{\hbar c}$ в том же виде, что и при определении калибровочных преобразований нерелятивистской волновой функции.

введение операторов

Мы хотим

На уравнениях движения можно ввести положительно и отрицательно частотные решения с амплитудами,

которые выражаются через амплитуды исходных вещественных полей

$$\alpha(k) = \frac{1}{\sqrt{2}}\left\{\alpha_1(k) + \mathrm{i}\alpha_2(k)\right\}, \quad \alpha_c^*(k) = \frac{1}{\sqrt{2}}\left\{\alpha_1^*(k) + \mathrm{i}\alpha_2^*(k)\right\}$$

Тогда элементарные подстановки, как в случае с вещественным полем, дают энергию

$$H = \int \frac{\mathrm{d}^3k}{2\hbar\omega(2\pi)^3}\hbar\omega\left\{\alpha^*(k)\alpha(k) + \alpha_c^*(k)\alpha_c(k)\right\}$$

и заряд $Q = \int d^3x j_0 = \mathrm{i}\frac{e}{c\hbar}\left\{\Phi^\dagger\partial_0\Phi - (\partial_0\Phi)^\dagger\Phi\right\}$ равен

$$Q = e\int \frac{\mathrm{d}^3k}{2\hbar\omega(2\pi)^3}\hbar\omega\left\{\alpha^*(k)\alpha(k) - \alpha_c^*(k)\alpha_c(k)\right\}$$

Введем для комплексного квантованного поля операторы, составленные из операторов рождения и уничтожения вещественных полей,

$$\hat{\mathfrak{a}}(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\left\{\hat{\mathfrak{a}}_1(\boldsymbol{k}) + \mathrm{i}\hat{\mathfrak{a}}_2(\boldsymbol{k})\right\}, \quad \hat{\mathfrak{a}}_c^\dagger(\boldsymbol{k}) = \frac{1}{\sqrt{2}}\left\{\hat{\mathfrak{a}}_1^\dagger(\boldsymbol{k}) + \mathrm{i}\hat{\mathfrak{a}}_2^\dagger(\boldsymbol{k})\right\}$$

Для этих операторов справедливы следующие коммутационные соотношения:

$$\left[\hat{\mathfrak{a}}(k), \hat{\mathfrak{a}}^\dagger(k')\right] = \left[\hat{\mathfrak{a}}_c(k), \hat{\mathfrak{a}}_c^\dagger(k')\right] = 2\hbar\omega(2\pi)^3\delta(k-k')$$
$$\left[\hat{\mathfrak{a}}(k), \hat{\mathfrak{a}}_c(k')\right] = \left[\hat{\mathfrak{a}}(k), \hat{\mathfrak{a}}_c^\dagger(k')\right] = 0$$

И теперь квантованное поле запишется через эти операторы в виде

$$\hat{\Phi}(x) = \int \frac{cd^3k}{2\omega(2\pi)^3}\left\{e^{\mathrm{i}kx}\hat{\mathfrak{a}}_c^\dagger(k) + e^{-\mathrm{i}kx}\hat{\mathfrak{a}}(k)\right\}$$

Энергия поля задается гамильтонианом

$$\hat{\mathcal{H}} = \int \frac{\mathrm{d}^3k}{2\hbar\omega(2\pi)^3}\hbar\omega\left\{\hat{\mathfrak{a}}^\dagger(\boldsymbol{k})\hat{\mathfrak{a}}(\boldsymbol{k}) + \hat{\mathfrak{a}}_c^\dagger(\boldsymbol{k})\hat{\mathfrak{a}}_c(\boldsymbol{k})\right\}$$

В то же время квантованный заряд - это нормальный символ

$$\hat{Q} = \int \mathrm{d}^3x : j_0(x)$$

который равен

$$\hat{Q} = e \int \frac{\mathrm{d}^3k}{2\hbar\omega(2\pi)^3} \left\{ \hat{\mathfrak{a}}^\dagger(\boldsymbol{k})\hat{\mathfrak{a}}(\boldsymbol{k}) - \hat{\mathfrak{a}}_c^\dagger(\boldsymbol{k})\hat{\mathfrak{a}}_c(\boldsymbol{k}) \right\}$$

где осциллирующие со временем вклады сократились.

Полученные выражения для полной энергии и заряда квантованного поля говорят о том, что в пространстве волнового вектора

$$n_+(\boldsymbol{k}) = \frac{1}{2\hbar\omega}\hat{\mathfrak{a}}^\dagger(\boldsymbol{k})\hat{\mathfrak{a}}(\boldsymbol{k}), \quad n_-(\boldsymbol{k}) = \frac{1}{2\hbar\omega}\hat{\mathfrak{a}}_c^\dagger(\boldsymbol{k})\hat{\mathfrak{a}}_c(\boldsymbol{k})$$

- это плотности квантов с зарядом e и $-e$ соответственно.

То есть в релятивистской нормировке $\hat{a}^\dagger(k)$ - оператор рождения кванта с зарядом e,

а $\hat{\mathfrak{a}}_c^\dagger(k)$ - оператор рождения античастицы с зарядом $-$.

зарядовое сопряжение

Посмотрим, какие свойства комплексного поля можно увидеть, производя над ним зарядовое сопряжение.

Если взять комплексное сопряжение классического поля

$$\Phi(x) = \int \frac{cd^3k}{2\omega(2\pi)^3} \left\{ e^{\mathrm{i}kx}\alpha_c^*(\boldsymbol{k}) + e^{-\mathrm{i}kx}\alpha(\boldsymbol{k}) \right\}$$

в явном виде

$$\Phi^*(x) = \int \frac{cd^3k}{2\omega(2\pi)^3} \left\{ e^{-\mathrm{i}kx}\alpha_c(k) + e^{\mathrm{i}kx}\alpha^*(k) \right\}$$

то это соответствует перестановке амплитуд $\alpha \leftrightarrow \alpha_c$, а значит, вспоминая явные формулы, мы понимаем,

что для комплексно сопряженного поля сохраняется выражение для энергии H, но меняется знак заряда Q.

В этом случае это то же, что и зарядовое сопряжение.

Для обобщения этой операции на квантованные поля построим антилинейный оператор зарядового сопряжения,

который оставляет инвариантным вакуум $\mathbb{C}|0\rangle = |0\rangle$ и переводит состояние частицы в состояние античастицы, сохраняя нормировку:

$$\mathbb{C}|\boldsymbol{k}\rangle = \eta\left(Q_\Phi\right)|\boldsymbol{k}_c\rangle, \quad |\eta| = 1$$

где $Q_\Phi = 1$, - знак заряда поля, и $|\boldsymbol{k}\rangle = \hat{\mathfrak{a}}^\dagger(\boldsymbol{k})|0\rangle$, $\quad |\boldsymbol{k}_c\rangle = \hat{\mathfrak{a}}_c^\dagger(\boldsymbol{k})|0\rangle$ и

$$\eta\left(Q_\Phi\right) = e^{\mathrm{i}\gamma Q_\Phi} \equiv \eta, \quad \gamma \in \mathbb{R}, \quad \mathbb{C}Q_\Phi\mathbb{C}^{-1} = -Q_\Phi$$

Поэтому (???)

$$\mathbb{C}\eta\mathbb{C}^{-1} = \eta, \quad \mathbb{C}\hat{\mathfrak{a}}^\dagger(\boldsymbol{k})\mathbb{C}^{-1}\mathbb{C}|0\rangle = \eta\hat{\mathfrak{a}}_c^\dagger(\boldsymbol{k})|0\rangle$$

и наконец, деиствие зарядового сопряжения на оператор рож

$$\mathbb{C}\hat{\mathfrak{a}}^\dagger(\boldsymbol{k})\mathbb{C}^{-1} = \eta\hat{\mathfrak{a}}_c^\dagger(\boldsymbol{k})$$

Поскольку двукратное применение зарядового сопряжения возвращает состояние в исходное с точностью до фазы,

$$\mathbb{C}^2 = \xi, |\xi| = 1 \to \mathbb{C}^{-2} = \xi^*$$

Тогда из ($\overline{3.19}$) следует, что

$$\eta\mathbb{C}\hat{\mathfrak{a}}_c^\dagger(\boldsymbol{k})\mathbb{C}^{-1} = \hat{\mathfrak{a}}^\dagger(\boldsymbol{k}) \leftrightarrow \mathbb{C}\hat{\mathfrak{a}}_c^\dagger(\boldsymbol{k})\mathbb{C}^{-1} = \eta^*\hat{\mathfrak{a}}^\dagger(\boldsymbol{k})$$

Совершенно аналогично из инвариантности вакуума получаем

$$\mathbb{C}\hat{\mathfrak{a}}(\boldsymbol{k})\mathbb{C}^{-1}\mathbb{C}|\boldsymbol{k}\rangle = \mathbb{C}\hat{\mathfrak{a}}(\boldsymbol{k})\mathbb{C}^{-1}\eta\,|\boldsymbol{k}_c\rangle$$

Поstomy

$$\mathbb{C}\hat{\mathfrak{a}}(\boldsymbol{k})\mathbb{C}^{-1} = \eta^*\hat{\mathfrak{a}}_c(\boldsymbol{k})$$

и, как следствие,

$$\mathbb{C}\hat{\mathfrak{a}}_c(\boldsymbol{k})\mathbb{C}^{-1} = \eta\hat{\mathfrak{a}}(\boldsymbol{k})$$

В итоге, очевидно, что зарядовое сопряжение меняет знак заряда $\hat{Q}$: $\mathbb{C}Q\mathbb{C}^{-1} = -Q$, оставляя инвариантным гамильтониан. Кроме того, легко заметить, что действие зарядового сопряжения на квантованное поле практически сводится к его эрмитовому сопряжению:

$$\mathbb{C}\hat{\Phi}(x)\mathbb{C}^{-1} = \eta^*\hat{\Phi}^\dagger(x)$$

При получении данного равенства необходимо было учесть, что в определение положительно и отрицательно частотных мод формально нужно ввести символ Q_Φ, который позволяет отличать, какой заряд рождается или уничтожается:

$$\hat{\Phi}(x) = \int \frac{cd^3k}{2\omega(2\pi)^3}\left\{e^{\mathrm{i}kxQ_\Phi}\hat{\mathfrak{a}}_c^\dagger(k) + e^{-\mathrm{i}kxQ_\Phi}\hat{\mathfrak{a}}(k)\right\}$$

Тогда квантовый вариант зарядового сопряжения с антилинейным оператором C дает тот же результат, что и классический вариант комплексного сопряжения поля. При этом, конечно, квантованное действие остается инвариантным.

и что???????

нахрена нам оно?

Инверсия времени

(пока что не актуально)

Как мы уже установили, оператору инверсии стрелы времени $t \xrightarrow{\mathrm{T}} t_\mathrm{T} = -t$ отвечает операция с комплексным сопряжением. (?????) Для классического поля преобразование

$$\Phi(t,\boldsymbol{x}) \mapsto \Phi_\mathrm{T}(t_\mathbb{T},\boldsymbol{x}) = \Phi^*(t,\boldsymbol{x}) \quad \to \quad \Phi_\mathrm{T}(t_\mathrm{T},\boldsymbol{x}) = \Phi^*(-t_\mathbb{T},\boldsymbol{x})$$

или, что то же, подстановка

$$\Phi(t,\boldsymbol{x}) \mapsto \Phi^*(-t,\boldsymbol{x})$$

не изменяет действие, а поле

$$\Phi^*(-t,\boldsymbol{x}) = \int \frac{cd^3k}{2\omega(2\pi)^3}\left\{e^{\mathrm{i}\omega t+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\alpha_c(\boldsymbol{k}) + e^{-\mathrm{i}\omega t-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\alpha^*(\boldsymbol{k})\right\}$$

после замены переменной интегрирования $k \mapsto -k$ сводится к

$$\Phi^*(-t,\boldsymbol{x}) = \int \frac{cd^3k}{2\omega(2\pi)^3}\left\{e^{\mathrm{i}kx}\alpha_c(-k) + e^{-\mathrm{i}kx}\alpha^*(-k)\right\}$$

которое, конечно, удовлетворяет уравнениям движения, и мы можем говорить об инвариантном преобразовании с инверсией стрелы времени. При этом, имеют место подстановки $\alpha(k) \mapsto \alpha^*(-k)$ и $\alpha_c^*(k) \mapsto \alpha_c(-k)$, что, конечно, оставляет инвариантными энергию и заряд, но, скажем, ток j меняет знак, поскольку вектор движения меняет направление на противоположное.

Точно такие же выводы и аналог (3.25) теперь нужно получить в квантовом случае, что должно соответствовать действию оператора

$$t \xrightarrow{\mathrm{T}} t_{\mathrm{T}} = -t$$

на поле

$$\mathbb{T}\hat{\Phi}(t,\boldsymbol{x})\mathbb{T}^{-1} \equiv \hat{\Phi}_{\mathbb{T}}\left(t_{\mathbb{T}},\boldsymbol{x}\right) = \zeta^*\hat{\Phi}^\dagger(t,\boldsymbol{x}) = \zeta^*\hat{\Phi}^\dagger\left(-t_{\mathbb{T}},\boldsymbol{x}\right)$$

Для полноты изложения отметим, что сохраняющий нормировку состояния антилинейный оператор называется антиунитарным. Действие такого оператора обладает следующими свойствами:

$$\mathbb{T}\left(c_1\left|\psi_1\right\rangle + c_2\left|\psi_2\right\rangle\right) = c_1^*\left|\mathbb{T}\psi_1\right\rangle + c_2^*\left|\mathbb{T}\psi_2\right\rangle$$

и

$$\langle\mathbb{T}\Phi|\mathbb{T}\Psi\rangle = \langle\Psi|\Phi\rangle = \langle\Phi|\Psi\rangle^*$$

Эрмитово сопряженный оператор для антилинейного определяется соотношением

$$\langle\Phi|\mathbb{T}\Psi\rangle = \left\langle\Psi|\mathbb{T}^\dagger\Phi\right\rangle$$

Движение вспять по времени нетривиально формулируется в терминах операторов рождени и уничтожения квантов. В самом деле, если оператор $\mathfrak{a}^\dagger(k)$ рождает квант с волновым вектором k в момент времени t, то рассмотрение этого же процесса в обратном направлении означает, что в момент времени $t_{\mathbb{T}} = -t$ происходит уничтожение кванта, который двигался с волновым вектором $-k$, оператором $\mathfrak{a}(-k)$. Эти взаимосвязи соответствуют операторной записи

$$\mathbb{T}e^{\mathrm{i}\omega t}\hat{\mathfrak{a}}^\dagger(k)\mathbb{T}^{-1} = \zeta e^{-\mathrm{i}\omega t_{\mathrm{T}}}\hat{\mathfrak{a}}(-k), \quad |\zeta| = 1$$

С учетом действия антилинейного оператора

$$\mathbb{T}e^{\mathrm{i}\omega t}\ \mathrm{T}^{-1} = e^{-\mathrm{i}\omega t_{\mathrm{T}}}$$

устанавливаем, что

$$\mathbb{T}\hat{\mathfrak{a}}^\dagger(k)\mathbb{T}^{-1} = \zeta\hat{\mathfrak{a}}(-k)$$

В этом тождестве двукратное действие операторами T дает

$$\hat{\mathfrak{a}}^\dagger(\boldsymbol{k}) = \mathbb{T}\zeta\hat{\mathfrak{a}}(-\boldsymbol{k})\mathbb{T}^{-1} = \mathbb{T}\zeta\mathbb{T}^{-1}\mathbb{T}\hat{\mathfrak{a}}(-\boldsymbol{k})\mathbb{T}^{-1}$$

Теперь необходимо установить, как преобразуется фазовый множитель ζ, и мы положим

$$\mathbb{T}\zeta\mathbb{T}^{-1} = \zeta$$

что возможно только, если по аналогии с зарядовым сопряжением, где мы вводили оператор знака заряда, ввести знак стрелы времени $\lambda_{\mathbb{T}} = 1$, чтобы

$$\mathbb{T}\lambda_{\mathbb{T}}\mathbb{T}^{-1} = -\lambda_{\mathbb{T}}$$

и

$$\zeta = e^{\mathrm{i}\gamma'\lambda_{\mathrm{T}}}, \quad \gamma' \in \mathbb{R}$$

При такой прескрипции, как мы увидим, эрмитов оператор числа квантов переходит в эрмитов оператор числа квантов, которые движутся в противоположном направлении.

Подытоживая выкладки, находим для антилинейного оператора инверсии времени

$$\begin{aligned}
\mathbb{T}\hat{\mathfrak{a}}(k)\mathbb{T}^{-1} &= \zeta^*\hat{\mathfrak{a}}^\dagger(-k) \\
\mathbb{T}\hat{\mathfrak{a}}_c^\dagger(k)\mathbb{T}^{-1} &= \zeta_c\hat{\mathfrak{a}}_c(-k) \\
\mathbb{T}\hat{\mathfrak{a}}_c(k)\mathbb{T}^{-1} &= \zeta_c^*\hat{a}_c^\dagger(-k)
\end{aligned}$$

где мы ввели аналогичный фазовый фактор ζ_c и для античастиц. Тогда в силу

$$\begin{aligned}
\mathbb{T}\hat{\mathfrak{a}}^\dagger(k)\hat{\mathfrak{a}}(k)\mathbb{T}^{-1} &= \hat{\mathfrak{a}}(-k)\hat{\mathfrak{a}}^\dagger(-k) \\
\mathbb{T}\hat{\mathfrak{a}}_c^\dagger(k)\hat{\mathfrak{a}}_c(k)\mathbb{T}^{-1} &= \hat{\mathfrak{a}}_c(-k)\hat{\mathfrak{a}}_c^\dagger(-k)
\end{aligned}$$

применение к этим эрмитово самосопряженным выражениям операции нормального упорядочивания позволяет установить, что гамильтониан $\hat{\mathcal{H}}$ и заряд поля $\hat{Q}$ останутся инвариантными при обрашении стрелы времени, как этого и следовало ожидать.

Действие оператора инверсии времени на квантовое поле при $t \mapsto t_{\mathrm{T}} = -t$ будет соответствовать классическим операциям, если в явном виде ввести соответствие положительно и отрицательно частотнъх мод и знака стрелы времени:

$$e^{\mp\mathrm{i}kx} \mapsto e^{\mp\mathrm{i}kx\lambda_{\mathrm{T}}}$$

а значит, с учетом $\mathbb{T}\left(\mathrm{i}\lambda_{\mathbb{T}}\right)\mathrm{T}^{-1} = \mathrm{i}\lambda_{\mathbb{T}}$ найдем после замены переменной интегрирования во втором из следующих равенств $k \mapsto -k$

$$\begin{aligned}
\mathbb{T}\hat{\Phi}(t,\boldsymbol{x})\mathbb{T}^{-1} = \int \frac{cd^3k}{2\omega(2\pi)^3}\left\{e^{\mathrm{i}(\omega t_{\mathrm{T}}-\boldsymbol{k}\cdot\boldsymbol{x})}\zeta_c\hat{\mathfrak{a}}_c(-\boldsymbol{k}) + e^{-\mathrm{i}(\omega t_{\mathrm{T}}-\boldsymbol{k}\cdot\boldsymbol{x})}\zeta^*\hat{\mathfrak{a}}^\dagger(-\boldsymbol{k})\right\} = \\
= \int \frac{cd^3k}{2\omega(2\pi)^3}\left\{e^{-\mathrm{i}(-\omega t_{\mathrm{T}}-\boldsymbol{k}\cdot\boldsymbol{x})}\zeta_c\hat{\mathfrak{a}}_c(\boldsymbol{k}) + e^{\mathrm{i}(-\omega t_{\mathrm{T}}-\boldsymbol{k}\cdot\boldsymbol{x})}\zeta^*\hat{\mathfrak{a}}^\dagger(\boldsymbol{k})\right\} \quad (9.1)
\end{aligned}$$

что, конечно, сводится к

$$\mathbb{T}\hat{\Phi}(t,\boldsymbol{x})\mathbb{T}^{-1} \equiv \hat{\Phi}_{\mathbb{T}}\left(t_{\mathbb{T}},\boldsymbol{x}\right) = \zeta^*\hat{\Phi}^\dagger\left(-t_{\mathbb{T}},\boldsymbol{x}\right)$$

ТОлько если

$$\zeta^* = \zeta_c$$

С учетом (3.33) квантовое действие свободного комплексного скалярного поля останется инвариантным относительно инверсии времени.

Инверсия пространства

(пока что не актуально)
Зеркальное отражение пространства

$$\boldsymbol{x} \xrightarrow{\mathbb{P}} \boldsymbol{x}_{\mathbb{P}} = -\boldsymbol{x}$$

в квантовой механике свободной релятивистской скалярной частицы соответствует линейному оператору, сохраняющему вакуум

$$\mathbb{P}|0\rangle = |0\rangle$$

и преобразующему, например, одночастичное состояние, сохраняя его норму:

$$\mathbb{P}|k\rangle = \xi|-k\rangle, \quad |\xi| = 1$$

и аналогично для античастицы.

Тогда та же схема рассуждений, что и в двух предыдущих разделах, приводит к следующим соотношениям:

$$\begin{array}{ll} \mathbb{P}\hat{\mathfrak{a}}^\dagger(k)\mathbb{P}^{-1} = \xi\hat{\mathfrak{a}}^\dagger(-k), & \mathbb{P}\hat{\mathfrak{a}}(k)\mathbb{P}^{-1} = \xi^*\hat{\mathfrak{a}}(-k) \\ \mathbb{P}\hat{\mathfrak{a}}_c^\dagger(k)\mathbb{P}^{-1} = \xi_c\hat{\mathfrak{a}}_c^\dagger(-k), & \mathbb{P}\hat{\mathfrak{a}}_c(k)\mathbb{P}^{-1} = \xi_c^*\hat{\mathfrak{a}}_c(-k) \end{array}$$

Если учесть элементарные соотношения типа

$$\mathbb{P}e^{\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k}}\mathbb{P}^{-1} = e^{\mathrm{i}\boldsymbol{x}_{\mathrm{P}}\cdot\boldsymbol{k}} = e^{-\mathrm{i}\boldsymbol{x}\cdot\boldsymbol{k}}$$

и в разложении поля по модам сделать очевидную замену переменных $k \mapsto -k$, то поле преобразуется согласно

$$\mathbb{P}\hat{\Phi}(t,\boldsymbol{x})\mathbb{P}^{-1} = \hat{\Phi}_{\mathbb{P}}(t,\boldsymbol{x}_{\mathbb{P}}) = \xi^*\hat{\Phi}(t,\boldsymbol{x})$$

если $\xi^* = \xi_c$, что дает также инвариантность действия. В итоге, преобразованное поле -

$$\hat{\Phi}_{\mathbb{P}}(t,\boldsymbol{x}) = \xi^*\hat{\Phi}(t,-\boldsymbol{x})$$

3.2.4.

Моды и пропагатор

Для квантованного комплексного скалярного поля

$$\hat{\Phi}(x) = \int \frac{\mathrm{d}^3k}{2\omega(2\pi)^3}\left\{e^{\mathrm{i}kx}\hat{\mathfrak{a}}_c^\dagger(k) + e^{-\mathrm{i}kx}\hat{\mathfrak{a}}(k)\right\}$$

необходимо подчеркнуть связь положительно и отрицательно частотных мод с пропагатором. В самом деле, для частиц положительно частотная мода получается при действии на состояние кванта оператора поля,

$$\langle 0|\hat{\Phi}(x)|k\rangle = e^{-\mathrm{i}kx}$$

а для античастиц положительно частотная мода получается при действии на состояние кванта оператора эрмитово сопряженного поля,

$$\left\langle 0\left|\hat{\Phi}^\dagger(x)\right|k_c\right\rangle = e^{-\mathrm{i}kx}$$

Аналогично для отрицательно частотных мод

$$\left\langle \boldsymbol{k} \left| \hat{\Phi}^\dagger(x) \right| 0 \right\rangle = e^{\mathrm{i}kx}, \quad \left\langle \boldsymbol{k}_c | \hat{\Phi}(x) | 0 \right\rangle = e^{\mathrm{i}kx}$$

Поэтому причинная функция Грина отвечает следующему матричному элементу

$$\left\langle 0 \left| \, \mathrm{T} \left\{ \hat{\Phi}(x) \hat{\Phi}^\dagger(y) \right\} \right| 0 \right\rangle = \mathrm{i} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} e^{-\mathrm{i}k(x-y)} \frac{1}{k^2 - m^2 + \mathrm{i}0}$$

Это значит, что при графическом изображении пропагатора заряженной частицы необходимо указывать стрелкой направление распространения заряда: частица распространяется, если $x_0 > y_0$, а если $y_0 > x_0$, то распространяется античастица.

Что касается коммутатора полей, который необходимо рассмотреть для выяснения зависимости операторов вне светового конуса, то необходимо напомнить, что комплексное поле построено из двух независимых вещественных полей $\Phi = \frac{1}{\sqrt{2}} (\varphi_1 + \mathrm{i}\varphi_2)$, так что

$$\left[\hat{\Phi}(x), \hat{\Phi}^\dagger(y) \right] = \frac{1}{2} \left[\hat{\varphi}_1(x), \hat{\varphi}_1(y) \right] + \frac{1}{2} \left[\hat{\varphi}_2(x), \hat{\varphi}_2(y) \right]$$

а значит, мы имеем дело всё с той же функцией Паули-Йордана, которая обращается в нуль вне светового конуса, и, стало быть, принцип релятивистской причинности, конечно, выполняется.

9.1.2 The Klein-Gordon field in space-time

Теория

В предыдущем разделе мы проквантовали поле Клейна-Гордона в представлении Шредингера и дали интерпретацию возникающей в результате теории в терминах релятивистских частиц. В этом разделе рассмотрим представление Гейзенберга, в котором будет легче обсуждать зависимость величин от времени и вопросы причинности. После вводных формул мы вернемся к вопросу о причинном распространении, поднятому в разделе 2.1. Мы также получим выражение для пропагатора Клейна-Гордона, играющего ключевую роль в разрабатываемых в гл. 4 правилах Фейнмана.

В представлении Гейзенберга операторы ϕ и π зависят от времени обычным обра30М:

$$\phi(x) = \phi(\mathbf{x}, t) = e^{iHt} \phi(\mathbf{x}) e^{-iHt}$$

и аналогично для $\pi(x) = \pi(\mathbf{x}, t)$. Уравнения движения Гейзенберга

$$i \frac{\partial}{\partial t} \mathscr{O} = [\mathscr{O}, H]$$

позволяют найти зависимость ϕ и π от времени:

$$
\begin{aligned}
i\frac{\partial}{\partial t}\phi(\mathbf{x},t) &= \left[\phi(\mathbf{x},t), \int d^3x' \left\{\frac{1}{2}\pi^2\left(\mathbf{x}',t\right) + \frac{1}{2}\left(\nabla\phi\left(\mathbf{x}',t\right)\right)^2 + \frac{1}{2}m^2\phi^2\left(\mathbf{x}',t\right)\right\}\right] = \\
&= \int d^3x' \left(i\delta^{(3)}\left(\mathbf{x}-\mathbf{x}'\right)\pi\left(\mathbf{x}',t\right)\right) = \\
&= i\pi(\mathbf{x},t); \\
i\frac{\partial}{\partial t}\pi(\mathbf{x},t) &= \left[\pi(\mathbf{x},t), \int d^3x' \left\{\frac{1}{2}\pi^2\left(\mathbf{x}',t\right) + \frac{1}{2}\phi\left(\mathbf{x}',t\right)\left(-\nabla^2+m^2\right)\phi\left(\mathbf{x}',t\right)\right\}\right] = \\
&= \int d^3x' \left(-i\delta^{(3)}\left(\mathbf{x}-\mathbf{x}'\right)\left(-\nabla^2+m^2\right)\phi\left(\mathbf{x}',t\right)\right) = \\
&= -i\left(-\nabla^2+m^2\right)\phi(\mathbf{x},t).
\end{aligned}
$$

Объединяя результаты, находим:

$$\frac{\partial^2}{\partial t^2}\phi = \left(\nabla^2 - m^2\right)\phi$$

т. е. получаем уравнение Клейна-Гордона. Мы можем лучше понять зависимость от времени $\phi(x)$ и $\pi(x)$, записывая их через операторы рождения и уничтожения. Сначала заметим, что

$$Ha_{\mathbf{p}} = a_{\mathbf{p}}\left(H - E_{\mathbf{p}}\right)$$

и, следовательно,

$$H^n a_{\mathbf{p}} = a_{\mathbf{p}}\left(H - E_{\mathbf{p}}\right)^n$$

для любого n. Подобное соотношение (с $-$, замененным на $+$) имеет место для $a^\dagger_{\mathbf{p}}$. Таким образом, мы получаем тождества

$$e^{iHt}a_{\mathbf{p}}e^{-iHt} = a_{\mathbf{p}}e^{-iE_{\mathbf{p}}t}, \quad e^{iHt}a^\dagger_{\mathbf{p}}e^{-iHt} = a^\dagger_{\mathbf{p}}e^{iE_{\mathbf{p}}t}$$

которые можно использовать в выражении (2.25) для $\phi(\mathbf{x})$, чтобы согласно (2.43) найти искомое выражение для гейзенберговского оператора $\phi(x)$. (Для обозначения независящих от времени повышающих и понижающих операторов в представлении Шредингера мы всегда будем использовать символы $a_{\mathbf{p}}$ и $a^\dagger_{\mathbf{p}}$.) Результат имеет вид

$$
\begin{aligned}
\phi(\mathbf{x},t) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)\Big|_{p^0=E_{\mathbf{p}}} \\
\pi(\mathbf{x},t) &= \frac{\partial}{\partial t}\phi(\mathbf{x},t)
\end{aligned}
$$

Стоит заметить, что такие же манипуляции можно выполнить с $\mathbf{P}$ вместо H, чтобы связать $\phi(\mathbf{x})$ и $\phi(0)$. По аналогии с (2.46) можно показать, что

$$e^{-i\mathbf{P}\cdot\mathbf{x}}a_{\mathbf{p}}e^{i\mathbf{P}\cdot\mathbf{x}} = a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}}, \quad e^{-i\mathbf{P}\cdot\mathbf{x}}a^\dagger_{\mathbf{p}}e^{i\mathbf{P}\cdot\mathbf{x}} = a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}},$$

и, следовательно,

$$\phi(x) = e^{i(Ht-\mathbf{P}\cdot\mathbf{x})}\phi(0)e^{-i(Ht-\mathbf{P}\cdot\mathbf{x})} = e^{iP\cdot x}\phi(0)e^{-iP\cdot x},$$

где $P^\mu = (H, \mathbf{P})$. (Обозначения здесь могут запутать, но они стандартны. Напомним, что $\mathbf{P}$ - оператор импульса, собственное значение которого является полным импульсом системы. С другой стороны, p - импульс одной фурье-моды поля, который мы интерпретируем как импульс частицы в этой моде. Для одночастичного состояния с определенным импульсом величина $\mathbf{p}$ является собственным значением оператора P.)

Уравнение (2.47) указывает на корпускулярно-волновой дуализм в интерпретации квантованного поля $\phi(x)$. С одной стороны, поле $\phi(x)$ представлено как оператор в гильбертовом пространстве, который рождает и уничтожает частицы, являющиеся квантами возбуждения этого поля. С другой стороны, поле $\phi(x)$ записывается как линейная комбинация решений ($e^{ip\cdot x}$ и $e^{-ip\cdot x}$) уравнения Клейна-Гордона. В экспоненте появляются оба знака для временнОй зависимости: присутствуют как $e^{-ip^0 t}$, так и $e^{+ip^0 t}$, причем p^0 всегда положителен. Для одночастичных волновых функций это соответствовало бы состояниям с положительной и отрицательной энергиями; далее будем называть эти состояния положительно- и отрицательно-частотными модами. Такая связь между операторами рождения частиц и видом рассматриваемых волновых функций всегда имеет место для свободных квантовых полей: положительно-частотное решение уравнения поля имеет в качестве коэффициента оператор уничтожения частицы в состоянии с соответствующей одночастичной волновой функцией. Отрицательно-частотное решение уравнения поля, являющееся эрмитово сопряженным к положительно-частотному решению, имеет коэффициентом оператор рождения частицы в состоянии с соответствующей одночастичной волновой функцией с положительной энергией. Таким образом, тот факт, что релятивистские волновые уравнения имеют и положительно-, и отрицательно-частотные решения, согласуется с требованием, чтобы разумная квантовая теория содержала только положительные энергии возбуждения.

9.1.3 Causality for Scalar Field

Обсудим подробно причинность.
(это очень подробно нужно написать!!!!)

Теория

Теперь вернемся к вопросу о причинности, поднятому в начале этой главы. В рамках рассматриваемого формализма и в представлении Гейзенберга амплитуда распространения частицы от точки y до точки x равна $\langle 0|\phi(x)\phi(y)|0\rangle$. Обозначим эту величину $D(x-y)$. Каждый оператор ϕ есть сумма операторов уничтожения a и рождения $a^\dagger$, но в выражении для этой амплитуды выживает только слагаемое $\left\langle 0 \left| a_{\mathbf{p}} a_q^\dagger \right| 0 \right\rangle = (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q})$. Легко проверить, что

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)}$$

Мы уже показали в (2.40), что интегралы такого вида лоренц-инвариантны. Теперь вычислим этот интеграл для конкретных значений $x-y$.

Сначала рассмотрим случай, когда отлична от нуля только временная компонента разности $x-y : x^0 - y^0 = t, \mathbf{x}-\mathbf{y} = 0$. (Если интервал от y до x времениподобен, то всегда найдется система отсчета, в которой он чисто временной.) Тогда имеем:

$$D(x-y) = \frac{4\pi}{(2\pi)^3}\int\limits_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}t} =$$
$$= \frac{1}{4\pi^2}\int\limits_m^\infty dE\sqrt{E^2-m^2}e^{-iEt} \underset{t\to\infty}{\sim} e^{-imt}$$

Далее рассмотрим случай, когда разность $x-y$ является чисто пространственной: $x^0-y^0 = 0, \mathbf{x}-\mathbf{y} = \mathbf{r}$. Амплитуда тогда имеет вид:

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}} e^{i\mathbf{p}\cdot\mathbf{r}} =$$
$$= \frac{2\pi}{(2\pi)^3}\int\limits_0^\infty dp \frac{p^2}{2E_{\mathbf{p}}}\frac{e^{ipr}-e^{-ipr}}{ipr} =$$
$$= \frac{-i}{2(2\pi)^2 r}\int\limits_{-\infty}^\infty dp \frac{pe^{ipr}}{\sqrt{p^2+m^2}}$$

Подынтегральное выражение, рассматриваемое как функция комплексной переменной p, имеет разрезы вдоль мнимой оси, начинающиеся в точках $\pm im$ (рис. 2.3). Для вычисления интеграла деформируем контур интегрирования в верхнюю полуплоскость так, чтобы он обошел верхний разрез. Полагая $\rho = -ip$, получаем

$$\frac{1}{4\pi^2 r}\int\limits_m^\infty d\rho \frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}} \underset{r\to\infty}{\sim} e^{-mr}.$$

Таким образом, оказывается, что вне светового конуса амплитуда распространения экспоненциально убывает, тем не менее, она остается отличной от нуля.

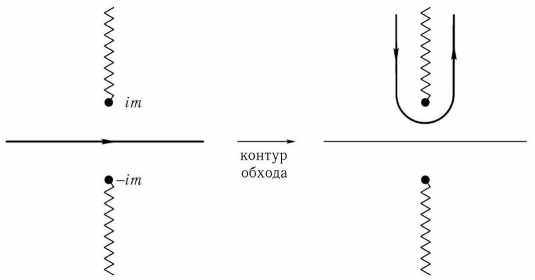


Рис. 2.3. Контур для вычисления амплитуды распространения $D(x-y)$ по пространственноподобному интервалу.

Однако для действительного понимания принципа причинности следует рассмотреть не столько вопрос о том, могут ли распространяться частицы на пространственноподобные расстояния, сколько вопрос о возможности воздействия измерения, проведенного в одной точке, на измерение, проводимое в другой точке, если эти точки разделены пространственноподобным интервалом. Простейшее, что можно попытаться измерить, - это поле $\phi(x)$. Для этого следует вычислить коммутатор $[\phi(x), \phi(y)]$; если этот коммутатор равен нулю, то одно измерение не может повлиять на другое. На самом деле, если

коммутатор равен нулю для $(x-y)^2 < 0$, то принцип причинности имеет место в общем случае, так как коммутаторы от любых функций $\phi(x)$, включая $\pi(x) = \partial\phi/\partial t$, также будут нулевыми. Из выражения (2.20) мы знаем, что этот коммутатор равен нулю при $x^0 = y^0$; теперь проведем более общее вычисление:

$$[\phi(x), \phi(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{q}}}} \left[\left(a_{\mathbf{p}} e^{-ip\cdot x} + a^\dagger_{\mathbf{p}} e^{ip\cdot x} \right), \left(a_{\mathbf{q}} e^{-iq\cdot y} + a^\dagger_{\mathbf{q}} e^{iq\cdot y} \right) \right] =$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) = D(x-y) - D(y-x).$$

Если $(x-y)^2 < 0$, то можно совершить преобразование Лоренца над вторым слагаемым (так как каждый член в отдельности лоренц-инвариантен), переводящее $(x-y) \to -(x-y)$ (рис. 2.4).

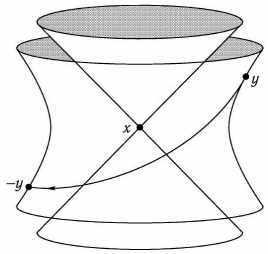


Рис. 2.4. Когда разность $x-y$ пространственноподобна, существует непрерывное преобразование Лоренца, переводящее $(x-y)$ в $-(x-y)$. (????? да почему????)

Таким образом, два слагаемых оказываются равными и сокращаются. Тем самым причинность сохраняется. Обратим внимание, что, если $(x-y)^2 > 0$, то не существует непрерывного преобразования Лоренца, которое переводит $(x-y) \to -(x-y)$. В этом случае согласно (2.51) амплитуда (к счастью) отлична от нуля, в частности, когда $\mathbf{x}-\mathbf{y}=0$, она ведет себя как $(e^{-imt} - e^{imt})$. Таким образом, мы заключаем, что никакое измерение в теории Клейна-Гордона не может повлиять на другое измерение вне светового конуса.

Как мы и предполагали в конце раздела 2.1, в теории Клейна-Гордона причинность имеет место. Чтобы лучше понять этот механизм, следует расширить рамки нашего обсуждения, включив в рассмотрение комплексное поле Клейна-Гордона, имеющее различающиеся возбуждения, отвечающие частицам и античастицам. Как было упомянуто при обсуждении (2.15), если поле $\phi(x)$ является не вещественным, а комплексным, в теории Клейна-Гордона существует сохраняющийся заряд. При квантовании комплексной скалярной теории поля (см. задачу 2.2) поле $\phi(x)$ рождает положительно заряженные и уничтожает отрицательно заряженные частицы, в то время как $\phi^\dagger(x)$ производит противоположные операции. Тогда коммутатор $\left[\phi(x), \phi^\dagger(y)\right]$ будет иметь отличные от нуля составляющие, которые для сохранения причинности должны точно сокращаться вне светового конуса. Два вклада с приписанными зарядами отвечают двум слагаемым в (2.53). Первое слагаемое представляет амплитуду распространения отрицательно заряженной частицы из точки y в точку x. Второе слагаемое представляет амплитуду распространения положительно заряженной частицы из точки x в точку y. Для того чтобы оба упомянутых процесса имели место и их амплитуды взаимно сокращались, обе эти частицы должны существовать в теории и иметь одинаковые массы. Таким образом, принцип причинности в квантовой теории поля требует, чтобы каждая частица имела соответствующую античастицу той же самой массы и с противоположными квантовыми числами (в данном

случае - электрическим зарядом). Для вещественного поля Клейна-Гордона частица является своей собственной античастицей.

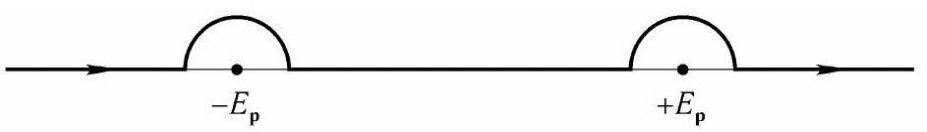
9.1.4 The Klein-Gordon Propagator

Теория

Изучим коммутатор $[\phi(x), \phi(y])$. Так как он является с-числом, можно записать: $[\phi(x), \phi(y)] = \langle 0|[\phi(x), \phi(y)]|0\rangle$. Предполагая, что $x^0 > y^0$, можно переписать это выражение в виде четырехмерного интеграла:

$$\begin{aligned}\langle 0|[\phi(x), \phi(y)]|0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) = \\ &= \int \frac{d^3p}{(2\pi)^3}\left\{\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}\Big|_{p^0=E_{\mathbf{p}}} + \frac{1}{-2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}\Big|_{p^0=-E_{\mathbf{p}}}\right\} x^0 > y^0 \\ &= {}_{x^0>y^0}\int \frac{d^3p}{(2\pi)^3}\int \frac{dp^0}{2\pi i}\frac{-1}{p^2-m^2}e^{-ip\cdot(x-y)}.\end{aligned}$$

В последнем интеграле интегрирование по p^0 выполняется вдоль контура:



Для $x^0 > y^0$ можно замкнуть контур в нижней полуплоскости, обходя оба полюса сверху, в результате чего получим предпоследнюю строку в формуле (2.54). Для $x^0 < y^0$ мы можем замкнуть контур в верхней полуплоскости, получив нуль. Таким образом, интеграл в последней строке (2.54) вместе с правилом обхода полюсов является чис- ленным выражением величины

$$D_R(x-y) \equiv \theta\left(x^0 - y^0\right)\langle 0|[\phi(x), \phi(y)]|0\rangle.$$

Чтобы лучше понять смысл этой величины, проделаем еще одно вычисление:

$$\begin{aligned}\left(\partial^2 + m^2\right)D_R(x-y) = &\left(\partial^2\theta\left(x^0-y^0\right)\right)\langle 0|[\phi(x),\phi(y)]|0\rangle + \\ &+ 2\left(\partial_\mu\theta\left(x^0-y^0\right)\right)\left(\partial^\mu\langle 0|[\phi(x),\phi(y)]|0\rangle\right) + \\ &+ \theta\left(x^0-y^0\right)\left(\partial^2+m^2\right)\langle 0|[\phi(x),\phi(y)]|0\rangle = \\ =&\delta\left(x^0-y^0\right)\langle 0|[\pi(x),\phi(y)]|0\rangle + \\ &+ 2\delta\left(x^0-y^0\right)\langle 0|[\pi(x),\phi(y)]|0\rangle + 0 = \\ =&-i\delta^{(4)}(x-y).\end{aligned}$$

Отсюда следует, что $D_R(x-y)$ является функцией Грина оператора Клейна-Гордона. Так как эта функция обращается в нуль для $x^0 < y^0$, то это запаздывающая функция Грина.

Если бы мы еще не получили выражение (2.54), его можно было бы получить с помощью преобразования Фурье. Записывая

$$D_R(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \widetilde{D}_R(p),$$

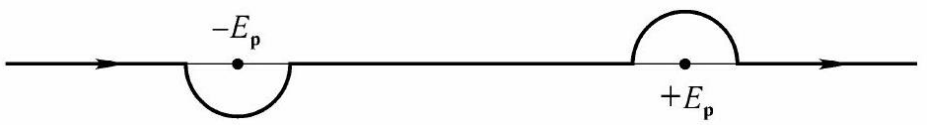
мы получаем алгебраическое выражение для $\widetilde{D}_R(p)$:

$$\left(-p^2 + m^2\right) \widetilde{D}_R(p) = -i.$$

Таким образом, сразу приходим к следующему результату:

$$D_R(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}$$

Интеграл по p^0 в (2.58) может быть вычислен по четырем различным контурам, из которых в (2.54) использовался только один. В гл. 4 мы найдем, что чрезвычайно полезным является другое правило обхода полюсов:



Это правило называется правилом Фейнмана. Чтобы запомнить его, удобно записать интеграл для функции Грина в виде:

$$D_F(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot(x-y)}$$

так как полюса теперь находятся в точках $p^0 = \pm\left(E_{\mathbf{p}} - i\varepsilon\right)$, смещенных соответственно ниже и выше вещественной оси. Когда $x^0 > y^0$, можно брать интеграл по p^0, замыкая контур интегрирования вниз, в результате получая в точности амплитуду распространения $D(x-y)$(2.50). Если $x^0 < y^0$, то контур интегрирования замыкается вверху, и в результате получается то же самое выражение, но с переставленными x и y. Таким образом,

$$\begin{aligned} D_F(x-y) &= \begin{cases} D(x-y) & \text{для } x^0 > y^0 \\ D(y-x) & \text{для } x^0 < y^0 \end{cases} \\ &= \theta\left(x^0 - y^0\right) \langle 0|\phi(x)\phi(y)|0\rangle + \theta\left(y^0 - x^0\right) \langle 0|\phi(y)\phi(x)|0\rangle \equiv \\ &\equiv \langle 0|T\phi(x)\phi(y)|0\rangle. \end{aligned}$$

Последняя строка определяет символ «хронологического упорядочивания» T, который указывает на то, что операторы, зависящие от более поздних моментов времени, располагаются левее тех, которые зависят от более ранних моментов времени. Применяя к последней строке оператор $(\partial^2 + m^2)$, можно непосредственно проверить, что D_F есть функция Грина оператора Клейна-Гордона.

С практической точки зрения выражения (2.59) и (2.60) являются наиболее важными результатами этой главы. Функция Грина $D_F(x-y)$ называется пропагатором Фейнмана для частицы Клейна-Гордона, так как она в конечном счете является амплитудой распространения. Действительно, фейнмановский пропагатор входит в правила Фейнмана: $D_F(x-y)$ (или $\widetilde{D}_F(p)\Big)$ является тем выражением, которое приписывается внутренним линиям диаграмм Фейнмана, представляя распространение виртуальных частиц.

Однако нам предстоит еще длинный путь, прежде чем мы сможем произвести какие-либо реальные вычисления, так как до сих пор мы работали только со свободной теорией Клейна-Гордона, в которой уравнение поля является линейным и взаимодействие отсутствует. Отдельные частицы живут в своих изолированных модах, не замечая существования друг друга и любых других сортов частиц. В рамках такой теории невозможно проделать какие-либо наблюдения при помощи рассеяния или любыми другими средствами. С другой стороны, разработанный формализм чрезвычайно важен, так как свободная теория формирует основу для выполнения вычислений по теории возмущений в теории с взаимодействием.

9.1.5 Production of particles by a classical source

Теория

Имеется один тип взаимодействия, к рассмотрению которого мы уже готовы. Рассмотрим поле Клейна-Гордона, связанное с внешним классическим полем-источником $j(x)$. Иными словами, рассмотрим уравнение поля:

$$\left(\partial^2 + m^2\right)\phi(x) = j(x),$$

где $j(x)$ - некоторая фиксированная известная функция пространственно-временных координат, которая отлична от нуля лишь на конечном временнОм интервале. Если в далеком прошлом система находится в вакуумном состоянии, что мы будем иметь после того, как поле $j(x)$ включается и затем вновь отключается?

Уравнение поля (2.61) вытекает из лагранжиана

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m^2\phi^2 + j(x)\phi(x).$$

Но если $j(x)$ включается только на конечное время, то проще всего решить поставленную задачу, непосредственно используя полевое уравнение. До того, как источник $j(x)$ включен, поле $\phi(x)$ имеет вид

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)$$

При отсутствии источника это поле было бы решением для всех моментов времени. При наличии источника решение уравнения движения может быть построено с помощью запаздывающей функции Грина:

$$\begin{aligned}\phi(x) &= \phi_0(x) + i\int d^4y D_R(x-y)j(y) = \\ &= \phi_0(x) + i\int d^4y \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\theta\left(x^0-y^0\right)\left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right)j(y)\end{aligned}$$

Если для наблюдателя моменты включения и выключения поля j находятся в прошлом, то тета-функция равняется единице во всей области интегрирования. Тогда поле $\phi(x)$ содержит лишь фурье-компоненты поля j:

$$\tilde{j}(p) = \int d^4y e^{ip\cdot y}j(y)$$

вычисленные при таких 4-импульсах p, что $p^2 = m^2$. Естественно сгруппировать положительно-частотные слагаемые вместе с $a_{\mathbf{p}}$ и отрицательно-частотные слагаемые вместе с $a_{\mathbf{p}}^\dagger$; в результате получим выражение:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left\{ \left(a_{\mathbf{p}} + \frac{i}{\sqrt{2E_{\mathbf{p}}}} \tilde{\jmath}(p) \right) e^{-ip\cdot x} + \text{ э.с.} \right\}$$

Теперь можно догадаться (или вычислить), какую форму будет иметь гамильтониан после действия $j(x)$. Чтобы получить ответ, остается лишь сделать замену $a_{\mathbf{p}}$ на $\left(a_{\mathbf{p}} + i\tilde{\jmath}(p)/\sqrt{2E_{\mathbf{p}}}\right)$:

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger - \frac{i}{\sqrt{2E_{\mathbf{p}}}} \tilde{\jmath}^*(p) \right) \left(a_{\mathbf{p}} + \frac{i}{\sqrt{2E_{\mathbf{p}}}} \tilde{\jmath}(p) \right)$$

Энергия системы после выключения источника равна

$$\langle 0|H|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} |\tilde{\jmath}(p)|^2$$

где $|0\rangle$ по-прежнему обозначает основное состояние свободной теории. Мы можем интерпретировать эти результаты в терминах частиц, рассматривая $|\tilde{\jmath}(p)|^2/2E_{\mathbf{p}}$ как плотность вероятности рождения частицы в моде p. Тогда общее число рожденных частиц равно

$$\int dN = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} |\tilde{\jmath}(p)|^2$$

При рождении частиц существенны лишь те фурье-компоненты источника $j(x)$, которые находятся в резонансе с волнами Клейна-Гордона на «массовой поверхности» (т. е. $p^2 = m^2$).

Мы вернемся к этому вопросу в задаче 4.1. В гл. 6 будет рассмотрена аналогичная задача о рождении фотонов ускоренным электроном (тормозное излучение).

9.1.6 Examples of Typical Transformations of a Scalar Field

ПШ-2.1. Об электродинамике

Классическая теория электромагнетизма (без источников) следует из выражения для действия

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad \text{где \%} \quad \%F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

(a) Получите уравнения Максвелла как уравнения Эйлера-Лагранжа для этого действия, трактуя компоненты $A_\mu(x)$ как динамические переменные. Запишите уравнения в стандартной форме, отождествляя $E^i = -F^{0i}$ и $\varepsilon^{ijk} B^k = -F^{ij}$.

(b) Постройте тензор энергии-импульса в этой теории. Обратите внимание, что обычная процедура приводит к несимметричному тензору. Чтобы исправить это, можно добавить к $T^{\mu\nu}$ слагаемое вида $\partial_\lambda K^{\lambda\mu\nu}$, где $K^{\lambda\mu\nu}$ антисимметричен по первым двум индексам. Такая добавка автоматически имеет нулевую дивергенцию, так что тензор

$$\widehat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

равносилен исходному тензору энергии-импульса с теми же самыми полными сохраняющимися энергией и импульсом. Покажите, что такое построение, где

$$K^{\lambda\mu\nu} = F^{\mu\lambda}A^{\nu},$$

приводит к симметричному тензору энергии-импульса $\widehat{T}$ и стандартным формулам для плотностей электромагнитной энергии и импульса:

$$\mathscr{E} = \frac{1}{2}\left(E^2 + B^2\right); \quad \mathbf{S} = \mathbf{E} \times \mathbf{B}.$$

ПШ-2.2. Комплексное скалярное поле.

Рассмотрите теорию комплексного скалярного поля, удовлетворяющего уравнению Клейна-Гордона. Действие этой теории имеет вид

$$S = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi\right).$$

Проще изучать эту теорию, рассматривая в качестве базисных динамических переменных $\phi(x)$ и $\phi^*(x)$, а не вещественную и мнимую части поля $\phi(x)$.

(a) Найдите сопряженные импульсы к $\phi(x)$ и $\phi^*(x)$ и канонические коммутационные соотношения. Покажите, что гамильтониан равен

$$H = \int d^3x \left(\pi^*\pi + \nabla\phi^* \cdot \nabla\phi + m^2\phi^*\phi\right).$$

Запишите гейзенберговское уравнение движения для $\phi(x)$ и покажите, что оно совпадает с уравнением Клейна-Гордона.

(b) Диагонализируйте гамильтониан H, вводя операторы рождения и уничтожения. Покажите, что теория содержит два типа частиц с массой m.

(c) Перепишите сохраняющийся заряд

$$Q = \int d^3x \frac{i}{2}\left(\phi^*\pi^* - \pi\phi\right)$$

в терминах операторов рождения и уничтожения и вычислите заряд частиц каждого типа. (d) Рассмотрите случай двух комплексных полей Клейна-Гордона с одинаковой массой. Обозначьте поля как $\phi_a(x)$, где $a = 1, 2$. Покажите, что теперь имеются четыре сохраняющихся заряда: один, задаваемый путем обобщения заряда, рассмотренного в (c), и три других, задаваемых формулами

$$Q^i = \int d^3x \frac{i}{2}\left(\phi_a^* \left(\sigma^i\right)_{ab} \pi_b^* - \pi_a \left(\sigma^i\right)_{ab} \phi_b\right)$$

где σ^i - матрицы Паули. Покажите, что последние три заряда имеют коммутационные соотношения, совпадающие с коммутаторами для углового момента ($SU(2)$). Обобщите эти результаты на случай n одинаковых комплексных скалярных полей.

ПШ-2.3.

Вычислите явно через функции Бесселя функцию

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)}$$

для случая пространственноподобной разности $(x-y)$, причем $(x-y)^2 = -r^2$.

9.2 Vector Field (photons)

9.2.1 Derivation of Vector Field Equations

9.2.2 Quantization of a Vector Field: Basics

Теория

We now apply the canonical formalism of Chapter 2 to quantize the free electromagnetic field, using the Lagrangian density (5.10) with $s_\mu(x) = 0$ and, in the first place, ignoring the Lorentz condition (5.13). With the fields $\pi^\mu(x)$ conjugate to $A_\mu(x)$ given by Eqs. (5.11), the equal-time commutation relations (2.31) become

$$\begin{aligned}
&\left[A^\mu(\mathbf{x},t), A^v(\mathbf{x}',t)\right] = 0,\\
&\left[\dot{A}^\mu(\mathbf{x},t), \dot{A}^v(\mathbf{x}',t)\right] = 0\\
&\left[A^\mu(\mathbf{x},t), \dot{A}^v(\mathbf{x}',t)\right] = -\mathrm{i}\hbar c^2 g^{\mu v}\delta(\mathbf{x}-\mathbf{x}')
\end{aligned}$$

Apart from the factor $(-g^{\mu\nu})$, these equations are identical with the commutation relations (3.6) of four independent Klein-Gordon fields, and each component $A^\mu(x)$ satisfies the wave equation (5.16) which is the limit of the Klein-Gordon equation (3.3) for particles of mass zero. [Both these points can be appreciated by comparing the Lagrangian densities (5.10) and (3.4).] This similarity enables us to take over earlier mathematical results, although their physical interpretation will have to be re-examined, taking into account the factor $(-g^{\mu\nu})$.

In Section 3.3, we derived the covariant commutation relations (3.42) for the Klein-Gordon field. From these we can at once write down the covariant commutation relations for the $A^\mu(x)$:

$$\left[A^\mu(x), A^v(x')\right] = \mathrm{i}\hbar c D^{\mu v}(x-x')$$

where

$$D^{\mu v}(x) = \lim_{m\to 0}\left[-g^{\mu v}\Delta(x)\right],$$

and $\Delta(x)$ is the invariant Δ-function (3.43). The Feynman photon propagator is similarly given by

$$\langle 0\,|\mathbf{T}\left\{A^\mu(x)A^v(x')\right\}|\,0\rangle = \mathrm{i}\hbar c D_{\mathrm{F}}^{\mu v}(x-x'),$$

where

$$D_{\mathrm{F}}^{\mu v}(x) = \lim_{m\to 0}\left[-g^{\mu v}\Delta_{\mathrm{F}}(x)\right] = \frac{-g^{\mu v}}{(2\pi)^4}\int \frac{\mathrm{d}^4k\mathrm{e}^{-\mathrm{i}kx}}{k^2+i\varepsilon},$$

as is seen from Eqs. (3.55) and (3.58). The photon propagator will be discussed fully in the next section.

To gain the photon interpretation of the quantized fields, we substitute the field expansions (5.16) in the commutation relations (5.23), with the result

$$\left.\begin{array}{c}\left[a_r(\mathbf{k}), a_s^\dagger(\mathbf{k}')\right] = \zeta_r \delta_{rs}\delta_{\mathbf{k}\mathbf{k}'} \\ \left[a_r(\mathbf{k}), a_s(\mathbf{k}')\right] = \left[a_r^\dagger(\mathbf{k}), a_s^\dagger(\mathbf{k}')\right] = 0\end{array}\right\}.$$

From Eq. (5.20) $\zeta_r = 1$ for $r = 1, 2, 3$ so that for these values of r, Eqs. (5.28) are the standard boson commutation relations (3.9) leading to the usual number representation for transverse photons ($r = 1, 2$) and longitudinal photons ($r = 3$). For $r = 0$ (scalar photons) $\zeta_0 = -1$, and it consequently looks as though the usual roles of absorption and creation operators must be interchanged for $a_0(\mathbf{k})$ and $a_0^\dagger(\mathbf{k})$. However, effecting only this change results in other difficulties, and the standard formalism must be modified more radically. Of the several procedures available, we shall follow that due to Gupta and to Bleuler. In the Gupta-Bleuler theory, the operators $a_r(\mathbf{k}), \mathrm{r} = 1, 2, 3$ and 0, are interpreted as absorption operators, $a_r^\dagger(\mathbf{k}), r = 1, 2, 3$ and 0, as creation operators for transverse, longitudinal and scalar photons. The vacuum state $|0\rangle$ is defined as the state in which there are no photons of any kind present, i.e.

$$a_r(\mathbf{k})|0\rangle = 0, \quad \text{all } \mathbf{k}, \quad r = 0..., 3,$$

 or, equivalently,

$$A^{\mu+}(x)|0\rangle = 0, \quad \text{all } x, \quad \mu = 0, ..., 3.$$

The operators $a_r^\dagger(\mathbf{k})$ operating on the vacuum state $|0\rangle$ create the one-photon states

$$|1_{\mathbf{k}r}\rangle = a_r^\dagger(\mathbf{k})|0\rangle$$

in which one transverse ($r = 1, 2$), longitudinal ($r = 3$) or scalar ($r = 0$) photon of momentum $\mathbf{k}$ is present.

To justify this interpretation of the operators a_r and $a_r^\dagger$, we consider the Hamiltonian operator of the field. From Eq. (2.51a) this is given by

$$H = \int \mathrm{d}^3\mathbf{x}\mathbf{N}\left[\pi^\mu(x)\dot{A}_\mu(x) - \mathscr{L}(x)\right],$$

which, as usual, is to be taken as a normal product. On substituting the free Lagrangian density corresponding to Eq. (5.10), Eq, (5.11) for $\pi^\mu(x)$ and the expansions (5.16) for the fields, Eq. (5.31) becomes

$$H = \sum_{r\mathbf{k}} \hbar\omega_{\mathbf{k}}\zeta_r a_r^\dagger(\mathbf{k})a_r(\mathbf{k}).$$

Despite the minus sign ($\zeta_0 = -1$) associated with the scalar photons in Eq. (5.32), this energy is positive definite. For example, for the one-photon states (5.30) one easily obtains, using the commutation relations (5.28),

$$\begin{aligned} H\,|1_{\mathbf{k}r}\rangle &= \sum_{\mathbf{q}s} \hbar\omega_{\mathbf{q}}\zeta_s a_s^\dagger(\mathbf{q})a_s(\mathbf{q})a_r^\dagger(\mathbf{k})|0\rangle \\ &= \hbar\omega_{\mathbf{k}} a_r^\dagger(\mathbf{k})|0\rangle, \quad r = 0, ..., 3, \end{aligned}$$

i.e. the energy has the positive value $\hbar\omega_{\mathbf{k}}$ for transverse, longitudinal and scalar photons. Correspondingly, we must define the number operators by

$$N_r(\mathbf{k}) = \zeta_r a_r^\dagger(\mathbf{k})a_r(\mathbf{k}),$$

and these definitions, together with the commutation relations (5.28), lead to consistent number representations for all types of photons.

Although the formalism, as far as we have developed it, seems satisfactory, there are some difficulties which show up if we calculate the normalization of photon states. For example, the norm of the state (5.30) is

$$\langle 1_{\mathrm{k}r}|1_{\mathrm{k}r}\rangle = \left\langle 0 \left| a_r(\mathbf{k})a_r^\dagger(\mathbf{k}) \right| 0 \right\rangle = \zeta_r\langle 0|0\rangle = \zeta_r$$

(if we normalize $|0\rangle$ to $\langle 0|0\rangle = 1$), and for a scalar photon this norm is negative. More generally, one can show that for any state containing an odd number of scalar photons the norm is negative. At first sight this looks like a serious difficulty, since the probability interpretation of quantum mechanics depends on states having positive norms. However, no scalar or longitudinal photons have ever been observed. Both these points are related to the fact that so far we have ignored the Lorentz condition (5.13), so that our theory is not yet equivalent to Maxwell's equations. We must now try and impose the Lorentz condition. 80 Quantum Field Theory Unfortunately, we cannot simply take the Lorentz condition (5.13) as an operator identity. Eq. (5.13) is incompatible with the commutation relations (5.24), since

$$[\partial_\mu A^\mu(x), A^v\,(x')] = \mathrm{i}\hbar c\partial_\mu D^{\mu v}\,(x - x')$$

and this is not identically zero. This problem was resolved by Gupta and Bleuler by replacing the Lorentz condition (5.13) by the weaker condition

$$\partial_\mu A^{\mu+}(x)|\Psi\rangle = 0,$$

involving absorption operators only. Eq. (5.34) is a restriction on the states which are allowed by the theory. From Eq. (5.34) and its adjoint

$$\langle\Psi|\partial_\mu A^{\mu-}(x) = 0$$

it follows that the Lorentz condition holds for expectation values:

$$\langle\Psi\,|\partial_\mu A^\mu(x)|\,\Psi\rangle = \left\langle\Psi\left|\partial_\mu A^{\mu+}(x) + \partial_\mu A^{\mu-}(x)\right|\,\Psi\right\rangle = 0.$$

This ensures that the Lorentz condition and hence Maxwell's equations hold as the classical limit of this theory.

In order to understand the meaning of the subsidiary condition (5.34), we express it in momentum space. On substituting Eqs. (5.16b) and (5.21), (5.22) for $A_\mu^+(x)$ and $\varepsilon_r^\mu(\mathbf{k})$, we obtain the conditions

$$[a_3(\mathbf{k}) - a_0(\mathbf{k})]\,|\Psi\rangle = 0, \qquad \text{all } \mathbf{k}.$$

This is a constraint on the linear combinations of longitudinal and scalar photons, for each value of $\mathbf{k}$, that may be present in a state. It places no restriction on the transverse photons that may be present.

The effect of the subsidiary condition (5.36) becomes apparent if we calculate the expectation value of the energy of an allowed state $|\Psi\rangle$. Since from Eq. (5.36) and its adjoint we have

$$\left\langle\Psi\left|a_3^\dagger(\mathbf{k})a_3(\mathbf{k}) - a_0^\dagger(\mathbf{k})a_0(\mathbf{k})\right|\,\Psi\right\rangle = \left\langle\Psi\left|a_3^\dagger(\mathbf{k})\,[a_3(\mathbf{k}) - a_0(\mathbf{k})]\right|\,\Psi\right\rangle = 0,$$

it follows from Eq. (5.32) that

$$\langle\Psi|H|\Psi\rangle = \left\langle \Psi \left| \sum_{\mathbf{k}} \sum_{r=1}^{2} \hbar\omega_k a_r^\dagger(\mathbf{k}) a_r(\mathbf{k}) \right| \Psi \right\rangle$$

i.e. only the transverse photons contribute to the expectation value of the energy as a consequence of the subsidiary condition. The same is true for all other observables.

Thus, as a result of the subsidiary condition, in free space, observable quantities will involve transverse photons only. This explains our earlier assertion that longitudinal and scalar photons are not observed as free particles. Only transverse photons are so observed, corresponding to the two degrees of freedom (for each $\mathbf{k}$) of the radiation field, which we found in the non-covariant formalism of Chapter 1 where we worked in the Coulomb gauge. In the covariant treatment, although they don't show up as free particles, the presence of longitudinal and scalar photons is not ruled out Photons: Covariant Theory 81 altogether. Of the resulting additional two degrees of freedom (for each $\mathbf{k}$), one is removed by the subsidiary condition (5.36). The other can be shown to correspond to the arbitrariness in choice of Lorentz gauge. More specifically, one can show that altering the allowed admixtures of longitudinal and scalar photons is equivalent to a gauge transformation between two potentials, both of which are in Lorentz gauges. (See Problems 5.2 and 5.3.)

For free fields (i.e. no charges present), it is then simplest to work in a gauge such that the vacuum is represented by the state $|0\rangle$ in which no photons of any kind are present [see Eq. (5.29a)]. But the vacuum could also be described by any state containing no transverse and only allowed admixtures of scalar and longitudinal photons. This description would merely correspond to a different choice of Lorentz gauge. The situation is entirely analogous for states containing transverse photons.

For the electromagnetic field in the presence of charges, the situation is more complicated. We can no longer ignore the longitudinal and scalar photons. When discussing the photon propagator in the next section, we shall see that longitudinal and scalar photons play an important role as virtual particles in intermediate states and provide a covariant description of the instantaneous Coulomb interaction of Chapter 1. However, in this case too, one need consider only transverse photons in initial and final states of scattering processes. This corresponds to a particular choice of gauge and the fact that one can consider particles initially and finally, when they are far apart, as free. In Section 6.2 we shall return to this idea of switching the interaction between colliding particles on and off adiabatically as they approach and as they move apart.

We have developed the Gupta-Bleuler formalism only to the limited extent to which it is needed in applications. It is possible to develop a more complete systematic formalism in which states with negative norm do not appear as a blemish in Hilbert space, but occur in a self-consistent manner in a function space with an indefinite metric. For most purposes, this complete formalism is not required.

Другое о квантовании векторного поля

> **Вот допустим у меня другое поле, искрилвенное, например, поле бублика, проквантуйте мне его! (!??!?!??!?)**

(и все, усели мы!!! я хз, напишу четко этот вопрос, перекину умным людям, я хз, что тут ответить вообще.)

9.2.3 Quantization of a Vector Field: Features

Первое, что приходит в голову после познания метода квантования, это идея проквантовать хорошо знакомое нам электроамгнитное поле.

итоговые методы

обзор квантования и сложностей

подготовимся

Квантование электромагнитного поля по Тонгу

In the following we shall quantize free Maxwell theory twice: once in Coulomb gauge, and again in Lorentz gauge. We'll ultimately get the same answers and, along the way, see that each method comes with its own subtleties.

The first of these subtleties is common to both methods and comes when computing the momentum π^μ conjugate to A_μ

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$$
$$\pi^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = -F^{0i} \equiv E^i$$

so the momentum π^0 conjugate to A_0 vanishes. This is the mathematical consequence of the statement we made above: A_0 is not a dynamical field. Meanwhile, the momentum conjugate to A_i is our old friend, the electric field. We can compute the Hamiltonian,

$$H = \int d^3x \pi^i \dot{A}_i - \mathcal{L} \quad = \int d^3x \frac{1}{2}\vec{E}\cdot\vec{E} + \frac{1}{2}\vec{B}\cdot\vec{B} - A_0(\nabla\cdot\vec{E})$$

So A_0 acts as a Lagrange multiplier which imposes Gauss' law

$$\nabla\cdot\vec{E} = 0$$

which is now a constraint on the system in which $\vec{A}$ are the physical degrees of freedom. Let's now see how to treat this system using different gauge fixing conditions.

6.2.1 Coulomb Gauge In Coulomb gauge, the equation of motion for $\vec{A}$ is

$$\partial_\mu\partial^\mu\vec{A} = 0$$

which we can solve in the usual way,

$$\vec{A} = \int \frac{d^3p}{(2\pi)^3}\vec{\xi}(\vec{p})e^{ip-x}$$

with $p_0^2 = |\vec{p}|^2$. The constraint $\nabla\cdot\vec{A} = 0$ tells us that $\vec{\xi}$ must satisfy

$$\vec{\xi}\cdot\vec{p} = 0$$

which means that $\vec{\xi}$ is perpendicular to the direction of motion $\vec{p}$. We can pick $\vec{\xi}(\vec{p})$ to be a linear combination of two orthonormal vectors $\vec{\epsilon}_r, r = 1, 2$, each of which satisfies $\vec{\epsilon}_r(\vec{p})\cdot\vec{p} = 0$ and

$$\vec{\epsilon}_r(\vec{p})\cdot\vec{\epsilon}_s(\vec{p}) = \delta_{rs}\% \quad \%r, s = 1, 2$$

These two vectors correspond to the two polarization states of the photon. It's worth pointing out that you can't consistently pick a continuous basis of polarization vectors for every value of $\vec{p}$ because you can't comb the hair on a sphere. But this topological fact doesn't cause any complications in computing QED scattering processes.

To quantize we turn the Poisson brackets into commutators. Naively we would write

$$[A_i(\vec{x}), A_j(\vec{y})] = [E^i(\vec{x}), E^j(\vec{y})] = 0 [A_i(\vec{x}), E^j(\vec{y})] = i\delta_i^j \delta^{(3)}(\vec{x} - \vec{y})$$

But this can't quite be right, because it's not consistent with the constraints. We still want to have $\nabla \cdot \vec{A} = \nabla \cdot \vec{E} = 0$, now imposed on the operators. But from the commutator relations above, we see

$$[\nabla \cdot \vec{A}(\vec{x}), \nabla \cdot \vec{E}(\vec{y})] = i\nabla^2 \delta^{(3)}(\vec{x} - \vec{y}) \neq 0$$

What's going on? In imposing the commutator relations (6.23) we haven't correctly taken into account the constraints. In fact, this is a problem already in the classical theory, where the Poisson bracket structure is already altered [4]. The correct Poisson bracket structure leads to an alteration of the last commutation relation,

$$[A_i(\vec{x}), E_j(\vec{y})] = i \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2} \right) \delta^{(3)}(\vec{x} - \vec{y})$$

To see that this is now consistent with the constraints, we can rewrite the right-hand side of the commutator in momentum space,

$$[A_i(\vec{x}), E_j(\vec{y})] = i \int \frac{d^3p}{(2\pi)^3} \left(\delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2} \right) e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$$

which is now consistent with the constraints, for example

$$[\partial_i A_i(\vec{x}), E_j(\vec{y})] = i \int \frac{d^3p}{(2\pi)^3} \left(\delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2} \right) i p_i e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = 0$$

We now write $\vec{A}$ in the usual mode expansion,

$$\vec{A}(\vec{x}) = \int \tfrac{d^3p}{(2\pi)^3} \tfrac{1}{\sqrt{2|\vec{p}|}} \sum_{r=1}^{2} \vec{\epsilon}_r(\vec{p}) \left[a^r_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a^{r\dagger}_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right]$$
$$\vec{E}(\vec{x}) = \int \tfrac{d^3p}{(2\pi)^3} (-i) \sqrt{\tfrac{|\vec{p}|}{2}} \sum_{r=1}^{2} \vec{\epsilon}_r(\vec{p}) \left[a_{\vec{p}} e^r e^{i\vec{p}\cdot\vec{x}} - a^{r\dagger}_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right]$$

where, as before, the polarization vectors satisfy

$$\vec{\epsilon}_r(\vec{p}) \cdot \vec{p} = 0 \quad \text{and} \quad \%\%\vec{\epsilon}_r(\vec{p}) \cdot \vec{\epsilon}_s(\vec{p}) = \delta_{rs}$$

It is not hard to show that the commutation relations (6.25) are equivalent to the usual commutation relations for the creation and annihilation operators,

$$\left[a^r_{\vec{p}}, a^s_{\vec{q}}\right] = \left[a^{r\dagger}_{\vec{p}}, a^{s\dagger}_{\vec{q}}\right] = 0$$
$$\left[a^r_{\vec{p}}, a^s_{\vec{q}}\right] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q})$$

where, in deriving this, we need the completeness relation for the polarization vectors,

$$\sum_{r=1}^{2} \epsilon^i_r(\vec{p}) \epsilon^j_r(\vec{p}) = \delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2}$$

You can easily check that this equation is true by acting on both sides with a basis of vectors $(\vec{\epsilon}_1(\vec{p}), \vec{\epsilon}_2(\vec{p}), \vec{p})$ We derive the Hamiltonian by substituting (6.28) into (6.17). The last term vanishes in Coulomb gauge. After normal ordering, and playing around with $\vec{\epsilon}_r$ polarization vectors, we get the simple expression

$$H = \int \frac{d^3p}{(2\pi)^3} |\vec{p}| \sum_{r=1}^{2} a_{\vec{p}}^{r\dagger} a_{\vec{p}}^{r}$$

The Coulomb gauge has the advantage that the physical degrees of freedom are manifest.

However, we've lost all semblance of Lorentz invariance.

One place where this manifests itself is in the propagator for the fields $A_i(x)$ (in the Heisenberg picture). In Coulomb gauge the propagator reads

$$D_{ij}^{\rm tr}(x-y) \equiv \langle 0 \left| T A_i(x) A_j(y) \right| 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 + i\epsilon} \left(\delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2} \right) e^{-ip\cdot(x-y)}$$

The *tr* superscript on the propagator refers to the "transverse" part of the photon. When we turn to the interacting theory, we will have to fight to massage this propagator into something a little nicer.

теории, следующие из этой, и схожие концепции

что еще есть?
где используется?
пойму - напишу

9.2.4 The Photon Propagator

The Photon Propagator by Mandl Shawl

In Section 3.4 we interpreted the Klein-Gordon propagator (3.56) as the exchange of a virtual meson in an intermediate state. We now expect a similar interpretation for the photon propagator (5.26) but, corresponding to the four-vector nature of the field $A^\mu(x)$ and the resulting four independent polarization states, we expect the exchange of four kinds of photons, two corresponding to transverse polarization and one each to longitudinal and scalar polarization. This description differs markedly from that of Chapter 1, where only transverse radiation occurred, but no longitudinal or scalar radiation. Instead we had the instantaneous Coulomb interaction between charges. We shall see that these two descriptions are indeed equivalent.

To establish this interpretation in terms of photon exchange we consider the momentum space propagator $D_{\rm F}^{\mu\nu}(k)$, related to the configuration space propagator $D_{\rm F}^{\mu\nu}(x)$ Eq. (5.27), by

$$D_{\rm F}^{\mu\nu}(x) = \frac{1}{(2\pi)^4} \int {\rm d}^4k D_{\rm F}^{\mu\nu}(k) {\rm e}^{-{\rm i}kx}.$$

From Eqs. (5.27) and (5.19) we obtain

$$D_{\rm F}^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2 + {\rm i}\varepsilon} = \frac{1}{k^2 + {\rm i}\varepsilon} \sum_r \zeta_r \varepsilon_r^\mu(\mathbf{k}) \varepsilon_r^v(\mathbf{k}).$$

In order to interpret this expression, we use the special frame of reference in which the polarization vectors $\varepsilon_r^\mu(\mathbf{k})$ are given by Eqs. (5.21) and (5.22). The last equation then becomes:

$$D_{\mathrm{F}}^{\mu\nu}(k) = \frac{1}{k^2 + \mathrm{i}\varepsilon}\left\{\sum_{r=1}^{2}\varepsilon_r^\mu(\mathbf{k})\varepsilon_r^v(\mathbf{k}) + \frac{\left[k^\mu - (kn)n^\mu\right]\left[k^\nu - (kn)n^\nu\right]}{(kn)^2 - k^2} + (-1)n^\mu n^\nu\right\}.$$

This equation exhibits the contributions to the photon propagator from transverse, longitudinal and scalar photons. By analogy with the meson case, we interpret the first term in Eq. (5.40),

$${}_{\mathrm{T}}D_{\mathrm{F}}^{\mu v}(k) \equiv \frac{1}{k^2 + \mathrm{i}\varepsilon}\sum_{r=1}^{2}\varepsilon_r^\mu(\mathbf{k})\varepsilon_r^\nu(\mathbf{k})$$

as the exchange of transverse photons. In the language of Chapter 1, it corresponds to the interaction of charges via the transverse radiation field.

The interpretation of the remaining two terms in Eq. (5.40) follows not from considering longitudinal and scalar photons separately, but from combining them into a term proportional to $n^\mu n^\nu$ plus the remainder. Eq. (5.40) then becomes

$$D_{\mathrm{F}}^{\mu\mathrm{v}}(k) = {}_{\mathrm{T}}D_{\mathrm{F}}^{\mu\mathrm{v}}(k) + {}_{\mathrm{C}}D_{\mathrm{F}}^{\mu\mathrm{v}}(k) + {}_{\mathrm{R}}D_{\mathrm{F}}^{\mu\mathrm{v}}(k),$$

where

$${}_{c}D_{\mathrm{F}}^{\mu\mathrm{v}}(k) \equiv \frac{n^\mu n^\nu}{(kn)^2 - k^2},$$

$${}_{\mathrm{R}}D_{\mathrm{F}}^{\mu\mathrm{v}}(k) \equiv \frac{1}{k^2 + \mathrm{i}\varepsilon}\left[\frac{k^\mu k^\nu - (kn)\left(k^\mu n^\nu + k^v n^\mu\right)}{(kn)^2 - k^2}\right],$$

and it is these linear combinations, both of which involve longitudinal and scalar photons, which allow a simple interpretation.

We first consider Eq. (5.41b) in configuration space. From Eqs. (5.38) and (5.21a) we obtain

$$\begin{aligned}{}_{c}D_{\mathrm{F}}^{\mu v}(x) &= \frac{g^{\mu 0}g^{v0}}{(2\pi)^4}\int\frac{\mathrm{d}^3\mathbf{k}\mathrm{e}^{\mathrm{ik}\cdot\mathrm{x}}}{|\mathbf{k}|^2}\int\mathrm{d}k^0\mathrm{e}^{-\mathrm{i}k^0x^0}\\ &= g^{\mu 0}g^{\mathrm{v}0}\frac{1}{4\pi|\mathbf{x}|}\delta\left(x^0\right).\end{aligned}$$

This expression has the time dependence $[\delta(x^0)]$ and the space dependence $[1/|\mathbf{x}|]$ characteristic of an instantaneous Coulomb potential. Thus we see that the exchange of longitudinal and scalar photons, represented by the term (5.43), corresponds to the instantaneous Coulomb interaction between charges. In Chapter 1, we quantized the transverse radiation field only and treated the instantaneous Coulomb interaction as a classical potential, corresponding to the fact that the instantaneous Coulomb field does not represent independent dynamical degrees of freedom, but is fully determined by the charges. In the present treatment, the longitudinal and scalar field components are also quantized and the instantaneous Coulomb interaction emerges as an exchange of longitudinal and scalar photons.

Finally, we must discuss the remainder term (5.41c). In Chapter 1, the complete electromagnetic interaction between charges was represented in terms of the interactions via the transverse radiation field and the instantaneous Coulomb fields. Both these have been accounted for in the present treatment, and for the two treatments to be equivalent, the contribution of the remainder

term (5.41c) to all observable quantities must vanish. This is indeed the case, the basic reason being that the electromagnetic field only interacts with the conserved charge-current density $s^\mu(x)$, Eqs. (5.2) and (5.4). We shall illustrate this for a simple example.

We shall see in Section 7.1, Eq. (7.14), that the scattering of charges by each other is, in lowest order of perturbation theory, given by the matrix element of the operator

$$\int \mathrm{d}^4x \int \mathrm{d}^4y s_1^\mu(x) D_{\mathrm{F}\mu v}(x-y) s_2^v(y).$$

Here $s_1^\mu(x)$ and $s_2^\nu(y)$ are the two interacting charge-current densities. It is clear from Eq. (5.43) that the contribution of $\mathrm{C_C}D_{\mathrm{F}\nu}(x-y)$ to (5.44) corresponds to the instantaneous Coulomb interaction between the charge densities $\rho_1(\mathbf{x}, x^0) = s_1^0(\mathbf{x}, x^0)/c$ and $\rho_2(\mathbf{y}, x^0) = s_2^0(\mathbf{y}, x^0)/c$. Similarly, the transverse propagator ${}_{\mathrm{T}}D_{\mathrm{F}\mu\nu}(x-y)$ accounts for the electromagnetic interaction between the current densities $\mathbf{j}_1(x) = \mathbf{s}_1(x)$ and $\mathbf{j}_2(y) = \mathbf{s}_2(y)$. The contribution to (5.44) of the remainder term ${}_{\mathrm{R}}D_{\mathrm{F}\mu\nu}(x-y)$ is easily shown to vanish, on account of current conservation. Transforming this contribution to expression (5.44) into momentum space, one obtains

$$\frac{1}{(2\pi)^4} \int \mathrm{d}^4k s_1^\mu(-k) {}_{\mathrm{R}}D_{\mathrm{F}\mu v}(k) s_2^v(k)$$

where the momentum transforms $s_r^\mu(k), r = 1, 2$, are defined, analogously to Eq. (5.38), by

$$s_r^\mu(x) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4k s_r^\mu(k) \mathrm{e}^{-\mathrm{i}kx}, \quad r = 1, 2.$$

The current conservation equations, $\partial_\mu s_r^\mu(x) = 0$, translated into momentum space, become

$$k_\mu s_r^\mu(k) = 0, \quad r = 1, 2.$$

We see from the explicit form (5.41c) that each term in ${}_{\mathrm{R}}D_{\mathrm{F}\mu\nu}(k)$ is proportional to either k_μ or k_ν or both. Hence it follows from Eq. (5.47) that the expression (5.45) vanishes. This completes our discussion of the equivalence of the two formulations of quantum electrodynamics. In doing this, we employed a special frame of reference leading to a division of the fields into transverse, longitudinal and scalar parts. In general, such a division is not required, and we shall work with manif estly covariant expressions involving summations over all four polarization states. In particular, the photon propagator, Eqs. (5.38) and (5.39), which will be very important in the development of quantum electrodynamics, has this property.

9.2.5 Covariant Gauges and the Photon Propagator

(мб в КЭД перенесу)

Theory

В электродинамике калибровочное условие Лоренца

$$X(\mathcal{A}) = \partial_\mu \mathcal{A}^\mu$$

приводит к лагранжиану, квадратичному по полю,

$$\mathcal{L}_F = -\frac{1}{4} F_{\mu\nu}^2 - \frac{1}{2\xi} (\partial_\mu \mathcal{A}^\mu)^2$$

В импульсном представлении дифференциальный оператор

$$\hat{\Gamma}_{\mu\nu} = k^2 \left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \right) - \frac{1}{\xi} k_\mu k_\nu$$

который определяет

$$\mathcal{L}_F = \frac{1}{2} \mathcal{A}^\mu(k) \hat{\Gamma}_{\mu\nu} \mathcal{A}^{*\nu}(k)$$

(что это такое???:))

может быть обращен, чтобы получить причинный пропагатор фотона

$$\mathcal{D}^{(2)}_{\mu\nu} = \frac{\mathrm{i}}{k^2 + \mathrm{i}0} \left(-g_{\mu\nu} + (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right)$$

Значение $\xi = 1$ отвечает калибровке Фейнмана, а $\xi = 0$ - поперечной калибровке Ландау.

При этом, конечно, амплитуда квантового перехода для асимптотических поперечных состояний никак не зависит от выбора ξ.

Подчеркнем, что вклад отдельных диаграмм может зависеть от калибровочного параметра и от выбора калибровочного условия.

Этим пользуются, чтобы выделять доминирующие диаграммы.

Однако суммирование диаграмм для заданной амплитуды никак не зависит от выбора калибровки и ее параметров.

Отметим, наконец, что калибровка Лоренца означает, что при калибровочном преобразовании $\partial_\mu \mathcal{A}^\mu \mapsto \partial_\mu \mathcal{A}^\mu - \partial^2 u$ матрица (как например она выглядит?????)

$$\frac{\partial X}{\partial u} = -\partial^2$$

совсем не зависит от поля.

Поэтому фактор

$$\mathrm{Det}\left(\mathcal{M}_X(\mathcal{A})\right)|_{X(\mathcal{A})=0} = -\mathcal{D}et\partial^2$$

можно вынести из под знака континуального интеграла и считать его несущественным нормировочным множителем,

который, конечно, сократится в $\hat{Z}(j) = Z(j)/Z(0)$. В этом смысле калибровка Лоренца весьма практична.

Вместе с тем, часто используют калибровки, в которых от детерминанта избавится невозможно, но это мы рассмотрим чуть позже в случае неабелевых теорий.

9.2.6 Quantum Vector Field in a Medium

9.3 Quantization of the Electromagnetic Field

9.3.1 The Continuum Integral for Degenerate Systems

Рассмотрим многомерную динамическую систему, гамильтониан которой зависит от n пар канонически сопряжённых переменных $\{p_\alpha, q^\alpha\}$ и в которой имеется $m < n$ связей первого рода $\varphi_i(p, q)$, а связи второго рода отсутствуют. Связи первого рода - это такие связи между динамическими переменными

$$\varphi_i(p, q) = 0, \quad i = 1, ..., m \tag{1}$$

что

$$[\varphi_i, \varphi_j] = c_{ij}^k \varphi_k, \quad [\varphi_i, \mathcal{H}] = 0 \tag{2}$$

В этом пункте $[..., ...]$ обозначает классическую скобку Пуассона (СП), а $\mathcal{H}$, как и везде гамильтониан. При вычислении всех СП связи игнорируются. В рассматриваемом случае полный гамильтониан системы $\mathcal{H}_T = \mathcal{H} + \sum_{i=1}^{m} v_i \varphi_i$ зависит от m произвольных функций времени v_i. Действительно, уравнения Гамильтона для связей

$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}t} = \left[\ \varphi_i, \mathcal{H}_T\right] = 0 \pmod{\varphi_j}$$

показывают, что при любых значениях множителей Лагранжа $\{v_i\}$ равенства (1) совместны с динамикой системы. Обозначим через Γ^{2n} фазовое $2n$-мерное пространство с координатами $\{p_\alpha, q^\alpha\}$.

На классическом уровне рассматриваемая обобщённая гамильтонова система эквивалентна обычной гамильтоновой системе с $(n-m)$ степенями свободы. Фазовое пространство $\Gamma^{*2(n-m)}$ последней системы реализуем следующим образом. Рассмотрим m дополнительных условий

$$\chi^j(p, q) = 0 \tag{3}$$

фиксирующих калибровку, для которых выполняются требования

$$\det \left\| \left[\varphi_i, \chi^j\right] \right\| \neq 0, \quad \left[\chi^i, \chi^j\right] = 0 \tag{4}$$

Фиксация калибровки означает определённый выбор набора функций $\{v_i\}$ (см. ниже (11)). Подпространство в Γ^{2n}, определяемое равенствами

$$\chi^i(p, q) = 0, \quad \varphi_i(p, q) = 0, \quad i = 1, ..., m \tag{5}$$

является пространством $\Gamma^{*2(n-m)}$. Вследствие (4) можно выбрать в Γ^{2n} канонические переменные так, чтобы первые m координатных переменных $q^1, ..., q^m$ совпали с величинами $\chi^1, ..., \chi^m$:

$$q = \left\{\chi^1, ..., \chi^m, q^*\right\} \tag{6}$$

Здесь q^* - остальные $(n-m)$ координатных переменных. Записывая лагранжиан в переменных (6), находим соответствующие канонически сопряжённые импульсные переменные. Обозначим их через

$$p = \{p_1, ..., p_m, p^*\} \tag{7}$$

В новых переменных условия (4) принимают вид

$$\det \left\| \frac{\partial \varphi_i}{\partial p_j} \right\| \neq 0 \tag{8}$$

Отсюода следует, что уравнения связей (1) можно разрешить относительно p_i с $i = 1, ..., m$. Эти решения обозначим $p_i\,(p^*, q^*)$. В результате подпространство $\Gamma^{*2(n-m)}$ задаётся уравнениями

$$\chi^i \equiv q^i = 0, \quad p_i = p_i\left(p^*, q^*\right), \quad i = 1, ..., m \tag{9}$$

Переменные $\{p^*, q^*\}$ являются каноническими, и гамильтонианом системы является функция

$$\mathcal{H}^*\left(p^*, q^*\right) \equiv \mathcal{H}(p, q)|_{\varphi=0, \chi=0} \tag{10}$$

Обозначим исходную систему с гамильтонианом $\mathcal{H}_T$ и фазовым пространством Γ^{2n} и новую систему с гамильтонианом $\mathcal{H}^*$ и фазовым пространством $\Gamma^{*2(n-m)}$ через Γ и Γ^*, соответственно.

Системы Γ и Γ^* эквивалентны и эквивалентность означает следующее.

Выберем в $\mathcal{H}_T$ лагранжевы множители v_i так, чтобы выполнялись равенства

$$\dot{q}^j = \left[\chi^j, \mathcal{H}_T\right] = \frac{\partial \mathcal{H}}{\partial p_j} + \sum_i v_i \frac{\partial \varphi_i}{\partial p_j} = 0, \quad j = 1, ..., m \tag{11}$$

Вследствие (4) эта задача имеет однозначное решение. При этом уравнения Гамильтона $\dot{g} = [g, \mathcal{H}_T]$ и связей (1), (9) совпадают с уравнениями Гамильтона для системы Γ^*.

Докажем этот факт в координатах (6). При помощи равенств (11) получаем следующие соотношения:

$$\begin{aligned}
&\sum_{i=1}^{m}\left(v_i \frac{\partial \varphi_i}{\partial q^*} - \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial p_i}{\partial q^*}\right) = \sum_{i=1}^{m} v_i\left(\frac{\partial \varphi_i}{\partial q^*} + \sum_{j=1}^{m}\frac{\partial \varphi_i}{\partial p_j}\frac{\partial p_j}{\partial q^*}\right) = 0 \\
&\sum_{i=1}^{m}\left(v_i \frac{\partial \varphi_i}{\partial p^*} - \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial p_i}{\partial p^*}\right) = \sum_{i=1}^{m} v_i\left(\frac{\partial \varphi_i}{\partial p^*} + \sum_{j=1}^{m}\frac{\partial \varphi_i}{\partial p_j}\frac{\partial p_j}{\partial p^*}\right) = 0
\end{aligned} \tag{12}$$

Последние равенства в (12) вытекают из того, что имеются тождества

$$\varphi_i\left(p\left(p^*, q^*\right), p^*, q^*\right) \equiv 0, \quad i = 1, ..., m$$

Из (12) следует, что уравнения

$$\dot{p}^* = -\left(\frac{\partial \mathcal{H}}{\partial q^*} + v_i \frac{\partial \varphi_i}{\partial q^*}\right)$$

и

$$\dot{p}^* = -\frac{\partial \mathcal{H}^*}{\partial q^*} = -\left(\frac{\partial \mathcal{H}}{\partial q^*} + \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial p_i}{\partial q^*}\right)$$

совпадают (соответствующие уравнения для переменных q^* не выписываются). Таким образом утверждение об эквивалентности систем Γ и Γ^* доказано.

Система Γ может быть проквантована в переменных $\{p^*, q^*\}$, в которых она является невырожденной. В этих переменных квантовый оператор эволюции задаётся континуальным интегралом (см. (12 Л1))

$$\left\langle q^{*\prime\prime}\left|U^*\left(t'', t'\right)\right|q^{*\prime}\right\rangle = \int \exp\left\{i\int_{t'}^{t''}\left(p^*\dot{q}^* - \mathcal{H}^*\left(p^*, q^*\right)\right)\mathrm{d}t\right\}\prod_t \frac{\mathrm{d}p^*(t)\mathrm{d}q^*(t)}{2\pi} \tag{13}$$

Чаще всего (как в электродинамике и многих других калибровочных теориях) нецелесообразно разрешать связи явно. Напротив, оператор эволюции полезно представить непосредственно в терминах переменных обобщённой системы Г. Покажем, что континуальный интеграл

$$\int \exp\left\{ i \int_{t'}^{t''} \left[p\dot{q} - \left(\mathcal{H}(p,q) + v_i \varphi_i(p,q) \right) \right] \mathrm{d}t \right\} \prod_t \prod_i \delta\left(\chi^i\right) \det \left\| \left[\varphi_k, \chi^j \right] \right\| \prod_\alpha \frac{\mathrm{d}p_\alpha(t) \mathrm{d}q^\alpha(t)}{2\pi} \frac{\mathrm{d}v_i(t)}{2\pi} \tag{14}$$

совпадает с интегралом (13). Для этого проинтегрируем (14) по лагранжеву множителю v, в результате чего он принимает вид

$$\int \exp\left\{ i \int_{t'}^{t''} \left[p\dot{q} - \mathcal{H}(p,q) \right] \mathrm{d}t \right\} \prod_t \prod_i \delta\left(\chi^i\right) \delta\left(\varphi_i\right) \det \left\| \left[\varphi_k, \chi^j \right] \right\| \prod_\alpha \frac{\mathrm{d}p_\alpha(t) \mathrm{d}q^\alpha(t)}{2\pi} \tag{15}$$

Мера в интеграле (15) инвариантна относительно канонических преобразований переменных интегрирования. Будем считать, что он вычисляется в переменных (6), (7), в которых

$$\prod_t \prod_i \delta\left(\chi^i\right) \delta\left(\varphi_i\right) \det \left\| \left[\varphi_k, \chi^j \right] \right\| = \prod_t \prod_i \delta\left(q^i\right) \delta\left(\varphi_i\right) \det \left\| \frac{\partial \varphi_k}{\partial p_j} \right\| = \prod_t \prod_i \delta\left(q^i\right) \delta\left[p_i - p_i\left(p^*, q^*\right) \right]$$

Теперь очевидно, что после интегрирования по переменным $\{p_i, q^i\}$ с $i = 1, ..., m$ интеграл (15) сводится к (13).

Заметим, что выражение в круглой скобке в экспоненте в (14) является полным гамильтонианом $\mathcal{H}_T$.

9.3.2 Electromagnetic Field as a Dynamic System

Свободное электромагнитное поле описывается действием

$$\mathcal{S} = -\frac{1}{16\pi} \int \mathrm{d}^{(4)}x F_{\mu\nu} F^{\mu\nu}, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{16}$$

Электрическое и магнитное поля равны соответственно

$$E^i = F^{i0} = -\partial_i A^0 - \dot{A}^i, \quad H^i = -\frac{1}{2} \varepsilon_{ijk} F^{jk} \tag{17}$$

Выпишем лагранжиан системы:

$$\mathcal{L} = \frac{1}{8\pi} \int \mathrm{d}^{(3)}x \left(F^{io}\right)^2 - \frac{1}{16\pi} \int \mathrm{d}^{(3)}x F_{ij}^2 \tag{18}$$

Проварьируем лагранжиан (18) относительно $\dot{A}^\mu$:

$$\delta_{\dot{A}} \mathcal{L} = -\frac{1}{4\pi} \int \mathrm{d}^{(3)}x F^{io} \delta\dot{A}^i = -\frac{1}{4\pi} \int \mathrm{d}^{(3)}x E^i \delta\dot{A}^i \tag{19}$$

Отсюда видно, что если поле A^μ рассматривается как совокупность координатных переменных, то соответствующие импульсные переменные π_μ равны

$$\pi_i(x) = -\frac{1}{4\pi}E^i(x) \tag{20}$$

$$\pi_0(x) = 0 \tag{21}$$

При помощи формул (17)-(21) находим гамильтониан:

$$\mathcal{H}_T \equiv \int \mathrm{d}^{(3)}x\pi_i\dot{A}^i - \mathcal{L} = \int \mathrm{d}^{(3)}x \left\{ \frac{\mathbf{E}^2 + \mathbf{H}^2}{8\pi} - \frac{1}{4\pi}A^0\nabla\mathbf{E} \right\} \tag{22}$$

Канонические одновременные СП имеют вид (выписаны лишь ненулевые СП):

$$\left[E^i(\mathbf{x}), A^j(\mathbf{y})\right] = 4\pi\delta^{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad \left[\pi^0(\mathbf{x}), A^0(\mathbf{y})\right] = -\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{23}$$

Мы имеем множество первичных связей (21). Вычислим их СП с гамильтонианом:

$$\left[\pi^0(\mathbf{x}), \mathcal{H}\right] = \frac{1}{4\pi}\nabla\mathbf{E}(\mathbf{x}) \tag{24}$$

Чтобы связи (21) сохранялись, необходимо наложить множество новых связей:

$$\varphi(\mathbf{x}) \equiv \nabla\mathbf{E}(\mathbf{x}) = 0 \tag{25}$$

При помощи (23) находим:

$$\left[A^i(\mathbf{x}), \frac{1}{4\pi}\int \mathrm{d}^3y A^0(\mathbf{y})\nabla\mathbf{E}(\mathbf{y})\right] = \partial_i A^0(\mathbf{x}) \tag{26}$$

Таким образом, связи (25) генерируют калибровочные преобразования. Нетрудно проверить, что положительная часть гамильтониана

$$\mathcal{H} = \int \mathrm{d}^{(3)}x \frac{\mathbf{E}^2 + \mathbf{H}^2}{8\pi} \tag{27}$$

калибровочно инвариантен:

$$[\nabla\mathbf{E}(\mathbf{x}), \mathcal{H}] = 0 \tag{28}$$

Кроме того, все взаимные СП связей (21), (25) обращаются в ноль. Поэтому все связи - первого рода и поле A^0 играет роль лагранжевого множителя, принимающего произвольные значения и гамильтониан (22) является полным гамильтонианом.

Равенство (26) показывает, что естественными дополнительными условиями, аналогичными (3), в рассматриваемом случае являются условия, фиксирующие калибровку:

$$\chi(x) \equiv \nabla\mathbf{A}(x) = 0 \tag{29}$$

Вычислим одновременные СП связей (25) и (29). При помощи СП (23) находим:

$$[\varphi(\mathbf{x}), \chi(\mathbf{y})] = [\nabla\mathbf{E}(\mathbf{x}), \nabla\mathbf{A}(\mathbf{y})] = -4\pi\nabla^2\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{30}$$

Таким образом, $\det \|[\varphi(\mathbf{x}), \chi(\mathbf{y})]\| = \mathrm{const}$, то есть этот определитель является величиной, не зависящей от динамических переменных, которая включается в общий нормировочный

множитель. (В более сложных калибровочных теориях, как теория Янга-Миллса, аналог оператора (30) зависит от полей и соответствующий определитель должен быть явно учтён.)

Обозначим через $\mathbf{A}^{\perp}(x)$ поперечное векторнозначное поле, удовлетворяющее условию

$$\nabla \mathbf{A}^{\perp} = 0 \tag{31}$$

В нашем случае система Γ описывается переменными $\{\mathbf{E}, \pi^0, A^\mu\}$, а система Γ* - переменными $\left\{\mathbf{E}^{\perp}, \mathbf{A}^{\perp}\right\}$. Однако удобнее взять в качестве исходной системы Γ несколько сокращённую систему с фазовым пространством, точками которого являются поля $\{\mathbf{E}, \mathbf{A}\}$, связями (25) и условиями (29), фиксирующими калибровку. При таком подходе роль лагранжевых множителей $\{v_i\}$ из предыдущего пункта берёт на себя поле, обозначаемое A^0.

При квантовании переменные системы Γ* удовлетворяют следующим одновременным КС (сравни с (23)):

$$\left[E^{\perp i}(\mathbf{x}), A^{\perp j}(\mathbf{y})\right] = 4\pi i \int \frac{d^{(3)}k}{(2\pi)^3}\left(\delta^{ij} - \frac{k^i k^j}{k^2}\right) e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \tag{32}$$

Последние КС реализуются, если полевые переменные разлагаются согласно (сравни с (34 Л2))

$$\begin{aligned}
&\mathbf{A}^{\perp}(x) = \sum_{\alpha=1,2}\int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\sqrt{\frac{2\pi}{|\mathbf{k}|}}\left(c_{\mathbf{k}\alpha}\mathbf{e}^{\perp}_{\mathbf{k}\alpha}e^{i\mathbf{k}\mathbf{x}} + c^{\dagger}_{\mathbf{k}\alpha}\mathbf{e}^{\perp}_{\mathbf{k}\alpha} * e^{-i\mathbf{k}\mathbf{x}}\right)\\
&\mathbf{E}^{\perp}(x) = i\sum_{\alpha=1,2}\int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\sqrt{2\pi|\mathbf{k}|}\left(c_{\mathbf{k}\alpha}\mathbf{e}^{\perp}_{\mathbf{k}\alpha}e^{i\mathbf{k}\mathbf{x}} - c^{\dagger}_{\mathbf{k}\alpha}\mathbf{e}^{\perp}_{\mathbf{k}\alpha}e^{-i\mathbf{k}\mathbf{x}}\right)\\
&\left[c_{\mathbf{k}\alpha}, c^{\dagger}_{\mathbf{p}\beta}\right] = (2\pi)^3\delta_{\alpha\beta}\delta^{(3)}(\mathbf{k}-\mathbf{p}), \quad \alpha,\beta = 1,2\\
&\mathbf{k}\cdot\mathbf{e}^{\perp}_{\mathbf{k}\alpha} = 0, \quad \mathbf{e}^{\perp *}_{\mathbf{k}\alpha}\cdot\mathbf{e}^{\perp}_{\mathbf{k}\beta} = \delta_{\alpha\beta}, \quad \sum_{\alpha=1,2} e^{\perp i}_{\mathbf{k}\alpha}e^{\perp *j}_{\mathbf{k}\alpha} = \left(\delta^{ij} - \frac{k^i k^j}{\mathbf{k}^2}\right)
\end{aligned} \tag{33}$$

Если в (27) подставить переменные (33), то мы получим гамильтониан Γ*-системы:

$$\mathcal{H}^*_0 = \frac{1}{2}\sum_{\alpha=1,2}\int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3}|\mathbf{k}|\left(c^{\dagger}_{\mathbf{k}\alpha}c_{\mathbf{k}\alpha} + c_{\mathbf{k}\alpha}c^{\dagger}_{\mathbf{k}\alpha}\right) \tag{34}$$

Фотонный вакуум определяется условиями

$$c_{\mathbf{k}\alpha}|0\rangle = 0 \tag{35}$$

Однофотонное состояние с импульсом $\mathbf{k}$ и поляризацией α в релятивистской нормировке представляется как (сравни с (38 Л2))

$$|\mathbf{k},\alpha\rangle = \sqrt{2|\mathbf{k}|}c^{\dagger}_{\mathbf{k}\alpha}|0\rangle \tag{36}$$

Двух- и т.д. фотонные состояния строятся из вакуумного очевидным образом при помощи операторов рождения $c^{\dagger}_{\mathbf{k}\alpha}$. Приведём гамильтониан (34) к нормальному виду (далее индекс *, отмечающую принадлежность к Γ*-системе, не выписываем):

$$\mathcal{H}_0 = \sum_{\alpha=1,2} \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3} |\mathbf{k}| c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha} + \mathcal{E}_0 \tag{37}$$

$$\mathcal{E}_0 = V \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3} |\mathbf{k}| \tag{38}$$

Здесь V - объём пространства. Величина (38) является энергией нулевых колебаний или энергией вакуума. Также как и в скалярной теории, в электродинамике она положительна и её плотность бесконечна. Существуют экспериментальные методы "тестирования"энергии вакуума (эффект Казимира, см. задачу к этой лекции).

Как известно, плотность импульса электромагнитного поля равна $(1/4\pi)[\mathbf{E}, \mathbf{H}]$. Поэтому оператор импульса электромагнитного поля (сравни с (42 Л2))

$$\mathcal{P} = \frac{1}{4\pi} \int \mathrm{d}^{(3)}x [\mathbf{E}, \mathbf{H}] = \sum_{\alpha=1,2} \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3} \mathbf{k} c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha}$$

Из данных определений следует, что (энергия вакуума вычеркнута из гамильтониана)

$$\mathcal{H}|0\rangle = \mathcal{P}|0\rangle = 0, \quad \mathcal{H}|\mathbf{k}, \alpha\rangle = |\mathbf{k}||\mathbf{k}, \alpha\rangle, \quad \mathcal{P}|\mathbf{k}, \alpha\rangle = \mathbf{k}|\mathbf{k}, \alpha\rangle$$

Таким образом, фотон является безмассовой частицей, распространяющейся со скоростью света.

Теперь при помощи формул из предыдущего пункта и лекции 2 мы можем, минуя детальные вычисления, выписать нормальный символ S-матрицы в фотодинамике в терминах переменных исходной Г-системы (сравни с (52 Л2)):

$$S(c^*, c) = \frac{1}{N} \int \exp\left\{ i \int \mathrm{d}^{(4)}x \left[-\frac{1}{4\pi} \left(\mathbf{E}\dot{\mathbf{A}} - A^0 \nabla \mathbf{E} \right) - \frac{\mathbf{E}^2 + \mathbf{H}^2}{8\pi} \right] \right\} \times$$
$$\times \prod_x \prod_{i=1}^{3} \delta(\nabla \mathbf{A}(x)) \mathrm{d}E^i(x) \mathrm{d}A^i(x) \mathrm{d}A^0(x) \tag{39}$$

Граничные условия в интеграле (39) таковы:

$$\mathbf{A}^\perp(x) \underset{t\to\pm\infty}{\longrightarrow} \mathbf{A}^\perp_{\substack{\text{out} \\ \text{in}}}(x) \qquad \mathbf{A}^\perp_{\text{in}}(x) = \sum_{\alpha=1,2} \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3} \sqrt{\frac{2\pi}{|\mathbf{k}|}} \left(c_{\mathbf{k}\alpha \ \substack{\text{out} \\ \text{in}}} \mathbf{e}^\perp_{\mathbf{k}\alpha} e^{-ikx} + c^*_{\mathbf{k}\alpha \ \substack{\text{out} \\ \text{in}}} \mathbf{e}^{\perp *}_{\mathbf{k}\alpha} e^{ikx} \right),$$

$$c_{\mathbf{k}\alpha \text{ in}} = c_{\mathbf{k}\alpha}, \quad c^*_{\mathbf{k}\alpha \text{ out}} = c^*_{\mathbf{k}\alpha}, \quad kx = k^0 x^0 - \mathbf{k}\mathbf{x}, \quad k^0 = |\mathbf{k}| \tag{40}$$

Обратим внимание на то, что граничные условия (40) имеют место лишь для физических степеней свободы, описываемых поперечной частью поля $\mathbf{A}^\perp(x)$ (31). Проинтегрируем в (39) по импульсным переменным $\mathbf{E}(x)$ и введём в теорию внешний источник. Это интегрирование проводится однозначно при помощи правила вычисления гауссовых интегралов, поскольку проблема граничных условий при этом интегрировании отсутствует. С учётом (16)-(18) получаем:

$$S_J\left(c^*,c\right)=\frac{1}{N}\int\exp\left\{i\int \mathrm{d}^{(4)}x\left[-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}+J^\mu A_\mu\right]\right\}\prod_x\delta\left(\partial_i A^i(x)\right)DA_\mu$$
$$DA_\mu\equiv\prod_x\prod_\mu \mathrm{d}A_\mu(x) \tag{41}$$

Интеграл (41) вычисляется с граничными условиями (40).

Из континуального интеграла (41) видно, что интегрирование идёт не по всем полям, а лишь по поперечным физическим полям $\mathbf{A}^\perp$.

Для вычисления пропагатора электромагнитного поля преобразуем интеграл (41) следующим образом:

$$S_J\left(c^*,c\right)=\frac{1}{N}\int\exp\left\{i\int \mathrm{d}^{(4)}x\left[-\frac{1}{16\pi}\left(\partial_\mu A_\nu-\partial_\nu A_\mu\right)^2+J^\mu A_\mu+\lambda\partial_i A_i\right]\right\}\prod_x\,\mathrm{d}\lambda(x)DA_\mu \tag{42}$$

Для вычисления гауссового интеграла следует найти экстремум показателя экспоненты в (42) относительно переменных $\{\lambda,A_\mu\}$. Этот экстремум находится при помощи системы уравнений

$$\partial_i A_i=0,\quad \Delta A^0=4\pi J^0,\quad \partial_\mu\partial^\mu A^i+\partial_i\dot{A}_0=4\pi\left(\partial_i\lambda-J^i\right) \tag{43}$$

Естественно считать, что источник удовлетворяет условию поперечности:

$$\partial_i J^i=0 \tag{44}$$

Вычисляя дивергенцию последнего из уравнений (43) и учитывая первые два, находим:

$$\Delta\lambda=\partial_0 J^0 \tag{45}$$

В результате система уравнений (43)-(44) сводится к уравнению (45), а также уравнениям

$$\Delta A^0=4\pi J^0,\quad \partial_\mu\partial^\mu\mathbf{A}^\perp=-4\pi\mathbf{J}^\perp \tag{46}$$

Теперь аналогично (53 Л2) мы можем представить электромагнитное поле, для которого действие в (42) экстремально:

$$A^\mu(x)=A_0^\mu(x)+\int \mathrm{d}^{(4)}y D_C^{\mu\nu}(x-y)J_\nu(y)$$
$$A_0^0=0,\quad \mathbf{A}_0(x)=\sum_{\alpha=1,2}\int\frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\sqrt{\frac{2\pi}{|\mathbf{k}|}}\left(c_{\mathbf{k}\alpha}\mathbf{e}^\perp_{\mathbf{k}\alpha}e^{-ikx}+c^*_{\mathbf{k}\alpha}\mathbf{e}^\perp_{\mathbf{k}\alpha}e^{ikx}\right),\quad k^0=|\mathbf{k}| \tag{47}$$
$$D_C^{ij}(x)=-4\pi\int\frac{d^{(4)}k}{(2\pi)^4}\frac{e^{-ikx}}{k^2+i\varepsilon}\left(\delta^{ij}-\frac{k^ik^j}{\mathbf{k}^2}\right),\quad D_C^{0i}=0,\quad D_C^{00}(x)=-4\pi\int\frac{d^{(4)}k}{(2\pi)^4}\frac{e^{-ikx}}{\mathbf{k}^2} \tag{48}$$

Явное выражение для $D_C^{\mu\nu}(x)$ показывает, что распространяются во времени только трёхмерно поперечные составляющие поля A_μ.

Недостатком диаграммной техники, развитой при помощи пропагатора (48), является отсутствие явной релятивистской инвариантности.

9.3.3 Covariant Quantization Rules and the Faddeev-Popov Trick

Чтобы восстановить явную релятивистскую инвариантность, поступим следующим образом. Обозначим через $a(x)$ некое скалярное поле и через A_μ^ω - калибровочно преобразованное электромагнитное поле:

$$A_\mu^\omega = A_\mu + \partial_\mu \omega \tag{49}$$

Определим величину Δ_L при помощи равенства

$$\Delta_L \int \prod_x \delta\left[\partial^\mu A_\mu^\omega - a(x)\right] \mathrm{d}\omega(x) = 1 \tag{50}$$

Очевидно, Δ_L не зависит от полей A_μ и $a(x)$: $\Delta_L = \det \partial^2$. Теперь вставим левую часть (50) в интеграл (41), что не изменит его значения:

$$S_J(c^*, c) = \frac{1}{N} \int \exp\left\{ i \int \mathrm{d}^{(4)}x \left[-\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu \right] \right\} \times$$
$$\times \prod_x \delta\left(\partial_i A^i(x)\right) \Delta_L \delta\left[\partial^\mu A_\mu^\omega - a(x)\right] \mathrm{d}\omega(x) DA_\mu \tag{51}$$

Теперь в интеграле (51) сделаем калибровочное преобразование, обратное к (49), и учтём, что мера DA_μ и плотность действия $F_{\mu\nu}^2$ при этом не изменяются, являясь калибровочными инвариантами. Поэтому после соответствующего переобозначения внешнего тока интеграл (51) принимает вид

$$S_J(c^*, c) = \frac{1}{N} \int \exp\left\{ i \int \mathrm{d}^{(4)}x \left[-\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu - J^\mu \partial_\mu \omega \right] \right\} \times$$
$$\times \prod_x \delta\left(\partial_i A_i^{\omega^{-1}}(x)\right) \Delta_L \delta\left[\partial^\mu A_\mu - a(x)\right] \mathrm{d}\omega(x) DA_\mu$$

За слагаемым вида $J^\mu \partial_\mu \omega$ в экспоненте нет смысла следить потому что асимптотическое поле (47) определено с точностью до слагаемого $\partial_\mu \omega$, где поле ω, так же как и в рассматриваемых функциональных интегралах - произвольное, удовлетворяющее уравнению д'Аламбера (см. ниже ур. (57)). Так как интеграл

$$\int \prod_x \delta\left[\partial_i A_i^{\omega^{-1}}(x)\right] \mathrm{d}\omega(x) = \det \Delta^{-1}$$

также не зависит от от вектор-потенциала, то его можно выбросить. Таким образом

$$S_J(c^*, c) = \frac{1}{N} \int \exp\left\{ i \int \mathrm{d}^{(4)}x \left[-\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + J^\mu A_\mu \right] \right\} \prod_x \delta\left[\partial^\mu A_\mu - a(x)\right] DA_\mu$$

Поскольку здесь левая часть не зависит от поля $a(x)$, мы можем проинтегрировать последнее выражение с весом

$$\exp\left\{ -\frac{i}{8\pi\alpha} \int a^2(x) \mathrm{d}^{(4)}x \right\}$$

что влияет лишь на нормировочный множитель:

$$S_J(c^*,c) = \frac{1}{N}\int \exp\left\{ i\int \mathrm{d}^{(4)}x \left[-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{8\pi\alpha}a^2(x) + J^\mu A_\mu \right] \right\} \times$$
$$\times \prod_x \delta\left[\partial^\mu A_\mu - a(x)\right] \mathrm{d}a(x) DA_\mu =$$
$$= \frac{1}{N}\int \exp\left\{ i\int \mathrm{d}^{(4)}x \left[-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{8\pi\alpha}\left(\partial_\mu A^\mu\right)^2 + J^\mu A_\mu \right] \right\} DA_\mu \tag{52}$$

Находя пропагатор таким же образом, как это было сделано выше, находим, что континуальный интеграл (52) приводит к ковариантному пропагатору

$$D^\alpha_{\mu\nu}(x) = 4\pi \int \frac{\mathrm{d}^{(4)}k}{(2\pi)^4}\frac{e^{-ikx}}{k^2+i\varepsilon}\left[\eta_{\mu\nu} - \frac{(1-\alpha)k_\mu k_\nu}{k^2+i\varepsilon}\right], \quad \eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1) \tag{53}$$

Калибровка, в которой пропагатор имеет вид (53), называется α-калибровкой. Частный случай этой калибровки $\alpha = 1$ называется калибровкой Фейнмана, а частный случай $\alpha = 0$ - калибровкой Ландау.

Правильность задания обхода полюсов в (53) легче всего устанавливается следующим образом. Пропагатор (48), конечно, как пропагатор, описывающий распространение реальных частиц в рамках теории S-матрицы, позволяет сделать поворот Вика в евклидово пространство. Иными словами, при вычислении S-матрицы можно использовать интеграл вида (41) в евклидовом пространстве, и лишь после вывода правил Фейнмана перейти в пространство Минковского. В евклидовом пространстве можно перейти к α-калибровке (52) при помощи описанного трюка Фаддеева-Попова и получить соответствующий евклидовский пропагатор. После этого обратный поворот Вика в пространство Минковского приводит к пропагатору (53) с указанным обходом полюсов.

Теперь мы можем выписать нормальный символ S-матрицы для свободного электромагнитного поля при наличии внешнего источника (сравни с (51 Л2). Значение экстремального значения электромагнитного поля задаётся формулой (см. (47))

$$A^\mu(x) = A^\mu_0(x) + \int \mathrm{d}^{(4)}y D^{\alpha\mu\nu}(x-y)J_\nu(y)$$

а нормальный символ S-матрицы имеет вид

$$S_J(c^*,c) = \frac{1}{N}\exp\left\{ i\int \mathrm{d}^{(4)}x J^\mu A_{0\mu} + \frac{i}{2}\iint \mathrm{d}^{(4)}x\ \mathrm{d}^{(4)}y J^\mu(x) D^\alpha_{\mu\nu}(x-y) J^\nu(y) \right\} \tag{54}$$

Последнее замечание касается вопроса об однозначности определения векторов поляризации фотонов $\mathbf{e}^\perp_{\mathbf{k}\alpha}$. Введём 4 -вектор поляризации фотона $e^\mu_{\mathbf{k}\alpha} = \left(0, \mathbf{e}^\perp_{\mathbf{k}\alpha}\right)$. Если $k^\mu = (|\mathbf{k}|, \mathbf{k}) - 4$ импульс фотона, то согласно (33)

$$k_\mu e^\mu_{\mathbf{k}\alpha} = 0, \quad \eta_{\mu\nu} e^\mu_{\mathbf{k}\alpha} e^{\nu *}_{\mathbf{k}\beta} = -\delta_{\alpha\beta} \tag{55}$$

Очевидно, что оба свойства (55) сохраняются, если сделать преобразование вектора поляризации согласно правилу

$$e^\mu_{\mathbf{k}\alpha} \to e^\mu_{\mathbf{k}\alpha} + f(\mathbf{k})k^\mu \tag{56}$$

где $f(\mathbf{k})$ - любая функция. Поэтому вектор поляризации фотона определён неоднозначно с точностью до преобразования (56), изменяющего четырёхмерно продольную часть фотона. Поэтому и асимптотическое поле (47) определено с точностью до произвольного калибровочного преобразования вида

$$A_{0\mu} \longrightarrow A_{0\mu} + \partial_\mu \omega, \quad \partial_\mu \partial^\mu \omega = 0 \tag{57}$$

В то же время, как мы увидим ниже, все физические величины инвариантны относительно преобразований (56). Эта инвариантность является следствием калибровочной инвариантности теории.

Задачи

1. Трёхмерный эффект Казимира.

а) В трёхмерном пространстве на расстоянии l друг от друга расположены две бесконечные плоские параллельные металлические пластины. Считая, что на пластинах обращаются в ноль тангенциальные компоненты электрического поля $\mathbf{E}$ и разность потенциалов между пластинами равна нулю, проквантовать фотонное поле и выразить его гамильтониан через лестничные операторы.

б) Вычислить силу на единицу площади пластин, с которой взаимодействуют эти пластины вследствие искажений нулевых колебаний фотонного поля и сдвига энергии вакуума.

Указание. При регуляризации интеграла типа (38), формально задающего энергию вакуума, использовать формулу размерностной регуляризации (3 Л4), а при регуляризации расходящихся сумм степеней целых чисел использовать аналитическое продолжение ζ-функции Римана.

2. Вычислить явно в координатном представлении в четырёхмерном пространстве фотонный пропагатор в α-калибровке.

9.3.4 Faddeev-Popov Trick: Detailed Overview

(??? discussion about is it important to know it, and what exactly happened and what is the idea????)

9.4 Spinor Field (fermions) 1: General Methods

(this shuold be rewritten, now structure does not match the title, I should go through this theory to find out what to delete and what to move????)

9.4.1 Fermion Propagator (after Mandl & Shaw)

Теория

In Section 3.4 we introduced the meson propagator. Corresponding to Eq. (3.55), we now define the Feynman fermion propagator as

$$\left\langle 0 \left| \mathbf{T} \left\{ \psi(x) \bar{\psi}\left(x'\right) \right\} \right| 0 \right\rangle$$

where spinor indices have again been suppressed. For fermion fields, the time-ordered product is defined by

$$T\left\{\psi(x)\bar{\psi}\left(x'\right)\right\} = \theta\left(t-t'\right)\psi(x)\bar{\psi}\left(x'\right) - \theta\left(t'-t\right)\bar{\psi}\left(x'\right)\psi(x)$$
$$= \left\{ \begin{array}{ll} \psi(x)\bar{\psi}\left(x'\right), & \text{if } t > t' \\ -\bar{\psi}\left(x'\right)\psi(x), & \text{if } t' > t \end{array} \right\}$$

The Dirac Field 67 (where $t = x^0/c$, etc.). This definition differs by a factor (-1) in the $t' > t$ term from the corresponding boson definition, Eqs. (3.52) and (3.54). This change in sign reflects the anticommutation property of fermion fields. (A similar difference occurred in the definition of the normal products of boson and fermion fields.) In order to calculate the fermion propagator (4.58), using Eq. (4.59), we note that

$$\left\langle 0\left|\psi(x)\bar{\psi}\left(x'\right)\right|0\right\rangle = \left\langle 0\left|\psi^{+}(x)\bar{\psi}^{-}\left(x'\right)\right|0\right\rangle$$
$$= \left\langle 0\left|\left[\psi^{+}(x),\bar{\psi}^{-}\left(x'\right)\right]_{+}\right|0\right\rangle = \mathrm{i}S^{+}\left(x-x'\right),$$

where we used Eq. (4.53a); similarly

$$\left\langle 0\left|\bar{\psi}\left(x'\right)\psi(x)\right|0\right\rangle = \mathrm{i}S^{-}\left(x-x'\right).$$

Combining Eqs. (4.58)-(4.60), we obtain the fermion propagator:

$$\left\langle 0\left|\ \mathrm{T}\left\{\psi(\mathrm{x})\bar{\psi}\left(x'\right)\right\}\right|0\right\rangle = \mathrm{i}S_{\mathrm{F}}\left(x-x'\right)$$

where $\mathrm{S_F}(x)$ is defined, analogously to Eqs. (3.56) and (4.54), by

$$S_{\mathrm{F}}(x) = \theta(t)S^{+}(x) - \theta(-t)S^{-}(x) = \left(\mathrm{i}\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} + \frac{mc}{\hbar}\right)\Delta_{\mathrm{F}}(x).$$

Corresponding to the integral representation (3.58) for $\Delta_{\mathrm{F}}(x), S_{\mathrm{F}}(x)$ can be written

$$S_{\mathrm{F}}(x) = \frac{\hbar}{(2\pi\hbar)^4}\int \mathrm{d}^4p\mathrm{e}^{-\mathrm{i}px/\hbar}\frac{\not{p}+mc}{p^2-m^2c^2+\mathrm{i}\varepsilon},$$

where the integration in the complex p_0-plane is along the whole real axis: $-\infty < p_0 < \infty$. (Compare Fig. 3.6.)

As for the meson propagator, it is useful to visualize the fermion propagator in terms of Feynman diagrams. (As mentioned before, one must not take this interpretation too literally.)

For $t' < t$, the contribution to the fermion propagator (4.61) stems from the term (4.60a) and thus leads to the interpretation of (4.61) as creation of an electron at x', its propagation to x and its annihilation at x. On the other hand, for $t < t'$, the contribution to (4.61) comes from (4.60b) and is pictured as the emission of a positron at x, and its propagation to x' where it is annihilated. The two cases are illustrated in Fig. 4.1. Note that in both diagrams the arrow on the fermion line points from the vertex associated with the $\bar{\psi}$-field (x') to the vertex associated with the ψ-field (x), i.e. the arrow runs in the same direction as time for electrons, in the opposite direction for positrons.

These ideas are illustrated in Fig. 4.2, which shows two of the leading contributions, in lowest order of perturbation theory, to Compton scattering by electrons. Fig. 4.2(a) represents an electron propagating in the direction of the arrow on the fermion line, emitting the final photon at the vertex x' and absorbing the initial photon at x. Fig. 4.2(b) represents the corresponding process for $t < t'$.

The initial photon is annihilated at x, creating an electron-positron pair, i.e. the final electron and a positron which propagates to x, where it annihilates with the initial electron to produce the final-state photon. Note that the arrow is always in the same sense along a fermion line, and in both diagrams is from the x'-vertex (associated with the $\bar{\psi}$ operator) to the x-vertex (associated with ψ) on the internal fermion line. Thus the two diagrams are topologically equivalent, i.e. they can be continuously deformed into each other.

The fermion propagator (4.61) includes both the contributions from Figs. 4.2(a) and (b), and we represent it by the single Feynman diagram in Fig. 4.3, in which no time ordering of the vertices x and x' is implied, and consequently no time direction is attached to the internal fermion line joining x' and x. The orientation of the line $x'x$ is of no significance in this diagram. However, we shall continue to interpret external lines (i.e. lines entering or leaving a diagram from outside) in the same way for Feynman graphs as for time-ordered graphs. A line entering a diagram from the left-hand side will be interpreted as a particle present initially, one leaving a diagram on the right-hand side as one present finally. With these conventions, Figs. 4.4(a) and (b) represent Compton scattering by electrons and positrons, without further labelling being necessary. The arrows on the fermion lines are required to distinguish electrons (arrows on external lines from left to right) from positrons (arrows on external lines from right to left). Although no arrows are required on the photon lines, we shall at times use arrows on external photon lines to emphasize initially and finally present quanta. In the following we shall frequently use such Feynman diagrams.

9.4.2 The Electromagnetic Interaction and Gauge Invariance

Теория

We shall now consider the interaction of relativistic electrons with an electromagnetic field, specified by the scalar and vector potentials $\phi(x)$ and $\mathrm{A}(x)$. For this purpose we shall take over the procedure which is successful in non-relativistic quantum mechanics. In the latter case, making the substitution

$$\mathrm{i}\hbar\frac{\partial}{\partial t} \to \mathrm{i}\hbar\frac{\partial}{\partial t} - q\phi(x), \quad -\mathrm{i}\hbar\nabla \to -\mathrm{i}\hbar\boldsymbol{\nabla} - \frac{q}{c}\mathbf{A}(x),$$

in the free-particle Schrödinger equation leads to the correct wave equation for a particle of charge q in this field. [The corresponding classical result is contained in Eq. (1.59).] The substitution (4.64a) is usually referred to as the 'minimal substitution'. In terms of the four-vector potential $A^\mu(x) = (\phi, \mathrm{A})$, the minimal substitution takes the explicitly covariant form

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \to D_\mu = \left[\partial_\mu + \frac{\mathrm{i}q}{\hbar c}A_\mu(x)\right].$$

We shall assume that this substitution also correctly introduces the electromagnetic interaction into the Dirac equation. With the replacement (4.64b), and $q = -e$ for electrons, the Dirac equation (4.14) and the Lagrangian density (4.20) become

$$(\mathrm{i}\hbar\gamma^\mu\partial_\mu - mc)\,\psi(x) = -\frac{e}{c}\gamma^\mu A_\mu(x)\psi(x)$$

and

$$\left.\begin{aligned}\mathscr{L} &= c\bar{\psi}(x)\,(\mathrm{i}\hbar\gamma^\mu D_\mu - mc)\,\psi(x)\\ &= \mathscr{L}_0 + \mathscr{G}_\mathrm{I}\end{aligned}\right\}$$

where $\mathscr{L}_0$ is the Lagrangian density of the free Dirac field, i.e.

$$\mathscr{L}_0 = c\bar{\psi}(x)\,(\mathrm{i}\hbar\gamma^\mu\partial_\mu - mc)\,\psi(x),$$

and $\mathscr{L}_{\mathrm{I}}$ is the interaction Lagrangian density

$$y' = e\bar{\psi}(x)\gamma^{\mu}\psi(x)A_{\mu}(x),$$

which couples the conserved current $s^{\mu}(x) = c(-e)\bar{\psi}\gamma^{\mu}\psi$, Eq. (4.28), to the electromagnetic field.

To obtain the complete Lagrangian density for electrodynamics, we must add to Eq. (4.66) the Lagrangian density $\mathscr{L}_{\mathrm{rad}}$ of the radiation field, i.e. of the electromagnetic field in the absence of charges. This division is analogous to that of the Hamiltonian in Chapter 1, Eqs. (1.61)-(1.63). $\mathscr{L}_{\mathrm{rad}}$ depends on the potential $A_{\mu}(x)$ only, and we shall study it in the next chapter.

We know that it is only the electromagnetic fields E and **B** which have physical significance, not the potential A_{μ} itself, i.e. the theory must be invariant under the gauge transformations of the potentials, Eqs. (1.3). The latter can be written in the covariant form

$$A_{\mu}(x) \to A'_{\mu}(x) = A_{\mu}(x) + \partial_{\mu}f(x)$$

where $f(x)$ is an arbitrary function. The invariance of the theory under gauge transformations follows from that of the Lagrangian density. $\mathscr{L}_{\mathrm{rad}}$ has this invariance property, as we shall see in the next chapter. However, applying (4.69a) to Eq. (4.66), we obtain

$$\to\to y' = y + e\bar{\psi}(x)\gamma^{\mu}\psi(x)\partial_{\mu}f(x)$$

i.e. $\mathscr{L}$ is not gauge invariant with respect to (4.69a). We can restore gauge invariance by demanding that coupled with the gauge transformation (4.69a) of the electromagnetic potentials, the Dirac fields transform according to

$$\left.\begin{array}{ll} \psi(x) \to \psi'(x) & = \psi(x)\mathrm{e}^{\mathrm{i}ef(x)/\hbar c} \\ \bar{\psi}(x) \to \bar{\psi}'(x) & = \bar{\psi}(x)\mathrm{e}^{-\mathrm{i}ef(x)/\hbar c} \end{array}\right\}.$$

Under the coupled transformations (4.69a) and (4.69b), the Lagrangian densities (4.67) and (4.68) transform according to

$$\begin{aligned} \mathscr{L}_0 \to \mathscr{L}'_0 &= \mathscr{L}_0 - e\bar{\psi}(x)\gamma^{\mu}\psi(x)\partial_{\mu}f(x) \\ \mathscr{L}_1 \to \mathscr{L}'_1 &= \mathscr{L}_1 + e\bar{\psi}(x)\gamma^{\mu}\psi(x)\partial_{\mu}f(x). \end{aligned}$$

Consequently, $\mathscr{G} = \mathscr{Z}_0 + \mathscr{Y}_1$ remains invariant under the coupled transformations. Eqs. (4.69b) are called a local phase transformation, since the phase factors depend on x. In the special case that $f(x) = \text{const.}$, Eqs. (4.69b) reduce to a global phase transformation, considered in Section 2.4, where we saw that invariance under a global phase transformation leads to a conserved charge. We have now seen that gauge invariance of the theory requires invariance when simultaneously transforming the electromagnetic potentials according to the gauge transformation (4.69a) and the Dirac fields according to the local phase transformation (4.69b), and we shall in future refer to these coupled transformations as gauge transformations. It can be shown that the application of Noether's theorem to the invariance with respect to these (coupled) gauge transformations does not lead to a new conservation law, but only reproduces the conservation of charge.

In what follows we shall assume that Eq. (4.68) gives the correct interaction of quantum electrodynamics. One could try and add other gauge-invariant and Lorentzinvariant local interaction terms, but these can be excluded if we impose another restriction - renomalizability of the theory - as shown in Section 11.3.2. The ultimate justif ication for taking Eq. (4.68) as the correct interaction lies, of course, in the complete agreement between some of physics' most precise experiments and theoretical predictions based on this interaction.

9.4.3 Зи.II.1 Dirac Equation

Смотря на огонь

Согласно физической легенде, по-видимому, соответствующей действительности, в один из вечеров 1928 года Дирак смотрел на огонь и вдруг осознал, что, по причинам, теперь не имеющим никакого значения, ему необходимо релятивистское волновое уравнение первого порядка по пространственно-временным производным $\partial_\mu \equiv \partial/\partial x^\mu$. Уже в то время хорошо было известно уравнение Клейна-Гордона $(\partial^2 + m^2)\,\varphi = 0$, которое описывает свободную частицу массой m и является квадратичным по пространственно-временным производным. Оно фактически является уравнением движения в теории скалярного поля, изученной нами ранее.

На первый взгляд желание Дирака не имеет смысла. Предположительно уравнение должно иметь форму «некая линейная комбинации ∂_μ, действующая на некое поле ψ, равна некоторой константе, умноженной на поле». Обозначим линейную комбинацию через $c^\mu\partial_\mu$. Если c^μ есть четыре обычных числа, то 4-вектор c^μ определяет некоторое направление и потому уравнение не может быть лоренц-инвариантным.

Тем не менее, следуя Дираку, запишем (воспользовавшись современными обозначениям):

$$(i\gamma^\mu\partial_\mu - m)\,\psi = 0.$$

На данном этапе четыре величины $i\gamma^\mu$ являются просто коэффициентами при ∂_μ, а m - просто константой. Мы уже знаем, что в качестве γ^μ не могут выступать просто четыре числа. Итак, давайте выясним, что должны представлять из себя эти четыре объекта, чтобы обозначенное уравнение правильно описывало физику.

Действуя на (1) выражением $(i\gamma^\mu\partial_\mu + m)$, Дирак получил $-\left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + \ + m^2\right)\psi = 0$. По традиции, определим в дополнение к коммутатору $[A, B] = AB - BA$, известному нам из квантовой механики, антикоммутатор $\{A, B\} = AB + BA$. Поскольку производные коммутируют друг с другом, $\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\,\partial_\mu\partial_\nu$, получаем $\left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\,\partial_\mu\partial_\nu + m^2\right)\psi = 0$. В момент озарения Дирак осознал, что если бы

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu},$$

где $\eta^{\mu\nu}$ является метрикой Минковского, то он пришел бы к выражению $(\partial^2 + m^2)\,\psi = 0$, которое описывает частицу массой m, и, стало быть, (1) также описывало бы частицу массой m.

Так как $\eta^{\mu\nu}$ есть диагональная матрица с диагональными элементами $\eta^{00} = 1$ и $\eta^{jj} = -1$, формула (2) гласит, что $(\gamma^0)^2 = 1, (\gamma^j)^2 = -1$ и $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ при $\mu \neq \nu$. В этой ситуации коэффициенты γ^μ антикоммутируют друг с другом, а это означает, что они и в самом деле не могут выражаться обычными числами. Рассуждения Дирака будут иметь смысл, если мы сможем найти четыре таких объекта.

Алгебра Клиффорда

Говорят, что множество объектов γ^μ (очевидно, что d из них в d-мерном пространстве-времени), удовлетворяющих соотношению (2), образует алгебру Клиффорда. Я изложу математику алгебры Клиффорда чуть позднее. А сейчас достаточно будет проверить, что следующие матрицы 4×4 удовлетворяют условию (2):

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = I \otimes \tau_3,$$
$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \sigma^i \otimes i\tau_2.$$

Здесь σ и τ обозначают стандартные матрицы Паули. По историческим причинам четыре матрицы γ^μ называют гамма-матрицами - не слишком образное название! (Согласно принятому соглашению, нам совершенно неважно, верхний или нижний индекс имеет матрица Паули. С другой стороны, мы определяем $\gamma_\mu \equiv \eta_{\mu\nu}\gamma^\nu$, и теперь нам не все равно, сверху или снизу стоит у гамма-матрицы индекс; его необходимо рассматривать подобно индексу любого лоренцевского вектора. Такое соглашение удобно, поскольку оно позволяет записать $\gamma^\mu\partial_\mu = \gamma_\mu\partial^\mu$.)

При расчетах удобным оказывается обозначение с использованием прямого произведения: например, $\gamma^i\gamma^j = (\sigma^i \otimes i\tau_2)(\sigma^j \otimes i\tau_2) = (\sigma^i\sigma^j \otimes\ i^2\tau_2\tau_2) = -(\sigma^i\sigma^j \otimes I)$, и поэтому $\{\gamma^i, \gamma^j\} = -\{\sigma^i, \sigma^j\} \otimes I = -2\delta^{ij}$, как и должно быть.

Можете убедиться в том, что размерность матриц γ^μ не может быть меньше, чем 4×4. С математической точки зрения спинор Дирака ψ должен иметь четыре компоненты! Физическое содержание уравнения Дирака (1) становится более понятным, если перейти в импульсное пространство: подставляем $\psi(x) = \int [d^4p/(2\pi)^4]\, e^{-ipx}\psi(p)$ в (1) и получаем

$$(\gamma^\mu p_\mu - m)\,\psi(p) = 0$$

В силу лоренц-инвариантности выражения (5), которую мы докажем позднее, его физическое содержание можно исследовать в любой системе координат. В частности, мы можем рассмотреть его в системе покоя $p^\mu = (m, \overrightarrow{0})$, где уравнение сводится к виду

$$\left(\gamma^0 - 1\right)\psi = 0$$

Так как $(\gamma^0 - 1)^2 = -2(\gamma^0 - 1)$, мы узнаем в $(\gamma^0 - 1)$ проектор с точностью до тривиальной нормировки. В самом деле, записав (3) в явной форме, мы видим, что в уравнении Дирака нет ничсго таинственного: в явном виде формула (6) равна

$$\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\psi = 0,$$

то есть две из четырех компонент в ψ равны нулю.

Это вполне осмысленеенное утверждение, поскольку известно, что электрон имеет 2 физические степени свободы, а не 4. В этом свете загадочное уравнение Дирака является ни больше ни меньше чем проекцией, избавляющей нас от нежелательных степеней свободы. Сравним полученный результат с уравнением движения массивной частицы со спином 1 (глава I.5). Для него мы также отбрасывали одну из четырех компонент A^μ. Действительно, уравнение Клейна-Гордона $(\partial^2 + m^2)\,\varphi(x) = 0$ выкидывает те фурье-компоненты $\varphi(k)$, которые не удовлетворяют условию массовой поверхности $k^2 = m^2$. Из наших рассуждений вытекает единый взгляд на уравнения движения в релятивистской физике: эти уравнения просто отбрасывают нефизические составляющие.

Удобное обозначение, введенное Фейнманом: $\not{a} \equiv \gamma^\mu a_\mu$ для произвольного 4-вектора a_μ, сейчас стало стандартным. Уравнение Дирака при этом имеет вид $(i\ \not{\partial} - m)\psi = 0$.

Родственники гамма-матриц

При преобразовании Лоренца $x'^{\nu} = \Lambda^{\nu}_{\mu} x^{\mu}$ четыре компоненты векторного поля A_{μ} преобразуются как вектор. А как преобразуются 4 компоненты ψ? Конечно же, не так, как A_{μ}, ведь даже при вращении поля ψ и A_{μ} преобразуются неодинаково: одно - как спин 1/2, а другое - как спин 1. Запишем $\psi(x) \to \psi'(x') \equiv S(\Lambda)\psi(x)$ и попытаемся определить 4×4-матрицу $S(\Lambda)$.

Для начала было бы неплохо выбрать (и обозначить) 16 линейно независимых матриц размерностью 4×4. Пять из них мы уже знаем: единичная матрица и четыре матрицы γ^{μ}. Стратегия заключается в простом умножении γ^{μ} друг на друга, в результате чего образуется больше 4×4-матриц. Перемножаем таким образом до тех пор, пока не получим все 16 матриц. Поскольку квадрат гамма-матрицы γ^{μ} равен \pm 1, а γ^{μ} антикоммутируют друг с другом, достаточно рассмотреть только $\gamma^{\mu}\gamma^{\nu}, \gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}$ и $\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\rho}$ с отличными друг от друга μ, ν, λ и ρ. Таким образом, единственным произведением четырех гамма-матриц, которое мы должны рассмотреть, является

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

Эта комбинация настолько важна, что она имеет свое собственное имя! (Происхождение сего странного имени связано с тем, что в некоторых старомодных обозначениях временная координата называлась x^4 с соответствующей γ^4.) Имеем

$$\gamma^5 = i\left(I \otimes \tau_3\right)\left(\sigma^1 \otimes i\tau_2\right)\left(\sigma^2 \otimes i\tau_2\right)\left(\sigma^3 \otimes i\tau_2\right) = i^4\left(I \otimes \tau_3\right)\left(\sigma^1\sigma^2\sigma^3 \otimes \tau_2\right)$$

и поэтому

$$\gamma^5 = I \otimes \tau_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

Восстановив множитель i, видим, что матрица γ^5 явно является эрмитовой. Ее важной особенностью является то, что γ^5 антикоммутирует со всеми γ^{μ}:

$$\left\{\gamma^5, \gamma^{\mu}\right\} = 0$$

Далее, видим, что произведение трех различных гамма-матриц может быть записано как $\gamma^{\mu}\gamma^5$ (например, $\gamma^1\gamma^2\gamma^3 = -i\gamma^0\gamma^5$). Наконец, используя (2), можно записать произведение двух гамма-матриц как $\gamma^{\mu}\gamma^{\nu} = \eta^{\mu\nu}$ - $-i\sigma^{\mu\nu}$, где

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right]$$

Всего имеется $4 \cdot 3/2 = 6$ таких $\sigma^{\mu\nu}$-матриц.

Подсчитаем их все и получим число 16. Множество 16 матриц $\{1, \gamma^{\mu}, \sigma^{\mu\nu}, \gamma^{\mu}\gamma^5, \gamma^5\}$ образует полный базис в пространстве всех 4×4-матриц, то есть любую 4×4-матрицу можно записать в виде линейной комбинации этих 16 матриц. Полезно записать $\sigma^{\mu\nu}$ в явном виде, используя представления (3) и (4):

$$\sigma^{0i} = i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$
$$\sigma^{ij} = \varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}.$$

Видим, что σ^{ij} являются просто удвоенными матрицами Паули, например,

$$\sigma^{12} = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}$$

Преобразование Лоренца

Вспомним из курса квантовой механики, что произвольное вращение может быть записано в форме $e^{i\vec{\theta}\vec{J}}$, где $\vec{J}$ - три генератора вращения, а $\vec{\theta}$ - три параметра вращения. Вспомним также, что группа Лоренца содержит, помимо вращений, еще и собственные преобразования Лоренца (бусты) с тремя соответствующими генераторами $\vec{K}$. Из курса электромагнетизма вспомним, что 6 генераторов $\{\vec{J}, \vec{K}\}$ преобразуются под действием группы Лоренца как компоненты антисимметричного тензора, подобно электромагнитному полю $F_{\mu\nu}$, и поэтому могут быть обозначены как $J_{\mu\nu}$. Более подробно обсудим это в главе II.3. В настоящий момент достаточно запомнить, что, используя такое обозначение, мы можем записать преобразование Лоренца в виде $\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}$, где J^{ij} порождают вращения, J^{0i} порождают бусты, а антисимметричный тензор равен $\omega_{\mu\nu} = -\omega_{\nu\mu}$, шесть составляющих которого ($6 = 4 \cdot 3/2$) соответствуют трем вращениям и трем бустам.

Опираясь на предыдущие рассуждения и тот факт, что у нас есть шесть матриц $\sigma^{\mu\nu}$, можно ожидать, что с точностью до общего численного множителя матрицы $\sigma^{\mu\nu}$ должны представлять собой 6 генераторов $J_{\mu\nu}$ группы Лоренца, действующей на спинор. Действительно, это предположение подтверждается анализом того, как действует вращение $e^{-\frac{1}{2}i\omega_{ij}J^{ij}}$. Возвращаясь к (12), видим, что если бы J^{ij} были равны $\frac{1}{2}\sigma^{ij}$, то это в точности соответствовало бы тому, как частица со спином 1/2 преобразуется в квантовой механике. Более точно, разобьем четыре компоненты дираковского спинора на две группы по две компоненты:

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

Из (12) следует, что при вращении вокруг третьей оси $\phi \to e^{-i\omega_{12}\frac{1}{2}\sigma^3}\phi$ и $\chi \to e^{-i\omega_{12}\frac{1}{2}\sigma^3}\chi$. Приятно видеть, что ϕ и χ преобразуются как 2-компонентные спиноры Паули.

Таким образом, мы пришли к выводу, что преобразование Лоренца Λ, действующее на ψ, представляется в виде $S(\Lambda) = e^{-(i/4)\omega_{\mu\nu}\sigma^{\mu\nu}}$, и поэтому, действуя на ψ, генераторы $J^{\mu\nu}$ в самом деле равны $\frac{1}{2}\sigma^{\mu\nu}$. Следовательно, можно ожидать, что если $\psi(x)$ удовлетворяет уравнению Дирака (1), то $\psi'(x') \equiv S(\Lambda)\psi(x)$ будет удовлетворять уравнению Дирака в штрихованной системе отсчета

$$\left(i\gamma^\mu\partial'_\mu - m\right)\psi'(x') = 0$$

где $\partial'_\mu \equiv \partial/\partial x'^\mu$. Чтобы это доказать, вычислим $\left[\sigma^{\mu\nu}, \gamma^\lambda\right] = 2i\left(\gamma^\mu\eta^{\nu\lambda} - \ -\gamma^\nu\eta^{\mu\lambda}\right]$ и, следовательно, для бесконечно малого ω получим $S\gamma^\lambda S^{-1} = \gamma^\mu - (i/4)\omega_{\mu\nu}\left[\sigma^{\mu\nu}, \gamma^\lambda\right] = \gamma^\lambda + \gamma^\mu\omega^\lambda_\mu$. Строя конечное преобразование Лоренца из комбинации бесконечно малых преобразований (так же, как при обсуждении группы вращений в квантовой механике), получаем $S\gamma^\lambda S^{-1} = \Lambda^\lambda_\mu\gamma^\mu$

Билинейные комбинации дираковских полей

Алгебра Клиффорда говорит нам, что $(\gamma^0)^2 = +1$ и $(\gamma^i)^2 = -1$, и, таким образом, указывает на необходимость i в (4). Одним из следствий такой необходимости в i является то, что γ^0 оказывается эрмитовой, в то время как γ^i являются антиэрмитовыми. Этот факт удобно выразить следующим образом:

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$

Таким образом, билинейная комбинация $\psi^\dagger\gamma^\mu\psi$ не является эрмитовой, как вы могли подумать; эрмитовой же является $\bar{\psi}\gamma^\mu\psi \text{c}\bar{\psi} \equiv \psi^\dagger\gamma^0$. Необходимость введения в релятивисткой физике параметра $\bar{\psi}$, помимо $\psi^\dagger$, обуславливается $(+,-,-,-)$-сигнатурой метрики Минковского.

Оказывается, $(\sigma^{\mu\nu})^\dagger = \gamma^0\sigma^{\mu\nu}\gamma^0$. Следовательно, $S(\Lambda)^\dagger = \gamma^0 e^{(i/4)\omega_{\mu\nu}\sigma^{\mu\nu}}\gamma^0$ (что, кстати, явно указывает на то, что S не является унитарным - этот факт нам известен из условия неэрмитовости σ_{0i}), и поэтому

$$\bar{\psi}'(x') = \psi(x)^\dagger S(\Lambda)^\dagger\gamma^0 = \bar{\psi}(x)e^{+(i/4)\omega_{\mu\nu}\sigma^{\mu\nu}}$$

Имеем

$$\bar{\psi}'(x')\,\psi'(x') = \bar{\psi}(x)e^{+(i/4)\omega_{\mu\nu}\sigma^{\mu\nu}}e^{-(i/4)\omega_{\mu\nu}\sigma^{\mu\nu}}\psi(x) = \bar{\psi}(x)\psi(x)$$

Вы, вероятно, привыкли писать $\psi^\dagger\psi$ в нерелятивистской физике. В релятивистской физике вы должны привыкнуть писать $\bar{\psi}\psi$. Именно $\bar{\psi}\psi$, а не $\psi^\dagger\psi$, преобразуется как лоренцевский скаляр.

Очевидно, что существует 16 билинейных комбинаций дираковских полей $\bar{\psi}\Gamma\psi$, соответствующих 16 линейно независимым Γ. Теперь можете рассмотреть, как преобразуются различные фермионные билинейные комбинации (упражнение II.1.1). Используемое обозначение очень удачно: различные объекты преобразуются так, как им и следовало бы. Мы просто смотрим на их лоренцевские индексы. Таким образом, $\bar{\psi}(x)\gamma^\mu\psi(x)$ преобразуется как лоренцевский вектор.

Четность

Важной дискретной симметрией в физике является четность, или зер-

кальное отражение [1]

$$x^\mu \to x^\mu = \left(x^0, -\vec{x}\right)$$

Умножим уравнение Дирака (1) на γ^0 : $\gamma^0\left(i\gamma^\mu\partial_\mu - m\right)\psi(x) = 0 = \left(i\gamma^\mu\partial'_\mu - -m\right)\gamma^0\psi(x)$, где $\partial'_\mu \equiv \partial/\partial x'^\mu$. Таким образом

$$\psi'(x') \equiv \eta\gamma^0\psi(x)$$

удовлетворяет уравнению Дирака в пространственно-отраженном мире (где η является произвольной фазой, которую можно положить равной 1).

Обратите внимание, что, например, $\bar{\psi}'(x')\,\psi'(x') = \bar{\psi}(x)\psi(x)$, но $\bar{\psi}'(x')\,\gamma^5\psi'(x') = \bar{\psi}(x)\gamma^0\gamma^5\gamma^0\psi(x) = -\bar{\psi}(x)\gamma^5\psi(x)$. При преобразовании

[1] Вращение состоит из всех линейных [2] преобразований $x^i \to R^{ij}x^j$, таких что $\det R = +1$. Преобразования с $\det R = -1$ состоят из отражения с последующим вращением. В

(3 + 1)-мерном пространстве-времени отражение можно определить как изменение знака одной из пространственных координат или всех трех пространственных координат. Эти 'Іве операции связаны вращением. Заметьте. что зеркальное отражение в нечетномерном пространстве-времени не эквивалентно инверсии пространства. при котором меняется знак всех пространственных координат (см. упражнение II.1.11).

[2] Точнее ортогональных. - Прим. ped. Лоренца $\bar{\psi}(x)\gamma^5\psi(x)$ и $\bar{\psi}(x)\psi(x)$ преобразуются одинаковым образом, но при пространственном отражении они преобразуются прямо противоположно друг другу; иными словами, в то время как $\bar{\psi}(x)\psi(x)$ преобразуется как скаляр, $\bar{\psi}(x)\gamma^5\psi(x)$ преобразуется как псевдоскаляр.

Теперь вы уже в состоянии выполнить крайне важные упражнения к этой главе.

Дираковский лагранжиан

Интересный вопрос: какой лагранжиан приводит к уравнению Дирака? Ответ таков:

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi.$$

Поскольку ψ - комплексное поле, для вывода уравнения движения Эйлера - Лагранжа можно варьировать ψ и $\bar{\psi}$ независимо друг от друга. Таким образом, $\partial_\mu\left(\partial\mathcal{L}/\delta\partial_\mu\psi\right) - \delta\mathcal{L}/\delta\psi = 0$ дает $\partial_\mu\left(i\bar{\psi}\gamma^\mu\right) + m\bar{\psi} = 0$, что в условиях эрмитова сопряжения и умножения на γ^0 приводит к уравнению Дирака (1). Другое вариационное уравнение $\partial_\mu\left(\delta\mathcal{L}/\delta\partial_\mu\bar{\psi}\right) - \delta\mathcal{L}/\partial\bar{\psi} = 0$ дает нам уравнение Дирака еще более непосредственно. (Если вам не нравится несимметричное рассмотрение ψ и $\bar{\psi}$, вы всегда можете проинтегрировать в действии по частям так, что ∂_μ в лагранжиане будет действовать на $\bar{\psi}$, а затем взять среднее от этих двух форм лагранжиана. В действие $S = \int d^4x\mathcal{L}$ поля ψ и $\bar{\psi}$ входят симметрично.)

Медленные и быстрые электроны

Для данного набора гамма-матриц легко решить уравнение Дирака

$$(\not{p} - m)\psi(p) = 0$$

относительно $\psi(p)$: оно представляет собой простое матричное уравнение (см. упражнение II.1.3).

Заметьте, что если кто-то использует гамма-матрицы γ^μ, вы вправе вместо этого использовать $\gamma'^\mu = W^{-1}\gamma^\mu W$ с произвольной обратимой 4×4-матрицей W. Очевидно, что γ'^μ также удовлетворяют алгебре Клиффорда. Такая свобода выбора соответствует простому изменению базиса. Физика не зависит от выбора базиса, зато выбор наиболее удобного базиса зависит от физики.

Например, предположим, что мы исследуем медленно движущиеся электроны. Используем для этого базис, определенный в (3) и (4), и двухкомпонентное разбиение ψ (13). Из (6) следует, что для покоящегося элек- трона $\chi(p) = 0$, поэтому можно ожидать, что для медленно движущегося электрона $\chi(p)$ будет много меньше, чем $\phi(p)$.

В противоположность этому, для импульсов, намного превышающих массу, можно приближенно записать (20) как $p\psi(p) = 0$. Умножая слева на γ^5, видим, что если $\psi(p)$ является решением, то $\gamma^5\psi(p)$ также является решением, поскольку γ^5 антикоммутирует с γ^μ. В силу того, что $\left(\gamma^5\right)^2 = 1$, мы можем построить два проекционных оператора $P_L \equiv \frac{1}{2}(1 - \gamma^5)$ и $P_R \equiv \frac{1}{2}(1+\gamma^5)$, удовлетворяющих условиям $P_L^2 = P_L, P_R^2 = P_R$ и $P_LP_R = 0$. Чрезвычайно полезно ввести две комбинации $\psi_L = \frac{1}{2}(1-\gamma^5)\psi$ и $\psi_R = \frac{1}{2}(1+\gamma^5)\psi$.

Обратите внимание, что $\gamma^5\psi_L = -\psi_L$ и $\gamma^5\psi_R = +\psi_R$. Говоря физически, релятивистский электрон имеет две степени свободы: он может вращаться либо по, либо против часовой стрелки вокруг направления движения. В качестве упражнения докажите, что ψ_L и ψ_R соответствуют именно этим двум возможностям. Индексы L и R указывают на то, левой или правой является частица. Таким образом, для быстро движущегося электрона более удобным оказывается базис, известный как базис Вейля, в котором диагональной оказывается матрица γ^5, а не γ^0. Вместо (3) выбираем

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = I \otimes \tau_1$$

γ^i остаются теми же самыми, что в (4). Так определяется базис Вейля. Вычислим теперь матрицу

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = i\left(I \otimes \tau_1\right)\left(\sigma^1\sigma^2\sigma^3 \otimes i^3\tau_2\right) = -\left(I \otimes \tau_3\right) = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix},$$

которая в самом деле оказывается диагональной, как мы того и хотели. Разложение на левые и правые поля, естественно, не зависит от нашего выбора базиса. Однако в базисе Вейля у нас есть одно хорошее свойство: ψ_L имеет две верхние компоненты, а ψ_R - две нижние. Спиноры ψ_L и ψ_R называются спинорами Вейля.

Обратите внимание, что при переходе от базиса Дирака к базису Вейля γ^0 и γ^5 меняются местами (с точностью до знака):

Дирак: γ^0 диагональна, Вейль: γ^5 диагональна.

Каким базисом пользоваться, зависит от физики: когда мы имеем дело с медленными частицами со спином $1/2$, то предпочтительнее иметь диа- гональной матрицу γ^0, в то время как для быстро движущихся частиц со спином $1/2$ удобнее иметь диагональной γ^5.

Замечу мимоходом, что если мы определим $\sigma^\mu \equiv (I, \vec{\sigma})$ и $\bar{\sigma}^\mu \equiv (I, -\vec{\sigma})$, то в базисе Вейля можно записать

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

в более компактной форме. (Более подробно данный вопрос обсуждается в приложении E.)

Киральность

Вне зависимости от того, является ли дираковское поле $\psi(x)$ массивным или безмассовым, чрезвычайно полезно разбить ψ на левое и правое поля $\psi(x) = \psi_L(x) + \psi_R(x) \equiv \frac{1}{2}\left(1-\gamma^5\right)\psi(x) + \frac{1}{2}\left(1+\gamma^5\right)\psi(x)$. В качестве упражнения покажите, что лагранжиан Дирака можно записать в виде

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi = \bar{\psi}_L i\not{\partial}\psi_L + \bar{\psi}_R i\not{\partial}\psi_R - m\left(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L\right)$$

Кинетическая энергия связывает левое поле с левым, а правое поле с правым, тогда как массовый член связывает левое поле с правым, а правое поле с левым.

Преобразование $\psi \to e^{i\theta}\psi$ оставляет лагранжиан $\mathcal{L}$ инвариантным. Применяя теорему Нётер, получаем связанный с этой симметрией сохраняющийся ток $J^\mu = \bar{\psi}\gamma^\mu\psi$. Проецируя на левые и правые поля, видим, что они преобразуются одинаково: $\psi_L \to e^{i\theta}\psi_L$ и $\psi_R \to e^{i\theta}\psi_R$.

Если $m = 0$, то $\mathcal{L}$ обладает дополнительной симметрией, известной как киральная симметрия, относительно которой $\psi \to e^{i\phi\gamma^5}\psi$. Теорема Нётер гласит, что киральный ток $J^{5\mu} \equiv \bar{\psi}\gamma^\mu\gamma^5\psi$ сохраняется. Левые и правые поля преобразуются противоположным образом: $\psi_L \to e^{-i\phi}\psi_L$ и $\psi_R \to e^{i\phi}\psi_R$. Все вышесказанное становится особенно очевидным, если выразить $\mathcal{L}$ через ψ_L и ψ_R, как в (24).

В 1956 году Ли и Янг выдвинули предположение, согласно которому слабое взаимодействие не сохраняет четность. Со временем стало понятно (мне пришлось опустить замечательную главу, в которой бы рассказывалось о развитии физики элементарных частиц, но я настоятельно советую вам прочитать об этом!), что лагранжиан слабого взаимодействия в общей форме записывается как

$$\mathcal{L} = G\bar{\psi}_{1L}\gamma^\mu\psi_{2L}\bar{\psi}_{3L}\gamma_\mu\psi_{4L}$$

где $\psi_{1,2,3,4}$ обозначают четыре дираковских поля, а G - константу связи Ферми. Очевидно, что лагранжиан нарушает четность: при пространственном отражении левые поля преобразуются в правые поля и наоборот.

Впредь, когда я буду говорить, что лагранжиан имеет определенную форму, я буду включать в лагранжиан только один или несколько значимых членов, как в (25). Остальные члены в лагранжиане, такие как $\bar{\psi}_1(i\partial\!\!\!/ - m_1)\,\psi_1$, будут только подразумеваться. Если член не является эрмитовым, то подразумевается, что мы также добавляем в лагранжиан эрмитово сопряженный.

Взаимодействия

Как мы видели в (25), что с помощью билинейных комбинаций по спинорному полю, классификацию которых вы получили в упражнении, легко ввести взаимодействия. В качестве другого примера можно связать скалярное поле φ с полем Дирака, добавив член $g\varphi\bar{\psi}\psi$ (с произвольной константой связи g) в лагранжиан $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi($ и, конечно же, добавив лагранжиан для φ). Аналогичным образом можно связать векторное поле A_μ, добавив член $eA_\mu\bar{\psi}\gamma^\mu\psi$. Обратите внимание, что в этом случае мы можем определить ковариантную производную $D_\mu = \partial_\mu - ieA_\mu$ и записать $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + eA_\mu\bar{\psi}\gamma^\mu\psi = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi$. Таким образом, лагранжиан дираковского поля, взаимодействующего с векторным полем массой μ, имеет вид

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\mu^2 A_\mu A^\mu$$

Если масса μ равна нулю, имеем лагранжиан квантовой электродинамики. Варьируя по $\bar{\psi}$, получаем уравнение Дирака в присутствии электромагнитного поля:

$$\left[i\gamma^\mu\left(\partial_\mu - ieA_\mu\right) - m\right]\psi = 0$$

Зарядовое сопряжение и антиматерия

Взаимодействие с электромагнитным полем связано с понятием заряда и, следовательно, зарядового сопряжения. Попробуем изменить знак заряда e. Возьмем комплексное

сопряжение от (27): $[-i\gamma^{\mu *}(\partial_\mu + ieA_\mu) - m]\psi^* = 0$. Комплексно сопрягая (2), мы видим, что $-\gamma^{\mu *}$ также удовлетворяют алгебре Клиффорда и, стало быть, должны выражаться матрицами γ^μ, записанными в другом базисе, то есть существует такая матрица $C\gamma^0$ (обозначение с явным множителем γ^0 является стандартным; см. ниже), что $-\gamma^{\mu *} = (C\gamma^0)^{-1}\gamma^\mu(C\gamma^0)$. Подставляя, получаем

$$[i\gamma^\mu(\partial_\mu + ieA_\mu) - m]\psi_c = 0,$$

где мы определили $\psi_c \equiv C\gamma^0\psi^*$. Таким образом, если ψ является полем электрона, то ψ_c является полем частицы с зарядом, противоположным заряду электрона, но с такой же, как у электрона, массой, то есть полем позитрона.

Открытие антиматерии было одним из самых значимых в физике двадцатого столетия. Более подробно об антиматерии мы поговорим в следующей главе.

Может оказаться полезным посмотреть на конкретную форму матрицы зарядового сопряжения C. Определяющее уравнение для C можно записать в виде $C\gamma^0\gamma^{\mu *}\gamma^0 C^{-1} = -\gamma^\mu$. Комплексно сопрягая уравнение $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, получаем $(\gamma^\mu)^T = \gamma^0\gamma^{\mu *}\gamma^0$, если γ^0 - вещественная матрица. Таким образом,

$$(\gamma^\mu)^T = -C^{-1}\gamma^\mu C$$

что объясняет, почему C необходимо задавать с добавлением γ^0.

В обоих базисах Дирака и Вейля γ^2 является единственной мнимой гамма-матрицей. Значит, определяющее уравнение для C свидетельствует о том, что $C\gamma^0$ коммутирует с γ^2, но антикоммутирует с тремя другими γ-матрицами. Отсюда очевидно, что $C = \gamma^2\gamma^0$ [с точностью до произвольной фазы, которая не фиксируется уравнением (29)] и действительно $\gamma^2\gamma^{\mu *}\gamma^2 = \gamma^\mu$. Заметьте, что имеет место простое (и удовлетворяющееся) соотношение

$$\psi_c = \gamma^2\psi^*.$$

Вы можете легко убедиться в том (упражнение 8), что зарядовое сопряжение левого поля является правым и наоборот. Этот факт, как мы увидим позднее, оказывается ключевым при построении теории великого объединения. Экспериментально установлено, что нейтрино является левым полем. Таким образом, уже сейчас можно предсказать, что антинейтрино - это правое поле.

Более того, ψ_c преобразуется как спинор. Давайте проверим: при преобразовании Лоренца $\psi \to e^{-(i/4)\omega_{\mu\nu}\sigma^{\mu\nu}}\psi$ с учетом комплексного сопряжения имеем $\psi^* \to e^{+(i/4)\omega_{\mu\nu}(\sigma^{\mu\nu})^*}\psi^*$; следовательно, $\psi_c \to \gamma^2 e^{+(i/4)\omega_{\mu\nu}(\sigma^{\mu\nu})^*}\psi^* = e^{-(i/4)\omega_{\mu\nu}\sigma^{\mu\nu}}\psi_c$. [Вспомните из (10), что в определении $\sigma^{\mu\nu}$ явно присутствует i.] Обратите внимание, что в обоих базисах Дирака и Вейля $C^T = \gamma^0\gamma^2 = -C$.

Майорановское нейтрино

Поскольку ψ_c преобразуется как спинор, Майорана [3] заметил, что лоренц-инвариантность допускает не только уравнение Дирака $i\not{\partial}\psi = m\psi$, но и уравнение Майораны

$$i\not{\partial}\psi = m\psi_c.$$

Комплексно сопрягая это уравнение и умножая на γ^2, мы получаем $-\gamma^2 i\gamma^{\mu *}\partial_\mu\psi^* = \gamma^2 m(-\gamma^2)\psi$, то есть $i\not{\partial}\psi_c = m\psi$. Таким образом,

$-\partial^2\psi = i\not{\partial}(i\not{\partial}) = i\not{\partial}\psi_c = m^2\psi$. Как мы и предполагали, m в самом деле является массой, называемой майорановской массой, частицы, связанной с полем ψ.

Уравнение Майораны (31) можно получить из лагранжиана [4]

$$\mathcal{L} = \bar{\psi} i\not{\partial}\psi - \frac{1}{2}m\left(\psi^T C\psi + \bar{\psi}C\bar{\psi}^T\right)$$

варьируя по $\bar{\psi}$.

Так как ψ и ψ_c несут противоположный заряд, уравнение Майораны, в отличие от уравнения Дирака, может применяться только к электрически нейтральным полям. Однако раз ψ_c является правым при условии, что ψ - левое поле, то уравнение Майораны, опять же в отличие от уравнения Дирака, сохраняет спиральность. Уравнение Майораны, таким образом, практически идеально подходит для описания нейтрино.

По замыслу нейтрино предполагалось безмассовым, но несколько лет назад экспериментаторы обнаружили, что оно имеет очень малую, но не нулевую массу. На момент написания этой книги остается неизвестным, является ли масса нейтрино дираковской или майорановской. В главе VII.7 мы увидим, что майорановская масса для нейтрино естественно возникает в $SO(10)$-теории великого объединения.

И наконец, может быть, что $\psi = \psi_c$, в этом случае ψ называется майорановским спинором.

[3] Карьера Этторе Майорана была блестящей, но трагически короткой. В возрасте 32 лет он пропал без вести около побережья Сицилии во время лодочной прогулки. Истинная причина его смерти остается загадкой.

[4] Вспоминая, что C является антисимметричной, можно предположить, что $\psi^T C\psi = C_{\alpha\beta}\psi_\alpha\psi_\beta$ обращается в нуль. В последующих главах мы узнаем, что ψ необходимо рассматривать как антикоммутирующие «грассмановы переменные».

Обращение времени

Наконец мы добрались до обращения времени [5], которое, как вы, вероятно, знаете, гораздо труднее понять, чем четность или зарядовое сопряжение. В знаменитой статье 1932 года Вигнер показал, что обращению времени соответствует антиунитарный оператор. Так как эта особенность появляется уже в нерелятивистской квантовой физике, то, в некотором смысле, книга по релятивистской теории поля не обязана разъяснять, почему обращение времени соответствует антиунитарному оператору. Тем не менее я постараюсь как можно более доступно это объяснить. Воспользуюсь для этого подходом: «пусть физика, а точнее, уравнения направляют нас».

Рассмотрим уравнение Шредингера $i(\partial/\partial t)\Psi(t) = H\Psi(t)$ (пусть, для определенности, $H = -(1/2m)\nabla^2 + V(\vec{x})$ описывает всего лишь простую одночастичную нерелятивистскую квантовую механику). Опускаем зависимость Ψ от $\vec{x}$. Рассмотрим преобразование $t \to t' = -t$. Необходимо найти такую $\Psi'(t')$, что $i\left(\partial/\partial t'\right)\Psi'(t') = H\Psi'(t')$. Запишем $\Psi'(t') = T\Psi(t)$, где T - некоторый оператор, подлежащий определению (с точностью до некоторого фазового множителя η). Подставляя, получаем $i[\partial/\partial(-t)]T\Psi(t) = HT\Psi(t)$. Умножая на T^{-1}, получаем $T^{-1}(-i)T(\partial/\partial t)\Psi(t) = T^{-1}HT\Psi(t)$. Так как H ни в каком виде не содержит t, мы хотим, чтобы $T^{-1}H = HT^{-1}$. Тогда $T^{-1}(-i)T(\partial/\partial t)\Psi(t) = H\Psi(t)$. Как и Вигнер, мы вынуждены сделать заключение, согласно которому

$$T^{-1}(-i)T = i$$

Проще говоря, в квантовой физике время появляется вместе с i, и поэтому, изменение знака времени также означает изменение знака i.

Пусть $T = UK$, где K комплексно сопрягает все, что стоит от него справа. Тогда $T^{-1} = KU^{-1}$ и (33) остается справедливым, если $U^{-1}iU = i$, то есть если U^{-1} является обычным (унитарным) оператором, который ничего не делает с i. В дальнейшем мы определим U. Присутствие K делает T «антиунитарным».

Проверим, что все это работает для бесспиновой частицы в состоянии плоской волны $\Psi(t) = e^{i(\vec{k}\cdot\vec{x}-Et)}$. Подставляя, получаем $\Psi'(t') = T\Psi(t) = UK\Psi(t) = U\Psi^*(t) = Ue^{-i(\vec{k}\cdot\vec{x}-Et)}$; так как Ψ состоит только

[5] Кстати, я не думаю, что мы до конца осознаем все следствия инвариантности относительно обращения времени. См. A. Zee. «Night thoughts on consciousness and time reversal» in: Art and Symmetry in Experimental Physics, pp. 246-249. из одной компоненты, U есть простой фазовый множитель [6]η, который мы можем выбрать равным 1. Переписывая, получаем $\Psi'(t) = e^{-i(\vec{k}\cdot\vec{x}+Et)} = e^{i(-\vec{k}\cdot\vec{x}-Et)}$. В самом деле, Ψ' описывает плоскую волну, распространяющуюся в противоположном направлении. Очень важно, что $\Psi'(t) \propto e^{-iEt}$ и, следовательно, имеет положительную энергию, как и должно быть. Обратите внимание, что при действии на бесспиновые частицы $T^2 = UKUK = UU^*K^2 = +1$

Далее рассмотрим нерелятивистский электрон со спином $1/2$. Действуя T на состояние $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ со спином, направленным вверх, мы хотим получить состояние со спином, направленным вниз, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Таким образом, нам нужна нетривиальная матрица $U = \eta\sigma_2$, изменяющая направление спина:

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = U\begin{pmatrix} 1 \\ 0 \end{pmatrix} = i\eta\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Аналогично при воздействии T на состояние со спином, направленным вниз, появляется состояние со спином, направленным вверх. Обратите внимание, что при действии на частицу со спином $1/2$

$$T^2 = \eta\sigma_2 K\eta\sigma_2 K = \eta\sigma_2\eta^*\sigma_2^* KK = -1$$

Этим объясняется вырождение Крамера: в системе с нечетным числом электронов в электрическом поле, неважно насколько сложным, каждый энергетический уровень является двукратно вырожденным. Доказательство очень простое: так как система инвариантна относительно обращения времени, Ψ и $T\Psi$ имеют одинаковую энергию. Предположим, что они фактически представляют одно и то же состояние. Тогда $T\Psi = e^{i\alpha}\Psi$, но при этом $T^2\Psi = T(T\Psi) = Te^{i\alpha}\Psi = e^{-i\alpha}T\Psi = \Psi \neq -\Psi$. Поэтому Ψ и $T\Psi$ должны представлять два различных состояния.

Все эти удивительные вещи вы, как я уже заметил, могли и должны были выучить из курса квантовой механики. Моя же обязанность здесь состоит в том, чтобы показать вам, как все это работает для уравнения Дирака. Умножая (1) слева на γ^0, имеем $i(\partial/\partial t)\psi(t) = H\psi(t)$ с $H = -i\gamma^0\gamma^i\partial_i + +\gamma^0 m$. Еще раз подчеркну, что мы хотим получить $i\left(\partial/\partial t'\right)\psi'\left(t'\right) = H\psi'\left(t'\right)$ с $\psi'\left(t'\right) = T\psi(t)$ и некоторым оператором T, который и предстоит определить. Предыдущие рассуждения остаются в силе, если $T^{-1}HT = H$, то

[6] Это именно фазовый множитель, а не произвольное комплексное число, потому что мы накладываем условие $|\Psi'|^2 = |\Psi|^2$. есть $KU^{-1}HUK = H$. Таким образом, мы требуем

$KU^{-1}\gamma^0 UK = \gamma^0$ и $KU^{-1}\left(i\gamma^0\gamma^i\right)UK = i\gamma^0\gamma^i$. Умножая на K слева и справа, мы видим, что необходимо найти такое U, для которого $U^{-1}\gamma^0 U = \gamma^{0*}$ и $U^{-1}\gamma^i U = -\gamma^{i*}$. Теперь ограничимся рассмотрением базисов Дирака и Вейля, в каждом из которых мнимым объектом является γ^2. Итак, что изменяет знак γ^1 и γ^3, но не изменяет знак γ^0 и γ^2? Ну, хорошо, подойдет $U = \eta\gamma^1\gamma^3$ (с произвольным фазовым множителем η):

$$\psi'(t') = \eta\gamma^1\gamma^3 K\psi(t)$$

Так как γ^i одинаковы в обоих базисах Дирака и Вейля, для каждого из них из (4) получаем

$$U = \eta\left(\sigma^1\otimes i\tau_2\right)\left(\sigma^3\otimes i\tau_2\right) = \eta i\sigma^2\otimes 1.$$

Как мы и предполагали, при действии на двухкомпонентные спиноры, содержащиеся в ψ, оператор обращения времени T умножает их на $i\sigma^2$. Заметьте также, что, как и в нерелятивистском случае, $T^2\psi = -\psi$.

От вашего внимания, вероятно, не ускользнуло то, что γ^0 присутствует в операторе четности (18), γ^2 в зарядовом сопряжении (30), а $\gamma^1\gamma^3$ в обращении времени (34). Если заменить дираковскую частицу на ее античастицу и обратить пространство-время, то появится γ^5.

CPT-теорема

Существует основополагающая теорема, утверждающая, что любая локальная лоренц-инвариантная теория поля должна быть инвариантна относительно [7]$\mathcal{CPT}$ - совместного действия зарядового сопряжения, четности и обращения времени. Наивное доказательство сводится к проверке того, что любое локальное лоренц-инвариантное взаимодействие, какое бы вы ни написали (к примеру, (25)), даже если оно нарушает зарядовое сопряжение, четность или обращение времени по отдельности, обязательно будет сохранять $\mathcal{CPT}$. Для более строго доказательства необходим гораздо более сложный формализм, который я не буду здесь развивать. Советую вам почитать о феноменологическом исследовании зарядового сопряжения, четности, обращении времени и $\mathcal{CPT}$, - несомненно одной из захватывающих глав истории физики.

[7] Довольно педантичный вопрос, но, возможно, неясный для некоторых студентов: я четко провожу различия между действием зарядового сопряжения $\mathcal{C}$ и матрицей C :: зарядовое сопряжение состоит в комплексном сопряжении ψ с последующим перемешиванием ее компонент с помощью $C\gamma^0$. Аналогично я различаю оператор обращения времени $\mathcal{T}$ и матрицу T.

[8] См.. например. J. J. Sakurai. Invariance Principles and Elementary particles и E. D. Commins. Weak Interactions.

Упражнения

II.1.1. Покажите, что следующие билинейные комбинации спинорного поля $\bar\psi\psi$, $\bar\psi\gamma^\mu\psi, \bar\psi\sigma^{\mu\nu}\psi, \bar\psi\gamma^\mu\gamma^5\psi$ и $\bar\psi\gamma^5\psi$ преобразуются под действием группы Лоренца и четности соответственно как скаляр, вектор, тензор, псевдоили аксиальный вектор и псевдоскаляр. [Указание: например, при бесконечно малом преобразовании Лоренца имеет место $\bar\psi\gamma^\mu\gamma^5\psi \to \bar\psi[1+ +(i/4)\omega\sigma]\gamma^\mu\gamma^5[1-(i/4)\omega\sigma]\psi$, а при четности $\to \bar\psi\gamma^0\gamma^\mu\gamma^5\gamma^0\psi$. Получите эти законы преобразования и покажите, что они определяют аксиальный вектор.]

II.1.2. Выразите все билинейные комбинации из предыдущего упражнения через ψ_L и ψ_R.

II.1.3. Решите явно $(\not{p}-m)\psi(p) = 0$ (в силу вращательной инвариантности достаточно решить это уравнение для $\vec{p}$ вдоль, например, третьей оси). Проверьте, что для медленно движущегося электрона χ действительно много меньше ϕ. Что происходит для быстро движущегося электрона?

II.1.4. Используя тот факт, что для медленно движущегося электрона χ много меньше ϕ, получите приближенное уравнение, которому удовлетворяет ϕ.

[9] C. N. Yang. Lectures at the Schwinger Memorial Session of the American Physical Society Meeting in Washington D.C., 1995. II.1.5. Выполните поворот вокруг оси z для релятивистского электрона, движущегося вдоль оси z. Другими словами, исследуйте действие $e^{-(i/4)\omega\sigma^{12}}$ на $\psi(p)$ и проверьте сделанное в тексте утверждение о ψ_L и ψ_R.

II.1.6. Явно покажите, что (25) нарушает четность.

II.1.7. Очевидно, что определяющее уравнение для C фиксирует C лишь с точностью до общей константы. Покажите, что значение константы определяется условием $(\psi_c)_c = \psi$.

II. 1.8. Покажите, что зарядовое сопряжение левого поля есть правое поле и наоборот.

II.1.9. Покажите, что $\psi C\psi$ есть лоренцевский скаляр.

II.1.10. Получите уравнение Дирака в $(1+1)$-мерном пространстве-времени.

II.1.11. Получите уравнение Дирака в $(2+1)$-мерном пространстве-времени. Покажите, что кажущийся безобидным массовый член нарушает четность и обращение времени. [Указание: три матрицы γ^μ являются просто тремя матрицами Паули с соответствующими множителями i.]

9.4.4 Зи.II.2 Quantization of the Dirac Field

Антикоммутирование

Для квантования дираковского поля воспользуемся каноническим формализмом из главы I.8.

Длительное и тщательное исследование атомной спектроскопии показало, что волновая функция двух электронов должна быть антисимметричной по отношению к перестановке их квантовых чисел. Отсюда следует, что мы не можем поместить два электрона на один и тот же энергетический уровень так, чтобы они имели одинаковые квантовые числа. В 1928 году Йордан и Вигнер продемонстрировали, как можно формализовать данное требование антисимметричности волновой функции с помощью операторов рождения и уничтожения электронов, удовлетворяющих антикоммутационным соотношениям, а не коммутационным соотношениям, как в (I.8.12).

Начнем с рассмотрения состояния без электронов $|0\rangle$ и обозначим через $b^\dagger_\alpha$ оператор рождения электрона с квантовыми числами α. Другими словами, состояние $b^\dagger_\alpha|0\rangle$ является состоянием с одним электроном, имеющим квантовые числа α. Предположим теперь, что мы хотим иметь другой электрон с квантовыми числами β, и поэтому образуем состояние $b^\dagger_\beta b^\dagger_\alpha|0\rangle$. Для того чтобы оно было антисимметрично относительно перестановки α и β, мы должны иметь

$$\left\{b^\dagger_\alpha, b^\dagger_\beta\right\} \equiv b^\dagger_\alpha b^\dagger_\beta + b^\dagger_\beta b^\dagger_\alpha = 0.$$

Эрмитово сопрягая, получаем $\{b_\alpha, b_\beta\} = 0$. В частности, $b^\dagger_\alpha b^\dagger_\alpha = 0$, так что мы не можем породить два электрона с одинаковыми квантовыми числами.

К полученному антикоммутационному соотношению добавим

$$\left\{b_\alpha, b_\beta^\dagger\right\} = \delta_{\alpha\beta}$$

Одним из аргументов в пользу этого является наше желание иметь оператор числа частиц, равный $N = \sum\limits_\alpha b_\alpha^\dagger b_\alpha$, как и в бозонном случае. По- кажите в одну строку, что $[AB, C] = A[B, C] + [A, C]B$ или $[AB, C] = A\{B, C\} - \{A, C\}B$. (Тот факт, что в антикоммутирующем случае стоит знак минус, можно запомнить эвристически: чтобы антикоммутировать C и A, мы должны протащить C через B.) Для того чтобы желаемый оператор числа частиц работал как положено, необходимо выполнение условия $\left[\sum\limits_\alpha b_\alpha^\dagger b_\alpha, b_\beta^\dagger\right] = +b_\beta^\dagger$ (так что, как и обычно, $N|0\rangle = 0$ и $Nb_\beta^\dagger|0\rangle = b_\beta^\dagger|0\rangle$), в результате чего приходим к (2).

Поле Дирака

Обратимся теперь к свободному лагранжиану Дирака

$$\mathcal{L} = \bar\psi(i\not\partial - m)\psi$$

Импульс, сопряженный с ψ, есть $\pi_\alpha = \delta\mathcal{L}/\delta\partial_t\psi_\alpha = i\psi_\alpha^\dagger$. Можно предвидеть, что корректная каноническая процедура требует наложения антикоммутационного соотношения:

$$\left\{\psi_\alpha(\vec x, t), \psi_\beta^\dagger(\overrightarrow{0}, t)\right\} = \delta^{(3)}(\vec x)\delta_{\alpha\beta}$$

Выведем его ниже.

Поле Дирака удовлетворяет условию

$$(i\not\partial - m)\psi = 0.$$

Подставляя плоские волны $u(p,s)e^{-ipx}$ и $v(p,s)e^{ipx}$ вместо ψ, получаем

$$(\not p - m)u(p,s) = 0$$

и

$$(\not p + m)v(p,s) = 0.$$

Индекс $s = \pm 1$ указывает на то, что каждое из двух уравнений имеет два решения: со спином, направленным вверх, и спином, направленным вниз. Очевидно, что при преобразовании Лоренца два спинора u и v преобразуются так же, как ψ. Таким образом, если определить $\bar u \equiv u^\dagger\gamma^0$ и $\bar v \equiv v^\dagger\gamma^0$, то $\bar u u$ и $\bar v v$ будут лоренцевскими скалярами.

Для этого сюжета характерно частое появление «особых» знаков, поэтому я буду действовать очень осторожно и раскрою вам смысл каждого из них. Во-первых, так как (6) и (7) линейны, мы должны зафиксировать нормировку u и v. В силу того что $\bar u(p,x)u(p,s)$ и $\bar v(p,s)v(p,s)$ являются лоренцевскими скалярами, условие нормировки, которое мы налагаем на них в системе покоя, будет справедливо в любой системе отсчета.

Наша стратегия состоит в том, чтобы, используя особый базис, проделать все в собственной системе покоя, а затем применить условие лоренц-инвариантности и независимости от базиса. В покоящейся системе координат (6) и (7) сводятся к

$(\gamma^0 - 1)\, u = 0$ и $(\gamma^0 + 1)\, v = 0$. В частности, в базисе Дирака $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, и поэтому два независимых спинора u (со спином $s = \pm 1$) имеют вид

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{и \%} \quad \% \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

а два независимых спинора v - вид

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \text{ и \% } \% \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Тогда условиями нормировки, неявно выбранными нами, будут $\bar{u}(p,s)\times\ \times u(p,s) = 1$ и $\bar{v}(p,s)v(p,s) = -1$. Обратите внимание на знак «минус». Очевидно, что мы также имеем условие ортогональности $\bar{u}v = 0$ и $\bar{v}u = 0$. Лоренц-инвариантность и независимость от базиса обеспечивают справедливость этих четырех соотношений и в общем случае.

Далее, в покоящейся системе координат

$$\sum_s u_\alpha(p,s)\bar{u}_\beta(p,s) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\beta} = \frac{1}{2}\left(\gamma^0 + 1\right)_{\alpha\beta}$$

И

$$\sum_s v_\alpha(p,s)\bar{v}_\beta(p,s) = \begin{pmatrix} 0 & 0 \\ 0 & -I \end{pmatrix}_{\alpha\beta} = \frac{1}{2}\left(\gamma^0 - 1\right)_{\alpha\beta}.$$

Таким образом, в общем случае

$$\sum_s u_\alpha(p,s)\bar{u}_\beta(p,s) = \left(\frac{\not{p} + m}{2m}\right)_{\alpha\beta}$$

И

$$\sum_s v_\alpha(p,s)\bar{v}_\beta(p,s) = \left(\frac{p - m}{2m}\right)_{\alpha\beta}$$

Есть и другой способ получить (8). Для этого необходимо заметить, что левая часть формулы есть 4×4-матрица (подобная вектор-столбцу, умноженному справа на вектор-строку) и поэтому должна быть линейной комбинацией шестнадцати матриц размерностью 4×4, перечисленных в главе II.1. Докажите, что γ^5 и $\gamma^\mu\gamma^5$ не могут быть среди них в силу четности, а $\sigma^{\mu\nu}$ - в силу лоренц-инвариантности и того факта, что в распоряжении имеется только один лоренцевский вектор, а именно p^μ. Следовательно, правая часть формулы должна быть линейной комбинацией p и m. Зафиксируем относительный коэффициент действием $p - m$ слева. Нормировка фиксируется условием $\alpha = \beta$ и суммированием по α. Аналогично и для (9). В частности, положив $\alpha = \beta$ и суммируя по α, мы восстанавливаем $\bar{v}(p,s)v(p,s) = -1$.

Теперь мы готовы преобразовать $\psi(x)$ в оператор. По аналогии с (I.8.11) разложим поле на плоские волны [1]

$$\psi(x) = \\ = \int \frac{d^3p}{(2\pi)^{3/2}\left(E_p/m\right)^{1/2}} \sum_s \left[b(p,s)u(p,s)e^{-ipx} + d^\dagger(p,s)v(p,s)e^{ipx}\right]$$

(Здесь $E_p = p_0 = +\sqrt{\vec{p}^2 + m^2}$ и $px = p_\mu x^\mu$.) Нормировочный множитель $(E_p/m)^{1/2}$ несколько отличен от нормировочного множителя в (I.8.11) по причинам, которые станут понятны вам ниже. Во всем остальном аргументация в пользу (10) аналогична той, что имеет место для (I.8.11). Мы интегрируем по импульсу $\vec{p}$, суммируем по спину s, разлагаем по плоским волнам и даем имена коэффициентам этого разложения. В силу того что ψ комплексно, имеем оператор b и оператор $d^\dagger$, как и в упражнении I.8.3, но в отличие от (I.8.11).

Как и в упражнении I.8.3, операторы b и $d^\dagger$ должны нести одинаковый заряд. Таким образом, если b уничтожает электрон с зарядом $e = -|e|$, оператор $d^\dagger$ должен уносить заряд e, то есть рождать позитрон с зарядом $-e = |e|$

Несколько слов об обозначениях: $b(p,s), d^\dagger(p,s), u(p,s)$ и $v(p,s)$ в (10) записаны как функции от 4-импульса p, но, строго говоря, они зависят только от $\vec{p}$, а p^0 всегда подразумевается равным $+\sqrt{\vec{p}^2 + m^2}$.

'Обозначение стандартно. См., например. J. A. Bjorken and S. D. Drell. Relativistic Quantum Mechanics (Д. А. Бьеркен и С. Д. Дрелл, Релятивистская квантовая механика). Итак, пусть $b^\dagger(p,s)$ и $b(p,s)$ являются операторами рождения и уничтожения электрона с импульсом p и спином s. Из наших предварительных рассуждений следует, что мы должны наложить условия

$$\left\{b(p,s), b^\dagger\left(p',s'\right)\right\} = \delta^{(3)}\left(\vec{p} - \vec{p}'\right)\delta_{ss'},$$
$$\left\{b(p,s), b\left(p',s'\right)\right\} = 0,$$
$$\left\{b^\dagger(p,s), b^\dagger\left(p',s'\right)\right\} = 0.$$

Для операторов рождения и уничтожения позитрона $d^\dagger(p,s)$ и $d(p,s)$ существует соответствующий набор соотношений, например:

$$\left\{d(p,s), d^\dagger\left(p',s'\right)\right\} = \delta^{(3)}\left(\vec{p} - \vec{p}'\right)\delta_{ss'}.$$

Теперь необходимо доказать, что мы в самом деле получаем (4). Запишем

$$\bar{\psi}(0) = \int \frac{d^3p'}{(2\pi)^{3/2}\left(E_{p'}/m\right)^{1/2}} \sum_{s'} \left[b^\dagger\left(p',s'\right)\bar{u}\left(p',s'\right) + d\left(p',s'\right)\bar{v}\left(p',s'\right)\right].$$

Ничего другого не остается, как двигаться вперед:

$$\{\psi(\vec{x},0), \bar{\psi}(0)\} = \\ = \int \frac{d^3p}{(2\pi)^3\left(E_p/m\right)} \sum_s \left[u(p,s)\bar{u}(p,s)e^{-i\vec{p}\cdot\vec{x}} + v(p,s)\bar{v}(p,s)e^{i\vec{p}\cdot\vec{x}}\right],$$

если ввести b и $b\dagger$, антикоммутирующие с d и $d^\dagger$. Используя (8) и (9), получаем

$$\{\psi(\vec{x},0),\bar{\psi}(0)\} = \int \frac{d^3p}{(2\pi)^3\,(2E_p)}\left[(p+m)e^{-i\vec{p}\cdot\vec{x}} + (\not{p}-m)e^{i\vec{p}\cdot\vec{x}}\right] =$$
$$= \int \frac{d^3p}{(2\pi)^3\,(eE_p)} 2p^0\gamma^0 e^{-i\vec{p}\cdot\vec{x}} = \gamma^0\delta^{(3)}(\vec{x}),$$

которое является слегка замаскированным соотношением (4).

Аналогично, записывая схематично, имеем $\{\psi,\psi\} = 0$ и $\left\{\psi^\dagger,\psi^\dagger\right\} = 0$. Несомненно, нормировочный множитель $(E_p/m)^{1/2}$ в (10) можно модифицировать путем включения соответствующих множителей в u и v. Порой сообразительные студенты спрашивают, что произойдет в случае безмассовой частицы со спином 1/2. Мы могли бы использовать нормировочный множитель $(2E_p)^{1/2}$ [подобный нормировочному множителю в (I.8.11)] и одновременно заменить (8) и (9) на $\sum\limits_s u\bar{u} = \not{p}$ и $\sum\limits_s v\bar{v} = \not{p}$.

Энергия вакуума

На данном этапе важно выполнить упражнение, заключающееся в вычислении гамильтониана, начиная с выражения для гамильтоновой плотности

$$\mathcal{H} = \pi\frac{\partial\psi}{\partial t} - \mathcal{L} = \bar{\psi}(i\vec{\gamma}\cdot\vec{\partial} + m)\psi$$

Подставляя в него (10) и интегрируя, получаем

$$H = \int d^3x\mathcal{H} = \int d^3x\bar{\psi}(i\vec{\gamma}\cdot\vec{\partial} + m)\psi = \int d^3x\bar{\psi}i\gamma^0\frac{\partial\psi}{\partial t}$$

что преобразуется к виду

$$H = \int d^3p\sum_s E_p\left[b^\dagger(p,s)b(p,s) - d(p,s)d^\dagger(p,s)\right]$$

Рассмотрим схематично появление в (17) важного знака минус: $\bar{\psi}$ в (16) дает множитель $\sim \left(b^\dagger + d\right)$, в то время как $\partial/\partial t$, действуя на ψ, привносит относительный знак минус, давая $\sim \left(b - d^\dagger\right)$, а все вместе дает нам $\sim \left(b^\dagger + d\right)\left(b - d^\dagger\right) \sim b^\dagger b - dd^\dagger$ (ортогональность между спинорами $\bar{v}u = 0$ сокращает перекрестные члены).

Чтобы переписать второй член в (17) в правильном порядке, антикоммутируем $-d(p,s)d^\dagger(p,s) = d^\dagger(p,s)d(p,s) - \delta^{(3)}(\overrightarrow{0})$ так, чтобы

$$H = \int d^3p\sum_s E_p\left[b^\dagger(p,s)b(p,s) + d^\dagger(p,s)d(p,s)\right] = -\delta^{(3)}(\overrightarrow{0})\int d^3p\sum_s E_p$$

Первые два члена указывают на то, что все электроны и все позитроны с импульсом p и спином s имеют одинаковую энергию E_p, как и должно быть. Но как насчет последнего члена? Эта $\delta^{(3)}(\overrightarrow{0})$ может вызывать опасения.

Все, однако, в порядке: замечая, что $\delta^{(3)}(\vec{p}) = [1/(2\pi)^3]\int d^3x e^{i\vec{p}\vec{x}}$, мы видим, что $\delta^{(3)}(\overrightarrow{0}) = [1/(2\pi)^3]\int d^3x$ (аналогичный маневр используется в упражнении I.8.2) и поэтому вклад последнего члена в H равен

$$E_0 = -\frac{1}{h^3}\int d^3x \int d^3p \sum_s 2\left(\frac{1}{2}E_p\right)$$

(поскольку в естественных единицах $\hbar = 1$ и, следовательно, $h = 2\pi$). Мы получили энергию $-\frac{1}{2}E_p$, приходящуюся на каждую элементарную ячейку фазового пространства $(1/h^3)\,d^3x d^3p$, с точки зрения статистической механики: на каждый спин и на каждый электрон и позитрон по отдельности (отсюда множитель 2). Этот бесконечный аддитивный член E_0 есть аналог энергии нулевых колебаний $\frac{1}{2}\hbar\omega$ гармонического осциллятора, известной вам из курса квантовой механики. Однако эта энергия появляется здесь со знаком минус, что кажется странным и непривычным!

Каждая мода дираковского поля дает вклад $-\frac{1}{2}\hbar\omega$ в энергию вакуума. В противоположность этому каждая мода скалярного поля, как мы видели в главе I.8, дает вклад $\frac{1}{2}\hbar\omega$. Этот факт имеет решающее значение для развития суперсимметрии, речь о которой пойдет в главе VIII.4.

Фермион, распространяющийся в пространстве-времени

По аналогии с (I.8.14), пропагатор для электрона определяется выражением $iS_{\alpha\beta}(x) \equiv \left\langle 0\left|T\psi_\alpha(x)\bar\psi_\beta(0)\right|0\right\rangle$, где аргумент функции $\bar\psi$ положен равным 0 в силу трансляционной инвариантности. Позднее мы увидим, что из-за антикоммутационного характера ψ мы должны определять упорядоченное по времени произведение со знаком минус, то есть

$$T\psi(x)\bar\psi(0) \equiv \theta\left(x^0\right)\psi(x)\bar\psi(0) - \theta\left(-x^0\right)\bar\psi(0)\psi(x)$$

Принимая во внимание (10), для $x^0 > 0$ получаем

$$iS(x) = \langle 0|\psi(x)\bar\psi(0)|0\rangle = \int \frac{d^3p}{(2\pi)^3\,(E_p/m)}\sum_s u(p,s)\bar u(p,s)e^{-ipx} = \\ = \int \frac{d^3p}{(2\pi)^3\,(E_p/m)}\frac{\not p + m}{2m}e^{-ipx}.$$

В случае $x^0 < 0$ мы должны быть более осторожными со спинорными индексами:

$$iS_{\alpha\beta}(x) = -\left\langle 0\left|\bar\psi_\beta(0)\psi_a(x)\right|0\right\rangle = \\ = -\int \frac{d^3p}{(2\pi)^3\,(E_p/m)}\sum_s \bar v_\beta(p,s)v_\alpha(p,s)e^{-ipx} = \\ = -\int \frac{d^3p}{(2\pi)^3\,(E_p/m)}\left(\frac{\not p - m}{2m}\right)_{\alpha\beta}e^{-ipx},$$

где используется тождество (9). Объединяя, получаем

$$iS(x) = \int \frac{d^3p}{(2\pi)^3\,(E_p/m)}\left[\theta\left(x^0\right)\frac{\not p + m}{2m}e^{-ipx} - \theta\left(-x^0\right)\frac{p - m}{2m}e^{+ipx}\right]$$

Теперь покажем, как можно этот фермионный пропагатор более изящно записать в виде 4-мерного интеграла:

$$iS(x) = i\int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \frac{\not p + m}{p^2 - m^2 + i\varepsilon} = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \frac{i}{\not p - m + i\varepsilon}.$$

Чтобы увидеть, что (22) на самом деле эквивалентно (21), проделаем, в сущности, те же шаги, что и после получения (I.8.14). В комплексной p^0-плоскости подынтегральное выражение имеет полюсы в точках $p^0 = \pm\sqrt{\vec{p}^2 + m^2 - i\varepsilon} \simeq (E_p - i\varepsilon)$. При $x^0 > 0$ множитель $e^{-ip^0x^0}$ указывает на необходимость замыкания контура интегрирования в нижней полуплоскости. Мы обходим по часовой стрелке полюс в $+(E_p - i\varepsilon)$ и получаем

$$iS(x) = (-i)i\int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot x} \frac{\not p + m}{2E_p}$$

что дает первый член в (21). При $x^0 < 0$ мы замыкаем контур в верхней полуплоскости и, таким образом, обходим полюс в $-(E_p - i\varepsilon)$ против часовой стрелки. Получаем

$$iS(x) = i^2\int \frac{d^3p}{(2\pi)^3} e^{+iE_px^0 + i\vec{p}\cdot\vec{x}} \frac{1}{-2E_p}\left(-E_p\gamma^0 - \vec{p}\vec{\gamma} + m\right)$$

и, обращая знак $\vec{p}$, приходим к выводу, что
$iS(x) = -\int \frac{d^3p}{(2\pi)^3} e^{ip\cdot x} \frac{1}{2E_p}\left(E_p\gamma^0 - \vec{p}\vec{\gamma} - m\right) = -\int \frac{d^3p}{(2\pi)^3} e^{ip\cdot x} \frac{1}{2E_p}(\not p - m)$
есть в точности второй член в (21) со знаком минус. Таким образом, мы обязаны определять упорядоченное по времени произведение со знаком минус, как в (20).

После всех этих вычислений заключаем, что фермионный пропагатор в импульсном пространстве имеет изящную форму

$$iS(p) = \frac{i}{\not p - m + i\varepsilon}$$

Она вполне логична: $S(p)$ соответствует обратному оператору Дирака $\not p - m$, так же как пропагатор скалярного бозона $D(k) = 1/\left(k^2 - m^2 - i\varepsilon\right)$ соответствует обратному оператору Клейна - Гордона $k^2 - m^2$.

Поэтические, но сбивающие с толку метафоры

В заключение этой главы позвольте мне задать риторический вопрос. Рассказывал ли я об электроне, распространяющемся назад во времени? Упоминал ли я море электронов с отрицательной энергией? Такой метафорический язык, будучи использованным блестящими умами, такими как Дирак и Фейнман, был легко запоминающимся и вдохновляющим, но, к сожалению, запутал не одно поколение студентов-физиков и самих физиков. Приведенное в книге изложение отвечает современному духу, при котором мы стремимся избежать этих потенциально приводящих к недоразумениям метафор.

Упражнения

- II.2.1. Используйте теорему Нётер для вывода сохраняющегося тока $J^\mu = \bar\psi\gamma^\mu\psi$. Вычислите $[Q, \psi]$, показав таким образом, что b и $d^\dagger$ несут одинаковый заряд.

- II.2.2. Проквантуйте дираковское поле в ящике объема V и покажите, что энергия вакуума E_0 действительно пропорциональна V. [Указание: интеграл по импульсам $\int d^3p$ необходимо заменить суммой по дискретным значениям импульса.]

9.4.5 Зи.II.3 Lorentz Group and Weyl Spinors

Алгебра Лоренца

В главе II.1 мы следовали оригинальным идеям Дирака при выводе его уравнения. Здесь мы более последовательно развиваем математическую теорию дираковского спинора. Более глубокое понимание дираковского спинора не только порождает чувство удовлетворения, но и является необходимым, как мы увидим далее, при изучении суперсимметрии - одной из фундаментальных концепций теории суперструн. И конечно, большинство фундаментальных частиц, таких как электрон и кварки, несут спин $1/2$ и описываются спинорными полями.

Для начала вспомним, как действует группа вращений. Три генератора $J_i (i = 1, 2, 3$ или $x, y, z)$ группы вращений удовлетворяют коммутационным соотношениям.

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

При действии на пространственно-временные координаты, записанные как вектор-столбец

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

генераторы вращения задаются эрмитовыми матрицами

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

а J_2 и J_3 находятся путем соответствующих перестановок. Вы можете проверить справедливость (1), старательно перемножая эти три матрицы. Обратите внимание, что знаки J_i фиксируются коммутационным соотношениеМ (1). Добавим теперь лоренцевский буст. Буст в направлении $x \equiv x^1$ преобразует пространственно-временные координаты следующим образом:

$$t' = (\operatorname{ch}\varphi)t + (\operatorname{sh}\varphi)x; \quad x' = (\operatorname{sh}\varphi)t + (\operatorname{ch}\varphi)x$$

или, для бесконечно малого φ,

$$t' = t + \varphi x; \quad x' = x + \varphi t.$$

Иными словами, бесконечно малый лоренцевский буст в направлении x задается эрмитовой матрицей (как обычно $x^0 \equiv t$)

$$iK_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Аналогично

$$iK_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Запишите K_3 самостоятельно.

Проверьте, что $[J_i, K_j] = i\epsilon_{ijk}K_k$. Данное условие означает, что генераторы бустов K_i преобразуются при вращении как 3-векторы, как и следовало ожидать.

Вы сейчас готовы проделать один из самых важных вычислений в истории физики двадцатого столетия. Получите прямым вычислением $[K_1, K_2]$. Вы обнаружите, что он равен $-iJ_3$. Два лоренцевых буста генерируют вращение! (Из курса электромагнетизма вспомним, что полученный математический факт ответственен за томасовскую прецессию.)

Математически генераторы группы Лоренца удовлетворяют следующей алгебре [известной знатокам как $SO(3,1)$]:

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk}J_k, \\ [J_i, K_j] &= i\epsilon_{ijk}K_k, \\ [K_i, K_j] &= -i\epsilon_{ijk}J_k. \end{aligned}$$

Обратите внимание на крайне важный знак минус.

Как нам изучать эту алгебру? Решающим наблюдением является то, что при образовании комбинаций $J_{\pm i} \equiv \frac{1}{2}(J_i \pm iK_i)$ алгебра разбивается на две части. Убедитесь, что

$$\begin{aligned} [J_{+i}, J_{+j}] &= i\epsilon_{ijk}J_{+k} \\ [J_{-i}, J_{-j}] &= i\epsilon_{ijk}J_{-k} \end{aligned}$$

и, что более важно,

$$[J_{+i}, J_{-j}] = 0.$$

Последнее коммутационное соотношение означает, что J_+и J_-образуют две независимые алгебры $SU(2)$. Иначе говоря, алгебра $SO(3,1)$ изоморфна $SU(2) \otimes SU(2)$.

От алгебры к представлению

Отсюда следует, что мы можем использовать наши знания о представлениях $SU(2)$ для построения представлений $SO(3,1)$. Как вы знаете, представления $SU(2)$ характеризуются $j = 0, \frac{1}{2}, 1\frac{3}{2}, \ldots$ Каждое представление можно рассматривать как набор $(2j+1)$ объектов ψ_m с $m = -j, -j+ +1, \ldots, j-1, j$, которые преобразуются друг в друга под действием $SU(2)$. Отсюда немедленно следует, что представления $SO(3,1)$ характеризуются (j^+, j^-), где j^+и j^-принимают значения $0, \frac{1}{2}, 1\frac{3}{2}, \ldots$ Каждое представление состоит из $(2j^+ + 1)(2j^- + 1)$ объектов $\psi_{m^+m^-}$с $m^+ = -j^+, -j^+ + +1, \ldots, j^+ - 1, j^+$и $m^- = -j^-, -j^- + 1, \ldots, j^- - 1, j^-$.

Таким образом, представлениями $SO(3,1)$ являются $(0,0)$, $\left(\frac{1}{2}, 0\right)$, $\left(0, \frac{1}{2}\right)$, $(1,0)$, $(0,1)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$ и так далее в порядке возрастания размерности. Ясно, что одномерное представление $(0,0)$ тривиально, т. е. соответствует лоренцевскому скаляру. Из размерных соображений можно ожидать, что 4-мерное представление $\left(\frac{1}{2}, \frac{1}{2}\right)$ соответствует лоренцевскому вектору, т. е. представлению наименьшей размерности группы Лоренца (см. упражнение II.3.1).

Спинорные представления

Что можно сказать о представлении $\left(\frac{1}{2},0\right)$? Запишем эти два объекта как ψ_α с $\alpha = 1,2$. Итак, что означает обозначение $\left(\frac{1}{2},0\right)$? Оно указывает на то, что $J_{+i} = \frac{1}{2}(J_i + iK)$ при действии на ψ_α соответствует $\frac{1}{2}\sigma_i$, тогда как $J_{-i} = \frac{1}{2}(J_i - iK_i)$ при действии на ψ_α соответствует 0. Складывая и вычитая, находим

$$J_i = \frac{1}{2}\sigma_i$$

и

$$iK_i = \frac{1}{2}\sigma_i$$

В этом контексте знак равенства означает «представляется как». (По традиции, мы не будем делать различий между верхними и нижними индексами 3-мерных величин J_i, K_i и σ_i.)

Аналогично обозначим два объекта в $\left(0,\frac{1}{2}\right)$ особенным символом $\bar{\chi}^{\dot{\alpha}}$. Обращаю ваше внимание на тривиальный, но порой сбивающий с толку факт: в отличие от фигурировавшей в главе II.1 черты, черта над $\bar{\chi}^{\dot{\alpha}}$ являстся типографическим элементом. По желанию можете рассматривать $\bar{\chi}$ как букву в хеттском алфавите. Точно так же символ $\dot{\alpha}$ не имеет никакого отношения к α: нельзя получить $\dot{\alpha}$ из α. Такое довольно странное обозначение, неформально называемое «пунктирным и непунктирным», а более формально - обозначением Ван дер Вардена, является для наших скромных целей несколько чрезмерным, но я ввел его потому, что именно такое обозначение используется в суперсимметричной физике и теории суперструн. (Между прочим, Дирак якобы жалел, что это не он придумал систему пунктирных обозначений.) Повторяя предыдущие шаги, вы обнаружите, что для представления $\left(0,\frac{1}{2}\right)$ имеет место $J_i = \frac{1}{2}\sigma_i$ и $iK_i = -\frac{1}{2}\sigma_i$. Знак минус критичен.

Двухкомпонентные спиноры ψ_α и $\bar{\chi}^{\dot{\alpha}}$ называются вейлевскими спинорами и образуют полноценные представления группы Лоренца. Почему же тогда дираковский спинор имеет 4 компоненты?

Причина в четности. При отражении имеем $\vec{x} \to -\vec{x}$ и $\vec{p} \to -\vec{p}$ и, следовательно, $\vec{J} \to \vec{J}$ и $\vec{K} \to -\vec{K}$, так что $\vec{J}_+ \leftrightarrow \vec{J}_-$. Другими словами, при отражении представления переходят друг в друга $\left(\frac{1}{2},0\right) \leftrightarrow \left(0,\frac{1}{2}\right)$. Таким образом, для описания электрона мы должны использовать оба этих двумерных представления или, в математических обозначениях, 4-мерное приводимое представление $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$.

Итак, из двух спиноров Вейля получаем спинор Дирака

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}$$

Конечно, спинор $\Psi(p)$ является функцией от 4-импульса p [и косвенно от $\psi_\alpha(p)$ и $\bar{\chi}^{\dot{\alpha}}(p)$], но мы будем опускать зависимость от p. Возвращаясь к (13) и (14), видим, что действующие на Ψ генераторы вращений задаются следующим образом:

$$\vec{J} = \begin{pmatrix} \frac{1}{2}\vec{\sigma} & 0 \\ 0 & \frac{1}{2}\vec{\sigma} \end{pmatrix},$$

где равенство опять-таки означает «представляется как», а генераторы бустов задаются как

$$i\vec{K} = \begin{pmatrix} \frac{1}{2}\vec{\sigma} & 0 \\ 0 & -\frac{1}{2}\vec{\sigma} \end{pmatrix}.$$

Снова обращаю ваше внимание на крайне важный знак минус.

В силу четности мы должны рассматривать 4-мерный спинор, но, с другой стороны, мы знаем, что электрон имеет только две физические степени свободы. Перейдем в покоящуюся систему координат. Мы должны избавиться от двух компонент, содержащихся в $\Psi(p_r)$ с импульсом покоя $p_r \equiv (m, \overrightarrow{0})$. Используя прошлый опыт, запишем проекционный оператор как $\mathcal{P} = \frac{1}{2}(1 - \gamma^0)$. Глядя на обозначения, вы, вероятно, уже догадались, что γ^0 оказывается одной из гамма-матриц, но на данном этапе логично предположить, что это всего лишь некоторая 4×4-матрица. Из условия $\mathcal{P}^2 = \mathcal{P}$ следует, что $(\gamma^0)^2 = 1$, так что собственные значения γ^0 равны \pm 1. Так как при отражении $\psi_\alpha \leftrightarrow \bar{\chi}^{\dot{\alpha}}$, естественно предположить, что ψ_α и $\bar{\chi}^{\dot{\alpha}}$ соответствуют левом и правым полям из главы II.1. Мы не можем просто использовать проекцию и положить, например, $\bar{\chi}^{\dot{\alpha}}$ равным 0. Четность подразумевает, что мы должны рассматривать ψ_α и $\bar{\chi}^{\dot{\alpha}}$ на равном основании. Выбираем

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

или, в более явном виде,

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

(Разный выбор γ^0 соответствует разному выбору базиса, обсуждавшемуся в главе II.1.) Другими словами, в покоящейся системе координат $\psi_\alpha - \bar{\chi}^{\dot{\alpha}} = 0$. Проекция на две степени свободы может быть записана как

$$\left(\gamma^0 - 1\right)\Psi(p_r) = 0$$

В самом деле, узнаем в полученном базис Вейля, речь о котором шла в глаВе II.1.

Уравнение Дирака

Мы только что вывели уравнение Дирака, правда в слегка завуалированной форме!

Поскольку наш вывод основан на поэтапном изучении спинорного представления группы Лоренца, мы знаем, как получить уравнение, которому удовлетворяет $\Psi(p)$ при любом p: необходимо совершить буст. Записывая $\Psi(p) = e^{-i\vec{\varphi}\vec{K}}\Psi(p_r)$, имеем $\left(e^{-i\vec{\varphi}\vec{K}}\gamma^0 e^{i\vec{\varphi}\vec{K}} - 1\right)\Psi(p) = 0$. Вводя обозначение $\gamma^\mu p_\mu/m \equiv e^{-i\vec{\varphi}\vec{K}}\gamma^0 e^{i\vec{\varphi}\vec{K}}$, получаем уравнение Дирака

$$(\gamma^\mu p_\mu - m)\Psi(p) = 0$$

Проделайте все детали этого вычисления в качестве упражнения.

Приведенный здесь вывод представляет собой глубокий теоретикогрупповой подход к уравнению Дирака: он заключается в проекции, преобразованной к произвольной системе отсчета.

Заметьте, что этот пример демонстрирует всю мощь симметрии, которой пропитана современная физика и эта книга: наше знание того, как поле электрона преобразуется под действием группы вращений, а именно то, что он имеет спин 1/2, позволяет нам определить, как оно преобразуется под действием группы Лоренца. Симметрия правит миром!

В приложении Е более подробно излагается пунктирная система обозначений, поскольку она понадобится нам при изучении главы, посвященной суперсимметрии.

Для более глубокого теоретико-группового понимания материала нелишним будет перечитать главу II.1 и сравнить ее с настоящей главой.

Упражнения

II.3.1. Прямым вычислением покажите, что $\left(\frac{1}{2},\frac{1}{2}\right)$ действительно является лоренцевым вектором. II.3.2. Установите, как преобразуются шесть объектов, содержащихся в $(1,0)$ и $(0,1)$, под действием группы Лоренца. Вспомните из курса по электродинамике, как преобразуются электрическое и магнитное поля $\vec{E}$ и $\vec{B}$. Покажите, что электромагнитное поле действительно преобразуется как $(1,0)\oplus(0,1)$. Покажите, что именно четность вынуждает нас использовать приводимое представление.

II.3.3. Покажите, что

$$e^{-i\vec{\varphi}\vec{K}}\gamma^0 e^{i\vec{\varphi}\vec{K}} = \begin{pmatrix} 0 & e^{-\vec{\varphi}\vec{\sigma}} \\ e^{\vec{\varphi}\vec{\sigma}} & 0 \end{pmatrix}$$

и

$$e^{\vec{\varphi}\vec{\sigma}} = \operatorname{ch}\varphi + \vec{\sigma}\cdot\widehat{\varphi}\operatorname{sh}\varphi$$

где $\vec{\varphi}\equiv\vec{\varphi}/\varphi$ - единичный вектор. Полагая $\vec{p} = m\widehat{\varphi}\operatorname{sh}\varphi$, выведите уравнение Дирака. Покажите, что

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

II.3.4. Покажите, как можно частицу со спином 3/2 описать вектор-спинором $\Psi_{\alpha\mu}$, то есть спинором Дирака, несущим лоренцевский индекс. Найдите соответствующее уравнение движения, известное как уравнение Рариты-Швингера. [Указание: объект $\Psi_{\alpha\mu}$ имеет 16 компонент, которые надо свести к количеству $2\cdot\frac{3}{2}+1=4$.]

9.4.6 Зи.II.4 Connection of Spin and Statistics

Степени интеллектуальной неполноты

Из курса нерелятивистской квантовой механики вы знаете о принципе Паули и его последующем обобщении, утверждающем, что частицы с полуцелыми спинами, такие как электроны, подчиняются статистике Ферми-Дирака и предпочитают находиться подальше друг от друга, а частицы с целыми спинами, такие как фотоны, или пары электронов, подчиняются статистике Бозе-Эйнштейна и любят собираться вместе. Все ослепительное богатство физических явлений, от микроскопической структуры атомов до макроскопической структуры нейтронных звезд, нельзя объяснить без правила спин-статистики. Многие элементы физики конденсированных сред, к примеру зонная

структура, теория ферми-жидкости, сверхтекучесть, сверхпроводимость, квантовый эффект Холла и т.д. и т.п., являются следствиями этого правила.

Квантовая статистика - одна из наиболее хрупких физических концепций - основывается на том факте, что в квантовом мире все элементарные частицы и, следовательно, все атомы абсолютно идентичны и потому неотличимы друг от друга [3]. То, что квантовая теория поля способна легко и естественно объяснить полную идентичность и неразличимость частиц, следует рассматривать как ее триумф. Каждый электрон во Вселенной есть возбуждение одного и того же электронного поля ψ. В противном случае можно было бы представить, что электроны, о которых мы знаем в настоящий момент, были выпущены сборочным конвейером где-то в ранней Вселенной и могли бы несколько отличаться друг от друга из-за некоторой халатности в процессе производства.

Несмотря на то что правило спин-статистики оказывает такое существенное влияние на квантовую механику, для его объяснения пришлось ждать формулировки релятивистской квантовой теории поля. Представьте себе цивилизацию, которая в силу каких-то причин сформулировала квантовую механику, но все еще ожидает открытия специальной теории относительности. Физики этой цивилизации со временем осознали бы, что им необходимо придумать некоторое правило для объяснения упомянутых выше явлений, ни одно из которых не связано с движением, скорость которого превышает скорость света. Физика была бы интеллектуально неудовлетворительной и неполной.

Интересным критерием сравнения различных областей физики является степень их интеллектуальной неполноты.

Несомненно, в физике мы часто принимаем правило, которое невозможно объяснить до тех пор, пока мы не перейдем на следующий уровень знаний. Например, в большинстве разделов физики мы воспринимаем как данное, что заряды протона и электрона в точности одинаковы по абсолютной величине и противоположны по знаку. Квантовая электродинамика сама по себе не в состоянии объяснить сей поразительный факт. Как мы увидим в главе VII.6, явление квантования заряда можно объяснить только путем рассмотрения квантовой электродинамики в рамках более общей структуры, такой как теория великого объединения. (В главе IV.4 мы узнаем, что существование магнитных монополей предполагает квантование заряда, но монополи не существуют в чистой квантовой электродинамике.)

Таким образом, обьяснение связи спина со статистикой (Фирцем и Паули в конце 1930-х, Людерсом и Зумино, а также Бергойном в конце 1950-х) расценивается как один из величайших триумфов релятивистской квантовой теории поля. Объем моей книти не позволяет привести тому общее и строгос доказательство 4. Опишу в общих чертах, какие серьезныс нарушения произойдут, если нарушить связь спина со статистикой.

Цена упрямства

Основной квантовый принцип утверждает, что если две наблюдаемые величины коммутируют, то их можно одновременно диагонализовать и, следовательно, наблюдать. Основной релятивистский принцип утверждает, что, если две пространственно-временные точки являются пространственно подобными по отношению друг к другу, то никакой сигнал не может распространиться между ними и, следовательно, измерение наблюдаемой величины в одной из точек не может влиять на измерение другой наблюдаемой величины во второй точке.

Рассмотрим плотность заряда $J_0 = i\left(\varphi^\dagger \partial_0 \varphi - \partial_0 \varphi^\dagger \varphi\right)$ в теории заряженного скалярного

поля. В соответствии с двумя только что провозглашенными фундаментальными принципами, $J_0(\vec{x}, t = 0)$ и $J_0(\vec{y}, t = 0)$ должны коммутировать при $\vec{x} \neq \vec{y}$. Вычисляя коммутатор $J_0(\vec{x}, t = 0)$ с $J_0(\vec{y}, t = 0)$, мы используем тот факт, что $\varphi(\vec{x}, t = 0)$ и $\partial_0\varphi(\vec{x}, t = 0)$ коммутируют с $\varphi(\vec{y}, t = 0)$ и $\partial_0\varphi(\vec{y}, t = 0)$, так что можно просто поменять местами поля в $\vec{x}$ и $\vec{y}$. Коммутатор обращается в нуль практически тривиально.

Теперь предположим, что мы упрямимся и квантуем операторы рождения и уничтожения в разложении (I.8.11)

$$\varphi(\vec{x}, t = 0) = \int \frac{d^D k}{\sqrt{(2\pi)^D 2\omega_k}} \left[a(\vec{k}) e^{i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right]$$

в соответствии с антикоммутационными соотношениями

$$\left\{ a(\vec{k}), a^\dagger(\vec{q}) \right\} = \delta^{(D)}(\vec{k} - \vec{q})$$

И

$$\{a(\vec{k}), a(\vec{q})\} = 0 = \left\{ a^\dagger(\vec{k}), a^\dagger(\vec{q}) \right\}$$

а не правильными коммутационными соотношениями.

Какова цена упрямства?

Теперь, когда мы пытаемся пронести $J_0(\vec{y}, t = 0)$ через $J_0(\vec{x}, t = 0)$, необходимо пронести поле в $\vec{y}$ через поле в $\vec{x}$, используя антикоммутатор

$$\begin{aligned}
\{\varphi(\vec{x}, t = 0), \varphi(\vec{y}, t = 0)\} &= \iint \frac{d^D k}{\sqrt{(2\pi)^D 2\omega_k}} \frac{d^D q}{\sqrt{(2\pi)^D 2\omega_q}} \times \\
&\times \left\{ \left[a(\vec{k}) e^{i\vec{k}\cdot\vec{x}} + a^\dagger(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right], \left[a(\vec{q}) e^{i\vec{q}\cdot\vec{y}} + a^\dagger(\vec{q}) e^{-i\vec{q}\cdot\vec{y}} \right] \right\} = \\
&= \int \frac{d^D k}{\sqrt{(2\pi)^D 2\omega_k}} \left(e^{i\vec{k}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \right)
\end{aligned}$$

Вы видите, в чем состоит проблема? В обычной теории скалярного поля, удовлетворяющего связи спина со статистикой, мы вычислили бы коммутатор, и тогда в последнем выражении в (2) получили бы $\left(e^{i\vec{k}\cdot(\vec{x}-\vec{y})} - \quad - e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \right)$, вместо $\left(e^{i\vec{k}\cdot(\vec{x}-\vec{y})} + e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \right)$. Интеграл

$$\int \frac{d^D k}{\sqrt{(2\pi)^D 2\omega_k}} \left(e^{i\vec{k}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \right)$$

очевидно обращался бы в ноль, что очевидно, и все было бы хорошо. Со знаком плюс мы получаем в (2) не равную нулю часть. Катастрофа, если мы квантуем скалярное поле как антикоммутирующее! Поле со спином 0 должно быть коммутирующим. Таким образом, теория относительности и квантовая физика объединяют свои усилия и обуславливают связь спина со статистикой.

Иногда говорят, что благодаря электромагнетизму вы не проваливаетесь сквозь пол, а благодаря гравитации - не воспаряете к потолку. К тому же, если бы не было слабого взаимодействия, регулирующего горение звезд, то вы бы проваливались или воспаряли в полной темноте. Без связи спина со статистикой электрон не подчинялся бы принципу Паули. Материя просто коллапсировала бы.

Упражнения

Зи-у.

II.4.1. Покажите появление аналогичной проблемы для случая, если квантовать поле Дирака с коммутаторами вместо антикоммутаторов. Вычислите $[J^0(\vec{x},0), J^0(0)]$

[5] Доказательство устойчивости материи. данное Дайсоном и Ленардом, существенно опирается на принцип Паули.

9.4.7 Зи.II.5 Grassmann Integrals and Fermion Diagrams

Фермионы таинственны

Я построил квантовую теорию скалярного поля $\varphi(x)$ сначала в формализме интеграла по траекториям, а затем в каноническом формализме. Квантовую же теорию свободного поля $\psi(x)$ со спином 1/2 я построил только в каноническом формализме. Мы узнали, что связь спина со статистикой вынуждает полевой оператор $\psi(x)$ удовлетворять антикоммутационным соотношениям. Из этого немедленно следует несколько странная запись интеграла по траекториям для спинорного поля ψ. В формализме интеграла по траекториям $\psi(x)$ является не оператором, а просто переменной интегрирования. Как отобразить тот факт, что операторный аналог в каноническом формализме антикоммутирует?

По-видимому, мы не сможем представить ψ как коммутирующую переменную, как в отношении с φ. В самом деле, мы обнаружим, что в формализме интеграла по траекториям ψ должно рассматриваться не как обыкновенное комплексное число, а как новое математическое понятие, известное под именем грассманова числа.

Если задуматься над этим, то, вероятно, придем к выводу, что здесь необходима новая математическая структура. В главе I.3 мы обобщили координаты точечной частицы $q_i(t)$ в квантовой механике до понятия скалярного поля $\varphi(\vec{x},t)$. Однако из квантовой механики известно, что частица со спином 1/2 имеет одно особое свойство: ее волновая функция меняет знак при повороте на 2π. В отличие от координат частицы, полуцелый спин не является интуитивным понятием.

Энергия вакуума

Чтобы мотивировать введение грассмановых полей, рассмотрим понятие энергии вакуума. Причина, по которой я выбрал эту на первый взгляд странную стратегию, вскоре будет понятна.

Квантовая теория поля вначале была построена для описания рассеяния фотонов и электронов, и позднее для рассеяния частиц. Вспомним, что в главе I.7 при рассмотрении рассеяния частиц мы строили диаграммы, описывающие вакуумные флуктуации, которыми мы просто пренебрегали (см. рис. I.7.12). Совершенно естественно, что специалисты в области физики элементарных частиц не придают этим флуктуациям особого значения. Экспериментально мы рассеиваем частицы друг на друге. Кого интересуют флуктуации где-то там в вакууме? И лишь в начале 1970-х физики в полной мере оценили значимость вакуумных флуктуаций. К вопросу о важности вакуума вернемся позже [1].

В главах I.8 и II.2 мы вычислили энергию вакуума свободного скалярного поля и свободного спинорного поля в каноническом формализме. Чтобы мотивировать использование грассмановых чисел при формулировке интеграла по траекториям для

спинорного поля, я прибегну к следующей стратегии. Сначала я воспользуюсь формализмом интеграла по траекториям и получу результат для свободного скалярного поля, найденный нами с помощью канонического формализма. Далее мы узнаем, что для получения уже известного нам результата для свободного спинорного поля мы должны модифицировать интеграл по траекториям.

По определению, вакуумные флуктуации происходят даже при отсутствии источников, способных создать частицы. Поэтому рассмотрим производящий функционал теории свободного скалярного поля при отсутствии источников [2]:

$$Z = \int D\varphi e^{i\int d^4x \frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right]} = C\left(\frac{1}{\det[\partial^2+m^2]}\right)^{1/2} = Ce^{-\frac{1}{2}\operatorname{Tr}\log\left(\partial^2+m^2\right)}$$

[1] На самом деле мы уже упоминали один из способов наблюдения за эффектами вакуумных флуктуаций в главе I.8.

[2] Строго говоря, чтобы приведенное здесь выражение являлось однозначно определенным. мы должны, как обсуждалось ранее, заменить m^2 на $m^2 - i\varepsilon$. Для первого равенства мы воспользовались выражением (I.2.15), в котором включили несущественные множители в константу C. Во втором равенстве мы воспользовались важным тождеством

$$\det M = e^{\operatorname{Tr}\log M}$$

с которым вы встречались в упражнении I.10.2.

Вспомним, что $Z = \left\langle 0\left|e^{-iHT}\right|0\right\rangle$ (где подразумевается $T \to \infty$, так что в (1) идет интегрирование по всему пространству-времени), в нашем случае равен e^{-iET}, где E - энергия вакуума. Вычисляя след в (1)

$$\operatorname{Tr}\mathcal{O} = \int d^4x\langle x|\mathcal{O}|x\rangle = \int d^4x \int \frac{d^4k}{(2\pi)^4}\int \frac{d^4q}{(2\pi)^4}\langle x|k\rangle\langle k\mathcal{O}|q\rangle\langle q|x\rangle$$

получаем

$$iET = \frac{1}{2}VT\int \frac{d^4k}{(2\pi)^4}\log\left(k^2 - m^2 + i\varepsilon\right) + A$$

где A - бесконечная константа, соответствующая множителю C в (1). Вспомним, что при выводе интеграла по траекториям у нас было много расходящихся множителей; все они собираются именно в множитель A. Польза от A выражается в том, что она позволяет решить проблему, которую вы, должно быть, уже заметили: аргумент логарифма не является безразмерным. Определим m', записав

$$A = -\frac{1}{2}VT\int \frac{d^4k}{(2\pi)^4}\log\left(k^2 - m'^2 + i\varepsilon\right)$$

Другими словами, мы вычисляем не энергию вакуума саму по себе, а только разницу между ней и энергией вакуума, которую мы имели бы, если бы частицы имели массу m', вместо m. Произвольно большое время T сокращается и, как и следовало ожидать, E оказывается пропорциональной объему пространства V. Таким образом, плотность энергии вакуума (точнее, разность плотностей энергии вакуума) равна

$$\begin{aligned}\frac{E}{V} &= -\frac{i}{2}\int \frac{d^4k}{(2\pi)^4}\log\left[\frac{k^2 - m^2 + i\varepsilon}{k^2 - m'^2 + i\varepsilon}\right] = \\ &= -\frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\int \frac{d\omega}{2\pi}\log\left[\frac{\omega^2 - \omega_k^2 + i\varepsilon}{\omega^2 - \omega_k'^2 + i\varepsilon}\right]\end{aligned}$$

где $\omega'_k \equiv +\sqrt{\vec{k}^2 + m'^2}$. Мы вычисляем (сходящийся) интеграл по ω путем интегрирования по частям:

$$\int \frac{d\omega}{2\pi} \int \frac{d\omega}{d\omega} \log\left[\frac{\omega^2 - \omega_k^2 + i\varepsilon}{\omega^2 - \omega_k'^2 + i\varepsilon}\right] =$$
$$= -2\int \frac{d\omega}{2\pi}\omega \left[\frac{\omega}{\omega^2 - \omega_k^2 + i\varepsilon} - (\omega_k \to \omega'_k)\right] =$$
$$= -i2\omega_k^2\left(\frac{1}{-2\omega_k}\right) - (\omega_k \to \omega'_k) = +i\left(\omega_k - \omega'_k\right).$$

Восстанавливая $\hbar$, мы действительно приходим к желаемому результату:

$$\frac{E}{V} = \int \frac{d^3k}{(2\pi)^3}\left(\frac{1}{2}\hbar\omega_k - \frac{1}{2}\hbar\omega'_k\right)$$

Чтобы получить этот результат, нам пришлось проделать несколько выкладок, но важным моментом здесь является то, что, используя формализм интеграла по траекториям, мы смогли получить результат, ранее полученный в каноническом формализме.

Особый знак для фермионов

Наша задача - записать интеграл по траекториям для спинорного поля. Вспомним из главы II.2, что вакуумная энергия спинорного поля появляется со знаком, противоположным знаку вакуумной энергии скалярного поля, и этот знак определенно находится в «верхней десятке» самых значимых знаков теоретической физики. Как его получить с помощью интеграла по траекториям?

Как говорилось в главе I.3, выражение (1) строится на основе простой формулы для гауссова интеграла

$$\int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}} = \sqrt{2\pi}e^{-\frac{1}{2}\log a}.$$

Грубо говоря, мы должны найти такой новый интеграл, чтобы аналогом гауссова интеграла было выражение типа $e^{+\frac{1}{2}\log a}$.

Грассманова математика

Оказывается, что необходимая для этих целей математика была изобретена Грассманом давным-давно. Введем новый тип числа, называемого грассмановым или антикоммутирующим числом, такой, что если η и ξ - грассмановы числа, то $\eta\xi = -\xi\eta$. В частности, $\eta^2 = 0$. Эвристически это отражает антикоммутационное соотношение, которому удовлетворяет спинорное поле. Грассман предположил, что любая функция от η может быть разложена в ряд Тейлора. Так как $\eta^2 = 0$, то наиболее общей функцией η является $f(\eta) = a + b\eta$.

Как нам определить интегрирование по η? Грассман заметил, что важным свойством обыкновенного интеграла является возможность сдвига фиктивной переменной интегрирования: $\int\limits_{-\infty}^{+\infty} dx f(x+c) = \int\limits_{-\infty}^{+\infty} dx f(x)$. Таким образом, мы должны требовать, чтобы

грассманов интеграл также удовлетворял правилу $\int d\eta f(\eta + \xi) = \int d\eta f(\eta)$, где ξ - произвольное грассманово число. Подставляя наиболее общую функцию, приведенную выше, находим, что $\int d\eta b\xi = 0$. Так как ξ произвольно, полученное выражение может выполняться только тогда, когда мы определяем $\int d\eta b = 0$ для любого обычного числа b, в частности, $\int d\eta \equiv \int d\eta 1 = 0$.

Поскольку для трех грассмановых чисел χ, η и ξ имеет место $\chi(\eta\xi) = (\eta\xi)\chi$, то есть произведение $(\eta\xi)$ коммутирует с любым грассмановым числом χ, приходим к заключению, что произведение двух антикоммутирующих чисел должно выражаться обычным числом. Таким образом, значение интеграла $\int d\eta\eta$ есть просто обычное число, которое можно положить равным 1: при этом фиксируется нормировка $d\eta$. Итак, грассманово интегрирование необычайно просто, оно определяется двумя правилами:

$$\int d\eta = 0$$

К

$$\int d\eta\eta = 1$$

Используя эти правила, можно проинтегрировать любую функцию от η:

$$\int d\eta f(\eta) = \int d\eta(a + b\eta) = b$$

если b - обычное число, то есть $f(\eta)$ является грассмановым, и

$$\int d\eta f(\eta) = \int d\eta(a + b\eta) = -b$$

если b - грассманово, то есть $f(\eta)$ - обычное число. Обратите внимание, что для грассманова интегрирования не существует понятия пределов интегрирования. Грассманово интегрирование гораздо легче освоить, чем обычное!

Пусть η и $\bar{\eta}$ - два независимых грассмановых числа, а a - произвольное обычное число. Тогда грассмановый аналог гауссова интеграла равен

$$\int d\eta \int d\bar{\eta} e^{\bar{\eta}a\eta} = \int d\eta \int d\bar{\eta}(1 + \bar{\eta}a\eta) = \int d\eta a\eta = a = e^{+\log a}.$$

В точности то, что нам и нужно!

Можно немедленно получить обобщение. Пусть $\eta = (\eta_1, \eta_2, ..., \eta_N)$ - N грассмановых чисел, и аналогично для $\bar{\eta}$; тогда имеем

$$\int d\eta \int d\bar{\eta} e^{\bar{\eta}A\eta} = \det A$$

для антисимметричной $N \times N$-матрицы $A = \{A_{ij}\}$. (Заметьте, что, в отличие от бозонного случая, матрицы, обратной матрице A, может и не существовать.) Можно построить дальнейшее обобщение до функционального интеграла.

Как мы вскоре увидим, у нас есть вся необходимая математика.

Грассманов интеграл по траекториям

По аналогии с производящим функционалом для скалярного поля

$$Z = \int D\varphi e^{iS(\varphi)} = \int D\varphi e^{i\int d^4x \frac{1}{2}\left[(\partial\varphi)^2 - \left(m^2 - i\varepsilon\right)\varphi^2\right]}$$

производящий функционал для спинорного поля естественно записать в виде:

$$Z = \int D\psi D\bar{\psi} e^{iS(\psi,\bar{\psi})} = \int D\psi \int D\bar{\psi} e^{i\int d^4x \bar{\psi}(i\not{\partial} - m + ie)\psi}.$$

Считая переменные интегрирования ψ и $\bar{\psi}$ дираковскими спинорами, значения которых выражаются грассмановыми числами, получаем:

$$\begin{aligned} Z &= \int D\psi \int D\bar{\psi} e^{i\int d^4x \bar{\psi}(i\not{\partial} + m + i\varepsilon)\psi} = C' \det(i\not{\partial} - m + i\varepsilon) = \\ &= C' e^{\operatorname{tr}\log(i\not{\partial} + m + i\varepsilon)}, \end{aligned}$$

где C' - некоторая мультипликативная константа. С учетом свойства цикличности следа видим, что (m берется равной $m - i\varepsilon$):

$$\begin{aligned} \operatorname{tr}\log(i\not{\partial} - m) &= \operatorname{tr}\log\gamma^5(i\not{\partial} - m)\gamma^5 = \operatorname{tr}\log(-i\not{\partial} - m) = \\ &= \frac{1}{2}[\operatorname{tr}\log(i\not{\partial} - m) + \operatorname{tr}\log(-i\not{\partial} - m)] = \\ &= \frac{1}{2}\operatorname{tr}\log\left(\partial^2 + m^2\right). \end{aligned}$$

То есть $Z = C' e^{\frac{1}{2}\operatorname{tr}\log\left(\partial^2 + m^2 - i\varepsilon\right)}$ [сравните с выражением (1)!].

Мы увидим, что получили ту же энергию вакуума, что и в главе II.2, с использованием канонического формализма, если вспомним, что здесь операция взятия следа содержит множитель 4 по сравнению с операцией взятия следа в уравнении (1), ведь $(i\not{\partial} - m)$ есть матрица 4×4.

Эвристически вы можете теперь видеть необходимость использования грассмановых переменных. Если бы мы рассматривали параметры ψ и $\bar{\psi}$ в уравнении (12) как комплексные числа, то получили бы что-то вроде $(1/\det[i\not{\partial} - m]) = e^{-\operatorname{tr}\log(i\not{\partial} - m}$ и получили бы энергию вакуума с неправильным знаком. Мы хотим, чтобы детерминант появлялся в числителс, а не в знаменателе.

Пропагатор Дирака

Мы знаем, что дираковское поле должно квантоваться грассмановым интегралом по траекториям, поэтому введем грассмановы спинорные источники η и $\bar{\eta}$:

$$Z(\eta,\bar{\eta}) = \int D\psi D\bar{\psi} e^{i\int d^4x[\bar{\psi}(i\not{\partial} - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta]}$$

и будем действовать, как и раньше. Дополняя до полного квадрата, как в случае скалярного поля, получаем:

$$\bar{\psi}K\psi + \bar{\eta}\psi + \bar{\psi}\eta = \left(\bar{\psi} + \bar{\eta}K^{-1}\right)K\left(\psi + K^{-1}\eta\right) - \bar{\eta}K^{-1}\eta$$

и, следовательно,

$$Z(\eta,\bar{\eta}) = C'' e^{-i\bar{\eta}(i\not{\partial}-m)^{-1}\eta}$$

Пропагатор $S(x)$ для дираковского поля соответствует обратному оператору $(i\not{\partial} - m)$. Другими словами, $S(x)$ определяется уравнением:

$$(i\not{\partial} - m)S(x) = \delta^{(4)}(x).$$

Как вы можете проверить, решение будет иметь вид:

$$iS(x) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ipx}}{p - m + i\varepsilon}$$

в согласии с (II.2.22).

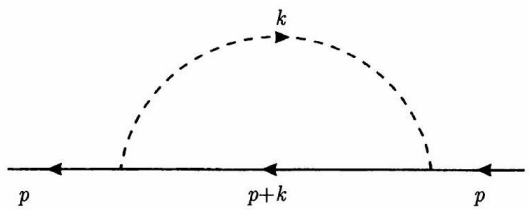


Рис. II.5.1

Правила Фейнмана для фермионов

Теперь получим правила Фейнмана для фермионов таким же образом, как и правила Фейнмана для скалярного поля. К примеру, воспользуемся теорией взаимодействия скалярного и дираковского полей

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi + \frac{1}{2}\left[(\partial\varphi)^2 - \mu^2\varphi^2\right] - \lambda\varphi^4 + f\varphi\bar{\psi}\psi$$

Производящий функционал

$$Z(\eta,\bar{\eta},J) = \int D\psi D\bar{\psi} D\varphi e^{iS(\psi,\bar{\psi},\varphi)+i\int d^4x(J\varphi+\bar{\eta}\psi+\bar{\psi}\eta)}$$

можно вычислить как двойной ряд по константам связи λ и f. Правила Фейнмана (исключая правила для чисто бозонного случая) формулируются следующим образом:

1. Нарисуйте диаграмму, используя сплошные линии для фермиона и пунктирные линии для бозона, и припишите каждой линии импульс, как, например, на рис. II.5.1.

2. Сопоставьте каждой фермионной линии пропагатор

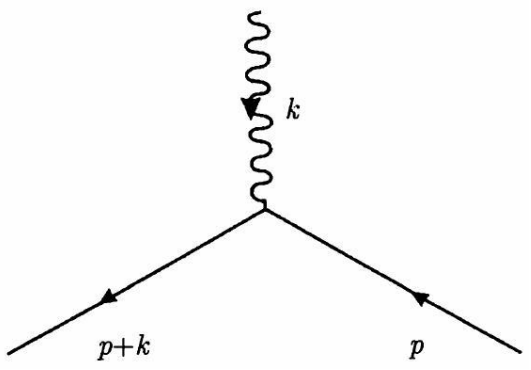
$$\frac{i}{p - m + i\varepsilon} = i\frac{\not{p} + m}{p^2 - m^2 + i\varepsilon}.$$

3. Сопоставьте каждую вершину взаимодействия с константой связи if и коэффициентом $(2\pi)^4\delta^{(4)}\left(\sum\limits_{\text{вх.}} p - \sum\limits_{\text{вых.}} p\right)$, выражающим закон сохранения импульса (суммируются входящие и выходящие импульсы соответственно).

4. По связанным с внутренними линиями импульсам проинтегрируйте с мерой $\int [d^4p/(2\pi)^4]$.

5. Внешние линии ампутируются. Для входящей фермионной линии пишем $u(p,s)$, а для выходящей фермионной линии $-\bar{u}(p',s')$. Источники и стоки должны различать спиновую поляризацию рожденных и поглощенных фермиона. [Для антифермионов имеем $v(p,s)$ и $\bar{v}(p',s')$].

6. Сопоставьте каждой замкнутой фермионной линии фактор (-1).



$ie\gamma^\mu$

Рис. II.5.2

Обратите внимание, что правило 6 справедливо только для фермионов, и в начале главы мы уже говорили об этом. Фейнмановская диаграмма, соответствующая вакуумным флуктуациям, не содержит внешних линий. Более подробно читайте в главе IV.3.

Для теории взаимодействия массивного векторного поля с дираковским полем, описываемой в главе II.1,

$$S(\psi,\bar{\psi},A)=\bar{\psi}\left[i\gamma^\mu\left(\partial_\mu-ieA_\mu\right)-m\right]\psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}\mu^2A_\mu A^\mu,$$

правила отличаются в следующем. Пропагатор векторного бозона равен:

$$\frac{i}{k^2-\mu^2}\left(\frac{k_\mu k_\nu}{\mu^2}-g_{\mu\nu}\right),$$

так что каждая векторная бозонная линия помечается не только импульсом, но и индексами μ и ν. Вершине (рис. II.5.2) соответствует $ie\gamma^\mu$.

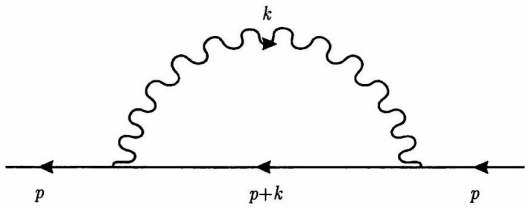


Рис. II.5.3

Упражнения

Зи-у.

II.5.1. Запишите фейнмановскую амплитуду для диаграммы на рис.

Зи-у.

II.5.1 для скалярной теории (уравнение (19)). Правильный ответ приведен в главе III.3.

Зи-у.

II.5.2. Применяя правила Фейнмана для векторной теории (22), покажите, что амплитуда для диаграммы на рис.

Зи-у.

II.5.3 будет иметь вид:

$$(ie)^2 i^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-\mu^2}\left(\frac{k_\mu k_\nu}{\mu^2}-g_{\mu\nu}\right)$$
$$\bar{u}(p)\gamma^\nu \frac{\not{p}+\not{k}+m}{(p+k)^2-m^2}\gamma^\nu u(p).$$

9.4.8 Зи.II.6 Electron Scattering and Gauge Invariance

Рассеяние электронов

Наконец, рассчитаем физический процесс, который можно провести и измерить экспериментальным путем. Рассмотрим два электрона, рассеивающихся друг на друге. На рис. II.6.1a,b приведены соответствующие фейнмановские диаграммы, в которых два электрона обмениваются фотоном.

Подождите, из главы I.5 мы знаем только пропагатор $iD_{\mu\nu} = [i/(k^2-\mu^2)](k_\mu k_\nu/\mu^2-\eta_{\mu\nu})$ для гипотетического массивного фотона. (Проведем очевидную замену: массу фотона будем обозначать μ, поскольку буквой m принято обозначать массу электрона.) В главе I.5 я изложил свою стратегию, согласно которой в последующих расчетах буду использовать ненулевое значение μ и надеяться на то, что в конце концов мы сможем положить μ равной нулю. Действительно, при вычислении потенциальной энергии между двумя внешними зарядами можно без проблем положить $\mu \to 0$ [см. уравнение (I.5.6)]. В этой и следующей главе мы проверим, всегда ли справедливо данное условие.

Применяя правила Фейнмана, находим амплитуду для диаграммы на рис. II.6.1a (передачу импульса при рассеянии берем равной $k = P_1 - p_1$):

$$A(P_1,P_2) =$$
$$= (-ie)^2 \frac{i}{(P_1-p_1)^2-\mu^2}\left(\frac{k_\mu k_\nu}{\mu^2}-\eta_{\mu\nu}\right)\bar{u}(P_1)\gamma^\mu u(p_1)\,\bar{u}(P_2)\gamma^\nu u(p_2) =$$
$$= (-ie)^2 \frac{(-i)}{(P_1-p_1)^2-\mu^2}\bar{u}(P_1)\gamma^\mu u(p_1)\,\bar{u}(P_2)\gamma_\mu u(p_2).$$

Мы пользовались тем, что $\bar{u}(P_1)$ и $u(p_1)$ удовлетворяют уравнениям движения, поэтому:

$$k_\mu \bar{u}\left(P_1\right)\gamma^\mu u\left(p_1\right) = \left(P_1 - p_1\right)_\mu \bar{u}\left(P_1\right)\gamma^\mu u\left(p_1\right) = \bar{u}\left(P_1\right)\left(\not{P}_1 - \not{p}_1\right) u\left(p_1\right) =$$
$$= \bar{u}\left(P_1\right)(m-m)u\left(p_1\right) = 0.$$

Это важное наблюдение означает, что член $k_\mu k_\nu/\mu^2$ в фотонном пропагаторе не дает вклада, следовательно, можно безболезненно положить массу фотона μ равной нулю. То есть

$$A\left(P_1, P_2\right) = \frac{ie^2}{\left(P_1 - p_1\right)^2}\bar{u}\left(P_1\right)\gamma^\mu u\left(p_1\right)\bar{u}\left(P_2\right)\gamma_\mu u\left(p_2\right)$$

Обратите внимание на то, что тождество, позволяющее нам положить μ равным нулю, является разновидностью условия сохранения электромагнитного тока в импульсном пространстве $\partial_\mu J^\mu = 0$, где $J^\mu = \bar{\psi}\gamma^\mu\psi$. Заметим, что это вычисление тесно связано с вычислением, которое мы провели, переходя от (I.5.4) к (I.5.5), только сейчас роль $J^\mu(k)$ играет $\bar{u}\left(P_1\right)\gamma^\mu u\left(p_1\right)$. Мы обозначили зависимость A исключительно от конечных импульсов. Согласно статистике Ферми, амплитуда для диаграммы на рис. II.6.1b будет равна $-A\left(P_2, P_1\right)$. Следовательно, амплитуда двух электронов, имеющих начальные импульсы p_1 и p_2 и рассеивающихся в два электрона с импульсами P_1 и P_2, равна

$$\mathcal{M} = A\left(P_1, P_2\right) - A\left(P_2, P_1\right)$$

Чтобы найти сечение, мы должны возвести в квадрат амплитуду:

$$\left|\mathcal{M}\right|^2 = \left[\left|A\left(P_1, P_2\right)\right|^2 + \left(P_1 \leftrightarrow P_2\right)\right] - 2\,\mathrm{Re}\,A\left(P_2, P_1\right)^* A\left(P_1, P_2\right)$$

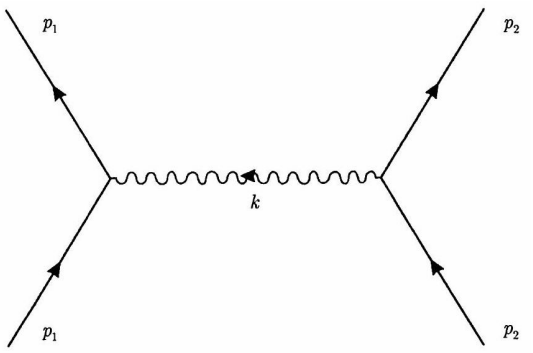
Теперь нам необходимо совершить довольно много выкладок, которые, однако, нельзя назвать сложными. Во-первых, член

$$\left|A\left(P_1, P_2\right)\right|^2 = \frac{e^4}{\left(P_1 - p_1\right)^4}\times$$
$$\times\left[\bar{u}\left(P_1\right)\gamma^\mu u\left(p_1\right)\bar{u}\left(p_1\right)\gamma^\nu u\left(P_1\right)\right]\left[\bar{u}\left(P_2\right)\gamma^\mu u\left(p_2\right)\bar{u}\left(p_2\right)\gamma^\nu u\left(P_2\right)\right]$$

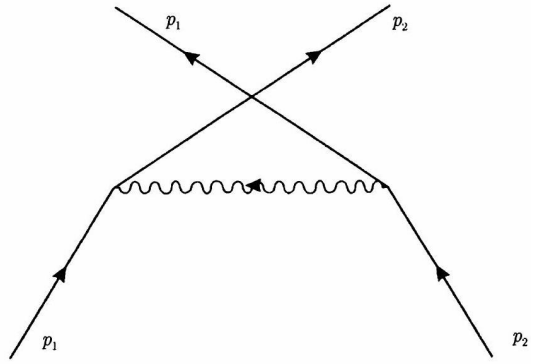
представляется в виде двух сомножителей так, что один из них содержит спиноры, несущие импульс с индексом 1, а другой - спиноры, несущие импульс с индексом 2. Напротив, интерференционный член $A\left(P_2, P_1\right)^* A\left(P_1, P_2\right)$ не факторизуется.

В простейших экспериментах начальные электроны не поляризованы и поляризация вылетающих электронов также не измеряется. Следовательно, можно провести усреднение по начальным спинам и суммирование по конечным спинам, используя соотношение (II.2.8):

$$\sum_s u(p,s)\bar{u}(p,s) = \frac{\not{p} + m}{2m}$$



(a)



(b)

Рис. II.6.1 Таким образом, усредняя и суммируя по спинам $|A(P_1,P_2)|^2$ приходим к выражению (спиновые значки выписаны явно):

$$\tau^{\mu\nu}(P_1,p_1)\equiv\sum_s\sum_S\bar{u}(P_1,S)\gamma^\mu u(p_1,s)\,\bar{u}(p_1,s)\gamma^\nu u(P_1,S)=$$
$$=\frac{1}{(2m)^2}\operatorname{tr}\left(\not{P}_1+m\right)\gamma^\mu\left(\not{p}_1+m\right)\gamma^\nu,$$

которое следует умножить на $\tau_{\mu\nu}(P_2,p_2)$. Аналогичным образом путем усреднения и суммирования $A(P_2,P_1)^*A(P_1,P_2)$ приходим к более сложному выражению

$$\varkappa\equiv\sum\sum\sum$$
$$\bar{u}(P_1)\gamma^\mu u(p_1)\,\bar{u}(P_2)\gamma_\mu u(p_2)\,\bar{u}(p_1)\gamma^\nu u(P_2)\,\bar{u}(p_2)\gamma_\nu u(P_1)$$

где для наглядности опущены спиновые значки. С учетом соотношения (6) можно записать $\varkappa$ в виде следа.

Теперь мы, точнее, вы должны научиться вычислять след произведения гамма-матриц. Ключевое наблюдение состоит в том, что квадрат гамма-матрицы равен либо +1, либо -1, а разные гамма-матрицы антикоммутируют. Отсюда очевидно, что след произведения нечетного количества гамма-матриц равен нулю. Таким образом,

$$\tau^{\mu\nu}(P_1,p_1)=\frac{1}{(2m)^2}\left[\operatorname{tr}\left(\not{P}_1\gamma^\mu\not{p}_1\gamma^\nu\right)+m^2\operatorname{tr}\left(\gamma^\mu\gamma^\nu\right)\right]$$

Если учесть, что

$$\mathrm{tr}\left(\not{P}_1\gamma^\mu\not{p}_1\gamma^\nu\right) = P_{1\rho}p_{1\lambda}\,\mathrm{tr}\left(\gamma^\rho\gamma^\mu\gamma^\lambda\gamma^\nu\right)$$

а также выражение для расчета следа произведения четного количества гамма-матриц, приведенное в приложении D, получим:

$$\tau^{\mu\nu}\left(P_1,p_1\right) = \frac{4}{(2m)^2}\left(P_1^\mu p_1^\nu - \eta^{\mu\nu}P_1\cdot p_1 + P_1^\nu p_1^\mu + m^2\eta^{\mu\nu}\right)$$

Более того, поскольку существуют лишь четыре различные гамма-матрицы, след произведения шести гамма-матриц всегда можно свести к следу произведения четырех гамма-матриц, ведь среди данных шести матриц всегда будут пары одинаковых матриц, которые можно поставить рядом антикоммутированием. Это справедливо и для следа произведения большего числа гамма-матриц. Расчет $\varkappa$ достаточно трудоемок, поскольку включает в себя вычисление следов произведения до восьми гамма-матриц. Мы ограничимся вычислением $\varkappa$ в релятивистском пределе, где m пренебрежимо мала по сравнению с импульсами:

$$\varkappa = \frac{1}{(2m)^4}\,\mathrm{tr}\left(\not{P}_1\gamma^\mu\not{p}_1\gamma^\nu\ \not{P}_2\gamma_\mu\not{p}_2\gamma_\nu\right)$$

Применяя к выражению (10) тождества, которые приведены в приложении D, получим:
$\mathrm{tr}\left(\not{P}_1\gamma^\mu\not{p}_1\gamma^\nu P_2\gamma_\mu\not{p}_2\gamma_\nu\right) = -2\,\mathrm{tr}\left(P_1\gamma^\mu\not{p}_1\not{p}_2\gamma_\mu\not{P}_2\right) = -32p_1\cdot p_2 P_1\cdot P_2$
В этом же пределе имеем

$$\tau^{\mu\nu}\left(P_1,p_1\right) = \frac{4}{(2m)^2}\left(P_1^\mu p_1^\nu + P_1^\nu p_1^\mu - \eta^{\mu\nu}P_1\cdot p_1\right)$$

и, следовательно,

$$\begin{aligned}
&\tau^{\mu\nu}\left(P_1,p_1\right)\tau_{\mu\nu}\left(P_2,p_2\right) = \\
&\quad = \frac{16}{(2m)^4}\left(P_1^\mu p_1^\nu + P_1^\nu p_1^\mu - \eta^{\mu\nu}P_1\cdot p_1\right)\left(2P_2^\mu p_2^\nu - \eta^{\mu\nu}P_2\cdot p_2\right) = \\
&\quad = \frac{16\cdot 2}{(2m)^4}\left(p_1\cdot p_2 P_1\cdot P_2 + p_1\cdot P_2 p_2\cdot P_1\right)
\end{aligned}$$

Хотя у меня нет цели обучить вас релятивистской кинематике и вычислснию сечений, чрезвычайно полезным иногда оказывается довести расчеты до логического завершения. В системе центра масс в релятивистском пределе: $p_1 = E(1,0,0,1), p_2 = E(1,0,0,-1), P_1 = E(1,\sin\theta,0,\cos\theta)$ и $P_2 = E(1,-\sin\theta,0,-\cos\theta)$. Следовательно, $p_1\cdot p_2 = P_1\cdot P_2 = -2E^2, p_1\cdot P_1 = p_2\cdot P_2 = 2E^2\sin^2(\theta/2)$ и $p_1\cdot P_2 = p_2\cdot P_1 = 2E^2\cos^2(\theta/2)$. Также в этом пределе выполняется условие $(P_1-p_1)^4 = \text{-}\ (-2p_1\cdot P_1)^2 = 16E^4\sin^4(\theta/2)$. Объединяя их вместе, приходим к выражснию $\frac{1}{2}\sum\limits_s\sum\limits_S|\mathcal{M}|^2 = (e^4/4m^4)\,f(\theta)$, в котором

$$f(\theta) = \frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)} + \frac{2}{\sin^2(\theta/2)\cos^2(\theta/2)} + \frac{1+\sin^4(\theta/2)}{\cos^4(\theta/2)}.$$

()изический смысл каждого из членов очевиден. Первый член характеризуст рассеяние, обусловленное фотонным пропагатором $\sim 1/k^2$. Третий член обусловлен неразличимостью двух вылетающих электронов: при $\theta\to\pi-\theta$ рассеяние должно быть симметрично, поскольку экспериментатор не в состоянии определить, какой из нелетающих электронов рассеивается вперед, а какой - в обратном направлении. Второй член представляет наибольший интерес:

он обусловлен квантовой интерференцией. Если ошибочно принять электроны за бозоны и учесть (3) со знаком плюс, второй член в выражении для $f(\theta)$ окажется со знаком минус. Разница огромна: например, $f(\pi/2)$ будет равна $5-8+5=2$, вместо $5+8+5=18$.

В связи с тем, что переход от квадрата амплитуды вероятности к эффективному сечению идейно аналогичен тому, как это делается в рамках нерелятивистской квантовой механики (делением на падающий поток и т. д.), я сошлюсь на соответствующую формулу в приложении С. В качестве упражнения попробуйте самостоятельно сделать последние шаги и вычислить дифференциальное сечение рассеяния:

$$\frac{d\sigma}{d\Omega}=\left(\frac{e^2}{4\pi}\right)^2\frac{1}{8E^2}f(\theta)$$

Эта удивительная физика

Если задуматься, теоретическая физика покажется удивительным занятием. Подготовив необходимое оборудование и осуществив процесс рассеяния электронов высокой энергии друг на друге, экспериментаторы действительно смогут определить дифференциальное сечение рассеяния, задаваемое формулой (12). В этом есть что-то почти магическое.

Зи-у.II.6.1. Проверьте утверждения из текста.

(??? напишу нормально вопрос тут.)

Зи-у.

II.6.2.
Вычислите сечение, приведенное в (12).

9.4.9 Зи.II.7 Diagrammatic Proof of Gauge Invariance

Калибровочная инвариантность

С концептуальной точки зрения нам важнее не вычислять сечения, а доказать тот факт, что мы действительно можем положить массу фотона μ равной нулю при вычислении любого физического процесса. При $\mu=0$ лагранжиан, который приведен в главе II.1, сводится к лагранжиану для квантовой электродинамики:

$$\mathcal{L}=\bar{\psi}\left[i\gamma^{\mu}\left(\partial_{\mu}-ieA_{\mu}\right)-m\right]\psi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Мы теперь готовы к одному из важнейших наблюдений в истории теоретической физики. Смотрите, лагранжиан левоинвариантен относительно калибровочного преобразования:

$$\psi(x)\to e^{i\Lambda(x)}\psi(x)$$

и

$$A_{\mu}(x)\to A_{\mu}(x)+\frac{1}{ie}e^{-i\Lambda(x)}\partial_{\mu}e^{i\Lambda(x)}=A_{\mu}(x)+\frac{1}{e}\partial_{\mu}\Lambda(x)$$

что означает

$$F_{\mu\nu}(x) \to F_{\mu\nu}(x)$$

Вы, конечно, уже знакомы с (3) и с инвариантностью $F_{\mu\nu}$ из классической электродинамики.

В современной теоретической физике калибровочная инвариантность [1] относится к числу фундаментальных и имеющих крайне важное значение, как мы увидим ниже. С современной точки зрения принято рассматривать лагранжиан (1) как следствие (2) и (3). Если мы хотим построить калибровочно-инвариантную релятивистскую теорию поля для полей со спинами $\frac{1}{2}$ и 1, мы с необходимостью придем к квантовой электродинамике.

Вы должны заметить, что в (3) я привел две эквивалентные формы. В большинстве учебных пособий пользуются второй формой, более простой, но не стоит забывать и про первую форму. Обратите внимание, что $\Lambda(x)$ и $\Lambda(x) + 2\pi$ приводят в точности к одинаковым преобразованиям. На математическом языке величины $e^{i\Lambda(x)}$ и $\partial_\mu \Lambda(x)$ хорошо определены, тогда как величина $\Lambda(x)$ - нет.

С учетом этих формальных, но физически важных замечаний мы можем приступить к работе над доказательством. Я оставляю вам самим дать общее доказательство, но покажу вам как это сделать, разобрав несколько характерных примеров.

Вспомним, что пропагатор гипотетического массивного фотона равен $iD_{\mu\nu} = i\left(k_\mu k_\nu/\mu^2 - g_{\mu\nu}\right)/\left(k^2 - \mu^2\right).\mu^2$ в знаменателе можно положить равной нулю, тогда выражение для фотонного пропагатора будет иметь вид $iD_{\mu\nu} = i\left(k_\mu k_\nu/\mu^2 - g_{\mu\nu}\right)/k^2$. Опасным является член $k_\mu k_\nu/\mu^2$. Мы хотим показать, что он выпадает.

Конкретный пример

Сначала рассмотрим электрон-электронное рассеяние с точностью до e^4. Среди всех возможных диаграмм сосредоточимся на двух, приведенных на рис. II.7.1a. Фейнмановская амплитуда равна:

$$\bar{u}\left(p'\right)\left(\gamma^\lambda \frac{1}{\not p + \not k - m}\gamma^\mu + \gamma^\mu \frac{1}{\not p' - \not k - m}\gamma^\lambda\right) u(p) \frac{i}{k^2}\left(\frac{k_\mu k_\nu}{\mu^2}\right)\Gamma^\nu_\lambda$$

где Γ^ν_λ - некоторый фактор, точная структура которого нас не интересует. Однако для конкретного случая, изображенного на рис. II.7.1a, мы, конечно, могли бы получить явное выражение для Γ^ν_λ. Обратите внимание на знак плюс, который обусловлен обменом двумя фотонами, подчиняющимися Бозе-статистике.

Обратимся к опасному члену. Сворачивая $\bar{u}\left(p'\right)(...)u(p)$ из (5) с k_μ, получаем

$$\bar{u}\left(p'\right)\left(\gamma^\lambda \frac{1}{\not p + \not k - m}\not k + \not k \frac{1}{\not p' - \not k - m}\gamma^\lambda\right) u(p)$$

Хитрость состоит в том, чтобы представить k в числителе первой дроби как $(\not p + k - m) - (\not p - m)$, а в числителе второй дроби как $(\not p - m) - (\not p$ - $-\not k - m)$. Поскольку $(\not p - m)u(p) = 0$ и $\bar{u}\left(p'\right)(\not p - m) = 0$, выражение (6) обращается в нуль, что доказывает теорему в этом простом случае. В связи с тем что в нашем доказательстве не фигурирует Γ^ν_λ в явном виде, оно будет справедливым даже в случае замены диаграммы на рис. II.7.1a более общей диаграммой на рис. II. 7.1b, в заштрихованной области которой могут протекать произвольно сложные процессы.

Можно обобщить доказательство на случай рис. II.7.1c. Помимо фотона, несущего импульс k, уже есть n фотонов на линии электрона. В доказательстве теоремы эти n фотонов

выступают в качестве «зрителей», точно так же, как фотон с импульсом k' на рис. II.7.1a не участвует в доказательстве утверждения о том, что (6) сокращается. Фотон с импульсом k может прицепляться к линии электрона в $n+1$ разных точках. Теперь в качестве упражнения попробуйте расширить доказательство на этот случай.

Попадание фотона на внутреннюю линию

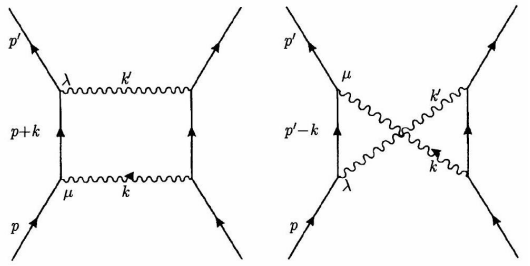
В только что рассмотренном нами примере фотонная линия оканчивается на внешней электронной линии. Ключевым моментом в доказательстве является тот факт, что линия «ограничена с двух концов» $\bar{u}(p')$ и $u(p)$. А что, если фотонная линия оканчивается на внутренней линии электрона?

На рис. II.7.2 изображен пример, соответствующий электрон-электронному рассеянию порядка e^8. Рисунок содержит три разные диаграммы. Электрон «слева» испускает три фотона, которые присоединяются ко внутренней электронной петле. Электрон «справа» испускает фотон с импульсом k, который может присоединиться к петле тремя разными способами.

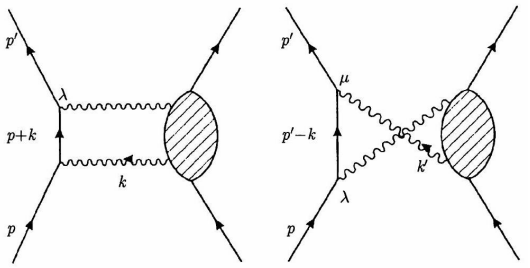
Поскольку нас интересует, исчезнет или нет член $k_\mu k_\rho/\mu^2$ из выражения для фотонного пропагатора $i\left(k_\mu k_\rho/\mu^2 - g_{\mu\rho}\right)/k^2$, заменим фотонный пропагатор на k_μ. Для удобства записи введем обозначения $p_1 = p+q_1$ и $p_2 = p_1+q_2$ (смотрите обозначения импульсов на рис. II.7.2). Рассмотрим лишь интересующие нас части трех диаграмм, обозначенных как A, B и C:

$$A = \int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^\nu \frac{1}{\not p_2 + k - m}\gamma^\sigma \frac{1}{\not p_1 + k - m}\gamma^\lambda \frac{1}{\not p + \not K - m}\not K \frac{1}{\not p - m}\right)$$

$$B = \int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^\nu \frac{1}{p_2 + \not K - m}\gamma^\sigma \frac{1}{\not p_1 + \not K - m}\not K \frac{1}{\not p_1 - m}\gamma^\lambda \frac{1}{\not p - m}\right)$$



(a)



(b)

Рис. II.7.1

И

$$C = \int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^\nu \frac{1}{\not p_2 + \not K - m}\not K \frac{1}{\not p_2 - m}\gamma^\sigma \frac{1}{\not p_1 - m}\gamma^\lambda \frac{1}{\not p - m}\right)$$

Создается впечатление, что с приведенными выражениями больше ничего нельзя сделать, однако это не так. Прибегнем к известному нам трюку

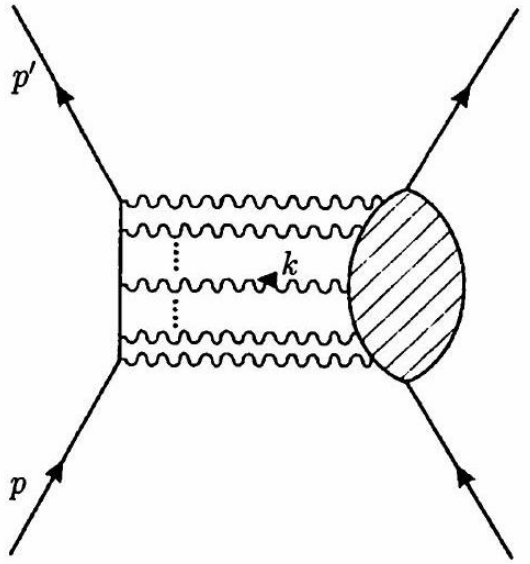


Рис. II.7.1. (Продолжение)

и представим $\not{k}$ в выражении для C как $\not{k} = (\not{p}_2 + \not{k} - m) - (\not{p}_2 - m)$ и получим:

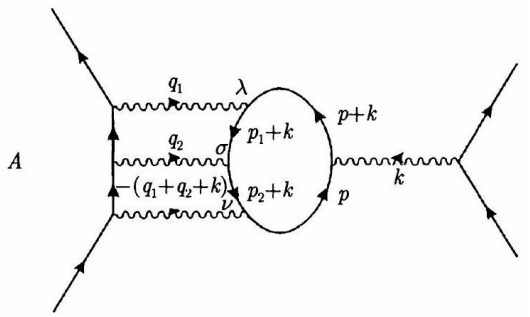
$$C = \int \frac{d^4p}{(2\pi)^4} \left[\mathrm{tr} \left(\gamma^\nu \frac{1}{p_2 - m} \gamma^\sigma \frac{1}{p_1 - m} \gamma^\lambda \frac{1}{\not{p}' - m} \right) - \right.$$
$$\left. - \mathrm{tr} \left(\gamma^\nu \frac{1}{p_2 + k - m} \gamma^\sigma \frac{1}{p_1 - m} \gamma^\lambda \frac{1}{p' - m} \right) \right].$$

В выражении для B запишем $k\prime = (p_1' + k\prime - m) - (p_1' - m)$, так что получим:

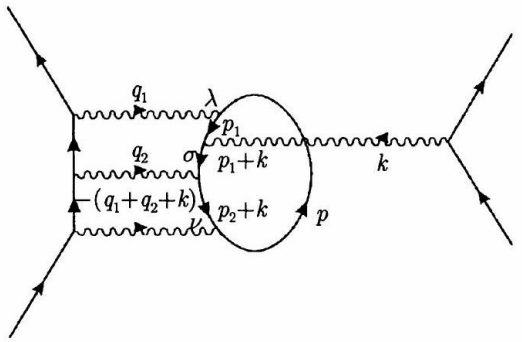
$$C = \int \frac{d^4p}{(2\pi)^4} \left[\mathrm{tr} \left(\gamma^\nu \frac{1}{p_2 + k - m} \gamma^\sigma \frac{1}{p_1 - m} \gamma^\lambda \frac{1}{p' - m} \right) - \right.$$
$$\left. - \mathrm{tr} \left(\gamma^\nu \frac{1}{p_2 + \not{l} - m} \gamma^\sigma \frac{1}{p_1 + k\prime - m} \gamma^\lambda \frac{1}{p' - m} \right) \right].$$

Наконец, в выражении для A запишем $\not{k} = (\not{p} + \not{k} - m) - (\not{p} - m)$, так что получим:

$$A = \int \frac{d^4p}{(2\pi)^4} \left[\mathrm{tr} \left(\gamma^\nu \frac{1}{p_2 + k - m} \gamma^\sigma \frac{1}{p_1 + k/ - m} \gamma^\lambda \frac{1}{p' - m} \right) - \right.$$
$$\left. - \mathrm{tr} \left(\gamma^\nu \frac{1}{p_2 + kb' - m} \gamma^\sigma \frac{1}{p_1 + k\prime - m} \gamma^\lambda \frac{1}{\not{l} + \not{l}' - m} \right) \right].$$



B



C

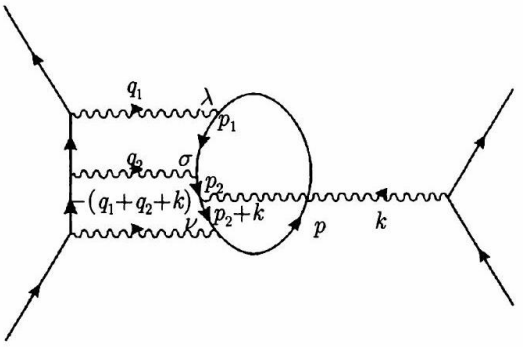


Рис. II.7.2 Теперь посмотрим, к чему это приведет. Если объединить все три диаграммы, члены попарно сократятся:

$$A+B+C=\int\frac{d^4p}{(2\pi)^4}\left[\mathrm{tr}\left(\gamma^\nu\frac{1}{\not p_2-m}\gamma^\sigma\frac{1}{\not p_1-m}\gamma^\lambda\frac{1}{\not p-m}\right)-\right.$$
$$\left.-\,\mathrm{tr}\left(\gamma^\nu\frac{1}{\not p_2+\not k-m}\gamma^\sigma\frac{1}{\not p_1+\not k-m}\gamma^\lambda\frac{1}{\not p+\not k-m}\right)\right]$$

Если сдвинуть (см. упражнение II.7.2) во втором члене переменную интегрирования $p\to p-k$, два члена взаимно сократятся. Действительно, в фотонном пропагаторе член $k_\mu k_\rho/\mu^2$ исчезает, так что можно положить $\mu=0$

Я оставляю вам найти общее доказательство. Мы получили его для одного конкретного процесса. Попробуте сами проделать то же самое для других процессов. Вы увидите, как оно работает.

Тождество Уорда-Такахаши

Подведем итоги. Если известна физическая амплитуда $T^{\mu\cdots}(k,...)$ со внешними электронами на массовой поверхности (т. е. все необходимые множители $u(p)$ и $\bar u(p)$ включены в $T^{\mu\cdots}(k,...)$), описывающая процесс испускания или поглощения фотона с импульсом k вершиной, обозначенной лоренцевым индексом μ, то выполняется условие:

$$k_\mu T^{\mu\cdots}(k,...)=0$$

Его иногда называют тождеством Уорда-Такахаши.

Решающим моментом служит представление фотонного пропагатора в виде $iD_{\mu\nu} = -ig^{\mu\nu}/k^2$. Раз мы можем исключить член $k_\mu k_\nu/\mu^2$ из фотонного пропагатора $i\left(k_\mu k_\nu/\mu^2 - g_{\mu\nu}\right)/k^2$, с тем же успехом мы можем добавить в него член $k_\mu k_\nu/k^2$ с произвольным коэффициентом. В результате получим фотонный пропагатор вида:

$$iD_{\mu\nu} = \frac{i}{k^2}\left[(1-\xi)\frac{k_\mu k_\nu}{k^2} - g_{\mu\nu}\right]$$

где для упрощения вычислений число ξ можно выбрать произвольным образом. Очевидно, что выбор значения ξ связан с выбором калибровки для электромагнитного поля. В частности, случай $\xi = 1$ известен под названием фейнмановской калибровки, а случай $\xi = 0$ - под названием калибровки Ландау. Если вы особенно удачно подберете иное значение ξ, можете назвать соответствующую калибровку в свою честь! Для простых вычислений разумно использовать произвольное значение ξ. Независимость конечного результата от ξ является полезной проверкой выкладок.

Итак, мы завершили вывод правил Фейнмана для квантовой электродинамики: они аналогичны правилам для теории массивного векторного бозона, которые даны в главе II.5, за исключением выражения (15) для фотонного пропагатора.

Таково диаграммное доказательство калибровочной инвариантности квантовой электродинамики. Впоследствии (в главе IV.7) мы увидим, что в некоторых случаях нельзя сдвигать, как при нашем доказательстве, импульс интегрирования.

Продольная мода

Вернемся к тому, что вызывало у нас беспокойство в главе I.5. Рассмотрим массивный мезон со спином 1, двигающийся в направлении z. Три вектора поляризации зафиксированы условием $k^\lambda\varepsilon_\lambda = 0$, где $k^\lambda = (\omega, 0, 0, k)$ (см. главу I.5), и нормировкой $\varepsilon^\lambda\varepsilon_\lambda = -1$, так что $\varepsilon_\lambda^{(1)} = (0,1,0,0), \varepsilon_\lambda^{(2)} = (0,0,1,0), \varepsilon_\lambda^{(3)} = (-k,0,0,\omega)/\mu$. Обратите внимание, что при $\mu \to 0$ вектор продольной поляризации $\varepsilon_\lambda^{(3)}$ пропорционален $k_\lambda = (\omega, 0, 0, -k)$. Амплитуда испускания мезона с продольной поляризацией в процессе, описываемом уравнением (14), равна $\varepsilon_\lambda^{(3)}T^{\lambda\cdots} = (-kT^{0\cdots} + +\omega T^{3\cdots})/\mu = \left(-kT^{0\cdots} + \sqrt{k^2+\mu^2}T^{3\cdots}\right)/\mu \simeq \left(-kT^{0\cdots} + \left(k + \frac{\mu^2}{2k}\right)T^{3\cdots}\right)/\mu$ (для $\mu \ll l$), т. е. $-k_\lambda T^{\lambda\cdots}/\mu + \frac{\mu}{2k}T^{3\cdots}$, где $k_\lambda = (k, 0, 0, -k)$. С учетом условия (14) приходим к тому, что при $\mu \to 0$ амплитуда $\frac{\mu}{2k}T^{3\cdots} \to 0$.

Продольная мода фотона не существует, так как она отщепляется во всех физических процессах.

Есть такой парадокс. Мистер Больцман утверждает, что при тепловом равновесии каждая степень свободы связана с $\frac{1}{2}T$. Значит, путем измерения некоторого теплового свойства (например, теплоемкости) ящика с фотонным газом с точностью 2/3 экспериментатор скорее подтвердит, что фотон действительно безмассов, чем скажет, что его масса равна одной миллиардной электрон-вольта.

Объяснение парадокса состоит в том, что по мере исчезновения продольной моды при $\mu \to 0$, время, необходимое продольной моде для уста- новления теплового равновесия, стремится к бесконечности. Экспериментатор из описанной парадоксальной ситуации должен в этом случае быть чрезвычайно терпеливым.

Упражнения

Зи-у.

II.7.1. Обобщите доказательство на случай, изображенный на рис. II.7.1c.
[Указание: обратите внимание на то, что рис. II.7.1b соответствует случаю $n = 1$.]

Зи-у.

II.7.2. Вы, наверное, хотели бы знать, когда можно сдвигать переменную интегрирования, а когда нельзя. Упрощая знаменатель в первом интеграле, в (13),

$$\int \frac{d^4p}{(2\pi)^4} \operatorname{tr}\left(\gamma^\nu \frac{1}{\not p_2 - m}\gamma^\sigma \frac{1}{\not p_1 - m}\gamma^\lambda \frac{1}{\not p - m}\right),$$

и вычислив след, вы можете убедиться, что этот интеграл расходится только логарифмически и, значит, сдвиг возможен. К этому вопросу мы еще раз вернемся в главе IV.7.

9.5 Spinor Field (fermions) 2: Other Properties and Conceptual Questions

(this shuold be rewritten, now structure does not match the title, I should go through this theory to find out what to delete and what to move????)

9.5.1 Lorentz Invariance of Wave Equations

(see field theory, there, there should be the same section)

Теория

В первую очередь мы должны поставить вопрос, к которому подошли в главе 2: что имеется в виду, когда говорят, что уравнение «релятивистски инвариантно»? Подходящим определением является следующее. Если ϕ - поле или совокупность полей и $\mathscr{D}$ - некоторый дифференциальный оператор, то утверждение, что уравнение $\mathscr{D}\phi = 0$ релятивистски инвариантно, означает, что если $\phi(x)$ удовлетворяет этому уравнению и мы совершаем преобразование Лоренца перехода к другой системе отсчета, то преобразованное поле в новой системе отсчета удовлетворяет тому же самому уравнению. Эквивалентно, можно применить одно и то же преобразование Лоренца ко всем частицам или полям, тогда уравнение $\mathscr{D}\phi = 0$ должно остаться справедливым и после преобразования. Далее мы будем придерживаться этой «активной» точки зрения на преобразования.

Лагранжева формулировка теории поля существенно облегчает вопрос о лоренц-инвариантности. Уравнение движения автоматически будет лоренц- инвариантным по данному выше определению, если это уравнение следует из лагранжиана, который является лоренцевым скаляром. Это - непосредственное следствие принципа наименьшего действия. Если преобразования Лоренца оставляют лагранжиан неизменным, то преобразованный экстремум действия будет снова экстремумом.

В качестве примера рассмотрим теорию Клейна-Гордона. Мы можем записать произвольное преобразование Лоренца как

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu$$

с какой-то 4×4-матрицей Λ. Что произойдет с полем Клейна-Гордона $\phi(x)$ при этом преобразовании? Представим себе поле ϕ как локальную меру некоторой величины, распределенной в пространстве. Если эта величина скапливается в точке $x = x_0$, тогда $\phi(x)$ будет иметь максимум в точке x_0. Если теперь подвергнуть первоначальное распределение преобразованию Лоренца, новое распределение будет иметь максимум в точке $x = \Lambda x_0$.

Это показано на рис. 3.1(a).

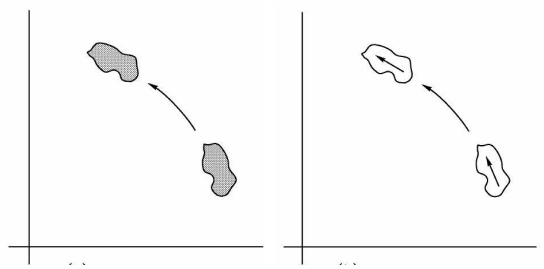


Рис. 3.1.

(a) скалярное поле (b) векторное поле

При вращении векторного поля изменяется ориентация вектора, а также расположение области локализации этого поля.

Соответствующее преобразование поля имеет вид:

$$\phi(x) \to \phi'(x) = \phi\left(\Lambda^{-1}x\right)$$

Иными словами, преобразованное поле, вычисленное в преобразованной точке, дает то же самое значение, что и первоначальное поле, вычисленное в исходной точке. Следует проверить, что это преобразование оставляет неизменной форму лагранжиана КлейнаГордона. Согласно (3.2) массовое слагаемое $\frac{1}{2}m^2\phi^2(x)$ просто смещается в точку $(\Lambda^{-1}x)$. Преобразование $\partial_\mu\phi(x)$ описывается формулой

$$\partial_\mu\phi(x) \to \partial_\mu\left(\phi\left(\Lambda^{-1}x\right)\right) = \left(\Lambda^{-1}\right)^\nu{}_\mu\left(\partial_\nu\phi\right)\left(\Lambda^{-1}x\right)$$

Так как метрический тензор $g^{\mu\nu}$ лоренц-инвариантен, матрицы Λ^{-1} удовлетворяют тождеству:

$$\left(\Lambda^{-1}\right)^\rho{}_\mu\left(\Lambda^{-1}\right)^\sigma{}_\nu g^{\mu\nu} = g^{\rho\sigma}.$$

Используя это соотношение, можно вычислить закон преобразования кинетического слагаемого в лагранжиане Клейна-Гордона:

$$\begin{aligned}\left(\partial_\mu\phi(x)\right)^2 &\to g^{\mu\nu}\left(\partial_\mu\phi'(x)\right)\left(\partial_\nu\phi'(x)\right) = \\ &= g^{\mu\nu}\left[\left(\Lambda^{-1}\right)^\rho{}_\mu\partial_\rho\phi\right]\left[\left(\Lambda^{-1}\right)^\sigma{}_\nu\partial_\sigma\phi\right]\left(\Lambda^{-1}x\right) = \\ &= g^{\rho\sigma}\left(\partial_\rho\phi\right)\left(\partial_\sigma\phi\right)\left(\Lambda^{-1}x\right) = \\ &= \left(\partial_\mu\phi\right)^2\left(\Lambda^{-1}x\right).\end{aligned}$$

Таким образом, весь лагранжиан преобразуется просто как скаляр:

$$\mathscr{L}(x) \to \mathscr{L}\left(\Lambda^{-1}x\right)$$

Действие S, полученное интегрированием $\mathscr{L}$ по пространству-времени, лоренц-инвариантно. Аналогичное вычисление показывает, что уравнение движения также лоренц-инвариантно:

$$\begin{aligned}\left(\partial^2 + m^2\right)\phi'(x) &= \left[\left(\Lambda^{-1}\right)_{\mu}^{\ \nu}\partial_\nu\left(\Lambda^{-1}\right)^{\sigma\mu}\partial_\sigma + m^2\right]\phi\left(\Lambda^{-1}x\right) = \\ &= \left(g^{\nu\sigma}\partial_\nu\partial_\sigma + m^2\right)\phi\left(\Lambda^{-1}x\right) = 0.\end{aligned}$$

Закон преобразования (3.2), использованный для поля ϕ, является простейшим из всех возможных законов преобразования полей. Для поля, имеющего одну компоненту, этот закон является единственно возможным. Но нам известны примеры многокомпонентных полей, которые преобразуются более сложным образом. Наиболее знакомый случай - векторное поле, например, плотность 4-тока $j^\mu(x)$ или вектор-потенциал $A^\mu(x)$. В этом случае величина, распределенная в пространстве-времени, имеет ориентацию, которая должна быть также преобразована. Как показано на рис. 3.1(б), ориентация должна вращаться в направлении перемещения точки, в которой вычисляется поле:

$$\begin{gathered}\text{при трехмерных вращениях } V^i(x) \to R^{ij}V^j\left(R^{-1}x\right); \\ \text{при преобразованиях Лоренца } V^\mu(x) \to \Lambda^\mu{}_\nu V^\nu\left(\Lambda^{-1}x\right).\end{gathered}$$

Тензоры произвольного ранга могут быть построены из векторов путем добавления большего количества индексов, и соответственно большего количества множителей Λ в законе преобразования. Используя такие векторные и тензорные поля, можно записать множество лоренц-инвариантных уравнений, например, уравнения Максвелла

$$\partial^\mu F_{\mu\nu} = 0 \quad \text{или \%} \quad \%\partial^2 A_\nu - \partial_\nu\partial^\mu A_\mu = 0,$$

которые следуют из лагранжиана

$$\mathscr{L}_{\text{Максвелл}} = -\frac{1}{4}\left(F_{\mu\nu}\right)^2 = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2.$$

Вообще, любое уравнение, каждый член которого имеет одинаковое число не просуммированных (висячих) лоренцевых индексов, будет инвариантным по отношению к преобразованиям Лоренца. Такой метод тензорных обозначений позволяет построить большой класс лоренц-инвариантных уравнений, но оказывается, что таких уравнений намного больше. Как их искать? Можно было бы попробовать систематически найти все возможные законы преобразований поля. Тогда было бы нетрудно выписать инвариантные лагранжианы. Для простоты ограничимся линейными преобразованиями, так что, если Φ_a является n-компонентным мультиплетом, закон лоренцевского преобразования определяется $n \times n$ матрицей $M(\Lambda)$:

$$\Phi_a(x) \to M_{ab}(\Lambda)\Phi_b\left(\Lambda^{-1}x\right).$$

Можно показать, что наиболее общие нелинейные законы преобразований могут быть построены из этих линейных преобразований, поэтому нет смысла рассматривать преобразования, более общие, чем (3.8). В последующих рассуждениях мы будем опускать изменение аргумента поля и записывать преобразование (3.8) в виде:

$$\Phi \to M(\Lambda)\Phi.$$

Каков допустимый вид матриц $M(\Lambda)$? Основное ограничение на $M(\Lambda)$ можно найти, рассматривая два последовательных преобразования Λ и Λ'. Результат должен быть снова преобразованием Лоренца Λ'', т. е. преобразования Лоренца образуют группу. Это дает условие согласованности, которому должны удовлетворять матрицы $M(\Lambda)$. При последовательном действии двух преобразований

$$\Phi \to M\left(\Lambda'\right) M(\Lambda)\Phi = M\left(\Lambda''\right)\Phi$$

где $\Lambda'' = \Lambda'\Lambda$. Таким образом, соответствие между матрицами M и преобразованиями Λ должно сохраняться при умножении. На математическом языке мы говорим, что матрицы M должны образовывать n-мерное представление группы Лоренца. Таким образом, на математическом языке наш вопрос перефразируется следующим образом: каковы (конечномерные) матричные представления группы Лоренца?

Прежде чем ответить на этот вопрос для группы Лоренца, рассмотрим более простую группу трехмерных вращений. Эта группа имеет представления любой размерности n, известные в квантовой механике как матрицы, которые вращают n-компонентные волновые функции частиц различных спинов. Размерность связана со спиновым квантовым числом s формулой $n = 2s + 1$. Наиболее важное нетривиальное представление - двумерное представление, соответствующее спину $1/2$. Матрицы этого представления - 2×2 унитарные матрицы с детерминантом, равным 1, которые могут быть представлены как

$$U = e^{-i\theta^i\sigma^i/2},$$

где θ^i - три произвольных параметра и σ^i - сигма-матрицы Паули.

Для любой непрерывной группы преобразования, бесконечно близкие к тождественному, определяют векторное пространство, называемое алгеброй Ли этой группы. Базисные векторы этого векторного пространства называются генераторами алгебры или группы Ли. Для группы вращений генераторы - это операторы углового момента J^i, которые удовлетворяют коммутационным соотношениям

$$\left[J^i, J^j\right] = i\varepsilon^{ijk}J^k.$$

Конечные преобразования вращения получаются экспоненцированием этих генераторов или операторов. В квантовой механике оператор

$$R = \exp\left[-i\theta^i J^i\right]$$

определяет вращение на угол $|\theta|$ вокруг оси $\hat{\theta}$. Коммутационные соотношения операторов J^i определяют законы умножения этих преобразований вращения. Таким образом, набор матриц, удовлетворяющих коммутационным соотношениям (3.12), дает при помощи экспоненцирования, как в (3.13), представление группы вращений. В примере из предыдущего раздела представление операторов углового момента

$$J^i \to \frac{\sigma^i}{2}$$

дает представление группы вращений (3.11). Имеет место общее утверждение: матричные представления непрерывной группы можно получить, зная матричные представления генераторов этой группы (которые должны удовлетворять соответствующим

коммутационным соотношениям) с последующим экспоненцированием этих бесконечно малых преобразований.

Для решения рассматриваемой задачи необходимо знать коммутационные соотношения генераторов преобразований группы Лоренца. Для группы вращений можно найти коммутационные соотношения, записывая генераторы, как дифференциальные операторы. Из выражения

$$\mathbf{J} = \mathbf{x} \times \mathbf{p} = \mathbf{x} \times (-i\nabla)$$

непосредственно следуют коммутационные соотношения (3.12) для углового момента. Использование векторного произведения в (3.15) характерно для трехмерного пространства. Однако можно также записать эти операторы в виде антисимметричных тензоров:

$$J^{ij} = -i\left(x^i\nabla^j - x^j\nabla^i\right)$$

так что $J^3 = J^{12}$ и так далее. Обобщение на четырехмерные преобразования Лоренца теперь очевидно:

$$J^{\mu\nu} = i\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right).$$

Вскоре мы увидим, что эти шесть операторов группы Лоренца генерируют три собственно лоренцевых преобразования и три преобразования пространственных вращений.

Чтобы определить коммутационные соотношения в алгебре Лоренца, можно просто вычислить коммутаторы дифференциальных операторов (3.16). В результате

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right)$$

Любые матрицы, которые представляют эту алгебру, должны удовлетворять таким же коммутационным соотношениям.

Чтобы увидеть, что мы получили, рассмотрим одно конкретное представление. Возьмем 4×4 матрицы

$$\left(\mathscr{J}^{\mu\nu}\right)_{\alpha\beta} = i\left(\delta^\mu_\alpha\delta_{\beta\beta} - \delta^\mu_\beta\delta^\nu_\alpha\right).$$

(Здесь μ и ν нумеруют каждую из шести матриц, в то время как α и β нумеруют матричные элементы.) Можно легко проверить, что эти матрицы удовлетворяют коммутационным соотношениям (3.17). Действительно, они являются ничем иным, как матрицами, действующими на обычные лоренцевы 4-векторы. Чтобы увидеть это, параметризуем бесконечно малые преобразования следующим образом:

$$V^\alpha \to \left(\delta^\alpha\beta - \frac{i}{2}\omega_{\mu\nu}\left(\mathscr{J}^{\mu\nu}\right)^\alpha{}_\beta\right)V^\beta,$$

где V – 4-вектор и $\omega_{\mu\nu}$ - антисимметричный тензор, задающий бесконечно малые углы. Например, рассмотрим случай, когда $\omega_{12} = -\omega_{21} = \theta$, а все остальные компоненты ω равны нулю. Тогда (3.19) дает

$$V \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V$$

что представляет бесконечно малое вращение в плоскости xy. Можно также проверить, что, полагая $\omega_{01} = -\omega_{10} = \beta$, получим

$$V \to \begin{pmatrix} 1 & \beta & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} V$$

и это является бесконечно малым бустом в направлении оси x. Другие компоненты ω генерируют оставшиеся бусты и вращения.

9.5.2 Dirac Equation

Теория

Теперь, когда мы нашли одно конечномерное представление группы Лоренца, следующим логическим шагом является разработка формализма для нахождения всех других представлений. Хотя это не очень трудно сделать (см. задачу 3.1), это едва ли необходимо для наших целей, так как нас главным образом интересуют представления, соответствующие спину 1/2. Мы можем найти такое представление, используя прием Дирака. Предположим, что мы имеем набор из четырех $n \times n$ матриц γ^μ, удовлетворяющих антикоммутационным соотношениям:

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \times \mathbf{1}_{n\times n}\% \quad \% \text{ (алгебра Дирака)}$$

Тогда можно немедленно записать n-мерное представление алгебры Лоренца. Оно имеет вид:

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

Повторно используя (3.22), легко проверить, что эти матрицы удовлетворяют коммутационным соотношениям (3.17).

Это вычисление можно выполнить в пространстве любой размерности с лоренцевой или евклидовой метрикой. В частности, оно должно быть верным в трехмерном евклидовом пространстве. Действительно, можно просто взять

$$\begin{aligned} &\gamma^j \equiv i\sigma^j\% \quad \% \text{ (сигма-матрицы Паули)}, \\ \text{так что} \quad &\{\gamma^i, \gamma^j\} = -2\delta^{ij}. \end{aligned}$$

Множитель i в первой строке и знак «минус» во второй строке полностью условны. Тогда матрицы, представляющие алгебру Лоренца, имеют вид:

$$S^{ij} = \frac{1}{2}\varepsilon^{ijk}\sigma^k,$$

и мы узнаем известное двумерное представление группы вращений.

Теперь найдем матрицы Дирака γ^μ для четырехмерного пространства Минковского. Оказывается, что они должны быть по крайней мере матрицами 4×4.(Не существует четвертой 2×2 матрицы, которая антикоммутирует с тремя сигма-матрицами Паули.) Далее, все 4×4 представления алгебры Дирака унитарно эквивалентны * Таким образом,

нужно выписать только одну явную реализацию алгебры Дирака. Одно представление в форме 2×2 блоков имеет вид:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

Это представление называется представлением Вейля или киральным представлением. Мы увидим, что это представление наиболее удобно, и будем использовать именно его на протяжении всей книги. (Однако будьте внимательными, так как многие учебники по теории поля выбирают другое представление, в котором γ^0 является диагональной. Кроме того, в книгах, использующих киральные представления, часто выбирают другие соглашения о знаках.)

В нашем представлении генераторы буста имеют вид:

$$S^{0i} = \frac{i}{4}\left[\gamma^0, \gamma^i\right] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix},$$

а генераторы пространственных вращений равны

$$S^{ij} = \frac{i}{4}\left[\gamma^i, \gamma^j\right] = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\varepsilon^{ijk}\Sigma^k.$$

*Это и предшествующее утверждения следуют из общей теории представлений группы Лоренца, изложенной в задаче 3.1. Четырехкомпонентное поле ψ, преобразующееся при бустах и вращениях согласно (3.26) и (3.27), называется дираковским спинором. Заметим, что генератор вращений S^{ij} представляет собой в точности трехмерную спинорную матрицу преобразования (3.24), взятую дважды. Генераторы бустов S^{0i} не являются эрмитовыми и, таким образом, наша реализация бустов неунитарна (это было верно и для векторного представления (3.18)). Действительно, так как группа Лоренца некомпактна, она не имеет однозначных конечномерных унитарных представлений. Но для нас это не имеет значения, поскольку ψ является не волновой функцией, а классическим полем.

Теперь, когда известен закон преобразования для ψ, следует найти соответствующее полевое уравнение. Одна из возможностей - просто взять уравнение КлейнаГордона:

$$\left(\partial^2 + m^2\right)\psi = 0.$$

Это уравнение годится, поскольку спинорные матрицы преобразования (3.26) и (3.27) действуют только во «внутреннем» пространстве; они коммутируют с дифференциальным оператором. Но можно написать более сильное уравнение первого порядка, содержащее дополнительную информацию и включающее (3.28). Для этого понадобится еще одно свойство γ-матриц. При помощи несложных вычислений можно проверить, что

$$[\gamma^\mu, S^{\rho\sigma}] = (\mathscr{J}^{\rho\sigma})^\mu_\nu \gamma^\nu,$$

или эквивалентно,

$$\left(1 + \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^\mu\left(1 - \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right) = \left(1 - \frac{i}{2}\omega_{\rho\sigma}\mathscr{J}^{\rho\sigma}\right)^\mu\gamma^\nu$$

Это уравнение есть инфинитизимальная форма соотношения

$$\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu_\nu\gamma^\nu$$

где

$$\Lambda_{\frac{1}{2}} = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)$$

является спинорным представлением преобразования Лоренца Λ (сравни с (3.19)). Уравнение (3.29) показывает, что γ-матрицы инвариантны относительно одновременного вращения по векторным и спинорным индексам (точно так же как σ^i инвариантны при пространственных вращениях). Другими словами, можно воспринимать векторный индекс μ в γ^μ всерьез и сворачивать γ^μ с ∂_μ, чтобы образовать лоренц-инвариантный дифференциальный оператор.

Теперь мы готовы записать уравнение Дирака. Оно имеет вид:

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi(x) = 0$$

Чтобы показать, что это уравнение лоренц-инвариантно, подвергнем лоренцевскому преобразованию его левую часть и проведем соответствующие вычисления:

$$\begin{aligned}
\left[i\gamma^\mu\partial_\mu - m\right]\psi(x) \to \left[i\gamma^\mu \left(\Lambda^{-1}\right)^\nu{}_\mu\partial_\nu - m\right]\Lambda_{\frac{1}{2}}\psi\left(\Lambda^{-1}x\right) = \\
= \Lambda_{\frac{1}{2}}\Lambda^{-1}_{\frac{1}{2}}\left[i\gamma^\mu \left(\Lambda^{-1}\right)^\nu{}_\mu\partial_\nu - m\right]\Lambda_{\frac{1}{2}}\psi\left(\Lambda^{-1}x\right) = \\
= \Lambda_{\frac{1}{2}}\left[i\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}}\left(\Lambda^{-1}\right)^\nu{}_\mu\partial_\nu - m\right]\psi\left(\Lambda^{-1}x\right) = \\
= \Lambda_{\frac{1}{2}}\left[i\Lambda^\mu_\sigma\gamma^\sigma\left(\Lambda^{-1}\right)^\nu_\mu\partial_\nu - m\right]\psi\left(\Lambda^{-1}x\right) = \\
= \Lambda_{\frac{1}{2}}\left[i\gamma^\nu\partial_\nu - m\right]\psi\left(\Lambda^{-1}x\right) = 0.
\end{aligned}$$

Чтобы увидеть, что из уравнения Дирака вытекает уравнение Клейна-Гордона, подействуем слева оператором $(-i\gamma^\mu\partial_\mu - m)$:

$$\begin{aligned}
0 = \left(-i\gamma^\mu\partial_\mu - m\right)\left(i\gamma^\nu\partial_\nu - m\right)\psi = \left(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2\right)\psi = \\
= \left(\frac{1}{2}\left\{\gamma^\mu,\gamma^\nu\right\}\partial_\mu\partial_\nu + m^2\right)\psi = \left(\partial^2 + m^2\right)\psi
\end{aligned}$$

Чтобы записать лагранжиан для теории Дирака, нужно установить, как умножать два дираковских спинора, чтобы получить лоренцевский скаляр. Простейшее предположение $\psi^\dagger\psi$ не проходит. При лоренцевском преобразовании получаем $\psi^\dagger\Lambda^\dagger_{\frac{1}{2}}\Lambda_{\frac{1}{2}}\psi$; если бы матрица буста была унитарна, мы бы имели $\Lambda^\dagger_{\frac{1}{2}} = \Lambda^{-1}_{\frac{1}{2}}$, и все было бы прекрасно. Но матрица $\Lambda_{\frac{1}{2}}$ неунитарна, потому что генераторы (3.26) неэрмитовы.

Решение заключается в том, чтобы определить

$$\bar{\psi} \equiv \psi^\dagger\gamma^0$$

При бесконечно малом преобразовании Лоренца, параметризованном величинами $\omega_{\mu\nu}$, $\bar{\psi} \to \psi^\dagger\left(1 + \frac{i}{2}\omega_{\mu\nu}\left(S^{\mu\nu}\right)^\dagger\right)\gamma^0$. Сумма по μ и ν имеет шесть различных отличных от нуля слагаемых. Во вращательных слагаемых, где μ и ν оба отличны от нуля, $\left(S^{\mu\nu}\right)^\dagger = S^{\mu\nu}$ и $S^{\mu\nu}$ коммутирует с γ^0. В слагаемых, отвечающих бусту, где индекс μ или ν равен 0, $\left(S^{\mu\nu}\right)^\dagger = -\left(S^{\mu\nu}\right)$, но при этом $S^{\mu\nu}$ антикоммутирует с γ^0. Поэтому перенос γ^0 налево снимает знак эрмитова сопряжения у $S^{\mu\nu}$, и в результате приходим к закону преобразования:

$$\bar{\psi} \to \bar{\psi}\Lambda^{-1}_{\frac{1}{2}}$$

Таким образом, величина $\bar{\psi}\psi$ есть лоренцевский скаляр. Аналогично можно показать при помощи (3.29), что $\bar{\psi}\gamma^\mu\psi$ является лоренцевским вектором.

Поэтому лоренц-инвариантный дираковский лагранжиан имеет вид:

$$\mathscr{L}_{\text{Dirac}} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi.$$

Уравнение Эйлера-Лагранжа для $\bar{\psi}$ (или $\psi^\dagger$) немедленно приводит к уравнению Дирака в форме (3.31); уравнение Эйлера-Лагранжа для ψ дает то же самое уравнение для эрмитово-сопряженной формы:

$$-i\partial_\mu\bar{\psi}\gamma^\mu - m\bar{\psi} = 0$$

9.5.3 Weyl Spinors

Теория

Из блочно-диагонального вида генераторов (3.26) и (3.27) видно, что представление Дирака группы Лоренца приводимо*. Мы можем образовать два 2-мерных представления, рассматривая каждый блок отдельно и записав

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

Кстати, при использовании другого представления гамма-матриц приводимость не столь явная; это, по существу, и является причиной использования кирального представления.

Двухкомпонентные объекты ψ_L и ψ_R называются левым и правым вейлевскими спинорами, соответственно. Можно легко проверить, что их законы преобразования при бесконечно малых вращениях $\boldsymbol{\theta}$ и бустах $\boldsymbol{\beta}$ имеют вид:

$$\psi_L \to \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_L$$
$$\psi_R \to \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_R$$

Эти законы преобразования связаны комплексным сопряжением. Используя тождество

$$\sigma^2\sigma^* = -\sigma\sigma^2,$$

нетрудно показать, что величина $\sigma^2\psi_L^*$ преобразуется подобно правому спинору.

С помощью ψ_L и ψ_R уравнение Дирака записывается в виде:

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi = \begin{pmatrix} -m & i\left(\partial_0 + \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right) \\ i\left(\partial_0 - \sigma\cdot\nabla\right. & \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

В уравнении Дирака два представления группы Лоренца ψ_L и ψ_R перемешиваются за счет массового слагаемого. Но если положить $m = 0$, то уравнения для ψ_L и ψ_R расцепляются:

$$i\left(\partial_0 - \sigma\cdot\nabla\right)\psi_L = 0$$
$$i\left(\partial_0 + \sigma\cdot\nabla\right)\psi_R = 0$$

Они называются уравнениями Вейля. Эти уравнения особенно важны при рассмотрении нейтрино и теории слабых взаимодействий.

Возможно некоторое усовершенствование введенных обозначений. Определим

$$\sigma^\mu \equiv (1, \sigma), \quad \bar{\sigma}^\mu \equiv (1, -\sigma),$$

так что

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

(Черта в $\bar{\sigma}$ не имеет никакого отношения к черте в $\bar{\psi}$.) Тогда уравнение Дирака может быть записано в виде

$$\begin{pmatrix} -m & i\sigma \cdot \partial \\ i\bar{\sigma} \cdot \partial & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

а уравнения Вейля примут вид:

$$i\bar{\sigma} \cdot \partial \psi_L = 0; \quad i\sigma \cdot \partial \psi_R = 0.$$

9.5.4 Solutions of the Dirac Equation for Free Particles

Теория

Чтобы ощутить физический смысл уравнения Дирака, обсудим его решения в виде плоских волн. Так как поле Дирака ψ удовлетворяет уравнению Клейна-Гордона, мы немедленно делаем вывод, что оно может быть записано как линейная комбинация плоских волн:

$$\psi(x) = u(p)e^{-ip\cdot x}, \quad \text{где } p^2 = m^2$$

На время сосредоточим внимание на решениях с положительной частотой, т. е. при $p^0 > 0$. Вектор-столбец $u(p)$ должен удовлетворять дополнительному ограничению, которое находится путем подстановки (3.45) в уравнение Дирака:

$$(\gamma^\mu p_\mu - m)\, u(p) = 0$$

Проще всего проанализировать это уравнение в системе покоя частицы, где $p = p_0 = (m, \mathbf{0})$. Решение для произвольного p может быть затем найдено при помощи буста с $\Lambda_{\frac{1}{2}}$. В системе покоя (3.46) принимает вид:

$$\left(m\gamma^0 - m\right) u\left(p_0\right) = m \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u\left(p_0\right) = 0$$

Его решения имеют вид

$$u\left(p_0\right) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

для любого числового двухкомпонентного спинора ξ. Мы, как обычно, нормируем ξ так, что $\xi^\dagger \xi = 1$; множитель $\sqrt{m}$ вставлен из соображений удобства. Спинор ξ можно

интерпретировать, рассматривая генератор вращения (3.27). Величина ξ преобразуется при вращениях как обычный двухкомпонентный спинор группы вращения и, следовательно, обычным образом определяет ориентацию спина частицы, описываемой этим дираковским решением. Например, когда $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, спин частицы ориентирован вверх вдоль третьей оси.

Заметим, что после использования уравнения Дирака можно произвольно выбрать только две из четырех компонент поля $u(p)$. Это как раз то, что нужно, так как частица со спином $1/2$ имеет только два физических состояния со спином вверх и вниз. (Конечно, здесь несколько преждевременно говорить о частицах и спине. Мы докажем в разделе 3.5 после квантования поля Дирака, что спиновый угловой момент дираковской частицы равен $\hbar/2$. Пока только обратим внимание, что для любого импульса p имеются два возможных решения $u(p)$.

Теперь, когда известен общий вид решения $u(p)$ в системе покоя, можно получить $u(p)$ в любой другой системе путем применения буста. Рассмотрим буст вдоль третьей оси. Сначала мы должны вспомнить, как буст действует на 4-вектор. В инфинитезимальной форме

$$\begin{pmatrix} E \\ p^3 \end{pmatrix} = \left[1 + \eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \begin{pmatrix} m \\ 0 \end{pmatrix}$$

где η - бесконечно малый параметр. Для конечного η следует написать:

$$\begin{aligned} \begin{pmatrix} E \\ p^3 \end{pmatrix} &= \exp\left[\eta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] \begin{pmatrix} m \\ 0 \end{pmatrix} = \\ &= \left[\operatorname{ch}\eta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \operatorname{sh}\eta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} m \\ 0 \end{pmatrix} = \begin{pmatrix} m \operatorname{ch}\eta \\ m \operatorname{sh}\eta \end{pmatrix} \end{aligned}$$

Параметр η называется быстротой. Эта величина является аддитивной при последовательных бустах. Теперь применим то же самое преобразование к $u(p)$. Соглас- но (3.26) и (3.30),

$$\begin{aligned} u(p) &= \exp\left[-\frac{1}{2}\eta \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right] \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} = \\ &= \left[\operatorname{ch}\left(\frac{1}{2}\eta\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \operatorname{sh}\left(\frac{1}{2}\eta\right) \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right] \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} = \\ &= \begin{pmatrix} e^{\eta/2}\left(\frac{1-\sigma^3}{2}\right) + e^{-\eta/2}\left(\frac{1+\sigma^3}{2}\right) & 0 \\ 0 & e^{\eta/2}\left(\frac{1+\sigma^3}{2}\right) + e^{-\eta/2}\left(\frac{1-\sigma^3}{2}\right) \end{pmatrix} \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} = \\ &= \begin{pmatrix} \left[\sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right)\right]\xi \\ \left[\sqrt{E+p^3}\left(\frac{1+\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1-\sigma^3}{2}\right)\right]\xi \end{pmatrix}. \end{aligned}$$

Последняя строка может быть упрощена, так что

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix},$$

где предполагается, что при взятии квадратного корня из матрицы мы извлекаем положительный корень из каждого собственного значения. Это выражение для $u(p)$ не

только более компактно, но и сохраняет силу для произвольного направления p. При работе с выражениями такого вида полезно знать тождество

$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = p^2 = m^2.$$

Теперь можно непосредственно проверить, что (3.50) есть решение уравнения Дирака в форме (3.43).

На практике часто удобно работать со специальными спинорами ξ. Полезным был бы выбор собственных состояний σ^3. Например, если $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (спин вверх вдоль третьей оси), имеем:

$$u(p) = \begin{pmatrix} \sqrt{E-p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E+p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \xrightarrow[\text{большой буст}]{} \sqrt{2E}\begin{pmatrix} 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix},$$

в то время как для $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (спин вниз вдоль третьей оси)

$$\boldsymbol{u(p) = \begin{pmatrix} \sqrt{E+p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E-p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \xrightarrow[\text{большой буст}]{} \sqrt{2E}\begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \end{pmatrix}.}$$

В пределе $\eta \to \infty$ состояния вырождаются в два двухкомпонентных спинора для безмассовой частицы. (Теперь видна причина введения множителя $\sqrt{m}$ в (3.47) - он сохраняет конечность выражений для спиноров в безмассовом пределе.)

Решения (3.52) и (3.53) являются собственными состояними оператора спиральности

$$h \equiv \hat{p} \cdot \mathbf{S} = \frac{1}{2}\hat{p}_i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

Частица с $h = +1/2$ называется правополяризованной, в то время как частица с $h = -1/2$ - левополяризованной. Спиральность массивной частицы зависит от системы отсчета, так как всегда можно перейти к системе отсчета, в которой импульс частицы противоположно направлен (но ее спин неизменен). Для безмассовой частицы, которая движется со скоростью света, такое преобразование выполнить нельзя.

Чрезвычайно простая форма $u(p)$ для безмассовой частицы в состоянии с определенной спиральностью делает понятным поведение такой частицы. В гл. 1 это дало возможность установить форму сечения рассеяния для процесса $e^+e^- \to \mu^+\mu^-$в безмассовом пределе. В последующих главах мы будем часто сначала проводить на первый взгляд бессмысленные вычисления, а затем рассматривать собственные состояния спиральности в пределе высоких энергий, чтобы понять, что было сделано.

Наконец, мы готовы понять происхождение обозначений ψ_L и ψ_R для вейлевских спиноров. Решения уравнений Вейля - это состояния с определенной спиральностью, отвечающие лево- и правополяризованным частицам, соответственно. Лоренц-инвариантность спиральности (для безмассовой частицы) явно видна в записи вейлевских спиноров, так как ψ_L и ψ_R относятся к разным представлениям группы Лоренца.

Условие нормировки для $u(p)$ удобно записывать в лоренц-инвариантном виде. Мы видели выше, что выражение $\psi^\dagger\psi$ не лоренц-инвариантно. Аналогично,

$$u^\dagger u = \left(\xi^\dagger\sqrt{p\cdot\sigma}, \xi^\dagger\sqrt{p\cdot\bar{\sigma}}\right)\cdot\begin{pmatrix}\sqrt{p\cdot\sigma}\xi\\ \sqrt{p\cdot\bar{\sigma}}\xi\end{pmatrix} = 2E_{\mathbf{p}}\xi^\dagger\xi$$

Чтобы получить лоренцевский скаляр, определим

$$\bar{u}(p) = u^\dagger(p)\gamma^0.$$

Затем при помощи почти такого же вычисления находим:

$$\bar{u}u = 2m\xi^\dagger\xi.$$

Это будет наше условие нормировки, причем мы также требуем, чтобы двухкомпонентный спинор ξ был нормирован как обычно: $\xi^\dagger\xi = 1$. Удобно также выбрать взаимно ортогональные базисные спиноры ξ^1 и ξ^2 (типа $\begin{pmatrix}1\\0\end{pmatrix}$ и $\begin{pmatrix}0\\1\end{pmatrix}$). Для безмассовой частицы выражение (3.57) тривиально, так что следует пользоваться условием нормировки в форме (3.55).

Подведем промежуточный итог. Общее решение уравнения Дирака может быть записано как линейная комбинация плоских волн. Волны с положительной частотой имеют вид:

$$\psi(x) = u(p)e^{-ip\cdot x}, \quad p^2 = m^2, \quad p^0 > 0.$$

Существуют два линейно независимых решения для $u(p)$:

$$u^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\ \sqrt{p\cdot\bar{\sigma}}\xi^s\end{pmatrix}, \quad s = 1, 2$$

которые нормированы условиями:

$$\bar{u}^r(p)u^s(p) = 2m\delta^{rs}\% \quad \% \text{ или } \% \quad \%u^{r\dagger}(p)u^s(p) = 2E_{\mathbf{p}}\delta^{rs}.$$

Точно так же можно найти отрицательно-частотные решения:

$$\psi(x) = v(p)e^{+ip\cdot x}, \quad p^2 = m^2, \quad p^0 > 0.$$

(Обратим внимание, что мы предпочли изменить знак в экспоненте, но не писать $p^0 < 0$). Имеются два линейно независимых решения для $v(p)$:

$$v^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\eta^s\\ -\sqrt{p\cdot\bar{\sigma}}\eta^s\end{pmatrix}, \quad s = 1, 2$$

где η^s - другой базис двухкомпонентных спиноров. Эти решения нормированы согласно

$$\bar{v}^r(p)v^s(p) = -2m\delta^{rs}\% \quad \% \text{ или } \% \quad \%v^{r\dagger}(p)v^s(p) = +2E_{\mathbf{p}}\delta^{rs}.$$

Столбцы u и v также взаимно ортогональны:

$$\bar{u}^r(p)v^s(p) = \bar{v}^r(p)u^s(p) = 0.$$

Следует быть внимательными, так как $u^{r\dagger}(p)v^s(p) \neq 0$ и $v^{r\dagger}(p)u^s(p) \neq 0$. Однако заметим, что

$$u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p}) = v^{r\dagger}(-\mathbf{p})u^s(\mathbf{p}) = 0,$$

где мы изменили знак 3-импульса в одном из сомножителей в каждом из спинорных произведений.

9.5.5 Spin Sums

Теория

При вычислении вкладов диаграмм Фейнмана часто будет необходимо производить суммирование по поляризациям фермионных состояний. С помощи простых вычислений можно вывести необходимые соотношения полноты:

$$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} \left(\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}}, \xi^{s\dagger}\sqrt{p\cdot\sigma}\right) =$$
$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma} \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}.$$

Во второй строке мы использовали формулу

$$\sum_{s=1,2} \xi^s\xi^{s\dagger} = \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Таким образом, приходим к желаемой формуле:

$$\sum_s u^s(p)\bar{u}^s(p) = \gamma\cdot p + m$$

Аналогично,

$$\sum_s v^s(p)\bar{v}^s(p) = \gamma\cdot p - m$$

Комбинация $\gamma\cdot p$ встречается настолько часто, что Фейнман ввел обозначение: $\not{p} \equiv \gamma^\mu p_\mu$. С этого момента мы будем часто использовать эту запись.

9.5.6 Dirac Matrices and Bilinear Forms

Теория

Мы видели в разделе 3.2, что величина $\bar{\psi}\psi$ есть лоренцевский скаляр. Легко показать, что $\bar{\psi}\gamma^\mu\psi$ является 4-вектором; мы использовали этот факт для записи дираковского лагранжиана (3.34). Зададим более общий вопрос. Рассмотрим выражение $\bar{\psi}\Gamma\psi$, где Γ - произвольная 4×4 постоянная матрица. Можем ли мы разложить это выражение на слагаемые, имеющие определенные трансформационные свойства по отношению к преобразованиям группы Лоренца? Ответ положителен, если представить матрицу Γ разложеной по базису, состоящему из шестнадцати 4×4 матриц, определенных как антисимметричные комбинации γ-матриц:

1	1 матрица
γ^μ	4 матрицы
$\gamma^{\mu\nu} = \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right] \equiv \gamma^{[\mu}\gamma^{\nu]} \equiv -i\sigma^{\mu\nu}$	6 матриц
$\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$	4 матрицы
$\gamma^{\mu\nu\rho\sigma} = \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]}$	1 матрица
	Всего 16 матриц

Легко установить свойства этих матриц относительно преобразований Лоренца. Например,

$$\bar{\psi}\gamma^{\mu\nu}\psi \to \left(\bar{\psi}\Lambda^{-1}_{\frac{1}{2}}\right)\left(\frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right]\right)\left(\Lambda_{\frac{1}{2}}\psi\right) =$$
$$= \frac{1}{2}\bar{\psi}\left(\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}}\Lambda^{-1}_{\frac{1}{2}}\gamma^\nu\Lambda_{\frac{1}{2}} - \Lambda^{-1}_{\frac{1}{2}}\gamma^\nu\Lambda_{\frac{1}{2}}\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}}\right)\psi = \Lambda^\mu_\alpha\Lambda^\nu_\beta\bar{\psi}\gamma^{\alpha\beta}\psi$$

Каждый набор матриц преобразуется каю антисимметричный тензор последовательно возрастающего ранга. Последние два набора матриц можно упростить, вводя дополнительную гамма-матрицу

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$$

Тогда $\gamma^{\mu\nu\rho\sigma} = -i\varepsilon^{\mu\nu\rho\sigma}\gamma^5$ и $\gamma^{\mu\nu\rho} = -i\varepsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5$. Матрица γ^5 имеет следующие свойства, которые могут быть проверены при помощи (3.68) и антикоммутационных соотношений (3.22):

$$\left(\gamma^5\right)^\dagger = \gamma^5$$
$$\left(\gamma^5\right)^2 = 1$$
$$\left\{\gamma^5, \gamma^\mu\right\} = 0$$

Из последнего свойства следует, что $[\gamma^5, S^{\mu\nu}] = 0$. Таким образом, представление Дирака должно быть приводимо, так как собственные векторы матрицы γ^5, чьи собственные значения различны, преобразуются не смешиваясь (этот критерий приводимости известен как лемма Шура). В нашем базисе матрица

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

т. е. является блочно-диагональной. Таким образом, дираковский спинор только с левой (правой) компонентой является собственным состоянием γ^5 с собственным значением $-1(+1)$. Эти спиноры, действительно, преобразуются без смешивания, как мы явно показали в разделе 3.2.

Теперь перепишем таблицу из 4×4 матриц, введя стандартную терминологию:

1	скаляр	1
γ^μ	вектор	4
$\sigma^{\mu\nu}\left[\gamma^\mu, \gamma^\nu\right]$	тензор	6
$\gamma^\mu\gamma^5$	псевдовектор	4
γ^5	псевдоскаляр	1

Термины псевдовектор и псевдоскаляр применяются к величинам, которые при непрерывных преобразованиях Лоренца преобразуются как вектор и скаляр, соответственно, но с дополнительным изменением знака при преобразованиях четности (как мы покажем в разделе 3.6).

Из векторных и псевдовекторных матриц можно построить два тока, билинейных относительно дираковских полей:

$$j^{\mu}(x) = \bar{\psi}(x)\gamma^{\mu}\psi(x); \quad j^{\mu 5}(x) = \bar{\psi}(x)\gamma^{\mu}\gamma^{5}\psi(x)$$

Вычислим дивергенции этих токов, предполагая, что ψ удовлетворяет уравнению Дирака:

$$\partial_{\mu}j^{\mu} = \left(\partial_{\mu}\bar{\psi}\right)\gamma^{\mu}\psi + \bar{\psi}\gamma^{\mu}\partial_{\mu}\psi = (im\bar{\psi})\psi + \bar{\psi}(-im\psi) = 0.$$

Таким образом, j^{μ} всегда сохраняется, если $\psi(x)$ удовлетворяет уравнению Дирака. Когда мы связываем поле Дирака с электромагнитным полем, ток j^{μ} приобретает смысл плотности электрического тока. Аналогичным образом можно вычислить

$$\partial_{\mu}j^{\mu 5} = 2im\bar{\psi}\gamma^{5}\psi$$

Если $m = 0$, этот ток (часто называемый аксиальным векторным током) также сохраняется. В этом случае полезно образовать следующие линейные комбинации:

$$j^{\mu}_{L} = \bar{\psi}\gamma^{\mu}\left(\frac{1-\gamma^{5}}{2}\right)\psi, \quad j^{\mu}_{R} = \bar{\psi}\gamma^{\mu}\left(\frac{1+\gamma^{5}}{2}\right)\psi$$

Когда $m = 0$, эти токи являются плотностями электрических токов левых и правых частиц, соответственно, и сохраняются по отдельности.

Два тока $j^{\mu}(x)$ и $j^{\mu 5}(x)$ являются нетеровскими токами, соответствующими двум преобразованиям

$$\psi(x) \to e^{i\alpha}\psi(x) \quad \text{и \%} \quad \%\psi(x) \to e^{i\alpha\gamma^{5}}\psi(x).$$

Первое из них представляет собой симметрию дираковского лагранжиана (3.34). Второе, называемое киральным преобразованием, является симметрией кинетического слагаемого в $\mathscr{L}$, но не массового слагаемого. Таким образом, теорема Нетер подтверждает, что аксиальный векторный ток сохраняется лишь в случае $m = 0$. Произведения дираковских билинейных форм подчиняются соотношениям, известным как тождества Фириа. Мы обсудим лишь простейшее из них, которое будет несколько раз использоваться в дальнейшем. Это простейшее тождество легче всего записывается через двухкомпонентные вейлевские спиноры, введенные в (3.36).

Суть этого соотношения - тождество для 2×2 матриц σ^{μ}, определенных в (3.41):

$$(\sigma^{\mu})_{\alpha\beta}(\sigma_{\mu})_{\gamma\delta} = 2\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}.$$

(Здесь α, β и т. д. - спинорные индексы и ε - антисимметричный символ.) Чтобы понять это соотношение, заметим, что индексы α, γ преобразуются по лоренцевскому представлению ψ_L, в то время как β, δ преобразуются по отдельному представлению ψ_R, при этом полная величина должна быть лоренц-инвариантной. Альтернативно, можно непосредственно проверить путем перебора все 16 компонент (3.77).

Заключая тождество (3.77) в обкладки из правых частей (то есть нижних половин) дираковских спиноров u_1, u_2, u_3, u_4, находим тождество

$$\begin{aligned}(\bar{u}_{1R}\sigma^{\mu}u_{2R})\,(\bar{u}_{3R}\sigma_{\mu}u_{4R}) &= 2\varepsilon_{\alpha\gamma}\bar{u}_{1R\alpha}\bar{u}_{3R\gamma}\varepsilon_{\beta\delta}u_{2R\beta}u_{4R\delta} = \\ &= -\,(\bar{u}_{1R}\sigma^{\mu}u_{4R})\,(\bar{u}_{3R}\sigma_{\mu}u_{2R})\end{aligned}$$

Это нетривиальное соотношение показывает, что произведение билинейных комбинаций в (3.78) антисимметрично относительно перестановок компонент с индексами 2 и 4, и также компонент с индексами 1 и 3. Тождество (3.77) имеет место также для $\bar{\sigma}^{\mu}$, таким образом, находим:

$$(\bar{u}_{1L}\bar{\sigma}^{\mu}u_{2L})\,(\bar{u}_{3L}\bar{\sigma}_{\mu}u_{4L}) = -\,(\bar{u}_{1L}\bar{\sigma}^{\mu}u_{4L})\,(\bar{u}_{3L}\bar{\sigma}_{\mu}u_{2L})$$

Иногда полезно объединить тождество Фирца (3.78) с тождеством, связывающим σ^{μ} и $\bar{\sigma}^{\mu}$:

$$\varepsilon_{\alpha\beta}\,(\sigma^{\mu})_{\beta\gamma} = \left(\bar{\sigma}^{\mu T}\right)_{\alpha\beta}\varepsilon_{\beta\gamma}.$$

Последнее соотношение также проверяется непосредственно. Используя (3.80), (3.79) и соотношение

$$\bar{\sigma}^{\mu}\sigma_{\mu} = 4,$$

мы можем упростить, например, громоздкие произведения билинейных комбинаций типа

$$\begin{aligned}\left(\bar{u}_{1L}\bar{\sigma}^{\mu}\sigma^{\nu}\bar{\sigma}^{\lambda}u_{2L}\right)(\bar{u}_{3L}\bar{\sigma}_{\mu}\sigma_{\nu}\bar{\sigma}_{\lambda}u_{4L}) &= 2\varepsilon_{\alpha\gamma}\bar{u}_{1L\alpha}\bar{u}_{3L\gamma}\varepsilon_{\beta\delta}\left(\sigma^{\nu}\bar{\sigma}^{\lambda}u_{2L}\right)_{\beta}(\sigma_{\nu}\bar{\sigma}_{\lambda}u_{4L})_{\delta} = \\ &= 2\varepsilon_{\alpha\gamma}\bar{u}_{1L\alpha}\bar{u}_{3L\gamma}\varepsilon_{\beta\delta}u_{2L\beta}\left(\sigma^{\lambda}\bar{\sigma}^{\nu}\sigma_{\nu}\bar{\sigma}_{\lambda}u_{4L}\right)_{\delta} = \\ &= 2\cdot(4)^2\cdot\varepsilon_{\alpha\gamma}\bar{u}_{1L\alpha}\bar{u}_{3L\gamma}\varepsilon_{\beta\delta}u_{2L\beta}u_{4L\delta} = \\ &= 16\,(\bar{u}_{1L}\bar{\sigma}^{\mu}u_{2L})\,(\bar{u}_{3L}\bar{\sigma}_{\mu}u_{4L})\,.\end{aligned}$$

Имеются также тождества Фирца для перестановок 4-компонентных дираковских спиноров и 4×4 матриц Дирака. Однако для их вывода полезно применить более систематический подход. В задаче 3.6 рассмотрен общий метод и некоторые примеры его применения.

9.5.7 Quantization of the Dirac Field

Теория

Мы теперь готовы построить квантовую теорию свободного дираковского поля. Из лагранжиана

$$\mathscr{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi = \bar{\psi}\,(i\gamma^{\mu}\partial_{\mu} - m)\,\psi$$

видим, что импульс, канонически сопряженный к ψ есть $i\psi^{\dagger}$, и, таким образом, гамильтониан имеет вид:

$$H = \int d^3x\bar{\psi}(-i\gamma\cdot\nabla + m)\psi = \int d^3x\psi^{\dagger}\left[-i\gamma^0\gamma\cdot\nabla + m\gamma^0\right]\psi$$

Если определить $\alpha = \gamma^0\gamma, \beta = \gamma^0$, то величину в скобках можно рассматривать как гамильтониан Дирака в одночастичной квантовой механике:

$$h_D = -i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + m\beta$$

Как нельзя квантовать дираковское поле: урок по связи спина и статистики

Чтобы проквантовать дираковское поле по аналогии с полем Клейна-Гордона, следует потребовать выполнения канонических коммутационных соотношений

$$\left[\psi_a(\mathbf{x}), \psi_b^\dagger(\mathbf{y})\right] = \delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{ab}, \qquad \text{(времена одинаковые)}$$

где a и b обозначают компоненты спинора ψ. Это выглядит довольно странно: если $\psi(x)$ было бы вещественным, то левая сторона была бы антисимметричной при замене $\mathbf{x} \longleftrightarrow \mathbf{y}$, в то время как правая сторона - симметричной. Но так как спинор ψ комплексный, пока еще противоречия нет. На самом деле мы скоро увидим, что при наложении коммутационных соотношений на дираковское поле возникают более серьезные проблемы. Но, чтобы лучше понять связь между спином и статистикой, полезно посмотреть, как далеко мы можем продвинуться в этом направлении. Так что продолжим; но не забудьте, что рассуждения на нескольких следующих страницах приведут в тупик.

Первая задача - найти представление коммутационных соотношений через операторы рождения и уничтожения, которое диагонализует H. Из формы гамильтониана (3.84) ясно, что полезно разложить поле $\psi(x)$ по базису собственных функций h_D. Мы уже знаем эти собственные функции из вычислений в разделе 3.3. Там было обнаружено, что

$$\left[i\gamma^0\partial_0 + i\gamma\cdot\nabla - m\right] u^s(p)e^{-ip\cdot x} = 0$$

так что $u^s(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}$ являются собственными функциями h_D с собственными значениями $E_\mathbf{p}$. Аналогично, функции $v^s(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}$ (или эквивалентно, $v^s(-\mathbf{p})e^{+i\mathbf{p}\cdot\mathbf{x}}$) - собственные функции h_D с собственными значениями $-E_\mathbf{p}$. Они образуют полный набор собственных функций, так как для любого $\mathbf{p}$ имеются два спинора u и два спинора v, дающих четыре собственных вектора 4×4 матрицы h_D.

Раскладывая ψ по этому базису, получаем:

$$\psi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}e^{i\mathbf{p}\cdot\mathbf{x}}\sum_{s=1,2}\left(a^s_\mathbf{p}u^s(\mathbf{p}) + b^s_{-\mathbf{p}}v^s(-\mathbf{p})\right)$$

где $a^s_\mathbf{p}$ и $b^s_\mathbf{p}$ - операторные коэффициенты. (Пока что мы используем представление Шредингера, где ψ не зависит от времени.) Постулируем следующие коммутационные соотношения:

$$\left[a^r_\mathbf{p}, a^{s\dagger}_\mathbf{q}\right] = \left[b^r_\mathbf{p}, b^{s\dagger}_\mathbf{q}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}.$$

Тогда нетрудно проверить, что коммутационные соотношения (3.86) для ψ и $\psi^\dagger$ имеют вид:

$$\begin{aligned}
\left[\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})\right] &= \int \frac{d^3pd^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_\mathbf{p}2E_\mathbf{q}}}e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\times \\
&\quad\times\sum_{r,s}\left(\left[a^r_\mathbf{p}, a^{s\dagger}_\mathbf{q}\right]u^r(\mathbf{p})\bar{u}^s(\mathbf{q}) + \left[b^r_{-\mathbf{p}}, b^{s\dagger}_{-\mathbf{q}}\right]v^r(-\mathbf{p})\bar{v}^s(-\mathbf{q})\right)\gamma^0 = \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\left[\left(\gamma^0E_\mathbf{p} - \gamma\cdot\mathbf{p} + m\right) + \left(\gamma^0E_\mathbf{p} + \gamma\cdot\mathbf{p} - m\right)\right]\gamma^0 = \\
&= \delta^{(3)}(\mathbf{x}-\mathbf{y})\times\mathbf{1}_{4\times4}.
\end{aligned}$$

На втором шаге мы использовали условия полноты (3.66) и (3.67) для сумм по спиновым состояниям. Теперь мы готовы написать H через операторы a и b. После еще одного небольшого вычисления (использующего соотношение ортогональности (3.60), (3.63) и (3.65)) находим:

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(E_{\mathbf{p}} a^{s\dagger}_{\mathbf{p}} a^s_{\mathbf{p}} - E_{\mathbf{p}} b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} \right)$$

Во втором слагаемом таится какая-то ужасная ошибка: рождая все новые и новые частицы при помощи $b^\dagger$, мы можем уменьшать энергию до бесконечности. (Замена $b \longleftrightarrow b^\dagger$ не помогла бы, так как это разрушило бы коммутационные соотношения (3.89).)

Похоже, мы столкнулись с серьезной трудностью, но попытаемся еще продвинуться вперед и исследуем причинность в этой теории. Для этого нам нужно вычислить $\left[\psi(x), \psi^\dagger(y)\right)$ (или удобнее, $\left[\psi(x), \bar\psi(y)\right])$ в разные моменты времени, в надежде получить нуль вне светового конуса. Сначала следует перейти к представлению Гейзенберга и восстановить зависимость от времени ψ и $\bar\psi$. Используя соотношения

$$e^{iHt} a^s_{\mathbf{p}} e^{-iHt} = a^s_{\mathbf{p}} e^{-iE_{\mathbf{p}}t}, \quad e^{iHt} b^s_{\mathbf{p}} e^{-iHt} = b^s_{\mathbf{p}} e^{+iE_{\mathbf{p}}t}$$

немедленно получаем

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^s_{\mathbf{p}} v^s(p) e^{ip\cdot x} \right)$$

$$\bar\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^{s\dagger}_{\mathbf{p}} \bar u^s(p) e^{ip\cdot x} + b^{s\dagger}_{\mathbf{p}} \bar v^s(p) e^{-ip\cdot x} \right)$$

Теперь можно вычислить общий коммутатор:

$$\begin{aligned}
\left[\psi_a(x), \bar\psi_b(y)\right] &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_s \left(u^s_a(p) \bar u^s_b(p) e^{-ip\cdot(x-y)} + v^s_a(p) \bar v^s_b(p) e^{ip\cdot(x-y)} \right) = \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left((\not p + m)_{ab} e^{-ip\cdot(x-y)} + (\not p - m)_{ab} e^{ip\cdot(x-y)} \right) = \\
&= (i\partial_x + m)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right) = \\
&= (i\partial_x + m)_{ab} \left[\phi(x), \phi(y)\right].
\end{aligned}$$

Так как коммутатор вещественного поля Клейна-Гордона $[\phi(x), \phi(y)]$ равен нулю вне светового конуса, эта величина также равна нулю.

Однако в этом решении проблемы причинности имеется что-то странное. Пусть $|0\rangle$ - состояние, которое уничтожается всеми операторами $a^s_{\mathbf{p}}$ и $b^s_{\mathbf{p}}$: $a^s_{\mathbf{p}}|0\rangle = b^s_{\mathbf{p}}|0\rangle = 0$. Тогда

$$\left[\psi_a(x), \bar\psi_b(y)\right] = \left\langle 0 \left| \left[\psi_a(x), \bar\psi_b(y)\right] \right| 0 \right\rangle = \left\langle 0 \left| \psi_a(x) \bar\psi_b(y) \right| 0 \right\rangle - \left\langle 0 \left| \bar\psi_b(y) \psi_a(x) \right| 0 \right\rangle$$

так же, как для поля Клейна-Гордона. Но в случае поля Клейна-Гордона в коммутатор давали вклады каждая из этих двух частей: вне светового конуса распространение частицы от y до x сокращалось с распространением античастицы от x до y. Здесь же оба вклада исходят от первой части $\langle 0|\psi(x)\bar\psi(y)|0\rangle$, так как вторая часть нулевая. Сокращение происходит между частицами с положительной энергией и частицами с отрицательной энергией, причем обе распространяются от y к x.

Это наблюдение может привести нас к разрешению проблемы отрицательной энергии. Одно из предположений, сделанных при квантовании теории Дирака, должно быть неправильным. Таким образом, забудем постулированные коммутационные соотношения (3.86) и (3.88) и посмотрим, можем ли мы найти способ обоснования распространения частиц с положительной энергией в обоих направлениях. Мы будем также вынуждены отбросить наше определение вакуума $|0\rangle$ как состояния, которое уничтожается всеми $a^s_{\mathbf{p}}$ и $b^s_{\mathbf{p}}$. Однако мы сохраним выражения (3.92) для $\psi(x)$ и $\bar\psi(x)$ как операторов Гейзенберга, так как, если $\psi(x)$ и $\bar\psi(x)$ являются решениями уравнения Дирака, то они должны разлагаться по таким плоско-волновым решениям.

Сначала рассмотрим амплитуду распространения $\langle 0|\psi(x)\bar\psi(y)|0\rangle$, которая должна представлять частицу с положительной энергией, распространяющуюся от y к x. В этом случае мы хотели бы, чтобы гейзенберговское состояние $\bar\psi(y)|0\rangle$ было линейной комбинацией состояний только с положительной или только с отрицательной энергией (так как гейзенберговское состояние равно $\Psi_H = e^{+iHt}\Psi_S$). Таким образом, вклад могут давать только слагаемые с $a^{s\dagger}_{\mathbf{p}}$ из $\bar\psi(y)$, что означает, что $b^{s\dagger}_{\mathbf{p}}$ должны при действии на вакуум давать нуль. Аналогично, $\langle 0|\psi(x)$ может содержать только положительночастотные компоненты. Таким образом, имеем:

$$\langle 0|\psi(x)\bar\psi(y)|0\rangle = \langle 0|\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_r a^r_{\mathbf{p}}u^r(p)e^{-ipx}\times \\ \times\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}\sum_s a^{s\dagger}_{\mathbf{q}}\bar u^s(q)e^{iqy}|0\rangle.$$

Можно кое-что сказать о матричных элементах $\left\langle 0\left|a^r_{\mathbf{p}}a^{s\dagger}_{\mathbf{q}}\right|0\right\rangle$, даже не зная перестановочных соотношений $a^r_{\mathbf{p}}$ и $a^{s\dagger}_{\mathbf{q}}$, а просто используя трансляционную и вращательную инвариантности. Если основное состояние $|0\rangle$ инвариантно относительно трансляций, то $|0\rangle = e^{i\mathbf{P}\cdot\mathbf{x}}|\mathbf{0}\rangle$. Далее, так как $a^{s\dagger}_{\mathbf{q}}$ порождает импульс $\mathbf{q}$, мы можем для вычисления использовать (2.48):

$$\left\langle 0\left|a^r_{\mathbf{p}}a^{s\dagger}_{\mathbf{q}}\right|0\right\rangle = \left\langle 0\left|a^r_{\mathbf{p}}a^{s\dagger}_{\mathbf{q}}e^{i\mathbf{P}\cdot\mathbf{x}}\right|0\right\rangle = e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\left\langle 0\left|e^{i\mathbf{P}\cdot\mathbf{x}}a^r_{\mathbf{p}}a^{s\dagger}_{\mathbf{q}}\right|0\right\rangle = e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\left\langle 0\left|a^r_{\mathbf{p}}a^{s\dagger}_{\mathbf{q}}\right|0\right\rangle.$$

Отсюда видно, что, если $\left\langle 0\left|a^r_{\mathbf{p}}a^{s\dagger}_{\mathbf{q}}\right|0\right\rangle$ отличнен от нуля, то $\mathbf{p}$ должен равняться $\mathbf{q}$. Аналогично можно показать, что из вращательной инвариантности $|0\rangle$ вытекает, что $r = s$. (Это должно быть интуитивно понятно и может быть проверено после изучения оператора углового момента далее в этом разделе.) Из этих рассуждений заключаем, что матричный элемент имеет вид:

$$\left\langle 0\left|a^r_{\mathbf{p}}a^{s\dagger}_{\mathbf{q}}\right|0\right\rangle = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}\cdot A(\mathbf{p}),$$

где $A(\mathbf{p})$ пока неопределено. Обратим внимание, что, если норма состояния всегда положительна (что должно выполняться в любом приемлемом гильбертовом пространстве), то $A(\mathbf{p})$ должно быть больше нуля. Теперь можно вернуться к (3.93) и записать:

$$\langle 0|\psi(x)\bar\psi(y)|0\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_s u^s(p)\bar u^s(p)A(\mathbf{p})e^{-ip(x-y)} = \\ = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}(\not p + m)A(\mathbf{p})e^{-ip(x-y)}$$

Это выражение явно инвариантно относительно бустов только тогда, когда $A(\mathbf{p})$ - лоренцевский скаляр, то есть $A(\mathbf{p}) = A(p^2)$. Так как $p^2 = m^2$, то A должно быть константой. В конечном итоге получаем:

$$\left\langle 0 \left| \psi_a(x)\bar{\psi}_b(y) \right| 0 \right\rangle = (i\partial_x + m)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip(x-y)} \cdot A$$

Аналогично, в амплитуде $\langle 0|\bar{\psi}(y)\psi(x)|0\rangle$ мы хотели бы оставить только вклады от положительно-частотных слагаемых в $\psi(y)$ и отрицательно-частотных слагаемых в $\psi(x)$. Таким образом, $a^s_{\mathbf{p}}$ все еще уничтожает вакуум, но $b^s_{\mathbf{p}}$ - нет. Согласно аргументам, аналогичным приведенным выше, имеем:

$$\left\langle 0 \left| \bar{\psi}_b(y)\psi_a(x) \right| 0 \right\rangle = -(i\not{\partial}_x + m)_{ab} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{ip(x-y)} \cdot B$$

где B - другая положительная константа. Знак «минус» важен; он происходит из соотношения полноты (3.67) для $\sum v\bar{v}$ и знака при x в экспоненте. Отсюда следует, что мы не можем иметь $\langle 0|[\psi(x), \bar{\psi}(y)]|0\rangle = 0$ вне светового конуса. Два слагаемых (3.94) и (3.95) действительно сократились бы в случае $A = -B$, но это невозможно, так как A и B должны быть оба положительными. Теперь мы близки к решению. Полагая $A = B = 1$, нетрудно получить (вне светового конуса):

$$\left\langle 0 \left| \psi_a(x)\bar{\psi}_b(y) \right| 0 \right\rangle = -\left\langle 0 \left| \bar{\psi}_b(y)\psi_a(x) \right| 0 \right\rangle .$$

Иными словами, спинорные поля антикоммутируют в точках, разделенных пространственноподобным интервалом. Этого достаточно, чтобы сохранить причинность, так как любые мыслимые наблюдаемые (такие, как энергия, заряд и число частиц) построены из четного числа спинорных полей; для любых наблюдаемых такого типа $\mathscr{O}_1$ и $\mathscr{O}_2$ все еще выполнено равенство $[\mathscr{O}_1(x), \mathscr{O}_2(y)] = 0$ для $(x-y)^2 < 0$. Примечательно, что постулируя антикоммутационные соотношения для дираковского поля, мы одновременно решаем проблему отрицательной энергии. ОдновременнЫе антикоммутационные соотношения примут вид:

$$\left\{ \psi_a(\mathbf{x}), \psi^\dagger_b(\mathbf{y}) \right\} = \delta^{(3)}(\mathbf{x}-\mathbf{y})\delta_{ab}$$
$$\{\psi_a(\mathbf{x}), \psi_b(\mathbf{y})\} = \left\{ \psi^\dagger_a(\mathbf{x}), \psi^\dagger_b(\mathbf{y}) \right\} = 0$$

Мы можем разложить $\psi(\mathbf{x})$ по $a^s_{\mathbf{p}}$ и $b^{s\dagger}_{\mathbf{p}}$, как и прежде (см. (3.87)). Операторы рождения и уничтожения должны теперь удовлетворять соотношениям

$$\left\{ a^r_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}} \right\} = \left\{ b^r_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}} \right\} = (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}$$

(все остальные антикоммутаторы равны нулю), чтобы удовлетворялись соотношения (3.96). Другое вычисление приводит к прежнему гамильтониану:

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(E_{\mathbf{p}} a^{s\dagger}_{\mathbf{p}} a^s_{\mathbf{p}} - E_{\mathbf{p}} b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} \right)$$

При этом $b^{s\dagger}_{\mathbf{p}}$ все еще рождает состояния с отрицательной энергией. Однако соотношение $\left\{ b^r_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}} \right\} = (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}$ является симметричным относительно $b^r_{\mathbf{p}}$ и $b^{s\dagger}_{\mathbf{q}}$. Поэтому просто переопределим:

$$\tilde{b}^s_{\mathbf{p}} \equiv b^{s\dagger}_{\mathbf{p}}; \quad \tilde{b}^{s\dagger}_{\mathbf{p}} \equiv b^s_{\mathbf{p}}.$$

Новые операторы удовлетворяют тем же антикоммутационным соотношениям, но теперь второе слагаемое в гамильтониане равно:

$$-E_{\mathbf{p}} b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} = +E_{\mathbf{p}} \tilde{b}^{\dagger\dagger}_{\mathbf{p}} \tilde{b}^s_{\mathbf{p}} - (\text{ const }).$$

Если определить $|0\rangle$ как состояние, которое уничтожается операторами $a^s_{\mathbf{p}}$ и $\tilde{b}^s_{\mathbf{p}}$, то все возбуждения над $|0\rangle$ имеют положительную энергию.

Что же случилось? Чтобы лучше понять этот прием, на время откажемся от теории поля и рассмотрим теорию с единственной парой операторов b и $b^\dagger$, удовлетворяющих соотношениям $\left\{b, b^\dagger\right\} = 1$ и $\{b, b\} = \left\{b^\dagger, b^\dagger\right\} = 0$. Выберем состояние $|0\rangle$ таким, что $b|0\rangle = 0$. Тогда $b^\dagger|0\rangle$ - новое состояние; обозначим его $|1\rangle$. Это состояние удовлетворяет условиям $b|1\rangle = |0\rangle$ и $b^\dagger|1\rangle = 0$. Таким образом, b и $b^\dagger$ действуют в гильбертовом пространстве из двух состояний $|0\rangle$ и $|1\rangle$. Мы могли бы сказать, что $|0\rangle$ представляет «пустое» состояние и что $b^\dagger$ «аполняет» это состояние. Но с таким же успехом можно было бы назвать $|1\rangle$ пустым состоянием и сказать, что $b = \tilde{b}^\dagger$ заполняет его. Два описания полностью эквивалентны, пока не определена некоторая наблюдаемая, которая позволяет нам физически различать эти состояния. В нашем случае правильным является такой выбор, когда состояние с более низкой энергией пусто. В результате оператором рождения объявляется тот оператор, который повышает энергию. Именно это и было сделано.

Заметим, между прочим, что так как $\left(\tilde{b}^\dagger\right)^2 = 0$, состояние не может быть заполнено дважды. Более общо, из антикоммутационных соотношений вытекает, что любое многочастичное состояние антисимметрично при перестановке двух частиц: $a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}}|0\rangle = -a^\dagger_{\mathbf{q}} a^\dagger_{\mathbf{p}}|0\rangle$. Таким образом, мы заключаем, что если лестничные операторы подчиняются антикоммутационным соотношениям, то соответствующие частицы удовлетворяют статистике Ферми-Дирака. Мы только что показали, что для того, чтобы быть уверенными, что обеспечено наличие только возбуждений с положительными энергиями, следует проквантовать дираковское поле с антикоммутационными соотношениями; при этих условиях частицы, связанные с дираковским полем, подчиняются статистике Ферми - Дирака. Этот вывод является частью более общего результата, впервые полученного Паули*: из совместного требования лоренц-инвариантности, положительности энергии, положительности нормы и причинности вытекает то, что частицы с целочисленным спином подчиняются статистике Бозе-Эйнштейна, в то время как частицы с полуцелыми спинами подчиняются статистике Ферми-Дирака.

Квантованное дираковское поле

Теперь систематизируем результаты квантования теории Дирака. Чтобы очистить записи от ненужной шелухи, следует улучшить обозначения. С этого места будем записывать $\tilde{b}_{\mathbf{p}}$ (оператор, понижающий энергию состояния) просто как $b_{\mathbf{p}}$, и $\tilde{b}^\dagger_{\mathbf{p}}$ как $b^\dagger_{\mathbf{p}}$. Все выражения, которые понадобятся в дальнейшем, выписаны ниже; соответствующие выражения, приведенные выше, должны быть забыты, если они отличаются от новых. Сначала выпишем полевые операторы:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{ip\cdot x} \right)$$
$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_s \left(b^s_{\mathbf{p}} \bar{v}^s(p) e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}} \bar{u}^s(p) e^{ip\cdot x} \right)$$

Операторы рождения и уничтожения удовлетворяют антикоммутационным правилам:

$$\left\{ a^r_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}} \right\} = \left\{ b^r_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}} \right\} = (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \delta^{rs},$$

причем все остальные антикоммутаторы равны нулю. Тогда одновременные антикоммутационные соотношения для ψ и $\psi^\dagger$ имеют вид:

$$\left\{ \psi_a(\mathbf{x}), \psi^\dagger_b(\mathbf{y}) \right\} = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \delta_{ab}$$
$$\{ \psi_a(\mathbf{x}), \psi_b(\mathbf{y}) \} = \left\{ \psi^\dagger_a(\mathbf{x}), \psi^\dagger_b(\mathbf{y}) \right\} = 0$$

Вакуум $|0\rangle$ определяется как такое состояние, что

$$a^s_{\mathbf{p}} |0\rangle = b^s_{\mathbf{p}} |0\rangle = 0.$$

Гамильтониан может быть записан в виде:

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\mathbf{p}} \left(a^{s\dagger}_{\mathbf{p}} a^s_{\mathbf{p}} + b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} \right)$$

где отброшено бесконечное постоянное слагаемое, возникающее от антикоммутирования $b^s_{\mathbf{p}}$ и $b^{s\dagger}_{\mathbf{p}}$. Отсюда видно, что вакуум является состоянием с наинизшей энергией, как мы того и хотели. Оператор импульса равен

$$\mathbf{P} = \int d^3x \psi^\dagger (-i\boldsymbol{\nabla}) \psi = \int \frac{d^3p}{(2\pi)^3} \sum_s \mathbf{p} \left(a^{s\dagger}_{\mathbf{p}} a^s_{\mathbf{p}} + b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} \right)$$

* W. Pauli, Phys. Rev. 58, 716 (1940), перепечатано в Schwinger (1958). Строгое рассмотрение дано в книге: R. F. Streater and A.S. Wightman, PCT, Spin and Statistics, and All That (Benjamin/Cummings, Reading, Mass., 1964). Таким образом, как $a^{s\dagger}_{\mathbf{p}}$, так и $b^{s\dagger}_{\mathbf{p}}$ рождают частицы с энергией $+E_{\mathbf{p}}$ и импульсом $\mathbf{p}$. Мы будем называть частицы, рожденные операторами $a^{s\dagger}_{\mathbf{p}}$, Фермионами, а рожденные операторами $b^{s\dagger}_{\mathbf{p}}$ - антифермионами.

Одночастичные состояния

$$|\mathbf{p}, s\rangle \equiv \sqrt{2E_{\mathbf{p}}} a^{s\dagger}_{\mathbf{p}} |0\rangle$$

определены так, что их скалярное произведение

$$\langle \mathbf{p}, r | \mathbf{q}, s \rangle = 2E_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}) \delta^{rs}$$

является лоренц-инвариантным. Отсюда следует, что оператор $U(\Lambda)$, осуществляющий преобразования Лоренца на состояниях гильбертова пространства, унитарен, хотя бусты $\Lambda_{\frac{1}{2}}$ не унитарны.

Для надежности удостоверимся, что $U(\Lambda)$ осуществляет правильное преобразование поля $\psi(x)$. Для этого вычислим

$$U\psi(x)U^{-1} = U\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s \left(a^s_{\mathbf{p}}u^s(p)e^{-ipx} + b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ipx}\right)U^{-1}$$

Рассмотрим первое слагаемое; второе рассматривается полностью аналогично. Из уравнения (3.106) следует, что $a^s_{\mathbf{p}}$ преобразуется согласно правилу:

$$U(\Lambda)a^s_{\mathbf{p}}U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda p}}{E_p}}a^s_{\Lambda p},$$

если принять, что ось квантования спина параллельна направлению буста или оси вращения. Для того, чтобы использовать это соотношение для вычисления (3.108), перепишем интеграл в виде:

$$\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}a^s_{\mathbf{p}} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\cdot\sqrt{2E_{\mathbf{p}}}a^s_{\mathbf{p}}$$

Второй множитель просто преобразуется под действием U, а первый является лоренц-инвариантным интегралом. Таким образом, если мы применим (3.109) и сделаем замену $\tilde{p} = \Lambda p$, то (3.108) принимает вид:

$$U(\Lambda)\psi(x)U^{-1}(\Lambda) = \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{2E_{\tilde{p}}}\sum_s u^s\left(\Lambda^{-1}\tilde{p}\right)\sqrt{2E_{\tilde{p}}}a^s_{\tilde{p}}e^{-i\tilde{p}\cdot\Lambda x} + \cdots$$

Но $u^s\left(\Lambda^{-1}\tilde{p}\right) = \Lambda^{-1}_{\frac{1}{2}}u^s(\tilde{p})$, поэтому действительно мы имеем:

$$U(\Lambda)\psi(x)U^{-1}(\Lambda) = \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{p}}}}\sum_s \Lambda^{-1}_{\frac{1}{2}}u^s(\tilde{p})a^s_{\tilde{p}}e^{-i\tilde{p}\cdot\Lambda x} + \cdots = \Lambda^{-1}_{\frac{1}{2}}\psi(\Lambda x)$$

Этот результат говорит о том, что преобразованное поле рождает и уничтожает частицы в точке Λx, как и должно быть. Однако обратим внимание, что это преобразование осуществляется в обратном направлении по сравнению с (3.2), где преобразованное поле ϕ было вычислено в точке $\Lambda^{-1}x$. Различие в том, что в разделе 3.1 мы считали, что преобразуется ранее существовавшее распределение поля, которое измерялось величиной $\phi(x)$. Здесь мы преобразовываем действие поля $\phi(x)$ при рождении или уничтожении частиц. Эти два пути выполнения преобразования Лоренца работают в противоположных направлениях. Тем не менее, отметим, что матрицы, действующие на ψ и преобразующие координаты x, имеют правильную относительную ориентацию, согласованную с (3.8).

Обсудим вопрос о спине дираковской частицы. Мы ожидаем, что дираковские фермионы будут иметь спин 1/2. Теперь можно вывести это свойство из нашего формализма. Мы уже показали, что частицы, рожденные операторами $a^{s\dagger}_{\mathbf{p}}$ и $b^{s\dagger}_{\mathbf{p}}$, имеют два "спиновых» состояния: $s = 1, 2$. Но мы все же не доказали, что этот «спин» имеет какое-либо отношение к угловому моменту. Чтобы сделать это, следует выписать оператор углового момента.

Напомним, что оператор обычного импульса в разделе 2.2 связан с трансляционной инвариантностью. Оператор момента импульса может быть найден аналогично, как следствие инвариантности относительно вращений. При вращении (или любом преобразовании Лоренца) дираковское поле ψ преобразуется (согласно исходному определению) как

$$\psi(x) \to \psi'(x) = \Lambda_{\frac{1}{2}}\psi\left(\Lambda^{-1}x\right).$$

Чтобы применить теорему Нетер, следует вычислить изменение поля в фиксированной точке, то есть

$$\delta\psi = \psi'(x) - \psi(x) = \Lambda_{\frac{1}{2}}\psi\left(\Lambda^{-1}x\right) - \psi(x).$$

Рассмотрим для определенности бесконечно малое вращение координат на угол θ вокруг оси z. Параметризация этого преобразования дана сразу после формулы (3.19): $\omega_{12} = -\omega_{21} = \theta$. Используя те же параметры, что и в (3.30), находим

$$\Lambda_{\frac{1}{2}} \approx 1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu} = 1 - \frac{i}{2}\theta\Sigma^3.$$

Теперь можно вычислить

$$\begin{aligned}\delta\psi(x) &= \left(1 - \frac{i}{2}\theta\Sigma^3\right)\psi(t, x+\theta y, y-\theta x, z) - \psi(x) = \\ &= -\theta\left(x\partial_y - y\partial_x + \frac{i}{2}\Sigma^3\right)\psi(x) \equiv \theta\Delta\psi\end{aligned}$$

ВременнАя компонента сохраняющегося нетеровского тока будет равна

$$j^0 = \frac{\partial\mathscr{L}}{\partial\left(\partial_0\psi\right)}\Delta\psi = -i\bar{\psi}\gamma^0\left(x\partial_y - y\partial_x + \frac{i}{2}\Sigma^3\right)\psi$$

Аналогичные выражения имеют место для вращений вокруг осей x и y. Таким образом, оператор углового момента имеет вид:

$$\mathbf{J} = \int d^3x\psi^\dagger\left(\mathbf{x}\times(-i\boldsymbol{\nabla}) + \frac{1}{2}\boldsymbol{\Sigma}\right)\psi$$

Для нерелятивистских фермионов первое слагаемое в (3.111) дает орбитальный угловой момент. Следовательно, второе слагаемое отвечает спиновому угловому моменту. К сожалению, для релятивистских фермионов разделение (3.111) на спиновую и орбитальную части не столь непосредственно очевидно, поэтому не так-то просто написать общее выражение для этой величины в терминах лестничных операторов.

Чтобы доказать, что дираковская частица имеет спин 1/2, достаточно рассмотреть покоящиеся частицы. В этом случае орбитальное слагаемое в (3.111) не дает вклада, и можно легко записать спиновое слагаемое через лестничные операторы. Проще всего использовать шредингеровское представление (3.87) для $\psi(\mathbf{x})$:

$$\begin{aligned}J_z = \int d^3x \int \frac{d^3pd^3p'}{(2\pi)^6}\frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{p}'}}}e^{-i\mathbf{p}'\cdot\mathbf{x}}e^{i\mathbf{p}\cdot\mathbf{x}}\times \\ \times\sum_{r,r'}\left(a^{r'\dagger}_{\mathbf{p}'}u^{r'\dagger}(\mathbf{p}') + b^{r'}_{-\mathbf{p}'}v^{r'\dagger}(-\mathbf{p}')\right)\frac{\Sigma^3}{2}\left(a^r_{\mathbf{p}}u^r(\mathbf{p}) + b^{r\dagger}_{-\mathbf{p}}v^r(-\mathbf{p})\right).\end{aligned}$$

Мы хотели бы применить этот оператор к одночастичному состоянию с нулевым импульсом $a_0^{s\dagger}|0\rangle$. Это легче всего выполнить, используя следующий прием. Так как J_z уничтожает вакуум, то $J_z a_0^{s\dagger}|0\rangle = \left[J_z, a_0^{s\dagger}\right]|0\rangle$. Единственное отличное от нуля слагаемое в последней величине имеет структуру $\left[a^{r\dagger}_{\mathbf{p}}a^{r'}_{\mathbf{p}}, a_0^{s\dagger}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p})a_0^{r\dagger}\delta^{r's}$; другие три слагаемых в коммутаторе или исчезают, или уничтожают вакуум. Таким образом,

$$J_z a_0^{s\dagger}|0\rangle = \frac{1}{2m}\sum_r \left(u^{r\dagger}(0)\frac{\Sigma^3}{2}u^s(0)\right) a_0^{r\dagger}|0\rangle = \sum_r \left(\xi^{r\dagger}\frac{\sigma^3}{2}\xi^s\right) a_0^{r\dagger}|0\rangle$$

Чтобы получить последнее выражение, здесь использована явная форма (3.47) для $u(0)$. Сумма по r проще всего берется, если выбрать спиноры ξ^r как собственные состояния σ^3. Тогда находим, что для $\xi^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ одночастичное состояние есть собственное состояние J_z с собственным значением $+1/2$, в то время как для $\xi^s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ оно является собственным состоянием J_z с собственным значением $-1/2$. Этот результат как раз совпадает с нашими ожиданиями для электронов.

Аналогичное вычисление определяет спин антифермиона с нулевым импульсом. Но в этом случае, так как порядок b и $b^\dagger$ в J_z обращен, при вычислении $\left[b_{\mathbf{p}}b_{\mathbf{p}}^\dagger, b_0^\dagger\right] = \left[-b_{\mathbf{p}}^\dagger b_{\mathbf{p}}, b_0^\dagger\right]$ получаем дополнительный знак "минус». Таким образом, для позитронов связь между спинорами ξ^s и спиновым угловым моментом противоположная: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ соответствует спину $-1/2$, в то время как $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ соответствует спину $+1/2$. Это обращение знака согласуется с предсказаниями дираковской теории дырок. С этой точки зрения позитрон есть отсутствие электрона с отрицательной энергией. Если пропавший электрон имел положительную проекцию J_z, то его отсутствие приводит к отрицательной проекции $-J_z$.

Итак, угловой момент фермионов с нулевым импульсом определяется равенствами:

$$J_z a_0^{s\dagger}|0\rangle = \pm\frac{1}{2}a_0^{s\dagger}|0\rangle, \quad J_z b_0^{s\dagger}|0\rangle = \mp\frac{1}{2}b_0^{s\dagger}|0\rangle,$$

где верхний знак берется для $\xi^s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, а нижний знак - для $\xi^s = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

В теории Дирака имеется еще одна важная сохраняющаяся величина. В разделе 3.4 мы видели, что ток $j^\mu = \bar{\psi}\gamma^\mu\psi$ сохраняется. Заряд, связанный с этим током, есть

$$Q = \int d^3x\psi^\dagger(x)\psi(x) = \int \frac{d^3p}{(2\pi)^3}\sum_s \left(a_{\mathbf{p}}^{s\dagger}a_{\mathbf{p}}^s + b_{-\mathbf{p}}^s b_{-\mathbf{p}}^{s\dagger}\right)$$

или, если мы отбрасываем еще одну бесконечную константу,

$$Q = \int \frac{d^3p}{(2\pi)^3}\sum_s \left(a_{\mathbf{p}}^{s\dagger}a_{\mathbf{p}}^s - b_{\mathbf{p}}^{s\dagger}b_{\mathbf{p}}^s\right)$$

Таким образом, операторы $a_{\mathbf{p}}^{s\dagger}$ рождают фермионы с зарядом $+1$, в то время как операторы $b_{\mathbf{p}}^{s\dagger}$ рождают антифермионы с зарядом -1. Когда мы связываем дираковское поле с электромагнитным, мы видим, что Q есть не что иное, как электрический заряд (с точностью до постоянного множителя, который зависит от того, какого типа частицы мы хотим описать; например, для электронов электрический заряд равен Qe).

В квантовой электродинамике спинорное поле ψ используется для описания электронов и позитронов. Частицы, рождаемые операторами $a_{\mathbf{p}}^{s\dagger}$, - электроны; они имеют энергию $E_{\mathbf{p}}$, импульс $\mathbf{p}$, спин 1/2 с поляризацией, соответствующей ξ^s, и заряд $+1$ (в единицах e). Частицы, рождаемые операторами $b_{\mathbf{p}}^{s\dagger}$, - позитроны; они имеют энергию $E_{\mathbf{p}}$, импульс

$\mathbf{p}$, спин 1/2 с поляризацией, обратной поляризации ξ^s, и заряд -1. Состояние $\psi_\alpha(x)|0\rangle$ содержит позитрон в точке x, чья поляризация соответствует выбранной компоненте спинора. Аналогично, $\bar\psi_\alpha(x)|0\rangle$ является состоянием одного электрона в точке x.

9.5.8 Dirac Propagator

Теория

Теперь вычисление амплитуд распространения для дираковского поля становится простым упражнением:

$$\begin{aligned}\left\langle 0\left|\psi_a(x)\bar\psi_b(y)\right|0\right\rangle &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_s u_a^s(p)\bar u_b^s(p)e^{-ip\cdot(x-y)} = \\ &= (i\not\partial_x + m)_{ab}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(x-y)} \\ \left\langle 0\left|\bar\psi_b(y)\psi_a(x)\right|0\right\rangle &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_s v_a^s(p)\bar v_b^s(p)e^{-ip\cdot(y-x)} = \\ &= -(i\not\partial_x + m)_{ab}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}e^{-ip\cdot(y-x)}.\end{aligned}$$

Так же, как для уравнения Клейна-Гордона, можно построить для уравнения Дирака функции Грина, удовлетворяющие различным граничным условиям. Например, запаздывающая функция Грина

$$S_R^{ab}(x-y)\equiv\theta\left(x^0-y^0\right)\left\langle 0\left|\left\{\psi_a(x),\bar\psi_b(y)\right\}\right|0\right\rangle.$$

Нетрудно проверить, что

$$S_R(x-y) = (i\not\partial_x + m)\,D_R(x-y)$$

так как в правой части слагаемое, включающее $\partial_0\theta\left(x^0-y^0\right)$, исчезает. Используя (3.117)

и то, что $\not\partial\not\partial = \partial^2$, получаем, что S_R — функция Грина оператора Дирака:

$$(i\not\partial_x - m)\,S_R(x-y) = i\delta^{(4)}(x-y)\cdot\mathbf{1}_{4\times4}.$$

Функция Грина оператора Дирака может также быть найдена при помощи преобразования Фурье. Раскладывая $S_R(x-y)$ в интеграл Фурье и действуя с обеих сторон оператором $(i\partial_x - m)$, находим:

$$i\delta^{(4)}(x-y) = \int\frac{d^4p}{(2\pi)^4}(\not p - m)e^{-ip\cdot(x-y)}\widetilde S_R(p)$$

следовательно,

$$\widetilde S_R(p) = \frac{i}{\not p - m} = \frac{i(\not p + m)}{p^2-m^2}.$$

Чтобы получить запаздывающую функцию Грина, следует вычислить интеграл по p^0 в (3.120) по контуру, показанному после формулы (2.54). Для $x^0 > y^0$ замыкаем контур внизу,

огибая оба полюса, получая в результате сумму (3.115) и (3.114). Для $x^0 < y^0$ замыкаем контур вверху и получаем ноль.

Функция Грина с граничными условиями Фейнмана определяется контуром, показанным после формулы (2.58):

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)} =$$
$$= \begin{cases} \langle 0|\psi(x)\bar{\psi}(y)|0\rangle & \text{для \% \%} x^0 > y^0 \text{ (контур замыкается внизу)} \\ -\langle 0|\bar{\psi}(y)\psi(x)|0\rangle & \text{для \% \%} x^0 < y^0 \text{ (контур замыкается вверху)} \end{cases} \equiv$$
$$\equiv \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle.$$

Здесь мы определили хронологическое произведение спинорных полей с дополнительным изменением знака при перестановке операторов. Это изменение знака чрезвычайно важно в квантовой теории поля фермионов; мы снова встретимся с этим фактом в разделе 4.7.

Как и в теории Клейна-Гордона, выражение (3.121) для фейнмановского пропагатора - наиболее полезный результат этой главы. Когда мы проводим вычисления по теории возмущений с помощью диаграмм Фейнмана, мы сопоставляем множитель $\widetilde{S}_F(p)$ каждой внутренней линии фермиона.

9.5.9 Discrete Symmetries in Dirac Theory

Теория

В последнем разделе обсуждалась реализация непрерывных преобразований Лоренца на гильбертовом пространстве в теории Дирака. Мы нашли, что для каждого преобразования Λ имеется унитарный оператор $U(\Lambda)$, который производит правильное преобразование полей:

$$U(\Lambda)\psi(x)U^{-1}(\Lambda) = \Lambda^{-1}_{\frac{1}{2}}\psi(\Lambda x)$$

В этом разделе мы рассмотрим аналогичные операторы, которые реализуют различные дискретные симметрии на дираковском поле.

В дополнение к непрерывным преобразованиям Лоренца имеются две других пространственно-временных операции, которые являются потенциальными симметриями лагранжиана: четность и обращение времени. Четность, обозначаемая P, преобразует $(t,\mathbf{x}) \to (t,-\mathbf{x})$, обращая ориентацию пространства. Обращение времени, обозначаемое T, преобразует $(t,\mathbf{x}) \to (-t,\mathbf{x})$, меняя местами будущий и прошлый световые конусы. Ни одна из этих операций не может быть получена при помощи непрерывных преобразований Лоренца. Обе, однако, сохраняют интервал Минковского $x^2 = t^2 - \mathbf{x}^2$. Согласно стандартной терминологии множество непрерывных преобразований Лоренца называется собственной ортохронной группой Лоренца и обозначается $\mathbf{L}^{\uparrow}_{+}$. Полная группа Лоренца разбивается на четыре несвязные компоненты:

$$\begin{array}{ccccc} \mathbf{L}^{\uparrow}_{+} & \overset{P}{\longleftrightarrow} & \mathbf{L}^{\uparrow}_{-} = P\mathbf{L}^{\uparrow}_{+} & & \text{«ортохронная»} \\ T\updownarrow & & \updownarrow T & & \\ \mathbf{L}^{\downarrow}_{+} = T\mathbf{L}^{\uparrow}_{+} & \overset{P}{\longleftrightarrow} & \mathbf{L}^{\downarrow}_{-} = PT\mathbf{L}^{\uparrow}_{+} & & \text{«неортохронная»} \\ \text{«собственная»} & & \text{«несобственная»} & & \end{array}$$

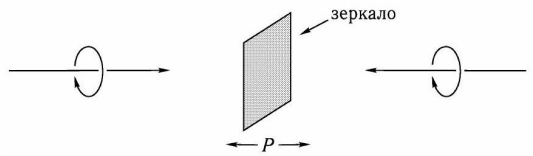
Одновременно с обсуждением симметрий P и T, удобно рассмотреть третью (не относящуюся к пространственно-временным) дискретную операцию - зарядовое сопряжение, обозначаемое C. Эта операция меняет местами частицы и античастицы.

Хотя любая релятивистская теория поля должна быть инвариантна относительно $\mathbf{L}_+^\uparrow$, она не обязана быть инвариантной относительно преобразований P, T или C. Каков статус этих операций симметрии в реальном мире? Из эксперимента известно, что три из сил природы - гравитационные, электромагнитные и сильные взаимодействия - симметричны относительно преобразований P, C и T. Слабые взаимодействия нарушают симметрии C и P по отдельности, но сохраняют CP и T. Одако некоторые редкие процессы (все до сих пор наблюдавшиеся включают нейтральные K-мезоны) также нарушают CP и T. Все наблюдения показывают, что комбинация CPT является точной симметрией природы.

Принятая в настоящее время теоретическая модель слабых взаимодействий - калибровочная теория Глэшоу-Вайнберга-Салама, описанная в гл. 20. В этой теории C и P нарушаются максимально. То, что C и P оказываются весьма хорошими симметриями в большинстве легко наблюдаемых процессов, неожиданно (хотя и не случайно). С другой стороны, никому неизвестна по-настоящему красивая теория, которая нарушает CP-симметрию. В существующей теории, где имеются три (или более) поколения фермионов, определен параметр, вызывающий CP-нарушение в случае, когда он отличен от нуля. Но величина этого параметра понята не лучше, чем величина массы электрона. Физическое происхождение CP-нарушения остается тайной. Мы обсудим этот вопрос далее в разделе 20.3.

9.5.10 Parity

После этого краткого вступления обсудим действие P-, T - и C-преобразований на дираковские частицы и поля. Сначала рассмотрим четность. Оператор P должен менять направление импульса частицы без изменения ее спина:



На математическом языке это означает, что операция P должна быть реализована унитарным оператором (правильнее обозначать его $U(P)$, но мы будем писать просто P), который преобразует, например, состояние $a^s_{\mathbf{p}}|0\rangle$ в $a^s_{-\mathbf{p}}|0\rangle$. Другими словами, мы хотим иметь следующий закон преобразования:

$$Pa^s_{\mathbf{p}}P = \eta_a a^s_{-\mathbf{p}}\% \quad \% \text{ и } \% \quad \%Pb^s_{\mathbf{p}}P = \eta_b b^s_{-\mathbf{p}}$$

где η_a и η_b - возможные фазы. Эти фазы ограничены условием, что двукратное применение оператора четности должно возвращать наблюдаемые к их первоначальным значениям. Так как наблюдаемые построены из четного числа фермионных операторов, то $\eta_a^2, \eta_b^2 = \pm 1$

Так же, как непрерывное преобразование Лоренца реализуется на дираковском поле как 4×4 постоянная матрица $\Lambda_{\frac{1}{2}}$, так и преобразование четности должно представляться 4×4 постоянной матрицей. Чтобы найти эту матрицу и определить η_a и η_b, вычислим действие P на $\psi(x)$. Используя (3.123), находим:

$$P\psi(x)P = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s \left(\eta_a a^s_{-\mathbf{p}}u^s(p)e^{-ipx} + \eta_b^* b^{s\dagger}_{-\mathbf{p}}v^s(p)e^{ipx}\right)$$

Теперь сделаем замену переменных на $\tilde{p} = (p^0, -\mathbf{p})$. Заметим, что $p\cdot x = \tilde{p}\cdot(t,-\mathbf{x})$. Также $\tilde{p}\cdot\sigma = p\cdot\bar{\sigma}$ и $\tilde{p}\cdot\bar{\sigma} = p\cdot\sigma$. Это позволяет записать:

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix} = \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\xi \\ \sqrt{\tilde{p}\cdot\sigma}\xi \end{pmatrix} = \gamma^0 u(\tilde{p})$$
$$v(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ -\sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix} = \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\xi \\ -\sqrt{\tilde{p}\cdot\sigma}\xi \end{pmatrix} = -\gamma^0 v(\tilde{p}).$$

Поэтому (3.124) будет иметь вид:

$$P\psi(x)P = \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{p}}}}\sum_s \left(\eta_a a^s_{\tilde{p}}\gamma^0 u^s(\tilde{p})e^{-i\tilde{p}(t,-\mathbf{x})} - \eta_b^* b^{s\dagger}_{\tilde{p}}\gamma^0 v^s(\tilde{p})e^{i\tilde{p}(t,-\mathbf{x})}\right)$$

Это выражение должно равняться некоторой постоянной матрице, умноженной на $\psi(t,-\mathbf{x})$, и это действительно выполняется, если положить $\eta_b^* = -\eta_a$. Отсюда следует, чТо

$$\eta_a\eta_b = -\eta_a\eta_a^* = -1$$

Таким образом, преобразование четности поля $\psi(x)$ в окончательной форме имеет вид:

$$P\psi(t,\mathbf{x})P = \eta_a\gamma^0\psi(t,-\mathbf{x}).$$

Нам важно знать (например, при построении лагранжианов), как преобразуются разные дираковские билинейные формы полей под действием преобразования четности. Имеется пять таких билинейных форм:

$$\bar{\psi}\psi, \quad \bar{\psi}\gamma^\mu\psi, \quad i\bar{\psi}\left[\gamma^\mu,\gamma^\nu\right]\psi, \quad \bar{\psi}\gamma^\mu\gamma^5\psi, \quad i\bar{\psi}\gamma^5\psi$$

Множитель i в некоторых из них присутствует для того, чтобы все эти величины были эрмитовыми, что легко проверяется. (Любое новое слагаемое, дающее вклад в лагранжиан, должно быть вещественным.) Сначала следует вычислить

$$P\bar{\psi}(t,\mathbf{x})P = P\psi^\dagger(t,\mathbf{x})P\gamma^0 = (P\psi(t,\mathbf{x})P)^\dagger\gamma^0 = \eta_a^*\bar{\psi}(t,-\mathbf{x})\gamma^0$$

Тогда скалярная билинейная форма преобразуется как

$$\boldsymbol{P}\bar{\psi}\psi\boldsymbol{P} = |\eta_a|^2\,\bar{\psi}(t,-\mathbf{x})\gamma^0\gamma^0\psi(t,-\mathbf{x}) = +\bar{\psi}\psi(t,-\mathbf{x}),$$

в то время как для векторной формы получаем:

$$P\bar{\psi}\gamma^\mu\psi\boldsymbol{P} = \bar{\psi}\gamma^0\gamma^\mu\gamma^0\psi(t,-\mathbf{x}) = \begin{cases} +\bar{\psi}\gamma^\mu\psi(t,-\mathbf{x}) & \text{для } \mu = 0, \\ -\bar{\psi}\gamma^\mu\psi(t,-\mathbf{x}) & \text{для } \mu = 1,2,3. \end{cases}$$

Обратим внимание, что вектор приобретает тот же самый знак «минус» при пространственных компонентах, как и вектор x^μ. Аналогично, найдем преобразования псевдоскаляра и псевдовектора:

$$\boldsymbol{P}\boldsymbol{i}\bar{\psi}\gamma^5\psi\boldsymbol{P} = i\bar{\psi}\gamma^0\gamma^5\gamma^0\psi(t,-\mathbf{x}) = -i\bar{\psi}\gamma^5\psi(t,-\mathbf{x})$$

$$P\bar{\psi}\gamma^\mu\gamma^5\psi P = \bar{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\psi(t,-\mathbf{x}) = \begin{cases} -\bar{\psi}\gamma^\mu\gamma^5\psi & \text{для } \mu = 0 \\ +\bar{\psi}\gamma^\mu\gamma^5\psi & \text{для } \mu = 1,2,3. \end{cases}$$

Как мы и предупреждали в разделе 3.4, приставка "псевдо» означает дополнительный знак «минус» при преобразовании четности. (Свойства преобразования $i\bar{\psi}\left[\gamma^\mu,\gamma^\nu\right]\psi = 2\bar{\psi}\sigma^{\mu\nu}\psi$ разбираются в задаче 3.7.) Заметим, что свойства преобразования фермионных билинейных форм не зависят от η_a, поэтому без потери общности можно положить с самого начала $\eta_a = -\eta_b = 1$.

Однако относительный знак «минус»в (3.125) в преобразованиях четности фермиона и антифермиона имеет важные следствия. Рассмотрим фермион-антифермионное состояние $a^{s\dagger}_{\mathbf{p}}b^{s'\dagger}_{\mathbf{q}}|0\rangle$. Примененяя P, мы находим $P\left(a^{s\dagger}_{\mathbf{p}}b^{s'\dagger}_{\mathbf{q}}|0\rangle\right) = -\left(a^{s\dagger}_{-\mathbf{p}}b^{s'\dagger}_{-\mathbf{q}}|0\rangle\right)$. Таким образом, состояние, содержащее фермионно-антифермионную пару, приобретает знак (-1) при преобразовании четности. Эта информация наиболее полезна при изучении связанных состояний, в которых импульсы фермиона и антифермиона интегрируются со шредингеровской волновой функцией, образуя систему, локализованную в пространстве. Мы подробно рассматриваем такие состояния в разделе 5.3, здесь же следует заметить, что если пространственная волновая функция симметрична при замене $x \to -x$, то состояние имеет отрицательную четность, в то время как если эта функция антисимметрична при замене $x \to -x$, то состояние имеет положительную четность. Например, связанное состояние с $L = 0$ имеет отрицательную четность; состояния с $J = 0$ преобразуется как псевдоскаляр, в то время как три состояния с $J = 1$ преобразуются как пространственные компоненты вектора. Эти свойства обнаруживаются в правилах отбора для распадов позитрония и системы кварк-антикварк (см. задачу 3.8).

9.5.11 Time Reversal

Теперь обратимся к реализации обращения времени. Мы хотели бы трактовать T как унитарный оператор, переводящий $a_{\mathbf{p}}$ в $a_{-\mathbf{p}}$ (и аналогично для $b_{\mathbf{p}}$) и $\psi(t,\mathbf{x})$ в $\psi(-t,\mathbf{x})$ (с точностью до постоянной матрицы). Однако удовлетворить этим требованиям чрезвычайно трудно, так как мы видели, что преобразование, переводящее $a_{\mathbf{p}}$ в $a_{-\mathbf{p}}$, эквивалентно преобразованию, переводящему $(T,\mathbf{x})$ в $(T,-\mathbf{x})$ в разложении ψ. Трудность еще более очевидна, когда мы требуем, чтобы операция обращения времени являлась симметрией в свободной теории Дирака, $[T,H]=0$. Тогда

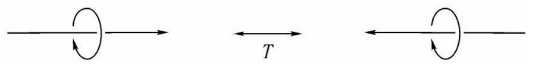
$$\psi(t,\mathbf{x}) = e^{iHt}\psi(\mathbf{x})e^{-iHt} \Rightarrow$$
$$\Rightarrow T\psi(t,\mathbf{x})T = e^{iHt}[T\psi(\mathbf{x})T]e^{-iHt} \Rightarrow$$
$$\Rightarrow T\psi(t,\mathbf{x})T|0\rangle = e^{iHt}[T\psi(\mathbf{x})T]|0\rangle,$$

в предположении, что $H|0\rangle = 0$. Здесь правая часть есть сумма лишь отрицательночастотных слагаемых. Но, если T обращает временную зависимость поля $\psi(t,\mathbf{x})$, тогда левая часть равна (с точностью до постоянной матрицы) $\psi(-t,\mathbf{x})|0\rangle = e^{-iHt}\psi(\mathbf{x})|0\rangle$, что является суммой положительно-частотных слагаемых. Таким образом, мы доказали, что T не может действовать как линейный унитарный оператор. Что можно сделать? Выход был бы в сохрании условия унитарности $T^\dagger = T^{-1}$, но при этом T должен действовать как на с-числа, так и на операторы по правилу:

$$T(\text{ с-число }) = (\text{ с-число })^* T.$$

Тогда, даже если $[T, H] = 0$, зависимость от времени всех множителей в показателе экспоненты обращается: $Te^{+iHt} = e^{-iHt}T$. Так как вся эволюция во времени в квантовой механике выполняется при помощи таких экспоненциальных множителей, это фактически изменяет знак t. Заметим, что операция комплексного сопряжения нелинейна, поэтому T называется антилинейным или антиунитарным оператором.

Кроме изменения импульса частицы, T также должен переворачивать спин:



Чтобы формально описать это, следует найти математическую операцию, которая переворачивает спинор ξ.

В предыдущих частях этой главы мы обозначали спиновое состояние фермиона индексом $s = 1, 2$. В оставшейся части этого раздела, мы свяжем s с физической проекцией спина фермиона на определенную ось. Если эта ось имеет полярные координаты θ, ϕ, то двухкомпонентные спиноры со спином вверх и вниз вдоль этой оси имеют вид:

$$\xi(\uparrow) = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \quad \xi(\downarrow) = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}.$$

Пусть $\xi^s = (\xi(\uparrow), \xi(\downarrow))$ для $s = 1, 2$. Определим также

$$\xi^{-s} = -i\sigma^2 (\xi^s)^* .$$

Эта величина есть перевернутый спинор; из явных формул находим:

$$\xi^{-s} = (\xi(\downarrow), -\xi(\uparrow)).$$

Вид формулы для обращения спина в более общем случае следует из тождества $\boldsymbol{\sigma}\sigma^2 = \sigma^2(-\sigma^*)$. Из него следует, что если ξ удовлетворяет $\mathbf{n}\cdot\boldsymbol{\sigma}\xi = +\xi$ для некоторой оси $\mathbf{n}$, то

$$(\mathbf{n}\cdot\sigma)\left(-i\sigma^2\xi^*\right) = -i\sigma^2(-\mathbf{n}\cdot\sigma)^*\xi^* = i\sigma^2(\xi^*) = -\left(-i\sigma^2\xi^*\right)$$

Обратим внимание, что, приняв такое определение переворота спина, находим, что два последовательных переворота приводят к умножению первоначального спинового состояния на (-1).

Теперь сопоставим различные фермионные спиновые состояния выписанным спинорам. Электронный оператор уничтожения $a^s_{\mathbf{p}}$ уничтожает электрон, описываемый спинором $u^s(p)$, содержащим ξ^s. Оператор уничтожения позитрона $b^s_{\mathbf{p}}$ уничтожает позитрон, чей спинор $v^s(p)$ содержит ξ^{-s}:

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^{-s} \\ -\sqrt{p\cdot\bar{\sigma}}\xi^{-s} \end{pmatrix}$$

Как и в (3.135), определяем:

$$a^{-s}_{\mathbf{p}} = \left(a^2_{\mathbf{p}}, -a^1_{\mathbf{p}}\right), \quad b^{-s}_{\mathbf{p}} = \left(b^2_{\mathbf{p}}, -b^1_{\mathbf{p}}\right).$$

Теперь можно вывести соотношение между дираковскими спинорами u и v и их обращениями во времени. Определим $\tilde{p} = (p^0, -\mathbf{p})$. Этот вектор удовлетворяет тождеству $\sqrt{\tilde{p}\cdot\sigma}\sigma^2 = \sigma^2\sqrt{p\cdot\sigma^*}$. Для доказательства разложим квадратный корень как в (3.49). При некотором выборе спина и импульса, связанных с дираковским спинором $u^s(p)$, пусть $u^{-s}(\tilde{p})$ обозначает спинор с обращенным импульсом и перевернутым спином. Эти величины связаны соотношением:

$$u^{-s}(\tilde{p}) = \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\,(-i\sigma^2\xi^{s*}) \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,(-i\sigma^2\xi^{s*}) \end{pmatrix} = \begin{pmatrix} -i\sigma^2\sqrt{p\cdot\sigma^*}\xi^{s*} \\ -i\sigma^2\sqrt{p\cdot\bar{\sigma}^*}\xi^{s*} \end{pmatrix} =$$
$$= -i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}[u^s(p)]^* = -\gamma^1\gamma^3\,[u^s(p)]^*$$

Аналогично для $v^s(p)$,

$$v^{-s}(\tilde{p}) = -\gamma^1\gamma^3\,[v^s(p)]^*$$

в этом соотношении v^{-s} содержит $\xi^{-(-s)} = -\xi^s$.

Используя (3.137), определим преобразование обращения времени фермионных операторов уничтожения следующим образом:

$$Ta^s_{\mathbf{p}}T = a^{-s}_{-\mathbf{p}}, \quad Tb^s_{\mathbf{p}}T = b^{-s}_{-\mathbf{p}}.$$

(Дополнительная общая фаза не имеет никакого влияния на дальнейшее изложение и для простоты опускается.) Соотношения (3.138) позволяют вычислить действие T на фермионное поле $\psi(x)$:

$$T\psi(t,\mathbf{x})T = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s T\left(a^s_{\mathbf{p}}u^s(p)e^{-ipx} + b^{s\dagger}_{\mathbf{p}}v^s(p)e^{ipx}\right)T =$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s \left(a^{-s}_{-\mathbf{p}}\,[u^s(p)]^*\,e^{ipx} + b^{-s\dagger}_{-\mathbf{p}}\,[v^s(p)]^*\,e^{-ipx}\right) =$$
$$= \left(-\gamma^1\gamma^3\right)\int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\tilde{p}}}}\sum_s \left(a^{-s}_{\tilde{p}}u^{-s}(\tilde{p})e^{i\tilde{p}(t,-\mathbf{x})} + b^{-s\dagger}_{\tilde{p}}v^{-s}(\tilde{p})e^{-i\tilde{p}(t,-\mathbf{x})}\right) =$$
$$= \left(-\gamma^1\gamma^3\right)\psi(-t,\mathbf{x}).$$

На последнем шаге мы использовали $\tilde{p}\cdot(t,-\mathbf{x}) = -\tilde{p}\cdot(-t,\mathbf{x})$. Как и для четности, мы получили простой закон преобразования для фермионного поля $\psi(x)$. Относительный знак «минус»в законах преобразования для частицы и античастицы неявно присутствует и здесь, в дважды перевернутом спиноре в v^{-s}.

Теперь можно установить результат действия T на различные билинейные формы. Нам потребуется формула:

$$T\bar{\psi}T = (T\psi T)^\dagger\left(\gamma^0\right)^* = \psi^\dagger(-t,\mathbf{x})\left[-\gamma^1\gamma^3\right]^\dagger\gamma^0 = \bar{\psi}(-t,\mathbf{x})\left[\gamma^1\gamma^3\right]$$

Тогда для преобразования скалярной билинейной формы имеем:

$$T\bar{\psi}\psi(t,\mathbf{x})T = \bar{\psi}\left(\gamma^1\gamma^3\right)\left(-\gamma^1\gamma^3\right)\psi(-t,\mathbf{x}) = +\bar{\psi}\psi(-t,\mathbf{x})$$

При протаскивании T через i, псевдоскаляр приобретает дополнительный знак «минус»:

$$Ti\bar{\psi}\gamma^5\psi T = -i\bar{\psi}\left(\gamma^1\gamma^3\right)\gamma^5\left(-\gamma^1\gamma^3\right)\psi = -i\bar{\psi}\gamma^5\psi(-t,\mathbf{x})$$

Для вектора нужно провести вычисления отдельно для каждого из четырех случаев $\mu = 0,1,2,3$. После небольшой работы находим:

$$T\bar{\psi}\gamma^\mu\psi T = \bar{\psi}\left(\gamma^1\gamma^3\right)\left(\gamma^\mu\right)^*\left(-\gamma^1\gamma^3\right)\psi = \begin{cases} +\bar{\psi}\gamma^\mu\psi(-t,\mathbf{x}) & \text{для } \mu = 0 \\ -\bar{\psi}\gamma^\mu\psi(-t,\mathbf{x}) & \text{для } \mu = 1,2,3 \end{cases}$$

Это как раз то свойство преобразования, которое нужно для таких векторов, как плотность тока. Можно проверить, что при обращении времени псевдовектор преобразуется таким же образом.

9.5.12 Charge Conjugation

Последняя из трех дискретных симметрий - это симметрия относительного зарядового сопряжения C, меняющая местами частицы и античастицы. Без труда можно реализавать C как унитарный линейный оператор. Зарядовое сопряжение обычно определяется как преобразование фермиона с заданной ориентацией спина в антифермион с той же спиновой ориентацией. Таким образом, удобным выбором для преобразования фермионных операторов уничтожения будет

$$Ca^s_{\mathbf{p}}C = b^s_{\mathbf{p}}; \quad Cb^s_{\mathbf{p}}C = a^s_{\mathbf{p}}.$$

Снова для простоты мы игнорируем возможные дополнительные фазы.

Далее мы хотим выяснить результат действия C на поле $\psi(x)$. Прежде всего нам понадобится соотношение между $v^s(p)$ и $u^s(p)$. Используя (3.136) и (3.134), находим:

$$\left(v^s(p)\right)^* = \begin{pmatrix} \sqrt{p\cdot\sigma}\left(-i\sigma^2\xi^*\right) \\ -\sqrt{p\cdot\bar{\sigma}}\left(-i\sigma^2\xi^*\right) \end{pmatrix}^* = \begin{pmatrix} -i\sigma^2\sqrt{p\cdot\bar{\sigma}^*}\xi^* \\ i\sigma^2\sqrt{p\cdot\sigma^*}\xi^* \end{pmatrix}^* = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix},$$

где ξ стоит за ξ^s. Это означает, что

$$u^s(p) = -i\gamma^2\left(v^s(p)\right)^*, \quad v^s(p) = -i\gamma^2\left(u^s(p)\right)^*.$$

Если подставить (3.144) в выражение для оператора фермионного поля и затем преобразовать этот оператор при помощи C, то

$$C\psi(x)C = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s\left(-i\gamma^2 b^s_{\mathbf{p}}\left(v^s(p)\right)^* e^{-ipx} - i\gamma^2 a^{s\dagger}_{\mathbf{p}}\left(u^s(p)\right)^* e^{ipx}\right) =$$
$$= -i\gamma^2\psi^*(x) = -i\gamma^2\left(\psi^\dagger\right)^T = -i\left(\bar{\psi}\gamma^0\gamma^2\right)^T$$

Обратим внимание, что C является линейным унитарным оператором, несмотря на то, что он переводит $\psi \to \psi^*$.

Снова мы хотели бы знать, как C действует на фермионные билинейные формы. Во-первых, нам понадобится формула:

$$C\bar{\psi}(x)C = C\psi^\dagger C\gamma^0 = \left(-i\gamma^2\psi\right)^T\gamma^0 = \left(-i\gamma^0\gamma^2\psi\right)^T$$

Вывод преобразований для билинейных форм несколько сложнее. Удобнее всего явно выписывать спинорные индексы. Для скаляра

$$C\bar{\psi}\psi C = \left(-i\gamma^0\gamma^2\psi\right)^T \left(-i\bar{\psi}\gamma^0\gamma^2\right)^T = -\gamma^0_{ab}\gamma^2_{bc}\psi_c\bar{\psi}_d\gamma^0_{de}\gamma^2_{ea} = \\ = +\bar{\psi}_d\gamma^0_{de}\gamma^2_{ea}\gamma^0_{ab}\gamma^2_{bc}\psi_c = -\bar{\psi}\gamma^2\gamma^0\gamma^0\gamma^2\psi = +\bar{\psi}\psi$$

(Знак "минус» на третьем шаге вытекает из антикоммутативности фермионов.) Для псевдоскаляра вывод не более сложен:

$$Ci\bar{\psi}\gamma^5\psi C = i\left(-i\gamma^0\gamma^2\psi\right)^T\gamma^5\left(-i\bar{\psi}\gamma^0\gamma^2\right)^T = i\bar{\psi}\gamma^5\psi$$

Преобразование для каждой компоненты вектора и псевдовектора следует делать отдельно. Учитывая, что γ^0 и γ^2 являются симметричными матрицами, в то время как γ^1 и γ^3 - антисимметричны, находим:

$$C\bar{\psi}\gamma^\mu\psi C = -\bar{\psi}\gamma^\mu\psi$$
$$C\bar{\psi}\gamma^\mu\gamma^5\psi C = +\bar{\psi}\gamma^\mu\gamma^5\psi$$

Хотя оператор C меняет местами ψ и $\bar{\psi}$, он реально не меняет порядка операторов уничтожения и рождения. Таким образом, если $\bar{\psi}\gamma^0\psi$ определена так, что в ней вычтена бесконечная константа, упомянутая перед (3.113), то эта константа не появляется вновь в результате зарядового сопряжения.

9.5.13 Summary of Formulas for C, P, and T Transformations

Свойства преобразования различных фермионных билинейных форм под действием преобразований $C, P,$ и T сведены вместе в приведенной ниже таблице. Здесь мы используем сокращения $(-1)^\mu \equiv 1$ для $\mu = 0$ и $(-1)^\mu \equiv -1$ для $\mu = 1, 2, 3$.

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	∂_μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
CPT	$+1$	$+1$	-1	-1	$+1$	-1

Мы включили сюда свойства преобразования тензорной билинейной формы (см. задачу 3.7), а также оператора производной.

Во-первых, заметим, что свободный дираковский лагранжиан $\mathscr{L}_0 = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi$ инвариантен относительно преобразований C, P и T по отдельности. Можно построить более общие квантовые системы, которые нарушают любые из этих симметрий, добавляя к $\mathscr{L}_0$ некоторые возмущения $\delta\mathscr{L}$. Но $\delta\mathscr{L}$ должен быть лоренцевым скаляром, и последняя строка таблицы показывает, что все комбинации $\bar{\psi}$ и ψ, являющиеся лоренцевыми скалярами, инвариантны относительно комбинированной симметрии CPT. На самом деле в общем случае справедливо утверждение, что невозможно построить лоренц-инвариантную квантовую теорию поля с эрмитовым гамильтонианом, в которой нарушается CPT симметрия*.

9.5.14 How does the electron know that some state is occupied and not go there by Pauli's principle?

(вопрос о том, как передается об этом информация? у нас же скорость света конечная. фотон что ли какой-то летит из таких занятых состояний? Или как-то волновая функция размыта и он чувствует, что там всё занято? это похоже на большой вопрос, вообще не знаю, как ответить на него???)

9.5.15 Examples of Problems and Solutions

ПШ-3.1. Группа Лоренца.

Напомним вытекающие из (3.17) коммутационные соотношения для генераторов группы Лоренца:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right)$$

(a) Определим генераторы вращений и собственно лоренцевых преобразований (бустов) как

$$L^i = \frac{1}{2}\varepsilon^{ijk}J^{jk}, \quad K^i = J^{0i}$$

где $i, j, k = 1, 2, 3$. Тогда бесконечно малое преобразование Лоренца можно записать в виде

$$\Phi \to (1 - i\boldsymbol{\theta}\cdot\mathbf{L} - i\boldsymbol{\beta}\cdot\mathbf{K})\Phi$$

Выпишите явно коммутационные соотношения для этих векторных операторов. (Например, $[L^i, L^j] = i\varepsilon^{ijk}L^k$.) Покажите, что комбинации

$$\mathbf{J}_+ = \frac{1}{2}(\mathbf{L} + i\mathbf{K}) \quad \text{и} \quad \%\quad \%\mathbf{J}_- = \frac{1}{2}(\mathbf{L} - i\mathbf{K})$$

коммутируют друг с другом и по отдельности удовлетворяют коммутационным соотношениям углового момента.

(b) Конечномерные представления группы вращений в точности соответствуют допустимым значениям углового момента, который может быть целым или полуцелым числом. Из результата части (a) следует, что все конечномерные представления группы Лоренца соответствуют парам целых или полуцелых чисел (j_+, j_-), соответствующих парам представлений группы вращений. Используя тот факт, что $J = \sigma/2$ в представлении углового момента со спином $1/2$, выпишите явно законы преобразования для двухкомпонентных объектов, преобразующихся по представлениям группы Лоренца, соответствующим парам $\left(\frac{1}{2}, 0\right)$ и $\left(0, \frac{1}{2}\right)$. Покажите, что они в точности соответствуют преобразованиям ψ_L и ψ_R, заданным в (3.37).

(c) Тождество $\sigma^T = -\sigma^2\sigma\sigma^2$ позволяет переписать преобразование для ψ_L в унитарно эквивалентной форме

$$\psi' \to \psi'\left(1 + i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)$$

где $\psi' = \psi_L^T\sigma^2$. Применяя этот закон, можно представить объект, преобразующийся по представлению $\left(\frac{1}{2}, \frac{1}{2}\right)$ в виде 2×2 матрицы, которая включает левый закон преобразования

для ψ_R вместе с правым законом преобразования для транспонированного ψ_L. Параметризуем эту матрицу как

$$\begin{pmatrix} V^0+V^3 & V^1-iV^2 \\ V^1+iV^2 & V^0-V^3 \end{pmatrix}$$

Покажите, что объект V^μ преобразуется как 4-вектор.

ПШ-3.2. Тождество Гордона

Получите тождество Гордона

$$\bar{u}\left(p'\right)\gamma^\mu u(p)=\bar{u}\left(p'\right)\left[\frac{p'^\mu+p^\mu}{2m}+\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p)$$

где $q=(p'-p)$. Мы будем использовать эту формулу в гл. 6.

ПШ-3.3. Произведения спиноров.

(Эта задача, вместе с задачами 5.3 и 5.6, дает эффективный вычислительный метод для процессов, включающих безмассовые частицы.) Пусть k_0^μ, k_1^μ - фиксированные 4-векторы, удовлетворяющие условиям $k_0^2=0,\ k_1^2=-1, k_0\cdot k_1=0$. Определим базисные спиноры следующим способом. Пусть u_{L0} - левополяризованный спинор для фермиона с импульсом k_0. Пусть $u_{R0}=\not{k}_1 u_{L0}$. Тогда для любого изотропного (светоподобного) $(p^2=0)$ 4-импульса p определим

$$u_L(p)=\frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{R0}\%\quad \%\text{ и }\%\quad \%u_R(p)=\frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{L0}$$

Этот набор условий однозначно определяет фазы спиноров (за исключением того случая, когда p параллелен k_0).

(a) Покажите, что $\not{k}_0 u_{R0}=0$. Покажите, что для любого изотропного $p\quad \not{p}u_L(p)=\not{p}u_R(p)=0$.

(b) Для случаев $k_0=(E,0,0,-E), k_1=(0,1,0,0)$, постройте явно $u_{L0}, u_{R0}, u_L(p)$ и $u_R(p)$.

(c) Определим произведение спиноров $s\left(p_1,p_2\right)$ и $t\left(p_1,p_2\right)$ для изотропных p_1,p_2 согласно формуле

$$s\left(p_1,p_2\right)=\bar{u}_R\left(p_1\right)u_L\left(p_2\right),\quad t\left(p_1,p_2\right)=\bar{u}_L\left(p_1\right)u_R\left(p_2\right)$$

Используя явный вид u_λ, заданный в (b), вычислите явно произведения спиноров и покажите, что $t\left(p_1,p_2\right)=\left(s\left(p_2,p_1\right)\right)^*$ и $s\left(p_1,p_2\right)=-s\left(p_2,p_1\right)$. Кроме того, покажите, что

$$\left|s\left(p_1,p_2\right)\right|^2=2p_1\cdot p_2.$$

Таким образом, произведение спиноров является квадратным корнем из скалярного произведения 4-векторов.

ПШ-3.4. Майорановские фермионы.

Напомним, что при помощи (3.40) можно написать релятивистское уравнение для безмассового 2-компонентного фермионного поля, преобразующегося как две верхних компоненты дираковского спинора (ψ_L). Обозначим такое двухкомпонентное поле $\chi_a(x), a = 1, 2$.

(a) Покажите, что уравнение для $\chi(x)$ можно записать как уравнение для массивного поля:

$$i\bar{\sigma} \cdot \partial\chi - im\sigma^2\chi^* = 0$$

Иначе говоря, во-первых, покажите, что это уравнение является релятивистски инвариантным и, во-вторых, что из него следует уравнение Клейна-Гордона $(\partial^2 + m^2)\,\chi = 0$. Этот вид массового слагаемого для фермиона назван майорановским массовым слагаемым.

(b) Следует ли уравнение Майорана из лагранжиана? Казалось бы, массовое слагаемое, получается варьированием выражения $(\sigma^2)_{ab}\,\chi_a^*\chi_b^*$; Однако, так как σ^2 - антисимметричная матрица, это выражение обратилось бы в нуль, если бы $\chi(x)$ было обычным с-числовым полем. Известно, что при переходе к квантовой теории поля $\chi(x)$ становится антикоммутирующим квантовым полем. Следовательно, имеет смысл развивать классическую теорию этого поля, рассматривая $\chi(x)$ как классическое антикоммутирующее поле, то есть как поле, которое принимает значения в грассмановых числах, удовлетворяющих условиям:

$$\alpha\beta = -\beta\alpha \quad \text{для любых } \alpha, \beta.$$

Заметим, что из этих соотношений следует, что $\alpha^2 = 0$. Грассманово поле $\xi(x)$ может быть разложено по базису функций как

$$\xi(x) = \sum_n \alpha_n \phi_n(x),$$

где $\phi_n(x)$ - ортогональные с-числовые функции и α_n - набор независимых грассмановых чисел. Определим комплексное сопряжение произведения грассмановых чисел как обращающее их порядок:

$$(\alpha\beta)^* \equiv \beta^*\alpha^* = -\alpha^*\beta^*.$$

Это правило по форме совпадает с эрмитовым сопряжением квантовых полей. Покажите, что классическое действие

$$S = \int d^4x \left[\chi^\dagger i\bar{\sigma} \cdot \partial\chi + \frac{im}{2}\left(\chi^T\sigma^2\chi - \chi^\dagger\sigma^2\chi^*\right)\right]$$

(где $\chi^\dagger = (\chi^*)^T$) является вещественным ($S^* = S$) и варьирование S относительно χ и χ^* приводит к уравнению Майорана.

(c) Запишем 4-компонентное дираковское поле как

$$\psi(x) = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

и напомним, что нижняя компонента ψ преобразуется по представлению, унитарно эквивалентному комплексно-сопряженному представлению ψ_L. Таким образом, можно переписать 4-компонентное дираковское поле в терминах двух 2-компонентных спиноров:

$$\psi_L(x) = \chi_1(x), \quad \psi_R(x) = i\sigma^2\chi_2^*(x).$$

Перепишите лагранжиан Дирака в терминах χ_1 и χ_2 и обратите внимание на форму массового слагаемого.

(d) Покажите, что действие пункта (c) имеет глобальную симметрию. Вычислите дивергенции токов

$$J^\mu = \chi^\dagger\bar\sigma^\mu\chi, \quad J^\mu = \chi_1^\dagger\bar\sigma^\mu\chi_1 - \chi_2^\dagger\bar\sigma^\mu\chi_2,$$

для теорий пунктов (b) и (c), соответственно, и свяжите результаты с симметриями этих теорий. Постройте теорию из N свободных массивных 2-компонентных фермионных полей с $O(N)$ симметрией (то есть с вращательной симметрией в N-мерном пространстве).

(e) Проквантуйте теорию Майорана пунктов (a) и (b). Иными словами, рассмотрите $\chi(x)$ как квантовое поле, удовлетворяющее каноническому антикоммутационному соотношению

$$\left\{\chi_a(\mathbf{x}), \chi_b^\dagger(\mathbf{y})\right\} = \delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

Постройте эрмитовый гамильтониан и найдите представление канонических коммутационных соотношений, которое диагонализует гамильтониан в терминах набора операторов рождения и уничтожения. (Указание: сравните поле $\chi(x)$ с двумя верхними компонентами проквантованного дираковского поля.)

ПШ-3.5. Суперсимметрия.

Можно написать теории поля с непрерывными симметриями, смешивающими фермионы и бозоны. Соответствующие преобразования названы суперсимметриями.

(a) Простейший пример суперсимметричной теории поля - теория свободных комплексного бозонного и фермионного вейлевского полей с лагранжианом

$$\mathscr{L} = \partial_\mu\phi^*\partial^\mu\phi + \chi^\dagger i\bar\sigma\cdot\partial\chi + F^*F.$$

Здесь F является дополнительным комплексным скалярным полем с уравнением движения $F = 0$. Покажите, что этот лагранжиан является инвариантным (с точностью до полной производной) относительно бесконечно малых преобразований вида

$$\begin{aligned}\delta\phi &= -i\varepsilon^T\sigma^2\chi,\\ \delta\chi &= \varepsilon F + \sigma\cdot\partial\phi\sigma^2\varepsilon^*,\\ \delta F &= -i\varepsilon^\dagger\bar\sigma\cdot\partial\chi,\end{aligned}$$

где параметр ε_a есть 2-компонентный грассмановский спинор.

(b) Покажите, что слагаемое

$$\Delta\mathscr{L} = \left[m\phi F + \frac{1}{2}im\chi^T\sigma^2\chi\right] + (\text{ комплексно-сопряженное })$$

является также лево-инвариантным под действием преобразования, заданного в части (a). Устраните F из полного лагранжиана $\mathscr{L}+\Delta\mathscr{L}$, решая его уравнение движения, и покажите, что бозонное и фермионное поля ϕ и χ имеют одинаковые массы.

(c) Можно написать суперсимметричные нелинейные уравнения поля, добавляя к лагранжиану слагаемые кубического и более высокого порядков. Покажите, что следующая, достаточно общая теория поля, содержащая поля $(\phi_i,\chi_i)\,, i=1,...,n$, является суперсимметричной:

$$\mathscr{L}=\partial_\mu\phi_i^*\partial^\mu\phi_i+\chi_i^\dagger i\bar\sigma\cdot\partial\chi_i+F_i^*F_i+\left(F_i\frac{\partial W[\phi]}{\partial\phi_i}+\frac{i}{2}\frac{\partial^2W[\phi]}{\partial\phi_i\partial\phi_j}\chi_i^T\sigma^2\chi_j+\text{ к.с. }\right)$$

где $W[\phi]$ - произвольная функция поля ϕ_i, называемая суперпотенциалом. Для простого случая $n=1$ и $W=g\phi^3/3$ выпишите уравнения движения для полей ϕ и χ (после устранения F).

ПШ-3.6. Преобразования Фирца.

Пусть $u_i, i=1,...,4$ - четыре 4-компонентных дираковских спинора. В тексте мы доказали формулы перестановки Фирца (3.78) и (3.79). Первая из этих формул может быть записана в 4-компонентных обозначениях как

$$\bar u_1\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_2\bar u_3\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)u_4=-\bar u_1\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u_4\bar u_3\gamma_\mu\left(\frac{1+\gamma^5}{2}\right)u_2.$$

На самом деле подобные формулы перестановки имеют место для любого произведения

$$\left(\bar u_1\Gamma^Au_2\right)\left(\bar u_3\Gamma^Bu_4\right),$$

где Γ^A,Γ^B - любая из 16 комбинаций матриц Дирака, перечисленных в разделе 3.4. (a) Для начала нормируем эти 16 матриц Γ^A по правилу

$$\text{tr}\left[\Gamma^A\Gamma^B\right]=4\delta^{AB}.$$

Это дает $\Gamma^A=\{1,\gamma^0,i\gamma^j,...\}$. Выпишите все 16 элементов этого набора.

(b) Выпишите главное тождество Фирца как уравнение

$$\left(\bar u_1\Gamma^Au_2\right)\left(\bar u_3\Gamma^Bu_4\right)=\sum_{C,D}C_{CD}^{AB}\left(\bar u_1\Gamma^Cu_4\right)\left(\bar u_3\Gamma^Du_2\right)$$

с неизвестными коэффициентами $C^{AB}{}_{CD}$. Используя условие полноты для шестнадцати матриц Γ^A, покажите, что

$$C_{CD}^{AB}=\frac{1}{16}\,\text{tr}\left[\Gamma^C\Gamma^A\Gamma^D\Gamma^B\right].$$

(c) Найдите явно законы преобразования Фирца для произведений $(\bar u_1u_2)\,(\bar u_3u_4)$ и $(\bar u_1\gamma^\mu u_2)\,(\bar u_3\gamma_\mu u_4)$

ПШ-3.7. О дискретных симметриях P, C, и T.

(a) Найдите свойства преобразования под действием P, C, и T преобразований антисимметричной тензорной фермионной билинейной формы $\bar{\psi}\sigma^{\mu\nu}\psi$, где $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Это завершает таблицу свойств преобразования билинейных форм, приведенную в конце главы.

(b) Пусть $\phi(x)$ - комплексное поле Клейна-Гордона, которое было рассмотрено в задаче 2.2. Найдите унитарные операторы P, C и антиунитарный оператор T (определенные в терминах их действия на операторы уничтожения a_p и b_p для частиц и античастиц Клейна-Гордона), которые дают следующие преобразования поля Клейна-Гордона:

$$P\phi(t,\mathbf{x})P = \phi(t,-\mathbf{x})$$
$$T\phi(t,\mathbf{x})T = \phi(-t,\mathbf{x})$$
$$C\phi(t,\mathbf{x})C = \phi^*(t,\mathbf{x})$$

Найдите свойства преобразования компонент тока

$$J^\mu = i\left(\phi^*\partial^\mu\phi - \partial^\mu\phi^*\phi\right)$$

при преобразованиях P, C и T.

(c) Покажите, что любой эрмитовый скалярный по отношению к преобразованиям Лоренца локальный оператор, построенный из полей $\psi(x), \phi(x)$ и их сопряженных, имеет $CPT = +1$.

ПШ-3.8. Связанные состояния.

Две частицы со спином 1/2 могут объединяться в состояние с полным спином, равным либо 0, либо 1. Волновые функции этих состояний при перестановке их спинов будут, соответственно, нечетные или четные.

(a) Используйте эту информацию, чтобы вычислить квантовые числа P - и C-четностей всех связанных состояний электрона и позитрона с S, P или D волновыми функциями. (b) Так как электрон-фотонное взаимодействие задается гамильтонианом

$$\Delta H = \int d^3x e A_\mu j^\mu$$

где j^μ - электрический ток, то электродинамика инвариантна относительно преобразований P и C, если компоненты векторного потенциала имеют такие же P - и C-четности, как и соответствующие компоненты j^μ. Покажите, что отсюда вытекает следующий удивительный факт: основное состояние позитрония со спином 0 может распадаться на 2 фотона, но основное состояние со спином 1 должно распадаться на 3 фотона. Найдите правила отбора для аннигиляции высших состояний позитрония, а также для однофотонных переходов между уровнями позитрония.

9.6 Spinor Field (fermions) 3: Mathematical Details

(this shuold be rewritten, now structure does not match the title, I should go through this theory to find out what to delete and what to move????)

9.6.1 1. Dirac Equation

В этой лекции рассмотрение идёт в четырёхмерном пространстве Минковского, $\mu, \nu, ... = 0, 1, 2, 3, \quad i, j, ... = 1, 2, 3.$

Рассмотрим операторы

$$J^i = -i\varepsilon_{ijk} x^j \partial_k + \frac{1}{2}\sigma^i \tag{1}$$

где

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

• сигма-матрицы Паули. Легко проверить, что

$$\left[J^i, J^j\right] = i\varepsilon_{ijk} J^k \tag{3}$$

Таким образом, оператор $\mathbf{J}$ является оператором полного момента импульса, включая спин частицы, который в рассмотренном случае равен (1/2). Как известно, операторы полного момента J являются генераторами вращений в 3 -мерном пространстве, что выражается формулами

$$\begin{gathered}(1 + i\boldsymbol{\omega}\mathbf{J})\mathbf{x}(1 - i\boldsymbol{\omega}\mathbf{J}) = \mathbf{x} + [\boldsymbol{\omega}, \mathbf{x}], \quad (1 + i\boldsymbol{\omega}\mathbf{J})\boldsymbol{\sigma}(1 - i\boldsymbol{\omega}\mathbf{J}) = \boldsymbol{\sigma} + [\boldsymbol{\omega}, \boldsymbol{\sigma}] \\ (1 + i\boldsymbol{\omega}\mathbf{J})\boldsymbol{\partial}(1 - i\boldsymbol{\omega}\mathbf{J}) = \boldsymbol{\partial} + [\boldsymbol{\omega}, \boldsymbol{\partial}], \quad (1 + i\boldsymbol{\omega}\mathbf{J})\boldsymbol{\sigma}\boldsymbol{\partial}(1 - i\boldsymbol{\omega}\mathbf{J}) = \boldsymbol{\sigma}\boldsymbol{\partial}\end{gathered} \tag{4}$$

где $\boldsymbol{\omega}$ - бесконечно малый вектор угла поворота и $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$.

Задача заключается в обобщении формул (1)-(4) на случай четырёхмерного пространства Минковского. На этом пути строится релятивистская теория частиц со спином (1/2).

Введём четыре комплексные матрицы γ^μ четвёртого порядка со следующими свойствами (сравни с соотношениями $\sigma^i\sigma^j + \sigma^j\sigma^i = 2\delta^{ij}$):

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{5}$$

Чаще всего используются

а) спинорное представление ($\gamma^\mu = (\gamma^0, \gamma^i)$):

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} \tag{6}$$

б) стандартное представление:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{7}$$

Из (5)-(7) вытекает, что

$$\left(\gamma^0\right)^\dagger = \gamma^0, \quad \left(\gamma^i\right)^\dagger = -\gamma^i, \quad \gamma^0\left(\gamma^i\right)^\dagger\gamma^0 = \gamma^i \tag{8}$$

Последние соотношения имеют место в любом представлении, поскольку γ-матрицы в разных представлениях различаются унитарным преобразованием.

Введенные матрицы (5) называются γ-матрицами Дирака.
Пусть $\gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3$. Из этого определения имеем:

$$\left(\gamma^5\right)^\dagger = \gamma^5, \quad \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0, \quad \left(\gamma^5\right)^2 = 1 \tag{9}$$

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} - - - \text{ спинорное представление;}$$

$$\gamma^5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} - - - \text{ стандаотное представление.}$$

Введём набор из шести матриц:

$$\sigma^{\mu\nu} \equiv \frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right] = -\sigma^{\nu\mu} \tag{10}$$

Вследствие (8)

$$\gamma^0 \left(\sigma^{\mu\nu}\right)^\dagger \gamma^0 = -\sigma^{\mu\nu} \tag{11}$$

Используя лишь алгебру (5), находим:

$$\left[\sigma^{\mu\nu}, \gamma^\lambda\right] = \eta^{\nu\lambda}\gamma^\mu - \eta^{\mu\lambda}\gamma^\nu, \quad \left[\sigma^{\mu\nu}, \sigma^{\lambda\rho}\right] = \eta^{\mu\rho}\sigma^{\nu\lambda} - \eta^{\nu\rho}\sigma^{\mu\lambda} - \eta^{\mu\lambda}\sigma^{\nu\rho} + \eta^{\nu\lambda}\sigma^{\mu\rho} \tag{12}$$

Генераторы четырёхмерных вращений $l_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$ удовлетворяют тем же перестановочным соотношениям, что и матрицы $\sigma^{\mu\nu}$. Поэтому для $J_{\mu\nu} = l_{\mu\nu} + \sigma_{\mu\nu}$ имеем:

$$[J_{\mu\nu}, J_{\lambda\rho}] = \eta_{\mu\rho}J_{\nu\lambda} - \eta_{\nu\rho}J_{\mu\lambda} - \eta_{\mu\lambda}J_{\nu\rho} + \eta_{\nu\lambda}J_{\mu\rho} \tag{13}$$

Мы видим, что матрицы $\sigma^{\mu\nu}$ являются обобщением нерелятивистских спиновых матриц $(1/2)\ \sigma^i$ на релятивистский случай.

Пусть поле

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix} = \begin{pmatrix} \varphi(x) \\ \chi(x) \end{pmatrix} \tag{14}$$

является матрицей-столбцом, состоящим из четырёх комплексных полей $\psi_a, a = 1, 2, 3, 4$ и

$$\bar{\psi}(x) = \psi^\dagger(x)\gamma^0 \tag{15}$$

- матрица-строка. Поля φ и χ в (14) являются двухрядными матрицами-столбцами, задающими верхнюю и нижнюю половины поля (14), соответственно. Поля ψ и $\bar{\psi}$ в (14) и (15) взаимно сопряжены (в смысле Дирака).

Уравнение Дирака во внешнем электромагнитном поле имеет вид

$$\left(i\hbar\gamma^\mu\nabla_\mu - mc\right)\psi(x) = 0, \quad \nabla_\mu = \frac{\partial}{\partial x^\mu} + i\frac{e}{\hbar c}A_\mu(x) \tag{16}$$

Далее, как обычно, полагаем $\hbar = c = 1$.

Прежде всего установим факт лоренц-инвариантности уравнения Дирака. Пуст $\omega_{\mu\nu} = -\omega_{\nu\mu}$ - шесть вещественных параметров преобразования Лоренца и

$$(\Omega_\omega)^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu + \frac{1}{2!}\omega^\mu_\lambda\omega^\lambda_\nu + ..., \quad \Lambda_\omega = \exp\left[\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu}\right] \tag{17}$$

Путём прямого вычисления находим:

$$\tilde{x}^\mu = \Lambda_\omega^{-1}x^\mu\Lambda_\omega = (\Omega_\omega)^\mu{}_\nu x^\nu \tag{18a}$$

$$\Lambda_\omega\frac{\partial}{\partial x^\mu}\Lambda_\omega^{-1} = (\Omega_\omega)^\nu{}_\mu\frac{\partial}{\partial x^\nu} \tag{18b}$$

$$\Lambda_\omega^{-1}\gamma^\mu\Lambda_\omega = (\Omega_\omega)^\mu{}_\nu \tag{18c}$$

Равенство (18 a) коротко можно записать в виде $\tilde{x} = \Omega_\omega x$. Из (18 b) и (18 c) следует, что

$$\Lambda_\omega\gamma^\mu\partial_\mu\Lambda_\omega^{-1} = \gamma^\mu\partial_\mu \tag{19}$$

Определим лоренц-преобразованное дираковское поле как

$$\tilde{\psi}(x) \equiv \Lambda_\omega\psi(x) = \left(\exp\frac{1}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\psi\left(\Omega_\omega^{-1}x\right) \tag{20}$$

Здесь второе равенство получено при помощи (18 a). Комбинируя (19) и (20), находим:

$$i\gamma^\mu\partial_\mu\psi(x) = i\Lambda_\omega^{-1}\left(\Lambda_\omega\gamma^\mu\partial_\mu\Lambda_\omega^{-1}\right)\left(\Lambda_\omega\psi(x)\right) = \Lambda_\omega^{-1}\left[i\gamma^\mu\partial_\mu\tilde{\psi}(x)\right] \tag{21}$$

Поэтому

$$\left(i\gamma^\mu\partial_\mu - m\right)\psi(x) = \Lambda_\omega^{-1}\left[\left(i\gamma^\mu\partial_\mu - m\right)\tilde{\psi}(x)\right] \tag{22}$$

откуда следует лоренц-инвариантность свободного уравнения Дирака. Уравнение Дирака во внешнем калибровочном поле также лоренц-инвариантно, поскольку наряду с (19) имеет место равенство

$$\Lambda_\omega\gamma^\mu\nabla_\mu\Lambda_\omega^{-1} = \gamma^\mu\tilde{\nabla}_\mu = \gamma^\mu\left(\partial_\mu + ie\tilde{A}_\mu\right) \tag{23}$$

если оператор лоренц-преобразования Λ_ω преобразует электромагнитное поле согласно формуле (18*b*), так что $\tilde{A}_\mu(x) = (\Omega_\omega)^\nu{}_\mu A_\nu\left(\Omega_\omega^{-1}x\right)$.

Умножим тождество (20) на оператор Λ_ω^{-1} и сделаем затем дираково сопряжение:

$$\bar{\psi}(x) = \tilde{\psi}^\dagger(x)\gamma^0\left(\gamma^0\Lambda_\omega^{-1\dagger}\gamma^0\right) \tag{24}$$

Поскольку вследствие (11)

$$\gamma^0\Lambda_\omega^{-1\dagger}\gamma^0 = \Lambda_\omega \tag{25}$$

то (24) означает, что

$$\bar{\psi}(x) = \bar{\tilde{\psi}}(x)\Lambda_\omega \tag{26}$$

Уравнение Дирака (16) может быть получено путём вариации фермионного действия

$$\mathcal{S}_\psi = \int \mathrm{d}^{(4)}x\bar{\psi}\left(i\gamma^\mu\nabla_\mu - m\right)\psi \tag{27}$$

в котором комплексные поля $\bar{\psi}, \psi$ следует рассматривать как независимые динамические переменные. При помощи формул (22) и (26) устанавливается, что действие (27) лоренц-инвариантно. Вещественность действия (27) устанавливается при помощи равенств (8) и (15).

По определению, вариация действия материи относительно электромагнитного поля даёт электромагнитный ток материи J^μ. В нашем случае имеем:

$$\delta_A\mathcal{S}_\psi \equiv -e\int \mathrm{d}^{(4)}xJ^\mu\delta A_\mu, \quad J^\mu = \bar{\psi}\gamma^\mu\psi$$

9.6.2 2. Nonrelativistic Limit

Перейдём к нерелятивистскому пределу в уравнении Дирака (16). Для этого подействуем на уравнение (16) оператором $(i\gamma^\mu\nabla_\mu + m)$ и учтём, что согласно уравнениям (5) и (10) $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + 2\sigma^{\mu\nu}$. Таким образом получим:

$$\left\{\nabla_\mu\nabla^\mu + \sigma^{\mu\nu}\left[\nabla_\mu, \nabla_\nu\right] + m^2\right\}\psi = 0$$

Но $[\nabla_\mu, \nabla_\nu] = ieF_{\mu\nu}$. При переходе к нерелятивистскому пределу следует сделать замену

$$\psi \to e^{-imt}\psi, \quad \nabla_0 = \frac{\partial}{\partial t} + ieA_0 \to \frac{\partial}{\partial t} + ieA_0 - im, \quad (\nabla_0)^2 \to -m^2 - 2im\left(\frac{\partial}{\partial t} + ieA_0\right)$$

Членами, не содержащими m (точнее $-mc^2$), мы вправе пренебречь в нерелятивистском пределе. С учётом этого последнее уравнение для дираковского поля принимает вид

$$i\dot{\psi} = \left\{-\frac{1}{2m}\nabla_i^2 + \frac{ie}{2m}\sigma^{\mu\nu}F_{\mu\nu} + eA_0\right\}\psi = \left\{-\frac{1}{2m}\nabla_i^2 - \frac{e}{m}\left(\frac{i}{2}\varepsilon_{ijk}\sigma^{jk}\right)H^i + \frac{ie}{m}\sigma^{0i}E^i + eA_0\right\}\psi \tag{28}$$

Здесь учтено, что согласно определению электрическое $\mathbf{E}$ и магнитное $\mathbf{H}$ поля связаны с тензором электромагнитного поля следующими формулами: $E^i = F^{io}, F_{ij} = -\varepsilon_{ijk}H^k$. Далее работаем в стандартном представлении (7), в котором

$$\sigma^{ij} = -\frac{i}{2}\varepsilon_{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}, \quad \sigma^{0i} = \frac{1}{2}\begin{pmatrix}0 & \sigma^i\\ \sigma^i & 0\end{pmatrix} \tag{29}$$

Рассмотрим плоскую волну

$$\psi_k = e^{-imt-ikx}u_k, \quad u_k = \text{ const}$$

Для покоящейся частицы ($k^\mu = 0$) свободное уравнение Дирака принимает вид

$$\left(\gamma^0 - 1\right)u = 0$$

то есть $\chi = 0$ (см. (14)). Нетрудно проследить, что при малых k вследствие уравнения Дирака имеет место оценка $\chi_k \sim (|\mathbf{k}|/m)\varphi_k$, и потому в нерелятивистском пределе χ следует

положить равным нулю. Поэтому из (28) для верхней компоненты φ следует уравнение Паули:

$$i\frac{\partial}{\partial t}\varphi = \left\{-\frac{1}{2m}\nabla_i^2 - \frac{e}{m}\left(\frac{\boldsymbol{\sigma}}{2}\right)\mathbf{H} + eA_0\right\}\varphi \tag{30}$$

Так как оператор спина электрона $\mathbf{s} = (1/2)\boldsymbol{\sigma}$ (см. задачу 2. в)), то уравнение Паули показывает, что в рассмотренном приближении магнитный момент электрона

$$\boldsymbol{\mu} = \frac{e}{m}\mathbf{s} \tag{31}$$

9.6.3 3. Quantization of the Dirac Field

Пусть $\dot{\psi} = (\partial/\partial t)\psi$. Определим стандартным образом обобщённый импульс дираковского поля:

$$\delta_{\dot{\psi}}\mathcal{S}_\psi = \int \mathrm{d}t \int \mathrm{d}^{(3)}x\pi_\psi\delta\dot{\psi}$$

Отсюда и из (27) имеем:

$$\pi_\psi = i\bar{\psi}\gamma^0 = i\psi^\dagger \tag{32}$$

Дираковский гамильтониан имеет вид

$$\mathcal{H}_\psi = \int \mathrm{d}^{(3)}x\pi_\psi\dot{\psi} - \mathcal{L} = \int \mathrm{d}^{(3)}x\psi^\dagger\left(-i\alpha^i\nabla_i + m\gamma^0 + eA_0\right)\psi, \quad \alpha^i \equiv \gamma^0\gamma^i, \quad \left(\alpha^i\right)^\dagger = \alpha^i \tag{33}$$

Оператор в круглых скобках в (33) - эрмитовский. Поэтому имеет смысл задача о собственных функциях этого оператора. Рассмотрим эту задачу для случая свободного дираковского поля, когда $A_\mu = 0$, что предполагается до конца этого пункта.

Уравнение для собственных функций оператора Гамильтона-Дирака в свободной теории имеет вид

$$\left(-i\alpha^i\partial_i + m\gamma^0\right)\psi_{\mathbf{k}} = \epsilon_{\mathbf{k}}\psi_{\mathbf{k}} \tag{34}$$

Решение этого уравнения ищется в виде бегущих волн $\psi_{\mathbf{k}}(x) = u_{\mathbf{k}}e^{i\mathbf{kx}}$, амплитуды которых удовлетворяют уравнениям

$$\left(\boldsymbol{\alpha}\mathbf{k} + m\gamma^0 - \epsilon_{\mathbf{k}}\right)u_{\mathbf{k}} = 0 \tag{35}$$

Умножая последнее уравнение на $(\boldsymbol{\alpha}\mathbf{k} + m\gamma^0 + \epsilon_{\mathbf{k}})$ и учитывая, что

$$\gamma^0\boldsymbol{\alpha} + \boldsymbol{\alpha}\gamma^0 = 0, \quad \alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij} \tag{36}$$

получаем

$$\left(\epsilon_{\mathbf{k}}^2 - m^2 - \mathbf{k}^2\right)u_{\mathbf{k}} = 0$$

Отсюда видно, что при $\epsilon_{\mathbf{k}} = \pm\sqrt{m^2 + \mathbf{k}^2}$ уравнение (35) имеет решение. Если $u_{\mathbf{k}}$ удовлетворяет уравнению (35), то $\tilde{u}_{\mathbf{k}} = \gamma^0\gamma^5 u_{\mathbf{k}}$ удовлетворяет уравнению

$$\left(\boldsymbol{\alpha}\mathbf{k} + m\gamma^0 + \epsilon_{\mathbf{k}}\right)\tilde{u}_{\mathbf{k}} = 0 \tag{37}$$

Поэтому имеется два независимых решения уравнения (35) с $\epsilon_{\mathbf{k}} = \sqrt{m^2 + \mathbf{k}^2} = \omega_k$ и два независимых решения с $\epsilon_{\mathbf{k}} = -\sqrt{m^2 + \mathbf{k}^2} = -\omega_k$. Эти решения называются положительно и отрицательно
частотными. Далее положительно частотные решения с амплитудой u_{k} обозначается u_k, а отрицательно частотные решения с амплитудой $u_{(-\mathbf{k})}$ обозначается u_{-k}. Для любого 4 -вектора p^μ введём обозначение $\gamma^\mu p_\mu \equiv \not{p}$. Тогда уравнение (35) для положительно и отрицательно частотных мод записывается как

$$(\not{k} - m)u_k = 0 \tag{38a}$$

$$(\not{k} + m)u_{-k} = 0 \tag{38b}$$

где $k^\mu = (\omega_k, \mathbf{k})$. Заметим, что если ψ_1 и ψ_2 - дираковские спиноры, то $\bar{\psi}_1\psi_2$ есть лоренцинвариант, а $\bar{\psi}_1\gamma^\mu\psi_2$ – 4-вектор. Поэтому мы примем лоренц-инвариантные условия нормировки:

$$\bar{u}_k u_k = 2m, \quad \bar{u}_{-k}u_{-k} = -2m \tag{39}$$

При этом предполагается, что спиноры $u_{\pm k}$ получаются (с точностью до фазового множителя) из спиноров в системе покоя частицы при помощи соответствующего преобразования Лоренца. Равенства (39) согласованы с равенствами

$$\bar{u}_k\gamma^\mu u_k = \bar{u}_{-k}\gamma^\mu u_{-k} = 2k^\mu \tag{40}$$

Для проверки равенств (39)-(40) заметим, что в системе покоя частицы, где $k^\mu = (m, \mathbf{0})$, уравнения (38) принимают вид $\gamma^0 u_{\pm k} = \pm u_{\pm k}$, из которых непосредственно вытекают следующие соотношения:

$$u^\dagger_{\pm k} = \pm\bar{u}_{\pm k}, \quad \gamma^0\gamma u_{\pm k} = \mp\gamma u_{\pm k}$$

Умножая скалярно соответственно левые и правые части последних уравнений, получаем:

$$\bar{u}_{\pm k}\gamma u_{\pm k} = -\bar{u}_{\pm k}\gamma u_{\pm k} = 0$$

Таким образом в системе покоя частицы

$$\bar{u}_{\pm k}\gamma^\mu u_{\pm k} = \left(u^\dagger_{\pm k}u_{\pm k}, 0, 0, 0\right)$$

Так как $u^\dagger_{\pm k}u_{\pm k} > 0$, то примем нормировку $u^\dagger_{\pm k}u_{\pm k} = 2m$. Тогда равенства (40) в любой системе отсчёта получаются путём преобразования Лоренца. Теперь умножим равенства (38) на $\bar{u}_{\pm k}$ соответственно и учтём (40):

$$m\bar{u}_{\pm k}u_{\pm k} = \pm\bar{u}_{\pm k}\gamma^\mu u_{\pm k}k_\mu = \pm 2m^2$$

Полученные равенства равносильны равенствам (39).

При помощи явных вычислений легко найти, что в стандартном представлении (7) решения уравнений (38) имеют вид

$$u_k = \begin{pmatrix} \sqrt{\omega_k + m}w \\ \sqrt{\omega_k - m}(\mathbf{n}\boldsymbol{\sigma})w \end{pmatrix}, \quad u_{-k} = \begin{pmatrix} \sqrt{\omega_k - m}(\mathbf{n}\boldsymbol{\sigma})w' \\ \sqrt{\omega_k + m}w' \end{pmatrix}$$
$$\mathbf{n} = \frac{\mathbf{k}}{|\mathbf{k}|}, \quad w^\dagger w = 1, \quad w'^\dagger w' = 1 \tag{41}$$

В системе покоя ($\omega_k = m$) имеем:

$$u_k = \sqrt{2m}\begin{pmatrix} w \\ 0 \end{pmatrix}, \quad u_{-k} = \sqrt{2m}\begin{pmatrix} 0 \\ w' \end{pmatrix} \tag{42}$$

то есть $w\,(w')$ представляет собой тот двухкомпонентный спинор в трёхмерном пространстве, к которому сводится в нерелятивистском пределе амплитуда волны. Сопрягая (41), получаем:

$$\bar{u}_k = \left(\sqrt{\omega_k + m}w^\dagger, -\sqrt{\omega_k - m}w^\dagger(\mathbf{n}\boldsymbol{\sigma})\right), \quad \bar{u}_{-k} = \left(\sqrt{\omega_k - m}w'^\dagger(\mathbf{n}\boldsymbol{\sigma}), -\sqrt{\omega_k + m}w'^\dagger\right) \tag{43}$$

При помощи формул (41) и (43) равенства (39) и (40) получаются непосредственно. Ортонормированный базис в спинорном пространстве $\{w\}\,(\{w'\})$ состоит из двух элементов, которые выбираются ортонормированными:

$$w^\alpha = \sum_{s=1,2} a_s w^\alpha_{(s)}, \quad \alpha = 1,2, \quad \sum_{\alpha=1,2} w^{\alpha *}_{(s)} w^\alpha_{(s')} = \delta_{ss'}, \quad \sum_{s=1,2} w^\alpha_{(s)} w^{\beta *}_{(s)} = \delta_{\alpha\beta} \tag{44}$$

Везде $\alpha, \beta, \ldots$ - спинорные индексы, а индексы $s, s', \ldots$ нумеруют спиноры. Аналогичные формулы имеют место для штрихованных спиноров. Обозначим через $u_{\pm k}s, s = 1,2$ дираковские спиноры (41), в которых сделаны подстановки $w \to w_{(s)}, w' \to w'_{(s)}$. Тогда при помощи (40) и (41) получаем:

$$u^\dagger_{ks} u_{ks'} = 2\omega_k \delta_{ss'}, \quad u^\dagger_{-ks} u_{-ks'} = 2\omega_k \delta_{ss'}, \quad u^\dagger_{ks} u_{-ps'} = 0 \tag{45}$$

Последнее равенство в (45) является следствием того, что $u^\dagger_{ks}$ и $u_{-ps'}$ являются собственными векторами эрмитовой матрицы (35) с разными собственными значениями.

Теперь мы можем выписать квантованное дираковское поле:

$$\psi(x) = \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{s=1,2} \left[a_{\mathbf{k}s} u_{ks} e^{i\mathbf{kx}} + b^\dagger_{\mathbf{k}s} u_{-ks} e^{-i\mathbf{kx}} \right]$$
$$\bar{\psi}(x) = \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{s=1,2} \left[b_{\mathbf{k}s} \bar{u}_{-ks} e^{i\mathbf{kx}} + a^\dagger_{\mathbf{k}s} \bar{u}_{ks} e^{-i\mathbf{kx}} \right] \tag{46}$$

Далее показывается, что множество величин $\left\{a_{\mathbf{k}s}, b_{\mathbf{k}s}, a^\dagger_{\mathbf{k}s}, b^\dagger_{\mathbf{k}s}\right\}$ являются фермиевскими операторами уничтожения и рождения. При помощи соотношений (38) и (45) вычисляем значение свободного гамильтониана (33) на полях (46):

$$\mathcal{H}_{0\psi} = \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3} \omega_k \sum_{s=1,2} \left(a^\dagger_{\mathbf{k}s} a_{\mathbf{k}s} - b_{\mathbf{k}s} b^\dagger_{\mathbf{k}s} \right) \tag{47}$$

Находим также оператор электрического заряда:

$$Q = e\int \mathrm{d}^{(3)}xJ^0 = e\int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\sum_{s=1,2}\left(a^\dagger_{\mathbf{k}s}a_{\mathbf{k}s} + b_{\mathbf{k}s}b^\dagger_{\mathbf{k}s}\right) \tag{48}$$

Обратим внимание на то, что условия квантования, определяющие фермионный вакуум, должны иметь вид

$$a_{\mathbf{k}s}|0\rangle = 0, \quad b_{\mathbf{k}s}|0\rangle = 0 \tag{49}$$

Действительно, пусть $x^0 > y^0$ и рассмотрим величину

$$\langle 0|\psi(x)\bar{\psi}(y)|0\rangle \tag{50}$$

где $\psi(x)$ и $\bar{\psi}(y)$ - дираковские поля в представлении Гейзенберга. Гейзенберговские поля представляются согласно (46) с той лишь поправкой, что в (46) производятся замены $\exp(\pm i\mathbf{kx}) \to \exp(\mp ikx)$, где $kx = \omega_k x^0 - \mathbf{kx}$. Действительно, вследствие (38) в результате этой замены экспонент поля (46) удовлетворяют свободному уравнению Дирака. Поэтому величина (50) равна, как и в скалярной теории, амплитуде распространения частицы, рождённой в точке y и распространяющейся в точку x. Вследствие условий (49) эта амплитуда является суперпозицией амплитуд с импульсом $\mathbf{k}$, содержащих множитель $\exp\left[-i\omega_k\left(x^0 - y^0\right)\right]$, то есть суперпозицией положительночастотных волн. Такая зависимость от времени означает, что распространяется частица с положительной энергией. Амплитуда распространения скалярной частицы из точки y в точку x (см. лекцию 2) $\langle 0|\phi(x)\phi(y)|0\rangle$ содержит такую же зависимость от времени. Условия квантования, отличные от (49), нарушили бы эту закономерность. Проведенное рассмотрение подтверждает правильность условий квантования (49).

Так как $\pi_\psi = i\psi^\dagger$, то одновременные перестановочные соотношения дираковских полей должны иметь вид либо

$$\left[\psi_a(x), \psi^\dagger_b(y)\right] = \delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{51}$$

либо

$$\left\{\psi_a(x), \psi^\dagger_b(y)\right\} \equiv \psi_a(x)\psi^\dagger_b(y) + \psi^\dagger_b(y)\psi_a(x) = \delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{52}$$

Вариант (51) реализуется, если (выписываются лишь ненулевые (анти)коммутаторы)

$$\left[a_{\mathbf{k}s}, a^\dagger_{\mathbf{p}s'}\right] = \left[b^\dagger_{\mathbf{k}s}, b_{\mathbf{p}s'}\right] = (2\pi)^3\delta_{ss'}\delta^{(3)}(\mathbf{k}-\mathbf{p}) \tag{53}$$

а вариант (52) - если

$$\left\{a_{\mathbf{k}s}, a^\dagger_{\mathbf{p}s'}\right\} = \left\{b_{\mathbf{k}s}, b^\dagger_{\mathbf{p}s'}\right\} = (2\pi)^3\delta_{ss'}\delta^{(3)}(\mathbf{k}-\mathbf{p}) \tag{54}$$

Действительно, для (51) при помощи (53) и для (52) при помощи (54) получаем:

$$\int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\frac{1}{2\omega_k}\sum_{s=1,2}\left[e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}u_{ks}\bar{u}_{ks} + e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}u_{-ks}\bar{u}_{-ks}\right]\gamma^0 \tag{55}$$

Теперь учтём, что

$$\sum_{s=1,2} u_{\pm ks}\bar{u}_{\pm ks} = \not{k} \pm m \tag{56}$$

Соотношения (56) следуют из равенств

$$\bar{u}_{\pm ks}u_{\pm ks'} = \pm 2m\delta_{ss'}, \quad \bar{u}_{ks}u_{-ks'} = 0 \tag{57}$$

уточняющих условия нормировки (39), и уравнений (38): при помощи (38) и (57) немедленно устанавливается, что все матричные элементы вида $\bar{u}_{\pm ks'}...u_{\pm ks''}4\times 4$ матриц в левых и правых частях равенств (56) совпадают. При помощи (56) выражение (55) преобразуется к виду

$$\int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\frac{1}{2\omega_k}\sum_{s=1,2}\left[e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\left(\omega_k\gamma^0 - \gamma\mathbf{k} + m\right) + e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\left(\omega_k\gamma^0 - \gamma\mathbf{k} - m\right)\right]\gamma^0 = \delta^{(3)}(\mathbf{x}-\mathbf{y})$$

что и требовалось.

Теперь покажем, что коммутационные соотношения (53) неприемлемы с физической точки зрения. Действительно, с учётом условий квантования (49) КС (53) приводят к индефинитной метрике в пространстве состояний. Например, если $|\mathbf{k}s\rangle = b^\dagger_{\mathbf{k}s}|0\rangle$, то $\langle\mathbf{k}s|\mathbf{k}s\rangle < 0$. Поэтому остаётся единственная возможность (52), (54). Выпишем дополняющие (52), (54) одновременные антикоммутационные соотношения:

$$\begin{gathered}\{\psi_a(x),\psi_b(y)\} = \left\{\bar{\psi}_a(x),\bar{\psi}_b(y)\right\} = 0\\ \{a_{\mathbf{k}s},a_{\mathbf{p}s'}\} = \{b_{\mathbf{k}s},b_{\mathbf{p}s'}\} = \left\{a^\dagger_{\mathbf{k}s},a^\dagger_{\mathbf{p}s'}\right\} = \left\{b^\dagger_{\mathbf{k}s},b^\dagger_{\mathbf{p}s'}\right\} = \{a_{\mathbf{k}s},b_{\mathbf{p}s'}\} = \left\{a_{\mathbf{k}s},b^\dagger_{\mathbf{p}s'}\right\} = 0\end{gathered} \tag{58}$$

Приведём оператор Гамильтона-Дирака (47) к нормальному виду при помощи (54) (сравни с (38 7)) :

$$\mathcal{H}_{0\psi} = \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\omega_k\sum_{s=1,2}\left(a^\dagger_{\mathbf{k}s}a_{\mathbf{k}s} + b^\dagger_{\mathbf{k}s}b_{\mathbf{k}s}\right) + \mathcal{E}_{0D}, \quad \mathcal{E}_{0D} = -2V\int\frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\omega_k \tag{59}$$

где V - объём пространства. Аналогично для электрического заряда (48) получаем:

$$Q = e\int\frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\sum_{s=1,2}\left(a^\dagger_{\mathbf{k}s}a_{\mathbf{k}s} - b^\dagger_{\mathbf{k}s}b_{\mathbf{k}s}\right) + Q_{0D}, \quad Q_{0D} = 2eV\int\frac{\mathrm{d}^{(3)}k}{(2\pi)^3} \tag{60}$$

Из (59) и (60) мы видим, что плотность энергии и заряда в теории Дирака равны $(-\infty)$ (поскольку $e<0$). Если эти постоянные значения $\mathcal{E}_{0D}$ и Q_{0D} вычеркнуть, то энергия любого фермионного состояния (за исключением вакуумного с нулевой энергией) положительна, а заряд равен $e\times$ (число частиц - число античастиц). Считается, что частицы порождаются операторами $a^\dagger$, а античастицы - операторами $b^\dagger$. Первые называются электронами, а вторые - позитронами. Например, одноэлектронное состояние

$$|\mathbf{p}s\rangle = \sqrt{2\omega_p}a^\dagger_{\mathbf{p}s}|0\rangle, \quad \langle\mathbf{q}s'|\mathbf{p}s\rangle = (2\pi)^3 2\omega_p\delta_{ss'}\delta^{(3)}(\mathbf{p}-\mathbf{q})$$

а однопозитронное состояние

$$|(+)\mathbf{p}s\rangle = \sqrt{2\omega_p} b^\dagger_{\mathbf{p}s}|0\rangle$$

Оператор импульса дираковского поля (сравни с (42 Л2))

$$\mathcal{P}_{0\psi} = -i\int \mathrm{d}^{(3)}x\psi^\dagger\nabla\psi = \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\mathbf{k}\sum_{s=1,2}\left(a^\dagger_{\mathbf{k}s}a_{\mathbf{k}s} + b^\dagger_{\mathbf{k}s}b_{\mathbf{k}s}\right)$$

Имеем:

$$\begin{aligned}
&\mathcal{H}_{0\psi}|0\rangle = \mathcal{P}_{0\psi}|0\rangle = 0\\
&\mathcal{H}_{0\psi}|\mathbf{p}s\rangle = \omega_p|\mathbf{p}s\rangle, \quad \mathcal{H}_{0\psi}|(+)\mathbf{p}s\rangle = \omega_p|(+)\mathbf{p}s\rangle\\
&\mathcal{P}_{0\psi}|\mathbf{p}s\rangle = \mathbf{p}|\mathbf{p}s\rangle, \quad \mathcal{P}_{0\psi}|(+)\mathbf{p}s\rangle = \mathbf{p}|(+)\mathbf{p}s\rangle
\end{aligned}$$

Таким образом, операторы рождения порождают состояния с теми импульсами, которые содержатся в обозначениях этих операторов. Из (58) следует, что $(a_{\mathbf{k}s})^2 = (b_{\mathbf{k}s})^2 = 0$. Последние равенства означают, что в фермионной системе в одном состоянии не может находиться более одной частицы. Таким образом, имеется фундаментальное различие между бозонными и фермионными системами.

Описанная ситуация в теории Дирака интерпретируется следующим образом. В теории имеется равное число состояний с положительной и отрицательной энергией. Вакуумное состояние - это такое состояние, в котором все состояния с отрицательной энергией заполнены, а все состояния с положительной энергией - пустые. Электронные состояния получаются путём заполнения положительноэнергетических уровней, а позитронные - путём освобождения (создания дырок) некоторых заселённых отрицательночастотных состояний. При этом энергия и заряд относительно основного состояния возрастают (помните, что $e < 0$). Поэтому энергия и заряд позитронов положительны. Заметим, что антикоммутационные соотношения симметричны относительно перестановок операторов рождения и уничтожения. Поэтому ответ на вопрос, какие операторы являются операторами рождения, а какие - операторами уничтожения, а также как заполняется вакуум, определяется общим знаком перед гамильтонианом.

Обозначим через $|f;0\rangle$ ложный вакуум, то есть такое состояние, в котором и положительно- и отрицательночастотные состояния не заполнены. Такое состояние удовлетворяет условиям $a_{\mathbf{k}s}|f;0\rangle = 0, b^\dagger_{\mathbf{k}s}|f;0\rangle = 0$. Это - действительно не заполненное состояние, поскольку оно обращает в нуль операторы Гамильтона (47) и заряда (48). Кроме того, $\psi|f;0\rangle = 0$. В теории с таким вакуумом следовало бы сделать переобозначение $b^\dagger_{\mathbf{k}s} \longleftrightarrow b_{\mathbf{k}s}$. Мы видим, что с физической точки зрения такая теория неудовлетворительна, поскольку в ней существуют и распространяются во времени состояния с отрицательной энергией.

9.6.4 4. Algebra of Dirac Matrices

Из соотношений (5) вытекает ряд соотношений между γ-матрицами, необходимыми при конкретных вычислениях. Напрмер,

$$\eta_{\mu\nu}\gamma^\mu\gamma^\nu \equiv \gamma_\mu\gamma^\mu = 4 \tag{61}$$

где $\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$. Далее:

$$\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu, \quad \gamma_\mu\gamma^\lambda\gamma^\nu\gamma^\mu = 4\eta^{\lambda\nu}, \quad \gamma_\mu\gamma^\lambda\gamma^\nu\gamma^\rho\gamma^\mu = -2\gamma^\rho\gamma^\nu\gamma^\lambda$$
$$\gamma_\mu\gamma^\lambda\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu = 2\left(\gamma^\sigma\gamma^\lambda\gamma^\nu\gamma^\rho + \gamma^\rho\gamma^\nu\gamma^\lambda\gamma^\sigma\right) \tag{62}$$

Введём обозначение

$$T^{\mu_1\mu_2\cdots\mu_n} = \frac{1}{4}\operatorname{tr}\left(\gamma^{\mu_1}\gamma^{\mu_2}...\gamma^{\mu_n}\right) \tag{63}$$

Очевидно

$$4T^\mu = \operatorname{tr}\gamma^\mu = 0 \tag{64}$$

так как из (18c) следует, что $\operatorname{tr}\gamma^\mu = (\Omega_\omega)^\mu{}_\nu \operatorname{tr}\gamma^\nu$. Из (5) находим, что

$$T^{\mu\nu} = \eta^{\mu\nu} \tag{65}$$

(след единичной 4×4-матрицы равен 4). Протаскивая в $T^{\lambda\mu\nu\rho}$ матрицу γ^λ под знаком следа слева направо и пользуясь при этом (5) и (65), получаем

$$T^{\lambda\mu\nu\rho} = \eta^{\lambda\mu}\eta^{\nu\rho} - \eta^{\lambda\nu}\eta^{\mu\rho} + \eta^{\lambda\rho}\eta^{\mu\nu} \tag{66}$$

Отсюда следует, что

$$\operatorname{tr}\gamma^5 = 0 \tag{67}$$

Имеем также:

$$\frac{1}{4}\operatorname{tr}\gamma^5\gamma^\lambda\gamma^\mu\gamma^\nu\gamma^\rho = i\varepsilon^{\lambda\mu\nu\rho} \tag{68}$$

Отметим, что след от любого нечётного числа γ-матриц равен нулю.

9.6.5 5. Discrete Symmetries of the Dirac Equation

A. Пространственная инверсия (-чётностъ).

В результате этой инверсии пространственные компоненты 4-векторов изменяют знак, а временные компоненты остаются неизменными. Пусть штрих сверху отмечает преобразованные величины. Тогда

$$x'^0 = x^0, \quad \mathbf{x}' = -\mathbf{x}, \quad J'^0\left(x^0,\mathbf{x}\right) = J^0\left(x^0,-\mathbf{x}\right), \quad \mathbf{J}'\left(x^0,\mathbf{x}\right) = -\mathbf{J}\left(x^0,-\mathbf{x}\right) \tag{69}$$

Кроме того мы потребуем, чтобы преобразованное дираковское поле так же, как и исходное поле, удовлетворяло уравнению Дирака.

Легко проверить, что если

$$\psi'\left(x^0,\mathbf{x}\right) \equiv \psi_P\left(x^0,\mathbf{x}\right) = \gamma^0\psi\left(x^0,-\mathbf{x}\right) \tag{70}$$

то все перечисленные требования удовлетворяются.

Действительно, для $J'^\mu(x) = \bar\psi'(x)\gamma^\mu\psi'(x)$ имеем соотношения (69) вследствие свойств γ матриц (5) и (8). Кроме того, из уравнения Дирака вытекает уравнение

$$\left(i\gamma^{\mu}\nabla'_{\mu} - m\right)\psi_P\left(x^0, \mathbf{x}\right) = 0 \tag{71}$$

где $\nabla'_0 = \nabla_0, \nabla'_i = -\nabla_i$. Наоборот, из последнего уравнения следует уравнение Дирака для поля $\psi(x)$. Аналогично показывается, что

$$(\bar{\psi}\psi)'\left(x^0, \mathbf{x}\right) = (\bar{\psi}\psi)\left(x^0, -\mathbf{x}\right)$$

то есть величина $(\bar{\psi}\psi)$ является скалярной относительно преобразования чётности.

Повторное применение преобразования чётности приводит к тождественному преобразованию.

Б. Обращение времени (Т-чётностъ).

При обращении времени имеем:

$$\begin{gathered} x'^0 = -x^0, \quad \mathbf{x}' = \mathbf{x}, \quad J'^0\left(x^0, \mathbf{x}\right) = J^0\left(-x^0, \mathbf{x}\right), \quad \mathbf{J}'\left(x^0, \mathbf{x}\right) = -\mathbf{J}\left(-x^0, \mathbf{x}\right) \\ A'^0\left(x^0, \mathbf{x}\right) = A^0\left(-x^0, \mathbf{x}\right), \quad \mathbf{A}'\left(x^0, \mathbf{x}\right) = -\mathbf{A}\left(-x^0, \mathbf{x}\right) \end{gathered} \tag{72}$$

Уравнение Дирака может сохранять свой вид лишь если дираковское поле сопрягается комплексно, что следует из уравнения Паули, в которое переходит уравнение Дирака в нерелятивистском пределе. О необходимости в данном случае эрмитовского сопряжения состояний и операторов говорит также тот факт, что при обращении времени начальное и конечное состояния меняются местами, но в амплитуды перехода первые входят в бра-представлении, а вторые - в кет-представлении.

Всем этим требованиям удовлетворяет преобразование дираковского поля вида

$$\psi'\left(x^0, \mathbf{x}\right) \equiv \psi_T\left(x^0, \mathbf{x}\right) = U_T\bar{\psi}^t\left(-x^0, \mathbf{x}\right), \quad \bar{\psi}_T\left(x^0, \mathbf{x}\right) = -\psi^t\left(-x^0, \mathbf{x}\right)U_T \tag{73}$$

где в спинорном и стандартном представлениях

$$U_T = \gamma^3\gamma^1\gamma^0, \quad U_T^2 = -1 \tag{74}$$

Имеем

$$U_T\gamma^{0t} = \gamma^0 U_T, \quad U_T\gamma^t = -\gamma U_T \tag{75}$$

Поскольку при обращении времени переставляются начальное и конечное состояния, то формулы преобразования (72)-(75) должны быть дополнены правилом, согласно которому всякое произведение операторов преобразуется в произведение преобразованных операторов в обратном порядке. При помощи формул (73)-(75) устанавливаем, что поле ψ_T удовлетворяет уравнению Дирака, если поле $\bar{\psi}$ удовлетворяет сопряжённому уравнению Дирака, и наоборот. Кроме того, согласно (73)-(75) и указанному правилу

$$\left(\bar{\psi}\gamma^{\mu}\psi\right)_T\left(x^0, \mathbf{x}\right) = -\left(\psi^t U_T\gamma^{\mu}U_T\bar{\psi}^t\right)^t\left(-x^0, \mathbf{x}\right) = \eta_{\mu\mu}\left(\psi^t\gamma^{\mu t}\bar{\psi}^t\right)^t\left(-x^0, \mathbf{x}\right) = \eta_{\mu\mu}\left(\bar{\psi}\gamma^{\mu}\psi\right)\left(-x^0, \mathbf{x}\right) \tag{76}$$

Здесь отсутствует суммирование по индексу μ.

В. Зарядовое сопряэнение (С-чётностъ).

Из (46) и (49) видно, что поля ψ и $\bar{\psi}$ уничтожают частицу и античастицу, соответственно, которые имеют взаимно противоположные заряды. Поэтому операция

зарядового сопряжения должна сопровождаться операцией дираковского сопряжения полей. При этом векторный ток должен менять знак:

$$J'^{\mu}(x) \equiv J_C(x) = -J^{\mu}(x) \tag{77}$$

Следует также считать, что при зарядовом сопряжении изменяет знак либо электрический заряд, либо 4-потенциал. Кроме того, исходные и зарядово-сопряжённые поля должны одновременно удовлетворять уравнению Дирака.

Все указанные требования выполнены, если

$$\psi'(x) \equiv \psi_C(x) = U_C\bar{\psi}^t(x), \quad \bar{\psi}_C = -\psi^t(x)U_C^{-1} \tag{78}$$

причём в спинорном и стандартном представлениях

$$U_C = i\gamma^2\gamma^0, \quad U_C^{-1}\gamma^{\mu}U_C = -\gamma^{\mu t}, \quad U_C^{-1}\gamma^5 U_C = \gamma^{5t}, \quad U_C^{-1}\sigma^{\mu\nu}U_C = -\sigma^{\mu\nu t} \tag{79}$$

При помощи (79) легко проверить, что после преобразования (78) и изменения знака 4 -потенциала уравнение Дирака переходит в уравнение Дирака для зарядово сопряжённого поля. Кроме того,

$$J'^{\mu} = -\psi^t U_C^{-1}\gamma^{\mu}U_C\bar{\psi}^t = \psi^t\gamma^{\mu t}\bar{\psi}^t$$

Отсюда и из свойства антикоммутативности (52) следует равенство (77).

Повторное применение операции зарядового сопряжения приводят к тождественному преобразованию.

Сформулируем понятие "вещественности"дираковского поля. "Вещественное"дираковское поле называется майорановским полем.

Определение. Дираковское поле называется майорановским, если операция зарядового сопряжения (78) не изменяет это поле, то есть если

$$\psi(x) = U_C\bar{\psi}^t(x) = i\gamma^2\psi^{\dagger t}(x) \tag{80}$$

Формула (80) имеет смысл в спинорном и стандартном представлениях.

Так как уравнение Дирака инвариантно относительно зарядового сопряжения, то майорановское поле может быть решением уравнения Дирака: если поле ψ удовлетворяет уравнению Дирака, то и поле $U_C\bar{\psi}^t$ также удовлетворяет уравнению Дирака. Поэтому и майорановское поле $\left(\psi + U_C\bar{\psi}^t\right)$ удовлетворяет уравнению Дирака.

9.6.6 6. Weyl Equation

Пусть поле ψ удовлетворяет уравнению Дирака с нулевой массой ($m = 0$). Тогда этому же уравнению удовлетворяет и поле $\gamma^5\psi$, а также поля

$$\psi_{\pm} = \frac{1}{2}\left(1 \mp \gamma^5\right)\psi \tag{81}$$

Два поля (81) называются правым и левым вейлевскими полями, соответственно.

В спинорном представлении (6)

$$\psi_+ = \begin{pmatrix}\varphi\\0\end{pmatrix}, \quad \psi_- = \begin{pmatrix}0\\\chi\end{pmatrix} \tag{82}$$

и уравнения Дирака для вейлевских полей имеют вид

$$i\left(\partial_0+\sigma^i\partial_i\right)\varphi=0,\quad i\left(\partial_0-\sigma^i\partial_i\right)\chi=0 \tag{83}$$

Уравнения (83) называются уравнениями Вейля.

Так как генераторы преобразования Лоренца $\sigma^{\mu\nu}$ (10) коммутируют с проекционными операторами $(1/2)\,(1\pm\gamma^5)$, то уравнения Вейля Лоренц-инвариантны.

9.6.7 7. Exact Solution of the Dirac Equation in a Coulomb Potential

Рассмотрим стационарное уравнение Гамильтона-Дирака в кулоновом потенциале (см. (33), (34)):

$$\left(-i\boldsymbol{\alpha}\boldsymbol{\partial}+m\gamma^0-\frac{Z\alpha}{r}\right)\psi=\epsilon\psi,\quad \alpha=\frac{e^2}{\hbar c} \tag{84}$$

Безразмерная величина α называется постоянной тонкой структуры.

Для решения уравнения Дирака (84) в центрально симметричном поле следует пользоваться нерелятивистскими двухкомпонентными спинорными состояниями Ω_{jl}, имеющими определённые значения квадрата орбитального момента $l_i=-i\varepsilon_{ijk}x^j\partial_k$, квадрата полного момента (1) и его проекции M, Из нерелятивистской квантовой механики известно, что

$$\Omega_{l+1/2,l,M}=\begin{pmatrix}\sqrt{\frac{j+M}{2j}}Y_{l,M-1/2}\\ \sqrt{\frac{j-M}{2j}}Y_{l,M+1/2}\end{pmatrix},\quad \Omega_{l-1/2,l,M}=\begin{pmatrix}\sqrt{\frac{j+1-M}{2j+2}}Y_{l,M-1/2}\\ -\sqrt{\frac{j+1+M}{2j+2}}Y_{l,M+1/2}\end{pmatrix}$$

$$\int\Omega^\dagger_{jlM'}\Omega_{j'l'M'}\mathrm{d}o=\delta_{jj'}\delta_{ll'}\delta_{MM'} \tag{85}$$

Так как

$$[\mathbf{j},(\boldsymbol{\sigma}\boldsymbol{\partial})]=[\mathbf{j},(\boldsymbol{\sigma}\mathbf{n})]=0,\quad \mathbf{n}=\mathbf{x}/r \tag{86}$$

то спиноры

$$(\boldsymbol{\sigma}\partial)\Omega_{jlM},\quad (\boldsymbol{\sigma}\mathbf{n})\Omega_{jlM} \tag{87}$$

имеют прежние значения чисел j и M. Однако числа орбитальных моментов спиноров (87) изменяются, поскольку чётности спинора Ω_{jl} и спиноров (87) относительно преобразования $\mathbf{x}\to-\mathbf{x}$ противоположны. Так как при заданном значении j для l имеется всего две возможности $l=j\pm(1/2)$, то ясно, что спиноры (87) пропорциональны друг другу:

$$(\boldsymbol{\sigma}\partial)\Omega_{jlM}=-\left[j(j+1)-l(l+1)-\frac{3}{4}\right]\frac{1}{r}(\boldsymbol{\sigma}\mathbf{n})\Omega_{jlM} \tag{88}$$

Коэффициент в равенстве (88) проверяется путём умножения этого равенства слева на $(\boldsymbol{\sigma}\mathbf{n})$.

Будем решать уравнение (84) в стандартном представлении (7) при помощи подстановки

$$\psi=\begin{pmatrix}f(r)\Omega_{jlM}(\theta,\phi)\\ -ig(r)(\boldsymbol{\sigma}\mathbf{n})\Omega_{jlM}(\theta,\phi)\end{pmatrix} \tag{89}$$

Спинор (89) имеет определённую чётность относительно операции пространственного сопряжения: согласно (7), (69) и (70) имеем для спинора (89) $\psi_P = (-1)^l\psi$. Так как оператор Дирака-Гамильтона в левой части уравнения (84) коммутирует с оператором P-чётности (см. (70)), то диагонализовать следует одновременно эти два оператора. Подставляя (89) в уравнение (84), пользуясь равенством (88), в результате несложных преобразований приходим к следующей системе уравнений:

$$\begin{cases} f' + \frac{1+\varkappa}{r} f - \left(m + \epsilon + \frac{Z\alpha}{r}\right) g = 0, \\ g' + \frac{1-\varkappa}{r} g - \left(m - \epsilon - \frac{Z\alpha}{r}\right) f = 0, \end{cases} \tag{90}$$

$$\varkappa = \begin{cases} -(j+1/2) = -(l+1), & \text{если } j = l + 1/2 \\ (j+1/2) = l, & \text{если } j = l - 1/2 \end{cases}. \tag{91}$$

Штрих сверху означает производную по r.

Исследуем поведение решений системы уравнений (90) при $r \to 0$ и $r \to \infty$.

При $r \to 0$ система (90) переходит в систему

$$\begin{cases} rf' + (1+\varkappa) f - Z\alpha g = 0 \\ rg' + (1-\varkappa) g + Z\alpha f = 0 \end{cases} \tag{92}$$

решение которой ищется в виде $f = a_0 r^\gamma, \quad g = b_0 r^\gamma$. Подстановка такого вида f и g в (92) приводит к уравнению

$$\begin{pmatrix} \gamma + 1 + \varkappa & -Z\alpha \\ Z\alpha & \gamma + 1 - \varkappa \end{pmatrix} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = 0 \tag{93}$$

Для существования ненулевого и нормируемого решения уравнения (93) необходимо и достаточно, чтобы

$$\gamma = -1 + \sqrt{\varkappa^2 - (Z\alpha)^2} \tag{94}$$

Соответствующее ненулевое решение с точностью до общего множителя имеет вид

$$a_0 = Z\alpha, \quad b_0 = \left(\varkappa + \sqrt{\varkappa^2 - (Z\alpha)^2}\right) \tag{95}$$

При $r \to \infty$ система (90) переходит в систему

$$\begin{cases} f' - (m+\epsilon) g = 0 \\ g' - (m-\epsilon) f = 0 \end{cases} \tag{96}$$

Мы будем интересоваться лишь дискретными уровнями, для которых $m^2 - \epsilon^2 > 0$. Тогда система уравнений (96) приводит к следующему поведению f и g нормируемых состояний при $r \to \infty$:

$$f \sim e^{-\lambda r}, \quad g \sim e^{-\lambda r}, \quad \lambda = \sqrt{m^2 - \epsilon^2} \tag{97}$$

Из проведенного рассмотрения видно, что решение системы уравнений (90) в случае дискретных уровней энергии следует искать в виде

$$f = e^{-\lambda r} \sum_{s=0}^{\infty} a_s r^{\gamma+s}, \quad g = e^{-\lambda r} \sum_{s=0}^{\infty} b_s r^{\gamma+s} \tag{98}$$

В (98) константы γ и λ задаются согласно (94) и (97) соответственно, а коэффициенты a_0 и b_0 - согласно (95).

Подставим анзац (98) в систему (90) и приравняем нулю по отдельности коэффициенты при одинаковых степенях r. Таким образом получается следующая система рекурсионных уравнений:

$$A_s u_{s+1} = B u_s, \quad s = 0, 1, ..., \quad u_s = \begin{pmatrix} a_s \\ b_s \end{pmatrix} \quad u_0 = \begin{pmatrix} Z\alpha \\ \varkappa + \sqrt{\varkappa^2 - (Z\alpha)^2} \end{pmatrix} \tag{99}$$

$$A_s = \begin{pmatrix} 1 + \sqrt{\varkappa^2 - (Z\alpha)^2} + s + \varkappa & -Z\alpha \\ Z\alpha & 1 + \sqrt{\varkappa^2 - (Z\alpha)^2} + s - \varkappa \end{pmatrix}, \quad B = \begin{pmatrix} \lambda & m + \epsilon \\ m - \epsilon & \lambda \end{pmatrix}$$

Заметим, что самое первое соотношение при степени r^{-1}, связывающее лишь значения a_0 и b_0 между собой, удовлетворяется автоматически вследствие специального выбора u_0 в соответствии с (95).

Так как

$$\det A_s = (1 + s)\left(1 + s + 2\sqrt{\varkappa^2 - (Z\alpha)^2}\right) \neq 0 \tag{100}$$

то уравнение (99) переписывается в виде

$$u_{s+1} = A_s^{-1} \cdot B \cdot u_s, \quad A_s^{-1} = (\det A_s)^{-1} \begin{pmatrix} 1 + \sqrt{\varkappa^2 - (Z\alpha)^2} + s - \varkappa & Z\alpha \\ -Z\alpha & 1 + \sqrt{\varkappa^2 - (Z\alpha)^2} + s + \varkappa \end{pmatrix}.$$

Собственные значения матрицы B равны нулю и 2λ и пусть v_1 и v_2 обозначают соответствующие собственные векторы:

$$Bv_1 = 0, \quad v_1 = \begin{pmatrix} \sqrt{m + \epsilon} \\ -\sqrt{m - \epsilon} \end{pmatrix}, \quad Bv_2 = 2\lambda v_2, \quad v_2 = \begin{pmatrix} \sqrt{m + \epsilon} \\ \sqrt{m - \epsilon} \end{pmatrix} \tag{102}$$

Согласно (100) и (101) при $s \to \infty$

$$A_s^{-1} \to \frac{1}{s} \tag{103}$$

Поэтому, если при достаточно больших s

$$u_s = pv_1 + qv_2, \quad q \neq 0 \tag{104}$$

где p и q - некие числа, то согласно (98), (102) и (103) при $r \to \infty$ функции f и g ведут себя как $e^{\lambda r}$. Волновые функции с таким поведением ненормируемы и потому неприемлемы. Чтобы избежать этой ситуации, для некого s мы должны иметь

$$A_s^{-1} v_2 = t v_1 \tag{105}$$

Здесь t - некое число. Действительно, для вектора общего вида (104) $Bu_s = 2\lambda q v_2$. Если имеет место равенство (105) для некого числа t, то вследствие (102) $u_{s+2} = 0$, рекурсионная цепочка обрывается и волновая функция оказывается нормируемой. При помощи (102) уравнение (105) переписывается как

$$B \cdot A_s^{-1} \cdot v_2 = 0, \quad s = 0, 1, ... \tag{106}$$

Ещё одна возможность обрыва рекурсионной цепочки (101), не учтённая в (106), реализуется при $\varkappa < 0$ в случае $u_0 = tv_1$ для некого числа t. Действительно, при $\varkappa < 0$ компоненты вектора u_0 разного знака (при $\varkappa > 0$ - одного знака) и уравнение $u_0 = tv_1$ может быть реализовано соответствующим подбором параметров ϵ и t. Следовательно, уравнение

$$Bu_0 = 0 \quad \text{при} \quad \varkappa < 0 \tag{107}$$

также является условием квантования энергии. Поскольку B - вырожденная матрица 2×2, то каждое из уравнений (106) и (107) содержит лишь одно независимое алгебраическое уравнение, решение совокупности которых приводит к следующим значениям уровней энергии связанных состояний:

$$\epsilon = mc^2 \left[1 + \frac{(Z\alpha)^2}{\left(n_r + \sqrt{(j+1/2)^2 - (Z\alpha)^2} \right)^2} \right]^{-1/2}, \qquad n_r = \begin{cases} 0, 1, 2, ..., \text{ если } j = l + 1/2 \\ 1, 2, ..., \text{ если } j = l - 1/2 \end{cases} \tag{108}$$

При получении формулы (108) было использовано соотношение (91). Энергия основного состояния получается из формулы (108), если в ней положить $l = 0, j = 1/2, n_r = 0$:

$$\epsilon_0 = mc^2 \sqrt{1 - (Z\alpha)^2} \tag{109}$$

Формулы (108), (109) являются точными и они остаются справедливыми при всех значениях заряда ядра, удовлетворяющих условию $(Z\alpha) < 1$. Поскольку $|\varkappa| = j + 1/2$, то согласно (108) имеет место двукратное вырождение (не считая вырождения относительно прекции полного момента M) уровней: энергии при $l = j \pm 1/2$ совпадают, что является точным результатом.

При малых зарядах ядра, когда $(Z\alpha) \ll 1$, представляет интерес разложение формулы (108) относительно $(Z\alpha)$. Поскольку $\alpha = e^2/\hbar c$, то это разложение можно трактовать как учёт релятивистских поправок. Имеем:

$$\epsilon = mc^2 - \frac{(Z\alpha)^2 mc^2}{2n^2} \left\{ 1 + \frac{(Z\alpha)^2}{n} \left[\frac{1}{j+1/2} - \frac{3}{4n} \right] \right\} + O\left((Z\alpha)^6 \right), \quad n = n_r + j + \frac{1}{2} = 1, 2, ... \tag{110}$$

Учёт поправок порядка $O\left((Z\alpha)^6\right)$ не имеет смысла, поскольку они перекрываются радиационными поправками, имеющими порядок $O\left(\alpha(Z\alpha)^4\right)$ (см. лекцию 13). Формула (110) может быть также получена при помощи ТВ в рамках нерелятивистской квантовой механики, когда спин-орбитальное взаимодействие и релятивистская поправка к кинетической энергии электрона воспринимаются как возмущение (см. задачу (5)).

Задачи

1. Выписать уравнение Дирака для сопряжённого поля $\bar{\psi}$.

2. а) Показать, что векторный ток $J^\mu = \bar{\psi}\gamma^\mu\psi$ сохраняется в силу уравнения Дирака (во внешнем электромагнитном поле), то есть $\partial_\mu J^\mu = 0$.

б) Доказать, что аксиально-векторный ток $J^{5\mu} \equiv \bar{\psi}\gamma^\mu\gamma^5\psi$ веществен и в силу уравнения Дирака

$$\partial_\mu J^{5\mu} = 2im\bar{\psi}\gamma^5\psi$$

Это означает, что в безмассовой теории сохраняется также и аксиально-векторный ток (в классической теории).

в) Показать, что полный 4 -момент

$$M^{\mu\nu} = \int \mathrm{d}^{(3)}x\psi^\dagger iJ^{\mu\nu}\psi$$

сохраняется, если поле ψ удовлетворяет свободному уравнению Дирака. Проверить, что в стандартном представлении

$$\frac{1}{2}\varepsilon_{ijk}M^{jk} = \int \mathrm{d}^{(3)}x\psi^\dagger J^i\psi$$

где J^i совпадает с оператором полного 3 -момента (1), если в (1) сделать замену

$$\frac{1}{2}\sigma^i \longrightarrow \Sigma^i \equiv \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} = -\frac{1}{2}\alpha^i\gamma^5 \tag{111}$$

Таким образом, в нерелятивистском пределе оператор $\mathbf{s} = (1/2)\boldsymbol{\sigma}$ является оператором спина электрона.

3. Используя закон преобразования дираковских полей при Лоренц-преобразованиях показать, что векторный и аксиально-векторный токи являются 4 -векторами.

4. Показать, что уравнение Дирака инвариантно относительно калибровочных преобразований:

$$\psi(x) \to e^{ie\alpha(x)}\psi(x), \quad A_\mu(x) \to A_\mu(x) - \partial_\mu\alpha(x)$$

где $\alpha(x)$ - произвольная вещественная функция.

5. Продолжить разложение уравнения Дирака до членов $\sim 1/c^2$ и получить соответствующий гамильтониан для шредингеровской двухкомпонентной волновой функции. Считать, что имеется только электрическое центрально-симметричное внешнее поле ($\mathbf{A} = 0, eA_0 = \Phi(r)$). Получить выражение для спин-орбитальной связи в случае атома водорода.

Ответ:

$$H = \left(\frac{\mathbf{p}^2}{2m} + \Phi(r)\right) + \left\{-\frac{\mathbf{p}^4}{8m^3c^2} + \frac{\hbar^2}{8m^2c^2}(\Delta\Phi) + \frac{\hbar^2}{2m^2c^2r}\frac{\mathrm{d}\Phi}{\mathrm{d}r}(\mathbf{ls})\right\}$$

6. а) Доказать, что при преобразовании P-чётности

$$J'^{5\mu}\left(x^0, \mathbf{x}\right) = -\eta_{\mu\mu}J'^{5\mu}\left(x^0, -\mathbf{x}\right), \quad \left(\bar{\psi}\gamma^5\psi\right)'\left(x^0, \mathbf{x}\right) = -\left(\bar{\psi}\gamma^5\psi\right)'\left(x^0, -\mathbf{x}\right)$$

Здесь нет суммирования по индексу μ.

б) Показать, что при T - и C-преобразованиях уравнение Дирака сохраняет свой вид.

в) Пусть ψ_{CPT} обозначает поле, возникающее из дираковского поля ψ при последовательном проведении операций обращения времени, затем пространственной инверсии и, наконец, зарядового сопряжения. Выразить ψ_{CPT} через ψ.

г) Показать, что векторный ток майорановского поля тождественно равен нулю.

7. Показать, что уравнения Вейля инвариантны относительно обращения времени и комбинированного CP-преобразования. Существуют ли вейлевско-майорановские (т.е. одновременно вейлевские и майорановские) поля?

Лекция 9

9.7 L9-1 Quantization of Fermions Using a Functional Integral

9.7.1 1. Fermionic Oscillator

Рассмотрим гармонический фермионный осциллятор, динамика которого описывается гамильтонианом

$$\mathcal{H} = \omega a^\dagger a \tag{1}$$

где $a^\dagger$ и a - взаимно сопряжённые фермионные операторы рождения и уничтожения:

$$\left\{a, a^\dagger\right\} = 1, \quad a^2 = 0, \quad \left(a^\dagger\right)^2 = 0 \tag{2}$$

Основное состояние $|0\rangle$ определяется условиями

$$a|0\rangle = 0, \quad \langle 0|0\rangle = 1 \tag{3}$$

Очевидно, что в теории имеется ещё лишь одно состояние, ортогональное к основному:

$$|1\rangle = a^\dagger|0\rangle, \quad \langle 1|0\rangle = \langle 0|a|0\rangle = 0, \quad \langle 1|1\rangle = \left\langle 0\left|aa^\dagger\right|0\right\rangle = \langle 0|0\rangle = 1 \tag{4}$$

Кроме того, в силу (1)-(3)

$$\mathcal{H}|0\rangle = 0, \quad \mathcal{H}|1\rangle = \omega|1\rangle \tag{5}$$

Простейшая фермионная система (1)-(5) интересна тем, что дираковская система (59 Л8) является суммой простейших ферми-систем.

Обратим внимание на следующий факт. Если бы гамильтониан был равен $\mathcal{H} = -\omega\alpha^\dagger\alpha$, где фермионные операторы $\alpha^\dagger$ и α удовлетворяют соотношениям (2), а основное состояние $|f;0\rangle$ удовлетворяет условию $\alpha|f;0\rangle = 0$ и $|f;1\rangle = \alpha^\dagger|f;0\rangle$, то $\mathcal{H}|f;0\rangle = 0, \mathcal{H}|f;1\rangle = -\omega|f;1\rangle$.

В этом случае энергия возбуждённого состояния меньше энергии основного. Однако путём переопределения $a = \alpha^\dagger, a^\dagger = \alpha, |0\rangle = |f;1\rangle, |1\rangle = |f;0\rangle$ мы приходим к ситуации, описанной формулами (1)-(5) с той лишь разницей, что $\mathcal{H} = \omega a^\dagger a - \omega$.

Здесь последнее слагаемое в гамильтониане соответствует энергии вакуума $\mathcal{E}_0$ (59 Л8). Таким образом, после переопределения новое основное состояние совпадает с исходным возбуждённым или заполненным состоянием. Описанная процедура переопределения называется операцией заполнения состояний с отрицательной энергией. В результате, хотя основное состояние имеет отрицательную энергию, все возбуждённые состояния имеют энергию большую, чем энергия основного состояния. Мы видели, что в квантовой теории Дирака все состояния с отрицательной энергией заполнены. На рис. 12 изображено основное состояние в теории Дирака. Заштрихованная область спектра с энергиями меньше $(-m)$ полностью заполнена.

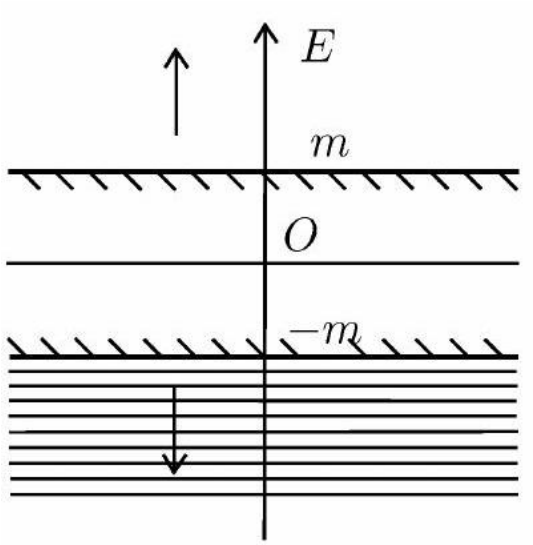


Рис. 12

9.7.2 2. Representation of Fermionic Amplitudes via Functional Integrals

Для решения этой задачи необходимо использование основ математического анализа над грассмановой алгеброй.

Пусть $\{a_\alpha, a_\alpha^*\}$ - образующие алгебры Грассмана над комплексными числами, $\alpha = 1, 2, ..., n$, причём

$$(a_\alpha)^* = a_\alpha^*, \quad (a_\alpha^*)^* = a_\alpha \tag{6}$$

По определению

$$a_\alpha a_\beta + a_\beta a_\alpha = a_\alpha^* a_\beta^* + a_\beta^* a_\alpha^* = a_\alpha a_\beta^* + a_\beta^* a_\alpha = 0, \quad c a_\alpha = a_\alpha c, \quad c a_\alpha^* = a_\alpha^* c \tag{7}$$

где c - комплексное число. Других соотношений, кроме (7), между образующими грассмановой алгебры нет. Из (7) следует, что

$$a_1^2 = a_2^2 = ... = a_n^2 = (a_1^*)^2 = (a_2^*)^2 = ... = (a_n^*)^2 = 0 \tag{8}$$

Отсюда видно, что общий элемент грассмановой алгебры представляется в виде

$$\begin{aligned} f(a, a^*) = f = & f_{0|0} + \sum_{\alpha_1} f_{\alpha_1|0} a_{\alpha_1} + \sum_{\alpha_1} f_{0|\alpha_1} a_{\alpha_1}^* + \sum_{\alpha_1 \alpha_2} f_{\alpha_1|\alpha_2} a_{\alpha_1} a_{\alpha_2}^* + ... + \\ & + f_{12...n|n(n-1)...1} a_1 a_2 ... a_n a_n^* a_{n-1}^* ... a_1^* \end{aligned} \tag{9}$$

где коэффициенты $f_{0|0}, f_{\alpha_1|0}$ и т. д. - комплексные числа. Определим производные на грассмановой алгебре при помощи следующих формул:

$$\begin{aligned} & \frac{\partial}{\partial a_\alpha} a_\beta = \delta_{\alpha\beta}, && \frac{\partial}{\partial a_\alpha^*} a_\beta^* = \delta_{\alpha\beta} \\ & \frac{\partial}{\partial a_\alpha} a_\beta^* = 0, && \frac{\partial}{\partial a_\alpha^*} a_\beta = 0. \end{aligned} \tag{10}$$

Представим элемент (9) в двух следующих видах:

$$f = f_0 + a_\alpha f_\alpha = \bar{f}_0 + a^*_\alpha \bar{f}_\alpha \tag{11}$$

где f_0 и f_α не зависят от a_α, а $\bar{f}_0$ и $\bar{f}_\alpha$ не зависят от a^*_α. (В (11) нет суммирования по α.) Вследствие (7) коэффициенты $f_0, f_\alpha, \bar{f}_0, \bar{f}_\alpha$ определены однозначно элементом (9). Согласно определению (10)

$$\frac{\partial}{\partial a_\alpha} f = f_\alpha, \quad \frac{\partial}{\partial a^*_\alpha} f = \bar{f}_\alpha \tag{12}$$

Из сказанного следует, что

$$\frac{\partial}{\partial a_\alpha}\frac{\partial}{\partial a_\beta} = -\frac{\partial}{\partial a_\beta}\frac{\partial}{\partial a_\alpha}, \quad \frac{\partial}{\partial a^*_\alpha}\frac{\partial}{\partial a^*_\beta} = -\frac{\partial}{\partial a^*_\beta}\frac{\partial}{\partial a^*_\alpha}, \quad \frac{\partial}{\partial a_\alpha}\frac{\partial}{\partial a^*_\beta} = -\frac{\partial}{\partial a^*_\beta}\frac{\partial}{\partial a_\alpha} \tag{13}$$

Интегрирование на алгебре Грассмана определяется с помощью правил

$$\int \mathrm{d}a_\alpha f = f_\alpha, \quad \int \mathrm{d}a^*_\alpha f = \bar{f}_\alpha \tag{14}$$

Кратный интеграл понимается как повторный. Например,

$$\int \mathrm{d}a^*_\alpha \, \mathrm{d}a_\beta f \equiv \int \mathrm{d}a^*_\alpha \left(\int \mathrm{d}a_\beta f \right) = \int \mathrm{d}a^*_\alpha f_\beta \tag{15}$$

Введенное правило интегрирования оправдывается тем, что интеграл от полной производной равен нулю:

$$\int \mathrm{d}a_\alpha \left(\frac{\partial}{\partial a_\alpha} f \right) = 0, \quad \int \mathrm{d}a^*_\alpha \left(\frac{\partial}{\partial a^*_\alpha} f \right) = 0 \tag{16}$$

Из данных определений видно, что дифференциалы грассмановых образующих антиперестановочны:

$$\mathrm{d}a_\alpha \mathrm{d}a_\beta = -\mathrm{d}a_\beta \mathrm{d}a_\alpha, \quad \mathrm{d}a^*_\alpha \, \mathrm{d}a^*_\beta = -\mathrm{d}a^*_\beta \, \mathrm{d}a^*_\alpha, \quad \mathrm{d}a_\alpha \mathrm{d}a^*_\beta = -\mathrm{d}a^*_\beta \, \mathrm{d}a_\alpha \tag{17}$$

Обозначим через

$$\mathrm{d}a^* \, \mathrm{d}a \equiv \mathrm{d}a^*_1 \, \mathrm{d}a^*_2 ... \mathrm{d}a^*_n \, \mathrm{d}a_n \, \mathrm{d}a_{n-1} ... \mathrm{d}a_1 \tag{18}$$

Имеем:

$$\int \mathrm{d}a^* \, \mathrm{d}a f\,(a, a^*) = f_{12...n|n(n-1)...1} \tag{19}$$

Пусть полное множество образующих грассмановой алгебры состоит из двух множеств $\{a_\alpha, a^*_\alpha\}$ и $\{\eta_\alpha, \eta^*_\alpha\}$. Будем называть чётными (нечётными) элементами грассмановой алгебры те элементы, которые разлагаются по чётным (нечётным) суммарным степеням образующих грассмановой
алгебры. Суммарная степень (или просто степень) произведения образующих алгебры Грассмана называется чётной (нечётной), если произведение составлено из чётного (нечётного) числа образующих. Таким образом, все образующие грассмановой алгебры являются нечётными элементами.

Пусть

$$\tilde{a}_\alpha = a_\alpha + q_\alpha, \quad \tilde{a}^*_\alpha = a^*_\alpha + p^*_\alpha \tag{20}$$

где q_α, p^*_α - нечётные элементы, разлагающиеся лишь по образующим $\{\eta_\alpha, \eta^*_\alpha\}$. Согласно данным определениям (сравни с (19))

$$\int \mathrm{d}a^* \ \mathrm{d}a f\left(\tilde{a}, \tilde{a}^*\right) = f_{12\ldots n|n(n-1)\ldots 1} = \int \mathrm{d}a^* \ \mathrm{d}a f\left(a, a^*\right) \tag{21}$$

то есть сдвиг переменных интегрирования согласно равенствам (20) не изменяет значения интеграла.

Сделаем линейную однородную замену образующих грассмановой алгебры:

$$a_\alpha = U_{\alpha\beta} b_\beta, \quad a^*_\alpha = U^*_{\alpha\beta} b^*_\beta, \tag{22}$$

или, сокращённо

$$a = Ub, \quad a^{*t} = b^{*t} U^\dagger$$

Для интеграла (19) имеем:

$$f_{12\ldots n|n(n-1)\ldots 1} = \int \mathrm{d}a^* \ \mathrm{d}a f\left(a, a^*\right) = \int \mathrm{d}a^* \ \mathrm{d}a f\left(Ub, b^{*t} U^\dagger\right) =$$

$$= \int \mathrm{d}a^* \ \mathrm{d}a f_{12\ldots n|n(n-1)\ldots 1} \left(U_{1\alpha_1} b_{\alpha_1}\right) \ldots \left(U_{n\alpha_n} b_{\alpha_n}\right) \left(U^*_{n\beta_n} b^*_{\beta_n}\right) \ldots \left(U^*_{1\beta_1} b^*_{\beta_1}\right) =$$

$$= f_{12\ldots n|n(n-1)\ldots 1} |\det U|^2 \int \mathrm{d}a^* \ \mathrm{d}a b_1 \ldots b_n b^*_n \ldots b^*_1$$

то есть

$$\int |\det U|^2 \ \mathrm{d}a^* \ \mathrm{d}a b_1 \ldots b_n b^*_n \ldots b^*_1 = 1$$

Отсюда следует заключение, что (!)

$$|\det U|^2 \ \mathrm{d}a^* \ \mathrm{d}a = \mathrm{d}b^* \ \mathrm{d}b. \tag{23}$$

Подчеркнём, что при замене переменных (22) в случае коммутирующих переменных мы имели бы $\mathrm{d}a^* \ \mathrm{d}a = |\det U|^2 \ \mathrm{d}b^* \ \mathrm{d}b$.

Рассмотрим гауссов интграл

$$I = \int \mathrm{d}a^* \ \mathrm{d}a \exp\left(a^*_\alpha A_{\alpha\beta} a_\beta\right) \tag{24}$$

Будем считать, что все элементы $A_{\alpha\beta}$ - чётные. Имеем:

$$\exp\left(a^*_\alpha A_{\alpha\beta} a_\beta\right) = 1 + \ldots + \frac{1}{n!} \left(a^*_\alpha A_{\alpha\beta} a_\beta\right)^n$$

Нас интересует лишь последнее слагаемое, которое равно

$$\left(a^*_{\alpha_1} A_{\alpha_1\beta_1} a_{\beta_1}\right) \ldots \left(a^*_{\alpha_n} A_{\alpha_n\beta_n} a_{\beta_n}\right) = n!(\det A) a^*_1 \ldots a^*_n a_n \ldots a_1 = (-1)^n n!(\det A) a_1 \ldots a_n a^*_n \ldots a^*_1 \tag{25}$$

Поэтому в соответствии с (19) (общий знак здесь нас не интересует)

$$I = (\det A) \tag{26}$$

Рассмотрим теперь интеграл

$$I(\eta, \eta^*) = \int \mathrm{d}a^* \; \mathrm{d}a \exp\left(a^*_\alpha A_{\alpha\beta} a_\beta + i\eta^*_\alpha a_\alpha + i a^*_\alpha \eta_\alpha\right) \tag{27}$$

где $\{\eta_\alpha, \eta^*_\alpha\}$ - образующие алгебры Грассмана, независимые от образующих $\{a_\alpha, a^*_\alpha\}$. Если сдвинуть переменные интегрирования в (27) согласно (20) с $q_\alpha = -iA^{-1}_{\alpha\beta}\eta_\beta, p^*_\alpha = -i\eta^*_\beta A^{-1}_{\beta\alpha}$, то интеграл (27) переписывается как

$$I(\eta, \eta^*) = \int \mathrm{d}a^* \; \mathrm{d}a \exp\left(a^*_\alpha A_{\alpha\beta} a_\beta\right) \exp\left(\eta^*_\alpha A^{-1}_{\alpha\beta}\eta_\beta\right)$$

С учётом (26) окончательно получаем:

$$I(\eta, \eta^*) = \int \mathrm{d}a^* \; \mathrm{d}a \exp\left(a^*_\alpha A_{\alpha\beta} a_\beta + i\eta^*_\alpha a_\alpha + i a^*_\alpha \eta_\alpha\right) = (\det A) \exp\left(\eta^*_\alpha A^{-1}_{\alpha\beta}\eta_\beta\right) \tag{28}$$

Далее мы будем использовать тот факт, что показатель экспоненты в правой части (28) равен экстремуму показателя экспоненты в левой части (28) относительно переменных интегрирования $\{a_\alpha, a^*_\alpha\}$, рассматриваемых как независимые переменные. Действительно, дифференцируя показатель экспоненты в левой части (28) и приравнивая результат нулю, находим экстремальные значения переменных:

$$a^{(cl)}_\alpha = -iA^{-1}_{\alpha\beta}\eta_\beta, \quad a^{*(cl)}_\alpha = -i\eta^*_\beta A^{-1}_{\beta\alpha} \tag{29}$$

Подставляя значения (29) в показатель экспоненты в левой части (28) мы получаем показатель экспоненты в правой части (28). В этом отношении фермионные гауссовы интегралы аналогичны бозонным.

Теперь разработаем аппарат для фермионных операторов и фермионных фоковских пространств в голоморфном представлении. Для простоты ограничимся одной степенью свободы. Обобщение на многомерный случай очевидно.

Голоморфными функциями назовём элементы грассмановой алгебры, зависящие только от a^*:

$$f(a^*) = f_0 + f_1 a^* \tag{30}$$

Множество таких функций образует двумерное пространство, и мы используем их для представления векторов состояний системы (1)-(5). Скалярное произведение в пространстве голоморфных функций определяется формулой

$$\langle f_1 | f_2 \rangle = \int \mathrm{d}a^* \; \mathrm{d}a \left(f_1(a^*)\right)^* f_2(a^*) e^{-a^* a}, \quad \left(f(a^*)\right)^* = f^*_0 + f^*_1 a \tag{31}$$

Скалярное произведение (31) положительно определено. Для этого покажем, что базис из двух одночленов $\{\psi_0 = 1, \psi_1 = a^*\}$ ортонормирован. Имеем:

$$\langle\psi_0|\psi_0\rangle = \int \mathrm{d}a^*\ \mathrm{d}a e^{-a^*a} = \int \mathrm{d}a^*\ \mathrm{d}a\,(1-a^*a) = 1$$
$$\langle\psi_0|\psi_1\rangle = \int \mathrm{d}a^*\ \mathrm{d}a a^* e^{-a^*a} = 0, \quad \langle\psi_1|\psi_1\rangle = \int \mathrm{d}a^*\ \mathrm{d}a a a^* e^{-a^*a} = 1 \tag{32}$$

Операторы $a^\dagger$ и a действуют в пространстве (30) следующим образом:

$$a^\dagger f(a^*) = a^* f(a^*), \quad a f(a^*) = \frac{\mathrm{d}}{\mathrm{d}a^*} f(a^*) \tag{33}$$

Если введенный базис представить базисными векторами в двумерном эрмитовом пространстве как

$$\psi_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{34}$$

то операторы $a^\dagger$ и a в этом базисе представляются матрицами

$$a^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{35}$$

откуда очевидна их взаимная сопряжённость, а также правила антикоммутации (2).

Опишем способы задания операторов в голоморфном представлении. Запишем оператор общего вида:

$$A = K_{00} + K_{10}a^\dagger + K_{01}a + K_{11}a^\dagger a \tag{36}$$

Оператору (36) сопоставим его нормальный символ

$$K(a^*, a) = K_{00} + K_{10}a^* + K_{01}a + K_{11}a^*a \tag{37}$$

и ядро

$$A(a^*, a) = A_{00} + A_{10}a^* + A_{01}a + A_{11}a^*a \tag{37}$$

По определению действие оператора (36) задаётся в гломорфном представлении согласно правилу (сравни с (10 Л2))

$$(Af)(a^*) \equiv \int A(a^*, \alpha) f(\alpha^*) e^{-\alpha^*\alpha} \mathrm{d}\alpha^*\ \mathrm{d}\alpha \tag{39}$$

Отсюда находим:

$$A\psi_0 = A_{00}\psi_0 + A_{10}\psi_1, \quad A\psi_1 = A_{01}\psi_0 + A_{11}\psi_1$$

откуда видно, что матрица ядра и матрица оператора

$$A_{mn} = \langle m|A|n\rangle, \quad m, n = 0, 1 \tag{40}$$

совпадают (сравни с (11 Л2) и (12 Л2)). Произведение операторов в голоморфном представлении задаётся формулой

$$(A_1A_2)(a^*,a)=\int A_1(a^*,\alpha)\,A_2(\alpha^*,a)\,e^{-\alpha^*\alpha}\mathrm{d}\alpha^*\;\mathrm{d}\alpha \tag{41}$$

Для записи (39) и (41) необходимо было ввести две новые образующие грассмановой алгебры (α^*,α), которые и далее будут вводиться по мере необходимости.

Подставляя оператор (36) в (40), находим:

$$A_{00}=K_{00},\quad A_{10}=K_{10},\quad A_{01}=K_{01},\quad A_{11}=K_{00}+K_{11}$$

Отсюда видно, что связь между ядром оператора и его нормальным символом такова (сравни с (13 Л2)):

$$A(a^*,a)=e^{a^*a}K(a^*,a) \tag{42}$$

Поскольку ядро S-матрицы для одномерного бозонного осциллятора (28 Л2) было получено при помощи лишь двух формул (12 Л2) и (13 Л2), аналоги которых в фермионном случае (41) и (42) имеют точно такой же вид, то выражение для S-матрицы фермионного гармонического осциллятора с внешним антикоммутирующим источником η и гамильтонианом

$$h=\omega a^*a-\frac{1}{\sqrt{2\omega}}(\eta^*a+a^*\eta) \tag{43}$$

может быть выписано при помощи справедливых также и в фермионном случае формул (15 Л2) и (26 Л2). Однако в фермионном случае следует учитывать антикоммутативность образующих грассмановой алгебры и правило вычисления гауссовых функциональных интегралов (28). В результате в фермионном случае мы получим следующее выражение для оператора S-матрицы:

$$S_\eta\left(a^\dagger,a\right)=\frac{1}{N}:\exp\left\{\frac{i}{\sqrt{2\omega}}\int \mathrm{d}t\left[\eta^*(t)ae^{-i\omega t}+a^\dagger\eta(t)e^{i\omega t}\right]+i\iint_{s<t}\mathrm{d}t\;\mathrm{d}s\eta^*(t)\frac{ie^{-i\omega(t-s)}}{2\omega}\eta(s)\right\} \tag{44}$$

Выражение (44) формально совпадает с оператором S-матрицы, следующим из формулы (27 Л2).

9.7.3 3. Functional Integral in Dirac Theory

В теории Дирака аналогом одночастичного гамильтониана (43) является гамильтониан (см. (33 Л8))

$$\mathcal{H}_\psi=\int \mathrm{d}^{(3)}x\left\{\bar\psi\left(-i\gamma^i\partial_i+m\right)\psi-(\bar\psi\eta+\bar\eta\psi)\right\} \tag{45}$$

в котором дираковские поля представляются согласно (46 Л8) и фермиевские операторы рождения и уничтожения удовлетворяют антикоммутационным соотношениям (54 Л8), (58 Л8), поля $\eta(x)$ $\bar\eta(x)$ - антикоммутирующие. Согласно (59 Л8) выражение (45) переписывается в виде

$$\mathcal{H}_\psi = \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\left\{\omega_k \sum_{s=1,2}\left(a^\dagger_{\mathbf{k}s}a_{\mathbf{k}s} + b^\dagger_{\mathbf{k}s}b_{\mathbf{k}s}\right) - \frac{1}{\sqrt{2\omega_k}}\left[\gamma_{ks}a_{\mathbf{k}s} + a^\dagger_{\mathbf{k}s}\delta_{ks} - \delta_{-ks}b_{\mathbf{k}s} - b^\dagger_{\mathbf{k}s}\gamma_{-ks}\right]\right\}$$

$$\gamma_{\pm ks}(t) = \tilde{\bar{\eta}}_{\pm\mathbf{k}}(t)u_{\pm ks}, \quad \delta_{\pm ks}(t) = \bar{u}_{\pm ks}\tilde{\eta}_{\pm\mathbf{k}}(t)$$

$$\tilde{\eta}_{\mathbf{k}}(t) = \int \mathrm{d}^{(3)}x\eta(x)e^{-i\mathbf{kx}}, \quad \tilde{\bar{\eta}}_{\mathbf{k}}(t) = \int \mathrm{d}^{(3)}x\bar{\eta}(x)e^{i\mathbf{kx}} \tag{46}$$

Представление (46) показывает, что дираковский гамильтониан (45) распадается на сумму одночастичных гамильтонианов вида (43). Поэтому оператор S-матрицы равен

$$S_\eta\left(a^\dagger, a\right) = \frac{1}{N} : \exp\left\{i\int \mathrm{d}^{(4)}x\left(\bar{\eta}\psi_0 + \bar{\psi}_0\eta\right) + \right.$$

$$\left. + i\iint_{s<t} \mathrm{d}t\ \mathrm{d}s \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\frac{ie^{-i\omega_k(t-s)}}{2\omega_k}\sum_{s=1,2}\left[\gamma_{ks}(t)\delta_{ks}(s) - \gamma_{-ks}(s)\delta_{-ks}(t)\right]\right\} : \tag{47}$$

При помощи формул (56 Л8) преобразуем суммы в экспоненте следующим образом:

$$\sum_{s=1,2}\gamma_{ks}(t)\delta_{ks}(s) = \tilde{\bar{\eta}}_{\mathbf{k}}(t)(\not{k} + m)\tilde{\eta}_{\mathbf{k}}(s), \quad -\sum_{s=1,2}\gamma_{-ks}(s)\delta_{-ks}(t) = \tilde{\bar{\eta}}_{-\mathbf{k}}(s)(-\not{k} + m)\tilde{\eta}_{-\mathbf{k}}(t)$$

где $k = (\omega_k, \mathbf{k})$. Последние равенства позволяют преобразовать второе слагаемое в экспоненте в (47) сначала к виду

$$i\iint \mathrm{d}x^0\ \mathrm{d}y^0 \int \frac{\mathrm{d}^{(4)}k}{(2\pi)^4}\frac{e^{-ik^0\left(x^0-y^0\right)}}{-k^2+m^2-i\varepsilon}\tilde{\bar{\eta}}_{\mathbf{k}}\left(x^0\right)(\not{k}+m)\tilde{\eta}_{\mathbf{k}}\left(y^0\right)$$

и затем к

$$i\int \mathrm{d}^{(4)}x\ \mathrm{d}^{(4)}y\bar{\eta}(x)\left(i\gamma^\mu\partial_\mu + m\right)D_c(x-y)\eta(y) \tag{48}$$

где $D_c(x-y)$ определена согласно (49 Л2). Определим дираковский пропагатор как

$$S_c(x-y) = -\left(i\gamma^\mu\partial_\mu + m\right)D_c(x-y) = \int \frac{\mathrm{d}^{(4)}k}{(2\pi)^4}\frac{\not{k}+m}{k^2-m^2+i\varepsilon}e^{-ik(x-y)} = \int \frac{\mathrm{d}^{(4)}k}{(2\pi)^4}S_c(k)e^{-ik(x-y)}$$

$$S_c(k) = \frac{\not{k}+m}{k^2-m^2+i\varepsilon} = (\not{k}-m+i\varepsilon)^{-1}, \quad \left(i\gamma^\mu\partial_\mu - m\right)S_c(x-y) = \delta^{(4)}(x-y) \tag{49}$$

Таким образом

$$S_\eta\left(a^\dagger, b^\dagger, a, b\right) = \frac{1}{N} : \exp\left\{i\int \mathrm{d}^{(4)}x\left(\bar{\eta}\psi_0 + \bar{\psi}_0\eta\right) - i\int \mathrm{d}^{(4)}x\ \mathrm{d}^{(4)}y\bar{\eta}(x)S_c(x-y)\eta(y)\right\} :$$

$$\psi_0(x) = \psi_0^{(+)}(x) + \psi_0^{(-)}(x), \quad \bar{\psi}_0(x) = \bar{\psi}_0^{(+)}(x) + \bar{\psi}_0^{(-)}(x)$$

$$\psi_0^{(+)}(x) = \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\sum_{s=1,2}a_{\mathbf{k}s}u_{ks}e^{-ikx}, \quad \psi_0^{(-)}(x) = \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\sum_{s=1,2}b^\dagger_{\mathbf{k}s}u_{-ks}e^{ikx}$$

$$\bar{\psi}_0^{(+)}(x) = \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\sum_{s=1,2}b_{\mathbf{k}s}\bar{u}_{-ks}e^{-ikx}, \quad \bar{\psi}_0^{(-)}(x) = \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\sum_{s=1,2}a^\dagger_{\mathbf{k}s}\bar{u}_{ks}e^{ikx} \tag{50}$$

где $k = (\omega_k, \mathbf{k})$.

Укажем, что правая часть уравнения (50) может быть получена также при помощи формулы (28) путём прямого вычисления фермионного функционального интеграла (сравни с (44 Л7))

$$\int D\psi D\bar{\psi} \exp\left\{ i \int \mathrm{d}^{(4)}x \left[\bar{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi + \bar{\eta}\psi + \bar{\psi}\eta \right] \right\} \tag{51}$$

с граничными условиями, фиксирующими при $t \to -\infty$ положительночастотные составляющие полей $\psi_0^{(+)}(x)$ и $\bar{\psi}_0^{(+)}(x)$ и при $t \to +\infty$ - отрицательночастотные составляющие полей $\psi_0^{(-)}(x)$ и $\bar{\psi}_0^{(-)}(x)$ согласно (50). В (51) дираковская функциональная мера

$$D\psi D\bar{\psi} \equiv \prod_x \prod_a \mathrm{d}\bar{\psi}_a(x) \mathrm{d}\psi_a(x) = \text{ const } \cdot \prod_t \prod_{\mathbf{k}} \prod_s \mathrm{d}a^*_{\mathbf{k}s}(t) \mathrm{d}a_{\mathbf{k}s}(t) \mathrm{d}b^*_{\mathbf{k}s}(t) \mathrm{d}b_{\mathbf{k}s}(t) \tag{52}$$

Это утверждение легко проверяется, если учесть, что экстремальное значение показателя экспоненты в (51) при указанных граничных условиях достигается на полях (сравни с (53 Л2))

$$\psi_{(cl)}(x) = \psi_0(x) - \int \mathrm{d}^{(4)}y S_c(x-y)\eta(y), \quad \bar{\psi}_{(cl)}(x) = \bar{\psi}_0(x) - \int \mathrm{d}^{(4)}y \bar{\eta}(y) S_c(y-x) \tag{53}$$

Подстановка полей (53) в экспоненту в (51) опять приводит к (50).

Второе равенство в (52) легко получается из соотношения для интегралов от квадратичных форм дифференциалов фермионных переменных:

$$\int \mathrm{d}^{(3)}x \ \mathrm{d}\bar{\psi}(x) \mathrm{d}\psi(x) = \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3} \sum_{s=1,2} \frac{m}{\omega_k} \left(\ \mathrm{d}a^*_{\mathbf{k}s} \ \mathrm{d}a_{\mathbf{k}s} + \mathrm{d}b^*_{\mathbf{k}s} \ \mathrm{d}b_{\mathbf{k}s} \right) \tag{54}$$

При выводе (54) фермионные операторы $\left\{ a_{\mathbf{k}s}, b_{\mathbf{k}s}, a^\dagger_{\mathbf{k}s}, b^\dagger_{\mathbf{k}s} \right\}$ в выражениях для дираковских полей (46 Л8) заменяются на соответствующие грассмановы переменные и используются формулы (57 Л8).

9.8 Other Quantum Fields (?)

9.8.1 Massive Quantum Vector Field (??)

(пока не встречал это)

9.8.2 Facts of life with γ_5 (by F. Jegerlehner)

(в другое место перенесу скорее всего)

Abstract

The increasing precision of many experiments in elementary particle physics leads to continuing interest in perturbative higher order calculations in the electroweak Standard Model or extensions of it. Such calculations are of increasing complexity because more loops and/or more legs are considered. Correspondingly efficient computational methods are mandatory

for many calculations. One problem which affects the feasibility of higher order calculations is the problem with γ_5 in dimensional regularization. Since the subject thirty years after its invention is still controversial I advocate here some ideas which seem not to be common knowledge but might shed some new light on the problem. I present arguments in favor of utilizing an anticommuting γ_5 and a simple 4–dimensional treatment of the hard anomalies.

Introduction

The electroweak Standard Model (SM) [1] has been extremely successful in the interpretation of LEP/SLC data and higher order effects typically amount to 10 σ deviations if not taken into account [2]. These precise predictions are only possible due to the renormalizability [3] of the SM and the by now very precise knowledge of the relevant input parameters. Last but not least the relevant coupling constants are small enough such that perturbation theory mostly works very well.

The formal proofs of renormalizability of the SM [4] often relied on the assumption that a gauge invariant regularization exists. The question whether such a regularization exists is non–trivial because of the chiral structure of the fermions involved. At present the only regularization, which makes elaborate computations of radiative corrections feasible, is the dimensional regularization (DR) scheme [5, 6] which is well-defined for field theories with vectorial gauge symmetries only. However, in theories exhibiting chiral fermions, like the electroweak SM, problems with the continuation of the Dirac matrix γ_5 to dimensions $D \neq 4$ remain open within this context and several modifications of the 't Hooft–Veltman DR have been proposed [7, 8, 9, 10, 11, 12, 13, 14, 15]. It turns out that starting from the standard SM-Lagrangian and using a γ_5, which does not anticommute with the other Dirac matrices γ^μ, leads to "spurious anomalies" which violate chiral symmetry and hence gauge invariance. These anomalies would spoil renormalizability if we would not get rid of them by imposing "by hand" the relevant Ward-Takahashi (WT) [16] and Slavnov-Taylor (ST) [17, 18] identities order by order in perturbation theory [13, 19, 20, 21, 22]. At first sight this might not look to be a serious problem, however, violating the symmetries of the SM makes practical calculations much more difficult and tedious than they are anyway.

The problems of course are related to the existence of the Adler–Bell–Jackiw (ABJ) anomaly [23], which must cancel in the SM in order not to spoil its renormalizability [24].

Surprisingly, the prescriptions proposed and/or used by many authors continue to be controversial [9, 11, 13, 15, 20, 25, 26, 27, 28, 29, 30, 31, 32], and hence it seems to be necessary to reconsider the problem once again. We shall emphasize, in particular, the advantage of working with chiral fields. The consequences of working as closely as possible with chiral fields, it seems to me, has not been stressed sufficiently in the literature so far.

As a matter of principle it is important to mention two other approaches which both work in $D = 4$ dimensions. i) In quantum field theories on the lattice a recent breakthrough was the discovery of exact chiral invariance on the lattice [33] which circumvents the Nielsen–Ninomiya no–go theorem [34]. A well defined regularization which preserves simultaneously chiral–and gauge–symmetries is thus known and could be applied to the SM. ii) The algebraic renormalization of the electroweak SM to all orders [35] within the Bogoliubov-Parasiuk-Hepp-Zimmermann (BPHZ) framework is a mathematically well defined scheme, which is much more involved because it breaks the symmetries at intermediate stages and hence leads to much longer expressions which are extremely tedious to handle in practice. In cases of doubt this is the only known scheme which is free of ambiguities and works directly in 4–dimensional continuum field theory.

For perturbative calculations in the continuum we have to stick as much as possible to the

more practical route of dimensional regularization. In the following tensor quantities in $D = 4$ dimensions are supposed to be defined by interpolation of $D = 2n$ ($n \geq 2$, integer) dimensions to dimensions below $D = 4$. It is well known that the γ–algebra, the so called "naive dimensional regularization" (NDR) [5]

$$\{\gamma^\mu, \gamma^\nu\} = 2g_{\mu\nu} \cdot \mathbf{1} \ , \quad g^\mu{}_\mu = D \ , \quad \mathrm{AC}(\mu) \equiv \{\gamma^\mu, \gamma_5\} = 0 \tag{9.2}$$

for dimensions of space–time $D = 4 - 2\epsilon$, $\epsilon \neq 0$ is inconsistent with

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) \neq 0 \ . \tag{9.3}$$

The latter condition is often considered to be necessary, however, for an acceptable regularization since at $D = 4$ we must find

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) = 4\mathrm{i}\varepsilon^{\mu\nu\rho\sigma} \ . \tag{9.4}$$

Generally, for γ_5 odd traces one obtains trace conditions from the cyclic property of traces. They are not fulfilled automatically, as we shall see, and hence the algebra is ill-defined in general. Considering $\mathrm{Tr}(\prod_{j=0}^{4} \gamma^{\mu_j}\gamma_5\gamma^\alpha)$ cyclicity requires

$$\mathrm{Tr}(\prod_{j=0}^{4} \gamma^{\mu_j}\mathrm{AC}(\alpha)) - 2\sum_{i=0}^{4}(-1)^i g^{\alpha\mu_i}\mathrm{Tr}(\prod_{j=0,\, j\neq i}^{4} \gamma^{\mu_j}\gamma_5) = 0 \ . \tag{9.5}$$

Contraction with the metric tensor $g_{\alpha\mu_0}$ yields

$$2\,(g^\alpha{}_\alpha - 4)\,\mathrm{Tr}(\prod_{j=1}^{4} \gamma^{\mu_j}\gamma_5) + \mathrm{Tr}(\prod_{j=1}^{4} \gamma^{\mu_j}\mathrm{AC}(\gamma)) = 0 \tag{9.6}$$

with $\mathrm{AC}(\gamma) \equiv \gamma_\alpha\mathrm{AC}(\alpha)$. Thus $g^\alpha{}_\alpha = D \neq 4$ together with (9.3) implies $\mathrm{AC}(\mu) \neq 0$. However, non-anti-commutativity of γ_5 is in conflict with the chiral structure and hence with gauge invariance of the SM, in general. It is the purpose of this note to study the possibility of restoring gauge invariance by employing chiral fields systematically.

Formally gauge invariant Feynman rules

Obviously only terms involving γ^μ in the standard SM Lagrangian can be affected by a non–anticommuting γ_5. As an example we consider the leptonic part, given by

$$\begin{aligned} \mathcal{L}_\ell &= \bar{\ell}_\mathrm{R} i\gamma^\mu\,(\partial_\mu + ig'B_\mu)\ell_\mathrm{R} + \bar{\nu}_{\ell\mathrm{R}} i\gamma^\mu\partial_\mu\nu_{\ell\mathrm{R}} \\ &+ \bar{L}_\ell i\gamma^\mu\,(\partial_\mu + i\frac{g'}{2}B_\mu - ig\frac{\tau_a}{2}W_{\mu a})\,L_\ell \end{aligned} \tag{9.7}$$

using standard notation. As usual the chiral fields

$$\ell_\mathrm{R} = \Pi_+\ell \ , \quad \nu_{\ell\mathrm{R}} = \Pi_+\nu_\ell \ , \quad L_\ell = \begin{pmatrix} \nu \\ \ell \end{pmatrix}_\mathrm{L} = \Pi_- \begin{pmatrix} \nu \\ \ell \end{pmatrix} \tag{9.8}$$

[5] I read it as "normal dimensional regularization"

may be represented in terms of the lepton fields $\ell(x)$ and the neutrino field $\nu_\ell(x)$ with the help of the chiral projectors

$$\Pi_\pm \equiv \frac{1}{2}\left(\mathbf{1} \pm \gamma_5\right) . \tag{9.9}$$

In order that $\Pi_\pm$ are Hermitean projection operators γ_5 must have the properties

$$\gamma_5^2 = \mathbf{1} \ , \quad \gamma_5^+ = \gamma_5 \ . \tag{9.10}$$

Furthermore, we demand $\Pi_\pm$ to be chiral projectors also for the adjoint $\bar{\psi} = \psi^+\gamma^0$ of a Dirac field ψ. This implies

$$\{\gamma^0, \gamma_5\} = 0 \ . \tag{9.11}$$

By Lorentz covariance in the 4–dimensional physical subspace the latter condition extends to

$$\{\gamma^\mu, \gamma_5\} = 0 \ \text{ for } \ \mu = 0, 1, 2, 3 \ . \tag{9.12}$$

It is easy to verify that $\mathcal{L}_\ell$ is invariant under local $SU(2)_{\rm L} \otimes U(1)_{\rm Y}$ gauge transformations, irrespective of $\mathrm{AC}(\mu) \neq 0$. Since the chiral fields have the simple transformation properties

$$\begin{aligned} L_\ell &\rightarrow \exp\{-i/2\,(g'\beta - g\tau_a\omega_a)\,\Pi_-\}L_\ell = \exp\{-i/2\,(g'\beta - g\tau_a\omega_a)\}L_\ell \\ \ell_{\rm R} &\rightarrow \exp\{-ig'\beta\Pi_+\}\ell_{\rm R} = \exp\{-ig'\beta\}\ell_{\rm R} \\ \nu_{\rm R} &\rightarrow \nu_{\rm R} \ , \end{aligned} \tag{9.13}$$

the invariance of $\mathcal{L}_\ell$ follows immediately from the properties of $\Pi_\pm$ alone.

We notice that in utilizing chiral fields there seems to be no conflict with the non–anticommutativity of γ_5 and the formal validity of the ST–identities.

Usually, one prefers to write Feynman rules in terms of the Dirac fields ℓ and ν_ℓ. The standard Feynman rules are obtained using the relations

$$\bar{\psi}\Pi_\mp\gamma^\mu\Pi_\pm\psi = \bar{\psi}\gamma^\mu\Pi_\pm\psi \ , \tag{9.14}$$

which are valid only, provided $\mathrm{AC}(\mu) = 0$.

If $\mathrm{AC}(\mu) \neq 0$ in $D \neq 4$ dimensional space–time, the above relations no longer hold and hence the standard Feynman rules manifestly violate gauge invariance. The correct relations, replacing (9.14), read

$$\bar{\psi}\Pi_\mp\gamma^\mu\Pi_\pm\psi = \frac{1}{2}\bar{\psi}\ \left(\Gamma^\mu \pm \Gamma_5^\mu\right)\ \psi \ , \tag{9.15}$$

with

$$\Gamma^\mu \equiv \gamma^\mu - \frac{1}{2}\mathrm{AC}(\mu)\gamma_5 = \frac{1}{2}\left(\gamma^\mu - \gamma_5\gamma^\mu\gamma_5\right) \tag{9.16}$$

and

$$\Gamma_5^\mu = \frac{1}{2}[\gamma^\mu, \gamma_5] = \Gamma^\mu\gamma_5 \ . \tag{9.17}$$

We notice that by definition all Γ's are anticommuting with γ_5

$$\{\Gamma^\mu, \gamma_5\} \equiv 0 \ . \tag{9.18}$$

According to (9.15) the proper expressions for the vector current and for the axial–vector current read

$$V^\mu(x) = \bar{\psi}\Gamma^\mu\psi = \bar{\psi}\gamma^\mu\psi - \frac{1}{2}\bar{\psi}\mathrm{AC}(\mu)\gamma_5\psi \tag{9.19}$$

and

$$A^\mu(x) = \bar\psi\Gamma^\mu\gamma_5\psi = \bar\psi\gamma^\mu\gamma_5\psi - \frac{1}{2}\bar\psi \mathrm{AC}(\mu)\psi \ , \tag{9.20}$$

respectively. It might be worthwhile to point out that the standard form of the axial current $\bar\psi\gamma^\mu\gamma_5\psi$ is not Hermitean when $\mathrm{AC}(\mu) \neq 0$. The above consideration also shows how anomalies may come about in the vector current when $\gamma_5\gamma^\mu\gamma_5 \neq -\gamma^\mu$.

The fermion kinetic term changes to

$$\bar\psi i\Gamma^\mu\partial_\mu\psi = \bar\psi i\gamma^\mu\partial_\mu\psi - \frac{1}{2}\bar\psi i\mathrm{AC}(\mu)\gamma_5\partial_\mu\psi \ . \tag{9.21}$$

Correspondingly, the free massless fermion fields must satisfy the field equation

$$\left(\gamma^\mu - \frac{1}{2}\mathrm{AC}(\mu)\gamma_5\right)\partial_\mu\psi = 0 \ . \tag{9.22}$$

This formally implies that the conserved canonical Noether currents are precisely the ones given above.

By the field equation the fermion spinors satisfy

$$\begin{aligned}\left(\not k - \frac{1}{2}\mathrm{AC}(k)\gamma_5 - m\right)\, u(k,s) &= 0\\ \left(\not k - \frac{1}{2}\mathrm{AC}(k)\gamma_5 + m\right)\, v(k,s) &= 0\end{aligned} \tag{9.23}$$

and the free fermion propagator reads ($\mathrm{AC}(k) \equiv k_\mu\mathrm{AC}(\mu)$)

$$S_F(k) = \frac{1}{\not k - \frac{1}{2}\mathrm{AC}(k)\gamma_5 - m + i0} = \frac{\not k - \frac{1}{2}\mathrm{AC}(k)\gamma_5 + m}{K^2 - m^2 + i0} \tag{9.24}$$

with

$$K^2 \equiv k^2 - \frac{1}{4}\mathrm{AC}(k)\mathrm{AC}(k) \ . \tag{9.25}$$

Formally, we have obtained chiral and gauge invariant Feynman rules for non-anticommuting γ_5. Eqs. (9.19), (9.20) and (9.24) replace the standard expressions valid for $\mathrm{AC}(\mu) = 0$.

Non-existence of a chirally invariant DR

The gauge invariant Feynman rules presented in the preceding section do not permit a regularization by continuation in the dimension D when $\mathrm{AC}(\mu)$ is chosen compatible with the trace condition (9.3). This can be proven as follows. First we consider the Dirac algebra extended to $D = 2n$ ($n \geq 2$, integer). In this case 2^n–dimensional representations of the γ–algebra are well known [8]. A basis for the algebra is given by the set of matrices $\mathbf{1}$, γ_5 and the antisymmetrized products $\gamma^{[\mu_1\cdots\mu_p]}$ associated with p–dimensional subspaces of M_D. We will split the $SO(1, D-1)$ vectors (tensors) into 4–dimensional vectors $p^\mu_\parallel = \hat p^\mu$ ($\mu = 0, 1, 2, 3$), in the physical subspace M_4, and their orthogonal complements $p^\mu_\perp = \bar p^\mu$ ($\mu = 4, ..., D-1$). If we impose the trace condition (9.3) in the physical subspace (see Eq. (9.12) above) we obtain the 't Hooft–Veltman algebra [5]:

$$\mathrm{AC}(\mu) = \begin{cases} 0 & ;\ \mu = 0, 1, 2, 3 \\ 2\bar\gamma^\mu\gamma_5 & ;\ \mu = 4, ..., D-1 \end{cases} \tag{9.26}$$

with $\gamma_5 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\hat{\gamma}^\mu\hat{\gamma}^\nu\hat{\gamma}^\rho\hat{\gamma}^\sigma$. Here, it is important to notice that $\mathrm{AC}(\mu)$ is a matrix of rank $\bar{\epsilon} \equiv D-4$. The matrix–elements themselves are of order $O(1)$. As a consequence higher products of AC-terms are not of higher order in $\bar{\epsilon}$ for $D \to 4$. This is the reason why the extra terms needed to restore the Ward-Takahashi identitiescannot be considered as perturbations. They affect the free part of the Lagrangian! and hence the form of the fermion propagators, as shown above. The symmetry at the end can only be there if the free and the interacting parts of the Lagrangian match appropriately.

We are now ready to reconsider the fermion propagator (9.24). Using (9.26), we get for the scalar product (32.63)

$$K^2 = k^2 - \bar{k}^2 = \hat{k}^2 \tag{9.27}$$

and thus

$$S_F(k) = \frac{\hat{\not{k}} + m}{\hat{k}^2 - m^2 + i0} \tag{9.28}$$

takes its 4–dimensional form, independent of D! It is then impossible to regularize fermion-loop integrals by continuation in D. The crucial point is that the consistency with the trace condition requires that in (32.63) the extra term proportional to $\mathrm{AC}(k)^2$ like AC is a matrix of rank $\bar{\epsilon} \equiv D-4$ and not a correction of order $O(\epsilon^2)$ in the ϵ–expansion!

The problem may be reconsidered in terms of the Γ–algebra defined by (9.16), which may be associated to any γ–algebra:

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu} \cdot \mathbf{1} - \frac{1}{4}\{\mathrm{AC}(\mu), \mathrm{AC}(\nu)\}\ , \quad \{\Gamma^\mu, \gamma_5\} = 0\ . \tag{9.29}$$

For any Γ–algebra in order to be closed, we must require

$$\{\Gamma^\mu, \Gamma^\nu\} \quad = \quad 2G^{\mu\nu} \cdot \mathbf{1} \tag{9.30}$$

for some symmetric $D \times D$–matrix G, which satisfies

$$g^\mu{}_\nu G^{\nu\rho} = G^{\mu\rho}\ . \tag{9.31}$$

The trace condition (9.5) must hold with the replacements

$$(\gamma^\mu,\ g^{\mu\nu},\ \mathrm{AC}(\mu) \neq 0) \to (\Gamma^\mu,\ G^{\mu\nu},\ \mathrm{AC}(\mu) = 0) \tag{9.32}$$

which implies

$$g_{\mu\nu}G^{\mu\nu} = G^\mu{}_\mu = 4\ . \tag{9.33}$$

Assuming G to have block–diagonal form

$$G = \left(\begin{array}{c|c} \hat{g} & 0 \\ \hline 0 & \bar{g} \end{array}\right) \tag{9.34}$$

the condition (9.3) can be satisfied with a singular metric G only:

$$\bar{g} = 0\ , \quad G = \hat{g} \tag{9.35}$$

where $\hat{g}$ must be the Minkowski metric. Thus, starting from the 't Hooft–Veltman scheme, we are lead to a dimensional reduction (DRED) scheme [10] by adding just some terms in the Feynman rules which vanish in $D = 4$.

As a result, the Γ–form of the 't Hooft–Veltman algebra is identical to the 4–dimensional Dirac algebra. In other words, using the 't Hooft–Veltman algebra (in its D–dimensional form) together with the chiral fields, which are adapted to the gauge symmetry, "non-regularization" of fermion–loops is implied. Again, a regularization can only be obtained by giving up either the trace condition (9.3) or gauge invariance.

This last statement, of course, is not terribly new. What we have shown is that the Dirac algebra assuming anticommuting γ_5 on the one hand and the 't Hooft–Veltman algebra on the other hand are not really different, since the latter can always be rewritten in the anticommuting Γ–form by means of the relations (9.16) and (9.17). In any case, for theories involving γ_5, "dimensional regularization" compatible with (9.5), does not provide well–defined integrals for loops involving fermion lines. This has been noticed by 't Hooft and Veltman in their original paper [5] where they state: "the usual ambiguity of choice of integration variables is replaced in our formalism by the ambiguity of location of γ_5 in the trace". Statements to the contrary, frequently found in the literature, are misleading. Usually, extra "prescriptions" about where to put the γ_5 in a particular calculation are proposed. These prescriptions, however, do not resolve the problem of mathematical inconsistencies, i.e., they still require an explicit check and the restoration of the Ward-Takahashi identities.

The use of chiral fields provides an unambiguous rule for the proper location of the γ_5–matrices before generalization to $D \neq 4$. Unfortunately, this has lead to the "non-regularization" by dimensional continuation when the $D \neq 4$ trace condition (9.3) is imposed, which in turn essentially implies the 't Hooft–Veltman scheme.

If we violate gauge invariance by the naive application of the 't Hooft–Veltman prescription, we have to restore the symmetry by imposing the relevant Ward–Takahashi identities and fixing appropriate counter terms. But this precisely amounts to including the extra $\mathrm{AC}(\mu)$ terms given in Eqs. (9.19) and (9.20). Which in turn is nothing but another way of utilizing the naive anticommuting γ_5.

Conclusion for the practitioner

According to our considerations above we are left with two possible strategies:

i) $\mathrm{AC}(\mu) \neq 0$: the chirally improved 't Hooft–Veltman scheme

If we insist on the trace condition (9.3) the gauge invariance must be manifestly broken in order to obtain the "pseudo regularization" by dimensional continuation. Again we start at the level of the chiral fields but must avoid the non–regularization by treating the AC–terms in the free part of the Lagrangian as interaction terms, i.e., we use the standard D–dimensional Fermi propagator

$$S_F(k) = \frac{\not{k} + m}{k^2 - m^2 + i0} \tag{9.36}$$

together with the chiral currents (9.19, 9.20) as our "chiral Feynman rules". Since $\mathrm{AC}(\mu) \neq 0$, the choice of the Fermi propagator (9.36) amounts to adding the symmetry breaking term

$$\Delta\mathcal{L}_{\mathrm{SB}} = \frac{1}{2}\bar{\psi} i \mathrm{AC}(\mu)\gamma_5 \partial_\mu \psi = \bar{\psi} i \bar{\gamma}^\mu \partial_\mu \psi \tag{9.37}$$

to the Lagrangian. Besides the fact that this operator has no 4–dimensional representation, it is not a higher order term for $D \neq 4$ as it would be necessary for treating it as a counter-term

(perturbation). Expanding $\Delta\mathcal{L}_{\rm SB}$ perturbatively amounts to the assumption that $\mathrm{AC}(\mu) = O(\epsilon)$ in the sense of matrix elements, which conflicts with (9.3). As we have mentioned earlier, (9.3) requires $\mathrm{AC}(\mu)$ to be a matrix of rank $\bar{\epsilon} = D - 4$ with matrix elements of order $O(1)$. A mathematically satisfactory way out of the dilemma within the framework of DR is not possible as a result of the existence of the ABJ–anomaly.

Our considerations show that "quasi gauge invariant" Feynman rules may be obtained for non-anticommuting γ_5 provided $\mathrm{AC}(\mu)$ is treated as a perturbation i.e. $\mathrm{AC}(\mu) = O(\epsilon)$. Examples are briefly considered in the Appendix. Results turn out to be AC–independent in this case. AC–invariance may be used as a helpful tool for checking the gauge invariance of fermionic loop contributions to amplitudes. Usually such checks are possible only by explicit consideration of WT- and/or ST-identities. We stress, once again, that any approach which treats the AC–term as a perturbation conflicts with the trace condition (9.3) at some point. Ignoring this point leads to "standard" confusions, frequently appearing in the literature. While working with the 't Hooft–Veltman prescription in the standard form requires the subsequent check of the Ward-Takahashi identities,after utilizing the chiral version of the Feynman rules we may restrict ourselves to check the hard anomaly diagrams.

Since amplitudes exhibiting spurious anomalies only may be chiralized either by our chirally improved Feynman rules or by imposing the Ward-Takahashi identitieswhich makes them AC–invariant we obviously may directly choose the scheme $\mathrm{AC}(\mu) = 0$, which is our second and preferred option:

ii) $\mathrm{AC}(\mu) = 0$**: the quasi self-chiral scheme**

From a practical point of view an acceptable computational scheme should avoid spurious anomalies in the first place. This is possible only if the trace condition (9.3) is given up. Gauge invariance can be preserved then by using an anticommuting γ_5. This has been noticed in [7] (see also [25, 26, 28, 38]).

We observe that taking chiral fields seriously on a formal level, the only consistent way to avoid the above non-regularization is the simple one: use anticommuting γ_5 from the very beginning, i.e., choose the NDR algebra (9.2). Since $\Gamma^\mu \equiv \gamma^\mu$ in this case we do not get the non-regularization of the fermion propagators. The ABJ–anomaly must be considered separately as we are going to discuss now [6].

In the gauge invariant approach, closed fermion loops exhibiting γ_5 odd traces and hard anomalies, cannot be obtained by dimensional continuation, merely, γ_5 odd traces are to be considered as intrinsically 4-dimensional quantities. Since charge conjugation properties and the related Bose symmetry are not automatically satisfied one has to account left- and right-circulation of the fermions in closed loops separately. In any case Adler's approach [39] can be utilized to resolve the remaining ambiguities. For this purpose, let us briefly consider the ABJ–anomaly [23] exhibited by the current correlator $< 0|T\{V^\mu(x_1)V^\nu(x_2)A^\lambda(y)\}|0 >$ of two vector currents and an axial–vector current. The one–loop diagrams are shown in Fig. 9.1.

In $D = 4$, working as usual in momentum space, we may perform a covariant decomposition of

[6] The terminology introduced in [9, 13] which calls a scheme "consistent" if it respects the trace condition (9.3) and "inconsistent" otherwise is definitely misleading by the considerations presented in this paper. Since we cannot satisfy the Ward-Takahashi identitiesand the trace condition simultaneously we have the choice which one we want to consider more fundamental. Something has to be restored at the end by hand in any case. To put into place the model independent ABJ–anomalies, is by far simpler, than restoring the chiral symmetry which is broken by non–NDR schemes.

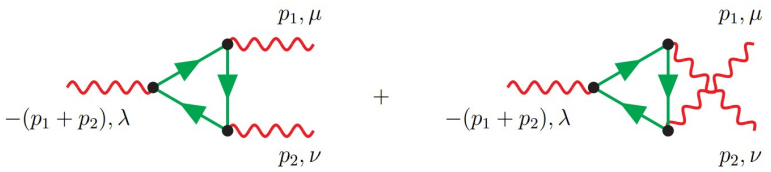


Figure 9.1: The VVA triangle diagrams.

the third rank pseudotensor which depends on the two independent momenta p_1 and p_2:

$$\begin{aligned} A^{\mu\nu\lambda}(p_1,p_2) &= \varepsilon^{\mu\nu\lambda\alpha}\ (p_{1\alpha}\, A_1 + p_{2\alpha}\, A_2) \\ &+ \varepsilon^{\mu\lambda\alpha\beta}\, p_{1\alpha}p_{2\beta}\ (p_1^\nu\, A_3 + p_2^\nu\, A_4) \\ &+ \varepsilon^{\nu\lambda\alpha\beta}\, p_{1\alpha}p_{2\beta}\ (p_1^\mu\, A_5 + p_2^\mu\, A_6) \\ &+ \varepsilon^{\mu\nu\alpha\beta}\, p_{1\alpha}p_{2\beta}\ \left(p_1^\lambda\, A_7 + p_2^\lambda\, A_8\right) \end{aligned} \tag{9.38}$$

where the amplitudes A_i are Lorentz scalars. We now impose

- Bose symmetry (i.e. consider the sum of the two diagrams of Fig. 9.1):

$$A^{\mu\nu\lambda}(p_1,p_2) = A^{\nu\mu\lambda}(p_2,p_1)$$

which implies

$$\begin{aligned} A_1(p_1,p_2) &= -A_2(p_2,p_1), \quad A_3(p_1,p_2) = -A_6(p_2,p_1), \\ A_4(p_1,p_2) &= -A_5(p_2,p_1), \quad A_7(p_1,p_2) = +A_8(p_2,p_1). \end{aligned} \tag{9.39}$$

- Vector current conservation:

$$p_{1\mu}A^{\mu\nu\lambda} = p_{2\nu}A^{\mu\nu\lambda} = 0$$

which implies

$$\begin{aligned} A_1 &= -\left(p_2^2\, A_4 + p_1p_2\, A_3\right) \\ A_2 &= -\left(p_1^2\, A_5 + p_1p_2\, A_6\right)\ . \end{aligned} \tag{9.40}$$

We thus find that the amplitudes A_1 and A_2 are determined uniquely in terms of the A_i, $i = 3, ..., 6$. The crucial observation, made by Adler long time ago [39], is that the amplitudes A_i, $i = 3, ..., 8$, have dimension $d_{\rm eff} = 1 - 3 = -2$ and hence are represented by convergent integrals. In contrast, A_i, $i = 1, 2$, have dimension $d_{\rm eff} = 1 - 1 = 0$ (logarithmically divergent) and thus require regularization and renormalization. However, imposing Bose symmetry and vector current conservation uniquely determines the two regularization/renormalization dependent amplitudes in terms of the other convergent and hence unambiguous ones, i.e., the result is unique without need to refer to a specific renormalization scheme. The divergence of the axial–vector current takes the form

$$-(p_1+p_2)_\lambda A^{\mu\nu\lambda} = 2mR^{\mu\nu} + 8\pi^2 p_{1\alpha}p_{2\beta}\varepsilon^{\alpha\beta\mu\nu} \neq 0$$

where the first term on the r.h.s. is the normal term which vanishes for vanishing fermion mass m while the second term is the mass independent anomaly. Formal axial–vector current conservation in the limit of vanishing fermion mass would require

$$A_1 - A_2 - (p_1^2 + p_1p_2)\, A_7 - (p_2^2 + p_1p_2)\, A_8 = 0\,?$$

with A_1 and A_2 fixed already by vector current conservation, this expression as we know does not vanish but yields the famous axial–vector current anomaly. All true anomalies, i.e.,quantum effects like the triangle anomaly which cannot be removed by adding a corresponding counter term to the Lagrangian, are well known to be related to the triangle diagram. Besides the triangle diagram itself they appear by tensor reduction from one–loop box and pentagon diagrams and diagrams which contain the one–loop anomalous graphs as subgraphs.

The Adler–Bardeen non-renormalization theorem [36] of the one–loop anomalies implies that matters are under control provided Bose symmetry and vector current conservation are imposed, if necessary by hand. In DR it has been reconsidered in [37, 38]. Last but not least we must have the anomaly cancelation, possible by virtue of the quark lepton duality, in order to have the SM renormalizable [24].

Summary: we have shown that different γ_5–schemes may be related by adding suitable terms in the D–dimensional Lagrangian which vanish at $D = 4$. In any scheme we can mimic chiral fields by the appropriate choice of the Feynman rules. We consider this to be crucial since the physical SM derives via a Higgs mechanism from a symmetric phase which exhibits chiral fermions only. The corresponding "chiral completion" (see (9.19,9.20)) of the Feynman rules cannot make a consistent scheme inconsistent or vice versa. Avoidable (often called "spurious") anomalies are then absent. Our arguments strongly support the application of the NDR scheme (9.2), i.e., the D–dimensional γ–algebra together with a strictly anticommuting γ_5, together with the simple 4–dimensional treatment of the hard anomalies discussed above. The NDR is easily implemented into computer codes and is by far the most convenient and efficient approach in calculations of radiative corrections. Removable anomalies are avoided and hence a tedious procedure of restoration of WT- and ST-identities is not needed.

The rules advocated here have been utilized successfully in the last twenty years by many authors at the one– and the two–loop level and beyond. Most SM calculations of higher order effects adopted the NDR scheme without encountering any inconsistencies. Of course, the NDR scheme has been advocated by several authors [7, 11, 25, 26, 28, 38] (see also [30]) in the past. I hope the present paper contributes to clarify part of the ongoing controversy.

Appendix: Calculations with $\mathrm{AC}(\mu) \neq 0$ in the SM: Two examples.

We [40] have verified explicitly that all spurious anomalies disappear from fermion propagators and fermion form factors at one-loop order for the case where we use Feynman rules as proposed in Sec. 9.8.2 in case $\mathrm{AC}(\mu) \neq 0$. As explained earlier, in order to avoid the "non-regularization" of fermion lines, we must treat AC as a perturbation $\mathrm{AC}(\mu) = O(\epsilon)$ and work to linear order in AC. All calculations have been performed in the 't Hooft gauge with an arbitrary gauge parameter ξ, which makes possible direct analytical checks of gauge invariance. We only summarize the structure of the results.

The irreducible self-energy $\Sigma(k)$ we obtained has the following form

$$\Sigma(k) = \left(\not{k} - m - \frac{1}{2}\mathrm{AC}(k)\gamma_5\right) A + \frac{1}{2}[\not{k}, \gamma_5]\, B + mC\,. \tag{9.41}$$

This implies that the mass- and wave-function renormalization are completely AC–independent:

$$\delta m = -mc_0\,, \quad \sqrt{Z_2} = 1 - \frac{1}{2}a_0 - \frac{1}{2}b_0\gamma_5\,. \tag{9.42}$$

Here the wave-function renormalization constant is given by the matrix

$$\sqrt{Z_2} = \sqrt{Z_R}\,\Pi_+ \;+\; \sqrt{Z_L}\,\Pi_- \tag{9.43}$$

where $\sqrt{Z_R}$ and $\sqrt{Z_L}$ are the independent wave–function renormalizations of the right–handed and left–handed fields, respectively. Thus the renormalized self–energy reads

$$\Sigma_r(k) = \left(\not{k} - m - \frac{1}{2}\mathrm{AC}(k)\gamma_5 \right) (A - a_0) + \frac{1}{2}[\not{k}, \gamma_5] (B - b_0) + m (C - c_0) \tag{9.44}$$

with $A - a_0$, $B - b_0$ and $C - c_0$ finite, and hence

$$\Sigma_r(k) = (\not{k} - m) (A - a_0) + \not{k}\gamma_5 (B - b_0) + m (C - c_0) + O(\epsilon) \tag{9.45}$$

By contrast, using standard Feynman rules, we obtain

$$\begin{aligned} \Sigma(k) &= (\not{k} - m) A + \frac{1}{2}[\not{k}, \gamma_5] B + mC \\ &+ \mathrm{AC}(k)\gamma_5 D + [\not{k}, \mathrm{AC}(\gamma)] E + [\not{k}, \mathrm{AC}(\gamma)] \gamma_5 F \end{aligned} \tag{9.46}$$

for the bare self-energy. In this case it is not possible to perform the renormalization in the standard way without imposing the Ward-Takahashi identitiesfirst, which must lead to the form (9.41).

Similar results can be found for form-factors. The following applies to the $\bar{\ell}\ell\gamma$ and $\bar{\ell}\ell Z$ vertices. The general form of the irreducible vertices reads

$$\Pi^\mu(p_1, p_2) = \left(\gamma^\mu - m - \frac{1}{2}\mathrm{AC}(\mu)\gamma_5 \right) F_1 + \frac{1}{2}[\gamma^\mu, \gamma_5] F_2 + p_1^\mu F_3 + p_2^\mu F_4 \,. \tag{9.47}$$

We notice that the only surviving AC–term is $\mathrm{AC}(\mu)\gamma_5$ which appears in the canonical from (9.16) as in the Born term. Thus the vertex renormalization can be performed in an AC–independent way, i.e., the renormalized vertex is given by

$$\Pi_r^\mu(p_1, p_2) = \left(\gamma^\mu - m - \frac{1}{2}\mathrm{AC}(\mu)\gamma_5 \right) (F_1 - c_1) + \frac{1}{2}[\gamma^\mu, \gamma_5] (F_2 - c_2) + p_1^\mu F_3 + p_2^\mu F_4 \tag{9.48}$$

with $F_1 - c_1$, $F_2 - c_2$, F_3 and F_4 finite. Hence, we have

$$\Pi_r^\mu(p_1, p_2) = \gamma^\mu (F_1 - c_1) + \gamma^\mu\gamma_5 (F_2 - c_2) + p_1^\mu F_3 + p_2^\mu F_4 + O(\epsilon) \tag{9.49}$$

independent of any AC–term. In contrast, by applying standard Feynman rules, we find additional terms of the form $\mathrm{AC}(\mu)\gamma_5$, $[\gamma^\mu, \mathrm{AC}(\gamma)]$ and $\{\gamma^\mu, \mathrm{AC}(\gamma)\}\gamma_5$ which cannot be removed by renormalization, unless we impose the Ward-Takahashi identitiesfirst. In the chiral scheme we obtain gauge invariant form factors directly without imposing Ward-Takahashi identitiesby hand. Calculations in this "chiral" scheme in fact look very similar to the ones performed with anticommuting γ_5.

As a result of these findings we decided to work with an anti-commuting γ_5 henceforth, first at the one–loop level [41, 42], later at the two–loop level [43, 44, 45]. In most of these calculations we worked in the 't Hooft gauge with a free gauge parameter which allowed us to check explicitly the gauge invariance of on-shell matrix elements.

10 Other Properties of Free Fields and Other Fields

10.0.1 Зи.I.10 Quantum Field Theory in Curved Spacetime

Общее преобразование координат

В теории гравитации Эйнштейна инвариантный пространственно-временной интервал Минковского $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = (dt)^2 - (d\vec{x})^2$ заменяется на $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, где

метрический тензор $g_{\mu\nu}(x)$ является функцией пространственно-временных координат x. Основополагающий принцип, известный как принцип общей ковариантности, гласит, что заключенная в действии S физика должна быть инвариантна относительно произвольного преобразования координат $x \to x'(x)$. Точнее, принцип [1] гласит, что с соответствующими ограничениями эффект гравитационного поля эквивалентен эффекту преобразования координат.

Так как

$$ds^2 = g'_{\lambda\sigma} dx'^{\lambda} dx'^{\sigma} = g'_{\lambda\sigma} \frac{\partial x'^{\lambda}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} dx^{\mu} dx^{\nu} = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

метрика преобразуется как

$$g'_{\lambda\sigma}\left(x'\right) \frac{\partial x'^{\lambda}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = g_{\mu\nu}(x)$$

Обратная метрика $g^{\mu\nu}$ определяется выражением $g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho}$.

Скалярное поле, в соответствии со своим названием, не преобразуется: $\varphi(x) = \varphi'\left(x'\right)$. Градиент скалярного поля преобразуется как

$$\partial_{\mu}\varphi(x) = \frac{\partial x'^{\lambda}}{\partial x^{\mu}} \frac{\partial \varphi'\left(x'\right)}{\partial x'^{\lambda}} = \frac{\partial x^{\lambda}}{\partial x^{\mu}} \partial'_{\lambda} \varphi'\left(x'\right)$$

[1] Точную формулировку принципа общей ковариантности смотрите в книге С. Вейнберга, Гравитачия и космология. с. 92. По определению (ковариантное) векторное поле преобразуется согласно

$$A_{\mu}(x) = \frac{\partial x'^{\lambda}}{\partial x^{\mu}} A'_{\lambda}\left(x'\right)$$

так что $\partial_{\mu}\varphi(x)$ является векторным полем. Если известны два векторных поля $A_{\mu}(x)$ и $B_{\nu}(x)$, можно свернуть их с $g^{\mu\nu}(x)$ в форму $g^{\mu\nu}(x) A_{\mu}(x) B_{\nu}(x)$, которая, как вы можете непосредственно проверить, является скаляром. В частности, $g^{\mu\nu}(x)\partial_{\mu}\varphi(x)\partial_{\nu}\varphi(x)$ является скаляром. Таким образом, если в лагранжиане $\mathcal{L} = \frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right] = \frac{1}{2}\left(\eta^{\mu\nu}\partial_{\mu}\varphi\partial_{\nu}\varphi - \right.$ $\left. -m^2\varphi^2\right)$ заменить метрику Минковского $\eta^{\mu\nu}$ на метрику Эйнштейна $g^{\mu\nu}$, то он станет координатно-инвариантным.

Действие находится интегрированием лагранжиана по пространствувремени. При координатном преобразовании имеет место $d^4x' = d^4x \times \ \times \det\left(\partial x'/\partial x\right)$. Вычисляя детерминант выражения (1), получаем

$$g \equiv \det g_{\mu\nu} = \det g'_{\lambda\sigma} \frac{\partial x'^{\lambda}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} = g' \left[\det\left(\frac{\partial x'}{\partial x}\right)\right]^2$$

Видим, что комбинация $d^4x\sqrt{-g} = d^4x'\sqrt{-g'}$ является координатно-инвариантной.

Таким образом, для произвольной квантовой теории поля можно немедленно записать соответствующую теорию в искривленном пространстве-времени. Все, что для этого требуется, - это заменить в нашем лагранжиане метрику Минковского $\eta^{\mu\nu}$ на метрику Эйнштейна $g^{\mu\nu}$ и включить множитель $\sqrt{-g}$ в меру интегрирования по пространству-времени [2]. При этом действие будет инвариантным относительно произвольных преобразований координат. Например, действие S для скалярного поля в искривленном пространстве-времени просто равно

$$S = \int d^4x\sqrt{-g}\frac{1}{2}\left(g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - m^2\varphi^2\right)$$

(Для поля со спином 1/2 имеется некоторая тонкость, о которой мы поговорим в части II. Со временем мы вернемся к этому в главе VIII.1.)

В квантовании скалярного поля в искривленном пространстве-времени нет никакой принципиальной трудности. Мы просто оперируем метрикой $g_{\mu\nu}$ как заданной [например, это может быть метрика Шварцшильда

[2] Мы также должны заменить обычные производные ∂_μ на ковариантные D_μ из теории относительности, но при действии на скалярное поле φ ковариантная производная совпадает с обычной $D_\mu\varphi = \partial_\mu\varphi$. в сферических координатах: $g_{00} = (1-2GM/r), g_{rr} = -(1-2GM/r)^{-1}$, $g_{\theta\theta} = -r^2$ и $g_{\phi\phi} = -r^2\sin\theta$] и изучаем интеграл по траекториям $\int D\varphi e^{iS}$, который по-прежнему является гауссовым и потому вычислимым. Можно получить пропагатор $D(x,y)$ скалярного поля и так далее, и тому подобное. (См. упражнение I.10.1.)

На данном этапе метрика $g_{\mu\nu}$ выглядит в точности как поле, если не считать того факта, что $G_{\mu\nu}(x)$ несет на себе лоренцевы индексы, а $\varphi(x)$ - нет, и на самом деле является классическим полем. Запишем действие мира $S = S_g + S_M$ в виде суммы двух членов: S_g, описывающего динамику гравитационного поля $g_{\mu\nu}$ (которое мы обсудим в главе VIII.1), и S_M, описывающего динамику всех остальных полей в мире [«материальных полей», как, например, φ в нашем простом примере с S_M, вычисляемым по (3)]. Мы могли бы проквантовать гравитацию, проинтегрировав также по $G_{\mu\nu}$, обобщая тем самым интеграл по траекториям до $\int DgD\varphi e^{iS}$.

Легчс сказать, чем сделать! Я уверен, вы слышали, что все попытки проинтегрировать по $g_{\mu\nu}(x)$ завершились непреодолимыми трудностями, так что в конечном итоге теоретикам пришлось искать утешение в теории струн. Когда придет время, я объясню, почему теорию Эйнштейна называют «неперенормируемой».

На что реагирует гравитон

Одним из наиболее глубоких результатов теории гравитации Эйнштейна является фундаментальное определение энергии и импульса. Чем же в точности являются энергия и импульс? Энергия и импульс - это то, на что реагирует гравитон. (Гравитон, конечно, есть частица, отвечающая полю $g_{\mu\nu}$)

Тензор энергии импульса $T^{\mu\nu}$ определяется как вариационная производная действия материи S по метрике $g_{\mu\nu}$ (при фиксированных координатах x^μ)

$$T^{\mu\nu}(x) = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}(x)}.$$

Энергия, по определению, равна $E = P^0 = \int d^3x\sqrt{-g}T^{00}(x)$, а импульс $P^i = \int d^3x\sqrt{-g}T^{0i}(x)$

Даже если нас не интересует само по себе искривленное пространство-время, выражение (4) по-прежнему дает нам простой (и фундаментальный) способ определения тензора энергии импульса для теории поля в плоском пространстве-времени. Мы просто варьируем метрику Минковского $\eta_{\mu\nu}$, записывая $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, и разлагаем S_M до первого порядка по h. Согласно (4) получаем [3]

$$S_M(h) = S_M(h=0) - \int d^4x \left[\frac{1}{2} h_{\mu\nu} T^{\mu\nu} + O\left(h^2\right)\right]$$

Симметричное тензорное поле $h_{\mu\nu}(x)$ является фактически полем гравитона (см. главы I.5 и VIII.1). Тензор энергии импульса $T^{\mu\nu}(x)$ - это то, с чем взаимодействует гравитон, точно так же, как электромагнитный ток $J^\mu(x)$ - это то, с чем взаимодействует фотон.

Рассмотрим общее выражение $S_M = \int d^4x\sqrt{-g}\left(A + g^{\mu\nu}B_{\mu\nu} + \ + g^{\mu\nu}g^{\lambda\rho}C_{\mu\nu\lambda\rho} + ...\right)$. Так как $-g = 1 + \eta^{\mu\nu}h_{\mu\nu} + O\left(h^2\right)$ и $g^{\mu\nu} = \eta^{\mu\nu}$ - $-h^{\mu\nu} + O\left(h^2\right)$, то в плоском пространстве-времени получаем

$$T_{\mu\nu} = 2\left(B_{\mu\nu} + 2C_{\mu\nu\lambda\rho}\eta^{\lambda\rho} + ...\right) - \eta_{\mu\nu}\mathcal{L}.$$

Заметим, что

$$T \equiv \eta^{\mu\nu}T_{\mu\nu} = -\left(4A + 2\eta^{\mu\nu}B_{\mu\nu} + 0\cdot\eta^{\mu\nu}\eta^{\lambda\rho}C_{\mu\nu\lambda\rho} + ...\right)$$

где специально подчеркивается тот факт, что $C_{\mu\nu\lambda\rho}$ не вносит вклада в след тензора энергии импульса.

Теперь продемонстрируем всю мощь подобного определения $T_{\mu\nu}$, получив давно известный результат для электромагнитного поля. Обобщая лагранжиан массивного поля со спином 1 до случая с искривленным пространством-временем, находим [4] $\mathcal{L} = \left(-\frac{1}{4}g^{\mu\nu}g^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho} + \frac{1}{2}m^2 g^{\mu\nu}A_\mu A_\nu\right)$ и, следовательно [5], $T_{\mu\nu} = -F_{\mu\nu}F^\lambda_\nu + m^2 A_\mu A_\nu - \eta_{\mu\nu}\mathcal{L}$.

Для электромагнитного поля полагаем $m = 0$. Во-первых, $\mathcal{L} = -\frac{1}{4}\times$ $\times F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\left(-2F_{0i}^2 + F_{ij}^2\right) = \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right)$. Таким образом, $T_{00} = -F_{0\lambda}\times$ $\times F_0^\lambda - \frac{1}{2}\left(\vec{E}^2 - \vec{B}^2\right) = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right)$. Успокаивает то, что мы получили

[3] Я использую обычное соглашение, согласно которому суммирование по индексам производится без учета какой-либо возможной симметрии; другими словами, $\frac{1}{2}h_{\mu\nu}T^{\mu\nu} = \frac{1}{2}\left(h_{01}T^{01} + h_{10}T^{10} + ...\right) = h_{01}T^{01} + ...$

[4] Здесь используется тот факт, что ковариантный ротор равен обычному ротору $D_\mu A_\nu$ - $-D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$, и поэтому $F_{\mu\nu}$ не включает в себя метрику.

[5] То. что координаты x^μ остаются фиксированными. означает, что фиксированным остается ∂_μ и, следовательно, A_μ. [Поскольку a_μ и ∂_μ связаны калибровочной инвариантностью здесь мы немного торопим события (см. главу II.7). но вы уже наверняка слышали о калибровочной инвариантности в нерелятивистском контексте.] известный «с детства» результат. Между прочим, становится ясно, что $\vec{E}^2$ можно рассматривать как кинетическую энергию, а $\vec{B}^2$ как потенциальную энергию. Далее, $T_{0i} = -F_{0\lambda}F_i^\lambda = F_{0j}F_{ij} = \varepsilon_{ijk}E_jB_k = (\vec{E}\times\vec{B})_i$ - это и есть вектор Пойнтинга.

Так как лагранжиан Максвелла $\mathcal{L} = -\frac{1}{4}g^{\mu\nu}g^{\lambda\rho}F_{\mu\lambda}F_{\nu\rho}$ включает в себя лишь C-член с $C_{\mu\nu\lambda\rho} = -\frac{1}{4}F_{\mu\nu}F_{\lambda\rho}$, из (7) следует, что тензор энергии импульса электромагнитного поля имеет нулевой след - это важный факт, который нам понадобится в главе VIII.1. Можно также непосредственно проверить, что $T = 0$ (упражнение I.10.4) [6].

Приложение: искривленное пространство-время - краткое введение

Общая теория относительности часто излагается более трудно и загадочно, чем следовало бы. Для дальнейшего использования я приведу здесь краткий обзор ее основных элементов.

Обозначим через X^μ пространственно-временные координаты точечной частицы. Для того чтобы построить для нее действие, заметим, что единственной координатно-инвариантной величиной является «длина» [7] мировой линии, вдоль которой движется частица (рис. I.10.1), а именно $\int ds = \int \sqrt{g_{\mu\nu} dX^\mu dX^\nu}$, где $G_{\mu\nu}$ вычисляется в точке нахождения частицы. Таким образом, действие для точечной частицы должно быть пропорционально

$$\int ds = \int \sqrt{g_{\mu\nu} dX^\mu dX^\nu} = \int \sqrt{g_{\mu\nu}[X(\zeta)] \frac{dX^\mu}{d\zeta} \frac{dX^\nu}{d\zeta}} d\zeta$$

где ζ - произвольный параметр, монотонно изменяющийся вдоль мировой линии. Длина, будучи геометрической величиной, явно является инвариантной относительно изменения параметризации, то есть не зависит от нашего выбора ζ, если он корректен. Это один из «более чем очевидных» фактов, так как $\int \sqrt{g_{\mu\nu} dX^\mu dX^\nu}$ явно не зависит от ζ. Если хотите, мы можем проверить параметризационную инвариантность $\int ds$.

[6] Мы видим, что условие бесследовости связано с тем. что электромагнитное поле не имеет масштаба. Как говорят, чистый электромагнетизм является масштабно-, или дилатационно-, инвариантным. Подробнее о дилатационной инвариантности можно узнать из работы Коулмена, Аспекты сияиметрии (S. Coleman. Aspects of Symmetry, p. 67).

[7] Мы употребили слово «длина» в кавычках потому, что если бы $g_{\mu\nu}$ имела евклидову сигнатуру, то величина $\int ds$ действительно соответствовала бы длине, и, минимизируя $\int ds$. мы бы получили кратчайшую траекторию (геодезическую) между конечными точками. но в нашем случае $g_{\mu\nu}$ имеет сигнатуру Минковского.

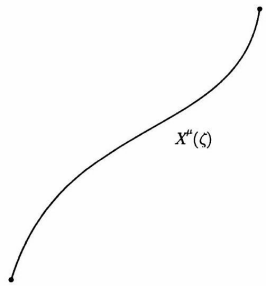


Рис. I.10.1

Очевидно, что степени $d\zeta$ согласуются. То есть если записать $\zeta = \zeta(\eta)$, то $dX^\mu/d\zeta = (d\eta/d\zeta)\,(dX^\mu/d\eta)$, то $d\zeta = (d\zeta/d\eta)d\eta$.

Для облегчения записи определим

$$K \equiv g_{\mu\nu}[X(\zeta)] \frac{dX^\mu}{d\zeta} \frac{dX^\nu}{d\zeta}$$

Считая изменение $\int d\zeta \sqrt{K}$ равным нулю, получаем выражение

$$\int d\zeta \frac{1}{\sqrt{K}} \left(2 g_{\mu\nu} \frac{dX^\mu}{d\zeta} \frac{d\delta X^\nu}{d\zeta} + \partial_\lambda g_{\mu\nu} \frac{dX^\mu}{d\zeta} \frac{dX^\nu}{d\zeta} \delta X^\lambda \right) = 0$$

которое после интегрирования по частям (при стандартном условии $\delta X^\lambda = 0$ на концах) приводит к уравнению движения

$$\sqrt{K}\frac{d}{d\zeta}\left(\frac{1}{\sqrt{K}}2g_{\mu\lambda}\frac{dX^\mu}{d\zeta}\right)-\partial_\lambda g_{\mu\nu}\frac{dX^\mu}{d\zeta}\frac{dX^\nu}{d\zeta}=0$$

Для упрощения (8) воспользуемся нашей свободой выбора ζ и положим $d\zeta = ds$ так, что $K = 1$. Получим

$$2g_{\mu\lambda}\frac{d^2X^\mu}{ds^2}+2\partial_\sigma g_{\mu\lambda}\frac{dX^\sigma}{ds}\frac{dX^\mu}{ds}-\partial_\lambda g_{\mu\nu}\frac{dX^\mu}{ds}\frac{dX^\nu}{ds}=0$$

Умножив на $g^{\rho\lambda}$, приходим к выражению

$$\frac{d^2X^\rho}{ds^2}+\frac{1}{2}g^{\rho\lambda}\left(2\partial_\nu g_{\mu\lambda}-\partial_\lambda g_{\mu\nu}\right)\frac{dX^\mu}{ds}\frac{dX^\nu}{ds}=0$$

То есть

$$\frac{d^2X^\rho}{ds^2}+\Gamma^\rho_{\mu\nu}[X(s)]\frac{dX^\mu}{ds}\frac{dX^\nu}{ds}=0$$

где мы определили символ Римана - Кристоффеля

$$\Gamma^\rho_{\mu\nu}\equiv\frac{1}{2}g^{\rho\lambda}\left(\partial_\mu g_{\nu\lambda}+\partial_\nu g_{\mu\lambda}-\partial_\lambda g_{\mu\nu}\right)$$

При заданном начальном положении $X^\mu(s_0)$ и скорости $(dX^\mu/ds)(s_0)$ имеем четыре дифференциальных уравнения второго порядка (9), определяющих геодезическую траекторию, по которой частица движется в искривленном пространстве-времени. Обратите внимание, что вопреки впечатлению, которое остается после прочтения некоторых книг, уравнение (9), в отличие от (8), не является репараметризационно инвариантным.

Для перехода к ньютоновской гравитации необходимо выполнение трех условий: (1) частицы двигаются медленно $dX^i/ds \ll dX^0/ds$; (2) гравитационное поле слабое, так что метрика близка к метрике Минковского $g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}$; и (3) гравитационное поле не зависит от времени. Условие (1) означает, что $d^2X^\rho/ds^2 + +\Gamma^\rho_{00}\left(dX^0/ds\right)^2 \simeq 0$, а (2) и (3) подразумевают, что $\Gamma^\rho_{00} \simeq -\frac{1}{2}\eta^{\rho\lambda}\partial_\lambda h_{00}$. Таким образом, (9) упрощается до вида $d^2X^0/ds^0 \simeq 0$ (и, как следствие, значение dX^0/ds остается неизменным) и $d^2X^i/ds^2 + \frac{1}{2}\partial_i h_{00}\left(dX^0/ds\right)^2 \simeq 0$, что, учитывая пропорциональность X^0 и s, сводится к виду $d^2X^i/dt^2 \simeq -\frac{1}{2}\partial_i h_{00}$. Таким образом, получаем уравнение Ньютона $\frac{d^2\vec{X}}{dt^2} \simeq -\vec{\nabla}\phi$, если определить гравитационный потенциал ϕ как $h_{00} = 2\phi$:

$$g_{00}\simeq 1+2\phi$$

С учетом метрики Шварцшильда приходим к выводу, что на больших расстояниях от массивного тела имеет место $\phi = -GM/r$, как мы и предполагали. (Заметьте, что процесс получения результата не зависит ни от h_{ij}, ни от h_{0j} до тех пор, пока они не зависят от времени.)

Таким образом, действие для точечной частицы равно

$$S=-m\int\sqrt{g_{\mu\nu}dX^\mu dX^\nu}=-m\int\sqrt{g_{\mu\nu}[X(\zeta)]\frac{dX^\mu}{d\zeta}\frac{dX^\nu}{d\zeta}}d\zeta$$

Множитель m следует из соображелий размерностей.

Быстрый способ найти S (в рамках которого мы, к тому же, получаем знак минус) состоит в том, чтобы начать с нерелятивистского действия для частицы в гравитационном потенциале ϕ, то есть с выражения $S = \int Ldt = \int \left(\frac{1}{2}mv^2 - \ -m - m\phi\right)dt$. Обратите внимание, что масса m входит со знаком минус, так как является частью потенциальной энергии в нерелятивистской физике. А теперь представим S в релятивистской форме: для малых v и ϕ

$$S = -m\int\left(1 - \frac{1}{2}v^2 + \phi\right)dt \simeq -m\int\sqrt{1 - v^2 + 2\phi}dt =$$
$$= -m\int\sqrt{(1+2\phi)(dt)^2 - (d\vec{x})^2}.$$

Видим [8], что происхождение 2 в (11) обуславливается квадратным корнем в множителе Лоренца-Фитцжеральда $\sqrt{1-v^2}$.

Теперь, зная S, с помощью (4) можно вычислить тензор энергии импульса точечной частицы:

$$T^{\mu\nu}(x) = \frac{m}{\sqrt{-g}}\int d\zeta K^{-1/2}\delta^{(4)}[x - X(\zeta)]\frac{dX^\mu}{d\zeta}\frac{dX^\nu}{d\zeta}$$

Полагая ζ равной s (которую в этом случае можно назвать собственным временем τ), получаем

$$T^{\mu\nu}(x) = \frac{m}{\sqrt{-g}}\int d\tau\delta^{(4)}[x - X(\tau)]\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}$$

В частности, как мы и ожидали, 4-импульс частицы равен

$$P^\nu = \int d^3x\sqrt{-g}T^{0\nu} = m\int d\tau\delta\left[x^0 - X^0(\tau)\right]\frac{dX^0}{d\tau}\frac{dX^\nu}{d\tau} = m\frac{dX^\nu}{d\tau}$$

Приведенное здесь действие (12) имеет два недостатка: (1) трудно работать с интегралом по траекториям $\int DXe^{-im\int d\zeta\sqrt{(dX^\mu/d\zeta)(dX_\mu/d\zeta)}}$, содержащим квадратный корень, и (2) S не имеет смысла для безмассовой частицы.

Для устранения указанных недостатков заметим, что в классическом случае S эквивалентно

$$S_{\text{imp}} = -\frac{1}{2}\int d\zeta\left(\frac{1}{\gamma}\frac{dX^\mu}{d\zeta}\frac{dX_\mu}{d\zeta} + \gamma m^2\right)$$

[8] Отсюда не следует, что $g_{ij} = \delta_{ij}$. Важно то, что в главном порядке по v/c наша частица чувствительна только к g_{00}. В самом деле, восстанавливая c в метрике Шварцшильда. получаем

$$ds^2 = \left(1 - \tfrac{2GM}{c^2r}\right)c^2dt^2 - \left(1 - \tfrac{2GM}{c^2r}\right)^{-1} \quad dr^2 - -r^2d\theta^2 - r^2\sin^2\theta d\phi^2 \to$$
$$\to c^2dt^2 - \vec{x}^2 - \tfrac{2GM}{r}dt^2 + O\left(1/c^2\right)$$

где $\left(dX^\mu/d\zeta\right)\left(dX_\mu/d\zeta\right) = g_{\mu\nu}(X)\left(dX^\mu/d\zeta\right)\left(dX^\nu/d\zeta\right)$. Варьируя по $\gamma(\zeta)$, получаем $m^2\gamma^2 = \left(dX^\mu/d\zeta\right)\left(dX_\mu/d\zeta\right)$. Исключая γ из S_{imp} , мы восстанавливаем S.

Интеграл по траекториям $\int DXe^{iS_{\text{imp}}}$ имеет стандартную квадратичную форму9. Квантовая механика релятивистской точечной частицы наиболее удобным образом формулируется в терминах $S_{\text{imp},}$, н не S. Более того, для случая $m = 0$ выражение

$S_{\rm imp} = -\frac{1}{2}\int d\zeta\left[\gamma^{-1}\left(dX^\mu/d\zeta\right)\left(dX_\mu/d\zeta\right)\right]$ совершенно осмысленно. Заметьте, что варьирование по γ приводит к хорошо известному факту, что для безмассовой частицы $g_{\mu\nu}(X)dX^\mu dX^\nu = 0$.

Действие $S_{\rm imp}$ послужит нам отправной точкой при обсуждении теории струн в главе VIII.5.

Упражнения

I.10.1. Проинтегрировав по частям, получите действие для скалярного поля

$$S = -\int d^4x\sqrt{-g}\frac{1}{2}\varphi\left(\frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu + m^2\right)\varphi$$

и выведите уравнение движения для φ в искривленном пространстве-времени. Обсудите пропагатор скалярного поля $D(x,y)$ (который, конечно, уже не является трансляционно инвариантным, то есть не является больше функцией от $x-y$).

I.10.2. Используя (4), найдите E для скалярного поля в плоском пространстве-времени. Покажите, что результат совпадает с тем, который вы бы получили при использовании канонического формализма из главы I.8.

I.10.3. Покажите, что в плоском пространстве-времени величина P^μ, полученная нами из тензора энергии импульса $T^{\mu\nu}$, будучи интерпретированной как оператор в каноническом формализме, удовлетворяет условию $[P^\mu, \varphi(x)] = -i\partial^\mu\varphi(x)$, то есть выполняют ту самую роль, которую мы ожидаем от операторов энергии и импульса, а именно является сопряженной со временем и пространством и, следовательно, имеет вид $-i\partial^\mu$.

I.10.4. Покажите, что для максвелловского поля $T_{ij} = -\left(E_iE_j + B_iB_j\right) + \frac{1}{2}\delta_{ij}\left(\vec{E}^2 + \ + \vec{B}^2\right)$ и, следовательно, $T = 0$.

[9] Техническая трудность, которую мы рассмотрим в главе III.4, состоит в том, что в интеграле по $X(\zeta)$ кажущиеся различными функции $X(\zeta)$ могут на самом деле быть физически тождественными, связанными друг с другом репараметризацией.

Зи.I. 11 Резюме теории поля

Что вы узнали?

Теперь, когда мы подошли к концу первой части, оценим критически то, что вы узнали. Квантовая теория поля вовсе не трудна; она сводится к вычислению одного-единственного, но большущего интеграла

$$Z(J) = \int D\varphi e^{i\int d^{D+1}x\left[\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \lambda\varphi^4 + J\varphi\right]}.$$

Многократным функциональным дифференцированием $Z(J)$ и последующим наложением условия $J = 0$ получаем

$$\int D\varphi\varphi\left(x_1\right)\varphi\left(x_2\right)...\varphi\left(x_n\right)e^{i\int d^{n+1}x\left[\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 - \lambda\varphi^4\right]},$$

что соответствует амплитуде для n частиц, ассоциированных с полем φ, которые возникают и исчезают в пространственно-временных точках $x_1, x_2, ..., x_n$, взаимодействуя друг с другом в интервалах между ними. Это есть рождение, смерть и, в некотором смысле, жизнь между ними.

Ах, если бы мы только могли вычислить интеграл (1)! Но мы не можем. Один из способов выйти из положения состоит в разложении этого интеграла в ряд по λ:

$$\sum_{k=0}^{\infty} \frac{(-i\lambda)^k}{k!} \int D\varphi\varphi\left(x_1\right)\varphi\left(x_2\right)...\varphi\left(x_n\right)\left[\int d^{D+1}y\varphi(y)^4\right]^k e^{i\int d^{D+1}x\left[\frac{1}{2}(\partial\varphi)^2-\frac{1}{2}m^2\varphi^2\right]}.$$

Для отслеживания членов полученного ряда мы рисуем маленькие диаграммы. Теоретики, занимающиеся квантовой теорией поля, пытаются придумать способы вычисления (1), но, не добившись успеха, изобретают разные трюки и методы, чтобы правдами или неправдами извлечь нужные физические результаты без явных вычислений (1).

Для того чтобы понять, что квантовая теория поля является непосредственным обобщением квантовой механики, посмотрим, как соответствующим образом упрощается (1). Мы записали теорию в $(D+1)$-мерном пространстве-времени, то есть в D пространственных измерениях и одном временном измерении. Рассмотрим (1) в $(0+1)$-мерном пространстве-времени, то есть в отсутствие пространства; получим

$$Z(J) = \int D\varphi e^{i\int dt\left[\frac{1}{2}\left(\frac{d\varphi}{dt}\right)^2-\frac{1}{2}m^2\varphi^2-\lambda\varphi^4+J\varphi\right]}$$

где мы теперь обозначаем пространственно-временную координату x просто временем t. Мы узнаем в полученном выражении квантовую механику ангармонического осциллятора, с обозначенным φ положением прикрепленной к пружине массы и внешней силой J, действующей на осциллятор.

Каждый член в действии квантовой теории поля (1) имеет физический смысл: первые два члена обобщают гармонический осциллятор до случая пространственных колебаний, третий член соответствует антармоничности, а последний член соответствует внешнему пробнику. Вы можете рассматривать квантовую теорию поля как бесконечный набор гармонических осцилляторов, по одному на каждую точку пространства.

Здесь у нас было скалярное поле φ. В предыдущих и последующих главах понятие поля было и будет несколько обобщено: поле может преобразовываться в соответствии с нетривиальным представлением группы Лоренца. Мы уже встречали поля, преобразующиеся как вектор и как тензор, а впоследствии встретим поле, преобразующееся как спинор. Лоренц-инвариантность или какие-либо другие имеющиеся симметрии накладывают ограничения на форму действия. Интеграл будет выглядеть более сложно, но подход к его вычислению останется в точности таким же.

Вот, собственно, и вся квантовая теория поля.

10.1 Symmetries (after Polyakov)

(у Полякова спишу)

[SEE NOTE ON FIELD THEORY FOR RUSSIAN VERION OF IT]

(????)

10.1.1 Global and Local Symmetries: Preliminary Description

[SEE NOTE ON FIELD THEORY FOR RUSSIAN VERION OF IT]

(????)

10.1.2 Discrete Global Symmetries

(????)

10.1.3 Continuous Abelian Global Symmetries

[SEE NOTE ON FIELD THEORY FOR RUSSIAN VERION OF IT]

10.1.4 Non-Abelian Global Symmetries

[SEE NOTE ON FIELD THEORY FOR RUSSIAN VERION OF IT]

(????)

10.1.5 Discrete Gauge Symmetries

10.1.6 O(2)-Gauge Systems

[SEE NOTE ON FIELD THEORY FOR RUSSIAN VERION OF IT]

(????)

10.1.7 Non-Abelian Gauge Symmetries

[SEE NOTE ON FIELD THEORY FOR RUSSIAN VERION OF IT]

(????)

10.2 Non-Abelian Fields, Ghosts, BRST Invariance (??)

(мб перекину в главу с методами описания полей, почему нет? больше это как теория..)

введение

В абелевой теории калибровочного поля, в лоренцевой калибровке детерминант калибровочного условия факторизовался как нормировочный множитель и был несущественнен для правил Фейнмана, потому что он не зависел от полей.

В неабелевой теории бесконечно малое калибровочное преобразование поля

$$\mathcal{A}_\mu \mapsto \mathcal{A}_\mu - \partial_\mu u + \mathrm{i}e\,[u, \mathcal{A}_\mu] = \mathcal{A}_\mu - [\nabla_\mu, u]$$

приводит (как????) к оператору $\mathcal{M}_L$ лоренцевского калибровочного условия $\partial^\mu \mathcal{A}_\mu = 0$ в виде

$$(\mathcal{M}_L(\mathcal{A}))^a_b|_{u\to 0} = \frac{\delta}{\delta u^b}\partial^\mu \left(-\partial_\mu u^a - e f^{ab'c} u^{b'} \mathcal{A}^c_\mu\right)\Big|_{u\to 0} = -\delta^{ab}\partial^2 - e f^{abc}\mathcal{A}^c_\mu \partial^\mu$$

Здесь в последнем равенстве мы учли, что $\partial^\mu \mathcal{A}_\mu = \mathcal{A}_\mu \partial^\mu + (\partial^\mu \mathcal{A}_\mu) = \mathcal{A}_\mu \partial^\mu$.

Детерминант этого оператора, явно зависящего от калибровочного поля, уже не удастся вынести из-под знака интеграла.

духи Фаддеева-Попова

Вспоминая свойства интегрирования по грассмановым полям???, мы можем переписать функциональный детерминант в виде функционального интеграла по вспомогательным грассмановым полям $\mathfrak{c}^b, \bar{\mathfrak{c}}_a$, которых называют духами Фаддева-Попова,

$$\mathrm{Det}\,(\mathcal{M}_L) = \int \mathcal{D}\bar{\mathfrak{c}}\mathcal{D}\mathfrak{c}\,\exp\left(\mathrm{i}\int \mathrm{d}^4x \bar{\mathfrak{c}}_a(x)(\mathcal{M}_L)^a_b \mathfrak{c}^b(x)\right)$$

Наименование «духи» становится более понятным, если, во-первых, обратить внимание на свободную кинетическую часть лагранжиана для полей $\mathfrak{c}(x)$ без калибровочного поля:

$$\bar{\mathfrak{c}}_a(x)\delta^a_b\left(-\partial^2\right)\mathfrak{c}^b(x)$$

который в точности воспроизводит лагранжиан скалярного бозонного поля, но мы вынуждены приписать этому полю статистику Ферми-Дирака.

А это означает, как мы знаем, что каждая петля духов добавляет в амплитуду фактор (-1), что весьма нетривиально.

Во-вторых, вспомогательные поля $\mathfrak{c}(x)$ не имеют никакого отношения к асимптотическим состояниям: внешние поля обусловлены только физическими модами полей исходного лагранжиана.

Стало быть, при построении амплитуд квантовых переходов необходимо рассматривать только диаграммы, в которых нет внешних линий с духами.

отличие методов для неабелевых теорий от абелевых

Необходимость особых мер в квантовой теории неабелевых калибровочных полей была показана Фейнманом.

Он установил, что из-за нелинейности теории по калибровочным полям, у этих полей возникает самодействие, причем вычисление петлевой поправки к пропагатору калибровочного поля странным образом приводит к противоречию:

при использовании правил Фейнмана с самодействием полей возникает одно выражение, а вычисление унитарным методом, когда в петле учитываются только физические моды поля, к другому выражению.

противоречие и его устранение (????)

(киселев чет сказал, ну и ладно)

Противоречие полностью исчезает, если учитываются правила Фейнмана при наличии духов, которые дают петлевой вклад, обеспечивающий согласование с унитарным методом.

тут нормально пороюсь, в чем же противоречие.

лагранжиан духов

в лоренцевской калибровке к действию необходимо вклад от лагранжиана духов

$$\mathcal{L}_{\mathrm{gh}} = -\delta^a_b \bar{\mathfrak{c}}_a \partial^2 \mathfrak{c}^b - e f^{abc} \bar{\mathfrak{c}}_a \mathcal{A}^c_\mu \partial^\mu \mathfrak{c}^b,$$

(???) После интегрирования по частям с учетом 4-мерной поперечности калибровочного поля дает эквивалентную запись

$$\mathcal{L}_{\text{gh}} = \delta^a_b \left(\partial_\mu \bar{\mathfrak{c}}_a\right) \partial^\mu \mathfrak{c}^b - e f^{abc} \mathcal{A}^c_\mu \frac{1}{2} \left(\bar{\mathfrak{c}}_a \partial^\mu \mathfrak{c}^b - \left(\partial^\mu \bar{\mathfrak{c}}_a\right) \mathfrak{c}^b\right)$$

(????) К этому лагранжиану можно, конечно, добавить источники для духов $\bar{\zeta}$ и антидухов ζ, а затем повторить стандартную технику построения диаграмм в теории возмущений.

правила Фейнмана для вклада духов

Отсюда сразу следуют правила Фейнмана для вклада духов.
(????? я тут полностью хз)
На внутренних линиях пропагатор духа

$$\mathrm{i}\frac{\delta^a_b}{p^2 + \mathrm{i}0}$$

Вершина духов с калибровочным полем легко вычисляется, если учесть, что стандартный фактор перед лагранжианом i при умножении на ∂_μ дает 4 -импульс p_μ, так что

$$-e f^{abc} \frac{1}{2} \left(p_{\mathfrak{c}} - p_{\bar{c}}\right)^\mu$$

(тут диаграммка)

примеры диаграмм для духов

пока ни один не могу привести.

аксиальная калибровка

Для справок укажем калибровочное условие, которое не требует введения духов: аксиальная калибровка

$$X_N = n^\mu \mathcal{A}_\mu(x) = 0$$

где 4-вектор n^μ не зависит от координат.

Тогда оператор

$$\mathcal{M}_N = -\delta^a_b n^\mu \partial_\mu$$

не зависит от поля, и его можно вынести из-под знака континуального интеграла по полям, как это было в случае лоренцевской калибровки в абелевой теории.

В аксиальной калибровке при добавлении в лагранжиан члена

$$\mathcal{L}_{\text{g.fix.}} = -\frac{\xi}{2}(n \cdot \mathcal{A})^2$$

пропагатор калибровочного поля принимает вид

пропагатор калибровочного поля принимает вид

$$\mathrm{i}\frac{\delta^a_b}{k^2 + \mathrm{i}0} \left(-g_{\mu\nu} + \frac{k_\mu n_\nu + k_\nu n_\mu}{k \cdot n} - \frac{k_\mu k_\nu}{(k \cdot n)^2} \frac{k^2 + \xi n^2}{\xi}\right)$$

Но поскольку физические результаты не зависят от ξ, часто проводят вычисления, полагая $\xi \to \infty$ при $n^2 \neq 0$, что, конечно, несколько упрощает вид пропагатора.

лагранжианы фиксации калибровки и гостов

Формально вклады в лагранжиан от члена, фиксирующего калибровку, и от духов Фаддеева-Попова

$$\begin{aligned}\mathcal{L}_{\text{g.fix.}}+\mathcal{L}_{\text{gh}} &= -\frac{1}{2\xi}\left(\partial_\mu \mathcal{A}^\mu\right)^2+\delta^a_b\left(\partial_\mu \bar{\mathfrak{c}}_a\right)\partial^\mu \mathfrak{c}^b - ef^{abc}\mathcal{A}^c_\mu\frac{1}{2}\left(\bar{\mathfrak{c}}_a\partial^\mu\mathfrak{c}^b-\mathfrak{c}^b\partial^\mu\bar{\mathfrak{c}}_a\right)\cong\\ &\cong -\frac{1}{2\xi}\left(\partial_\mu\mathcal{A}^\mu\right)^2+\bar{\mathfrak{c}}\left(\mathcal{M}_L(\mathcal{A})\mathfrak{c}\right)\cong\\ &\cong -\frac{1}{2\xi}\left(\partial_\mu\mathcal{A}^\mu\right)^2+\left(\partial_\mu\bar{\mathfrak{c}}\right)\left[\nabla^\mu,\mathfrak{c}\right]\end{aligned}$$

не обладают инвариантностью относительно калибровочной группы.
и что делать с ними?

бесконечно малые преобразования

бесконечно малые преобразования калибровочного поля и духа равны

$$\begin{aligned}\delta\mathcal{A}_\mu &= \varepsilon\left(-\partial_\mu\mathfrak{c}+\mathrm{i}e\left[\mathfrak{c},\mathcal{A}_\mu\right]\right)=-\varepsilon\left[\nabla_\mu,\mathfrak{c}\right]\\ \delta\mathfrak{c} &= Y_{\mathrm{c}}\varepsilon\mathrm{i}e[\mathfrak{c},\mathfrak{c}]\equiv Y_{\mathrm{c}}\varepsilon\mathrm{i}\mathfrak{c}^a\mathfrak{c}^b\left[t_a,t_b\right]\end{aligned}$$

где $Y_{\mathfrak{c}}$ - заряд духа
(????) потому что что если записать параметр локального калибровочного преобразования в виде произведения грассмановых переменных

$$u^a(x)=\varepsilon\mathfrak{c}^a(x)$$

с глобальным параметром ε и локальным духом $\mathfrak{c}(x)$.
Заряд нужно найти, поскольку

$$\mathfrak{c}\mapsto e^{\mathrm{i}eY_{\mathrm{c}}\varepsilon\mathfrak{c}}\mathfrak{c}e^{-\mathrm{i}eY_{\mathrm{c}}\varepsilon\mathfrak{c}}$$

а коммутаторы вычисляются в матричной форме, так что, например,

$$[\mathfrak{c},\mathfrak{c}]=\mathfrak{c}^a\mathfrak{c}^b\left[t_a,t_b\right]=\mathrm{i}f_{abc}t_c\mathfrak{c}^a\mathfrak{c}^b$$

причем для грассмановых переменных конструкция $\mathfrak{c}^a\mathfrak{c}^b$ антисимметрична по перестановке индексов $a\leftrightarrow b$, так что за счет полной антисимметрии структурных констант группы f_{abc} коммутатор отличен от нуля.

вариация полного калибровочного лагранжиана

Тогда вариация калибровочно инвариантных вкладов, конечно, равна нулю, а

$$\delta\left(\mathcal{L}_{\text{g.fix.}}+\mathcal{L}_{\text{gh}}\right)=-\frac{1}{\xi}(\partial_\mu\mathcal{A}^\mu)\varepsilon\left(\mathcal{M}_L(\mathcal{A})\mathfrak{c}\right)+\left(\delta\bar{\mathfrak{c}}\right)\left(\mathcal{M}_L(\mathcal{A})\mathfrak{c}\right)+\left(\partial^\mu\bar{\mathfrak{c}}\right)\delta\left[\nabla_\mu,\mathfrak{c}\right]$$

Покажем, что вариация $\delta\left[\nabla_\mu,\mathfrak{c}\right]\equiv 0$ при действии (9.24).
В общем выражении

$$\delta\left[\nabla_\mu,\mathfrak{c}\right]=\partial_\mu\delta\mathfrak{c}-\mathrm{i}e\left[\delta\mathfrak{c},\mathcal{A}_\mu\right]-\mathrm{i}e\left[\mathfrak{c},\delta\mathcal{A}_\mu\right]$$

первый член равен

$$\mathrm{i}e\varepsilon\, Y_{\mathfrak{c}}\partial_\mu[\mathfrak{c},\mathfrak{c}] = \mathrm{i}\, e\varepsilon 2Y_{\mathfrak{c}}\,[\partial_\mu\mathfrak{c},\mathfrak{c}]$$

второй -

$$-(\mathrm{i}e)^2\varepsilon Y_{\mathrm{c}}\,[[\mathfrak{c},\mathfrak{c}],\mathcal{A}_\mu]$$

третий -

$$-\mathrm{i}\, e\varepsilon\,[\mathfrak{c},-[\nabla_\mu,\mathfrak{c}]] = \mathrm{i}e\varepsilon\,[\mathfrak{c},\partial_\mu\mathfrak{c}] - (\mathrm{i}e)^2\varepsilon\,[\mathfrak{c},[\mathfrak{c},\mathcal{A}_\mu]]$$

Как видим, при $Y_{\mathfrak{c}} = -\frac{1}{2}$ сокращаются вклады с производной поля духов $\partial_\mu\mathfrak{c}$. Что же касается, членов с двойными коммутаторами, то

$$-\frac{1}{2}\Big[[\mathfrak{c},\mathfrak{c}] + [\mathfrak{c},[\mathfrak{c},\mathcal{A}_\mu]] = -\frac{1}{2}\mathfrak{c}^a\mathfrak{c}^b\mathcal{A}^c_\mu\left([[t_a,t_b]\,,t_c] - 2\,[t_a,[t_b,t_c]]\right)$$

Но в силу антисимметрии грассманова фактора

$$\mathfrak{c}^a\mathfrak{c}^b 2\,[t_a,[t_b,t_c]] = \mathfrak{c}^a\mathfrak{c}^b\left([t_a,[t_b,t_c]] - [t_b,[t_a,t_c]]\right.$$

так что, в итоге, используя антисимметрию коммутатора $[A,B] = -[B,F]$, находим выражение

$$\frac{1}{2}\mathfrak{c}^a\mathfrak{c}^b\mathcal{A}^c_\mu\left([t_c,[t_a,t_b]] + [t_a,[t_b,t_c]] + [t_b,[t_c,t_a]]\right) \equiv 0$$

в силу тождества Якоби для коммутатора.

BRST-суперпреобразования

Таким образом, действие, которое включает в себя также вклад от фиксирующего калибровку члена и поля духов Фаддеева-Попова локально инвариантно относительно преобразований

$$\begin{aligned}\delta\mathcal{A}_\mu &= \varepsilon\left(-\partial_\mu\mathfrak{c} + \mathrm{i}e\,[\mathfrak{c},\mathcal{A}_\mu]\right) = -\varepsilon\,[\nabla_\mu,\mathfrak{c}]\\ \delta\mathfrak{c} &= -\frac{1}{2}\varepsilon\mathrm{i}e[\mathfrak{c},\mathrm{c}]\\ \delta\overline{\mathfrak{c}} &= \frac{1}{\xi}\left(\partial_\mu\mathcal{A}^\mu\right)\varepsilon\end{aligned}$$

которые называют BRST-суперпреобразованиями по именам четырех авторов Becchi-Rouet-Stora-Tyutin. Суперсимметричность преобразований обусловлена грассмановой переменной ε, а так как эта переменна не зависит от координат, симметрия является глобальной.

примеры BRST-инвариантности

свойства интегрирования по грассмановым полям

хз, но там где-то они были нужны.

10.3 Free Quantum Fields at Finite Temperature

(this is also a fundamental properties of fields, so I'll add theory about it here. se cond matter for now. I haven't heard that it is used in particles physics and somewhere except phase transitions, condensed matter...)

Part IV
Practical Methods in QFT

11 Setups for Phenomena

(пока тут этот раздел, в 1 части очень четкое указание на важные тут формулы будет!!!!)

11.1 L9-2 S-Matrix

11.1.1 4. General Formula for the S-Matrix in QED and the Simplest Amplitudes

Мы можем, наконец, собрать все формулы вместе и выписать общее выражение для S-матрицы в КЭД при помощи континуального интеграла по бозонным и фермионным полям:

$$S_{J,\eta,\bar{\eta}} = \frac{1}{N}\int \exp\left\{ i\int \mathrm{d}^{(d)}x \left[-\frac{1}{16\pi}F_{\mu\nu}^2 - \frac{1}{8\pi\alpha}\left(\partial_\mu A^\mu\right)^2 + \right.\right.$$
$$\left.\left. +\bar{\psi}\left(i\gamma^\mu\nabla_\mu - m\right)\psi + J^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta \right] \right\} DA_\mu D\bar{\psi}D\psi \tag{55}$$

Граничные условия для полей при $|t|\to\infty$ указаны в лекции 7 и предыдущем пункте. Как обычно, для развития ТВ действие в экспоненте в (55) разбивается на квадратичную (линейную) и нелинейную части. Последняя описывает взаимодействие:

$$S = S_0\{A,\bar{\psi},\psi;J,\bar{\eta},\eta\} + S_{int}, \quad iS_{int} = -ie\int \mathrm{d}^{(d)}x A_\mu \bar{\psi}_a \gamma^\mu_{ab}\psi_b \tag{56}$$

Представим S_{int} в виде дифференциального оператора

$$i\hat{S}_{int} = -e\int \mathrm{d}^{(d)}x \frac{\delta}{\delta J^\mu}\frac{\delta}{\delta\eta_a}\gamma^\mu_{ab}\frac{\delta}{\delta\bar{\eta}_b} \tag{57}$$

который может находиться как под знаком функционального интеграла, так и вне его. Это позволяет переписать выражение (55) в виде

$$S = \frac{1}{N}\exp\left\{ -e\int \mathrm{d}^{(4)}x \frac{\delta}{\delta J^\mu(x)}\frac{\delta}{\delta\eta_a(x)}\gamma^\mu_{ab}\frac{\delta}{\delta\bar{\eta}_b(x)} \right\} : \exp\left\{ i\int \mathrm{d}^{(4)}x \left(-\mathbf{J}\mathbf{A}_0 + \bar{\eta}\psi_0 + \bar{\psi}_0\eta\right) + \right.$$
$$\left. +i\int \mathrm{d}^{(4)}x\, \mathrm{d}^{(4)}y \left[\frac{1}{2}J^\mu(x)D^\alpha_{\mu\nu}(x-y)J^\nu(y) - \bar{\eta}(x)S_c(x-y)\eta(y) \right] \right\} : |_{J,\bar{\eta},\eta=0} \tag{58}$$

где пропагаторы $D^\alpha_{\mu\nu}$ и S_c задаются формулами (53 Л7) и (49), соответственно. Асимптотические поля $\mathbf{A}_0,\bar{\psi}_0,\psi_0$ разлагаются согласно (см. (47 Л7) и (46 Л8))

$$\mathbf{A}_0(x) = \sum_{\alpha=1,2} \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3} \sqrt{\frac{2\pi}{|\mathbf{k}|}} \left(c_{\mathbf{k}\alpha} \mathbf{e}_{\mathbf{k}\alpha} e^{-ikx} + c^\dagger_{\mathbf{k}\alpha} \mathbf{e}^*_{\mathbf{k}\alpha} e^{ikx} \right)$$

$$\psi_0(x) = \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{s=1,2} \left(a_{\mathbf{k}s} u_{ks} e^{-ikx} + b^\dagger_{ks} u_{-ks} e^{ikx} \right)$$

$$\bar\psi_0(x) = \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_{s=1,2} \left(b_{\mathbf{k}s} \bar u_{-ks} e^{-ikx} + a^\dagger_{ks} \bar u_{ks} e^{ikx} \right) \tag{59}$$

Здесь вектора поляризации фотонов $\mathbf{e}_{\mathbf{k}}\alpha$ удовлетворяют условиям (55 J 7), а спиноры u_{ks}, u_{-ks} - уравнениям (38 Л8) и условиям (57 Л8). Коммутационные или антикоммутационные соотношения операторов рождения и уничтожения имеют вид (33 Л7) и (54 Л8), (58 Л8). Вакуумное состояние определяется условиями

$$a_{\mathbf{k}s}|0\rangle = b_{\mathbf{k}s}|0\rangle = c_{\mathbf{k}\alpha}|0\rangle = 0$$

a, например, состояние электрона с импульсом $\mathbf{p}$ и спином s и позитрона с импульсом $\mathbf{q}$ и спином s' и фотона с импульсом $\mathbf{k}$ и поляризацией α имеет вид

$$|\mathbf{p}s, \mathbf{q}s', \mathbf{k}\alpha\rangle = \sqrt{2\omega_p 2\omega_q 2|\mathbf{k}|} a^\dagger_{\mathbf{p}s} b^\dagger_{\mathbf{q}s'} c^\dagger_{\mathbf{k}\alpha}|0\rangle$$

и т.д.

Из выражения (58) получаем также амплитуду перехода из вакуума в вакуум в присутствии внешних источников:

$$Z(J,\eta,\bar\eta) = \frac{1}{N} \exp\left\{ -e \int \mathrm{d}^{(4)}x \frac{\delta}{\delta J^\mu} \frac{\delta}{\delta \eta_a} \gamma^\mu_{ab} \frac{\delta}{\delta \bar\eta_b} \right\}$$
$$\cdot \exp\left\{ i \int \mathrm{d}^{(4)}x \; \mathrm{d}^{(4)}y \left[\frac{1}{2} J^\mu(x) D^\alpha_{\mu\nu}(x-y) J^\nu(y) - \bar\eta(x) S_c(x-y) \eta(y) \right] \right\} \tag{60}$$

Укажем здесь, что при комплексном значении параметра размерности пространства d базовое соотношение для матриц Дирака (5 Л8) остаётся в силе, в то время как ряд других принимают следующий вид:

$$g_{\mu\nu} g^{\mu\nu} = d, \quad \gamma_\mu \gamma^\mu = d, \quad \mathrm{tr}\, \gamma^\mu \gamma^\nu = 2^{d/2} \eta^{\mu\nu} \tag{61}$$

Поэтому если в (63 Л8) определить

$$2^{d/2} T^{\mu_1 \mu_2 \ldots \mu_n} = \mathrm{tr}\,(\gamma^{\mu_1} \gamma^{\mu_2} \ldots \gamma^{\mu_n})$$

то все формулы для этой величины, полученные в лекции 8, сохраняются. Парадоксы возникают при комплексном d лишь при работе с γ^5-матрицей, однако с этим парадоксом нам придётся столкнуться лишь в лекции, посвящённой аксиальной аномалии.

При комплексном значении размерности пространства d выражения для S-матрицы (58) и производящего функционала (60) регуляризованы. При этом сохранена калибровочная инвариантность теории, поскольку эта симметрия имеет место при любом d.

Рассмотрим простейший процесс рассеяния двух электронов в два электрона. Для этого следует выражение для оператора S-матрицы (58) поместить в обкладки между

бра-состоянием $\langle 0|a_{\mathbf{p}_1' s_1'} a_{\mathbf{p}_2' s_2'}$ и кет-состоянием $a^\dagger_{\mathbf{p}_2 s_2} a^\dagger_{\mathbf{p}_1 s_1}|0\rangle$. Здесь $\mathbf{p}_i, s_i \quad$ и $p_i', s_i', \quad i = 1, 2$ - импульсы и спины начальных и конечных электронов. Чтобы сократить запись формул, мы будем писать u_1 вместо $u_{p_1 s_1}$ и $\bar{u}_2'$ вместо $\bar{u}_{\mathrm{p}_2' s_2'}, u_{-1}$ вместо $u_{-p_1 s_1}$, и т.д. Наинизший порядок, в котором амплитуда этого процесса отличается от нуля, получается при разложении первой экспоненты в (58) до второго порядка, что даёт вклад в S-матрицу

$$\frac{ie^2}{2!} : \int \mathrm{d}^{(4)}x \; \mathrm{d}^{(4)}y \left[\bar{\psi}_0(x)\gamma^\mu \psi_0(x)\right] \left[\bar{\psi}_0(y)\gamma^\nu \psi_0(y)\right] : D^\alpha_{\mu\nu}(x-y) \tag{62}$$

Поскольку поле ψ_0 удовлетворяет уравнению Дирака, то $\partial_\mu \left[\bar{\psi}_0(x)\gamma^\mu \psi_0(x)\right] = 0$, и потому продольная часть фотонного пропагатора в (62) выпадает, в качестве фотонного пропагатора можно взять $D^1_{\mu\nu} \sim \eta_{\mu\nu}$. Мы видим, что физические величины не зависят от калибровки. Это утверждение имеет место во всех порядках теории возмущений. Помещая выражение (62) в указанные выше обкладки, в результате простых вычислений получаем для амплитуды iM:

$$iM = 4\pi i e^2 \left\{ \frac{1}{t} \left(\bar{u}_2' \gamma^\mu u_2\right) \left(\bar{u}_1' \gamma_\mu u_1\right) - \frac{1}{u} \left(\bar{u}_2' \gamma^\mu u_1\right) \left(\bar{u}_1' \gamma_\mu u_2\right) \right\} \tag{63}$$

где используются введенные согласно (27 Л4) инварианты

$$\begin{gathered} s = (p_1 + p_2)^2, \quad t = (p_1 - p_1')^2, \quad u = (p_1 - p_2')^2 \\ p_1^\mu + p_2^\mu = p_1'^\mu + p_2'^\mu \end{gathered} \tag{64}$$

Формула для сечения рассеяния σ двух частиц в две частицы в системе центра инерции содержится в Приложении (см. (А19)). Пусть $\varepsilon = p_1^0 + p_2^0$ - полная энергия, $\mathbf{p}_1 = -\mathbf{p}_2 \equiv \mathbf{p}$ и $\mathbf{p}_1' = -\mathbf{p}_2' \equiv \mathbf{p}'$ - начальный и конечный импульсы. Тогда в нашем случае

$$\mathrm{d}\sigma = \frac{|M|^2}{64\pi^2 \varepsilon^2} \, \mathrm{d}o' \tag{65}$$

где $\mathrm{d}o'$ - дифференциал телесного угла конечного импульса $\mathbf{p}'$. В системе центра инерции

$$t = -(\mathbf{p}_1 - \mathbf{p}_1')^2 = -4\mathbf{p}^2 \sin^2 \frac{\theta}{2}, \quad u = -4\mathbf{p}^2 \cos^2 \frac{\theta}{2}, \quad -\mathrm{d}t = 2\mathbf{p}^2 |\, \mathrm{d}\cos\theta| = \frac{\mathbf{p}^2}{\pi} \mathrm{d}o' \tag{66}$$

где θ - угол рассеяния.

Рассмотрим это сечение в нерелятивистском пределе, когда согласно (42 Л8)

$$\bar{u}' \gamma^\mu u = 2m w'^\dagger w \delta_0^\mu \tag{67}$$

Подставим (63) с учётом (67) в (65) и усредним по начальным спинам электронов (то есть заменим везде $w_\alpha w_\beta^* \to \frac{1}{2}\delta_{\alpha\beta}$ для начальных частиц):

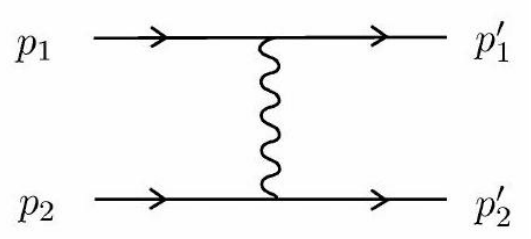
$$\mathrm{d}\sigma = e^4 m^2 \left\{ \frac{1}{4t^2} + \frac{1}{4u^2} - 2\frac{\left(u_1'^\dagger u_2\right)^2}{4tu} \right\} \mathrm{d}o' \tag{68}$$

Теперь просуммируем независимо по спинам конечных частиц. В результате получим:

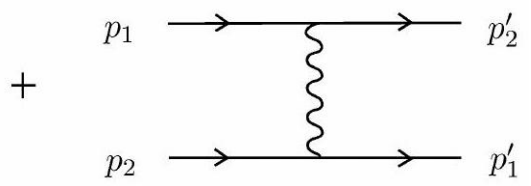
$$\mathrm{d}\sigma = e^4 m^2 \left(\frac{1}{t^2} + \frac{1}{u^2} - \frac{1}{tu}\right) \mathrm{d}o' = \left(\frac{e^2}{mv^2}\right)^2 \left(\frac{1}{\sin^4\frac{\theta}{2}} + \frac{1}{\cos^4\frac{\theta}{2}} - \frac{1}{\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}\right) \mathrm{d}o' \qquad (69)$$

где $\mathbf{v} = 2\mathbf{p}/m$ - относительная скорость электронов. Сечение (69) совпадает с известным результатом в нерелятивистской теории.

Графически амплитуда (63) представляется в виде суммы двух следующих диаграмм Фейнмана:



(a)



(b)

Рис. 13

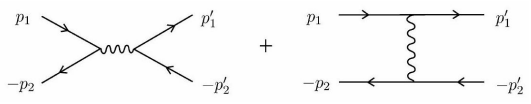
Укажем, что в используемой системе единиц

$$\frac{e^2}{\hbar c} = \frac{1}{137,04} \qquad (70)$$

Вычислим теперь в наинизшем порядке упругую амплитуду рассеяния электрона на позитроне. Для этого следует вычислить выражение (62) в обкладках $\langle 0|a_{\mathbf{p}_1' s_1'} b_{\mathbf{p}_2' s_2'}$ и $b^\dagger_{\mathbf{p}_2 s_2} a^\dagger_{\mathbf{p}_1 s_1}|0\rangle$:

$$iM = 4\pi i e^2 \left\{ \frac{1}{s} \left(\bar{u}_1' \gamma^\mu u_{-2}'\right) \left(\bar{u}_{-2} \gamma_\mu u_1\right) - \frac{1}{t} \left(\bar{u}_{-2} \gamma^\mu u_{-2}'\right) \left(\bar{u}_1' \gamma_\mu u_1\right) \right\} \qquad (71)$$

где s, t и u задаются согласно (64). Амплитуда (71) представляется двумя диаграммами:



(a)

(b)

Рис. 14

Легко увидеть, что амплитуда (71) получается из амплитуды (63) (и наоборот) путём замены $p_2 \to -p_2', p_2' \to -p_2$, При этом происходят замены $t \leftrightarrow t, u \leftrightarrow s$, диаграмма (a) на рис. 13 переходит в диаграмму (b) на рис. 14, диаграмма (b) на рис. 13 переходит в диаграмму (a) на рис. 14.

11.1.2 Introduction to Mathematical Perturbation Theory

Теория

Мы достаточно подробно обсудили квантование двух свободных теорий поля, приближенно описывающих многие из найденных в природе свободных частиц. Однако до сих пор состояния свободных частиц были собственными состояниями гамильтониана. У нас не было ни взаимодействия, ни рассеяния. Чтобы получить описание, более близкое к реальности, следует включить в гамильтониан (или лагранжиан) новые, нелинейные слагаемые, связывающие друг с другом различные фурье-моды (и частицы, которые их населяют). Для сохранения причинности потребуем, чтобы новые слагаемые содержали произведения полей только в одной и той же точке пространства-времени. Так, выражение $[\phi(x)]^4$ прекрасно подходит, а выражение $\phi(x)\phi(y)$ не годится. Таким образом, слагаемые, описывающие взаимодействия, будут иметь вид:

$$H_{\text{int}} = \int d^3x \mathscr{H}_{\text{int}}[\phi(x)] = -\int d^3x \mathscr{L}_{\text{int}}[\phi(x)].$$

Пока ограничимся рассмотрением теорий, в которых $\mathscr{H}_{\text{int}}$ $(= -\mathscr{L}_{\text{int}})$ является функцией только самих полей, но не их производных.

В этой главе мы обсудим три важных примера взаимодействующих теорий поля. Первый из них - это теория ϕ^4:

$$\mathscr{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4,$$

где λ - безразмерная константа связи. (Взаимодействие ϕ^3 было бы немного проще, но в этом случае энергия не будет положительно определена, до тех пор, пока не добавить более высокую четную степень ф.) Хотя мы рассматриваем эту теорию в чисто педагогических целях (так как она - самая простая из всех квантовых теорий с взаимодействием), модели реального мира содержат взаимодействия ϕ^4. Наиболее важный пример такого взаимодействия в физике частиц - самодействие хиггсовского поля в стандартной электрослабой теории. В части II мы увидим, что теория ϕ^4 возникает также в статистической механике. Уравнение движения для теории ϕ^4 имеет вид

$$\left(\partial^2 + m^2\right)\phi = -\frac{\lambda}{3!}\phi^3.$$

Оно не может быть решено с помощью фурье-разложения по примеру свободного уравнения Клейна-Гордона. В квантовой теории мы налагаем одновременные коммутационные соотношения

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$

на которые не влияет $\mathscr{L}_{\text{int.}}$. (Заметим, однако, что если бы в $\mathscr{L}_{\text{int}}$ содержалось $\partial_\mu\phi$, то определение $\pi(\mathbf{x})$ было бы иным.) Нетрудно записать гамильтониан этой теории и найти гейзенберговские уравнения движения для оператора $\phi(x)$. Так же, как это было в свободной теории, результат совпадает с классическим уравнением движения (4.2).

Нашим вторым примером взаимодействующей теории поля будет квантовая электродинамика:

$$\begin{aligned}\mathscr{L}_{\text{КЭд}} &= \mathscr{L}_{\text{Дирак}} + \mathscr{L}_{\text{Максвелл}} + \mathscr{L}_{\text{int}} = \\ &= \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}\left(F_{\mu\nu}\right)^2 - e\bar{\psi}\gamma^\mu\psi A_\mu,\end{aligned}$$

где A_μ - электромагнитный векторный потенциал, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ - тензор электромагнитного поля и $e = -|e|$ - электрический заряд. (Чтобы описать фермион с зарядом Q, заменим e на Q. Если мы хотим рассмотреть сразу несколько разновидностей заряженных частиц, мы просто продублируем $\mathscr{L}_{\text{Дирак и}}\,\mathscr{L}_{\text{int}}$ для каждой отдельной разновидности.) Удивительно, что этот простой лагранжиан описывает почти все наблюдаемые явления от макроскопических масштабов до масштабов 10^{-13}cm. На самом деле лагранжиан в КЭД может быть записан еще проще:

$$\mathscr{L}_{\text{КЭд}} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}\left(F_{\mu\nu}\right)^2,$$

где D_μ - калибровочно-ковариантная производная,

$$D_\mu \equiv \partial_\mu + ieA_\mu(x).$$

Определяющее свойство лагранжиана КЭД состоит в том, что он инвариантен относительно калибровочного преобразования

$$\psi(x) \to e^{i\alpha(x)}\psi(x), \quad A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x)$$

которое действует на дираковское поле как локальное фазовое вращение. Эта инвариантность при локальных фазовых вращениях имеет фундаментальное геометрическое значение, которое мотивирует появление ковариантной производной. Однако для наших целей достаточно того факта, что преобразование (4.6) является симметрией теории.

Уравнения движения получаются из (4.3) при помощи канонической процедуры. Уравнение Эйлера-Лагранжа для ψ имеет вид:

$$(i\not{\partial} - m)\psi(x) = 0$$

Это уравнение является уравнением Дирака, связанным с электромагнитным полем по рецепту минимальной связи, отвечающей замене $\partial \to D$. Уравнение Эйлера-Лагранжа для поля A_ν имеет вид:

$$\partial_\mu F^{\mu\nu} = e\bar{\psi}\gamma^\nu\psi = ej^\nu.$$

Это неоднородные уравнения Максвелла с плотностью тока $j^\nu = \bar{\psi}\gamma^\nu\psi$, определяемой сохраняющимся дираковским векторным током (3.73). Так же, как и в теории ϕ^4, уравнения движения могут быть получены как гейзенберговские уравнения для операторов $\psi(x)$ и $A_\mu(x)$. Это нетрудно проверить для поля $\psi(x)$, но мы до сих пор не провели квантование электромагнитного поля.

На самом деле мы вообще не будем обсуждать в этой книге каноническое квантование электромагнитного поля. Это весьма запутанный вопрос, главным образом из-за калибровочной инвариантности. Заметим, что, так как $\dot{A}^0$ не содержится в лагранжиане (4.3), то импульс, сопряженный к A^0, тождественно равен нулю. Это противоречит каноническому коммутационному соотношению $[A^0(\mathbf{x}), \pi^0(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y})$. Одно из решений

состоит в квантовании в кулоновской калибровке, когда $\nabla \cdot \mathbf{A} = 0$ и компонента A^0 является не динамической переменной, а подчиняется некой связи. Но в этом случае неочевидна явная лоренц-инвариантность. С другой стороны, можно квантовать поле в лоренцевской калибровке $\partial_\mu A^\mu = 0$. Тогда можно модифицировать лагранжиан, добавляя слагаемое с $\dot{A}^0$. При таком подходе возникают коммутационные соотношения $\left[A^\mu(\mathbf{x}), \dot{A}^\nu(\mathbf{y})\right] = -ig^{\mu\nu}\delta(\mathbf{x}-\mathbf{y})$, по существу совпадающие с аналогичными соотношениями для четырех полей Клейна-Гордона. Но дополнительный знак «минус» в коммутаторе $\left[A^0, \dot{A}^0\right]$ приводит к другой (преодолимой) трудности: состояния, рожденные оператором $a_p^0\dagger$, имеют отрицательную норму*

Правила Фейнмана для вычисления амплитуд рассеяния, включающих фотоны, легче получаются в рамках формулировки теории поля на языке функционального интегрирования, излагаемом в главе 9. Как мы увидим в части III, этот метод имеет дополнительное преимущество обобщения на случай неабелевых калибровочных полей. В данной главе мы просто попытаемся угадать правила Фейнмана для фотонов. Сделать это будет совсем нетрудно после того, как мы получим правила Фейнмана для аналогичной, но более простой теории - теории Юкавы:

$$\mathscr{L}_{\text{Юкава}} = \mathscr{L}_{\text{Дирак}} + \mathscr{L}_{\text{Клейн - Гордон}} - g\bar{\psi}\psi\phi.$$

Это будет нашим третьим примером. Теория Юкавы подобна КЭД, но в ней фотон заменен скалярной частицей ϕ. Слагаемое с взаимодействием содержит безразмерную константу связи g, аналогичную электрическому заряду e. Вначале Юкава предложил эту теорию для описания взаимодействия нуклонов (ψ) и пионов (ϕ). В современной теории частиц стандартная модель содержит слагаемое с взаимодействием Юкавы, связывающий скалярное поле Хиггса с кварками и лептонами; большинство свободных параметров в стандартной модели - это юкавские константы связи.

Выписав три основополагающих взаимодействия, сделаем паузу и обсудим, какие другие взаимодействия можно найти в природе. Сначала может показаться, что список будет бесконечен; даже для скалярной теории можно было бы записать взаимодействия вида ϕ^n при любом n. Однако одна простая и убедительная аксиома устраняет почти все из возможных взаимодействий, кроме нескольких. Эта аксиома заключается в том, что теория должна быть перенормируемой, и возникает следующим образом. Как упоминалось в главе 1, слагаемые более высокого порядка в теории возмущений включают интегралы по 4-импульсам промежуточных («виртуальных») частиц. Эти интегралы часто формально расходятся, поэтому в общем случае необходимо применять некоторую процедуру обрезания. Простейший метод заключается в обрезании интегралов при некотором большом, но конечном импульсе Λ. В конце вычисления переходят к пределу $\Lambda \to \infty$, ожидая, что физические величины окажутся независящими от Λ. Если это действительно имеет место, то теория называется перенормируемой. Предположим, однако, что теория включает взаимодействия с константами связи, имеющими размерность массы в некоторой отрицательной степени. Тогда, для получения безразмерной амплитуды рассеяния, эта константа связи должна быть умножена на некоторую величину положительной массовой размерности. Эта обезразмеривающая величина оказывается ни чем иным, как параметром обрезания Λ. Такое слагаемое расходится при $\Lambda \to \infty$, так что теория оказывается неперенормируемой.

*Легко найти превосходное изложение обеих процедур квантования. Так, для квантования в кулоновской калибровке см. Bjorken, Drell (1965), глава 14; для квантования

в лоренцевой калибровке см. Mandl и Shaw (1984), глава 5. Мы обсудим эти вопросы подробно в главе 10. Пока что просто обратим внимание на то, что любая теория, содержащая константу связи с отрицательной массовой размерностью, неперенормируема. Размерный анализ позволяет легко отвергнуть почти всех кандидатов на взаимодействие. Так как действие $S = \int \mathscr{L} d^4x$ безразмерно, то лагранжиан $\mathscr{L}$ должен иметь размерность (масса) ^{4} (или просто размерность 4). Из вида кинетических слагаемых различных свободных лагранжианов следует, что скалярное и векторное поля ϕ и A^μ имеют размерность 1, в то время как спинорное поле ψ имеет размерность 3/2. Теперь можно свести в таблицу все допустимые перенормируемые взаимодействия. Для теорий, включающих только скаляры, допустимые слагаемые, описывающие взаимодействия, имеют вид

$$\mu\phi^3 \quad \text{и} \quad \lambda\phi^4.$$

Константа связи μ имеет размерность 1, в то время как λ безразмерна. Члены вида ϕ^n для $n > 4$ не подходят, так как их константы связи имели бы размерность $4 - n$. Конечно, более интересные теории могут быть получены включением нескольких вещественных или комплексных скалярных полей (см. задачу 4.3).

Теперь добавим спинорные поля. Спинорные самодействия не допускаются, так как ψ^3 (не говоря уже о нарушении лоренцевской инвариантности) имеет размерность 9/2. Таким образом, единственное новое допустимое взаимодействие имеет вид взаимодействия Юкавы

$$g\bar{\psi}\psi\phi$$

Правда, подобные взаимодействия могут быть построены также из вейлевских и майорановских спиноров.

Много новых взаимодействий можно получить, добавляя векторные поля. Наиболее знакомое из них - это векторно-спинорное взаимодействие в КЭД:

$$e\bar{\psi}\gamma^\mu\psi A_\mu$$

И вновь нетрудно построить аналогичные слагаемые с помощью вейлевских и майорановских спиноров. Менее важной теорией является скалярная КЭД с лагранжианом

$$\mathscr{L} = |D_\mu\phi|^2 - m^2|\phi|^2,$$

который содержит слагаемые

$$eA^\mu\phi\partial_\mu\phi^*, \quad e^2|\phi|^2A^2.$$

Это - наш первый пример взаимодействия с производными; квантование такой теории будет намного проще в формализме функционального интегрирования, поэтому мы откладываем ее рассмотрение до главы 9. Другие возможные лоренц-инвариантные слагаемые, включающие векторные поля, имеют вид

$$A^2\left(\partial_\mu A^\mu\right) \quad \text{и} \quad A^4.$$

Хотя это далеко не очевидно, выписанные слагаемые приводят к противоречиям, если только их константы связи не выбраны на основе специального типа симметрии, которая должна включать несколько векторных полей. Эта симметрия лежит в основе неабелевых

калибровочных теорий, которые будут основным предметом изучения в части III. Массовое слагаемое $\frac{1}{2}m^2A^2$ для векторных полей также приводит к противоречию за исключением отдельного случая, когда оно добавляется к КЭД; в любом случае оно нарушает (абелеву или неабелеву) калибровочную инвариантность. Сказанное выше исчерпывает список возможных лагранжианов, включающих скалярные, спинорные и векторные частицы. Интересно отметить, что принятые в настоящее время модели сильного, слабого и электромагнитного взаимодействий включают все перечисленные выше типы взаимодействий. Три типа взаимодействий, которые будут изучаться в этой главе, покрывают почти половину всех возможностей. Другие взаимодействия мы подробно рассмотрим далее в этой книге.

Предположение, что реалистичные теории должны быть перенормируемыми, конечно, удобно, так как неперенормируемая теория имела бы минимальную предсказательную силу. Однако возникает вопрос, почему природа так добра, что использует только перенормируемые взаимодействия. Можно было бы ожидать, что истинная теория природы является квантовой теорией гораздо более общего вида. Однако можно показать, что какой бы сложной не была фундаментальная теория при очень высоких энергиях, низкоэнергетическое приближение к этой теории, наблюдаемое в экспериментах, должно быть перенормируемой квантовой теорией поля. Мы покажем это в разделе 12.1.

На более практическом уровне предыдущий анализ высвечивает большую методологическую разницу в построении нерелятивистской квантовой механики и релятивистской квантовой теории поля. Так как потенциал $V(\mathbf{r})$, появляющийся в уравнении Шредингера, абсолютно произволен, нерелятивистская квантовая механика не накладывает ограничений на вид взаимодействий в реальном мире. Но мы только что видели, что квантовая теория поля налагает очень жесткие ограничения на природу (или наоборот). Образно говоря, из нашего обсуждения вытекает, что единственная оставшаяся задача в физике частиц состоит в перечислении существующих элементарных частиц, измерении их масс и констант связи. Может быть, эта точка зрения несколько радикальна, но тот факт, что она существует, по-видимому, означает, что физика частиц находится на верном пути к фундаментальной теории.

Имея данный набор частиц и взаимодействий, нужно еще получить экспериментальные следствия. Как мы должны анализировать квантовую механику взаимодействующей теории поля? Было бы хорошо, если бы мы могли явно решить по крайней мере несколько примеров (то есть точно найти собственные значения и собственные векторы, как мы это делали в свободных теориях), чтобы получить представление о свойствах взаимодействующих теорий. К сожалению, это проще сказать, чем сделать. Точно решаемые (интегрируемые) взаимодействующие теории поля в пространстве-времени с размерностью, больше двух, не известны. Даже интегрируемые модели, содержащие специальные симметрии, чрезвычайно сложны*. Рассмотрение этих теорий было бы интересно, но едва ли стоит усилий на этом уровне изучения. Вместо этого мы вернемся к более простому и применимому в большинстве случаев подходу. Будем рассматривать гамильтониан взаимодействия H_{int} как возмущение и вычислим производимые им эффекты с помощью теории возмущений, предполагая, что константа связи достаточно мала для того, чтобы получалось разумное приближение к точному ответу. На самом деле ряд теории возмущений, который мы получим, окажется очень простым по своей структуре. Эффекты взаимодействий в произвольно высоком порядке удастся, по крайней мере, сделать наглядными с помощью диаграмм Фейнмана.

Такое упрощение ряда теории возмущений для релятивистских теорий поля было

огромным достижением Томонаги, Швингера и Фейнмана. Чтобы достичь этого упрощения, каждый независимо нашел способ, как переформулировать квантовую механику, устранив выделенную роль времени, и затем применил новую точку зрения, чтобы представить каждый член ряда теории возмущений как пространственно-временной процесс. В главе 9 мы разовьем квантовую теорию поля с пространственно-времен-

*Краткий обзор интегрируемых квантовых теорий поля дан в Эпилоге. ной точки зрения, используя метод функционального интегрирования Фейнмана. В данной главе мы следуем путем, более пригодным для новичка и впервые разработанным Дайсоном, чтобы получить пространственно-временную картину теории возмущений с помощью обычной техники квантовой механики †.

11.2 Description of Decays

11.2.1 Basics of Decays

параметры распада

Посмотрим, какие вообще есть параметры и что они означают?
Есть ширина распада.

алгоритм расчета параметров

я пока хз, как их считать....

Ширина распада по Киселеву

постановка задачи

Рассмотрим для примера взаимодействие массивного поля φ_M с комплексным безмассовым полем Φ,

$$V_{\text{int}} = -\mathcal{L}_{\text{int}} = \Lambda \varphi_M \Phi^\dagger \Phi$$

То есть распад нейтральной массивной частицы на 2 безмассовых заряженных частицы.

Считаем, что исчезновению нейтральной частице в прошлом (поле φ_M) сопоставить положительно частотную моду $e^{-\mathrm{i}px}$, и появлению заряженной частицы в будущем (поле $\Phi^\dagger$) сопоставить отрицательно частотную моду $e^{\mathrm{i}k_1 x}$, и появлению античастицы в будущем (поле Φ) - отрицательно частотную моду $e^{\mathrm{i}k_2 x}$

амплитуда распада

В этой модели считаем амплитуду распада
(?????)

$$\mathrm{i}\mathcal{M}^c_{(3)} = -\mathrm{i}\Lambda(2\pi)^4 \delta\left(p - k_1 - k_2\right)$$

где p - 4-импульс распадающейся частицы, а $k_{1,2}$ - 4-импульсы частицы и античастицы, соответственно, как это графически показано ниже

Возведение модуля амплитуды в квадрат дает

$$\left|\mathcal{M}^c_{(3)}\right|^2 = \Lambda^2(2\pi)^4\delta\left(p-k_1-k_2\right)\left\{(2\pi)^4\delta\left(k'\right)\right\}\Big|_{k'\to 0}$$

(????)

Стандартным приемом найдем «сингулярное значение»

$$\left\{(2\pi)^4\delta\left(k'\right)\right\}\Big|_{k'\to 0} = \int \mathrm{d}^4x e^{\mathrm{i}k'x}\Big|_{k'\to 0} = \delta t V_{[3]}$$

где интервал времени наблюдения за системой δt и объем 3 -мерной области пространства $V_{[3]}$ мы считаем стремящимися к бесконечности.

Подчеркнем, что релятивистски инвариантное выражение для потенциала приводит к релятивистски инвариантной амплитуде с учетом,

что мы следуем релятивистски инвариантной нормировке квантовых состояний и мод поля.

Напомним, что состояния нормированы на величину $2EV_{[3].}$

вероятность распада

Поэтому для расчета вероятности квантового распада в единицу времени необходимо модуль квадрата инвариантной амплитуду поделить на нормировку начального состояния и умножить на плотность состояний частиц в конечном состоянии в непрерывном спектре,

$$\mathrm{d}\Gamma = \frac{\left|\mathcal{M}^c_{(3)}\right|^2}{\delta t}\frac{1}{2EV_{[3]}}\frac{\mathrm{d}^3k_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3k_2}{(2\pi)^3 2E_2}$$

где временные компоненты 4-импульсов $E = p^0, E_1 = k_1^0, E_2 = k_2^0$. Интегрирование дифференциальной вероятности распада в единицу времени дает ширину распада, которую мы вычислим в системе покоя исходной частицы, $E = M$

$$\Gamma = \int \frac{\Lambda^2}{2M}(2\pi)^4\delta\left(p-k_1-k_2\right)\frac{\mathrm{d}^3k_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3k_2}{(2\pi)^3 2E_2}$$

Преобразуем с помощью уравнения массовой поверхности для частицы в конечном состоянии,

$$\int (2\pi)^4\delta\left(p-k_1-k_2\right)\frac{\mathrm{d}^3k_1}{(2\pi)^3 2E_1} = \int (2\pi)^4\delta\left(p-k_1-k_2\right)2\pi\delta\left(k_1^2\right)\frac{\mathrm{d}^4k_1}{(2\pi)^4}\Big|_{E_1>0}$$

так что интеграл по 4-импульсу снимается законом сохранения 4-импульса, и

$$\Gamma = \frac{\Lambda^2}{2M}\int \delta\left(\left[p-k_2\right]^2\right)\frac{\mathrm{d}^3k_2}{(2\pi)^2 2E_2}$$

В системе покоя распадающейся частицы $\left(p-k_2\right)^2 = M^2 - 2ME_2$,

так что

$$\delta\left(\left[p-k_2\right]^2\right) = \frac{1}{2M}\delta\left(E_2 - \frac{M}{2}\right)$$

и с учетом $|k_2|\,\mathrm{d}\,|k_2| = E_2 dE_2$ и независимости квадрата амплитуды от углов

$$\frac{\mathrm{d}^3 k_2}{(2\pi)^2 2E_2} = \frac{4\pi}{4\pi^2}\frac{|k_2|^2\, d\,|k_2|}{2E_2} = \frac{|k_2|\,\mathrm{d}E_2}{2\pi}$$

В итоге, релятивистски инвариантный фазовый объем

$$\int \delta\left([p-k_2]^2\right)\frac{\mathrm{d}^3 k_2}{(2\pi)^2 2E_2} = \frac{1}{4\pi}\frac{|k_2|}{M} = \frac{1}{8\pi}$$

Ширина найдена

$$\Gamma = \frac{\Lambda^2}{16\pi M}, \quad [\Gamma] = [m]$$

(!!!!????????)

В теории возмущений необходимо, чтобы

$$\Gamma \ll M \quad \to \quad \Lambda \ll M$$

(???)

Важно подчеркнуть, что «вспомогательные» величины: время и объем области, в которых велись наблюдения, конечно, не вошли в ответ, так что их можно было не записывать в явном виде.

Они потребовались только для того,

чтобы согласовать рассуждения с нормировками.

Поэтому можно использовать правила,

согласно которым матричный элемент процесса $\mathcal{M}$ выражается через редуцированный матричный элемент $\mathfrak{M}$ на поверхности закона сохранения 4-имспульса

$$\mathcal{M} = (2\pi)^4\delta\left(k_{(0)} - \sum_{n=1}^{n_f} k_{(n)}\right)\mathfrak{M}$$

так что дифференциальная ширина распада частицы с массой M и импульсом $k_{(0)}$

$$\mathrm{d}\Gamma = \frac{|\mathfrak{M}|^2}{2M} d\mathfrak{p}_f$$

где релятивистски инвариантный фазовый объем частиц в конечном состоянии

$$\mathrm{d}\mathfrak{p}_f = (2\pi)^4\delta\left(k_{(0)} - \sum_{n=1}^{n_f} k_{(n)}\right)\prod_{n=1}^{n_f}\frac{\mathrm{d}^3 k_{(n)}}{2E_{(n)}}$$

(место итогов и выводов?)

11.2.2 Other Aspects of Decays (??????)

(много вопросов будет, все это тут будет)

11.3 Description of Scatterings

соберем всю теорию, которую будем использовать для рассеяний.

11.3.1 Scattering Cross-Section

простейшая модель рассеяния

Для описания расчета сечения рассеяния в квантовой теории поля определим похожую модель,

что и при распаде, но с двумя типами безмассового заряженного скалярного поля

$$\mathcal{L}_{\text{int}} = -\Lambda_1 \varphi_M \Phi_1^\dagger \Phi_1 - \Lambda_2 \varphi_M \Phi_2^\dagger \Phi_2$$

Рассмотрим аннигиляцию частицы с зарядом e_1 и 4-импульсом k_1 и ее античастицы с 4-импульсом k_1' в частицу с зарядом e_2 и 4-импульсом k_2 и ее античастицу с 4-импульсом k_2' в ведущем приближении по $\Lambda_{1,2.}$

Ясно, что в этом случае экспонента от S_{int} сведется к

$$e^{\mathrm{i}S_{\text{int}}} \approx -\mathrm{i}\int \mathrm{d}^4x \Lambda_1 \varphi_M \Phi_1^\dagger \Phi_1 \times (-\mathrm{i})\int \mathrm{d}^4y \Lambda_2 \varphi_M \Phi_2^\dagger \Phi_2$$

где заряженные поля в теории возмущений перейдут в заданные внешние поля, а массивное нейтральное поле необходимо спарить, т. е. заменить пропагатором.

Более детально.

При вычислении трехточечных функций Грина сначала введем действие с внешними источниками для всех квантовых полей в исходном действии,

$$\begin{aligned} S_{\text{ext}}\left(\varphi_M, \Phi_1, \Phi_1^\dagger, \Phi_2, \Phi_2^\dagger\right) = \int \mathrm{d}^4x \Big(& j^M(x)\varphi_M(x) \\ & + J_1^\dagger(x)\Phi_1(x) + \Phi_1(x)J_1^\dagger(x) + \\ & + J_2^\dagger(x)\Phi_2(x) + \Phi_1(x)J_2^\dagger(x)\Big) \end{aligned}$$

Теперь в первом порядке по $\Lambda_{1,2}$ для амплитуды квантовых переходов запишем

$$\begin{aligned} Z \approx & \exp\left\{-\frac{1}{2}\hat{j}_{x'}^M \mathcal{D}_{x'y'}^M \hat{j}_{y'}^M - \hat{J}_1^\dagger(x_1')\,\mathcal{D}_{x_1'y_1'}^{[1]}\hat{J}_1(y_1') - \hat{J}_2^\dagger(x_2')\,\mathcal{D}_{x_2'y_2'}^{[1]}\hat{J}_2(y_2')\right\} \times \\ & \times \left(-\mathrm{i}\Lambda_1 \int \mathrm{d}^4x \tilde{\varphi}_M \tilde{\Phi}_1^\dagger \tilde{\Phi}_1 - \mathrm{i}\Lambda_2 \int \mathrm{d}^4y \tilde{\varphi}_M \tilde{\Phi}_2^\dagger \tilde{\Phi}_2\right) \times \\ & \times \exp\left\{\mathrm{i}S_{\text{ext}}\left(\tilde{\varphi}_M, \tilde{\Phi}_1, \tilde{\Phi}_1^\dagger, \tilde{\Phi}_2, \tilde{\Phi}_2^\dagger\right)\right\}\Big|_{\tilde{\varphi}_M = \ldots = 0} \end{aligned}$$

где внешние токи со шляпками - это операторы вариационного дифференцирования по полям с тильдой

$$\hat{j}_x^M = \frac{\delta}{\delta\tilde{\varphi}_M(x)}, \quad \hat{J}_{1,2}(x) = \frac{\delta}{\delta\tilde{\Phi}_{1,2}^\dagger(x)} \quad \hat{J}_{1,2}^\dagger(x) = \frac{\delta}{\delta\tilde{\Phi}_{1,2}(x)}$$

при эти вариационные производные действуют только на последний фактор $\exp\{\mathrm{i}S_{\text{ext}}\}$, который только и зависит от полей с тильдой.

А спаривания определяются в следующем виде: для заряженных безмассовых полей -

$$\mathcal{D}_{1x}^{[1]} = \mathcal{D}_{1x}^{[2]} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2 + \mathrm{i}0}e^{-\mathrm{i}k(x_1 - x)}$$

для массивного вещественного поля -

$$\mathcal{D}^{M}_{xy} = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}0} e^{-\mathrm{i}k(x-y)}$$

Легко сообразить, что в такой записи тильду у полей вполне можно опустить (?????), при этом опять-таки подчеркивая, что внешние токи содержатся только в действии S_{ext}

Тогда трехточечная функция Грина для взаимодействия вещественного поля с заряженным полем типа 1 (??????) в первом порядке теории возмущений сведется с одной стороны к Т-произведению полей,

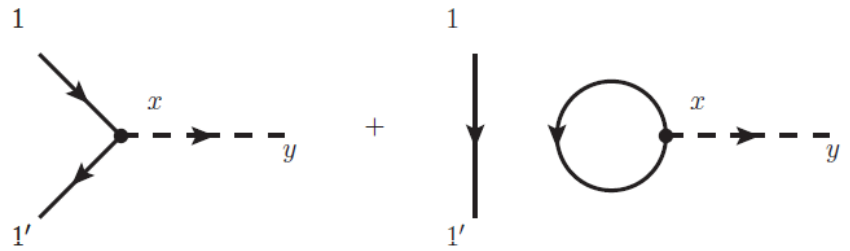
$$\mathcal{D}^{[1]}_{11'y} = (-\mathrm{i})^3 \frac{\delta^3 Z}{\delta j^M(y) \delta J_1(x_1) \delta J_1^{\dagger}(x_1')} = \langle 0| \ \mathrm{T} \left\{ \hat{\varphi}_M(y) \hat{\Phi}_1^{\dagger}(x_1) \hat{\Phi}_1(x_1') \right\} |0\rangle$$

а с другой стороны, к спариваниям в выражении (???)

$$\mathcal{D}^{[1]}_{11'y} = \sum_{\text{спар.}} -\mathrm{i}\Lambda_1 \int \mathrm{d}^4 x \varphi_M(x) \Phi_1^{\dagger}(x) \Phi_1(x) \varphi_M(y) \Phi_1^{\dagger}(x_1) \Phi_1(x_1')$$

Для вещественного поля есть только один вариант спаривания, что дает фактор $\mathcal{D}^{M}_{xy}$, а для заряженного поля - 2 варианта: $\mathcal{D}^{[1]}_{xx}\mathcal{D}^{[1]}_{11'}$ и $\mathcal{D}^{[1]}_{x1}\mathcal{D}^{[1]}_{1'x}$.

Первый вариант отвечает петле заряженного поля с вершиной в точке x и свободному распространению заряженного поля от внешний точки 1 к внешней точке $1'$, а второй вариант - вершине, в которой сходятся все линии, как это показано на рис.



Связный и не связный вклад с 3-точечную функцию $\mathcal{D}^{[1]}_{11'y}$.

(???? что происходит?????)

мы получим следующие выражения для связной 3 -точечной функции в координатном представлении для взаимодействия вещественного поля с безмассовыми заряженными частицами первого сорта

$$\mathcal{D}^{[1c]}_{11'y} = -\mathrm{i}\Lambda_1 \mathcal{D}^{[1]}_{1x} \mathcal{D}^{[1]}_{xx'} \mathcal{D}^{M}_{xy}$$

Ясно, что ровно такие же рассуждения приведут к связной функции Грина для взаимодействия вещественного поля с заряженным полем второго сорта $\mathcal{D}^{[2c]}_{11'y}$.

(???? что за сорт частиц кстати?)

Если теперь поставить вопрос о связной 4-точечной функции для перехода частицы и античастицы сорта 1 в частицу и античастицу сорта 2

$$\mathcal{D}_{11'22'} = (-\mathrm{i})^4 \frac{\delta^4 Z}{\delta J_1(x_1) \delta J_1^{\dagger}(x_1') \delta J_2(x_2) \delta J_2^{\dagger}(x_2')} = \\ = \langle 0| \ \mathrm{T} \left\{ \hat{\Phi}_1^{\dagger}(x_1) \hat{\Phi}_1(x_1') \hat{\Phi}_2^{\dagger}(x_2) \hat{\Phi}_2(x_2') \right\} |0\rangle$$

то в ведущем порядке теории возмущений по константам связи отличный от нуля вклад возникает для произведения $\Lambda_1\Lambda_2$, а значит, мы рассмотрим спаривания в выражении

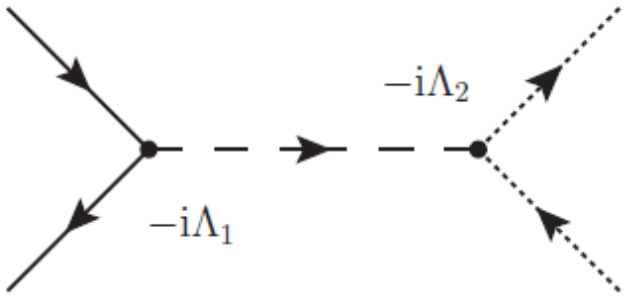
$$\mathcal{D}_{11'22'} = \sum_{\text{спар.}} (-\mathrm{i}\Lambda_1) \int \mathrm{d}^4x \varphi_M(x) \Phi_1^\dagger(x) \Phi_1(x) \Phi_1^\dagger(x_1)\, \Phi_1(x_1') \times$$
$$\times (-\mathrm{i}\Lambda_2) \int \mathrm{d}^4y \varphi_M(y) \Phi_2^\dagger(y) \Phi_2(y) \Phi_2^\dagger(x_2)\, \Phi_2(x_2')$$

где спаривание вещественного поля даст фактор $\mathcal{D}^M_{xy}$, а связный вклад для заряженных полей приведет к выражению

$$\mathcal{D}^c_{11'22'} = (-\mathrm{i}\Lambda_1)(-\mathrm{i}\Lambda_2)\, \mathcal{D}^{[1]}_{1x} \mathcal{D}^{[1]}_{x1'} \mathcal{D}^M_{xy} \mathcal{D}^{[2]}_{y2} \mathcal{D}^{[1]}_{2'y}$$

где, конечно ведется интегрирование по повторяющимся координатным индексам. В теории возмущений получается диаграмма рассеяния на рис.

(?????)



Эта 4-точечная функция задает связный вклад в вариационный ряд для амплитуды квантовых переходов в заданном порядке теории возмущений в виде

$$Z = \mathrm{i}^4 \int \mathrm{d}^4x_1 d^4x_1' \mathrm{d}^4x_2 d^4x_2' \mathcal{D}^c_{11'22'} J_1(x_1)\, J_1^\dagger(x_1')\, J_2(x_2)\, J_2^\dagger(x_2') + \ldots$$

Для расчета амплитуды квантовых переходов в асимптотически свободной области необходимо перейти от Z к $\mathcal{S}$ -матрице $S = \mathbb{1} + \mathrm{i}T$ и от внешних токов к внешним полям, когда

$$\mathrm{i} \int \mathrm{d}^4x_1 J_1(x_1)\, \mathcal{D}^{[1]}_{1x} = \Phi_1^{\mathrm{cl}}(x), \quad \mathrm{i} \int \mathrm{d}^4x_2 J_2(x_2)\, \mathcal{D}^{[2]}_{2y} = \Phi_2^{\mathrm{cl}}(y)$$

а также

$$\mathrm{i} \int \mathrm{d}^4x_1' J_1^\dagger(x_1')\, \mathcal{D}^{[1]}_{x1'} = \left(\Phi_1^{\mathrm{cl}}(x)\right)^\dagger, \quad \mathrm{i} \int \mathrm{d}^4x_2' J_2^\dagger(x_2')\, \mathcal{D}^{[2]}_{y2'} = \left(\Phi_2^{\mathrm{cl}}(y)\right)^\dagger$$

Но это означает, что в выражении для амплитуды в асимптотически свободной области $\mathcal{D}^c_{11'22'}$ все двухточечные функции при интегрировании с внешними токами заменяются на классические внешние поля, и остается только пропагатор виртуального вещественного поля

$$\mathrm{i}\mathcal{M} = (-\mathrm{i}\Lambda_1)(-\mathrm{i}\Lambda_2) \int \mathrm{d}^4x d^4y \mathcal{D}^M_{xy} \Phi_1^{\mathrm{cl}}(x) \left(\Phi_1^{\mathrm{cl}}(x)\right)^\dagger \Phi_2^{\mathrm{cl}}(y) \left(\Phi_2^{\mathrm{cl}}(y)\right)^\dagger$$

В задаче аннигиляции частицы сорта 1 со своей античастицей начальным состояниям внешних полей отвечают положительно частотные моды полей $\Phi_1^{\rm cl}(x)$ и $\left(\Phi_1^{\rm cl}(x)\right)^\dagger$, соответственно, поскольку в точке x эти частица античастица исчезают, так что

$$\Phi_1^{\rm cl}(x) \mapsto e^{-{\rm i}k_1 x}, \quad \left(\Phi_1^{\rm cl}(x)\right)^\dagger \mapsto e^{-{\rm i}k_1' x}$$

где мы присвоили внешним полям в начальном состоянии вполне определенные 4-импульсы k_1 для частицы и k_1' для античастицы.

Аналогично, внешним полям в конечном состоянии соответствуют отрицательно частотные моды полей второго сорта $\Phi_2^{\rm cl}(y)$ и $\left(\Phi_2^{\rm cl}(y)\right)^\dagger$, соответственно поскольку в точке эти частица и античастица рождаются, так что

$$\Phi_2^{\rm cl}(y) \mapsto e^{{\rm i}k_2' y}, \quad \left(\Phi_2^{\rm cl}(2)\right)^\dagger \mapsto e^{{\rm i}k_2 y}$$

где мы присвоили внешним полям в конечном состоянии вполне определенные 4-импульсы k_2 для частицы и k_2' для античастицы.

Наконец, подстановка пропагатора виртуальной частицы в виде интеграла по импульсу дает

$${\rm i}\mathcal{M} = (-{\rm i}\Lambda_1)\,(-{\rm i}\Lambda_2) \int {\rm d}^4x d^4y \frac{{\rm d}^4p}{(2\pi)^4} \mathcal{D}^M(p) e^{{\rm i}x\left(k_1+k_1'-p\right)+{\rm i}y\left(p-k_1-k_2'\right)}$$

Как видим, в каждой вершине имеет место закон сохранения 4-импульса.

При этом отметим, что в качестве внешнего поля в начальном и конечном состоянии можно рассматривать и волновые пакеты,

в прямом соответствии с тем, как ставится экспериментальная задача.

В итоге, матричный элемент процесса в импульсном представлении примет вид

$${\rm i}\mathcal{M} = (2\pi)^4 \delta\left(k_1 + k_1' - k_2 - k_2'\right) (-{\rm i}\Lambda_1)\,(-{\rm i}\Lambda_2)\, \mathcal{D}^M\left(k_1 + k_1'\right)$$

(?????)

где мы учли, что 4-импульс массивной частицы по закону сохранения равен сумме 4-импульсов частиц в начальном состоянии, так что

$$\mathcal{D}^M\left(k_1 + k_1'\right) = \frac{\rm i}{\left(k_1 + k_1'\right)^2 - M^2 + {\rm i}0} = \frac{\rm i}{s - M^2 + {\rm i}0}$$

где мы ввели стандартную характеристику для столкновения двух частиц квадрат инвариантной массы системы s. В реальности, как мы уже знаем,

массивная частица может иметь ненулевую ширину распада Γ за счет того же взаимодействия, как минимум.

Это значит, что вместо массы следует использовать величину $M \to M - {\rm i}\Gamma/2$, так что в силу $\Gamma \ll M$ пропагатор принимает вид

$$\mathcal{D}^M\left(k_1 + k_1'\right) = \frac{\rm i}{s - M^2 + {\rm i}\Gamma M}$$

Дифференциальная вероятность процесса в единицу времени вычисляется возведением в квадрат модуля амплитуды и делением его на время наблюдения за системой,

а также делением на нормировку состояния мишени,

которой мы можем считать, например, античастицу,

с последующим домножением на плотность состояний в конечном состоянии,
то есть:

$$\mathrm{d}w = \frac{|\mathcal{M}|^2}{\delta t}\frac{1}{2E_1' V_{[3]}}\frac{\mathrm{d}^3 k_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3 k_2'}{(2\pi)^3 2E_2'}$$

откуда

$$\mathrm{d}w = \frac{\Lambda_1^2\Lambda_2^2}{\left(s-M^2\right)^2+M^2\Gamma^2}\frac{1}{2E_1'}(2\pi)^4\delta\left(k_1+k_1'-k_2-k_2'\right)\frac{\mathrm{d}^3 k_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3 k_2'}{(2\pi)^3 2E_2'}$$

Число событий определяется потоком частиц, которые падают на античастицу.

Этот поток мы вычисляли для неподвижной мишени.

(и что там было?)

Его обобщение очевидно, так как наблюдаемый объем увеличивается для наблюдателя за счет того,

что сама мишень движется навстречу пучку, и в нашем случае

$$j = 2E_1\left(v_1+v_1'\right)$$

Поэтому дифференциальное сечение

$$\mathrm{d}\sigma = \frac{\mathrm{d}w}{j}$$

и в него войдет величина

$$2E_1' 2E_1\left(v_1+v_1'\right)$$

которая является релятивистским инвариантом.

В самом деле, покажем это в самом общем случае,

и для краткости дальнейших выкладок по этому вопросу будем рассматривать столкновение частиц с массой m и $\tilde{m}$:

$$2E2\tilde{E}(v+\tilde{v})$$

Элементарные преобразования квадрата суммы скоростей дает

$$(v+\tilde{v})^2 = v^2+2v\tilde{v}+\tilde{v}^2 = 1+2v\tilde{v}+v^2\tilde{v}^2-\left(1-v^2\right)\left(1-\tilde{v}^2\right)$$

а, стало быть, умножение на квадраты энергий приводит к формуле

$$E^2\tilde{E}^2(v+\tilde{v})^2 = (E\tilde{E}+E\tilde{E}v\tilde{v})^2-E^2\left(1-v^2\right)\tilde{E}^2\left(1-\tilde{v}^2\right) = (p\cdot\tilde{p})^2-m^2\tilde{m}^2$$

где $p\cdot\tilde{p}$ - произведение 4-импульсов частиц, которые движутся навстречу друг другу.

В итоге,

$$2E2\tilde{E}(v+\tilde{v}) = 4\sqrt{(p\cdot\tilde{p})^2-m^2\tilde{m}^2}$$

В нашей задаче массы сталкивающихся частиц равны нулю, так что

$$2E_1' 2E_1\left(v_1+v_1'\right) = 8E_1'E_1 = 4\sqrt{\left(\left(k_1+k_1'\right)^2/2\right)^2} = 2s$$

Таким образом,

$$\mathrm{d}\sigma = \frac{\Lambda_1^2\Lambda_2^2}{\left(s-M^2\right)^2+M^2\Gamma^2}\frac{1}{2s}(2\pi)^4\delta\left(k_1+k_1'-k_2-k_2'\right)\frac{\mathrm{d}^3 k_2}{(2\pi)^3 2E_2}\frac{\mathrm{d}^3 k_2'}{(2\pi)^3 2E_2'}$$

Вычисление инвариантного фазового объема для безмассовых частиц при фиксированной инвариантной массе мы уже проводили (??????)

$$\int (2\pi)^4 \delta\left(k_1 + k_1' - k_2 - k_2'\right) \frac{\mathrm{d}^3 k_2}{(2\pi)^3 2E_2} \frac{\mathrm{d}^3 k_2'}{(2\pi)^3 2E_2'} = \frac{1}{8\pi}$$

Полное сечение аннигилляции

$$\sigma = \frac{1}{16\pi s} \frac{\Lambda_1^2 \Lambda_2^2}{\left(s - M^2\right)^2 + M^2\Gamma^2}, \quad [\sigma] = \frac{1}{[m]^2}$$

и оно в данном случае имеет резонансный пик Брейта-Вигнера при инвариантной массе, равной массе промежуточной частицы.

Отметим также, что матричный элемент процесса при скалярном обмене не зависит от углов, т. е. это рассеяние изотропно.

Резюмируя,

стоит записать общее выражение для дифференциального сечения в терминах редуцированного матричного элемента $\mathfrak{M}$ так же, как мы это сделали для дифференциальной ширины распада,

$$\mathrm{d}\sigma = \frac{|\mathfrak{M}|^2}{4\sqrt{(p\cdot\tilde{p})^2 - m^2\tilde{m}^2}} d\mathfrak{p}_f$$

где фазовый объем в конечном состоянии по аналогии с (5.2)

$$\mathrm{d}\mathfrak{p}_f = (2\pi)^4 \delta\left(p + \tilde{p} - \sum_{n=1}^{n_f} k_{(n)}\right) \prod_{n=1}^{n_f} \frac{\mathrm{d}^3 k_{(n)}}{2E_{(n)}}$$

а $p, \tilde{p}$ - 4-импульсы сталкивающихся частиц.

Приведенные нами примеры показывает, что связный вклад определяет матричный элемент процесса распада или рассеяния, и в теории возмущений он находится элементарно с помощью диаграмм.

Поэтому стоит систематизировать общие утверждения о получении всех связных диаграмм и вне теории возмущений.

Ширина распада

И наконец, повторим, как связана амплитуда с дифференциальной шириной распада частицы массой M с 4 -импульсм p

$$\mathrm{d}\Gamma = \frac{|\mathfrak{M}|^2}{2M} d\mathfrak{p}_f$$

где релятивистски инвариантный фазовый объем частиц в конечном состоя- нии

$$\mathrm{d}\mathfrak{p}_f = (2\pi)^4 \delta\left(p - \sum_{n=1}^{n_f} k_{(n)}\right) \prod_{n=1}^{n_f} \frac{\mathrm{d}^3 k_{(n)}}{2E_{(n)}}$$

и дифференциальным сечением рассеяния частиц с 4-импульсами p и $\tilde{p}$

$$\mathrm{d}\sigma = \frac{|\mathfrak{M}|^2}{4\sqrt{(p\cdot\tilde{p})^2 - m^2\tilde{m}^2}} d\mathfrak{p}_f$$

где фазовый объем в конечном состоянии

$$\mathrm{d}\mathfrak{p}_f = (2\pi)^4 \delta\left(p + \tilde{p} - \sum_{n=1}^{n_f} k_{(n)}\right) \prod_{n=1}^{n_f} \frac{\mathrm{d}^3 k_{(n)}}{2E_{(n)}}$$

Отметим, что эти правила относятся к случаю, когда одноточечная функция Грина поля равна нулю. Однако, легко сообразить, что ненулевое

вакуумное значение поля $v = \langle 0|\hat{\varphi}(x)|0\rangle$ отвечает просто случаю внешнего поля с нулевым импульсом. Такую внешнюю линию обычно изображают с символом $\otimes$ на конце. При этом вершины взаимодействия внешнего вакуумного поля легко определить, если в исходном лагранжиане произвести замену поля, скажем, $\varphi \mapsto v + \varphi$, так что новое поле уже не имеет вакуумного среднего.

11.3.2 Particle Reactions (!?!??)

я вообще хз, как описывать их с помощью КТП, но буду изучать.

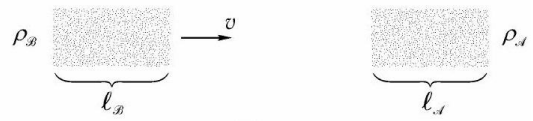
11.3.3 Scattering Cross-Sections and the S-Matrix by Peskin & Schroeder

Теперь в нашем распоряжении чрезвычайно красивая формула (4.57) для вычисления совершенно абстрактной величины - n-точечной корреляционной функции. Следующая задача - найти в равной степени красивые пути вычисления величин, которые могут быть фактически измерены, - сечений рассеяния и вероятностей распада. В этом разделе после краткого обзора определений этих величин мы свяжем их (без излишней строгости, но вполне аккуратным образом) с более примитивной величиной S-матрицей. В следующем разделе мы узнаем, как вычислять элементы S-матрицы, используя диаграммы Фейнмана.

Сечение рассеяния по Пескину Шредеру

Эксперименты, которые исследуют поведение элементарных частиц, особенно в релятивистском режиме, являются экспериментами по рассеянию. Сталкивают два пучка частиц с известными импульсами и наблюдают за тем, что происходит. Вероятность обнаружения любого отдельного конечного состояния может быть выражена через *се* - чение рассеяния - величину, которая характеризует внутренние свойства сталкивающихся частиц и, следовательно, позволяет провести сравнение двух различных экспериментов с пучками разных размеров и интенсивностей.

Сечение рассеяния определяется следующим образом. Рассмотрим покоящуюся мишень, состоящую из частиц типа $\mathscr{A}$ с плотностью $\rho_{\mathscr{A}}$ (число частиц на единицу объема). Направим на эту мишень пучок частиц типа $\mathscr{B}$ с плотностью числа частиц $\rho_{\mathscr{B}}$ и скоростью v:



Пусть $\ell_{\mathscr{A}}$ и $\ell_{\mathscr{B}}$ - длины пучков частиц. Тогда мы ожидаем, что полное число событий рассеяния (или событий рассеяния любого заданного типа) должно быть пропорционально

$\rho_{\mathscr{A}}, \rho_{\mathscr{B}}, \ell_{\mathscr{A}}, \ell_{\mathscr{B}}$ и площади пересечения пучков рассеяния A, общей для двух пучков. Сечение рассеяния, обозначаемое σ, равно общему числу событий (любого заданного типа), деленному на произведение всех этих величин:

$$\sigma \equiv \frac{\text{Число событий рассеяния}}{\rho_{\mathscr{A}}\ell_{\mathscr{A}}\rho_{\mathscr{B}}\ell_{\mathscr{B}}A}.$$

Данное определение симметрично относительно $\mathscr{A}$ и $\mathscr{B}$, поэтому можно было бы считать покоящимися частицы типа $\mathscr{B}$ или проводить рассмотрение в любой другой системе отсчета. Сечение рассеяния имеет размерность площади. Действительно, это есть эффективная площадь куска, выбиваемого из одного пучка каждой частицей другого пучка и впоследствии переходящего в интересующее нас конечное состояние.

В реальных пучках $\rho_{\mathscr{A}}$ и $\rho_{\mathscr{B}}$ не постоянны - плотность частиц в центре пучка обычно больше, чем на краях. Однако мы будем полагать, что как радиус взаимодействия частиц, так и ширина отдельных волновых пакетов частиц малы по сравнению с диаметром пучка. Поэтому в дальнейшем можно считать $\rho_{\mathscr{A}}$ и $\rho_{\mathscr{B}}$ постоянными, не забывая при этом, что для вычисления вероятности событий в реальном ускорителе нужно проинтегрировать по сечению пучка:

$$\text{Число событий } = \sigma\ell_{\mathscr{A}}\ell_{\mathscr{B}}\int d^2x\rho_{\mathscr{A}}(x)\rho_{\mathscr{B}}(x).$$

Если плотности постоянны или если мы используем эту формулу для вычисления эффективной площади A сечения пучков, то

$$\text{Число событий } = \frac{\sigma N_{\mathscr{A}}N_{\mathscr{B}}}{A},$$

где $N_{\mathscr{A}}$ и $N_{\mathscr{B}}$ - полное число частиц типа $\mathscr{A}$ и $\mathscr{B}$.

Сечения многих различных процессов могут быть получены в одном эксперименте рассеяния. Например, в столкновениях e^+e^-можно измерять сечения рождения $\mu^+\mu^-$, $\tau^+\tau^-, \mu^+\mu^-\gamma, \mu^+\mu^-\gamma\gamma$ и т. д. и бесчисленных процессов, включающих рождение адронов, не говоря уже о простом e^+e^-рассеянии. Обычно мы хотим определить не только тип частиц в конечном состоянии, но также и их импульсы. Определение (4.59) сечения σ все еще применимо и в этом случае, но если мы выделяем точные значения импульсов, то σ будет бесконечно малым. Решение состоит в том, чтобы определить дифференциальное сечение рассеяния $d\sigma/\left(d^3p_1...d^3p_n\right)$. Это величина, которая в результате интегрирования по любому малому объему $d^3p_1...d^3p_n$ определяет сечение рассеяния в этот объем импульсов конечного состояния. Импульсы в конечном состоянии не являются независимыми. Четыре компоненты будут всегда связаны условием сохранения 4-импульса. В самом простом случае, когда имеется только две частицы в конечном состоянии, это условие оставляет только две независимых компоненты импульса, в качестве которых обычно выбираются углы θ и ϕ импульса одной из частиц. Интегрирование $d\sigma/\left(d^3p_1d^3p_2\right)$ по четырем ограниченным законом сохранения компонентам импульса приводит к обычному дифференциальному сечению рассеяния $d\sigma/d\Omega$.

Более простая измеримая величина - это вероятность распада нестабильной частицы $\mathscr{A}$ (которая считается покоящейся) в определенное конечное состояние (из двух или более частиц). Эта вероятность определяется как

$$\Gamma \equiv \frac{\text{Число распадов в единицу времени}}{\text{Число имеющихся частиц типа } \mathscr{A}}.$$

Тогда время жизни τ частицы является обратным сумме ее вероятностей распадов во все возможные конечные состояния. (Время полураспада частицы $-\tau \cdot \ln 2$.)

В нерелятивистской квантовой механике нестабильное атомное состояние проявляет себя в экспериментах по рассеянию как резонанс. Вблизи резонансной энергии E_0 амплитуда рассеяния определяется формулой Брейта-Вигнера:

$$f(E) \propto \frac{1}{E - E_0 + i\Gamma/2}$$

Следовательно, сечение рассеяния имеет пик вида

$$\sigma \propto \frac{1}{(E - E_0)^2 + \Gamma^2/4}$$

Ширина пика резонанса равна вероятности распада нестабильного состояния.

Формула Брейта-Вигнера (4.63) имеет место также и в релятивистской квантовой механике. В частности, она определяет амплитуду рассеяния для процессов, в которых начальные частицы образуют составную частицу, которая затем распадается. Нестабильная частица, рассматриваемая как возбужденное состояние вакуума, является прямым аналогом нестабильного нерелятивистского атомного состояния. Если задать 4-импульс p нестабильной частицы массой m, то можно получить релятивистски инвариантное обобщение формулы (4.63):

$$\frac{1}{p^2 - m^2 + im\Gamma} \approx \frac{1}{2E_{\mathbf{p}}\left(p^0 - E_{\mathbf{p}} + i\left(m/E_{\mathbf{p}}\right)\Gamma/2\right)}.$$

Вероятность распада нестабильной частицы в произвольной системе отсчета равна $(m/E_{\mathbf{p}})\,\Gamma$, что согласуется с релятивистским растяжением времени. Хотя обе части равенства в (4.64) равны в окрестности резонанса, запись слева, которая является явно лоренц-инвариантной, намного удобнее.

11.3.4 Expressing Process Probabilities via Amplitudes

Рассмотрим вопрос о выражении сечений рассеяния и вероятностей распадов частиц через инвариантные амплитуды рассеяния в трёхмерном пространстве.

Сначала проясним вопрос о нормировке r-частичных состояний. Согласно (39 Л2)

$$\langle k|k\rangle = 2\omega_k(2\pi)^3\delta^{(3)}(0) \tag{A1}$$

Так как объём пространства представляется как

$$V = \int \mathrm{d}^{(3)}x e^{i(\mathbf{p}-\mathbf{k})\mathbf{x}}\Big|_{\mathbf{p}=\mathbf{k}} = (2\pi)^3\delta^{(3)}(0)$$

то (A1) означает, что принятая нами лоренц-инвариантная нормировка состояний (38 Л2) такова, что в единице объёма содержится

$$n_{\mathbf{k}} = 2\omega_k = 2\sqrt{m^2 + \mathbf{k}^2} \tag{A2}$$

частиц.

Построим r-частичное нормированное состояние. Пусть G - малая область в импульсном пространстве частиц вблизи значений импульсов $(\mathbf{p}_1, ..., \mathbf{p}_r)$, так что объём G равен:

$$G = \int\limits_G \mathrm{d}^{(3)}p_1 ... \mathrm{d}^{(3)}p_r = \prod_{j=1}^{r} \Delta^{(3)} p_j \tag{A3}$$

Состояние $\left(\omega_j \equiv \sqrt{m_j^2 + \mathbf{p}_j^2}\right)$

$$|\tilde{f}\rangle = \prod_{j=1}^{r} \int\limits_G \frac{\mathrm{d}^{(3)}p_j}{[(2\pi)^3 2\omega_j]} \, |\mathbf{p}_1, ..., \mathbf{p}_r\rangle$$

имеет норму

$$\begin{aligned} \langle \tilde{f}|\tilde{f}\rangle &= \prod_{j=1}^{r} \int\limits_G \frac{\mathrm{d}^{(3)}p_j}{[(2\pi)^3 2\omega_j]} \int\limits_G \frac{\mathrm{d}^{(3)}p'_j}{\left[(2\pi)^3 2\omega'_j\right]} \, \langle \mathbf{p}'_1, ..., \mathbf{p}'_r | \mathbf{p}_1, ..., \mathbf{p}_r\rangle = \\ &= \prod_{j=1}^{r} \int\limits_G \frac{\mathrm{d}^{(3)}p_j}{[(2\pi)^3 2\omega_j]} \int\limits_G \mathrm{d}^{(3)}p'_j \delta^{(3)}\left(\mathbf{p}_j - \mathbf{p}'_j\right) = \prod_{j=1}^{r} \frac{\Delta^{(3)} p_j}{[(2\pi)^3 2\omega_j]} \end{aligned} \tag{A4}$$

При получении (А4) была использована формула (45 Л2), а также тот факт, что импульсы $\mathbf{p}_j$ и $\mathbf{p}'_j$ почти совпадают при всех j, в то время как импульсы $\mathbf{p}_j$ и $\mathbf{p}'_i$ при $i \neq j$ принадлежат непересекающимся областям Δp_j и Δp_i. Следует заметить, что (А4) есть лоренц-инвариант. Таким образом состояние

$$|f\rangle = \left[\prod_{j=1}^{r} \frac{\Delta^{(3)} p_j}{[(2\pi)^3 2\omega_j]}\right]^{-1/2} |\tilde{f}\rangle \tag{A5}$$

нормировано, причём это свойство лоренц-инвариантно.

Пусть $|i\rangle$ - n-частичное состояние (43 Л2) и $|f\rangle$ - состояние (А5). Тогда для недиагональных матричных элементов (см. (2 ЛЗ)-(4 ЛЗ))

$$\langle f|S|i\rangle = \langle f|iT|i\rangle = \left[\prod_{j=1}^{r} \frac{\Delta^{(3)} p_j}{[(2\pi)^3 2\omega_j]}\right]^{-1/2} \prod_{j=1}^{r} \int\limits_G \frac{\mathrm{d}^{(3)}p_j}{[(2\pi)^3 2\omega_j]} \, \langle \mathbf{p}_1, ..., \mathbf{p}_r | iT | \mathbf{k}_1, ..., \mathbf{k}_n\rangle \tag{A6}$$

Согласно вышесказанному выражение (А6) лоренц-инвариантно. Перепишем (А6) при помощи амплитуды M_{fi} (см. (4 ЛЗ) и вычислим вероятность перехода из состояния $|i\rangle$ в состояние $|f\rangle$. Так как $|f\rangle$ - нормированное состояние, то согласно общим правилам квантовой механики эта вероятность

$$|\langle f|S|i\rangle|^2 = \prod_{j=1}^{r} \int\limits_G \frac{\mathrm{d}^{(3)}p_j}{[(2\pi)^3 2\omega_j]} \left[(2\pi)^4 \delta^{(4)}\left(P_f - P_i\right)\right]^2 |M_{fi}|^2 \tag{A7}$$

При получении последней формулы была учтена малость объёма G, а также равенство

$$\left[\prod_{j=1}^{r} \frac{\Delta^{(3)} p_j}{[(2\pi)^3 2\omega_j]}\right]^{-1} \prod_{j=1}^{r} \int\limits_G \frac{\mathrm{d}^{(3)}p_j}{[(2\pi)^3 2\omega_j]} = 1$$

В(А7)

$$P_i = k_1 + ... + k_n, \quad P_f = p_1 + ... + p_r$$

Положим

$$\left[(2\pi)^4\delta^{(4)}\left(P_f - P_i\right)\right]^2 = VT(2\pi)^4\delta^{(4)}\left(P_f - P_i\right)$$

где V - объём пространства, а T-полный промежуток времени, в течение которого рассматривается процесс. Таким образом, деля выражение (А7) на 4 -объём VT, мы получаем вероятность перехода в единице пространства в единицу времени:

$$\mathrm{d}w_{fi} = (2\pi)^4\delta^{(4)}\left(P_f - P_i\right)\left]\,|M_{fi}|^2\prod_{j=1}^{r}\frac{\mathrm{d}^{(3)}p_j}{[(2\pi)^3 2\omega_j]}\right. \tag{А8}$$

В (А8) опущен знак интеграла по импульсам конечных частиц. Поэтому вероятность перехода (А8) является дифференциальной вероятностью перехода в бесконечно малый элемент импульсного пространства конечных состояний.

Ещё раз подчеркнём, что выражение (А8) лоренц-инвариантно и в случае применимости ТВ амплитуда M_{fi} вычисляется при помощи диаграммной техники Фейнмана согласно вышеизложенным правилам.

Теперь рассмотрим частные случаи

1. Процессы распада.

В этом процессе в начальном состоянии имеется одна частица с массой M. Чтобы получить вероятность распада одной частицы, следует разделить (А8) на плотность начальных частиц $n_{\mathbf{k}} = 2\sqrt{M^2 + \mathrm{k}^2} = 2\omega_k$. Таким образом вероятность распада частицы в единицу времени

$$\mathrm{d}\Gamma = \frac{(2\pi)^4}{2\sqrt{M^2 + \mathbf{k}^2}}\delta^{(4)}\left(P_f - k\right)|M_{fi}|^2\prod_{j=1}^{r}\frac{\mathrm{d}^{(3)}p_j}{[(2\pi)^3 2\omega_j]} \tag{А9}$$

2. Взаимное рассеяние двух частиц

В системе отсчёта, в которой одна из рассеивающихся частиц покоится, вероятность (А8) представляется в виде

$$\mathrm{d}w_{fi} \equiv \mathrm{d}\sigma \cdot v_{\text{ОТН}} n_1 n_2 \tag{А10}$$

где n_1 и n_2 - плотности частиц сортов 1 и 2, соответственно, $v_{\text{отн}}$ - относительная скорость частиц и $\mathrm{d}\sigma$ - лоренц-инвариантное дифференциальное сечение рассеяния. Из определения (А10) видно, что сечение имеет размерность площади, если n_1 и n_2 в (А10) нормируются так же как и начальные состояния при вычислении амплитуды рассеяния, т.е. согласно (А2).

Для проведения лоренц-инвариантных вычислений при нахождении сечений введём инвариантную величину, которую можно назвать инвариантной плотностью потока:

$$j = \sqrt{(k_1 k_2)^2 - m_1^2 m_2^2} \tag{А11}$$

В системе отсчёта, в которой частица 2 покоится

$$j = \frac{m_1 m_2 v_{\text{OTH}}}{\sqrt{1 - v_{\text{OTH}}^2}} = \frac{1}{4} v_{\text{OTH}} n_1 n_2 \tag{A12}$$

где

$$n_1 = \frac{2m_1}{\sqrt{1 - v_{\text{OTH}}^2}}, \quad n_2 = 2m_2$$

Если же налетающая частица 1 является безмассовой, то выражение для j при помощи правой части (A12) остаётся справедливым, если положить $v_{\text{O}} = 1$. Таким образом лоренц-инвариантное дифференциальное сечение рассеяния представляется как

$$\mathrm{d}\sigma = \frac{\mathrm{d}w_{fi}}{4j} \tag{A13}$$

Легко проверить следующую формулу:

$$2j = \sqrt{\left[s - (m_1 + m_2)^2\right]\left[s - (m_1 - m_2)^2\right]}, \quad s = (k_1 + k_2)^2 \tag{A14}$$

В системе центра инерции частиц

$$k_1 = (\omega_1, \mathbf{k}), \quad k_2 = (\omega_2, -\mathbf{k})$$

и плотность потока

$$j = |\mathbf{k}| (\omega_1 + \omega_2) \tag{A15}$$

Рассмотрим подробнее формулу (A13) в системе центра инерции в том случае, когда в конечном состоянии имеется также две частицы с массами m_1' и m_2'. Имеем:

$$p_1 = (\omega_1', \mathbf{p}), \quad p_2 = (\omega_1', -\mathbf{p}), \quad \omega_1'^2 - m_1'^2 = \omega_2'^2 - m_2'^2 = |\mathbf{p}|^2 \tag{A16}$$

Отсюда

$$\omega_1' \mathrm{d}\omega_1' = \omega_2' \mathrm{d}\omega_2' = |\mathbf{p}| \mathrm{d}|\mathbf{p}| \longrightarrow \mathrm{d}(\omega_1' + \omega_2') = (\omega_1' + \omega_2') \frac{|\mathbf{p}| \mathrm{d}|\mathbf{p}|}{\omega_1' \omega_2'} \tag{A17}$$

При помощи обозначений

$$\omega = \omega_1 + \omega_2, \quad \omega' = \omega_1' + \omega_2'$$

формулы (A15) и (A17) переписываются как

$$j = |\mathbf{k}| \omega, \quad |\mathbf{p}| \mathrm{d}|\mathbf{p}| = \frac{\omega_1' \omega_2'}{\omega'} \mathrm{d}\omega' \tag{A18}$$

что позволяет представить сечение (A13) в виде

$$\mathrm{d}\sigma = \frac{|M_{fi}|^2 |\mathbf{p}| \mathrm{d}o'}{64\pi^2 |\mathbf{k}| \omega^2} \tag{A19}$$

Здесь d o' - элемент телесного угла, в котором заключён импульс рассеянной частицы:

$$\mathrm{d}o' = \sin\theta \mathrm{d}\theta \mathrm{d}\phi \tag{A20}$$

так что θ есть угол между векторами $\mathbf{k}$ и $\mathbf{p}$, а угол ϕ является азимутальным углом вектора $\mathbf{p}$ относительно некого фиксированного ортогонального репера, третья ось которого совпадает с $\mathbf{k}$. Таким образом

$$t = (k_1 - p_1)^2 = m_1^2 + m_1'^2 - 2\omega\omega' + 2|\mathbf{k}||\mathbf{p}|\cos\theta$$

Поэтому при фиксированной энергии ω

$$\mathrm{d}t = 2|\mathbf{k}||\mathbf{p}|\mathrm{d}\cos\theta, \quad \mathrm{d}o' = \mathrm{d}\phi\mathrm{d}(-\cos\theta) = \frac{\mathrm{d}\phi\mathrm{d}(-t)}{2|\mathbf{k}||\mathbf{p}|}$$

так что

$$\mathrm{d}\sigma = \frac{|M_{fi}|^2 \ \mathrm{d}t}{64\pi|\mathbf{k}|^2|\omega^2} \cdot \frac{\mathrm{d}\phi}{2\pi} = \frac{|M_{fi}|^2 \ \mathrm{d}t}{64\pi j^2} \cdot \frac{\mathrm{d}\phi}{2\pi} \tag{A21}$$

Так как в физически интересных случаях амплитуда не зависит от угла ϕ, то, интегрируя (A21) по этому углу, находим:

$$\mathrm{d}\sigma = \frac{|M_{fi}|^2 \ \mathrm{d}t}{64\pi j^2} \tag{A22}$$

Поскольку в последнем равенстве инвариантное дифференциальное сечение представлено через инвариантные же величины, то этим равенством можно пользоваться в любой системе отсчёта.

11.3.5 Spin Density Matrices

Выпишем некоторые формулы, относящиеся к спиновым матрицам плотности. Спиновые матрицы плотности необходимы при вычислении сечений и вероятностей распадов, т.к. в процессе таких вычислений приходится делать замены выражений, билинейных относительно спиновых волновых функций начальных и конечных частиц, согласно правилу

$$e_{\mathbf{k}\mu}e^*_{\mathbf{k}\nu} \longrightarrow \rho_{\mu\nu}(k), \quad u_{\pm ka}\bar{u}_{\pm kb} \longrightarrow \rho^{(\pm)}_{ab}(k) \tag{Б1}$$

где $\rho_{\mu\nu}(k)$ - спиновая матрица плотности фотона, $\rho^{(+)}_{ab}(k)$ $\left(\rho^{(-)}_{ab}(k)\right)$ - спиновая матрица плотности электрона (позитрона), причём импульс всех этих частиц равен $\mathbf{k}$.

1. Photon Spin Density Matrices

Для неполяризованного фотона в трёхмерно-поперечной калибровке (см. (33 ЛП7))

$$\rho_{ij} = \frac{1}{2}\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right), \quad \rho_{0i} = \rho_{i0} = \rho_{00} = 0 \tag{Б2}$$

Структура матрицы плотности неполяризованного фотона (Б2) очевидна, а общий коэффициент определяется требованием того, что сумма вероятностей нахождения спина фотона во всех состояниях равна единице:

$$\rho^\mu_\mu = -1 \tag{Б3}$$

Преобразуем выражение (Б2) к более удобному для использования виду, воспользовавшись свойством калибровочной инвариантности теории. В отношении фотонной матрицы плотности последнее свойство означает, что с физической точки зрения матрица плотности не изменяется при замене (сравни с (56Ω7))

$$\rho_{\mu\nu}(k) \longrightarrow \rho_{\mu\nu}(k) + (\chi_\mu k_\nu + \chi_\nu k_\mu) \tag{Б4}$$

где $\chi_\mu(k)$ - произвольный 4 -вектор. Действительно, каждый из векторов k^μ в круглых скобках в (Б4) сворачивается по векторному индексу с хронологическими корреляторами токов, из которых строятся все амплитуды. При этом импульс фотона k^μ есть входящий (или выходящий) импульс в соответствующую вершину. Вследствие равенств (17 Л10) описанные слагаемые обращаются в нуль и потому слагаемое в круглых скобках в (Б4) несущественно. При помощи преобразования (Б4) матрица плотности (Б2)приводится к виду

$$\rho_{\mu\nu} = -\eta_{\mu\nu}/2 \tag{Б5}$$

если

$$\chi^\mu = \left(-\frac{1}{4\omega}, \frac{\mathbf{k}}{4\omega^2}\right)$$

В общем случае матрица плотности представляется в виде

$$\rho_{\mu\nu} = \rho^*_{\nu\mu} = \sum_{\alpha,\beta=1}^{2} \rho^{\alpha\beta} e_{\alpha\mu} e^*_{\beta\nu} \tag{Б6}$$

Здесь $(e_{1\mu}, e_{2\mu})$ - базис векторов поляризации фотона. Эрмитовость матрицы (Б6) обеспечивается эрмитовостью матрицы $\rho^{\alpha\beta}$:

$$\rho^{\alpha\beta} = \rho^{\beta\alpha *} \tag{Б7}$$

Очевидно, в рассмотренном случае неполяризованного фотона

$$\rho^{\alpha\beta} = (1/2)\delta^{\alpha\beta} \tag{Б8}$$

Из (Б8) находим, что

$$\det \rho^{\alpha\beta} = 1/4 \tag{Б9}$$

В случае полностью поляризованного фотона матрица (Б6) должна представляться как тензорное произведение двух взаимно сопряжённых 4 -векторов. Это возможно лишь в том случае когда

$$\rho^{\alpha\beta} = v^\alpha v^{\beta *} \tag{В10}$$

Очевидно, что в этом случае

$$\det \rho^{\alpha\beta} = 0 \tag{Б11}$$

В общем случае матрица $\rho^{\alpha\beta}$ подчиняется условию (Б7), условию

$$\sum_{\alpha=1}^{2} \rho^{\alpha\alpha} = 1 \tag{Б12}$$

вытекающему из (Б3), а также условию

$$0 \leq \det \rho^{\alpha\beta} \leq 1/4 \tag{Б13}$$

Общий вид матрицы ранга 2, удовлетворяющий условиям (Б7) и (Б12), таков:

$$\rho = \frac{1}{2}(1 + \boldsymbol{\xi}\boldsymbol{\sigma}), \quad \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \tag{Б14}$$

Здесь $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ - матрицы Паули. Имеем:

$$\det \rho = \frac{1}{4}\left(1 - \xi_1^2 - \xi_2^2 - \xi_3^2\right) \tag{Б15}$$

Из условия (Б13) следует, что

$$0 \leq \xi_1^2 + \xi_2^2 + \xi_3^2 \equiv \boldsymbol{\xi}^2 \leq 1 \tag{Б16}$$

Из сказанного выше вытекает, что в неполяризованном случае

$$\xi^2 = 0 \tag{Б17}$$

а в полностью поляризованном случае

$$\xi^2 = 1 \tag{Б18}$$

2. Spinor Density Matrices

Из определения (Б1) и (39 Л8) вытекает, что

$$\operatorname{tr} \rho^{(\pm)}(k) = \pm 2m \tag{Б19}$$

Спинорные матрицы плотности удовлетворяют уравнениям (ср. с (38 Л8))

$$(\not{k} \mp m)\rho^{(\pm)}(k) = \rho^{(\pm)}(k)(\not{k} \mp m) = 0 \tag{Б20}$$

При заданной матрице плотности среднее значение спина вычисляется при помощи формулы (определение спинового оператора $\boldsymbol{\Sigma}$ см. в (111 Л8))

$$\langle \mathbf{s} \rangle = \frac{1}{2\omega_k} \operatorname{tr}\left(\rho^{(\pm)}\gamma^0\boldsymbol{\Sigma}\right) = \frac{1}{4\omega_k} \operatorname{tr}\left(\rho^{(\pm)}\gamma^5\gamma\right) \tag{Б21}$$

Матрица плотности для неполяризованного состояния получается при помощи суммы (56 Л8), делённой на 2. Действительно, в неполяризованном состоянии обе проекции спина имеют одинаковую вероятность (1/2). Таким образом в неполяризованном состоянии

$$\rho^{(\pm)}(k) = (1/2)(\not{k} \pm m) \tag{Б22}$$

Матрица плотности (Б22) удовлетворяет условиям (Б19) и (Б20).

В общем случае спинорные матрицы плотности представляются в виде

$$\rho^{(\pm)}(k) = (1/2)(\not{k} \pm m)\left(1 - \gamma^5 \not{a}\right) \tag{Б23}$$

где a^μ есть 4-вектор, характеризующий поляризацию и удовлетворяющий условию

$$k_\mu a^\mu = 0 \longleftrightarrow \not{a}\not{k} + \not{k}\not{a} = 0 \tag{Б24}$$

Непосредственно проверяется, что при условии выполнения равенств (Б24) матрица (Б23) также удовлетворяет условиям (Б19) и (Б20). Имеем также:

$$\pm a^\mu = \frac{1}{2m}\,\mathrm{tr}\left(\rho^{(\pm)}\gamma^5\gamma^\mu\right) \tag{b25}$$

Сравнение формулы (Б21), (Б24) и (Б25) показывает, что в системе покоя частицы

$$a^\mu = (0, \pm 2\langle \mathbf{s}\rangle) \tag{Б26}$$

Поскольку

$$0 \le \langle \mathbf{s}\rangle^2 \le 1/4$$

то 4-вектор a^μ удовлетворяет ограничению

$$-1 \le a^2 \le 0 \tag{Б27}$$

Напоследок укажем, что в случае рассеяния неполяризованных частиц без фиксации поляризации рассеянных частиц каждой начальной и конечной частице в сечении рассеяния соответствуют матрицы (Б5) либо (Б22) (в зависимости от сорта частиц), причём в сечение рассеяния следует внести дополнительный множитель 2^r, где r - суммарное число электронов, позитронов и фотонов в конечном состоянии. Этот множитель означает учёт обоих состояний спина каждой конечной частицы.

11.4 Scattering Matrix by Landau 4

11.4.1 Scattering Amplitude

(?? I'll check what was here in the book and move it to the previous part most likely)

11.4.2 Reactions with Polarized Particles

(??? see LL4)

11.4.3 Kinematic Invariants

(??? see LL4)

11.4.4 Physical Regions

(??? see LL4)

11.4.5 Expansion in Partial Amplitudes

(??? see LL4)

11.4.6 Symmetry of Spiral Scattering Amplitudes

(??? see LL4)

11.4.7 Invariant Amplitudes

(??? see LL4)

11.4.8 Unitarity Condition

(??? see LL4)

11.5 Taking Into Account Features of Real Setups

11.5.1 Distorted Wave Approximation

(this method is known to take into account that electron in a wave packet, not a plane wave. no one was telling me how it works, but if needed, I'll look at articles about it and then understand)

Part V
Theoretical Methods and Fundamental Effects of QFT

12 Operator Formalism

(не уверен в такой структуре)

12.0.1 Quantization with Operators

Метод

Что конеретно из себя представляет замена моды на оператор (??????)

(вот и продумаю это!!!)

Другое о квантовании

(потом пойму, что еще тут супер важно.)

12.0.2 Quantization with Operators: Detailed Overview

Мотивация работать с операторами

Минусы работы с каноническим квантованием

примеры

примеры

12.0.3 Interactions of Fields in the Operator Formalism

(пока тут)

спинорные поля

квантование спинорного поля

12.0.4 Other Aspects of the Operator Formalism

(разные вопросы тут соберу потом)

Каноническое квантование для полей не лоренц инвариантно (??)

(проверю! не проверял, преобразую при лоренцевых бустах его, посмотрю! вроде да, потому что время у нас не входит в то, чему равен коммутатор.)

13 Path Integral Formalism

(тут по идее про то, как работать с ним, конкретные закаулки - раздел про диаграммы Фейнмана.)

13.0.1 Simplified Mathematics Used in the Path Integral (!?!??)

(полезные иллюстрации для раскрытия темы тут будут!)

Интеграл по траекториям для гауссовых интегралов (???)

13.0.2 Mathematical Identities for Constructing the Path Integral (!?!??)

(всякие леммы добавлю потом)

13.0.3 Path Integral for Particles

(nearly the same is in QM)

9.1 Path Integrals in Quantum Mechanics

Feynman once said that* "every theoretical physicist who is any good knows six or seven different theoretical representations for exactly the same physics." Following his advice, we introduce in this chapter an alternative method of deriving the Fcynman rules for an interacting quantum field theory: the method of functional integration.

Aside from Feynman's general principle, we have several specific reasons for introducing this formalism. It will provide us with a relatively easy derivation of our expression for the photon propagator, completing the proof of the Feynman rules for QED given in Section 4.8. The functional method generalizes more readily to other interacting theories, such as scalar QED (Problem 9.1), and especially the non-Abelian gauge theories (Part III). Since it uses the Lagrangian, rather than the Hamiltonian, as its fundamental quantity, the functional formalism explicitly preserves all symmetries of a theory. Finally, the functional approach reveals the close analogy between quantum field theory and statistical mechanics. Exploiting this analogy, we will turn Feynman's advice upside down and apply the same theoretical representation to two completely different areas of physics.

We begin by applying the functional integral (or path integral) method to the simplest imaginable system: a nonrelativistic quantum-mechanical particle moving in one dimension. The Hamiltonian for this system is

$$H = \frac{p^2}{2m} + V(x)$$

Suppose that we wish to compute the amplitude for this particle to travel from one point (x_a) to another (x_b) in a given time (T). We will call this amplitude $U(x_a, x_b; T)$; it is the position representation of the Schrdinger time-evolution operator. In the canonical Hamiltonian formalism, U is given by

$$U(x_a, x_b; T) = \langle x_b | e^{-iHT/h} | x_a \rangle \tag{9.1}$$

(For the next few pages we will display all factors of $\hbar$ explicitly.)
In the path-integral formalism, U is given by a very different-looking expression. We will first try to motivate that expression, then prove that it is equivalent to (9.1).

Recall that in quantum mechanics there is a superposition principle: When a process can take place in more than one way, its total amplitude is the coherent sum of the amplitudes for each way. A simple but nontrivial example is the famous double-slit experiment, shown in Fig. 9.1. The total amplitude for an electron to arrive at the detector is the sum of the amplitudes for the two paths shown. Since the paths differ in length, these two amplitudes generally differ, causing interference.

For a general system, we might therefore write the total amplitude for traveling from x_a to x_b as

$$U\left(x_a, x_b; T\right) = \sum_{\text{all paths}} e^{i\cdot(\text{ phase })} = \int \mathcal{D}x(t) e^{i\cdot(\text{ phase })} \tag{9.2}$$

To be democratic, we have written the amplitude for each particular path as a pure phase, so that no path is inherently more important than any other. The symbol $\int \mathcal{D}x(t)$ is simply another way of writing "sum over all paths"; since there is one path for every function $x(t)$ that begins at x_a and ends at x_b, the sum is actually an integral over this continuous space of functions.

We can define this integral as part of a natural generalization of the calculus to spaces of functions. A function that maps functions to numbers is called a functional. The integrand in (9.2) is a functional, since it associates a complex amplitude with any function $x(t)$. The argument of a functional $F[x(t)]$ is conventionally written in square brackets rather than parentheses. Just as an ordinary function $y(x)$ can be integrated over a set of points x, a functional $F[x(t)]$ can be integrated over a set of functions $x(t)$; the measure of such a functional integral is conventionally written with a script capital $\mathcal{D}$, as in (9.2). A functional can also be differentiated with respect to its argument (a function), and this functional derivative is denoted by $\delta F/\delta x(t)$. We will develop more precise definitions of this new integral and derivative in the course of this section and the next.

What should we use for the "phase" in Eq. (9.2)? In the classical limit, we should find that only one path, the classical path, contributes to the total amplitude. We might therefore hope to evaluate the integral in (9.2) by the method of stationary phase, identifying the classical path $x_{\text{cl}}(t)$ by the stationary condition,

$$\frac{\delta}{\delta x(t)}(\text{phase}[x(t)])\bigg|_{x_{\text{cl}}} = 0$$

But the classical path is the one that satisfies the principle of least action,

$$\frac{\delta}{\delta x(t)}(S[x(t)])\bigg|_{x_{\text{cl}}} = 0$$

[6]*The Character of Physical Law (MIT Press, 1965), p. 168.

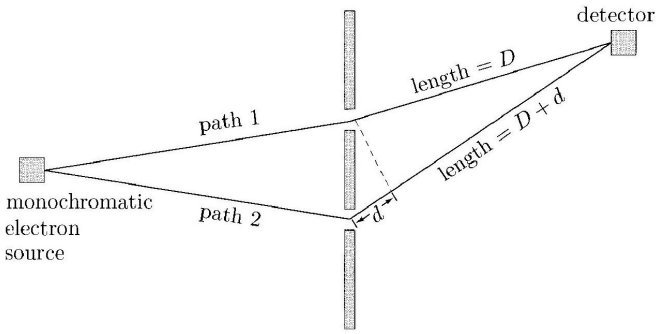


Figure 9.1. The double-slit experiment. Path 2 is longer than path 1 by an amount d, and therefore has a phase that is larger by $2\pi d/\lambda$, where $\lambda = 2\pi\hbar/p$ is the particle's de Broglie wavelength. Constructive interference occurs when $d = 0, \lambda, ...$, while destructive interference occurs when $d = \lambda/2, 3\lambda/2, ...$

where $S = \int L dt$ is the classical action. It is tempting, therefore, to identify the phase with S, up to a constant. Since the stationary-phase approximation should be valid in the classical limit-that is, when $S \gg \hbar$-we will use $S/\hbar$ for the phase. Our final formula for the propagation amplitude is thus

$$\langle x_b | \, e^{-iHT/\hbar} \, | x_a \rangle = U\left(x_a, x_b; T\right) = \int \mathcal{D}x(t) e^{iS[x(t)]/\hbar} \tag{9.3}$$

We can easily verify that this formula gives the correct interference pattern in the double-slit experiment. The action for either path shown in Fig. 9.1 is just $(1/2)mv^2t$, the kinetic energy times the time. For path 1 the velocity is $v_1 = D/t$, so the phase is $mD^2/2\hbar t$. For path 2 we have $v_2 = (D+d)/t$, so the phase is $m(D+d)^2/2\hbar t$. We must assume that $d \ll D$, so that $v_1 \approx v_2$ (i.e., the electrons have a well-defined velocity). The excess phase for path 2 is then $mDd/\hbar t \approx pd/\hbar$, where p is the momentum. This is exactly what we would expect from the de Broglie relation $p = h/\lambda$, so we must be doing something right.

To evaluate the functional integral more generally, we must define the symbol $\int \mathcal{D}x(t)$ in the case where the number of paths $x(t)$ is more than two (and, in fact, continuously infinite). We will use a brute-force definition, by discretization. Break up the time interval from 0 to T into many small pieces of duration ϵ, as shown in Fig. 9.2. Approximate a path $x(t)$ as a sequence of straight lines, one in each time slice. The action for this discretized path is

$$S = \int_0^T dt \left(\frac{m}{2}\dot{x}^2 - V(x) \right) \longrightarrow \sum_k \left[\frac{m}{2} \frac{\left(x_{k+1} - x_k\right)^2}{\epsilon} - \epsilon V\left(\frac{x_{k+1} + x_k}{2} \right) \right].$$

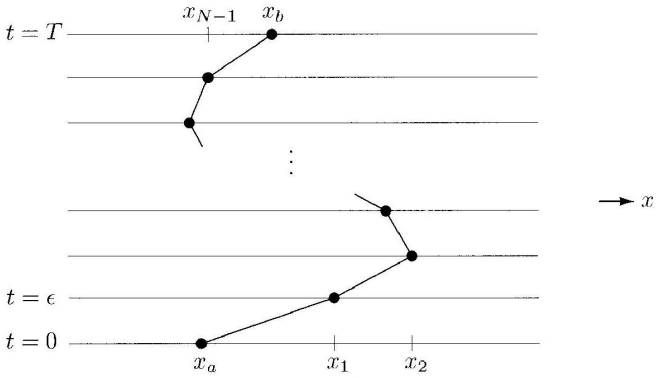


Figure 9.2. We define the path integral by dividing the time interval into small slices of duration ϵ, then integrating over the coordinate x_k of each slice.

We then define the path integral by

$$\int \mathcal{D}x(t) \equiv \frac{1}{C(\epsilon)} \int \frac{dx_1}{C(\epsilon)} \int \frac{dx_2}{C(\epsilon)} \cdots \int \frac{dx_{N-1}}{C(\epsilon)} = \frac{1}{C(\epsilon)} \prod_k \int_{-\infty}^{\infty} \frac{dx_k}{C(\epsilon)} \tag{9.4}$$

where $C(\epsilon)$ is a constant, to be determined later. (We have included one factor of $C(\epsilon)$ for each of the N time slices, for reasons that will be clear below.) At the end of the calculation we take the limit $\epsilon \to 0$. (As in Sections 4.5 and 6.2, the $\prod$ symbol is an instruction to write what follows once for each k.)

Using (9.4) as the definition of the right-hand side of (9.3), we will now demonstrate the validity of (9.3) for a general one-particle potential problem. To do this, we will show that the left- and right-hand sides of (9.3) are obtained by integrating the same differential cquation, with the same initial condition. In the process, we will determine the constant $C(\epsilon)$.

To derive the differential equation satisfied by (9.4), consider the addition of the very last time slice in Fig. 9.2. According to (9.3) and the definition (9.4), we should have

$$U(x_a, x_b; T) = \int_{-\infty}^{\infty} \frac{dx'}{C(\epsilon)} \exp\left[\frac{i}{\hbar} \frac{m(x_b - x')^2}{2\epsilon} - \frac{i}{\hbar} \epsilon V\left(\frac{x_b + x'}{2}\right)\right] U(x_a, x'; T - \epsilon)$$

The integral over x' is just the contribution to $\int \mathcal{D}x$ from the last time slice, while the exponential factor is the contribution to $e^{iS/\hbar}$ from that slice. All contributions from previous slices are contained in $U(x_a, x'; T - \epsilon)$.

As we send $\epsilon \to 0$, the rapid oscillation of the first term in the exponential constrains x' to be very close to x_b. We can therefore expand the above

13.0.4 Path Integral for Fields

(a place for main theory)

13.0.5 Path Integral for Fields: Easy Examples

(??? I'll write later good ones)

описание векторного мезона

Теория поля векторного мезона определяется интегралом по траекториям:

$$Z = \int DA e^{iS(A)} \equiv e^{iW(J)}$$

И действие:

$$S(A) = \int d^4x \mathcal{L} = \int d^4x \{A_\mu[(\partial^2 + m^2)g^{\mu\nu} - \partial^\mu \partial^\nu]A_\nu + A_\mu J^\mu\}$$

какая-то теория Киселева

пока не использую это.

При введении в действие полиномиального взаимодействия по полям степени выше 2

$$S_{\rm int}(\varphi) = -\int {\rm d}^4 x V(\varphi)$$

та же методика спаривания полей трансформирует амплитуду вероятности квантовых переходов при наличии источников

$$Z(j) = \exp\left\{\frac{\rm i}{\hbar} S_{\rm int}\left(-{\rm i}\hbar\frac{\delta}{\delta j}\right)\right\} \exp\left\{\frac{\rm i}{\hbar}\frac{1}{2} j_x \mathcal{G}^{(2)}_{xy} j_y\right\}$$

в выражение

$$Z(j) = \exp\left\{-\frac{{\rm i}\hbar}{2}\mathcal{G}^{(2)}_{xy}\frac{\delta^2}{\delta\tilde\varphi_x\delta\tilde\varphi_y}\right\} e^{\frac{\rm i}{\hbar}\{S_{\rm int}(\tilde\varphi)+j_{x'}\tilde\varphi_{x'}\}}\Big|_{\tilde\varphi=0}$$

Значит, для многоточечной амплитуды находим

$$\mathcal{D}^{(n)}_{1\ldots n} = \exp\left\{-\frac{{\rm i}\hbar}{2}\mathcal{G}^{(2)}_{xy}\frac{\delta^2}{\delta\tilde\varphi_x\delta\tilde\varphi_y}\right\} e^{\frac{\rm i}{\hbar}S_{\rm int}(\tilde\varphi)}\tilde\varphi_1\ldots\tilde\varphi_n\Big|_{\tilde\varphi=0}$$

В этом выражении нужно учитывать все спаривания полей, включая поля в $S_{\rm int}$.

Формула (4.53) представляет собой основу для построения теории возмущений,

поскольку все многоточечные функции выражаются через спаривания «свободных» полей,

часть из которых возникает при разложении экспоненты с действием взаимодействия.

В свою очередь, действие взаимодействия обычно включает в себя полиномы по полям с коэффициентами - константами взаимодействия.

Подчеркнем, что в рассмотренном нами простом случае, когда нет вакуумных полей, т. е. $\mathcal{G}^{(1)} \equiv 0$, вклад вакуумных амплитуд

$$\mathcal{D}^{(0)} = \exp\left\{-\frac{{\rm i}\hbar}{2}\mathcal{G}^{(2)}_{xy}\frac{\delta^2}{\delta\tilde\varphi_x\delta\tilde\varphi_y}\right\} e^{\frac{\rm i}{\hbar}S_{\rm int}(\tilde\varphi)}\Big|_{\tilde\varphi=0}$$

очевидно, факторизуется для всех многоточечных функций в виде суммы всех диаграмм без внешних концов.

Теория возмущений для потенциала ϕ^3 (??)

(??? суть еще не понял??)

Рассмотрим теперь взаимодействие с модельным потенциалом

$$V = \frac{\Lambda}{3!}\varphi^3,$$

где $[\Lambda] = [m]$.

Вычислим трехточечный причинный коррелятор в ведущем приближении по константе взаимодействия Λ,

$$\begin{aligned}\mathcal{D}^{(3)}_{123} &= \left\langle 0\left|\hat{\rm T}\left\{\varphi_1\varphi_2\varphi_3\right\}\right|0\right\rangle \approx \\ &\approx \exp\left\{-\frac{{\rm i}\hbar}{2}\mathcal{G}^{(2)}_{xy}\frac{\delta^2}{\delta\tilde\varphi_x\delta\tilde\varphi_y}\right\}\tilde\varphi_1\tilde\varphi_2\tilde\varphi_3\left(-\frac{\rm i}{\hbar}\right)\int {\rm d}^4x_4\frac{\Lambda}{3!}\tilde\varphi^3(x_4)\Big|_{\tilde\varphi=0}\end{aligned}$$

В этом выражении отличный от нуля вклад получается, если имеет место спаривание всех шести полей, а стало быть, экспоненту необходимо разложить до третьей степени ($\exp x \mapsto x^3/3!$)

$$\mathcal{D}^{(3)}_{123} \approx \hbar^2 \frac{\Lambda}{3!} \left(\frac{1}{2} \int \mathrm{d}^4x d^4y \mathcal{G}^{(2)}_{xy} \frac{\delta^2}{\delta\tilde{\varphi}_x \delta\tilde{\varphi}_y} \right)^3 \frac{1}{3!} \tilde{\varphi}_1 \tilde{\varphi}_2 \tilde{\varphi}_3 \int \mathrm{d}^4x_4 \tilde{\varphi}^3 (x_4) \Bigg|_{\tilde{\varphi}=0}$$

В системе единиц $\hbar = c = 1$, как мы знаем, $\mathcal{G}^{(2)}_{xy} = \mathrm{i}\mathcal{D}^{(2)}_{xy}$, так что

$$\mathcal{D}^{(3)}_{123} \approx -\,\mathrm{i}\frac{\Lambda}{3!} \left(\frac{1}{2} \int \mathrm{d}^4x d^4y \mathcal{D}^{(2)}_{xy} \frac{\delta^2}{\delta\tilde{\varphi}_x \delta\tilde{\varphi}_y} \right)^3 \frac{1}{3!} \tilde{\varphi}_1 \tilde{\varphi}_2 \tilde{\varphi}_3 \int \mathrm{d}^4x_4 \tilde{\varphi}^3 (x_4) \Bigg|_{\tilde{\varphi}=0}$$

Связный вклад $\mathcal{D}^c_{123}$ получается, если каждое их полей $\tilde{\varphi}_{1,2,3}$ спарить с полями $\tilde{\varphi}_4$, причем, в силу симметричности варьирования по $\tilde{\varphi}_x$ и $\tilde{\varphi}_y$ фактор $1/2$ компенсируется перестановкой $x \leftrightarrow y$, а комбинаторика по выборке $\tilde{\varphi}_4$ из $\tilde{\varphi}^3_4$ компенсирует фактор $1/3!$ и комбинаторика выбора вариации из $\tilde{\varphi}_1\tilde{\varphi}_2\tilde{\varphi}_3$ также компенсирует фактор $1/3!$, так что

$$\mathcal{D}^c_{123} \approx -\mathrm{i}\Lambda \int \mathrm{d}^4x_4 \mathcal{D}^{(2)}_{14} \mathcal{D}^{(2)}_{24} \mathcal{D}^{(2)}_{34}$$

На этой диаграмме мы показали также несвязный вклад с одной петлей. Он получается, если, например, произвести спаривание полей в точках 1 и 2, а спаривание других точек дает вклад, который за счет комбинаторики повторяется трижды, так что

$$\mathcal{D}^{nc}_{123} \approx -\mathrm{i}\frac{\Lambda}{2} \mathcal{D}^{(2)}_{12} \int \mathrm{d}^4x_4 \mathcal{D}^{(2)}_{34} \mathcal{D}^{(2)}_{44}$$

Многоточие на диаграмме означает аналогичные вклады с перестановкой точек 1,2,3

внутренние линии

На диаграмме многоточечной функции Грина встречаются внешние линии,

когда одна из точек пропагатора относится ко множеству точек-аргументов самой функции Грина,

а также внутренние линии, когда обе точки пропагатора не принадлежат множеству аргументов функции Грина.

При этом, по координатам внутренних линий проводится интегрирование.

Среди внутренних линий встречаются и петли, когда конец одной из внутренних линий совпадает с началом другой или,

как в примере, той же внутренней линии.

Говорят, что частицы на внутренних линиях диаграммы являются виртуальными:

квантовые флуктуации приводят к тому, что квадрат 4-импульса виртуально частицы отличается от квадрата массы, т.е. виртуальная частицы находится вне массовой поверхности.

Виртуальные частицы появляются только при наличии взаимодействий: как мы видели, согласно теореме Вика для свободных частиц внутренние линии не возникают, а стало быть, не возникают и условия схода частиц с массовой поверхности.

Если внутренняя линия, тем не менее, при каких-то определенных условиях попадает на массовую поверхность,

то говорят, что возникают сингулярности Ландау.

В этой части курса квантовой теории поля мы не рассматриваем ни петли, ни сингулярности Ландау.

Рассмотрим, какой вклад в амплитуду $\hat{Z}(j)$ дает связная диаграмма,

$$Z^c_{(3)} = \frac{1}{3!}\mathrm{i}^3\mathcal{D}^c_{123}j_1j_2j_3$$

Учтем, что в асимптотической области внешние токи связаны с внешними классическими полями,

например,

$$\mathrm{i}\mathcal{D}^{(2)}_{xy}j_y = \varphi_{\mathrm{cl}}(x) \to j_y = -\mathrm{i}\left(\mathcal{D}^{(2)}_{xy}\right)^{-1}\varphi_{\mathrm{cl}}(x)$$

$\mathcal{T}$-матрица

При рассмотрении амплитуды квантовых переходов в асимптотической области для Z вводят специальное обозначение $\mathcal{S}$-матрицы, а для отличия $\mathcal{S}$-матрицы от единицы:

$$\mathcal{S} = \mathbb{1} + \mathrm{i}\mathcal{T}.$$

Матричные элементы $\mathcal{T}$ -матрицы для конкретных процессов обычно обозначают либо символом $\mathcal{M}$, либо $\mathcal{A}$.

Тогда в выражении для связного вклада в 3 -точку

(4.54)

подстановка внешних токов через внешние поля в асимптотически свободной области

(4.55)

в точности приводит к сокращению двухточечных пропагаторов, и

$$\mathrm{i}\mathcal{M}^c_{(3)} = \frac{1}{3!}(-\mathrm{i}\Lambda)\int \mathrm{d}^4x_4\varphi^3_{\mathrm{cl}}(x_4)$$

что, конечно, в точности совпадает с выражением для затравочного действия взаимодействия, как и должно быть в теории возмущений.

(???)

В это выражение мы теперь можем подставить вполне определенные моды внешнего классического поля,

которые соответствуют поставленной задаче.

Например, если речь идет о переходе одной частицы в две: $\varphi_{\mathrm{cl}}(x_n) \mapsto e^{\mp\mathrm{i}k_nx_n}$ в зависимости от того, какие моды распространяются в будущее, а какие в прошлое,

интегрирование по координате дает амплитуду в импульсном представлении

$$\mathrm{i}\mathcal{M}^c_{(3)} = \frac{1}{3!}(-\mathrm{i}\Lambda)(2\pi)^4\delta(k_1-k_2-k_3)\,3!\frac{1}{\sqrt{2!}}$$

где мы ввели комбинаторный фактор 3! по счету вариантов прескрипции импульсов для квантов поля φ_{cl} в силу их тождественности и фактор $1/\sqrt{2!}$ для нормировки амплитуды по перестановке тождественных частиц в конечном состоянии.

Этот пример в данном случае тривиален, поскольку сохранение 4-импульса с учетом того,

что все эти 4-импульсы находятся на массовой поверхности, приводит при $m \neq 0$ к нулевому результату,

так как под аргументом дельта-функции стоит ненулевая величина, т. е. переход запрещен при $m \neq 0$.

В месте с тем, отметим следующее: при вычислении амплитуды квантового перехода пропагаторы от внешних точек функции Грина к внутренним вершинам сокращаются, когда мы выражаем внешние токи через внешние поля в асимптотически свободной области наблюдений! Такая операция называется редукцией или усечением (ампутацией) внешних ног диаграммы, а получающиеся диаграммы называются усеченными. При этом обычно внешние линии перечеркивают, обозначая тем самым, что процедура усечения произведена.

редукция внешних ног диаграммы

Вместе с тем, отметим следующее: при вычислении амплитуды квантового перехода пропагаторы от внешних точек функции Грина к внутренним вершинам сокращаются,

когда мы выражаем внешние токи через внешние поля в асимптотически свободной области наблюдений!

Такая операция называется редукцией или усечением (ампутанией) внешних ног диаграммы, а получающиеся диаграммы называются усеченными.

При этом обычно внешние линии перечеркивают, обозначая тем самым, что процедура усечения произведена.

13.0.6 Infinite Series Sums and Expansions in QFT (!?!?)

(тут про тот гениальный ход, что так берем и вариационную производную выводим через экспоненту. кто бы мог подумать, что так можно вообще делать???? про это тут подробно.)

(??? maybe it will be written later, in special section)

13.0.7 Perturbative Expansion – Making Sense of Divergent Series

By How I Learned to Stop Worrying and Love QFT 2011 Robert C. Helling

Before we take a look at divergent series, we will first give a brief review of how perturbative expansion is used in quantum field theory.

Brief overview on path integrals

A quantum field theory in Minkowski spacetime is described by a Lagrangian density

$\mathcal{L}(\phi, \partial\phi)$ and a generating functional of correlation functions[1]

$$\mathcal{Z}[J] = \int \mathcal{D}\phi e^{i\int \mathrm{d}^4x(\mathcal{L}+J\phi)}$$

The correlation functions can be obtained by functional derivatives of (1) with respect to J

$\langle \phi(x_1)\,\phi(x_2)\,...\phi(x_n)\rangle = \frac{1}{\mathcal{Z}[0]}\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)...\left(-i\frac{\delta}{\delta J(x_n)}\right)\mathcal{Z}[J]\Big|_{J=0}$

In this lecture we will use Euclidean signature for the metric instead of Minkowski. The change between the metrics can be performed as rotation of the time axis in the complex plane $t \to -i\tau$ if all expressions are analytic. In Euclidean metric, the exponent in the generating functional is real and falls of at large field values. This gives the path integral a chance to have a mathematical definition in terms of Wiener measures but that will not concern us in these notes.

$$\mathcal{Z}[J] = \int \mathcal{D}\phi e^{\int \mathrm{d}^4x(\mathcal{L}+J\phi)}$$

In general, the integral (3) cannot be computed exactly. For a scalar quantum field theory in Euclidean space the Lagrangian has the form

$$\mathcal{L} = \frac{1}{2}\phi\left(\Box - m^2\right)\phi - V(\phi)$$

with $\Box \equiv (\partial_\tau)^2 + (\nabla)^2$

If the potential $V(\phi)$ vanishes, equation (3) can be formally computed as it becomes an integral of Gaussian type. One therefore arbitrarily splits the Lagrangian into its "kinetic part" $\frac{1}{2}\phi\left(\Box - m^2\right)\phi$ and its "interaction part" $-V(\phi)$.

$$\begin{aligned}\mathcal{Z}[J] &= \int \mathcal{D}\phi e^{\frac{1}{2}\int \mathrm{d}^4x\phi\left(\Box-m^2\right)\phi} e^{-\int \mathrm{d}^4xV(\phi)} e^{-\int \mathrm{d}^4xJ\phi} \\ &= e^{-\int \mathrm{d}^4xV\left(\frac{\delta}{\delta J}\right)} \int \mathcal{D}\phi e^{\int \mathrm{d}^4x\frac{1}{2}\phi\left(\Box-m^2\right)\phi - J\phi}\end{aligned}$$

To obtain the Gaussian integral one has to complete the square in the exponent. This is achieved by shifting the field ϕ:

$$\phi' = \phi + \left(\Box - m^2\right)^{-1} J$$

[1] This subsection displays some standard expressions to set the context. For many more details see for example [1. The inverse of $\Box - m^2$, called "Green's function" $G(x-y)$, is a distribution defined by

$$\left(\Box - m^2\right) G(x-y) = \delta(x-y)$$

Changing variables in the functional integral (5) leads to

$$\mathcal{Z}[J] = e^{-\int \mathrm{d}^4xV\left(\frac{\delta}{\delta J}\right)} \int \mathcal{D}\phi' e^{\int \mathrm{d}^4x\frac{1}{2}\phi'\left(\Box-m^2\right)\phi'} e^{-\int \mathrm{d}^4x\int \mathrm{d}^4y\frac{1}{2}J(x)G(x-y)J(y)}$$

The complicated expression in the middle of equation (8) does not depend on J and will in fact cancel out in equation (2) for the correlation function, so we will just denote it C and forget about it:

$$\mathcal{Z}[J] = Ce^{-\int \mathrm{d}^4xV\left(\frac{\delta}{\delta J}\right)} e^{-\int \mathrm{d}^4x\int \mathrm{d}^4y\frac{1}{2}J(x)G(x-y)J(y)}$$

Now let us take a look on a specific example for a quantum field theory by choosing a potential for the scalar field. We will consider our favorite ϕ^4 theory given by the potential

$$V(\phi) = \lambda\phi^4$$

The next step is to insert this potential in equation (8) and write the exponential as a power series in the coupling strength λ.

$$\mathcal{Z}[J] = C\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\int \mathrm{d}^4x_1\frac{\delta^4}{\delta J\left(x_1\right)^4}\cdots\int \mathrm{d}^4x_k\frac{\delta^4}{\delta J\left(x_k\right)^4}e^{-\int \mathrm{d}^4x\int \mathrm{d}^4y\frac{1}{2}J(x)G(x-y)J(y)}$$

We now found an expression for any general correlation function in terms of an power series expansion in the coupling strength.

$$\begin{aligned}\left\langle\phi\left(y_1\right)...\phi\left(y_n\right)\right\rangle =& \frac{\delta}{\delta J\left(y_1\right)}\cdots\frac{\delta}{\delta J\left(y_1\right)}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!}\int \mathrm{d}^4x_1\frac{\delta^4}{\delta J\left(x_1\right)^4}\cdots\int \mathrm{d}^4x_k\frac{\delta^4}{\delta J\left(x_k\right)^4}\times\\ & e^{-\int \mathrm{d}^4x\int \mathrm{d}^4y\frac{1}{2}J(x)G(x-y)J(y)}\Big|_{J=0}\end{aligned}$$

The combinatorics of the occurring expressions in terms of integrals over interaction points x_i, Green's functions and external fields can be summarized in terms of Feynman diagrams each standing for a single term in the power series in the coupling constant λ^2 In the following, we want to study the convergence behavior of this power series.

Radius of convergence of correlation functions

Let us briefly review the definition of the radius of convergence for a power series from introductory analyis. It is useful to think of a power series to be defined in the complex plane:

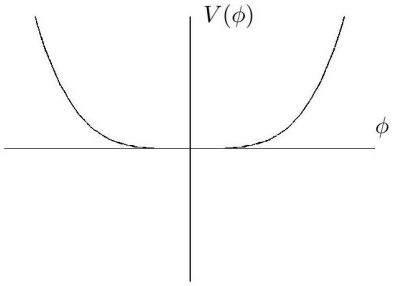
$$\sum_k^{\infty}\lambda^k(...)\quad \lambda\in\mathbb{C}$$

[2] The careful reader wishing to avoid ill-defined expressions using path-integrals, can use this formula as the definition of the terms in the perturbative series. (If one does not like the idea of a complex coupling strength in a quantum field theory, just restrict to the special $\lambda\in\mathbb{C}$ that happen to be real.). Every power series has a radius of convergence $R\in[0,\infty]$ such that

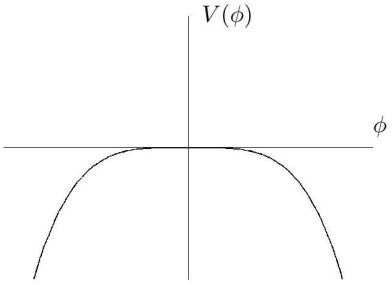
$$\sum_k^{\infty}\lambda^k(...)\begin{cases}\text{converges} & \forall|\lambda|<R\\ \text{diverges} & \forall|\lambda|>R\end{cases}$$

Now we want to find out the radius of convergence for the correlation functions (11) in a quantum field theory. A physicist's argument was given by Freeman Dyson in 1952 22]. Let us take a look on the potential, for example in our ϕ^4 theory as shown in figure 1 For positive coupling strength λ the potential is bounded from below and large values of ϕ are strongly disfavored. This behavior, however, gets radically different in case of a negative λ. The potential becomes unbounded from below and the field ϕ will want to run off to $\phi=\pm\infty$. Obviously, such a behavior is highly unphysical, since ever increasing values of ϕ would lead to an infinite energy gain. It is thus clear that such a theory cannot lead to healthy correlation functions, in other words for any negative λ the power series (12 will diverg]][3] From this we can conclude the radius of convergence being $R=0$!

$$\sum_k^{\infty}\lambda^k(...)\quad \text{diverges } \forall\lambda>0$$



(a) $\lambda > 0$



(b) $\lambda < 0$

Figure 1: Potentials of ϕ^4 theory

For readers not satisfied by this argument using physics of unstable potentials for determining the radius of convergence, let us mention an alternative line of argument. Again, consider equation (11), this time, however, we will focus on the Feynman diagrams. At any order k in the perturbation expansion there is a sum of different Feynman diagrams expressing the integrals in (11), where k counts the number of vertices. The combinatorics of all Feynman diagrams shows that the

[3] We expect at least a phase transition when λ is changed from positive to negative values.

number of Feynman diagrams grows like k !. The power series, therefore, will behave like

$$\sum_k^\infty \lambda^k k!(...)$$

Assuming that (...) is not surprisingly suppressed for large k, the coefficients of λ^k grow faster than any power, we again find the radius of convergence $R = 0$.

In the following, we want to give an example, how one can nevertheless make sense of (some) divergent power series.

Non-perturbative corrections

In order to get a feeling for the problem of divergent power series, we will consider a one dimensional toy problem (rather than the infinite dimensional problem of a path integral):

$$\mathcal{Z}(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}x e^{-x^2-\lambda x^4}$$

We take $\lambda \geq 0$, so this integral yields some finite, positive number. For $\lambda = 0$ the solution is well known

$$\mathcal{Z}(0) = \sqrt{\pi}$$

In general, equation (16) can be expressed in terms of special functions, e.g. Mathematica gives the solution

$$\mathcal{Z}(\lambda) = \frac{e^{\frac{1}{8\lambda}} K_{1/4}(1/8\lambda)}{2\sqrt{\lambda}}$$

with $K_n(x)$ being the modified Bessel function of the second kind. We call solution (18) the "full, non-perturbative answer". Now we will do the same as in quantum field theory and split the integral into a "kinetic" and an "interaction" part, respectively.

Treating the toy model perturbatively

Following the same procedure, we will again expand the "interaction part" $-\lambda x^4$ in a power series:

$$\mathcal{Z}(\lambda) = \int_{-\infty}^{\infty} \mathrm{d}x e^{-x^2-\lambda x^4} = \int_{-\infty}^{\infty} \mathrm{d}x e^{-x^2} \sum_{k=0}^{\infty} \frac{\left(-\lambda x^4\right)^k}{k!}$$

Now comes the crucial step and "root of all evil". Following precisely the same steps leading towards equation (11) for correlation functions in quantum field theory, we will change the order of integration and summation, leading to the interpretation of a power series of Feynman diagrams:

$$\mathcal{Z}(\lambda)" = " \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \int_{-\infty}^{\infty} \mathrm{d}x x^{4k} e^{-x^2}$$

From this step, as we will see later, the problems arise. Although this step is forbidden (as roughly speaking, we are changing the behavior of the integrand at $x = \pm\infty$), we are interested in to what extent a "perturbative solution" obtained from equation (20) will agree with the full, non-perturbative solution (18). Carrying on, we observe that the integral in (20) is now of the type "polynomial times Gaussian" and can be computed with standard methods. We smuggle an addtional factor a into the exponent allowing us to write the integrand as derivatives of e^{-ax^2} with respect to a at the point $a = 1$.

$$\mathcal{Z}(\lambda) = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \int_{-\infty}^{\infty} \mathrm{d}x \frac{\partial^{2k}}{\partial a^{2k}} e^{-ax^2} \bigg|_{a=1} = \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \frac{\partial^{2k}}{\partial a^{2k}} \sqrt{\frac{\pi}{a}} \bigg|_{a=1}$$

Of course we can easily evaluate the derivatives:

$$\frac{\partial^{2k}}{\partial a^{2k}} a^{-\frac{1}{2}} \bigg|_{a=1} = \underbrace{\frac{1}{2}\frac{3}{2} \cdot \frac{5}{2}\frac{7}{2} \cdot \frac{9}{2}\frac{11}{2} \cdots}_{\text{total of } 2k \text{ factors}}$$

In order to find an explicit expression for (22) one can insert factors of 1 between all factors, such that the nominator becomes $(4k)!$:

$$\left.\frac{\partial^{2k}}{\partial a^{2k}} a^{-\frac{1}{2}}\right|_{a=1} = \underbrace{\frac{1}{2}\frac{2}{2}\frac{3}{2}\frac{4}{4}\frac{5}{2}\frac{6}{6}\frac{7}{2}\frac{8}{8}\frac{9}{2}\frac{10}{10}\frac{11}{2}\frac{12}{12}\cdots}_{\text{total of } 4k \text{ factors}} = \frac{(4k)!}{2^{2k}} \underbrace{\frac{1}{2}\frac{1}{4}\frac{1}{6}\frac{1}{8}\frac{1}{10}\frac{1}{12}\cdots}_{\text{total of } 2k \text{ factors}}$$
$$= \frac{(4k)!}{2^{2k}} \frac{1}{2^{2k}(2k)!} = \frac{(4k)!}{2^{4k}(2k)!}$$

Thus we obtain the "perturbative solution" of problem 16p

$$\mathcal{Z}(\lambda) = \sum_{k=0}^{\infty} \sqrt{\pi} \frac{(-\lambda)^k (4k)!}{2^{4k}(2k)!k!}$$

Let us take a closer look at this expression. By observing that the denominator of the summand eventually contains smaller factors than the nominator for all k larger than a critical integer, we can realize that the series is divergent. More carefully we can apply Stirling's formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ for large values of k:

$$\frac{(4k)!}{2^{4k}(2k)!k!} \approx \frac{4^k}{\sqrt{\pi k}} \left(\frac{k}{e}\right)^k \approx \frac{1}{\sqrt{2}\pi} 4^k k!$$

We already know that the sum

$$\sum_{k=0}^{\infty} (-4\lambda)^k k!$$

will diverge. This shows that the power series (24) is divergent and in particular it is not the finite number that we are looking for as an expression for (16).

The perturbative and the full solution compared

Even though the perturbative series will diverge, we want to study its numerical usefulness at finite order. After all, one usually computes only a finite number of Feynman diagrams to obtain only the first few summands of the perturbative expansion. Is there a way to approximate the full, non-perturbative solution 18p from 24. ? Let us choose one value for λ, e.g. $\frac{1}{50}$, and evaluate 18 numerically:

$$\mathcal{Z}\left(\frac{1}{50}\right) = 1.7478812...$$

For the same value of λ the evaluation of the first few terms of the infinite sum 24

$$\mathcal{Z}_N(\lambda) = \sum_{k=0}^{N} \sqrt{\pi} \frac{(-\lambda)^k (4k)!}{2^{4k}(2k)!k!}$$

gives

$$\mathcal{Z}_5 = 1.7478728...$$
$$\mathcal{Z}_{10} = 1.7478818...$$

The first terms of the perturbative solution agree up to six digits! We can use conveniently a figure for plotting higher orders of the perturbative series. Figure 2 shows that the perturbative

solution gets in a certain regime very close to the result of the full solution, before the series starts to diverge. We can use a figure as well

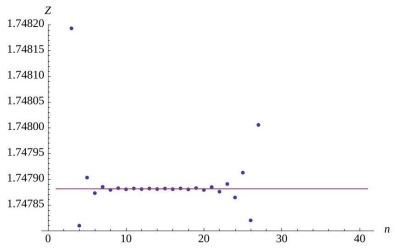


Figure 2: Values of the perturbative series 24 evaluated to order N

to compare the solutions for variable λ. Figure 3 shows nicely the non-perturbative solution and compares it to the perturbative solution for orders of one to twelve. We can see that at some point all approximations given by the perturbative solution will disagree strongly from the full solution!

The question that arises is, how long does the perturbative solution become better before it starts to diverge? Obviously, the fact that it approximates the non-perturbative solution to high precision leads to the great success of quantum field theory, even if for higher orders the series diverges! As we will see now, the perturbative solution 24 is a good approximation as long as we only consider terms up to order $N = O\left(\frac{1}{\lambda}\right)$. Remembering the dimensionless coupling strength of Quantum Electrodynamics being the Sommerfeld finestructure constant $\alpha \approx \frac{1}{137}$ we can be ensured that perturbation theory will lead to great precision given that the most elaborate QED calculations for $(g-2)$ are to order $N = 7$!

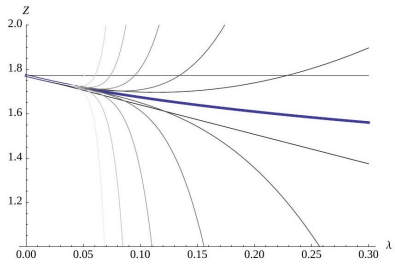


Figure 3: $\mathcal{Z}(\lambda)$ obtained from the full solution (thick) and first approximations from the perturbative series

The method of steepest descend

But what is the origin of the eventual divergence and complete loss of numerical accuracy? It turns out that there are "non-perturbative" terms that do not show up in a Taylor expansion but that become dominant when the perturbative expansion breaks down. To see this, let us substitute $x^2 \equiv \frac{u^2}{\lambda}$ in equation 16:

$$\mathcal{Z}(\lambda) = \frac{1}{\sqrt{\lambda}} \int \mathrm{d}u e^{-\frac{u^2+u^4}{\lambda}}$$

The exponent is strictly negative and its absolute value becomes very large in the limit of small λ. This allows to perform the method of steepest descent: The main contribution to the integral, as $\lambda \to 0$ comes from the extrema of the integrand

$$u^2 + u^4$$

In general the method works as follow: For $\Lambda \to \infty$ we want to solve an integral of the general form

$$\int \mathrm{d}x A(x) e^{(i)\phi(x)\Lambda}$$

One expands now around its extrema $\overset{4}{\frown} \phi'(x_0) = 0$ and obtains again an integral of "Gaussian times polynomial" typt 5^5

$$\begin{aligned} &= \sum_{x_0:\phi'(x_0)=0} \int \mathrm{d}x \left(A(x_0) + (x - x_0) A'(x_0) ...\right) e^{\Lambda\left(\phi(x_0)+(x-x_0)^2\phi''(x_0)+...\right)} \\ &= \sum_{x_0:\phi'(x_0)=0} A(x_0) e^{\Lambda\phi(x_0)} \sqrt{\frac{2\pi}{\phi''(x_0)\Lambda}} \left(1 + O\left(\frac{1}{\Lambda}\right)\right). \end{aligned}$$

[4] Notice that in field theory $\phi'(x) = 0$ is the equation of motion

[5] Corrections from $(x - x_0) A'(x_0)$ can be obtained by doing again the trick of smuggling an a into the exponent and write the term as derivative with respect of a evaluated at $a = 1$. In our case, the extrema of 31 are $u = 0$ and $u = \pm i/\sqrt{2}$. Expansion around the first yields the perturbative expansion of above. The other two yield contributions like $e^{\frac{1}{4\lambda}}$ that are invisible to a Taylor expansion around $\lambda = 0$, as all derivative vanish here. We have found an example of a "non-perturbative contribution".

The perturbative solution, however, gives meaningful results, as long as its terms are bigger than to the non-perturbative contributions. This allows an estimate, to what order in the perturbative series the expansion around $\lambda = 0$ dominates. This happens also to be the order at which the divergence from the exat solution starts as we are missing the non-perturbative terms:

$$\begin{aligned} e^{-\frac{1}{4\lambda}} &\approx \lambda^k \\ -\frac{1}{4\lambda} &\approx k \cdot \ln(\lambda) \\ k &\approx \frac{1}{\lambda} \end{aligned}$$

We have seen that the perturbative analysis of 30 requires as well expansions around the other extrema, besides $\lambda = 0$! Combining all power series together, the resulting perturbative solution has a chance to converge. Before we continue to the mathematical discussion of the problem how finite results can be obtained from divergent series, we will take a look on examples of non-perturbative contributions in physics.

Instantons

"Field configurations" contributing $e^{-\frac{1}{\sim\lambda}}$ are called "instantons". Usually these contributions are hard to calculate, in some situations, however, one can find the result. Consider for example a gauge theory [6]

$$S = \int \mathcal{L} = \int \frac{1}{g^2} \mathrm{tr}(F \wedge *F)$$

The stationary point we use for the expansion is given by the equations of motion

$$\mathrm{d}F = 0$$
$$\mathrm{d} * F = 0$$

The first equation (36a) is automatically fulfilled once we express the field-strength in terms of a vector potential $F = \mathrm{d}A$. The second 36 b is automatically solved if it happens that

$$F = *F.$$

One calls solutions to (37) instantons. In terms of the vector potential A (37) is a first order partial differential equation as compared to 336 b which is second order. One can easily see that there exist no solution in Lorentzian metric as the Hodge star squares to -1 on 2 -forms.

$$F = *F = * * F = -F.$$

In Euclidean metric, however, such solutions exist because of $* * F = F$. As it turns out (as one can for example argue using the Atiya-Singer index theorem), for

[6] Hodge $*$ operator: $*F^{\mu\nu} \equiv \epsilon^{\mu\nu\sigma\tau} F_{\sigma\tau}$. More details can be found in chapters 1.10 and 10.5 of 3]. a compact manifold M, the action in the instanton case yields an integer (up to a pre-factor):

$$\int_M \mathrm{tr}(F \wedge F) \in 8\pi^2 \mathbb{Z}$$

This leads to

$$e^{\frac{1}{g^2}\int \mathrm{tr}(F \wedge *F)} = e^{\frac{1}{g^2}\int \mathrm{tr}(F \wedge F)} = e^{-\frac{1}{g^2} 8\pi^2 N}$$

Dual theories

Sometimes a quantum field theory with coupling constant λ can be rewritten in terms of another (or possibly the same) quantum field theory with coupling $\tilde{\lambda} = 1/\lambda$. One calls such a relation between two theories a "duality". In many examples, such theories arise from string theory constructions, where the coupling λ can be given a geometric meaning. Imagine for example a problem of a quantum field theory on a torus. A torus can be viewed as $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with $\tau \in \mathbb{C}\backslash\mathbb{R}$. The torus has a basis of two non-contractible circles, one that goes along the real axis from 0 to 1 and one that goes from 0 to τ. This choice of basis, however, is not unique: For example, swapping these cycles corresponds to a substitution $\tau \to -1/\tau$. If the torus parameter τ is identified with the coupling strength a duality has been found since both τ and $-1/\tau$ describe geometricaly the same torus! To make contact with our discussion above, we should identify λ with the imaginary part of τ. The duality allows for a Taylor expansion of the non-perturbative contributions via

$$e^{-\frac{1}{4\lambda}} = e^{-\frac{\tilde{\lambda}}{4}} = \sum_k^\infty \frac{(-\tilde{\lambda})^k}{k! 4^k}$$

Troublesome terms in one theory are therefore perfectly defined in the dual theory. The caveat however is the difficulty of actually proving that $\lambda \to 1/\lambda$ is a symmetry of the quantum field theory at hand.

Asymptotic series and Borel summation

In the following, we take a look on the mathematical situation of asymptotic series. This discussion is based on chapter XII of 4].

Definition 1 Let $f : \mathbb{R}_{\geq 0} \to \mathbb{C}$. The series $\sum_n^\infty a_n z^n$ is called asymptotic to f as $z \searrow 0$ iff

$$\forall N \in \mathbb{N} : \lim_{z \searrow 0} \frac{f(z) - \sum_n^N a_n z^n}{z^n} = 0$$

For $z \in \mathbb{C}$ a analog definition is possible.

Obviously, every function can have at most one asymptotic expansion. This can be seen by assuming two asymptotic expansions a_n and $\tilde{a}_n$.42 requires that $a_n = \tilde{a}_n$. Otherwise, let n be the smallest index for which $a_n \neq \tilde{a}_n$ and

$$\lim_{z \searrow 0} \frac{\sum_k (a_k - \tilde{a}_k) z^k}{z^n} = a_n - \tilde{a}_n \stackrel{!}{=} 0.$$

The other way around is not true, as can be seen by $f(z) = e^{-\frac{1}{z}}$ and $\tilde{f}(z) = 0$ having both the asymptotic series $\sum_k^\infty 0 \cdot z^k$. This means that knowing the asymptotic series of a function tells us nothing about $f(z)$ for a non vanishing z, we only know how $f(z)$ approaches $f(0)$ as $z \searrow 0$.

We try to find a stronger definition of an asymptotic series, allowing us to uniquely recover one function. The following theorem helps us to find the necessary conditon:

Theorem 2 (Carleman's theorem) Let g be an analytic function in the interior of $S = \left\{z \in \mathbb{C} || z| \leq B, |\arg z| \leq \frac{\pi}{2}\right\}$ and continous on S. If for all $n \in \mathbb{N}$ and $z \in S$ we have $|g(z)| \leq b_n |z|^n$ and $\sum_n^\infty b_n^{-\frac{1}{n}} = \infty$, then g is identically zero.

A simpler special case of the theorem is found by considering g an analytic function in the interior of $S_\epsilon = \left\{z \in \mathbb{C} || z| \leq R,\ |\arg z| \leq \frac{\pi}{2} + \epsilon\right\}$ for some $\epsilon > 0$ and continous on S_ϵ. If there exist C and B so that $|g(z)| < CB^n n! |z|^n \forall z \in S$ and $\forall n$, then g is identically zero.

In order to find a unique function for an asymptotic series, we use Carleman's theorem to define "strong asymptotic series".

Definition 3 Let f be an analytic function on the interior of $S_\epsilon = \left\{z \in \mathbb{C} || z| \leq R, |\arg z| \leq \frac{\pi}{2} + \epsilon\right\} \to \mathbb{R}$. The series $\sum_n^\infty a_n z^n$ is a strong asymptotic series if there exist C, σ so that $\forall N \in \mathbb{N}, z \in S_\epsilon$ the strong asymptotic condition

$$\left| f(z) - \sum_n^N a_n z^n \right| \leq C \sigma^{N+1} (N+1)! |z|^{N+1}$$

is fulfilled.

This means, if we are given a strong asymptotic series, we can recover by theorem 2 the function! Assume for example $\sum_n^\infty a_n z^n$ is a strong asymptotic series for two functions f and g, respectively. Then

$$|f(z) - g(z)| \leq 2C\sigma^{N+1}(N+1)!|z|^{N+1} \Rightarrow f = g$$

The strong asymptotic condition (44) implies $|a_n| \leq C\sigma^n n\,!$. This is precisely the growth behavior of 24 we found in our toy example, where $C = \frac{1}{\sqrt{2\pi}}$ and $\sigma = 4$. The necessary conditions, therefore, are fulfilled in our toy model (assuming analyticity away from 0 of course).

By now, we learned that a strong asymptotic series (in particular the type we obtain in quantum field theory) although not converging has the chance to be a unique approximation to one function. The final question is, how one can obtain this function f from its strong asymptotic series. In the last theorem we introduce the method of "Borel summation" to obtain a final result. We can define a convergent series by taking out a factor of $n\,!$ from the coefficients:

Theorem 4 (Watson's theorem) If $f : S_\epsilon \to \mathbb{R}$ has a strong asymptotic series $\sum_n^\infty a_n z^n$, we define the Borel transform

$$g(z) = \sum_n^\infty \frac{a_n}{n!} z^n$$

The Borel transform converges for $|z| < \frac{1}{|\sigma|}$. We obtained a convergent power series with finite radius of convergence, which, as it turns out, can be analytically continued to all complex $z \in \mathbb{C}$ with $|\arg z| < \epsilon$. Then the function f is given by the Laplace transform

$$f(z) = \int_0^\infty \mathrm{d}b g(bz) e^{-b}.$$

This Laplace transform is called "inverse Borel transform" and the method outlined here is known as "Borel summability method". It describes how to obtain a finite answer from divergent series, that is formally a sum for the series.

Let us make a sanity check. Using $\int_0^\infty \mathrm{d}x x^k e^{-x} = k\,!$ we can plug the definition of the Borel transform into 47), formally interchange the sum and the integration and obtain

$$f(z) = \int_0^\infty \mathrm{d}b g(bz) e^{-b} = \int_0^\infty \mathrm{d}b \sum_n \frac{a_n}{n!} b^n z^n e^{-b} \text{"} = \text{"} \sum_n a_n z^n.$$

So at least for analytic functions we do recover the original function.

Summary

We have learned why for $N < \mathcal{O}\left(\frac{1}{\lambda}\right)$ the sum of the first N terms of the perturbation expansion is numerically good, even when the original series $\sum_n a_n z^n = \infty$ diverges. This way we approximate the true function up to instantonic terms of the order of $e^{1/\lambda}$ which a Taylor expansion cannot resolve. Given that the coefficients a_n obey the strong asymptotic condition $|a_n| \leq C\sigma^n n\,!$, which is usually the case when using Feynman diagrams, the Borel transform exists and one can compute the Borel summation. Unfortunately this is more a theoretical assurance that perturbation theory can be given a mathematical meaning even though it does not converge, since in order to really compute the integral (47) one has to know the analytic continuation of g which requires knowledge of all coefficients a_n and not just the first N.

13.0.8 Regularization and Renormalization as Extensions of Distributions

from "How I Learned to Stop Worrying and Love QFT" 2011 by Robert C. Helling

In the previous section, we learned how to make sense of (some) divergent series of the form $\sum_{k=0}^{\infty} a_k = \infty$, but in QFT the factors a_k are typically complicated mathematical expressions described by Feynman diagramms, and generically, these expressions diverge themselves, creating a need for renormalization techniques.

A typical example of a divergent diagramm (in 4 dimensions) is shown in Figure 4

The term described by this diagramm reads (without unimportant factors):

$$\int d^4k \frac{1}{k^2+m^2} \frac{1}{(p_1+p_2+k)^2+m^2} \overset{k\gg p,m}{\longrightarrow} \int \frac{k^3 dk}{k^4} = \infty$$

where m denotes the mass of the scalar particles we are scattering. This integral obviously diverges logarithmically for $k \to \infty$ as shown above. The most straightforward approach to this problem is to introduce a cut-off energy-scale Λ, such that the divergence at the upper boundary becomes

$$\int^{\Lambda} \frac{k^3 dk}{k^4} \sim \log(\Lambda)$$

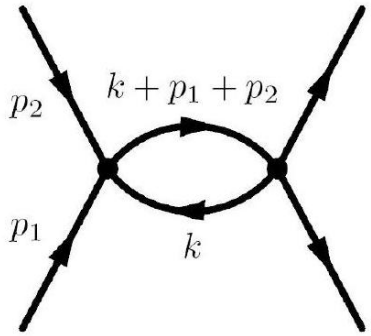


Figure 4: Divergent 1-loop diagramm

Usually, such blunt cut-off regularization is incompatible with the symmetries of the theory at hand and is thus only useful to estimate "how divergent" a diagram is (a notion we will below formalize as the "singular degree") and has to be replaced by more sophisticated methods like dimensional or Pauli-Villars regularization in more practical applications.

In these notes, instead of momentum representation, we will work in position space where instead of loop momenta one integrates over the position of the interaction vertices.

What was $1/(k^2+m^2)$, is now the propagator G defined by the equation

$$\left(\Box + m^2\right) G(x) = \delta(x)$$

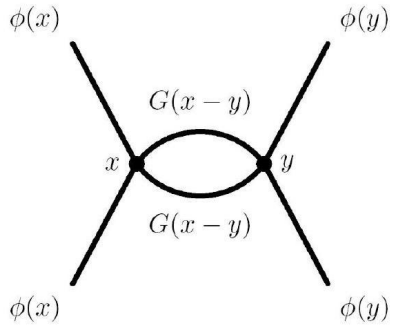


Figure 5: Divergent 1-loop diagramm in position-space language

we can compute the same diagramm in position space language, which then reads (see Figure 5):

$$\int d^4x \int d^4y \phi_0^2(x) G^2(x-y) \phi_0^2(y) = \int d^4x \int d^4u \phi_0^2(x) G^2(u) \phi_0^2(x-u)$$

The approach of "causal perturbation theory" or "Epstein-Glaser regularization" is to take seriously the fact that the propagator is really a distribution and in the above expression, we are trying to multiply distributions which in general is undefined. This approach is advocated in the book by Scharf [5]. Here, we will follow (a simplified, flat space version) of [6] and in particular [7.

Specifically, in the defining equation δ is not a function but a distribution (physicists writing $\delta(x)$ are trying to imply that this is the kernel of the distribution δ, i.e. that δ arises by muliplying the testfunction by a function $\delta(x)$ and then integrating over x, which of course does not exist). Thus, we should interpret $G(x)$ as a distribution as well (a priori it is only a weak solution of the differential equation (49)). But as it is in general not possible to multiply distributions, as we will see later, we do not have a naive way to obtain " G^2 " as a distribution. In this chapter, our goal will be to understand renormalization techniques in terms of distributions. Our route will be led by the question how to define the product of two distributions that are almost everywhere functions (which can be multiplied). We will first, therefore, recapitulate what distributions actually are. Then, we will see in which cases it is possible to multiply distributions and in which it is not. This will lead us to the renormalization techniques we are searching for.

Recapitulation of distributions

Distributions are generalized functions. Like in many other cases of generalizations this is done via dualization: Starting from an ordinatry function f (in our case locally integrable, that is $f : \mathbb{R}^n \to \mathbb{C}$ with $\int\limits_K |f| < \infty$ for each compact $K \subset \mathbb{R}^n$, so that "divergence of the integral $\int |f|$ at infinity" is tolerated) one can view it as a linear functional T_f (called a "regular distribution") on the functions of compact support via

$$T_f : \phi \mapsto \int f\phi$$

As the map $f \mapsto T_f$ is injective we can use the T_f 's to distinguish the different f 's and view T_f in place of f. This suggests to generalize the construction to all linear functionals $T : \phi \mapsto T(\phi)$ called distributions of which the regular ones arising from functions f as above are a subset.

Specifically, distributions are defined to be linear and continuous functionals on the space of test functions $D(\mathbb{R}^n) = \mathcal{C}_0^\infty(\mathbb{R}^n)$ (the subscript 0 meaning compact support) equipped with an

appropriate topology that will not concern us here. So formally, we can denote the distributions to be elements of a space:

$$D'(\mathbb{R}^n) = \{T : D(\mathbb{R}^n) \to \mathbb{C} | T \text{ is linear and continuous } \}$$

Besides the regular distibutions T_f encountered above (of which the function f is called the kernel) the typical example of a singular distribution is the δ-distribution: if we take a given test function $\phi(x) \in D$, then the δ-distribution is defined to be the functional $\delta[\phi] = \phi(0)$. This distribution is not regular even though physicists pretend it to be with a kernel $\delta(x)$ that is so singular at $x = 0$ that $\int \delta(x) = 1$ even though it vanishes for all $x \neq 0$.

Later, we will make use of the fact that distributions can be differentiated. Using integration by parts in the integral representation of a regular distribution, we easily obtain $T_{f'}[\phi] = -T_f[\phi']$ which enables us to define the derivative of a distribution to be $T'[\phi] \equiv -T[\phi']$. Thus we can take the derivative of a regular distribution T_f even if the kernel f is not differentiable.

The only operation defined on functions that does not directly carry over to distributions is (pointwise) multiplication $(f \cdot g)(x) = f(x)g(x)$. Already L^1_{loc} is not closed under multiplication (recall that in order for a function to be in L^1_{loc} it must not have singularities that go like $1/x^\alpha$ with $\alpha \geq 1$, a property not stable under multiplication) and in general the product of distributions is not defined. Of course, as long as with f and g also $f \cdot g \in L^1_{\text{loc}}$ we still have the regular distribution $T_{f \cdot g}$ and, from a technical perspective, in this sections, we will deal with the problem to extend a distribution that can be written as $T_{f \cdot g}$ for a subset of test functions (those that vanish where $f \cdot g$ is too singular to be in L^1_{loc}) to all test functions.

To this end, for any distribution $T \in D'$ we define the singular support of T (singsupp(T)) as the smallest closed set in $\mathbb{R}^n$ such that there exists a function $f \in L^1_{\text{loc}}$ with $T[\phi] = T_f[\phi]$ for all $\phi \in D$ with $\operatorname{supp}\phi \cap \text{singsupp}(T) = \emptyset$. For example singsupp$(\delta) = \{0\}$, and the corresponding function $f \in L^1_{\text{loc}}$ is simply $f(x) = 0$. So the idea behind this definition is that every distribution can be written as regular distributions as long as it is only applied to test functions which vanish in a neighbourhood of the distribution's singular support, which enables us to multiply distributions if we manage to take care of the singular support.

Definition of G^2 in $D'(\mathbb{R}^4 \backslash \{0\})$

Coming back to our concrete field-theoretic problem for a moment, we now have to answer one important question: what is the singular support of G? Here, we can utilize the fact that $\Box + m^2$ is an elliptic operator $\Box$[7] and it can be shown that if two distributions T and S are related by $\sigma T = S$ with an elliptic operator σ, then singsupp$(T) \subset$ singsupp(S), so we immediately see singsupp$(G) \subset \{0\}$.

It is clear that the singularity of $G(x)$ at $x = 0$ corresponds to the divergence of the momentum-integral at high energies $\Lambda \to \infty$, because in order to probe small distances, short wavelengths which correspond to high momenta are needed, therefore we speak of UV-divergencies. In this regime, we can set $m^2 \approx 0$, and so (49) simplifies to $\Box G(x) = \delta(x) \Rightarrow G(x) \sim \frac{1}{x^2}$ for small $x\Box$[8]

Using this, we see that the position-space integral $\int\limits_{|x| > \frac{1}{\Lambda}} d^4x G^2(x)$ again diverges as $\log(\Lambda)$. Because of this divergence $G^2(x) \notin L^1_{\text{loc}}(\mathbb{R}^4)$, so G^2 is still not defined as distribution in $D'(\mathbf{R}^4)$. Nevertheless, we can use $G^2(x)$ as kernel of a distribution in $D'(\mathbf{R}^4 \backslash \{0\}) = \{T : D(\mathbb{R}^4 \backslash \{0\}) \to \mathbb{C} \mid$ linear and continuous $\}$ where $D(\mathbb{R}^4 \backslash \{0\})$ is the set of test functions with $\{0\} \notin \operatorname{supp}\phi$.

We now managed to define a distribution G^2, but we still have to extend it from $D(\mathbb{R}^4 \backslash \{0\})$ to $D(\mathbb{R}^4)$. Formally, as a linear map, we have to say what values the extension takes on

$D(\mathbb{R}^4)/D(\mathbb{R}^4\backslash\{0\})$ which is still an infinite dimensional vector space. To control this infinity, we will use the scaling degree.

The scaling degree and extensions of distributions

Consider a scaling-map Λ acting on test functions:

$$\begin{aligned}\mathbb{R}_{>0}\times D(\mathbb{R}^n)&\to D(\mathbb{R}^n)\\(\lambda,\phi)&\mapsto \phi_\lambda(x)\equiv\lambda^{-n}\phi\left(\lambda^{-1}x\right)\end{aligned}$$

[7] Explaining it without going into details, a differential operator which is defined as polynomial of $\vec{\partial}$ (with possible coordinate dependent coefficients) is elliptic if it is non-zero if we replace $\vec{\partial}$ with any non-zero vector $\vec{y}$. In our euclidian examples, $\Box=\sum_{i=1}^{4}\partial_i^2\to|\vec{y}|^2>0$ foy any non-zero $\vec{y}$

[8] The relation $\delta(\vec{x})=-\frac{1}{4\pi}\Box\frac{1}{|\vec{x}|}$ is well known to hold in 3 dimensions. In general, $\Box|x|^{2-n}\propto\delta$ in n dimensions The pullback of this map to the space distributions reads

$$(\Lambda^*T)[\phi]=T[\phi_\lambda]\equiv T_\lambda[\phi]$$

which for regular distributions gives

$$T_{f,\lambda}[\phi]=\int\frac{d^nx}{\lambda^n}f(x)\phi\left(\frac{x}{\lambda}\right)=\int d^nxf(\lambda x)\phi(x),$$

so power of λ in the scaling map acting on test functions is chosen such that the kernel f transforms in a simple manner without prefactor. We now define the scaling degree (sd) of $T\in D'(M\subset\mathbb{R}^n)$

$$sd(T)=\inf\left\{\omega\in\mathbb{R}|\lim_{\lambda\searrow0}\lambda^\omega T_\lambda=0\right\}$$

To understand this definition, we have to note several properties:

- $sd(T)\in[-\infty,\infty[$
- For regular distributions $sd(T_f)\leq0$
- $sd(\delta)=n$
- $sd(\partial^\alpha T)\leq\mathrm{sd}(T)+|\alpha|$ with some multi-index α
- $sd(x^\alpha T)\leq sd(T)-|\alpha|$ with some multi-index a[9]
- $sd(T_1+T_2)=\max\{sd(T_1),sd(T_2)\}$

This leads us to the following important theorem:

Theorem 5 If $T_0\in D'(\mathbb{R}^n\backslash\{0\})$ is a distribution with $\mathrm{dd}(T_0)<n$, then there is a unique distribution $T\in D'(\mathbb{R}^n)$ with $sd(T)=\mathrm{sd}(T_0)$ extending T_0.

The proof of uniqueness is quite easy: We do it by assuming the existence of two solutions T and $\tilde{T}$ extending T_0, and showing a contradiction. Obviously supp(T - $\tilde{T}$) = $\{0\}$ and from this it follows that $T-\tilde{T}=P(\partial)\delta$ with some polynomial P. As can be seen from the above notes, $sd(P(\partial)\delta)\geq n$ and this would be a contradiction to $sd(T)=sd(\tilde{T})=sd(T_0)<n$. Existence is

shown constructively using a smooth cut-off function $c_\epsilon(x)$ that is 1 outside a ball of radius 2ϵ and and vanishes in a ball of radius ϵ. Then we can define

$$T[\phi] = \lim_{\epsilon \searrow 0} T_0 \left[c_\epsilon \phi\right]$$

where one still has to show that the above limit exists in the sense of distributions.

The theorem above now enables us to uniquely extend distributions of low scaling degree to the full space $D'\left(\mathbb{R}^n\right)$, but what about distributions with scaling degree $\geq n$? We will solve this problem in the next section, and afterwards we will be able to return to our field-theoretic problem of understanding the nature of G^2.

But first we have to determine what the scaling degree of the massive propagator G, defined by $\delta = (\Box + m^2)\, G$. We know that $\mathrm{sd}(\delta) = n$, and therefore $\mathrm{sd}((\Box + m^2)\, G) = n$ too. If we denote $\mathrm{sd}(G)$ by w, from the above items it follows that $sd(\Box G) = w + 2, sd\left(m^2 G\right) = w$ and therefore $sd\left((\Box + m^2)\, G\right) = w + 2$. From this it follows that $w = n - 2$ even for the massive propagator.

[9] Remember that distributions do not depend on coordinates, only their kernels. Here we used the definition $\left(x^\alpha T\right)[\phi] \equiv T\left[x^\alpha \phi\right]$

Case of distributions with high scaling degree

Considering now a distribution $T_0 \in D'\left(\mathbb{R}^n \backslash \{0\}\right)$ with $sd\left(T_0\right) \geq n$, uniqueness as in the above theorem does not hold anymore. But if we take a test function $\phi \in D(\mathbb{R})$ which vanishes of order $\omega \equiv sd\left(T_0\right) - n$ ("singular order") at $x = 0$, i.e. which can be written as $\phi(x) = \sum_{|\alpha| = \lfloor \omega \rfloor + 1} x^\alpha \phi_\alpha(x)$ where $\phi_\alpha(0)$ is finite and $\lfloor \omega \rfloor$ denotes the largest integer not bigger than ω, we can define $T[\phi] \equiv \sum_{|\alpha| = [\omega] + 1} \left(x^\alpha T_0\right)[\phi_\alpha]$. Then the distribution $x^\alpha T_0$ has scaling degree less than n and can thus be uniquely extended.

A general test function can of course be written as a sum of a function vanishing of order ω and a polynomial of degree at most ω by subtracting and adding the order ω Taylor polynomial at $x = 0$:

$$\phi_s(x) \equiv \phi(x) - \sum_{|\alpha| \leq \omega} \frac{x^\alpha}{|\alpha|!} \partial^\alpha \phi(0)$$

This procedure of subtracting the terms leading to divergencies is the regularization in this framework. Since the extended distribution T beeing applied to ϕ_s is unique, by linearity, we still have to define T only on the monomials in x of maximal degree ω. There is no further restriction on doing this and this ambiguity in the extension T is what one would have expected: Changing the value of T on a monomial x^α correspond to adding a multiple of $\partial^\alpha \delta$ to T.

Note well that the arbitratry values of $T\left[x^\alpha\right]$ are exactly those where $T_0\left[x^\alpha\right]$ was undefined (divergent in physicists' parlance) and selecting a certain value corresponds to picking a counter term. procedure known as renormalization, as formally infinite values are replaced by finte ones (that have to be fixed by further physical input like the measurement of the "physical mass" or the physical "coupling constant"). In the following small sections, we will try out this method in a few easy, concrete examples.

Example in $n = 1$

In order to let our steps so far become clearer, we are going to apply them to a simple example in $n = 1$. In fact, this example shows already the full regularization and renormalization procedure.

As can be easily checked, the function $f(x) = \frac{1}{|x|}$ is not an element of $L^1_{loc}(\mathbb{R})$ because of its pole at $x = 0$ is not integrable (it is of course log-divergent), so we can not a priori use it as kernel of a distribution $T_f \in D'(\mathbb{R})$ as we have seen in section 3.1. But $f(x) \in L^1_{loc}(\mathbb{R}\backslash\{0\})$ and $sd\,(T_f) = 1 = n$, therefore $\omega = 0$. This means that for a test function $\phi(x)$ with $\phi(0) = 0$ we can define $T_f[\phi] \equiv \int dx \frac{\phi(x)}{|x|}$ which gives a finite result: Using l'Hôpital's rule, we see $\lim_{x\to\pm 0} \frac{\phi(x)}{|x|} = \lim_{x\to\pm 0} \frac{\phi'(x)}{\mathrm{sign}(x)} =$ finite and thus the integrand is finite everywhere. This is similar to what we have done in sections 3.2 and 3.3. For other test functions, we can again (as in this section above) define $\phi_s(x) \equiv \phi(x) - \phi(0)$. Afterwards, we write the general extension for a distribution acting on ϕ as $T_f[\phi] \equiv T_f\,[\phi_s] + c\phi(0)$ with one arbitrary constant c of our choice.

The careful reader will have realised that there is still a problem as ϕ_s fails to have compact support when $\phi(0) \neq 0$ and thus the integration now diverges at the boundary $x \to \pm\infty$. We will deal with this problem below but the important observation is that the divergence in the ultraviolet, that is at small x is cured.

Example in $n = 4$

In our field theoretic problem 50 from above, we have $G^2 \sim \frac{1}{x^4}$ in $\mathbb{R}^4$ which is quite similar to the previous example, as it is the kernel of a distribution $T_{G^2} = T_{\frac{1}{x^4}} \in D'\left(\mathbb{R}^4\backslash\{0\}\right)$. Again, we are looking for an extension. Once more, we have $sd\,(G^2) = 4 = n$. Regularization and renormalization are as in the example above and yield

$$T^r_{\frac{1}{x^4}}[\phi] = \int d^4x \frac{\phi(x) - \phi(0)}{x^4} + c\delta[\phi]$$

with $T^r_{\frac{1}{x^4}} \in D'\left(\mathbb{R}^4\right)$ and arbitrary c. Again, we successfully got rid of the problems at $x = 0$ (at the cost of introducing one constant c).

This concludes our calculation of the fish diagram Fig. 4 that computes a contribution of the form $\phi(x)^2\phi(y)^2$ to the effective action of the theory. Since the ambiguous term we found is $c\delta(x-y)$, the ambiguity in the effective action is indeed $\phi(x)^4\delta(x-y)$. We see, that it corresponds to the counter term Fig. 6 and renormalizes the coupling constant (the coefficient of the ϕ^4-term in the action).

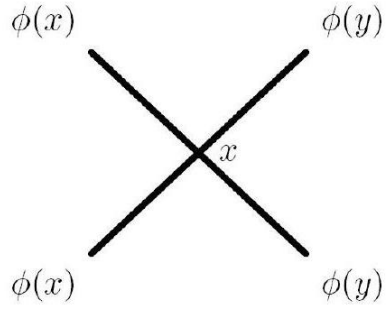


Figure 6: Counter-term diagramm.

Example with $sd(T) > n$ and preservation of symmetry

In the two examples above, we had both times distributions T with $sd(T) = n$ which led us to the introduction of one arbitrary constant c. This amount of ambiguity increases with $sd(T)$, but not all possible polynomials $P(\partial)\delta$ allowed by the counting can arise physically. In particular, we require that our theory is still Lorentz invariant after renormalization and if it has a local gauge symmetry before that needs to be maintained as well (otherwise one has an anomaly that renders the theory illdefined at the quantum level since the number of degrees of freedom changes upon renormalization).

Let us consider one example where SO(4) invariance (the euclidian version of the Lorentz group SO(3, 1)) selects a subset of the possible counter terms.

In a theory with potential $\propto \phi^4$ (quartic interaction), there cannot only be diagramms like Figure 4 but also such ones like Figure 7 known as the setting sun diagram.

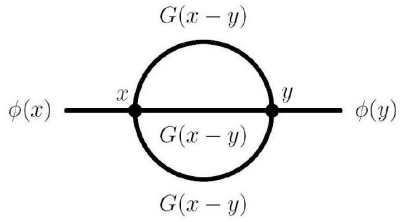


Figure 7: Setting sun diagramm in quartic interaction

The term encoded by this diagramm obviously contains $G^3 \sim \frac{1}{x^6}$, which has $sd\left(G^3\right) = 6 > n = 4$. By performing the same steps as above, in this case we a priori get an ambiguity $c_1\delta + c_2^i\partial_i\delta + c_3^{ij}\partial_i\partial_j\delta$ with in total $1 + 4 + \frac{4(4+1)}{2} = 15$ arbitrary constants, but upon imposing SO(4)-invariance this reduces to $c_1\delta + c_3\Delta\delta$ with only 2 arbitrary constants.

In the effective action, as above, they contribute to the quadratic terms (as the diagram Fig. 7 has two external lines) $\phi(x)\left(c_1\delta(x-y) + c_3\Delta\delta(x-y)\right)\phi(y) = \phi(x)\left(c_3\Delta + c_1\right)\phi(x)\delta(x-y)$. We recognize that c_3 is a wave function renormalization while c_1 renormalizes the mass-term $m^2\phi^2$.

The fact that ϕ^4-theory is renormalizable in $n = 4$ means that these two counter terms and the one in the previous subsection are the only ambiguities that arise when any Feynman diagram of the theory is renormalized, a proof of being well beyond the scope of these notes.

Regaining compact support and RG flow

In the above calculations, we ignored an important problem: $\phi_s(x) = \phi(x) - \phi(0)$ is not necessarily a test function, as it obviously has $\lim_{x\to\infty} = -\phi(0)$, therefore for example the integral $\int dx \frac{\phi(x)-\phi(0)}{|x|}$ that we encountered in section 3.4.1 may diverge at infinity. We can solve this by introducing a function $w(x) \in D\left(\mathbb{R}^n\right)$ with (without loss of generality) $w(0) = 1$. We then change the regularized part (i.e. the part without arbitrary constants) of the integral in (54) to

$$T_{\frac{1}{|x|}}[\phi] \equiv \int dx \frac{\phi(x) - w(x)\frac{\phi(0)}{w(0)}}{|x|}.$$

This is a special case of the general formula

$$\phi_s(x) \equiv \phi(x) - \sum_{|\alpha|\leq\omega} \frac{x^\alpha w(x)}{|\alpha|!}\left(\partial^\alpha \frac{\phi(x)}{w(x)}\right)\Bigg|_{x=0}$$

which replaces equation (53). Starting from (55) we can now write

$$T_{\frac{1}{|x|}}[\phi] = \int dx \frac{w(x)(\phi(x)-\phi(0))}{|x|} + \underbrace{\int dx \frac{(1-w(x))\phi(x)}{|x|}}_{=T_{\frac{1-w(x)}{|x|}}[\phi]}.$$

The second term already is a perfectly fine distribution, the first term can be manipulated in the following way:

$$\int dx \frac{w(x)(\phi(x)-\phi(0))}{|x|} = \int dx \frac{w(x)}{|x|} \int_0^x du \phi'(u)$$

Now, in the inner integral we can substitute $u = tx$ and afterwards interchange the integrals:

$$\int dx \frac{w(x)}{|x|} \int_0^1 dt x \phi'(tx) = \int_0^1 dt \int dx \, \mathrm{sign}(x) w(x) \phi'(tx)$$

After this, we can substitute $y = tx$ in the inner integral, giving us

$$\begin{aligned}
&\int_0^1 dt \int \frac{dy}{t} w\left(\frac{y}{t}\right) \mathrm{sign}\left(\frac{y}{t}\right) \phi'(y) \\
&= \int dy \underbrace{\int_0^1 \frac{dt}{t} w\left(\frac{y}{t}\right) \mathrm{sign}(y)}_{\equiv f(y)} \phi'(y) \\
&= T_f[\partial\phi] = -\left(\partial T_f\right)[\phi].
\end{aligned}$$

So also the first term is a good distribution. The function $f(y)$ defined above as function of y is well behaved, as $w(x)$ which enters its definition is a test function, and therefore extremely well behaved, in particular vanishes for large arguments, so taking $t \to 0$ does not introduce problems.

As an example, let us now set $w(x) = \theta(1 - M|x|)$ (or actually a smoothed out version of this non-continuous function)

$$\begin{aligned}
f(y) &= \int_0^1 \frac{dt}{t} \theta\left(1 - M\frac{|y|}{t}\right) \mathrm{sign}(y) \\
&= \int_{M|y|}^1 \frac{dt}{t} \theta(1 - M|y|) \, \mathrm{sign}(y) \\
&= -\ln(M|y|)\theta(1 - M|y|) \, \mathrm{sign}(y)
\end{aligned}$$

Morally, we regularized our distribution with non-integrable kernel $\propto \frac{1}{|x|}$ by substituting the derivative of a distribution with kernel $\propto \log(|y|)$, which is integrable.

In the above calculations, we introduced a mass/energy-scale M. It is now an important question to ask how the distribution changes under transformations of this scale, i.e. renormalization

group (RG) transformations generated by $M\frac{\partial}{\partial M}$, so called RG flows. We will now show that it is only the part const $\delta(x)$, i.e. the part that is fixed by arbitrary renormalization constants that will change.

First of all, using $\partial_x \operatorname{sign}(x) = 2\delta(x)$, we see:

$$f'(x) = \frac{-1}{x}\theta(1 - M|x|)\operatorname{sign}(x) - \log(M|x|)\theta(1 - M|x|)2\delta(x)$$

Then, we start with:

$$M\frac{\partial}{\partial M}T_{\frac{1}{|x|}}[\phi] = M\frac{\partial}{\partial M}\left(\int dx\frac{(1 - w(x))\phi(x)}{|x|} + T_{-\partial f}[\phi]\right)$$

Because of $\frac{1-w(x)}{|x|} = \frac{\theta(M|x|-1)}{|x|}$ in our example, the first term becomes a distribution with kernel

$$M\frac{\partial}{\partial M}\frac{\theta(M|x| - 1)}{|x|} = M\delta(M|x| - 1).$$

The second term in contrast becomes a distribution with kernel

$$M\frac{\partial}{\partial M}\left(-f'(x)\right) = -M\delta(M|x| - 1) + M\frac{\partial}{\partial M}[2\log(M|x|)\theta(1 - M|x|)\delta(x)]$$

The first term of this expression obviously cancels with the contribution from (56), so $M\frac{\partial}{\partial M}T_{\frac{1}{|x|}}$ turns out to be a distribution with kernel:

$$\begin{aligned}
&M\frac{\partial}{\partial M}[2\log(M|x|)\theta(1 - M|x|)\delta(x)] \\
&= 2\delta(x)[\theta(1 - M|x|) - \log(M|x|)M\delta(1 - M|x|)|x|] \\
&= 2\delta(x)
\end{aligned}$$

In the last step, we used the presence of the factor $\delta(x)$ (under an integral!) to set $\log(M|x|)|x| = 0$ and $\theta(1 - M|x|) = 1$. So, under a renormalization group transformation, the distribution changes by $\delta T \propto \text{const} \cdot\delta(x)$, that means that a change of energy-scale corresponds to a change of the (at the beginning) arbitrarily selected renormalization coefficients.

What we have achieved in this section

We have seen a way to recast what looks like divergent Feynman diagrams as to what looks like distributions for non-integrable functions. We could then turn these into proper distributions by first restricting the space of test-functions and then extend them to a full distribution, possibly at the price of a finite number of undetermined numerical constants. Those have to be determined by a finite number of measurements.

In order for the number of introduced parameters for all Feynman diagrams of the theory to be finite, the scaling degrees of all appearing distributions in all diagramms have to be below some maximum, otherwise the theory is not renormalizable.

Summary

The material in these notes will not be useful for any concrete calculation in quantum field theory that a physicist might be interested in. But they might give him or her some confidence that the calculation envisaged has a chance to be meaningful.

We tried to present material that is in no sense original but still is probably not covered in most introductions to quantum field theory. Hopefully, it helps to refute some of the prejudices against (perturbative) quantum field theory that mathematically minded people may have and helps others to better understand how far the hand waving arguments that we use in our daily work can carry.

In particular, we put our emphasis on two points: Even if the perturbative expansion is divergent as a power series it can serve two purposes: The first terms do provide a numerically good approximation to the true, non-perturbative result and all terms taken together can indeed recover the full result but only in terms of Borel resummation rather than as a power series. Second, unphysical in

nite momentum integrals in the computation of Feynman diagrams can be avoided when properly expressed in terms of distributions. The renormalization of coupling constants is then expressed as the problem to extend a distribution from a subspace to all test functions. The language of distribution theory allows one to avoid mathematically ill-de

ned divergent expressions altogether.

13.0.9 On the Strict Formalism (??)

(тут суть всякой теории меры напишу, пока не до нее)

Пример, когда знать строгий формализм важно (!??!)

(хз, важный вопрос для определения, стоит ли этим заниматься?)

13.0.10 On Something Else (??)

(много дополнений будет, пока не до них)

13.1 Holomorphic Path Integral, S-Matrix, Scalar Field Theory

13.1.1 1. Creation and Annihilation Operators

Далее полагаем $\hbar = c = 1$.

Действие осциллятора (19 Л1) в случае нулевого источника приводит к следующему гамильтониану:

$$\mathcal{H}(p,q) = \frac{1}{2}p^2 + \frac{\omega^2}{2}q^2 \tag{1}$$

Введём комплексные взаимно сопряжённые переменные a, a^*:

$$q = \sqrt{\frac{1}{2\omega}}\,(a + a^*)\,, \quad p = -i\sqrt{\frac{\omega}{2}}\,(a - a^*) \tag{2}$$

В квантовой теории формулы (2) сохраняются для операторов и коммутационные соотношения

$$[q,q] = [p,p] = 0, \quad [q,p] = i \tag{3}$$

и

$$[a,a]=\left[a^{\dagger},a^{\dagger}\right]=0,\quad \left[a,a^{\dagger}\right]=1 \tag{4}$$

эквивалентны.

Перестановочные соотношения (4) имеют представление в пространстве аналитических и, более того, голоморфных функций $f\left(a^{*}\right)$ со скалярным произведением

$$\langle f_1|f_2\rangle=\int\left(f_1\left(a^{*}\right)\right)^{*}f_2\left(a^{*}\right)e^{-a^{*}a}\frac{\mathrm{d}a^{*}\ \mathrm{d}a}{2\pi i} \tag{5}$$

Пояснение. Можно считать, что

$$a=x+iy,\quad (a)^{*}=a^{*}=x-iy,\quad a^{*}a=x^{2}+y^{2}$$

Тогда

$$\left|\begin{array}{cc}\frac{\partial a^{*}}{\partial x} & \frac{\partial a^{*}}{\partial y}\\ \frac{\partial a}{\partial x} & \frac{\partial a}{\partial y}\end{array}\right|=\left|\begin{array}{cc}1 & -i\\ 1 & i\end{array}\right|=2i$$

Поэтому

$$\frac{\mathrm{d}a^{*}\ \mathrm{d}a}{2\pi i}=\frac{\mathrm{d}x\ \mathrm{d}y}{\pi}=\frac{\rho\mathrm{d}\rho\mathrm{d}\varphi}{\pi} \tag{6}$$

где (ρ,φ) - полярные координаты в плоскости (x,y)

Произведение (5) положительно определено. Действительно, произвольная голоморфная функция $f\left(a^{*}\right)$ является линейной комбинацией одночленов $\psi_n\left(a^{*}\right)=\left(a^{*}\right)^{n}/\sqrt{n!},n\geq 0$. Имеем:

$$\langle\psi_n|\psi_m\rangle=\frac{1}{\sqrt{n!m!}}\int a^{n}\left(a^{*}\right)^{m}e^{-a^{*}a}\frac{\mathrm{d}a^{*}\ \mathrm{d}a}{2\pi i}=\frac{1}{\sqrt{n!m!}}\int\limits_{0}^{\infty}\rho^{m+n+1}e^{-\rho^{2}}\ \mathrm{d}\rho\int\limits_{0}^{2\pi}e^{i(n-m)\varphi}\frac{\mathrm{d}\varphi}{\pi}=$$

$$=\frac{\delta_{n,m}}{n!}\int\limits_{0}^{\infty}\left(\rho^{2}\right)^{n}e^{-\rho^{2}}2\rho\mathrm{d}\rho=\delta_{n,m} \tag{7}$$

Поэтому множество функций $\left\{\psi_n\right\},n=0,1,...$ образует полный ортонормированный базис (ОНБ).

По определению, операторы $a^{\dagger}$ и a действуют согласно правилу

$$a^{\dagger}f\left(a^{*}\right)=a^{*}f\left(a^{*}\right),\quad af\left(a^{*}\right)=\frac{\mathrm{d}}{\mathrm{d}a^{*}}f\left(a^{*}\right) \tag{8}$$

Заметим, что

$$\frac{\mathrm{d}}{\mathrm{d}a^{*}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right),\quad \frac{\mathrm{d}}{\mathrm{d}a}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\quad\longrightarrow\quad \frac{\mathrm{d}}{\mathrm{d}a^{*}}a^{*}=1,\quad \frac{\mathrm{d}}{\mathrm{d}a^{*}}a=0 \tag{9}$$

Согласно определениям скалярного произведения (5) и операторов $a^{\dagger},a$ (8) эти операторы взаимно сопряжены. Действительно,

$$a^{*}e^{-a^{*}a}=-\frac{\mathrm{d}}{\mathrm{d}a}e^{-a^{*}a},\quad \frac{\mathrm{d}}{\mathrm{d}a}f\left(a^{*}\right)=0$$

и потому

$$\left\langle f_1|a^\dagger f_2\right\rangle = \int \left(f_1\left(a^*\right)\right)^* a^* f_2\left(a^*\right) e^{-a^*a}\frac{\mathrm{d}a^*\ \mathrm{d}a}{2\pi i} = -\int \left(f_1\left(a^*\right)\right)^* f_2\left(a^*\right)\left(\frac{\mathrm{d}}{\mathrm{d}a}e^{-a^*a}\right)\frac{\mathrm{d}a^*\ \mathrm{d}a}{2\pi i} =$$
$$= \int \left[\frac{\mathrm{d}}{\mathrm{d}a}\left(f_1\left(a^*\right)\right)^* f_2\left(a^*\right)\right] e^{-a^*a}\frac{\mathrm{d}a^*\ \mathrm{d}a}{2\pi i} = \int \left[\frac{\mathrm{d}}{\mathrm{d}a^*}f_1\left(a^*\right)\right]^* f_2\left(a^*\right) e^{-a^*a}\frac{\mathrm{d}a^*\ \mathrm{d}a}{2\pi i} =$$
$$= \int \left[af_1\left(a^*\right)\right]^* f_2\left(a^*\right) e^{-a^*a}\frac{\mathrm{d}a^*\ \mathrm{d}a}{2\pi i} = \left\langle af_1|f_2\right\rangle$$

13.1.2 2. General Operators

В голоморфном представлении операторы общего вида описываются при помощи ядра. Пусть A - некий оператор и

$$A_{mn} = \langle\psi_m|A|\psi_n\rangle \tag{10}$$

- матрица этого оператора относительно введенного выше ОНБ. Ядром оператора A называется следующая функция:

$$A\left(a^*, a\right) = \sum_{n,m} A_{nm}\frac{\left(a^*\right)^n}{\sqrt{n!}}\frac{a^m}{\sqrt{m!}} \equiv \sum_{n,m} A_{nm}\left|\psi_n\right\rangle\left\langle\psi_m\right| \tag{11}$$

Основываясь на условии полноты $\sum\limits_n \left|\psi_n\right\rangle\left\langle\psi_n\right| = 1$, получаем следующие очевидные формулы:

$$\left(Af\right)\left(a^*\right) = \int A\left(a^*, \alpha\right) f\left(\alpha^*\right) e^{-\alpha^*\alpha}\frac{\mathrm{d}\alpha^*\ \mathrm{d}\alpha}{2\pi i}$$
$$\left(A_1A_2\right)\left(a^*, a\right) = \int A_1\left(a^*, \alpha\right) A_2\left(\alpha^*, a\right) e^{-\alpha^*\alpha}\frac{\mathrm{d}\alpha^*\ \mathrm{d}\alpha}{2\pi i}. \tag{12}$$

Теперь представим оператор A в виде нормально упорядоченного ряда относительно операторов рождения и уничтожения: $A = \sum\limits_{n,m} K_{n,m}\left(a^\dagger\right)^n a^m$. Функция операторов рождения и уничтожения называется нормально упорядоченной, если в этой функции все операторы рождения расположены левее всех операторов уничтожения. Сопоставим оператору A функцию

$$K\left(a^*, a\right) = \sum_{n,m} K_{nm}\left(a^*\right)^n\left(a\right)^m$$

которая называется нормальным символом этого оператора.

Докажем, что

$$A\left(a^*, a\right) = e^{a^*a}K\left(a^*, a\right) \tag{13}$$

Достаточно проверить (13) в частном случае $A = \left(a^\dagger\right)^k a^l$. Имеем:

$$A_{nm} = \frac{1}{\sqrt{n!m!}} \left\langle 0 \left| a^n \left(a^\dagger\right)^k a^l \left(a^\dagger\right)^m \right| 0 \right\rangle$$

Это выражение отлично от нуля лишь при $m \geq l$ и $n \geq k$ и $m - l = n - k$. Обозначим

$$\theta(m \geq l) = \begin{cases} 1, & \text{если } m \geq l \\ 0, & \text{если } m < l \end{cases}$$

При $m \geq l$

$$a^l \left(a^\dagger\right)^m |0\rangle = m(m-1)...(m-l+1) \left(a^\dagger\right)^{m-l} |0\rangle$$

Поэтому

$$A_{nm} = \delta_{m-l,n-k}\theta(m \geq l)\theta(n \geq k)\frac{\sqrt{m \cdot (m-1)...(m-l+1) \cdot n \cdot (n-1)...(n-k+1)}}{\sqrt{(m-l)!(n-k)!}} \times$$
$$= \sqrt{m \cdot (m-1)...(m-l+1)} \cdot \sqrt{n \cdot (n-1)...(n-k+1)} \cdot \theta(m \geq l)\theta(n \geq k) \cdot \delta_{m-l,n-k}$$

Теперь имеем:

$$A\left(a^*, a\right) = \sum_{m,n \geq 0} A_{mn} \frac{\left(a^*\right)^n}{\sqrt{n!}} \cdot \frac{a^m}{\sqrt{m!}} = \sum_{m \geq l} \sum_{n \geq k} \delta_{m-l,n-k} \frac{\left(a^*\right)^n a^m}{\sqrt{(n-k)!(m-l)!}} =$$
$$= \sum_{n \geq 0} \frac{1}{n!} \left(a^*\right)^{k+n} a^{l+n} = \left(a^*\right)^k a^l e^{a^* a} = e^{a^* a} K\left(a^*, a\right)$$

Тем самым формула (13) доказана.

13.1.3 3. Functional Integral in the Holomorphic Representation

Пусть функция $h\left(a^*, a\right)$ является нормальным символом оператора Гамильтона $\mathcal{H}\left(a^\dagger, a\right)$. Тогда ядро оператора эволюции

$$U = \exp(-i\Delta t \mathcal{H}), \quad \Delta t \to 0$$

согласно сказанному выше представляется как

$$U\left(a^*, a; \Delta t\right) = \exp\left\{a^* a - i h\left(a^*, a\right) \Delta t\right\} \tag{14}$$

Для произвольного интервала времени $t'' - t' = N\Delta t$:

$$U\left(a^*, a; t'', t'\right) = \int \exp\left\{\left[a_N^* a_{N-1} - a_{N-1}^* a_{N-1} + ... - a_1^* a_1 + a_1^* a_0\right] - \right.$$
$$\left. - i\Delta t \left[h\left(a_N^*, a_{N-1}\right) + ... + h\left(a_1^*, a_0\right)\right]\right\} \prod_{k=1}^{N-1} \frac{\mathrm{d}a_k^* \, \mathrm{d}a_k}{2\pi i}$$

Мы обозначили $a_0 = a, a_N^* = a^*$.
Теперь перейдём к пределу $N \to \infty, \Delta t \to 0$:

$$U\left(a^*,a;t'',t'\right)=\int e^{a^*(t'')a(t'')}\exp\left\{\int\limits_{t'}^{t''}\mathrm{d}t\left[-a^*\dot{a}-ih\left(a^*,a\right)\right]\right\}\prod_t\frac{\mathrm{d}a^*\ \mathrm{d}a}{2\pi i} \tag{15}$$

Граничные условия таковы:

$$a^*\left(t''\right)=a^*,\quad a\left(t'\right)=a \tag{16}$$

причём по переменным $a\left(t''\right)$ и $a^*\left(t'\right)$ идёт интегрирование. С учётом того что

$$-\frac{1}{2}\int\limits_{t'}^{t''}\mathrm{d}ta^*\dot{a}=\frac{1}{2}\int\limits_{t'}^{t''}\mathrm{d}t\dot{a}^*a-\frac{1}{2}a^*\left(t''\right)a\left(t''\right)+\frac{1}{2}a^*\left(t'\right)a\left(t'\right)$$

имеем для ядра оператора эволюции:

$$U\left(a^*,a;t'',t'\right)=\int\exp\left\{\frac{1}{2}\left[a^*\left(t''\right)a\left(t''\right)+a^*\left(t'\right)a\left(t'\right)\right]\right\}\times$$

$$\times\exp\left\{i\int\limits_{t'}^{t''}\mathrm{d}t\left[\frac{1}{2i}\left(\dot{a}^*a-a^*\dot{a}\right)-h\left(a^*,a\right)\right]\right\}\prod_t\frac{\mathrm{d}a^*\ \mathrm{d}a}{2\pi i}$$

Обратим внимание на то, что здесь интеграл по времени во второй экспоненте есть действие системы.

В случае гармонического осциллятора интеграл (15) как гауссов интеграл легко вычисляется. Так как

$$\eta q=\frac{\eta}{\sqrt{2\omega}}\left(a+a^*\right)$$

то гамильтониан гармонического осциллятора при наличии источника

$$h\left(a^*,a\right)=\omega\left(a^*a+1/2\right)-\frac{\eta}{\sqrt{2\omega}}\left(a+a^*\right) \tag{17}$$

Из (16) имеем:

$$\delta a^*\left(t''\right)=\delta a\left(t'\right)=0 \tag{18}$$

Поэтому для вариации показателя экспоненты в континуальном интеграле (15) находим:

$$\delta\left\{a^*\left(t''\right)a\left(t''\right)+\int\limits_{t'}^{t''}\mathrm{d}t\left[-a^*\dot{a}-i\omega a^*a+\frac{i\eta}{\sqrt{2\omega}}\left(a+a^*\right)\right]\right\}=$$

$$=a^*\left(t''\right)\delta a\left(t''\right)+\int\limits_{t'}^{t''}\mathrm{d}t\left[-a^*\delta\dot{a}+\delta a\left(-i\omega a^*+\frac{i\eta}{\sqrt{2\omega}}\right)-\delta a^*\left(\dot{a}+i\omega a-\frac{i\eta}{\sqrt{2\omega}}\right)\right]=$$

$$=\int\limits_{t'}^{t''}\mathrm{d}t\left[\delta a\left(\dot{a}^*-i\omega a^*+\frac{i\eta}{\sqrt{2\omega}}\right)-\delta a^*\left(\dot{a}+i\omega a-\frac{i\eta}{\sqrt{2\omega}}\right)\right]$$

Отсюда следует, что точка экстремума показателя экспоненты, которая в теории поля называется массовой поверхностью, есть решение уравнений движения:

$$\dot{a} = -i\omega a + \frac{i\eta}{\sqrt{2\omega}}, \quad \dot{a}^* = i\omega a^* - \frac{i\eta^*}{\sqrt{2\omega}} \tag{19}$$

(Здесь для дальнейшего удобства сделана замена $(\eta/\sqrt{2\omega})(a + a^*) \to (1/\sqrt{2\omega})(\eta^* a + \eta a^*)$, хотя в рассматриваемом случае $\eta^* = \eta$.)

Рассмотрим случай $\eta = 0$, когда уравнения (19) с граничными условиями (16) тривиально решаются:

$$a(t) = e^{-i\omega(t-t')} a(t'), \quad a^*(t) = e^{i\omega(t-t'')} a^*(t'') \tag{20}$$

Далее вместо $a(t')$ и $a^*(t'')$ пишем a и a^*, соответственно. Находим экстремальное значение показателя экспоненты в интеграле (15): $a^* a \exp(-i\omega(t'' - t'))$. Таким образом

$$U_0(a^*, a; t'' - t') = \exp\left\{a^* a e^{-i\omega(t''-t')}\right\} \tag{21}$$

Вследствие унитарности оператора (21) нормировочный множитель $1/N$ равен единице. Если $f(a^*)$ - произвольная функция, то

$$(U_0 f)(a^*) = \int \exp\left\{a^* \alpha e^{-i\omega t}\right\} f(\alpha^*) e^{-\alpha^* \alpha} \frac{\mathrm{d}\alpha^* \, \mathrm{d}\alpha}{2\pi i} = f\left(a^* e^{-i\omega t}\right) \tag{22}$$

(Проверить эту формулу для $f(a^*) = (a^*)^n$.)

Функция (21) является ядром оператора $e^{-it\mathcal{H}_0}$, где $\mathcal{H}_0$ - гамильтониан свободного осциллятора.

Замечание. Здесь мы выбросили "нулевую"энергию из гамильтониана. В противном случае имели бы

$$U_0(a^*, a; t) = \exp\left\{a^* a e^{-i\omega t} - \frac{i}{2}\omega t\right\}$$

Пусть A - оператор с ядром $A(a^*, a)$. Нам понадобится обобщение формулы (22), то есть ядро оператора $e^{it''\mathcal{H}_0} A e^{-it'\mathcal{H}_0}$:

$$[U_0(-t'') A U_0(t')](a^*, a) = A\left(a^* e^{i\omega t''}, a e^{-i\omega t'}\right) \tag{23}$$

13.1.4 4. Definition of the S-Matrix

Пусть $\mathcal{H}$ - гамильтониан теории, $\mathcal{H}_0$ - свободный гамильтониан и $U(t'', t')$ - оператор эволюции взаимодействующей теории. S-матрица определяется при помощи следующей формулы:

$$S = \lim e^{it''\mathcal{H}_0} U(t'', t') e^{-it'\mathcal{H}_0}, \quad t'' \to +\infty, \quad t' \to -\infty \tag{24}$$

Обкладки $e^{it''\mathcal{H}_0} ... e^{-it'\mathcal{H}_0}$ в (24) приводят к исключению свободного движения, которое не представляет интереса. Действительно, если $\mathcal{H} = \mathcal{H}_0$, то S-матрица (24) равна единичному оператору.

Вычислим S-матрицу гармонического осциллятора при наличии внешнего источника. Согласно определению (24) и формуле (23) теперь граничные условия (16) должны быть заменены на следующие:

$$a^*\left(t''\right) = a^* e^{i\omega t''}, \quad a\left(t'\right) = a e^{-i\omega t'} \tag{25}$$

Легко находим решения уравнений (19) с граничными условиями (25):

$$a^*(t) = e^{i\omega t}\left[a^* + i\int\limits_t^{t''} \mathrm{d}s \frac{e^{-i\omega s}\eta^*(s)}{\sqrt{2\omega}}\right], \quad a(t) = e^{-i\omega t}\left[a + i\int\limits_{t'}^{t} \mathrm{d}s \frac{e^{i\omega s}\eta(s)}{\sqrt{2\omega}}\right] \tag{26}$$

Подставим решение (26) в показатель экспоненты в (15) и раздвинем пределы интегрирования ($t' \to -\infty, t'' \to +\infty$). Таким образом мы получим ядро S-матрицы для гармонического осциллятора в присутствии внешнего источника:

$$S_\eta\left(a^*, a\right) = \frac{1}{N}\exp\left\{a^* a + \frac{i}{\sqrt{2\omega}}\int\limits_{-\infty}^{+\infty} \mathrm{d}t\left[\eta^*(t) a e^{-i\omega t} + a^* e^{i\omega t}\eta(t)\right] + \right.$$
$$\left. + i\int\limits_{-\infty}^{+\infty} \mathrm{d}t \int\limits_{-\infty}^{t} \mathrm{d}s \eta^*(t) i \frac{e^{-i\omega(t-s)}}{2\omega}\eta(s)\right\} \tag{27}$$

При помощи (23 Л1) выражение (27) переписывается как

$$S_\eta\left(a^*, a\right) = \frac{1}{N}\exp\left\{a^* a + \frac{i}{\sqrt{2\omega}}\int\limits_{-\infty}^{+\infty} \mathrm{d}t\left[\eta^*(t) a e^{-i\omega t} + a^* e^{i\omega t}\eta(t)\right] + \right.$$
$$\left. + \frac{i}{2}\int\limits_{-\infty}^{+\infty+\infty} \int\limits_{-\infty} \mathrm{d}t\ \mathrm{d}s \eta^*(t) D_c(t-s)\eta(s)\right\} \tag{28}$$

Подчеркнём, что в (28) используется именно причинная гриновская функция, так как уже в определении S-матрицы (24) сделано предположение, что при $|t| \to \infty$ взаимодействие выключается адиабатически и система описывается свободным гамильтонианом $\mathcal{H}_0$.

В дальнейшем нам понадобится формула (26), записанная в несколько изменённом виде:

$$q(t) \equiv \frac{1}{\sqrt{2\omega}}\left[a(t) + a^*(t)\right] = \frac{1}{\sqrt{2\omega}}\left[a e^{-i\omega t} + a^* e^{i\omega t}\right] + \int\limits_{t'}^{t''} \mathrm{d}s D_c(t-s)\eta(s)$$

13.1.5 5. S-Matrix in Scalar Field Theory

Рассмотрим скалярную теорию поля в d-мерном пространстве-времени. В свободном случае в присутствии внешнего источника действие такой теории имеет вид

$$\mathcal{S} = \int \mathrm{d}^{(d)}x \left\{\frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2 + \eta\phi\right\} \tag{29}$$

Отсюда получаем лагранжиан:

$$\mathcal{L} = \int \mathrm{d}^{(d-1)}x \left\{ \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 + \eta\phi \right\} \tag{30}$$

где $\dot{\phi} = \partial\phi(t,\mathbf{x})/\partial t$. Варьируя лагранжиан (30) относительно поля $\dot{\phi}$ находим обобщённый импульс поля ϕ:

$$\delta_{\dot{\phi}}\mathcal{L} = \int \mathrm{d}^{(d-1)}x\pi\delta\dot{\phi} \quad \longrightarrow \quad \pi = \dot{\phi} \tag{31}$$

Теперь путём канонической процедуры получаем гамильтониан теории:

$$\mathcal{H} = \int \mathrm{d}^{(d-1)}x\pi\dot{\phi} - \mathcal{L} \longrightarrow \mathcal{H} = \int \mathrm{d}^{(d-1)}x \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 - \eta\phi \right\} \tag{32}$$

Выпишем лишь ненулевые одновременные коммутационные соотношения (КС):

$$[\phi(t,\mathbf{x}), \pi(t,\mathbf{y})] = i\delta^{(d-1)}(\mathbf{x}-\mathbf{y}) \tag{33}$$

Как правило, реализация КС(33) осуществляется согласно следующей схеме:

$$\begin{gathered}
\phi(t,\mathbf{x}) = \int \frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}} \frac{1}{\sqrt{2\omega_k}} \left(a_{\mathbf{k}}(t)e^{i\mathbf{kx}} + a^\dagger_{\mathbf{k}}(t)e^{-i\mathbf{kx}} \right) \\
\pi(t,\mathbf{x}) = -i \int \frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}} \sqrt{\frac{\omega_k}{2}} \left(a_{\mathbf{k}}(t)e^{i\mathbf{kx}} - a^\dagger_{\mathbf{k}}(t)e^{-i\mathbf{kx}} \right) \\
[a_{\mathbf{k}}(t), a_{\mathbf{p}}(t)] = \left[a^\dagger_{\mathbf{k}}(t), a^\dagger_{\mathbf{p}}(t) \right] = 0, \quad \left[a_{\mathbf{k}}(t), a^\dagger_{\mathbf{p}}(t) \right] = (2\pi)^{d-1}\delta^{(d-1)}(\mathbf{k}-\mathbf{p})
\end{gathered} \tag{34}$$

Разложение (34) является естественным в случае свободной теории. Действительно, если

$$\omega_k = \sqrt{\mathbf{k}^2 + m^2} \tag{35}$$

то гамильтониан (32) распадается на сумму гамильтонианов гармонических осцилляторов (мы здесь полагаем $\eta = 0$):

$$\mathcal{H}_0 = \frac{1}{2}\int \frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}}\omega_k \left(a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + a_{\mathbf{k}}a^\dagger_{\mathbf{k}} \right) = \int \frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}}\omega_k a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + \delta^{(d-1)}(0)\int \mathrm{d}^{(d-1)}k\frac{\omega_k}{2}$$

где $(2\pi)^{d-1}\delta^{(d-1)}(0) = \int \mathrm{d}^{(d-1)}x = V$ - объём пространства. Второе слагаемое в правой части последнего равенства является энергией нулевых колебаний осцилляторов. Хотя плотность этой энергии бесконечна, она как правило не играет роли и потому в обычной квантовой теории поля вычёркивается. Энергия нулевых колебаний (или энергия вакуума) является серьёзной проблемой в квантовой теории гравитации, однако эта тема лежит за рамками нашего курса лекций. Отметим лишь, что рассматриваемый далее эффект Казимира существует вследствие реальности энергии вакуума.

Сделаем замечание о нормировке, используемой в этих лекциях. Везде интегрирование в координатном пространстве осуществляется по правилу $\int \mathrm{d}^{(d)}x$, в то время как интегрирование в импульсном пространстве - по правилу $\int \mathrm{d}^{(d)}k/(2\pi)^d$. Это правило согласовано с правилами коммутации (33) и (34). Иными словами, единичный оператор в

линейном пространстве функций в координатном представлении есть $\delta^{(d)}(x-y)$, а в импульсном представлении - $(2\pi)^d\delta^{(d)}(k-p)$.

Основное состояние или вакуум в теории (32)-(35) обозначается $|0\rangle$ и определяется условиями

$$a_{\mathbf{k}}|0\rangle = 0, \quad \langle 0|0\rangle = 1 \tag{36}$$

Очевидно,

$$\mathcal{H}_0|0\rangle = 0 \tag{37}$$

Возбуждённые состояния строятся при помощи операторов рождения $a^\dagger_{\mathbf{k}}$. Например, одночастичное состояние имеет вид

$$|\mathbf{k}\rangle = \sqrt{2\omega_k}a^\dagger_{\mathbf{k}}|0\rangle \tag{38}$$

Вследствие (34) и (36) скалярное произведение двух одночастичных состояний

$$\langle \mathbf{k}|\mathbf{p}\rangle = 2\omega_k(2\pi)^{d-1}\delta^{(d-1)}(\mathbf{k}-\mathbf{p}) \tag{39}$$

Можно показать (см. задачи), что правая часть (39) лоренц-инвариантна. Нормировка состояний (38) оправдывается именно лоренц-инвариантностью скалярных произведений одночастичных состояний. Заметим также, что лоренц-инвариантная мера в пространстве одночастичных состояний равна

$$\frac{\mathrm{d}^{(d-1)}k}{2\omega_k(2\pi)^{d-1}} \tag{40}$$

Мера (40) согласована со скалярным произведением (39), то есть

$$\int \frac{\mathrm{d}^{(d-1)}k}{2\omega_k(2\pi)^{d-1}}|\mathbf{k}\rangle\langle\mathbf{k}| = 1 \tag{41}$$

где единица в правой части равенства (40) обозначает единичный оператор в пространстве одночастичных состояний.

Энергия состояния (38) легко вычисляется при помощи равенства (37) и КС(34):

$$\mathcal{H}_0|\mathbf{k}\rangle = \omega_k|\mathbf{k}\rangle$$

Оператор импульса в скалярной теории поля имеет вид

$$\mathcal{P} = -\int \mathrm{d}^{(d-1)}x\pi(x)\nabla\phi(x) \tag{42}$$

что оправдывается значениями следующих коммутаторов:

$$[\phi(x),\mathcal{P}] = -i\nabla\phi(x), \quad [\pi(x),\mathcal{P}] = -i\nabla\pi(x)$$

и так далее. Подставляя разложение (34) в (42), получаем:

$$\mathcal{P} = \int \frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}}\mathbf{k}a^\dagger_{\mathbf{k}}a_{\mathbf{k}}$$

Легко найти, что

$$\mathcal{P}|0\rangle = 0, \quad \mathcal{P}|\mathbf{k}\rangle = \mathbf{k}|\mathbf{k}\rangle$$

Таким образом, состояние $|\mathbf{k}\rangle$ имеет определённый импульс $\mathbf{k}$.

Пусть $\{\mathbf{k}_i\}, i = 1, ..., n$ - такой набор импульсов, что попарно $\mathbf{k}_i \neq \mathbf{k}_j$. Тогда многочастичное состояние представляется в виде

$$|\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_n\rangle = \prod_{i=1}^{n} \left(\sqrt{2\omega_{k_i}} a^\dagger_{\mathbf{k}_i}\right)|0\rangle \tag{43}$$

Имеем:

$$\begin{aligned}\mathcal{H}_0 |\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_n\rangle &= \left(\sum_{i=1}^{n} \omega_{k_2}\right) |\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_n\rangle \\ \mathcal{P} |\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_n\rangle &= \left(\sum_{i=1}^{n} \mathbf{k}_i\right) |\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_n\rangle\end{aligned} \tag{44}$$

а также

$$\begin{aligned}\langle \mathbf{p}_1, \mathbf{p}_2, ..., \mathbf{p}_n | \mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_n\rangle = \prod_{i=1}^{n} \left[2\omega_{k_i}(2\pi)^{d-1}\right] \\ \sum_{\text{все перестановки } p_i} \delta^{(d-1)}(\mathbf{k}_1 - \mathbf{p}_{i_1}) ... \delta^{(d-1)}(\mathbf{k}_n - \mathbf{p}_{i_n})\end{aligned} \tag{45}$$

Естественно, существуют также состояния вида $\left(a^\dagger_{\mathbf{k}}\right)^n |0\rangle$ (n частиц в одном состоянии), однако в теории S-матрицы эти состояния интереса не представляют.

Пространство состояний, построенное описанным способом, называется фоковским пространством состояний.

Пусть

$$\eta_{\mathbf{k}}(t) = \int \mathrm{d}^{(d-1)} x \eta(t, \mathbf{x}) e^{-i\mathbf{k}\mathbf{x}} = \eta^*_{-\mathbf{k}} \tag{46}$$

Тогда для гамильтониана (32) имеем:

$$\mathcal{H} = \int \frac{\mathrm{d}^{(d-1)} k}{(2\pi)^{d-1}} \left\{\omega_k a^\dagger_{\mathbf{k}}(t) a_{\mathbf{k}}(t) - \frac{1}{\sqrt{2\omega_k}} \left(\eta^*_{\mathbf{k}}(t) a_{\mathbf{k}}(t) + \eta_{\mathbf{k}}(t) a^\dagger_{\mathbf{k}}(t)\right)\right\} \tag{47}$$

Отсюда видно, что в данном случае гамильтониан является суммой гамильтонианов вида (17) и потому ядро S-матрицы системы (29) получается непосредственно из ядра S-матрицы (27). Первое слагаемое в экспоненте в (27) переходит в

$$\int \frac{\mathrm{d}^{(d-1)} k}{(2\pi)^{d-1}} a^*_{\mathbf{k}} a_{\mathbf{k}}$$

Второе - в

$$i\int \mathrm{d}^{(d)}x\eta(x)\phi_0(x),\quad \phi_0(x)=\int\frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}}\frac{1}{\sqrt{2\omega_k}}\left(a^*_{\mathbf{k}}e^{ikx}+a_{\mathbf{k}}e^{-ikx}\right)$$
$$kx=k^0x^0-\mathbf{kx},\quad k^0=\omega_k \tag{48}$$

а третье - в

$$i\int\limits_{-\infty}^{+\infty}\mathrm{d}t\int\limits_{-\infty}^{t}\mathrm{d}s\int\frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}}\eta^*_{\mathbf{k}}(t)\left[i\frac{e^{-i\omega_k(t-s)}}{2\omega_k}\right]\eta_{\mathbf{k}}(s)=\frac{i}{2}\iint\mathrm{d}^{(d)}x\ \mathrm{d}^{(d)}y\eta(x)D_c(x-y)\eta(y)$$

где

$$D_c(x)=i\int\frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}}\frac{e^{i\mathbf{kx}-i\omega_k\left|x^0\right|}}{2\omega_k}=\int\frac{\mathrm{d}^{(d)}k}{(2\pi)^d}\frac{e^{-ikx}}{-k^2+m^2-i\varepsilon}$$
$$\left(\partial_\mu\partial^\mu+m^2\right)D_c(x)=\delta^{(d)}(x) \tag{49}$$

Из правой части равенства (49) очевидны лоренц-инвариантность причинной гриновской функции $D_c(x)$ и то, что везде $\mathrm{Im}\,\omega_k=-\varepsilon,\varepsilon\to+0$. Для ядра S-матрицы имеем:

$$S\left(a^*,a\right)=\frac{1}{N}\exp\left\{\int\frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}}a^*_{\mathbf{k}}a_{\mathbf{k}}+i\int\mathrm{d}^{(d)}x\eta(x)\phi_0(x)+\right.$$
$$\left.+\frac{i}{2}\iint\mathrm{d}^{(d)}x\ \mathrm{d}^{(d)}y\eta(x)D_c(x-y)\eta(y)\right\} \tag{50}$$

Теперь перейдём от ядра к нормальному символу S-матрицы:

$$S\left(a^*,a\right)=\frac{1}{N}\exp\left\{i\int\mathrm{d}^{(d)}x\eta(x)\phi_0(x)+\frac{i}{2}\iint\mathrm{d}^{(d)}x\ \mathrm{d}^{(d)}y\eta(x)D_c(x-y)\eta(y)\right\}\equiv S_0\left(\phi_0;\eta\right) \tag{51}$$

Поле (48) удовлетворяет свободному уравнению

$$\left(\partial_\mu\partial^\mu+m^2\right)\phi_0=0$$

которое называется уравнением Клейна-Гордона-Фока.

Укажем, что нормальный символ S-матрицы представляется также как функциональный интеграл

$$S\left(a^*,a\right)=\frac{1}{N'}\int\exp\left\{i\int\mathrm{d}t\left[\ \mathrm{d}^{(d-1)}x\pi\dot{\phi}-\mathcal{H}\right]\right\}D\pi D\phi=\frac{1}{N}\int\exp(i\mathcal{S})D\phi$$
$$D\pi\equiv\prod_x\ \mathrm{d}\pi(x),\quad D\phi\equiv\prod_x\ \mathrm{d}\phi(x)$$

$$\phi(x)\overrightarrow{}_{t\to\pm\infty}\phi_{\mathbf{k}\ \substack{\mathrm{out}\\ \mathrm{in}}}(x)\qquad \phi_{\mathbf{k}\ \substack{\mathrm{out}\\ \mathrm{in}}}(x)=\int\frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}}\frac{1}{\sqrt{2\omega_k}}\left(a_{\mathbf{k}\ \substack{\mathrm{out}\\ \mathrm{in}}}e^{-ikx}+a^*_{\mathbf{k}\ \substack{\mathrm{out}\\ \mathrm{in}}}e^{ikx}\right),$$

$$a_{\mathbf{k}\ \mathrm{in}} = a_{\mathbf{k}}, \quad a^*_{\mathbf{k}\ \mathrm{out}} = a^*_{\mathbf{k}}, \quad kx = k^0x^0 - \mathbf{kx}, \quad k^0 = \omega_k \tag{52}$$

Действие $\mathcal{S}$ в (52) может быть как свободным действием (29), так и включающим взаимодействие (см. ниже (54)).

В справедливости представления (52) нетрудно убедиться следующим образом. В случае свободной теории с внешним источником, описывающейся действием (29), интеграл (52) как гауссов вычисляется точно. В этом случае действие в экспоненте функционального интеграла экстремально при условии $(\partial_\mu\partial^\mu + m^2)\,\phi = \eta$. Решение последнего уравнения с необходимыми граничными условиями типа (25) есть (сравни с одномерным случаем (26′))

$$\phi(x) = \phi_0(x) + \int \mathrm{d}^{(d)}y D_c(x-y)\eta(y) \tag{53}$$

Подстановка (53) в экспоненту функционального интеграла приводит к выражению (51). Очевидно, полученный результат не случаен, поскольку, согласно лекции 1, амплитуда перехода вообще, и S-матрица в частности, представляется в виде функционального интеграла (52). Поэтому единственный остающийся вопрос при вычислении нормального символа S-матрицы - это вопрос о граничных условиях динамического поля при $|t| \to \infty$. Важность голоморфного представления функционального интеграла заключается именно в том, что это представление даёт ответ на вопрос, каковы должны быть граничные условия при вычислении нормального символа S-матрицы: положительно частотные волны в ϕ_{in} и отрицательно частотные волны в ϕ_{out} совпадают с соответствующими волнами в ϕ_0 (48). Здесь уместно ещё раз подчеркнуть, что такие граничные условия и сама теория S-матрицы существуют лишь при справедливости предположения об адиабатическом выключении взаимодействия при $|t| \to \infty$.

13.1.6 6. S-Matrix in Interacting Field Theory

Рассмотрим теорию скалярного поля с самодействием, описывающуюся действием

$$\mathcal{S} = \int \mathrm{d}^{(d)}x \left\{ \frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{1}{2}m_0^2\phi^2 - V(\phi) + \eta\phi \right\} \tag{54}$$

Наибольший интерес для нас представляет случай

$$V(\phi) = \frac{1}{4!}\lambda_0\phi^4 \tag{55}$$

Размерность полей и констант определяется исходя из того, что действие имеет размерность постоянной Планка, а в случае $\hbar = 1$ - безразмерно. Например, из безразмерности действия (54) имеем:

$$[m] = \mathrm{cm}^{-1}, \quad [\phi] = \mathrm{cm}^{(1-d/2)}, \quad [\lambda] = \mathrm{cm}^{d-4}$$

Теперь континуальный интеграл не может быть вычислен точно, но существует вычислительная техника, основанная на теории возмущений (ТВ). Идея заключается в разложении S-матрицы по потенциалу взаимодействия $\left(\int \mathrm{d}^{(d)}xV(\phi)\right)$. Тогда в нулевом приближении мы имеем точный результат (51), а при разложении по $\left(\int \mathrm{d}^{(d)}xV(\phi)\right)$ возникают интегралы от гауссовых средних вида $\langle 0\,|T\phi_1...\phi_n|\,0\rangle$, где $\phi_i = \phi\,(x_i)$.

Замечание. Рассмотрим интеграл

$$I(\lambda) = \int\limits_{-\infty}^{+\infty} \mathrm{d}x \exp\left(-\frac{1}{2}x^2 - \lambda x^4\right)$$

который точно не вычисляется. Очевидно также, что функция $I(\lambda)$ при $\lambda = 0$ не аналитична, она имеет особенность. Тем не менее при $\lambda \ll 1$ имеет смысл разложение

$$I(\lambda) = \sqrt{2\pi} - \lambda \int \mathrm{d}x x^4 e^{-(1/2)x^2} + \ldots \tag{56}$$

Это разложение имеет смысл лишь до тех пор пока следующий член разложения меньше предыдущего. Интегралы вида

$$I_n = \int \mathrm{d}x x^n e^{-(1/2)x^2} \tag{57}$$

вычисляются при помощи формул

$$x^n e^{-(1/2)x^2} = \left.\frac{\mathrm{d}^n}{\mathrm{d}\eta^n} e^{-(1/2)x^2+\eta x}\right|_{\eta=0} \tag{58}$$

$$\int \mathrm{d}x e^{-(1/2)x^2+\eta x} = \sqrt{2\pi} e^{(1/2)\eta^2} \tag{59}$$

$$I_n = \left[\int \mathrm{d}x \frac{\mathrm{d}^n}{\mathrm{d}\eta^n} e^{-(1/2)x^2+\eta x}\right]\Bigg|_{\eta=0} = \left[\frac{\mathrm{d}^n}{\mathrm{d}\eta^n}\int \mathrm{d}x e^{-(1/2)x^2+\eta x}\right]\Bigg|_{\eta=0} =$$
$$= \sqrt{2\pi}\left[\frac{\mathrm{d}^n}{\mathrm{d}\eta^n} e^{(1/2)\eta^2}\right]\Bigg|_{\eta=0}$$

так что

$$I(\lambda) = \sqrt{2\pi}(1 - 3\lambda + \ldots)$$

Если $\Phi(x)$ - произвольная функция, то

$$\Phi(x) = \left[\Phi\left(\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}\eta}\right) e^{i\eta x}\right]\Bigg|_{\eta=0}$$

Это равенство позволяет свести вычисление негауссова интеграла к следующей формуле:

$$\int \mathrm{d}x \exp\left\{i\left(\frac{1}{2}x^2 - V(x)\right)\right\} =$$
$$= \left[\exp\left\{-iV\left(\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}\eta}\right)\right\}\int \mathrm{d}x \exp\left\{i\left(\frac{1}{2}x^2 + \eta x\right)\right\}\right]\Bigg|_{\eta=0} \tag{60}$$

Интеграл в правой части (60) гауссов и потому легко вычисляется. После этого мы можем развить ТВ, разлагая правую часть (60) относительно V и пользуясь теоремой Вика.

В теории поля действуем аналогично. Имеем:

$$\phi\left(x_1\right)...\phi\left(x_n\right)=\frac{1}{i}\frac{\delta}{\delta\eta\left(x_1\right)}\cdots\frac{1}{i}\frac{\delta}{\delta\eta\left(x_n\right)}\exp\left\{i\int\phi(x)\eta(x)\mathrm{d}x\right\}\Bigg|_{\eta=0}$$

Для любого функционала $\Phi\{\phi(x)\}$

$$\Phi\{\phi(x)\}=\Phi\left\{\frac{1}{i}\frac{\delta}{\delta\eta(x)}\right\}\exp\left\{i\int\phi(x)\eta(x)\mathrm{d}x\right\}\Bigg|_{\eta=0} \tag{61}$$

и, в частности,

$$\exp\left\{-i\int V(\phi)\mathrm{d}x\right\}=\exp\left\{-i\int V\left(\frac{1}{i}\frac{\delta}{\delta\eta(x)}\right)\mathrm{d}x\right\}\exp\left\{i\int\phi(x)\eta(x)\mathrm{d}x\right\}\Bigg|_{\eta=0} \tag{62}$$

Теперь, прямолинейно обобщая одномерную формулу (60) на случай теории поля и пользуясь формулой (51) для значения гауссова интеграла в задаче об S-матрице, выпишем нормальный символ S-матрицы в скалярной теории поля со взаимодействием:

$$\begin{aligned}S\left(a^*,a\right)=S\left(\phi_0\right)=\frac{1}{N}\exp\left\{-i\int V\left(\frac{1}{i}\frac{\delta}{\delta\eta(x)}\right)\mathrm{d}x\right\}\times\\ \times\exp\left\{i\int\mathrm{d}x\eta(x)\phi_0(x)+\frac{i}{2}\iint\mathrm{d}x\ \mathrm{d}y\eta(x)D_c(x-y)\eta(y)\right\}\Bigg|_{\eta=0}\end{aligned} \tag{63}$$

Здесь и далее $D_c(x)$ задаётся согласно (49),

Разложим функционал (63) по ϕ_0 и перейдём от нормального символа оператора S-матрицы к самому оператору S-матрицы:

$$\begin{aligned}S\left(\phi_0\right)&=\sum_n\frac{1}{n!}\int S_n\left(x_1,...,x_n\right):\phi_0\left(x_1\right)...\phi_0\left(x_n\right):\mathrm{d}x_1...\mathrm{d}x_n\\ \phi_0(x)&=\int\frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}}\frac{1}{\sqrt{2\omega_k}}\left(a_{\mathbf{k}}e^{-ikx}+a_{\mathbf{k}}^{\dagger}e^{ikx}\right),\quad k^0=\omega_k\end{aligned} \tag{64}$$

Функции $S_n\left(x_1,...,x_n\right)$ являются коэффициентами при степенях поля $\phi_0(x)$ в разложении функционала (63) по степеням этого поля и они называются коэффициентными функциями S матрицы, а $S\left(\phi_0\right)$ - производящим функционалом для функций S_n. Поле ϕ_0 в (64) является свободным квантованным полем в представлении Гейзенберга. Согласно определению, размещение любого оператора, являющегося функцией операторов рождения и уничтожения, между двумя двоеточиями означает приведение оператора к нормальному виду, причём в процессе этого приведения все операторы рождения и уничтожения считаются коммутативными. Например, $:a^{\dagger}a:=a^{\dagger}a,:aa^{\dagger}:=a^{\dagger}a,:a^m\left(a^{\dagger}\right)^n:=\left(a^{\dagger}\right)^n a^m,:aa^{\dagger}a:=a^{\dagger}a^2$, и т.д.

Задачи

1. Проверить в скалярной теории поля, что уравнения Гейзенберга $i\dot{\mathcal{O}}=[\mathcal{O},\mathcal{H}]$ совпадают по форме с лагранжевыми уравнениями.

2. Рассмотрите теорию комплексного скалярного поля с действием

$$S = \int \mathrm{d}^4x \left(\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi\right)$$

Изучите эту теорию, рассматривая в качестве базисных динамических переменных $\phi(x)$ и $\phi^*(x)$, а не вещественную и мнимую части поля $\phi(x)$.

а) Выписать канонически сопряжённые импульсы к $\phi(x)$ и $\phi^*(x)$ и канонические коммутационные соотношения. Показать, что гамильтониан

$$\mathcal{H} = \int \mathrm{d}x \left(\pi^* \pi + \nabla \phi^* \nabla \phi + m^2 \phi^* \phi\right)$$

Выписать уравнение Гейзенберга для $\phi(x)$ и показать, что оно совпадает с уравнением КлейнаГордона-Фока.

б) Диагонализуйте гамильтониан, вводя операторы рождения и уничтожения. Покажите, что теория содержит два типа частиц с массой m.

в) Докажите, что заряд

$$Q = \int \mathrm{d}^3x \frac{i}{2} \left(\phi^* \pi^* - \pi \phi\right)$$

сохраняется и перепишите его в терминах операторов рождения и уничтожения и вычислите заряд частиц каждого типа.

3. а) Доказать, что величина

$$\sqrt{m^2 + \mathbf{k}^2} \delta^{(d-1)}(\mathbf{k} - \mathbf{p})$$

лоренц-инвариантна.

б) Доказать, что мера

$$\frac{\mathrm{d}^{(d-1)}k}{\sqrt{m^2 + \mathbf{k}^2}}$$

лоренц-инвариантна.

14 Interacting Fields in Series, Feynman Diagrams

14.1 Expansion of Correlation Functions in Perturbation Theory

Теория

Начнем изучение теории возмущений для взаимодействующих полей с построения формализма, который позволит сделать наглядными ряды теории возмущений как пространственно-временные процессы. Хотя нам не потребуется переформулировать всю квантовую механику, мы вновь построим зависящую от времени теорию возмущений в форме, удобной для наших целей. Ясно, что в конечном счете мы хотим вычислить вероятности распадов и сечения рассеяния. Поставим пока менее амбициозную задачу вычислить простейшую (но более абстрактную) величину - двухточечную корреляционную функцию или двухточечную функцию Грина

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle,$$

в теории ϕ^4. Символом $|\Omega\rangle$ мы обозначим основное состояние взаимодействующей теории, которое в общем случае отличается от основного состояния свободной теории $|0\rangle$. Символ хронологического упорядочения T введен для дальнейшего удобства. Корреляционную функцию можно физически интерпретировать как амплитуду распространения частицы или возбуждения между точками y и x.В свободной теории - это фейнмановский пропагатор:

$$\langle 0|T\phi(x)\phi(y)|0\rangle_{\text{free}} = D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\varepsilon}.$$

Мы хотели бы знать, как это выражение изменяется во взаимодействующей теории. Как только мы проанализируем двухточечную корреляционную функцию, будет легко обобщить результаты на корреляционные функции более высокого порядка, в которых содержится более двух полевых операторов. В разделах 4.3 и 4.4 мы продолжим анализ корреляционных функций и разработаем в конце концов формализм фейнмановских диаграмм для вычисления этих функций по теории возмущений. Затем в разделах 4.5 и 4.6 мы научимся с помощью этой же техники вычислять сечения рассеяния и вероятности распадов.

Чтобы приступить к задаче, запишем гамильтониан теории ϕ^4 в виде:

$$H = H_0 + H_{\text{int}} = H_{\text{Клейн-Гордон}} + \int d^3x \frac{\lambda}{4!}\phi^4(\mathbf{x}).$$

Мы хотим получить выражение для двухточечной корреляционной функции (4.10) в виде ряда по степеням константы λ. Гамильтониан взаимодействия H_{int} встречается в (4.10) в двух местах: во-первых, в определении гейзенберговского поля

$$\phi(x) = e^{iHt}\phi(\mathbf{x})e^{-iHt},$$

и, во-вторых, в определении основного состояния $|\Omega\rangle$. Мы должны выразить как $\phi(x)$, так и $|\Omega\rangle$ через величины, с которыми мы знаем, как обращаться, а именно, через операторы свободного поля и вакуум в свободной теории $|0\rangle$.

† Историческую справку о вкладах Томонаги, Швингера, Фейнмана и Дайсона см. в Schweber (1994). Проще начать с $\phi(x)$. В любой фиксированный момент времени t_0 мы можем, как и прежде, разложить ϕ по лестничным операторам:

$$\phi(t_0,\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right)$$

Тогда, чтобы получить $\phi(t,\mathbf{x})$ при $t \neq t_0$, мы, как обычно, переходим в гейзенберговское представление:

$$\phi(t,\mathbf{x}) = e^{iH(t-t_0)}\phi(t_0,\mathbf{x})\,e^{-iH(t-t_0)}.$$

При $\lambda = 0$ гамильтониан H совпадает с H_0, и последняя формула сводится к

$$\phi(t,\mathbf{x})|_{\lambda=0} = e^{iH_0(t-t_0)}\phi(t_0,\mathbf{x})\,e^{-iH_0(t-t_0)} \equiv \phi_I(t,\mathbf{x}).$$

Если λ мало, то это выражение будет все еще определять самую важную часть временной зависимости поля $\phi(x)$, поэтому удобно присвоить этой величине название: поле в представлении взаимодействия $\phi_I(t,\mathbf{x})$. Так как мы можем диагонализовать H_0, легко выписать ϕ_I явно:

$$\phi_I(t,\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-ip\cdot x} + a^\dagger_{\mathbf{p}}e^{ip\cdot x}\right)\Bigg|_{x^0=t-t_0}$$

Это знакомое выражение для свободного поля из гл. 2.

Задача теперь состоит в том, чтобы выразить полное гейзенберговское поле ϕ в терминах ϕ_I. Формально

$$\begin{aligned}\phi(t,\mathbf{x}) &= e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\phi_I(t,\mathbf{x})e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \equiv \\ &\equiv U^\dagger(t,t_0)\,\phi_I(t,\mathbf{x})U(t,t_0)\end{aligned}$$

где мы определили унитарный оператор

$$U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)},$$

известный как пропагатор в представлении взаимодействия или оператор временной эволюции. Мы хотели бы выразить $U(t,t_0)$ полностью через поле ϕ_I, для которого существует явное выражение через лестничные операторы. Чтобы сделать это, заметим, что $U(t,t_0)$ - единственное решение с начальным условием $U(t_0,t_0) = 1$ обыкновенного дифференциального уравнения (уравнения Шредингера):

$$\begin{aligned}i\frac{\partial}{\partial t}U(t,t_0) &= e^{iH_0(t-t_0)}(H-H_0)\,e^{-iH(t-t_0)} = \\ &= e^{iH_0(t-t_0)}(H_{\text{int}})\,e^{-iH(t-t_0)} = \\ &= e^{iH_0(t-t_0)}(H_{\text{int}})\,e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}e^{-iH(t-t_0)} = \\ &= H_I(t)U(t,t_0),\end{aligned}$$

где

$$H_I(t) = e^{iH_0(t-t_0)}(H_{\text{int}})\,e^{-iH_0(t-t_0)} = \int d^3x\frac{\lambda}{4!}\phi_I^4$$

является гамильтонианом взаимодействия, записанным в представлении взаимодействия. Решение этого дифференциального уравнения для $U(t,t_0)$ должно иметь вид, похожий на выражение $U\sim\exp(-iH_It)$; это было бы искомой формулой для U через ϕ_I. Более аккуратно покажем, что настоящее решение является следующим степенным рядом по λ:

$$\begin{aligned}U(t,t_0) = 1 + (-i)\int_{t_0}^{t}dt_1H_I(t_1) + (-i)^2\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2H_I(t_1)\,H_I(t_2) + \\ +(-i)^3\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\int_{t_0}^{t_2}dt_3H_I(t_1)\,H_I(t_2)\,H_I(t_3) + \ldots\end{aligned}$$

Чтобы проверить это, вычислим производную по времени. При этом каждый член ряда дает предыдущий член, умноженный на $-iH_I(t)$. Очевидно, что начальное условие $U(t,t_0) = 1$ для $t = t_0$ Удовлетворено.

Обратим внимание, что различные множители H_I в (4.20) расположены в хронологическом порядке, т. е. отвечающий более позднему времени расположен всегда левее. Это позволяет существенно упростить выражение, используя символ хронологического упорядочения T. Например, слагаемое порядка H_I^2 записывается в виде

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) = \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 T\{H_I(t_1) H_I(t_2)\}$$

Двойной интеграл справа учитывает все дважды, так как в $t_1 t_2$-плоскости подынтегральное выражение $T\{H_I(t_1) H_I(t_2)\}$ симметрично относительно прямой $t_1 = t_2$ (см. рис. 4.1).

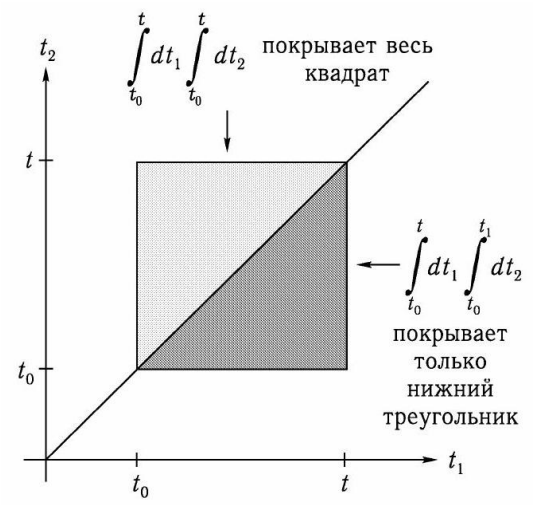


Рис. 4.1. Геометрическая интерпретация формулы (4.21).

Аналогичное тождество имеет место для высших слагаемых:

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \ldots \int_{t_0}^{t_{n-1}} dt_n H_I(t_1) \ldots H_I(t_n) = \frac{1}{n!} \int_{t_0}^{t} dt_1 \ldots dt_n T\{H_I(t_1) \ldots H_I(t_n)\}$$

Хотя этот случай несколько труднее изобразить наглядно, но нетрудно убедиться, что формула верна. С помощью последнего тождества можно записать $U(t, t_0)$ в чрезвычайно компактной форме:

$$U(t, t_0) = 1 + (-i) \int_{t_0}^{t} dt_1 H_I(t_1) + \frac{(-i)^2}{2!} \int_{t_0}^{t} dt_1 dt_2 T\{H_I(t_1) H_I(t_2)\} + \ldots \equiv$$

$$\equiv T\left\{\exp\left[-i \int_{t_0}^{t} dt' H_I(t')\right]\right\},$$

где хронологически упорядоченная экспонента определяется как ряд Тейлора, в котором каждый член хронологически упорядочен. В реальных вычислениях мы оставляем лишь несколько первых членов этого ряда; хронологически упорядоченная экспонента является лишь компактным способом записи и запоминания нужного выражения.

Теперь мы контролируем поле $\phi(t, \mathbf{x})$ в следующем смысле: мы записали его, как и хотели, полностью через поля ϕ_I. Прежде чем перейти к рассмотрению $|\Omega\rangle$, удобно обобщить определение оператора U, допуская, что его второй аргумент может принимать значения, отличающиеся от нашей «точки отсчета» t_0. Правильное определение вполне естественно:

$$U(t, t') \equiv T\left\{\exp\left[-i\int_{t'}^{t} dt'' H_I(t'')\right]\right\} \quad (t \geqslant t').$$

Проверим несколько свойств, непосредственно следующих из этого определения. Во-первых, $U(t, t')$ удовлетворяет тому же самому дифференциальному уравнению (4.18):

$$i\frac{\partial}{\partial t}U(t, t') = H_I(t)U(t, t'),$$

но теперь с начальным условием $U = 1$ для $t = t'$. Из этого уравнения следует:

$$U(t, t') = e^{iH_0(t-t_0)}e^{-iH(t-t')}e^{-iH_0(t'-t_0)},$$

что доказывает унитарность оператора U. Наконец, $U(t, t')$ удовлетворяет следующим тождествам (для $t_1 \geqslant t_2 \geqslant t_3$):

$$U(t_1, t_2)\,U(t_2, t_3) = U(t_1, t_3)\,; U(t_1, t_3)\,[U(t_2, t_3)]^{\dagger} = U(t_1, t_2).$$

Теперь мы можем перейти к обсуждению состояния $|\Omega\rangle$. Так как $|\Omega\rangle$ - основное состояние гамильтониана H, можно выделить его при помощи следующей процедуры. Допустим, что мы начинаем с $|0\rangle$, являющегося основным состоянием H_0, и рассматриваем его эволюцию во времени, определяющуюся полным гамильтонианом H:

$$e^{-iHT}|0\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|0\rangle,$$

где E_n - собственные значения H. Мы должны предположить, что $|\Omega\rangle$ перекрывается с $|0\rangle$, то есть $\langle\Omega|0\rangle \neq 0$ (если это не так, то H_I ни в каком смысле не может считаться малым возмущением). Тогда выписанный выше ряд содержит $|\Omega\rangle$, и мы можем записать:

$$e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle,$$

где $E_0 \equiv \langle\Omega|H|\Omega\rangle$. (Нулевая энергия определяется как $H_0|0\rangle = 0$.) Так как $E_n > E_0$ для всех $n \neq 0$, можно избавиться от всех слагаемых в разложении с $n \neq 0$, устремляя T к ∞ в направлении с малой мнимой добавкой в комплексной плоскости: $T \to \infty(1 - i\varepsilon)$. Тогда экспоненциальный множитель e^{-iE_nT} вымирает медленнее всего для $n = 0$:

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle.$$

Так как теперь T очень велико, можно сдвинуть его на малую константу:

$$\begin{aligned}
|\Omega\rangle &= \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-iH(T+t_0)}|0\rangle = \\
&= \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} e^{-iH(t_0-(-T))}e^{-iH_0(-T-t_0)}|0\rangle = \\
&= \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} U(t_0, -T)\,|0\rangle.
\end{aligned}$$

Во второй строке мы использовали равенство $H_0|0\rangle = 0$. Если отбросить с-числовой множитель, видим, что состояние $|\Omega\rangle$ может быть получено простой эволюцией состояния $|0\rangle$ во времени от $-T$ до t_0 при помощи оператора U. Точно так же можно представить состояние $\langle\Omega|$ как

$$\langle\Omega| = \lim_{T\to\infty(1-i\varepsilon)} \langle 0|U\left(T,t_0\right)\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}.$$

Соберем вместе отдельные слагаемые двухточечной корреляционной функции. На время предположим, что $x^0 > y^0 > t_0$. Тогда

$$\begin{aligned}\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = &\lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1}\langle 0|U\left(T,t_0\right)\times\\ &\times\left[U\left(x^0,t_0\right)\right]^\dagger\phi_I(x)U\left(x^0,t_0\right)\left[U\left(y^0,t_0\right)\right]^\dagger\phi_I(y)U\left(y^0,t_0\right)\times\\ &\times U\left(t_0,-T\right)|0\rangle\left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} =\\ =&\lim_{T\to\infty(1-i\varepsilon)}\left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1}\times\\ &\times\left\langle 0\left|U\left(T,x^0\right)\phi_I(x)U\left(x^0,y^0\right)\phi_I(y)U\left(y^0,-T\right)\right|0\right\rangle.\end{aligned}$$

Это выражение выглядело бы простым, если бы не громоздкий множитель впереди. Чтобы избавиться от него, разделим это выражение на 1, представленную в виде

$$1 = \langle\Omega|\Omega\rangle = \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1}\langle 0\left|U\left(T,t_0\right)U\left(t_0,-T\right)\right|0\rangle.$$

Тогда наша формула все еще для $x^0 > y^0$ принимает вид:

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)}\frac{\langle 0\left|U\left(T,x^0\right)\phi_I(x)U\left(x^0,y^0\right)\phi_I(y)U\left(y^0,-T\right)\right|0\rangle}{\langle 0|U(T,-T)|0\rangle}$$

Теперь обратим внимание на то, что все поля в обеих частях этого выражения упорядочены по времени. Это сохраняет силу и для случая $y^0 > x^0$. Таким образом, получаем окончательное выражение, справедливое теперь для любых x^0 и y^0:

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)1-i}\frac{\left\langle 0\left|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int\limits_{-T}^{T}dtH_I(t)\right]\right\}\right|0\right\rangle}{\left\langle 0\left|T\left\{\exp\left[-i\int\limits_{-T}^{T}dtH_I(t)\right]\right\}\right|0\right\rangle}$$

Преимущество рассмотрения хронологически упорядоченного произведения состоит в следующем. Это позволяет вставлять что угодно под знак общего оператора T-упорядочения. Подобная формула имеет место и для корреляционных функций более высокой степени от произвольного числа полей: для каждого дополнительного множителя ϕ слева поместим дополнительный множитель ϕ_I справа. Пока что полученное выражение остается точным. Но теперь оно идеально подходит для выполнения вычислений по теории возмущений. Нужно только сохранить столько членов, сколько требуется, в ряде Тейлора для экспоненты.

14.2 Basics of Feynman Diagrams

14.2.1 Connected Diagrams

Связной компонентой диаграммы мы назовем непрерывную цепочку внутренних линий. Если у диаграммы нет внутренних линий, то это также предельный случай связной компоненты с нулем внутренних линий. Если у диаграммы только одна связная компонента, то диаграмма называется связной.

Разложение в ряд по источникам для амплитуды $\tilde{Z}(j) = Z(j)/Z(0)$ начинается с единицы:

$$\tilde{Z}(j) = 1 + \sum_{k=1}^{\infty} \left(\frac{i}{\hbar}\right)^k \int \mathcal{D}^{(k)}_{1\ldots k}(x_1, \ldots, x_k) \frac{1}{k!} \prod_{k'=1}^{k} j(x_{k'}) \, \mathrm{d}^4 x_{k'}$$

В функциях Грина $\mathcal{D}$, как мы видели, есть несвязные вклады, что соответствует произведениям связных функций $\mathcal{D}^c[c = \text{connected}]$, которые уже не распадаются на произведения других связных функций. Для краткости обозначений запишем k -точечные связные вклады в виде

$$a_k = \left(\frac{\mathrm{i}}{\hbar}\right)^k \int \mathcal{D}^{c(k)}_{1\ldots k}(x_1, \ldots, x_k) \frac{1}{k!} \prod_{k'=1}^{k} j(x_{k'}) \, \mathrm{d}^4 x_{k'}$$

Тогда

$$\tilde{Z}(j) = 1 + \sum_{N=1}^{\infty} \prod_{k=1}^{\infty} \frac{1}{(n_k)!} a_k^{n_k} \Bigg|_{\sum n_k = N}$$

где N - число произведений связных вкладов, а n_k - количество вкладов с k -точечными связными функциями Грина, которые входят симметрично и, стало быть, допускают $(n_k)!$ перестановок, дающих одинаковые вклады, причем $\sum n_k = N$ Заметим теперь, что разложение экспоненты в ряд

$$\exp a_1 = 1 + \sum_{N=1}^{\infty} \prod_{k=1}^{1} \frac{1}{(n_1)!} a_1^{n_1} \Bigg|_{n_1 = N}$$

для двух слагаемых принимает вид

$$\exp\{a_1 + a_2\} = 1 + \sum_{N=1}^{\infty} \prod_{k=1}^{2} \frac{1}{(n_k)!} a_k^{n_k} \Bigg|_{\sum n_k = N}$$

и, как нетрудно сообразить,

$$\exp\left\{\sum_{k=1}^{\infty} a_k\right\} = 1 + \sum_{N=1}^{\infty} \prod_{k=1}^{\infty} \frac{1}{(n_k)!} a_k^{n_k} \Bigg|_{\sum n_k = N}$$

Таким образом, мы показали, что факторизация несвязных амплитуд в произведение связных действительно приводит к экспоненциированию связных амплитуд, и подстановка $Z(j) = \exp\{\mathrm{i}G(j)/\hbar\}$ - это переход к производящему функционалу связных функций Грина $G(j)$. Этот вывод справедлив и при наличии ненулевых вакуумных полей, поскольку вклад вакуума факторизован в $Z(0)$

14.2.2 Loop Expansion (????)

Теория по Киселеву (???)

Амплитуду квантовых переходов при наличии взаимодействий

$$\hat{Z}(j) = \int \mathcal{D}\varphi \exp\left\{\frac{\mathrm{i}}{\hbar}\left(S_0(\varphi) + j_{x'}\varphi_{x'}\right) + \frac{\mathrm{i}}{\hbar}S_{\mathrm{int}}(\varphi)\right\}$$

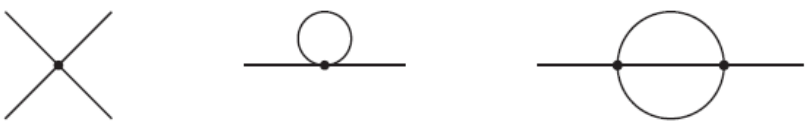
можно выразить с помощью действия оператора поля на функционал, построенный для свободных полей,

$$\hat{Z}(j) = \exp\left\{\frac{\mathrm{i}}{\hbar}S_{\mathrm{int}}\left(-\mathrm{i}\hbar\frac{\delta}{\delta j}\right)\right\}\int \mathcal{D}\varphi \exp\left\{\frac{\mathrm{i}}{\hbar}\left(S_0(\varphi) + j_{x'}\varphi_{x'}\right)\right\}$$

Этот функционал позволяет найти общее соотношение для числа петель,
степени возмущения и числа внутренних линий в связной диаграмме.
Действительно, если N_V - степень вклада возмущения в функцию Грина,
и значит, N_V - это число внутренних вершин в диаграмме,
то вклад вершин в число интегрирований по координатам равен N_V (vertex, number of vertecies).
Если в диаграмме есть внутренняя линия, то она соединяет 2 точки, т. е.
каждой внутренней линии отвечает одно интегрирование по координатам на конце внутренней линии,
так что при наличии N_I внутренних линий (internal lines) N_I интегрирований относятся именно к этим линиям,
и нужно добавить еще одну начальную точку, так что при последовательном счете,
если рассматривать связную цепочку внутренних линий,
имеется $N_I + 1$ число интегрирований в цепочке.
При этом, поскольку внутренняя линия соединяет именно вершины,
по сути мы полагаем, что в диаграмме столько же и вершин.
Однако если диаграмма связная, то некоторые начальные и конечные точки в цепочке внутренних линий могут совпасть.
Каждое такое совпадение означает наличие замкнутой петли.
Число петель N_L (loop, number of loops) определяет число точек посчитанных дважды, так что общее число интегрирований по внутренним вершинам или просто число вершин в действительности равно

$$N_V = N_I + 1 - N_L, \quad N_L = N_I + 1 - N_V$$

В справедливости данной формулы читатель может наглядно убедиться на примере четырех- и двухточечной функции Грина в теории «фи-в-четвертой», как это показано на рис.



Заметим теперь, что связные функции Грина $\mathrm{i}G(j)/\hbar$ с фиксированным числом петель получаются суммированием диаграмм с N_V степенями возмущения и N_I внутренними линиями, причем каждое возмущение дает фактор $1/\hbar$, а каждая внутренняя линия отвечает обратному оператору на квадратичном по траектории свободном действии,

т. е. вносит фактор $\hbar$(cm. связь связной двухточечной функции Грина со связной частью двухточечной функции Грина для амплитуды вероятности $Z(j)$).

В итоге связные функции Грина с петлями имеют следующий масштаб по $\hbar$:

$$\frac{1}{\hbar}G^{(L)} \sim \sum_{N_V,N_I} \left(\frac{1}{\hbar}\right)^{N_V} \hbar^{N_I}\Bigg|_{N_L \text{ fixed}} \sim \hbar^{N_L-1}$$

Таким образом, разложение связных функций Грина по петлям совпадает с разложением по степеням $\hbar$! Это означает, что в классическом пределе $\hbar \to 0$ среди связных диаграмм остаются только диаграммы без петель или, как говорят, древесные диаграммы.

Подчеркнем также, что полученные выше закономерности, связанные с числом петель, никак не зависят от того, как устроено возмущение (кубическое по полям или высшей степени по полям) и сколько внешних линий имеют связные функции Грина.

14.2.3 One-Particle Irreducible Diagrams (1PI)

Теория по Киселеву (???)

Эффективное действие $\Gamma(\varphi_{\mathrm{cl}})$, которое получается из производящего функционала связных функций Грина $G(j)$ посредством преобразования Лежандра, является производящим функционалом сильно связных (proper) функций Грина или, как говорят, вершин (vertex).

Под термином «сильно связный» понимают одночастичную неприводимость (one-particle irreducible): диаграмма остается связной даже после разрезания (усечения, удаления, ампутации) любой одной внутренней линии в этой диаграмме. Для доказательства этого утверждения воспользуемся методом Сиднея Коулмена и Стивена Вайнберга: построим производяший функционал функций Грина не с исходным действием S, а с эффективным действием Γ,

$$Z_\Gamma(j) = \exp\left\{\frac{\mathrm{i}}{\hbar_\Gamma}G_\Gamma(j)\right\} = \mathcal{N}\int \mathcal{D}\varphi \exp\left\{\frac{\mathrm{i}}{\hbar_\Gamma}\left(\Gamma(\varphi) + \varphi_{x'}j_{x'}\right)\right\}$$

где мы также провели формальную замену $\hbar \to \hbar_\Gamma$. Тогда $G_\Gamma(j)$ - это производящий функционал связных функций Грина с действием Γ. При этом, как мы установили, в пределе при $\hbar_\Gamma \to 0$ величина $G_\Gamma(j) \to G^{(0)}_\Gamma(j)$ не будет содержать петель, т. е. будет представлять собой, как говорят, лишь набор связных древесных диаграмм. Но в этом же пределе функциональный интеграл сводится к своему значению в стационарной точке:

$$\begin{aligned}\exp\left\{\frac{\mathrm{i}}{\hbar_\Gamma}G^{(0)}_\Gamma(j)\right\} &= \lim_{\hbar_\Gamma\to 0}\int \mathcal{D}\varphi \exp\left\{\frac{\mathrm{i}}{\hbar_\Gamma}\left(\Gamma(\varphi) + \varphi_{x'}j_{x'}\right)\right\} = \\ &= \exp\left\{\frac{\mathrm{i}}{\hbar_\Gamma}\left(\Gamma + \int \mathrm{d}^4x'\varphi\left(x'\right)j\left(x'\right)\right)\right\}\Bigg|_{\delta[\Gamma(\varphi)+\varphi_{x'}j_{x'}]/\delta\varphi(x)=0}\end{aligned}$$

Условие стационарности означает, что

$$\frac{\delta\Gamma}{\delta\varphi(x)} = -j(x), \quad G^{(0)}_\Gamma = \Gamma(\varphi) + \varphi_{x'}j_{x'}$$

т. е. предел производящего функционала в точности совпадает с производящим функционалом связных функций Грина в исходной теории:

$$G_{\Gamma}^{(0)} = G(j)$$

Таким образом, производящий функционал связных функций Грина соответствует связным древесным диаграммам с пропагаторами и вершинами, которые задаются эффективным действием Γ.

Остается справедливым соотношение

$$\Gamma_{13}^{(2)}\mathcal{G}_{32}^{(2)} = -\delta_{12}$$

так что связная двухточечная функция Грина $\mathcal{G}^{(2)}$ представляет собой двухточечную вершину $\Gamma^{(2)}$ с внешними линиями:

$$\mathrm{i}\ \mathcal{G}_{12}^{(2)} = \mathrm{i}\mathcal{G}_{13}^{(2)}\mathrm{i}\Gamma_{33'}^{(2)}\mathrm{i}\mathcal{G}_{3'2}^{(2)}$$

Более того, многократное повторение (вставка) блоков $\mathrm{i}\Gamma_{33'}^{(2)}\mathrm{i}\mathcal{G}_{3'2}^{(2)}$ в эту диаграмму сводится к единичному оператору, т. е. все цепочки с многократной вставкой двухточечной вершины идентичны древесной диаграмме с одной двухточечной вершинной функцией!

связные функции Грина в $G(j)$ для исходной теории с затравочным действием S являются в то же время диаграммами, построенными в древесном приближении в теории, где затравочным является уже эффективное действие Γ. Это значит, во-первых, что все такие древесные диаграммы, построенные по вершинам с эффективным действием, являются одночастично приводимыми: удаление любой внутренней линии делает диаграмму несвязной. Во-вторых, быть представлена как древесная, поскольку разложение такой вершины на сумму одночастично приводимых вкладов и одночастично неприводимый вклад означал бы, что вклад суммы одночастично приводимых диаграмм уже учтен прежде в виде такого же древесного вклада (см. илллюстрацию этого утверждения на примере 4-точки на рис.). Таким образом, по построению, если учтены все древесные вклады с одночастично неприводимыми вершинами низшего порядка, то вершины более высокого порядка уже не содержат одночастично приводимых вкладов. Отсюда следует, что все вершинные функции должны быть одночастично неприводимыми.

Покажем теперь, что вершинные функции - это усеченные сильносвязные функции Грина (одночастично неприводимые функции Грина с «ампутированными ногами»). В самом деле, поскольку

$$G(j) = \Gamma\left(\varphi_{\mathrm{cl}}\right) - \int \mathrm{d}^4x\varphi_{\mathrm{cl}}(x)\frac{\delta\Gamma}{\delta\varphi_{\mathrm{cl}}(x)}$$

мы видим, что первый член справа - эффективное действие Γ - задает все многоточечные одночастично неприводимые связные функции Грина, а во втором члене уже содержатся одночастично приводимые связные диаграммы. Особый случай - двухточечная связная функция Грина, поскольку это всего одна линия. Значит, для n -точечных функций

$$\Gamma_{1\ldots n}^{(n)}\varphi_{\mathrm{cl}}^{(1)}\cdots\varphi_{\mathrm{cl}}^{(n)} \cong \mathcal{G}_{1\ldots n}^{p(n)}j_1\ldots j_n, \qquad \text{высшие вершины } n > 2$$

где справа уже содержатся только сильно связные диаграммы (proper), за исключением двухточечной связной функции Грина, которая, с точностью до знака, является обратной к двухточечной вершине. Тогда вариационное дифференцирование источников по φ_{cl} при нулевых значениях этих внешних

полей справа дает

$$\Gamma^{(n)}_{1\ldots n} = \frac{\delta^n G^p}{\delta j_{1'}\ldots j_{n'}} \frac{\delta j_{1'}}{\delta\varphi_{\rm cl}(1)}\ldots\frac{\delta j_{n'}}{\delta\varphi_{\rm cl}(n)}, \quad n>2$$

Но вариационные производные при нулевых внешних классических полях

$$\frac{\delta j_{k'}}{\delta\varphi_{\rm cl}(k)} = -\frac{\delta^2\Gamma}{\delta\varphi_{\rm cl}(k)\delta\varphi_{\rm cl}\,(k')} = \left(\mathcal{G}^{(2)}_{kk'}\right)^{-1}$$

так что

$$\Gamma^{(n)}_{1\ldots n} = \frac{\delta^n G^{1-p}}{\delta j_{1'}\ldots j_{n'}} \prod_{k=1}^{n}\left(\mathcal{G}^{(2)}_{kk'}\right)^{-1}, \quad n>2$$

Это и означает, что вершины представляют собой связные одночастично неприводимые диаграммы, в которых внешние линии ампутированы, т. е. пропагаторы этих линий удалены за счет домножения на обратный двухточечный связный пропагатор.

В принципе, эта процедура срабатывает и с двухточечным эффективным действием: нужно взять двухточечную связную функции Грина и домножить ее дважы, по числу внешних точек, на обратную функцию, но добавить еще и множитель, равный -1. Об этом приходится помнить!

Если вариационные производные $Z(j)$ мы условились изображать в виде линий, исходящих из овала, а овал можно затем представить в виде сумм диаграмм в теории возмущений с распространением частиц вдоль тех же линий, то в той же логике можно условиться изображать вариационные производные $G(j)$ в виде линий, исходяших из круга, где круг можно затем представить

в виде сумм связных диаграмм с вершинами в теории возмущений и с распространением частиц вдоль линий, пересекающихся в вершинах. Тогда принято изображать вариационные производные эффективного действия Γ в виде круга с исходящими из него перечеркнутыми линиями, которые

символизируют усеченные внешние пропагаторы)амплитудные ноги)

как это показано на рис.

(рисунки снова)

14.2.4 Reduction Formulas

Теория по Киселеву (???)

Для того, чтобы получить из производящего функционала $\angle(j)$ S-матрицу

необходимо выразить внешние токи j через внешние поля в асимптотической области.

Но в силу

$$j(x) = -\frac{\delta\Gamma}{\delta\varphi_{\rm cl}(x)}, \quad \left.\frac{\delta\Gamma}{\delta\varphi_{\rm cl}(x)}\right|_{\varphi_{\rm cl}=0} = 0$$

в самом общем случае можно записать, что разложение тока j в вариационный ряд начинается со второго члена,

$$j(x) = -\int \frac{\delta^2\Gamma}{\delta\varphi_{\rm cl}(x)\delta\varphi_{\rm cl}(y)}\varphi_{\rm cl}(y)\mathrm{d}^4y + \ldots = \left(\mathcal{G}^{(2)}_{xy}\right)^{-1}\varphi^y_{\rm cl} + \ldots$$

Но в асимптотической области внешние поля стремятся к полям, которые, как мы помним, не содержат флуктуаций, т. е. удовлетворяют классическим уравнениям, а значит, все высшие вклады в (5.9) пренебрежимо малы, и

$$j(x) \mapsto \left(\mathcal{G}^{(2)}_{xy}\right)^{-1} \varphi^{y}_{\mathrm{cl}}$$

а вариаационный ряд для $Z(j)$ преобразуется в вариационный ряд $\mathcal{S}$ -матрицы:

$$\mathcal{D}^{(n)}_{1\ldots n} j_1 \ldots j_n \mapsto \mathcal{D}^{(n)}_{1\ldots n} \prod_{k=1}^{n} \left(\mathcal{G}^{(2)}_{kk'}\right)^{-1} \varphi^{(k')}_{\mathrm{cl}}$$

где асимптотически свободные поля. Эти поля были сгенерированы источниками в обобщенном смысле этого слова: источником можно считать как устройство, которое производит начальное состояние поля, так и детектор, который производит отсчет попавших в него частиц вместе с характеристиками этих частиц, т. е. регистрирует конечное состояние.

Вариационные n -точечные функции $\mathcal{S}$ -матрицы связаны с n -точечными причинными корреляторами редукционными формулами:

$$\mathcal{S}^{(n)}_{1'\ldots n'} = \mathcal{D}^{(n)}_{1\ldots n} \prod_{k=1}^{n} \left(\mathcal{G}^{(2)}_{kk'}\right)^{-1}$$

и мы опять имеем дело с усеченными диаграммами. Для расчета амплитуды вероятности квантового перехода с заданными асимптотическими внешними полями необходимо вычислить величину

$$\frac{1}{n!}\left(\frac{\mathrm{i}}{\hbar}\right)^{n} \mathcal{D}^{(n)}_{1\ldots n} \prod_{k=1}^{n} \left(\mathcal{G}^{(2)}_{kk'}\right)^{-1} \varphi^{(k')}_{\mathrm{cl}}$$

с учетом возможных перестановок тождественных частиц в начальном и конечном состояниях.

Примеры таких расчетов с переходом в импульсное представление мы уже приводили выше, например, для распада скаляра на частицу и античастицу и для аннигиляции скалярных безмассовых частиц в частицу и античастицу другого сорта.

14.2.5 Feynman Rules by Kiselev

Теория по Киселеву (???)

Резюмируя проведенное нами рассмотрение, мы можем теперь сформулировать правила Фейнмана для вычисления амплитуд квантовых переходов

в теории со скалярными полями при заданных начальных и конечных состояниях. Эти правила зависят, конечно, от исходного (bare) лагранжиана, что подразумевает определение состава полей и их взаимодействий. Поэтому опишем типовую модель с нейтральным (вещественным) полем φ и заряженным (комплексным) полем Φ, так что лагранжиан свободных полей имеет стандартный вид

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu\varphi\partial^\mu\varphi - M^2\varphi^2\right) + \partial_\mu\Phi^\dagger\partial^\mu\Phi - m^2\Phi^\dagger\Phi$$

где M -масса нейтрального поля, а m - масса заряженного поля, а взаимодействия включают в себя

$$\mathcal{L}_{\text{int}} = -\Lambda\varphi\Phi^\dagger\Phi - \frac{\lambda}{4}\left(\Phi^\dagger\Phi\right)^2$$

так что Λ - размерная постоянная юкавского взаимодействия полей, а λ константа самодействия заряженного поля.

В импульсном представлении амплитуда квантового перехода строится по диаграмме с учетом закона сохранения 4-импульса, которые приписываются всем линиям,

$$\mathrm{i}\mathcal{M} = (2\pi)^4\delta\left(\sum p\right)\mathrm{i}\mathfrak{M}$$

причем в $\mathfrak{M}$ внутренним линиям отвечают следующие пропагаторы: вещественное поля φ - штриховая линия -

(вот тут хороший список правил Фейнмана!!!!)

$$\frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}0}, \quad - - - - -$$

комплексное поле Φ - сплошная линия со стрелкой по направлению движения заряда -

$$\frac{\mathrm{i}}{k^2 - m^2 + \mathrm{i}0},$$

Вершина Юкавского взаимодействия:

....

вершина самодействия заряженного поля

...

Вершина, как мы убедились, получаются из затравочного действия путем дифференцирования лагранжиана $\mathcal{L}_{\text{int}}$ по полям и домножением на мнимую единицу. Внешним линиям в конечном состоянии отвечают следующие моды:

заряженная частица - отрицательно частотная мода поля $\hat{\Phi}^\dagger$,

заряженная античастица - отрицательно частотная мода поля $\hat{\Phi}$,

нейтральная частица - отрицательно частотная мода поля $\hat{\varphi}$.

Внешним линиям в начальном состоянии отвечают следующие моды: заряженная античастица - положительно частотная мода поля $\hat{\Phi}^\dagger$, заряженная частица - положительно частотная мода поля $\hat{\Phi}$, нейтральная частица - положительно частотная мода поля $\hat{\varphi}$. Для скалярных частиц в принятой релятивистски ковариантной нормировке состояний все эти моды в координатном представлении отвечают волнам де Бройля с соответствующей прескрипцией для импульсов и энергий в начальных и конечных состояниях, чтобы выполнялся закон сохранения 4 импульса. В импульсном же представлении эти моды оказываются равными 1. Однако понимание того, как они были получены, упростит нам задачу в случае полей с ненулевым спином, когда моды - это многокомпонентные строки или столбцы, а не единицы.

При наличии в диаграмме петель закон сохранения 4-импульса приводит к дополнительному интегрированию по 4-импульсу, который может принимать произвольные значения,

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$$

для каждой петли. Подчеркнем, что при наличии тождественных полей необходимо учесть симметрийный фактор: для внешних полей провести симметризацию по тождественным

частицам и сосчитать варианты прескрипции 4-импульсов этим полям, для внутренних линий сосчитать число графических вариантов возникновения одних и тех же диаграмм по заданному набору линий и вершин. Пример расчета симметрийного фактора мы видели в теории кубического самодействия нейтрального поля.

14.2.6 Coupling Constant Expansion by Vergeles (!!!!)

(вроде у него круто это написано!)
(!!!! в 1ю часть перенесу многое!!!)

Теория

Рассмотрим случай потенциала самодействия вида $V(\phi) = \frac{1}{4!}\lambda_0\phi^4$ и разложим первую экспоненту в выражении

$$S(a^*, a) = S(\phi_0) = \frac{1}{N}\exp\left\{-i\int V\left(\frac{1}{i}\frac{\delta}{\delta\eta(x)}\right)\mathrm{d}x\right\}\times \\ \times\exp\left\{i\int \mathrm{d}x\eta(x)\phi_0(x) + \frac{i}{2}\iint \mathrm{d}x\ \mathrm{d}y\eta(x)D_c(x-y)\eta(y)\right\}\Bigg|_{\eta=0}.$$

в ряд по потенциалу V.

Члены этого ряда содержат произведения четырёхкратных дифференцирований по $\eta(x_i)$ с последующим интегрированием по x_i. Кратное применение операторов $\delta/\delta\eta(x_i)$ к $S_0(\phi_0;\eta)$ и последующее обращение в нуль поля η приводит к появлению в S-матрице множителей $\phi_0(x_i)$ и $D_c(x_i - x_j)$, причём последняя функция соответствует дифференциальному оператору второго порядка $(-i)\delta^2/\delta\eta(x_i)\,\delta\eta(x_j)$.

(?? выше не совсем еще понимаю.)

Изобразим графически каждый член этого разложения. Эти графики называются диаграммами Фейнмана.

Члену разложения n-ого порядка, содержащему дифференциальный оператор

$$\frac{(-i)^n}{n!}\int \mathrm{d}x_1...\mathrm{d}x_n V\left(\frac{1}{i}\frac{\delta}{\delta\eta(x_1)}\right)...V\left(\frac{1}{i}\frac{\delta}{\delta\eta(x_n)}\right),$$

сопоставляется диаграмма, состоящая из n вершин в точках $x_i, i = 1, ..., n$, находящихся под n-кратным интегралом по пространству-времени.

В рассматриваемом случае $V(\phi) = \frac{1}{4!}\lambda_0\phi^4$ из каждой вершины выходит 4 линии (в теории с потенциалом $\lambda\phi^s$ из каждой вершины выходит s линий). Эти линии могут либо замыкаться, соединяя пару вершин (такие линии называются внутренними), либо выходить из диаграммы, соответствуя полю $\phi_0(x_i)$ (такие линии - внешние).

Каждой внутренней линии, соединяющей точки x_i и x_j, сопоставляется функция $D_c(x_i - x_j)$.

Каждой внешней линии, выходящей из точки x_i, сопоставляется поле $\phi_0(x_i)$.

По всем точкам $\{x_i\}$ производится пространственно-временное интегрирование.

Полное выражение для $S(\phi_0)$ получается суммированием по всем диаграммам.

Приступим к конкретным вычислениям. Представим S-матрицу в виде

$$S = 1 + iT.$$

Пусть начальное и конечное состояния имеют вид

$$|\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_n\rangle = \prod_{i=1}^{n} \left(\sqrt{2\omega_{k_i}} a^\dagger_{\mathbf{k}_i}\right)|0\rangle$$

обозначим через $|i\rangle$ и $|f\rangle$ начальное и конечное состояния, соответственно:

$$|i\rangle = |\mathbf{k}_1, \mathbf{k}_2, ...\rangle\,, \quad |f\rangle = |\mathbf{p}_1, \mathbf{p}_2, ...\rangle\,.$$

Вследствие общей трансляционной инвариантности теории справедлива формула

$$iT_{fi} \equiv \langle \mathbf{p}_1, \mathbf{p}_2, ...|iT|\mathbf{k}_1, \mathbf{k}_2, ...\rangle = (2\pi)^d \delta^{(d)}\left(P_f - P_i\right) iM_{fi}$$

Здесь $P_i = k_1 + k_2 + ...$ - суммарный начальный и $P_f = p_1 + p_2 + ...$ - суммарный конечный 4-импульсы. Введенная выше величина M_{fi} называется амплитудой рассеяния. Информация о взаимодействии содержится в T-матрице и в амплитуде M_{fi}, через которую выражаются сечения рассеяния, вероятности распадов частиц, и т.п. (в зависимости от постановки задачи). Из общих принципов квантовой механики следует, что эти величины квадратичны по M_{fi}.

Вычислим S-матрицу

$$S\left(\phi_0\right) = \sum_n \frac{1}{n!} \int S_n\left(x_1, ..., x_n\right) : \phi_0\left(x_1\right) ... \phi_0\left(x_n\right) : \mathrm{d}x_1 ... \mathrm{d}x_n$$

$$\phi_0(x) = \int \frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}} \frac{1}{\sqrt{2\omega_k}} \left(a_{\mathbf{k}} e^{-ikx} + a^\dagger_{\mathbf{k}} e^{ikx}\right), \quad k^0 = \omega_k$$

с точностью до второго порядка по λ_0 и четвёртого по ϕ_0. Будем пользоваться формулами

$$S\left(a^*, a\right) = \frac{1}{N} \exp\left\{i \int \mathrm{d}^{(d)}x \eta(x)\phi_0(x) + \frac{i}{2} \iint \mathrm{d}^{(d)}x\ \mathrm{d}^{(d)}y \eta(x) D_c(x-y)\eta(y)\right\} \equiv S_0\left(\phi_0; \eta\right).$$

и

$$\frac{1}{i}\frac{\delta}{\delta\eta\left(x_i\right)} S_0\left(\phi_0; \eta\right) =: \left[\phi_0\left(x_i\right) + \int \mathrm{d}y D_c\left(x_i - y\right)\eta(y)\right] S_0\left(\phi_0; \eta\right) :\equiv: \left[x_i\right] S_0\left(\phi_0; \eta\right) :$$

$$\frac{1}{i}\frac{\delta}{\delta\eta\left(x_j\right)}\left[x_i\right] = -iD_c\left(x_i - x_j\right).$$

В первом порядке по λ_0 находим (нормировочную константу $1/N$ можно опускать, так как она всегда сокращается в окончательном результате):

$$-i\frac{\lambda_0}{4!} \int \mathrm{d}x : \left(\frac{1}{i}\frac{\delta}{\delta\eta(x)}\right)^4 S_0\left(\phi_0; \eta\right) : |_{\eta=0} = -i\frac{\lambda_0}{4!} \int \mathrm{d}x : \left\{-3D_c^2(0) - 6iD_c(0)\phi_0^2(x) + \phi_0^4(x)\right\} :$$

Во втором порядке следует вычислить величину

$$-\frac{1}{2}\left(\frac{\lambda_0}{4!}\right)^2 \int \mathrm{d}x_1\ \mathrm{d}x_2 : \left(\frac{1}{i}\frac{\delta}{\delta\eta\left(x_2\right)}\right)^4 \left(\frac{1}{i}\frac{\delta}{\delta\eta\left(x_1\right)}\right)^4 S_0\left(\phi_0; \eta\right) : |_{\eta=0} =$$

$$= -\frac{1}{2}\left(\frac{\lambda_0}{4!}\right)^2 \int \mathrm{d}x_1\ \mathrm{d}x_2 \left(\frac{1}{i}\frac{\delta}{\delta\eta\left(x_2\right)}\right)^4 : \left\{-3D_c^2(0) - 6iD_c(0)\left[x_1\right]^2 + \left[x_1\right]^4\right\} S_0\left(\phi_0; \eta\right) : |_{\eta=0}.$$

Далее целесообразно использовать формулу кратного дифференцирования

$$(fg)'''' = f''''g + 4f'''g' + 6f''g'' + 4f'g''' + fg''''$$

подразумевая под f - фигурную скобку в последнем равенстве, а под g - оператор $S_0(\phi_0;\eta)$. Таким образом находим искомую величину:

$$\frac{1}{2}\left(\frac{\lambda_0}{4!}\right)^2 \int \mathrm{d}x_1\ \mathrm{d}x_2 : \Big\{ -\left[9D_c^4(0) + 72D_c^2(0)D_c^2(x_1-x_2) + 24D_c^4(x_1-x_2)\right] -$$
$$-iD_c(x_1-x_2)\left[96D_c^2(x_1-x_2) + 144D_c^2(0)\right]\phi_0(x_1)\,\phi_0(x_2) -$$
$$-iD_c(0)\left[72D_c^2(x_1-x_2) + 18D_c^2(0)\right]\left(\phi_0^2(x_1)+\phi_0^2(x_2)\right) + \left[72D_c^2(x_1-x_2) + 36D_c^2(0)\right]\phi_0^2(x_1)\,\phi_0^2(x_2) +$$
$$+48D_c(0)D_c(x_1-x_2)\left[\phi_0(x_1)\,\phi_0^3(x_2) + \phi_0^3(x_1)\,\phi_0(x_2)\right] + 3D_c^2(0)\left[\phi_0^4(x_1)+\phi_0^4(x_2)\right]\Big\} :$$

При помощи теоремы Вика, а также разложения для ϕ_0 и определения векторов состояния находим следующие значения для элементов S-матрицы в случае перехода одночастичного состояния $|\mathbf{k}\rangle$ в одночастичное состояние $|\mathbf{p}\rangle$ (заметим, что все частоты $\omega_{\mathbf{k}}$, содержащиеся в формулах, определяющих вектора-состояния, и в лоренц-инвариантной мере, взаимно сокращаются):

(?? про это потом подумаю, пока что не вижу это.)

В нулевом приближении по λ_0:

$$\langle \mathbf{p}|S|\mathbf{k}\rangle = 2\omega_k(2\pi)^{d-1}\delta^{(d-1)}(\mathbf{k}-\mathbf{p})$$

(нет взаимодействия).

В первом порядке по λ_0:

$$2\omega_k(2\pi)^{d-1}\delta^{(d-1)}(\mathbf{k}-\mathbf{p})\int \mathrm{d}^{(d)}x\left[\frac{i\lambda_0}{8}D_c^2(0)\right] + (2\pi)^d\delta^{(d)}(k-p)\left[-\frac{\lambda_0}{2}D_c(0)\right].$$

Во втором порядке по λ_0:

$$2\omega_k(2\pi)^{d-1}\delta^{(d-1)}(\mathbf{k}-\mathbf{p})\int \mathrm{d}^{(d)}x_1\ \mathrm{d}^{(d)}x_2\left[-\frac{\lambda_0^2}{2^7}D_c^4(0) - \frac{\lambda_0^2}{2^4}D_c^2(0)D_c^2(x_1-x_2) - \frac{\lambda_0^2}{2^4\cdot 3}D_c^4(x_1-x_2)\right] -$$
$$-i\frac{\lambda_0^2}{6}\int \mathrm{d}^{(d)}x_1\ \mathrm{d}^{(d)}x_2 D_c^3(x_1-x_2)\,e^{ipx_1-ikx_2} - i\frac{\lambda_0^2}{4}\int \mathrm{d}^{(d)}x_1\ \mathrm{d}^{(d)}x_2 D_c^2(0)D_c(x_1-x_2)\,e^{ipx_1-ikx_2} -$$
$$-i\frac{\lambda_0^2}{4}\int \mathrm{d}^{(d)}x_1\ \mathrm{d}^{(d)}x_2 D_c(0)D_c^2(x_1-x_2)\,e^{i(p-k)x_1} - i\frac{\lambda_0^2}{16}\int \mathrm{d}^{(d)}x_1\ \mathrm{d}^{(d)}x_2 D_c^3(0)e^{i(p-k)x_1}.$$

Согласно $(\partial_\mu\partial^\mu + m^2)\,D_c(x) = \delta^{(d)}(x)$,

$$D_c(x) = \int \frac{\mathrm{d}^{(d)}k}{(2\pi)^d}D_c(k)e^{-ikx}, \quad D_c(k) = \frac{1}{-k^2+m_0^2-i\varepsilon}, \quad \varepsilon\to+0.$$

В частности, $D_c(0) = \int \frac{\mathrm{d}^{(d)}k}{(2\pi)^d}D_c(k)$.

Преобразуем некоторые интегралы при помощи этого:

$$\int \mathrm{d}^{(d)}x_1\ \mathrm{d}^{(d)}x_2 D_c^2\left(x_1-x_2\right)=\int \mathrm{d}^{(d)}x\int\frac{\mathrm{d}^{(d)}k}{(2\pi)^d}D_c^2(k),$$

$$\int \mathrm{d}^{(d)}x_1\ \mathrm{d}^{(d)}x_2 D_c^4\left(x_1-x_2\right)=\int \mathrm{d}^{(d)}x\int\frac{\mathrm{d}^{(d)}k_1}{(2\pi)^d}\frac{\mathrm{d}^{(d)}k_2}{(2\pi)^d}\frac{\mathrm{d}^{(d)}k_3}{(2\pi)^d}D_c\left(k_1\right)D_c\left(k_2\right)D_c\left(k_3\right)D_c\left(k_1+k_2+k_3\right),$$

$$\int \mathrm{d}^{(d)}x_1\ \mathrm{d}^{(d)}x_2 D_c^3\left(x_1-x_2\right)e^{ipx_1-ikx_2}=(2\pi)^d\delta^{(d)}(k-p)\int\frac{\mathrm{d}^{(d)}k_1\mathrm{d}^{(d)}k_2}{(2\pi)^{2d}}D_c\left(k_1\right)D_c\left(k_2\right)D_c\left(k-k_1-k_2\right),$$

$$\int \mathrm{d}^{(d)}x_1\ \mathrm{d}^{(d)}x_2 D_c\left(x_1-x_2\right)e^{ipx_1-ikx_2}=(2\pi)^d\delta^{(d)}(k-p)D_c(k),$$

$$\int \mathrm{d}^{(d)}x_1\ \mathrm{d}^{(d)}x_2 e^{i(p-k)x_1}D_c^2\left(x_1-x_2\right)=(2\pi)^d\delta^{(d)}(k-p)\int\frac{\mathrm{d}^{(d)}k}{(2\pi)^d}D_c(k)D_c(-k),$$

Теперь при помощи выписанных формул вычислим вклад в амплитуду перехода $|\mathbf{k}\rangle\to|\mathbf{p}\rangle$, содержащийся в сумме членов (8)-(10), с точностью $O\left(\lambda_0^2\right)$:

$$2\omega_k(2\pi)^{d-1}\delta^{(d-1)}(\mathbf{k}-\mathbf{p})\left\{1+\frac{i\lambda_0}{8}\left(\int\frac{\mathrm{d}^{(d)}q}{(2\pi)^d}D_c(q)\right)^2\int\mathrm{d}^dx-\right.$$

$$-\lambda_0^2\left[\frac{1}{2^7}\left(\int\frac{\mathrm{d}^{(d)}q}{(2\pi)^d}D_c(q)\right)^4\int\mathrm{d}^{(d)}x_1\int\mathrm{d}^{(d)}x_2+\frac{1}{2^4}\int\mathrm{d}^{(d)}x\left(\int\frac{\mathrm{d}^{(d)}k}{(2\pi)^d}D_c(k)\right)^2\int\frac{\mathrm{d}^{(d)}q}{(2\pi)^d}D_c(q)D_c(-q)+\right.$$

$$\left.\left.+\frac{1}{2^43}\int\mathrm{d}^{(d)}x\int\frac{\mathrm{d}^{(d)}k_1}{(2\pi)^d}\frac{\mathrm{d}^{(d)}k_2}{(2\pi)^d}\frac{\mathrm{d}^{(d)}k_3}{(2\pi)^d}D_c\left(k_1\right)D_c\left(k_2\right)D_c\left(k_3\right)D_c\left(k_1+k_2+k_3\right)\right]\right\}+$$

$$+(2\pi)^d\delta^{(d)}(k-p)\left\{-\frac{\lambda_0}{2}\int\frac{\mathrm{d}^{(d)}q}{(2\pi)^d}D_c(q)-i\lambda_0^2\left[\frac{1}{6}\int\frac{\mathrm{d}^{(d)}q_1}{(2\pi)^d}\frac{\mathrm{d}^{(d)}q_2}{(2\pi)^d}D_c\left(q_1\right)D_c\left(q_2\right)D_c\left(k-q_1-q_2\right)+\right.\right.$$

$$\left.\left.+\frac{1}{16}\int\mathrm{d}x\left(\int\frac{\mathrm{d}^{(d)}q}{(2\pi)^d}D_c(q)\right)^3\right]\right\}$$

2. Диаграммы Фейнмана и правила Фейнмана

Аналитическому выражению

$$2\omega_k(2\pi)^{d-1}\delta^{(d-1)}(\mathbf{k}-\mathbf{p})\left\{1+\frac{i\lambda_0}{8}\left(\int\frac{\mathrm{d}^{(d)}q}{(2\pi)^d}D_c(q)\right)^2\int\mathrm{d}^dx-\right.$$

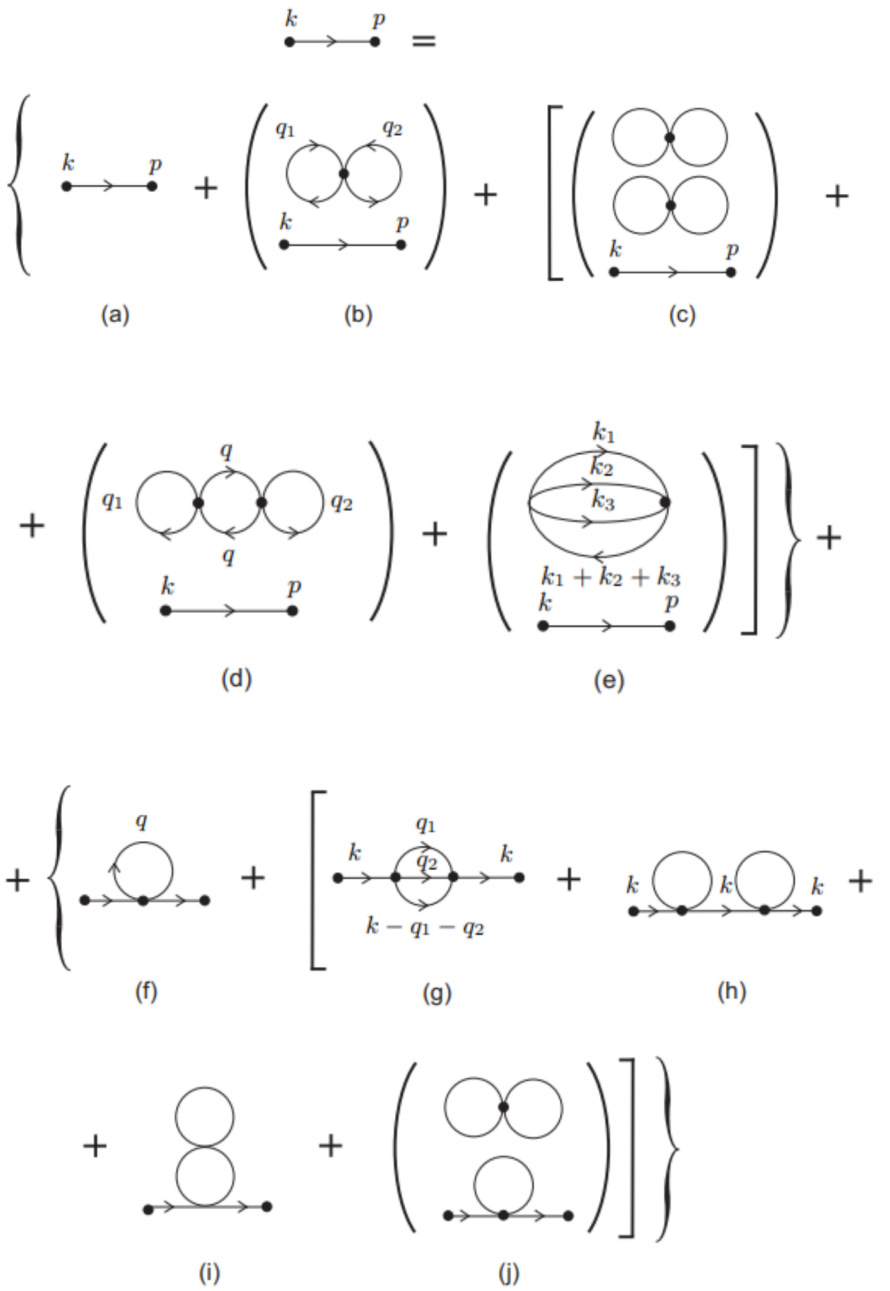
$$-\lambda_0^2\left[\frac{1}{2^7}\left(\int\frac{\mathrm{d}^{(d)}q}{(2\pi)^d}D_c(q)\right)^4\int\mathrm{d}^{(d)}x_1\int\mathrm{d}^{(d)}x_2+\frac{1}{2^4}\int\mathrm{d}^{(d)}x\left(\int\frac{\mathrm{d}^{(d)}k}{(2\pi)^d}D_c(k)\right)^2\int\frac{\mathrm{d}^{(d)}q}{(2\pi)^d}D_c(q)D_c(-q)+\right.$$

$$\left.\left.+\frac{1}{2^43}\int\mathrm{d}^{(d)}x\int\frac{\mathrm{d}^{(d)}k_1}{(2\pi)^d}\frac{\mathrm{d}^{(d)}k_2}{(2\pi)^d}\frac{\mathrm{d}^{(d)}k_3}{(2\pi)^d}D_c\left(k_1\right)D_c\left(k_2\right)D_c\left(k_3\right)D_c\left(k_1+k_2+k_3\right)\right]\right\}+$$

$$+(2\pi)^d\delta^{(d)}(k-p)\left\{-\frac{\lambda_0}{2}\int\frac{\mathrm{d}^{(d)}q}{(2\pi)^d}D_c(q)-i\lambda_0^2\left[\frac{1}{6}\int\frac{\mathrm{d}^{(d)}q_1}{(2\pi)^d}\frac{\mathrm{d}^{(d)}q_2}{(2\pi)^d}D_c\left(q_1\right)D_c\left(q_2\right)D_c\left(k-q_1-q_2\right)+\right.\right.$$

$$\left.\left.+\frac{1}{16}\int\mathrm{d}x\left(\int\frac{\mathrm{d}^{(d)}q}{(2\pi)^d}D_c(q)\right)^3\right]\right\}$$

соответствует следующая сумма диаграмм Фейнмана на рисунке ниже в том же порядке:



Сформулируем правила Фейнмана применительно к рассматриваемой модели.

Внутренним и внешним линиям приписывается D-импульс, а свободным линиям - два D-импульса (по одному каждому концу), причём изменение направления линий влечёт изменение знака соответствующих D-импульсов. Внутренние линии называются пропагаторами.

Сформулируем правила Фейнмана для вычисления амплитуды перехода iM, сопоставляющие диаграммам аналитические выражения:

0. каждому свободному отрезку (см. рис. a)

(отрезок)

$$= (2\pi)^{d-1}\delta^{(d-1)}(\mathbf{k}-\mathbf{p}),\ \text{где } k \text{ и } p - d\text{-импульсы концов отрезка;}$$

1. каждому пропагатору (тупо линия) $\longrightarrow \vec{k} = -iD_c(k)$, где $k - d$-импульс пропагатора;

2. каждой вершине
точка $= -i\lambda$
3. каждой внешней линии
$\longrightarrow \quad = 1$
4. потребовать выполнения закона сохранения d-импульса в каждой вершине;
5. проинтегрировать по каждому не зафиксированному законом сохранения импульсу: $\int \frac{\mathrm{d}^{(d)}p}{(2\pi)^d}$;
6. разделить на множитель D_n - порядок группы симметрии диаграммы;
7. каждый связный вакуумный пузырёк (как "восьмёрка"на рис. 3b) содержит множитель $\left(\int \mathrm{d}^{(d)}x\right)$.

Все интегрирования интерпретируются как следствие принципа суперпозиции, согласно которому амплитуды суммируются.

Правило 0 имеет значение лишь для тривиальной части амплитуды рассеяния. Важная для физики частиц амплитуда M (см. (4)) содержится во второй фигурной скобке в (13), а также во второй фигурной скобке на на рис. 3. Правило 7 имеет значение, как мы увидим, лишь для амплитуды перехода из вакуума в вакуум.

Укажем порядки групп симметрии D_n диаграмм на рис. 3:

(b) $D_n = 8$ - по две перестановки каждого из концов двух пропагаторов и две перестановки самих пропагаторов $(2 \cdot 2 \cdot 2 = 8)$;

(c) $D_n = 2 \cdot 8 \cdot 8 = 2^7$;

(d) $D_n = 2 \cdot 2 \cdot 2 \cdot 2 = 2^4$;

(e) $D_n = 2 \cdot 4!$ - две перестановки вершин и 4 ! перестановки пропагаторов вакуумного пузырька;

(f) $D_n = 2$ - перестановка концов пропагатора;

(g) $D_n = 3$! - число перестановок пропагаторов;

(h) $D_n = 2 \cdot 2 = 4$;

(i) $D_n = 2 \cdot 2 = 4$ - две перестановки пропагаторов в нижней петле и две перестановки концов пропагатора в верхней петле;

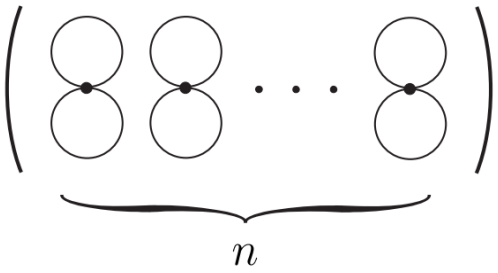
(j) $D_n = 8 \cdot 2 = D'_n \cdot 2 = 16 - D'_n = 8$ для вакуумного пузырька и две перестановки концов пропагатора.

В случае сомнения при вычислении D_n следует непосредственно вычислить коэффициент при диаграмме.

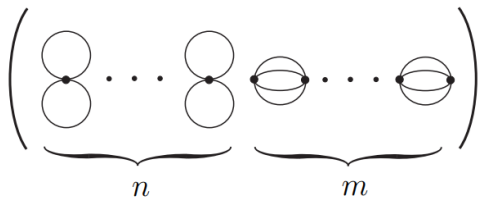
Нетрудно понять происхождение правил Фейнмана. Действительно, каждой вершине сопоставляется множитель $\int \mathrm{d}^{(d)}x(-i\lambda/4!)$ и в случае n-вершинной диаграммы имеется общий множитель $(1/n!)$ из ряда Тейлора. Но множитель $1/n$! всегда сокращается множителем $n!$ от перестановок вершин. Кроме того, каждая вершина имеет 4 линии, приходящие из четырёх различных мест. Перестановки относительно этих мест (что делается автоматически при вычислении $(\delta/i\delta\eta(x))^4$), дают множитель 4 !, сокращающий знаменатель в $(-i\lambda/4!)$. Поэтому каждой вершине сопоставляется множитель $\int \mathrm{d}^{(d)}x(-i\lambda)$. Если же мы имеем, например, вакуумный пузырёк с двумя вершинами такой, как на рис. 3e, то из двух исходных множителей $1/4!$ один остаётся. Не сокращается также общий множитель $(1/2)$. Но в данном случае как раз $D_n = 2 \cdot 4!$. Действительно, при действии дифференциального оператора $[(\delta/i\delta\eta(x))^4]\,[(\delta/i\delta\eta(y))^4]$ на величину $S_0\left(\phi_0;\eta\right)$ та квадратная скобка, которая действует первая, не даёт множителя 4!.

Для подсчета комбинаторного коэффициента диаграммы удобно представлять вычисление так, как будто все операторы $[-i\delta/\delta\eta(x)]$ либо "спариваются"попарно, либо "спариваются"с полем ϕ_0. В первом случае каждой "спарке" $[-i\delta/\delta\eta\left(x_i\right)]$ и $[-i\delta/\delta\eta\left(x_j\right)]$

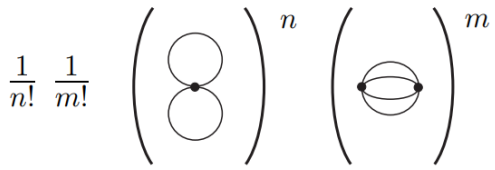
соответствует множитель $[-iD_c(x_i - x_j)]$, а во втором случае - $\phi_0(x)$. Это правило является одним из вариантов теоремы Вика. Легко сообразить, что, например, при вычислении диаграммы на рис.



множитель $(1/n!)$ в дифференциальном операторе (1) не сокращается: каждый множитель $\left[\frac{1}{4!}\int \mathrm{d}x_i \left(\frac{1}{i}\frac{\delta}{\delta\eta(x_i)}\right)^4\right]$ даёт в амплитуде множитель, соответствующий одной диаграмме-восьмёрке. Таким образом, в случае диаграммы на рис. 4 порядок группы симметрии $D_n = n!(8)^n$. Аналогично, если мы имеем несвязную вакуумную диаграмму вида



то она равна



Из сказанного видно, что любая диаграмма, не содержащая несвязных вакуумных компонент, умноЖается на ряд

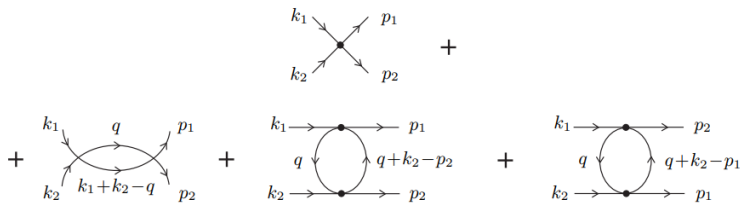
$$\sum_{\{n_i\}} \left(\prod_i \frac{1}{n_i!} V_i^{n_i} \right) = \exp\left(\sum_i V_i \right).$$

Таким образом, сумма всех диаграмм с заданным числом внешних линий равна сумме всех связных диаграмм с этими же внешними линиями, умноженной на экспоненту от суммы всех разных несвязных диаграмм без внешних линий (вакуумных пузырьков). Эта экспонента интерпретируется как фаза, набегающая при переходе вакуума в вакуум за бесконечное время. Заметим, что каждая диаграмма V_i содержит множитель $\left(\int \mathrm{d}^{(d)}x\right)$. Эта вакуумная фаза не представляет интереса, поскольку сокращается в физических величинах. Поэтому её не учитывают.

Во второй фигурной скобке на рис. 3 диаграммы $(f),(g),(h),(i)$ дают поправки к амплитуде распространения одной частицы в вакууме. Все эти поправки могут лишь привести к изменению (перенормировке) тех параметров частицы, которые могут изменяться непрерывно. Например, это может быть масса. Эти поправки далее рассматриваются в меру необходимости. Здесь лишь заметим, что буквальный учёт диаграмм типа рис. $3(f)-(i)$ затруднителен, поскольку, например, диаграммы (f) и (g) квадратично расходятся, диаграмма (h) содержит множитель

$$D_c(k)=\frac{1}{-k^2+m_0^2-i\varepsilon}.$$

Однако $k^2=m_0^2$, и потому $D_c(k)=\frac{1}{0}=\infty$. Мы видим, что поправки к внешним линиям, 4-импульсы которых k_i удовлетворяют условию $k_i^2=m^2$, где m - наблюдаемая (или физическая) масса, требуют отдельного рассмотрения. Такое рассмотрение далее проводится в более изящной теории поля квантовой электродинамике. В скалярной теории поля квантовые поправки к внешним линиям здесь не изучаются. (Говорят, что если импульс частицы удовлетворяет условию $k_i^2=m^2$, то её импульс находится на массовой поверхности; в этой ситуации говорят также, что частица находится на массовой поверхности.) Таким образом, из двух диаграмм левая будет учитываться, а правая - нет.



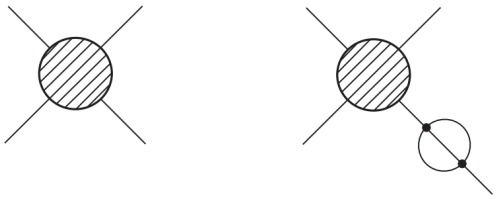
Будем говорить, что у диаграммы ноги на массовой поверхности ампутированы, если её нельзя рассечь на две несвязных диаграммы, разрезая одну внутреннюю линию, и при этом одна из двух диаграмм имеет две внешних линии на массовой поверхности.

Таким образом, при проведении вычислений по ТВ следует нарисовать все диаграммы процесса, имеющие физический смысл, с заданным числом вершим, определяемым степенью необходимой точности вычисления, сопоставить диаграммам аналитические выражения согласно правилам Фейнмана и вычислить их.

(???? все, забили что ли, итоговый ответ не нашли???????)

взаимное рассеяние двух начальных частиц

Например, рассмотрим взаимное рассеяние двух начальных частиц с импульсами $\mathbf{k}_1$ и $\mathbf{k}_2$, в результате чего при $t\to\infty$ имеем две частицы с импульсами $\mathbf{p}_1$ и $\mathbf{p}_2$. Этот процесс с точностью $O\left(\lambda_0^2\right)$ описывается следующими диаграммами:



Аналитическое выражение этих диаграмм таково:

$$iM = -i\lambda_0 + \left\{ \frac{1}{2}\lambda_0^2 \int \frac{\mathrm{d}^{(d)}q}{(2\pi)^d} D_c(q) \left[D_c\left(k_1 + k_2 - q\right) + D_c\left(k_2 - p_2 + q\right) + D_c\left(k_2 - p_1 + q\right) \right] \right\}.$$

Амплитуду (15) легко также получить непосредственно при помощи выражений (6) и (7).

3. Производящий функционал гриновских функций $Z(\eta)$

Если в

$$S\left(a^*, a\right) = S\left(\phi_0\right) = \frac{1}{N} \exp\left\{ -i \int V\left(\frac{1}{i} \frac{\delta}{\delta\eta(x)} \right) \mathrm{d}x \right\} \times$$
$$\times \exp\left\{ i \int \mathrm{d}x \eta(x)\phi_0(x) + \frac{i}{2} \iint \mathrm{d}x\ \mathrm{d}y \eta(x) D_c(x-y)\eta(y) \right\} \Bigg|_{\eta=0}. \quad (14.1)$$

не полагать $\eta = 0$, то полученный функционал обозначим $S\left(\phi_0; \eta\right)$; он является нормальным символом S-матрицы для рассеяния взаимодействующих частиц в присутствии источника $\eta(x)$. Рассмотрим также функционал

$$Z(\eta) = \exp\left\{ -i \int V\left(\frac{1}{i} \frac{\delta}{\delta\eta(z)} \right) \mathrm{d}z \right\} \cdot \exp\left\{ \frac{i}{2} \int \eta(x) D_c(x-y)\eta(y) \mathrm{d}x\ \mathrm{d}y \right\},$$

имеющий смысл амплитуды перехода из вакуума в вакуум в присутствии внешнего источника. Коэффициентные функции $G_n\left(x_1, ..., x_n\right)$ в разложении функционала $Z(\eta)$ в ряд по $\eta(x)$

$$Z(\eta) = \sum_n \frac{1}{n!} \int G_n\left(x_1, ..., x_n\right) \eta\left(x_1\right) ... \eta\left(x_n\right) \mathrm{d}x_1 ... \mathrm{d}x_n$$

определяют функции Грина, то есть вакуумные средние от хронологических произведений гейзенберговых операторов поля (см. конец лекции 1). Поэтому функционал $Z(\eta)$ называется производящим функционалом.

Функционал $Z(\eta)$ содержит в себе больше информации, чем функционал $S\left(\phi_0\right)$ (63 Л2): зная $Z(\eta)$, можно восстановить $S\left(\phi_0\right)$. Чтобы получить явные формулы, введём расширенный функционал $S(\phi)$, заменив в (63 Л2) поле $\phi_0(x)$ произвольной функцией четырёх переменных $\phi(x)$. Коэффициентные расширенные функции $\tilde{S}_n$ суть вариационные производные (под расширенными функциями мы понимаем коэффициентные функции для $S(\phi)$):

$$\tilde{S}_n\left(x_1, ..., x_n\right) = \left(\frac{1}{i} \frac{\delta}{\delta\phi\left(x_1\right)} \right) \cdots \left(\frac{1}{i} \frac{\delta}{\delta\phi\left(x_n\right)} \right) S(\phi) \Big|_{\phi=0} =$$
$$= \exp\left\{ -i \int V\left(\frac{1}{i} \frac{\delta}{\delta\eta(z)} \right) \mathrm{d}z \right\} \eta\left(x_1\right) ... \eta\left(x_n\right) \cdot \exp\left\{ \frac{i}{2} \int \eta(x) D_c(x-y)\eta(y) \mathrm{d}x\ \mathrm{d}y \right\}.$$

Пусть

$$\tilde{\eta}(x) = \int \mathrm{d}y D_c(x-y)\eta(y) \to \eta(x) = (\partial_\mu \partial^\mu +$$
$$\frac{\delta}{\delta\tilde{\eta}(x)} = \left(\partial_\mu \partial^\mu + m^2\right)_x \frac{\delta}{\delta\eta(x)}$$

Очевидно

$$\left(\frac{1}{i}\frac{\delta}{\delta\tilde{\eta}(x)}\right)Z(\eta)=\exp\left\{-i\int V\left(\frac{1}{i}\frac{\delta}{\delta\eta(z)}\right)\mathrm{d}z\right\}\eta(x)\exp\left\{\frac{i}{2}\int\eta(y)D(y-z)\eta(z)\mathrm{d}y\ \mathrm{d}z\right\},$$

$$\left(\frac{1}{i}\frac{\delta}{\delta\tilde{\eta}\left(x_1\right)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\tilde{\eta}\left(x_2\right)}\right)Z(\eta)=\exp\left\{-i\int V\left(\frac{1}{i}\frac{\delta}{\delta\eta(z)}\right)\mathrm{d}z\right\}\cdot$$

$$\cdot\left(\frac{1}{i}\frac{\delta}{\delta\tilde{\eta}\left(x_1\right)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\tilde{\eta}\left(x_2\right)}\right)Z(\eta)=\exp\left\{-i\int V\left(\frac{1}{i}\frac{\delta}{\delta\eta(z)}\right)\mathrm{d}z\right\}$$

$$\left[\eta\left(x_1\right)\eta\left(x_2\right)+\frac{1}{i}\left(\partial_\mu\partial^\mu+m^2\right)_{x_2}\delta^{(d)}\left(x_2-x_1\right)\right]\exp\left\{\frac{i}{2}\int\eta(y)D_c(y-z)\eta(z)\mathrm{d}y\ \mathrm{d}z\right\},$$

и т.д. Очевидно, применение оператора

$$\left(\frac{1}{i}\frac{\delta}{\delta\tilde{\eta}\left(x_1\right)}\right)...\left(\frac{1}{i}\frac{\delta}{\delta\tilde{\eta}\left(x_n\right)}\right)$$

к функционалу (16) даёт функционал того же вида, что в правой части (20), где квадратная скобка заменяется на величину

$$\left[\eta\left(x_1\right)...\eta\left(x_n\right)+\varkappa\left(x_1,...,x_n\right)\right],$$

причём

$$\int\phi_0\left(x_1\right)...\phi_0\left(x_n\right)\varkappa\left(x_1,...,x_n\right)\mathrm{d}x_1...\mathrm{d}x_n=0.$$

Теперь при помощи (18)-(21) находим:

$$\int\mathrm{d}x_1...\mathrm{d}x_n\phi_0\left(x_1\right)...\phi_0\left(x_n\right)\left\{\left(\frac{1}{i}\frac{\delta}{\delta\phi\left(x_1\right)}\right)\cdots\left(\frac{1}{i}\frac{\delta}{\delta\phi\left(x_n\right)}\right)S(\phi)\Big|_{\phi=0}-\right.$$

$$\left.-\left(\frac{1}{i}\frac{\delta}{\delta\tilde{\eta}\left(x_1\right)}\right)\cdots\left(\frac{1}{i}\frac{\delta}{\delta\tilde{\eta}\left(x_n\right)}\right)Z(\eta)\right\}=0$$

Таким образом, для вычисления нормального символа S-матрицы нужно вычислить вариационные производные

$$\left(\frac{1}{i}\frac{\delta}{\delta\eta\left(x_1\right)}\right)\cdots\left(\frac{1}{i}\frac{\delta}{\delta\eta\left(x_n\right)}\right)Z(\eta)\Big|_{\eta=0},$$

т.е. функции Грина $G_n\left(x_1,...,x_n\right)$, подействовать на эти функции дифференциальным оператором

$$\prod_{i=1}^{n}\left(\partial_\mu\partial_{x_i}^\mu+m^2\right)$$

умножить результат на произведение

$$\frac{1}{n!}\prod_i\phi_0\left(x_i\right),$$

проинтегрировать по всем x_i и просуммировать по n.

14.2.7 Combinatoric Factors

(??? discussion of how to obtain it with examples?)

14.2.8 Feynman Rules: Derivation from Weinberg

(??? когда-то выгружу то, что было до этого, пока так.)

Теория

Мы начнем с формулы для S -матрицы, полученной из разложения Дайсона (3.5.10) и выражения (4.2.2) для свободных полей:

$$S_{\mathbf{p}'_1\sigma'_1n'_1;\mathbf{p}'_2\sigma'_2n'_2;...,\mathbf{p}'_1\sigma_1n_1;\mathbf{p}_2\sigma_2n_2;...} =$$

$$= \sum_{N=0}^{\infty} \frac{(-i)^N}{N!} \int d^4x_1 ... d^4x_N \left(\Phi_0, ... a\left(\mathbf{p}'_2\sigma'_2n'_2\right) a\left(\mathbf{p}'_1\sigma'_1n'_1\right) \times \right.$$

$$\left. \times T\left\{\mathscr{H}\left(x_1\right) ... \mathscr{H}\left(x_N\right)\right\} a^{\dagger}\left(\mathbf{p}_1\sigma_1n_1\right) a^{\dagger}\left(\mathbf{p}_2\sigma_2n_2\right) ... \Phi_0\right)$$

Напомним, что $\mathbf{p}, \sigma$ и n обозначают импульс, спин и сорт частиц; штрихи обозначают конечные состояния частиц; Φ_0 - вакуум свободных частиц; $a, a^{\dagger}$ - операторы уничтожения, рождения; T - знак хронологического упорядочивания, который располагает множители $\mathscr{H}(x)$ в таком порядке, что аргументы x^0 уменьшаются слева направо; наконец, $\mathscr{H}(x)$ - гамильтониан (плотность функции Гамильтона) взаимодействия, который считается полиномом от полей и сопряженных полей

$$\mathscr{H}(x) = \sum_i g_i \mathscr{H}_i(x)$$

а каждое слагаемое $\mathscr{H}_i$ является произведением определенного числа полей и сопряженных им величин каждого типа. Поле каждой частицы сорта n, преобразуещееся по определенному представлению однородной группы Лоренца (с пространственными инверсиями или без них), дается формулой

$$\psi_l(x) = \sum_{\sigma} (2\pi)^{-3/2} \int d^3p \left[u_l(\mathbf{p}, \sigma, n) a(\mathbf{p}, \sigma, n) e^{ip\cdot x} + + v_l(\mathbf{p}, \sigma, n) a^{\dagger}\left(\mathbf{p}, \sigma, n^c\right) e^{-ip\cdot x}\right]$$

Здесь n^c обозначает античастицу сорта n, а в экспоненте $e^{\pm ip\cdot x}$ энергия p^0 равна $\sqrt{\mathrm{p}^2 + m_n^2}$. Функции-коэффициенты u_l, v_l зависят от лоренц-трансформационных свойств поля и спина частицы, которую это поле описывает; они были вычислены в гл. 5. (Например, в случае скалярного поля для частицы с энергией E функция u_l есть просто множитель $(2E)^{-1/2}$, а для дираковского поля u_l, v_l являются нормированными дираковскими спинорами, введенными в \$5.5). Индекс l у полей означает не только сорт частицы, но и представление группы Лоренца, по которому преобразуется поле, а также обозначает все "бегушие" индексы многокомпонентного поля в этом представлении. Нет необходимости отдельно рассматривать взаимодействия с производными от полей, поскольку в нашей записи производная от поля (6.1.3) снова является полем того же вида, но с другими u_l, v_l. Мы будем отличать некие произвольные частицы, например электроны, протоны и т.п., от их античастиц, таких, как позитроны и антипротоны. Операторы поля,

которые уничтожают частицы и создают античастицы, у нас называются просто "поля"; сопряженные им операторы, уничтожающие античастицы и создающие частицы, называются "сопряженные поля". Разумеется, некоторые частицы могут совпадать с их античастицами, например фотон γ и нейтральный пи-мезон π^0; для таких полей сопряженные поля пропорциональны самим полям.

Теперь переставим в формуле (6.1.1) все операторы уничтожения направо, применяя нужное число раз (анти)коммутационные соотношения:

$$
\begin{aligned}
&a(\mathbf{p}\sigma n)a^{\dagger}\left(\mathbf{p}'\sigma' n'\right) = \pm a^{\dagger}\left(\mathbf{p}'\sigma' n'\right) a(\mathbf{p}\sigma n) + \delta^3\left(p'-p\right)\delta_{\sigma'\sigma}\delta_{n'n} \\
&a(\mathbf{p}\sigma n)a\left(\mathbf{p}'\sigma' n'\right) = \pm a\left(\mathbf{p}'\sigma' n'\right) a(\mathbf{p}\sigma n) \\
&a^{\dagger}(\mathbf{p}\sigma n)a^{\dagger}\left(\mathbf{p}'\sigma' n'\right) = \pm a^{\dagger}\left(\mathbf{p}'\sigma' n'\right) a^{\dagger}(\mathbf{p}\sigma n)
\end{aligned}
$$

и аналогично для античастиц, причем знак " $\pm$ " равен " - ", если обе частицы n, n' - фермионы, и равен " + ",

если одна или обе частицы - бозоны. Как только оператор уничтожения оказывается крайним справа (или оператор рождения крайним слева), соответствующий вклад в (6.1.1) пропадает, поскольку эти операторы аннулируют вакуумное состояние:

$$
\begin{aligned}
a(\mathbf{p}\sigma n)\Phi_0 &= 0 \\
\Phi_0^{\dagger}a^{\dagger}(\mathbf{p}\sigma n) &= 0
\end{aligned}
$$

В уравнении (6.1.1) остается только вклад от δ -функционных слагаемых (6.1.4), остающихся при каждой перестановке оператора уничтожения с оператором рождения из начального или конечного состояния или гамильтониана взаимодействия, которые, таким образом, должны присутствовать парами.

вклад заданного порядка в (6.1.1) в каждое из слагаемых $\mathscr{H}_i$ полинома $\mathscr{H}\left(\psi(x), \psi^{\dagger}(x)\right)$ дается суммой (по всем способам спаривания операторов рождения и уничтожения [2]) интегралов от произведений множителей, согласно следующим правилам:

а) Спаривание частицы в конечном состоянии с квантовыми числами $\mathbf{p}', \sigma', n'$ с сопряженным полем $\psi_l^{\dagger}(x)$ в $\mathscr{H}_i(x)$ дает множитель

$$
\left[a\left(\mathbf{p}'\sigma' n'\right), \psi_l^{\dagger}(x)\right]_{\mp} = (2\pi)^{-3/2} e^{-ip'\cdot x} u_l^{*}\left(\mathbf{p}'\sigma' n'\right)
$$

б) Спаривание античастицы в конечном состоянии с квантовыми числами $\mathbf{p}', \sigma', n'^c$ с полем $\psi_l(x)$ в $\mathscr{H}_i(x)$ дает множитель

$$
\left[a\left(\mathbf{p}'\sigma' n'^c\right), \psi_l(x)\right]_{\mp} = (2\pi)^{-3/2} e^{-ip'\cdot x} v_l\left(\mathbf{p}'\sigma' n'\right)
$$

в) Спаривание частицы в начальном состоянии с квантовыми числами $\mathbf{p}, \sigma, n$ с полем $\psi_l(x)$ в $\mathscr{H}_i(x)$ дает множитель

$$
\left[\psi_l(x), a^{\dagger}(\mathbf{p}\sigma n)\right]_{\mp} = (2\pi)^{-3/2} e^{ip\cdot x} u_l^{*}(\mathbf{p}\sigma n)
$$

г) Спаривание античастицы в начальном состоянии с квантовыми числами $\mathbf{p}, \sigma, n^c$ с сопряженным полем $\psi_l^{\dagger}(x)$ в $\mathscr{H}_i(x)$ дает множитель

$$
\left[\psi_l^{\dagger}(x), a^{\dagger}\left(\mathbf{p}\sigma n^c\right)\right]_{\mp} = (2\pi)^{-3/2} e^{ip\cdot x} v_l^{*}(\mathbf{p}\sigma n)
$$

д) Спаривание (анти)частицы в конечном состоянии с квантовыми числами $\mathbf{p}', \sigma', n'$ с полем (анти)частицы в начальном состоянии с квантовыми числами $\mathbf{p}, \sigma, n$ дает множитель

$$\left[a\left(\mathbf{p}'\sigma'n'\right), a(\mathbf{p}\sigma n)\right]_{\mp} = \delta^3\left(\mathbf{p}' - \mathbf{p}\right)\delta_{\sigma'\sigma}\delta_{n'n}$$

е) Спаривание поля $\psi_l(x)$ в $\mathscr{H}_i(x)$ с сопряженным полем $\psi_m^{\dagger}(x)$ в $\mathscr{H}_j(y)$ дает множитель)

$$\theta(x-y)\left[\psi_l^{+}(x), \left[\psi_m^{+\dagger}(y)\right]_{\mp} \pm \theta(y-x)\left[\psi_m^{-\dagger}(y), \left[\psi_l^{-}(x)\right]_{\mp} \equiv \right.\right. \\ \equiv -i\delta_{lm}(x,y)$$

где ψ^{+} и ψ^{-} - слагаемые в ψ, которые соответственно уничтожают частицы и создают античастицы:

$$\psi_l^{+}(x) = (2\pi)^{-3/2}\int d^3p \sum_{\sigma} u_l(\mathbf{p}\sigma n)e^{ip\cdot x}a(\mathbf{p}\sigma n)$$
$$\psi_l^{-}(x) = (2\pi)^{-3/2}\int d^3p \sum_{\sigma} v_l(\mathbf{p}\sigma n)e^{-ip\cdot x}a^{\dagger}\left(\mathbf{p}\sigma n^c\right)$$

Напомним, что $\theta(x-y)$ - функция Хевисайда ("ступенька"), равная $+1$ при $x^0 > y^0$ и нулю при $x^0 < y^0$. Эти функции возникают в (6.1.14) из-за хронологизации в (6.1.1); спаривание оператора уничтожения $\psi^{+}(x)$ из $\mathscr{H}(x)$ с оператором рождения $\psi^{+\dagger}(y)$ из $\mathscr{H}(y)$ может произойти, только если множитель $\mathscr{H}(x)$ изначально располагался слева от $\mathscr{H}(y)$ в уравнении (6.1.1), т.е., если $x^0 > y^0$; аналогично, спаривание оператора уничтожения $\psi^{-\dagger}(y)$ из $\mathscr{H}(y)$ с оператором рождения $\psi^{-}(x)$ из $\mathscr{H}(x)$ может произойти, только если из $\mathscr{H}(y)$ изначально располагался слева от $\mathscr{H}(x)$ в уравнении (6.1.1), т.е., если $y^0 > x^0$. Знак $\pm$ во втором слагаемом в (6.1.14) будет объяснен чуть позже. Величина (6.1.14) известна под названием пропагатор.

Она S -матрица получается в результате перемножения всех этих множителей, наряду с дополнительными числовыми множителями, которые мы обсудим далее, интегрирования по $x_1, ..., x_N$, суммирования по всем спариваниям, а затем по всем видам взаимодействий. Прежде чем мы уточним все детали, полезно описать диаграммный формализм, который позволяет отслеживать все возможные спаривания.

Правила вычисления S -матрицы удобно сформулировать в терминах фейнмановских диаграмм (см. рис. 6.1).

Диаграммы состоят из точек, называемых вериинами, каждая из которых представляет один их членов $\mathscr{H}_i(x)$, и линий, которые соответствуют спариванию операторов уничтожения и рождения. Более точно, а) Спаривание частицы в конечном состоянии с сопряженным полем из некоторого $\mathscr{H}(x)$ представляется линией, идущей из вер-

(и тут у Вайнберга Диаграммы, которые мб когда-нибудь перерисую)

шины, отвечаюшей этому $\mathscr{H}(x)$, вверх из диаграммы, со стрелкой вверх.

б) Спаривание античастицы в конечном состоянии с полем из некоторого $\mathscr{H}(x)$ также представляется линией, идущей из верши- ны, отвечаюшей этому $\mathscr{H}(x)$, вверх из диаграммы, но со стрелкой вниз. Для истинно нейтральных (самосопряженных) полей, например фотонов, стрелки опускаются.

в) Спаривание частицы в начальном состоянии с полем из некото- рого $\mathscr{H}(x)$ представляется линией, идущей в вершину, отвечающую этому $\mathscr{H}(x)$, внутрь диаграммы, со стрелкой вверх.

г) Спаривание античастицы в начальном состоянии с полем из некоторого $\mathscr{H}(x)$ представляется линией, идушей в вершину, отвечаЮшую этому $\mathscr{H}(x)$, внутрь диаграммы, со стрелкой вниз.

д) Спаривание конечной (анти)частицы с начальной (анти)частицей представляется линией, идущей через диаграмму снизу вверх, не входяшую ни в какие вершины, со стрелкой вверх или вниз для частицы и античастицы соответственно.

е) Спаривание поля в $\mathscr{H}(x)$ с сопряженным полем из $\mathscr{H}(y)$ представляется линией, соединяюшей вершины, отвечающие этим $\mathscr{H}(x)$ и $\mathscr{H}(y)$, со стрелкой, направленной от y к x.

Заметим, что стрелка направлена в сторону движения частицы и против движения античастицы (как уже было сказано, стрелки для истинно нейтральных частиц следует опускать). Направление стрелки по правилу е) согласуется с этой условностью, потому что сопряженное поле из $\mathscr{H}_j(y)$ может или создать частицу, уничтоженную полем из $\mathscr{H}_i(x)$, или уничтожить античастицу, созданную полем из $\mathscr{H}_i(x)$. Отметим также, что, поскольку каждое (сопряженное) поле должно спариваться с другим полем, полное число линий в вершине типа i, отвечающей члену $\mathscr{H}_i(x)$ в (6.1.2), равно полному числу множителей (полей и сопряженных полей) в $\mathscr{H}_i(x)$.

Число линий со стрелкой внутрь вершины или наружу равно соответственно числу полей или сопряженных полей в соответствуюшем члене взаимодействия.

Чтобы вычислить вклад данного процесса в S -матрицу для данного порядка N_i по каждому из взаимодействий $\mathscr{H}_i(x)$в уравнении (6.1.2), нужно проделать следующие шаги:

(i) Нарисовать все диаграммы Фейнмана, которые содержат N_i вершин каждого типа i, входяшие снизу линии (анти)частиц в начальном состоянии, выходящие сверху линии (анти)частиц в конечном состоянии, а также любое число внутренних линий, соединяющих вершины, так, чтобы в каждую вершину входило нужное число линий. На линиях указываются стрелки вверх или вниз по вышеописанным правилам. Каждая вершина помечается типом взаимодействия и 4-координатой x^μ. Каждая внутренняя или внешняя линия помечается индексом поля l на том конце, где она входит в вершину (что отвечает полю $\psi_l(x)$ или $\psi_l^\dagger(x)$, которое создает или уничтожает (анти)частицу в этой вершине). Каждая внешняя линия помечается квантовыми числами $\mathbf{p}, \sigma, n$ или $\mathbf{p}', \sigma', n'$ начального или конечного состояния (анти)частицы. (ii) Для каждой вершины типа i нужно добавить множитель $-i$, который происходит от множителя $(-i)^N$ в формуле (6.1.1), а также множитель g_i - константу связи для данного взаимодействия $\mathscr{H}_i(x)$. Для каждой выходяшей вверх линии дополнительно вводится множитель (6.1.9) или (6.1.10), в зависимости от направления стрелки (вверх или вниз). Для каждой входяшей снизу линии добавляется множи-

тель (6.1.11), если стрелка указывает вверх, или (6.1.12) - в противном случае. Для каждой линии, проходящей насквозь диаграммы, добавляется множитель (6.1.13). Для каждой внутренней линии, соединяющей вершины, вводится множитель (6.1.14). (iii) Проинтегрировать полученное произведение по всем координатам $x_1, x_2, ...$ каждой вершины. (iv) Сложить результаты для всех диаграмм Фейнмана. Полный ряд теории возмущений для S -матрицы получается сложением вкладов каждого порядка по каждому взаимодействию.

Обратим внимание, что в перечисленных правилах не учитывается множитель 1/N! из формулы (6.1.1), поскольку хронологически упорядоченное произведение есть сумма по $N!$ перестановкам координат $x_1, ..., x_N$, а каждая перестановка дает одинаковый вклад в окончательный результат. Другими словами, каждая диаграмма Фейнмана с N вершинами - это одна из $N!$ идентичных диаграмм, отличающихся лишь перестановками обозначений вершин, что дает множитель $N!$, сокращающий 1/N! в формуле (6.1.1). (Из этого правила есть исключения, о которых см. ниже.) По этой причине мы не учитываем диаграммы

Фейнмана, отличающиеся лишь переобозначением вершин.

В некоторых случаях возникают дополнительные комбинаторные множители или знаки, которые необходимо включить во вклад отдельных диаграмм: (v) Предположим, что взаимодействие $\mathscr{H}_i(x)$ содержит M мно-

жителей одного и того жсе поля. Допустим, каждое из них спарено с сопряженным полем из другого взаимодействия или из начального (конечного) состояния. Первое из этих сопряженных полей может быть спарено с любым из M идентичных полей в $\mathscr{H}_i(x)$, второе только с любым из оставшегося $M-1$ поля, и т.д., что дает дополнительный множитель $M!$ Для его компенсации константы связи g_i обычно определяют так, чтобы множитель 1/M! явным образом появился в гамильтониане $\mathscr{H}_i(x)$, который содержит M идентичных полей (или сопряженных полей). Например, самодействие скалярного поля M -го порядка следует представлять в виде $g\phi^M/M!$ Вообще, множитель 1/M! часто выделяют в случаях, когда взаимодействие содержит сумму M произведений из одного мультиплета группы симметрии или когда по этой или иной причине набор констант взаимодействия полностью симметричен или антисимметричен при перестановках M бозонных или фермионных полей. Однако сокращение множителей $M!$ может быть неполным. Например, рассмотрим фейнмановскую диаграмму, в которой M идентичных полей в одном взаимодействии $\mathscr{H}_i(x)$ спариваются с M соответствующими сопряженными полями в другом взаимодействии $\mathscr{H}_j(y)$(cm. рис. 6.2). Рассуждая аналогично, находим $M!$ различных спариваний, так что сокрашается лишь один из множителей 1/M!

двух различных взаимодействий. Следовательно, нужно добавить множитель 1/M! для этой диаграммы Фейнмана. Другие комбинаторные множители возникают, когда некоторые перестановки вершин не меняют диаграмму Фейнмана. Мы уже об- судили сокращение множителя $1/N!$ в разложении (6.1.1) за счет суммы по идентичным диаграммам, отличающимся лишь переобо- значением N вершин. О днако это сокращение будет неполным, если перестановка вершин не меняет диаграмму. Обычно это наблюдается при вычислении вакуум-вакуумных матричных элементов S -матрицы

в теории с квадратичным взаимодействием $\mathscr{H} = \psi_1^\dagger M_{ll'}\psi_{l'}$, где M может зависеть от внешних полей. (Физический смысл таких вакуумных флуктуаций подробно обсуждается в т. II.) Диаграмма Фейнмана N -го порядка по $\mathscr{H}$ представляет собой кольцо с N вершинами (см. рис. 6.3). Имеется лишь $(N-1)!$ различных диаграмм этого вида, поскольку перестановка значков со сдвигом в соседние вершины

по кругу дает ту же диаграмму. Значит, этой диаграмме отвечает дополнительный множитель

$$\frac{(N-1)!}{N!} = \frac{1}{N}$$

(vi) В теориях с фермионными полями применение формул (6.1.4) - (6.1.6) для перемешения операторов уничтожения и рождения вправо или влево дает дополнительные знаки минус в различных спариваниях. Более точно, знак минус получается, когда для того, чтобы поставить спаренные операторы друг рядом с другом (так, чтобы слева в паре стоял оператор уничтожения), в формуле (6.1.1) нужно сделать нечетное число перестановок местами фермионных операторов. Это можно объяснить так: для вычисления вклада от данного спаривания нужно расположить все операторы уничтожения в формуле (6.1.1) слева от спаренных с каждым из них оператором рождения, игнорируя все (анти)коммутаторы неспаренных операторов, а затем заменить каждую пару спаренных полей на их (анти)коммутатор. Отсюда немедленно следует, что относительный знак двух слагаемых в (6.1.14) для фермионных полей является минусом. Какая бы перестановка ни

помешала уничтожающую часть $\psi^{+}(x)$ поля в $\mathscr{H}(x)$ слева от рождающей части $\psi^{+\dagger}(y)$ сопряженного поля в $\mathscr{H}(y)$, при перестановке уничтожающей части $\psi^{-\dagger}(y)$ сопряженного поля налево от рождающей части $\psi^{-}(x)$ поля происходит одна дополнительная замена местами фермионных операторов, что дает знак " - " для фермионов во втором слагаемом левой части уравнения (6.1.14).

Помимо этого, диаграмма Фейнмана может приобрести дополнительный знак " - ". В качестве примера рассмотрим теорию с единственным взаимодействием фермионов вида

$$\mathscr{H}(x) = \sum_{lmk} g_{lmk} \psi_l^{\dagger}(x) \psi_m(x) \phi_k(x)$$

где $g_{lmk} - 1$ нты общего вида, $\psi_l(x)$ - набор комплексных фермионных полей, а $\phi_m(x)$ - набор вещественных бозонных (необязательно скалярных) полей. Заметим, что не только электродинамика, но и вся стандартная модель слабых, электромагнитных и сильных взаимодействий соответствует взаимодействиям такого вида. Для начала рассмотрим процесс рассеяния фермиона на фермионе, $12 \to 1'2'$, с точностью до второго порядка по $\mathscr{H}$. Фермионные операторы во вкладе второго порядка в формуле (6.1.1) оказываются расположенными в порядке (с очевидными сокращениями обозначений)

$$a\left(2'\right) a\left(1'\right) \psi^{\dagger}(x) \psi(x) \psi^{\dagger}(y) \psi(y) a^{\dagger}(1) a^{\dagger}(2)$$

Имеются две связные диаграммы этого порядка, отвечающие спариваниям

$$\left[a\left(2'\right) \psi^{\dagger}(x)\right] \left[a\left(1'\right) \psi^{\dagger}(y)\right] \left[\psi(y) a^{\dagger}(1)\right] \left[\psi(x) a^{\dagger}(2)\right]$$

и

$$\left[a\left(1'\right) \psi^{\dagger}(x)\right] \left[a\left(2'\right) \psi^{\dagger}(y)\right] \left[\psi(y) a^{\dagger}(1)\right] \left[\psi(x) a^{\dagger}(2)\right]$$

(см. рис. 6.4). Переход от (6.1.19) к (6.1.20) требует четной перестановки фермионных операторов. Например, можно переместить $\psi(x)$

через три оператора направо, а затем переместить $a\left(1'\right)$ через один оператор направо. Поэтому дополнительного минуса не возникает во вкладе от спаривания (6.1.20). Само по себе это было бы неважно, поскольку общий знак матричного элемента не влияет на скорость реакции, и в любом случае зависит от соглашений о знаке начального и конечного состояний. Но вклады спариваний (6.1.20) и (6.1.21) имеют разный знак, что легко заметить, поскольку разница между этими спариваниями компенсируется одной перестановкой местами фермионных операторов: $a\left(1'\right) \leftrightarrow a\left(2'\right)$. На самом деле, относительный знак минус требуется правилами статистики Ферми: амплитуда рассеяния должна быть антисимметрична относительно перестановки одинаковых фермионов ($1', 2'$ или $1, 2$).

Однако не всегда знаковые факторы можно обосновать таким простым образом, даже в наинизшем порядке теории возмущений. Чтобы убедиться в этом, рассмотрим рассеяние фермиона на антифермионе, $12^{c} \to 1'2'^{c}$, во втором порядке по тому же взаимодействию (6.1.18). Фермионные операторы во вкладе второго порядка оказываЮтся в следующем порядке:

$$a\left(2'^{c}\right) a\left(1'\right) \psi^{\dagger}(x) \psi(x) \psi^{\dagger}(y) \psi(y) a^{\dagger}(1) a^{\dagger}\left(2^{c}\right)$$

Здесь снова возникают две диаграммы Фейнмана, отвечающие спари- ваниям

$$\left[a\left(2'^{c}\right) \psi(x)\right] \left[a\left(1'\right) \psi^{\dagger}(x)\right] \left[\psi(y) a^{\dagger}(1)\right] \left[\psi^{\dagger}(y) a^{\dagger}\left(2^{c}\right)\right]$$

И

$$\left[a\left(2'^{c}\right)\psi(x)\right]\left[a\left(1'\right)\psi^{\dagger}(y)\right]\left[\psi(y)a^{\dagger}(1)\right]\left[\psi^{\dagger}(x)a^{\dagger}\left(2^{c}\right)\right]$$

(см. рис. 6.5). Переход от (6.1.22) к (6.1.23) требует четной перестановки фермионных операторов (например, можно переместить $\psi(x)$ через

два оператора налево, а затем переместить $\psi^{\dagger}(y)$ через два оператора направо), поэтому дополнительного минуса не возникает во вкладе от спаривания (6.1.23). С другой стороны, для перехода от (6.1.22) к (6.1.24) требуется нечетная перестановка фермионных операторов (то же, что для (6.1.23), плюс одна перестановка $\psi^{\dagger}(x) \leftrightarrow \psi^{\dagger}(y)\big)$,

поэтому вклад этого спаривания получает дополнительный минус [2]). Дополнительные знаки могут появиться во вкладах высших порядков. В теориях, подобных рассмотренным здесь, со взаимодействием фермионов вида (6.1.18), фермионные линии в диаграммах общего вида образуют либо цепочки, проходящие через диаграмму с произвольным числом взаимодействий с бозонами, как на рис. 6.6, либо фермионные петли, как на рис. 6.7. Рассмотрим вклад фермионной

петли с M вершинами в любой процесс. Она отвечает спариванию операторов

$$\left[\psi\left(x_1\right)\bar{\psi}\left(x_2\right)\right]\left[\psi\left(x_2\right)\bar{\psi}\left(x_3\right)\right]...\left[\psi\left(x_M\right)\bar{\psi}\left(x_1\right)\right]$$

С другой стороны, эти операторы входят в (6.1.1) в следующем порядке:

$$\bar{\psi}\left(x_1\right)\psi\left(x_1\right)\bar{\psi}\left(x_2\right)\psi\left(x_2\right)...\bar{\psi}\left(x_M\right)\psi\left(x_M\right)$$

Для перехода от (6.1.26) к (6.1.25) требуется нечетная перестановка фермионных операторов (перемещение $\bar{\psi}\left(x_1\right)$ направо через $2M-1$ оператор), так что вклад каждой такой петли сопровождается знаком МиНус.

Эти правила позволяют вычислить полную S -матрицу, включая вклады от процессов, в которых различные группы частиц взаимодействуют в удаленных по времени и пространству областях. В гл. 4 уже говорилось, что следует брать лишь связные диаграммы Фейнмана, чтобы исключить вклады от таких групповых взаимодействий.

В частности, это исключает диаграммы с линиями, идушими напрямую без взаимодействия, которые дали бы множители (6.1.13).

Чтобы прояснить правила Фейнмана, вычислим вклад низшего порядка в S -матрицу для рассеяния частиц в двух различных теориях.

теория с фермионами и зарядово-самосопряженными (нейтральными) бозонами, со взаимодействием

Рассмотрим теорию с фермионами и зарядово-самосопряженными (нейтральными) бозонами, со взаимодействием (6.1.18). Связные диаграммы наинизшего порядка фермион-бозонного рассеяния изображены на рис.

6.6. Согласно правилам, приведенным на рис. 6.1, соответствующий матричный элемент равен

$$S_{\mathbf{p}_1'\sigma_1'n_1'\mathbf{p}_2'\sigma_2'n_2',\mathbf{p}_1\sigma_1n_1\mathbf{p}_2\sigma_2n_2} =$$

$$= (2\pi)^{-6}\sum_{k'l'm'klm}(-i)^2 g_{l'm'k'}g_{mlk}u_{l'}^{*}\left(\mathbf{p}_1'\sigma_1'n_1'\right)u_l\left(\mathbf{p}_1\sigma_1n_1\right)\times$$

$$\times \int d^4x \int d^4y \left(-i\delta_{m'm}(y-x)\right) e^{-ip_1'\cdot y} e^{ip_1\cdot x} \times$$

$$\times \Big[e^{-ip_2'\cdot y} u_{k'}^* \left(\mathbf{p}_2'\sigma_2'n_2'\right) e^{ip_2\cdot x} u_k \left(\mathbf{p}_2\sigma_2 n_2\right) +$$

$$+ e^{-ip_2'\cdot x} u_k^* \left(\mathbf{p}_2'\sigma_2'n_2'\right) e^{ip_2\cdot y} u_{k'} \left(\mathbf{p}_2\sigma_2 n_2\right) \Big]$$

Индексы 1 и 2 обозначают здесь соответственно фермионы и бозоны. Для фермион-фермионного рассеяния имеются две диаграммы второго порядка, представленные на рис. 6.4. Они дают S -матричный элемент

$$S_{\mathbf{p}_1'\sigma_1'n_1'\mathbf{p}_2'\sigma_2'n_2',\mathbf{p}_1\sigma_1 n_1\mathbf{p}_2\sigma_2 n_2} = (2\pi)^{-6} \sum_{k'l'm'klm} (-i)^2 g_{m'mk'} g_{l'lk} \times$$

$$\times u_{m'}^* \left(\mathbf{p}_2'\sigma_2'n_2'\right) u_{l'}^* \left(\mathbf{p}_1'\sigma_1'n_1'\right) u_m \left(\mathbf{p}_2\sigma_2 n_2\right) u_l \left(\mathbf{p}_1\sigma_1 n_1\right) \times$$

$$\times \int d^4x \int d^4y e^{-ip_2'\cdot x} e^{-ip_1'\cdot y} e^{ip_2\cdot x} e^{ip_1\cdot y} (-i)\delta_{k'k}(x-y) -$$

$$- \left[1' \rightleftharpoons 2'\right]$$

где последнее слагаемое сокращенно обозначает предыдущее слагаемое, в котором переставлены частицы $1', 2'$ (или, что то же самое, частицы 1,2). Во втором порядке нет диаграмм бозон-бозонного рассеяния; наинизший порядок, в котором они появляются, равен 4 (см. рис. 6.7). Более конкретные примеры формул типа (6.1.27), (6.1.28) будут приведены в \$6.3, после попытки вычисления пропагатора и перехода в импульсное пространство.

Во взаимодействии (6.1.18) все три сомножителя - различные поля. Было бы поучительно рассмотреть трилинейное взаимодействие, в котором все три поля одинаковы или входят во взаимодействие симметрично.

теория 2

Теперь рассмотрим в качестве плотности гамильтониана трилинейную форму

$$\mathscr{H}(x) = \frac{1}{3!} \sum_{lmn} g_{lmn} \phi_l(x) \phi_m(x) \phi_n(x)$$

где $\phi_l(x)$ - вещественные бозонные поля, а g_{lmn} - полностью симметричный набор констант связи. Допустим, мы хотим рассмотреть процесс рассеяния $12 \to 1'2'$ с точностью до второго порядка по этим константам связи. В каждую из двух вершин должны входить две из четырех внешних линий. (Единственная альтернатива - когда одна из внешних линий входит в одну вершину, а три других - в другую, но тогда получается несвязная диаграмма, поскольку у второй вершины не останется места для внутренней, соединяющей линии.)

Остающаяся у каждой вершины линия должна соединять их между собой. Возможны три рисунка такого вида, отличающиеся лишь тем, какие пары внешних линий входят в вершины (см. рис. 6.8). По правилам Фейнмана, приведенным выше, получаем вклад в S -матрицу ОТ ЭТих трех диаграмм:

$$S_{\mathbf{p}_1'\sigma_1'n_1'\mathbf{p}_2'\sigma_2'n_2',\mathbf{p}_1\sigma_1 n_1\mathbf{p}_2\sigma_2 n_2} =$$

$$= (-i)^2(2\pi)^{-6} \sum_{u'l''mm'm''} g_{ll'l''}g_{mm'm''} \int d^4x \int d^4y \left(-i\delta_{l''m''}(x,y)\right) \times$$

$$\times \Big[u_l^* \left(\mathbf{p}_1'\sigma_1'n_1'\right) e^{-ip_1'\cdot x} u_{l'}^* \left(\mathbf{p}_2'\sigma_2'n_2'\right) e^{-ip_2^2\cdot x} \times$$

$$\times u_m \left(\mathbf{p}_1\sigma_1 n_1\right) e^{ip_1\cdot y} u_{m'} \left(\mathbf{p}_2\sigma_2 n_2\right) e^{ip_2\cdot y} +$$

$$+ u_{l'}^* \left(\mathbf{p}_1'\sigma_1'n_1'\right) e^{-ip_1'\cdot x} u_l \left(\mathbf{p}_1\sigma_1 n_1\right) e^{ip_1\cdot x} \times$$

$$\times u_{m'}^* \left(\mathbf{p}_2'\sigma_2'n_2'\right) e^{-ip_2'\cdot y} u_m \left(\mathbf{p}_2\sigma_2 n_2\right) e^{ip_2\cdot y} +$$

$$+ u_{l'}^* \left(\mathbf{p}_2'\sigma_2'n_2'\right) e^{-ip_2'\cdot x} u_l \left(\mathbf{p}_1\sigma_1 n_1\right) e^{ip_1\cdot x} \times$$

$$\times u_{m'}^* \left(p_1'\sigma_1'n_1'\right) e^{-ip_1'\cdot y} u_m \left(p_2\sigma_2 n_2\right) e^{ip_2\cdot y} \Big]$$

В еще более частном случае, когда бозоны являются бесспиновыми частицами одного сорта,

взаимодействие (6.1.29) принимает простой вид

$$\mathscr{H} = g\phi^3/3!$$

а матричный элемент для скаляр-скалярного рассеяния равен

$$S_{\mathbf{p}_1'\mathbf{p}_2',\mathbf{p}_1\mathbf{p}_2} = \frac{ig^2}{(2\pi)^6\sqrt{16E_1'E_2'E_1E_2}} \int d^4x \int d^4y \delta_F(x-y) \times$$

$$\times \left[\exp\left(-i\left(p_1'+p_2'\right)\cdot x\right)\exp\left(i\left(p_1+p_2\right)\cdot y\right) +\right.$$

$$+ \exp\left(i\left(p_1-p_1'\right)\cdot x\right)\exp\left(i\left(p_2-p_2'\right)\cdot y\right) +$$

$$\left. + \exp\left(i\left(p_1-p_2'\right)\cdot x\right)\exp\left(i\left(p_2-p_1'\right)\cdot y\right)\right]$$

где $\delta_F(x-y)$ - пропагатор скалярного поля, который мы вычислим в следующем параграфе. В этой теории отсутствуют диаграммы третьего, и вообще нечетного, порядка.

Вычисление пропагатора

Теперь мы перейдем к вычислению пропагатора (6.1.14),

важного компонента правил Фейнмана,

который возникает от спаривания поля $\psi_l(x)$ с сопряженным полем $\psi_m^\dagger(y)$.

Подставляя формулы (6.1.15) и (6.1.16) в уравнение (6.1.14) и применяя (анти)коммутационные соотношения для операторов рождения-уничтожения,

немедленно получаем:

$$-\delta_{lm}(x,y) = \theta(x-y)(2\pi)^{-3} \int d^3p \sum_\sigma u_l(\mathbf{p}\sigma n) u_m^*(\mathbf{p}\sigma n) e^{ip\cdot(x-y)} \pm$$

$$\pm\theta(y-x)(2\pi)^{-3}\int d^3p\sum_\sigma v_m^*(\mathbf{p}\sigma n)v_l^*(\mathbf{p}\sigma n)e^{ip\cdot(y-x)}$$

При вычислении (анти)коммутаторов в гл. 5 было показано, что

$$\sum_\sigma u_l(\mathbf{p}\sigma n)u_m^*(\mathbf{p}\sigma n)=\left(2\sqrt{\mathbf{p}^2+m_n^2}\right)^{-1}P_{lm}\left(\mathbf{p},\sqrt{\mathbf{p}^2+m_n^2}\right)$$
$$\sum_\sigma v_l(\mathbf{p}\sigma n)v_m^*(\mathbf{p}\sigma n)=\pm\left(2\sqrt{\mathbf{p}^2+m_n^2}\right)^{-1}P_{lm}\left(-\mathbf{p},-\sqrt{\mathbf{p}^2+m_n^2}\right)$$

где $P_{lm}(\mathbf{p},\omega)$ - полином по $\mathbf{p}$ и ω. Здесь, как и в формуле (6.2.1), верхний и нижний знаки отвечают соответственно бозонным и фермионным полям. Например, если $\psi_l(x)$ и $\psi_m(y)$ суть скалярные поля (частицы со спином 0) $\phi(x),\phi(y)$, то получается просто

$$P(p)=1$$

Если $\psi_l(x)$ и $\psi_m(y)$ - дираковские поля (частицы спина 1/2), то

$$P_{lm}(p)=\left[(-i\gamma_\mu p^\mu+m)\,\beta\right]_{lm}$$

где l,m обозначают индексы дираковских 4-компонентных спиноров. (Матрица β здесь появилась, поскольку мы рассматриваем спаривание $\psi_l(x)$ с $\psi_m^\dagger(y)$.) Ее не будет в спаривании $\psi_l(x)$ и $\bar\psi_m(y)\equiv\psi_m^\dagger(y)\beta$.) Если $\psi_l(x)$ - векторные поля (частицы спина 1), то

$$P_{\mu\nu}(p)=\eta_{\mu\nu}+m^{-2}p_\mu p_\nu$$

В общем случае, если $\psi_l(x)$ и $\psi_m(y)$ - компоненты полей $\psi_{ab}(x)$ и $\psi_{\tilde a\tilde b}(y)$ частицы со спином j в неприводимых представлениях (A,B) и $(\widetilde A,\widetilde B)$ однородной группы Лоренца, то

$$P_{ab,\tilde a\tilde b}(p)=\sum_{a'b'}\sum_{\tilde a'\tilde b'}\sum_\sigma C_{AB}\left(j\sigma,a'b'\right)C_{\tilde A\tilde B}\left(j\sigma,\widetilde a'\widetilde b'\right)\times$$
$$\times\left[\exp\left(-\theta\hat p\cdot\mathbf{J}^{(A)}\right)\right]_{aa'}\left[\exp\left(+\theta\hat p\cdot\mathbf{J}^{(B)}\right)\right]_{bb'}\times$$
$$\times\left[\exp\left(-\theta\widehat p\cdot\mathbf{J}^{(\tilde A)}\right)\right]_{\tilde a\tilde a'}\left[\exp\left(+\theta\hat p\cdot\mathbf{J}^{(\tilde B)}\right)\right]_{bb'}$$

где $\operatorname{sh}\theta=|\mathbf{p}|/m$, а индексы $a,b,\tilde a,\tilde b$ пробегают с шагом 1 значения от $-A$ до A, от $-B$ до B, от $-\widetilde A$ до $\widetilde A$, от $-\widetilde B$ до $\widetilde B$ соответственно и аналогично для индексов со штрихами. Подстановка (6.2.2) и (6.2.3) в уравнение (6.2.1) дает

$$-i\delta_{lm}(x,y)=\theta(x-y)P_{lm}\left(-i\tfrac{\partial}{\partial x}\right)\delta_+(x-y)+$$
$$+\theta(y-x)P_{lm}\left(-i\tfrac{\partial}{\partial x}\right)\delta_+(y-x)$$

где $\delta_+(x)$ - функция, введенная в гл. 5:

$$\delta_+(x)=(2\pi)^{-3}\int d^3p\left(2p^0\right)^{-1}e^{ip\cdot x}$$

где $p^0=+\sqrt{\mathbf{p}^2+m^2}$ Перед тем как продолжить вычисления, следует пояснить, как можно обобщить определение полинома $P(p)$. Уравнения (6.2.2) и (6.2.3) определяют его только

на так называемой массовой оболочке, т.е. при $p^0 = \pm\sqrt{\mathbf{p}^2 + m^2}$. Любой многочлен от 4-импульса на массо-

вой оболочке можно считать линейным по p^0, поскольку любая степень $(p^0)^{2\nu}$ или $(p^0)^{2\nu+1}$ может быть записана в виде $(\mathbf{p}^2 + m^2)^{2\nu}$ или $p^0 (\mathbf{p}^2 + m^2)^{2\nu}$ соответственно. Поэтому мы зададим полином $P^{(L)}(q)$ следующими условиями:

$$P^{(L)}(p) = P(p) \quad \left(\text{где } p^0 = \sqrt{\mathbf{p}^2 + m^2} \right)$$
$$P^{(L)}(q) = P^{(0)}(\mathbf{q}) + q^0 P^{(1)}(\mathbf{q}) \quad (\text{ для произвольного } q^\mu)$$

где $P^{(0,1)}$ - полиномы, зависяшие только от q. Используем теперь соотношения

$$\frac{\partial}{\partial x^0}\theta\left(x^0 - y^0\right) = -\frac{\partial}{\partial x^0}\theta\left(y^0 - x^0\right) = \delta\left(x^0 - y^0\right)$$

(вспомнив, что функция $\theta(x)$ имеет скачок на единицу в x^0 и постоянна в других местах), чтобы переместить операторы дифференцирования налево от θ -функций в формуле (6.2.8)

$$\begin{aligned}\delta_{lm}(x,y) =& P^{(L)}_{lm}\left(-i\frac{\partial}{\partial x}\right)\delta_F(x-y) + \\ &+ \delta\left(x^0 - y^0\right) P^{(1)}_{lm}(-i\nabla)\left[\delta_+(x-y) - \delta_+(y-x)\right]\end{aligned}$$

где δ_F - "фейнмановский пропагатор"

$$-i\delta_F(x) \equiv \theta(x)\delta_+(x) + \theta(-x)\delta_+(-x)$$

Однако при $x^0 = 0$ функция $\delta_+(x)$ четна по x, поскольку замену

$\mathbf{x} \to -\mathbf{x}$ в (6.2.9) можно компенсировать заменой $\mathbf{p} \to -\mathbf{p}$ переменной интегрирования. По этой причине можно опустить второе слагаемое в (6.2.12) и получить простое выражение

$$\delta_{lm}(x,y) = P^{(L)}_{lm}\left(-i\frac{\partial}{\partial x}\right)\delta_F(x-y)$$

Для дальнейших вычислений нам понадобится фурье-представление фейнмановского пропагатора. Функции-ступеньки в (6.2.13) записываются в виде интеграла Фурье)

$$\theta(t) = \frac{-1}{2\pi i}\int\limits_{-\infty}^{+\infty}\frac{\exp(-ist)}{s + i\epsilon}ds$$

Это можно объединить с интегралом Фурье (6.2.9) для функции $\delta_+(x)$. Введем новые переменные интегрирования, $\mathbf{q} = \mathbf{p}, q^0 = p^0 = s$ в первом слагаемом уравнения (6.2.13), что дает

$$-i\delta_F(x) = -\frac{1}{2\pi i}\int d^3q \int dq^0 \frac{\exp\left(i\mathbf{q}\cdot\mathbf{x} - iq^0x^0\right)}{(2\pi)^3 2\sqrt{\mathbf{q}^2 + m^2}} \times$$
$$\times\left[\left(q^0 - \sqrt{\mathbf{q}^2 + m^2} + i\epsilon\right)^{-1} + \left(-q^0 - \sqrt{\mathbf{q}^2 + m^2} + i\epsilon\right)^{-1}\right]$$

Приводя подынтегральное выражение к обшему знаменателю и используя 4-мерные обозначения, элементарно получаем

$$\delta_F(x) = (2\pi)^{-4}\int d^4q \frac{\exp(iq\cdot x)}{q^2 + m^2 - i\epsilon}$$

где $q^2 \equiv \mathbf{q}^2 - (q^0)^2$. В знаменателе мы заменили $2\epsilon\sqrt{\mathbf{q}^2+m^2}$ на ϵ, поскольку важно лишь то, что это положительная бесконечно малая величина. Отсюда видно, что δ_F - функция Грина оператора КлейнаГордона, в том смысле, что

$$\left(\Box - m^2\right)\delta_F(x) = -\delta^4(x)$$

а граничные условия определяются добавкой - $i\epsilon$ в знаменателе: как видно из формулы (6.2.13), при $x^0 \to \infty$ или $x^0 \to -\infty$ функция $\delta_F(x)$ содержит только положительные или отрицательные частоты, соответственно $\exp\left(-ix^0\sqrt{\mathbf{p}^2+m^2}\right)$ или $\exp\left(-ix^0\sqrt{\mathbf{p}^2+m^2}\right)$ В результате подстановки (6.2.16) в (6.1.14) получаем пропагатор

в виде

$$\delta_{lm}(x,y) = (2\pi)^{-4}\int d^4q \frac{P^{(L)}_{lm}(q)e^{iq\cdot(x-y)}}{q^2+m^2-i\epsilon}$$

Здесь возникает следующая проблема. Многочлен $P(p)$ лоренц-ковариантен, если p лежит на массовой оболочке, т.е. $p^2 = -m^2$, но в уравнении (6.2.18) производится интегрирование по всем q^μ, а не только по массовой оболочке. Для произвольных q^μ многочлен $P^{(L)}(q)$ по определению линеен по q^0, что явно нарушает лоренц-ковариантность, если многочлен не является линейным по каждой пространственной компоненте q^i. Вместо такого обобщения многочлена $P(p)$ можно ввести его лоренц-ковариантное обобщение на произвольные q^μ (которое мы обозначим просто $P(q)$), в том смысле, что

$$P_{lm}(\Lambda q) = D_{ll'}(\Lambda)D^*_{mm'}(\Lambda)P_{l'm'}(q)$$

где Λ^μ_ν - преобразование Лоренца общего вида, а $D(\Lambda)$ - некоторое представление группы Лоренца. К примеру, в случае скалярных, дираковских и векторных полей такое ковариантное обобщение получается простой заменой импульса p^μ на импульс общего вида q^μ в формулах (6.2.4), (6.2.5) и (6.2.6). В случае скалярных и дираковских

полей пропагатор и так линеен по q^0, а потому нет разницы меж-

ду $P^{(L)}(q)$ и $P(q)$:

$$P^{(L)}_{lm}(q) = P_{lm}(q) \qquad \text{(скалярное поле, дираковское поле)}$$

С другой стороны, в случае векторного (спин-1) поля 00-компонента ковариантного полинома $P_{\mu\nu}(q) \equiv \eta_{\mu\nu} + m^{-2}q_\mu q_\nu$ квадратична по q^0, так что здесь есть разница между $P^{(L)}(q)$ и $P(q)$:

$$\begin{aligned} P^{(L)}_{\mu\nu}(q) = \eta_{\mu\nu} + m^{-2}\left[q_\mu q_\nu - \delta^0_\mu\delta^0_\nu\left(q_0^2 - \mathbf{q} - m^2\right)\right] = \\ = P_{\mu\nu}(q) + m^{-2}\left(q^2+m^2\right)\delta^0_\mu\delta^0_\nu \end{aligned}$$

Дополнительное слагаемое здесь фиксировано двумя условиями - оно должно сокращать слагаемое $(q^0)^2$ в $P_{00}(q)$, и должно зануляться на массовой оболочке. Подстановка этого выражения в формулу (6.2.18) дает пропагатор векторного поля

$$\delta_{\mu\nu}(x,y) = (2\pi)^{-4}\int d^4q \frac{P_{\mu\nu}(q)e^{iq\cdot(x-y)}}{q^2+m^2-i\epsilon} + m^{-2}\delta^{-4}(x-y)\delta^0_\mu\delta^0_\nu$$

Первое слагаемое, очевидно, ковариантно, а второе - не ковариантно, но является локальным, так что его можно сократить путем добавления локального нековариантного члена в гамильтониан. Конкретно, если $V_\mu(x)$ взаимодействует с другими полями

посредством члена $V_\mu(x)J^\mu(x)$ в гамильтониане, то вклад добавочного слагаемого в (6.2.21) равносилен добавлению взаимодействия

$$\mathscr{H}_{NC}(x) = -\mathscr{H}_{eff}(x) = \frac{1}{2m^2}\left[J^0(x)\right]^2$$

Именно сингулярность (при нулевом расстоянии) одновременных коммутаторов векторных полей вынуждает нас использовать более широкий класс взаимодействий, чем просто скалярную плотность гамильтониана. Подробное непертурбативное доказательство лоренцинвариантности S -матрицы в этой теории будет представлено в следующей главе.

Не стоит думать, что это явление специфично для полей спина $j \geqslant 1$. Например, рассмотрим векторное поле, построенное из скалярного поля, равное (см. гл. 5) его производной $\partial_\lambda\phi(x)$. В случае

спаривания этого поля со скалярным полем, $\phi^\dagger(y)$, полином $P(p)$ на массовой оболочке равен

$$P_\lambda(p) = ip_\lambda$$

а спаривание того же поля с $\partial_\eta\phi^\dagger(y)$ дает многочлен

$$P_{\lambda,\eta} = p_\lambda p_\eta$$

Ковариантные многочлены для импульса q^μ вне массовой оболочки опять получаются заменой q^μ на p^μ в формулах (6.2.13) и (6.2.24). Из формулы (6.2.23) мы видим, что $P_\lambda(q)$ линеен по q_0, значит, полиномы $P_\lambda(q)$ и $P^{(L)}_\lambda(q)$ совпадают. Но в случае формулы (6.2.24) разница между ними есть:

$$P^{(L)}_{\lambda,\eta}(q) = q_\lambda q_\eta - \left(q_0^2 - \mathbf{q}^2 - m^2\right)\delta^0_\lambda\delta^0_\eta = \\ = P_{\lambda,\eta}(q) + \left(q^2 + m^2\right)\delta^0_\lambda\delta^0_\eta$$

так что пропагатор равен

$$\delta_{\lambda,\eta}(x,y) = (2\pi)^{-4}\int d^4q \frac{q_\lambda q_\eta e^{iq\cdot x}}{q^2 + m^2 - i\epsilon} + \delta^0_\lambda\delta^0_\eta\delta^4(x-y)$$

Так же как и в предыдущем примере, нековариантный вклад второго слагаемого можно убрать, добавив в гамильтониан нековариантное взаимодействие

$$\mathscr{H}_{NC}(x) = \frac{1}{2}\left[J^{(0)}(x)\right]^2$$

где $J^\mu(x)$ - это ток, на который умножается градиент $\partial_\mu\phi(x)$ в ковариантной части $\mathscr{H}(x)$.

Теперь должно стать понятно, что, по крайней мере для массивных частиц, вклады от нековариантных слагаемых в пропагаторе всегда можно уничтожить путем добавления в гамильтониан нековариантных локальных членов. Это объясняется тем, что числитель $P^{(L)}_{lm}(q)$ в пропагаторе должен быть равен ковариантному полиному $P'_{lm}(q)$, когда q^μ лежит на массовой оболочке, так что разность между ними должна содержать множитель $q^2 + m^2$. Этот множитель сокращается со знаменателем $(q^2 + m^2 - i\epsilon)$ во вкладе этой разности в формулу (6.2.18), следовательно, уравнение (6.2.18) всегда состоит из ковариантного члена и слагаемого, пропорционального дельта-функции $\delta^4(x-y)$ или ее производным. Вклад последнего слагаемого можно убрать, добавив в гамильтониан взаимодействия член, квадратичный по току, с которым взаимодействуют спаренные поля, или по производным

этих полей. В дальнейшем мы будем неявно предполагать, что такое слагаемое уже введено в гамильтониан, а потому будем использовать ковариантный полином $P_{lm}(q)$ в пропагаторе (6.2.18), и будем опускать индекс L.

Может показаться, что такая процедура несколько искусственна. К счастью, в каноническом формализме, который мы рассмотрим в следующей главе, нековариантное слагаемое, нужное для сокрашения нековариантных членов в пропагаторе, появляется автоматически. Это, в частности, может служить мотивацией для развития канонического формализма.

* * *

Прежде чем закончить этот параграф, следует упомянуть некоторые другие часто встречающиеся в литературе определения пропагатора, эквивалентные уравнению (6.2.1). Во-первых, взятие вакуумного среднего от обеих частей уравнения (6.1.14) дает

$$-\delta_{lm}(x,y) = \theta(x-y)\left\langle \left[\psi_l^+(x), \psi_m^{+\dagger}(y)\right]_{\mp}\right\rangle_0 \pm \\ \pm\theta(y-x)\left\langle \left[\psi_m^{-\dagger}(y), \psi_l^-(x)\right]_{\mp}\right\rangle_0$$

Здесь $\langle AB...\rangle_0$ обозначает вакуумное среднее $(\Phi_0, AB...\Phi_0)$. Оба оператора $\psi_l^+(x)$ и $\psi_m^{-\dagger}(y)$ уничтожают вакуум, значит, остается только одно слагаемое в каждом (анти)коммутаторе в уравнении (6.2.28), даюшее вклад в пропагатор:

$$-i\delta_{lm}(x,y) = \theta(x-y)\left\langle \psi_l^+(x)\psi_m^{+\dagger}(y)\right\rangle_0 \pm \theta(y-x)\left\langle \psi_m^{-\dagger}(y)\psi_l^-(x)\right\rangle_0$$

Кроме того, $\psi^{-\dagger}$ и ψ^+ уничтожают вакуумное состояние справа, аψ^- и $\psi^{+\dagger}$ - слева, поэтому ψ^+ и ψ^- повсюду в уравнении (6.2.29)

можно заменить на полное поле $\psi = \psi^+ + \psi^-$:

$$-i\delta_{lm}(x,y) = \theta(x-y)\left\langle \psi_l(x)\psi_m^{\dagger}(y)\right\rangle_0 \pm \theta(y-x)\left\langle \psi_m^{\dagger}(y)\phi_l(x)\right\rangle_0$$

Последнюю формулу можно переписать в виде

$$-i\delta_{lm}(x,y) = \left\langle T\left\{\psi_l(x)\psi_m^{\dagger}(y)\right\}\right\rangle_0$$

где T - хролологическое упорядочение (или символ хронологизации), обобщенное теперь [2]) для любых полей, со знаком минус для любой нечетной перестановки фермионных операторов.

14.2.9 Examples of Feynman Rules in Different Theories

(in cond matter, in QFT, there are many of its)

Feynman Rules in QED

(see Peskin Shroder for now)

14.2.10 Furry's theorem about Fermionic loop

теорема Фари о зануленииfермионной петли с нечетным числом вершин

Если в диаграмме есть фермионная петля с нечетным числом фотонных вершин, то вклад этой петли равен нулю.

В самом деле, поскольку электродинамика инвариантна относительно зарядового сопряжения,

а указанная петля содержит в качестве внешних к ней линий только фотонные, можно вычислить вклад этой петли в зарядово симметричной теории, т. е. просто заменив в вершинах заряд e на $-$.

Тогда петля в точности совпадет сама с собой, но с обратным знаком, а значит, вклад тождественно равен нулю.

Это можно наглядно интерпретировать графически: весь вклад пели соответствует графику с определенным направлением стрелки, отвечающему направлению движения заряда, и графику с противоположным направлением стрелки, а их сумма тождественно обращается в нуль в силу зарядовой инвариантности электродинамики и нечетного числа k произведения факторов $(-1)^k$ для второго графика с обратным направлением стрелки.

(пример применения теоремы Фари нужен)

14.2.11 Feynman Diagrams by Zi.I.7 1

Ангармоничность в теории поля

Для теории свободного поля, которую мы рассматривали в последних главах, найти решение легко, поскольку определяющий интеграл по траекториям (I.3.14) является гауссовым, так что можно просто применить (I.2.15). (Это соответствует решению задачи о гармоническом осцилляторе в квантовой механике.) Как говорилось в главе I.3, в рамках гармонического приближения колебательные моды на матраце можно линейно складывать, и таким образом они просто проходят друг сквозь друга. Частицы, представленные волновыми пакетами, составленными из этих мод, не взаимодействуют [1]: отсюда и название «теория свободного поля». Для того чтобы моды начали рассеиваться друг на друге, мы должны включить в лагранжиан ангармонические члены так, чтобы уравнения движения перестали быть линейными. Для простоты добавим к нашей свободной теории только один ангармонический член $-\frac{\lambda}{4!}\varphi^4$ и с учетом (I.3.11) попробуем вычислить

$$Z(J) = \int D\varphi e^{i\int d^4x\left\{\frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right] - \frac{\lambda}{4!}\varphi^4 + J\varphi\right\}}$$

(Мы опустили зависимость Z от λ.)

Решать квантовую теорию поля совсем нетрудно, скажете вы, для этого надо всего лишь вычислить функциональный интеграл (1). Но интеграл-то не из легких! Если бы вы смогли его вычислить, это была бы сенсация.

[1] Потенциальный источник путаницы. Вследствие распространения φ, источники. связанные с φ, как мы видели в главе I.4, взаимодействуют, но частицы, ассоциированные с полем φ, не взаимодействуют друг с другом. Такая ситуация аналогична утверждению, что заряженные частицы. связанные с фотоном, взаимодействуют. но фотоны (в главном приближении) не взаимодействуют друг с другом.

Диаграммы Фейнмана облегчают задачу

Будучи студентом, я слышал об этих таинственных маленьких картинках, называемых диаграммами Фейнмана, и очень хотел о них узнать. Я уверен, что и вы интересуетесь этими забавными диаграммами. Итак, я покажу вам, что в диаграммах Фейнмана нет ничего такого: на самом деле мы уже рисовали маленькие пространственно-временные картинки в главах I.3 и I.4), демонстрирующие, как частицы появляются, распространяются и исчезают.

Для того, кто впервые изучает квантовую теорию поля, диаграммы Фейнмана кажутся несколько сложными. Для их вывода в стандартных учебниках используется канонический формализм (который я введу в следующей главе), вместо формализма интеграла по траекториям, используемого здесь. Впоследствии мы увидим, что в каноническом формализме поля являются квантовыми операторами. Для вывода диаграмм Фейнмана мы должны были бы решить уравнения движения для полевых операторов с применением теории возмущений относительно λ. Для этого требуется развить довольно сложную технику.

По мнению тех, кто предпочитает формализм интеграла по траекториям, построение диаграмм в нем значительно проще (и это естественно!). Тем не менее вывод диаграмм может быть довольно сложным, и студент легко может потерять нить. Так или иначе, придется потрудиться.

Я постараюсь максимально облегчить вам задачу, применив один замечательный педагогический прием, состоящий в том, чтобы дать вам возможность самим открыть диаграммы Фейнмана. Буду придерживаться следующей стратегии: дам вам решить две задачи на возрастающую сложность, которые я называю задачей для младенца и задачей для ребенка соответственно. После того как вы их одолеете, вычисление (1) покажется вам намного проще.

Задача для младенца

Задача для младенца состоит в вычислении обычного интеграла

$$Z(J)=\int\limits_{-\infty}^{+\infty}dqe^{-\frac{1}{2}m^2q^2-\frac{\lambda}{4!}q^4+Jq}$$

который, очевидно, является более простой версией (1). Сначала сделаю тривиальное замечание: мы всегда можем заменить $q\to q/m$ так, чтобы $Z=m^{-1}\mathcal{F}\left(\frac{\lambda}{m^4},\frac{J}{m}\right)$, но делать этого не будем.

Для $\lambda=0$ получаем всего лишь один из гауссовых интегралов, описываемых в приложении к главе I.2. Вы скажете, что $Z(J)$ достаточно просто вычислить как ряд по λ: разложив

$$Z(J)=\int\limits_{-\infty}^{+\infty}dqe^{-\frac{1}{2}m^2q^2+Jq}\left[1-\frac{\lambda}{4!}+\frac{1}{2}\left(\frac{\lambda}{4!}\right)^2q^8+...\right]$$

и почленно проинтегрировав. Вы, возможно, даже знаете один из нескольких приемов вычисления $\int\limits_{-\infty}^{+\infty}dqe^{-\frac{1}{2}m^2q^2+Jq}q^{4n}$: можно записать интеюрал в виде $\left(\frac{d}{dJ}\right)\int\limits_{-\infty}^{+\infty}dqe^{-\frac{1}{2}m^2q^2+Jq}$ и применить (I.2.11). Итак:

$$Z(J)=\left(1-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4+\frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\left(\frac{d}{dJ}\right)^8+...\right)\int\limits_{-\infty}^{+\infty}dqe^{-\frac{1}{2}m^2q^2+.I!}$$

$$=e^{-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4}\int\limits_{-\infty}^{+\infty}dqe^{-\frac{1}{2}m^2q^2+Jq}=\left(\frac{2\pi}{m^2}\right)^{1/2}e^{-\frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4}e^{\frac{1}{2mm^2},'}$$

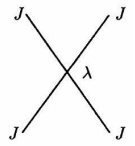
(Есть и другие способы, например, многократное дифферснцировап $\int\limits_{-\infty}^{+\infty}dqe^{-\frac{1}{2}m^2q^2+Jq}$ по m^2, но я хочу использовать трюк, который также ра ботает и в теории поля.)

Разложим обе экспоненты и получим любой члс двойного степенного разложения $Z(J)$ по λ и J. [Часто общий множитл. $(2\pi/m^2)^{1/2}=Z(J=0,\lambda=0)\equiv Z(0,0)$ опускается, поскольку он одынаков для всех членов ряда. Когда нам захочется точности, мы опредслим $\widetilde{Z}=Z(J)/Z(0,0)\cdot$

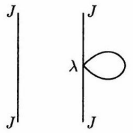
Предположим, например, что нам нужен член в $\widetilde{Z}$ порядка λ и J^4. Выделим член порядка J^8 в $e^{J^2/2m^2}$, то есть $\left[1/4!\left(2m^2\right)^4\right]J^8$, замсиим $e^{-(\lambda/4!)(d/dJ)^4}$ на $-(\lambda/4!)(d/dJ)^4$ и продифференцируем, чтобы получить $\left[8(-\lambda)/(4!)^3\left(2m^2\right)^4\right]J^4$.

Другой пример: член порядка λ^2 и J^6 разен $\frac{1}{2}(\lambda/4!)^2(d/dJ)^8\left[1/7!\left(2m^2\right)^7\right]J^{14}=\left[14!(-\lambda)^2/(4!)^2 6!7!2\left(2m^2\right)^7\right]J^6$. Третий пример: член порядка λ^2 и J^4 имеет вид $\left[12!(-\lambda)^2/(4!)^3 2(6!)\left(2m^2\right)^6\right]J^4$.

И наконец, для члена порядка λ и J^0 получаем $\left[1/2\left(2m^2\right)^2\right](-\lambda)$.



(a)



(b)

$$\left(\frac{1}{m^2}\right)^4\lambda J^4$$

(c)

Рис. I.7.1

Вы можете делать это не хуже меня! Получите еще несколько членов ряда, и вы быстро поймете их структуру. Со временем вы узнаете, что можно отождествлять члены ряда с диаграммами и формулировать для них правила. Рассмотренные нами четыре примера приводятся в соответствие диаграммам, изображенным на рисунках I.7.1-I.7.4. Можно видеть, что по некоторой причине, которую вы потом поймете, каждый член сопоставляется с несколькими диаграммами. В качестве упражнения постарайтесь точно сформулировать соответствующие правила, чтобы получить правильные численные коэффициенты (но поверьте мне, «будущее демократии» от них не зависит.) Правила должны звучать примерно так: (1) диаграммы строятся из линий и вершин, в которых встречаются четыре линии; (2) каждой вершине соответствует множитель $(-\lambda)$; (3) каждой линии соответствует $1/m^2$; и (4) каждому внешнему концу соответствует J (например, рис. I.7.2 имеет семь линий, две вершины и шесть внешних концов, что дает $\sim \left[(-\lambda)^2/\left(m^2\right)^7\right] J^6$). (Вы заметили, что удвоенное число линий равно числу вершин, помноженному на 4, плюс число внешних концов? Соотношения подобного рода встретятся нам в главе III.2.)

По очевидным причинам некоторые диаграммы (например, рис. I.7.1 a, I.7.2a) называются древесными [2], а другие (например, рис. I.7.1b и I.7.3a) петлевыми.

[2] Китайский иероглиф, обозначающий дерево (A. Zi, Swallowing Clouds). показан на рис. I.7.5. В качестве упражнения подумайте, почему эта диаграмма не появляется в разложении $Z(J)$ (?????)

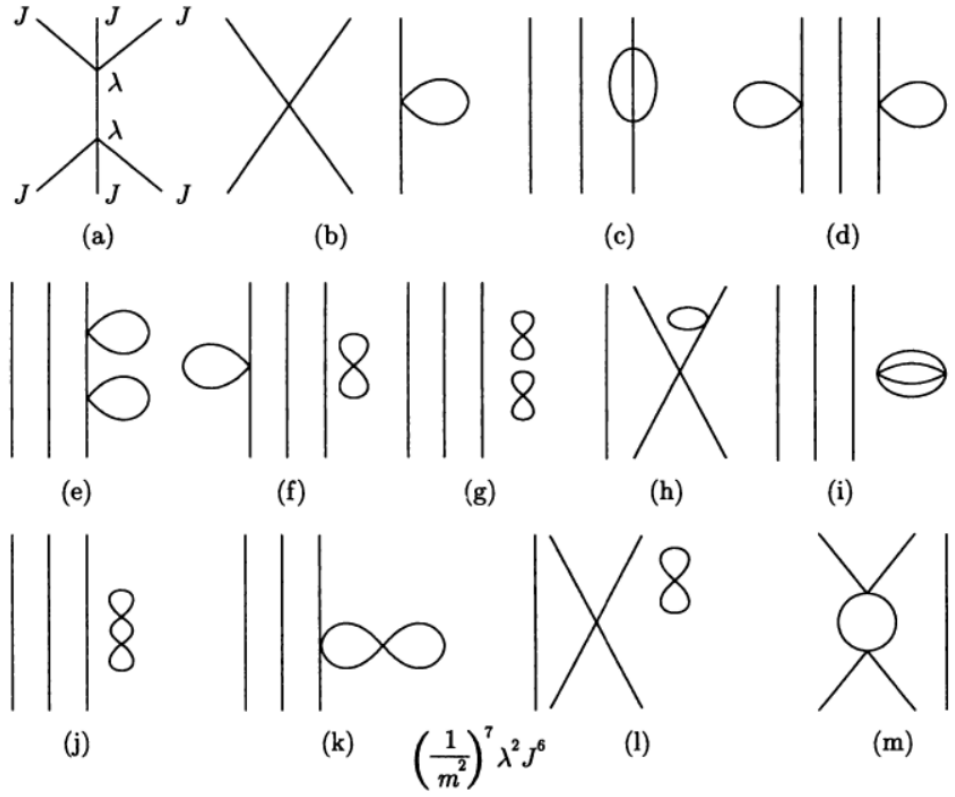
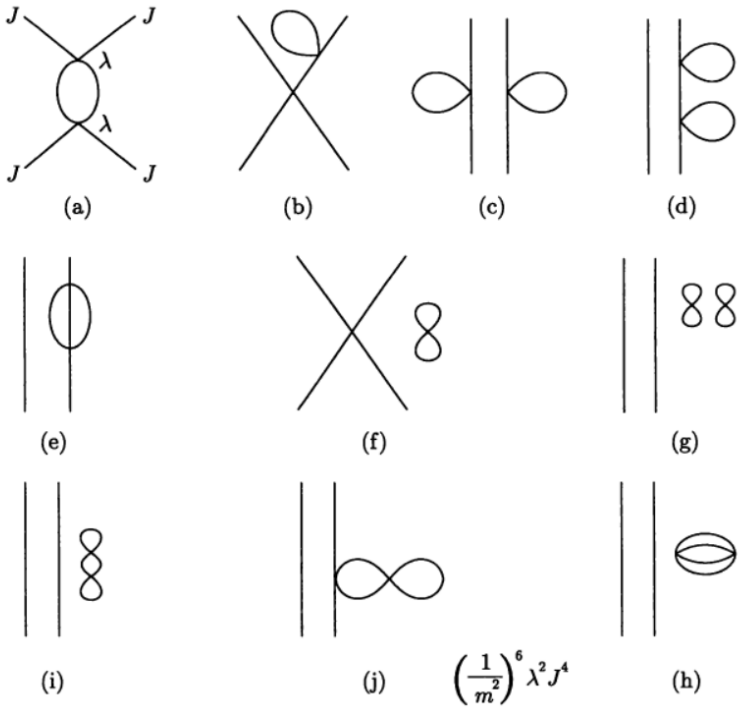


Рис. I.7.2

Рассмотрите столько примеров, сколь вам нужно, чтобы добиться должного понимания происходящего, так как мы будем делать в точности то же самое в квантовой теории поля.

Там это будет выглядеть более запутанным, но лишь на первый взгляд. Прежде чем двигаться дальше, удостоверьтесь, что вы понимаете, как использовать диаграммы для представления двойного степенного разложения Z(J). Из своей преподавательской практики я знаю, что у студентов, основательно не разобравшихся с разложением $\widetilde{Z}(J)$, нет шансов понять то, что мы собираемся делать в контексте теории поля.



(??? не уверен, что это тут лучше ставить, но пусть пока тут побудут.)

Виковская свертка

Совершенно очевидно, что $Z(J)$ можно при желании разложить в ряд по степеням J, а не λ. Далее вы увидите, что специалисты в области физики элементарных частиц предпочитают классификацию по степеням J. В рамках нашей задачи для младенца можно записать

$$Z(J) = \sum_{s=0}^{\infty} \frac{1}{s!} J^s \int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4} q^s \equiv Z(0,0) \sum_{s=0}^{\infty} \frac{1}{s!} J^s G^{(s)}$$

Коэффициент $G^{(s)}$, аналоги которого в квантовой теории поля называются функциями Грина, можно вычислить как ряд по λ, каждый член которого определяется виковской сверткой (I.2.10). Например, член $O(\lambda)$ в $G^{(4)}$ имеет вид

$$\frac{-\lambda}{4!Z(0,0)} \int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2q^2} q^8 = \frac{-7!!\lambda}{4!m^8}$$

и который, конечно, равен [3] значению, полученному нами ранее для члена λJ^4 в $\widetilde{Z}$. Таким образом, существует два способа вычисления Z: с разложением сначала по λ или по J.

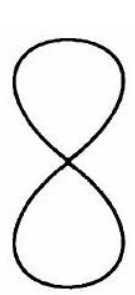


Рис. I.7.4

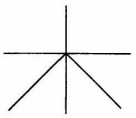


Рис. I.7.5

Связанные против несвязанных

Вы, должно быть, заметили, что некоторые диаграммы Фейнмана являются связанными, а другие - нет. Так, диаграмма на рис. I.7.2a связанная, а на рис. I.7.2b нет. Я предсказывал это в конце главы I.4 и на рис. I.4.2 и I.4.3. Запишем

$$Z(J,\lambda) = Z(J=0,\lambda)e^{W(J,\lambda)} = Z(J=0,\lambda)\sum_{N=0}^{\infty}\frac{1}{N!}[W(J,\lambda)]^N.$$

По определению, $Z(J=0,\lambda)$ состоит из диаграмм без внешнего источника J, таких как диаграмма на рис. I.7.4. Мы утверждаем, что W является суммой связанных диаграмм, тогда как Z содержит как связанные, так и несвязанные диаграммы. Таким образом, рисунок I.7.2b состоит из двух несвязанных частей и определяется членом $(1/2!)[W(J,\lambda)]^2$ в (6); множитель 2 ! указывает на то, что нам абсолютно не важно, какую из двух частей мы помещаем «слева или справа». Аналогично рис. I.7.2c определяется членом $(1/3!)[W(J,\lambda)]^3$. Значит, необходимо вычислять W, а не Z. Если вам хорошо преподавали статистическую механику, вы должны признать, что за всеми этими рассуждениями о связных и несвязных графах лежит то, что служит основой связи между свободной энергией и статистическкой суммой.

Распространение: отсюда туда

Все эти особенности задачи для младенца по своей структуре совершенно аналогичны особенностям теории поля, и мы можем немедленно

[3] В качестве проверки законов арифметики удостоверимся, что действительно $7!!/(4!)^2 = 8!/(4!)^3 2^4$. обратиться к нашим рассуждениям. Однако, прежде чем переходить к теории поля, рассмотрим так называемую задачу для ребенка, т. е. задачу на кратные (а не обычные) интегралы:

$$Z(J) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} dq_1 dq_2 ... dq_N e^{-\frac{1}{2}q\cdot A\cdot q - \frac{\lambda}{4!}q^4 + J\cdot q}$$

где $q^4 \equiv \sum_i q_i^4$. Обобщая последовательность действий, приведших нас к (3), получаем

$$Z(J)=\left[\frac{(2\pi)^N}{\det A}\right]^{1/2} e^{-\frac{\lambda}{4!}\sum_i(\partial/\partial J_i)^4} e^{\frac{1}{2}J\cdot A^{-1}\cdot J}$$

В качестве альтернативы мы могли бы, как в (5), разложить по степеням J

$$Z(J)=\sum_{s=0}^{\infty}\frac{1}{s!}J_{i_1}...J_{i_s}\int\limits_{-\infty}^{+\infty}\left(\prod_l dq_l\right)e^{-\frac{1}{2}q\cdot A\cdot q-\frac{\lambda}{4!}q^4}q_{i_1}...q_{i_s}=$$
$$=Z(0,0)\sum_{s=0}^{\infty}\frac{1}{s!}J_{i_1}...J_{i_s}G^{(s)}_{i_1}...i_2$$

затем разложить по степеням λ и вычислить виковские свертки.

Одна из особенностей задачи для ребенка, которая отсутствует в задаче для младенца, состоит в распространении «отсюда туда». Вспомним обсуждение пропагатора в главе I.3. Как и в (I.3.16), мы можем рассматривать индекс i как метку узла решетки. В самом деле, в (I.3.16) мы, по сути, вычислили «двухточечную функцию Грина» $G^{(2)}_{ij}$ в нулевом порядке по λ (дважды продифференцируйте (I.3.16) по J):

$$G^{(2)}_{ij}(\lambda=0)=\left[\int\limits_{-\infty}^{+\infty}\left(\prod_l dq_l\right)e^{-\frac{1}{2}q\cdot A\cdot q}q_iq_j\right]/Z(0,0)=\left(A^{-1}\right)_{ij}$$

(см. также приложение к главе I.2). Матричный элемент $\left(A^{-1}\right)_{ij}$ описывает распространение из i в j. В задаче для младенца каждый член в разложении $Z(J)$ можно ассоциировать с несколькими диаграммами, но при наличии распространения этого сделать нельзя. Теперь вычислим «четырехточечную функцию Грина» $G^{(4)}_{ijkl}$ в первом порядке по λ:

$$G^{(4)}_{ijkl}=\int\limits_{-\infty}^{+\infty}\left(\prod_m dq_m\right)e^{-\frac{1}{2}q\cdot A\cdot q}q_iq_jq_kq_l\left[1-\frac{\lambda}{4!}\sum_n q_n^4+O\left(\lambda^2\right)\right]/Z(0,0)=$$
$$=\left(A^{-1}\right)_{ij}\left(A^{-1}\right)_{kl}+\left(A^{-1}\right)_{ik}\left(A^{-1}\right)_{jl}+\left(A^{-1}\right)_{il}\left(A^{-1}\right)_{jk}-$$
$$-\lambda\sum_n\left(A^{-1}\right)_{in}\left(A^{-1}\right)_{jn}\left(A^{-1}\right)_{kn}\left(A^{-1}\right)_{ln}+O\left(\lambda^2\right). \quad (10)$$

Первые три члена описывают распространение одного возбуждения из i в j, а другого - из k в l, плюс две возможные перестановки этой «истории». Наличие члена, порядок которого равен λ, указывает на то, что четыре возбуждения, распространяющиеся из i в n, из j в n, из k в n и из l в n, встречаются в n и взаимодействуют с амплитудой пропорциональной λ, где n может находиться где угодно на решетке или матраце. Между прочим, видно, почему удобно определять взаимодействие $(\lambda/4!)\varphi^4$c1/4! : q_i имеет на выбор четыре q_n для свертывания, q_j - три q_n и так далее, что дает множитель 4 !, который сокращается с (1/4!).

Пертурбативная теория поля

Теперь вы должны быть готовы перейти к теории поля!

В самом деле, функциональный интеграл в (1) (который я здесь снова привожу)

$$Z(J)=\int D\varphi e^{i\int d^4x\left\{\frac{1}{2}\left[(\partial\varphi)^2-m^2\varphi^2\right]-(\lambda/4!)\varphi^4+J\varphi\right\}}$$

имеет такую же форму, что и обыкновенный интеграл в (2) и кратный интеграл в (7). Имеется одна незначительная разница: в (2) и (7) отсутствует i, но, как говорилось в главе I.2, мы можем совершить над (11) виковский поворот и избавиться от i, но делать этого не будем. Существенное отличие состоит в том, что J и φ в (11) являются функциями от непрерывной переменной x, в то время как J и q в (2) ни от чего не зависят, а в (7) они зависят от дискретной переменной. Если не считать этого, все остальное совсршснно идснтично.

Как в (3) и (8) имеем

$$Z(J)=e^{-\frac{i}{4!}\lambda\int d^4w[\delta/i\delta J(w)]^4}\int D\varphi e^{i\int d^4x\left\{\frac{1}{2}\left[(\partial\varphi)^2-m^2\varphi^2\right]+J\varphi\right\}}=$$

$$=Z(0,0)e^{-\frac{i}{4!}\lambda\int d^4w[\delta/i\delta J(w)]^4}e^{-\frac{i}{2}\iint d^4xd^4yJ(x)D(x-y)J(y)}$$

Полное структурное подобие!

Роль $1/m^2$ в (3) и A^{-1} в (8) теперь играет пропагатор

$$D(x-y)=\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(x-y)}}{k^2-m^2+i\varepsilon}$$

Кстати, если вернуться к главе I.3, то можно увидеть, что в d-мерном пространстве-времени выражение для $D(x-y)$ имеет такую же форму, но с заменой $d^4k/(2\pi)^4$ на $d^dk/(2\pi)^d$. Обыкновенный интеграл (2) подобен теории поля в 0 -мерном пространстве-времени: если положить $d=0$, то распространение будет отсутствовать и выражение для $D(x-y)$ сведется к виду $-1/m^2$. Согласитесь, что все это вполне логично.

Мы тажже знаем, что $J(x)$ соответствует источнику или стоку. Таким образом, если бы мы разложили $Z(J)$ по J, то степени J отвечали бы числу частиц, вовлеченных в процесс. (Обратите внимание, что в используемой системе обозначений процесс рассеяния $\varphi+\varphi\to\varphi+\varphi$ рассматривается как 4-частичный процесс: мы подсчитываем общее число входящих и выходящих частиц.) Поэтому в теории поля часто разумнее определять степень J. Аналогично тому, как это делалось в младенческой и детской задачах, можно начать с разложения по J:

$$Z(J)=Z(0,0)\sum_{s=0}^{\infty}\frac{1}{s!}\int d^4x_1...d^4x_sJ\left(x_1\right)...J\left(x_s\right)G^{(s)}\left(x_1,...,x_s\right)=$$

$$=\sum_{s=0}^{\infty}\frac{1}{s!}\int d^4x_1...d^4x_sJ\left(x_1\right)...J\left(x_s\right)\times$$

$$\times\int D\varphi e^{i\int d^4x\left\{\frac{1}{2}\left[(\partial\varphi)^2-m^2\varphi^2\right]-\frac{\lambda}{4!}\varphi^4\right\}}\varphi\left(x_1\right)...\varphi\left(x_s\right)$$

В частности, имеем двухточечную функцию Грина:

$$G\left(x_1,x_2\right)\equiv\frac{1}{Z(0,0)}\int D\varphi e^{i\int d^4x\left\{\frac{1}{2}\left[(\partial\varphi)^2-m^2\varphi^2\right]-\frac{\lambda}{4!}\varphi^4\right\}}\varphi\left(x_1\right)\varphi\left(x_2\right)$$

четырехточечную функцию Грина:

$$G\left(x_1,x_2x_3x_4\right)\equiv\frac{1}{Z(0,0)}\int D\varphi e^{i\int d^4x\left\{\frac{1}{2}\left[(\partial\varphi)^2-m^2\varphi^2\right]-\frac{\lambda}{4!}\varphi^4\right\}}\varphi\left(x_1\right)\varphi\left(x_2\right)\varphi\left(x_3\right)\varphi\left(x_4\right)$$

и так далее.

[Иногда $Z(J)$ называют производящим функционалом, так как он порождает функции Грина.]

Очевидно, что в силу трансляционной инвариантности $G(x_1, x_2)$ зависит лишь от $x_1 - x_2$, а не от x_1 и x_2 по отдельности. Аналогично, $G(x_1, x_2, x_3, x_4)$ зависит только от $x_1 - x_4$, $x_2 - x_4$ и $x_3 - x_4$. При $\lambda = 0$ выражение для $G(x_1, x_2)$ сводится к виду $iD(x_1 - x_2)$, т. е. к пропагатору из главы I.3. В то время как $D(x_1 - x_2)$ описывает распространение частицы между x_1 и x_2 в отсутствие взаимодействия, пропагатор $G(x_1 - x_2)$ отвечает за распространение частицы между x_1 и x_2 в присутствии взаимодействия. Если вы поняли, как строится $G^{(4)}_{ijkl}$, то должны знать, что $G(x_1, x_2, x_3, x_4)$ описывает рассеяние частиц.

В некотором смысле существует два способа построения теории поля, которые я мог бы назвать способом Швингера (12) и способом Вика (13).

Подведем итог: с помощью диаграмм Фейнмана очень удобно представлять члены двойного степенного разложения $Z(J)$ по λ и J.

Как говорилось в предисловии, я не стремлюсь сделать из вас специалистов по вычислению фейнмановских диаграмм. В любом случае это может прийти только с опытом. К тому же, существует много прекрасных книг, посвященных вычислению диаграмм. Я постарался дать вам максимально ясное представление о концепции, скрывающейся за этим выдающимся изобретением Фейнмана, которое, как довольно едко заметил Швингер, позволило практически каждому стать теоретиком-полевиком. А пока не думайте слишком много о множителях 4 ! и 2 !.

Столкновения между частицами

Как я уже отмечал, в главе I.4 описывается стратегия введения источников и стоков с целью наблюдения за распространением частицы (которую я буду называть мезоном), связанной с полем φ. Давайте введем два источника и два стока для наблюдения за рассеянием двух мезонов друг на друге (рис. I.7.6). Оба источника, локализованных в областях 1 и 2, порождают по одному мезону, и эти два мезона со временем исчезают в стоках, локализованных в областях 3 и 4. В этом случае достаточно найти в Z член, содержащий $J(x_1) J(x_2) J(x_3) J(x_4)$. Но он в точности совпадает с $G(x_1, x_2, x_3, x_4)$.

Ограничимся вычислением в первом порядке по λ. Следуя Вику, мы должны вычислить

$$\frac{1}{Z(0,0)} \left(-\frac{i\lambda}{4!}\right) \int d^4w \int D\varphi e^{i\int d^4x \left\{\frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right]\right\}}{}_{\times}$$
$$\times \varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\varphi(w)^4.$$

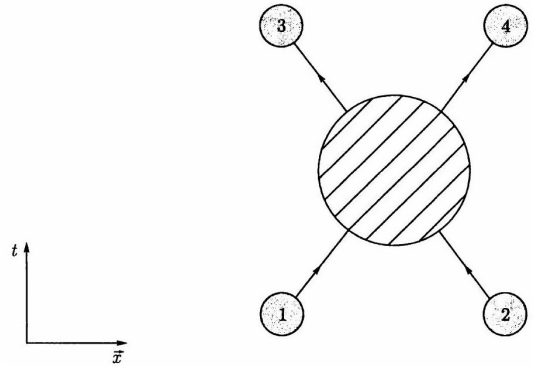


Рис. I.7.6

Как и в (10), мы образуем виковские свертки и получаем

$$(-i\lambda)\int d^4w D\left(x_1-w\right)D\left(x_2-w\right)D\left(x_3-w\right)D\left(x_4-w\right)$$

В качестве проверки получим это выражение способом Швингера. Заменим $e^{-(i/4!)\lambda\int d^4w(\delta/\delta J(w))^4}$% % на $-(i/4!)\lambda\int d^4w(\delta/\delta J(w))^4$% % и $e^{-(i/2)\iint d^4d^4yJ(x)D(x-y)J(y)}$ на

$$\frac{i^4}{4!2^4}\left[\iint d^4d^4yJ(x)D(x-y)J(y)\right]^4$$

Для сокращения записи благоразумным будет обозначать $J\left(x_a\right)$ как J_a, $\int d^4x_a$ как $\int\limits_a$, а $D\left(x_a-x_b\right)$ как D_{ab}. Опуская общие численные коэффициенты, которые я предлагаю вам вписать самостоятельно, получаем

$$\sim i\lambda\int\limits_w\left(\frac{\delta}{\delta J_w}\right)^4\int\cdots\int D_{ae}D_{bf}D_{cg}D_{dh}J_aJ_bJ_cJ_dJ_eJ_fJ_gJ_h$$

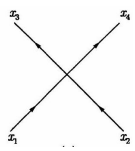
Четыре $(\delta/\delta J_w)$ действуют на восемь J всеми возможными способами, порождая при этом много членов, которые я снова предлагаю вам получить самостоятельно. Связный член равен

$$\sim i\lambda\iiint\iint D_{aw}D_{bw}D_{cw}D_{dw}J_aJ_bJ_cJ_d$$

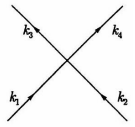
Очевидно, что это выражение является результатом действия четырех $(\delta/\delta J_w)$ на J_e,J_f,J_g и J_h, при котором x_e,x_f,x_g и x_h становятся равными w. Сравните (19) с $\left[8!(-\lambda)/(4!)^3\left(2m^2\right)^4\right]J^4$ из задачи для младенца.

Вспомним, что мы принялись за вычисления с целью породить два мезона источниками, локализованными в областях 1 и 2, пронаблюдать за их рассеянием друг на друге и уничтожить их в стоках, локализованных в областях 3 и 4. Другими словами, мы задаем функцию источника $J(x)$ равной сумме дельта-функций, имеющих пики в точках x_1, x_2,x_3 и x_4. Из (19) немедленно следует, что амплитуда рассеяния равна $-i\lambda\int\limits_w D_{1w}D_{2w}D_{3w}D_{4w}$, что в точности совпадает с (17).

Полученный результат очень легко объяснить (см. рис. I.7.7a). Два мезона распространяются с амплитудой $D\left(x_1-w\right)D\left(x_2-w\right)$ из мест рождения x_1 и x_2 до пространственно-временной точки w, рассеиваются с амплитудой $-i\lambda$ и затем распространяются с амплитудой $D\left(w-x_3\right)D\left(w-x_4\right)$ до своих мест поглощения в x_3 и x_4 [заметим, что $D(x)=D(-x)$]. Интегрирование по w означает, что точка взаимодействия может находиться где угодно. Все как в задаче для ребенка.



(a)



(b)

Рис. I.7.7

Все действительно довольно просто, если понимать, что происходит. Все еще путаетесь? Тогда может оказаться полезным представить $e^{-(i/4!)\lambda\int d^4w[\delta/\delta J(w)]^4}$ как некую машину, действующую на

$$Z(J,\lambda=0)=e^{-(i/2)\iint d^4xd^4yJ(x)D(x-y)J(y)}$$

Будучи разложенной в ряд, $Z(J,\lambda=0)$ представляет собой несколько J, разбросанных в пространстве-времени, а каждая пара J связана посредством D. Вообразите пучок струн, концы которых соответствуют J. Что делает «машина»? Машина состоит из суммы членов, например, члена

$$\sim\lambda^2\int d^4w_1\int d^4w_2\left[\frac{\delta}{\delta J\left(w_1\right)}\right]^4\left[\frac{\delta}{\delta J\left(w_2\right)}\right]^4.$$

Когда этот член действует на какой-либо член в $Z(J,\lambda=0)$, он хватает четыре струнных конца и склеивает их вместе в точке w_2; затем другие четыре струнных конца и склеивает их вместе в точке w_1. Затем по положениям w_1 и w_2 проводит интегрирование. Два примера показаны на рис. I.7.8. В эту игру можно играть даже с ребенком. Эта детская игра, заключающаяся в склеивании четырех струнных концов, генерирует все диаграммы Фейнмана в нашей теории скалярного поля.

14.2.12 Feynman Diagrams by Zi.I.7 2

Сделай это раз и навсегда

А теперь появляется Фейнман и говорит, что глупо каждый раз повторять эту утомительную болтовню. Надо просто понять правила раз и навсегда.

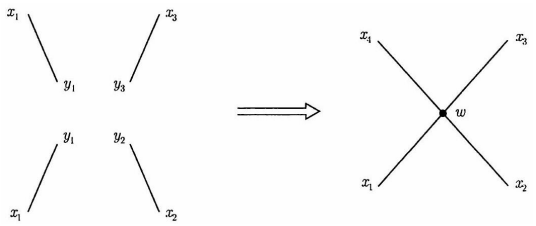
Например, для диаграммы на рис. I.7.7a сопоставим рассеянию множитель $-i\lambda$, распространению из x_1 в w множитель $D\left(x_1-w\right)$ и так далее - что с концептуальной точки зрения аналогично нашей младенческой задаче. Смотрите, вы сами могли бы изобрести диаграммы Фейнмана. (Ну, хорошо, может и могли бы, а может и нет.)

Точно так же, как при переходе от (I.4.1) к (I.4.2), легче работать в импульсном пространстве. И действительно, эксперименты так и проводят. Мезон с импульсом k_1 и мезон с импульсом k_2 сталкиваются и после рассеяния разлетаются с импульсами k_3 и k_4 (см. рис. I.7.7b). Каждый пропагатор пространства-времени дает нам

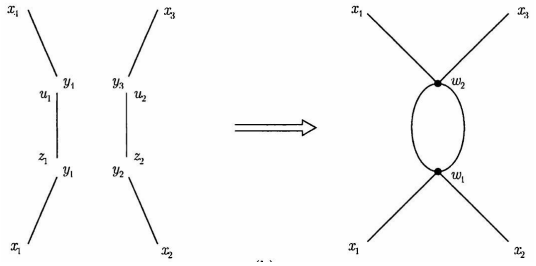
$$D\left(x_a-w\right)=\int\frac{d^4k_a}{(2\pi)^4}\frac{e^{\pm ik_a(x_a-w)}}{k_a^2-m^2+i\varepsilon}.$$

Обратите внимание, что мы свободны выбирать знак (либо плюс, либо минус) фиктивной переменной интегрирования в экспоненте. После интегрирования в (17) по w получаем

$$\int d^4we^{-i(k_1+k_2-k_3-k_4)w}=(2\pi)^4\delta^{(4)}\left(k_1+k_2-k_3-k_4\right).$$



(a)



(b)

Рис. I.7.8

Тот факт, что взаимодействие могло произойти в любой точке пространства-времени, отражается законом сохранения импульса $k_1 + k_2 = k_3 + k_4$. (Мы включили соответствующие знаки минус в два пропагатора D, с тем чтобы рассматривать k_3 и k_4 как выходящие импульсы.)

Итак, существуют диаграммы Фейнмана как в реальном пространстве-времени, так и импульсном пространстве. Пространственно-временные диаграммы Фейнмана буквальным образом представляют картину того, что происходит. (Тривиальное замечание: ориентация диаграмм Фейнмана зависит от личного выбора человека. Кто-то рисует их так, что время изменяется в вертикальном направлении, а кто-то - так, что в горизонтальном.)

Правила

Итак, мы вывели знаменитые правила Фейнмана в импульсном пространстве для нашей теории скалярного поля, которые формулируются следующим образом:

1. Нарисовать диаграмму Фейнмана процесса (рис. I.7.7b для только что рассмотренного процесса).

2. Обозначить каждую линию импульсом k и сопоставить с ней пропагатор $i/\left(k^2 - m^2 + i\varepsilon\right)$.

3. Сопоставить каждой вершине взаимодействия константу связи $-i\lambda$ и $(2\pi)^4\delta^{(4)}\left(\sum_i k_i - \sum_j k_j\right)$, в результате чего сумма входящих в вершину импульсов $\sum_i k_i$ окажется равной сумме выходящих из нее импульсов $\sum_j k_j$.

4. Проинтегрировать по импульсам внутренних линий с мерой $\frac{d^4k}{(2\pi)^4}$. Данное действие соответствует суммированию по промежуточным состояниям в обычной теории возмущений.

5. Наконец, существует правило для вычисления «противных» симметрийных множителей. Они являются аналогами численных коэффициентов в рассмотренной нами младенческой задаче. Как следствие, некоторые диаграммы необходимо умножить на симметрийный множитель, такой как $\frac{1}{2}$. Эти множители обуславливаются различными комбинационными множителями, которые учитывают различные варианты воздействия $(\delta/\delta J)$ на J в выражениях типа (18). Пример вычисления симметрийного множителя вы найдете в упражнении I.7.2.

Проиллюстрируем на примерах, что означают эти правила (а заодно и понятие внутренней линии).

В качестве первого примера рассмотрим вычисленную нами диаграмму (рис. I.7.7b). Применяя правила, получаем

$$-i\lambda(2\pi)^4\delta^{(4)}\left(k_1+k_2-k_3-k_4\right)\prod_{a=1}^{4}\left(\frac{i}{k_a^2-m^2+i\varepsilon}\right).$$

Вы, надеюсь, согласитесь, что довольно глупо таскать с собой множитель $\prod_{a=1}^{4}\left(\frac{i}{k_a^2-m^2+i\varepsilon}\right)$, общий для всех диаграмм, в которых два мезона рассеиваются в два мезона. Поэтому мы дополним правила Фейнмана еще одним, утверждающим, что с внешними линиями пропагатор ассоциировать не надо. (На профессиональном языке это называется «ампутацией внешних концов».) Поскольку всегда присутствует общий множитель, отвечающий за сохранение импульса, нам также не стоит таскать за собой общую дельта-функцию. Таким образом, получаем еще два правила:

6. Со внешними линиями пропагатор не сопоставлять.

7. Предполагать наличие дельта-функции, отвечающей за сохранение общего импульса.

Амплитуда для диаграммы на рис. I.7.7b равна $-i\lambda$.

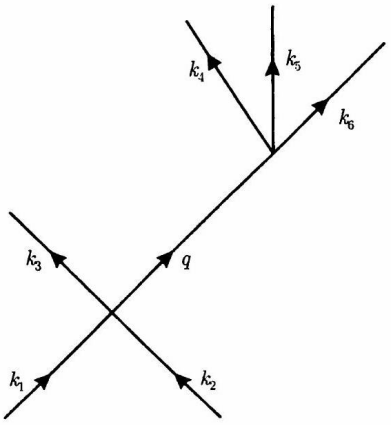


Рис. I.7.9

Рождение частиц

Как объясняется в главе I.1, одной из мотиваций для построения квантовой теории поля явилась необходимость описать рождение частиц. Теперь мы в состоянии описать, как два мезона образуют четыре. Диаграмма Фейнмана на рис. I.7.9 (сравните с рис. I.7.2a) может появиться в теории возмущений с порядком λ^2. Ампутируя внешние концы, мы опускаем множитель $\prod_{a=1}^{6}\left[i/k_a^2 - m^2 + i\varepsilon\right]$, сопоставленный с шестью внешними линиями, оставляя при этом лишь пропагатор, связанный с одной внутренней линией. Для каждой вершины включаем множитель $(-i\lambda)$ и дельта-функцию, отвечающую за сохранение импульса (правило 3). Затем интегрируем по внутреннему импульсу q (правило 4) и получаем

$$(-i\lambda)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\varepsilon} (2\pi)^4 \delta^{(4)}\left(k_1 + k_2 - k_3 - q\right) \times \\ \times (2\pi)^4 \delta^{(4)}\left[q - (k_4 + k_5 + k_6)\right]$$

Интеграл по q трудности нс вызывает, его вычисление приводит к

$$(-i\lambda)^2 \frac{i}{\left(k_4 + k_5 + k_6\right)^2 - m^2 + i\varepsilon} (2\pi) \delta^{(4)}\left[k_1 + k_2 - (k_3 + k_4 + k_5 + k_6)\right]$$

Ранее мы договорились (правило 7) не писать каждый раз общую дельта-функцию. Из примера видно, что нет необходимости писать дельта-функции, которые затем исчезают (все кроме одной) при интегрировании. На рис. I.7.9 мы могли с самого начала использовать закон сохранения импульса и обозначить внутреннюю линию суммой $k_4+k_5+k_6$, а не символом q.

Если бы у вас были практические навыки, вы бы легко записали амплитуду

$$(-i\lambda)^2 \frac{i}{\left(k_4 + k_5 + k_6\right)^2 - m^2 + i\varepsilon}$$

с самого начала: просто запомните, что мы ассоциируем константу $(-i\lambda)$ с каждой вершиной и пропагатор - с каждой внутренней (но не внешней) линией. Довольно просто, если понять суть.

Чего стоит быть ненастоящим

Лежащая в основе наших действий физика довольно проста: внутренней линии соответствует виртуальная частица, у которой квадрат релятивистского 4-импульса $k_4 + k_5 + k_6$ не обязательно равен m^2, как должно быть в случае с реальной частицей. Чем дальше импульс виртуальной частицы от массовой поверхности, тем меньше амплитуда. Такова плата за нереальность.

Согласно квантовым правилам для тождественных частиц, полная амплитуда находится симметризацией по четырем конечным импульсам. Другими словами, мы должны заметить, что линия, помеченная как k_3 на рис. I.7.9, могла бы быть также помечена как k_4, k_5 или k_6, так что необходимо добавить все четыре вклада. Чтобы удостовериться в том, что вы понимаете правила Фейнмана, я настоятельно рекомендую вам получить (21) из (12) и (13), вычислив интеграл по траекториям.

Еще раз повторю (думаю, это того стоит), что в диаграммах Фейнмана нет ничего магического.

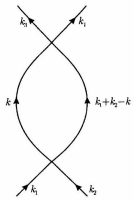


Рис. I.7.10

Петли и первое знакомство с расходимостями

Как и в нашей младенческой задаче, существуют древесные и петлевые диаграммы. До сих пор мы рассматривали только древесные диаграммы. Нашим следующим примером будет петлевая диаграмма на рис. I.7.10 (сравните с рис. I.7.3а). Применяя правила Фейнмана, получаем

$$\frac{1}{2}(-i\lambda)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \frac{i}{(k_1+k_2-k)^2-m^2+i\varepsilon}.$$

Как и ранее, лежащая в основе (23) физика проста: при изменении k подынтегральное выражение является большим только тогда, когда первая или вторая (или обе вместе) виртуальные частицы, ассоциированные с двумя внутренними линиями, близки к реальным. Еще раз повторяю, что за свою нереальность частицы должны платить (см. упражнение I.7.4). На больших k подынтегральное выражение ведет себя как $1/k^4$. Интеграл оказывается бесконечным! Он расходится как $\int d^4k\,(1/k^4)$. Мы вернемся к этому явному бедствию в главе III.1.

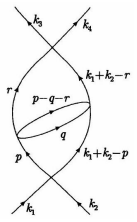


Рис. I.7.11

Получив некоторые практические навыки, вы сможете записывать амплитуду с первого взгляда. В качестве другого примера рассмотрим трехпетлевую диаграмму на рис. I.7.11, дающую вклад $O(\lambda^4)$ в мезон-мезонное рассеяние. Для начала выберем в каждой петле по одной внутренней линии и обозначим бегущий по ней импульс (символами p, q и r в нашем примере). На этом этапе у нас есть большая свобода выбора - ваш выбор может отличаться от моего, но физика, разумеется, от этого не зависит. После этого импульсы, бегущие по другим внутренним линиям, фиксируются законом сохранения импульса, как указано на рисунке. Запишем для каждой вершины константу связи, а для каждой внутренней линии - пропагатор и проинтегрируем по внутренним импульсам p, q и r. Таким образом, без учета симметрийных множителей получаем амплитуду

$$(-i\lambda)^4 \int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\frac{i}{p^2-m^2+i\varepsilon}\frac{i}{(k_1+k_2-p)^2-m^2+i\varepsilon}\times$$
$$\times\frac{i}{q^2-m^2+i\varepsilon}\frac{i}{(p-q-r)^2-m^2+i\varepsilon}\times$$
$$\times\frac{i}{r^2-m^2+i\varepsilon}\frac{i}{(k_1+k_2-r)^2-m^2+i\varepsilon}.$$

Этот тройной интеграл снова расходится: он ведет себя как $\int d^{12}P\,(1/P^{12})$.

Мое заверение

Когда я преподаю квантовую теорию поля, именно в этом месте некоторые студенты по непонятным причинам испытывают страх перед диаграммами Фейнмана. Мне бы хотелось заверить читателей, что все это на самом деле довольно просто. Диаграммы Фейнмана можно представлять себе как пространственно-временные картинки поведения частиц, которые собираются вместе, сталкиваются и порождают другие частицы, и так далее. Один студент был озадачен тем, что частицы не распространяются по прямым линиям. Помните, что квантовая частица распространяется как волна: $D(x-y)$ описывает амплитуду распространения частицы из x в y. Очевидно, что гораздо удобнее рассматривать частицы в импульсном пространстве: так нам сказал Фурье. Мы увидим еще много примеров фейнмановских диаграмм, и вскоре вы с ними хорошо познакомитесь. Другой студент был обеспокоен вычислением интегралов (23) и (24). Пока я вас этому еще не учил, но со временем обучу. Хорошей новостью является то, что на передовых рубежах современных исследований лишь немногим теоретикам приходится вычислять диаграммы Фейнмана с учетом всех множителей 2. В большинстве случаев достаточно понять лишь общее поведение интеграла. Но, конечно, всегда почетно получить точный результат. В главах II.6, III.6 и III.7 я приведу детальные вычисления диаграмм Фейнмана, правильно получив численные множители, чтобы можно было сравнить с экспериментами.

Вакуумные флуктуации

Давайте вернемся к членам, которые я опустил в (18) и которые вы должны были найти сами. Например, четыре $[\delta/\delta J(w)]$ должны были подействовать на J_c, J_d, J_g и J_h в (18) и дать что-то вроде

$$-i\lambda \iiiint \int D_{ae}D_{bf}J_aJ_bJ_eJ_f\left(\int_w D_{ww}D_{ww}\right).$$

Коэффициент при $J(x_1)\,J(x_2)\,J(x_3)\,J(x_4)$ оказывается равным $D_{13}\times$ $\times D_{24}\left(-i\lambda\int_w D_{ww}D_{ww}\right)$ плюс члены, полученные с помощью перестановок.

Соответствующий физический процесс легко описать как словесно, так и графически (см. рис. I.7.12). Источник в x_1 порождает частицу, которая безо всякого взаимодействия свободно распространяется до x_3, где

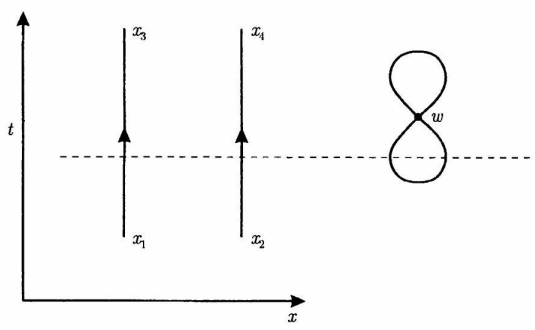


Рис. I.7.12

она спокойно умирает. Частица, рожденная в x_2, живет той же небогатой событиями жизнью, прежде чем исчезнуть в x_4. Эти две частицы совсем не взаимодействуют. Где-то там в точке w, которая может находиться абсолютно где угодно во Вселенной, происходит взаимодействие с амплитудой $-i\lambda$. Это называется вакуумной флуктуацией: как говорилось в главе I.1, квантовая механика и специальная теория относительности неизбежно ведут к появлению частиц из вакуума, которые могут даже взаимодействовать, прежде чем вновь исчезнуть в вакууме. Посмотрите на различные временные срезы (один из которых обозначен пунктирной линией) на рис. I.7.12. В далеком прошлом во Вселенной частицы отсутствовали. Потом там было две, потом четыре, потом снова две и, наконец, в отдаленном будущем, ни одной частицы. Более подробно об этих флуктуациях мы поговорим в главе VIII.2. Отмечу, что вакуумные флуктуации также присутствуют в задачах для младенца и ребенка (см., например, рис. I.7.1c, I.7.2f,g, I.7.3f,g и так далее).

Упражнения

I.7.1. Вычислите амплитуду, соответствующую рисунку I.7.12 в (18).

I.7.2. Выведите (23) из первых принципов, то есть с выражения (12). Это несколько утомительно, хотя и прямолинейно. Найдите симметрийный множитель $\frac{1}{2}$.

I.7.3. Нарисуйте все диаграммы, описывающие процесс рождения четырех мезонов из двух, до порядка λ^2 включительно. Выпишите соответствующие фейнмановские амплитуды.

1.7.4. В силу лоренц-инвариантности мы всегда можем выбрать $k_1 + k_2 = (E, \vec{0})$ в (23). Интеграл можно изучать как функцию от E. Покажите, что, для того чтобы обе внутренние частицы стали реальными, E должна быть больше $2m$. Дайте этому физическую интерпретацию.

14.2.13 4.3 Wick's Theorem

Wick's Theorem

We have now reduced the problem of calculating correlation functions to that of evaluating expressions of the form

$$\langle 0|T\left\{\phi_I\left(x_1\right)\phi_I\left(x_2\right)\cdots\phi_I\left(x_n\right)\right\}|0\rangle$$

that is, vacuum expectation values of time-ordered products of finite (but arbitrary) numbers of free field operators. For $n = 2$ this expression is just the Feynman propagator. For higher n you could evaluate this object by brute force, plugging in the expansion of ϕ_I in terms of ladder

operators. In this section and the next, however, we will see how to simplify such calculations immensely.

Consider again the case of two fields, $\langle 0|T\{\phi_I(x)\phi_I(y)\}|0\rangle$. We already know how to calculate this quantity, but now we would like to rewrite it in a form that is easy to evaluate and also generalizes to the case of more than two fields. To do this we first decompose $\phi_I(x)$ into positive- and negativefrequency parts:

$$\phi_I(x) = \phi_I^+(x) + \phi_I^-(x), \tag{4.32}$$

where

$$\phi_I^+(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}a_{\mathbf{p}}e^{-ip\cdot x}; \quad \phi_I^-(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}a_{\mathbf{p}}^\dagger e^{+ip\cdot x}$$

This decomposition can be done for any free field. It is useful because

$$\phi_I^+(x)|0\rangle = 0 \quad \text{and} \quad \langle 0|\phi_I^-(x) = 0.$$

For example, consider the case $x^0 > y^0$. The time-ordered product of two fields is then

$$\begin{aligned} T\phi_I(x)\phi_I(y) &\underset{x^0>y^0}{=} \phi_I^+(x)\phi_I^+(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) \\ &= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) \\ &\quad + \left[\phi_I^+(x), \phi_I^-(y)\right]. \end{aligned} \tag{4.33}$$

In every term except the commutator, all the $a_{\mathbf{p}}$'s are to the right of all the $a_{\mathbf{p}}^\dagger$'s. Such a term (e.g., $a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger a_{\mathbf{k}} a_{\mathbf{l}}$) is said to be in normal order, and has vanishing vacuum expectation value. Let us also define the normal ordering symbol $N()$ to place whatever operators it contains in normal order, for example,

$$N\left(a_{\mathbf{p}}a_{\mathbf{k}}^\dagger a_{\mathbf{q}}\right) \equiv a_{\mathbf{k}}^\dagger a_{\mathbf{p}} a_{\mathbf{q}} \tag{4.34}$$

The order of $a_{\mathbf{p}}$ and $a_{\mathbf{q}}$ on the right-hand side makes no difference since they commute.*

If we had instead considered the case $y^0 > x^0$, we would get the same four normal-ordered terms as in (4.33), but this time the final commutator would be $\left[\phi_I^+(y), \phi_I^-(x)\right]$. Let us therefore define one more quantity, the contraction of two fields, as follows:

This quantity is exactly the Feynman propagator:

$$\overrightarrow{\phi(x)}\phi\,(y) = D_F(x-y) \tag{4.36}$$

(From here on we will often drop the subscript I for convenience; contractions will always involve interaction-picture fields.)

The relation between time-ordering and normal-ordering is now extremely simple to express, at least for two fields:

$$T\{\phi(x)\phi(y)\} = N\{\phi(x)\phi(y) + \overbrace{\phi(x)}^{\square}\phi(y)\} \tag{4.37}$$

[6]* In the literature one often sees the notation: $\phi_1\phi_2$: instead of $N(\phi_1\phi_2)$.

But now that we have all this new notation, the generalization to arbitrarily many fields is also easy to write down:

$$\begin{aligned} &T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\cdots\phi\left(x_m\right)\right\} \\ &\quad = N\left\{\phi\left(x_1\right)\phi\left(x_2\right)\cdots\phi\left(x_m\right) + \text{ all possible contractions }\right\} \end{aligned} \tag{4.38}$$

This identity is known as Wick's theorem, and we will prove it in a moment. For $m = 2$ it is identical to (4.37). The phrase all possible contractions means there will be one term for each possible way of contracting the m fields in pairs. Thus for $m = 4$ we have (writing $\phi\left(x_a\right)$ as ϕ_a for brevity)

$$\begin{aligned} T\left\{\phi_1\phi_2\phi_3\phi_4\right\} = N\Big\{&\phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \sqrt{\phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4} \\ &+\sqrt{\square} \\ &+\phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 \\ &\square \\ &+\phi_1\phi_2\phi_3\phi_4 + \sqrt{\square}\phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4\Big\} \end{aligned} \tag{4.39}$$

When the contraction symbol connects two operators that are not adjacent, we still define it to give a factor of D_F. For example,

$$N\Big\{\phi_1\phi_2\phi_3\phi_4\Big\} \quad \text{means} \quad \overrightarrow{D_F}\left(x_1 - x_3\right)\cdot N\left\{\phi_2\phi_4\right\}\ldots\,.$$

In the vacuum expectation value of (4.39), any term in which there remain uncontracted operators gives zero (since $\langle 0|N$ (any operator) $|0\rangle = 0$). Only the three fully contracted terms in the last line survive, and they are all cnumbers, so we have

$$\begin{aligned} \langle 0|T\left\{\phi_1\phi_2\phi_3\phi_4\right\}|0\rangle = &D_F\left(x_1 - x_2\right)D_F\left(x_3 - x_4\right) \\ &+ D_F\left(x_1 - x_3\right)D_F\left(x_2 - x_4\right) \\ &+ D_F\left(x_1 - x_4\right)D_F\left(x_2 - x_3\right) \end{aligned} \tag{4.40}$$

Now let us prove Wick's theorem. Naturally the proof is by induction on m, the number of fields. We have already proved the case $m = 2$. So assume the theorem is true for $m - 1$ fields, and let's try to prove it for m fields. Without loss of generality, we can restrict ourselves to the case $x_1^0 \geq x_2^0 \geq \cdots x_m^0$; if this is not the case we can just relabel the points, without affecting either side of (4.38). Then applying Wick's theorem to $\phi_2\cdots\phi_m$, we have

$$\begin{aligned} T\left\{\phi_1\cdots\phi_m\right\} &= \phi_{\mathbf{1}}\cdots\phi_m \\ &= \phi_{\mathbf{1}}N\left\{\phi_2\cdots\phi_m + \begin{pmatrix}\text{all contractions}\\ \text{not involving }\phi_1\end{pmatrix}\right\} \\ &= \left(\phi_1^+ + \phi_1^-\right)N\left\{\phi_2\cdots\phi_m + \begin{pmatrix}\text{all contractions}\\ \text{not involving }\phi_1\end{pmatrix}\right\}. \end{aligned} \tag{4.41}$$

We want to move the $\phi_1^+$and ϕ_1^-inside the $N\{\}$. For the ϕ_1^-term this is easy: Just move it in, since (being on the left) it is already in normal order. The term with ϕ_1^+must be put in normal order by commuting it to the right past all the other ϕ 's. Consider, for example, the term with no contractions:

$$\begin{aligned}\phi_1^+ N\left(\phi_2 \cdots \phi_m\right) =& N\left(\phi_2 \cdots \phi_m\right) \phi_1^+ + \left[\phi_1^+, N\left(\phi_2 \cdots \phi_m\right)\right] \\ =& N\left(\phi_1^+ \phi_2 \cdots \phi_m\right) \\ &+ N\left(\left[\phi_1^+, \phi_2^-\right] \phi_3 \cdots \phi_m + \phi_2\left[\phi_1^+, \phi_3^-\right] \phi_4 \cdots \phi_m + \cdots\right) \\ =& N\left(\phi_1^+ \phi_2 \cdots \phi_m + \phi_1 \phi_2 \phi_3 \cdots \phi_m + \phi_1 \phi_2 \phi_3 \cdots + \cdots\right).\end{aligned}$$

The first term in the last line combines with part of the ϕ_1^-term from (4.41) to give $N\{\phi_1\phi_2\cdots\phi_m\}$, so we now have the first term on the right-hand side of Wick's theorem, as well as all possible terms involving a single contraction of ϕ_1 with another field. Similarly, a term in (4.41) involving one contraction will produce all possible terms involving both that contraction and a contraction of ϕ_1 with one of the other fields. Doing this with all the terms of (4.41), we eventually get all possible contractions of all the fields, including ϕ_1. Thus the induction step is complete, and Wick's theorem is proved.

суть теоремы Вика

Тут интересная теория, думаю, важно понять, как она работает: https://physics.stackexchange.com/questions/304869/systematic-way-to-draw-all-inequivalent-feynman-diagrams (по идее должен понимать, но что-то пробелы еще остаются в понимании)

https://physics.stackexchange.com/questions/297035/symmetry-factor-via-wicks-theorem я что это тоже не понимаю еще?

вывод теоремы Вика

В случае рассмотрения вакуум-вакуумных переходов для затравочного действия,

квадратичного по полю,

в формализме интеграла по траекториям легко получить выражение для функции Грина функционала $Z(j)$

$$\mathcal{D}^{(n)}\left(x_1, ..., x_n\right) = \langle \text{in}|\hat{\text{T}}\left\{\hat{\varphi}\left(x_1\right) ... \hat{\varphi}\left(x_n\right)\right\}|\text{in}\rangle$$

где мы опустили индекс,

обозначающий представление Гейзенберга для операторов, зависящих от времени.

Для квадратичного действия,

$$Z(j) = \exp\left\{\frac{\mathrm{i}}{\hbar}\frac{1}{2} j_x \mathcal{G}^{(2)}_{xy} j_y\right\}$$

или при помощи введения вспомогательного поля $\tilde{\varphi}$

$$Z(j) = \exp\left\{-\frac{\mathrm{i}\hbar}{2}\mathcal{G}^{(2)}_{xy}\frac{\delta^2}{\delta\tilde{\varphi}_x \delta\tilde{\varphi}_y}\right\} e^{\frac{\mathrm{i}}{\hbar} j_{x'}\tilde{\varphi}_{x'}}\Big|_{\tilde{\varphi}=0}$$

Для (4.49) находим

$$\mathcal{D}^{(n)}_{1\ldots n} = (-\mathrm{i}\hbar)^n \frac{\delta^n Z(j)}{\delta j_1 \ldots \delta j_n}\bigg|_{j=0} = \exp\left\{-\frac{\mathrm{i}\hbar}{2}\mathcal{G}^{(2)}_{xy}\frac{\delta^2}{\delta\tilde{\varphi}_x\delta\tilde{\varphi}_y}\right\}\tilde{\varphi}_1\ldots\tilde{\varphi}_n\bigg|_{\tilde{\varphi}=0}$$

При нечетном $n = 2p+1$, (????????),

$$\mathcal{D}^{(n)}_{1\ldots n} = 0.$$

Введем понятие «спаривания полей»

$$\underbrace{\tilde{\varphi}_k\tilde{\varphi}_{k'}} \cong -\mathrm{i}\hbar\mathcal{G}^{(2)}_{kk'} = \mathcal{D}^{(2)}_{kk'}$$

Тогда при четном $n = 2p$ функция Грина сводится к сумме всевозможных спариваний полей что и составляет суть теоремы Вика.

В приведенном выводе оказывается, что знак тильды не меняет дела: спаривание полей в точности отвечает расчету пропагатора частиц поля.

физический смысл спаривания и овалы (??)

Однако в операторной формулировке задается физический смысл спаривания.

Это позволит нам установить правила графического изображения для корреляторов.

В самом деле, для свободных полей,

первая ненулевая функция Грина $\mathcal{D}^{(2)}_{12}$ отвечает траектории частицы между точками 1 и 2, так что естественно изобразить эту функцию Грина в виде линии,

соединяющей эти точки,

причем,

графическому изображению соответствует вполне определенное функциональное выражение.

Для высшей функции Грина каждая точка в аргументе этой функции также отвечает концу или началу линии распространения частицы,

поскольку все возможные спаривания также отвечают соединениям точек линиями движения частиц.

(переформулирую) Всю совокупность этих спариваний можно обозначить в виде множества точек,

отвечающих аргументам функции,

причем,

из этих точек исходят линии, которые входят в область, скажем, овал, обозначающий все варианты попарного соединения точек линиями.

При отсутствии взаимодействий, овалы расшифровываются в указанном элементарном смысле.

пример: двухточечная функция свободных полей

Для примера, у свободных полей двухточечная функция -

(тут у Киселева картинки-расшифровки)

В последнем случае диаграммы не являются связными,

и эти несвязные диаграммы отсутствуют для функций Грина порядка выше двух для производящего функционала $G(j)$.

Важно подчеркнуть, что вариационная производная производяшего функционала - это линия, которая отвечает распространению частицы из овала в точку прескрипции вариационной производной!

При наличии взаимодействий можно начать рассмотрение с построения ведущего приближения теории возмущений и его графического представления, чтобы выяснить, какие варианты могут скрываться за овалом, из которого исходят линии распространения частиц, кроме элементарного соединения двух точек в свободной теории.

примеры использования теоремы Вика

(?)

14.2.14 Wick's Theorem by Mandl & Shawl

Обсудим подробно теорему Вика.

$$\underbrace{A(x_1)B}(x_2) = \langle 0|T....$$

Теория

We must now see how to obtain from the S-matrix expansion (6.23) the transition amplitude $\langle f|S|i\rangle$ for a particular transition $|i\rangle \to |f\rangle$ in a given order of perturbation theory. The Hamiltonian density $\mathscr{H}_{\rm I}(x)$ in Eq. (6.23) involves the interacting fields, each linear in creation and absorption operators. Hence the expansion (6.23) will describe a large number of different processes. However, only certain terms of the S-matrix will contribute to a given transition $|i\rangle \to |f\rangle$. For these terms must contain just the right absorption operators to destroy the particles present in $|i\rangle$, and they must contain the right creation operators to emit the particles present in $|f\rangle$. They may also contain additional creation and absorption operators, which create particles which are subsequently re-absorbed. These particles are only present in intermediate states and are called virtual particles.

Calculations can be greatly simplified by avoiding the explicit introduction of virtual intermediate particles. This can be achieved by writing the S-matrix expansion as a sum of normal products, since, in a normal product, all absorption operators stand to the right of all creation operators. Such an operator first absorbs a certain number of particles and then emits some particles. It does not cause emission and re-absorption of intermediate particles. Each of these normal products will effect a particular transition $|i\rangle \to |f\rangle$, which can be represented by a Feynman graph, similar to those introduced in Chapters 3 and 4. Consider, for example, Compton scattering ($e^- + \gamma \to e^- + \gamma$). The QED interaction Hamiltonian density is, from Eq. (6.10),

$$\mathscr{C}_{\rm I}(x) = -\mathscr{L}_{\rm I}(x) = -e\ {\rm N}\left[\bar{\psi}(x)A_A(x)\psi(x)\right].$$

Since the negative (positive) frequency parts $A^-, \bar{\psi}^-, \psi^-$ $\left(A^+, \psi^+\bar{\psi}^+\right)$are linear in creation (absorption) operators for photons, electrons and positrons respectively, the only normal product which contributes to Compton scattering is

$$\bar{\psi}^- A^- \psi^+ A^+$$

The method for expanding the S-matrix as a sum of normal products, which we shall now describe, is due to Dyson and Wick.

We first of all summarize the general definition of a normal product. Let $Q, R, ..., W$ be operators of the type $\psi^{\pm}, A^{\pm}$, etc., i.e. each is linear in either creation or absorption operators, then

$$\mathrm{N}(QR...W) = (-1)^P (Q'R'...W')$$

The S-Matrix Expansion 95 Here $Q', R', ..., W'$ are the operators $Q, R, ..., W$ reordered, so that all absorption operators (i.e. positive frequency parts) stand to the right of all creation operators (i.e. negative frequency parts). The exponent P is the number of interchanges of neighbouring fermion operators required to change the order $(QR...W)$ into $Q'R'...W')$. We generalize the definition (6.25a) by requiring the normal product to obey the distributive law

$$\mathrm{N}(RS... + VW...) = \mathrm{N}(RS...) + \mathrm{N}(VW...).$$

The QED interaction (6.24) is a normal product of field operators. We shall find that in other cases too, the interaction Hamiltonian density can be written as a normal product, i.e.

$$"\mathrm{I}(x) = \mathrm{N}\{A(x)B(x)...\}$$

where each of the fields $A(x), B(x), ...$, is linear in creation and absorption operators. Hence, we must consider the expansion into a sum of normal product of 'mixed' T-product (i.e. a T-product whose factors are normal products), such as occurs in the S-matrix expansion (6.23).

From the definition of the normal product, we have, for two field operators $A \equiv A(x_1)$ and $B \equiv B(x_2)$, that

$$AB - \mathrm{N}(AB) = \left\{ \begin{array}{ll} [A^+, B^-]_+ . & \text{for two fermion fields} \\ [A^+, B^-], & \text{otherwise} \end{array} \right\}$$

For two fermion fields, the anticommutators, and in all other cases, the commutators, are c-numbers, i.e. they do not involve creation or annihilation operators. [We had examples in Eqs. (3.40) and (4.53a).] Hence, the right-hand side of Eq. (6.27) is always a c-number. It is given by $\langle 0|AB|0\rangle$, as follows by taking the vacuum expectation value of Eq. (6.27). Hence Eq. (6.27) becomes:

$$AB = \mathrm{N}(AB) + \langle 0|AB|0\rangle.$$

Since

$$\mathrm{N}(AB) = \pm \mathrm{N}(BA),$$

the minus sign applying in the case of two fermion fields, the plus sign in all other cases, it follows from Eq. (6.28) that for $x_1^0 \neq x_2^0$

$$\mathrm{T}\{A(x_1) B(x_2)\} = \mathrm{N}\{A(x_1) B(x_2)\} + \langle 0 |\ \mathrm{T}\{A(x_1) B(x_2)\} |\, 0\rangle .$$

The case of equal times, $x_1^0 = x_2^0$, will be considered below.

The special notation

$$\underbrace{A(x_1)B}(x_2) \equiv \langle 0 |\ \mathrm{T}\{A(x_1) B(x_2)\} |\, 0\rangle$$

contractions are, of course, just the Feynman propagators, e.g. Eqs. (3.59), (3.60), (4.61) and (5.26):

$$\phi(x_1)\phi(x_2) = \mathrm{i}\Delta_{\mathrm{F}}(x_1 - x_2)$$
$$\psi_\alpha(x_1)\bar{\psi}_\beta(x_2) = -\bar{\psi}_\beta(x_2)\psi_\alpha(x_1) = \mathrm{i}S_{\mathrm{F}\alpha\beta}(x_1 - x_2)$$
$$\phi^\dagger(x_2)\phi(x_1) = \mathrm{i}\Delta_{\mathrm{F}}(x_1 - x_2)$$
$$A^\mu(x_1) A^v(x_2) = \mathrm{i}D_{\mathrm{F}}^{\mu v}(x_1 - x_2)$$

To generalize Eq. (6.30) to several operators $A \equiv A(x_1), ..., M \equiv M(x_m), ...$, the generalized normal product is defined by where P is the number of interchanges of neighbouring fermion operators required to change the order $(ABC...)$ to $(AKB...)$; for example

$$\begin{aligned}&\mathrm{N}\left(\psi_\alpha(x_1)\ \psi_\beta(x_2)\, A^\mu(x_3)\, \bar{\psi}_\gamma(x_4)\, \bar{\psi}_\delta(x_5)\right)\\ &\qquad = (-1)\psi_\beta(x_2)\, \bar{\psi}_\delta(x_5)\, \mathrm{N}\left(\psi_\alpha(x_1)\, A^\mu(x_3)\, \bar{\psi}_\gamma(x_4)\right).\end{aligned}$$

On the right-hand side of this equation appears the sum of all possible generalized normal products that can be formed from $(ABCD...WXYZ)$, the first, second and third lines representing all terms with no, one and two contractions, and so on. Each term on the righthand side of this equation contains all the factors in the same order in which they occur in the T-product on the left-hand side.

Eq. (6.35) states Wick's theorem. We shall not reproduce its proof, which is by induction, and so not very illuminating. [3] [3] G. C. Wick, Phys. Rev. 80 (1950) 268. The S-Matrix Expansion 97 With the interaction (6.26), the S-matrix expansion (6.23) contains the mixed T-products

$$\mathrm{T}\left\{\mathscr{H}_{\mathrm{I}}(x_1) ... \mathscr{X}_{\mathrm{I}}(x_n)\right\} = \mathrm{T}\left\{\mathrm{N}(AB...)_{x_1} ... \mathrm{N}(AB...)_{x_n}\right\}.$$

Wick extended the theorem (6.35) to include such mixed T-products. In each factor $\mathrm{N}(AB...)_{x_r}$ we replace $x_r = (x_r^0, \mathbf{x}_r)$ by $\xi_r = (x_r^0 \pm \varepsilon, \mathbf{x}_r), (\varepsilon < 0)$, depending on whether the substitution is made in the creation or absorption part of the field. Hence

$$\mathrm{T}\left\{\mathrm{N}(AB...)_{x_1} ... \mathrm{N}(AB...)_{x_n}\right\} = \lim_{\varepsilon \to 0}\ \mathrm{T}\left\{(AB...)_{\xi_1} ... (AB...)_{\xi_n}\right\},$$

the normal and chronological orderings within each group $(AB...)_{\xi_r}$ being the same on account of the $\pm\varepsilon$ in ξ_r^0. On expanding the right-hand side of Eq. (6.37) by Wick's theorem before going to the limit $\varepsilon \to 0$, contractions within one group $(AB...)\xi_t$ (i.e. over equaltimes operators when $\varepsilon \to 0$) vanish as the group is already in normal order. We thus have the desired result: the mixed T-product (6.36) can be expanded according to Eq. (6.35), provided contractions over equal times are omitted:

$$\mathrm{T}\left\{\mathrm{N}(AB...)_{x_1} ... \mathrm{N}(AB...)_{x_n}\right\} = \mathrm{T}\left\{(AB...)_{x_1} ... (AB...)_{x_n}\right\}_{\text{no e.t.c.}}$$

where 'no e.t.c.' stands for 'no equal-times contractions'. Eqs. (6.35) and (6.38) represent the desired result, enabling us to expand each term in the S-matrix expansion (6.23) into a sum of generalized normal products. Each of these normal products corresponds to a definite process, characterized by the operators not contracted, which absorb and create the particles present in the initial and final states respectively. The non-vanishing contractions which occur in these generalized normal products are the Feynman propagators (6.32), corresponding to virtual particles being emitted and re-absorbed in intermediate states. In the next chapter we shall see how to evaluate these individual contributions to $\langle f|S|i\rangle$, which result from the application of Wick's theorem.

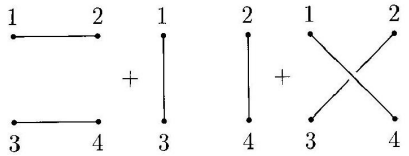
14.3 4.4 Feynman Diagrams

Wick's theorem allows us to turn any expression of the form

$$\langle 0|T\left\{\phi_I(x_1)\, \phi_I(x_2) \cdots \phi_I(x_n)\right\}|0\rangle$$

into a sum of products of Feynman propagators. Now we are ready to develop a diagrammatic interpretation of such expressions. Consider first the case of four fields, all at different spacetime

points, which we worked out in Eq. (4.40). Let us represent each of the points x_1 through x_4 by a dot, and each factor
$D_F(x-y)$ by a line joining x to y. Then Eq. (4.40) can be represented as the sum of three diagrams (called Feynman diagrams):

$\langle 0|\, T\{\phi_1\phi_2\phi_3\phi_4\}\,|0\rangle =$ 

Although this isn't exactly a measurable quantity, the diagrams do suggest an interpretation: Particles are created at two spacetime points, each propagates to one of the other points, and then they are annihilated. This can happen in three ways, corresponding to the three ways to connect the points in pairs, as shown in the three diagrams. The total amplitude for the process is the sum of the three diagrams.

Things get more interesting when the expression contains more than one field at the same spacetime point. So let us now return to the evaluation of the two-point function $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$, and put formula (4.31) to use. We will ignore the denominator until the very end of this section. The numerator, with the exponential expanded as a power series, is

$$\langle 0|T\left\{\phi(x)\phi(y)+\phi(x)\phi(y)\left[-i\int dtH_I(t)\right]+\cdots\right\}|0\rangle. \tag{4.43}$$

The first term gives the free-field result, $\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = D_F(x-y)$. The second term, in ϕ^4 theory, is

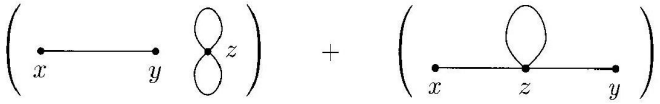
$$\begin{aligned}&\langle 0|T\left\{\phi(x)\phi(y)(-i)\int dt\int d^3z\frac{\lambda}{4!}\phi^4\right\}|0\rangle\\ &\quad=\langle 0|T\left\{\phi(x)\phi(y)\left(\frac{-i\lambda}{4!}\right)\int d^4z\phi(z)\phi(z)\phi(z)\phi(z)\right\}|0\rangle\end{aligned}$$

Now apply Wick's theorem. We get one term for every way of contracting the six ϕ operators with each other in pairs. There are 15 ways to do this, but (fortunately) only two of them are really different. If we contract $\phi(x)$ with $\phi(y)$, then there are three ways to contract the four $\phi(z)$'s with each other, and all three give identical expressions. The other possibility is to contract $\phi(x)$ with one of the $\phi(z)$'s (four choices), $\phi(y)$ with one of the others (three choices), and the remaining two $\phi(z)$'s with each other (one choice). There are twelve ways to do this, and all give identical expressions. Thus we have

$$\begin{aligned}&\langle 0|T\left\{\phi(x)\phi(y)(-i)\int dt\int d^3z\frac{\lambda}{4!}\phi^4\right\}|0\rangle\\ &\quad=3\cdot\left(\frac{-i\lambda}{4!}\right)D_F(x-y)\int d^4zD_F(z-z)D_F(z-z)\\ &\qquad+12\cdot\left(\frac{-i\lambda}{4!}\right)\int d^4zD_F(x-z)D_F(y-z)D_F(z-z).\end{aligned} \tag{4.44}$$

We can understand this expression better if we represent each term as a Feynman diagram. Again we draw each contraction D_F as a line, and each point as a dot. But this time we must distinguish between the "external" points, x and y, and the "internal" point z; each internal point

is associated with a factor of $(-i\lambda)\int d^4z$. We will worry about the constant factors in a moment. Using these rules, we see that the above expression (4.44) is equal to the sum of two diagrams:



We refer to the lines in these diagrams as propagators, since they represent the propagation amplitude D_F. Internal points, where four lines meet, are called vertices. Since $D_F(x-y)$ is the amplitude for a free Klein-Gordon particle to propagate between x and y, the diagrams actually interpret the analytic formula as a process of particle creation, propagation, and annihilation which takes place in spacetime.

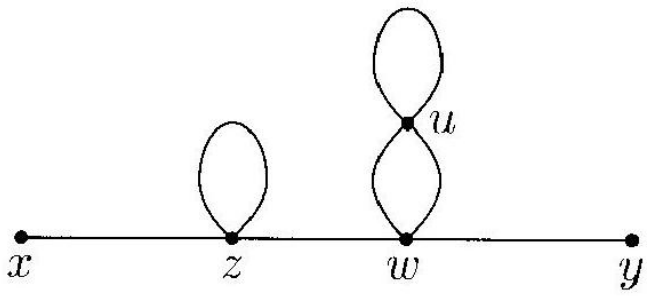
Now let's try a more complicated contraction, from the λ^3 term in the expansion of the correlation function:

$$\begin{aligned}&\langle 0|\phi(x)\phi(y)\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int d^4z\phi\phi\phi\phi\int d^4w\phi\phi\phi\phi\int d^4u\phi\phi\phi\phi|0\rangle\\&=\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int d^4zd^4wd^4uD_F(x-z)D_F(z-z)D_F(z-w)\\&\qquad\times D_F(w-y)D_F^2(w-u)D_F(u-u)\end{aligned}\tag{4.45}$$

The number of "different" contractions that give this same expression is large:

$$\underbrace{3!}_{\substack{\text{interchange}\\\text{of vertices}}}\times\underbrace{4\cdot3}_{\substack{\text{placement of}\\\text{contractions}\\\text{into } z \text{ vertex}}}\times\underbrace{4\cdot3\cdot2}_{\substack{\text{placement of}\\\text{contractions}\\\text{into } w \text{ vertex}}}\times\underbrace{4\cdot3}_{\substack{\text{placement of}\\\text{contractions}\\\text{into } u \text{ vertex}}}\times\underbrace{1/2}_{\substack{\text{interchange}\\\text{of } w-u\\\text{contractions}}}$$

The product of these combinatoric factors is 10,368 , roughly 1/13 of the total of 135,135 possible full contractions of the 14 operators. The structure of this particular contraction can be represented by the following "cactus" diagram:



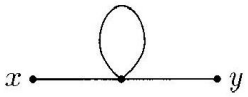
It is conventional, for obvious reasons, to let this one diagram represent the sum of all 10,368 identical terms.

In practice one always draws the diagram first, using it as a mnemonic device for writing down the analytic expression. But then the question arises, What is the overall constant? We could, of course, work it out as above: We could associate a factor $\int d^4z(-i\lambda/4\ !)$ with each vertex, put in the $1/n\ !$ from the Taylor series, and then do the combinatorics by writing out the product of fields as in (4.45) and counting. But the $1/n\ !$ from the Taylor series will almost always cancel the $n\ !$

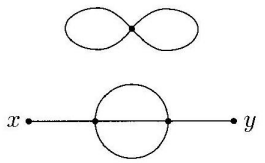
from interchanging the vertices, so we can just forget about both of these factors. Furthermore, the generic vertex has four lines coming in from four different places, so the various placements of these contractions into $\phi\phi\phi\phi$ generates a factor of 4! (as in the w vertex above), which cancels the denominator in $(-i\lambda/4\,!)$. It is therefore conventional to associate the expression $\int d^4z(-i\lambda)$ with each vertex. (This was the reason for the factor of 4 ! in the ϕ^4 coupling.)

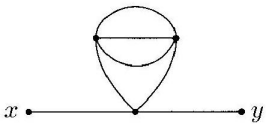
In the above diagram, this scheme gives a constant that is too large by a factor of $8 = 2 \cdot 2 \cdot 2$, the symmetry factor of the diagram. Two factors of 2 come from lines that start and end on the same vertex: The diagram is symmetric under the interchange of the two ends of such a line. The other factor of 2 comes from the two propagators connecting w to u: The diagram is symmetric under the interchange of these two lines with each other. A third possible type of symmetry is the equivalence of two vertices. To get the correct overall constant for a diagram, we divide by its symmetry factor, which is in general the number of ways of interchanging components without changing the diagram.

Most people never need to evaluate a diagram with a symmetry factor greater than 2 , so there's no need to worry too much about these technicalities. But here are a few examples, to make some sense out of the above rules:



$S = 2$





$S = 2 \cdot 2 \cdot 2 = 8$

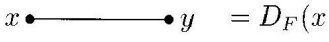
$S = 3! = 6$

$S = 3! \cdot 2 = 12$

When in doubt, you can always determine the symmetry factor by counting equivalent contractions, as we did above.

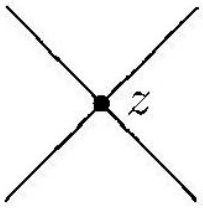
We are now ready to summarize our rules for calculating the numerator of our expression (4.31) for $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$:
$\langle 0|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int dtH_I(t)\right]\right\}|0\rangle = \left(\begin{smallmatrix}\text{sum of all possible diagrams}\\ \text{with two external points}\end{smallmatrix}\right)$,
where each diagram is built out of propagators, vertices, and external points. The rules for associating analytic expressions with pieces of diagrams are called the Feynman rules. In ϕ^4 theory the rules are:

1. For each propagator,



$= D_F(x-y);$

2. For each vertex,



$$= (-i\lambda) \int d^4z$$

3. For each external point, x

4. Divide by the symmetry factor.

One way to interpret these rules is to think of the vertex factor $(-i\lambda)$ as the amplitude for the emission and/or absorption of particles at a vertex. The integral $\int d^4z$ instructs us to sum over all points where this process can occur. This is just the superposition principle of quantum mechanics: When a process can happen in alternative ways, we add the amplitudes for each possible way. To compute each individual amplitude, the Feynman rules tell us to multiply the amplitudes (propagators and vertex factors) for each independent part of the process.

Since these rules are written in terms of the spacetime points x, y, etc., they are sometimes called the position-space Feynman rules. In most calculations, it is simpler to express the Feynman rules in terms of momenta, by introducing the Fourier expansion of each propagator:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} \tag{4.46}$$

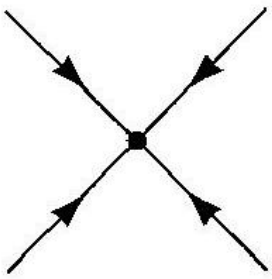
Represent this in the diagram by assigning a 4 -momentum p to each propagator, indicating the direction with an arrow. (Since $D_F(x-y) = D_F(y-x)$, the direction of p is arbitrary.) Then when four lines meet at a vertex, the z-dependent factors of the diagram are

$$\longleftrightarrow \quad \begin{array}{l} \int d^4z e^{-ip_1 z} e^{-ip_2 z} e^{-ip_3 z} e^{+ip_4 z} \\ = (2\pi)^4 \delta^{(4)}\left(p_1 + p_2 + p_3 - p_4\right) \end{array} , \$p_2 \tag{4.47}$$

In other words, momentum is conserved at each vertex. The delta functions
from the vertices can now be used to perform some of the momentum integrals from the propagators. We are left with the following momentum-space Feynman rules:

1. For each propagator, $\longrightarrow p = \frac{i}{p^2 - m^2 + i\epsilon}$;

2. For each vertex,



$$= -i\lambda$$

3. For each external point, $x\bullet \longleftarrow\!= e^{-ip\cdot x}$;

4. Impose momentum conservation at each vertex;

5. Integrate over each undetermined momentum: $\int \frac{d^4p}{(2\pi)^4}$;

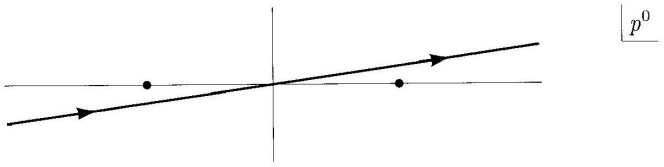
6. Divide by the symmetry factor.

Again, we can interpret each factor as the amplitude for that part of the process, with the integrations coming from the superposition principle. The exponential factor for an external point is just the amplitude for a particle at that point to have the needed momentum, or, depending on the direction of the arrow, for a particle with a certain momentum to be found at that point.

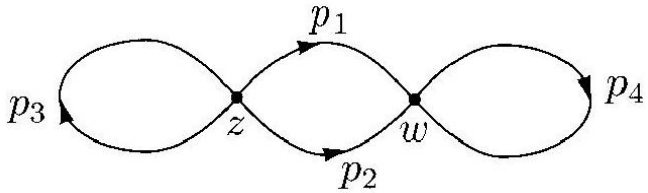
This nearly completes our discussion of the computation of correlation functions, but there are still a few loose ends. First, what happened to the large time T that was taken to $\infty(1-i\epsilon)$? We glossed over it completely in this section, starting with Eq. (4.43). The place to put it back is Eq. (4.47), where instead of just integrating over d^4z, we should have

$$\lim_{T\to\infty(1-i\epsilon)} \int_{-T}^{T} dz^0 \int d^3z e^{-i(p_1+p_2+p_3-p_4)\cdot z}.$$

The exponential blows up as $z^0 \to \infty$ or $z^0 \to -\infty$ unless its argument is purely imaginary. To achieve this, we can take each p^0 to have a small imaginary part: $p^0 \propto (1+i\epsilon)$. But this is precisely what we do in following the Feynman boundary conditions for computing D_F: We integrate along a contour that is rotated slightly away from the real axis, so that $p^0 \propto (1+i\epsilon)$:



The explicit dependence on T seems to disappear when we take the limit $T \to \infty$ in (4.46). But consider the diagram

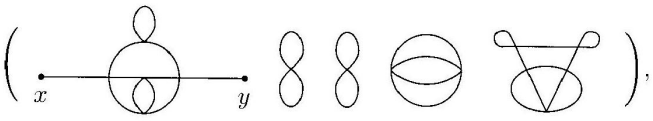


The delta function for the left-hand vertex is $(2\pi)^4\delta(p_1+p_2)$, so momentum conservation at the right-hand vertex is automatically satisfied, and we get $(2\pi)^4\delta(0)$ there. This awkward factor is easy to understand by going back to position space. It is simply the integral of a constant over d^4w:

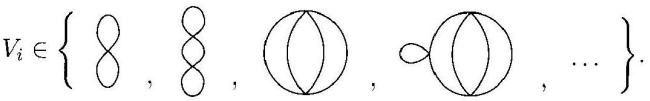
$$\int d^4w(\text{ const }) \propto (2T)\cdot(\text{ volume of space }) \tag{4.49}$$

This just tells us that the spacetime process (4.48) can happen at any place in space, and at any time between $-T$ and T. Every disconnected piece of a diagram, that is, every piece that is not connected to an external point, will have one such $(2\pi)^4\delta(0) = 2T\cdot V$ factor.

The contributions to the correlation function coming from such diagrams can be better understood with the help of a very pretty identity, the exponentiation of the disconnected diagrams. It works as follows. A typical diagram has the form



with a piece connected to x and y, and several disconnected pieces. (Since each vertex has an even number of lines coming into it, x and y must be connected to each other.) Label the various possible disconnected pieces by V_i:



The elements V_i are connected internally, but disconnected from external points. Suppose that a diagram (such as (4.50)) has n_i pieces of the form V_i, for each i, in addition to its one piece that is connected to x and y. (In any given diagram, only finitely many of the n_i will be nonzero.) If we also let V_i denote the value of the piece V_i, then the value of such a diagram is

$$(\text{value of connected piece}) \prod_i \frac{1}{n_i!} (V_i)^{n_i}$$

The $1/n_i$! is the symmetry factor coming from interchanging the n_i copies of V_i. The sum of all diagrams, representing the numerator of our formula for
the two-point correlation function, is then

$$\sum_{\substack{\text{all possible} \\ \text{connected} \\ \text{pieces}}} \sum_{\{n_i\}} \left(\begin{array}{c} \text{value of} \\ \text{connected piece} \end{array} \right) \times \left(\prod_i \frac{1}{n_i!} (V_i)^{n_i} \right)$$

where "all $\{n_i\}$ " means "all ordered sets $\{n_1, n_2, n_3, ...\}$ of nonnegative integers." The sum of the connected pieces factors out of this expression, giving

$$= \left(\sum \text{ connected} \right) \times \sum_{\text{all } \{n_i\}} \left(\prod_i \frac{1}{n_i!} (V_i)^{n_i} \right)$$

where $(\sum$ connected$)$ is an abbreviation for the sum of the values of all connected pieces. It is not too hard to see that the rest of the expression can also be factored (try working backwards):

$$\begin{aligned} &= \left(\sum \text{ connected} \right) \times \left(\sum_{n_1} \frac{1}{n_1!} V_1^{n_1} \right) \left(\sum_{n_2} \frac{1}{n_2!} V_2^{n_2} \right) \left(\sum_{n_3} \frac{1}{n_3!} V_3^{n_3} \right) \cdots \\ &= \left(\sum \text{ connected} \right) \times \prod_i \left(\sum_{n_i} \frac{1}{n_i!} V_i^{n_i} \right) \\ &= \left(\sum \text{ connected} \right) \times \prod_i \exp(V_i) \\ &= \left(\sum \text{ connected} \right) \times \exp\left(\sum_i V_i \right) \end{aligned} \tag{4.52}$$

We have just shown that the sum of all diagrams is equal to the sum of all connected diagrams, times the exponential of the sum of all disconnected diagrams. (We should really say "pieces" rather than "diagrams" on the righthand side of the equality, but from now on we will often just call a single piece a "diagram.") Pictorially, the identity is

$$\lim_{T\to\infty(1-i\epsilon)}\langle 0|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dtH_I(t)\right]\right\}|0\rangle$$
$$= \left(\begin{matrix}\dot{y}\\ x\end{matrix} + \underset{\dot{y}}{\overset{\dot{x}}{\bigotimes}} + \cdots\right)$$
$$\times \exp[\therefore + 8 + (\square). \tag{4.53}$$

Now consider the denominator of our formula (4.31) for the two-point function. By an argument identical to the above, it is just

$$\langle 0|T\left\{\exp\left[-i\int_{-T}^{T}dtH_I(t)\right]\right\}|0\rangle = \exp[\}+\Big\} + (\square) + \cdots\Big]$$

which cancels the exponential of the disconnected diagrams in the numerator. This is the final simplification of the formula, which now reads

$$\langle\Omega|T[\phi(x)\phi(y)]|\Omega\rangle$$
$$= \text{ sum of all connected diagrams with two external points} \tag{4.54}$$

We have come a long way from our original formula, Eq. (4.31).
Having gotten rid of the disconnected diagrams in our formula for the correlation function, we might pause a moment to go back and interpret them physically. The place to look is Eq. (4.30), which can be written

$$\lim_{T\to\infty(1-i\epsilon)}\langle 0|T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dtH_I(t)\right]\right\}|0\rangle$$
$$= \langle\Omega|T\phi(x)\phi(y)|\Omega\rangle\cdot\lim_{T\to\infty(1-i\epsilon)}\left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)$$

Looking only at the T-dependent parts of both sides, this implies

$$\exp\left[\sum_i V_i\right] \propto \exp\left[-iE_0(2T)\right] \tag{4.55}$$

Since each disconnected diagram V_i contains a factor of $(2\pi)^4\delta^{(4)}(0) = 2T\cdot V$, this gives us a formula for the energy density of the vacuum (relative to the zero of energy set by $H_0|0\rangle = 0$):

$$\frac{E_0}{\text{volume}} = i[\bigcup + \zeta + (\square)]/\left[(2\pi)^4\delta^{(4)}(0)\right] \tag{4.56}$$

We should emphasize that the right-hand side is independent of T and (volume); in particular it is reassuring to see that E_0 is proportional to the volume of space. In Chapter 11 we will find that this formula is actually useful.

This completes our present analysis of the two-point correlation function. The generalization to higher correlation functions is easy:

$$\langle\Omega|T\left[\phi\left(x_1\right)\cdots\phi\left(x_n\right)\right]|\Omega\rangle=\left(\begin{array}{c}\text{sum of all connected diagrams}\\ \text{with } n \text{ external points}\end{array}\right) \tag{4.57}$$

The disconnected diagrams exponentiate, factor, and cancel as before, by the same argument. There is now a potential confusion in terminology, however. By "disconnected" we mean "disconnected from all external points"-exactly the same diagrams as in (4.51). (They are sometimes called "vacuum bubbles"
or "vacuum to vacuum transitions".) In higher correlation functions, diagrams can also be disconnected in another sense. Consider, for example, the fourpoint function:

$$\begin{aligned}&\langle\Omega|T\phi_1\phi_2\phi_3\phi_4|\Omega\rangle\\ &=-\quad+1+\quad 1+\lambda+\underline{0}+0\mid+\cdots\\ &+X+\varnothing\mid+\underline{00}+\cdots\\ &+\wp+\cdots+\not{+}\cdots+\cdots\,.\end{aligned} \tag{4.58}$$

In many of these diagrams, external points are disconnected from each other. Such diagrams do not exponentiate or factor; they contribute to the amplitude just as do the fully connected diagrams (in which any point can be reached from any other by traveling along the lines).

Note that in ϕ^4 theory, all correlation functions of an odd number of fields vanish, since it is impossible to draw an allowed diagram with an odd number of external points. We can also see this by going back to Wick's theorem: The interaction Hamiltonian H_I contains an even number of fields, so all terms in the perturbation expansion of an odd correlation function will contain an odd number of fields. But it is impossible to fully contract an odd number of fields in pairs, and only fully contracted terms have nonvanishing vacuum expectation value.

14.4 4.5 Cross Sections and the S-Matrix

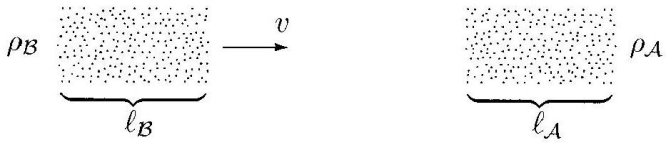
We now have an extremely beautiful formula, Eq. (4.57), for computing an extremely abstract quantity: the n-point correlation function. Our next task is to find equally beautiful ways of computing quantities that can actually be measured: cross sections and decay rates. In this section, after briefly reviewing the definitions of these objects, we will relate them (via a rather technical but fairly careful derivation) to a more primitive quantity, the S-matrix. In the next section we will learn how to compute the matrix elements of the S-matrix using Feynman diagrams.

The Cross Section

The experiments that probe the behavior of elementary particles, especially in the relativistic regime, are scattering experiments. One collides two beams of particles with well-defined momenta, and observes what comes out. The likelihood of any particular final state can be expressed in terms of the cross section, a quantity that is intrinsic to the colliding particles, and therefore
allows comparison of two different experiments with different beam sizes and intensities.

The cross section is defined as follows. Consider a target, at rest, of particles of type $\mathcal{A}$, with density $\rho_{\mathcal{A}}$ (particles per unit volume). Aim at this target a bunch of particles of type $\mathcal{B}$, with

number density $\rho_{\mathcal{B}}$ and velocity v:



Let $\ell_{\mathcal{A}}$ and $\ell_{\mathcal{B}}$ be the lengths of the bunches of particles. Then we expect the total number of scattering events (or scattering events of any particular desired type) to be proportional to $\rho_{\mathcal{A}}, \rho_{\mathcal{B}}, \ell_{\mathcal{A}}, \ell_{\mathcal{B}}$, and the cross-sectional area A common to the two bunches. The cross section, denoted by σ, is just the total number of events (of whatever type desired) divided by all of these quantities:

$$\sigma \equiv \frac{\text{Number of scattering events}}{\rho_{\mathcal{A}}\ell_{\mathcal{A}}\rho_{\mathcal{B}}\ell_{\mathcal{B}}A} \tag{4.59}$$

The definition is symmetric between the $\mathcal{A}$'s and $\mathcal{B}$'s, so of course we could have taken the $\mathcal{B}$'s to be at rest, or worked in any other reference frame.

The cross section has units of area. In fact, it is the effective area of a chunk taken out of one beam, by each particle in the other beam, that subsequently becomes the final state we are interested in.

In real beams, $\rho_{\mathcal{A}}$ and $\rho_{\mathcal{B}}$ are not constant; the particle density is generally larger at the center of the beam than at the edges. We will assume, however, that both the range of the interaction between the particles and the width of the individual particle wavepackets are small compared to the beam diameter. We can then consider $\rho_{\mathcal{A}}$ and $\rho_{\mathcal{B}}$ to be constant in what follows, and remember that, to compute the event rate in an actual accelerator, one must integrate over the beam area:

$$\text{Number of events } = \sigma\ell_{\mathcal{A}}\ell_{\mathcal{B}} \int d^2x \rho_{\mathcal{A}}(x)\rho_{\mathcal{B}}(x) \tag{4.60}$$

If the densities are constant, or if we use this formula to compute an effective area A of the beams, then we have simply

$$\text{Number of events } = \frac{\sigma N_{\mathcal{A}} N_{\mathcal{B}}}{A}, \tag{4.61}$$

where $N_{\mathcal{A}}$ and $N_{\mathcal{B}}$ are the total numbers of $\mathcal{A}$ and $\mathcal{B}$ particles.

Cross sections for many different processes may be relevant to a single scattering experiment. In e^+e^-collisions, for example, one can measure the cross sections for production of $\mu^+\mu^-, \tau^+\tau^-, \mu^+\mu^-\gamma, \mu^+\mu^-\gamma\gamma$, etc., and countless processes involving hadron production, not to mention simple e^+e^- scattering. Usually, of course, we wish to measure not only what the final-state particles are, but also the momenta with which they come out. In this case our definition (4.59) of σ still works, but if we specify the exact momenta desired, σ will be infinitesimal. The solution is to define the differential cross section, $d\sigma/\left(d^3p_1 \cdots d^3p_n\right)$. It is simply the quantity that, when integrated over any small $d^3p_1 \cdots d^3p_n$, gives the cross section for scattering into that region of final-state momentum space. The various final-state momenta are not all independent: Four components will always be constrained by 4 -momentum conservation. In the simplest case, where there are only two final-state particles, this leaves only two unconstrained momentum components, usually taken to be the angles θ and ϕ of the momentum of one of the particles. Integrating $d\sigma/\left(d^3p_1 d^3p_2\right)$ over the four constrained momentum components then leaves us with the usual differential cross section $d\sigma/d\Omega$.

A somewhat simpler measurable quantity is the decay rate Γ of an unstable particle $\mathcal{A}$ (assumed to be at rest) into a specified final state (of two or more particles). It is defined as

$$\Gamma \equiv \frac{\text{Number of decays per unit time}}{\text{Number of } \mathcal{A} \text{ particles present}} \tag{4.62}$$

The lifetime τ of the particle is then the reciprocal of the sum of its decay rates into all possible final states. (The particle's half-life is $\tau \cdot \ln 2$.)

In nonrelativistic quantum mechanics, an unstable atomic state shows up in scattering experiments as a resonance. Near the resonance energy E_0, the scattering amplitude is given by the Breit-Wigner formula

$$f(E) \propto \frac{1}{E - E_0 + i\Gamma/2}. \tag{4.63}$$

The cross section therefore has a peak of the form

$$\sigma \propto \frac{1}{(E - E_0)^2 + \Gamma^2/4}$$

The width of the resonance peak is equal to the decay rate of the unstable state.

The Breit-Wigner formula (4.63) also applies in relativistic quantum mechanics. In particular, it gives the scattering amplitude for processes in which initial particles combine to form an unstable particle, which then decays. The unstable particle, viewed as an excited state of the vacuum, is a direct analogue of the unstable nonrelativistic atomic state. If we call the 4 -momentum of the unstable particle p and its mass m, we can make a relativistically invariant generalization of (4.63):

$$\frac{1}{p^2 - m^2 + im\Gamma} \approx \frac{1}{2E_{\mathbf{p}}\left(p^0 - E_{\mathbf{p}} + i\left(m/E_{\mathbf{p}}\right)\Gamma/2\right)} \tag{4.64}$$

The decay rate of the unstable particle in a general frame is $(m/E_{\mathbf{p}})\,\Gamma$, in accord with relativistic time dilation. Although the two expressions in (4.64) are equal in the vicinity of the resonance, the left-hand side, which is manifestly Lorentz invariant, is much more convenient.

The S-Matrix

How, then, do we calculate a cross section? We must set up wavepackets representing the initial-state particles, evolve this initial state for a very long time with the time-evolution operator $\exp(-iHt)$ of the interacting field theory, and overlap the resulting final state with wavepackets representing some desired set of final-state particles. This gives the probability amplitude for producing that final state, which is simply related to the cross section. We will find that, in the limit where the wavepackets are very narrow in momentum space, the amplitude depends only on the momenta of the wavepackets, not on the details of their shapes. †

A wavepacket representing some desired state $|\phi\rangle$ can be expressed as

$$|\phi\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}}}} \phi(\mathbf{k})|\mathbf{k}\rangle \tag{4.65}$$

where $\phi(\mathbf{k})$ is the Fourier transform of the spatial wavefunction, and $|\mathbf{k}\rangle$ is a one-particle state of momentum $\mathbf{k}$ in the interacting theory. In the free theory, we would have $|\mathbf{k}\rangle = \sqrt{2E_{\mathbf{k}}}\, a^\dagger_{\mathbf{k}}|0\rangle$. The factor of $\sqrt{2E_{\mathbf{k}}}$ converts our relativistic normalization of $|\mathbf{k}\rangle$ to the conventional normalization in which the sum of all probabilities adds up to 1:

$$\langle\phi|\phi\rangle = 1 \quad \text{if} \quad \int \frac{d^3k}{(2\pi)^3}|\phi(\mathbf{k})|^2 = 1 \tag{4.66}$$

The probability we wish to compute is then

$$\mathcal{P} = |\langle\underbrace{\phi_1\phi_2\cdots}_{\text{future}}|\underbrace{\phi_{\mathcal{A}}\phi_{\mathcal{B}}}_{\text{past}}\rangle|^2, \tag{4.67}$$

where $|\phi_{\mathcal{A}}\phi_{\mathcal{B}}\rangle$ is a state of two wavepackets constructed in the far past and $\langle\phi_1\phi_2\cdots|$ is a state of several wavepackets (one for each final-state particle) constructed in the far future. The wavepackets are localized in space, so each can be constructed independently of the others. States constructed in this way are called in and out states. Note that we use the Heisenberg picture: States are time-independent, but the name we give a state depends on the eigenvalues or expectation values of time-dependent operators. Thus states with different names constructed at different times have a nontrivial overlap, which depends on the time dependence of the operators.

If we set up $|\phi_{\mathcal{A}}\phi_{\mathcal{B}}\rangle$ in the remote past, and then take the limit in which the wavepackets $\phi_i(\mathbf{k}_i)$ become concentrated about definite momenta $\mathbf{p}_i$, this defines an *in* state $|\mathbf{p}_{\mathcal{A}}\mathbf{p}_{\mathcal{B}}\rangle_{\text{in}}$ with definite initial momenta. It is useful to view $|\phi_{\mathcal{A}}\phi_{\mathcal{B}}\rangle$ as a linear superposition of such states. It is important, however, to

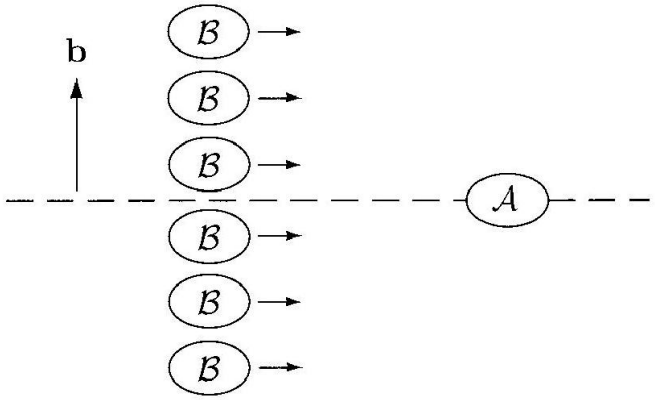


Figure 4.2. Incident wavepackets are uniformly distributed in impact parameter $\mathbf{b}$.

take into account the transverse displacement of the wavepacket $\phi_{\mathcal{B}}$ relative to $\phi_{\mathcal{A}}$ in position space (see Fig. 4.2). Although we could leave this implicit in the form of $\phi_{\mathcal{B}}(\mathbf{k}_{\mathcal{B}})$, we instead adopt the convention that our reference momentum-space wavefunctions are collinear (that is, have impact parameter $\mathbf{b} = 0$), and write $\phi_{\mathcal{B}}(\mathbf{k}_{\mathcal{B}})$ with an explicit factor $\exp(-i\mathbf{b}\cdot\mathbf{k}_{\mathcal{B}})$ to account for the spatial translation. Then, since $\phi_{\mathcal{A}}$ and $\phi_{\mathcal{B}}$ are constructed independently at different locations, we can write the initial state as

$$|\phi_{\mathcal{A}}\phi_{\mathcal{B}}\rangle_{\text{in}} = \int \frac{d^3k_{\mathcal{A}}}{(2\pi)^3}\int \frac{d^3k_{\mathcal{B}}}{(2\pi)^3}\frac{\phi_{\mathcal{A}}(\mathbf{k}_{\mathcal{A}})\,\phi_{\mathcal{B}}(\mathbf{k}_{\mathcal{B}})\,e^{-i\mathbf{b}\cdot\mathbf{k}_{\mathcal{B}}}}{\sqrt{(2E_{\mathcal{A}})(2E_{\mathcal{B}})}}|\mathbf{k}_{\mathcal{A}}\mathbf{k}_{\mathcal{B}}\rangle_{\text{in}} \tag{4.68}$$

We could expand $\langle\phi_1\phi_2\cdots|$ in terms of similarly defined out states of definite momentum formed in the asymptotic future: ‡

[6†] Much of this section is based on the treatment of nonrelativistic scattering given in Taylor (1972), Chapters 2, 3, and 17. We concentrate on the additional complications of the relativistic theory, glossing over many subtleties, common to both cases, which Taylor explains carcfully.

$$_{\text{out}}\,\langle\phi_1\phi_2\cdots| = \left(\prod_f \int \frac{d^3p_f}{(2\pi)^3}\frac{\phi_f\left(\mathbf{p}_f\right)}{\sqrt{2E_f}}\right) \text{out}\,\langle\mathbf{p}_1\mathbf{p}_2\cdots|\,.$$

It is much easier, however, to use the out states of definite momentum as the final states in the probability amplitude (4.67), and to multiply by the various normalization factors after squaring the amplitude. This is physically reasonable as long as the detectors of final-state particles mainly measure momentum - that is, they do not resolve positions at the level of de Broglie wavelengths.

We can now relate the probability of scattering in a real experiment to an idealized set of transition amplitudes between the asymptotically defined in and out states of definite momentum,

$$_{\text{out}}\,\langle\mathbf{p}_1\mathbf{p}_2\cdots|\mathbf{k}_\mathcal{A}\mathbf{k}_\mathcal{B}\rangle_{\text{in}}\ . \tag{4.69}$$

To compute the overlap of in states with out states, we note that the conventions for defining the two sets of states are related by time translation:

$$\begin{aligned} _{\text{out}}\,\langle\mathbf{p}_1\mathbf{p}_2\cdots|\mathbf{k}_\mathcal{A}\mathbf{k}_\mathcal{B}\rangle_{\text{in}} &= \lim_{T\to\infty}\langle\underbrace{\mathbf{p}_1\mathbf{p}_2\cdots}_{T}|\,\underbrace{\mathbf{k}_\mathcal{A}\mathbf{k}_\mathcal{B}}_{-T}\rangle \\ &= \lim_{T\to\infty}\langle\mathbf{p}_1\mathbf{p}_2\cdots|\,e^{-iH(2T)}\,|\mathbf{k}_\mathcal{A}\mathbf{k}_\mathcal{B}\rangle \end{aligned} \tag{4.70}$$

In the last line, the states are defined at any common reference time. Thus, the in and out states are related by the limit of a sequence of unitary operators. This limiting unitary operator is called the S-matrix:

$$_{\text{out}}\,\langle\mathbf{p}_1\mathbf{p}_2\cdots|\mathbf{k}_\mathcal{A}\mathbf{k}_\mathcal{B}\rangle_{\text{in}} \equiv \langle\mathbf{p}_1\mathbf{p}_2\cdots|\,S\,|\mathbf{k}_\mathcal{A}\mathbf{k}_\mathcal{B}\rangle \tag{4.71}$$

The S-matrix has the following structure: If the particles in question do not interact at all, S is simply the identity operator. Even if the theory contains interactions, the particles have some probability of simply missing one another. To isolate the interesting part of the S-matrix-that is, the part due to interactions-we define the T-matrix by

$$S = \mathbf{1} + iT \tag{4.72}$$

Next we note that the matrix elements of S should reflect 4-momentum conservation. Thus S or T should always contain a factor $\delta^{(4)}\left(k_\mathcal{A}+k_\mathcal{B}-\sum p_f\right)$. Extracting this factor, we define the invariant matrix element $\mathcal{M}$, by

$$\langle\mathbf{p}_1\mathbf{p}_2\cdots|\,iT\,|\mathbf{k}_\mathcal{A}\mathbf{k}_\mathcal{B}\rangle = (2\pi)^4\delta^{(4)}\left(k_\mathcal{A}+k_\mathcal{B}-\sum p_f\right)\cdot i\mathcal{M}\left(k_\mathcal{A},k_\mathcal{B}\to p_f\right) \tag{4.73}$$

We have written this expression in terms of 4-momenta p and k, but of course all 4-momenta are on mass-shell: $p^0=E_\mathbf{p}, k^0=E_\mathbf{k}$. (Note that our entire treatment is specific to the case where the initial state contains only two particles. For $3\to$ many or many $\to$ many interactions, one can invent analogous constructions, but we will not consider such complicated experiments in this book.)

The matrix element $\mathcal{M}$ is analogous to the scattering amplitude f of one-particle quantum mechanics. It is useful because it allows us to separate all the physics that depends on the details

[6]‡ Here and below, the product symbol applies (symbolically) to the integral as well as the other factors in parentheses; the integrals apply to what is outside the parentheses as well.

of the interaction Hamiltonian ("dynamics") from all the physics that doesn't ("kinematics"). In the next section we will discuss how to compute $\mathcal{M}$ using Feynman diagrams. But first, we must figure out how to reconstruct the cross section σ from $\mathcal{M}$.

To do this, let us calculate, in terms of $\mathcal{M}$, the probability for the initial state $|\phi_A\phi_B\rangle$ to scatter and become a final state of n particles whose momenta lie in a small region $d^3p_1 \cdots d^3p_n$. In our normalization, this probability is

$$\mathcal{P}(\mathcal{AB} \to 12...n) = \left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\Bigg|_{\text{out}} \langle \mathbf{p}_1 \cdots \mathbf{p}_n | \phi_A\phi_B\rangle_{\text{in}}\Bigg|^2 \tag{4.74}$$

For a single target ($\mathcal{A}$) particle and many incident ($\mathcal{B}$) particles with different impact parameters $\mathbf{b}$, the number of scattering events is

$$N = \sum_{\substack{\text{all incident}\\ \text{particles } i}} \mathcal{P}_i = \int d^2b n_{\mathcal{B}} \mathcal{P}(\mathbf{b})$$

where $n_{\mathcal{B}}$ is the number density (particles per unit area) of $\mathcal{B}$ particles. Since we are assuming that this number density is constant over the range of the interaction, $n_{\mathcal{B}}$ can be taken outside the integral. The cross section is then

$$\sigma = \frac{N}{n_{\mathcal{B}} N_{\mathcal{A}}} = \frac{N}{n_{\mathcal{B}} \cdot 1} = \int d^2b \mathcal{P}(\mathbf{b}) \tag{4.75}$$

Deriving a simple expression for σ in terms of $\mathcal{M}$ is now a fairly straightforward calculation. Combining (4.75), (4.74), and (4.68), we have (writing $d\sigma$ rather than σ since this is an infinitesimal quantity)

$$\begin{aligned} d\sigma = &\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\ \frac{1}{2E_f}\right)\int d^2b \left(\prod_{i=\mathcal{A},\mathcal{B}} \int \frac{d^3k_i}{(2\pi)^3}\frac{\phi_i(\mathbf{k}_i)}{\sqrt{2E_i}} \int \frac{d^3\bar{k}_i}{(2\pi)^3}\frac{\phi_i^*(\bar{\mathbf{k}}_i)}{\sqrt{2\bar{E}_i}}\right) \\ &\times e^{i\mathbf{b}\cdot\left(\bar{\mathbf{k}}_{\mathcal{B}} - \mathbf{k}_{\mathcal{B}}\right)} \left({}_{\text{out}}\ \langle\{\mathbf{p}_f\} \mid \{\mathbf{k}_i\}\rangle_{\text{in}}\right)\left({}_{\text{out}}\ \left\langle\{\mathbf{p}_f\} \mid \left\{\bar{\mathbf{k}}_i\right\}\right\rangle_{\text{in}}\right)^* \end{aligned} \tag{4.76}$$

where we have used $\bar{k}_{\mathcal{A}}$ and $\bar{k}_{\mathcal{B}}$ as dummy integration variables in the second half of the squared amplitude. The d^2b integral can be performed to give a factor of $(2\pi)^2\delta^{(2)}\left(k_{\mathcal{B}}^{\perp} - \bar{k}_{\mathcal{B}}^{\perp}\right)$. We get more delta functions by writing the final two factors of (4.76) in terms of $\mathcal{M}$. Assuming that we are not interested in the trivial case of forward scattering where no interaction takes place, we can drop the 1 in Eq. (4.72) and write these factors as

$$\begin{aligned} \left({}_{\text{out}}\ \left(\{\mathbf{p}_f\} \mid \{\mathbf{k}_i\}\right\rangle_{\text{in}}\right) &= i\mathcal{M}\left(\{k_i\} \to \{p_f\}\right)(2\pi)^4\delta^{(4)}\left(\sum k_i - \sum p_f\right) \\ \left({}_{\text{out}}\ \left\langle\{\mathbf{p}_f\} \mid \left\{\bar{\mathbf{k}}_i\right\}\right\rangle_{\text{in}}\right)^* &= -i\mathcal{M}^*\left(\left\{\bar{k}_i\right\} \to \{p_f\}\right)(2\pi)^4\delta^{(4)}\left(\sum \bar{k}_i - \sum p_f\right) \end{aligned}$$

We can use the second of these delta functions, together with the $\delta^{(2)}\left(k_{\mathcal{B}} - \bar{k}_{\mathcal{B}}\right)$, to perform all six of the $\bar{k}$ integrals in (4.76). Of the six integrals, only those over $\bar{k}_{\mathcal{A}}^z$ and $\bar{k}_{\mathcal{B}}^z$ require some work:

$$\begin{aligned}
&\int d\bar{k}^z_{\mathcal{A}} d\bar{k}^z_{\mathcal{B}} \delta\left(\bar{k}^z_{\mathcal{A}} + \bar{k}^z_{\mathcal{B}} - \sum p^z_f\right) \delta\left(\bar{E}_{\mathcal{A}} + \bar{E}_{\mathcal{B}} - \sum E_f\right) \\
&= \int d\bar{k}^z_{\mathcal{A}} \delta\left(\sqrt{\bar{k}^2_{\mathcal{A}} + m^2_{\mathcal{A}}} + \sqrt{\bar{k}^2_{\mathcal{B}} + m^2_{\mathcal{B}}} - \sum E_f\right)\Big|_{\bar{k}^z_{\mathcal{B}} = \Sigma p^z_f - \bar{k}^z_{\mathcal{A}}} \\
&= \frac{1}{\left|\frac{\bar{k}^z_{\mathcal{A}}}{\bar{E}_{\mathcal{A}}} - \frac{\bar{k}^z_{\mathcal{B}}}{\bar{E}^z_{\mathcal{B}}}\right|} \equiv \frac{1}{|v_{\mathcal{A}} - v_{\mathcal{B}}|}
\end{aligned} \tag{4.77}$$

In the last line and in the rest of Eq. (4.76) it is understood that the constraints $\bar{k}^z_{\mathcal{A}} + \bar{k}^z_{\mathcal{B}} = \sum p^z_f$ and $\bar{E}_{\mathcal{A}} + \bar{E}_{\mathcal{B}} = \sum E_f$ now apply (in addition to the constraints $\bar{k}^\perp_{\mathcal{A}} = k^\perp_{\mathcal{A}}$ and $\bar{k}^\perp_{\mathcal{B}} = k^\perp_{\mathcal{B}}$ coming from the other four integrals).

The difference $|v_A - v_B|$ is the relative velocity of the beams as viewed from the laboratory frame.

Now recall that the initial wavepackets are localized in momentum space, centered on $\mathbf{p}_{\mathcal{A}}$ and $\mathbf{p}_{\mathcal{B}}$. This means that we can evaluate all factors that are smooth functions of $\mathbf{k}_{\mathcal{A}}$ and $\mathbf{k}_{\mathcal{B}}$ at $\mathbf{p}_{\mathcal{A}}$ and $\mathbf{p}_{\mathcal{B}}$, pulling them outside the integrals. These factors include $E_{\mathcal{A}}, E_{\mathcal{B}}, |v_{\mathcal{A}} - v_{\mathcal{B}}|$, and $\mathcal{M}$-everything except the remaining delta function. After doing this, we arrive at the expression

$$\begin{aligned}
d\sigma = &\left(\prod_f \tfrac{d^3p_f}{(2\pi)^3} \tfrac{1}{2E_f}\right) \tfrac{\left|\mathcal{M}\left(p_{\mathcal{A}}, p_{\mathcal{B}} \to \{p_f\}\right)\right|^2}{2E_{\mathcal{A}} 2E_{\mathcal{B}} |v_{\mathcal{A}} - v_{\mathcal{B}}|} \int \tfrac{d^3k_{\mathcal{A}}}{(2\pi)^3} \int \tfrac{d^3k_{\mathcal{B}}}{(2\pi)^3} \\
&\times \left|\phi_{\mathcal{A}}\left(\mathbf{k}_{\mathcal{A}}\right)\right|^2 \left|\phi_{\mathcal{B}}\left(\mathbf{k}_{\mathcal{B}}\right)\right|^2 (2\pi)^4 \delta^{(4)}\left(k_{\mathcal{A}} + k_{\mathcal{B}} - \sum p_f\right)
\end{aligned} \tag{4.78}$$

To simplify this formula further, we should think a bit more about the properties of real particle detectors. We have already noted that real detectors project mainly onto eigenstates of momentum. But real detectors have finite resolution; that is, they sum incoherently over momentum bites of finite size. Normally, the measurement of the final-state momentum is not of such high quality that it can resolve the small variation of this momentum that results from the momentum spread of the initial wavepackets $\phi_{\mathcal{A}}, \phi_{\mathcal{B}}$. In that case, we may treat even the momentum vector $k_{\mathcal{A}} + k_{\mathcal{B}}$ inside the delta function as being well approximated by its central value $p_{\mathcal{A}} + p_{\mathcal{B}}$. With this further approximation, we can perform the integrals over $k_{\mathcal{A}}$ and $k_{\mathcal{B}}$ using the normalization condition (4.66). This produces the final form of the relation between S-matrix elements and cross sections,

$$\begin{aligned}
d\sigma = &\frac{1}{2E_{\mathcal{A}} 2E_{\mathcal{B}} \left|v_{\mathcal{A}} - v_{\mathcal{B}}\right|} \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f}\right) \\
&\times \left|\mathcal{M}\left(p_{\mathcal{A}}, p_{\mathcal{B}} \to \{p_f\}\right)\right|^2 (2\pi)^4 \delta^{(4)}\left(p_{\mathcal{A}} + p_{\mathcal{B}} - \sum p_f\right)
\end{aligned} \tag{4.79}$$

All dependence on the shapes of the wavepackets has disappeared.
The integral over final-state momenta in (4.79) has the structure

$$\int d\Pi_n = \left(\prod_f \int \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f}\right) (2\pi)^4 \delta^{(4)}\left(P - \sum p_f\right) \tag{4.80}$$

with P the total initial 4-momentum. This integral is manifestly Lorentz invariant, since it is built up from invariant 3 -momentum integrals constrained by a 4 -momentum delta function. This integral is known as relativistically invariant n-body phase space. Of the other ingredients in

(4.79), the matrix element $\mathcal{M}$ is also Lorentz invariant. The Lorentz transformation property of (4.79) therefore comes entirely from the prefactor

$$\frac{1}{E_{\mathcal{A}}E_{\mathcal{B}}\left|v_{\mathcal{A}}-v_{\mathcal{B}}\right|}=\frac{1}{\left|E_{\mathcal{B}}p_{\mathcal{A}}^{z}-E_{\mathcal{A}}p_{\mathcal{B}}^{z}\right|}=\frac{1}{\left|\epsilon_{\mu xy\nu}p_{\mathcal{A}}^{\mu}p_{\mathcal{B}}^{\nu}\right|}$$

This is not Lorentz invariant, but it is invariant to boosts along the z-direction. In fact, this expression has exactly the transformation properties of a crosssectional area.

For the special case of two particles in the final state, we can simplify the general expression (4.79) by partially evaluating the phase-space integrals in the center-of-mass frame. Label the momenta of the two final particles p_1 and p_2. We first choose to integrate all three components of $\mathbf{p}_2$ over the delta functions enforcing 3 -momentum conservation. This sets $\mathbf{p}_2=-\mathbf{p}_1$ and converts the integral over two-body phase space to the form

$$\int d\Pi_2=\int\frac{dp_1p_1^2d\Omega}{(2\pi)^3 2E_1 2E_2}(2\pi)\delta\left(E_{\mathrm{cm}}-E_1-E_2\right) \tag{4.81}$$

where $E_1=\sqrt{p_1^2+m_1^2}, E_2=\sqrt{p_1^2+m_2^2}$, and E_{cm} is the total initial energy. Integrating over the final delta function gives

$$\begin{aligned}\int d\Pi_2&=\int d\Omega\frac{p_1^2}{16\pi^2E_1E_2}\left(\frac{p_1}{E_1}+\frac{p_1}{E_2}\right)^{-1}\\&=\int d\Omega\frac{1}{16\pi^2}\frac{|\mathbf{p}_1|}{E_{\mathrm{cm}}}\end{aligned} \tag{4.82}$$

For reactions symmetric about the collision axis, two-body phase space can be written simply as an integral over the polar angle in the center-of-mass frame:

$$\int d\Pi_2=\int d\cos\theta\frac{1}{16\pi}\frac{2\left|\mathbf{p}_1\right|}{E_{\mathrm{cm}}} \tag{4.83}$$

The last factor tends to 1 at high energy.
Applying this simplification to (4.79), we find the following form of the cross section for two final-state particles:

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}}=\frac{1}{2E_{\mathcal{A}}2E_{\mathcal{B}}\left|v_{\mathcal{A}}-v_{\mathcal{B}}\right|}\frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{\mathrm{cm}}}\left|\mathcal{M}\left(p_{\mathcal{A}},p_{\mathcal{B}}\to p_1,p_2\right)\right|^2 \tag{4.84}$$

In the special case where all four particles have identical masses (including the commonly seen limit $m\to 0$), this reduces to the formula quoted in Chapter 1,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}}=\frac{|\mathcal{M}|^2}{64\pi^2E_{\mathrm{cm}}^2}\quad\text{(all four masses identical).} \tag{4.85}$$

To conclude this section, we should derive a formula for the differential decay rate, $d\Gamma$, in terms of $\mathcal{M}$. The correct expression is only a slight modification of (4.79), and is quite easy to guess: Just remove from (4.79) the factors that do not make sense when the initial state consists of a single particle. The definition of Γ assumes that the decaying particle is at rest, so the normalization factor $(2E_{\mathcal{A}})^{-1}$ becomes $(2m_{\mathcal{A}})^{-1}$. (In any other frame, this factor would give the usual time dilation.) Thus the decay rate formula is

$$d\Gamma = \frac{1}{2m_{\mathcal{A}}}\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)|\mathcal{M}\left(m_{\mathcal{A}} \to \{p_f\}\right)|^2 (2\pi)^4\delta^{(4)}\left(p_{\mathcal{A}} - \sum p_f\right) \tag{4.86}$$

Unfortunately, the meaning of this formula is far from clear. Since an unstable particle cannot be sent into the infinitely distant past, our definition (4.73)
of $\mathcal{M}\left(m_{\mathcal{A}} \to \{p_f\}\right)$ in terms of the S-matrix makes no sense in this context. Nevertheless formula (4.86) is correct, when $\mathcal{M}$ is computed according to the Feynman rules for S-matrix elements that we will present in the following section. We postpone the further discussion of these matters, and the proof of Eq. (4.86), until Section 7.3. Until then, an intuitive notion of $\mathcal{M}$ as a transition amplitude should suffice.

Equations (4.79) and (4.86) are completely general, whether or not the final state contains several identical particles. (The computation of $\mathcal{M}$, of course, will be quite different when identical particles are present, but that is another matter.) When integrating either of these formulae to obtain a total cross section or decay rate, however, we must be careful to avoid counting the same final state several times. If there are n identical particles in the final state, we must either restrict the integration to inequivalent configurations, or divide by n ! after integrating over all sets of momenta.

14.5 4.6 Computing S-Matrix Elements from Feynman Diagrams

Now that we have formulae for cross sections and decay rates in terms of the invariant matrix element $\mathcal{M}$, the only remaining task is to find a way of computing $\mathcal{M}$ for various processes in various interacting field theories. In this section we will write down (and try to motivate) a formula for $\mathcal{M}$ in terms of Feynman diagrams. We postpone the actual proof of this formula until Section 7.2 , since the proof is somewhat technical and will be much easier to understand after we have seen how the formula is used.

Recall from its definition, Eq. (4.71), that the S-matrix is simply the time-evolution operator, $\exp(-iHt)$, in the limit of very large t:

$$\langle \mathbf{p}_1\mathbf{p}_2\cdots|\, S\,|\mathbf{k}_{\mathcal{A}}\mathbf{k}_{\mathcal{B}}\rangle = \lim_{T\to\infty}\langle \mathbf{p}_1\mathbf{p}_2\cdots|\, e^{-iH(2T)}\,|\mathbf{k}_{\mathcal{A}}\mathbf{k}_{\mathcal{B}}\rangle \tag{4.87}$$

To compute this quantity we would like to replace the external plane-wave states in (4.87), which are eigenstates of H, with their counterparts in the unperturbed theory, which are eigenstates of H_0. We successfully made such a replacement for the vacuum state $|\Omega\rangle$ in Eq. (4.27):

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)}\left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

This time we would like to find a relation of the form

$$|\mathbf{k}_{\mathcal{A}}\mathbf{k}_{\mathcal{B}}\rangle \propto \lim_{T\to\infty(1-i\epsilon)} e^{-iHT}\,|\mathbf{k}_{\mathcal{A}}\mathbf{k}_{\mathcal{B}}\rangle_0 \tag{4.88}$$

where we have omitted some unknown phases and overlap factors like those in (4.27). To find such a relation would not be easy. In (4.27), we used the fact that the vacuum was the state of absolute lowest energy. Here we can use only the much weaker statement that the external states with well-separated initial and final particles have the lowest energy consistent with the predetermined
nonzero values of momentum. The problem is a deep one, and it is associated with one of the most

fundamental difficulties of field theory, that interactions affect not only the scattering of distinct particles but also the form of the single-particle states themselves.

If the formula (4.88) could somehow be justified, we could use it to rewrite the right-hand side of (4.87) as

$$\begin{aligned}&\lim_{T\to\infty(1-i\epsilon)} {}_0\left\langle \mathbf{p}_1\cdots\mathbf{p}_n\right| e^{-iH(2T)}\left|\mathbf{p}_{\mathcal{A}}\mathbf{p}_{\mathcal{B}}\right\rangle_0 \\ &\quad\propto \lim_{T\to\infty(1-i\epsilon)} {}_0\left\langle \mathbf{p}_1\cdots\mathbf{p}_n\right| T\left(\exp\left[-i\int_{-T}^{T} dt H_I(t)\right]\right)\left|\mathbf{p}_{\mathcal{A}}\mathbf{p}_{\mathcal{B}}\right\rangle_0 \end{aligned} \tag{4.89}$$

In the evaluation of vacuum expectation values, the awkward proportionality factors between free and interacting vacuum states cancelled out of the final formula, Eq. (4.31). In the present case those factors are so horrible that we have not even attempted to write them down; we only hope that a similar dramatic cancellation will take place here. In fact such a cancellation does take place, although it is not easy to derive this conclusion from our present approach. Up to one small modification (which is unimportant for our present purposes), the formula for the nontrivial part of the S-matrix can be simplified to the following form:

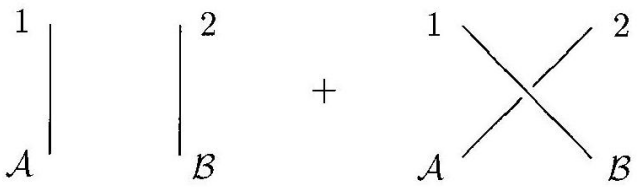
$$\begin{aligned}&\left\langle \mathbf{p}_1\cdots\mathbf{p}_n\right| iT\left|\mathbf{p}_{\mathcal{A}}\mathbf{p}_{\mathcal{B}}\right\rangle \\ &\quad=\lim_{T\to\infty(1-i\epsilon)}\left({}_0\left\langle \mathbf{p}_1\cdots\mathbf{p}_n\right| T\left(\exp\left[-i\int_{-T}^{T} dt H_I(t)\right]\right)\left|\mathbf{p}_{\mathcal{A}}\mathbf{p}_{\mathcal{B}}\right\rangle_0\right)_{\substack{\text{connected,}\\ \text{amputated}}}\end{aligned} \tag{4.90}$$

The attributes "connected" and "amputated" refer to restrictions on the class of possible Feynman diagrams; these terms will be defined in a moment. We will prove Eq. (4.90) in Section 7.2. In the remainder of this section, we will explain this formula and motivate the new restrictions that we have added.

First we must learn how to represent the matrix element in (4.90) as a sum of Feynman diagrams. Let us evaluate the first few terms explicitly, in ϕ^4 theory, for the case of two particles in the final state. The first term is

$$\begin{aligned}\left\langle \mathbf{p}_1\mathbf{p}_2|\mathbf{p}_{\mathcal{A}}\mathbf{p}_{\mathcal{B}}\right\rangle_0 =&\sqrt{2E_1 2E_2 2E_{\mathcal{A}} 2E_{\mathcal{B}}}\langle 0|a_1 a_2 a^\dagger_{\mathcal{A}} a^\dagger_{\mathcal{B}}|0\rangle \\ =&2E_{\mathcal{A}}2E_{\mathcal{B}}(2\pi)^6\left(\delta\left(\mathbf{p}_{\mathcal{A}}-\mathbf{p}_1\right)\delta\left(\mathbf{p}_{\mathcal{B}}-\mathbf{p}_2\right)\right. \\ &\left.+\delta\left(\mathbf{p}_{\mathcal{A}}-\mathbf{p}_2\right)\delta\left(\mathbf{p}_{\mathcal{B}}-\mathbf{p}_1\right)\right)\end{aligned} \tag{4.91}$$

The delta functions force the final state to be identical to the initial state, so this term is part of the ' 1 ' in $S = 1 + iT$, and does not contribute to the
scattering matrix element $\mathcal{M}$. We can represent it diagrammatically as



The next term in $\left\langle \mathbf{p}_1\mathbf{p}_2\right| S\left|\mathbf{p}_{\mathcal{A}}\mathbf{p}_{\mathcal{B}}\right\rangle$ is

$$
\begin{aligned}
&{}_0\langle \mathbf{p}_1\mathbf{p}_2|\, T\left(-i\frac{\lambda}{4!}\int d^4x\phi_I^4(x)\right)|\mathbf{p}_\mathcal{A}\mathbf{p}_\mathcal{B}\rangle_0 \\
&\quad = {}_0\langle \mathbf{p}_1\mathbf{p}_2|\, N\left(-i\frac{\lambda}{4!}\int d^4x\phi_I^4(x) + \text{ contractions }\right)|\mathbf{p}_\mathcal{A}\mathbf{p}_\mathcal{B}\rangle_0
\end{aligned}
\tag{4.92}
$$

using Wick's theorem. Since the external states are not $|0\rangle$, terms that are not fully contracted do not necessarily vanish; we can use an annihilation operator from $\phi_I(x)$ to annihilate an initial-state particle, or a creation operator from $\phi_I(x)$ to produce a final-state particle. For example,

$$
\begin{aligned}
\phi_I^+(x)|\mathbf{p}\rangle_0 &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}a_\mathbf{k}e^{-ik\cdot x}\sqrt{2E_\mathbf{p}}a_\mathbf{p}^\dagger|0\rangle \\
&= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{k}}}e^{-ik\cdot x}\sqrt{2E_\mathbf{p}}(2\pi)^3\delta(\mathbf{k}-\mathbf{p})|0\rangle \\
&= e^{-ip\cdot x}|0\rangle
\end{aligned}
\tag{4.93}
$$

An uncontracted ϕ_I operator inside the N-product of (4.92) has two terms: ϕ_I^+on the far right and ϕ_I^-on the far left. We get one contribution to the S-matrix element for each way of commuting the a of ϕ_I^+past an initial-state $a^\dagger$, and one contribution for each way of commuting the $a^\dagger$ of ϕ_I^-past a finalstate a. It is natural, then, to define the contractions of field operators with external states as follows:

$$
(wickcontrlosthere)\ \phi_I(x)|\mathbf{p}\rangle = e^{-ip\cdot x};\quad {}^{p|\phi_I}(x) = e^{+ip\cdot x}
\tag{4.94}
$$

To evaluate an S-matrix element such as (4.92), we simply write down all possible full contractions of the ϕ_I operators and the external-state momenta.

To see that this prescription is correct, let us evaluate (4.92) in detail. The N-product contains terms of the form

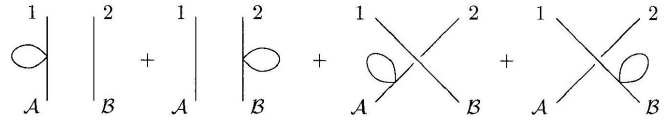
$$
\phi\phi\phi\phi; \qquad \overset{\sqcap}{\phi\phi\phi\phi}; \qquad \overset{\sqcap\ \sqcap}{\phi\phi\phi\phi}.
$$

The last term, in which the ϕ operators are fully contracted with each other, is equal to a vacuum bubble diagram times the value of (4.91) calculated above:

$$
\begin{aligned}
-i\tfrac{\lambda}{4!}\textstyle\int d^4x_0\,\langle \mathbf{p}_1\mathbf{p}_2|\,\phi\phi\phi\phi\,|\mathbf{p}_\mathcal{A}\mathbf{p}\mathcal{B}\rangle_0 \quad &= \bigotimes_\mathcal{A}^{\sqcap\sqcap}\times\big(|^1_\mathcal{B}\,| \\
&\times\,|^2_\mathcal{A}+1 \\
\mathcal{B}\qquad\qquad\qquad

$$\phi_I(\vec{x})|\mathbf{p}\Big\rangle = \Big\rangle_x \qquad \vec{p} \qquad \langle\mathbf{p}|\phi_I(x) = \overset{x\backslash}{\underset{x}{\overset{p}{}}} \tag{4.97}$$

Feynman diagrams for S-matrix elements will always contain external lines, rather than the external points of diagrams for correlation functions. The second term of (4.95) thus yields four diagrams:



The integration $\int d^4x$ produces a momentum-conserving delta function at each vertex (including the external momenta), so these diagrams again describe trivial processes in which the initial and final states are identical. This illustrates a general principle: Only fully connected diagrams, in which all external lines are connected to each other, contribute to the T-matrix.

Finally, consider the term of (4.95) in which none of the ϕ operators are contracted with each other. Our prescription tells us to contract two of the ϕ 's with $|\mathbf{p}_\mathcal{A}\mathbf{p}_\mathcal{B}\rangle$ and the other two with $\langle\mathbf{p}_1\mathbf{p}_2|$. There are 4 ! ways to do this.

Thus we obtain the diagram

$$\begin{aligned} &= (4!)\cdot\left(-i\tfrac{\lambda}{4!}\right)\int d^4x e^{-i(p_\mathcal{A}+p_\mathcal{B}-p_1-p_2)\cdot x} \\ &= -i\lambda(2\pi)^4\delta^{(4)}\left(p_\mathcal{A}+p_\mathcal{B}-p_1-p_2\right) \end{aligned} \tag{4.98}$$

This is exactly of the form $i\mathcal{M}(2\pi)^4\delta^{(4)}\left(p_\mathcal{A}+p_\mathcal{B}-p_1-p_2\right)$, with $\mathcal{M}=-\lambda$.
Before continuing our discussion of Feynman diagrams for S-matrix elements, we should certainly pause to turn this result into a cross section. For scattering in the center-of-mass frame, we can simply plug $|\mathcal{M}|^2=\lambda^2$ into Eq. (4.85) to obtain

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{\lambda^2}{64\pi^2E_{\text{cm}}^2} \tag{4.99}$$

We have just computed our first quantum field theory cross section. It is a rather dull result, having no angular dependence at all. (This situation will be remedied when we consider fermions in the next section.) Integrating over $d\Omega$, and dividing by 2 since there are two identical particles in the final state, we find the total cross section,

$$\sigma_{\text{total}} = \frac{\lambda^2}{32\pi E_{\text{cm}}^2} \tag{4.100}$$

In practice, one would probably use this result to measure the value of λ.
Returning to our general discussion, let us consider some higher-order contributions to the T-matrix for the process $\mathcal{A},\mathcal{B}\to 1,2$. If we ignore, for the moment, the "connected and amputated" prescription, we have the formula

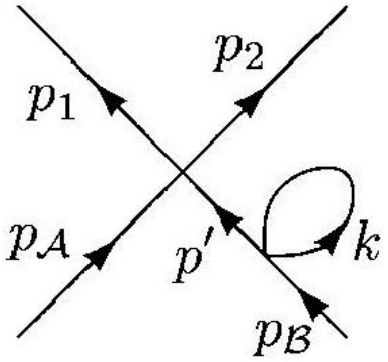
$$\begin{aligned} &\langle\mathbf{p}_1\mathbf{p}_2|\,iT\,|\mathbf{p}_\mathcal{A}\mathbf{p}_\mathcal{B}\rangle \overset{?}{=} \searrow + \chi + \chi + \infty + \cdots \\ &+ (\searrow 8) + \cdots + (\searrow 0) + \cdots \\ &+ > 0 + \cdots \end{aligned} \tag{4.101}$$

plus diagrams in which the four external lines are not all connected to each other. We have already seen that this last class of diagrams gives no contribution to the T-matrix. The first

diagram shown in (4.101) gives the lowest-order contribution to T, which we calculated above. The next three diagrams give
expected corrections to this amplitude, involving creation and annihilation of additional "virtual" particles.

The diagrams in the second line of (4.101) contain disconnected "vacuum bubbles". By the same argument as at the end of Section 4.4, the disconnected pieces exponentiate to an overall phase factor giving the shift of the energy of the interacting vacuum state upon which the scattering takes place. Thus they are irrelevant to S. We have now seen that only fully connected diagrams give sensible contributions to S-matrix elements.

The last diagram is more problematical; let us evaluate it. After integrating over the two vertex positions, we obtain

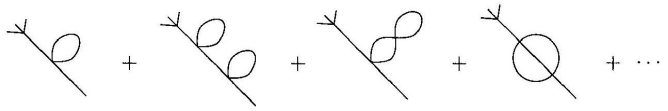


$$
\begin{aligned}
= \frac{1}{2} \int \frac{d^4p'}{(2\pi)^4} \frac{i}{p'^2 - m^2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \\
\times (-i\lambda)(2\pi)^4 \delta^{(4)}\left(p_{\mathcal{A}} + p' - p_1 - p_2\right) \\
\times (-i\lambda)(2\pi)^4 \delta^{(4)}\left(p_{\mathcal{B}} - p'\right).
\end{aligned}
\tag{4.102}
$$

We can integrate over p' using the second delta function. It tells us to evaluate

$$
\left. \frac{1}{p'^2 - m^2} \right|_{p'=p_{\mathcal{B}}} = \frac{1}{p_{\mathcal{B}}^2 - m^2} = \frac{1}{0}.
$$

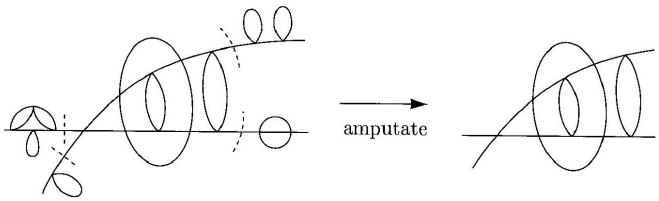
We get infinity, since $p_{\mathcal{B}}$, being the momentum of an external particle, is onshell: $p_{\mathcal{B}}^2 = m^2$. This is a disaster. Clearly, our formula for S makes sense only if we exclude diagrams of this form, that is, diagrams with loops connected to only one external leg. Fortunately, this is physically reasonable: In the same way that the vacuum bubble diagrams represent the evolution of $|0\rangle$ into $|\Omega\rangle$, these external leg corrections,



represent the evolution of $|\mathbf{p}\rangle_0$ into $|\mathbf{p}\rangle$, the single-particle state of the interacting theory. Since these corrections have nothing to do with the scattering process, we should exclude them from the computation of S.

For a general diagram with external legs, we define amputation in the following way. Starting from the tip of each external leg, find the last point at which the diagram can be cut by removing a single propagator, such that this operation separates the leg from the rest of the diagram. Cut

there. For example:

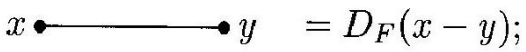


Let us summarize our prescription for calculating scattering amplitudes. Our formula for S-matrix elements, Eq. (4.90), can be rewritten

$$\begin{aligned} &i\mathcal{M}\cdot(2\pi)^4\delta^{(4)}\left(p_{\mathcal{A}}+p_{\mathcal{B}}-\sum p_f\right) \\ &\qquad = \left(\begin{array}{l}\text{sum of all connected, amputated Feynman} \\ \text{diagrams with } p_{\mathcal{A}}, p_{\mathcal{B}} \text{ incoming, } p_f \text{ outgoing}\end{array}\right). \end{aligned} \tag{4.103}$$

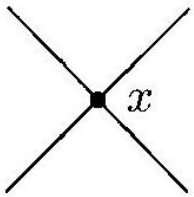
By 'connected', we now mean fully connected, that is, with no vacuum bubbles, and all external legs connected to each other. The Feynman rules for scattering amplitudes in ϕ^4 theory are, in position space,

1. For each propagator,



$$= D_F(x-y);$$

2. For each vertex,



$$= (-i\lambda)\int d^4x$$

3. For each external line,

$$= e^{-ip\cdot x};$$

4. Divide by the symmetry factor.

Notice that the factor for an ingoing line is just the amplitude for that particle to be found at the vertex it connects to, i.e., the particle's wavefunction. Similarly, the factor for an outgoing line is the amplitude for a particle produced at the vertex to have the desired final momentum.

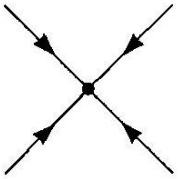
Just as with the Feynman rules for correlation functions, it is usually simpler to introduce the momentum-space representation of the propagators, carry out the vertex integrals to obtain

momentum-conserving delta functions, and use these delta functions to evaluate as many momentum integrals as possible. In a scattering amplitude, however, there will always be an overall delta function, which can be used to cancel the one on the left-hand side of Eq. (4.103). We are then left with

$$i\mathcal{M} = \text{ sum of all connected, amputated diagrams,} \tag{4.104}$$

where the diagrams are evaluated according to the following rules:

1. For each propagator, $\longrightarrow p = \frac{i}{p^2 - m^2 + i\epsilon}$;
2. For each vertex,



$= -i\lambda;$

3. For each external line,

$= 1;$

4. Impose momentum conservation at each vertex;
5. Integrate over each undetermined loop momentum: $\int \frac{d^4p}{(2\pi)^4}$;
6. Divide by the symmetry factor.

This is our final version of the Feynman rules for ϕ^4 theory; these rules are also listed in the Appendix, for reference.

Actually, Eq. (4.103) still isn't quite correct. One more modification is necessary, involving the proportionality factors that were omitted from Eq. (4.89). But the modification affects only diagrams containing loops, so we postpone its discussion until Chapters 6 and 7 , where we first evaluate such diagrams. We will prove the corrected formula (4.103) in Section 7.2, by relating S matrix elements to correlation functions, for which we have actually derived a formula in terms of Feynman diagrams.

14.6 4.7 Feynman Rules for Fermions

So far in this chapter we have discussed only ϕ^4 theory, in order to avoid unnecessary complication. We are now ready to generalize our results to theories containing fermions.

Our treatment of correlation functions in Section 4.2 generalizes without difficulty. Lorentz invariance requires that the interaction Hamiltonian H_I be a product of an even number of spinor fields, so no difficulties arise in defining the time-ordered exponential of H_I.

To apply Wick's theorem, however, we must generalize the definitions of the time-ordering and normal-ordering symbols to include fermions. We saw at the end of Section 3.5 that the time-ordering operator T acting on two spinor fields is most conveniently defined with an additional minus sign:

$$T(\psi(x)\bar{\psi}(y)) \equiv \begin{cases} \psi(x)\bar{\psi}(y) \text{ for } x^0 > y^0 \\ -\bar{\psi}(y)\psi(x) \text{ for } x^0 < y^0 \end{cases} \tag{4.105}$$

With this definition, the Feynman propagator for the Dirac field is

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(x-y)} = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle. \tag{4.106}$$

For products of more than two spinor fields, we generalize this definition in the natural way: The time-ordered product picks up one minus sign for each interchange of operators that is necessary to put the fields in time order. For example,

$$T\left(\psi_1\psi_2\psi_3\psi_4\right) = (-1)^3\psi_3\psi_1\psi_4\psi_2 \quad \text{if} \quad x_3^0 > x_1^0 > x_4^0 > x_2^0$$

The definition of the normal-ordered product of spinor fields is analogous: Put in an extra minus sign for each fermion interchange. The anticommutation properties make it possible to write a normal-ordered product in several ways, but with our conventions these are completely equivalent:

$$N\left(a_{\mathbf{p}}a_{\mathbf{q}}a_{\mathbf{r}}^\dagger\right) = (-1)^2 a_{\mathbf{r}}^\dagger a_{\mathbf{p}} a_{\mathbf{q}} = (-1)^3 a_{\mathbf{r}}^\dagger a_{\mathbf{q}} a_{\mathbf{p}}$$

Using these definitions, it is not hard to generalize Wick's theorem. Consider first the case of two Dirac fields, say $T[\psi(x)\bar{\psi}(y)]$. In analogy with (4.37), define the contraction of two fields by

$$T[\psi(x)\bar{\psi}(y)] = N[\psi(x)\bar{\psi}(y)] + \overrightarrow{\psi(x)}\,\bar{\psi}(y) \tag{4.107}$$

Explicitly, for the Dirac field,

$$\overrightarrow{\psi(x)}\bar{\psi}\,(y) \equiv \left\{ \begin{array}{ll} \left\{\psi^+(x), \bar{\psi}^-(y)\right\} & \text{for } x^0 > y^0 \\ -\left\{\bar{\psi}^+(y), \psi^-(x)\right\} & \text{for } x^0 < y^0 \end{array} \right\} = S_F(x-y); \tag{4.108}$$

$$\overrightarrow{\psi(x)}\psi\,(y) = \overrightarrow{\bar{\psi}}(x)\bar{\psi}(y) = 0 \tag{4.109}$$

Define contractions under the normal-ordering symbol to include minus signs for operator interchanges:

$$N\left(\psi_1\psi_2\bar{\psi}_3\bar{\psi}_4\right) = -\overset{\psi}{\bar{\psi}}_{13} N\left(\psi_2\bar{\psi}_4\right) = -S_F\left(x_1 - x_3\right) N\left(\psi_2\bar{\psi}_4\right) \tag{4.110}$$

With these conventions, Wick's theorem takes the same form as before:

$$T\left[\psi_1\bar{\psi}_2\psi_3\cdots\right] = N\left[\psi_1\bar{\psi}_2\psi_3\cdots + \text{ all possible contractions }\right] \tag{4.111}$$

The proof is essentially unchanged from the bosonic case, since all extra minus signs are accounted for by the above definitions.

Yukawa Theory

Writing down the Feynman rules for fermion correlation functions would now be easy, but instead let's press on and discuss scattering processes. For definiteness, we begin by analyzing the Yukawa theory:

$$H = H_{\text{Dirac}} + H_{\text{Klein-Gordon}} + \int d^3x g\bar{\psi}\psi\phi \tag{4.112}$$

This is a simplified model of Quantum Electrodynamics. In this section we will carefully work out the rules of calculation for Yukawa theory, so that in the next section we can guess the rules for QED without too much difficulty.

To be even more specific, consider the two-particle scattering reaction

$$\text{fermion } (p) + \text{ fermion } (k) \longrightarrow \text{ fermion } (p') + \text{ fermion } (k')$$

The leading contribution comes from the H_I^2 term of the S-matrix:

$${}_0\langle \mathbf{p}', \mathbf{k}'| T\left(\frac{1}{2!}(-ig)\int d^4x \bar{\psi}_I\psi_I\phi_I \quad (-ig)\int d^4y \bar{\psi}_I\psi_I\phi_I\right)|\mathbf{p},\mathbf{k}\rangle_0 \tag{4.113}$$

To evaluate this expression, use Wick's theorem to reduce the T-product to an N -product of contractions, then act the uncontracted fields on the initialand final-state particles. Represent this latter process as the contraction

$$\begin{aligned} {}^{\ulcorner\psi_I(x)|\mathbf{p},s\rangle} &= \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}'}}}\sum_{s'} a^{s'}_{\mathbf{p}'}u^{s'}(p')\, e^{-ip'\cdot x}\sqrt{2E_{\mathbf{p}}}a^{s\dagger}_{\mathbf{p}}|0\rangle \\ &= e^{-ip\cdot x}u^s(p)|0\rangle \end{aligned} \tag{4.114}$$

Similar expressions hold for the contraction of $\bar{\psi}_I$ with a final-state fermion, and for contractions of ψ_I and $\bar{\psi}_I$ with antifermion states. Note that ψ_I can be contracted with a fermion on the right or an antifermion on the left; the opposite is true for $\bar{\psi}_I$.

We can write a typical contribution to the matrix element (4.113) as the contraction

$$\langle \mathbf{p}', \mathbf{k}'|\tfrac{1}{2!}(-ig)\textstyle\int d^4x\, \bar{\psi}\psi\phi\ (-ig)\int d^4y\, \bar{\psi}\psi\phi\, |\mathbf{p},\mathbf{k}\rangle.$$

Up to a possible minus sign, the value of this quantity is

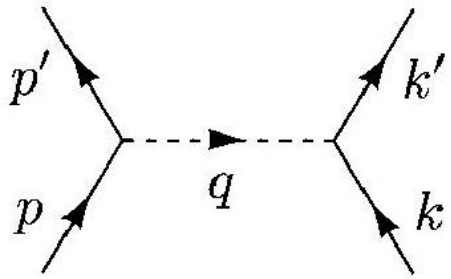
$$\begin{aligned} &(-ig)^2\int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m_\phi^2}(2\pi)^4\delta(p'-p-q) \\ &\times(2\pi)^4\delta(k'-k+q)\,\bar{u}(p')u(p)\bar{u}(k')u(k) \end{aligned}$$

(We have dropped the factor 1/2 ! because there is a second, identical term that comes from interchanging x and y in (4.115).) Using either delta function to perform the integral, we find that this expression takes the form $i\mathcal{M}(2\pi)^4\delta(\Sigma p)$, with

$$i\mathcal{M} = \frac{-ig^2}{q^2-m_\phi^2}\bar{u}(p')u(p)\bar{u}(k')u(k) \tag{4.116}$$

When writing it in this way, we must remember to impose the constraints $p-p'=q=k'-k$.

Instead of working from (4.115), we could draw a Feynman diagram:



We denote scalar particles by dashed lines, and fermions by solid lines. The S matrix element could then be obtained directly from the following momentumspace Feynman rules.

1. Propagators:

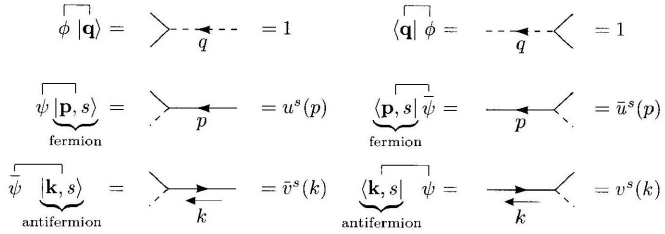
$$\phi(\overrightarrow{x})\phi\,(y) = \cdots = \frac{i}{q^2 - m_\phi^2 + i\epsilon}$$

$$\sqrt{\psi(x)\bar{\psi}(y)} = \xrightarrow[p]{p} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

ces:

2. Vertices:

3. External leg contractions:



4. Impose momentum conservation at each vertex.

5. Integrate over each undetermined loop momentum.

6. Figure out the overall sign of the diagram.

Several comments are in order regarding these rules.
First, note that the $1/n\,!$ from the Taylor series of the time-ordered exponential is always canceled by the $n\,!$ ways of interchanging vertices to obtain the same contraction. The diagrams of Yukawa theory never have symmetry factors, since the three fields $(\bar{\psi}\psi\phi)$ in H_I cannot substitute for one another in contractions.

Second, the direction of the momentum on a fermion line is always significant. On external lines, as for bosons, the direction of the momentum is always
ingoing for initial-state particles and outgoing for final-state particles. This follows immediately from the expansions of ψ and $\bar{\psi}$, where the annihilation operators $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ both multiply $e^{-ip\cdot x}$ and the creation operators $a_{\mathbf{p}}^\dagger$ and $b_{\mathbf{p}}^\dagger$ both multiply $e^{+ip\cdot x}$. On internal fermion lines (propagators), the momentum must be assigned in the direction of particle-number flow (for electrons, this is the

direction of negative charge flow). This requirement is most easily seen by working out an example from first principles. Consider the annihilation of a fermion and an antifermion into two bosons:

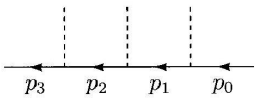
$$\ddots$$

$$\sim \int d^4x \int d^4y \bar{v}\,(p')\, e^{-ip'\cdot x} \int \frac{d^4q}{(2\pi)^4} \frac{i(d+m)}{q^2-m^2} e^{-iq\cdot(x-y)} u(p) e^{-ip\cdot y}$$

The integrals over x and y give delta functions that force q to flow from y to x, as shown. On internal boson lines the direction of the momentum is irrelevant and may be chosen for convenience, since $D_F(x-y) = D_F(y-x)$.

It is conventional to draw arrows on fermion lines, as shown, to represent the direction of particle-number flow. The momentum assigned to a fermion propagator then flows in the direction of this arrow. For external antiparticles, however, the momentum flows opposite to the arrow; it helps to show this explicitly by drawing a second arrow next to the line.

Third, note that in our examples the Dirac indices contract together along the fermion lines. This will also happen in more complicated diagrams:



$$\sim \bar{u}(p_3) \cdot \frac{i(\not{p}_2 + m)}{p_2^2 - m^2} \cdot \frac{i(\not{p}_1 + m)}{p_1^2 - m^2} \cdot u(p_0).$$

Finally, let's take a moment to worry about fermion minus signs. Return to the example of the fermion-fermion scattering process. We adopt a sign convention for the initial and final states:

$$|\mathbf{p},\mathbf{k}\rangle \sim a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{k}} |0\rangle, \quad \langle \mathbf{p}',\mathbf{k}'| \sim \langle 0| a_{\mathbf{k}'} a_{\mathbf{p}'} \tag{4.118}$$

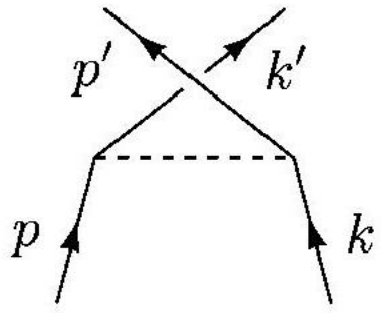
so that $(|p,k\rangle)^\dagger = \langle p,k|$. Then the contraction

$$\langle \mathbf{p}',\mathbf{k}'|(\bar\psi\psi)_x\,(\bar\psi\psi)_y|\mathbf{p},\mathbf{k}\rangle \sim \langle 0|\, a_{\mathbf{k}'} a_{\mathbf{p}'}\, \bar\psi_x \psi_x\, \bar\psi_y \psi_y\, a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{k}}\, |0\rangle$$

can be untangled by moving $\bar\psi_y$ two spaces to the left, and so picks up a factor of $(-1)^2 = +1$. But note that in the contraction

$$\langle \mathbf{p}',\mathbf{k}'|\,(\bar\psi\psi)_x(\bar\psi\psi)_y|\mathbf{p},\mathbf{k}\rangle \sim \langle 0|a_{\mathbf{k}'}a_{\mathbf{p}'}\bar\psi_x\psi_x\bar\psi_y\psi_y a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{k}}|0\rangle$$

it is sufficient to move the $\bar\psi_y$ one space to the left, giving a factor of -1 . This contraction corresponds to the diagram



The full result, to lowest order, for the S-matrix element for this process is therefore

$$
\begin{aligned}
i\mathcal{M} &= p' \\
&= \left(-ig^2\right)\left(\bar{u}\left(p'\right)u(p)\frac{1}{\left(p'-p\right)^2-m_\phi^2}\bar{u}\left(k'\right)u(k)\right. \\
&\quad \left.-\bar{u}\left(p'\right)u(k)\frac{1}{\left(p'-k\right)^2-m_\phi^2}\bar{u}\left(k'\right)u(p)\right).
\end{aligned} \tag{4.119}
$$

The minus sign difference between these diagrams is a reflection of Fermi statistics. Turning this expression into an explicit cross section would require some additional work; we postpone such calculations until Chapter 5 , when we can work with QED instead of the less interesting Yukawa theory.

In complicated diagrams, one can often simplify the determination of the minus signs by noting that the product $(\bar{\psi}\psi)$, or any other pair of fermions, commutes with any operator. Thus,

$$
\begin{aligned}
\cdots(\bar{\psi}\psi)_x(\bar{\psi}\psi)_y(\bar{\psi}\psi)_z(\bar{\psi}\psi)_w\cdots &= \cdots(+1)(\bar{\psi}\psi)_x(\bar{\psi}\psi)_z(\bar{\psi}\psi)_y(\bar{\psi}\psi)_w\cdots \\
&= \cdots S_F(x-z)S_F(z-y)S_F(y-w)\cdots
\end{aligned}
$$

But note that in a closed loop of n fermion propagators we have

$$
\begin{aligned}
=&\sqrt{\psi}\psi\sqrt{\psi}\sqrt{}\sqrt{\psi}\psi\sqrt{\psi}\psi \\
&= (-1)\,\mathrm{tr}[\sqrt{\psi}\bar{\psi}\bar{\psi}\psi\bar{\psi}\psi\bar{\psi}] \\
&= (-1)\,\mathrm{tr}\left[S_F S_F S_F S_F\right]
\end{aligned}
$$

A closed fermion loop always gives a factor of -1 and the trace of a product of Dirac matrices.

The Yukawa Potential

We now have all the formal rules we need to compute scattering amplitudes in Yukawa theory. Before going on to discuss QED, let us briefly descend from abstraction to concrete physics, and consider one very simple application of these rules: the scattering of distinguishable fermions, in the nonrelativistic limit. By comparing the amplitude for this process to the Born approximation formula from nonrelativistic quantum mechanics, we can determine the potential $V(r)$ created by the Yukawa interaction.

If the two interacting particles are distinguishable, only the first diagram in (4.119) contributes. To evaluate the amplitude in the nonrelativistic limit, we keep terms only to lowest order in the 3 -momenta. Thus, up to $\mathcal{O}\left(\mathbf{p}^2,\mathbf{p}'^2,\ldots\right)$,

$$
\begin{aligned}
p &= (m,\mathbf{p}), & k &= (m,\mathbf{k}), \\
p' &= (m,\mathbf{p}'), & k' &= (m,\mathbf{k}')
\end{aligned} \tag{4.121}
$$

Using these expressions, we have

$$
\begin{aligned}
\left(p'-p\right)^2 &= -\left|\mathbf{p}'-\mathbf{p}\right|^2+\mathcal{O}\left(\mathbf{p}^4\right), \\
u^s(p) &= \sqrt{m}\begin{pmatrix}\xi^s\\ \xi^s\end{pmatrix}, \quad \text{etc.}
\end{aligned}
$$

where ξ^s is a two-component constant spinor normalized to $\xi^{s'\dagger}\xi^s = \delta^{ss'}$. The spinor products in (4.119) are then

$$\begin{aligned} \bar{u}^{s'}(p')\,u^s(p) &= 2m\xi^{s'\dagger}\xi^s = 2m\delta^{ss'} \\ \bar{u}^{r'}(k')\,u^r(k) &= 2m\xi^{r'\dagger}\xi^r = 2m\delta^{rr'} \end{aligned} \tag{4.122}$$

So our first physical conclusion is that the spin of each particle is separately conserved in this nonrelativistic scattering interaction-a pleasing result.

Putting together the pieces of the scattering amplitude (4.119), we find

$$i\mathcal{M} = \frac{ig^2}{|\mathbf{p}'-\mathbf{p}|^2 + m_\phi^2} 2m\delta^{ss'} 2m\delta^{rr'} \tag{4.123}$$

This should be compared with the Born approximation to the scattering amplitude in nonrelativistic quantum mechanics, written in terms of the potential function $V(\mathbf{x})$:

$$\langle p'|\, iT|p\rangle = -i\widetilde{V}(\mathbf{q})(2\pi)\delta\left(E_{\mathbf{p}'} - E_{\mathbf{p}}\right), \quad (\mathbf{q} = \mathbf{p}' - \mathbf{p}) \tag{4.124}$$

So apparently, for the Yukawa interaction,

$$\widetilde{V}(\mathbf{q}) = \frac{-g^2}{|\mathbf{q}|^2 + m_\phi^2} \tag{4.125}$$

(The factors of $2m$ in (4.123) arise from our relativistic normalization conventions, and must be dropped when comparing to (4.124), which assumes conventional nonrelativistic normalization of states. The additional $\delta^{(3)}(\mathbf{p}'-\mathbf{p})$ goes away when we integrate over the momentum of the target.)

Inverting the Fourier transform to find $V(\mathbf{x})$ requires a short calculation:

$$\begin{aligned} V(\mathbf{x}) &= \int \frac{d^3q}{(2\pi)^3} \frac{-g^2}{|\mathbf{q}|^2 + m_\phi^2} e^{i\mathbf{q}\cdot\mathbf{x}} \\ &= \frac{-g^2}{4\pi^2} \int_0^\infty dq q^2 \frac{e^{iqr} - e^{-iqr}}{iqr} \frac{1}{q^2 + m_\phi^2} \\ &= \frac{-g^2}{4\pi^2 ir} \int_{-\infty}^\infty dq \frac{qe^{iqr}}{q^2 + m_\phi^2} \end{aligned} \tag{4.126}$$

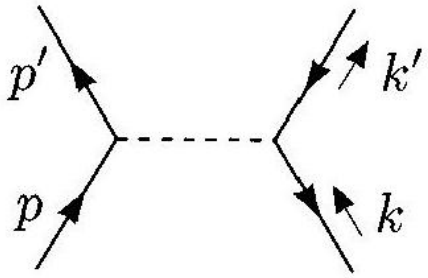
The contour of this integral can be closed above in the complex plane, and we pick up the residue of the simple pole at $q = +im_\phi$. Thus we find

$$V(r) = -\frac{g^2}{4\pi}\frac{1}{r}e^{-m_\phi r} \tag{4.127}$$

an attractive "Yukawa potential", with range $1/m_\phi = \hbar/m_\phi c$, the Compton wavelength of the exchanged boson. Yukawa made this potential the basis for his theory of the nuclear force, and worked backwards from the range of the force (about 1 fm) to predict the mass (about 200 MeV) of the required boson, the pion.

What happens if instead we scatter particles off of antiparticles? For the process

$$f_1(p)\bar{f}_2(k) \longrightarrow f_1(p')\,\bar{f}_2(k')$$



we need to evaluate (nonrelativistically)

$$\bar{v}^s(k)v^{s'}(k') \approx m\left(\xi^{s\dagger}, -\xi^{s\dagger}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi^{s'} \\ -\xi^{s'} \end{pmatrix} = -2m\delta^{ss'} \tag{4.128}$$

We must also work out the fermion minus sign. Using $|\mathbf{p},\mathbf{k}\rangle = a^\dagger_{\mathbf{p}} b^\dagger_{\mathbf{k}}|0\rangle$ and $\langle \mathbf{p}',\mathbf{k}'| = \langle 0|b_{\mathbf{k}'}a_{\mathbf{p}'}$, we can write the contracted matrix element as

$$\langle \mathbf{p}',\mathbf{k}'|\,\bar{\psi}\psi\;\bar{\psi}\psi\,|\mathbf{p},\mathbf{k}\rangle = \langle 0|\,b_{\mathbf{k}'}a_{\mathbf{p}'}\;\bar{\psi}\psi\;\bar{\psi}\psi\;a^\dagger_{\mathbf{p}}b^\dagger_{\mathbf{k}}\,|0\rangle\,.$$

To untangle the contractions requires three operator interchanges, so there is an overall factor of -1 . This cancels the extra minus sign in (4.128), and therefore we see that the Yukawa potential between a fermion and an antifermion is also attractive, and identical in strength to that between two fermions.

The remaining case to consider is scattering of two antifermions. It should not be surprising that the potential is again attractive; there is an additional minus sign from changing the other $\bar{u}u$ into $\bar{v}v$, and the number of interchanges necessary to untangle the contractions is even. Thus we conclude that the Yukawa potential is universally attractive, whether it is between a pair of fermions, a pair of antifermions, or one of each.

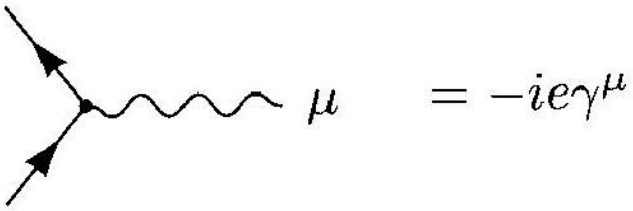
14.7 4.8 Feynman Rules for Quantum Electrodynamics

Now we are ready to step from Yukawa theory to Quantum Electrodynamics. To do this, we replace the scalar particle ϕ with a vector particle A_μ, and replace the Yukawa interaction Hamiltonian with

$$H_{\text{int}} = \int d^3x e\bar{\psi}\gamma^\mu\psi A_\mu \tag{4.129}$$

How do the Feynman rules change? The answer, though difficult to prove, is easy to guess. In addition to the fermion rules from the previous section, we have

New vertex:



Photon propagator:

$$\mu \smile\!\!\!\!\!\!\sim\!\!\!\!\!\!\sim \nu = \frac{-ig_{\mu\nu}}{q^2 + i\epsilon}$$

External photon lines:

$$\overleftarrow{A_\mu|\mathbf{p}\rangle} = \widetilde{\sim}\mu = \epsilon_\mu(p)$$
$$\rightarrow_{\mathbf{p}|A_\mu} = \mu\widetilde{\sim} \,| = \epsilon^*_\mu(p)$$

Photons are conventionally drawn as wavy lines. The symbol $\epsilon_\mu(p)$ stands for the polarization vector of the initial- or final-state photon.

To justify these rules, recall that in Lorentz gauge (which we employ to retain explicit relativistic invariance) the field equation for A_μ is

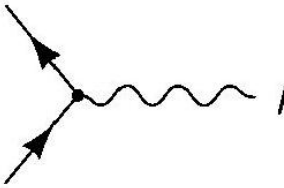
$$\partial^2 A_\mu = 0 \tag{4.130}$$

Thus each component of A separately obeys the Klein-Gordon equation (with $m = 0$). The momentum-space solutions of this equation are $\epsilon_\mu(p)e^{-ip\cdot x}$, where $p^2 = 0$ and $\epsilon_\mu(p)$ is any 4 - vector. The interpretation of ϵ as the polarization vector of the field should be familiar from classical electromagnetism. If we expand the quantized electromagnetic field in terms of classical solutions of the wave equation, as we did for the Klein-Gordon field, we find

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{r=0}^{3}\left(a^r_{\mathbf{p}}\epsilon^r_\mu(p)e^{-ip\cdot x} + a^{r\dagger}_{\mathbf{p}}\epsilon^{r*}_\mu(p)e^{ip\cdot x}\right) \tag{4.131}$$

where $r = 0, 1, 2, 3$ labels a basis of polarization vectors. The external line factors in the Feynman rules above follow immediately from this expansion, just as we obtained u 's and v 's as the external line factors for Dirac particles. The only subtlety is that we must restrict initial- and final-state photons to be transversely polarized: Their polarization vectors are always of the form $\epsilon^\mu = (0, \boldsymbol{\epsilon})$, where $\mathbf{p}\cdot\boldsymbol{\epsilon} = 0$. For $\mathbf{p}$ along the z-axis, the right- and left-handed polarization vectors are $\epsilon^\mu = (0, 1, \pm i, 0)/\sqrt{2}$.

The form of the QED vertex factor is also easy to justify, by simply looking at the interaction Hamiltonian (4.129). Note that the γ matrix in a QED amplitude will sit between spinors or other γ matrices, with the Dirac indices contracted along the fermion line. Note also that this interaction term is specific to the case of an electron (and its antiparticle, the positron). In general, for a Dirac particle with electric charge $Q|e|$,



$$= -iQ|e|\gamma^\mu.$$

For example, an electron has $Q = -1$, an up quark has $Q = +2/3$, and a down quark has $Q_d = -1/3$.

There is no easy way to derive the form of the photon propagator, so for now we will settle for a plausibility argument. Since the electromagnetic field in Lorentz gauge obeys the massless Klein-Gordon equation, it should come as no surprise that the photon propagator is nearly identical to the massless Klein-Gordon propagator. The factor of $-g_{\mu\nu}$, however, requires explanation.

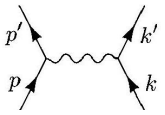
Lorentz invariance dictates that the photon propagator be an isotropic secondrank tensor that can dot together the γ^μ and γ^ν from the vertices at each end. The simplest candidate is $g^{\mu\nu}$. To understand the overall sign of the propagator, evaluate its Fourier transform:

$$\int \frac{d^4q}{(2\pi)^4}\frac{-ig_{\mu\nu}}{q^2+i\epsilon}e^{-iq\cdot(x-y)} = \int \frac{d^3q}{(2\pi)^3}\frac{1}{2|\mathbf{q}|}e^{-iq\cdot(x-y)}\cdot(-g_{\mu\nu}) \tag{4.132}$$

Presumably this is equal to $\langle 0|T\left[A_\mu(x)A_\nu(y)\right]|0\rangle$. Now set $\mu=\nu$, and take the limit $x^0 \to y^0$ from the positive direction. Then this quantity becomes the norm of the state $A_\mu(x)|0\rangle$, which should be positive. We see that our choice of signs in the propagator implies that the three states created by A_i, with with $i=1,2,3$, indeed have positive norm. These states include all real (nonvirtual) photons, which always have spacelike polarizations. Unfortunately, because $g_{\mu\nu}$ is not positive definite, the states created by A_0 inevitably have negative norm. This is potentially a serious problem for any theory with vector particles. For Quantum Electrodynamics, we will show in Section 5.5 that the negative-norm states created by A_0 are never produced in physical processes. In Section 9.4 we will give a careful derivation of the photon propagator.

The Coulomb Potential

As a simple application of these Feynman rules, and to better understand the sign of the propagator, let us repeat the nonrelativistic scattering calculation of the previous section, this time for QED. The leading-order contribution is

$i\mathcal{M} =$ 

$= (-ie)^2\bar{u}(p')\gamma^\mu u(p)\frac{-ig_{\mu\nu}}{(p'-p)^2}\bar{u}(k')\gamma^\nu u(k).$

In the nonrelativistic limit,

$$\bar{u}\left(p'\right)\gamma^0 u(p) = u^\dagger\left(p'\right)u(p) \approx +2m\xi'^\dagger\xi$$

You can easily verify that the other terms, $\bar{u}\left(p'\right)\gamma^i u(p)$, vanish if $p=p'=0$; they can therefore be neglected compared to $\bar{u}\left(p'\right)\gamma^0 u(p)$ in the nonrelativistic limit. Thus we have

$$\begin{aligned} i\mathcal{M} &\approx \frac{+ie^2}{-\left|\mathbf{p}'-\mathbf{p}\right|^2}\left(2m\xi'^\dagger\xi\right)_p\left(2m\xi'^\dagger\xi\right)_k\cdot g_{00} \\ &= \frac{-ie^2}{\left|\mathbf{p}'-\mathbf{p}\right|^2}\left(2m\xi'^\dagger\xi\right)_p\left(2m\xi'^\dagger\xi\right)_k \end{aligned} \tag{4.134}$$

Comparing this to the Yukawa case (4.123), we see that there is an extra factor of -1 ; the potential is a repulsive Yukawa potential with $m=0$, that is, a repulsive Coulomb potential:

$$V(r) = \frac{e^2}{4\pi r} = \frac{\alpha}{r} \tag{4.135}$$

where $\alpha = e^2/4\pi \approx 1/137$ is the fine-structure constant.
For particle-antiparticle scattering, note first that

$$\bar{v}(k)\gamma^0 v\left(k'\right) = v^\dagger(k)v\left(k'\right) \approx +2m\xi^\dagger\xi'$$

The presence of the γ^0 eliminates the minus sign that we found in the Yukawa case. The nonrelativistic scattering amplitude is therefore

$$i\mathcal{M} = p^{p'}{}_{k'} = (-1) \cdot \frac{-ie^2}{|\mathbf{p}' - \mathbf{p}|^2} \left(+2m\xi'^{\dagger}\xi\right)_p \left(+2m\xi^{\dagger}\xi'\right)_k \tag{4.136}$$

where the (-1) is the same fermion minus sign we saw in the Yukawa case. This is an attractive potential. Similarly, for antifermion-antifermion scattering one finds a repulsive potential. We have just verified that in quantum field theory, when a vector particle is exchanged, like charges repel while unlike charges attract.

Note that the repulsion in fermion-fermion scattering came entirely from the extra factor $-g_{00} = -1$ in the vector boson propagator. A tensor boson, such as the graviton, would have a propagator

$$\mu\nu \curvearrowright\sim\sim\sim \rho\sigma = \frac{1}{2}\left((-g_{\mu\rho})(-g_{\nu\sigma}) + (-g_{\mu\sigma})(-g_{\nu\rho})\right)\left(\frac{i}{q^2 + i\epsilon}\right)$$

which in nonrelativistic collisions gives a factor $(-g_{00})^2 = +1$; this will result in a universally attractive potential. It is reassuring to see that quantum field theory does indeed reproduce the obvious features of the electric and gravitational forces:

Exchanged particle	ff and $\bar{f}\bar{f}$	$f\bar{f}$
scalar (Yukawa)	attractive	attractive
vector (electricity)	repulsive	attractive
tensor (gravity)	attractive	attractive

Problems

4.1 Let us return to the problem of the creation of Klein-Gordon particles by a classical source. Recall from Chapter 2 that this process can be described by the Hamiltonian

$$H = H_0 + \int d^3x(-j(t,\mathbf{x})\phi(x))$$

where H_0 is the free Klein-Gordon Hamiltonian, $\phi(x)$ is the Klein-Gordon field, and $j(x)$ is a c-number scalar function. We found that, if the system is in the vacuum state before the source is turned on, the source will create a mean number of particles

$$\langle N\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|\tilde{j}(p)|^2$$

In this problem we will verify that statement, and extract more detailed information, by using a perturbation expansion in the strength of the source.
(a) Show that the probability that the source creates no particles is given by

$$P(0) = \left|\langle 0|T\left\{\exp\left[i\int d^4x j(x)\phi_I(x)\right]\right\}\right| 0\Big\rangle\,|^2$$

(b) Evaluate the term in P(0) of order j^2, and show that $P(0) = 1 - \lambda + \mathcal{O}(j^4)$, where λ equals the expression given above for $\langle N\rangle$.
(c) Represent the term computed in part (b) as a Feynman diagram. Now represent the whole pertubation series for $P(0)$ in terms of Feynman diagrams. Show that this series exponentiates, so that it can be summed exactly: $P(0) = \exp(-\lambda)$.
(d) Compute the probability that the source creates one particle of momentum k. Perform this

computation first to $\mathcal{O}(j)$ and then to all orders, using the trick of part (c) to sum the series.
(e) Show that the probability of producing n particles is given by

$$P(n) = (1/n!)\lambda^n \exp(-\lambda).$$

This is a Poisson distribution.
(f) Prove the following facts about the Poisson distribution:

$$\sum_{n=0}^{\infty} P(n) = 1; \quad \langle N \rangle = \sum_{n=0}^{\infty} nP(n) = \lambda$$

The first identity says that the $P(n)$'s are properly normalized probabilities, while the second confirms our proposal for $\langle N \rangle$. Compute the mean square fluctuation $\langle (N - \langle N \rangle)^2 \rangle$.
4.2 Decay of a scalar particle. Consider the following Lagrangian, involving two real scalar fields Φ and ϕ:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \Phi\right)^2 - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2 - \mu\Phi\phi\phi.$$

The last term is an interaction that allows a Φ particle to decay into two ϕ 's, provided that $M > 2m$. Assuming that this condition is met, calculate the lifetime of the Φ to lowest order in μ.
4.3 Linear sigma model. The interactions of pions at low energy can be described by a phenomenological model called the linear sigma model. Essentially, this model consists of N real scalar fields coupled by a ϕ^4 interaction that is symmetric under rotations of the N fields. More specifically, let $\Phi^i(x), i = 1, ..., N$ be a set of N fields, governed by the Hamiltonian

$$H = \int d^3x \left(\frac{1}{2}\left(\Pi^i\right)^2 + \frac{1}{2}\left(\nabla\Phi^i\right)^2 + V\left(\Phi^2\right) \right)$$

where $\left(\Phi^i\right)^2 = \mathbf{\Phi} \cdot \mathbf{\Phi}$, and

$$V\left(\Phi^2\right) = \frac{1}{2}m^2\left(\Phi^i\right)^2 + \frac{\lambda}{4}\left(\left(\Phi^i\right)^2\right)^2$$

is a function symmetric under rotations of $\mathbf{\Phi}$. For (classical) field configurations of $\Phi^i(x)$ that are constant in space and time, this term gives the only contribution to H; hence, V is the field potential energy.
(What does this Hamiltonian have to do with the strong interactions? There are two types of light quarks, u and d. These quarks have identical strong interactions, but different masses. If these quarks are massless, the Hamiltonian of the strong interactions is invariant to unitary transformations of the 2-component object (u, d):

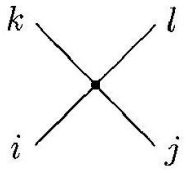
$$\begin{pmatrix} u \\ d \end{pmatrix} \to \exp(i\boldsymbol{\alpha} \cdot \boldsymbol{\sigma}/2) \begin{pmatrix} u \\ d \end{pmatrix}$$

This transformation is called an isospin rotation. If, in addition, the strong interactions are described by a vector "gluon" field (as is true in QCD), the strong interaction Hamiltonian is invariant to the isospin rotations done separately on the left-handed and right-handed components of the quark fields. Thus, the complete symmetry of QCD with two massless quarks is $SU(2) \times SU(2)$. It happens that $SO(4)$, the group of rotations in 4 dimensions, is isomorphic to $SU(2) \times SU(2)$, so for $N = 4$, the linear sigma model has the same symmetry group as the strong interactions.)
(a) Analyze the linear sigma model for $m^2 > 0$ by noticing that, for $\lambda = 0$, the Hamiltonian given

above is exactly N copies of the Klein-Gordon Hamiltonian. We can then calculate scattering amplitudes as perturbation series in the parameter λ. Show that the propagator is

$$^{\Phi^i(x)}\,\Phi^j(y) = \delta^{ij} D_F(x-y)$$

where D_F is the standard Klein-Gordon propagator for mass m, and that there is one type of vertex given by



$$= -2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}).$$

(That is, the vertex between two Φ^1 S_S and two Φ^2 S has the value $(-2i\lambda)$; that between four $\Phi^{\mathbf{1}}$s has the value $(-6i\lambda)$.) Compute, to leading order in λ, the differential cross sections $d\sigma/d\Omega$, in the center-of-mass frame, for the scattering processes

$$\Phi^1\Phi^2 \to \Phi^1\Phi^2, \quad \Phi^1\Phi^1 \to \Phi^2\Phi^2, \quad \text{and} \quad \Phi^1\Phi^1 \to \Phi^1\Phi^1$$

as functions of the center-of-mass energy.

(b) Now consider the case $m^2 < 0 : m^2 = -\mu^2$. In this case, V has a local maximum, rather than a minimum, at $\Phi^i = 0$. Since V is a potential energy, this implies that the ground state of the theory is not near $\Phi^i = 0$ but rather is obtained by shifting Φ^i toward the minimum of V. By rotational invariance, we can consider this shift to be in the N th direction. Write, then,

$$\Phi^i(x) = \pi^i(x), \quad i = 1, ..., N-1,$$
$$\Phi^N(x) = v + \sigma(x),$$

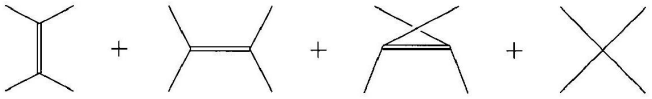
where v is a constant chosen to minimize V. (The notation π^i suggests a pion field and should not be confused with a canonical momentum.) Show that, in these new coordinates (and substituting for v its expression in terms of λ and μ), we have a theory of a massive σ field and $N-1$ massless pion fields, interacting through cubic and quartic potential energy terms which all become small as $\lambda \to 0$. Construct the Feynman rules by assigning values to the propagators and vertices:

$$\bar{\pi}^i \pi^j = i \longrightarrow \quad j$$

(c) Compute the scattering amplitude for the process

$$\pi^i(p_1)\,\pi^j(p_2) \to \pi^k(p_3)\,\pi^l(p_4)$$

to leading order in λ. There are now four Feynman diagrams that contribute:



Show that, at threshold ($\mathbf{p}_i = 0$), these diagrams sum to zero. (Hint: It may be easiest to first consider the specific process $\pi^1\pi^1 \to \pi^2\pi^2$, for which only the first and fourth diagrams are nonzero, before tackling the general case.) Show that, in the special case $N = 2$ (1 species of pion), the term of $\mathcal{O}(p^2)$ also cancels.

(d) Add to V a symmetry-breaking term,

$$\Delta V = -a\Phi^N$$

where a is a (small) constant. (In QCD, a term of this form is produced if the u and d quarks have the same nonvanishing mass.) Find the new value of v that minimizes V, and work out the content of the theory about that point. Show that the pion acquires a mass such that $m_\pi^2 \sim a$, and show that the pion scattering amplitude at threshold is now nonvanishing and also proportional to a.

4.4 Rutherford scattering. The cross section for scattering of an electron by the Coulomb field of a nucleus can be computed, to lowest order, without quantizing the electromagnetic field. Instead, treat the field as a given, classical potential $A_\mu(x)$. The interaction Hamiltonian is

$$H_I = \int d^3x e\bar{\psi}\gamma^\mu\psi A_\mu$$

where $\psi(x)$ is the usual quantized Dirac field.

(a) Show that the T-matrix element for electron scattering off a localized classical potential is, to lowest order,

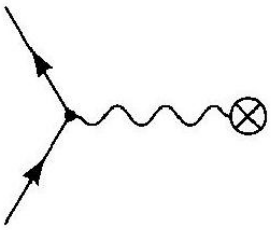
$$\langle p'|\, iT|p\rangle = -ie\bar{u}\,(p')\,\gamma^\mu u(p)\cdot\widetilde{A}_\mu\,(p'-p)\,,$$

where $\widetilde{A}_\mu(q)$ is the four-dimensional Fourier transform of $A_\mu(x)$.

(b) If $A_\mu(x)$ is time independent, its Fourier transform contains a delta function of energy. It is then natural to define

$$\langle p'|\, iT|p\rangle \equiv i\mathcal{M}\cdot(2\pi)\delta\,(E_f - E_i)$$

where E_i and E_f are the initial and final energies of the particle, and to adopt a new Feynman rule for computing $\mathcal{M}$:



$$= -ie\gamma^\mu\widetilde{A}_\mu(\mathbf{q}),$$

where $\widetilde{A}_\mu(\mathbf{q})$ is the three-dimensional Fourier transform of $A_\mu(x)$. Given this definition of $\mathcal{M}$, show that the cross section for scattering off a time-independent,
localized potential is

$$d\sigma = \frac{1}{v_i}\frac{1}{2E_i}\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\left|\mathcal{M}\,(p_i\to p_f)\right|^2(2\pi)\delta\,(E_f - E_i)$$

where v_i is the particle's initial velocity. This formula is a natural modification of (4.79). Integrate over $|p_f|$ to find a simple expression for $d\sigma/d\Omega$.

(c) Specialize to the case of electron scattering from a Coulomb potential ($A^0 = Ze/4\pi r$). Working in the nonrelativistic limit, derive the Rutherford formula,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2Z^2}{4m^2v^4\sin^4(\theta/2)}$$

(With a few calculational tricks from Section 5.1, you will have no difficulty evaluating the general cross section in the relativistic case; see Problem 5.1.)

14.7.1 5. Feynman Rules for QED

Рассмотренные два примера для амплитуд (63) и (71) помогают понять правила Фейнмана для диаграммной техники при вычислении амплитуды рассеяния iM (в импульсном представлении). Эти правила выводятся при помощи выражения для S-матрицы (58).

1. Приближению n-ого порядка ТВ отвечают диаграммы с n вершинами, в каждой из которых сходятся одна входящая и одна выходящая электронные (сплошные) и одна фотонная (волнистая) линии. В амплитуду процесса рассеяния входят все диаграммы, имеющие свободные концы (внешние линии) в числе, равном числу начальных и конечных частиц.

2. Каждой внешней входящей сплошной линии сопоставляется амплитуда начального электрона u_p или конечного позитрона u_{-p} (p − 4-импульс частицы). Каждой выходящей сплошной линии сопоставляется амплитуда конечного электрона $\bar{u}_p$ или начального позитрона $\bar{u}_{-p}$.

3. Каждой вершине сопоставляется 4 -вектор $(-ie\gamma^\mu)$.

4. Каждой внешней входящей волнистой линии сопоставляется амплитуда начального фотона $\sqrt{4\pi}e^\mu$, а выходящей линии - амплитуда $\sqrt{4\pi}e^*_\mu$ конечного фотона (e − 4-вектор поляризации фотона). Векторный индекс μ совпадает с индексом матрицы γ^μ в соответствующей вершине (так что возникает скалярное произведение $\gamma^\mu e_\mu$ или $\gamma^\mu e^*_\mu$).

5. Каждой внутренней сплошной линии сопоставляется множитель $iS_c(p) = i(\not{p} - m + i\varepsilon)^{-1}$, а внутренней волнистой линии - множитель $\left(-iD^\alpha_{\mu\nu}(p)\right) = (-i)\frac{4\pi}{p^2+i\varepsilon}\left[\eta_{\mu\nu} - \frac{(1-\alpha)p_\mu p_\nu}{p^2+i\varepsilon}\right]$. Тензорные индексы $\mu\nu$ совпадают с индексами матриц γ^μ, γ^ν в вершинах, соединяемых волнистой линией.

6. Вдоль каждой непрерывной последовательности электронных линий стрелки имеют неизменное направление, а расположение биспинорных индексов вдоль них соответствует записи матриц слева направо при движении против стрелок. Замкнутой электронной петле отвечает след произведения расположенных вдоль неё матриц.

7. В каждой вершине 4-импульсы пересекающихся в ней линий удовлетворяют закону сохранения, то есть сумма импульсов входящих линий равна сумме импульсов выходящих линий. Импульсы свободных концов - заданные (с соблюдением общего закона сохранения 4-импульса) величины, причём позитронной линии приписывается 4 -импульс $(-p)$. По импульсам внутренних линий, остающимся нефиксированными после учёта законов сохранения во всех вершинах, производится интегрирование (по $\mathrm{d}^{(4)}p/(2\pi)^4$).

8. Входящему свободному концу, отвечающему внешнему электромагнитному полю, сопоставляется множитель $A^{(e)}(q) = \int \mathrm{d}^{(4)}x e^{ikx} A^{(e)}(x)$; 4-вектор q связан с 4-импульсами сплошных электрон-позитронных линий в этой же вершине законом

сохранения. Если поле не зависит от времени, свободному концу сопоставляется множитель $A^{(e)}(\mathbf{q}) = \int \mathrm{d}^{(3)}x e^{-i\mathbf{qx}} A^{(e)}(\mathbf{x})$, а по остающимся нефиксированными трёхмерным импульсам внутренних линий производится интегрирование по $\mathrm{d}^{(3)}p/(2\pi)^3$.

9. Дополнительный множитель (-1) привносится в выражение для iM каждой замкнутой электронной петлёй и каждой парой позитронных внешних концов, если эти концы - начало и конец одной последовательности сплощных линий. Если среди начальных или среди конечных частиц имеется несколько электронов или позитронов, то относительный знак диаграмм, различающихся нечётным числом перестановок пар тождественных частиц (т. е. соответствующих им внешних концов), должен быть противоположным.

Последнее правило означает, в частности, что одинаковыми знаками должны обладать диаграммы с одинаковыми сплошными линиями, т. е. диаграммы, которые оказались бы тождественными после снятия с них всех фотонных линий. Укажем также, что при наличии тождественных фермионов общий знак амплитуды условен. Однонаправленность стрелок вдоль фермионных линий (правило 6) очевидно из того, что любая фермионная линия даёт в диаграмму множитель вида

$$...\gamma^{\mu} S_c(x-y)\gamma^{\nu} S_c(y-z)\gamma^{\lambda} S_c(z-u)...$$

Если определить направление стрелки фермионногой линии совпадающим с направлением распространения электрона, то в пропагаторе $S_c(x-y)$ стрелка направлена от точки y к точке x. Действительно, согласно задаче 1.в) при $x^0 > y^0$ величина $S_c(x-y)$ является амплитудой распространения электрона из точки y к точке x и при $x^0 < y^0$ - амплитудой распространения позитрона от точки x к точке y. Так как направление стрелки пропагатора не зависит от значения его аргументов, то мы видим, что вдоль стрелок электрон-позитронной линии распространяется электрон, а против стрелок -позитрон.

Задачи

1. Определим хронологически упорядоченное произведение от любого числа дираковских полей в представлении Гейзенберга

$$\left(T\psi_{a_1}(x_1)...\psi_{a_l}(x_l)\,\bar{\psi}_{a_{l+1}}(x_{l+1})...\psi_{a_m}(x_m)\right)$$

как произведение всех этих операторов, расположенных справа налево в порядке возрастания их временных аргументов, причём знак этого произведения определяется чётностью перестановки, которую нужно произвести, чтобы получить этот порядок из порядка, указанного под знаком T произведения.

а) Показать, что

$$\left\langle 0 \left| \left(T\psi_{a_1}(x_1)...\psi_{a_l}(x_l)\,\bar{\psi}_{a_{l+1}}(x_{l+1})...\bar{\psi}_{a_m}(x_m)\right) \right| 0 \right\rangle =$$
$$= \frac{1}{Z}\left(\frac{1}{i}\right)^m (-1)^{m-l} \frac{\delta}{\delta\bar{\eta}_{a_1}(x_1)} \cdots \frac{\delta}{\delta\bar{\eta}_{a_l}(x_l)} \frac{\delta}{\delta\eta_{a_{l+1}}(x_{l+1})} \cdots \frac{\delta}{\delta\eta_{a_m}(x_m)} Z \Big|_{J,\bar{\eta},\eta=0}$$

б) Показать, что этот коррелятор не равен нулю лишь если $m = 2l$.

Указание. В функциональном интеграле (55) (под знаком которого имеются усредняемые поля) сделать глобальное преобразование полей

$$\psi \to e^{i\alpha}\psi, \quad \bar{\psi} \to e^{-i\alpha}\bar{\psi}, \quad \eta \to e^{i\alpha}\eta, \quad \bar{\eta} \to e^{-i\alpha}\bar{\eta}$$

и учесть, что при этом мера $D\bar{\psi}D\psi$ инвариантна.

в) Показать, что в свободной теории

$$iS_c(x-y) = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle$$

2. Сформулировать и доказать теорему Вика для фермиевских операторов рождения и уничтожения.

3*. Вычислить сечение рассеяния фотона на электроне (эффект Комптона) в наинизшем приближении, считая начальные частицы неполяризованными и не фиксируя спины конечных частиц. Используя формулу (A22) выписать ответ через инвариантные переменные Мандельстама и затем перейти в систему отсчёта покоя электрона. Рассмотреть случай $\omega \ll m$ (здесь ω - энергия начального фотона, m - энергия покоя электрона) и сравнить ответ с классической нерелятивистской формулой Томсона.

Ответ:
$\mathrm{d}\sigma = 8\pi \left(\frac{e^2}{m}\right)^2 \frac{m^2\ \mathrm{d}t}{(s-m^2)^2} \left\{ \left(\frac{m^2}{s-m^2} + \frac{m^2}{u-m^2}\right)^2 + \left(\frac{m^2}{s-m^2} + \frac{m^2}{u-m^2}\right) - \frac{1}{4}\left(\frac{s-m^2}{u-m^2} + \frac{u-m^2}{s-m^2}\right) \right\}$.

4. Используя ответ предыдущей задачи, получить ответ для дифференциального сечения процесса аннигиляции электрон-позитронной пары в два фотона. Считать начальные частицы неполяризованными и спины конечных частиц нефиксированными.

14.8 Series Expansion in QFT

(I'll add theory here from the above!)

14.8.1 Mathematics of Series Expansion for QFT and Regions of Convergence (!!!)

(очень хочется именно такой раздел, где четко математически обосновывается, где граница сходимости тех рядов, в которые мы раскладываем? Почему бы в другие ряды не разложить, а не только в Тейлора?)

14.8.2 Quantum Amplitudes and External Fields by Kiselev (!!!!)

общие соображения

(перекину в запись про популярное КТП эти идеи)

После квантования поля в операторном подходе можно ставить задачу об эволюции квантовой системы, которая описывается состояниями в фоковском пространстве.

Для этого необходимо задать гамильтониан с учетом взаимодействий, чтобы эволюция не оставалась элементарной, каковой она является для свободных полей.

Тогда при наличии взаимодействий самих полей и взаимодействий с внешними источниками нужно будет построить оператор эволюции и вычислить амплитуды квантовых переходов между асимптотическими состояниями.

Но мы так делать не будем и просто забьем на все это и построим новую теорию.

Наряду с операторным подходом можно пойти по пути, который предложили Швингер и Фейнман, что мы и будем делать в этом курсе квантовой теории поля.

Этот подход, совершенно эквивалентен операторному подходу, но идеологически он опирается на рассмотрение взаимодействия полей с внешними источниками и на диаграммную технику Фейнмана при вычислении все тех же амплитуд квантовых переходов.

На примере вещественного скалярного поля $\varphi(x)$ мы сначала рассмотрим классические уравнения поля при наличии внешнего источника и покажем, как при помощи преобразования Лежандра от действия $G(j)$, зависящего от источников, к эффективному действию $\Gamma(\varphi_{\mathrm{cl}})$ происходит переход от понятия внешнего источника $j(x)$ к эквивалентному понятию классического внешнего поля $\varphi_{\mathrm{cl}}(x)$.

Такое новое понятие соответствует более прозрачной постановке задачи о воздействии на систему полей внешних сил или о зависимости характеристик системы от внешних параметров, которые как раз допускают стандартное описание в терминах классического внешнего поля φ_{cl}.

Например, в задаче рассеяния внешнее поле может описывать частицы в пучке, которые падают на мишень, или частицы, которые попадают в детектор после рассеяния.

При таком описании, внешние источники j и внешние поля φ_{cl} - это дуальные понятия в смысле преобразования Лежандра.

При этом для классического поля φ, которое является решением уравнений движения при наличии источников, мы вводим понятие амплитуды вероятности наблюдать это поле $Z(j)$.

Вероятность такого наблюдения, конечно, равна единице, $|Z(j)|^2 = 1$, но для этой амплитуды мы сформулируем вариационные уравнения Швингера-Дайсона, которые совершенно эквивалентны уравнениям Эйлера-Лагранжа для классического поля φ.

В квантовом случае поле флуктуирует, так что при вычислении амплитуды вероятности $Z(j)$ наблюдать поле с флуктуациями необходимо провести суммирование амплитуд по всем возможным флуктуациям с учетом того, что эти флуктуации согласуются с теми же уравнениями Эйлера-Лагранжа, так что в итоге мы придем к квантовым уравнениям Швингера-Дайсона для $Z(j)$.

В принципе, эти квантовые уравнения как раз и являются методом построения оператора эволюции, а поскольку оператор эволюции строится с помощью упорядочивания по времени бесконечно малых интервалов эволюции, которые следуют друг за другом, мы установим связь вариационных производных квантовой амплитуды $Z(j)$ с T-произведением операторов поля, которые, как мы уже видели, имеют вполне определенную интерпретацию в терминах рождения и уничтожения частиц, являющихся квантами локального поля.

Изящность подхода Швингера-Фейнмана заключается в том, что для вычисления квантовых амплитуд перехода можно зафиксировать внешние классические поля в начальном и конечном состояниях φ_{cl}, которые отвечают условиям поставленной задачи, и тогда для квантовых полей необходимо положить, что они при эволюции остаются в вакуумном состоянии. Поэтому говорят, что рассматривается вакуум-вакуумная амплитуда квантового перехода $Z(j)$ при наличии внешних источников j, которые, как мы уже понимаем, совершенно эквивалентны (по Лежандру) соответствующим внешним классическим полям φ_{cl}.

Суммирование по флуктуциям поля можно описывать вполне конструктивно, точно так же, как это было сделано в квантовой механике для одной нерелятивистской частицы.

При этом, в силу того, что матричный элемент оператора эволюции вычисляется по вакууму, необходимо учесть это условие в явном виде при формулировке континуального интеграла Фейнмана.

Важно, что суммирование по флуктуациям существенно влияет на форму эффективного действия $\Gamma(\varphi_{\mathrm{cl}})$. Само эффективное действие можно раскладывать в вариационный ряд по внешним полям (аналог ряда Тейлора для функций), что приводит в рамках квантовой операторной интерпретации коэффициентов разложения - функций Грина - к построению диаграммной техники Фейнмана.

Классическое поле и флуктуации

Рассмотрим классическую теорию вещественного скалярного поля с действием

$$S = \int \mathrm{d}^4 x \mathcal{L}\left(\varphi, \partial_\mu \varphi\right)$$

где можно выделить вклад свободного поля

$$S_0 = \int \mathrm{d}^4 x \frac{1}{2}\left\{\left(\partial_\mu \varphi\right)^2 - m^2 \varphi^2\right\}$$

и взаимодействия

$$S_{\mathrm{int}} = -\int \mathrm{d}^4 x V(\varphi)$$

так что

$$S = S_0 + S_{\mathrm{int}} \,,$$

а $V(\varphi)$ - это потенциал самодействия поля, который построен в виде полинома с младшей степенью, больше двух, чтобы можно было рассуждать о S_0 в терминах свободных полей, т. е. квадратичного по полю лагранжиана. К примеру, в теории «лямбда-фи-в-четвертой» $V = \lambda\varphi^4/4!$

Добавим к действию член с внешним током $j(x)$

$$S_{\mathrm{ext}}(j) = \int \mathrm{d}^4 x j(x)\varphi(x)$$

Для краткости записи договоримся о стандартных обозначениях для аргументов: поле или ток с номером - это поле или ток, которые зависят от набора координат с тем же номером, например, $\varphi\left(x_1\right) \equiv \varphi_1, j\left(x_1\right) \equiv j_1$.

Более того, повторение номера аргумента - это интегрирование по этому аргументу, так что $S_{\mathrm{ext}}(j) \equiv j_1\varphi_1$, например.

Ясно, что номер набора координат следует отличать от компонент в наборе, так что номер набора бы станем обозначать либо конкретной цифрой, либо латинской буквой, в отличие от греческих символов для компонент координат.

15 Other Fundamental Methods of QFT

(если тут окажется что-то типичное, что очень важно знать, то перенесу в 2ю часть. пока тут, потому что скорее всего это только в специфических вопросах понадобится.)

15.1 Some Well-known General Properties

15.1.1 Ward–Takahashi–Slavnov–Taylor Identities

мотивация

Theory

Квантовые флуктуации могут быть несовместимы с симметрией классического действия.

Эта несовместимость бывает обусловлена необходимостью регуляризации петлевых вкладов.

Напомним, что число петель отвечает степени $\hbar$, так что квантовые поправки могут потребовать наложения условий, которые позволят трактовать эти поправки как конечные величины.

Однако эти условия могут вступить в неустранимое противоречие с калибровочной инвариантностью,

например, из-за необходимости удовлетворять исходным принципам,

которые не зависят от модели взаимодействий, в том числе на квантовом уровне, скажем, принципу запрета Паули для фермионов. Тогда возникает ситуация квантовой аномалии: флуктуации нарушают закон сохранения тока!

В физических моделях требование сокращения квантовых аномалий является одним из ключевых.

Это может быть условие на сумму квантовых чисел полей, например.

Поэтому мы будем считать, что имеем дело с теорией, в которой нет квантовых аномалий. Тогда квантовые флуктуации оставляют амплитуды калибровочно инвариантными.

На примере неабелевой теории калибровочного поля в калибровке Лоренца с внешними источниками рассмотрим вариацию подынтегрального выражения для $Z(j)$ по локальному калибровочному параметру $u(x)$.

Возникающий при этом фактор нам уже известен, и в силу калибровочной инвариантности он равен нулю

$$-\frac{1}{\xi}(\partial^\mu \mathcal{A}_\mu)(\mathcal{M}_L(\mathcal{A})u) + j^\mu\left(-\left[\nabla_\mu, u\right]\right) = 0$$

Подставим в это выражение $u(x) = \mathcal{M}_L^{-1}(\mathcal{A})_{xy}\tilde{u}(y)$, где обратный оператор, как мы понимаем, задает пропагатор для духов при ненулевых источниках:

$$\langle 0|\ \mathrm{T}\{\mathfrak{c}(x)\overline{\mathfrak{c}}(y)\}|0\rangle = (-\mathrm{i})^2\frac{\delta^2 Z(j,\zeta,\bar{\zeta})}{\delta\bar{\zeta}(x)\delta\zeta(y)} = \mathcal{D}_{\mathfrak{c}}(x,y) = \mathrm{i}M_L^{-1}\left(-\mathrm{i}\frac{\delta Z}{\delta j^\mu}\right)$$

причем обратный оператор - это интегральный оператор с определенным ядром, зависящим от калибровочного поля,

$$\mathcal{M}_L(\mathcal{A})_x\mathcal{M}_L^{-1}(\mathcal{A})_{xy}\tilde{u}(y) = \delta_{xy}\tilde{u}(y) = \tilde{u}(x)$$

Тогда

$$\frac{1}{\xi}\left(\partial^\mu\mathcal{A}_\mu\right)\tilde{u} + j^\mu\left[\nabla_\mu(\mathcal{A}), \mathcal{M}_L^{-1}(\mathcal{A})_{xy}\tilde{u}(y)\right] = 0$$

Если теперь вынести это подынтегральное равенство из-под знака континуального интеграла и продифференцировать по $\tilde{u}$, то получится следующее уравнение в

вариационных производных

$$\frac{1}{\xi}\partial_x^\mu\left(-\mathrm{i}\frac{\delta Z}{\delta j^\mu}\right)+j^\mu(x')\left[\nabla_\mu^{x'}\left(-\mathrm{i}\frac{\delta Z}{\delta j^\mu}\right),\mathcal{M}_L^{-1}\left(-\mathrm{i}\frac{\delta Z}{\delta j^\mu}\right)_{x'x}\right]=0$$

Это - обобщенное тождество Уорда-Такахаши-Славнова-Тейлора в неабелевых калибровочных теориях без спинорных и других полей.

Оно содержит бесконечное число связей между многоточечными корреляторами.

Для иллюстрации продифференцируем обобщенное тождество Уорда-Такахаши-Славнова-Тейлора по $\mathrm{i}j^\nu(y)$ и положим $j=0$, так что в краткой записи

$$\frac{1}{\xi}\partial_x^\mu(-\mathrm{i})^2\frac{\delta^2 Z}{\delta j^\nu(y)\delta j^\mu(x)}-\mathrm{i}\delta_\nu^\mu\nabla_\mu\mathcal{M}_L^{-1}=0,\quad\text{при } j=0$$

В это выражение входит причинный пропагатор калибровочного поля

$$(-\mathrm{i})^2\frac{\delta^2 Z}{\delta j^\nu(y)\delta j^\mu(x)}=\mathcal{D}_{\nu\mu}(y,x)$$

так что, если взять 4-дивергенцию и по второму индексу этого пропагатора, то найдем

$$\frac{1}{\xi}\partial_y^\nu\partial_x^\mu\mathcal{D}_{\nu\mu}(y,x)-\mathrm{i}\delta_\nu^\mu\partial^\nu\nabla_\mu\mathcal{M}_L^{-1}=0,\quad\text{при } j=0.$$

Но поскольку $\mathcal{M}_L=-\partial^\mu\nabla_\mu$

$$-\partial^\mu\nabla_\mu\mathcal{M}_L^{-1}=\delta_{xy}$$

так что

$$\partial_y^\nu\partial_x^\mu\mathcal{D}_{\nu\mu}(y-x)=-\mathrm{i}\xi\delta_{xy}$$

что в импульсном пространстве означает равенство продольной части пропагатора его исходному выражению в исходном свободном лагранжиане, несмотря на квантовые флуктуации,

$$k^\mu k^\nu\frac{\mathrm{i}}{k^2+\mathrm{i}0}\left(Z_{\mathcal{A}}^\perp(k)\left(-g_{\mu\nu}+\frac{k_\mu k_\nu}{k^2}\right)+Z_{\mathcal{A}}^\parallel(k)\frac{k_\mu k_\nu}{k^2}\right)=-\mathrm{i}\xi\quad\to\quad Z_{\mathcal{A}}^\parallel(k)=-\xi$$

В случае электродинамики с внешними токами оператор $\mathcal{M}_{L_4}$ становится очень простым $\mathcal{M}_L^{\mathrm{QED}}=-\partial^2=-\Box$, так что тождество (10.1) означает следующее: продольное поле остается нулевым при внешних токах, которые сохраняются, $\partial_\mu j^\mu=0$. Это упражнение мы предлагаем читателю проделвть самостоятельно с использованием подстановки для продольного поля $\mathcal{A}_\mu^\parallel=\partial_\mu\mathcal{B}$

Тождества (10.1) могут быть получены и путем применения BRST-преобразования.

Воспользуемся эти способом для случая с фермионными полями ψ, например.

Тогда единственными членами в действии, которые получают преобразования, - это слагаемые с источниками,

$$\delta\mathcal{L}_{\text{ext.}}=\delta\left(j^\mu\mathcal{A}_\mu+\bar{\mathfrak{c}}_\zeta+\bar{\zeta}\mathfrak{c}+\bar{\psi}\eta+\bar{\eta}\psi\right)=j^\mu\delta\mathcal{A}_\mu+\delta\bar{\mathfrak{c}}\zeta+\bar{\zeta}\delta\mathfrak{c}+\delta\bar{\psi}\eta+\bar{\eta}\delta\psi$$

Выпишем в явном виде преобразования полей

$$\begin{aligned}
\delta\mathcal{A}_\mu &= \varepsilon\left(-\partial_\mu\mathfrak{c} + \mathrm{i}e\left[\mathfrak{c},\mathcal{A}_\mu\right]\right) = -\varepsilon\left[\nabla_\mu,\mathfrak{c}\right]\\
\delta\mathfrak{c}_c &= -\frac{1}{2}\varepsilon\mathrm{i}e\mathfrak{c}^a\mathfrak{c}^b\mathrm{i}f_{abc} = \frac{1}{2}\varepsilon e f_{abc}\mathfrak{c}^a\mathfrak{c}^b\\
\delta\bar{\mathfrak{c}} &= \frac{1}{\xi}\varepsilon\left(\partial_\mu\mathcal{A}^\mu\right)\\
\delta\psi &= \mathrm{i}e_\psi\varepsilon\mathfrak{c}^a t_a\\
\delta\bar{\psi} &= -\mathrm{i}e_\psi\bar{\psi}\varepsilon\mathfrak{c}^a t_a
\end{aligned}$$

откуда

$$\delta\mathcal{L}_{\text{ext.}} = -j^\mu\varepsilon\left[\nabla_\mu,\mathfrak{c}\right] + \frac{1}{\xi}\varepsilon\left(\partial_\mu\mathcal{A}^\mu\right)_\zeta + \bar{\zeta}^c\frac{1}{2}\varepsilon e f_{abc}\mathfrak{c}^a\mathfrak{c}^b - \mathrm{i}e_\psi\bar{\psi}\varepsilon\mathfrak{c}^a t_a\eta + \bar{\eta}\mathrm{i}e_\psi\varepsilon\mathfrak{c}^a t_a\psi$$

Переставляя грассманов параметр ε налево с учетом антикоммутации с другими грассмановыми переменными, мы может затем продифференцировать по ε и получить равенство

$$\frac{\partial}{\partial\varepsilon}\delta\mathcal{L}_{\text{ext.}} = -j^\mu\left[\nabla_\mu,\mathfrak{c}\right] + \frac{1}{\xi}\left(\partial_\mu\mathcal{A}^\mu\right)\zeta + \bar{\zeta}^c\frac{1}{2}e f_{abc}\mathfrak{c}^a\mathfrak{c}^b + \mathrm{i}e_\psi\bar{\psi}\mathfrak{c}^a t_a\eta - \bar{\eta}\mathrm{i}e_\psi\mathfrak{c}^a t_a\psi$$

Далее, положим теперь источник $\bar{\zeta} = 0$, что даст

$$\frac{\partial}{\partial\varepsilon}\delta\mathcal{L}_{\text{ext.}}\Big|_{\bar{\zeta}=0} = -j^\mu\left[\nabla_\mu,\mathfrak{c}\right] + \frac{1}{\xi}\left(\partial_\mu\mathcal{A}^\mu\right)\zeta + \mathrm{i}e_\psi\bar{\psi}\mathfrak{c}^a t_a\eta - \bar{\eta}\mathrm{i}e_\psi\mathfrak{c}^a t_a\psi$$

Если вместо полей теперь иметь ввиду соответствующие вариационные производные, то действие приведенного выражения на амплитуду Z,конечно, обращается в нуль вследствие калибровочной инвариантности континуального интеграла,

$$\frac{\partial}{\partial\varepsilon}\delta\mathcal{L}_{\text{ext.}}\Big|_{\bar{\zeta}=0} Z(j,\zeta,\eta,\bar{\eta}) = 0$$

при любых значениях источников. Избавимся от источников духов с. Для этого возьмем вариационную производную по i c и положим их равными нулю. тогда в подынтегральном выражении появятся вклады

$$\bar{\mathfrak{c}}\frac{\partial}{\partial\varepsilon}\delta\mathcal{L}_{\text{ext.}}\Big|_{\bar{\zeta}=\zeta=0} - \mathrm{i}\frac{\partial^2}{\delta\zeta\partial\varepsilon}\delta\mathcal{L}_{\text{ext.}}\Big|_{\bar{\zeta}=\mathrm{s}=0} = 0$$

что эквивалентно

$$-j^\mu\left[\nabla_\mu,\mathfrak{c}\right]\bar{\mathfrak{c}} + \mathrm{i}e_\psi\bar{\psi}\mathfrak{c}^a t_a\eta\bar{\mathfrak{c}} - \bar{\eta}\mathrm{i}e_\psi\mathfrak{c}^a t_a\psi\bar{\mathfrak{c}} - \mathrm{i}\frac{1}{\xi}\left(\partial_\mu\mathcal{A}^\mu\right) = 0$$

Спаривание полей духов дает

$$\underbrace{\mathfrak{c}\bar{\mathfrak{c}}} = \mathrm{i}\mathcal{M}_L^{-1}$$

так что с учетом перестановок антикоммутирующих переменных найдем обобщенные тождества Уорда-Такахаши-Славнова-Тейлора для калибровочной теории с фермионами, если иметь ввиду операторное действие полей на амплитуду Z,

$$\begin{aligned}
j_a^\mu\left(\nabla_\mu\mathcal{M}_L^{-1}\right)^{ab} + \frac{1}{\xi}\left(\partial_\mu\mathcal{A}^\mu\right)^b -\\
- e_\psi\bar{\psi}t_a\eta\left(\mathcal{M}_L^{-1}\right)^{ab} + e_\psi\bar{\eta}t_a\psi\left(\mathcal{M}_L^{-1}\right)^{ab} = 0
\end{aligned}$$

Если в этом выражении положить равными нулю источники для фермионных полей, то мы, как и следовало ожидать, воспроизведем тождества Уорда для случая динамики калибровочных полей с источниками, проведя вывод теперь с использованием BRST-инвариантности.

Соотношения (10.2) позволяют связать квантовые поправки для функций Грина в разном порядке по константе взаимодействия при условии калибровочной инвариантности теории без квантовых аномалий.

Это становится особенно существенным,

когда речь идет о регуляризации петлевых вкладов в теории перенормировок.

(???????)

примеры использования

В этой лекции предполагается, что функциональный интеграл (55 Л9) вычисляется при $d = 4 - \varepsilon$ и, таким образом, является регуляризованным.

15.1.2 1. Furry's Theorem

Докажем теорему Фарри, утверждающую, что диаграммы, содержащие хотя бы один нечётный замкнутый спинорный цикл (петлю), взаимно аннулируются. Рассмотрим некую фермионную петлю, изображённую на рис. 15.

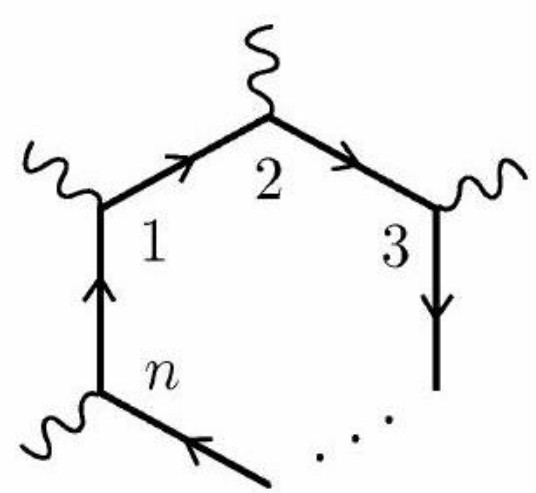


Рис. 15

Пусть при обходе петли в направлении стрелок индекс j следует за индексом i и k_{ji} есть 4 -импульс линии, соединяющей вершины i и j и направленный от вершины i к вершине j. Кроме петли на рис. 15, содержащейся в некой диаграмме, обозначаемой L_+, имеется диаграмма L_-, отличающаяся от L_+лишь тем, что она содержит такую же петлю с обращёнными стрелками. Сумма этих двух диаграмм (а точнее - самих петель) равна нулю при нечётном числе n вершин этих петель.

Множитель в диаграмме L_+, соответствующий петле на рис. 15 (до проведения интегрирования по импульсам), равен

$$l_+ = (-1)\,\mathrm{tr}\left[\gamma S_c\left(k_{n(n-1)}\right)...\gamma S_c\left(k_{21}\right)\gamma S_c\left(k_{1n}\right)\right] \tag{1}$$

Воспользуемся тем, что ни один результат теории Дирака не изменяется при замене (см. (79 Л9))

$$\gamma^{\mu} \to -\gamma^{\mu t} = U_C^{-1}\gamma^{\mu}U_C \tag{2}$$

Действительно, при замене (2) остаются инвариантными как фундаментальные соотношения γ матриц (5 Л8), так и вся алгебра матриц Дирака, включая значения следов произведения матриц Дирака, Поэтому в результате замены (2) находим:

$$\begin{gathered} l_+ = -(-1)^n \operatorname{tr}\left[\gamma^t S_c^t\left(k_{(n-1)n}\right) ... \gamma^t S_c^t\left(k_{12}\right)\gamma^t S_c^t\left(k_{n1}\right)\right] = \\ = -(-1)^n \operatorname{tr}\left[\gamma S_c\left(k_{n1}\right)\gamma S_c\left(k_{12}\right) ... \gamma S_c\left(k_{(n-1)n}\right)\right] = (-1)^n l_- \end{gathered}$$

где l_-−соответствующая петля в диаграмме L_-. Отсюда видно, что при нечётном n $l_+ + l_- = 0 \longrightarrow L_+ + L_- = 0$, что и требовалось доказать.

15.1.3 2. Ward–Takahashi Identity

Сделаем в функциональном интеграле (55 Л9) замену переменных

$$\psi(x) = e^{i\alpha(x)}\psi'(x), \quad \bar{\psi}(x) = \bar{\psi}'(x)e^{-i\alpha(x)} \tag{3}$$

и будем считать поле $\alpha(x)$ бесконечно малым. Учитывая вклад в интеграл лишь первого порядка относительно поля α и тот факт, что $D\bar{\psi}D\psi = D\bar{\psi}'D\psi'$, получаем следующее равенство

$$\frac{1}{N}\int (\exp iS\{A,\psi,\bar{\psi};J,\eta,\bar{\eta}\})DAD\bar{\psi}D\psi =$$

$$= \frac{1}{N}\int \left(\exp iS\left\{A,\psi',\bar{\psi}';J,\eta,\bar{\eta}\right\}\right)\left\{1 + i\int \mathrm{d}^{(d)}x\alpha\left[\partial_\mu J'^{\mu} + i\left(\bar{\eta}\psi' - \bar{\psi}'\eta\right)\right]\right\} DAD\bar{\psi}'D\psi' \tag{4}$$

Здесь $S\left\{A,\psi',\bar{\psi}';J,\eta,\bar{\eta}\right\}$ - действие в экспоненте в интеграле (55 Л9), $J'^{\mu} = \bar{\psi}'\gamma^{\mu}\psi'$, а нормировочный множитель N - такой же функциональный интеграл, как в (4), но с нулевыми источниками. Сравнивая обе части последнего равенства, мы видим, что равна нулю та величина, которая пропорциональна полю $\alpha(x)$. Так как поле $\alpha(x)$ - произвольно, то мы имеем в каждой точке x следующее равенство:

$$\frac{\partial}{\partial x^{\mu}}\langle J^{\mu}(x)\rangle = -i\bar{\eta}(x)\langle\psi(x)\rangle + i\langle\bar{\psi}(x)\rangle\eta(x) \tag{5}$$

Теперь вычислим повторные функциональные производные $\left(\delta/\delta\bar{\eta}\left(x_1\right)\right)\left(\delta/\delta\eta\left(x_2\right)\right)$ от равенства (5), затем положим источники равными нулю. При этом следует учесть, что обе части равенства (5) являются рядами по источникам, коэффициентные функции при степенях которых есть хронологически упорядоченные средние, и операторы частных производных $\partial/\partial x^{\mu}$, $\delta/\delta\eta\left(x_1\right)$, $\delta/\delta\bar{\eta}\left(x_2\right)$ взаимно (анти)коммутируют. В результате находим:

$$\begin{gathered} \frac{\partial}{\partial x^{\mu}}\left\langle 0\left|TJ^{\mu}(x)\psi\left(x_1\right)\bar{\psi}\left(x_2\right)\right|0\right\rangle = \\ = -\delta^{(d)}\left(x - x_1\right)\left\langle 0\left|T\psi\left(x_1\right)\bar{\psi}\left(x_2\right)\right|0\right\rangle + \delta^{(d)}\left(x - x_2\right)\left\langle 0\left|T\psi\left(x_1\right)\bar{\psi}\left(x_2\right)\right|0\right\rangle \end{gathered} \tag{6}$$

Заметим, что вследствие трансляционной инвариантности все корреляторы зависят лишь от разностей координат. Поэтому

$$\left\langle 0 \left| T J^{\mu}(x) \psi\left(x_1\right) \bar{\psi}\left(x_2\right) \right| 0 \right\rangle \equiv$$
$$\equiv \int \mathrm{d}^{(d)} y_1 \; \mathrm{d}^{(d)} y_2 \left[i \mathfrak{S}\left(x_1 - y_1\right) \right] \Gamma^{\mu}\left(y_1 - x, x - y_2\right) \left[i \mathfrak{S}\left(y_2 - x_2\right) \right] \tag{7}$$

где

$$\mathfrak{S}(x-y) = -i \langle 0 | T \psi(x) \bar{\psi}(y) | 0 \rangle \tag{8}$$

- точный электрон-позитронный пропагатор, а $\Gamma^{\mu}\left(y_1 - x, x - y_2\right)$ - вериинный оператор (в координатном пространстве).

Приведём все диаграммы вершинного оператора с точностью до величин пятого порядка:

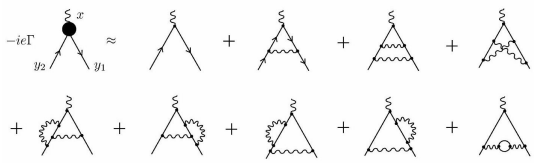


Рис. 16

Из определения вершинного оператора (7) видно, что первое слагаемое справа на рис. 16 равно $-ie\gamma^{\mu}\delta^{(d)}\left(y_1 - x\right)\delta^{(d)}\left(x - y_2\right)$. Таким образом, вершинный оператор есть сумма всех диаграмм с двумя фермионными и одной фотонной внешними ампутированными линиями.

Выразим обе части уравнения (6) при помощи формул (7) и (8) и затем проинтегрируем с весом $\int \mathrm{d}^{(4)} x_1 \; \mathrm{d}^{(4)} x_2 \exp\left[i(p+k)x_1 - ipx_2\right]$. С учётом того, что

$$\mathfrak{S}(k) = \int \mathrm{d}^{(d)} x \mathfrak{S}(x) e^{ikx}, \quad \Gamma^{\mu}(q,p) \equiv \int \mathrm{d}^{(d)} x_1 \; \mathrm{d}^{(d)} x_2 e^{iqx_1 + ipx_2} \Gamma^{\mu}\left(x_1, x_2\right) \tag{9}$$

(так что в наинизшем порядке $\Gamma^{\mu}(q,p) = \gamma^{\mu}$) находим соотношение

$$\mathfrak{S}(p) - \mathfrak{S}(p+k) = \mathfrak{S}(p+k)\left[k_{\mu}\Gamma^{\mu}(p+k,p)\right]\mathfrak{S}(p)$$

откуда получается равенство

$$k_{\mu}\Gamma^{\mu}(p+k,p) = \mathfrak{S}^{-1}(p+k) - \mathfrak{S}^{-1}(p) \tag{10}$$

При $k \to 0$ из (10) получаем также:

$$\frac{\partial \mathfrak{S}^{-1}(p)}{\partial p_{\mu}} = \Gamma^{\mu}(p,p) \tag{11}$$

15.1.4 3. Index of Divergence of Diagrams

Для произвольной диаграммы введём следующие обозначения:
L_e - число внешних электронных линий;
L_γ - число внешних фотонных линий;
N_e - число внутренних электронных линий;
N_γ - число внутренних фотонных линий;
n - число вершин;
l - число петель.
Тогда индекс расходимости диаграммы

$$\omega = dl - N_e - 2N_\gamma \tag{12}$$

Для числа петель имеем:

$$l = N_e + N_\gamma - n + 1 \tag{13}$$

так как каждому пропагатору соответствует интегрирование (по своему d-импульсу), но каждая вершина содержит δ-функцию по входящим импульсам, однако одна δ-функция даёт общий закон сохранения. Для числа вершин имеем:

$$n = L_\gamma + 2N_\gamma = \frac{1}{2}\left(L_e + 2N_e\right) \tag{14}$$

Теперь подставим l из (13) в (12) и при помощи уравнений (14) исключим N_γ и N_e. Таким образом получим:

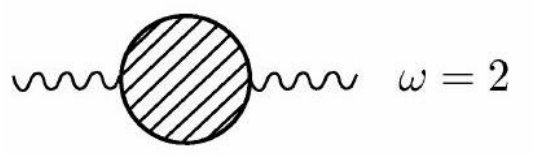
$$\omega = \frac{n}{2}(d-4) + d - \frac{d-1}{2}L_e - \frac{d-2}{2}L_\gamma \tag{15}$$

Отсюда видно, что при $d > 4$ теория заведомо неперенормируема, а при $d = 4$

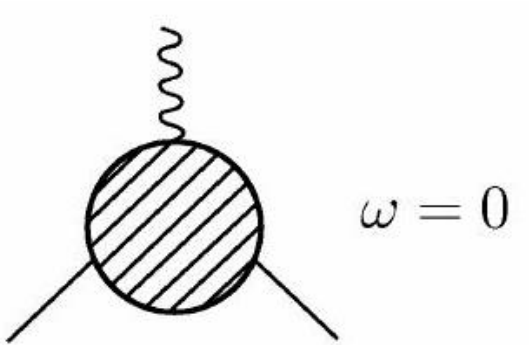
$$\omega = 4 - L_\gamma - \frac{3}{2}L_e \tag{16}$$

то есть формальная расходимость диаграммы зависит лишь от числа внешних концов. Диаграмма расходится при $\omega \geq 0$. Подчеркнём, что импульсы внешних концов диаграмм не предполагаются находящимися на массовой поверхности.

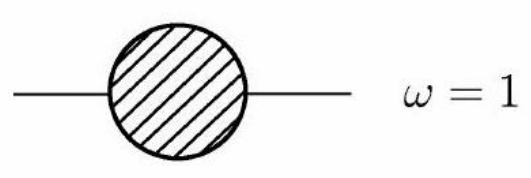
Укажем сразу, что суммы диаграмм при фиксированном порядке по e с нечётным числом L_γ и $L_e = 0$ согласно теореме Фарри равны нулю. Действительно, в таких диаграммах полное число вершин нечётно (см. (14)), и потому имеется хотя бы одна фермионная петля с нечётным числом вершин. Вакуумные пузырьки ($L_\gamma = L_e = 0$) нет смысла рассматривать, так как они дают ненаблюдаемый сдвиг энергии вакуумного состояния. Поэтому имеется всего 4 расходящихся диаграммы:



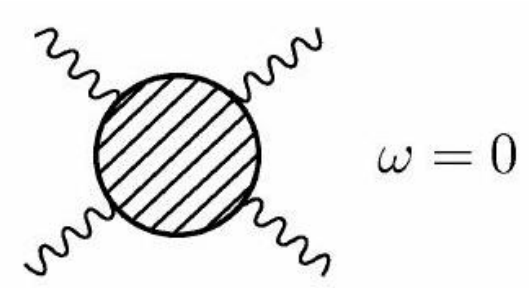
(a)



(*c*)



(b)



(*d*)
Рис. 17
Покажем, что диаграмма (*d*) на рис. 17 конечна.
Прежде всего заметим, что имеют место равенства

$$\frac{\partial}{\partial x^{\mu}} \langle 0 \left| T J^{\mu}(x) J^{\mu_1}(x_1) \ldots J^{\mu_n}(x_n) \right| 0 \rangle = 0 \tag{17}$$

Действительно, если бы мы рассмотрели вместо интеграла (55 Л9) этот же интеграл со вставкой под функциональный интеграл выражения $(J^{\mu_1}(x_1) \ldots J^{\mu_n}(x_n))$ и проделали бы манипуляции, приводящие к (5), то мы получили бы вместо (5) равенство, отличающееся лишь вставкой во все средние выражения $(J^{\mu_1}(x_1) \ldots J^{\mu_n}(x_n))$. Таким образом после зануления внешних источников получается равенство (17). В импульсном пространстве равенство (17) означает, что

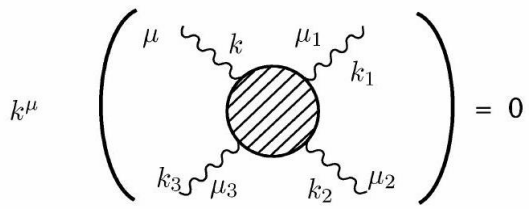


Рис. 18

Равенство, изображённое на рис. 18, возможно, лишь если диаграмма в круглых скобках на рис. 18 пропорциональна структуре вида $(\eta^{\mu\mu_1}k^{\mu_2} - \eta^{\mu\mu_2}k^{\mu_1})$. Аналогичные множители содержатся в остальных трёх вершинах. Каждый из указанных множителей линеен и однороден относительно импульсов, которые "вытягиваются"из пропагаторов внутренних линий. Поэтому некоторым внутренним линиям соответствуют множители, убывающие быстрее при возрастании импульсов, чем при формальном подсчёте. В результате оказывается, что степень расходимости диаграммы (d) на рис. 17$\omega = -4$ и она конечна.

Подчеркнём, что внешние импульсы входят в диаграммы лишь в неотрицательных степенях, поскольку по их отношениям к большим промежуточным импульсам интегрирования производится разложение.

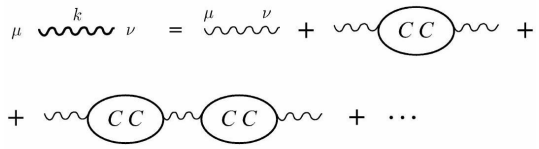
Рассмотрим диаграммы вида (a) на рис. 17. Обозначим через $i\mathcal{P}^{\mu\nu}/4\pi$ сумму всех сильно связанных диаграмм вида (a) на рис. 17. Функция $\mathcal{P}_{\mu\nu}(k)$ называется поляризационным оператором. Рассуждения, аналогичные применённым при рассмотрении диаграмм вида (d) на рис. 17, показывают, что

$$\underbrace{\mu}_{k} \sim CC \sim\sim\sim \frac{\mathcal{P}^{\mu\nu}(k)}{4\pi} = i\frac{\mathcal{P}(k)}{4\pi}\left(k^2\eta^{\mu\nu} - k^\mu k^\nu\right) \tag{18}$$

где оператор $\mathcal{P}\left(k^2\right)$ расходится лишь логарифмически. Обозначим точный фотонный пропагатор

$$\mathcal{D}_{\mu\nu}(x-y) = i\left\langle 0\left|TA_\mu(x)A_\nu(y)\right|0\right\rangle \tag{19}$$

жирной волнистой линией. Тогда на диаграммном языке имеем:



а аналитически (в импульсном пространстве)

$$\mathcal{D}^\alpha_{\mu\nu}(k) = D^\alpha_{\mu\nu}(k) + D^\alpha_{\mu\mu_1}(k)\frac{\mathcal{P}^{\mu_1\mu_2}(k)}{4\pi}D^\alpha_{\mu_2\mu}(k) + ...$$

При помощи формул (53 Л7) и (18) эта геометрическая прогрессия легко суммируется:

$$\mathcal{D}^\alpha_{\mu\nu}(k) = \frac{4\pi}{k^2(1-\mathcal{P}(k))}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + \frac{4\pi\alpha k_\mu k_\nu}{\left(k^2\right)^2} \tag{20}$$

Обозначим через $(-i\mathcal{M})$ сумму всех сильно связанных диаграмм вида (b) на рис. 17 и назовём эту функцию массовым оператором. Для точного фермионного пропагатора (8) имеет место разложение, аналогичное разложению точного фотонного пропагатора:

$$\mathfrak{S}(k) = S_c(k) + S_c(k)\mathcal{M}(k)S_c(k) + ... = S_c(k)(1 + \mathcal{M}(k)\mathfrak{S}(k)) = \frac{1}{\not{k} - m_0 - \mathcal{M}(k)} \tag{21}$$

где m_0 - затравочная масса

Общий вид массового оператора таков:

$$\mathcal{M}(k) = a_0 m_0 + a_1\not{k} + m(k) \tag{22}$$

где $m(k)$ - конечная функция. Из размерностных соображений очевидно, что константы a_0 и a_1 расходятся логарифмически. Важно, что в правой части (22) не может быть слагаемого a_2, не обращающегося в нуль при $m_0 = 0$ и $k^\mu = 0$ и потому расходящегося линейно. Согласно (21) такое слагаемое a_2 давало бы ненулевую массу фермионам даже в случае $m_0 = 0$ и потому нарушало бы глобальную γ^5-симметрию

$$\psi \to e^{i\alpha\gamma^5}\psi, \quad \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma^5}, \quad \alpha = \text{ const} \tag{23}$$

При $m_0 = 0$ имеет место глобальная γ^5-симметрия (23) и она не может быть нарушена квантовыми поправками, т.е. точный фермионный пропагатор антикоммутирует с γ^5:

$$\gamma^5\mathfrak{S}(k) + \mathfrak{S}(k)\gamma^5 = 0 \longleftrightarrow \gamma^5\mathcal{N}(k) + \mathcal{M}(k)\gamma^5 = 0$$

Действительно, любая диаграмма, дающая вклад в точный пропагатор, представляется как одна сплошная фермионная линия с внешними концами и некоторое количество фермионных петель, соединённых между собою фотонными пропагаторами. Так как каждая фермионная петля находится под знаком спинорного следа, то спинорная структура всей диаграммы определяется произведением свободных пропагаторов S_c и γ-матриц вдоль единственной фермионной линии с внешними концами. Очевидно, что спинорная структура этой линии определяется произведением

$$S_c(k)\left[\gamma^{\mu_1}S_c\left(p_1\right)\right]...\left[\gamma^{\mu_n}S_c\left(p_n\right)\right]\left[\gamma^{\mu}S_c(k)\right], \quad n = 1, 2, ...$$

которое антикоммутирует с γ^5 в случае $m_0 = 0$. Отсюда видно, что в случае $m_0 = 0$ упомянутого слагаемого a_2 не может появиться при учёте квантовых поправок.

Мы видим, что в квантовой электродинамике имеются лишь логарифмические расходимости, которые, как мы покажем, могут быть устранены при помощи перенормировки полей и затравочных констант m_0 и e_0.

Пусть исходный (затравочный) лагранжиан КЭД равен

$$\mathcal{L} = -\frac{1}{16\pi}F_{\mu\nu}^2 + \bar{\psi}\left(i\partial\!\!\!/ - m_0\right)\psi - e_0\bar{\psi}\gamma^\mu\psi A_\mu \tag{24}$$

Вычислим с помощью этого лагранжиана в размерности $d = 4 - \varepsilon$ точные электронный (8) и фотонный (19) пропагаторы, которые вблизи полюсов ведут себя как

$$\mathfrak{S}(k) = \frac{Z_2}{\not{k} - m} + s(k) \tag{25}$$

$$\mathcal{D}_{\mu\nu}^{\perp}(k) = \frac{4\pi Z_3}{k^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + d_{\mu\nu}(k) \tag{26}$$

В (25) и (26) операторы $s(k)$ и $d_{\mu\nu}(k)$ конечны при стремлении $k^2 \to m^2$ и $k^2 \to 0$, соответственно.

15.1.5 4. Renormalization Conditions

Займёмся перенормировкой внешних линий. Рассмотрим выходящую внешнюю линию электрона с импульсом p и спиновым состоянием s. Пусть эта линия является единственной выходящей линией амплитуды $M_{(ps)i}$, изображённой на рис. 20.

Здесь все входящие и выходящие линии - жирные, которым соответствуют точные величины с учётом всех радиационных

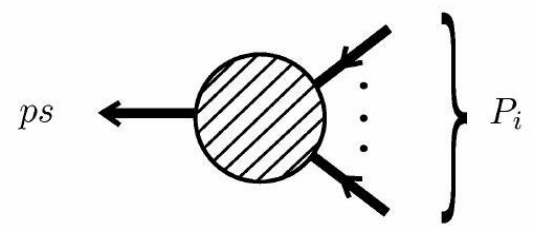


Рис. 20 поправок. В КЭД все 4-импульсы входящих линий в амплитуде на рис. 20 не могут находиться на массовой поверхности, если выходящий электрон - реальный с $p^2 = m^2$, поскольку законы сохранения 4 -импульса и заряда не допускают этого. Однако именно случай $p^2 = m^2$ нас интересует, поэтому входящие линии амплитуды $M_{(ps)i}$ не соответствуют, вообще говоря, реальным частицам. Такая ситуация может быть получена путём аналитического продолжения некоторой амплитуды с массовой поверхности в нужную область её переменных. Множитель в $M_{(ps)i}$, соответствующий выходящему электрону и обозначаемый $\overline{\mathfrak{U}}_{ps}$, получается так же, как точный электронный пропагатор с той лишь разницей, что в разложении (21) во всех слагаемых самый левый пропагатор S_c заменяется на биспинор $\bar{u}_p$, который содержится в разложении затравочного дираковского поля в (24) и удовлетворяет условиям нормировки (20 Л8), (56 Л8) и (57 Л8):

$$\overline{\mathfrak{U}}_p = \bar{u}_p(1 + \mathcal{M}\mathfrak{S}) \tag{27}$$

Таким образом имеем:

$$M_{(ps)i} = \overline{\mathfrak{U}}_{ps} M'_{pi} \tag{28}$$

где M'_{pi} является дираковским спинором - 4-рядной матрицей-столбцом. Сопрягая комплексно равенство (28) и заменяя состояние $|i\rangle$ на $|f\rangle$, получаем:

$$M^*_{(ps)f} = \bar{M}'_{pf} \mathfrak{U}_{ps} \tag{29}$$

Теперь воспользуемся формулой (38 Л5) для значения амплитуды M_{fi} при $P_i^2 = P_f^2 = p^2 \to m^2$, которая была получена из общего требования унитарности теории:

$$iM_{fi} = -\bar{M}'_{pf} \frac{i \sum\limits_s \mathfrak{U}_{ps} \overline{\mathfrak{U}}_{ps}}{p^2 - m^2 + i\varepsilon} M'_{pi} \tag{30}$$

Амплитуда (30) является аналитическим выражением процесса, изображённого на рис. 21:

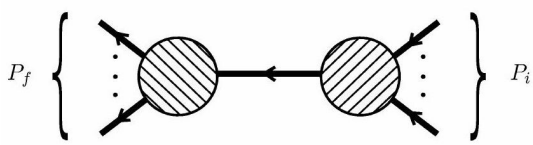


Рис. 21

В силу Лоренц-инвариантности для биспиноров $\mathfrak{U}_{ps}$ имеют место те же соотношения, что и для затравочных биспиноров u_{ps}, кроме нормировки. Следовательно (сравни с (56 Л8)):

$$\sum_s \mathfrak{U}_{ps} \overline{\mathfrak{U}}_{ps} = Z'(\not{p} + m)$$

и потому при $p^2 \to m^2$

$$iM_{fi} = -\bar{M}'_{pf} \frac{iZ'(\not{p} + m)}{p^2 - m^2 + i\varepsilon} M'_{pi} \tag{31}$$

С другой стороны, вычисление этой же амплитуды непосредственно по диаграмме на рис. 21 даёт (учтём, что в рассматриваемой области переменных у амплитуды $M_{f(ps)}$ нет реальных промежуточных состояний, и потому согласно (36 Л5) имеем $M_{f(ps)} = M^*_{(ps)f} = \bar{M}'_{pf}\mathfrak{U}_{ps}$; следовательно, левой части амплитуды на рис. 21 сопоставляется амплитуда $\left(i\bar{M}'_{pf}\right)$, пропагатору $-(i\mathfrak{S}(p))$, а правой части $-\left(iM'_{pi}\right)$)

$$iM_{fi} = \left(i\bar{M}'_{pf}\right)(i\mathfrak{S}(p))iM'_{pi} \tag{32}$$

Сопоставляя формулы (31), (32) и (25), находим $Z' = Z_2$. Поэтому

$$\mathfrak{U}_{ps} = \sqrt{Z_2}u_{ps} \tag{33}$$

где u_{ps} - дираковский биспинор, удовлетворяющий всем соотношениям, приведенным в лекции 8.

Аналогичные вычисления показывают, что точный вектор поляризации фотона $\mathcal{E}_{\mathrm{k}\alpha}$ связан с нормированным согласно (55Ω7) вектором поляризации $\mathbf{e_k}\alpha$ формулой

$$\mathcal{E}_{\mathbf{k}\alpha} = \sqrt{Z_3}\mathbf{e}_{\mathbf{k}\alpha} \tag{34}$$

Теперь введём перенормированные поля при помощи формул

$$\psi = \sqrt{Z_2}\psi_r, \quad A^\mu = \sqrt{Z_3}A^\mu_r \tag{35}$$

В результате этой замены лагранжиан принимает вид

$$\mathcal{L} = -\frac{1}{16\pi}Z_3\left(F^{\mu\nu}_r\right)^2 + Z_2\bar{\psi}_r\left(i\not{\partial} - m_0\right)\psi_r - e_0Z_2\sqrt{Z_3}\bar{\psi}_r\gamma^\mu\psi_rA_{r\mu} \tag{36}$$

Константа m в (25) является физической массой. Пусть e обозначает физический заряд, измеряемый на больших расстояниях. Определим масштабный множитель Z_1 как

$$e_0Z_2\sqrt{Z_3} = eZ_1 \tag{37}$$

Теперь разобьём лагранжиан (36) на две части:

$$\begin{gathered}\mathcal{L} = \left[-\frac{1}{16\pi}\left(F^{\mu\nu}_r\right)^2 + \bar{\psi}_r(i\not{\partial} - m)\psi_r - e\bar{\psi}_r\gamma^\mu\psi_rA_{r\mu}\right] + \\ + \left[-\frac{1}{16\pi}\delta_3\left(F^{\mu\nu}_r\right)^2 + \bar{\psi}_r\left(i\delta_2\not{\partial} - \delta_m\right)\psi_r - e\delta_1\bar{\psi}_r\gamma^\mu\psi_rA_{r\mu}\right] \\ \delta_3 = Z_3 - 1, \quad \delta_2 = Z_2 - 1, \quad \delta_m = Z_2m_0 - m, \quad \delta_1 = Z_1 - 1 = \left(\frac{e_0}{e}\right)Z_2\sqrt{Z_3} - 1\end{gathered} \tag{38}$$

Здесь первое слагаемое в лагранжиане даёт правила Фейнмана, описанные в лекции 9, а слагаемое в квадратной скобке - контрчлены.

Перепишем равенство (7) для перенормированных полей:

$$\left\langle 0\left|TJ^{\mu}(x)\psi\left(x_1\right)\bar{\psi}\left(x_2\right)\right|0\right\rangle = Z_2^2\left\langle 0\left|TJ_r^{\mu}(x)\psi_r\left(x_1\right)\bar{\psi}_r\left(x_2\right)\right|0\right\rangle =$$
$$= Z_2^2\int \mathrm{d}^{(4)}y_1\ \mathrm{d}^{(4)}y_2\left[i\mathfrak{S}_r\left(x_1-y_1\right)\right]\Gamma_r^{\mu}\left(y_1-x,x-y_2\right)\left[i\mathfrak{S}_r\left(y_2-x_2\right)\right]$$

где $\mathfrak{S}_r(x)$ - точный перенормированный фермионный пропагатор, построенный из перенормированных ферми-полей при помощи формулы (8), а Γ_r^{μ} - перенормированный вершинный оператор. Так как $Z_2\mathfrak{S}_r = \mathfrak{S}$, то из сравнения последнего равенства с равенством (7) мы видим, что $\Gamma_r^{\mu} = \Gamma^{\mu}$.

Теперь точный электронный (8) и точный фотонный (19) пропагаторы вблизи полюсов ведут себя как

$$\mathfrak{S}_r(k) = \frac{1}{\not{k} - m - \mathcal{M}_r(k)} \approx \frac{1}{\not{k} - m}$$
$$\mathfrak{D}_{r\mu\nu}^{\perp}(k) = \frac{4\pi}{k^2\left(1-\mathcal{P}_r\left(k^2\right)\right)}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \approx \frac{4\pi}{k^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \tag{39}$$

Отсюда вытекают следующие условия нормировки:

$$\mathcal{M}_r(k) = (\not{k} - m)g(k)(\not{k} - m) \tag{40a}$$
$$\mathcal{P}_r\left(k^2 = 0\right) = 0 \tag{40b}$$

При помощи (39), (40) и (38 а Л8) получаем:

$$\bar{u}_p \frac{\partial}{\partial p_\mu}\left[Z_2^{-1}\mathfrak{S}_r^{-1}(p)\right] u_p\Big|_{p^2=m^2} = Z_2^{-1}\bar{u}_p\gamma^{\mu}u_p \tag{41}$$

Так как тождества Уорда (11) имеют место для неперенормированных величин, то при помощи (41) находим тождество Уорда для перенормированных величин:

$$\bar{u}_p\Gamma^{\mu}(p,p)u_p|_{p^2=m^2} = Z_2^{-1}\bar{u}_p\gamma^{\mu}u_p \tag{42}$$

Рассмотрим диаграмму

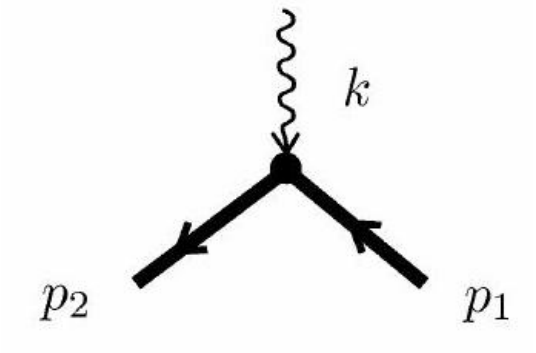


Рис. 22

описывающую рассеяние электрона во внешнем поле $A^{(e)}(k)$ (в первом порядке по полю) с учётом всех радиационных поправок. В пределе $k \to 0, p_2 \to p_1 \equiv p$ этой диаграмме соответствует амплитуда

$$iM_{fi} = -ie\bar{u}_p\gamma^\mu u_p A^{(e)}_{r\mu}(k\to 0) \tag{43}$$

Здесь все величины - перенормированные и e - физически наблюдаемый заряд на больших расстояниях.

Теперь рассмотрим эту же амплитуду в неперенормированной теории:

$$\begin{aligned} iM_{fi} &= -ie_0\overline{\mathfrak{U}}_p\Gamma^\mu(p,p)\big|_{p^2=m^2}\,\mathfrak{U}_p A^{(e)}_\mu(k\to 0) = \\ &= -ie_0 Z_2 Z_3^{1/2}\bar{u}_p\Gamma^\mu(p,p)\Big|_{p^2=m^2}\,u_p A^{(e)}_{r\mu}(k\to 0) = -iZ_1Z_2^{-1}e\bar{u}_p\gamma^\mu u_p A^{(e)}_{r\mu}(k\to 0) \end{aligned} \tag{44}$$

Мы воспользовались определением (37) тождеством Уорда (42). Сравнивая (43) и (44), мы видим, что для того чтобы константа e действительно была физическим зарядом, необходимо

$$Z_1 = Z_2, \quad e = \sqrt{Z_3}e_0 \tag{45}$$

Воспользуемся неопределённостью, возникающей из-за расходимостей, содержащихся в Γ, и потребуем, чтобы имело место равенство

$$\Gamma^\mu(p,p)\big|_{\not p=m} = \gamma^\mu \tag{46}$$

то есть $Z_1 = Z_2 = 1$. (Подразумевается, что равенство (46) имеет место в обкладках $\bar{u}_p ... u_p$, когда $\not p = m$.) Удобство этого определения состоит в том, что отпадает необходимость введения поправок во внешние электронные линии, так как теперь (см (33)) $\mathfrak{U}_{ps} = u_{ps}$, и перенормированный фермионный пропагатор совпадает с неперенормированным. Кроме того, в силу (34) и (45) $e\mathbf{e}_{\mathbf{k}\alpha} = e_0\mathcal{E}_{\mathbf{k}\alpha}$, а также $[e^2\mathfrak{D}_{r\mu\nu}] = e_0^2\mathfrak{D}_{\mu\nu}$ (здесь справа стоят неперенормированные величины). Поэтому, хотя в неперенормированной теории используется затравочный заряд e_0, внешней фотонной линии соответствует вектор поляризации $\mathcal{E}_{\mathbf{k}\alpha}$, а в качестве точного фотонного пропагатора используется функция $\mathfrak{D}_{\mu\nu}$, вместо них можно использовать физический заряд e, стандартные
вектора поляризации (см. лекцию 7) $\mathbf{e}_{\mathbf{k}}\alpha$ и перенормированные пропагаторы $\mathfrak{D}_{r\mu\nu}$. В результате такой замены значения амплитуд не изменяются.

Таким образом, если в диаграммной технике используется физический заряд, физическая масса и перенормированные величины с условиями нормировки (40), (46), то внешним линиям сопоставляются стандартные биспиноры $u_{\pm ps}, \bar{u}_{\pm ps}$ и вектора поляризации $\sqrt{4\pi}\mathbf{e}_{\mathbf{k}}\alpha$, удовлетворяющие естественным условиям нормировки, данным в лекциях 7 и 8. Указанное правило предполагает, что если какая либо внешняя линия диаграммы находится на массовой поверхности, то соответствующая "нога"ампутирована (поскольку все радиационные поправки уже учтены). Асимптотические состояния нормированы обычным способом. Например, для одноэлектронного состояния $|\mathbf{p}s\rangle$ имеем $\langle\mathbf{q}s'|\mathbf{p}s\rangle = (2\pi)^3\omega_{\mathbf{p}}\delta_{ss'}\delta^{(3)}(\mathbf{q}-\mathbf{p})$.

Далее индекс r у перенормированных величин опускается.

Задачи

1. а) Сформулировать теорию Дираковских спиноров в двумерном пространстве-времени Минковского.

Указание. Исходить из необходимости реализации алгебры Дирака $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$ в двумерном пространстве Минковского.

б) Рассмотреть в двумерном пространстве Минковского модель Тирринга с действием

$$\mathcal{S} = \int \mathrm{d}^{(2)}x \left\{ \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - \frac{g}{2}\left(\bar{\psi}\gamma^\mu\psi\right)\left(\bar{\psi}\gamma_\mu\psi\right) \right\}$$

Здесь g - константа связи. Сформулировать правила диаграммной техники Фейнмана для модели Тирринга. Исследовать модель на предмет перенормируемости.

2. Доказать, что

$$\begin{gathered}(\partial/\partial x^\mu)\,\langle 0|T J^\mu(x) J^{\mu_1}(x_1)\ldots J^{\mu_m}(x_m)\, J^{5\nu_1}(y_1)\ldots J^{5\nu_n}(y_n) \\ (\bar{\psi}\psi)(z_1)\ldots(\bar{\psi}\psi)(z_l)\,|0\rangle = \\ = \langle 0|T\left(\partial_\mu J^\mu(x)\right) J^{\mu_1}(x_1)\ldots J^{\mu_m}(x_m)\, J^{5\nu_1}(y_1)\ldots J^{5\nu_n}(y_n) \\ (\bar{\psi}\psi)(z_1)\ldots(\bar{\psi}\psi)(z_l)\,|0\rangle = 0\end{gathered}$$

15.2 Schwinger–Dyson Equations

(потом сгруппирую в одно место про эти уравнения, скорее всего в одину из первой главы в этом раздеел)

Уравнения Швингера-Дайсона по Киселеву

Уравнения Эйлера-Лагранжа с учетом всех вкладов в лагранжиан, включая член с внешним током, т. е. для суммы лагранжианов $\mathcal{L}(j) = \mathcal{L} + \mathcal{L}_{\text{int}} + \mathcal{L}_{\text{ext}}$

$$\partial_\mu\left(\frac{\partial\mathcal{L}(j)}{\partial\partial_\mu\varphi}\right) = \frac{\partial\mathcal{L}_{(j)}}{\partial\varphi}$$

примут вид

$$\left(\partial^2 + m^2\right)\varphi(x) + V'(\varphi) = j(x)$$

Если поле удовлетворяет уравнениям Эйлера-Лагранжа, то можно ввести амплитуду вероятности наблюдать такое поле

$$Z(j) = e^{\mathrm{i}S + \mathrm{i}S_{\text{ext}}(j)}$$

причем в силу вещественности действия,

$$|Z(j)| = 1$$

так что вероятность наблюдать классическое поле, которое удовлетворяет уравнениям движения при наличии источников, $W = |Z(j)|^2$ равна единице.

Уравнения движения легко переписать в эквивалентном виде как уравнения для амплитуды $Z(j)$, если заметить, что поле выражается через вариационную производную

$$\frac{\delta Z(j)}{\delta j(x)} = \mathrm{i}\varphi(x)Z(j) \quad \leftrightarrow \quad \varphi(z)Z(j) = -\mathrm{i}\frac{\delta Z(j)}{\delta j(x)}$$

Тогда, например, если $S_{\text{int}} = 0$, то

$$\left(\partial^2 + m^2\right)(-\mathrm{i})\frac{\delta Z(j)}{\delta j(x)} = j(x)Z(j) \quad \leftrightarrow \quad \left(\partial^2 + m^2\right)\varphi(x)Z(j) = j(x)Z(j)$$

При наличии степенного самодействия поля уравнения движения в вариационных производных принимают вид

$$-\mathrm{i}\left(\partial^2 + m^2\right)\frac{\delta Z(j)}{\delta j(x)} + V'\left(-\mathrm{i}\frac{\delta}{\delta j(x)}\right)Z = j(x)Z(j)$$

Это - уравнения Швингера-Дайсона.

(применять бы их уметь...)

(запишу позже алгоритм их решений)

При подстановке в (3.40) амплитуды $Z(j)$ в виде $Z(j) = e^{\mathrm{i}S + \mathrm{i}S_{\text{ext}}(j)}$ уравнения эквивалентны уравнениям Эйлера-Лагранжа в классической теории поля с внешним источником, конечно.

Если эту подстановку не делать, то решение уравнений приобретает более общий смысл, который требует определенной физической интерпретации.

Мы установим, что решение уравнений Швингера-Дайсона $Z(j)$ - это амплитуда квантовых переходов при воздействии на систему внешних источников, если провести суммирование $Z(j) = e^{\mathrm{i}S + \mathrm{i}S_{\text{ext}}(j)}$ по всем флуктуациям, которые имеют место для полей в квантовой механике.

Но прежде, чем заняться описанием квантовых флуктуаций, введем еще несколько понятий, которые также обобщаются с классической теории на квантовую.

Именно, запишем

$$Z(j) = e^{\mathrm{i}G(j)}$$

где $G(j)$ называют производящим функционалом связных функций Грина.

Подставляя $Z(j)$ в уравнения Швингера-Дайсона и умножая их на Z^{-1}, получим эквивалентные уравнения для $G(j)$

$$\left(\partial^2 + m^2\right)\frac{\delta G(j)}{\delta j(x)} + e^{-\mathrm{i}G}V'\left(-\mathrm{i}\frac{\delta}{\delta j(x)}\right)e^{\mathrm{i}G} = j(x)$$

В классической теории поля элементарно находим, что

$$G(j) = S + S_{\text{ext}}$$

а это значит, что в классической теории отлична от нуля только первая вариационная производная, для которой в общем случае вводят обозначение

$$\frac{\delta G}{\delta j(x)} = \varphi_{\text{cl}}(x)$$

Тогда в классике уравнение Швингера-Дайсона в форме (3.42) сводится к виду

$$\left(\partial^2 + m^2\right)\varphi_{\text{cl}}(x) + V'\left(\varphi_{\text{cl}}\right) = j(x)$$

что оправдывает такое наименование $\varphi_{\text{cl}}(x)$, поскольку это поле удовлетворяет классическим уравнениям Эйлера-Лагрнажа.

Определение (3.44) позволяет построить преобразование Лежандра с переходом от функциональных переменных внешних источников j к классическим внешним полям $\varphi_{\rm cl}$, так что вводится функционал эффективного действия

$$\Gamma\left(\varphi_{\rm cl}\right) = G(j) - \int \mathrm{d}^4 x j(x)\varphi_{\rm cl}(x)$$

причем, конечно,

$$\frac{\delta\Gamma\left(\varphi_{\rm cl}\right)}{\delta\varphi_{\rm cl}(x)} = -j(x)$$

В классической теории поля с подстановкой (3.43) эффективное действие сводится к исходному действию на внешних классических полях

$$\Gamma\left(\varphi_{\rm cl}\right) \mapsto S\left(\varphi_{\rm cl}\right)$$

Отметим, что вместо внешних источников мы теперь можем оперировать внешними классическими полями. Эти поля в асимптотической области становятся свободными.

Эффективное действие $\Gamma\left(\varphi_{\rm cl}\right)$ называют также производящим функционалом сильносвязных функций Грина: когда речь идет о производящем функционале имеют ввиду разложение этого функционала в вариационный ряд Тейлора, причем, вариационные коэффициенты в этом разложении и являются функциями Грина,

$$\Gamma\left(\varphi_{\rm cl}\right) = \Gamma_0 + \int \mathrm{d}^4 x_1 \Gamma_1\left(x_1\right)\varphi_{\rm cl}\left(x_1\right) + \frac{1}{2!}\varphi_{\rm cl}^{(1)}\Gamma_{12}\varphi_{\rm cl}^{(2)} + \ldots$$

где, напомним, повторяющиеся номера отвечают интегрированию по соответствующим наборам координат.

Для физической интерпретации введенных понятий нам потребуется сформулировать их определения в терминах операторов и квантов поля - частиц. Однако для начала в качестве наводящих соображений проведем неформальные рассуждения, опираясь на концепцию флуктуаций наблюдаемых величин при их квантовом описании. При таком подходе амплитуду (3.39) следует просуммировать по квантовым флуктуациям поля, а значит, определить суперпозицию амплитуд с различными конфигурациями поля

$$Z(j) = \sum_{\varphi_{\rm conf}} \mathcal{A}_{\rm conf}\, e^{\mathrm{i}S_{\rm conf} + \mathrm{i}S_{\rm ext}\,(j)} = \sum_{\varphi_{\rm conf}} Z_{\rm conf}\,(j)$$

где индекс conf в явном виде указывает номер конфигурации, от которого зависят и действие, и относительная амплитуда $\mathcal{A}$.

Сама сумма по континууму полевых конфигураций будет нами определена позже в терминах когерентных состояний поля.

Важно, что эта сумма существует и конечна: это утверждение эквивалентно выделению класса физически осмысленных теорий поля среди всех возможных.

Следовательно, амплитуду для физической теории поля всегда можно нормировать, так что мы сейчас не будем уделять пристальное внимание нормировке.

Каждая амплитуда под знаком суммы в (3.47) должна описывать поле, удовлетворяющее уравнениям Эйлера-Лагрнажа, как это мы уже установили выше.

Но это означает что для каждого вклада в $Z(j)$вариационная производная даст фактор поля, специфического для каждой конфигурации, так что резонно определить оператор поля в следующем виде.

$$\hat{\varphi}(x)Z(j) = -\mathrm{i}\frac{\delta Z(j)}{\delta j(x)} = \sum_{\varphi_{\rm conf}} -\mathrm{i}\frac{\delta Z_{\rm conf}\,(j)}{\delta j(x)} = \sum_{\varphi_{\rm conf}} \varphi_{\rm conf}\,(x) Z_{\rm conf}\,(j)$$

Поскольку все эти построения линейны по $Z(j)$ уравнения Швингера-

функционала связных функций Грина $G(j)$ остаются в силе! Но теперь эти уравнения становятся сутью задачи квантовой теории поля: их нужно решить, чтобы определить сумму по полям с квантовыми флуктуациями в (3.47).

В частности, теперь, за исключением простейших случаев без взаимодействий, не удастся утверждать, что эффективное действие совпадает с исходным действием, а также, что высшие вариационные производные $G(j)$ равны нулю и т.

п.

Амплитуду $Z(j)$ можно записать после применения преобразования Лежандра в терминах внешних классических полей

$$Z = \exp\left\{\mathrm{i}\Gamma\left(\varphi_{\mathrm{cl}}\right) - \mathrm{i}\int \mathrm{d}^4x\varphi_{\mathrm{cl}}(x)\frac{\delta\Gamma}{\delta\varphi_{\mathrm{cl}}(x)}\right\}$$

а это значит, что в ней присутствует элементарный вклад в отсутствие всяких внешних полей при $\varphi_{\mathrm{cl}} = 0$, а также вклады, каждый из которых характеризуется строго фиксированным числом этих полей, если мы применяем разложение в вариационный ряд Тейлора.

Если мы теперь в качестве граничьх условий для квантовых полей установим вакуум, то все квантовые амплитуды будут соответствовать точно известным условиям в терминах внешних полей, которые порождают из вакуума квантовые поля в начальном состоянии и уничтожают квантовые поля, чтобы конечное квантовое состояние также было вакуумом.

В частности, при $\varphi_{\mathrm{cl}} = 0$ происходит эволюция вакуума в вакуум без какого-либо внешнего воздействия, а значит, при условии стабильности вакуума, которое выделяет физически осмысленные теории поля, мы должны получать амплитуду в виде фазового фактора

$$Z(0) = e^{\mathrm{i}\Gamma(0)} = e^{\mathrm{i}G(0)}$$

Поэтому часто рассматривают величину со шляпкой

$$\hat{Z}(j) = \frac{Z(j)}{Z(0)}$$

чтобы разложение в ряд по источникам или внешним полям начиналось с единицы.

Отметим также, что Z в (3.49) включает в себя, как мы увидим, вклад разного графического типа: связные и не связные диаграммы, а также сильносвязные, которые мы определим ниже, но выражение в терминах внешнего поля позволит нам при операторной интерпретации определить связь между этими типами в удобной форме, которая носит название редукционных соотношений.

Затем мы установим связь амплитуды $Z(j)$ и ее вариационных производных с Т-произведением операторов.

И в качестве вводной части повторим, как был построен матричный элемент оператора эволюции в квантовой механике нерелятивистских частиц с заданными граничными точками траектории, а затем, рассмотрим вакуумные переходы в голоморфном представлении, т.е.

в терминах когерентных состоняний осциллятора, что допускает прямое обощение для когерентных состояний свободного поля.

Это позволит нам как установить общие свойства и интерпретацию в терминах квантов поля для функций Грина, так и сформулировать фейнмановские правила для вычисления амплитуд квантовых переходов,

используя диаграммы.

мотивация

Theory

Как мы помним, классические уравнения движения полей в квантовом случае за счет квантовых флуктуаций переходят в вариационные уравнения для амплитуды Z, причем в эти уравнения можно включить и внешние источники.

Например, уравнения для электромагнитного поля уравнения $\mathcal{A}_\mu$ с внешними источниками j^μ и источниками за счет фермионного дираковского поля ψ имеют вид

$$\partial_\nu F^{\nu\mu} + \frac{1}{\xi}\partial^\mu\partial_\nu\mathcal{A}^\nu = -j^\mu + e\bar{\psi}\gamma^\mu\psi, \quad F^{\nu\mu} = \partial^\nu\mathcal{A}^\mu - \partial^\mu\mathcal{A}^\nu$$

Согласно общей процедуре, это уравнение необходимо домножить на $Z(j,\eta,\bar{\eta})$,

а поля необходимо заменить на вариационные операторы

$$\hat{\mathcal{A}}^\mu = -\mathrm{i}\frac{\delta}{\delta j_\mu}, \quad \hat{\bar{\psi}} = +\mathrm{i}\frac{\delta}{\delta\eta}, \quad \hat{\psi} = -\mathrm{i}\frac{\delta}{\delta\bar{\eta}}$$

Тогда с учетом стандартного обозначения для дифференциальных операторов $\hat{k}_\mu = \mathrm{i}\partial_\mu$ получим

$$\hat{k}^2\left(-g^{\mu\nu} + \frac{\hat{k}^\mu\hat{k}^\nu}{\hat{k}^2}\left(1-\frac{1}{\xi}\right)\right)\hat{\mathcal{A}}_\nu Z = -j^\mu Z + e\hat{\bar{\psi}}\gamma^\mu\hat{\psi}Z$$

Дифференциальный оператор нам в левой части нам хорошо известен, поскольку пропагатор фотона связан с обратным к нему элементарной формулой

$$\mathcal{D}^{(2)}_{\mu\nu} = \mathrm{i}\left\{\hat{k}^2\left(-g^{\mu\nu} + \frac{\hat{k}^\mu\hat{k}^\nu}{\hat{k}^2}\left(1-\frac{1}{\xi}\right)\right)\right\}^{-1} = \frac{\mathrm{i}}{k^2+\mathrm{i}0}\left(-g^{\mu\nu} + \frac{\hat{k}^\mu\hat{k}^\nu}{\hat{k}^2}(1-\xi)\right)$$

так что умножение операторного уравнения движения на $-\mathrm{i}\mathcal{D}^{(2)}$ дает опера- торное тождество

$$\hat{\mathcal{A}}_\nu Z = \mathrm{i}\mathcal{D}^{(2)}_{\mu\nu}j^\mu Z - \mathrm{i}e\mathcal{D}^{(2)}_{\mu\nu}\hat{\bar{\psi}}\gamma^\mu\hat{\psi}Z$$

которое представляет собой вариационное уравнение Швингера-Дайсона, которое можно бесконечное число раз вариационно дифференцировать по внешним источникам. К примеру, умножение на вариационный оператор $\hat{\mathcal{A}}_\mu = -\mathrm{i}\delta/\delta j^\mu$ и условие $j=\eta=\bar{\eta}=0$ дают

$$\hat{\mathcal{A}}_\mu\hat{\mathcal{A}}_\nu Z\Big|_{j=\eta=\bar{\eta}=0} = \mathcal{D}^{(2)}_{\mu\nu} - \mathrm{i}e\mathcal{D}^{(2)}_{\mu'\nu}\hat{\bar{\psi}}\gamma^{\mu'}\hat{\psi}\hat{\mathcal{A}}_\mu Z\Big|_{j=\eta=\bar{\eta}=0^*}$$

В этом выражении мы замечаем пропагатор фотона в теории возмущений)норным током - ie γ^μ, а также полный причинный двухточечный корелятор фотона

$$\mathcal{D}_{\mu\nu}(x,y) = \hat{\mathcal{A}}_\mu(x)\hat{\mathcal{A}}_\nu(y)Z\Big|_{j=\eta=\bar{\eta}=0} = (-\mathrm{i})^2\frac{\delta^2 Z}{\delta j^\mu(x)\delta j^\nu(y)}\Big|_{j=\eta=\bar{\eta}=0}$$

и полную трехточечную функцию взаимодействия фотона с фермионами

$$(\Gamma_\mu)^a_b(x,x',y) = (-\mathrm{i})^3\frac{\delta^3 Z}{\delta\eta^b(x)\delta j^\mu(y)\delta\bar{\eta}_a(x')}\Big|_{j=\eta=\bar{\eta}=0}$$

где мы указали координатные и спинорные аргументы. Тогда уравнение (10.7) приобретает вид

$$\mathcal{D}_{\mu\nu}(x,y) = \mathcal{D}^{(2)}_{\mu\nu}(x,y) - \mathrm{i}e\left(\gamma^{\mu'}\right)^{b}_{a}\int \mathrm{d}^4x'\mathcal{D}^{(2)}_{\mu'\nu}\left(x,x'\right)\left(\Gamma_{\mu}\right)^{a}_{b}\left(x',x',y\right)$$

Сущность этого уравнения легко понять: полная причинная двухточечная функция Грина фотона представляет собой сумму пропагатора Фейнмана и вклада за счет квантовых поправок с учетом «голой» и полной трехточечной функции Грина взаимодействия фотона с фермионами. При этом, согласно принятых нами правил, распространение частиц изображается следующим обраЗом:

(рисунок)

Петлевой вклад в пропагатор фотона после перехода к квадратичному по полю вкладу в эффективном действии называют поляризацией вакуума $\Pi\left(k^2\right)$. Как видим, поляризация вакуума отвечает, вообще говоря, изменению закона дисперсии фотона: $k^2 \mapsto k^2 + \Pi\left(k^2\right)$, - вне массовой поверхности, конечно.

Действуя совершенно аналогично с уравнением для поля Дирака с подстановкой вариационных операторов поля

$$\left(\gamma^{\mu}\mathrm{i}\partial_{\mu} - m\right)\hat{\psi}Z - e\gamma^{\mu}\hat{\mathcal{A}}_{\mu}\hat{\psi}Z + \eta Z = 0$$

после действия обратным оператором, связанным с «голым" пропагатором поля Дирака

$$\mathcal{S}^{(2)} = \mathrm{i}\left(\gamma^{\mu}\mathrm{i}\partial_{\mu} - m\right)^{-1}$$

найдем уравнение Швингера-Дайсона в квантовой электродинамике для поля Дирака

$$\hat{\psi}Z = \mathrm{i}\mathcal{S}^{(2)}\eta Z - \mathrm{i}e\mathcal{S}^{(2)}\gamma^{\mu}\hat{\mathcal{A}}_{\mu}\hat{\psi}Z$$

Умножение на оператор сопряженного поля Дирака $\hat{\bar{\psi}} = \mathrm{i}\delta/\delta\eta$ после наложения условия нулевых значений внешних полей даст уравнение на полный пропагатор поля Дирака

$$\mathcal{S}(x,y) = \mathcal{S}^{(2)}(x,y) - \mathrm{i}e\int \mathrm{d}^4x'\mathcal{S}^{(2)}\left(x,x'\right)\gamma^{\mu}\Gamma_{\mu}\left(x',y,x'\right)$$

где мы подразумеваем тамричное умножение величин со спинорными индексами. Очевидно, чтоэто уравнение графически представляется в следующем виде:

(рисунок)

Опять же, как и в случае фотона, квантовые поправки возникают за счет петли с голой и полной вершиной взаимодействия поля Дирака с фотонами. Подчеркнем, что полную вершину можно построить, например, в теории возмущений. Петлевой вклад в эффективное действие, квадратичное по полю Дирака, называют собственной энергией, например, электрона Σ: $\gamma^{\mu}p_{\mu} - m \mapsto \gamma^{\mu}p_{\mu} - m - \Sigma(p)$, - причем, собственная энергия - это матрица.

примеры использования

15.3 Bethe–Salpeter Equation

мотивация

Theory

Родственником уравнений Швингера-Дайсона является уравнение БетеСолпитера, которое описывает «4-точку» для фермионного поля. Физически «4-точка» - это амплитуда

рассеяния, скажем, электрона на электроне или электрона на позитроне в зависимости от исследуемой задачи. Уравнение

Бете-Солпитера, конечно, можно вывести из общего уравнения ШвингераДайсона с источниками, если в нем провести достаточное количество вариационного дифференцирования по внешним источникам. В самом деле, указывая аргументы в виде численных индексов, из уравнения $\sqrt{10.9}$)

$$\hat{\psi}_1 Z = \mathrm{i}\mathcal{S}^{(2)}_{11'}\eta_{1'}Z - \mathrm{i}e\mathcal{S}^{(2)}_{11'}\gamma^{\mu}_{1'}\hat{\mathcal{A}}^{1'}_{\mu}\hat{\psi}_{1'}Z$$

умножение на вариационные операторы поля с учетом смены знака при перестановке $\psi_1\hat{\bar{\psi}}_2$, с которой мы уже встречались при выводе уравнения Швингера-Дайсона для поля Дирака, дает

$$\hat{\psi}_3\hat{\bar{\psi}}_4\hat{\psi}_1\hat{\bar{\psi}}_2 Z = -\mathrm{i}\mathcal{S}^{(2)}_{11'}\hat{\psi}_3\hat{\bar{\psi}}_4\hat{\bar{\psi}}_2\eta_{1'}Z - \mathrm{i}e\mathcal{S}^{(2)}_{11'}\gamma^{\mu}_{1'}\hat{\mathcal{A}}^{1'}_{\mu}\hat{\psi}_3\hat{\bar{\psi}}_4\hat{\psi}_{1'}\hat{\bar{\psi}}_2 Z$$

Если теперь занулить источники $j = \eta = \bar{\eta} = 0$, то в левой части этого равенства окажется 4-точечная функция Грина для рассеяния полей в процессе $13 \mapsto 24$, т. е. заявленный нами объект исследования в уравнении Бете - Солпитера.

В правой части равенства есть вариационное дифференцирование источника $\eta_{1'}$. Тогда, если это дифференцирование производится за счет $\hat{\bar{\psi}}_2 = \mathrm{i}\delta/\delta\eta_2$, то возникает, во-первых, $\delta_{1'2}$, которая снимает интегрирование по $\mathrm{d}^4x_{1'}$, а вовторых, другие операторы поля действуют на Z, что дает полный пропагатор фермиона с учетом квантовых поправок. Если же дифференцирование $\eta_{1'}$ производится за счет $\hat{\bar{\psi}}_2 = \mathrm{i}\delta/\delta\eta_4$, то возникает дополнительный минус за счет перестановки грассмановых переменных, а также аналогичный вклад от полного пропагатора фермиона с другими конечными точками. В итоге, эти вклады группируются в виде

$$\mathcal{S}^{(2)}_{12}\mathcal{S}_{34} - \mathcal{S}^{(2)}_{14}\mathcal{S}_{32}$$

и возникает диаграмма на рис. 9, где мы включили также и вклад 5-точечной функции Грина.

Однако 5 -точечная функция, очевидно, содержит и несвязные вклады, которые в точности совпадают с петлевыми вкладами в полный пропагатор дираковского поля, как это становится ясно из графического представления на рис. 10

Это означает, что в действительности несвязные вклады - линии невзаимодействующих фермионов - на рис. 9 уже отвечают полным пропагаторам в силу уравнения Швингера-Дайсона для двухточечного фермионного коррелятора:

$$\mathcal{S}^{(2)}_{12}\mathcal{S}_{34} - \mathcal{S}^{(2)}_{14}\mathcal{S}_{32} \mapsto \mathcal{S}_{12}\mathcal{S}_{34} - \mathcal{S}_{14}\mathcal{S}_{32}$$

На основе анализа теории возмущений Бете и Солпитер пришли к выводу, что оставшаяся связная часть 5-точечной функции представима в виде свертки 4-точечного ядра V с исследуемой 4-точечной функцией Грина. Ядро на рис. 10 обозначено прямоугольником. Утверждение о ядре в некотором смысле элементарно: с одной стороны, связные диаграммы с двумя входящими внешними и двумя исходящими внешними линиями всегда можно отнести к свертке ядра со свободным вкладом от 4-точки, а с другой стороны, повторение таких диаграмм, соединенных двумя внутренними линиями, как раз, отвечает вкладу свертки ядра уравнения Бете- Солпиитера с 4-точкой. В итоге, уравнение Бете-Солпитера приобретает симовлический вид

$$\mathcal{S}_{13\to 24} = \mathcal{S}_{12}\mathcal{S}_{34} - \mathcal{S}_{14}\mathcal{S}_{32} + V_{13\to 2'4'}\mathcal{S}_{2'4'\to 24}$$

Неприятности с определением ядра можно обойти в теории возмущений или задавая ядро моделью, имеющую под собой какие-либо основания физического или математического рода.

примеры использования

15.4 Quartic Interaction or $\lambda\phi^4$ theory

(?? I'll study it for 4 days at least later)

15.4.1 Main Constructions

https://www.imperial.ac.uk/media/imperial-college/research-centres-and-groups/theoretical-physics/msc/current/qft/problem-sheets-and-solutions/qftProblemSheet7Answers.pdf тоже вроде хорошее саммари.

15.4.2 Overview of ϕ^4 theory

(??? answer, why is it important??? why there are such big books about it???)

15.4.3 Other Constructions

15.5 Other about Perturbation Theory in QFT

(all other comments, questions are here)

Сравнение с теорией возмущений в квантовой механике

(напишу, такой вопрос может возникнуть.)

15.6 Exact Propagators and Vertex Parts by Landau (?)

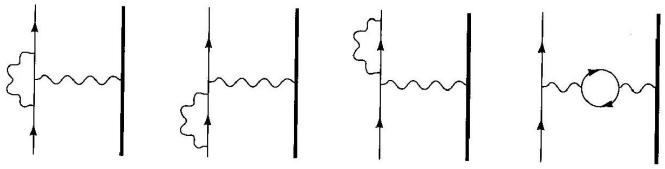
(пока скорее всего это в предыдущую часть перенесу, пока тут зависла, все равно не так много это читал еще.)

102.Операторы полей в гейзенберговском представлении
103.Точный фотонный пропагатор 511
104.Собственно-энергетическая функция фотона518
105.Точный электронный пропагатор522
106.Вершинный оператор 526
107.Уравнения Дайсона 530
108.Тождество Уорда533
109.Электронный пропагатор во внешнем поле536
110.Физические условия перенормировки 543
111.Аналитические свойства фотонного пропагатора 549
112.Регуляризация интегралов Фейнмана553

16 Radiative Corrections: Introduction

Now that we have acquired some experience at performing QED calculations, let us move on to some more complicated problems. Chapter 5 dealt only with tree-level processes, that is, with diagrams that contain no loops. But all such processes receive higher-order contributions, known as radiative corrections, from diagrams that do contain loops. Another source of radiative corrections in QED is bremsstrahlung, the emission of extra final-state photons during a reaction. In this chapter we will investigate both types of radiative corrections, and find that it is inconsistent to include one without also including the other.

Throughout this chapter, in order to illustrate these ideas in the simplest possible context, we will consider the process of electron scattering from another, very heavy, particle. We analyzed this process at tree level in Section 5.4 and Problem 5.1. At the next order in perturbation theory, we encounter the following four diagrams:



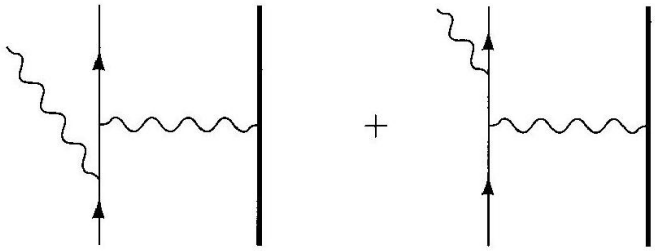
The order- α correction to the cross section comes from the interference term between these diagrams and the tree-level diagram. There are six additional one-loop diagrams involving the heavy particle in the loop, but they can be neglected in the limit where that particle is much heavier than the electron, since the mass appears in the denominator of the propagator. (Physically, the heavy particle accelerates less, and therefore radiates less, during the collision.)

Of the four diagrams in (6.1), the first (known as the vertex correction) is the most intricate and gives the largest variety of new effects. For example, it gives rise to an anomalous magnetic moment for the electron, which we will compute in Section 6.3.

The next two diagrams of (6.1) are external leg corrections. We will neglect them in this chapter because they are not amputated, as required by our formula (4.90) for S-matrix elements. We will discuss these diagrams in more
detail when we prove that formula in Section 7.2.
The final diagram of (6.1) is called the vacuum polarization. Since it requires more computational machinery than the others, we will not evaluate this diagram until Section 7.5.

Our study of these corrections will be complicated by the fact that they are ill-defined. Each diagram of (6.1) involves an integration over the undetermined loop momentum, and in each case the integral is divergent in the $k \to \infty$ or ultraviolet region. Fortunately, the infinite parts of these integrals will always cancel out of expressions for observable quantities such as cross sections.

The first three diagrams of (6.1) also contain infrared divergences: infinities coming from the $k \to 0$ end of the loop-momentum integrals. We will see in Section 6.4 that these divergences are canceled when we also include the following bremsstrahlung diagrams:



These diagrams are divergent in the limit where the energy of the radiated photon tends to zero.

In this limit, the photon cannot be observed by any physical detector, so it makes sense to add the cross section for producing these low-energy photons to the cross section for scattering without radiation. The bremsstrahlung diagrams are thus an essential part of the radiative correction, in this and any other QED process.

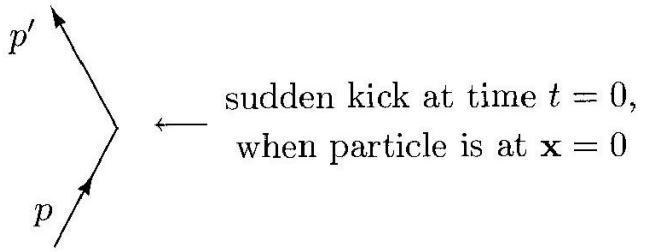
Our main goals in the present chapter are to understand bremsstrahlung of low-energy photons, the vertex correction diagram, and the cancellation of infrared divergences between these two types of radiative corrections.

16.1 6.1 Soft Bremsstrahlung

Let us begin our study of radiative corrections by analyzing the bremsstrahlung process. In this section we will first do a classical computation of the intensity of the low-frequency bremsstrahlung radiation when an electron undergoes a sudden acceleration. We will then compute a closely related quantity in quantum field theory: the cross section for emission of one very soft photon, given by diagrams (6.2). We would like to understand how the classical result arises as a limiting case of the quantum result.

Classical Computation

Suppose that a classical electron receives a sudden kick at time $t = 0$ and position $\mathbf{x} = 0$, causing its 4 -momentum to change from p to p'. (An infinitely sudden change of momentum is of course an unrealistic idealization. The precise form of the trajectory during the acceleration does not affect the low-frequency radiation, however. Our calculation will be valid for radiation with a frequency less than the reciprocal of the scattering time.)



We can find the radiation field by writing down the current of this electron, and considering that current as a source for Maxwell's equations.

What is the current density of such a particle? For a charged particle at rest at $\mathbf{x} = 0$, the current would be

$$\begin{aligned} j^\mu(x) &= (1,\mathbf{0})^\mu \cdot e\delta^{(3)}(\mathbf{x}) \\ &= \int dt (1,\mathbf{0})^\mu \cdot e\delta^{(4)}(x - y(t)), \qquad \text{with } y^\mu(t) = (t,\mathbf{0})^\mu \end{aligned}$$

From this we can guess the current for an arbitrary trajectory $y^\mu(\tau)$:

$$j^\mu(x) = e\int d\tau \frac{dy^\mu(\tau)}{d\tau}\delta^{(4)}(x - y(\tau)) \tag{6.3}$$

Note that this expression is independent of the precise way in which the curve $y^\mu(\tau)$ is parametrized: Changing variables from τ to $\sigma(\tau)$ gives a factor of $d\tau/d\sigma$ in the integration measure, which combines with $dy^\mu/d\tau$ via the chain rule to give $dy^\mu/d\sigma$. We can also prove from

(6.3) that the current is automatically conserved: For any "test function" $f(x)$ that falls off at infinity, we have

$$\begin{aligned}\int d^4x f(x)\partial_\mu j^\mu(x) &= \int d^4x f(x) e \int d\tau \frac{dy^\mu(\tau)}{d\tau}\partial_\mu \delta^{(4)}(x-y(\tau)) \\ &= -e\int d\tau \frac{dy^\mu(\tau)}{d\tau}\frac{\partial}{\partial x^\mu} f(x)\Big|_{x=y(\tau)} \\ &= -e\int d\tau \frac{d}{d\tau} f(y(\tau)) \\ &= -ef(y(\tau))|_{-\infty}^{\infty} = 0\end{aligned}$$

For our process the trajectory is

$$y^\mu(\tau) = \begin{cases} (p^\mu/m)\,\tau & \text{for } \tau < 0 \\ (p'^\mu/m)\,\tau & \text{for } \tau > 0 \end{cases}$$

Thus the current can be written

$$j^\mu(x) = e\int_0^\infty d\tau \frac{p'^\mu}{m}\delta^{(4)}\left(x - \frac{p'}{m}\tau\right) + e\int_{-\infty}^0 d\tau \frac{p^\mu}{m}\delta^{(4)}\left(x-\frac{p}{m}\tau\right)$$

In a moment we will need to know the Fourier transform of this function. Inserting factors of $e^{-\epsilon\tau}$ and $e^{\epsilon\tau}$ to make the integrals converge, we have

$$\begin{aligned}\tilde{j}^\mu(k) &= \int d^4x e^{ik\cdot x} j^\mu(x) \\ &= e\int_0^\infty d\tau \frac{p^\mu}{m} e^{i(kp'/m + i\epsilon)\tau} + e\int_{-\infty}^0 d\tau \frac{p^\mu}{m} e^{i(kp/m - i\epsilon)\tau} \\ &= ie\left(\frac{p'^\mu}{k\cdot p' + i\epsilon} - \frac{p^\mu}{k\cdot p - i\epsilon}\right)\end{aligned} \tag{6.4}$$

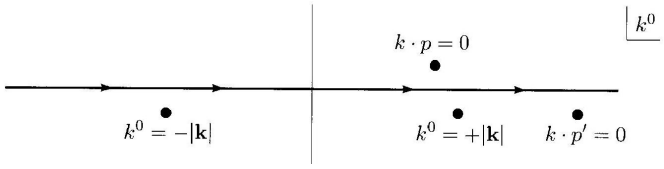
We are now ready to solve Maxwell's equations. In Lorentz gauge ($\partial^\mu A_\mu = 0$) we must solve $\partial^2 A^\mu = j^\mu$, or in Fourier space,

$$\widetilde{A}^\mu(k) = -\frac{1}{k^2}\tilde{j}^\mu(k).$$

Plugging in (6.4), we obtain a formula for the vector potential:

$$A^\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\frac{-ie}{k^2}\left(\frac{p'^\mu}{k\cdot p' + i\epsilon} - \frac{p^\mu}{k\cdot p - i\epsilon}\right). \tag{6.5}$$

The k^0 integral can be performed as a contour integral in the complex plane. The locations of the poles are as follows:



We place the poles at $k^0 = \pm|\mathbf{k}|$ below the real axis so that (as we shall soon confirm) the radiation field will satisfy retarded boundary conditions.

For $t < 0$ we close the contour upward, picking up the pole at $k \cdot p = 0$, that is, $k^0 = \mathbf{k} \cdot \mathbf{p}/p^0$. The result is

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i(\mathbf{k}\cdot\mathbf{p}/p^0)t} \frac{(2\pi i)(+ie)}{(2\pi)k^2} \frac{p^\mu}{p^0}.$$

In the reference frame where the particle is initially at rest, its momentum vector is $p^\mu = (p^0, \mathbf{0})$ and the vector potential reduces to

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \frac{e}{|\mathbf{k}|^2} \cdot (1, \mathbf{0})$$

This is just the Coulomb potential of an unaccelerated charge. As we would expect, there is no radiation field before the particle is scattered.

After scattering ($t > 0$), we close the contour downward, picking up the three poles below the real axis. The pole at $k^0 = \mathbf{k} \cdot \mathbf{p}'/p'^0$ gives the Coulomb potential of the outgoing particle. Thus the other two poles are completely responsible for the radiation field. Their contribution gives

$$\begin{aligned} A^\mu_{\text{rad}}(x) &= \int \frac{d^3k}{(2\pi)^3} \frac{-e}{2|\mathbf{k}|} \left\{ e^{-ik\cdot x} \left(\frac{p'^\mu}{k \cdot p'} - \frac{p^\mu}{k \cdot p} \right) + \text{ c.c. } \right\} \Bigg|_{k^0 = |\mathbf{k}|} \\ &= \text{Re} \int \frac{d^3k}{(2\pi)^3} \mathcal{A}^\mu(\mathbf{k}) e^{-ik\cdot x} \end{aligned} \tag{6.6}$$

where the momentum-space amplitude $\mathcal{A}(\mathbf{k})$ is given by

$$\mathcal{A}^\mu(\mathbf{k}) = \frac{-e}{|\mathbf{k}|} \left(\frac{p'^\mu}{k \cdot p'} - \frac{p^\mu}{k \cdot p} \right) \tag{6.7}$$

(The condition $k^0 = |\mathbf{k}|$ is implicit here and in the rest of this calculation.)
To calculate the energy radiated, we must find the electric and magnetic fields. It is easiest to write $\mathbf{E}$ and $\mathbf{B}$ as the real parts of complex Fourier integrals, just as we did for A^μ:

$$\begin{aligned} \mathbf{E}(x) &= \text{Re} \int \frac{d^3k}{(2\pi)^3} \mathcal{E}(\mathbf{k}) e^{-ik\cdot x} \\ \mathbf{B}(x) &= \text{Re} \int \frac{d^3k}{(2\pi)^3} \boldsymbol{\mathcal{B}}(\mathbf{k}) e^{-ik\cdot x} \end{aligned} \tag{6.8}$$

The momentum-space amplitudes $\mathcal{E}(\mathbf{k})$ and $\mathcal{B}(\mathbf{k})$ of the radiation fields are then simply

$$\begin{aligned} \mathcal{E}(\mathbf{k}) &= -i\mathbf{k}\mathcal{A}^0(\mathbf{k}) + ik^0 \mathcal{A}(\mathbf{k}) \\ \mathcal{B}(\mathbf{k}) &= i\mathbf{k} \times \mathcal{A}(\mathbf{k}) = \hat{k} \times \mathcal{E}(\mathbf{k}) \end{aligned} \tag{6.9}$$

Using the explicit form (6.7) of $\mathcal{A}^\mu(\mathbf{k})$, you can easily check that the electric field is transverse: $\mathbf{k} \cdot \mathcal{E}(\mathbf{k}) = 0$.

Having expressed the fields in this way, we can compute the energy radiated:

$$\text{Energy } = \frac{1}{2}\int d^3x \left(|\mathbf{E}(x)|^2 + |\mathbf{B}(x)|^2\right) \tag{6.10}$$

The first term is

$$\begin{aligned}&\frac{1}{8}\int d^3x \int \frac{d^3k}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3}\left(\mathcal{E}(\mathbf{k})e^{-ikx} + \mathcal{E}^*(\mathbf{k})e^{ikx}\right)\cdot\left(\mathcal{E}\left(\mathbf{k}'\right)e^{-ik'x} + \mathcal{E}^*\left(\mathbf{k}'\right)e^{ik'x}\right)\\ &\quad = \frac{1}{8}\int \frac{d^3k}{(2\pi)^3}\left(\mathcal{E}(\mathbf{k})\cdot\mathcal{E}(-\mathbf{k})e^{-2ik^0t} + 2\mathcal{E}(\mathbf{k})\cdot\mathcal{E}^*(\mathbf{k}) + \mathcal{E}^*(\mathbf{k})\cdot\mathcal{E}^*(-\mathbf{k})e^{2ik^0t}\right)\end{aligned}$$

A similar expression involving $\boldsymbol{\mathcal{B}}(\mathbf{k})$ holds for the second term. Using (6.9) and the fact that $\mathcal{E}(\mathbf{k})$ is transverse, you can show that the time-dependent terms cancel between $\mathcal{E}$ and $\mathcal{B}$, while the remaining terms add to give

$$\text{Energy } = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\mathcal{E}(\mathbf{k})\cdot\mathcal{E}^*(\mathbf{k}) \tag{6.11}$$

Since $\mathcal{E}(\mathbf{k})$ is transverse, let us introduce two transverse unit polarization vectors $\boldsymbol{\epsilon}_\lambda(\mathbf{k}), \lambda = 1, 2$. We can then write the integrand as

$$\mathcal{E}(\mathbf{k})\cdot\mathcal{E}^*(\mathbf{k}) = \sum_{\lambda=1,2}|\boldsymbol{\epsilon}_\lambda(\mathbf{k})\cdot\mathcal{E}(\mathbf{k})|^2 = |\mathbf{k}|^2\sum_{\lambda=1,2}|\epsilon_\lambda(\mathbf{k})\cdot\mathcal{A}(\mathbf{k})|^2$$

Using the explicit form of $\boldsymbol{\mathcal{A}}(\mathbf{k})$ (6.7), we finally arrive at an expression for the energy radiated*:

$$\text{Energy } = \int \frac{d^3k}{(2\pi)^3}\sum_{\lambda=1,2}\frac{e^2}{2}\left|\boldsymbol{\epsilon}_\lambda(\mathbf{k})\cdot\left(\frac{\mathbf{p}'}{k\cdot p'} - \frac{\mathbf{p}}{k\cdot p}\right)\right|^2 \tag{6.12}$$

We can freely change $\boldsymbol{\epsilon}, \mathbf{p}'$, and $\mathbf{p}$ into 4 -vectors in this expression. Then, noting that substituting k^μ for ϵ^μ would give zero,

$$k_\mu\left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p}\right) = 0$$

we find that we can perform the sum over polarizations using the trick of Section 5.5, replacing $\sum\epsilon_\mu\epsilon_\nu^*$ by $-g_{\mu\nu}$. Our result then becomes

$$\begin{aligned}\text{Energy } &= \int \frac{d^3k}{(2\pi)^3}\frac{e^2}{2}\left(-g_{\mu\nu}\right)\left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p}\right)\left(\frac{p'^\nu}{k\cdot p'} - \frac{p^\nu}{k\cdot p}\right)\\ &= \int \frac{d^3k}{(2\pi)^3}\frac{e^2}{2}\left(\frac{2p\cdot p'}{(k\cdot p')\,(k\cdot p)} - \frac{m^2}{(k\cdot p')^2} - \frac{m^2}{(k\cdot p)^2}\right)\end{aligned} \tag{6.13}$$

To make this formula more explicit, choose a frame in which $p^0 = p'^0 = E$. Then the momenta are

$$k^\mu = (k, \mathbf{k}), \quad p^\mu = E(1, \mathbf{v}), \quad p'^\mu = E\,(1, \mathbf{v}')$$

*This result is also derived in Jackson (1975), p. 703.

In such a frame our formula becomes

$$\text{Energy } = \frac{e^2}{(2\pi)^2}\int dk\mathcal{I}\left(\mathbf{v},\mathbf{v}'\right) \tag{6.14}$$

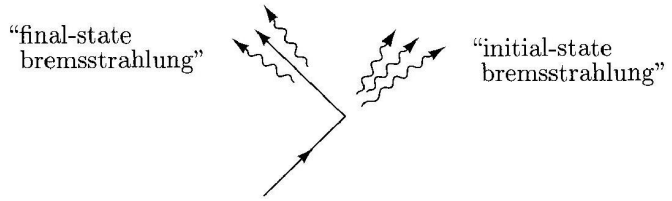
where $\mathcal{I}\left(\mathbf{v},\mathbf{v}'\right)$ (which is essentially the differential intensity d (Energy) $/dk$) is given by

$$\mathcal{I}\left(\mathbf{v},\mathbf{v}'\right)=\int\frac{d\Omega_{\hat{k}}}{4\pi}\left(\frac{2\left(1-\mathbf{v}\cdot\mathbf{v}'\right)}{\left(1-\hat{k}\cdot\mathbf{v}\right)\left(1-\hat{k}\cdot\mathbf{v}'\right)}-\frac{m^2/E^2}{\left(1-\hat{k}\cdot\mathbf{v}'\right)^2}-\frac{m^2/E^2}{(1-\hat{k}\cdot\mathbf{v})^2}\right) \tag{6.15}$$

Since $\mathcal{I}\left(\mathbf{v},\mathbf{v}'\right)$ does not depend on k, we see that the integral over k in (6.14) is trivial but divergent. This divergence comes from our idealization of an infinitely sudden change in momentum. We expect our formula to be valid only for radiation whose frequency is less than the reciprocal of the scattering time. For a relativistic electron, another possible cutoff would take effect when individual photons carry away a sizable fraction of the electron's energy. In either case our formula is valid in the low-frequency limit, provided that we cut off the integral at some maximum frequency $k_{\max}$. We then have

$$\text{Energy } = \frac{\alpha}{\pi}\cdot k_{\max}\cdot\mathcal{I}\left(\mathbf{v},\mathbf{v}'\right) \tag{6.16}$$

The integrand of $\mathcal{I}\left(\mathbf{v},\mathbf{v}'\right)$ peaks when $\hat{k}$ is parallel to either $\mathbf{v}$ or $\mathbf{v}'$:



In the extreme relativistic limit, most of the radiated energy comes from the two peaks in the first term of (6.15). Let us evaluate $\mathcal{I}\left(\mathbf{v},\mathbf{v}'\right)$ in this limit, by concentrating on the regions around these peaks. Break up the integral into a piece for each peak, and let $\theta=0$ along the peak in each case. Integrate over a small region around $\theta=0$, as follows:

$$\begin{aligned}\mathcal{I}\left(\mathbf{v},\mathbf{v}'\right)\approx&\int_{\hat{k}\cdot\mathbf{v}=\mathbf{v}'\cdot\mathbf{v}}^{\cos\theta=1}d\cos\theta\frac{1-\mathbf{v}\cdot\mathbf{v}'}{\left(1-v\cos\theta\right)\left(1-\mathbf{v}\cdot\mathbf{v}'\right)}\\&+\int_{\hat{k}\cdot\mathbf{v}'=\mathbf{v}'\cdot\mathbf{v}}^{\cos\theta=1}d\cos\theta\frac{1-\mathbf{v}\cdot\mathbf{v}'}{\left(1-\mathbf{v}\cdot\mathbf{v}'\right)\left(1-v'\cos\theta\right)}\end{aligned}$$

(The lower limits on the integrals are not critical; an equally good choice would be $\hat{k}\cdot\mathbf{v}=1-x\left(1-\mathbf{v}\cdot\mathbf{v}'\right)$, as long as x is neither too close to 0 nor too much bigger than 1 . It is then easy to show that the leading term in the
relativistic limit does not depend on x.) The integrals are easy to perform, and we obtain

$$\begin{aligned}\mathcal{I}\left(\mathbf{v},\mathbf{v}'\right)&\approx\log\left(\frac{1-\mathbf{v}'\cdot\mathbf{v}}{1-|\mathbf{v}|}\right)+\log\left(\frac{1-\mathbf{v}'\cdot\mathbf{v}}{1-|\mathbf{v}'|}\right)=\log\left(\frac{\left(E^2-\mathbf{p}\cdot\mathbf{p}'\right)^2}{E^2(E-p)^2}\right)\\&\approx 2\log\left(\frac{p\cdot p'}{\left(E^2-p^2\right)/2}\right)=2\log\left(\frac{-q^2}{m^2}\right)\end{aligned} \tag{6.17}$$

where $q^2 = (p' - p)^2$.
In conclusion, we have found that the radiated energy at low frequencies is given by

$$\text{Energy } = \frac{\alpha}{\pi} \int_0^{k_{\max}} dk \mathcal{I}\left(\mathbf{v}, \mathbf{v}'\right) \underset{E \gg m}{\longrightarrow} \frac{2\alpha}{\pi} \int_0^{k_{\max}} dk \log\left(\frac{-q^2}{m^2}\right) \tag{6.18}$$

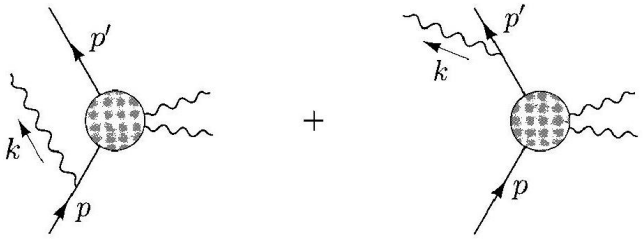
If this energy is made up of photons, each photon contributes energy k. We would then expect

$$\text{Number of photons } = \frac{\alpha}{\pi} \int_0^{k_{\max}} dk \frac{1}{k} \mathcal{I}\left(\mathbf{v}, \mathbf{v}'\right) \tag{6.19}$$

We hope that a quantum-mechanical calculation will confirm this result.

Quantum Computation

Consider now the quantum-mechanical process in which one photon is radiated during the scattering of an electron:



Let $\mathcal{M}_0$ denote the part of the amplitude that comes from the electron's interaction with the external field. Then the amplitude for the whole process is

$$\begin{aligned} i\mathcal{M} = -ie\bar{u}\left(p'\right) \Bigg(& \mathcal{M}_0\left(p', p-k\right) \frac{i(\not{p} - \not{k} + m)}{(p-k)^2 - m^2} \gamma^\mu \epsilon_\mu^*(k) \\ & + \gamma^\mu \epsilon_\mu^*(k) \frac{i\left(\not{p}' + \not{k} + m\right)}{\left(p'+k\right)^2 - m^2} \mathcal{M}_0\left(p'+k, p\right) \Bigg) u(p) \end{aligned} \tag{6.20}$$

Since we are interested in connecting with the classical limit, assume that the photon radiated is soft: $|\mathbf{k}| \ll |\mathbf{p}' - \mathbf{p}|$. Then we can approximate

$$\mathcal{M}_0\left(p', p-k\right) \approx \mathcal{M}_0\left(p'+k, p\right) \approx \mathcal{M}_0\left(p', p\right) \tag{6.21}$$

and we can ignore $\not{\angle} x$ in the numerators of the propagators. The numerators can be further simplified with some Dirac algebra. In the first term we have

$$\begin{aligned} (\not{p} + m)\gamma^\mu \epsilon_\mu^* u(p) &= \left[2p^\mu \epsilon_\mu^* + \gamma^\mu \epsilon_\mu^* (-\not{p} + m)\right] u(p) \\ &= 2p^\mu \epsilon_\mu^* u(p) \end{aligned}$$

Similarly, in the second term,

$$\bar{u}\left(p'\right) \gamma^\mu \epsilon_\mu^* \left(\not{p}' + m\right) = \bar{u}\left(p'\right) 2p'^\mu \epsilon_\mu^*.$$

The denominators of the propagators also simplify:

$$(p-k)^2 - m^2 = -2p \cdot k; \quad \left(p'+k\right)^2 - m^2 = 2p' \cdot k$$

So in the soft-photon approximation, the amplitude becomes

$$i\mathcal{M} = \bar{u}\left(p'\right)\left[\mathcal{M}_0\left(p', p\right)\right] u(p) \cdot \left[e\left(\frac{p' \cdot \epsilon^*}{p' \cdot k} - \frac{p \cdot \epsilon^*}{p \cdot k}\right)\right]. \tag{6.22}$$

This is just the amplitude for elastic scattering (without bremsstrahlung), times a factor (in brackets) for the emission of the photon.

The cross section for our process is also easy to express in terms of the elastic cross section; just insert an additional phase-space integration for the photon variable k. Summing over the two photon polarization states, we have

$$d\sigma\left(p \to p' + \gamma\right) = d\sigma\left(p \to p'\right) \cdot \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k} \sum_{\lambda=1,2} e^2 \left|\frac{p' \cdot \epsilon^{(\lambda)}}{p' \cdot k} - \frac{p \cdot \epsilon^{(\lambda)}}{p \cdot k}\right|^2 \tag{6.23}$$

Thus the differential probability of radiating a photon with momentum k, given that the electron scatters from p to p', is

$$d(\text{ prob }) = \frac{d^3k}{(2\pi)^3} \sum_{\lambda} \frac{e^2}{2k} \left|\boldsymbol{\epsilon}_\lambda \cdot \left(\frac{\mathbf{p}'}{p' \cdot k} - \frac{\mathbf{p}}{p \cdot k}\right)\right|^2 \tag{6.24}$$

This looks very familiar; if we multiply by the photon energy k to compute the expected energy radiated, we recover the classical expression (6.12).

But there is a problem. Equation (6.24) is an expression not for the expected number of photons radiated, but for the probability of radiating a single photon. The problem becomes worse if we integrate over the photon momentum. As in (6.16), we can integrate only up to the energy at which our soft-photon approximations break down; a reasonable estimate for this energy is $|\mathbf{q}| = |\mathbf{p} - \mathbf{p}'|$. The integral is therefore

$$\text{Total probability } \approx \frac{\alpha}{\pi} \int_0^{|\mathbf{q}|} dk \frac{1}{k} \mathcal{I}\left(\mathbf{v}, \mathbf{v}'\right) \tag{6.25}$$

Since $\mathcal{I}\left(\mathbf{v}, \mathbf{v}'\right)$ is independent of k, the integral diverges at its lower limit (where all our approximations are well justified). In other words, the total
probability of radiating a very soft photon is infinite. This is the famous problem of infrared divergences in QED perturbation theory.

We can artificially make the integral in (6.25) well-defined by pretending that the photon has a very small mass μ. This mass would then provide a lower cutoff for the integral, allowing us to write the result of this section as

$$\begin{aligned} d\sigma\left(p \to p' + \gamma(k)\right) &= d\sigma\left(p \to p'\right) \cdot \frac{\alpha}{2\pi} \log\left(\frac{-q^2}{\mu^2}\right) \mathcal{I}\left(\mathbf{v}, \mathbf{v}'\right) \\ &\underset{-q^2 \to \infty}{\approx} d\sigma\left(p \to p'\right) \cdot \frac{\alpha}{\pi} \log\left(\frac{-q^2}{\mu^2}\right) \log\left(\frac{-q^2}{m^2}\right). \end{aligned} \tag{6.26}$$

The q^2 dependence of this result, known as the Sudakov double logarithm, is physical and will appear again in Section 6.4. The dependence on μ, however, presents a problem that we must solve. It is not hard to guess that the resolution of this problem will involve reinterpreting (6.24) as the expected number of radiated photons, rather than the probability of radiating a single photon. We will see in Sections 6.4 and 6.5 how this reinterpretation follows from the Feynman diagrams.

To prepare for that discussion, however, we need to improve our understanding of the amplitude for scattering without radiation.

16.2 6.2 The Electron Vertex Function: Formal Structure

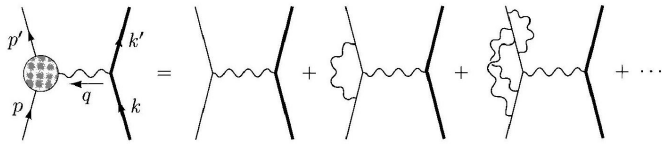
Having briefly discussed QED radiative corrections due to emission of photons (bremsstrahlung), let us now study the correction to electron scattering that comes from the presence of an additional virtual photon:

(add the diagram here!!)

This will be our first experience with a Feynman diagram containing a loop. Such diagrams give rise to significant and profound complications in quantum field theory.

The result of computing this diagram will be rather complicated, so it will be useful to think ahead about what form we expect this correction to take and how to interpret its various possible terms. In this section, we will consider the general properties of vertex correction diagrams. We will see that the basic requirements of Lorentz invariance, the discrete symmetries of QED, and the Ward identity strongly constrain the form of the vertex.

Consider, then, the class of diagrams



where the gray circle indicates the sum of the lowest-order electron-photon vertex and all amputated loop corrections. We will call this sum of vertex diagrams $-ie\Gamma^{\mu}\left(p',p\right)$. Then, according to our master formula (4.103) for S matrix elements, the amplitude for electron scattering from a heavy target is

$$i\mathcal{M}=ie^{2}\left(\bar{u}\left(p'\right)\Gamma^{\mu}\left(p',p\right)u(p)\right)\frac{1}{q^{2}}\left(\bar{u}\left(k'\right)\gamma_{\mu}u(k)\right). \tag{6.28}$$

More generally, the function $\Gamma^{\mu}\left(p',p\right)$ appears in the S-matrix element for the scattering of an electron from an external electromagnetic field. As in Problem 4.4, add to the Hamiltonian of QED the interaction

$$\Delta H_{\text{int}}=\int d^{3}xeA_{\mu}^{\text{cl}}j^{\mu} \tag{6.29}$$

where $j^{\mu}(x)=\bar{\psi}(x)\gamma^{\mu}\psi(x)$ is the electromagnetic current and A_{μ}^{cl} is a fixed classical potential. In the leading order of perturbation theory, the S-matrix element for scattering from this field is

$$i\mathcal{M}(2\pi)\delta\left(p^{0\prime}-p^{0}\right)=-ie\bar{u}\left(p'\right)\gamma^{\mu}u(p)\cdot\widetilde{A}_{\mu}^{\text{cl}}\left(p'-p\right)$$

where $\widetilde{A}_{\mu}^{\text{cl}}(q)$ is the Fourier transform of $A_{\mu}^{\text{cl}}(x)$. The vertex corrections modify this expression to

$$i\mathcal{M}(2\pi)\delta\left(p^{0\prime}-p^{0}\right)=-ie\bar{u}\left(p'\right)\Gamma^{\mu}\left(p',p\right)u(p)\cdot\widetilde{A}_{\mu}^{\text{cl}}\left(p'-p\right) \tag{6.30}$$

In writing (6.28) and (6.30), we have deliberately omitted the contribution of vacuum polarization diagrams, such as the fourth diagram of (6.1). The reason for this omission is that these

diagrams should be considered corrections to the electromagnetic field itself, while the diagrams included in Γ^μ represent corrections to the electron's response to a given applied field. †

We can use general arguments to restrict the form of $\Gamma^\mu(p', p)$. To lowest order, $\Gamma^\mu = \gamma^\mu$. In general, Γ^μ is some expression that involves p, p', γ^μ, and constants such as m, e, and pure numbers. This list is exhaustive, since no other objects appear in the Feynman rules for evaluating the diagrams that contribute to Γ^μ. The only other object that could appear in any theory is $\epsilon^{\mu\nu\rho\sigma}$ (or equivalently, γ^5); but this is forbidden in any parity-conserving theory.

We can narrow down the form of Γ^μ considerably by appealing to Lorentz invariance. Since Γ^μ transforms as a vector (in the same sense that γ^μ does), it must be a linear combination of the vectors from the list above: γ^μ, p^μ, and p'^μ. Using the combinations $p' + p$ and $p' - p$ for convenience, we have

$$\Gamma^\mu = \gamma^\mu \cdot A + (p'^\mu + p^\mu) \cdot B + (p'^\mu - p^\mu) \cdot C \tag{6.31}$$

The coefficients A, B, and C could involve Dirac matrices dotted into vectors, that is, $\not{p}$ or $\not{p}'$. But since $\not{p}u(p) = m \cdot u(p)$ and $\bar{u}(p')\not{p}' = \bar{u}(p') \cdot m$, we can write the coefficients in terms of ordinary numbers without loss of generality. The only nontrivial scalar available is $q^2 = -2p' \cdot p + 2m^2$, so A, B, and C must be functions only of q^2 (and of constants such as m).

The list of allowed vectors can be further shortened by applying the Ward identity (5.79): $q_\mu \Gamma^\mu = 0$. (Note that our arguments for this identity in Section 5.5 - and the proof in Section 7.4 - do not require $q^2 = 0$.) Dotting q_μ into (6.31), we find that the second term vanishes, as does the first when sandwiched between $\bar{u}(p')$ and $u(p)$. The third term does not automatically vanish, so C must be zero.

We can make no further simplifications of (6.31) on general principles. It is conventional, however, to rewrite (6.31) by means of the Gordon identity (see Problem 3.2):

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) \tag{6.32}$$

This identity allows us to swap the $(p' + p)$ term for one involving $\sigma^{\mu\nu}q_\nu$. We write our final result as

$$\Gamma^\mu(p', p) = \gamma^\mu F_1\left(q^2\right) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2\left(q^2\right) \tag{6.33}$$

where F_1 and F_2 are unknown functions of q^2 called form factors.
To lowest order, $F_1 = 1$ and $F_2 = 0$. In the next section we will compute the one-loop (order- α) corrections to the form factors, due to the vertex correction diagram (6.27). In principle, the form factors can be computed to any order in perturbation theory.

Since F_1 and F_2 contain complete information about the influence of an electromagnetic field on the electron, they should, in particular, contain the electron's gross electric and magnetic couplings. To identify the electric charge of the electron, we can use (6.30) to compute the amplitude for elastic Coulomb scattering of a nonrelativistic electron from a region of nonzero electrostatic potential. Set $A^{\rm cl}_\mu(x) = (\phi(\mathbf{x}), \mathbf{0})$. Then $\widetilde{A}^{\rm cl}_\mu(q) = \Big((2\pi)\delta(q^0)\,\tilde{\phi}(\mathbf{q}), \mathbf{0}\Big)$. Inserting this into (6.30), we find

$$i\mathcal{M} = -ie\bar{u}(p')\,\Gamma^0(p', p)\,u(p) \cdot \tilde{\phi}(\mathbf{q})$$

[6†] To justify this statement, we must give a careful definition of an applied external field in a quantum field theory. We will do this in Chapter 11.

If the electrostatic field is very slowly varying over a large (perhaps macroscopic) region, $\phi(\mathbf{q})$ will be concentrated about $\mathbf{q} = 0$; then we can take the
limit $\mathbf{q} \to 0$ in the spinor matrix element. Only the form factor F_1 contributes. Using the nonrelativistic limit of the spinors,

$$\bar{u}\,(p')\,\gamma^0 u(p) = u^\dagger\,(p')\,u(p) \approx 2m\xi'^\dagger\xi$$

the amplitude for electron scattering from an electric field takes the form

$$i\mathcal{M} = -ieF_1(0)\tilde{\phi}(\mathbf{q}) \cdot 2m\xi'^\dagger\xi \tag{6.34}$$

This is the Born approximation for scattering from a potential

$$V(\mathbf{x}) = eF_1(0)\phi(\mathbf{x})$$

Thus $F_1(0)$ is the electric charge of the electron, in units of e. Since $F_1(0) = 1$ already in the leading order of perturbation theory, radiative corrections to $F_1\,(q^2)$ should vanish at $q^2 = 0$.

By repeating this analysis for an electron scattering from a static vector potential, we can derive a similar connection between the form factors and the electron's magnetic moment. ‡ Set $A^{\rm cl}_\mu(x) = \big(0, \mathbf{A}^{\rm cl}(\mathbf{x})\big)$. Then the amplitude for scattering from this field is

$$i\mathcal{M} = +ie\left[\bar{u}\,(p')\left(\gamma^i F_1 + \frac{i\sigma^{i\nu}q_\nu}{2m}F_2\right)u(p)\right]\widetilde{A}^i_{\rm cl}(\mathbf{q}). \tag{6.35}$$

The expression in brackets vanishes at $\mathbf{q} = 0$, so we must carefully extract from it a contribution linear in q^i. To do this, insert the nonrelativistic expansion of the spinors $u(p)$, keeping terms through first order in momenta:

$$u(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\xi\\ \sqrt{p\cdot\bar{\sigma}}\xi\end{pmatrix} \approx \sqrt{m}\begin{pmatrix}(1-\mathbf{p}\cdot\boldsymbol{\sigma}/2m)\xi\\ (1+\mathbf{p}\cdot\boldsymbol{\sigma}/2m)\xi\end{pmatrix}. \tag{6.36}$$

Then the F_1 term can be simplified as follows:

$$\bar{u}\,(p')\,\gamma^i u(p) = 2m\xi'^\dagger\left(\frac{\mathbf{p}'\cdot\boldsymbol{\sigma}}{2m}\sigma^i + \sigma^i\frac{\mathbf{p}\cdot\boldsymbol{\sigma}}{2m}\right)\xi$$

Applying the identity $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$, we find a spin-independent term, proportional to $(\mathbf{p}' + \mathbf{p})$, and a spin-dependent term, proportional to $(\mathbf{p}' - \mathbf{p})$. The first of these terms is the contribution of the operator $[\mathbf{p}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{p}]$ in the standard kinetic energy term of nonrelativistic quantum mechanics. The second is the magnetic moment interaction we are seeking. Retaining only the latter term, we have

$$\bar{u}\,(p')\,\gamma^i u(p) = 2m\xi'^\dagger\left(\frac{-i}{2m}\epsilon^{ijk}q^j\sigma^k\right)\xi$$

The F_2 term already contains an explicit factor of q, so we can evaluate it using the leading-order term of the expansion of the spinors. This gives

$$\bar{u}\,(p')\left(\frac{i}{2m}\sigma^{i\nu}q_\nu\right)u(p) = 2m\xi'^\dagger\left(\frac{-i}{2m}\epsilon^{ijk}q^j\sigma^k\right)\xi$$

[6‡] The following argument contains numerous factors of (-1) from raising and lowering spacelike indices. Be careful in verifying the algebra.

Thus, the complete term linear in q^j in the electron-photon vertex function is

$$\bar{u}(p')\left(\gamma^i F_1 + \frac{i\sigma^{i\nu}q_\nu}{2m}F_2\right)u(p) \underset{q\to 0}{\approx} 2m\xi'^\dagger\left(\frac{-i}{2m}\epsilon^{ijk}q^j\sigma^k\left[F_1(0)+F_2(0)\right]\right)\xi$$

Inserting this expression into (6.35), we find

$$i\mathcal{M} = -i(2m)\cdot e\xi'^\dagger\left(\frac{-1}{2m}\sigma^k\left[F_1(0)+F_2(0)\right]\right)\xi\tilde{B}^k(\mathbf{q})$$

where

$$\widetilde{B}^k(\mathbf{q}) = -i\epsilon^{ijk}q^i\widetilde{A}^j_{\rm cl}(\mathbf{q})$$

is the Fourier transform of the magnetic field produced by $\mathbf{A}^{\rm cl}(\mathbf{x})$.
Again we can interpret $\mathcal{M}$ as the Born approximation to the scattering of the electron from a potential well. The potential is just that of a magnetic moment interaction,

$$V(\mathbf{x}) = -\langle\boldsymbol{\mu}\rangle\cdot\mathbf{B}(\mathbf{x})$$

where

$$\langle\boldsymbol{\mu}\rangle = \frac{e}{m}\left[F_1(0)+F_2(0)\right]\xi'^\dagger\frac{\boldsymbol{\sigma}}{2}\xi.$$

This expression for the magnetic moment of the electron can be rewritten in the standard form

$$\boldsymbol{\mu} = g\left(\frac{e}{2m}\right)\mathbf{S}$$

where $\mathbf{S}$ is the electron spin. The coefficient g, called the Land g-factor, is

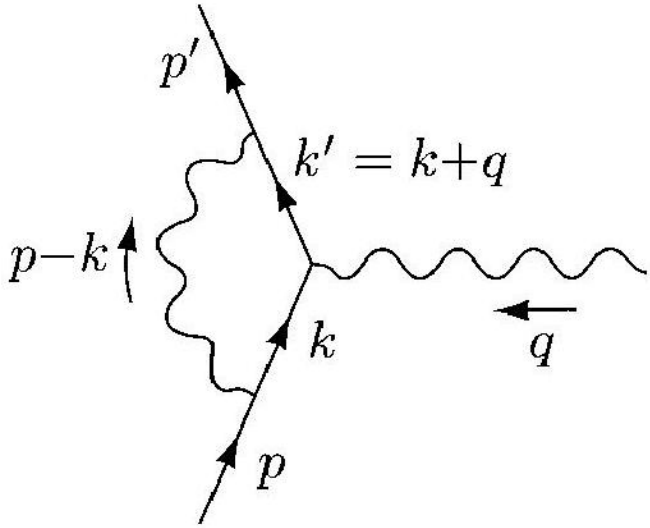
$$g = 2\left[F_1(0)+F_2(0)\right] = 2+2F_2(0). \tag{6.37}$$

Since the leading order of perturbation theory gives no F_2 term, QED predicts $g = 2+\mathcal{O}(\alpha)$. The leading term is the standard prediction of the Dirac equation. In higher orders, however, we will find a nonzero F_2 and thus a small difference between the electron's magnetic moment and the Dirac value. We will compute the order- α contribution to this anomalous magnetic moment in the next section.

Since our derivation of the structure (6.33) for the vertex function used only general symmetry principles, we expect this formula to apply not only to the electron but to any fermion with electromagnetic interactions. For example, the electromagnetic scattering amplitude of the proton should also be described by two invariant functions of q^2. Since the proton is not an elementary particle, we should not expect the Dirac equation values $F_1 = 1$ and $F_2 = 0$ to be good approximations to the form factors of the proton. In fact, both proton form factors depend strongly on q^2. However, the description of the vertex function in term of form factors provides a useful summary of data on scattering at many energies and angles. The precise transcription between form factors and cross sections is worked out in Problem 6.1. In addition, the general constraints at $q^2 = 0$ that we have just derived apply to the proton: $F_1(0) = 1$, and $2F_2(0) = (g_p - 2)$, though the g-factor of the proton differs by 40% from the Dirac value.

16.3 6.3 The Electron Vertex Function: Evaluation

Now that we know what form the answer is to take (Eq. (6.33)), we are ready to evaluate the one-loop contribution to the electron vertex function. Assign momenta on the diagram as follows:



Applying the Feynman rules, we find, to order α, that $\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu$, where

$$\begin{aligned}
&\bar{u}(p')\,\delta\Gamma^\mu(p',p)\,u(p) \\
&\quad = \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon}\bar{u}(p')(-ie\gamma^\nu)\frac{i(\not{l}'+m)}{k'^2-m^2+i\epsilon}\gamma^\mu\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}(-ie\gamma^\rho)\,u(p) \\
&\quad = 2ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}(p')\left[\not{k}\gamma^\mu\ \not{l}'+m^2\gamma^\mu-2m(k+k')^\mu\right]u(p)}{((k-p)^2+i\epsilon)(k'^2-m^2+i\epsilon)(k^2-m^2+i\epsilon)}.
\end{aligned} \tag{6.38}$$

In the second line we have used the contraction identity $\gamma^\nu\gamma^\mu\gamma_\nu = -2\gamma^\mu$. Note that the $+i\epsilon$ terms in the denominators cannot be dropped; they are necessary for proper evaluation of the loop-momentum integral.

The integral looks impossible, and in fact it will not be easy. The evaluation of such integrals requires another piece of computational technology, known as the method of Feynman parameters (although a very similar method was introduced earlier by Schwinger).

Feynman Parameters

The goal of this method is to squeeze the three denominator factors of (6.38) into a single quadratic polynomial in k, raised to the third power. We can then shift k by a constant to complete the square in this polynomial and evaluate the remaining spherically symmetric integral without difficulty. The price will be the introduction of auxiliary parameters to be integrated over.

It is easiest to begin with the simpler case of two factors in the denominator. We would then use the identity

$$\frac{1}{AB} = \int_0^1 dx\frac{1}{[xA+(1-x)B]^2} = \int_0^1 dxdy\delta(x+y-1)\frac{1}{[xA+yB]^2} \tag{6.39}$$

An example of its use might look like this:

$$\frac{1}{(k-p)^2\left(k^2-m^2\right)} = \int_0^1 dxdy\delta(x+y-1)\frac{1}{\left[x(k-p)^2+y\left(k^2-m^2\right)\right]^2}$$
$$= \int_0^1 dxdy\delta(x+y-1)\frac{1}{\left[k^2-2xk\cdot p+xp^2-ym^2\right]^2}$$

If we now let $\ell \equiv k - xp$, we see that the denominator depends only on ℓ^2. Integrating over d^4k would now be much easier, since $d^4k = d^4\ell$ and the integrand is spherically symmetric with respect to ℓ. The variables x and y that make this transformation possible are called Feynman parameters.

Our integral (6.38) involves a denominator with three factors, so we need a slightly better identity. By differentiating (6.39) with respect to B, it is easy to prove

$$\frac{1}{AB^n} = \int_0^1 dxdy\delta(x+y-1)\frac{ny^{n-1}}{[xA+yB]^{n+1}} \tag{6.40}$$

But this still isn't quite good enough. The formula we need is

$$\frac{1}{A_1A_2\cdots A_n} = \int_0^1 dx_1\cdots dx_n\delta\left(\sum x_i - 1\right)\frac{(n-1)!}{\left[x_1A_1+x_2A_2+\cdots x_nA_n\right]^n} \tag{6.41}$$

The proof of this identity is by induction. The case $n = 2$ is just Eq. (6.39); the induction step is not difficult and involves the use of (6.40).

By repeated differentiation of (6.41), you can derive the even more general identity

$$\frac{1}{A_1^{m_1}A_2^{m_2}\cdots A_n^{m_n}} = \int_0^1 dx_1\cdots dx_n\delta\left(\sum x_i - 1\right)\frac{\prod x_i^{m_i-1}}{\left[\sum x_iA_i\right]^{\Sigma m_i}}\frac{\Gamma\left(m_1+\cdots+m_n\right)}{\Gamma\left(m_1\right)\cdots\Gamma\left(m_n\right)} \tag{6.42}$$

This formula is true even when the m_i are not integers; in Section 10.5 we will apply it in such a case.

Evaluation of the Form Factors

Now let us apply formula (6.41) to the denominator of (6.38):

$$\frac{1}{\left((k-p)^2+i\epsilon\right)\left(k'^2-m^2+i\epsilon\right)\left(k^2-m^2+i\epsilon\right)} = \int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{D^3},$$

where the new denominator D is

$$D = x\left(k^2-m^2\right)+y\left(k^2-m^2\right)+z(k-p)^2+(x+y+z)i\epsilon$$
$$= k^2+2k\cdot(yq-zp)+yq^2+zp^2-(x+y)m^2+i\epsilon. \tag{6.43}$$

In the second line we have used $x + y + z = 1$ and $k' = k + q$. Now shift k to complete the square:

$$\ell \equiv k+yq-zp$$

After a bit of algebra we find that D simplifies to

$$D = \ell^2 - \Delta + i\epsilon,$$

where

$$\Delta \equiv -xyq^2 + (1-z)^2 m^2 \tag{6.44}$$

Since $q^2 < 0$ for a scattering process, Δ is positive; we can think of it as an effective mass term.

Next we must express the numerator of (6.38) in terms of ℓ. This task is simplified by noting that since D depends only on the magnitude of ℓ,

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^\mu}{D^3} = 0 \tag{6.45}$$

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^\mu \ell^\nu}{D^3} = \int \frac{d^4\ell}{(2\pi)^4}\frac{\frac{1}{4}g^{\mu\nu}\ell^2}{D^3} \tag{6.46}$$

The first identity follows from symmetry. To prove the second, note that the integral vanishes by symmetry unless $\mu = \nu$. Lorentz invariance therefore requires that we get something proportional to $g^{\mu\nu}$. To check the coefficient, contract each side with $g_{\mu\nu}$. Using these identities, we have

$$\begin{aligned}
&\text{Numerator } = \bar{u}(p')\left[\not{\ell}\gamma^\mu \not{\ell}'\right.\\
&\left.+m^2\gamma^\mu - 2m(k+k')^\mu\right] u(p)\\
&\to \bar{u}(p')\left[-\frac{1}{2}\gamma^\mu\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p})\right.\\
&\left.+m^2\gamma^\mu - 2m((1-2y)q^\mu + 2zp^\mu)\right] u(p).
\end{aligned}$$

(Remember that $k' = k + q$.)

Putting the numerator into a useful form is now just a matter of some tedious Dirac algebra (about a page or two). This is where our work in the last section pays off, since it tells us what kind of an answer to expect. We eventually want to group everything into two terms, proportional to γ^μ and $i\sigma^{\mu\nu}q_\nu$. The most straightforward way to accomplish this is to aim instead for an expression of the form

$$\gamma^\mu \cdot A + (p'^\mu + p^\mu)\cdot B + q^\mu \cdot C$$

just as in (6.31). Attaining this form requires only the anticommutation relations (for example, $\not{p}\gamma^\mu = 2p^\mu - \gamma^\mu \not{p}$) and the Dirac equation ($\not{p}u(p) = mu(p)$

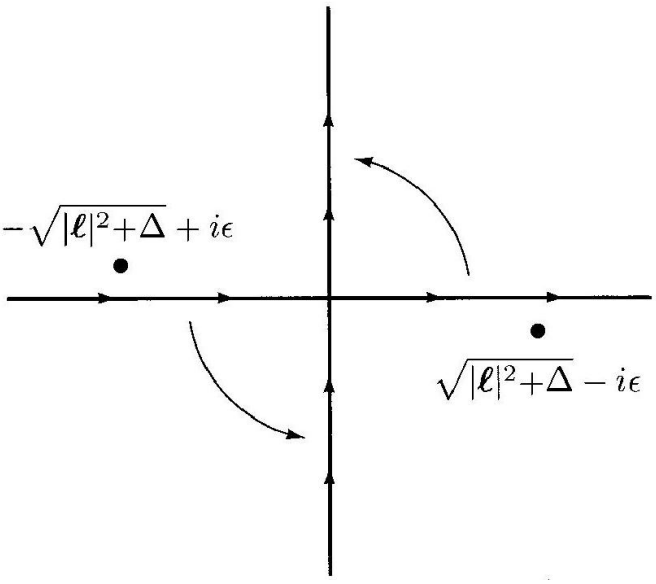


Figure 6.1. The contour of the ℓ^0 integration can be rotated as shown.

and $\bar{u}(p')\not{p}'' = \bar{u}(p')\,m$; note that this implies $\bar{u}(p')\not{q}u(p) = 0$). It is also useful to remember that $x + y + z = 1$. When the smoke clears, we have

$$\begin{aligned}\text{Numerator} &= \bar{u}(p')\left[\gamma^\mu \cdot \left(-\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + \left(1-2z-z^2\right)m^2\right)\right.\\ &\left. + (p'^\mu + p^\mu)\cdot mz(z-1) + q^\mu \cdot m(z-2)(x-y)\right]u(p)\end{aligned}$$

The coefficient of q^μ must vanish according to the Ward identity, as discussed after Eq. (6.31). To see that it does, note from (6.44) that the denominator is symmetric under $x \leftrightarrow y$. The coefficient of q^μ is odd under $x \leftrightarrow y$ and therefore vanishes when integrated over x and y.

Still following our work in the previous section, we now use the Gordon identity (6.32) to eliminate $(p' + p)$ in favor of $i\sigma^{\mu\nu}q_\nu$. Our entire expression for the $\mathcal{O}(\alpha)$ contribution to the electron vertex then becomes

$$\begin{aligned}\bar{u}(p')\,\delta\Gamma^\mu(p',p)\,u(p) &= 2ie^2\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{D^3}\\ &\times\bar{u}(p')\left[\gamma^\mu\cdot\left(-\frac{1}{2}\ell^2+(1-x)(1-y)q^2+\left(1-4z+z^2\right)m^2\right)\right.\\ &\left.+\frac{i\sigma^{\mu\nu}q_\nu}{2m}\left(2m^2z(1-z)\right)\right]u(p)\end{aligned}\tag{6.47}$$

where as before,

$$D = \ell^2 - \Delta + i\epsilon, \quad \Delta = -xyq^2 + (1-z)^2m^2 > 0$$

The decomposition into form factors is now manifest.

With most of the work behind us, our main remaining task is to perform the momentum integral. It is not difficult to evaluate the ℓ^0 integral as a

contour integral, then do the spatial integrals in spherical coordinates. We will use an even easier method, making use of a trick called Wick rotation. Note that if it were not for the minus signs in the Minkowski metric, we could perform the entire four-dimensional integral in four-dimensional

"spherical" coordinates. To remove the minus signs, consider the contour of integration in the ℓ^0-plane (see Fig. 6.1). The locations of the poles, and the fact that the integrand falls off sufficiently rapidly at large $|\ell^0|$, allow us to rotate the contour counterclockwise by 90°. We then define a Euclidean 4 -momentum variable ℓ_E:

$$\ell^0 \equiv i\ell_E^0; \quad \ell = \ell_E \tag{6.48}$$

Our rotated contour goes from $\ell_E^0 = -\infty$ to ∞. By simply changing variables to ℓ_E, we can now evaluate the integral in four-dimensional spherical coordinates.

Let us first evaluate

$$\begin{aligned}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta]^m} &= \frac{i}{(-1)^m}\frac{1}{(2\pi)^4}\int d^4\ell_E \frac{1}{[\ell_E^2+\Delta]^m} \\ &= \frac{i(-1)^m}{(2\pi)^4}\int d\Omega_4 \int_0^\infty d\ell_E \frac{\ell_E^3}{[\ell_E^2+\Delta]^m}\end{aligned}$$

(Here we need only the case $m = 3$, but the more general result will be useful for other loop calculations.) The factor $\int d\Omega_4$ is the surface "area" of a fourdimensional unit sphere, which happens to equal $2\pi^2$. (One way to compute this area is to use four-dimensional spherical coordinates,

$$x = (r\sin\omega\sin\theta\cos\phi, r\sin\omega\sin\theta\sin\phi, r\sin\omega\cos\theta, r\cos\omega)$$

The integration measure is then $d^4x = r^3\sin^2\omega\sin\theta d\phi d\theta d\omega dr$.) The rest of the integral is straightforward, and we have

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta]^m} = \frac{i(-1)^m}{(4\pi)^2}\frac{1}{(m-1)(m-2)}\frac{1}{\Delta^{m-2}} \tag{6.49}$$

Similarly,

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{[\ell^2-\Delta]^m} = \frac{i(-1)^{m-1}}{(4\pi)^2}\frac{2}{(m-1)(m-2)(m-3)}\frac{1}{\Delta^{m-3}} \tag{6.50}$$

Note that this second result is valid only when $m > 3$. When $m = 3$, the Wick rotation cannot be justified, and the integral is in any event divergent. But it is just this case that we need for (6.47).

We will eventually explore the physical meaning of this divergence, but for the moment we simply introduce an artificial prescription to make our integral finite. Go back to the original expression for the Feynman integral in
(6.38), and replace in the photon propagator

$$\frac{1}{(k-p)^2+i\epsilon} \longrightarrow \frac{1}{(k-p)^2+i\epsilon} - \frac{1}{(k-p)^2-\Lambda^2+i\epsilon} \tag{6.51}$$

where Λ is a very large mass. The integrand is unaffected for small k (since Λ is large), but cuts off smoothly when $k \gtrsim \Lambda$. We can think of the second term as the propagator of a fictitious heavy photon, whose contribution is subtracted from that of the ordinary photon. In terms involving the heavy photon, the numerator algebra is unchanged and the denominator is altered by

$$\Delta \longrightarrow \Delta_\Lambda = -xyq^2 + (1-z)^2m^2 + z\Lambda^2 \tag{6.52}$$

The integral (6.50) is then replaced with a convergent integral, which can be Wick-rotated and evaluated:

$$\int \frac{d^4\ell}{(2\pi)^4}\left(\frac{\ell^2}{\left[\ell^2-\Delta\right]^3}-\frac{\ell^2}{\left[\ell^2-\Delta_\Lambda\right]^3}\right)=\frac{i}{(4\pi)^2}\int_0^\infty d\ell_E^2\left(\frac{\ell_E^4}{\left[\ell_E^2+\Delta\right]^3}-\frac{\ell_E^4}{\left[\ell_E^2+\Delta_\Lambda\right]^3}\right)$$
$$=\frac{i}{(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right) \tag{6.53}$$

The convergent terms in (6.47) are modified by terms of order Λ^{-2}, which we ignore.

This prescription for rendering Feynman integrals finite by introducing fictitious heavy particles is known as Pauli-Villars regularization. Please note that the fictitious photon has no physical significance, and that this method is only one of many for defining the divergent integrals. (We will discuss other methods in the next chapter; see especially Problem 7.2.) We must hope that the new parameter Λ will not appear in our final results for observable cross sections.

Using formulae (6.49) and (6.53) to evaluate the integrals in (6.47), we obtain an explicit, though complicated, expression for the one-loop vertex correction:

$$\left\{\begin{array}{l}\left\{\begin{array}{l}2\pi\dfrac{\alpha}{2}dxdydz\delta(x+y+z-1)\\ \quad\times\bar{u}\left(p'\right)\left(\gamma^\mu\left[\log\dfrac{z\Lambda^2}{\Delta}+\dfrac{1}{\Delta}\left((1-x)(1-y)q^2+\left(1-4z+z^2\right)m^2\right)\right]\right.\\ \qquad\left.+\dfrac{i\sigma^{\mu\nu}q_\nu}{2m}\left[\dfrac{1}{\Delta}2m^2z(1-z)\right]\right)u(p).\end{array}\right.\end{array}\right.$$

The bracketed expressions are our desired corrections to the form factors.

Before we try to interpret this result, let us summarize the calculational methods we used. The techniques are common to all loop calculations:

1. Draw the diagram(s) and write down the amplitude.

2. Introduce Feynman parameters to combine the denominators of the propagators.

3. Complete the square in the new denominator by shifting to a new loop momentum variable, ℓ.

4. Write the numerator in terms of ℓ. Drop odd powers of ℓ, and rewrite even powers using identities like (6.46).

5. Perform the momentum integral by means of a Wick rotation and fourdimensional spherical coordinates.
   The momentum integral in the last step will often be divergent. In that case we must define (or regularize) the integral using the Pauli-Villars prescription or some other device.

Now that we have parametrized the ultraviolet divergence in (6.54), let us try to interpret it. Notice that the divergence appears in the worst possible place: It corrects $F_1\left(q^2=0\right)$, which should (according to our discussion at the end of the previous section) be fixed at the value 1 . But this is the only effect of the divergent term. We will therefore adopt a simple but completely ad hoc fix for this difficulty: Subtract from the above expression a term proportional to the zeroth-order

vertex function $(\bar{u}(p')\gamma^\mu u(p))$, in such a way as to maintain the condition $F_1(0) = 1$. In other words, make the substitution

$$\delta F_1\left(q^2\right) \to \delta F_1\left(q^2\right) - \delta F_1(0) \tag{6.55}$$

(where δF_1 denotes the first-order correction to F_1). The justification of this procedure involves the minor correction to our S-matrix formula (4.103) mentioned in Section 4.6. In brief, the term we are subtracting corrects for our omission of the external leg correction diagrams of (6.1). We postpone the justification of this statement until Section 7.2.

There is also an infrared divergence in $F_1(q^2)$, coming from the $1/\Delta$ term. For example, at $q^2 = 0$ this term is

$$\begin{aligned}\int_0^1 dxdydz\delta(x+y+z-1)\frac{1-4z+z^2}{\Delta(q^2=0)} &= \int_0^1 dz\int_0^{1-z} dy\frac{-2+(1-z)(3-z)}{m^2(1-z)^2} \\ &= \int^1 dz\frac{-2}{m^2(1-z)} + \text{ finite terms.}\end{aligned}$$

We can cure this disease by pretending that the photon has a small nonzero mass μ. Then in the denominator of the photon propagator, $(k-p)^2$ would become $(k-p)^2-\mu^2$. This denominator was multiplied by z in (6.43), so the net effect is to add a term $z\mu^2$ to Δ. We will discuss the infrared divergence further in the next two sections.

With both of these provisional modifications, the form factors are

$$\begin{aligned}&F_1(q^2) = 1+ \\ &+\tfrac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1) \\ &\times\left[\log\left(\tfrac{m^2(1-z)^2}{m^2(1-z)^2-q^2xy}\right) + \tfrac{m^2\left(1-4z+z^2\right)+q^2(1-x)(1-y)}{m^2(1-z)^2-q^2xy+\mu^2z}\right. \\ &\left.\quad -\tfrac{m^2\left(1-4z+z^2\right)}{m^2(1-z)^2+\mu^2z}\right] + \mathcal{O}(\alpha^2)\end{aligned} \tag{6.56}$$

Note that neither the ultraviolet nor the infrared divergence affects $F_2(q^2)$. We can therefore evaluate unambiguously

$$\begin{aligned}F_2\left(q^2=0\right) &= \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2m^2z(1-z)}{m^2(1-z)^2} \\ &= \frac{\alpha}{\pi}\int_0^1 dz\int_0^{1-z} dy\frac{z}{1-z} = \frac{\alpha}{2\pi}\end{aligned} \tag{6.58}$$

Thus, we get a correction to the g-factor of the electron:

$$a_e \equiv \frac{g-2}{2} = \frac{\alpha}{2\pi} \approx .0011614 \tag{6.59}$$

This result was first obtained by Schwinger in 1948.* Experiments give $a_e = .0011597$. Apparently, the unambiguous value that we obtained for $F_2(0)$ is also, up to higher orders in α, unambiguously correct.

Precision Tests of QED

Building on the success of the order- α QED prediction for a_e, successive generations of physicists have improved the accuracy of both the theoretical and the experimental determination of this quantity. The coefficients of the QED formula for a_e are now known through order α^4. The calculation of the order- α^2 and higher coefficients requires a systematic treatment of ultraviolet divergences.

These challenging theoretical calculations have been matched by increasingly imaginative experiments. The most recent measurement of a_e uses a technique, developed by Dehmelt and collaborators, in which individual electrons are trapped in a system of electrostatic and magnetostatic fields and

excited to a spin resonance. [†] Today, the best theoretical and experimental values of a_e agree to eight significant figures.

High-order QED calculations have also been carried out for several other quantities. These include transition energies in hydrogen and hydrogen-like atoms, the anomalous magnetic moment of the muon, and the decay rates of singlet and triplet positronium. Many of these quantities have also been measured to high precision. The full set of these comparisons gives a detailed test of the validity of QED in a variety of settings. The results of these precision tests are summarized in Table 6.1.

There is some subtlety in reporting the results of precision comparisons between QED theory and experiment, since theoretical predictions require an extremely precise value of α, which can only be obtained from another precision QED experiment. We therefore quote each comparison between theory and experiment as an independent determination of α. Each value of α is assigned an error that is the composite of the expected uncertainties from theory and experiment. QED is confirmed to the extent that the values of α from different sources agree.

The first nine entries in Table 6.1 refer to QED calculations in atomic physics settings. Of these, the hydrogen hyperfine splitting, measured using Ramsey's hydrogen maser, is the most precisely known quantity in physics. Unfortunately, the influence of the internal structure of the proton leads to uncertainties that limit the accuracy with which this quantity can be predicted theoretically. The same difficulty applies to the Lamb shift, the splitting between the $j = 1/22S$ and $2P$ levels of hydrogen. The most accurate QED tests now come from systems that involve no strongly interacting particles, the electron $g - 2$ and the hyperfine splitting in the $e^-\mu^+$atom, muonium. The last entry in this group gives a new method for determining α, by converting a very accurate measurement of the neutron Compton wavelength, using accurately known mass ratios, to a value of the electron mass. This can be combined with the known value of the Rydberg energy and accurate QED formulae to determine α. The only serious discrepancy among these numbers comes in the triplet positronium decay rate; however, there is some evidence that diagrams of relative order α^2 give a large correction to the value quoted in the table.

The next two entries are determinations of α from higher-order QED reactions at high-energy electron colliders. These high-energy experiments typically achieve only percent-level accuracy, but their results are consistent with the precise information available at lower energies.

Finally, the last two entries in the table give two independent measurements of α from exotic quantum interference phenomena in condensed-matter systems. These two effects provide a standard resistance and a standard frequency, respectively, which are believed to measure the charge of the electron

[6]*J. Schwinger, Phys. Rev. 73, 416L (1948).

[6†] R. Van Dyck, Jr., P. Schwinberg, and H. Dehmelt, Phys. Rev. Lett. 59, 26 (1987).

Table 6.1. Values of α^{-1} Obtained from Precision QED Experiments

Low-Energy QED:

Electron $(g-2)$

Muon $(g-2)$

Muonium hyperfine splitting

Lamb shift

Hydrogen hyperfine splitting

$2^3S_1 - 1^3S_1$ splitting in positronium

$^1S_0$ positronium decay rate

$^3S_1$ positronium decay rate

Neutron compton wavelength

High-Energy QED:

$$\sigma\left(e^+e^- \to e^+e^-e^+e^-\right)$$

$$\sigma\left(e^+e^- \to e^+e^-\mu^+\mu^-\right)$$

Condensed Matter:

Quantum Hall effect

AC Josephson effect

137.03599235 (73)

137.0355 (11)

137.035994 (18)

137.0368 (7)

137.0360 (3)

137.034 (16)

137.00 (6)

136.971 (6)

137.0360101 (54)

136.5 (2.7)

139.9 (1.2)

137.0359979(32)

137.0359770(77)

Each value of α displayed in this table is obtained by fitting an experimental measurement to a theoretical expression that contains α as a parameter. The numbers in parentheses are the standard errors in the last displayed digits, including both theoretical and experimental uncertainties. This table is based on results presented in the survey of precision QED of Kinoshita (1990). That book contains a series of lucid reviews of the remarkable theoretical and experimental technology that has been developed for the detailed analysis of QED processes. The five most accurate values are updated as given by T. Kinoshita in History of Original Ideas and Basic Discoveries in Particle Physics, H. Newman and T. Ypsilantis, eds. (Plenum Press, New York, 1995). This latter paper also gives an interesting perspective on the future of precision QED experiments.

with corrections that are strictly zero for macroscropic systems. ‡

The entire picture fits together well beyond any reasonable expectation. On the evidence presented in this table, QED is the most stringently tested- and the most dramatically successful-of all physical theories.

[6‡] For a discussion of these effects, and their exact relation to α, see D. R. Yennie, Rev. Mod. Phys. 59, 781 (1987).

16.4 6.4 The Electron Vertex Function: Infrared Divergence

Now let us confront the infrared divergence in our result (6.56) for $F_1(q^2)$. The dominant part, in the $\mu \to 0$ limit, is

$$F_1(q^2) \approx \tfrac{\alpha}{2\pi} \int_0^1 dxdydz\delta(x+y+z-1) \left[\frac{m^2\left(1-4z+z^2\right)+q^2(1-x)(1-y)}{m^2(1-z)^2-q^2xy+\mu^2 z} - \frac{m^2\left(1-4z+z^2\right)}{m^2(1-z)^2+\mu^2 z} \right] \tag{6.60}$$

To understand this expression we must do some work to simplify it, extracting and evaluating the divergent part of the integral. Throughout this section we will retain only terms that diverge in the limit $\mu \to 0$.

First note that the divergence occurs in the corner of Feynman-parameter space where $z \approx 1$ (and therefore $x \approx y \approx 0$). In this region we can set $z = 1$ and $x = y = 0$ in the numerators of (6.60). We can also set $z = 1$ in the μ^2 terms in the denominators. Using the delta function to evaluate the x-integral, we then have

$$F_1\left(q^2\right) = \frac{\alpha}{2\pi} \int_0^1 dz \int_0^{1-z} dy \left[\frac{-2m^2+q^2}{m^2(1-z)^2 - q^2 y(1-z-y)+\mu^2} - \frac{-2m^2}{m^2(1-z)^2+\mu^2} \right]$$

(The lower limit on the z-integral is unimportant.) Making the variable changes

$$y = (1-z)\xi, \quad w = (1-z)$$

this expression becomes

$$\begin{aligned} F_1\left(q^2\right) &= \frac{\alpha}{2\pi} \int_0^1 d\xi \frac{1}{2} \int_0^1 d\left(w^2\right) \left[\frac{-2m^2+q^2}{\left[m^2 - q^2\xi(1-\xi)\right] w^2 + \mu^2} - \frac{-2m^2}{m^2 w^2 + \mu^2} \right] \\ &= \frac{\alpha}{4\pi} \int_0^1 d\xi \left[\frac{-2m^2+q^2}{m^2 - q^2\xi(1-\xi)} \log\left(\frac{m^2 - q^2\xi(1-\xi)}{\mu^2} \right) + 2\log\left(\frac{m^2}{\mu^2} \right) \right] \end{aligned}$$

In the limit $\mu \to 0$ we can ignore the details of the numerators inside the logarithms; anything proportional to m^2 or q^2 is effectively the same. We therefore write

$$F_1\left(q^2\right) = 1 - \frac{\alpha}{2\pi} f_{\text{IR}}\left(q^2\right) \log\left(\frac{-q^2 \text{ or } m^2}{\mu^2} \right) + \mathcal{O}\left(\alpha^2\right) \tag{6.61}$$

where the coefficient of the divergent logarithm is

$$f_{\text{IR}}\left(q^2\right) = \int_0^1 \left(\frac{m^2 - q^2/2}{m^2 - q^2\xi(1-\xi)} \right) d\xi - 1 \tag{6.62}$$

Since q^2 is negative and $\xi(1-\xi)$ has a maximum value of $1/4$, the first term is greater than 1 and hence $f_{\text{IR}}(q^2)$ is positive.

How does this infinite term affect the cross section for electron scattering off a potential? Since $F_1(q^2)$ is just the quantity that multiplies γ^μ in the matrix element, we can find the new cross section by making the replacement $e \to e \cdot F_1(q^2)$. The cross section for the process $\mathbf{p} \to \mathbf{p}'$ is therefore

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{d\sigma}{d\Omega}\right)_0 \cdot \left[1 - \frac{\alpha}{\pi} f_{\mathrm{IR}}\left(q^2\right) \log\left(\frac{-q^2 \text{ or } m^2}{\mu^2}\right) + \mathcal{O}\left(\alpha^2\right)\right] \tag{6.63}$$

where the first factor is the tree-level result. Note that the $\mathcal{O}(\alpha)$ correction to the cross section is not only infinite, but negative. Something is terribly wrong.

To gain a better understanding of the divergence, let us evaluate the coefficient of the divergent logarithm, $f_{\mathrm{IR}}\left(q^2\right)$, in the limit $-q^2 \to \infty$. In this limit, we find a second logarithm:

$$\begin{aligned}\int_0^1 d\xi \frac{-q^2/2}{-q^2\xi(1-\xi)+m^2} &\simeq \frac{1}{2}\int_0 d\xi \frac{-q^2}{-q^2\xi+m^2} + \begin{pmatrix}\text{equal contribution}\\ \text{from } \xi \approx 1\end{pmatrix} \\ &= \log\left(\frac{-q^2}{m^2}\right)\end{aligned} \tag{6.64}$$

The form factor in this limit is therefore

$$F_1\left(-q^2 \to \infty\right) = 1 - \frac{\alpha}{2\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}\left(\alpha^2\right) \tag{6.65}$$

Note that the numerator in the second logarithm is $-q^2$, not m^2; this expression contains not only the correct coefficient of $\log\left(1/\mu^2\right)$, but also the correct coefficient of $\log^2\left(q^2\right)$.

The same double logarithm of $-q^2$ appeared in the cross section for soft bremsstrahlung, Eq. (6.26). This correspondence points to a resolution of the infrared divergence problem. Comparing (6.65) with (6.26), we find in the limit $-q^2 \to \infty$

$$\begin{aligned}\frac{d\sigma}{d\Omega}\left(p \to p'\right) &= \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}\left(\alpha^2\right)\right] \\ \frac{d\sigma}{d\Omega}\left(p \to p' + \gamma\right) &= \left(\frac{d\sigma}{d\Omega}\right)_0 \left[+\frac{\alpha}{\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{\mu^2}\right) + \mathcal{O}\left(\alpha^2\right)\right]\end{aligned} \tag{6.66}$$

The separate cross sections are divergent, but their sum is independent of μ and therefore finite.

In fact, neither the elastic cross section nor the soft bremsstrahlung cross section can be measured individually; only their sum is physically observable. In any real experiment, a photon detector can detect photons only down to
some minimum limiting energy E_ℓ. The probability that a scattering event occurs and this detector does not see a photon is the sum

$$\frac{d\sigma}{d\Omega}\left(p \to p'\right) + \frac{d\sigma}{d\Omega}\left(p \to p' + \gamma\left(k < E_\ell\right)\right) \equiv \left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} \tag{6.67}$$

The divergent part of this "measured" cross section is

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} \approx \left(\frac{d\sigma}{d\Omega}\right)_0 \Bigg[1 &- \frac{\alpha}{\pi} f_{\mathrm{IR}}\left(q^2\right)\log\left(\frac{-q^2 \text{ or } m^2}{\mu^2}\right) \\ &+ \frac{\alpha}{2\pi}\mathcal{I}\left(\mathbf{v},\mathbf{v}'\right)\log\left(\frac{E_\ell^2}{\mu^2}\right) + \mathcal{O}\left(\alpha^2\right)\Bigg]\end{aligned}$$

We have just seen that $\mathcal{I}\left(\mathbf{v},\mathbf{v}'\right) = 2f_{\mathrm{IR}}\left(q^2\right)$ when $-q^2 \gg m^2$. If the same relation holds for general q^2, the measured cross section becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} \approx \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} f_{\text{IR}}\left(q^2\right) \log\left(\frac{--q^2 \text{ or } m^2}{E_\ell^2}\right) + \mathcal{O}\left(\alpha^2\right)\right] \tag{6.68}$$

which depends on the experimental conditions, but no longer on μ^2. The infrared divergences from soft bremsstrahlung and from $F_1\left(q^2\right)$ cancel each other, yielding a finite cross section for the quantity that can actually be measured.

We must still verify the identity $\mathcal{I}\left(\mathbf{v}, \mathbf{v}'\right) = 2 f_{\text{IR}}\left(q^2\right)$ for arbitrary values of q^2. From (6.13) we have

$$\mathcal{I}\left(\mathbf{v}, \mathbf{v}'\right) = \int \frac{d\Omega_{\mathbf{k}}}{4\pi} \left(\frac{2p \cdot p'}{\left(\hat{k} \cdot p'\right)\left(\hat{k} \cdot p\right)} - \frac{m^2}{\left(\hat{k} \cdot p'\right)^2} - \frac{m^2}{\left(\hat{k} \cdot p\right)^2} \right) \tag{6.69}$$

The last two terms are easy to evaluate:

$$\int \frac{d\Omega_{\mathbf{k}}}{4\pi} \frac{1}{\left(\hat{k} \cdot p\right)^2} = \frac{1}{2} \int_{-1}^{1} d\cos\theta \frac{1}{\left(p^0 - p\cos\theta\right)^2} = \frac{1}{p^2} = \frac{1}{m^2}$$

In the first term, we can combine the denominators with a Feynman parameter and perform the integral in the same way:

$$\begin{aligned} \int \frac{d\Omega_{\mathbf{k}}}{4\pi} \frac{1}{\left(\hat{k} \cdot p'\right)\left(\hat{k} \cdot p\right)} &= \int_0^1 d\xi \int \frac{d\Omega_{\mathbf{k}}}{4\pi} \frac{1}{\left[\xi \hat{k} \cdot p' + (1-\xi)\hat{k} \cdot p\right]^2} \\ &= \int_0^1 d\xi \frac{1}{\left[\xi p' + (1-\xi)p\right]^2} = \int_0^1 d\xi \frac{1}{m^2 - \xi(1-\xi)q^2} \end{aligned}$$

(In the last step we have used $2p \cdot p' = 2m^2 - q^2$.) Putting all the terms of (6.69) together, we find

$$\mathcal{I}\left(\mathbf{v}, \mathbf{v}'\right) = \int_0^1 \left(\frac{2m^2 - q^2}{m^2 - \xi(1-\xi)q^2} \right) d\xi - 2 = 2 f_{\text{IR}}\left(q^2\right) \tag{6.70}$$

just what we need to cancel the infrared divergence.

Although Eq. (6.68) demonstrates the cancellation of the infrared divergence, this result has little practical use. An experimentalist would want to know the precise dependence on q^2, which we did not evaluate carefully. Recall from (6.65), however, that we were careful to obtain the correct coefficient of $\log^2\left(-q^2\right)$ in the limit $-q^2 \gg m^2$. In that limit, therefore, (6.68) becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{measured}} \approx \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - \frac{\alpha}{\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{E_\ell^2}\right) + \mathcal{O}\left(\alpha^2\right)\right] \tag{6.71}$$

This result is unambiguous and useful. Note that the $\mathcal{O}(\alpha)$ correction again involves the Sudakov double logarithm.

16.5 6.5 Summation and Interpretation of Infrared Divergences

The discussion of infrared divergences in the previous section suffices for removing the infinities from our bremsstrahlung and vertex-correction calculations. There are still, however, three points that we have not addressed:

1. We have not demonstrated the cancellation of infrared divergences beyond the leading order.

2. The correction to the measured cross section that we found after the infrared cancellation (Eqs. (6.68) and (6.71)) can be made arbitrarily negative by making photon detectors with a sufficiently low threshold E_ℓ.

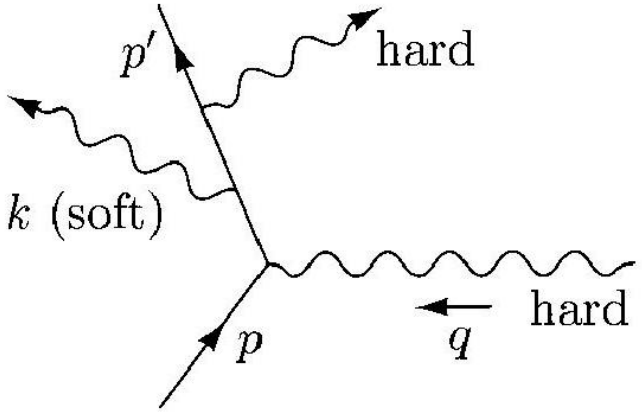
3. We have not yet reproduced the classical result (6.19) for the number of photons radiated during a collision.
   The solutions of the second and third problems will follow immediately from that of the first, to which we now turn.

A complete treatment of infrared divergences to all orders is beyond the scope of this book.* We will discuss here only the terms with the largest logarithmic enhancement at each order of perturbation theory. In general, these terms are of order

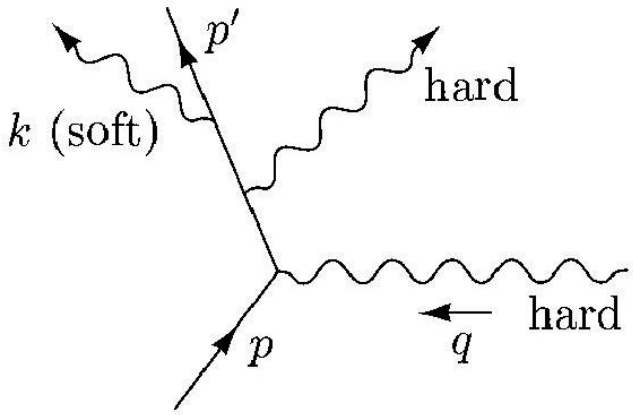
$$\left[\frac{\alpha}{\pi}\log\left(\frac{-q^2}{\mu^2}\right)\log\left(\frac{-q^2}{m^2}\right)\right]^n \tag{6.72}$$

in the nth order of perturbation theory. Our final physical conclusions were first presented by Bloch and Nordsieck in a prescient paper written before the invention of relativistic perturbation theory. † We will follow a modern, and simplified, version of the analysis due to Weinberg. ‡

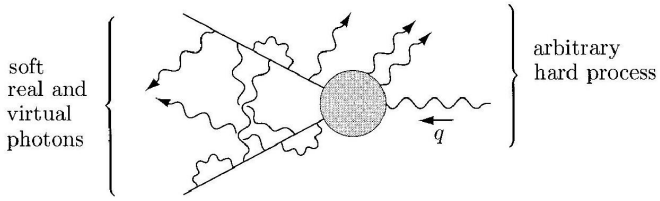
Infrared divergences arise from photons with "soft" momenta: real photons with energy less than some cutoff E_ℓ, and virtual photons with (after Wick rotation) $k^2 < E_\ell^2$. A typical higher-order diagram will involve numerous real and virtual photons. But to find a divergence, we need more than a soft photon; we need a singular denominator in an electron propagator. Consider, for example, the following two diagrams:



[6]*The definitive treatment is given in D. Yennie, S. Frautschi, and H. Suura, Ann. Phys. 13, 379 (1961).
† F. Bloch and A. Nordsieck, Phys. Rev. 52, 54 (1937).
‡ S. Weinberg, Phys. Rev. 140, B516 (1965).

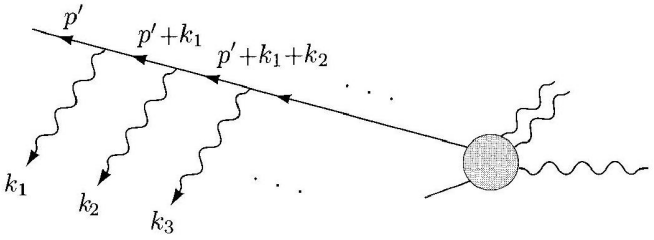


The first diagram, in which the electron emits a soft photon followed by a hard photon, has no infrared divergence, since the momenta in both electron propagators are far from the mass shell. If the soft photon is emitted last, however, the denominator of the adjacent propagator is $(p'+k)^2 - m^2 = 2p'\cdot k$, which vanishes as $k \to 0$. Thus the second diagram does contain a divergence. We would like, then, to consider diagrams in which an arbitrary hard process, possibly involving emission of hard and soft photons, is modified by the addition of soft real and virtual photons on the electron legs:



Following Weinberg, we will add up the contributions of all such diagrams. The only new difficulty in this calculation will be in the combinatorics of counting all the ways in which the photons can appear.

First consider the outgoing electron line:



We attach n photons to the line, with momenta $k_1 \ldots k_n$. For the moment we do not care whether these are external photons, virtual photons connected to each other, or virtual photons connected to vertices on the incoming electron line. The Dirac structure of this diagram is

$$\begin{aligned}
&\bar{u}(p')(-ie\gamma^{\mu_1})\frac{i(\not{p}'+\not{k}_1+m)}{2p'\cdot k_1}(-ie\gamma^{\mu_2})\frac{i(\not{p}'+\not{k}_1+\not{k}_2+m)}{2p'\cdot(k_1+k_2)+\mathcal{O}(k^2)} \\
&\cdots(-ie\gamma^{\mu_n})\frac{i(\not{p}'+\not{k}_1+\cdots+\not{k}_n+m)}{2p'\cdot(k_1+\cdots k_n)+\mathcal{O}(k^2)}(i\mathcal{M}_{\text{hard}})\cdots
\end{aligned} \tag{6.73}$$

We will assume that all the k_i are small, dropping the $\mathcal{O}(k^2)$ terms in the denominators. We will also drop the $\not{p}_i$ terms in the numerators, just as in our treatment of bremsstrahlung

in Section 6.1. Also, as we did there, we can push the factors of $(\not{p}' + m)$ to the left and use $\bar{u}(p')(-\not{p}' + m) = 0$:

$$\begin{aligned}\bar{u}(p')\gamma^{\mu_1}(\not{p}' + m)\gamma^{\mu_2}(\not{p}' + m)\cdots &= \bar{u}(p')\,2p^{\mu_1}\gamma^{\mu_2}(\not{p}' + m)\cdots \\ &= \bar{u}(p')\,2p'^{\mu_1}2p'^{\mu_2}\cdots\end{aligned}$$

This turns expression (6.73) into

$$\bar{u}(p')\left(e\frac{p'^{\mu_1}}{p'\cdot k_1}\right)\left(e\frac{p'^{\mu_2}}{p'\cdot(k_1+k_2)}\right)\cdots\left(e\frac{p'^{\mu_n}}{p'\cdot(k_1+\cdots+k_n)}\right)\cdots \tag{6.74}$$

Still working with only the outgoing electron line, we must now sum over all possible orderings of the momenta $k_1 \ldots k_n$. (This procedure will overcount when two of the photons are attached together to form a single virtual photon. We will deal with this overcounting later.) There are $n\,!$ different diagrams to sum, corresponding to the $n\,!$ permutations of the n photon momenta. Let π denote one such permutation, so that $\pi(i)$ is the number between 1 and n that i is taken to. (For example, if π denotes the permutation that takes $1 \to 3$, $2 \to 1$ and $3 \to 2$, then $\pi(1) = 3, \pi(2) = 1$, and $\pi(3) = 2$.)

Armed with this notation, we can perform the sum over permutations by means of the following identity:

$$\begin{aligned}\sum_{\substack{\text{all permu-}\\ \text{tations } \pi}} &\frac{1}{p\cdot k_{\pi(1)}}\frac{1}{p\cdot\left(k_{\pi(1)}+k_{\pi(2)}\right)}\cdots\frac{1}{p\cdot\left(k_{\pi(1)}+k_{\pi(2)}+\cdots+k_{\pi(n)}\right)} \\ &= \frac{1}{p\cdot k_1}\frac{1}{p\cdot k_2}\cdots\frac{1}{p\cdot k_n}\end{aligned} \tag{6.75}$$

The proof of this formula proceeds by induction on n. For $n = 2$ we have

$$\begin{aligned}\sum_{\pi}\frac{1}{p\cdot k_{\pi(1)}}\frac{1}{p\cdot\left(k_{\pi(1)}+k_{\pi(2)}\right)} &= \frac{1}{p\cdot k_1}\frac{1}{p\cdot(k_1+k_2)} + \frac{1}{p\cdot k_2}\frac{1}{p\cdot(k_2+k_1)} \\ &= \frac{1}{p\cdot k_1}\frac{1}{p\cdot k_2}\end{aligned}$$

For the induction step, notice that the last factor on the left-hand side of (6.75) is the same for every permutation π. Pulling this factor outside the sum, the left-hand side becomes

$$\text{LHS} = \frac{1}{p\cdot\sum k}\sum_{\pi}\frac{1}{p\cdot k_{\pi(1)}}\frac{1}{p\cdot\left(k_{\pi(1)}+k_{\pi(2)}\right)}\cdots\frac{1}{p\cdot\left(k_{\pi(1)}+\cdots+k_{\pi(n-1)}\right)}.$$

For any given π, the quantity being summed is independent of $k_{\pi(n)}$. Letting $i = \pi(n)$, we can now write

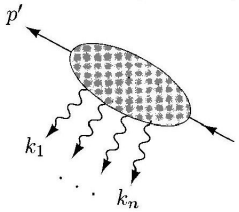
$$\sum_{\pi} = \sum_{i=1}^{n}\sum_{\pi'(i)}$$

where $\pi'(i)$ is the set of all permutations on the remaining $n-1$ integers. Assuming by induction that (6.75) is true for $n-1$, we have

$$\text{LHS} = \frac{1}{p \cdot \sum k} \sum_{i=1}^{n} \frac{1}{p \cdot k_1} \frac{1}{p \cdot k_2} \cdots \frac{1}{p \cdot k_{i-1}} \frac{1}{p \cdot k_{i+1}} \cdots \frac{1}{p \cdot k_n}$$

If we now multiply and divide each term in this sum by $p \cdot k_i$, we easily obtain our desired result (6.75).

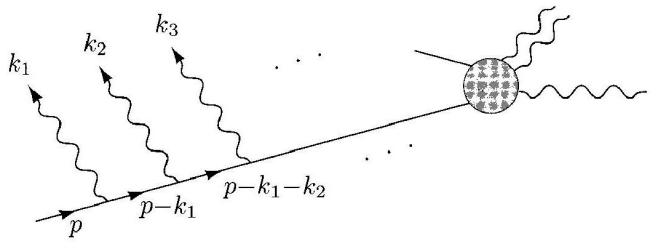
Applying (6.75) to (6.74), we find



$$= \bar{u}(p')\Big(e\frac{p'^{\mu_1}}{p' \cdot k_1}\Big)\Big(e\frac{p'^{\mu_2}}{p' \cdot k_2}\Big) \cdots \Big(e\frac{p'^{\mu_n}}{p' \cdot k_n}\Big),$$

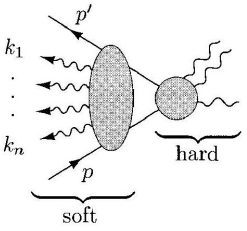
where the blob denotes a sum over all possible orders of inserting the n photon lines.

A similar set of manipulations simplifies the sum over soft photon insertions on the initial electron line. There, however, the propagator momenta are $p - k_1, p - k_1 - k_2$, and so on:



We therefore get an extra minus sign in the factor for each photon, since $(p - \Sigma k)^2 - m^2 \approx -2p \cdot \Sigma k$.

Now consider diagrams containing a total of n soft photons, connected in any possible order to the initial or final electron lines. The sum over all such diagrams can be written



$$= \bar{u}(p')\, i\mathcal{M}_{\text{hard}}\, u(p)$$
$$\cdot e\Big(\frac{p'^{\mu_1}}{p' \cdot k_1} - \frac{p^{\mu_1}}{p \cdot k_1}\Big) \cdot e\Big(\frac{p'^{\mu_2}}{p' \cdot k_2} - \frac{p^{\mu_2}}{p \cdot k_2}\Big)$$
$$\cdots e\Big(\frac{p'^{\mu_n}}{p' \cdot k_n} - \frac{p^{\mu_n}}{p \cdot k_n}\Big).$$

By multiplying out all the factors, you can see that we get the correct term for each possible way of dividing the n photons between the two lines.

Next we must decide which photons are real and which are virtual.
We can make a virtual photon by picking two photon momenta k_i and k_j, setting $k_j = -k_i \equiv k$, multiplying by the photon propagator, and integrating over k. For each virtual photon we then obtain the expression

$$\frac{e^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + i\epsilon} \left(\frac{p'}{p' \cdot k} - \frac{p}{p \cdot k}\right) \cdot \left(\frac{p'}{-p' \cdot k} - \frac{p}{-p \cdot k}\right) \equiv \mathbf{X} \tag{6.78}$$

The factor of $1/2$ is required because our procedure has counted each Feynman diagram twice: interchanging k_i and k_j gives back the same diagram. It is possible to evaluate this expression by careful contour integration, but there is an easier way. Notice that this approximation scheme assigns to the diagram with one loop and no external photons the value

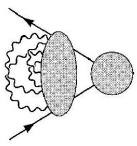
$$\bar{u}\left(p'\right)\left(i\mathcal{M}_{\text{hard}}\right)u(p)\cdot\mathbf{X}$$

Thus, $\mathbf{X}$ must be precisely the infrared limit of the one-loop correction to the form factor, as displayed in (6.61):

$$\mathbf{X}=-\frac{\alpha}{2\pi}f_{\text{IR}}\left(q^2\right)\log\left(\frac{-q^2}{\mu^2}\right) \tag{6.79}$$

A direct derivation of this result from (6.78) is given in Weinberg's paper cited above. Note that result (6.79) followed in our argument of the previous section only after the subtraction at $q^2=0$, and so we should worry whether (6.79) is consistent with the corresponding subtraction of the n th-order diagram. In addition, some of the diagrams we are summing contain external-leg corrections, which we have not discussed. Here we simply remark that neither of these subtleties affects the final answer; the proof requires the heavy machinery in the paper of Yennie, Frautschi, and Suura.

If there are m virtual photons we get m factors like (6.79), and also an additional symmetry factor of $1/m$! since interchanging virtual photons with each other does not change the diagram. We can then sum over m to obtain
the complete correction due to the presence of arbitrarily many soft virtual photons:


$$=\quad\times\sum_{m=0}^{\infty}\frac{\mathbf{X}^m}{m!}=\bar{u}(p')\big(i\mathcal{M}_{\text{hard}}\big)u(p)\exp(\mathbf{X}).$$

If in addition to the m virtual photons we also emit a real photon, we must multiply by its polarization vector, sum over polarizations, and integrate the squared matrix element over the photon's phase space. This gives an additional factor

$$\int\frac{d^3k}{(2\pi)^3}\frac{1}{2k}e^2\left(-g_{\mu\nu}\right)\left(\frac{p'^{\mu}}{p'\cdot k}-\frac{p^{\mu}}{p\cdot k}\right)\left(\frac{p'^{\nu}}{p'\cdot k}-\frac{p^{\nu}}{p\cdot k}\right)\equiv\mathbf{Y} \tag{6.81}$$

in the cross section. Assuming that the energy of the photon is greater than μ and less than E_ℓ (the detector threshold), this expression is simply

$$\mathbf{Y}=\frac{\alpha}{\pi}\mathcal{I}\left(\mathbf{v},\mathbf{v}'\right)\log\left(\frac{E_\ell}{\mu}\right)=\frac{\alpha}{\pi}f_{\text{IR}}\left(q^2\right)\log\left(\frac{E_\ell^2}{\mu^2}\right) \tag{6.82}$$

If n real photons are emitted we get n such factors, and also a symmetry factor of $1/n!$ since there are n identical bosons in the final state. The cross section for emission of any number of soft photons is therefore

$$\sum_{n=0}^{\infty}\frac{d\sigma}{d\Omega}\left(\mathbf{p}\to\mathbf{p}'+n\gamma\right)=\frac{d\sigma}{d\Omega}\left(\mathbf{p}\to\mathbf{p}'\right)\cdot\sum_{n=0}^{\infty}\frac{1}{n!}\mathbf{Y}^n=\frac{d\sigma}{d\Omega}\left(\mathbf{p}\to\mathbf{p}'\right)\cdot\exp(\mathbf{Y}) \tag{6.83}$$

Combining our results for virtual and real photons gives our final result for the measured cross section, to all orders in α, for the process $\mathbf{p}\to\mathbf{p}'+$ (any number of photons with $k<E_\ell$):

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)_{\text{meas.}} &= \left(\frac{d\sigma}{d\Omega}\right)_0 \times \exp(2\mathbf{X}) \times \exp(\mathbf{Y}) \\ &= \left(\frac{d\sigma}{d\Omega}\right)_0 \times \exp\left[-\frac{\alpha}{\pi} f_{\text{IR}}\left(q^2\right) \log\left(\frac{-q^2}{\mu^2}\right)\right] \times \exp\left[\frac{\alpha}{\pi} f_{\text{IR}}\left(q^2\right) \log\left(\frac{E_\ell^2}{\mu^2}\right)\right] \\ &= \left(\frac{d\sigma}{d\Omega}\right)_0 \times \exp\left[-\frac{\alpha}{\pi} f_{\text{IR}}\left(q^2\right) \log\left(\frac{-q^2}{E_\ell^2}\right)\right] \end{aligned} \tag{6.84}$$

The correction factor depends on the detector sensitivity E_ℓ, but is independent of the infrared cutoff μ. Note that if we expand this result to $\mathcal{O}(\alpha)$, we recover our earlier result (6.68). Now, however, the correction factor is controlled in magnitude-always between 0 and 1 .

In the limit $-q^2 \gg m^2$, our result becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{meas.}} = \left(\frac{d\sigma}{d\Omega}\right)_0 \times \left|\exp\left[-\frac{\alpha}{2\pi} \log\left(\frac{-q^2}{m^2}\right) \log\left(\frac{-q^2}{E_\ell^2}\right)\right]\right|^2 \tag{6.85}$$

In this limit, the probability of scattering without emitting a hard photon decreases faster than any power of q^2. The exponential correction factor, containing the Sudakov double logarithm, is known as the Sudakov form factor.

To conclude this section, let us calculate the probability, in the same approximation, that some hard scattering process is accompanied by the production of n soft photons, all with energies between $E_-$and E_+. The phase-space integral for these photons gives $\log\left(E_+/E_-\right)$instead of $\log\left(E_\ell/\mu\right)$. If we assign photons with energy greater than $E_+$to the "hard" part of the process, we find that the cross section is given by (6.84), times the additional factor

$$\begin{aligned}\text{Prob}\left(n\gamma \text{ with } E_- < E < E_+\right) = \frac{1}{n!}\Big[\frac{\alpha}{\pi}\ f_{\text{IR}}\left(q^2\right) \log\left(\frac{E_+^2}{E_-^2}\right)\Big]^n \\ \times \exp\left[-\frac{\alpha}{\pi} f_{\text{IR}}\left(q^2\right) \log\left(\frac{E_+^2}{E_-^2}\right)\right]\end{aligned} \tag{6.86}$$

This expression has the form of a Poisson distribution,

$$P(n) = \frac{1}{n!}\lambda^n e^{-\lambda},$$

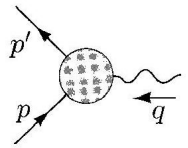
with

$$\lambda = \langle n \rangle = \frac{\alpha}{\pi} \log\left(\frac{E_+}{E_-}\right) \mathcal{I}\left(\mathbf{v}, \mathbf{v}'\right)$$

This is precisely the semiclassical estimate of the number of radiated photons that we made in Eq. (6.19).

Problems

6.1 Rosenbluth formula. As discussed Section 6.2, the exact electromagnetic interaction vertex for a Dirac fermion can be written quite generally in terms of two form factors $F_1\left(q^2\right)$ and $F_2\left(q^2\right)$:



$$= \bar{u}(p')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)\right]u(p),$$

where $q = p' - p$ and $\sigma^{\mu\nu} = \frac{1}{2}i\left[\gamma^\mu, \gamma^\nu\right]$. If the fermion is a strongly interacting particle such as the proton, the form factors reflect the structure that results from the strong interactions and so are not easy to compute from first principles. However, these form factors can be determined experimentally. Consider the scattering of an electron with energy $E \gg m_e$ from a proton initially at rest. Show that the above expression for the vertex leads to the following expression (the Rosenbluth formula) for the elastic scattering cross section, computed to leading order in α but to all orders in the strong interactions:

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2\left[\left(F_1^2 - \frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}\left(F_1 + F_2\right)^2\sin^2\frac{\theta}{2}\right]}{2E^2\left[1 + \frac{2E}{m}\sin^2\frac{\theta}{2}\right]\sin^4\frac{\theta}{2}},$$

where θ is the lab-frame scattering angle and F_1 and F_2 are to be evaluated at the q^2 associated with elastic scattering at this angle. By measuring $(d\sigma/d\cos\theta)$ as a function of angle, it is thus possible to extract F_1 and F_2. Note that when $F_1 = 1$ and $F_2 = 0$, the Rosenbluth formula reduces to the Mott formula (in the massless limit) for scattering off a point particle (see Problem 5.1).

6.2 Equivalent photon approximation. Consider the process in which electrors of very high energy scatter from a target. In leading order in α, the electron is connected to the target by one photon propagator. If the initial and final encrgies of the electron are E and E', the photon will carry momentum q such that $q^2 \approx -2EE'(1 - \cos\theta)$. In the limit of forward scattering, whatcver the energy loss, the photon momentum approaches $q^2 = 0$; thus the reaction is highly peaked in the forward direction. It is tempting to guess that, in this limit, the virtual photon becomes a real photon. Let us investigate in what sense that is true.

(a) The matrix element for the scattering process can be written as

$$\mathcal{M} = (-ie)\bar{u}\left(p'\right)\gamma^\mu u(p)\frac{-ig_{\mu\nu}}{q^2}\widehat{\mathcal{M}}^\nu(q),$$

where $\widehat{\mathcal{M}}^\nu$ represents the (in general, complicated) coupling of the virtual photon to the target. Let us analyze the structure of the piece $\bar{u}\left(p'\right)\gamma^\mu u(p)$. Let $q = (q^0, \mathbf{q})$, and define $\tilde{q} = (q^0, -\mathbf{q})$. We can expand the spinor product as:

$$\bar{u}\left(p'\right)\gamma^\mu u(p) = A \cdot q^\mu + B \cdot \tilde{q}^\mu + C \cdot \epsilon_1^\mu + D \cdot \epsilon_2^\mu,$$

where A, B, C, D are functions of the scattering angle and energy loss and ϵ_i are two unit vectors transverse to $\mathbf{q}$. By dotting this expression with q_μ, show that the coefficient B is at most of order θ^2. This will mean that we can ignore it in the rest of the analysis. The coefficient A is large, but it is also irrelevant, since, by the Ward identity, $q^\mu\widehat{\mathcal{M}}_\mu = 0$.

(b) Working in the frame where $p = (E, 0, 0, E)$, compute explicitly

$$\bar{u}\left(p'\right)\boldsymbol{\gamma}\cdot\boldsymbol{\epsilon}_i u(p)$$

using massless electrons, $u(p)$ and $u\left(p'\right)$ spinors of definite helicity, and $\boldsymbol{\epsilon}_1, \boldsymbol{\epsilon}_2$ unit vectors parallel and perpendicular to the plane of scattering. We need this quantity only for scattering near the forward direction, and we need only the term of order θ. Note, however, that for ϵ in the plane of scattering, the small $\hat{3}$ component of ϵ also gives a term of order θ which must be taken

into account.
(c) Now write the expression for the electron scattering cross section, in terms of $\left|\widehat{\mathcal{M}}^{\mu}\right|^2$ and the integral over phase space on the target side. This expression must be integrated over the final electron momentum p'. The integral over p'^3 is an integral over the energy loss of the electron. Show that the integral over $p'_\perp$ diverges logarithmically as $p'_\perp$ or $\theta \to 0$.
(d) The divergence as $\theta \to 0$ appears because we have ignored the electron mass in too many places. Show that reintroducing the electron mass in the expression for q^2,

$$q^2 = -2\left(EE' - pp'\cos\theta\right) + 2m^2,$$

cuts off the divergence and yields a factor of $\log\left(s/m^2\right)$ in its place.
(e) Assembling all the factors, and assuming that the target cross sections are independent of the photon polarization, show that the largest part of the electrontarget scattering cross section is given by considering the electron to be the source of a beam of real photons with energy distribution ($x = E_\gamma/E$):

$$N_\gamma(x)dx = \frac{dx}{x}\frac{\alpha}{2\pi}\left[1 + (1-x)^2\right]\log\left(\frac{s}{m^2}\right)$$

This is the Weizsacker-Williams equivalent photon approximation. This phenomenon allows us, for example, to study photon-photon scattering using e^+e^- collisions. Notice that the distribution we have found here is the same one that appeared in Problem 5.5 when we considered soft photon emission before electron scattering. It should be clear that a parallel general derivation can be constructed for that case.
6.3 Exotic contributions to $\boldsymbol{g-2}$. Any particle that couples to the electron can produce a correction to the electron-photon form factors and, in particular, a correction to $g-2$. Because the electron $g-2$ agrees with QED to high accuracy, these corrections allow us to constrain the properties of hypothetical new particles.
(a) The unified theory of weak and electromagnetic interactions contains a scalar particle h called the Higgs boson, which couples to the electron according to

$$H_{\text{int}} = \int d^3x \frac{\lambda}{\sqrt{2}} h\bar{\psi}\psi$$

Compute the contribution of a virtual Higgs boson to the electron $(g-2)$, in terms of λ and the mass m_h of the Higgs boson.
(b) QED accounts extremely well for the electron's anomalous magnetic moment. If $a = (g-2)/2$,

$$|a_{\text{expt.}} - a_{\text{QED}}| < 1 \times 10^{-10}$$

What limits does this place on λ and m_h? In the simplest version of the electroweak theory, $\lambda = 3\times10^{-6}$ and $m_h > 60\text{GeV}$. Show that these values are not excluded. The coupling of the Higgs boson to the muon is larger by a factor (m_μ/m_e): $\lambda = 6 \times 10^{-4}$. Thus, although our experimental knowledge of the muon anomalous magnetic moment is not as precise,

$$|a_{\text{expt.}} - a_{\text{QED}}| < 3 \times 10^{-8},$$

one can still obtain a stronger limit on m_h. Is it strong enough?
(c) Some more complex versions of this theory contain a pseudoscalar particle called the axion, which couples to the electron according to

$$H_{\rm int} = \int d^3x \frac{i\lambda}{\sqrt{2}} a\bar{\psi}\gamma^5\psi$$

The axion may be as light as the electron, or lighter, and may couple more strongly than the Higgs boson. Compute the contribution of a virtual axion to the $g-2$ of the electron, and work out the excluded values of λ and m_a.

16.6 K-l.I: Radiative corrections. General analysis of divergent integrals

Preface

Today there exist many excellent textbooks on quantum field theory. The most popular ones are listed in the bibliography to the present lectures. Nevertheless, everyone who gives lectures on quantum field theory faces the problem of selection of material and writing the lecture notes for students. The present text is just the lecture notes devoted to the radiative corrections in QFT. On this way, one encounters two problems, namely, the ultraviolet and the infrared divergences. Our task is to demonstrate how one can get rid of these divergences and obtain finite corrections to the cross-sections of elementary processes. During the course we describe the methods of Feynman diagram evaluation and regularization of divergences. In more detail, we consider the renormalization theory and elimination of ultraviolet divergencies in the Green functions off mass shell, as exemplified by scalar and gauge theories. In connection with the renormalization procedure we describe also the renormalization group formalism in QFT. As for the infrared divergences, in the literature one can find mainly the discussion of the IR divergencies in quantum electrodynamics. In non-Abelian theories as well as in QED with massless particles the situation is much more involved as there arise collinear divergences as well. In the last lecture, we show how one can get rid of these divergences using the methods developed in quantum chromodynamics. One more topic also related to divergences is the so-called anomalies. They also lead to unwanted ultraviolet divergent contributions. Therefore, a separate lecture is dedicated to the axial and conformal anomalies.

The presented text overlaps with many textbooks and is partly borrowed from there. However, the composition of the material and most of the calculations belong to the author, so we omit the direct references to any textbooks. It should be admitted that the style of presentation in different textbooks varies very much and the reader can choose the book according to his preferences. We mostly used the classical monograph by N.Bogoliubov and D.Shirkov when describing the renormalization theory and more modern book by M.Peskin and D.Schreder which we followed when discussing the infrared divergences.

Our experience in giving lectures on quantum field theory, the renormalization theory and the renormalization group tells us that this material is still complicated for perception and is not always presented clearly enough. One often meets with the lack of understanding of the complicated structure of the field theory which manifests itself in renormalization theory. Sometimes the nonrenormalizable theories are simplistically treated as the field theories with a dimensional coupling constant which otherwise have no difference from the renormalizable ones. The collinear divergences arising in theories with massless particles, despite a long history, have not also become the well-known part of the QFT course. Here we make an attempt of a simplified presentation of this complicated material. Of course, this means that one has to sacrifice some rigorousness and completeness. We hope that together with the existing literature the present lectures will serve the goal of clarification and mastering of quantum field theory and its applications to particle physics.

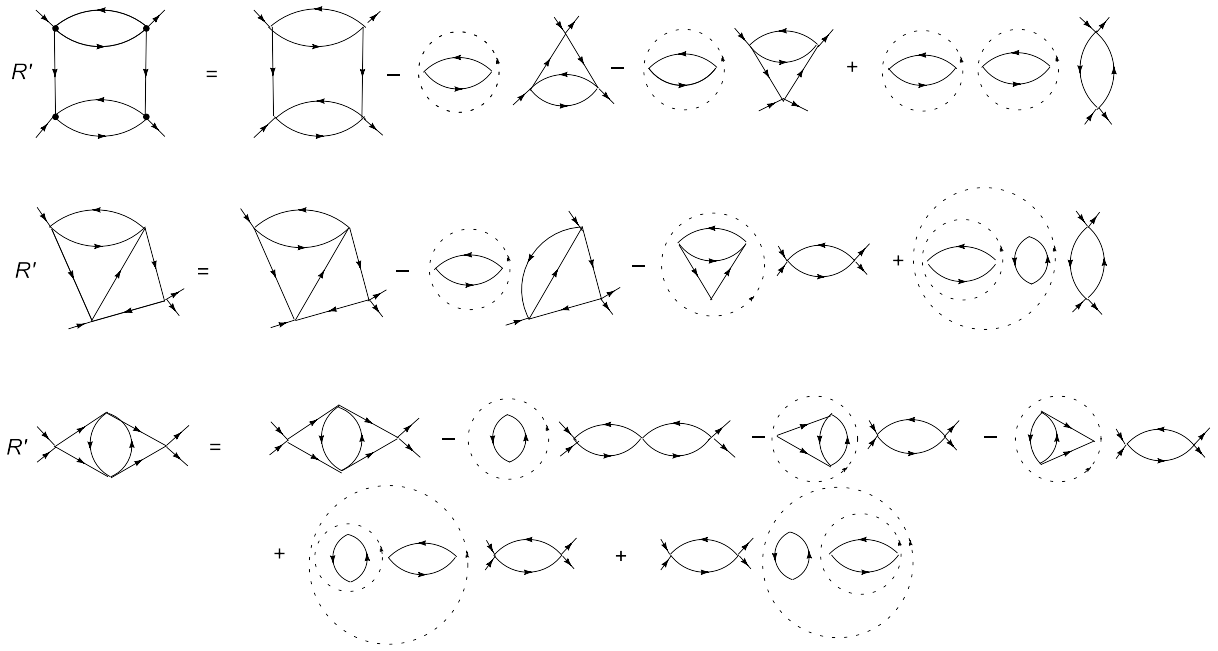


Figure 16.1: The $\mathcal{R}'$-operation for the multiloop diagrams

Afterword

Local quantum field theory, being the mathematical basis of elementary particle physics, is the logical continuation of quantum mechanics. It exploits the same basic ideas, but describing the system with an infinite number of degrees of freedom permits the creation and annihilation of particles in the course of the interaction. The modern formulation is based on the interaction representation which assumes the existence of the asymptotic states of the free fields. In the S-matric approach we presume that these fields interact in a local way in the space-time, and calculating the S-matrix elements one can find the probabilities of various processes. The most developed and reliable method of these calculations is the perturbation theory in the coupling constant which is similar to the one in quantum mechanics. However, due to a much more complicated structure of the field theory, the methods of perturbation theory encounter problems which have no analogy in quantum mechanics, namely the divergence of the appearing integrals for the radiative corrections. We have shown in these lectures how one can deal with these divergences which have the ultraviolet and the infrared nature and how to get the finite answers for the probabilities of the physical processes. We did not aim to prove the main theorems like the Bogoliubov-Parasiuk or the Kinoshita-Lee-Nauenberg theorem, but have exemplified how they work. The explicit calculations allow one to convince himself in the validity of the final conclusions.

It should be noted that the formalism of quantum field theory contains the physical principles which we have to follow sometimes not realizing it. Thus, for example, the ultraviolet divergences restrict the type of the interaction and, contrary to quantum mechanics, there are only a few types of allowed Lagrangians. Not without reason the renormalizability played such an important role in the formation of the Standard Model. The other example is the notion of the asymptotic states. Even starting with the free fields within the perturbation theory, from the requirement of the cancellation of the infrared divergences we come to the definition of the physical initial and final states which are essentially the coherent states.

The very fact that the gravitational interaction does not fit to the general scheme probably

means that local quantum field theory has a limited applicability and should be replaced by a more general construction. It might be nonlocal like in the string theory, or multidimensional one like in the brane-world theory. However, in any case, in the low energy limit one has the local quantum field theory though possibly going beyond the Standard Model that we considered here.

16.6.1 Radiative corrections

The formalism of quantum field theory, being the generalization of quantum mechanics to the case of an infinite number of degrees of freedom with nonconservation of the number of particles, allows one to describe the processes of scattering, annihilation, creation and decay of particles with the help of the set of well-defined rules. As in quantum mechanics the cross-section of any process is given by the square of the modulus of the probability amplitude calculated according to the Feynman rules for the corresponding Lagrangian integrated over the phase space. Since the exact calculations of the probability amplitudes seem to be impossible, one is bound to use the perturbation theory with a small parameter - the coupling constant - and get the result in the form of a power series. The leading terms of this series can be presented by Feynman diagrams without loops, the so-called tree diagrams. The examples of such diagrams for some typical processes in QED are shown in Fig.16.2.

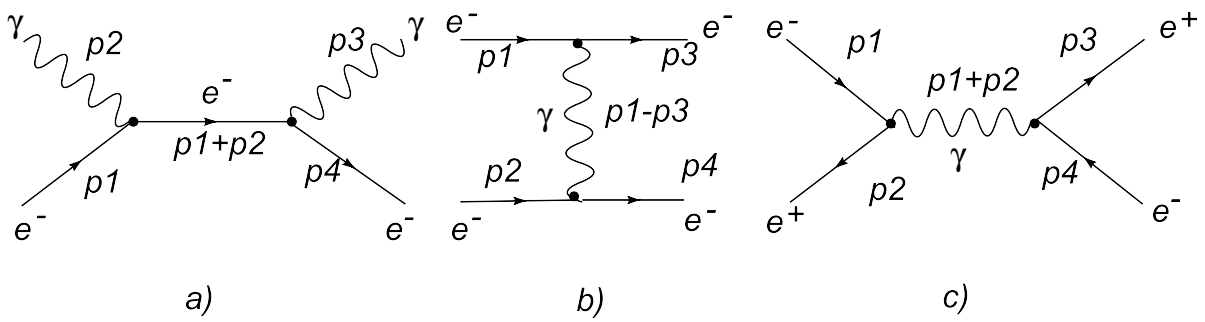


Figure 16.2: The examples of tree diagrams of different processes in QED:) the Compton scattering, b) the Mueller scattering, c) the annihilation of the particle-antiparticle pair. Shown are the momenta of external (real) and internal (virtual) particles

All the diagrams shown in Fig.16.2 are proportional to the square of the coupling constant e^2. They are constructed according to the well-known Feynman rules and do not contain any integration over momenta (when working in momentum representation) since due to the conservation of four-momentum all momenta are defined uniquely.

The situation changes when considering the next order of perturbation theory. As an example, in Fig.16.3 we show the corresponding diagrams for the Compton scattering.

They got the name of *radiative corrections* since in electrodynamics they correspond to the emission and absorption of photons. This name is also accepted in other theories for perturbative corrections. All these diagrams are proportional to the fourth power of the coupling constant e^4 and, hence, are the next order perturbations with respect to the tree diagrams. However, contrary to the tree diagrams, they contain a closed loop which requires the integration over the four-momenta running through the loop. Any loop corresponds to the bifurcation of momenta similarly to the bifurcation of the electric current, according to the Kirchhoff rules, so that the total momentum is conserved but the momentum running along each line is arbitrary. Therefore, one has to integrate over it.

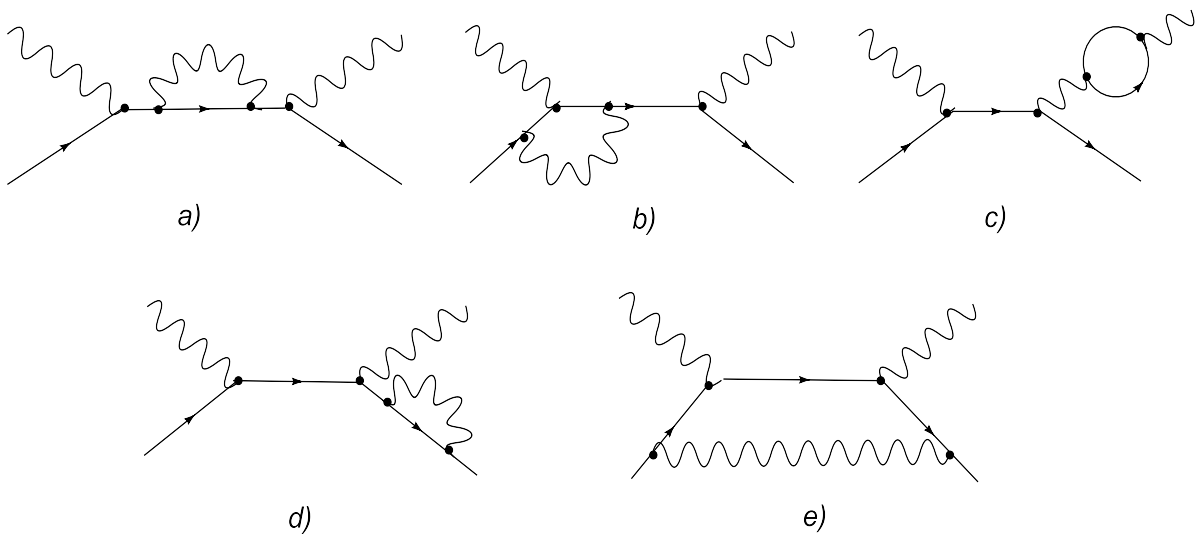


Figure 16.3: The one-loop diagrams for the process of the Compton scattering

16.6.2 Divergence of integrals

Prior to calculating the radiative corrections let us consider the behaviour of the integrand and the integral as a whole. As an example we take the diagrams of the Compton scattering shown in Fig.16.3. The integral corresponding to the diagram shown in Fig.16.3.a) has the form

$$\int d^4k \frac{\gamma^\mu(\hat{p}-\hat{k}+m)\gamma^\mu}{[k^2+i\varepsilon][(p-k)^2-m^2+i\varepsilon]}, \tag{16.1}$$

where the photon propagator is written in Feynman gauge and the integration takes place in Minkowskian space. We shall not calculate explicitly this integral now (we shall do it later) but consider the integrand from the point of view of the presence of singularities as well as the behaviour at small and large momenta.

The presence of poles in the propagators for momentum equal to the mass squared does not create any problem for the integration since according to the Feynman rules the denominator contains the infinitesimal imaginary term $\sim \varepsilon \to 0$, which defines the way to bypass the pole. The choice accepted in (16.1) corresponds to the causal Green function.

Consider now the case of $k_\mu \to 0$, the so-called *infrared* behaviour. Despite the presence of k^2 in the denominator, the singularity is absent due to the measure of the 4-dimensional integration which is also proportional to k^4. This is true for all such integrals. The singularities appear only for certain external momenta which are on mass shell and have a physical reason. Off shell the singularities are absent. For this reason we shall not discuss the infrared behaviour of the integrals so far.

Consider at last the case of $k_\mu \to \infty$, the so-called *ultraviolet* behaviour. Notice that in the denominator one has 4 powers of momenta, while in the numerator one has 1 plus 4 powers in the measure of integration. Hence one has 5-4=1, i.e. the integral is linearly divergent as $k_\mu \to \infty$. Is it the property of a particular integral or is it a general situation? What happens with the other diagrams?

Consider the integral corresponding to the diagram shown in Fig.16.3.). One has, according to

the Feynman rules

$$\int d^4k \frac{\gamma^\mu(\hat{p_1}-\hat{k})+m)\gamma^\nu(\hat{p_2}-\hat{k})+m)\gamma^\mu}{k^2[(p_1-k)^2-m^2][(p_2-k)^2-m^2]}. \tag{16.2}$$

We are again interested in the behaviour for $k_\mu \to \infty$. The counting of the powers of momenta in the numerator and the denominator gives: 6 in the denominator and 2 in the numerator plus 4 in the integration measure. Altogether one has 6-6=0, i.e., the integral is logarithmically divergent as $k_\mu \to \infty$.

Here we met the difficulty called the ultraviolet divergence of the integrals for the radiative corrections. The examples considered above are not exceptional but the usual ones. The corrections are infinite, which makes perturbation theory over a small parameter meaningless. The way out of this trouble was found with the help of the renormalization theory which will be considered later and now we try to estimate the divergence of the integrals in a theory with an arbitrary Lagrangian.

16.6.3 General analysis of ultraviolet divergences

Consider an arbitrary Feynman diagram G shown in Fig.16.4. and try to find out whether

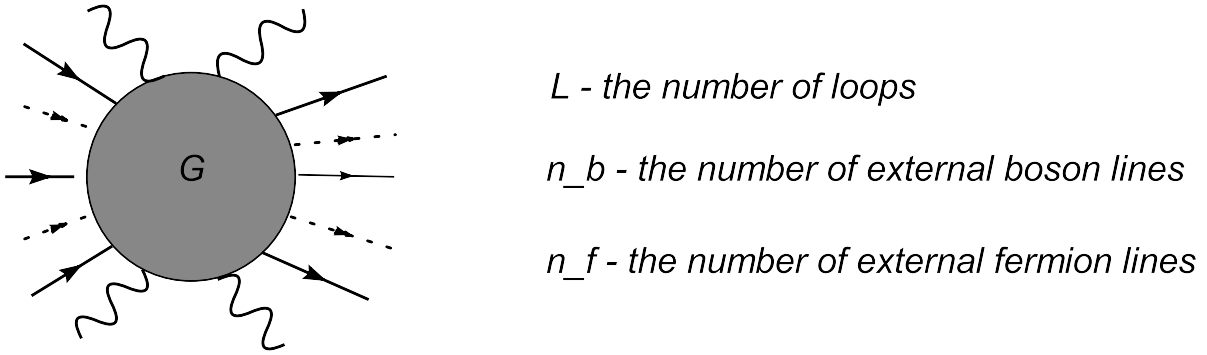


Figure 16.4: An arbitrary diagram containing L integrations

it is ultraviolet divergent or not. For this purpose we have to calculate the number of powers of momenta in the integrand: each internal loop leads to integration d^4p that gives 4 powers of momenta; each derivative in the vertex gives the momentum in p-space, i.e., 1; each internal line gives a propagator which behaves as p^{r_l}/p^2, i.e., $r_l - 2$ powers of momenta, where $r_l = 0, 1, 2$ for various fields. Combining all these powers together we get the quantity called *the index of divergence of the diagram* (UV)

$$\omega(G) = 4L + \sum_{vertices} \delta_v + \sum_{internal\ lines} (r_l - 2), \tag{16.3}$$

where L is the number of loops and δ_v is the number of derivatives in a vertex v.

The absence of the ultraviolet divergences means that $\omega(G) < 0$. However, one has to be careful, there might be subdivergences in some subgraphs. Hence, the necessary condition for finiteness is

$$\text{The finiteness condition (UV):}\ \ \omega(\gamma_i) < 0, \quad \forall \gamma_i \subset G,$$

where γ_i are all possible subgraphs of the graph G including the graph G itself.

There exists, however, a simpler way to answer the same question which does not need to analyse all the diagrams. One can see it directly from the form of the Lagrangian. To see this, let us introduce the quantity called *the index of the vertex* (UV)

$$\omega_v = \delta_v + b_v + \frac{3}{2} f_v - 4, \tag{16.4}$$

where δ_v, b_v and f_v are the number of derivatives, internal boson and fermion lines, respectively. Then the index of a diagram (16.3) can be written as

$$\omega(G) = \sum_{vertices} \omega_v^{max} + 4 - n_b - \frac{3}{2} n_f, \tag{16.5}$$

where ω_v^{max} corresponds to the vertex where all the lines are internal, n_b and n_f are the number of external boson and fermion lines, and we have used the fact that usually $r_l(boson) = 0$ and $r_l(fermion) = 1$.

Equation (16.5) tells us that the finiteness ($\omega(G) < 0$) can take place if $\omega_v \leq 0$ and the number of external lines is big enough. Prior to the formulation of conditions when it happens, let us consider some examples.

Example 1: The scalar theory $\mathcal{L}_{int} = -\lambda\varphi^4$.

In this case $\delta_v = 0$, $f_v = 0$, $b_v = 4$ and, hence, $\omega_v^{max} = 0$. Thus, according to (16.5), $\omega(G) = 4 - n_b - \frac{3}{2} n_f$ and everything is defined by the number of external lines. The situation is illustrated in Fig.16.5.

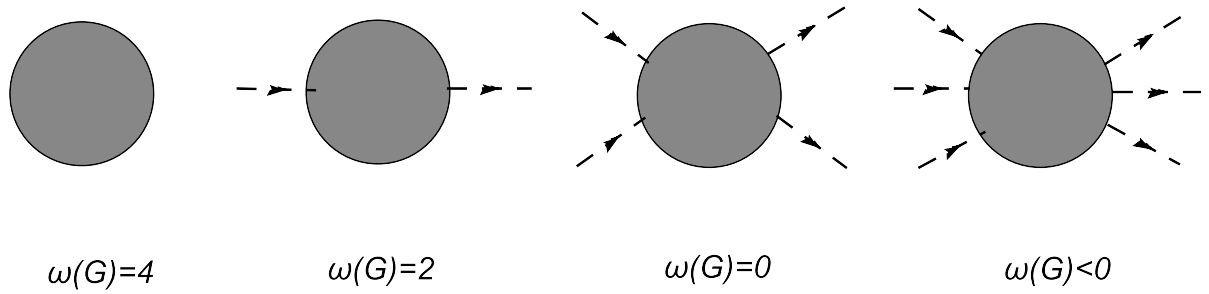


Figure 16.5: The indices of divergence of the diagrams in the scalar theory

We see that there exists a limited number of divergent structures in the φ^4 theory. These are the vacuum graphs, the two- and four-point functions. All the other diagrams having more than 4 external lines are convergent (though may have divergent subgraphs).

Example 2: Quantum Electrodynamics $\mathcal{L}_{int} = \bar{\psi}\hat{A}\psi$.

In this case $\delta_v = 0, f_v = 2, b_v = 1$, $\omega_v^{max} = 0$. Hence, $\omega(G) = 4 - n_b - \frac{3}{2} n_f$ and the situation is similar to the previous example, everything is defined by external lines. Divergent are the vacuum diagrams ($\omega(G) = 4$), the photon propagator ($\omega(G) = 2$), the electron propagator ($\omega(G) = 1$) and the triple vertex ($\omega(G) = 0$). All the other diagrams are convergent.

Example 3: Four-fermion interaction $\mathcal{L}_{int} = G\bar{\psi}\psi\bar{\psi}\psi$.

Here $\delta_v = 0, f_v = 4, b_v = 0,\ \omega_v^{max} = 2$ and, hence, $\omega(G) = 2N - \frac{3}{2}n_f$. Therefore, increasing the number of vertices we get new divergent diagrams independently of the number of external lines. The number of divergent structures happens to be infinite.

Thus, the key role is played by the maximal index of the vertex. All the theories may be classified according to the value of $\omega_v^{\max}$:

$$\omega_v^{\max} = \begin{cases} < 0 & \textit{Finite number of divergent diagrams}, \\ 0 & \textit{Finite number of divergent structures}, \\ > 0 & \textit{Infinite number of divergent structures}. \end{cases} \tag{16.6}$$

Below we show that for the first two types of theories we can handle the ultraviolet divergences with the help of the renormalization procedure. The theories with $\omega_v^{\max} = 0$ are called *renormalizable*, the theories with $\omega_v^{\max} > 0$ are called *nonrenormalizable*, and the theories with $\omega_v^{\max} < 0$ are called *superrenormalizable*.

16.6.4 The analysis of dimensions

The property of a theory with respect to ultraviolet divergences can be reformulated in terms of dimensions. Consider for this purpose an arbitrary term of the interaction Lagrangian which is the product of the field operators and their derivatives

$$\mathcal{L}_I(x) = g \prod_{i,j} \varphi_i(x) \partial \varphi_j(x). \tag{16.7}$$

Consider the action which is the four-dimensional integral of the Lagrangian density

$$A = \int d^4x \mathcal{L}(x), \tag{16.8}$$

and find the dimensions of parameters in eq.(16.7). As a unit of measure we take the dimension of a mass equal to 1. Then the dimension of length $[L] = -1$, the dimension of time is also $[T] = -1$, the dimension of derivative $[\partial_\mu] = 1$, the dimension of momenta $[p_\mu] = 1$. Since the action is dimensionless (we use the natural units where $\hbar = c = 1$)

$$[A] = 0,$$

the dimension of the Lagrangian is

$$[\mathcal{L}] = 4, \quad (D - \text{in D dimensional space.})$$

This gives us the dimensions of the fields. Indeed, from the kinetic term for the scalar field one finds

$$[(\partial\phi)^2] = 4 \to [\phi] = 1, \quad (\frac{D-2}{2} \text{ in D dimensional space}),$$

for the spinor field

$$[\bar{\psi}\hat{\partial}\psi] = 4 \to [\psi] = \frac{3}{2}, \quad (\frac{D-1}{2} \text{ in D dimensional space}),$$

for the vector field

$$[(\partial_\mu A_\nu - \partial_\nu A_\mu)^2] = 4 \to [A_\mu] = 1, \quad (\frac{D-2}{2} \text{ in D dimensional space}).$$

This allows one to find the dimension of the coupling constant in (16.7)

$$[g] = 4 - \delta_v - b_v - \frac{3}{2} f_v = -\omega_v^{max}. \tag{16.9}$$

Then the classification of interactions (16.6) can be written as

$$[g] = \begin{cases} > 0 & \textit{Superrenormalizable theories}, \\ 0 & \textit{Renormalizable theories}, \\ < 0 & \textit{Nonrenormalizable thoeries}. \end{cases} \tag{16.10}$$

Consider which category various theories belong to. For this purpose we have to calculate the dimensions of the couplings.

Illustration

$$\begin{array}{lll} \mathcal{L}_{\varphi^3} = -\lambda\varphi^3 & \Rightarrow & [\lambda] = 1, \quad SuperRen. \\ \mathcal{L}_{\varphi^4} = -\lambda\varphi^4 & \Rightarrow & [\lambda] = 0, \quad Ren. \\ \mathcal{L}_{QED} = e\overline{\psi}\gamma^\mu A_\mu \psi & \Rightarrow & [e] = 0, \quad Ren. \\ \mathcal{L}_{gauge} = -\frac{1}{4}F_{\mu\nu}^2 = -\frac{1}{4}\left[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c\right]^2 & \Rightarrow & [g] = 0, \quad Ren. \\ \mathcal{L}_{Yukawa} = y\overline{\psi}\varphi\psi & \Rightarrow & [y] = 0. \quad Ren. \end{array}$$

Thus, all these models are renormalizable.

$$\begin{array}{lll} \mathcal{L} = -h\varphi^6 & \Rightarrow & [h] = -2, \quad Nonren. \\ \mathcal{L} = G\overline{\psi}\psi\overline{\psi}\psi & \Rightarrow & [G] = -2 \quad Nonren. \\ \mathcal{L} = \kappa\overline{\psi}\partial_\mu V_\mu \psi & \Rightarrow & [\kappa] = -1 \quad Nonren. \\ \mathcal{L} = \gamma\overline{\psi}\partial_\mu\varphi\gamma^\mu\psi & \Rightarrow & [\gamma] = -1. \quad Nonren. \end{array}$$

All these models on the contrary are nonrenormalizable. Notice that they include the four-fermion or current-current interaction which was previously used in the theory of weak interactions.

Hence, we come to the following conclusion: the only renormalizable interactions in four dimensions are:

i) the φ^4 interaction;

ii) the Yukawa interaction;

iii) the gauge interaction;

iv) the theory φ^3 is superrenormalizable. It contains only two divergent diagrams shown in Fig.16.6.

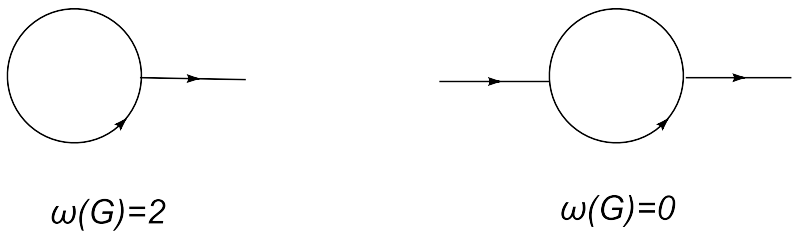


Figure 16.6: The only divergent diagrams in the ϕ^3 theory

If one looks at the spins of particles involved in the interactions, one finds out that they are strongly restricted. The renormalizable interactions contain only the fields with spins 0, 1/2 and

1. All the models with spins 3/2, 2, etc. are nonrenormalizable. The latter include also gravity. Indeed, the coupling constant in this case is the Newton constant which has dimension equal to $[G] = -2$, i.e., quantum gravity is nonrenormalizable.

Since we do not know how to handle the nonrenormalizable interactions because the ultraviolet divergences are out of control, there are only three types of interactions which are used in the construction of the Standard Model of fundamental interactions, namely the φ^4, the Yukawa and the gauge interactions with the scalar, spinor and vector particles.

Here one has to make a comment concerning the vector fields with $M \neq 0$. Remind the form of the propagator of the massive vector field

$$\overline{V_\mu V_\nu} = i\frac{g_{\mu\nu} - k_\mu k_\nu/M^2}{M^2 - k^2 - i\epsilon}.$$

It gives $r_l = 2$, which leads to some modification of the formulas used above and finally to the nonrenormalizability of the theory. The only known way to avoid this difficulty is the spontaneous breaking of symmetry. In this case,

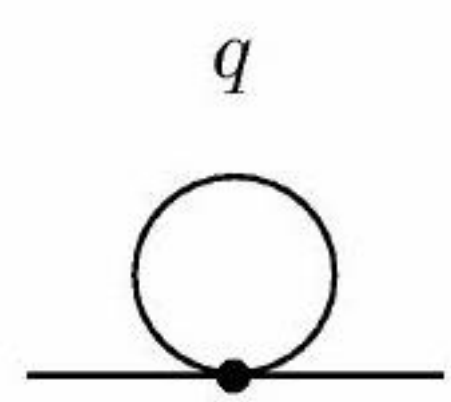
$$\overline{V_\mu V_\nu} = i\frac{g_{\mu\nu} - k_\mu k_\nu/k^2}{M^2 - k^2 - i\epsilon},$$

that gives $r_l = 0$ and the theory happens to be renormalizable. This mechanism is used in the Standard Model to give masses to the intermediate weak bosons without breaking the renormalizability of the theory.

16.7 L4 Ultraviolet Divergences and Regularization

16.7.1 1. Dimensional Regularization of Divergences

Попробуем вычислить простейшую диаграмму (f) на рис. 3:



Соответствующий вклад в iM, обозначаемый $-i\Sigma$, равен

$$-i\Sigma^{(1)} = -\frac{\lambda_0\mu^{4-d}}{2}\int\frac{\mathrm{d}^{(d)}k}{(2\pi)^d}\frac{1}{-\left(k^0\right)^2 + \mathbf{k}^2 + m_0^2 - i\varepsilon}$$

Здесь μ - константа, имеющая размерность массы, так что λ_0 - безразмерная константа. При интегрировании по k^0 сделаем поворот Вика (см. рис. 1, 2 в лекции 1) $k^0 = ik^d$:

$$-i\Sigma^{(1)} = -\frac{i\lambda_0\mu^{4-d}}{2}\int\limits_E \frac{\mathrm{d}^{(d)}k}{(2\pi)^d}\frac{1}{k^2+m_0^2} \tag{1}$$

Обозначение $\int\limits_E \mathrm{d}^{(d)}k$ указывает на то, что интегрирование идёт по d-мерному евклидову пространству, в котором $k^2 = \left(k^1\right)^2 + \ldots + \left(k^d\right)^2$. Интеграл (1) при $d \geq 2$ расходится, при $d < 2$ - сходится. Мы видим, что сходимость диаграмм Фейнмана существенно зависит от размерности d пространства-времени. С другой стороны, большинство симметрий теории поля не зависят от d. Важнейшими из таких симметрий являются лоренц- и калибровочная инвариантности. Поэтому возникла идея (Ж. 't Хоофт, М. Вельтман) вычислять диаграммы Фейнмана при произвольной комплексной размерности пространства, устремляя в конце вычислений параметр d к нужному значению.

Для дальнейших вычислений нам понадобятся следующие формулы и интегралы.

Пусть $V_R^{(d)} = S^{(d)}R^d$ - объём шара радиуса R в d-мерном евклидовом пространстве. Вычислим $S^{(d)}$ - объём единичного шара в d-мерном пространстве. Имеем:

$$\pi^{d/2} = \int \mathrm{d}^{(d)}x e^{-\left(x_1^2+\ldots+x_d^2\right)} = \int\limits_0^\infty e^{-R^2}\,\mathrm{d}V_R^{(d)} = dS^{(d)}\int\limits_0^\infty e^{-R^2}R^{d-1}\,\mathrm{d}R =$$

$$= \frac{d}{2}S^{(d)}\int\limits_0^\infty e^{-t}t^{d/2-1}\,\mathrm{d}t = S^{(d)}\Gamma\left(\frac{d}{2}\right)\left(\frac{d}{2}\right)$$

Таким образом

$$S^{(d)} = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)\left(\frac{d}{2}\right)} = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)} \tag{2}$$

Везде $\Gamma(z)$ - гамма-функция. При $c > 0$ и $\alpha > d/2$ имеем:

$$\int\limits_E \frac{\mathrm{d}^{(d)}k}{\left(k^2+c\right)^\alpha} = dS^{(d)}\int\limits_0^\infty \frac{k^{d-1}\,\mathrm{d}k}{\left(k^2+c\right)^\alpha} = \left(\frac{d}{2}\right)S^{(d)}\int\limits_0^\infty \frac{t^{d/2-1}\,\mathrm{d}t}{(t+c)^\alpha} = \left(\frac{d}{2}\right)S^{(d)}\left[c^{d/2-\alpha}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\alpha-\frac{d}{2}\right)}{\Gamma(\alpha)}\right]$$

Теперь при помощи (2) получаем:

$$\int\limits_E \frac{\mathrm{d}^{(d)}k}{\left(k^2+c\right)^\alpha} = \pi^{d/2}c^{d/2-\alpha}\frac{\Gamma\left(\alpha-\frac{d}{2}\right)}{\Gamma(\alpha)} \tag{3}$$

Интеграл (3) был вычислен при условии $d < 2\alpha$. Однако правая часть в (3) является аналитической функцией всех параметров. Поэтому при произвольных комплексных значениях параметров d. α и с будем считать правую часть уравнения (3) определением интеграла, стоящего в левой части равенства (3).

При помощи (3) вычисляем интеграл (1):

$$-i\Sigma^{(1)} = -\frac{i\lambda_0\mu^{4-d}m_0^{d-2}}{2^{d+1}\pi^{d/2}}\Gamma\left(1-\frac{d}{2}\right) \tag{4}$$

Нас интересует значение $d = 4$ или значения d, близкие к числу 4. Поэтому введём обозначения

$$d = 4 - \varepsilon, \quad \ln \frac{\Lambda}{\mu} = \frac{1}{\varepsilon}, \quad \varepsilon \to +0 \tag{5}$$

Как будет видно из дальнейших вычислений, параметр $\Lambda \to \infty$ играет роль максимально допустимого импульса в теории - импульса обрезания. Перепишем (4) в новых обозначениях:

$$-i\Sigma^{(1)} = -\frac{i\lambda_0 \mu^{\varepsilon} m_0^{2-\varepsilon}}{2^{(5-\varepsilon)}\pi^{(2-\varepsilon/2)}} \Gamma\left(-1 + \frac{\varepsilon}{2}\right) \tag{6}$$

При помощи формул

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \quad \Gamma(1) = 1 \tag{7}$$

находим, что величина (6)

$$-i\Sigma^{(1)} \underset{\varepsilon \to +0}{} \frac{i\lambda_0 m_0^2}{2^4 \pi^2} \left(\frac{1}{\varepsilon}\right) + \text{ const} \tag{8}$$

где const - конечная величина при $\varepsilon \to 0$.

Теперь займёмся вычислением однопетлевых диаграмм на рис. 7 (n-петлевой диаграммой называется такая диаграмма, у которой имеется n нефиксированных импульсов, по которым идёт интегрирование). Каждая из этих трёх диаграмм представляется интегралом

$$I = \frac{1}{2}\lambda_0^2 \mu^{2(4-d)} \int \frac{\mathrm{d}^{(d)} q}{(2\pi)^d} D_c(q) D_c(p-q) \tag{9}$$

Сначала оценим интеграл (9) непосредственно в 4 -мерном пространстве-времени. Рассмотрим этот интеграл в ультрафиолетовой области ($q \to \infty$). Так как внешние импульсы и масса фиксированы, то ими можно пренебречь при вычислении расходящейся части. Это можно сделать не всегда, но в данном случае, когда степень расходимости интеграла не выше логарифмической, внешними импульсами и массами можно пренебречь: учёт поправок к интегралу (9), пропорциональных m и p, приводит к сходящимся интегралам.

Сделаем поворот Вика. Тогда расходящаяся часть суммы трёх интегралов в (15 Л3) равна

$$iM_{div}^{(2)} = \frac{3i\lambda_0^2}{32\pi^4} \int\limits_E \frac{\mathrm{d}^{(4)} q}{(q^2)^2} = \frac{3i\lambda_0^2}{16\pi^2} \int\limits_{\mu}^{\Lambda} \frac{\mathrm{d}q}{q}, \quad \Lambda \to \infty \tag{10}$$

Здесь Λ - промежуточный импульс обрезания, μ - нижний предел интегрирования, имеющий порядок физической массы частицы или внешних импульсов, который называется точкой нормировки.

Мы опять видим, что амплитуды перехода в квантовой теории поля содержат расходящиеся вклады.

Приведём ещё один пример.

Диаграмма (g) на рис. 3 даёт вклад в амплитуду перехода частицы в эту же частицу, аналогично диаграмме (f) на рис. 3:

$$-i\Sigma^{(2)} = -\frac{i\lambda_0^2}{6}\int \frac{\mathrm{d}^{(4)}q_1\ \mathrm{d}^{(4)}q_2}{(2\pi)^8} D_c\left(q_1\right) D_c\left(q_2\right) D_c\left(k-q_1-q_2\right) \tag{11}$$

Интеграл (11), так же как и интеграл (1), при $d = 4$ расходится квадратично.

16.7.2 2. Index of Divergence of Diagrams

Вычислим индекс расходимости произвольной диаграммы. Для этого сделаем масштабное преобразование всех импульсов в диаграмме, а также массы частицы: $k_i, p_f, q, m \ \rightarrow \ \alpha k_i, \alpha p_f, \alpha q, \alpha m$. При таком преобразовании подынтегральное выражение умножается на α^ω, где ω и есть индекс диаграммы. Очевидно, что все внутренние линии дают вклад в число ω, равный $(-2L_{in})$. Легко увидеть, что число d-мерных дифференциалов от внутренних независимых импульсов равно $L = [L_{in} - (n-1)]$, где n - число вершин диаграммы. Действительно, независимых внутренних импульсов в диаграмме с n вершинами имеется на $(n-1)$ меньше, чем внутренних линий, так как в каждой из n вершин имеет место закон сохранения d-импульса. Однако совокупность n законов сохранения содержит в себе один закон сохранения совокупности внешних входящих и выходящих d-импульсов. Этим объясняется уменьшение независимых d-импульсов не на n, а на $(n-1)$. Поэтому имеем для числа ω

$$\omega = d\left(L_{in} - n + 1\right) - 2L_{in} \tag{12}$$

Обозначим через L_{ex} суммарное число внешних линий. Имеем:

$$L_{ex} + 2L_{in} = 4n \tag{13}$$

Действительно, к каждой вершине подходит 4 линии, но каждая внутренняя линия подходит к двум вершинам. Исключим из (12) число n при помощи равенства (13):

$$\omega = \frac{1}{2}(d-4)L_{in} + d\left(1 - \frac{L_{ex}}{4}\right) \tag{14}$$

Диаграмма является формально сходящейся, если $\omega < 0$.

Мы рассмотрели расходимости диаграмм f и g на рис. 3 с $L_{ex} = 2$ и однопетлевые диаграммы на рис. 7 с $L_{ex} = 4$ при $d = 4$. Согласно (14) при $d = 4$ диаграммы с двумя внешними линиями расходятся квадратично, однако в них содержатся также и логарифмические расходимости. Это будет показано ниже. Диаграммы с четырьмя внешними линиями расходятся лишь логарифмически, что и было выяснено на примере однопетлевых диаграмм на рис. 7. Подчеркнём, что в рассматриваемой модели при $d = 4$ степень расходимости диаграммы зависит не от её числа петелъ, а от числа внешних линий. Фактически это свойство является следствием безразмерности константы связи λ_0, по которой идёт разложение. Действительно, лишь при $d = 4$ константа λ_0 безразмерна. При $d > 4$ её размерность такова, что степень расходимости диаграммы возрастает по мере её усложнения (см. (14)). Такие теории называются неперенормируемыми. При $d < 4$ теория ϕ^4 относится к так называемым сверхперенормируемым, поскольку в теории содержится лишь конечное число расходящихся диаграмм.

16.7.3 3. Diagram with Four External Lines

Теперь займёмся вычислением интеграла (9) при произвольном комплексном значении параметра d. Причинные гриновские функции допускают следующее представление:

$$D_c(q) = i\int\limits_0^\infty \exp\left[i\alpha\left(q^2 - m_0^2 + i\varepsilon\right)\right]\mathrm{d}\alpha \tag{15}$$

Делая поворот Вика и используя (15), переписываем (9) в виде

$$I = -i\frac{\lambda_0^2\mu^{2(4-d)}}{2(2\pi)^d}\int\limits_0^\infty \mathrm{d}\alpha\int\limits_0^\infty \mathrm{d}\beta\int\limits_E \mathrm{d}^{(d)}q\exp\left\{-i\alpha\left(q^2+m_0^2\right) - i\beta\left[(p-q)^2+m_0^2\right] - \varepsilon(\alpha+\beta)\right\}$$

Здесь интеграл по q является гауссовым и вычисляется при помощи формулы (13 Л1), где следует сделать замены $N \to d, \mathcal{M} \to -[2(\alpha+\beta)]^{-1}, \xi \to 2\beta p$. После вычисления этого интеграла возвращаемся в пространство Минковского:

$$I = -\frac{i}{2}\lambda_0^2\mu^{2(4-d)}(4\pi i)^{-d/2}\int\limits_0^\infty \mathrm{d}\alpha\int\limits_0^\infty \mathrm{d}\beta\frac{1}{(\alpha+\beta)^{d/2}}\exp\left\{i\frac{\alpha\beta}{\alpha+\beta}p^2 - i(\alpha+\beta)\left(m_0^2 - i\varepsilon\right)\right\} \tag{16}$$

Перепишем интеграл (16) в новых переменных интегрирования $\alpha = \lambda t, \beta = (1-\lambda)t, \quad 0 < \lambda < 1,\ 0 < t < \infty$:

$$I = -\frac{i}{2}\lambda_0^2\mu^{2(4-d)}(4\pi i)^{-d/2}\int\limits_0^1 \mathrm{d}\lambda\int\limits_0^\infty \mathrm{d}t t^{(1-d/2)}\exp\left\{-t\left[-i\lambda(1-\lambda)p^2 + im_0^2 + \varepsilon\right]\right\} \tag{17}$$

Внутренний интеграл в (17) вычисляется при помощи формулы

$$\int\limits_0^\infty \mathrm{d}t t^{z-1}e^{-kt} = k^{-z}\Gamma(z), \quad \operatorname{Re} k > 0 \tag{18}$$

В нашем случае $z = 2 - d/2, \quad k = i\left[m_0^2 - \lambda(1-\lambda)p^2\right] + \varepsilon$. Поэтому имеем:

$$\frac{I}{\mu^{4-d}} = \frac{i\lambda_0^2\mu^{(4-d)}\Gamma\left(2-\frac{d}{2}\right)}{2(4\pi)^{d/2}}\int\limits_0^1 \mathrm{d}\lambda\left[m_0^2 - \lambda(1-\lambda)p^2\right]^{d/2-2} \tag{19}$$

Разложим (с учётом (7)) правую часть (19) в ряд Лорана вблизи точки $d = 4$:

$$\frac{I}{\mu^{4-d}} = \frac{i\lambda_0^2}{(4\pi)^2(4-d)} + \frac{i\lambda_0^2[\psi(1) + \ln 4\pi]}{8(2\pi)^2} -$$
$$-\frac{i\lambda_0^2}{8(2\pi)^2}\int\limits_0^1 \mathrm{d}\lambda\ln\frac{m_0^2 - \lambda(1-\lambda)p^2 - i\varepsilon}{\mu^2}, \quad \psi(\alpha) = \frac{\mathrm{d}}{\mathrm{d}\alpha}\ln\Gamma(\alpha) \tag{20}$$

Отсюда мы видим, что расходящаяся часть в $iM^{(2)}$ равна

$$iM^{(2)}_{div} = \frac{3i\lambda_0^2}{16\pi^2}\frac{1}{4-d} \tag{21}$$

Сравнивая (21) и (10), мы оправдываем соотношение (5):

$$\frac{1}{4-d} = \frac{1}{\varepsilon} = \ln\frac{\Lambda}{\mu} \tag{22}$$

Укажем на следующее важное обстоятельство: метод размерностной регуляризации учитывает лишь логарифмические расходимости. Действительно, если $d > 2\alpha$, то полагая $c = 0$ в (3), получаем:

$$\int\limits_E \frac{\mathrm{d}^d k}{(k^2)^\alpha} = 0 \quad \text{при } d > 2\alpha \tag{23}$$

Равенство (23) означает, что в методе размерностной регуляризации при $d > 2\alpha$ интеграл (23) определён и он равен нулю. Отсюда следует автоматическое устранение расходимостей степени выше логарифмической. Это же видно, в частности, из формулы (8).

16.7.4 4. Elimination of Divergences

Предположим, что в теории возмущений, описываемой рядом диаграмм Фейнмана, содержатся расходимости в диаграммах с ограниченным числом внешних линий. Таким примером является рассматриваемая теория ϕ^4 при $d = 4$. Более того, в этой теории расходятся лишь диаграммы с двумя и четырьмя внешними линиями, а действие содержит слагаемые, пропорциональные ϕ^2 и ϕ^4. В такой ситуации для устранения расходимостей вводится промежуточная регуляризация, смысл которой заключается в обрезании петлевых интегралов по нефиксированным импульсам неким параметром Λ, стремящимся к бесконечности, Это обрезание проводится таким образом, чтобы была сохранена Лоренц-инвариантность, а также, по возможности, все прочие симметрии, имеющиеся в теории. Размерностная регуляризация является наиболее используемым методом такой регуляризации. Оказывается, что в некоторых случаях (как в теории ϕ^4 при $d = 4$) исходные параметры
теории (затравочные константы) можно выбрать такими функциями параметра обрезания Λ, что при $\Lambda \to \infty$ все расходимости в теории сокращаются. В рассматриваемой нами скалярной теории поля способ действий таков: если в неком порядке теории возмущений расходимости уже устранены, то в следующем порядке ТВ расходятся диаграммы с двумя и с четырьмя внешними линиями. В первом случае расходимость устраняется переделкой (перенормировкой) затравочной массы, а во втором случае - перенормировкой затравочной константы связи. При этом может оказаться, что затравочные константы не имеют конечного предела при устремлении промежуточного импульса обрезания к бесконечности. Но в процедуре перенормировки наблюдаемые (или физические, или перенормированные) величины (массы частиц, константы взаимодействия физических частиц, амплитуды перехода) должны стремиться к определённому пределу при устремлении промежуточного импульса обрезания к бесконечности. Затравочные константы теории разлагаются в ряд по перенормированным константам, причём коэффициенты в этом разложении являются функциями параметра обрезания.

Теории, допускающие подобную процедуру перенормировки, называются перенормируемыми теориями.

Продемонстрируем сказанное на примере перенормировки констант в теории ϕ^4 в четырёхмерном пространстве-времени. Мы увидим, что эта теория является перенормируемой. Пусть затравочная константа связи λ_0, входящая в исходное (затравочное) действие, разлагается в ряд по физической константе связи λ. Далее мы увидим, что физическая константа связи может быть определена через значение амплитуды перехода двух частиц в две частицы при некоторых значениях импульсов частиц. Пусть во втором порядке

$$\lambda_0 = \lambda + \frac{3\lambda^2}{16\pi^2} \ln \frac{\Lambda}{\mu}, \quad \lambda_0^2 = \lambda^2 + O\left(\lambda^3\right) \tag{24}$$

Тогда согласно (10) и (24) в сумме диаграмм на рис. 7 расходимости сокращаются по крайней мере с точностью $O\left(\lambda^3\right)$ и амплитуда $2 \to 2$ является конечным выражением, зависящим от физической безразмерной константы λ.

16.7.5 5. Mandelstam Invariant Variables

В заключение лекции дадим определение удобных лоренц-инвариантных переменных, при помощи которых удобнее всего описывать амплитуды перехода двух частиц в две частицы.

Прежде всего заметим, что построенная ТВ явно лоренц-инвариантна. Это значит, что все амплитуды перехода как функции внешних 4-импульсов являются лоренц-инвариантами, то есть они зависят лишь от лоренц-инвариантов, построенных из внешних 4-импульсов. Это замечание остаётся в силе для диаграмм Фейнмана, внешние импульсы которых не находятся на массовой поверхности.

Пусть в реакции участвует всего 4 частицы с массами m_1, m_2, m_3, m_4, причём при $t \to -\infty$ имелось 2 частицы и при $t \to +\infty$ имеется также 2 частицы. Обобщённый закон сохранения 4 -импульса запишем в виде

$$k_1 + k_2 + k_3 + k_4 = 0 \tag{25}$$

При этом предполагается, что, если, например, идёт реакция $1 + 2 \to 3 + 4$, то

$$k_1^0 > 0, k_2^0 > 0, k_3^0 < 0, k_4^0 < 0 \tag{26}$$

и т.д.

Из четырёх 4-импульсов, находящихся на массовой поверхности и удовлетворяющих уравнению (25), можно составить всего 2 независимых инвариантных переменных. Действительно, в силу (25) всего три импульса независимы, например, k_1, k_2, k_3. Из них можно составить всего 6 инвариантов: $k_1^2 = m_1^2, \quad k_2^2 = m_2^2, k_3^2 = m_3^2, k_1k_2, k_1k_3, k_2k_3$. Лишь последние 3 из них являются переменными. Кроме того, так как $\left(k_1 + k_2 + k_3\right)^2 = k_4^2 = m_4^2$, то независимых переменных остаётся 2.

Удобно ввести три лоренц-инвариантных переменных Мандельстама:

$$\begin{aligned} s &= \left(k_1 + k_2\right)^2 = \left(k_3 + k_4\right)^2 \\ t &= \left(k_1 + k_3\right)^2 = \left(k_2 + k_4\right)^2 \\ u &= \left(k_1 + k_4\right)^2 = \left(k_2 + k_3\right)^2 \end{aligned} \tag{27}$$

Легко проверить, что

$$s+t+u=m_1^2+m_2^2+m_3^2+m_4^2 \tag{28}$$

Теперь рассмотрим самый простой случай $m_1=m_2=m_3=m_4=m$ и реакцию

$$1+2\longrightarrow 3+4 \tag{29}$$

для которой переменные мандельстама принимают значения в следующих областях:

$$s\geq 4m^2,\quad t\leq 0,\quad u\leq 0 \tag{30}$$

О реакции (29) говорят, что это - s-канальная реакция. Амплитуда реакции в t-канале

$$1+3\longrightarrow 2+4 \tag{31}$$

получается аналитическим продолжением амплитуды в s-канале в область переменных

$$t\geq 4m^2,\quad s\leq 0,\quad u\leq 0 \tag{32}$$

Это возможно, так как в теории S-матрицы амплитуды являются аналитическими функциями инвариантных переменных.

Введём обозначение

$$iM^{(2)}(s)=\frac{1}{2}\lambda_0^2\int\frac{\mathrm{d}^dq}{(2\pi)^d}D_c(q)D_c\left(k_1+k_2-q\right) \tag{33}$$

Тогда амплитуда в (15 ЛЗ) представляется симметричным образом:

$$iM^{(2)}=i\left[M^{(2)}(s)+M^{(2)}(t)+M^{(2)}(u)\right] \tag{34}$$

Укажем ещё один удобный способ параметризации петлевых интегралов, отличный от параметризации (14)-(15).

Очевидно имеем:

$$\frac{1}{AB}=\int\limits_0^1\frac{\mathrm{d}x}{[xA+(1-x)B]^2}=\int\limits_{x,y\geq 0}\frac{\mathrm{d}x\ \mathrm{d}y\delta(x+y-1)}{(xA+yB)^2} \tag{35}$$

Дифференцируя (35) по B, получаем

$$\frac{1}{AB^{n-1}}=\int\limits_{x,y\geq 0}\mathrm{d}x\ \mathrm{d}y\delta(x+y-1)\frac{(n-1)y^{n-2}}{[xA+yB]^n} \tag{36}$$

Задачи

1. Доказать по индукции, что

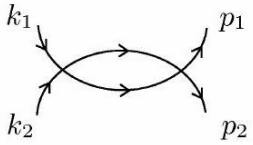
$$\frac{1}{A_1A_2...A_n}=\int\limits_{x_i\geq 0}\mathrm{d}x_1...\mathrm{d}x_n\delta\left(\sum_{i=1}^n x_i-1\right)\frac{(n-1)!}{(x_1A_1+...+x_nA_n)^n} \tag{37}$$

Переменные $\{x_i\}$ в этом интеграле называются фейнмановскими параметрами.

Указание. Использовать равенства (35) и (36).

Вергелес 4.2

Вычислить s-канальную однопетлевую диаграмму



при помощи параметризации Фейнмана. Указать, при каких значениях s эта диаграмма вещественна.

Ответ:

$$iM = \frac{i\lambda_0^2 \mu^{4-d} \Gamma(2-d/2)}{2(4\pi)^{d/2}} \int\limits_0^1 \mathrm{d}x \left[m^2 - x(1-x)s - i\varepsilon\right]^{d/2-2}$$

M вещественна при $s \leq 4m^2$.

16.8 L6 Renormalization Group

16.8.1 1. Wilsonian Approach to Renormalization Theory

Вильсоновский метод основан на описании теории поля с помощью функционального интеграла, в котором степени свободы квантового поля являются переменными интегрирования. При этом подходе происходит разделение переменных интегрирования на длинноволновые и коротковолновые. Предполагается, что длинноволновые флуктуации описывают реальные физические процессы, а учёт коротковолновых флуктуаций (то есть интегрирование по коротковолновым степеням свободы) даёт эффективное действие для описания длинноволновой физики.

Такая идеология возникла из понятия перенормируемости в квантовой теории поля. Мы видели, что суть перенормируемой квантовой теории в том, что в ней на процессы с энергией $\sim M$ степени свободы с энергией $\sim \Lambda \gg M$ влияют лишь косвенно: через перенормировку констант теории. В канонической квантовой теории поля импульс обрезания Λ является конечным лишь в промежуточных вычислениях и в конце вычислений он устремляется к бесконечности. Вильсоновский анализ основан на противоположной точке зрения, согласно которой любая квантовая теория на фундаментальном уровне определена с обрезанием Λ, которое имеет некоторое физическое значение. В приложениях к статистической механике этот масштаб импульсов равен обратному межатомному расстоянию. В КЭД и других фундаментальных квантовых теориях поля обрезание может быть связано с фундаментальной зернистостью пространства-времени, природа которой, по-видимому, может быть изучена лишь вместе с построением квантовой теории гравитации. Во всяком случае этот элементарный масштаб очень далёк от масштабов, экспериментально достижимых в настоящее время.

Продемонстрируем вильсоновский подход на примере теории ϕ^4 в евклидовом d-мерном пространстве, поскольку в пространстве Минковского ограниченное значение квадрата d-импульса не означает ограниченности его компонент, если d-импульс изотропен, что внесло бы дополнительные усложнения.

Разложим вещественное скалярное поле в интеграл Фурье:

$$\phi(x) = \int \frac{\mathrm{d}^{(d)}k}{(2\pi)^d}\phi(k)e^{ikx} \tag{1}$$

Равенство

$$\int \mathrm{d}^{(d)}x\phi_1(x)\phi_2(x) = \int \frac{\mathrm{d}^{(d)}k}{(2\pi)^d}\phi_1^*(k)\phi_2(k) \tag{2}$$

показывает, что преобразование Фурье является унитарным в пространстве полей с метрикой (2). Поэтому функциональные меры $\prod_x \mathrm{d}\phi(x)$ и $\prod_k \mathrm{d}\phi(k)$ можно считать равными. Это даёт возможность определения регуляризованной функциональной меры:

$$[D\phi]_\Lambda = \prod_{|k|<\Lambda} \mathrm{d}\phi(k) \tag{3}$$

Регуляризованная статистическая сумма скалярного поля (сравни с (34 Л1)) представляется интегралом

$$Z = \int [D\phi]_\Lambda \exp\left\{-\int \mathrm{d}^{(d)}x \left[\frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right]\right\} \tag{4}$$

Здесь мы положили внешний источник равным нулю. В (4) m и λ являются голыми параметрами или параметрами на масштабе Λ.

Разделим в (4) переменные интегрирования на две группы. Пусть $0 < b < 1$ и

$$\hat{\phi}(k) = \begin{cases} \phi(k), & \text{для} \quad b\Lambda \le |k| < \Lambda \\ 0, & \text{для} \quad |k| < b\Lambda, \end{cases}$$

$$\phi(k) = \begin{cases} 0, & \text{для} \quad b\Lambda \le |k| < \Lambda \\ \phi(k), & \text{для} \quad |k| < b\Lambda \end{cases} \tag{5}$$

Тогда в (4) следует заменить $\phi(k)$ на $(\phi(k) + \hat{\phi}(k))$:

$$\begin{aligned} Z = \int [D\phi]_{b\Lambda} \int D\hat{\phi} \exp\left\{-\int \mathrm{d}^{(d)}x \left[\frac{1}{2}\left(\partial_\mu\phi + \partial_\mu\hat{\phi}\right)^2 + \frac{m^2}{2}(\phi+\hat{\phi})^2 + \frac{\lambda}{4!}(\phi+\hat{\phi})^4\right]\right\} = \\ = \int [D\phi]_{b\Lambda} e^{-\int \mathcal{L}(\phi)} \int D\hat{\phi} \exp\left\{-\int \mathrm{d}^{(d)}x \left[\frac{1}{2}\left(\partial_\mu\hat{\phi}\right)^2 + \frac{m^2}{2}(\hat{\phi})^2 + \right.\right. \\ \left.\left. + \lambda\left(\frac{1}{6}\phi^3\hat{\phi} + \frac{1}{4}\phi^2\hat{\phi}^2 + \frac{1}{6}\phi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4\right)\right]\right\} \end{aligned} \tag{6}$$

В (6) лагранжиан $\mathcal{L}(\phi)$ зависит лишь от поля ϕ и он имеет тот же вид, что и в (4). Квадратичные слагаемые вида $\phi\hat{\phi}$ исчезают в (6) вследствие равенства (2) и определения полей ϕ и $\hat{\phi}$ согласно (5).

Проинтегрируем (6) по полю $\hat{\phi}$:

$$Z = \int [D\phi]_{b\Lambda} \exp\left\{-\int \mathrm{d}^{(d)}x\mathcal{L}_{eff}(\phi)\right\} \tag{7}$$

Здесь $\mathcal{L}_{eff}(\phi)$ содержит только те фурье-компоненты $\phi(k)$, для которых $|k| < b\Lambda$. Если вычислять интеграл по $\hat{\phi}$ при помощи ТВ, разлагая по константе λ, то окажется, что $\mathcal{L}_{eff}(\phi)$ отличается от $\mathcal{L}(\phi)$ на поправки, пропорциональные степеням λ и ϕ и их производным. Эти поправочные члены компенсируют отсутствующие фурье-компоненты $\hat{\phi}$.

Поправки в $\mathcal{L}_{eff}$ могут быть представлены в диаграммной форме. Все слагаемые в (6), пропорциональные степеням λ, рассматриваются как возмущение. Поскольку предполагается, что $m^2 \ll \Lambda^2$, то слагаемое $(1/2)m^2\hat{\phi}^2$ также считается возмущением. Тогда слагаемое старшего порядка в последней экспоненте в (6) имеет вид

$$\int \hat{\mathcal{L}}_0 = \int\limits_{b\Lambda<|k|<\Lambda} \frac{\mathrm{d}^{(d)}k}{(2\pi)^d}\hat{\phi}^*(k)k^2\hat{\phi}(k) \tag{8}$$

Это слагаемое приводит к пропагатору

$$\langle\hat{\phi}(k)\hat{\phi}(p)\rangle = \frac{\int D\hat{\phi}e^{-\int\hat{\mathcal{L}}_0}\hat{\phi}(k)\hat{\phi}(p)}{\int D\hat{\phi}e^{-\int\hat{\mathcal{L}}_0}} = \frac{1}{k^2}(2\pi)^d\delta^{(d)}(k+p)\theta(k)$$
$$\theta(k) = \begin{cases}1, & \text{если } b\Lambda < |k| < \Lambda\\ 0, & \text{если } |k| < b\Lambda\end{cases} \tag{9}$$

Оставшиеся в (6) $\hat{\phi}$-слагаемые рассматриваются как возмущения и их учёт осуществляется путём разложения и использования теоремы Вика (см. конец лекции 1). При этом правильная нормировка достигается, когда интеграл по полю $\hat{\phi}$ в (6) заменяется на

$$1 + \text{ поправки.} \tag{10}$$

Разложим сначала (6) относительно $\phi^2\hat{\phi}^2$ и учтём, что

$$\langle\hat{\phi}(x)\hat{\phi}(y)\rangle = \int\limits_{b\Lambda<|k|<\Lambda} \frac{\mathrm{d}^{(d)}k}{(2\pi)^d}\frac{e^{ik(x-y)}}{k^2} \tag{11}$$

При помощи (11) легко находим эту поправку:

$$-\frac{\lambda}{4}\int \mathrm{d}^{(d)}x\phi^2\left\langle\hat{\phi}^2\right\rangle = -\frac{1}{2}\delta m^2\int \mathrm{d}^{(d)}x\phi^2 \tag{12}$$

$$\delta m^2 = \frac{\lambda}{2}\int\limits_{b\Lambda<|k|<\Lambda}\frac{\mathrm{d}^{(d)}k}{(2\pi)^d}\frac{1}{k^2} = \frac{\lambda\Lambda^{d-2}}{(4\pi)^{d/2}\Gamma(d/2)}\frac{1-b^{d-2}}{d-2} \tag{13}$$

Представим пропагатор (9) двойной линией. Поля ϕ представим как одинарные внешние линии.

Тогда вклад (12) изображается следующей диаграммой:

Рис. 10

Второй порядок относительно $\phi^2\hat{\phi}^2$ даёт вклад, соответствующий двум диаграммам:

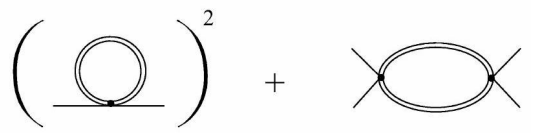


Рис. 11

Первое слагаемое на рис. 11 равно (1/2) (величина (12))^2, причём множитель (1/2) - комбинаторного происхождения. Если учесть все степени величины (12), возникающие во всех порядках, то окажется, что все эти слагаемые соберутся в экспоненту. Аналогичное утверждение справедливо в отношении любой связной диаграммы. Поэтому имеем следующий результат:

$$\mathcal{L}_{eff} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 + (\text{ сумма связных диаграмм }). \tag{14}$$

Вычислим вторую диаграмму на рис. 11. При помощи теоремы Вика находим:

$$\frac{\lambda^2}{16}\int \mathrm{d}^{(d)}x_1\ \mathrm{d}^{(d)}x_2\phi^2\left(x_1\right)\left[\int\limits_{b\Lambda<|k|<\Lambda}\frac{\mathrm{d}^{(d)}k}{(2\pi)^d}\frac{e^{ik(x_1-x_2)}}{k^2}\right]^2\phi^2\left(x_2\right) \tag{15}$$

Введём новые переменные

$$x_1 = x + \frac{y}{2}, \quad x_2 = x - \frac{y}{2} \tag{16}$$

в которых (15) принимает вид

$$\frac{\lambda^2}{16}\int \mathrm{d}^{(d)}x\ \mathrm{d}^{(d)}y\phi^2\left(x+\frac{y}{2}\right)\left[\int\limits_{b\Lambda<|k|<\Lambda}\frac{\mathrm{d}^{(d)}k}{(2\pi)^d}\frac{e^{iky}}{k^2}\right]^2\phi^2\left(x-\frac{y}{2}\right) \tag{17}$$

Рассмотрим предельный случай, когда импульсы, переносимые полем ϕ, исчезающе малы по сравнению с $(b\Lambda)$. Тогда (17) переходит в

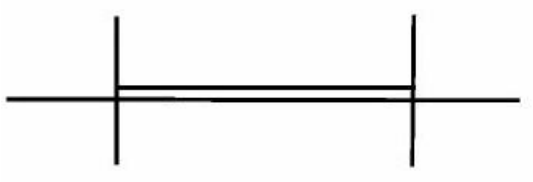
$$-\frac{\Delta\lambda}{4!}\int \mathrm{d}^{(d)}x\phi^4 \tag{18}$$

$$\Delta\lambda = -4!\frac{\lambda^2}{16}\int\limits_{b\Lambda<|k|<\Lambda}\frac{\mathrm{d}^{(d)}k}{(2\pi)^d}\frac{1}{\left(k^2\right)^2} = -\frac{3\lambda^2\Lambda^{d-4}\left(1-b^{d-4}\right)}{(4\pi)^{d/2}\Gamma(d/2)(d-4)} \quad \xrightarrow[d\to 4]{} -\frac{3\lambda^2}{16\pi^2}\ln\frac{1}{b} \tag{19}$$

Описанная диаграммная теория возмущений генерирует не только вклады в $\mathcal{L}_{eff}$, пропорциональные ϕ^2 и ϕ^4, но и вклады высших степеней ϕ, а также вклады от производных поля ϕ. Например, в (17) содержится слагаемое, пропорциональное

$$\int \mathrm{d}^{(d)}x\phi^2\left(\partial_\mu\phi\right)^2 \tag{20}$$

Вклад, пропорциональный $\left(\frac{\lambda}{\Lambda}\right)^2\int \mathrm{d}^{(d)}x\phi^6$, получается из диаграммы



Вообще говоря, вильсоновская процедура генерирует все возможные взаимодействия поля ϕ и их производных.

При помощи формулы (6) легко увидеть, что поправки с точностью $O(\lambda)$ дают

$$\mathcal{L}_{eff} = \frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{(\tilde{m})^2}{2}\phi^2 + \frac{\tilde{\lambda}}{4!}\phi^4, \quad \tilde{m}^2 = m^2 + \Delta m^2, \quad \tilde{\lambda} = \lambda - \frac{3\lambda^2}{16\pi^2}\ln\frac{1}{b} \tag{21}$$

Здесь Δm^2 задаётся согласно (13). (Последнее равенство в (21) сравни с (24 Л4).)

Для более детального сравнения нового интеграла (7) с исходным интегралом (4) изменим в (7) масштаб расстояний и импульсов по следующему правилу:

$$x' = bx, \quad k' = \frac{k}{b}, \quad |k'| < \Lambda \tag{22}$$

Теперь импульсы в (7) изменяются в тех же пределах, что и в (4). Имеем (сравни с (29 Л5)):

$$\begin{gathered}\int \mathrm{d}^{(d)}x\mathcal{L}_{\text{eff}} = \\ = \int \mathrm{d}^{(d)}x\left[\frac{1}{2}(1+\Delta Z)\left(\partial_\mu\phi\right)^2 + \frac{1}{2}\left(m^2+\Delta m^2\right)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \Delta B\left(\partial_\mu\phi\right)^4 + \Delta C\phi^6 + \ldots\right] = \\ = \int \mathrm{d}^{(d)}x' b^{-d}\left[\frac{1}{2}(1+\Delta Z)b^2\left(\partial'_\mu\phi\right)^2 + \frac{1}{2}\left(m^2+\Delta m^2\right)\phi^2 + \frac{1}{4!}(\lambda+\Delta\lambda)\phi^4 + \right. \\ \left. + \Delta B b^4\left(\partial'_\mu\phi\right)^4 + \Delta C\phi^6 + \ldots\right]\end{gathered} \tag{23}$$

Чтобы в новом функциональном интеграле пропагатор сохранил прежний вид (9), изменим масштаб поля ϕ по правилу

$$\phi' = \left[b^{2-d}(1+\Delta Z)\right]^{1/2}\phi \tag{24}$$

после чего эффективное действие приобретает первоначальную форму

$$\int \mathrm{d}^{(d)}x\mathcal{L}_{eff} = \int \mathrm{d}^{(d)}x'\left[\frac{1}{2}\left(\partial'_\mu\phi'\right)^2 + \frac{1}{2}m'^2\phi'^2 + \frac{1}{4!}\lambda'\phi'^4 + B'\left(\partial'_\mu\phi'\right)^4 + C'\phi'^6 + \ldots\right] \tag{25}$$

с новыми параметрами

$$\begin{gathered}m'^2 = \left(m^2+\Delta m^2\right)(1+\Delta Z)^{-1}b^{-2} \\ \lambda' = (\lambda+\Delta\lambda)(1+\Delta Z)^{-2}b^{d-4} \\ B' = (B+\Delta B)(1+\Delta Z)^{-2}b^{d} \\ C' = (C+\Delta C)(1+\Delta Z)^{-3}b^{2d-6}\end{gathered} \tag{26}$$

и т.д. Заметим, что в исходном лагранжиане $B = C = 0$, но если бы эти константы отличались от нуля, то мы имели бы формулы (26).

Теперь повторим ещё и ещё раз описанную процедуру интегрирования по коротковолновым флуктуациям с последующими масштабными преобразованиями (22)-(26). Получаемые таким образом преобразования лагранжиана называются преобразованиями под действием ренормализаиионной группы (сокращённо - ренормгруппы). Это не есть группа в математическом смысле, поскольку при усреднении по коротковолновым флуктуациям теряется информация, эта процедура необратима.

Рассмотрим сначала с этой точки зрения простейший пример свободной теории, когда в затравочном лагранжиане $m^2 = \lambda = B = C = ... = 0$ и все возмущения исчезают. Тогда на всех этапах описанной рекурсионной процедуры Вильсона эффективный лагранжиан остаётся свободным. Поэтому говорят, что свободный лагранжиан

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu \phi\right)^2 \tag{27}$$

является фиксированной точкой преобразований ренормгруппы.

Если мы находимся в непосредственной близости фиксированной точки, то можно отбросить в итерационных формулах $\Delta m^2, \Delta\lambda$, и т.д. Тогда при помощи (26) получаем:

$$m'^2 = m^2 b^{-2}, \quad \lambda' = \lambda b^{d-4}, \quad B' = B b^d, \quad C' = C b^{2d-6} \tag{28}$$

и т.д. Так как $b < 1$, то параметры, которые умножаются на отрицательные степени b, растут, и наоборот. Если лагранжиан содержит растущие коэффициенты, они уводят его от точки $\mathcal{L}_0$.

Отдельные слагаемые в квадратной скобке в (25) называются локальными операторами. Операторы, коэффициенты которых растут (вымирают) при рекурсионной процедуре, называются *cy* щественными (несущественными). Если коэффициент оператора умножается на b^0 (например, оператор ϕ^4 при $d = 4$), мы называем его маргиналъным. Для выяснения того, растёт ли или уменьшается его коэффициент, требуется вычисление высших поправок.

Из (26) и (28) видно, что массовый оператор ϕ^2 является существенным, а два последних слагаемых в (25) при $d = 4$ являются несущественными операторами. Этим и объясняется, что теория ϕ^4 является перенормируемой при $d = 4$, так как фактически и при вильсоновской процедуре перенормировок эта теория восстанавливает свой исходный вид. Это же утверждение справедливо в отношении любой перенормируемой теории.

Так как в порядке $O(\lambda) \quad \Delta Z = 0$, то при помощи (26) и (21) находим, что с точностью $O\left(\lambda^2\right)$ при $d = 4$

$$\lambda' = \lambda - \frac{3\lambda^2}{16\pi^2} \ln \frac{1}{b} \tag{29}$$

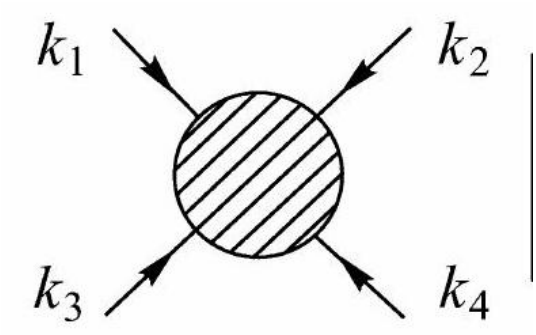
Отсюда видно, что коэффициент при маргинальном операторе ϕ^4 уменьшается, то есть константа связи уменьшается по логарифмическому закону. Соотношение (29) является аналогом соотношения (24 Л4), разница лишь в том, что в (29) и (24 Л4) представлены взаимно обратные функции. Из сказанного следует, что при $d = 4$ теория ϕ^4 в результате вильсоновской процедуры перенормировок стремится к свободной теории. Эта картина справедлива, конечно, лишь до тех пор, пока массовый оператор не становится преобладающим.

16.8.2 2. Callan–Symanzik Equation

На языке Вильсона перенормируемая квантовая теория поля с произвольно большим обрезанием отвечает эффективной теории, испытавшей большое число ренормгрупповых операций Вильсона и которая проэволюционировала до большой величины массового параметра. Такая траектория должна пройти произвольно близко от фиксированной точки. В результате этой эволюции несущественные операторы в первоначальном лагранжиане вымирают и остаются только существенные и маргинальные операторы. Коэффициенты при оставшихся операторах находятся во взаимно однозначном соответствии с наблюдаемыми параметрами, а информация об эволюции несущественных операторов утрачивается полностью в перенормируемой теории поля. Информацию же об эволюции параметров при существенных и маргинальных операторах можно восстановить, изучая их зависимость от нормировочного масштаба.

Для конкретности рассмотрим теорию ϕ^4 в четырёхмерном пространстве Минковского, изученную нами, и ограничимся случаем $m^2 = 0$. Поскольку физический размерностный оператор отсутствует, то для проведения программы перенормировок необходимо ввести размерностный параметр $\mu^2 > 0$. Поскольку условия перенормировки (30Ω5) приводят к сингулярностям при $m^2 \to 0$ (см. (32 Л5)), то наложим эти же условия перенормировки, но при пространственноподобных значениях импульсов $p^2 = -\mu^2$:

$$\Sigma\left(k^2\right)\big|_{k^2=0} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}k^2}\Sigma\left(k^2\right)\Big|_{k^2=-\mu^2} = 0$$



амputированная

$$= -i\lambda \tag{30}$$

$(k_1 + k_2)^2 = (k_1 + k_3)^2 = (k_1 + k_4)^2 = -\mu^2$

Заметим, что в (30) импульсы k_a не предполагаются находящимися на массовой поверхности и потому условия (30) совместны. Таким образом, теория определена на масштабе μ^2.

Обозначим через $\phi_0(x)$ и $\phi(x)$ - затравочное и перенормированное поля, соответственно, и пусть $G_0^{(n)}(x_1, ..., x_n)$ - связная корреляционная функция (хронологическое среднее) неперенормированных полей, определяемая согласно (19 Л5), а

$$G^{(n)}(x_1, ..., x_n) = \langle 0 \,|T\phi(x_1) ...\phi(x_n)|\, 0\rangle \tag{31}$$

- связный коррелятор перенормированных полей (для удобства мы сменили обозначения, сделав замены $\phi \to \phi_0, \phi_r \to \phi$). Так как согласно (26 Л5) $\phi_0 = \sqrt{Z}\phi$, то

$$Z^{n/2}G^{(n)}\left(x_1,...,x_n;\mu,\lambda(\mu)\right)=G_0^{(n)}\left(x_1,...,x_n\right) \tag{32}$$

Обозначение $G^{(n)}\left(x_1,...,x_n;\mu,\lambda(\mu)\right)$ подчёркивает, что этот коррелятор зависит от точки нормировки μ и перенормированной константы связи $\lambda(\mu)$.

Теперь изменим точку нормировки $\mu\to\mu'$, константу связи λ и константу перенормировки Z, но оставим неизменными импульс обрезания Λ, а также затравочные константы. Это означает, что мы продвинулись вдоль траектории ренормгруппы. Ясно, что при этом "голые" (неперенормированные) гриновские функции (32) остаются неизменными, поскольку они зависят лишь от затравочных параметров и импульса обрезания Λ. Поэтому производная по параметру μ от обоих частей равенства (32) равна нулю:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+n\gamma(\lambda)\right]G^{(n)}\left(\{x_i\};\mu,\lambda\right)=0 \tag{33}$$

Здесь

$$\beta(\lambda)\equiv\mu\frac{\partial\lambda}{\partial\mu},\quad \gamma(\lambda)\equiv\frac{1}{2Z}\mu\frac{\partial}{\partial\mu}Z \tag{34}$$

Функции $\beta(\lambda)$ и $\gamma(\lambda)$ - одни и те же для всех корреляторов в теории и они не могут зависеть от импульса обрезания, поскольку перенормированные корреляторы не зависят от обрезания. Следовательно, по размерным соображениям, они не могут зависеть и от μ. Поэтому функции (34) могут зависеть лишь от перенормированной константы связи λ.

Уравнение (33) называется уравнением Каллана-Симанчика.

Вычислим функцию β в наинизшем неисчезающем приближении при $d=4$. Так как $Z=1+O\left(\lambda^2\right)$, то с точностью $O\left(\lambda^2\right)$ перенормированный четырёхточечный коррелятор в импульсном представлении (см. рис. 7 и (20 Л4)) равен

$$G^{(4)}\left(k_1,...,k_4\right)=\left[-i\lambda-\frac{i\lambda^2}{32\pi^2}\left(\ln\frac{-s-i\varepsilon}{\mu^2}+\ln\frac{-t-i\varepsilon}{\mu^2}+\ln\frac{-u-i\varepsilon}{\mu^2}\right)\right]\prod_{i=1}^{4}\frac{i}{k_i^2} \tag{35}$$

Так как $\gamma=O\left(\lambda^2\right)$, то в порядке $O\left(\lambda^2\right)$ функция γ выпадает из уравнения Каллана-Симанчика. Поэтому, подставляя (35) в (33), получаем:

$$\beta(\lambda)=\frac{3\lambda^2}{16\pi^2}+O\left(\lambda^3\right) \tag{36}$$

16.8.3 3. Evolution of the Running Coupling Constant

Сопоставим уравнения (34) и (34):

$$\frac{\mathrm{d}\lambda}{\mathrm{d}\ln\mu}=\beta(\lambda)=\frac{3\lambda^2}{16\pi^2} \tag{37}$$

Здесь первое равенство является точным по определению, а второе - приближённое. Проинтегрируем явно второе равенство в (37):

$$\lambda(\mu)=\frac{\lambda(m)}{1-\frac{3\lambda(m)}{16\pi^2}\ln\frac{\mu}{m}} \tag{38}$$

Будем считать, что m - некий масштаб порядка физической массы. Тогда следует считать, что $\mu \gg m$, так что на масштабах $\sim \mu$ теория фактически безмассовая и уравнения (34)-(37) справедливы. Тогда уравнение (38) показывает, что эффективная константа связи растёт при возрастании импульса согласно формуле (38). При этом $\lambda(m)$ - константа связи при импульсах, стремящихся к нулю.

Мы видим, что квантовая теория поля приводит к новому понятию: бегущей константе связи. При этом в теории ϕ^4 при $d = 4$ бегущая (эффективная) константа связи уменьшается, если нас интересуют всё меньшие и меньшие энергии взаимодействия (вплоть до масштаба физической массы частиц). Мы увидим, что константа взаимодействия в электродинамике (электрический заряд) также уменьшается при понижении энергии взаимодействия. Такие теории называются нульзарядными и в них β-функции положительны. Однако существуют такие теории (например, теория Янга-Миллса и построенная на её основе теория сильного взаимодействия - квантовая хромодинамика), в которых эффективная константа связи растёт при уменьшении энергии взаимодействующих частиц. Такие теории называются асимптотически свободными, β-функции в этих теориях отрицательны.

β-функция в уравнениях (33), (34) называется функиией Гелл-мана и Лоу.

Напоследок сделаем следующую оценку. Рассмотрим двухточечный коррелятор при высоких энергиях, когда можно пренебречь массой. Согласно сказанному выше имеем:

$$G(k) \sim \frac{Z\left(k^2\right)}{k^2} \tag{39}$$

Но при помощи (34) находим, что

$$Z\left(k^2\right) \sim \left(\frac{k^2}{m^2}\right)^{\gamma} \tag{40}$$

если $\gamma\left(k^2\right)$ изменяется медленно. Комбинируя (39) и (40), получаем:

$$G(k) \sim \frac{1}{\left(m^2\right)^{\gamma}\left(k^2\right)^{1-\gamma}}, \tag{41}$$

откуда

$$G(x) \sim \int \mathrm{d}^{(d)} k e^{ikx} G(k) \sim \frac{1}{|x|^{d-2+2\gamma}} \tag{42}$$

Таким образом, функции γ ответственны за появление аномальных размерностей полей.

Задачи

1. Показать, что при $d < 4$ в теории ϕ^4 имеется фиксированная точка ренормгруппы с ненулевой константой связи λ_0 и найти это значение константы связи в однопетлевом приближении.

Ответ:

$$\lambda_0 = \frac{(4-d)(4\pi)^{d/2}\Gamma(D/2)}{3}(\Lambda b)^{4-d}$$

2. Используя уравнение Каллана-Симанчика, вычислить в безмассовой теории ϕ^4 при $d = 4$ функцию $\gamma(\lambda)$ в наинизшем приближении.

3. Вычислить β-функцию в первом неисчезающем порядке в теории с лагранжианом (см. задачу 3 к лекции 3)

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n} \partial_\mu \phi^i \partial^\mu \phi^i - \frac{\lambda}{4}\left(\sum_{i=1}^{n} \left(\phi^i\right)^2\right)^2$$

16.9 L5 General Scheme of Renormalizations

16.9.1 1. Källén–Lehmann Spectral Representation of the Two-Point Function

(???? why is it here, what does it has to do with renormalization????)

Согласно (27 ЛЗ) фурье-образ вакуумного среднего $\langle 0|T\phi(x)\phi(0)|0\rangle$ для свободного квантованного поля ϕ имеет вид

$$\int \mathrm{d}x e^{ikx} \langle 0|T\phi(x)\phi(0)|0\rangle = \frac{i}{k^2 - m_0^2 + i\varepsilon} = -iD_c(k) \tag{1}$$

Теперь попробуем вычислить это же среднее во взаимодействующей теории. Мы будем опираться только на общие принципы теории относительности и квантовой механики; при этом вид взаимодействия, специфика ряда ТВ и другие детали не существенны. Предполагается лишь, что в теории имеются частицы с определённой массой (и, возможно, другими квантовыми числами), динамика которых лоренц-инвариантна и описывается согласно законам квантовой механики. Поэтому результаты, полученные в этой лекции, справедливы для очень широкого класса квантовых теорий поля.

Обозначим через $\mathcal{P}^\mu$ оператор d-импульса теории, так что $\mathcal{P}^0 = \mathcal{H}$ - гамильтониан теории, а $\mathcal{P}^i, \quad i = 1, ..., (d-1)$ - компоненты импульса. В свободной скалярной теории оператор $\mathcal{P}^\mu$ представляется согласно (32 Л2) и (42 Л2). Согласно самому определению d-импульса для любого локального поля $\phi(x)$ имеем:

$$\phi(x) = e^{i\mathcal{P}x}\phi(0)e^{-i\mathcal{P}x} \tag{2}$$

По определению основного состояния

$$\mathcal{P}^\mu|0\rangle = 0 \tag{3}$$

Согласно алгебре Пуанкаре (см. Ю. Швингер, "Частицы, источники, поля, Т.1) все компоненты d импульса взаимно коммутируют. Поэтому можно считать, что каждое состояние имеет определённое значение d-импульса. Обозначим через $|\lambda_{\mathbf{k}}\rangle$ какое либо из таких состояний, так что

$$\mathcal{H}\,|\lambda_{\mathbf{k}}\rangle = E_{\mathbf{k}}(\lambda)\,|\lambda_{\mathbf{k}}\rangle\,, \quad \mathcal{P}\,|\lambda_{\mathbf{k}}\rangle = \mathbf{k}\,|\lambda_{\mathbf{k}}\rangle \tag{4}$$

Вследствие Лоренц-инвариантности

$$E_{\mathbf{k}}(\lambda) = \sqrt{m_\lambda^2 + \mathbf{k}^2} > 0 \tag{5}$$

где m_λ - энергия состояния в той системе отсчёта, в которой суммарный пространственный импульс равен нулю. Неравенство в (5) означает, что энергия вакуума является наименьшей энергией в теории, которая согласно (3) равна нулю. Мы часто будем выделять из совокупности состояний $\{|\lambda_{\mathbf{k}}\rangle\}$ одночастичные состояния $\{|\mathbf{k}\rangle\}$. Тогда $|\lambda_{\mathbf{k}}\rangle$ обозначает многочастичное состояние. Состояние $|\mathbf{k}\rangle$ имеет импульс $\mathbf{k}$ и энергию $\omega_{\mathbf{k}} = \sqrt{m^2 + \mathbf{k}^2}$. Подчеркнём, что в случае одночастичного состояния импульс $\mathbf{k}$ полностью определяет энергию состояния, в то время как энергия неодночастичного состояния $|\lambda_{\mathbf{k}}\rangle$ не определяется параметром $\mathbf{k}$. Предположим, что все возбуждённые состояния принадлежат линейной оболочке одно-, двух- и т.д. частичных состояний, то есть состояния с определённым числом частиц образуют базис пространства состояний. Если в теории имеется всего один сорт частиц с массой m, то для n-частичного состояния

$$m_\lambda^2 \geq (nm)^2 \tag{6}$$

причём m_λ^2 - непрерывный параметр. Мы предполагаем также, что состояния $|\lambda_{\mathbf{k}}\rangle$ нормированы лоренц-инвариантным образом (см. (43 Л2), (45 Л2)), так что разложение единичного оператора имеет вид

$$\mathbf{1} = |0\rangle \left\langle 0 \right| + \sum_\lambda \int \frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}} \frac{1}{2E_{\mathbf{k}}(\lambda)} \left| \lambda_{\mathbf{k}} \right\rangle \langle \lambda_{\mathbf{k}}| \tag{7}$$

где суммирование идёт по всем многочастичным состояниям с нулевым суммарным импульсом $|\lambda_0\rangle$. Справедливость формулы (7) очевидна, если заметить, что в одночастичном секторе, согласно (38 Л2), (39 Л2)

$$\boldsymbol{I} = \int \frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}} \frac{1}{2\omega_{\mathbf{k}}} |\mathbf{k}\rangle\langle\mathbf{k}| \tag{8}$$

Будем считать, что $x^0 > y^0$ и вставим разложение единицы (7) между полями в среднее $\langle 0|\phi(x)\phi(y)|0\rangle$. Пренебрежём постоянной величиной $\langle 0|\phi(x)|0\rangle\langle 0|\phi(y)|0\rangle$, которая в большинстве случаев равна нулю. Таким образом получаем:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \sum_\lambda \int \frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}} \frac{1}{2E_{\mathbf{k}}(\lambda)} \langle 0|\phi(x)|\lambda_{\mathbf{k}}\rangle \langle \lambda_{\mathbf{k}}|\phi(y)|0\rangle \tag{9}$$

Преобразуем матричные элементы в (9) при помощи формул (2)-(4):

$$\begin{aligned} \langle 0|\phi(x)|\lambda_{\mathbf{k}}\rangle = \left\langle 0 \left| e^{i\mathcal{P}x}\phi(0)e^{-i\mathcal{P}x} \right| \lambda_{\mathbf{k}} \right\rangle = \langle 0|\phi(0)|\lambda_{\mathbf{k}}\rangle\, e^{-ikx}\big|_{k^0=E_{\mathbf{k}}(\lambda)} = \\ = \langle 0|\phi(0)|\lambda_0\rangle\, e^{-ikx}\big|_{k^0=E_{\mathbf{k}}(\lambda)} \end{aligned} \tag{10}$$

Последнее равенство является следствием следующих соотношений:

$$|\lambda_{\mathbf{k}}\rangle = U\,|\lambda_0\rangle\,, \quad U^{-1}\phi(0)U = \phi(0), \quad \langle 0|U^{-1} = \langle 0|$$

где унитарный оператор U генерирует соответствующее преобразование Лоренца. Такие операторы называются бустами. Подставим (10) в (9):

$$\langle 0|\phi(x)\phi(y)|0\rangle = \sum_\lambda \int \frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}} \frac{1}{2E_{\mathbf{k}}(\lambda)} e^{-ik(x-y)}\, |\langle 0|\phi(0)|\lambda_0\rangle|^2 \Big|_{k^0=E_{\mathbf{k}}(\lambda)}$$

Так как $x_0 > y_0$, то можно воспользоваться формулой (23 Л1). Таким образом получаем:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \sum_\lambda \int \frac{\mathrm{d}^{(d)}k}{(2\pi)^d} \frac{ie^{-ik(x-y)}}{k^2 - m_\lambda^2 + i\varepsilon} \left|\langle 0|\phi(0)|\lambda_0\rangle\right|^2$$

В случае $x^0 < y^0$ имеет место аналогичная формула. Обе формулы объединяются в следующем выражении:

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \int\limits_0^\infty \frac{\mathrm{d}M^2}{2\pi} \rho\left(M^2\right) \left[-iD_c\left(x-y; M^2\right)\right]$$

$$D_c\left(x; M^2\right) = \int \frac{\mathrm{d}^{(d)}k}{(2\pi)^d} \frac{e^{-ikx}}{-k^2 + M^2 - i\varepsilon} \tag{11}$$

где $\rho\left(M^2\right)$ - положительная спектральная плотность

$$\rho\left(M^2\right) = \sum_\lambda (2\pi)\delta\left(M^2 - m_\lambda^2\right) \left|\langle 0|\phi(0)|\lambda_0\rangle\right|^2 \tag{12}$$

График спектральной плотности $\rho\left(M^2\right)$ для типичной теории поля показан на рис. 8.

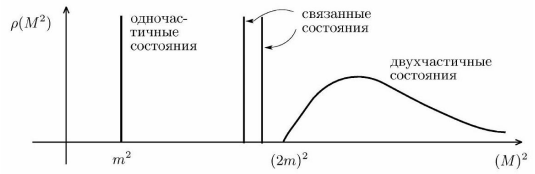


Рис. 8.

Одночастичные состояния вносят в спектральную плотность вклад в виде изолированных δ функций:

$$\rho\left(M^2\right) = 2\pi Z\delta\left(M^2 - m^2\right) + \left(\text{ ничего другого вплоть до } M^2 \gtrsim (2m)^2\right) \tag{13}$$

где

$$Z = \left|\langle 0|\phi(0)|\mathbf{k} = 0\rangle\right| \tag{14}$$

Константа Z называется константой перенормировки поля ϕ. Величина m есть физически наблюдаемая масса элементарных возбуждений поля ϕ, а m_0 в (53 Л2) является затравочной или "голой"массой, причём $m \neq m_0$. Непосредственно наблюдаемой является физическая масса m, а не затравочная масса.

С помощью (11) и (13) находим во взаимодействующей теории фурье-образ хронологического парного среднего, являющегося аналогом величины (1) в свободной теории:

$$\int \mathrm{d}x e^{ikx} \langle 0|T\phi(x)\phi(0)|0\rangle = \int\limits_0^\infty \frac{\mathrm{d}M^2}{2\pi} \rho\left(M^2\right) \frac{i}{k^2 - M^2 + i\varepsilon} =$$

$$= \frac{iZ}{k^2 - m^2 + i\varepsilon} + \int\limits_{\sim 4m^2}^\infty \frac{\mathrm{d}M^2}{2\pi} \rho\left(M^2\right) \frac{i}{k^2 - M^2 + i\varepsilon} \tag{15}$$

Аналитическая структура этой функции в комплексной k^2-плоскости показана на рис. 9.

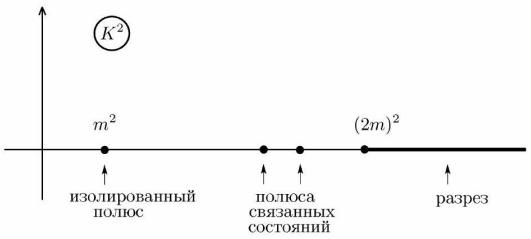


Рис. 9.

Первое слагаемое в (15) соответствует изолированному полюсу при $k^2 = m^2$, а второе - разрезу, начинающемуся в точке $4\,\mathrm{m}^2$. Если существуют двухчастичные связанные состояния, то в (15) возникают дополнительные полюса.

Сравнивая (1) и (15) мы видим, что взаимодействующее поле, в отличие от свободного, рождает из вакуума не только одночастичное, но и многочастичные состояния. Кроме того, в первом полюсном слагаемом содержится константа перенормировки поля (14), то есть вероятность того, что $\phi(0)$ родит данное одночастичное состояние из вакуума.

Проводя вычисления, аналогичные приведенным выше, легко получаем также:

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = \int\limits_0^\infty \frac{\mathrm{d}M^2}{2\pi}\rho\left(M^2\right)\int \frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}}\frac{e^{-ik(x-y)}-e^{ik(x-y)}}{2k^0}\Bigg|_{k^0=\sqrt{M^2+\mathbf{k}^2}} =$$

$$= \int\limits_0^\infty \frac{\mathrm{d}M^2}{2\pi}\rho\left(M^2\right)\left[\frac{1}{i}D(x-y;M)\right] \tag{16}$$

Здесь $x^0 \neq y^0$ и введена перестановочная функция Паули-Йордана

$$D(x;M) \equiv i\int \frac{\mathrm{d}^{(d-1)}k}{(2\pi)^{d-1}}\frac{e^{-ikx}-e^{ikx}}{2k^0}\Bigg|_{k^0=\sqrt{M^2+\mathbf{k}^2}} = i\int \frac{\mathrm{d}^{(d)}k}{(2\pi)^{d-1}}\theta\left(k^0\right)\delta\left(k^2-M^2\right)\left(e^{-ikx}-e^{ikx}\right)$$

которая удовлетворяет однородному уравнению Клейна-Гордона

$$\left(\partial_\mu\partial^\mu + M^2\right)D(x;M) = 0$$

а также соотношению

$$\frac{\partial}{\partial x^0}D(x;M)\Big|_{x^0=0} = \delta^{(3)}(\mathbf{x})$$

Поэтому из (16) находим, что

$$\langle 0|[\phi(x),\dot{\phi}(y)]|0\rangle\Big|_{x^0=y^0} = \int\limits_0^\infty \frac{\mathrm{d}M^2}{2\pi}\rho\left(M^2\right)\left[i\delta^{(3)}(\mathbf{x}-\mathbf{y})\right]$$

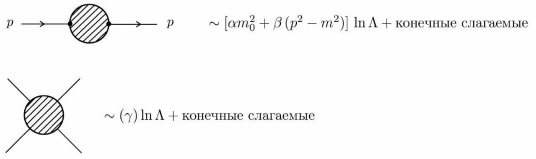
С другой стороны, левая часть этого равенства по определению равна $\left[i\delta^{(3)}(\mathbf{x}-\mathbf{y})\right]$, и потому

$$\int_0^\infty \frac{\mathrm{d}M^2}{2\pi}\rho\left(M^2\right) = 1 \longrightarrow 0 < Z < 1 \tag{17}$$

Последние неравенства получаются при учёте представления (13).

16.9.2 2. Renormalizations

Теперь опишем индуктивную процедуру, при помощи которой из расходящихся диаграмм выделяются различные физические величины, остающиеся конечными в пределе $\Lambda \to \infty$. Будем пользоваться размерностной регуляризацией, которая выделяет лишь логарифмические расходимости. Кроме того, вакуумные пузырьки, как ненаблюдаемые поправки, мы не рассматриваем. Таким образом, расходящимися являются лишь два типа диаграмм:



$\sim (\gamma)\ln\Lambda+$ конечные слагаемые

Здесь α, β, γ - безразмерные ряды по λ. Мы не предполагаем, что импульсы внешних концов диаграмм лежат на массовой поверхности. Очевидно, степень расходимости диаграмм не зависит от значений внешних импульсов.

Рассмотрим сначала n-точечную корреляционную функцию

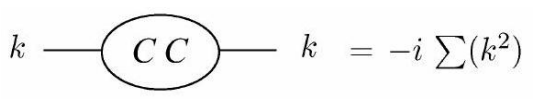
$$G\left(x_1, ..., x_n\right) = \left\langle 0 \left| T\phi\left(x_1\right) ...\phi\left(x_n\right) \right| 0 \right\rangle \tag{19}$$

Любая диаграмма, дающая вклад в коррелятор (19), содержит расходящиеся поддиаграммы указанных двух типов. Для устранения расходимостей действуют последовательно. этот процесс начинается с однопетлевых диаграмм. Так, согласно (8 Л4) в $\Sigma^{(1)}$ имеется расходящаяся часть. Она устраняется путём введения в затравочную массу соответствующей добавки:

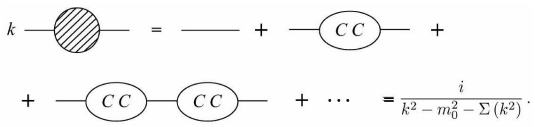
$$m_0^2 = m^2 + \delta_m, \quad \delta_m + \Sigma^{(1)} = 0 \tag{20}$$

Здесь m_0^2 - квадрат затравочной массы в однопетлевом приближении. Легче всего это установить следующим образом.

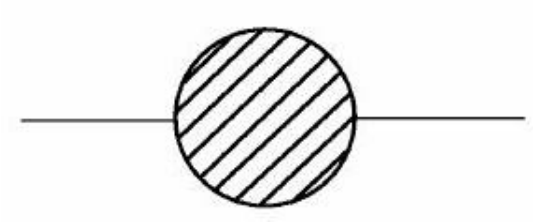
Назовём диаграмму сильно связной, если её нельзя разделить на две отдельные диаграммы путём разрезания всего лишь одной внутренней линии. Обозначим через



сумму всех сильно связных вставок в двухточечную корреляционную функцию (13) или сумму всех сильно связных диаграмм с двумя внешними линиями и ампутированными ногами. Тогда полная
функция (13) определяется геометрической прогрессией



В первом порядке по λ величина Σ задаётся согласно (6 Л4). Поэтому с учётом (20) в этом приближении



$$= \frac{i}{k^2 - m^2 + i\varepsilon} \tag{23}$$

и полюс корреляционной функции (13) находится при $k^2 = m^2$, то есть там, где и должен быть.

Устранение логарифмической расходимости в однопетлевом приближении из четырёхточечного коррелятора было проведено в предыдущей лекции (см. (24 Л4)).

Предположим, что во всех s-петлевых диаграммах при описанном способе действий устранены ультрафиолетовые расходимости. Покажем, что действуя так же, можно устранить расходимости в $(s+1)$-петлевых диаграммах. Действительно, по предположению в любой $(s+1)$-петлевой диаграмме все s-петлевые поддиаграммы уже конечны. Поэтому $(s+1)$-петлевые диаграммы с двумя и четырьмя внешними линиями расходятся лишь в той области интегрирования, где все $(s+1)$ промежуточных импульсов стремятся к бесконечности. Эти новые расходимости можно устранить и сделать диаграммы конечными путём соответствующего подбора величин $\delta_m^{(s+1)}(\lambda), \lambda_0^{(s+1)}(\lambda)$ и константы перенормировки поля $Z^{(s+1)}(\lambda)$, являющихся полиномами относительно физической константы связи λ. Очевидно, что $\delta_m^{(s+1)}(\lambda) \quad \left(\lambda_0^{(s+1)}(\lambda), Z^{(s+1)}(\lambda)\right)$ отличается от $\delta_m^{(s)}(\lambda)\left(\lambda_0^{(s)}(\lambda), Z^{(s)}(\lambda)\right)$ лишь более высокими степенями константы λ, чем степень полинома $\delta_m^{(s)}(\lambda)\left(\lambda_0^{(s)}(\lambda), Z^{(s)}(\lambda)\right)$. Действительно, диаграммы с четырьмя внешними концами расходятся лишь логарифмически и притом $(s+1)$-петлевые расходимости набегают в области, где все импульсы интегрирования стремятся к бесконечности. Поэтому расходящиеся части таких диаграмм являются числами, не зависящими от внешних импульсов и их можно поглотить за счёт подбора $\lambda_0^{(s+1)}(\lambda)$. Аналогично (см. первую диаграмму в (18)), для устранения $(s+1)$-петлевых расходимостей диаграмм с двумя внешними концами достаточно подобрать соответствующим образом константы $m_0^2(\lambda)$ и $Z(\lambda)$ в нужном порядке по физической константе связи, причём второе слагаемое в первой диаграмме в (18) вносит вклад в константу перенормировки поля. Оказывается, что n-точечный коррелятор полей (19) равен $Z^{n/2}\times$ (конечная часть) (см. ниже). Таким образом устраняются расходимости во всех $(s+1)$ петлевых диаграммах.

Замечание. Если бы оказалось, что в теории ϕ^4 имеются расходящиеся диаграммы, например, с шестью внешними линиями, то такие расходимости было бы невозможно устранить путём переопределения массы и константы связи. Для устранения таких

расходимостей потребовалось бы введение в теорию членов нового типа, пропорциональных ϕ^6. Это означало бы, что теория неперенормируема.

Можно показать, что имеет место соотношение

$$\prod_{i=1}^{n}\int \mathrm{d}^4x_i e^{ip_ix_2}\prod_{j=1}^{m}\int \mathrm{d}^4y_j e^{-ik_jy_j}\left\langle 0\left|\phi\left(x_1\right)...\phi\left(x_n\right)\phi\left(y_1\right)...\phi\left(y_m\right)\right|0\right\rangle \sim$$
$$\sim\left(\prod_{i=1}^{n}\frac{i\sqrt{Z}}{p_i^2-m^2+i\varepsilon}\right)\left(\prod_{j=1}^{m}\frac{i\sqrt{Z}}{k_j^2-m^2+i\varepsilon}\right)\langle \mathbf{p}_1...\mathbf{p}_n|S|\mathbf{k}_1...\mathbf{k}_m\rangle \tag{24}$$

Здесь в обоих частях равенства $k_j^0, p_i^0 \to \omega_k, \omega_p\left(\omega_k=\sqrt{m^2+\mathrm{k}^2}\right), Z$ - константа перенормировки поля. Знак $\sim$ в (24) означает, что обе части равенства ведут себя одинаково вблизи полюсов. Формула (24) называется редукиионной формулой Лемана-Симанчика-Циммермана.

Описанную процедуру перенормировки можно технически упростить и работать непосредственно с перенормированными физическими величинами. Начнём рассуждение с записи затравочного лагранжиана:

$$\mathcal{L}=\frac{1}{2}\left(\partial_\mu\phi\right)^2-\frac{1}{2}m_0^2\phi^2-\frac{\lambda_0}{4!}\phi^4 \tag{25}$$

Устраним множитель Z в значении вычета в полюсе хронологического парного среднего поля ϕ (см. (17)). Для этого введём перенормированное поле ϕ_r согласно равенству

$$\phi=\sqrt{Z}\phi_r \tag{26}$$

Корреляторы, построенные из полей ϕ_r, содержат столько дополнительных (по сравнению с аналогичными корреляторами полей ϕ) множителей $Z^{-1/2}$, сколько полей стоит под знаком коррелятора. Теперь лагранжиан (25) принимает вид

$$\mathcal{L}=\frac{1}{2}Z\left(\partial_\mu\phi_r\right)^2-\frac{1}{2}m_0^2Z\phi_r^2-\frac{\lambda_0}{4!}Z^2\phi_r^4 \tag{27}$$

Определим константы

$$\delta_Z=Z-1,\quad \delta_m=m_0^2Z-m^2,\quad \delta_\lambda=\lambda_0Z^2-\lambda \tag{28}$$

где m и λ - физические константы. При помощи введенных констант лагранжиан переписывается следующим образом:

$$\mathcal{L}=\left[\frac{1}{2}\left(\partial_\mu\phi_r\right)^2-\frac{1}{2}m^2\phi_r^2-\frac{\lambda}{4!}\phi_r^4\right]+\left[\frac{1}{2}\delta_Z\left(\partial_\mu\phi_r\right)^2-\frac{1}{2}\delta_m\phi_r^2-\frac{\delta_\lambda}{4!}\phi_r^4\right] \tag{29}$$

Первая квадратная скобка в правой части (29) теперь выглядит как стандартный лагранжиан в теории ϕ^4, но записанный через перенормированные величины и поля. Слагаемые во второй квадратной скобке называются контрчленами, они поглощают расходимости в ТВ. Всё что было сделано - это проведено разбиение затравочного действия (25) так, чтобы было удобно проводить дальнейшие вычисления. Мы имеем следующие условия перенормировки

$$= \int \mathrm{d}x e^{ikx} \langle 0 \,|T\phi_r(x)\phi_r(0)|\, 0\rangle = \frac{i}{k^2 - m^2 - \Sigma\left(k^2\right)} =$$

$$= \frac{i}{k^2 - m^2 + i\varepsilon} + \left(\text{ слагаемые, регулярные при } k^2 = m^2\right)$$

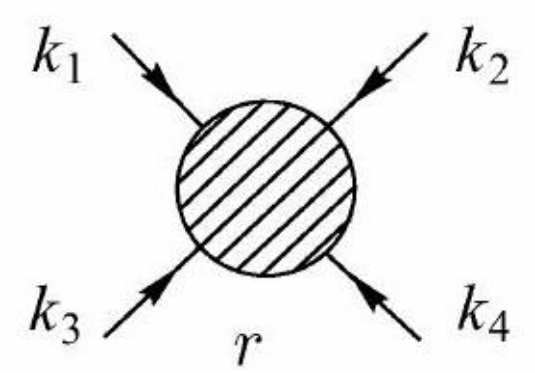
Последнее равенство означает, что

$$\Sigma\left(k^2\right)\big|_{k^2=m^2} = 0, \quad \frac{\mathrm{d}}{\mathrm{d}k^2}\Sigma\left(k^2\right)\bigg|_{k^2=m^2} = 0 \tag{30a}$$

Индекс r на диаграммах указывает на то, что эти величины - перенормированные. Однако этот факт далее как правило не отмечается, что не ведёт к недоразумениям. Условия (30a) означают, что на каждом следующем шаге индуктивной вычислительной процедуры следует сделать замену (провести вычитание)

$$\Sigma\left(k^2\right) \longrightarrow \Sigma\left(k^2\right) - \Sigma\left(m^2\right) - \left(k^2 - m^2\right)\left(\frac{\mathrm{d}}{\mathrm{d}k^2}\Sigma\left(k^2\right)\bigg|_{k^2=m^2}\right)$$

Имеет место также условие, фиксирующее константу связи:



$$\big|_{\text{амputированная}} = -\,i\lambda\big|_{\text{при } s=4m^2, t=u=0} \tag{30b}$$

Описываемая процедура перенормировки называется перенормировкой на xy.

Продемонстрируем эту процедуру перенормировки при вычислении $iM^{(2)}$. Согласно (20 Л4), (314) и (34 Л4) без учёта контрчленов в (29)

$$i\tilde{M}^{(2)}(s,t,u) = \text{ const } - i\frac{\lambda^2}{32\pi^2}\int\limits_0^1 \mathrm{d}x \left\{\ln\frac{m^2 - x(1-x)s - i\varepsilon}{\mu^2} + \ln\frac{m^2 - x(1-x)t - i\varepsilon}{\mu^2} + \right.$$
$$\left. + \ln\frac{m^2 - x(1-x)u - i\varepsilon}{\mu^2}\right\} \tag{31}$$

Здесь const - расходящаяся константа, не зависящая от переменных Мандельстама. Для выполнения условия нормировки (30 b) следует провести одно вычитание:

$$iM^{(2)}(s,t,u) = i\left[\tilde{M}^{(2)}(s,t,u) - \tilde{M}^{(2)}\left(4m^2,0,0\right)\right] =$$

$$= -i\frac{\lambda^2}{32\pi^2}\int\limits_0^1 \mathrm{d}x\left\{\ln\frac{m^2-x(1-x)s-i\varepsilon}{m^2[1-4x(1-x)-i\varepsilon]} + \ln\frac{m^2-x(1-x)t-i\varepsilon}{m^2} + \ln\frac{m^2-x(1-x)u-i\varepsilon}{m^2}\right\} \tag{32}$$

16.9.3 3. Unitarity Condition

Согласно (2 ЛЗ), S-матрица представляется как $S = 1 + iT$. Из условия унитарности $S^\dagger S = 1$ следуют важные соотношения для амплитуд. Имеем:

$$\left(T - T^\dagger\right) = iT^\dagger T \tag{33}$$

Пусть бра-вектор $\langle f|$ - конечное, а кет-вектор $|i\rangle$ - начальное состояние. Поместим равенство (33) в обкладки $\langle f|...|i\rangle$ и между операторами $T^\dagger$ и T в правой части (33) вставим единичный оператор в форме (7). Кроме того, перейдём к амплитуде M_{fi} согласно (4 ЛЗ). Таким образом получим:

$$M_{fi} - M^*_{if} = (2\pi)^4 i\sum_\lambda \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3 2E_{\mathbf{k}}(\lambda)}\delta^{(4)}\left(P_f - k\right) M^*_{\lambda_{\mathbf{k}}f} M_{\lambda_{\mathbf{k}}i} \tag{34}$$

Множитель $\delta^{(4)}\left(P_f - P_i\right)$ в обеих частях равенства (34) сокращается.

Равенство (34) полезно переписать также в виде

$$M_{fi} - M^*_{if} = (2\pi)^4 i\sum_n \sum_s \frac{1}{D}\prod_{\alpha=1}^n \left(\int \frac{\mathrm{d}^{(3)}k_\alpha}{(2\pi)^3 2\omega^{(\alpha)}_{k_\alpha}}\right)\delta^{(4)}\left(P_f - \sum_{\alpha=1}^n k_\alpha\right) M^*_{\lambda_{\mathbf{k}}f} M_{\lambda_{\mathbf{k}}i}$$

где n - число частиц в состоянии $|\lambda_{\mathbf{k}}\rangle, k_\alpha$ – 4-импульс, $\omega^{(\alpha)}_{k_\alpha} = k^0_\alpha$ - частота частицы с номером α, $\sum\limits_s$ - сумма по спинам промежуточных частиц и D - множитель, учитывающий тождественность частиц. Например, если промежуточное состояние содержит всего две тождественные частицы, то $D = 2$.

Рассмотрим важный частный случай, когда промежуточное состояние является одночастичным с массой M. Тогда суммирование $\sum\limits_\lambda$ в (34) ограничивается суммой по спинам промежуточной частицы $\sum\limits_s, E_{\mathbf{k}}(\lambda) = \sqrt{\mathbf{k}^2 + M^2}$ и

$$\frac{\mathrm{d}^3k}{2E_{\mathbf{k}}(\lambda)} = \theta\left(k^0\right)\delta\left(k_\mu k^\mu - M^2\right)\mathrm{d}^4k \tag{35}$$

Подставляя (35) в (34) и интегрируя по d^4k, находим:

$$M_{fi} - M^*_{if} = 2\pi i\delta\left(p^2 - M^2\right)\sum_s M^*_{(ps)f} M_{(ps)i} \tag{36}$$

где $|ps\rangle$ - состояние частицы с массой M, спином s и импульсом $p = P_i = P_f$.

Предположим, что теория T - и P-инвариантна. Тогда $M_{if} = M_{f'i'}$, где штрихованные состояния могут отличаться от соответствующих нештрихованных не более чем ориентацией

спина. Если начальное и конечное состояния усреднены по спину, то $M_{if} = M_{fi}$, и равенство (36) означает:

$$\operatorname{Im} M_{fi} = \pi\delta\left(p^2 - M^2\right) \sum_s M^*_{(ps)f} M_{(ps)i} \tag{37}$$

Отсюда следует, что при $p^2 \to M^2$

$$M_{fi} \to -\frac{\sum\limits_s M^*_{(ps)f} M_{(ps)i}}{p^2 - M^2 + i\varepsilon} \tag{38}$$

В этом случае говорят, что при переходе из состояния $|i\rangle$ в состояние $|f\rangle$ процесс идёт через промежуточную виртуальную частицу с массой M.

Задачи

1. Вычислить в безмассовой теории ϕ^4 при $d \to 4$ оператор собственной энергии $-i\Sigma\left(k^2\right)$ (21) в двухпетлевом приближении.

Указание. В безмассовом случае ($m^2 = 0$) вместо условий нормировки (30) при $k^2 = 0$, которые оказываются бессмысленными, используют следующие нормировочные условия:

$$\Sigma\left(k^2\right)\big|_{k^2=0} = 0 \quad \text{(отсутствие массы)} , \quad \frac{\mathrm{d}}{\mathrm{d}k^2}\Sigma\left(k^2\right)\bigg|_{k^2=-M^2} = 0$$

Применить параметризацию Фейнмана.

2. Вычислить при помощи условия унитарности мнимую часть амплитуды рассеяния $2 \to 2$ при $s > 4m^2$ в однопетлевом приближении.

17 Radiative Corrections: Some Formal Developments 1

We cheated four times in the last three chapters,* stating (and sometimes motivating) a result but postponing its proof. Those results were:

1. The formula for decay rates in terms of S-matrix elements, Eq. (4.86).
2. The master formula for S-matrix elements in terms of Feynman diagrams, Eq. (4.103).
3. The Ward identity, Eq. (5.79).
4. The *adhoc* subtraction to remove the ultraviolet divergence in the vertexcorrection diagram, Eq. (6.55).
   It is time now to return to these issues and give them a proper treatment. In Sections 7.2 through 7.4 we will derive all four of these results. The knowledge we gain along the way will help us interpret the three remaining loop corrections for electron scattering from a heavy target shown in (6.1): the external leg corrections and the vacuum polarization. We will evaluate the former in Section 7.1 and the latter in Section 7.5.

This chapter will be more abstract than the two preceding ones. Its main theme will be the singularities of Feynman diagrams viewed as analytic functions of their external momenta. We will find, however, that this apparently esoteric subject is rich in physical implications, and that it illuminates the relation between Feynman diagrams and the general principles of quantum theory.

17.1 7.1 Field-Strength Renormalization

In this section we will investigate the analytic structure of the two-point correlation function,

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle \quad \text{or} \quad \langle\Omega|T\psi(x)\bar{\psi}(y)|\Omega\rangle$$

In a free field theory, the two-point function $\langle 0|T\phi(x)\phi(y)|0\rangle$ has a simple interpretation: It is the amplitude for a particle to propagate from y to x. To what extent does this interpretation carry over into an interacting theory?

Our analysis of the two-point function will rely only on general principles of relativity and quantum mechanics; it will not depend on the nature of the interactions or on an expansion in perturbation theory. We will, however, restrict our consideration to scalar fields. Similar results can be obtained for correlation functions of fields with spin; we will display the analogous result for Dirac fields at the end of the analysis.

To dissect the two-point function $\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle$ we will insert the identity operator, in the form of a sum over a complete set of states, between $\phi(x)$ and $\phi(y)$. We choose these states to be eigenstates of the full interacting Hamiltonian, H. Since the momentum operator $\mathbf{P}$ commutes with H, we can also choose the states to be eigenstates of $\mathbf{P}$. But we can also make a stronger use of Lorentz invariance. Let $|\lambda_0\rangle$ be an eigenstate of H with momentum zero: $\mathbf{P}\,|\lambda_0\rangle = 0$. Then all the boosts of $|\lambda_0\rangle$ are also eigenstates of H, and these have all possible 3 -momenta. Conversely, any eigenstate of H with definite momentum can be written as the boost of some zero-momentum eigenstate $|\lambda_0\rangle$. The eigenvalues of the 4 -momentum operator $P^\mu = (H, \mathbf{P})$ organize themselves into hyperboloids, as shown in Fig. 7.1.

Recall from Chapter 2 that the completeness relation for the one-particle states is

$$(\mathbf{1})_{1-\text{ particle}} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}|\mathbf{p}\rangle\langle\mathbf{p}| \tag{7.1}$$

We can write an analogous completeness relation for the entire Hilbert space with the aid of a bit of notation. Let $|\lambda_{\mathbf{p}}\rangle$ be the boost of $|\lambda_0\rangle$ with momentum $\mathbf{p}$, and assume that the states $|\lambda_{\mathbf{p}}\rangle$, like the one-particle states $|\mathbf{p}\rangle$, are relativistically normalized. Let $E_{\mathbf{p}}(\lambda) \equiv \sqrt{|\mathbf{p}|^2 + m_\lambda^2}$, where m_λ is the "mass" of the states $|\lambda_{\mathbf{p}}\rangle$, that is, the energy of the state $|\lambda_0\rangle$. Then the desired completeness relation is

$$\mathbf{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}(\lambda)}\,|\lambda_{\mathbf{p}}\rangle\,\langle\lambda_{\mathbf{p}}| \tag{7.2}$$

where the sum runs over all zero-momentum states $|\lambda_0\rangle$.

We now insert this expansion between the operators in the two-point function. Assume for now that $x^0 > y^0$. Let us drop the uninteresting constant term $\langle\Omega|\phi(x)|\Omega\rangle\langle\Omega|\phi(y)|\Omega\rangle$. (This term is usually zero by symmetry; for higher-spin fields, it is zero by Lorentz invariance.) The two-point function is then

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}(\lambda)}\langle\Omega|\phi(x)\,|\lambda_{\mathbf{p}}\rangle\,\langle\lambda_{\mathbf{p}}|\,\phi(y)|\Omega\rangle \tag{7.3}$$

[6] *A fifth cheat, postulating rather than deriving the photon propagator, will be remedied in Chapter 9.

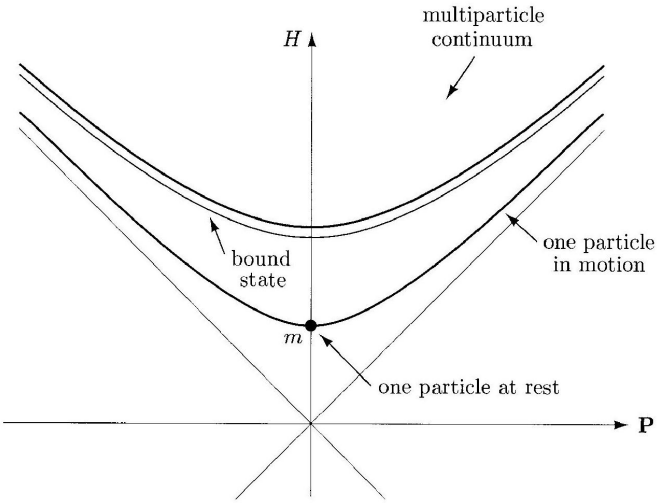


Figure 7.1. The eigenvalues of the 4 -momentum operator $P^\mu = (H, \mathbf{P})$ occupy a set of hyperboloids in energy-momentum space. For a typical theory the states consist of one or more particles of mass m. Thus there is a hyperboloid of one-particle states and a continuum of hyperboloids of two-particle states, three-particle states, and so on. There may also be one or more boundstate hyperboloids below the threshold for creation of two free particles.

We can manipulate the matrix elements as follows:

$$\begin{aligned} \langle\Omega|\phi(x)\,|\lambda_{\mathbf{p}}\rangle &= \langle\Omega|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}\,|\lambda_{\mathbf{p}}\rangle \\ &= \langle\Omega|\phi(0)\,|\lambda_{\mathbf{p}}\rangle\, e^{-ip\cdot x}\big|_{p^0=E_{\mathbf{p}}} \\ &= \langle\Omega|\phi(0)\,|\lambda_0\rangle\, e^{-ip\cdot x}\big|_{p^0=E_{\mathbf{p}}} \end{aligned} \tag{7.4}$$

The last equality is a result of the Lorentz invariance of $\langle\Omega|$ and $\phi(0)$: Insert factors of $U^{-1}U$, where U is the unitary operator that implements a Lorentz boost from $\mathbf{p}$ to 0 , and use $U\phi(0)U^{-1} = \phi(0)$. (For a field with spin, we would need to keep track of its nontrivial Lorentz transformation.) Introducing an integration over p^0, our expression for the two-point function (still for $x^0 > y^0$) becomes

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_\lambda \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m_\lambda^2 + i\epsilon}e^{-ip\cdot(x-y)}|\langle\Omega|\phi(0)|\lambda_0\Big\rangle|^2 \tag{7.5}$$

Note the appearance of the Feynman propagator, $D_F(x-y)$, but with m replaced by m_λ.

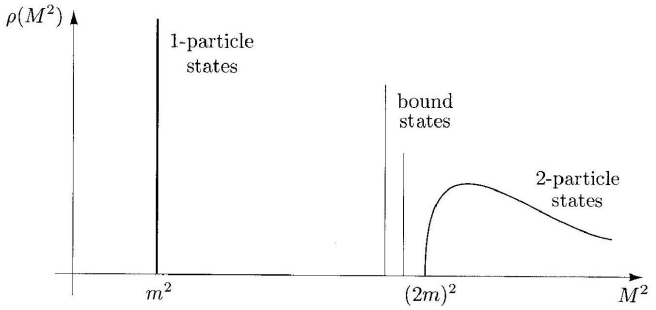


Figure 7.2. The spectral function $\rho\,(M^2)$ for a typical interacting field theory. The one-particle states contribute a delta function at m^2 (the square of the particle's mass). Multiparticle states have a continuous spectrum beginning at $(2m)^2$. There may also be bound states.

Analogous expressions hold for the case $y^0 > x^0$. Both cases can be summarized in the following general representation of the two-point function (the Klln-Lehmann spectral representation):

$$\langle \Omega | T\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\rho\left(M^2\right) D_F\left(x-y; M^2\right) \tag{7.6}$$

where $\rho\left(M^2\right)$ is a positive spectral density function,

$$\rho\left(M^2\right) = \sum_\lambda (2\pi)\delta\left(M^2 - m_\lambda^2\right) |\langle \Omega|\phi(0)|\lambda_0\rangle|^2 \tag{7.7}$$

The spectral density $\rho\left(M^2\right)$ for a typical theory is plotted in Fig. 7.2. Note that the one-particle states contribute an isolated delta function to the spectral density:

$$\rho\left(M^2\right) = 2\pi\delta\left(M^2 - m^2\right)\cdot Z + \left(\text{ nothing else until } M^2 \gtrsim (2m)^2\right) \tag{7.8}$$

where Z is some number given by the squared matrix element in (7.7). We refer to Z as the field-strength renormalization. The quantity m is the exact mass of a single particle - the exact energy eigenvalue at rest. This quantity will in general differ from the value of the mass parameter that appears in the Lagrangian. We will refer to the parameter in the Lagrangian as m_0, the bare mass, and refer to m as the physical mass of the ϕ boson. Only the physical mass m is directly observable.

The spectral decomposition (7.6) yields the following form for the Fourier

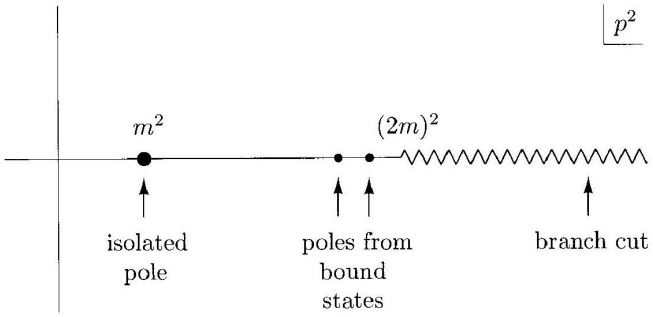


Figure 7.3. Analytic structure in the complex p^2-plane of the Fourier transform of the two-point function for a typical theory. The one-particle states contribute an isolated pole at the square of the particle mass. States of two or more free particles give a branch cut, while bound states give additional poles.

transform of the two-point function:

$$\int d^4x e^{ip\cdot x}\langle \Omega|T\phi(x)\phi(0)|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\rho\left(M^2\right)\frac{i}{p^2 - M^2 + i\epsilon} \tag{7.9}$$
$$= \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{\sim 4m^2}^\infty \frac{dM^2}{2\pi}\rho\left(M^2\right)\frac{i}{p^2 - M^2 + i\epsilon}$$

The analytic structure of this function in the complex p^2-plane is shown in Fig. 7.3. The first term gives an isolated simple pole at $p^2 = m^2$, while the second term contributes a branch cut beginning at $p^2 = (2m)^2$. If there are any two-particle bound states, these will appear as additional delta functions in $\rho\left(M^2\right)$ and thus as additional poles below the cut.

In Section 2.4, we found an explicit result for the two-point correlation function in the theory of a free scalar field:

$$\int d^4x e^{ip\cdot x}\langle 0|T\phi(x)\phi(0)|0\rangle = \frac{i}{p^2 - m^2 + i\epsilon} \tag{7.10}$$

We interpreted this formula, for $x^0 > 0$, as the amplitude for a particle to propagate from 0 to x. Equation (7.9) shows that the two-point function takes a similar form in the most general theory of an interacting scalar field. The general expression is essentially a sum of scalar propagation amplitudes for states created from the vacuum by the field operator $\phi(0)$. There are two differences between (7.9) and (7.10). First, Eq. (7.9) contains the fieldstrength renormalization factor $Z = |\langle\lambda_0|\,\phi(0)|\,\Omega\rangle\,|^2$, the probability for $\phi(0)$ to create a given state from the vacuum. In (7.10), this factor is included implicitly, since $\langle p|\phi(0)|0\rangle = 1$ in free field theory. Second, Eq. (7.9) contains
contributions from multiparticle intermediate states with a continuous mass spectrum. In free field theory, $\phi(0)$ can create only a single particle from the vacuum. With these two differences, (7.9) is a direct generalization of (7.10).

It will be important in our later analysis that the contributions to (7.9) from one-particle and multiparticle intermediate states can be distinguished by the strength of their analytic singularities. The poles in p^2 come only from one-particle intermediate states, while multiparticle intermediate states give weaker branch cut singularities. We will see in the next section that this rather formal observation generalizes to higher-point correlation functions and plays a crucial role in our derivation of the diagrammatic formula for S-matrix elements.

The analysis of this section generalizes directly to two-point functions of higher-spin fields. The main complication comes in the generalization of the manipulation (7.4), since now the field has a nontrivial transformation law under boosts. In general, several invariant spectral functions are required to represent the multiparticle states. But this complication does not affect the major result that a pole in p^2 can arise only from the contribution of a singleparticle state created by the field operator. The two-point function of Dirac fields, for example, has the structure

$$\begin{aligned}\int d^4x e^{ip\cdot x}\langle\Omega|T\psi(x)\bar\psi(0)|\Omega\rangle &= \frac{iZ_2\sum_s u^s(p)\bar u^s(p)}{p^2 - m^2 + i\epsilon} + \cdots \\ &= \frac{iZ_2(\not p + m)}{p^2 - m^2 + i\epsilon} + \cdots\end{aligned} \tag{7.11}$$

where the omitted terms give the multiparticle branch cut. As in the scalar case, the constant Z_2 is the probability for the quantum field to create or annihilate an exact one-particle eigenstate of H:

$$\langle\Omega|\psi(0)|p,s\rangle = \sqrt{Z_2}u^s(p) \tag{7.12}$$

(For an antiparticle, replace u with $\bar v$.) At the pole, the Dirac two-point function is exactly that of a free field with the physical mass, aside from the rescaling factor Z_2.

An Example: The Electron Self-Energy

This nonperturbative analysis of the two-point correlation function has been very different from our usual direct analysis of Feynman diagrams. But since this derivation was done in complete generality, the singularity structure of the two-point function that it implies ought also to be visible in a Feynman diagram computation. In the rest of this section we will explicitly check our results for the electron two-point function in QED.

The electron two-point function is equal to the sum of diagrams

$$\langle\Omega|T\psi(x)\bar{\psi}(y)|\Omega\rangle = \bar{x} + \frac{r\sim}{x} + \cdots \tag{7.13}$$

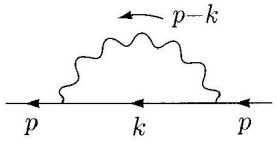
Each of these diagrams, according to the Feynman rules for correlation functions, contains a factor of $e^{ip\cdot(x-y)}$ for the two external points and an integration $\int (d^4p/(2\pi)^4)$ over the momentum p carried by the initial and final propagators. We will consistently omit these factors in this section; in other words, each diagram will denote the corresponding term in the Fourier transform of the two-point function.

The first diagram is just the free-field propagator:

$$\bar{p} = \frac{i\,(\not{p} + m_0)}{p^2 - m_0^2 + i\epsilon} \tag{7.14}$$

Throughout this calculation, we will write m_0 instead of m as the mass in the electron propagator. This makes explicit the fact noted above that the mass appearing in the Lagrangian differs, in general, from the observable rest energy of a particle. However, if a perturbation expansion is applicable, the leadingorder expression for the propagator should approximate the exact expression. Indeed, the function (7.14) has a pole, of just the form of (7.11), at $p^2 = m_0^2$. We therefore expect that the complete expression for the two-point function also has a pole of this form, at a slightly shifted location $m^2 = m_0^2 + \mathcal{O}(\alpha)$.

The second diagram in (7.13), called the electron self-energy, is somewhat more complicated:



$$= \frac{i(\not{p} + m_0)}{p^2 - m_0^2}\left[-i\Sigma_2(p)\right]\frac{i(\not{p} + m_0)}{p^2 - m_0^2},$$

where

$$-i\Sigma_2(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\gamma^\mu \frac{i\,(\not{k} + m_0)}{k^2 - m_0^2 + i\epsilon}\gamma_\mu \frac{-i}{(p-k)^2 - \mu^2 + i\epsilon}. \tag{7.16}$$

(The notation Σ_2 indicates that this is the second-order (in e) contribution to a quantity Σ that we will define below.) The integral Σ_2 has an infrared divergence, which we have regularized by adding a small photon mass μ. Outside this integral, the diagram seems to have a double pole at $p^2 = m_0^2$. All in all, the form of this correction is quite unpleasant. But let us press on and try to evaluate $\Sigma_2(p)$ using the calculational techniques developed for the vertex correction in the Section 6.3.

First introduce a Feynman parameter to combine the two denominators:

$$\frac{1}{k^2 - m_0^2 + i\epsilon}\frac{1}{(p-k)^2 - \mu^2 + i\epsilon} = \int_0^1 dx \frac{1}{\left[k^2 - 2xk\cdot p + xp^2 - x\mu^2 - (1-x)m_0^2 + i\epsilon\right]^2}$$

Now complete the square and define a shifted momentum $\ell \equiv k - xp$. Dropping the term linear in ℓ from the numerator, we have

$$-i\Sigma_2(p) = -e^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{-2x\ \not{p} + 4m_0}{\left[\ell^2 - \Delta + i\epsilon\right]^2} \tag{7.17}$$

where $\Delta = -x(1-x)p^2 + x\mu^2 + (1-x)m_0^2$. The integral over ℓ is divergent. To evaluate it, we first regulate it using the Pauli-Villars procedure (6.51):

$$\frac{1}{(p-k)^2-\mu^2+i\epsilon} \to \frac{1}{(p-k)^2-\mu^2+i\epsilon} - \frac{1}{(p-k)^2-\Lambda^2+i\epsilon}$$

The second term will have the same form as (7.17), but with μ replaced by Λ. As in Section 6.3, we now Wick-rotate and substitute the Euclidean variable $\ell_E^0 = -i\ell^0$. This gives

$$\begin{aligned}\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\left[\ell^2-\Delta\right]^2} &\to \frac{i}{(4\pi)^2}\int_0^\infty d\ell_E^2 \left(\frac{\ell_E^2}{\left[\ell_E^2+\Delta\right]^2} - \frac{\ell_E^2}{\left[\ell_E^2+\Delta_\Lambda\right]^2}\right) \\ &= \frac{i}{(4\pi)^2}\log\left(\frac{\Delta_\Lambda}{\Delta}\right) \end{aligned} \tag{7.18}$$

where

$$\Delta_\Lambda = -x(1-x)p^2 + x\Lambda^2 + (1-x)m_0^2 \underset{\Lambda\to\infty}{\longrightarrow} x\Lambda^2$$

The final result is therefore

$$\Sigma_2(p) = \frac{\alpha}{2\pi}\int_0^1 dx\,(2m_0 - x\not{p})\log\left(\frac{x\Lambda^2}{(1-x)m_0^2 + x\mu^2 - x(1-x)p^2}\right) \tag{7.19}$$

Before discussing the divergences in this expression, let us work out its analytic behavior as a function of p^2. The logarithm in (7.19) has a branch cut when its argument becomes negative, and for any fixed x this will occur for sufficiently large p^2. More exactly, the cut begins at the point where

$$(1-x)m_0^2 + x\mu^2 - x(1-x)p^2 = 0$$

Solving this equation for x, we find

$$\begin{aligned} x &= \frac{1}{2} + \frac{m_0^2}{2p^2} - \frac{\mu^2}{2p^2} \pm \sqrt{\frac{\left(p^2+m_0^2-\mu^2\right)^2}{4p^4} - \frac{m_0^2}{p^2}} \\ &= \frac{1}{2} + \frac{m_0^2}{2p^2} - \frac{\mu^2}{2p^2} \pm \frac{1}{2p^2}\sqrt{\left[p^2-(m_0+\mu)^2\right]\left[p^2-(m_0-\mu)^2\right]} \end{aligned} \tag{7.20}$$

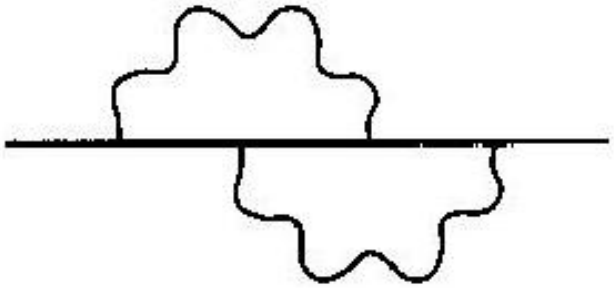
The branch cut of $\Sigma_2\left(p^2\right)$ begins at the minimum value of p^2 such that this equation has a real solution for x between 0 and 1 . This occurs when $p^2 = (m_0+\mu)^2$, that is, at the threshold for creation of a two-particle (electron
plus photon) state. In fact, it is a simple exercise in relativistic kinematics to show that the square root in (7.20), written in the form

$$k = \frac{1}{2\sqrt{p^2}}\sqrt{\left[p^2-(m_0+\mu)^2\right]\left[p^2-(m_0-\mu)^2\right]}$$

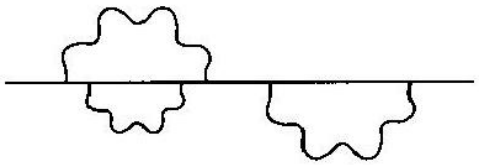
is precisely the momentum in the center-of-mass frame for two particles of mass m_0 and μ and total energy $\sqrt{p^2}$. It is natural that this momentum becomes real at the two-particle threshold. The location of the branch cut is exactly where we would expect from the Klln-Lehmann spectral representation. †

We have now located the two-particle branch cut predicted by the KllnLehmann representation, but we have not found the expected simple pole at $p^2 = m^2$. To find it we must actually include an infinite series of Feynman diagrams. Fortunately, this series will be easily summed.

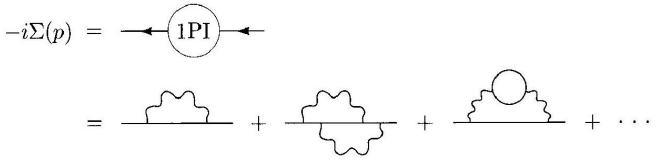
Let us define a one-particle irreducible (1PI) diagram to be any diagram that cannot be split in two by removing a single line:



is 1 PI , while



Let $-i\Sigma(p)$ denote the sum of all 1PI diagrams with two external fermion lines:



(Although each diagram has two external lines, the Feynman propagators for these lines are not to be included in the expression for $\Sigma(p)$.) To leading order in α we see that $\Sigma = \Sigma_2$.

The Fourier transform of the two-point function can now be written as

$$\begin{aligned}
&\int d^4x \langle \Omega | T\psi(x)\bar{\psi}(0)|\Omega\rangle e^{ip\cdot x} = \\
&= \frac{\text{1PI}}{\leftarrow} + \cdots + \text{1PI} - \text{1PI} + \cdots \\
&= \frac{i\,(\not{p} + m_0)}{p^2 - m_0^2} + \frac{i\,(\not{p} + m_0)}{p^2 - m_0^2}(-i\Sigma)\frac{i\,(\not{p} + m_0)}{p^2 - m_0^2} + \cdots
\end{aligned} \tag{7.22}$$

The first diagram has a simple pole at $p^2 = m_0^2$. Each diagram in the second class has a double pole at $p^2 = m_0^2$. Each diagram in the third class has a triple pole. The behavior near $p^2 = m_0^2$ gets worse and worse as we include more and more diagrams. But fortunately, the sum of all the diagrams forms a geometric series. Note that $\Sigma(p)$ commutes with $\not{p}$, since $\Sigma(p)$ is a function only of pure numbers and $\not{p}$. In fact, we can consider $\Sigma(p)$ to be a function of $\not{p}$, writing $p^2 = (\not{p})^2$. Then we can rewrite each electron propagator as $i/(\not{p} - m_0)$ and express the above series as

[6†] In real QED, $\mu = 0$ and the two-particle branch cut merges with the one-particle pole. This subtlety plays a role in the full treatment of the cancellation of infrared divergences, but it is beyond the scope of our present analysis.

$$\int d^4x \langle \Omega | T\psi(x)\bar{\psi}(0)|\Omega\rangle e^{ip\cdot x}$$
$$= \frac{i}{\not p - m_0} + \frac{i}{\not p - m_0}\left(\frac{\Sigma(\not p)}{\not p - m_0}\right) + \frac{i}{\not p - m_0}\left(\frac{\Sigma(\not p)}{\not p - m_0}\right)^2 + \cdots$$
$$= \frac{i}{\not p - m_0 - \Sigma(\not p)}. \tag{7.23}$$

The full propagator has a simple pole, which is shifted away from m_0 by $\Sigma(\not p)$. The location of this pole, the physical mass m, is the solution of the equation

$$[\not p - m_0 - \Sigma(\not p)]\big|_{\not p = m} = 0 \tag{7.24}$$

Notice that, if $\Sigma(pp)$ is defined by the convention (7.21), then a positive contribution to Σ yields a positive shift of the electron mass. Close to the pole, the denominator of (7.23) has the form

$$(\not p - m)\cdot\left(1 - \frac{d\Sigma}{d\not p}\bigg|_{\not p = m}\right) + \mathcal{O}\left((\not p - m)^2\right). \tag{7.25}$$

Thus the full electron propagator has a single-particle pole of just the form (7.11), with m given by (7.24) and

$$Z_2^{-1} = 1 - \frac{d\Sigma}{d\not p}\bigg|_{\not p = m} \tag{7.26}$$

Our explicit calculation of Σ_2 allows us to compute the first corrections to m and Z_2. Let us begin with m. To order α, the mass shift is

$$\delta m = m - m_0 = \Sigma_2(\not p = m) \approx \Sigma_2(\not p = m_0) \tag{7.27}$$

Thus, using (7.19),

$$\delta m = \frac{\alpha}{2\pi} m_0 \int_0^1 dx (2-x) \log\left(\frac{x\Lambda^2}{(1-x)^2 m_0^2 + x\mu^2}\right) \tag{7.28}$$

The mass shift is ultraviolet divergent; the divergent term has the form

$$\delta m \underset{\Lambda\to\infty}{\longrightarrow} \frac{3\alpha}{4\pi} m_0 \log\left(\frac{\Lambda^2}{m_0^2}\right) \tag{7.29}$$

Is it a problem that m differs from m_0 by a divergent quantity? This question has two levels, those of concept and practice.

On the conceptual level, we should fully expect the mass of the electron to be modified by its coupling to the electromagnetic field. In classical electrodynamics, the rest energy of any charge is increased by the energy of its electrostatic field, and this energy shift diverges in the case of a point charge:

$$\int d^3r \frac{1}{2}|\mathbf{E}|^2 = \int d^3r \frac{1}{2}\left(\frac{e}{4\pi r^2}\right)^2 = \frac{\alpha}{2}\int \frac{dr}{r^2} \sim \alpha\Lambda \tag{7.30}$$

In fact, it is puzzling why the divergence in (7.29) is so weak, logarithmic in Λ rather than linear as in (7.30). To understand this feature, suppose that m_0 were set to 0 . Then the two helicity components of the electron field ψ_L and ψ_R would not be coupled by any term in the QED Hamiltonian. This would imply that perturbative corrections could never induce a coupling of ψ_L and ψ_R, nor, in particular, an electron mass term. In other words, δm must vanish when $m_0 = 0$. The mass shift must therefore be proportional to m_0, and so, by dimensional analysis, it can depend only logarithmically on Λ.

On a practical level, the infinite mass shift casts doubt on our perturbative calculations. For example, all of the theoretical results in Chapter 5 should technically involve m_0 rather than m. To compare theory to experiment we must eliminate m_0 in favor of m, using the relation $m_0 = m + \mathcal{O}(\alpha)$. Since the "small" $\mathcal{O}(\alpha)$ correction is infinite, the validity of this procedure is far from obvious. The validity of perturbation theory would be more plausible if we could compute Feynman diagrams using the propagator $i/(pp - m)$, which has the correct pole location, instead of $i/(\not{p} - m_0)$. In Chapter 10 we will see how to rearrange the perturbation series in such a way that m_0 is systematically eliminated in favor of m and the zeroth-order propagator has its pole at the physical mass. In the remainder of this chapter, we will continue to simply replace m_0 by m in expressions for order- α corrections.

Finally, let us examine the perturbative correction to Z_2. From (7.26), we find that the order-α correction $\delta Z_2 = (Z_2 - 1)$ is

$$\begin{aligned}\delta Z_2 &= \left.\frac{d\Sigma_2}{d\ \not{p}p}\right|_{\not{p}=m} \\ &= \frac{\alpha}{2\pi}\int_0^1 dx\left[-x\log\frac{x\Lambda^2}{(1-x)^2m^2+x\mu^2} + 2(2-x)\frac{x(1-x)m^2}{(1-x)^2m^2+x\mu^2}\right] \end{aligned} \tag{7.31}$$

This expression is also logarithmically ultraviolet divergent. We will discuss the observability of this divergent term at the end of Section 7.2. However, it is interesting to note, even before that discussion, that (7.31) is very similar in form to the value of the ad hoc subtraction that we made in our calculation of the electron vertex correction in Section 6.3. From Eq. (6.56), the value of this subtraction was

$$\begin{aligned}\delta F_1(0) =& \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1) \\ &\times\left[\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right) + \frac{(1-4z+z^2)\,m^2}{(1-z)^2m^2+z\mu^2}\right] \\ =& \frac{\alpha}{2\pi}\int_0^1 dz(1-z)\left[\log\left(\frac{z\Lambda^2}{(1-z)^2m^2+z\mu^2}\right) + \frac{(1-4z+z^2)\,m^2}{(1-z)^2m^2+z\mu^2}\right] \end{aligned} \tag{7.32}$$

Using the integration by parts

$$\begin{aligned}\int_0^1 dz(1-2z)\log\left(\frac{\Lambda^2}{(1-z)^2m^2+z\mu^2}\right) &= -\int_0^1 dzz(1-z)\frac{2(1-z)m^2-\mu^2}{(1-z)^2m^2+z\mu^2} \\ &= \int_0^1 dz\left[(1-z) - \frac{(1-z)\,(1-z^2)\,m^2}{(1-z)^2m^2+z\mu^2}\right]\end{aligned}$$

it is not hard to show that $\delta F_1(0) + \delta Z_2 = 0$. This identity will play a crucial role in justifying the ad hoc subtraction of Section 6.3.

17.2 7.2 The LSZ Reduction Formula

In the last section we saw that the Fourier transform of the two-point correlation function, considered as an analytic function of p^2, has a simple pole at the mass of the one-particle state:

$$\int d^4x e^{ip\cdot x}\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle \underset{p^2\to m^2}{\sim} \frac{iZ}{p^2-m^2+i\epsilon} \tag{7.33}$$

(Here and throughout this section we use the symbol $\sim$ to mean that the poles of both sides are identical; there are additional finite terms, given in this case by Eq. (7.9).) In this section we will generalize this result to higher correlation functions. We will derive a general relation between correlation functions and S-matrix elements first obtained by Lehmann, Symanzik, and Zimmermann and known as the LSZ reduction formula. ‡ This result, combined with our Feynman rules for computing correlation functions, will justify Eq. (4.103), our master formula for S-matrix elements in terms of Feynman diagrams. For simplicity, we will carry out the whole analysis for the case of scalar fields.

The strategy of the argument will be as follows. To calculate the S-matrix element for a 2-body $\to$ n-body scattering process, we begin with the correlation function of $n+2$ Heisenberg fields. Fourier-transforming with respect to the coordinate of any one of these fields, we will find a pole of the form (7.33) in the Fourier-transform variable p^2. We will argue that the one-particle states associated with these poles are in fact asymptotic states, that is, states given by the limit of well-separated wavepackets as they become concentrated around definite momenta. Taking the limit in which all $n+2$ external particles go on-shell, we can then interpret the coefficient of the multiple pole as an S-matrix element.

To begin, let us Fourier-transform the $(n+2)$-point correlation function with respect to one argument x. We must then analyze the integral

$$\int d^4x e^{ip\cdot x}\langle\Omega|T\left\{\phi(x)\phi\left(z_1\right)\phi\left(z_2\right)\cdots\right\}|\Omega\rangle$$

We would like to identify poles in the variable p^0. To do this, divide the integral over x^0 into three regions:

$$\int dx^0 = \int_{T_+}^{\infty} dx^0 + \int_{T_-}^{T_+} dx^0 + \int_{-\infty}^{T_-} dx^0 \tag{7.34}$$

where $T_+$is much greater than all the z_i^0 and $T_-$is much less than all the z_i^0. Call these three intervals regions I, II, and III. Since region II is bounded and the integrand depends on p^0 through the analytic function $\exp\left(ip^0x^0\right)$, the contribution from this region will be analytic in p^0. However, regions I and III, which are unbounded, may develop singularities in p^0.

Consider first region I. Here x^0 is the latest time, so $\phi(x)$ stands first in the time ordering. Insert a complete set of intermediate states in the form of (7.2):

$$\mathbf{1} = \sum_{\lambda}\int \frac{d^3q}{(2\pi)^3}\frac{1}{2E_{\mathbf{q}}(\lambda)}\left|\lambda_{\mathbf{q}}\right\rangle\left\langle\lambda_{\mathbf{q}}\right|$$

The integral over region I then becomes

[6‡] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento 1, 1425 (1955).

$$\int_{T_+}^{\infty} dx^0 \int d^3x e^{ip^0x^0} e^{-i\mathbf{p}\cdot\mathbf{x}} \sum_\lambda \int \frac{d^3q}{(2\pi)^3} \frac{1}{2E_{\mathbf{q}}(\lambda)} \langle\Omega|\phi(x)\,|\lambda_{\mathbf{q}}\rangle \times \langle\lambda_{\mathbf{q}}|\,T\left\{\phi\left(z_1\right)\phi\left(z_2\right)\cdots\right\}|\Omega\rangle \tag{7.35}$$

Using Eq. (7.4),

$$\langle\Omega|\phi(x)\,|\lambda_{\mathbf{q}}\rangle = \langle\Omega|\phi(0)\,|\lambda_0\rangle\, e^{-iq\cdot x}\big|_{q^0=E_{\mathbf{q}}(\lambda)}$$

and including a factor $e^{-\epsilon x^0}$ to insure that the integral is well defined, this integral becomes

$$\begin{aligned}
&\sum_\lambda \int_{T_+}^{\infty} dx^0 \int \frac{d^3q}{(2\pi)^3} \frac{1}{2E_{\mathbf{q}}(\lambda)} e^{ip^0x^0} e^{-iq^0x^0} e^{-\epsilon x^0} \langle\Omega|\phi(0)\,|\lambda_0\rangle\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \\
&\times \langle\lambda_{\mathbf{q}}|\,T\left\{\phi\left(z_1\right)\cdots\right\}|\Omega\rangle \\
&= \sum_\lambda \frac{1}{2E_{\mathbf{p}}(\lambda)} \frac{ie^{i\left(p^0-E_{\mathbf{p}}+i\epsilon\right)T_+}}{p^0-E_{\mathbf{p}}(\lambda)+i\epsilon} \langle\Omega|\phi(0)\,|\lambda_0\rangle\,\langle\lambda_{\mathbf{p}}|\,T\left\{\phi\left(z_1\right)\cdots\right\}|\Omega\rangle
\end{aligned} \tag{7.36}$$

The denominator is just that of Eq. (7.5): $p^2 - m_\lambda^2$. There is an analytic singularity in p^0; as in Section 7.1 , this singularity will be either a pole or a branch cut depending upon whether or not the rest energy m_λ is isolated. The one-particle state corresponds to an isolated energy value $p^0 = E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$, and at this point Eq. (7.36) has a pole:

$$\begin{aligned}
&\int d^4x e^{ip\cdot x} \langle\Omega|T\left\{\phi(x)\phi\left(z_1\right)\cdots\right\}|\Omega\rangle \\
&\qquad \underset{p^0\to+E_{\mathbf{p}}}{\sim} \frac{i}{p^2-m^2+i\epsilon} \sqrt{Z}\langle\mathbf{p}|T\left\{\phi\left(z_1\right)\cdots\right\}|\Omega\rangle
\end{aligned} \tag{7.37}$$

The factor $\sqrt{Z}$ is the same field strength renormalization factor as in Eq. (7.8), since it replaces the same matrix element as in (7.7).

To evaluate the contribution from region III, we would put $\phi(x)$ last in the operator ordering, then insert a complete set of states between $T\left\{\phi\left(z_1\right)\cdots\right\}$ and $\phi(x)$. Repeating the above argument produces a pole as $p^0 \to -E_{\mathbf{p}}$:

$$\begin{aligned}
&\int d^4x e^{ip\cdot x} \langle\Omega| \\
&\qquad T\left\{\phi(x)\phi\left(z_1\right)\cdots\right\}|\Omega\rangle \\
&\qquad \sim \underset{p^0\to-E_{\mathbf{p}}}{\sim} \langle\Omega|T\left\{\phi\left(z_1\right)\cdots\right\}|-\mathbf{p}\rangle\sqrt{Z}\frac{i}{p^2-m^2+i\epsilon}
\end{aligned} \tag{7.38}$$

Next we would like to Fourier-transform with respect to the remaining field coordinates. To keep the various external particles from interfering, however, we must isolate them from each other in space. Let us therefore repeat the preceding calculation using a wavepacket rather than a simple Fourier transform. In Eq. (7.35), replace

$$\int d^4x e^{ip^0x^0} e^{-i\mathbf{p}\cdot\mathbf{x}} \to \int \frac{d^3k}{(2\pi)^3} \int d^4x e^{ip^0x^0} e^{-i\mathbf{k}\cdot\mathbf{x}} \varphi(\mathbf{k}) \tag{7.39}$$

where $\varphi(\mathbf{k})$ is a narrow distribution centered on $\mathbf{k} = \mathbf{p}$. This distribution constrains x to lie within a band, whose spatial extent is that of the wavepacket, about the trajectory of a particle with momentum $\mathbf{p}$. With this modification, the right-hand side of (7.36) has a more complicated singularity structure:

$$\begin{aligned}\sum_\lambda \int \frac{d^3k}{(2\pi)^3} \varphi(\mathbf{k}) \frac{1}{2E_{\mathbf{k}}(\lambda)} \frac{i}{p^0 - E_{\mathbf{k}}(\lambda) + i\epsilon} \langle\Omega|\phi(0)\,|\lambda_0\rangle\,\langle\lambda_{\mathbf{k}}|\,T\left\{\phi\left(z_1\right)\cdots\right\}|\Omega\rangle \\ \underset{p^0\to+E_{\mathbf{p}}}{\sim} \int \frac{d^3k}{(2\pi)^3} \varphi(\mathbf{k}) \frac{i}{\tilde{p}^2 - m^2 + i\epsilon} \sqrt{Z} \langle\mathbf{k}|T\left\{\phi\left(z_1\right)\cdots\right\}|\Omega\rangle,\end{aligned} \tag{7.40}$$

where, in the second line, $\tilde{p} = (p_0, \mathbf{k})$. The one-particle singularity is now a branch cut, whose length is the width in momentum space of the wavepacket $\varphi(\mathbf{k})$. However, if $\varphi(\mathbf{k})$ defines the momentum narrowly, this branch cut is very short, and (7.40) has a well-defined limit in which $\varphi(\mathbf{k})$ tends to $(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{p})$ and the singularity of (7.40) sharpens up to the pole of (7.36). The singularity due to single-particle states in the far past, Eq. (7.38), is modified in the same way.

Now consider integrating each of the coordinates in the $(n+2)$-point correlation function against a wavepacket, to form*

$$\left(\prod_i \int \frac{d^3k_i}{(2\pi)^3} \int d^4x_i e^{i\tilde{p}_i\cdot x_i} \varphi_i\left(\mathbf{k}_i\right)\right) \langle\Omega|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\cdots\right\}|\Omega\rangle \tag{7.41}$$

The wavepackets should be chosen to overlap in a region around $x = 0$ and to separate in the far past and the far future. To analyze this integral, we choose a large positive time T_+such that all of the wavepackets are well separated for $x_i^0 > T_+$, and we choose a large negative time T_-such that all of the wavepackets are well separated for $x_i^0 < T_-$. Then we can break up each of of the integrals over x_i^0 into three regions as in (7.34). The integral of any x_i^0 over the bounded region II leads to an expression analytic in the corresponding energy p_0^i, so we can concentrate on the case in which all of the x_i^0 are placed at large past or future times.

For definiteness, consider the contribution in which only two of the time coordinates, x_1^0 and x_2^0, are in the future. In this case, the fields $\phi\left(x_1\right)$ and $\phi\left(x_2\right)$ stand to the left of the other fields in time order. Inserting a complete set of states $|\lambda_{\mathbf{K}}\rangle$, the integrations in (7.41) over the coordinates of these two fields take the form

$$\begin{aligned}\sum_\lambda \int \frac{d^3K}{(2\pi)^3} \frac{1}{2E_{\mathbf{K}}} \left(\prod_{i=1,2} \int \frac{d^3k_i}{(2\pi)^3} \int d^4x_i e^{i\tilde{p}_i\cdot x_i} \varphi_i\left(\mathbf{k}_i\right)\right) \\ \times \langle\Omega|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\right\}|\lambda_{\mathbf{K}}\rangle\,\langle\lambda_{\mathbf{K}}|\,T\left\{\phi\left(x_3\right)\cdots\right\}|\Omega\rangle\end{aligned}$$

The state $|\lambda_{\mathbf{K}}\rangle$ is annihilated by two field operators constrained to lie in distant wavepackets. It must therefore consist of two distinct excitations of the vacuum at two distinct locations. If these excitations are well separated,

they should be independent of one another, so we can approximate

$$

[6]*As in Section 4.5, the product symbol applies symbolically to the integrations as well as to the other factors within the parentheses; the x_i integrals apply to what lies outside the parentheses as well.

$$\sum_{\lambda} \int \frac{d^3K}{(2\pi)^3} \frac{1}{2E_{\mathbf{K}}} \langle \Omega | T \left\{ \phi(x_1) \phi(x_2) \right\} | \lambda_{\mathbf{K}} \rangle \langle \lambda_{\mathbf{K}} |$$
$$= \sum_{\lambda_1, \lambda_{\mathbf{2}}} \int \frac{d^3q_1}{(2\pi)^3} \frac{1}{2E_{\mathbf{q}_1}} \int \frac{d^3q_2}{(2\pi)^3} \frac{1}{2E_{\mathbf{q}_2}} \langle \Omega | \phi(x_1) | \lambda_{\mathbf{q}_1} \rangle \langle \Omega | \phi(x_2) | \lambda_{\mathbf{q}_2} \rangle \langle \lambda_{\mathbf{q}_1} \lambda_{\mathbf{q}_2} |$$

$$

The sums over λ_1 and λ_2 in the this equation run over all zero-momentum states, but only single-particle states will contribute the poles we are looking for. In this case, the integrals over x_1^0 and $\mathbf{q}_1$ produce a sharp singularity in p_1^0 of the form of (7.40), and the integrals over x_2^0 and $\mathbf{q}_2$ produce the same singular behavior in p_2^0. The term in (7.41) with both singularities is

$$\left(\prod_{i=1,2} \int \frac{d^3k_i}{(2\pi)^3} \varphi_i(\mathbf{k}_i) \frac{i}{\tilde{p}_i^2 - m^2 + i\epsilon} \cdot \sqrt{Z} \right) \langle \mathbf{k}_1 \mathbf{k}_2 | T \left\{ \phi(x_3) \cdots \right\} | \Omega \rangle$$

In the limit in which the wavepackets tend to delta functions concentrated at definite momenta $\mathbf{p}_1$ and $\mathbf{p}_2$, this expression tends to

$$\left(\prod_{i=1,2} \frac{i}{p_i^2 - m^2 + i\epsilon} \cdot \sqrt{Z} \right)_{\text{out}} \langle \mathbf{p}_1 \mathbf{p}_2 | T \left\{ \phi(x_3) \cdots \right\} | \Omega \rangle$$

The state $\langle \mathbf{p}_1 \mathbf{p}_2 |$ is precisely an out state as defined in Section 4.5, since it is the definite-momentum limit of a state of particles constrained to wellseparated wavepackets. Applying the same analysis to the times x_i^0 in the far past gives the result that the coefficient of the maximally singular term in the corresponding p_i^0 is a matrix element with an in state. This most singular term in (7.41) thus has the form

$$\left(\prod_{i=1,2} \frac{i}{p_i^2 - m^2 + i\epsilon} \cdot \sqrt{Z} \right) \left(\prod_{i=3,\ldots} \frac{i}{p_i^2 - m^2 + i\epsilon} \cdot \sqrt{Z} \right)_{\text{out}} \langle \mathbf{p}_1 \mathbf{p}_2 \mid -\mathbf{p}_3 \cdots \rangle_{\text{in}}$$

The last factor is just an S-matrix element.

We have now shown that we can extract the value of an S-matrix element by folding the corresponding vacuum expectation value of fields with wavepackets, extracting the leading singularities in the energies p_i^0, and then taking the limit as these wavepackets become delta functions of momenta. However, the computation would be made much simpler if we could do these operations in the reverse order-first letting the wavepackets become delta functions, returning us to the simple Fourier transform, and then extracting the singularities. In fact, the result for the leading singularity is not changed by this switch of the order of operations. It is, however, rather subtle to argue this point. Roughly, the explanation is the following: In the language of the analysis just completed, new singularities might arise because, in the Fourier transform, x_1 and x_2 can become close together in the far future. However, in this region, the exponential factor is close to $\exp\left[i\left(p_1 + p_2\right) \cdot x_1\right]$, and thus the new singularities are single poles in the variable $(p_1^0 + p_2^0)$, rather than

being products of poles in the two separate energy variables. A more careful argument (unfortunately, couched in a rather different language) can be found in the original paper of Lehmann, Symanzik, and Zimmermann cited at the beginning of this section.

Given the ability to make this reversal in the order of operations, we obtain a precise relation between Fourier transforms of correlation functions and S-matrix elements. This is the LSZ reduction formula:

$$\begin{array}{c}\prod_1^n \int d^4x_i e^{ip_i\cdot x_i} \prod_1^m \int d^4y_j e^{-ik_j\cdot y_j} \langle\Omega|T\left\{\phi\left(x_1\right)\cdots\phi\left(x_n\right)\phi\left(y_1\right)\cdots\phi\left(y_m\right)\right\}|\Omega\rangle \\ \underset{\substack{\text{each } p_i^0\to+E_{\mathbf{p}_i}\\ \text{each } k_j^0\to+E_{\mathbf{k}_j}}}{\sim} \left(\prod_{i=1}^n \frac{\sqrt{Z}i}{p_i^2-m^2+i\epsilon}\right)\left(\prod_{j=1}^m \frac{\sqrt{Z}i}{k_j^2-m^2+i\epsilon}\right)\langle\mathbf{p}_1\cdots\mathbf{p}_n|\,S\,|\mathbf{k}_1\cdots\mathbf{k}_m\rangle\end{array}$$

The quantity Z that appears in this equation is exactly the field-strength renormalization constant, defined in Section 7.1 as the residue of the singleparticle pole in the two-point function of fields. Each distinct particle will have a distinct renormalization factor Z, obtained from its own two-point function. For higher-spin fields, each factor of $\sqrt{Z}$ comes with a polarization factor such as $u^s(p)$, as in Eq. (7.12). The polarization s must be summed over in the second line of (7.42).

In words, the LSZ formula says that an S-matrix element can be computed as follows. Compute the appropriate Fourier-transformed correlation function, look at the region of momentum space where the external particles are near the mass shell, and identify the coefficient of the multiparticle pole. For fields with spin, one must also multiply by a polarization spinor (like $u^s(p)$) or vector (like $\epsilon^r(k)$) to project out the desired spin state.

Our next goal is to express this procedure in the language of Feynman diagrams. Let us analyze the relation between the diagrammatic expansion of the scalar field four-point function and the S-matrix element for 2-particle $\to$ 2-particle scattering. We will consider explicitly the fully connected Feynman diagrams contributing to the correlator. By a similar analysis, it is easy to confirm that disconnected diagrams should be disregarded because they do not have the singularity structure, with a product of four poles, indicated on the right-hand side of (7.42).

The exact four-point function

$$\left(\prod_1^2 \int d^4x_i e^{ip_i\cdot x_i}\right)\left(\prod_1^2 \int d^4y_i e^{-ik_i\cdot y_i}\right)\langle\Omega|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(y_1\right)\phi\left(y_2\right)\right\}|\Omega\rangle$$

has the general form shown in Fig. 7.4. In this figure, we have indicated explicitly the diagrammatic corrections on each leg; the shaded circle in the center represents the sum of all amputated four-point diagrams.

We can sum up the corrections to each external leg just as we did for the electron propagator in the previous section. Let $-iM^2\left(p^2\right)$ denote the sum of

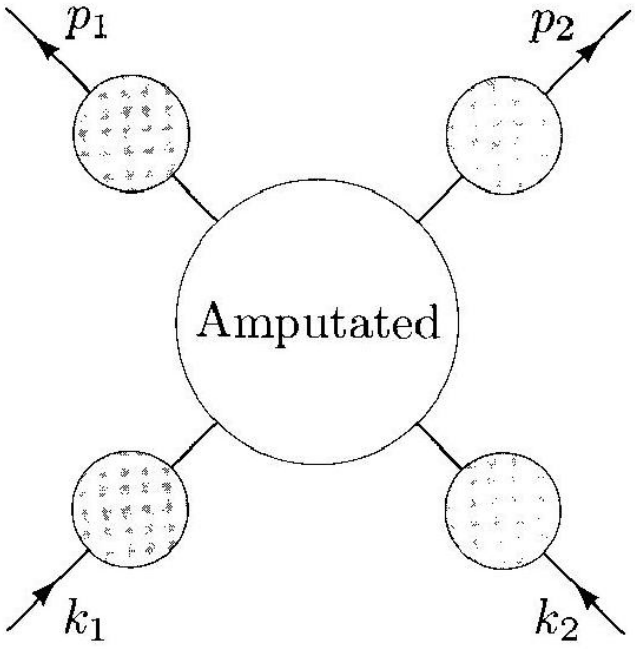


Figure 7.4. Structure of the exact four-point function in scalar field theory.
all one-particle-irreducible (1PI) insertions into the scalar propagator:

$$-iM^2\left(p^2\right) = -Q + \square + \cdots =$$

Then the exact propagator can be written as a geometric series and summed as in Eq. (7.23):

$$\begin{aligned} & 1 - 1\mathrm{PI} + 1\mathrm{PI} - 1\mathrm{PI} - \cdots \\ & = \frac{i}{p^2 - m_0^2} + \frac{i}{p^2 - m_0^2}\left(-iM^2\right)\frac{i}{p^2 - m_0^2} + \cdots \\ & = \frac{i}{p^2 - m_0^2 - M^2\left(p^2\right)} \end{aligned} \tag{7.43}$$

Notice that, as in the case of the electron propagator, our sign convention for the 1PI self-energy $M^2\left(p^2\right)$ implies that a positive contribution to $M^2\left(p^2\right)$ corresponds to a positive shift of the scalar particle mass. If we expand each resummed propagator about the physical particle pole, we see that each external leg of the four-point amplitude contributes

$$\frac{i}{p^2 - m_0^2 - M^2} \underset{p^0 \to E_{\mathbf{p}}}{\sim} \frac{iZ}{p^2 - m^2} + (\text{ regular }) \tag{7.44}$$

Thus, the sum of diagrams contains a product of four poles:

$$\frac{iZ}{p_1^2 - m^2}\frac{iZ}{p_2^2 - m^2}\frac{iZ}{k_1^2 - m^2}\frac{iZ}{k_2^2 - m^2}$$

This is exactly the singularity on the second line of (7.42). Comparing the coefficients of this product of poles, we find the relation

$$\left\langle \mathbf{p}_1\mathbf{p}_2\right| S\left|\mathbf{k}_1\mathbf{k}_2\right\rangle = (\sqrt{Z})^4 \underbrace{p_1}_{k_1}$$

where the shaded circle is the sum of amputated four-point diagrams and Z is the field-strength renormalization factor.

An identical analysis can be applied to the Fourier transform of the $(n+2)$-point correlator in a general field theory. The relation between S-matrix elements and Feynman diagrams then takes the form

$$\langle \mathbf{p}_1 \ldots \mathbf{p}_n | \, S \, | \mathbf{k}_1 \mathbf{k}_2 \rangle = (\sqrt{Z})^{n+2} \tag{7.45}$$

(If the external particles are of different species, each has its own renormalization factor $\sqrt{Z}$; if the particles have nonzero spin, there will be additional polarization factors such as $u^s(k)$ on the right-hand side.) This is almost precisely the diagrammatic formula for the S-matrix that we wrote down in Section 4.6. The only new feature is the appearance of the renormalization factors $\sqrt{Z}$. The Z factors are irrelevant for calculations at the leading order of perturbation theory, but are important in the calculation of higher-order corrections.

Up to this point, we have performed only one full calculation of a higherorder correction, the computation of the order- α corrections to the electron form factors. We did not take into account the effects of the electron fieldstrength renormalization. Let us now add in this factor and examine its effects.

Since the expressions (6.28) and (6.30) for electron scattering from a heavy target were derived using our previous, incorrect formula for S-matrix elements, we should correct these formulae by inserting factors of $\sqrt{Z_2}$ for the initial and final electrons. Equation (6.33) for the structure of the exact vertex should then read

$$Z_2 \Gamma^\mu (p', p) = \gamma^\mu F_1 \left(q^2\right) + \frac{i \sigma^{\mu\nu} q_\nu}{2m} F_2 \left(q^2\right) \tag{7.46}$$

with $\Gamma^\mu (p', p)$ the sum of amputated electron-photon vertex diagrams.
We can use this equation to reevaluate the form factors to order α. Since $Z_2 = 1 + \mathcal{O}(\alpha)$ and F_2 begins in order α, our previous computation of F_2 is unaffected. To compute F_1, write the left-hand side of (7.46) as

$$Z_2 \Gamma^\mu = (1 + \delta Z_2) (\gamma^\mu + \delta \Gamma^\mu) = \gamma^\mu + \delta \Gamma^\mu + \gamma^\mu \cdot \delta Z_2$$

where δZ_2 and $\delta \Gamma^\mu$ denote the order- α corrections to these quantities. Comparing to the right-hand side of (7.46), we see that $F_1 (q^2)$ receives a new
contribution equal to δZ_2. Now let $\delta F_1 (q^2)$ denote the (unsubtracted) correction to the form factor that we computed in Section 6.3, and recall from the end of Section 7.1 that $\delta Z_2 = -\delta F_1(0)$. Then

$$F_1 \left(q^2\right) = 1 + \delta F_1 \left(q^2\right) + \delta Z_2 = 1 + \left[\delta F_1 \left(q^2\right) - \delta F_1(0)\right]$$

This is exactly the result we claimed, but did not prove, in Section 6.3. The inclusion of field-strength renormalization justifies the subtraction procedure that we applied on an ad hoc basis there.

At this level of analysis, it is difficult to see how the cancellation of divergences in F_1 will persist to higher orders. Worse, though we argued in Section 6.3 for the general result $F_1(0) = 1$, our verification of this result in order α seems to depend on a numerical coincidence.

We can state this problem carefully as follows: Define a second rescaling factor Z_1 by the relation

$$\Gamma^\mu (q = 0) = Z_1^{-1} \gamma^\mu \tag{7.47}$$

where Γ^μ is the complete amputated vertex function. To find $F_1(0) = 1$, we must prove the identity $Z_1 = Z_2$, so that the vertex rescaling exactly compensates the electron field-strength renormalization. We will prove this identity to all orders in perturbation theory at the end Section 7.4.

We conclude our discussion of the LSZ reduction formula with one further formal observation. The LSZ formula distinguishes in and out particles only by the sign of the Fourier transform momentum p_i^0 or k_i^0. This means that, by analytically continuing the residue of the pole in p^2 from positive to negative p^0, we can convert the S-matrix element with $\phi(\mathbf{p})$ in the final state into the S-matrix element with the antiparticle $\phi^*(-\mathbf{p})$ in the initial state. This is exactly the statement of crossing symmetry, which we derived diagrammatically in Section 5.4:

$$\langle \cdots \phi(p)|S|\cdots\rangle\big|_{p=-k} = \langle \cdots |S\,|\phi^*(k)\cdots\rangle$$

Since the proof of the LSZ formula does not depend on perturbation theory, we see that the crossing symmetry of the S-matrix is a general result of quantum theory, not merely a property of Feynman diagrams.

17.3 7.3 The Optical Theorem

In Section 7.1 we saw that the two-point correlation function of quantum fields, viewed as an analytic function of the momentum p^2, has branch cut singularities associated with multiparticle intermediate states. This conclusion should not be surprising to those familiar with nonrelativistic scattering theory, since it is already true there that the scattering amplitude, as a function of energy, has a branch cut on the positive real axis. The imaginary part of the scattering amplitude appears as a discontinuity across this branch cut. By the optical theorem, the imaginary part of the forward scattering amplitude is

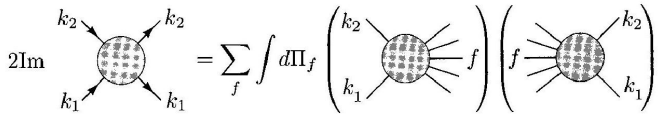


Figure 7.5. The optical theorem: The imaginary part of a forward scattering amplitude arises from a sum of contributions from all possible intermediatestate particles.

proportional to the total cross section. We will now prove the field-theoretic version of the optical theorem and illustrate how it arises in Feynman diagram calculations.

The optical theorem is a straightforward consequence of the unitarity of the S-matrix: $S^\dagger S = 1$. Inserting $S = 1 + iT$ as in (4.72), we have

$$-i\left(T - T^\dagger\right) = T^\dagger T \tag{7.48}$$

Let us take the matrix element of this equation between two-particle states $|\mathbf{p}_1\mathbf{p}_2\rangle$ and $|\mathbf{k}_1\mathbf{k}_2\rangle$. To evaluate the right-hand side, insert a complete set of intermediate states:

$$\langle \mathbf{p}_1\mathbf{p}_2|\,T^\dagger T\,|\mathbf{k}_1\mathbf{k}_2\rangle = \sum_n \left(\prod_{i=1}^n \int \frac{d^3q_i}{(2\pi)^3}\frac{1}{2E_i}\right) \langle \mathbf{p}_1\mathbf{p}_2|\,T^\dagger\,|\{\mathbf{q}_i\}\rangle\,\langle\{\mathbf{q}_i\}|\,T\,|\mathbf{k}_1\mathbf{k}_2\rangle$$

Now express the T-matrix elements as invariant matrix elements $\mathcal{M}$ times 4 -momentum-conserving delta functions. Identity (7.48) then becomes

$$
\begin{aligned}
&- i\left[\mathcal{M}\left(k_1 k_2 \to p_1 p_2\right) - \mathcal{M}^*\left(p_1 p_2 \to\ k_1 k_2\right)\right] \\
&= \sum_n \left(\prod_{i=1}^{n} \int \frac{d^3 q_i}{(2\pi)^3} \frac{1}{2E_i} \right) \mathcal{M}^*\left(p_1 p_2 \to \{q_i\}\right) \mathcal{M}\left(k_1 k_2 \to \{q_i\}\right) \\
&\times (2\pi)^4 \delta^{(4)} \left(k_1 + k_2 - \sum_i q_i \right)
\end{aligned}
$$

times an overall delta function $(2\pi)^4 \delta^{(4)}\left(k_1 + k_2 - p_1 - p_2\right)$. Let us abbreviate this identity as

$$
-i\left[\mathcal{M}(a \to b) - \mathcal{M}^*(b \to a)\right] = \sum_f \int d\Pi_f \mathcal{M}^*(b \to f) \mathcal{M}(a \to f) \tag{7.49}
$$

where the sum runs over all possible sets f of final-state particles. Although we have so far assumed that a and b are two-particle states, they could equally well be one-particle or multiparticle asymptotic states.

For the important special case of forward scattering, we can set $p_i = k_i$ to obtain a simpler identity, shown pictorially in Fig. 7.5. Supplying the kinematic factors required by (4.79) to build a cross section, we obtain the standard form of the optical theorem,

$$
\operatorname{Im} \mathcal{M}\left(k_1, k_2 \to k_1, k_2\right) = 2 E_{\mathrm{cm}} p_{\mathrm{cm}} \sigma_{\mathrm{tot}}\left(k_1, k_2 \to\ \text{anything }\right) \tag{7.50}
$$

where E_{cm} is the total center-of-mass energy and p_{cm} is the momentum of either particle in the center-of-mass frame. This equation relates the forward scattering amplitude to the total cross section for production of all final states. Since the imaginary part of the forward scattering amplitude gives the attenuation of the forward-going wave as the beam passes through the target, it is natural that this quantity should be proportional to the probability of scattering. Identity (7.50) gives the precise connection.

The Optical Theorem for Feynman Diagrams

Let us now investigate how this identity for the imaginary part of an S matrix element arises in the Feynman diagram expansion. It is easily checked (in QED, for example) that each diagram contributing to an S-matrix element $\mathcal{M}$ is purely real unless some denominators vanish, so that the $i\epsilon$ prescription for treating the poles becomes relevant. A Feynman diagram thus yields an imaginary part for $\mathcal{M}$ only when the virtual particles in the diagram go onshell. We will now show how to isolate and compute this imaginary part.

For our present purposes, let us define $\mathcal{M}$ by the Feynman rules for perturbation theory. This allows us to consider $\mathcal{M}(s)$ as an analytic function of the complex variable $s = E_{\mathrm{cm}}^2$, even though S-matrix elements are defined only for external particles with real momenta.

We first demonstrate that the appearance of an imaginary part of $\mathcal{M}(s)$ always requires a branch cut singularity. Let s_0 be the threshold energy for production of the lightest multiparticle state. For real s below s_0 the intermediate state cannot go on-shell, so $\mathcal{M}(s)$ is real. Thus, for real $s < s_0$, we have the identity

$$
\mathcal{M}(s) = \left[\mathcal{M}\left(s^*\right)\right]^* \tag{7.51}
$$

Each side of this equation is an analytic function of s, so it can be analytically continued to the entire complex s plane. In particular, near the real axis for $s > s_0$, Eq. (7.51) implies

$$\mathrm{Re}\,\mathcal{M}(s+i\epsilon) = \mathrm{Re}\,\mathcal{M}(s-i\epsilon)$$
$$\mathrm{Im}\,\mathcal{M}(s+i\epsilon) = -\,\mathrm{Im}\,\mathcal{M}(s-i\epsilon)$$

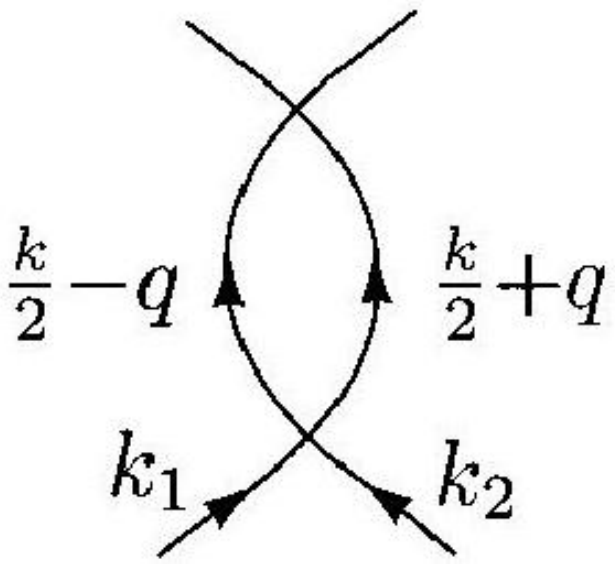
There is a branch cut across the real axis, starting at the threshold energy s_0; the discontinuity across the cut is

$$\mathrm{Disc}\,\mathcal{M}(s) = 2i\,\mathrm{Im}\,M(s+i\epsilon)$$

Usually it is easier to compute the discontinuity of a diagram than to compute the imaginary part directly. The $i\epsilon$ prescription in the Feynman propagator indicates that physical scattering amplitudes should be evaluated above the cut, at $s + i\epsilon$.

We already saw in Section 7.1 that the electron self-energy diagram has a branch cut beginning at the physical electron-photon threshold. Let us now study more general one-loop diagrams, and show that their discontinuities
give precisely the imaginary parts required by (7.49). The generalization of this result to multiloop diagrams has been proven by Cutkosky, † who showed in the process that the discontinuity of a Feynman diagram across its branch cut is always given by a simple set of cutting rules. ‡

We begin by checking (7.49) in ϕ^4 theory. Since the right-hand side of (7.49) begins in order λ^2, we expect that $\mathrm{Im}\,\mathcal{M}$ should also receive its first contribution from higher-order diagrams. Consider, then, the order- λ^2 diagram



$$k = k_1 + k_2$$

with a loop in the s-channel. (It is easy to check that the corresponding t - and u-channel diagrams have no branch cut singularities for s above threshold.) The total momentum is $k = k_1 + k_2$, and for simplicity we have chosen the symmetrical routing of momenta shown above. The value of this Feynman diagram is

$$i\delta\mathcal{M} = \frac{\lambda^2}{2}\int \frac{d^4q}{(2\pi)^4}\,\frac{1}{(k/2-q)^2 - m^2 + i\epsilon}\,\frac{1}{(k/2+q)^2 - m^2 + i\epsilon} \tag{7.52}$$

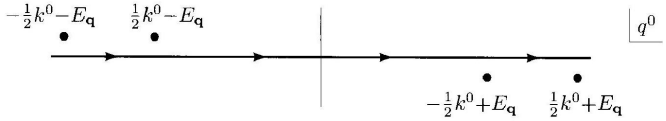
When this integral is evaluated using the methods of Section 6.3, the Wick rotation produces an extra factor of i, so that, below threshold, $\delta\mathcal{M}$ is purely real.

We would like to verify that the integral (7.52) has a discontinuity across the real axis in the physical region $k^0 > 2m$. It is easiest to identify this discontinuity by computing the integral for $k^0 < 2m$, then increasing k^0 by analytic continuation. It is not difficult to compute the integral directly using Feynman parameters (see Problem 7.1). However, it is illuminating to use a less direct approach, as follows.

Let us work in the center-of-mass frame, where $k = (k^0, \mathbf{0})$. Then the integrand of (7.52) has four poles in the integration variable q^0, at the locations

$$q^0 = \frac{1}{2}k^0 \pm (E_{\mathbf{q}} - i\epsilon)\,, \quad q^0 = -\frac{1}{2}k^0 \pm (E_{\mathbf{q}} - i\epsilon)$$

Two of these poles lie above the real q^0 axis and two lie below, as shown:



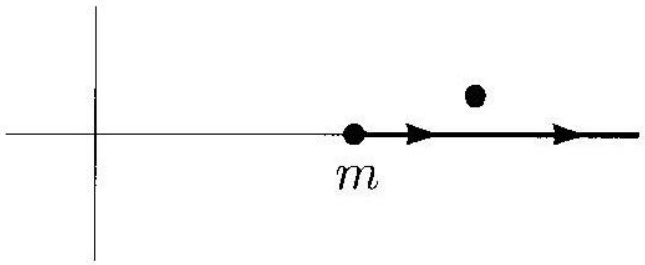
We will close the integration contour downward and pick up the residues of the poles in the lower half-plane. Of these, only the pole at $q^0 = -(1/2)k^0 + E_{\mathbf{q}}$ will contribute to the discontinuity. Note that picking up the residue of this pole is equivalent to replacing

$$\frac{1}{(k/2+q)^2 - m^2 + i\epsilon} \to -2\pi i\delta\left((k/2+q)^2 - m^2\right) \tag{7.53}$$

under the dq^0 integral.
The contribution of this pole yields the integral

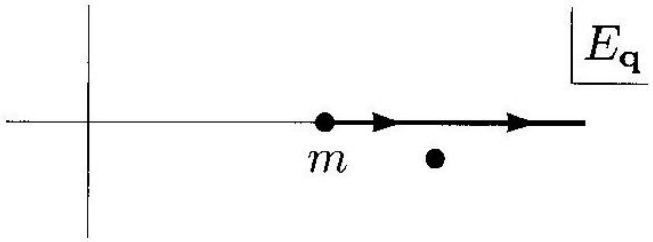
$$\begin{aligned} i\delta\mathcal{M} &= -2\pi i\frac{\lambda^2}{2}\int\frac{d^3q}{(2\pi)^4}\frac{1}{2E_{\mathbf{q}}}\frac{1}{\left(k^0 - E_{\mathbf{q}}\right)^2 - E_{\mathbf{q}}^2} \\ &= -2\pi i\frac{\lambda^2}{2}\frac{4\pi}{(2\pi)^4}\int_m^\infty dE_{\mathbf{q}}E_{\mathbf{q}}|\mathbf{q}|\frac{1}{2E_{\mathbf{q}}}\frac{1}{k^0\left(k^0 - 2E_{\mathbf{q}}\right)} \end{aligned} \tag{7.54}$$

The integrand in the second line has a pole at $E_{\mathbf{q}} = k^0/2$. When $k^0 < 2m$, this pole does not lie on the integration contour, so $\delta\mathcal{M}$ is manifestly real. When $k^0 > 2m$, however, the pole lies just above or below the contour of integration, depending upon whether k^0 is given a small positive or negative imaginary part:



[6†] R. E. Cutkosky, J. Math. Phys. 1, 429 (1960).
‡ These rules are simple only for singularities in the physical region. Away from the physical region, the singularities of three- and higher-point amplitudes can become quite intricate. This subject is reviewed in R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, The Analytic S-Matrix (Cambridge University Press, 1966).

or



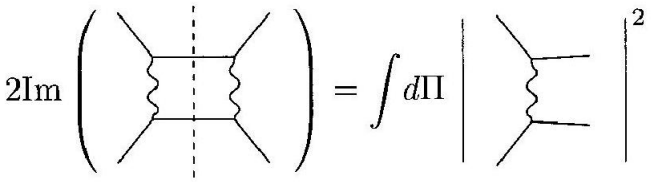
Thus the integral acquires a discontinuity between $k^2 + i\epsilon$ and $k^2 - i\epsilon$. To compute this discontinuity, apply

$$\frac{1}{k^0 - 2E_{\mathbf{q}} \pm i\epsilon} = P\frac{1}{k^0 - 2E_{\mathbf{q}}} \mp i\pi\delta\left(k^0 - 2E_{\mathbf{q}}\right)$$

(where P denotes the principal value). The discontinuity is given by replacing the pole with a delta function. This in turn is equivalent to replacing the original propagator by a delta function:

$$\frac{1}{(k/2-q)^2 - m^2 + i\epsilon} \to -2\pi i\delta\left((k/2-q)^2 - m^2\right). \tag{7.55}$$

(a)



(b)

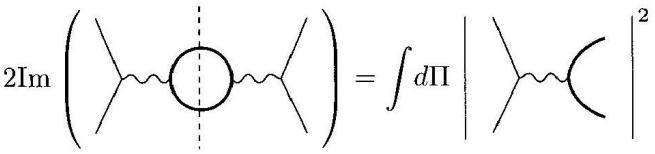


Figure 7.6. Two contributions to the optical theorem for Bhabha scattering.

Let us now retrace our steps and see what we have proved. Go back to the original integral (7.52), relabel the momenta on the two propagators as p_1, p_2 and substitute

$$\int \frac{d^4q}{(2\pi)^4} = \int \frac{d^4p_1}{(2\pi)^4}\int \frac{d^4p_2}{(2\pi)^4}(2\pi)^4\delta^{(4)}\left(p_1 + p_2 - k\right)$$

We have shown that the discontinuity of the integral is computed by replacing each of the two propagators by a delta function:

$$\frac{1}{p_i^2 - m^2 + i\epsilon} \to -2\pi i\delta\left(p_i^2 - m^2\right) \tag{7.56}$$

The discontinuity of $\mathcal{M}$ comes only from the region of the d^4q integral in which the two delta functions are simultaneously satisfied. By integrating over the delta functions, we put the momenta p_i on shell and convert the integrals d^4p_i into integrals over relativistic phase space. What is left over in expression (7.52) is just the factor λ^2, the square of the leading-order scattering amplitude, and the symmetry factor (1/2), which can be reinterpreted as the symmetry factor for identical

bosons in the final state. Thus we have shown that, to order λ^2 on each side of the equation, $\text{Disc}\,\mathcal{M}(k) = 2i\,\text{Im}\,\mathcal{M}(k)$

$$= \frac{i}{2}\int \frac{d^3p_1}{(2\pi)^3}\frac{1}{2E_1}\frac{d^3p_2}{(2\pi)^3}\frac{1}{2E_2}|\mathcal{M}(k)|^2(2\pi)^4\delta^{(4)}\left(p_1+p_2-k\right).$$

This explicitly verifies (7.49) to order λ^2 in ϕ^4 theory.
The preceding argument made no essential use of the fact that the two propagators in the diagram had equal masses, or of the fact that these propagators connected to a simple point vertex. Indeed, the analysis can be applied to an arbitrary one-loop diagram. Whenever, in the region of momentum integration of the diagram, two propagators can simultaneously go on-shell, we can follow the argument above to compute a nonzero discontinuity of $\mathcal{M}$. The value of this discontinuity is given by making the substitution (7.56) for
each of the two propagators. For example, in the order- α^2 Bhabha scattering diagrams shown in Fig. 7.6, we can compute the imaginary parts by cutting through the diagrams as shown and putting the cut propagators on shell using (7.56). The poles of the additional propagators in the diagrams do not contribute to the discontinuities. By integrating over the delta functions as in the previous paragraph, we derive the indicated relations between the imaginary parts of these diagrams and contributions to the total cross section.

Cutkosky proved that this method of computing discontinuities is completely general. The physical discontinuity of any Feynman diagram is given by the following algorithm:

1. Cut through the diagram in all possible ways such that the cut propagators can simultaneously be put on shell.

2. For each cut, replace $1/\left(p^2-m^2+i\epsilon\right) \to -2\pi i\delta\left(p^2-m^2\right)$ in each cut propagator, then perform the loop integrals.

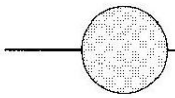
3. Sum the contributions of all possible cuts.

Using these cutting rules, it is possible to prove the optical theorem (7.49) to all orders in perturbation theory.

Unstable Particles

The cutting rules imply that the generalized optical theorem (7.49) is true not only for S-matrix elements, but for any amplitudes $\mathcal{M}$ that we can define in terms of Feynman diagrams. This fact is extremely useful for dealing with unstable particles, which never appear in asymptotic states.

Recall from Eq. (7.43) that the exact two-point function for a scalar particle has the form



$$= \frac{i}{p^2-m_0^2-M^2(p^2)}.$$

We defined the quantity $-iM^2\left(p^2\right)$ as the sum of all 1PI insertions into the boson propagator, but we can equally well think of it as the sum of all amputated diagrams for 1-particle $\to$ 1-particle "scattering". The LSZ formula then implies

$$\mathcal{M}(p\to p) = -ZM^2\left(p^2\right) \tag{7.57}$$

We can use this relation and the generalized optical theorem (7.49) to discuss the imaginary part of $M^2\left(p^2\right)$.

First consider the familiar case where the scalar boson is stable. In this case, there is no possible final state that can contribute to the right-hand side of (7.49). Thus $M^2(p^2)$ is real. The position of the pole in the propagator is determined by the equation $m^2 - m_0^2 - M^2(m^2) = 0$, which has a real-valued solution m. The pole therefore lies on the real p^2 axis, below the multiparticle branch cut.

Often, however, a particle can decay into two or more lighter particles. In this case $M^2(p^2)$ will acquire an imaginary part, so we must modify our
definitions slightly. Let us define the particle's mass m by the condition

$$m^2 - m_0^2 - \operatorname{Re} M^2\left(m^2\right) = 0 \tag{7.58}$$

Then the pole in the propagator is displaced from the real axis:

$$-\left(\sim \frac{iZ}{p^2 - m^2 - iZ \operatorname{Im} M^2(p^2)}.\right.$$

If this propagator appears in the s channel of a Feynman diagram, the cross section one computes, in the vicinity of the pole, will have the form

$$\sigma \propto \left| \frac{1}{s - m^2 - iZ \operatorname{Im} M^2(s)} \right|^2 \tag{7.59}$$

This expression closely resembles the relativistic Breit-Wigner formula (4.64) for the cross section in the region of a resonance:

$$\sigma \propto \left| \frac{1}{p^2 - m^2 + im\Gamma} \right|^2 \tag{7.60}$$

The mass m defined by (7.58) is the position of the resonance. If $\operatorname{Im} M^2(m^2)$ is small, so that the resonance in (7.59) is narrow, we can approximate $\operatorname{Im} M^2(s)$ as $\operatorname{Im} M^2(m^2)$ over the width of the resonance; then (7.59) will have precisely the Breit-Wigner form. In this case, we can identify

$$\Gamma = -\frac{Z}{m} \operatorname{Im} M^2\left(m^2\right) \tag{7.61}$$

If the resonance is broad, it will show deviations from the Breit-Wigner shape, generally becoming narrower on the leading edge and broader on the trailing edge.

To compute $\operatorname{Im} M^2$, and hence Γ, we could use the definition of M^2 as the sum of 1PI insertions into the propagator. The imaginary parts of the relevant loop diagrams give the decay rate. But the optical theorem (7.49), generalized to Feynman diagrams by the Cutkosky rules, simplifies this procedure. If we take (7.57) as the definition of the matrix element $\mathcal{M}(p \to p)$, and similarly define the decay matrix elements $\mathcal{M}(p \to f)$ through their Feynman diagram expansions, then (7.49) implies

$$Z \operatorname{Im} M^2\left(p^2\right) = -\operatorname{Im} \mathcal{M}(p \to p) = -\frac{1}{2} \sum_f \int d\Pi_f |\mathcal{M}(p \to f)|^2 \tag{7.62}$$

where the sum runs over all possible final states f. The decay rate is therefore

$$\Gamma = \frac{1}{2m} \sum_f \int d\Pi_f |\mathcal{M}(p \to f)|^2 \tag{7.63}$$

as quoted in Eq. (4.86).
We stress once again that our derivation of this equation applies only to the case of a long-lived unstable particle, so that $\Gamma \ll m$. For a broad resonance, the full energy dependence of $M^2(p^2)$ must be taken into account.

17.4 7.4 The Ward-Takahashi Identity

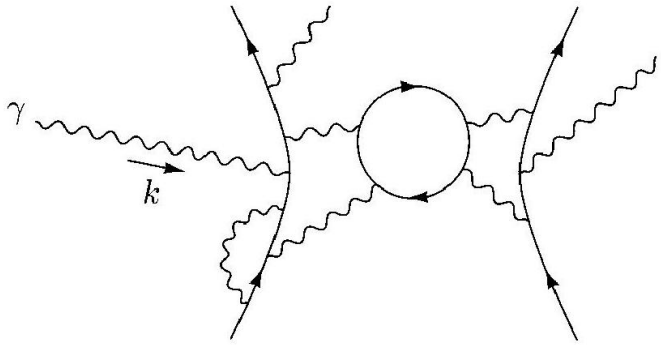
Of the loose ends listed at the beginning of this chapter, only one remains, the proof of the Ward identity. Recall from Section 5.5 that this identity states the following: If $\mathcal{M}(k) = \epsilon_\mu(k)\mathcal{M}^\mu(k)$ is the amplitude for some QED process involving an external photon with momentum k, then this amplitude vanishes if we replace ϵ_μ with k_μ:

$$k_\mu \mathcal{M}^\mu(k) = 0 \tag{7.64}$$

To prove this assertion, it is useful to prove a more general identity for QED correlation functions, called the Ward-Takahashi identity. To discuss this more general case we will let $\mathcal{M}$ denote a Fourier-transformed correlation function, in which the external momenta are not necessarily on-shell. The right-hand side of (7.64) will contain nonzero terms in this case; but when we apply the LSZ formula to extract an S-matrix element, those terms will not contribute.

We will prove the Ward-Takahashi identity order by order in α, by looking directly at the Feynman diagrams that contribute to $\mathcal{M}(k)$. The identity is generally not true for individual Feynman diagrams; we must sum over the diagrams for $\mathcal{M}(k)$ at any given order.

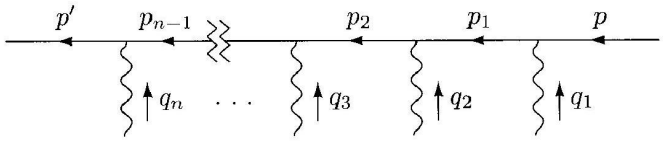
Consider a typical diagram for a typical amplitude $\mathcal{M}(k)$:



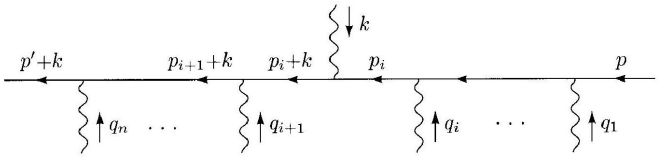
If we remove the photon $\gamma(k)$, we obtain a simpler diagram which is part of a simpler amplitude $\mathcal{M}_0$. If we reinsert the photon somewhere else inside the simpler diagram, we again obtain a contribution to $\mathcal{M}(k)$. The crucial observation is that by summing over all the diagrams that contribute to $\mathcal{M}_0$, then summing over all possible points of insertion in each of these diagrams, we obtain $\mathcal{M}(k)$. The Ward-Takahashi identity is true individually for each diagram contributing to $\mathcal{M}_0$, once we sum over insertion points; this is what we will prove.

When we insert our photon into one of the diagrams of $\mathcal{M}_0$, it must attach either to an electron line that runs out of the diagram to two external points, or to an internal electron loop. Let us consider each of these cases in turn.

First suppose that the electron line runs between external points. Before we insert our photon $\gamma(k)$, the line looks like this:



The electron propagators have momenta $p, p_1 = p + q_1, p_2 = p_1 + q_2$, and so on up to $p' = p_{n-1} + q_n$. If there are n vertices, we can insert our photon in $n + 1$ different places. Suppose we insert it after the i th vertex:



The electron propagators to the left of the new photon then have their momenta increased by k.

Let us now look at the values of these diagrams, with the polarization vector $\epsilon_\mu(k)$ replaced by k_μ. The product of k_μ with the new vertex is conveniently written:

$$-iek_\mu\gamma^\mu = -ie\left[(\not{p}_i + \not{k} - m) - (\not{p}_i - m)\right].$$

Multiplying by the adjacent electron propagators, we obtain

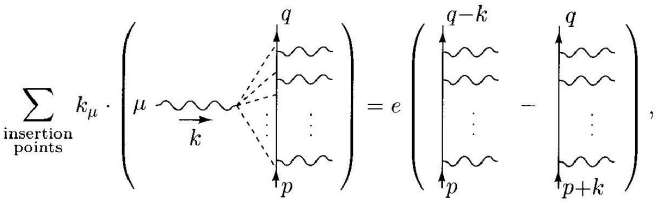
The diagram with the photon inserted at position i therefore has the structure

$$\times \left(\frac{i}{\not{p}_{i-1} - m}\right)\gamma^{\lambda_{i-1}} \cdots .$$

Similarly, the diagram with the photon inserted at position $i - 1$ has the structure

$$\times \left(\frac{i}{\not{p}_{i-1} - m} - \frac{i}{\not{p}_{i-1} + \not{k} - m}\right)\gamma^{\lambda_{i-1}} \cdots .$$

Note that the first term of this expression cancels the second term of the previous expression. A similar cancellation occurs between any other pair of
diagrams with adjacent insertions. When we sum over all possible insertion points along the line, everything cancels except the unpaired terms at the ends. The unpaired term coming from insertion after the last vertex (on the far left) is just e times the value of the original diagram; the other unpaired term, from insertion before the first vertex, is identical except for a minus sign and the replacement of p by $p + k$ everywhere. Diagrammatically, our result is



where we have renamed $p' + k \to q$ for the sake of symmetry.

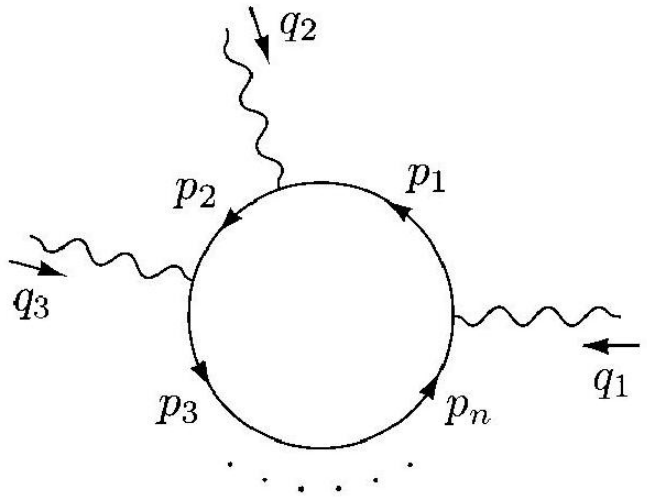
In each diagram on the left-hand side of (7.66), the momentum entering the electron line is p and the momentum exiting is q. According to the LSZ formula, we can extract from each diagram a contribution to an S-matrix element by taking the coefficient of the product of poles

$$\left(\frac{i}{\not{q} - m}\right)\left(\frac{i}{\not{p} - m}\right)$$

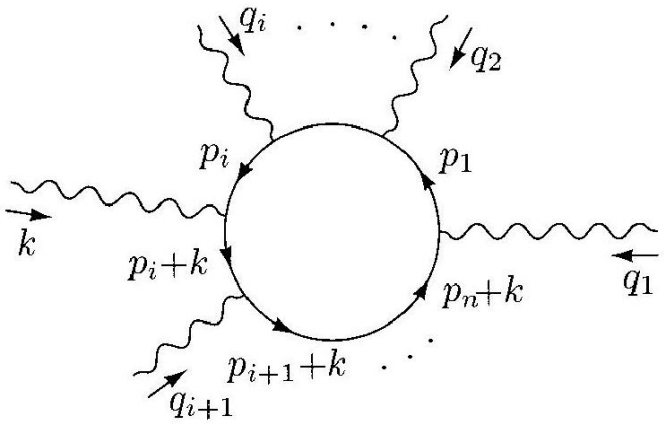
The terms on the right-hand side of (7.66) each contain one of these poles, but neither contains both poles. Thus the right-hand side of (7.66) contributes nothing to the S-matrix.*

To complete the proof of the Ward-Takahashi identity, we must consider the case in which our photon attaches to an internal electron loop. Before the insertion of the photon, a typical loop

looks like this:



The electron propagators have momenta $p_1, p_1 + q_2 = p_2$, and so on up to p_n. Suppose now that we insert the photon $\gamma(k)$ between vertices i and $i+1$:



We now have an additional momentum k running around the loop from the new vertex; by convention, this momentum exits at vertex 1 .

To evaluate the sum over all such insertions into the loop, apply identity (7.65) to each diagram. For the diagram in which the photon is inserted between vertices 1 and 2 , we obtain

$$-e\int\frac{d^4p_1}{(2\pi)^4}\,\mathrm{tr}\left[\left(\frac{i}{\not p_n+\not k-m}\right)\gamma^{\lambda_n}\cdots\left(\frac{i}{\not p_2+\not q_2-m}\right)\gamma^{\lambda_2}\right.$$
$$\left.\times\left(\frac{i}{\not p_1-m}-\frac{i}{\not p_1+\not k-m}\right)\gamma^{\lambda_1}\right].$$

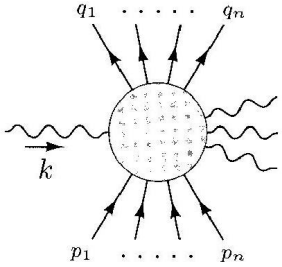
The first term will be canceled by one of the terms from the diagram with the photon inserted between vertices 2 and 3 . Similar cancellations take place between terms from other pairs of adjacent insertions. When we sum over all n insertion points we are left with

[6]*This step of the argument is straightforward only if we have arranged the perturbation series so that the propagator contains m rather than m_0. We will do this in Chapter 10.

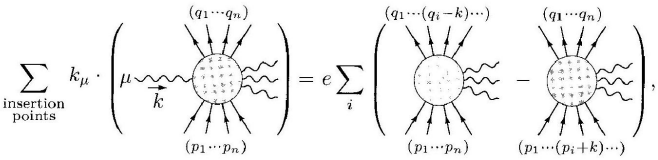
$$-e\int\frac{d^4p_1}{(2\pi)^4}\,\mathrm{tr}\left[\left(\frac{i}{\not p_n-m}\right)\gamma^{\lambda_n}\left(\frac{i}{\not p_{n-1}-m}\right)\gamma^{\lambda_{n-1}}\cdots\left(\frac{i}{\not p_1-m}\right)\gamma^{\lambda_1}\right.$$
$$\left.-\left(\frac{i}{\not p_n+\not k-m}\right)\gamma^{\lambda_n}\left(\frac{i}{\not p_{n-1}+\not b-m}\right)\gamma^{\lambda_{n-1}}\cdots\left(\frac{i}{\not p_1+\not b-m}\right)\gamma^{\lambda_1}\right]. \tag{7.67}$$

Shifting the integration variable from p_1 to p_1+k in the second term, we see that the two remaining terms cancel. Thus the diagrams in which the photon is inserted along a closed loop add up to zero.

We are now ready to assemble the pieces of the proof. Suppose that the amplitude $\mathcal{M}$ has $2n$ external electron lines, n incoming and n outgoing. Label
the incoming momenta p_i and the outgoing momenta q_i:

$\mathcal{M}(k;p_1\cdots p_n;q_1\cdots q_n)=$ 

(The amplitude can also involve an arbitrary number of additional external photons.) The amplitude $\mathcal{M}_0$ lacks the photon $\gamma(k)$ but is otherwise identical. To form $k_\mu\mathcal{M}^\mu$ from $\mathcal{M}_0$ we must sum over all diagrams that contribute to $\mathcal{M}_0$, and for each diagram, sum over all points at which the photon could be inserted. Summing over insertion points along an internal loop in any diagram gives zero. Summing over insertion points along a through-going line in any diagram gives a contribution of the form (7.66). Summing over all insertion points for any particular diagram, we obtain

$\sum_{\substack{\text{insertion}\\\text{points}}} k_\mu\cdot$ 

where the shaded circle represents any particular diagram that contributes to $\mathcal{M}_0$. Summing over all such diagrams, we finally obtain

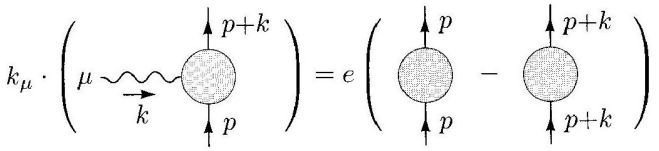
$$k_\mu\mathcal{M}^\mu(k;p_1\cdots p_n;q_1\cdots q_n)=e\sum_i[\mathcal{M}_0(p_1\cdots p_n;q_1\cdots(q_i-k)\cdots) \tag{7.68}$$
$$-\mathcal{M}_0(p_1\cdots(p_i+k)\cdots;q_1\cdots q_n)].$$

This is the Ward-Takahashi identity for correlation functions in QED. We saw below (7.66) that the right-hand side does not contribute to the S-matrix; thus in the special case where $\mathcal{M}$ is an S-matrix element, Eq. (7.68) reduces to the Ward identity (7.64).

Before discussing this identity further, we should mention a potential flaw in the above proof. In order to find the necessary cancellation in Eq. (7.67), we had to shift the integration variable by a constant. If the integral diverges, however, this shift is not permissible. Similarly, there may be divergent loopmomentum integrals in the expressions leading to Eq. (7.66). Here there is no explicit shift in the proof, but in practice one does generally perform a shift while evaluating the

integrals. In either case, ultraviolet divergences can potentially invalidate the Ward-Takahashi identity. We will see an example of this problem, as well as a general solution to it, in the next section.

The simplest example of the Ward-Takahashi identity involves, on the lefthand side, the three-point function with one entering and one exiting electron and one external photon:



The quantities on the right-hand side are exact electron propagators, evaluated at p and $(p+k)$ respectively. Label these quantities $S(p)$ and $S(p+k)$; from Eq. (7.23),

$$S(p) = \frac{i}{\not{p} - m - \Sigma(p)}.$$

The full three-point amplitude on the left-hand side can be rewritten, just as in Eq. (7.44), as a product of full propagators for the entering and exiting electrons, times an amputated scattering diagram. In this case, the amputated function is just the vertex $\Gamma^\mu(p+k, p)$. Then the Ward-Takahashi identity reads:

$$S(p+k)\left[-ie k_\mu \Gamma^\mu(p+k, p)\right] S(p) = e(S(p) - S(p+k))$$

To simplify this equation, multiply, on the left and right respectively, by the Dirac matrices $S^{-1}(p+k)$ and $S^{-1}(p)$. This gives

$$-ik_\mu \Gamma^\mu(p+k, p) = S^{-1}(p+k) - S^{-1}(p) \tag{7.69}$$

Often the term Ward-Takahashi identity is used to mean only this special case.

We can use identity (7.69) to obtain the general relation between the renormalization factors Z_1 and Z_2. We defined Z_1 in (7.47) by the relation

$$\Gamma^\mu(p+k, p) \to Z_1^{-1}\gamma^\mu \quad \text{as} \quad k \to 0$$

We defined Z_2 as the residue of the pole in $S(p)$:

$$S(p) \sim \frac{iZ_2}{\not{p} - m}$$

Setting p near mass shell and expanding (7.69) about $k = 0$, we find for the first-order terms on the left and right

$$-iZ_1^{-1} \not{k} = -iZ_2^{-1} \not{k},$$

that is,

$$Z_1 = Z_2 \tag{7.70}$$

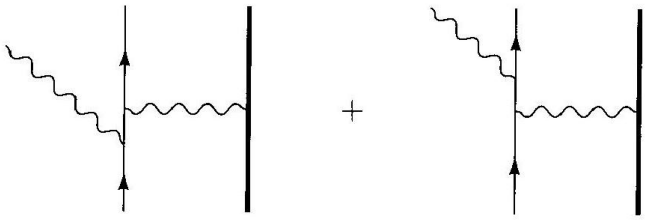
Thus, the Ward-Takahashi identity guarantees the exact cancellation of infinite rescaling factors in the electron scattering amplitude that we found at the end of Section 7.2. When combined with the correct formal expression

(7.46) for the electron form factors, this identity guarantees that $F_1(0) = 1$ to all orders in perturbation theory.

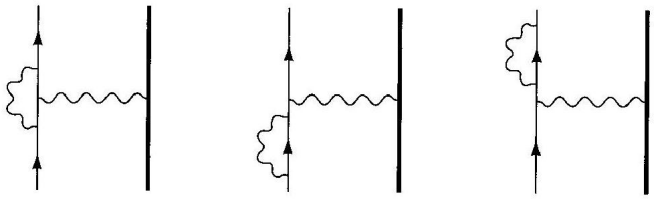
Often, in the literature, the terms Ward identity, current conservation, and gauge invariance are used interchangeably. This is quite natural, since the Ward identity is the diagrammatic expression of the conservation of the electric current, which is in turn a consequence of gauge invariance. In this book, however, we will distinguish these three concepts. By gauge invariance we mean the fundamental symmetry of the Lagrangian; by current conservation we mean the equation of motion that follows from this symmetry; and by the Ward identity we mean the diagrammatic identity that imposes the symmetry on quantum mechanical amplitudes.

17.5 7.5 Renormalization of the Electric Charge

At the beginning of Chapter 6 we set out to study the order- α radiative corrections to electron scattering from a heavy target. We evaluated (at least in the classical limit) the bremsstrahlung diagrams,

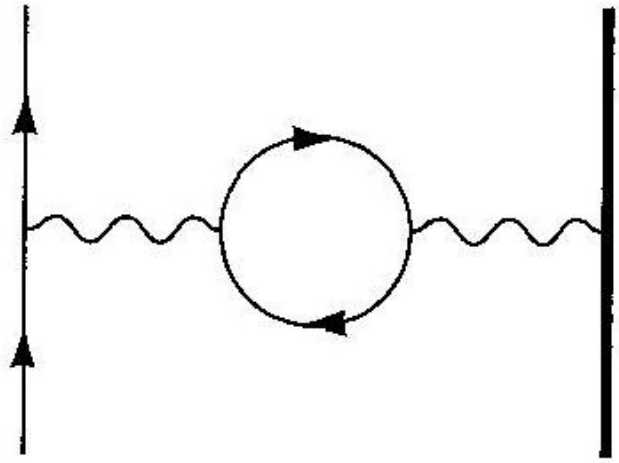


and also the corrections due to virtual photons:



Our discussion of field-strength renormalization in this chapter has finally clarified the role of the last two diagrams. In particular, we have seen that the Ward identity, through the relation $Z_1 = Z_2$, insures that the sum of the virtual photon corrections vanishes as the momentum transfer q goes to zero.

There is one remaining type of radiative correction to this process:

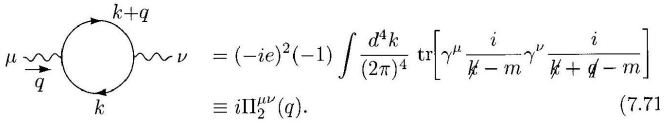


This is the order- α vacuum polarization diagram, also known as the photon self-energy. It can

be viewed as a modification to the photon structure by a
virtual electron-positron pair. This diagram will alter the effective field $A^\mu(x)$ seen by the scattered electron. It can potentially shift the overall strength of this field, and can also change its dependence on x (or in Fourier space, on q). In this section we will compute this diagram, and see that it has both of these effects.

Overview of Charge Renormalization

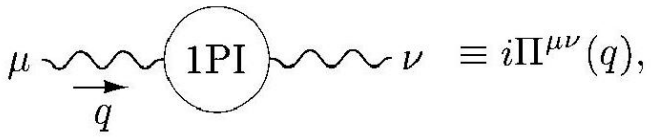
Before beginning a detailed calculation, let's ask what kind of an answer we expect and what its interpretation will be. The interesting part of the diagram is the electron loop:



$$= (-ie)^2(-1)\int \frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left[\gamma^\mu \frac{i}{\not k - m}\gamma^\nu \frac{i}{\not k + \not q - m}\right]$$

$$\equiv i\Pi_2^{\mu\nu}(q). \qquad (7.71$$

(The fermion loop factor of (-1) was derived in Eq. (4.120).) More generally, let us define $i\Pi^{\mu\nu}(q)$ to be the sum of all 1-particle-irreducible insertions into the photon propagator,



$$\equiv i\Pi^{\mu\nu}(q),$$

so that $\Pi_2^{\mu\nu}(q)$ is the second-order (in e) contribution to $\Pi^{\mu\nu}(q)$.
The only tensors that can appear in $\Pi^{\mu\nu}(q)$ are $g^{\mu\nu}$ and $q^\mu q^\nu$. The Ward identity, however, tells us that $q_\mu \Pi^{\mu\nu}(q) = 0$. This implies that $\Pi^{\mu\nu}(q)$ is proportional to the projector $(g^{\mu\nu} - q^\mu q^\nu/q^2)$. Furthermore, we expect that $\Pi^{\mu\nu}(q)$ will not have a pole at $q^2 = 0$; the only obvious source of such a pole would be a single-massless-particle intermediate state, which cannot occur in any 1 PI diagram. [†] It is therefore convenient to extract the tensor structure from $\Pi^{\mu\nu}$ in the following way:

$$\Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi\left(q^2\right) \qquad (7.73)$$

where $\Pi\left(q^2\right)$ is regular at $q^2 = 0$.
Using this notation, the exact photon two-point function is

$$\begin{aligned}
\sim_\mu = &\sim\sim\sim\sim \text{ 1PI } \sim + \sim \text{ 1PI } \sim \text{ 1PI } \sim + \cdots \\
&= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left[i\left(q^2 g^{\rho\sigma} - q^\rho q^\sigma\right)\Pi\left(q^2\right)\right]\frac{-ig_{\sigma\nu}}{q^2} + \cdots \\
&= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\Delta_\nu^\rho \Pi\left(q^2\right) + \frac{-ig_{\mu\rho}}{q^2}\Delta_\sigma^\rho \Delta_\nu^\sigma \Pi^2\left(q^2\right) + \cdots
\end{aligned}$$

where $\Delta_\nu^\rho \equiv \delta_\nu^\rho - q^\rho q_\nu/q^2$. Noting that $\Delta_\sigma^\rho \Delta_\nu^\sigma = \Delta_\nu^\rho$, we can simplify this expression to
$$

$$\begin{aligned}
\widetilde{\mu}_\nu &= \frac{-ig_{\mu\nu}}{q^2} + \frac{-ig_{\mu\rho}}{q^2}\left(\delta_\nu^\rho - \frac{q^\rho q_\nu}{q^2}\right)\left(\Pi\left(q^2\right) + \Pi^2\left(q^2\right) + \cdots\right) \\
&= \frac{-i}{q^2\left(1 - \Pi\Pi\left(q^2\right)\right)}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\left(\frac{q_\mu q_\nu}{q^2}\right) \qquad (7.74)
\end{aligned}$$

[6†] One can prove that there is no such pole, but the proof is nontrivial. Schwinger has shown that, in two spacetime dimensions, the singularity in Π_2 due to a pair of massless fermions is a pole rather than a cut; this is a famous counterexample to our argument. There is no such problem in four dimensions.

$$

In any S-matrix element calculation, at least one end of this exact propagator will connect to a fermion line. When we sum over all places along the line where it could connect, we must find, according to the Ward identity, that terms proportional to q_μ or q_ν vanish. For the purposes of computing S-matrix elements, therefore, we can abbreviate

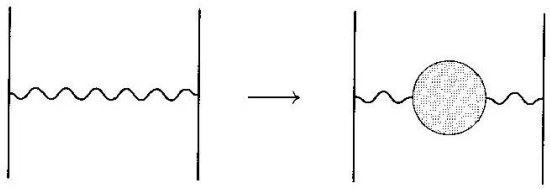
$$\widetilde{\mu}\sim\sim = \frac{-ig_{\mu\nu}}{q^2\left(1-\Pi\left(q^2\right)\right)} \tag{7.75}$$

Notice that as long as $\Pi\left(q^2\right)$ is regular at $q^2=0$, the exact propagator always has a pole at $q^2=0$. In other words, the photon remains absolutely massless at all orders in perturbation theory. This claim depends strongly on our use of the Ward identity in (7.73). If, for example, $\Pi^{\mu\nu}(q)$ contained a term $M^2g^{\mu\nu}$ (with no compensating $q^\mu q^\nu$ term), the photon mass would be shifted to M.

The residue of the $q^2=0$ pole is

$$\frac{1}{1-\Pi(0)} \equiv Z_3$$

The amplitude for any low- q^2 scattering process will be shifted by this factor, relative to the tree-level approximation:



or $\quad \cdots \dfrac{e^2 g_{\mu\nu}}{q^2} \cdots \longrightarrow \cdots \dfrac{Z_3\, e^2 g_{\mu\nu}}{q^2} \cdots .$

Since a factor of e lies at each end of the photon propagator, we can conveniently account for this shift by making the replacement $e \to \sqrt{Z_3}e$. This replacement is called charge renormalization; it is in many ways analogous to the mass renormalization introduced in Section 7.1. Note in particular that the "physical" electron charge measured in experiments is $\sqrt{Z_3}e$. We will therefore shift our notation and call this quantity simply e. From now on we
will refer to the "bare" charge (the quantity that multiplies $A_\mu\bar{\psi}\gamma^\mu\psi$ in the Lagrangian) as e_0. We then have

$$\text{(physical charge)} \;= e = \sqrt{Z_3}e_0 = \sqrt{Z_3}\cdot(\text{ bare charge }). \tag{7.76}$$

To lowest order, $Z_3=1$ and $e=e_0$.
In addition to this constant shift in the strength of the electric charge, $\Pi\left(q^2\right)$ has another effect. Consider a scattering process with nonzero q^2, and suppose that we have computed $\Pi\left(q^2\right)$ to leading order in α : $\Pi\left(q^2\right) \approx \Pi_2\left(q^2\right)$. The amplitude for the process will then involve the quantity

$$\frac{-ig_{\mu\nu}}{q^2}\left(\frac{e_0^2}{1-\Pi\left(q^2\right)}\right) \underset{\mathcal{O}(\alpha)}{=} \frac{-ig_{\mu\nu}}{q^2}\left(\frac{e^2}{1-\left[\Pi_2\left(q^2\right)-\Pi_2(0)\right]}\right)$$

(Swapping e^2 for e_0^2 does not matter to lowest order.) The quantity in parentheses can be interpreted as a q^2-dependent electric charge. The full effect of replacing the tree-level photon propagator with the exact photon propagator is therefore to replace

$$\alpha_0 \to \alpha_{\text{eff}}\left(q^2\right) = \frac{e_0^2/4\pi}{1-\Pi\left(q^2\right)} \underset{\mathcal{O}(\alpha)}{=} \frac{\alpha}{1-\left[\Pi_2\left(q^2\right)-\Pi_2(0)\right]} \tag{7.77}$$

(To leading order we could just as well bring the Π-terms into the numerator; but we will see in Chapter 12 that in this form, the expression is true to all orders when Π_2 is replaced by Π.)

Computation of Π_2

Having worked so hard to interpret $\Pi_2\left(q^2\right)$, we had better calculate it. Going back to (7.71), we have

$$\begin{aligned} i\Pi_2^{\mu\nu}(q) &= -(-ie)^2 \int \frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left[\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2}\gamma^\nu \frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2}\right] \\ &= -4e^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}\left(k\cdot(k+q)-m^2\right)}{\left(k^2-m^2\right)\left((k+q)^2-m^2\right)} \end{aligned} \tag{7.78}$$

We have written e and m instead of e_0 and m_0 for convenience, since the difference would give only an order- α^2 contribution to $\Pi^{\mu\nu}$.

Now introduce a Feynman parameter to combine the denominator factors:

$$\begin{aligned} \frac{1}{\left(k^2-m^2\right)\left((k+q)^2-m^2\right)} &= \int_0^1 dx \frac{1}{\left(k^2+2xk\cdot q+xq^2-m^2\right)^2} \\ &= \int_0^1 dx \frac{1}{\left(\ell^2+x(1-x)q^2-m^2\right)^2} \end{aligned}$$

where $\ell = k + xq$. In terms of ℓ, the numerator of (7.78) is

$$\begin{aligned} \text{Numerator } &= 2\ell^\mu\ell^\nu - g^{\mu\nu}\ell^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}\left(m^2+x(1-x)q^2\right) \\ &\quad + (\text{ terms linear in } \ell) \end{aligned}$$

Performing a Wick rotation and substituting $\ell^0 = i\ell_E^0$, we obtain

$$\begin{aligned} i\Pi_2^{\mu\nu}(q) = -4ie^2 \int_0^1 dx \int \frac{d^4\ell_E}{(2\pi)^4} \\ \times \frac{-\frac{1}{2}g^{\mu\nu}\ell_E^2 + g^{\mu\nu}\ell_E^2 - 2x(1-x)q^\mu q^\nu + g^{\mu\nu}\left(m^2+x(1-x)q^2\right)}{\left(\ell_E^2+\Delta\right)^2} \end{aligned} \tag{7.79}$$

where $\Delta = m^2 - x(1-x)q^2$. This integral is very badly ultraviolet divergent. If we were to cut it off at $\ell_E = \Lambda$, we would find for the leading term,

$$i\Pi_2^{\mu\nu}(q) \propto e^2\Lambda^2 g^{\mu\nu}$$

with no compensating $q^\mu q^\nu$ term. This result violates the Ward identity; it would give the photon an infinite mass $M \propto e\Lambda$.

Our proof of the Ward identity has failed, in precisely the way anticipated at the end of the previous section: The shift of the integration variable in (7.67) is not permissible when the integral is divergent. In our present calculation, the failure of the Ward identity is catastrophic: It leads

to an infinite photon mass, [‡] in conflict with experiment. Fortunately, there is a way to rescue the Ward identity.

In the above analysis we regulated the divergent integral in the most straightforward and most naive way: by cutting it off at a large momentum Λ. But other regulators are possible, and some will in fact preserve the Ward identity. In our computations of the vertex and electron self-energy diagrams, we used a Pauli-Villars regulator. This regulator preserved the relation $Z_1 = Z_2$, a consequence of the Ward identity; a naive cutoff does not (see Problem 7.2). We could fix our present computation by introducing Pauli-Villars fermions. Unfortunately, several sets of fermions are required, making the method rather complicated.* We will use a simpler method, dimensional regularization, due to 't Hooft and Veltman. [†] Dimensional regularization preserves the symmetries of QED and also of a wide class of more general theories.

The question of which regulator to use has no a priori answer in quantum field theory. Often the choice has no effect on the predictions of the theory.

When two regulators give different answers for observable quantities, it is generally because some symmetry (such as the Ward identity) is being violated by one (or both) of them. In these cases we take the symmetry to be fundamental and demand that it be preserved by the regulator. But the validity of this choice cannot be proven; we are adopting the symmetry as a new axiom.

Dimensional Regularization

The idea of dimensional regularization is very simple to state: Compute the Feynman diagram as an analytic function of the dimensionality of spacetime, d. For sufficiently small d, any loop-momentum integral will converge and therefore the Ward identity can be proved. The final expression for any observable quantity should have a well-defined limit as $d \to 4$.

Let us do a practice calculation to see how this technique works. We consider spacetime to have one time dimension and $(d-1)$ space dimensions. Then we can Wick-rotate Feynman integrals as before, to give integrals over a d-dimensional Euclidean space. A typical example is

$$\int \frac{d^d \ell_E}{(2\pi)^d} \frac{1}{\left(\ell_E^2 + \Delta\right)^2} = \int \frac{d\Omega_d}{(2\pi)^d} \cdot \int_0^\infty d\ell_E \frac{\ell_E^{d-1}}{\left(\ell_E^2 + \Delta\right)^2} \tag{7.80}$$

The first factor in (7.80) contains the area of a unit sphere in d dimensions. To compute it, use the following trick:

$$\begin{aligned}
(\sqrt{\pi})^d &= \left(\int dx e^{-x^2}\right)^d = \int d^d x \exp\left(-\sum_{i=1}^d x_i^2\right) \\
&= \int d\Omega_d \int_0^\infty dx x^{d-1} e^{-x^2} = \left(\int d\Omega_d\right) \cdot \frac{1}{2} \int_0^\infty d\left(x^2\right) \left(x^2\right)^{\frac{d}{2}-1} e^{-\left(x^2\right)} \\
&= \left(\int d\Omega_d\right) \cdot \frac{1}{2} \Gamma(d/2)
\end{aligned}$$

So the area of a d-dimensional unit sphere is

[6‡] We could still make the observed photon mass zero by adding a compensating infinite photon mass term to the Lagrangian. More generally, we could add terms to the Lagrangian to make the Ward identity valid for any n-point correlation function. This procedure would give the same results as the one we are about to follow, but would be much more complicated.

*This method is presented in Bjorken and Drell (1964), p. 154.

[†] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B44, 189 (1972).

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{7.81}$$

This formula reproduces the familiar special cases:

d	$\Gamma(d/2)$	$\int d\Omega_d$
1	$\sqrt{\pi}$	2
2	1	2π
3	$\sqrt{\pi}/2$	4π
4	1	$2\pi^2$

The second factor in (7.80) is

$$\begin{aligned}\int_0^\infty d\ell \frac{\ell^{d-1}}{\left(\ell^2+\Delta\right)^2} &= \frac{1}{2}\int_0^\infty d\left(\ell^2\right)\frac{\left(\ell^2\right)^{\frac{d}{2}-1}}{\left(\ell^2+\Delta\right)^2} \\ &= \frac{1}{2}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\int_0^1 dx x^{1-\frac{d}{2}}(1-x)^{\frac{d}{2}-1}\end{aligned}$$

where we have substituted $x = \Delta/\left(\ell^2+\Delta\right)$ in the second line. Using the definition of the beta function,

$$\int_0^1 dx x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{7.82}$$

we can easily evaluate the integral over x. The final result for the d-dimensional integral is

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{\left(\ell_E^2+\Delta\right)^2} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Gamma(2)}\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

Since $\Gamma(z)$ has isolated poles at $z = 0, -1, -2, ...$, this integral has isolated poles at $d = 4, 6, 8, ...$ To find the behavior near $d = 4$, define $\epsilon = 4 - d$, and use the approximation ‡

$$\Gamma\left(2-\frac{d}{2}\right) = \Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \tag{7.83}$$

where $\gamma \approx .5772$ is the Euler-Mascheroni constant. (This constant will always cancel in observable quantities.) The integral is then

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{\left(\ell_E^2+\Delta\right)^2} \underset{d\to 4}{\longrightarrow} \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \log\Delta - \gamma + \log(4\pi) + \mathcal{O}(\epsilon)\right) \tag{7.84}$$

When we defined this integral with a Pauli-Villars regulator in Eq. (7.18), we found

$$\int \frac{d^4\ell_E}{(2\pi)^4}\frac{1}{\left(\ell_E^2+\Delta\right)^2} \underset{\Lambda\to\infty}{\longrightarrow} \frac{1}{(4\pi)^2}\left(\log\frac{x\Lambda^2}{\Delta} + \mathcal{O}\left(\Lambda^{-1}\right)\right)$$

Thus the $1/\epsilon$ pole in dimensional regularization corresponds to a logarithmic divergence in the momentum integral. Note the curious fact that (7.84)

‡ This expansion follows immediately from the infinite product representation

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

involves the logarithm of Δ, a dimensionful quantity. The scale of the logarithm is hidden in the $1/\epsilon$ term, and appears explicitly when the divergence is canceled.

You can easily verify the more general integration formulae,

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma\left(n-\frac{d}{2}\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}} \tag{7.85}$$

$$\int \frac{d^d\ell_E}{(2\pi)^d}\frac{\ell_E^2}{(\ell_E^2+\Delta)^n} = \frac{1}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma\left(n-\frac{d}{2}-1\right)}{\Gamma(n)}\left(\frac{1}{\Delta}\right)^{n-\frac{d}{2}-1} \tag{7.86}$$

In d dimensions, $g^{\mu\nu}$ obeys $g_{\mu\nu}g^{\mu\nu} = d$. Thus, if the numerator of a symmetric integrand contains $\ell^\mu\ell^\nu$, we should replace

$$\ell^\mu\ell^\nu \to \frac{1}{d}\ell^2 g^{\mu\nu} \tag{7.87}$$

in analogy with Eq. (6.46). In QED, the Dirac matrices can be manipulated as a set of d matrices satisfying

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}, \quad \mathrm{tr}[\mathbf{1}] = 4 \tag{7.88}$$

In manipulating Eq. (7.78), these rules give the same result as the purely four-dimensional rules. However, in the evaluation of other diagrams, there are additional contributions of order ϵ. In particular, the contraction identities (5.9) are modified in $d = 4-\epsilon$ to

$$\begin{aligned}\gamma^\mu\gamma^\nu\gamma_\mu &= -(2-\epsilon)\gamma^\nu\\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4g^{\nu\rho} - \epsilon\gamma^\nu\gamma^\rho\\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu + \epsilon\gamma^\nu\gamma^\rho\gamma^\sigma\end{aligned} \tag{7.89}$$

These extra terms can contribute to the final value of the Feynman diagram if they multiply a factor ϵ^{-1} from a divergent integral. In QED at one-loop order, such extra terms appear in the vertex and self-energy diagrams but cancel when these diagrams are combined to compute an observable quantity.

Computation of Π_2, Continued

Now let us apply these dimensional regularization formulae to the momentum integral in (7.79). The unpleasant terms with ℓ^2 in the numerator give

$$\begin{aligned}\int \frac{d^d\ell_E}{(2\pi)^d}\frac{\left(-\frac{2}{d}+1\right)g^{\mu\nu}\ell_E^2}{\left(\ell_E^2+\Delta\right)^2} &= \frac{-1}{(4\pi)^{d/2}}\left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}}g^{\mu\nu}\\ &= \frac{1}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\cdot(-\Delta g^{\mu\nu})\end{aligned}$$

We would have expected a pole at $d = 2$, since the quadratic divergence in 4 dimensions becomes a logarithmic divergence in 2 dimensions. But the pole cancels. The Ward identity is working.

Evaluating the remaining terms in (7.79) and using $\Delta = m^2 - x(1-x)q^2$, we obtain

$$\begin{aligned} i\Pi_2^{\mu\nu}(q) =& -4ie^2 \int_0^1 dx \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}} \\ & \times \left[g^{\mu\nu}\left(-m^2 + x(1-x)q^2\right) + g^{\mu\nu}\left(m^2 + x(1-x)q^2\right) - 2x(1-x)q^\mu q^\nu\right] \\ =& \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \cdot i\Pi_2\left(q^2\right) \end{aligned}$$

where

$$\begin{aligned} \Pi_2\left(q^2\right) =& \frac{-8e^2}{(4\pi)^{d/2}} \int_0^1 dx x(1-x) \frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-d/2}} \\ \xrightarrow[d\to 4]{} & -\frac{2\alpha}{\pi} \int_0^1 dx x(1-x) \left(\frac{2}{\epsilon} - \log\Delta - \gamma + \log(4\pi)\right) \quad (\epsilon = 4-d) \end{aligned} \tag{7.90}$$

With dimensional regularization, $\Pi_2^{\mu\nu}(q)$ indeed takes the form required by the Ward identity. But it is still logarithmically divergent.

We can now compute the order- α shift in the electric charge:

$$\frac{e^2 - e_0^2}{e_0^2} = \delta Z_3 \overset{\mathcal{O}(\alpha)}{=} \Pi_2(0) \approx -\frac{2\alpha}{3\pi\epsilon}$$

The bare charge is infinitely larger than the observed charge. But this difference is not observable. What can be observed is the q^2 dependence of the effective electric charge (7.77). This quantity depends on the difference

$$\widehat{\Pi}_2\left(q^2\right) \equiv \Pi_2\left(q^2\right) - \Pi_2(0) = -\frac{2\alpha}{\pi} \int_0^1 dx x(1-x) \log\left(\frac{m^2}{m^2 - x(1-x)q^2}\right) \tag{7.91}$$

which is independent of ϵ in the limit $\epsilon \to 0$. For the rest of this section we will investigate what physics this expression contains.

Interpretation of Π_2

First consider the analytic structure of $\widehat{\Pi}_2\left(q^2\right)$. For $q^2 < 0$, as is the case when the photon propagator is in the t - or u-channel, $\widehat{\Pi}_2\left(q^2\right)$ is manifestly real and analytic. But for an s-channel process, q^2 will be positive. The logarithm function has a branch cut when its argument becomes negative, that is, when

$$m^2 - x(1-x)q^2 < 0$$

The product $x(1-x)$ is at most $1/4$, so $\widehat{\Pi}_2\left(q^2\right)$ has a branch cut beginning at

$$q^2 = 4m^2$$

at the threshold for creation of a real electron-positron pair.

Let us calculate the imaginary part of $\widehat{\Pi}_2$ for $q^2 > 4m^2$. For any fixed q^2, the x-values that contribute are between the points $x = \frac{1}{2} \pm \frac{1}{2}\beta$, where $\beta = \sqrt{1 - 4m^2/q^2}$. Since $\mathrm{Im}[\log(-X \pm i\epsilon)] = \pm\pi$, we have

$$\begin{aligned}
\mathrm{Im}\left[\widehat{\Pi}_2\left(q^2 \pm i\epsilon\right)\right] &= -\frac{2\alpha}{\pi}(\pm\pi)\int_{\frac{1}{2}-\frac{1}{2}\beta}^{\frac{1}{2}+\frac{1}{2}\beta} dx x(1-x) \\
&= \mp 2\alpha \int_{-\beta/2}^{\beta/2} dy \left(\frac{1}{4} - y^2\right) \quad \left(y \equiv x - \frac{1}{2}\right) \\
&= \mp\frac{\alpha}{3}\sqrt{1 - \frac{4m^2}{q^2}}\left(1 + \frac{2m^2}{q^2}\right)
\end{aligned} \tag{7.92}$$

This dependence on q^2 is exactly the same as in Eq. (5.13), the cross section for production of a fermion-antifermion pair. That is just what we would expect from the unitarity relation shown in Fig. 7.6(b); the cut through the diagram for forward Bhabha scattering gives the total cross section for $e^+e^- \to f\bar{f}$. The parameter β is precisely the velocity of the fermions in the center-of-mass frame.

Next let us examine how $\widehat{\Pi}_2(q^2)$ modifies the electromagnetic interaction, as determined by Eq. (7.77). In the nonrelativistic limit it makes sense to compute the potential $V(r)$. For the interaction between unlike charges, we have, in analogy with Eq. (4.126),

$$V(\mathbf{x}) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \frac{-e^2}{|\mathbf{q}|^2\left[1 - \widehat{\Pi}_2\left(-|\mathbf{q}|^2\right)\right]} \tag{7.93}$$

Expanding $\widehat{\Pi}_2$ for $|q^2| \ll m^2$, we obtain

$$V(\mathbf{x}) = -\frac{\alpha}{r} - \frac{4\alpha^2}{15m^2}\delta^{(3)}(\mathbf{x}) \tag{7.94}$$

The correction term indicates that the electromagnetic force becomes much stronger at small distances. This effect can be measured in the hydrogen atom, where the energy levels are shifted by

$$\Delta E = \int d^3x |\psi(\mathbf{x})|^2 \cdot \left(-\frac{4\alpha^2}{15m^2}\delta^{(3)}(\mathbf{x})\right) = -\frac{4\alpha^2}{15m^2}|\psi(0)|^2$$

The wavefunction $\psi(\mathbf{x})$ is nonzero at the origin only for s-wave states. For the $2S$ state, the shift is

$$\Delta E = -\frac{4\alpha^2}{15m^2} \cdot \frac{\alpha^3 m^3}{8\pi} = -\frac{\alpha^5 m}{30\pi} = -1.123 \times 10^{-7}\mathrm{eV}$$

This is a (small) part of the Lamb shift splitting listed in Table 6.1.

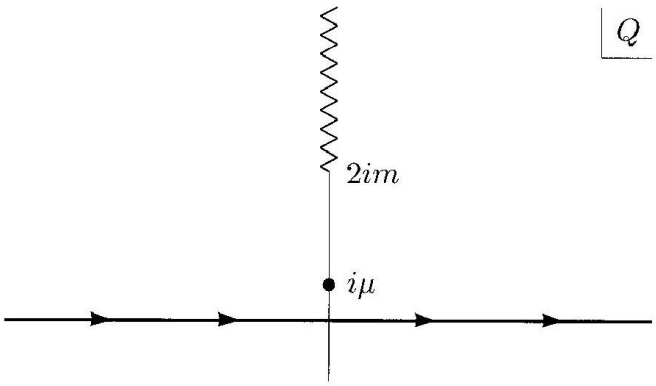


Figure 7.7. Contour for evaluating the effective strength of the electromagnetic interaction in the nonrelativistic limit. The pole at $Q = i\mu$ gives the Coulomb potential. The branch cut gives the order- α correction due to vacuum polarization.

The delta function in Eq. (7.94) is only an approximation; to find the true range of the correction term, we write Eq. (7.93) in the form

$$V(\mathbf{x}) = \frac{ie^2}{(2\pi)^2 r} \int_{-\infty}^{\infty} dQ \frac{Qe^{iQr}}{Q^2 + \mu^2} \left[1 + \widehat{\Pi}_2\left(-Q^2\right)\right] \quad (Q \equiv |\mathbf{q}|)$$

where we have inserted a photon mass μ to regulate the Coulomb potential. To perform this integral we push the contour upward (see Fig. 7.7). The leading contribution comes from the pole at $Q = i\mu$, giving the Coulomb potential, $-\alpha/r$. But there is an additional contribution from the branch cut, which begins at $Q = 2mi$. The real part of the integrand is the same on both sides of the cut, so the only contribution to the integral comes from the imaginary part of $\widehat{\Pi}_2$. Defining $q = -iQ$, we find that the contribution from the cut is

$$\begin{aligned}
\delta V(r) &= \frac{-e^2}{(2\pi)^2 r} \cdot 2 \int_{2m}^{\infty} dq \frac{e^{-qr}}{q} \operatorname{Im}\left[\widehat{\Pi}_2\left(q^2 - i\epsilon\right)\right] \\
&= -\frac{\alpha}{r}\frac{2}{\pi} \int_{2m}^{\infty} dq \frac{e^{-qr}}{q} \frac{\alpha}{3} \sqrt{1 - \frac{4m^2}{q^2}} \left(1 + \frac{2m^2}{q^2}\right)
\end{aligned}$$

When $r \gg 1/m$, this integral is dominated by the region where $q \approx 2m$. Approximating the integrand in this region and substituting $t = q - 2m$, we find

$$\begin{aligned}
\delta V(r) &= -\frac{\alpha}{r} \cdot \frac{2}{\pi} \int_0^{\infty} dt \frac{e^{-(t+2m)r}}{2m} \frac{\alpha}{3} \sqrt{\frac{t}{m}} \left(\frac{3}{2}\right) + \mathcal{O}(t) \\
&\approx -\frac{\alpha}{r} \cdot \frac{\alpha}{4\sqrt{\pi}} \frac{e^{-2mr}}{(mr)^{3/2}}
\end{aligned}$$

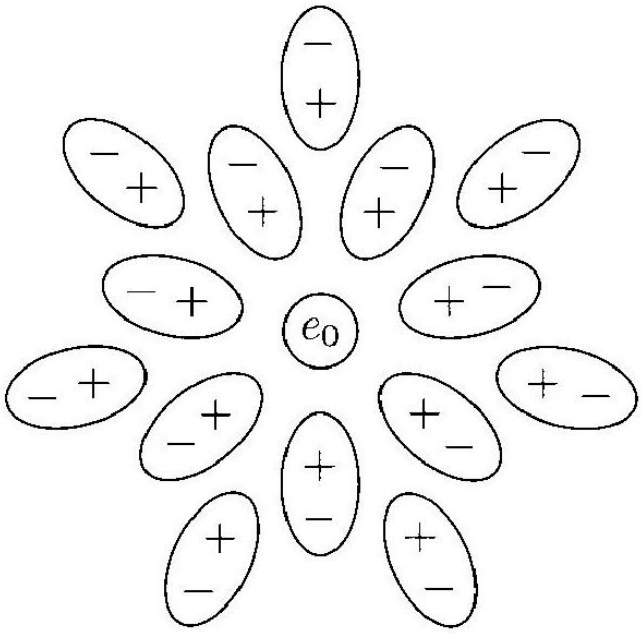


Figure 7.8. Virtual e^+e^-pairs are effectively dipoles of length $\sim 1/m$, which screen the bare charge of the electron.

so that

$$V(r) = -\frac{\alpha}{r}\left(1 + \frac{\alpha}{4\sqrt{\pi}}\frac{e^{-2mr}}{(mr)^{3/2}} + \cdots\right) \tag{7.95}$$

Thus the range of the correction term is roughly the electron Compton wavelength, $1/m$. Since hydrogen wavefunctions are nearly constant on this scale, the delta function in Eq. (7.94) was a good approximation. The radiative correction to $V(r)$ is called the Uehling potential.

We can interpret the correction as being due to screening. At $r \gtrsim 1/m$, virtual e^+e^-pairs make the vacuum a dielectric medium in which the apparent charge is less than the true charge (see Fig. 7.8). At smaller distances we begin to penetrate the polarization cloud and see the bare charge. This phenomenon is known as vacuum polarization.

Now consider the opposite limit: small distance or $-q^2 \gg m^2$. Equation (7.91) then becomes

$$\begin{aligned}\widehat{\Pi}_2\left(q^2\right) &\approx \frac{2\alpha}{\pi}\int_0^1 dx x(1-x)\left[\log\left(\frac{-q^2}{m^2}\right) + \log(x(1-x)) + \mathcal{O}\left(\frac{m^2}{q^2}\right)\right] \\ &= \frac{\alpha}{3\pi}\left[\log\left(\frac{-q^2}{m^2}\right) - \frac{5}{3} + \mathcal{O}\left(\frac{m^2}{q^2}\right)\right].\end{aligned}$$

The effective coupling constant in this limit is therefore

$$\alpha_{\text{eff}}\left(q^2\right) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\log\left(\frac{-q^2}{Am^2}\right)} \tag{7.96}$$

where $A = \exp(5/3)$. The effective electric charge becomes much larger at small distances, as we penetrate the screening cloud of virtual electronpositron pairs.

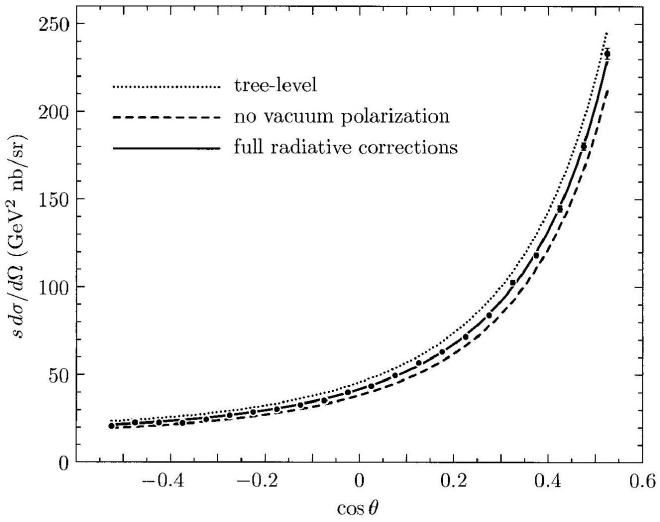


Figure 7.9. Differential cross section for Bhabha scattering, $e^+e^- \to e^+e^-$, at $E_{\rm cm} = 29$GeV, as measured by the HRS collaboration, M. Derrick, et. al., Phys. Rev. D34, 3286 (1986). The upper curve is the order- α^2 prediction derived in Problem 5.2, plus a very small ($\sim 2\%$) correction due to the weak interaction. The lower curve includes all QED radiative corrections to order α^3 except the vacuum polarization contribution; note that these corrections depend on the experimental conditions, as explained in Chapter 6. The middle curve includes the vacuum polarization contribution as well, which increases the effective value of α^2 by about 10% at this energy.

The combined vacuum polarization effect of the electron and of heavier quarks and leptons causes the value of $\alpha_{\rm eff}\ (q^2)$ to increase by about 5% from $q = 0$ to $q = 30$GeV, and this effect is observed in high-energy experiments. Figure 7.9 shows the cross section for Bhabha scattering at $E_{\rm cm} = 29$GeV, and a comparison to QED with and without the vacuum polarization diagram.

We can write $\alpha_{\rm eff}$ as a function of r by setting $q = 1/r$. The behavior of $\alpha_{\rm eff}\,(r)$ for all r is sketched in Fig. 7.10. The idea of a distance-dependent (or "scale-dependent" or "running") coupling constant will be a major theme of the rest of this book.

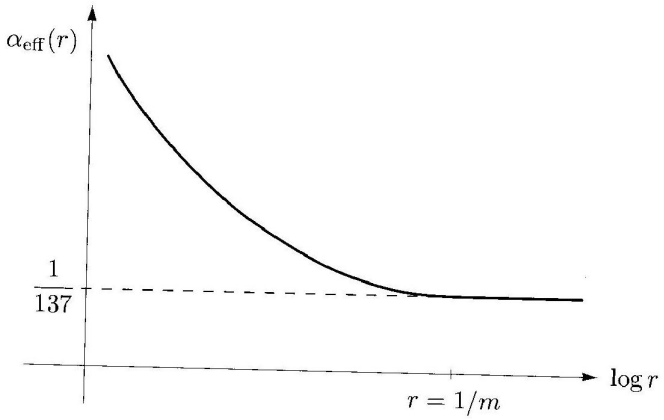


Figure 7.10. A qualitative sketch of the effective electromagnetic coupling constant generated by the one-loop vacuum polarization diagram, as a function of distance. The horizontal scale covers many orders of magnitude.

Problems

7.1 In Section 7.3 we used an indirect method to analyze the one-loop s-channel diagram for boson-boson scattering in ϕ^4 theory. To verify our indirect analysis, evaluate all three one-loop

diagrams, using the standard method of Feynman parameters. Check the validity of the optical theorem.

7.2 Alternative regulators in QED. In Section 7.5, we saw that the Ward identity can be violated by an improperly chosen regulator. Let us check the validity of the identity $Z_1 = Z_2$, to order α, for several choices of the regulator. We have already verified that the relation holds for Pauli-Villars regularization.

(a) Recompute δZ_1 and δZ_2, defining the integrals (6.49) and (6.50) by simply placing an upper limit Λ on the integration over ℓ_E. Show that, with this definition, $\delta Z_1 \neq \delta Z_2$.

(b) Recompute δZ_1 and δZ_2, defining the integrals (6.49) and (6.50) by dimensional regularization. You may take the Dirac matrices to be 4×4 as usual, but note that, in d dimensions,

$$g_{\mu\nu}\gamma^\mu\gamma^\nu = d.$$

Show that, with this definition, $\delta Z_1 = \delta Z_2$.

7.3 Consider a theory of elementary fermions that couple both to QED and to a Yukawa field ϕ:

$$H_{\text{int}} = \int d^3x \frac{\lambda}{\sqrt{2}}\phi\bar{\psi}\psi + \int d^3x e A_\mu \bar{\psi}\gamma^\mu\psi$$

(a) Verify that the contribution to Z_1 from the vertex diagram with a virtual ϕ equals the contribution to Z_2 from the diagram with a virtual ϕ. Use dimensional regularization. Is the Ward identity generally true in this theory?

(b) Now consider the renormalization of the $\phi\bar{\psi}\psi$ vertex. Show that the rescaling of this vertex at $q^2 = 0$ is not canceled by the correction to Z_2. (It suffices to compute the ultraviolet-divergent parts of the diagrams.) In this theory, the vertex and field-strength rescaling give additional shifts of the observable coupling constant relative to its bare value.

Radiation of Gluon Jets

Although we have discussed QED radiative corrections at length in the last two chapters, so far we have made no attempt to compute a full radiatively corrected cross section. The reason is of course that such calculations are quite lengthy. Nevertheless it would be dishonest to pretend that one understands radiative corrections after computing only isolated effects as we have done. This "final project" is an attempt to remedy this situation. The project is the computation of one of the simplest, but most important, radiatively corrected cross sections. You should finish Chapter 6 before starting this project, but you need not have read Chapter 7.

Strongly interacting particles-pions, kaons, and protons-are produced in e^+e^-annihilation when the virtual photon creates a pair of quarks. If one ignores the effects of the strong interactions, it is easy to calculate the total cross section for quark pair production. In this final project, we will analyze the first corrections to this formula due to the strong interactions.

Let us represent the strong interactions by the following simple model: Introduce a new massless vector particle, the gluon, which couples universally to quarks:

$$\Delta H = \int d^3x g \bar{\psi}_{fi}\gamma^\mu\psi_{fi}B_\mu$$

Here f labels the type ("flavor") of the quark (u, d, s, c, etc.) and $i = 1, 2, 3$ labels the color. The strong coupling constant g is independent of flavor and color. The electromagnetic coupling of quarks depends on the flavor, since the u and c quarks have charge $Q_f = +2/3$ while the d and s quarks have charge $Q_f = -1/3$. By analogy to α, let us define

$$\alpha_g = \frac{g^2}{4\pi}$$

In this exercise, we will compute the radiative corrections to quark pair production proportional to α_g.

This model of the strong interactions of quarks does not quite agree with the currently accepted theory of the strong interactions, quantum chromodynamics (QCD). However, all of the results that we will derive here are also
correct in QCD with the replacement

$$\alpha_g \to \frac{4}{3}\alpha_s$$

We will verify this claim in Chapter 17.
Throughout this exercise, you may ignore the masses of quarks. You may also ignore the mass of the electron, and average over electron and positron polarizations. To control infrared divergences, it will be necessary to assume that the gluons have a small nonzero mass μ, which can be taken to zero only at the end of the calculation. However (as we discussed in Problem 5.5), it is consistent to sum over polarization states of this massive boson by the replacement:

$$\sum \epsilon^\mu \epsilon^{\nu *} \to -g^{\mu\nu}$$

this also implies that we may use the propagator

$$B^\mu \overset{\Gamma}{B^\nu} = \frac{-ig^{\mu\nu}}{k^2 - \mu^2 + i\epsilon}$$

(a) Recall from Section 5.1 that, to lowest order in α and neglecting the effects of gluons, the total cross section for production of a pair of quarks of flavor f is

$$\sigma\left(e^+e^- \to \bar{q}q\right) = \frac{4\pi\alpha^2}{3s} \cdot 3Q_f^2$$

Compute the diagram contributing to $e^+e^- \to \bar{q}q$ involving one virtual gluon. Reduce this expression to an integral over Feynman parameters, and renormalize it by subtraction at $q^2 = 0$, following the prescription used in Eq. (6.55). Notice that the resulting expression can be considered as a correction to $F_1\left(q^2\right)$ for the quark. Argue that, for massless quarks, to all orders in α_g, the total cross section for production of a quark pair unaccompanied by gluons is

$$\sigma\left(e^+e^- \to \bar{q}q\right) = \frac{4\pi\alpha^2}{3s} \cdot 3\left|F_1\left(q^2 = s\right)\right|^2,$$

with $F_1\left(q^2 = 0\right) = Q_f$.
(b) Before we attempt to evaluate the Feynman parameter integrals in part (a), let us put this contribution aside and study the process $e^+e^- \to \bar{q}qg$, quark pair production with an additional gluon emitted. Before we compute the cross section, it will be useful to work out some kinematics. Let q be the total 4 -momentum of the reaction, let k_1 and k_2 be the 4 momenta of the final quark and antiquark, and let k_3 be the 4 -momentum of the gluon. Define

$$x_i = \frac{2k_i \cdot q}{q^2}, \quad i = 1, 2, 3$$

this is the ratio of the center-of-mass energy of particle i to the maximum available energy. Then show (i) $\sum x_i = 2$, (ii) all other Lorentz scalars involving only the final-state momenta can be computed in terms of the x_i and the particle masses, and (iii) the complete integral over 3-body phase space can be written as

$$\int d\Pi_3 = \prod_i \int \frac{d^3k_i}{(2\pi)^3}\frac{1}{2E_i}(2\pi)^4\delta^{(4)}\left(q - \sum_i k_i\right) = \frac{q^2}{128\pi^3}\int dx_1 dx_2$$

Find the region of integration for x_1 and x_2 if the quark and antiquark are massless but the gluon has mass μ.

(c) Draw the Feynman diagrams for the process $e^+e^- \to \bar{q}qg$, to leading order in α and α_g, and compute the differential cross section. You may throw away the information concerning the correlation between the initial beam axis and the directions of the final particles. This is conveniently done as follows: The usual trace tricks for evaluating the square of the matrix element give for this process a result of the structure

$$\int d\Pi_3 \frac{1}{4}\sum |\mathcal{M}|^2 = L_{\mu\nu}\int d\Pi_3 H^{\mu\nu}$$

where $L_{\mu\nu}$ represents the electron trace and $H^{\mu\nu}$ represents the quark trace. If we integrate over all parameters of the final state except x_1 and x_2, which are scalars, the only preferred 4 -vector characterizing the final state is q^μ. On the other hand, $H_{\mu\nu}$ satisfies

$$q^\mu H_{\mu\nu} = H_{\mu\nu}q^\nu = 0$$

Why is this true? (There is an argument based on general principles; however, you might find it a useful check on your calculation to verify this property explicitly.) Since, after integrating over final-state vectors, $\int H^{\mu\nu}$ depends only on q^μ and scalars, it can only have the form

$$\int d\Pi_3 H^{\mu\nu} = \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\cdot H$$

where H is a scalar. With this information, show that

$$L_{\mu\nu}\int d\Pi_3 H^{\mu\nu} = \frac{1}{3}\left(g^{\mu\nu}L_{\mu\nu}\right)\cdot\int d\Pi_3\left(g^{\rho\sigma}H_{\rho\sigma}\right)$$

Using this trick, derive the differential cross section

$$\frac{d\sigma}{dx_1 dx_2}\left(e^+e^- \to \bar{q}qg\right) = \frac{4\pi\alpha^2}{3s}\cdot 3Q_f^2\cdot\frac{\alpha_g}{2\pi}\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

in the limit $\mu \to 0$. If we assume that each original final-state particle is realized physically as a jet of strongly interacting particles, this formula gives the probability for observing three-jet events in e^+e^-annihilation and the kinematic distribution of these events. The form of the distribution in the x_i is an absolute prediction, and it agrees with experiment. The
normalization of this distribution is a measure of the strong-interaction coupling constant.

(d) Now replace $\mu \neq 0$ in the formula of part (c) for the differential cross section, and carefully integrate over the region found in part (b). You may assume $\mu^2 \ll q^2$. In this limit, you will find infrared-divergent terms of order $\log(q^2/\mu^2)$ and also $\log^2(q^2/\mu^2)$, finite terms of order 1 , and terms explicitly suppressed by powers of (μ^2/q^2). You may drop terms of the last type throughout this calculation. For the moment, collect and evaluate only the infrared-divergent terms.

(e) Now analyze the Feynman parameter integral obtained in part (a), again working in the limit $\mu^2 \ll q^2$. Note that this integral has singularities in the region of integration. These should be controlled by evaluating the integral for q spacelike and then analytically continuing into the physical region. That is, write $Q^2 = -q^2$, evaluate the integral for $Q^2 > 0$, and then carefully analytically continue the result to $Q^2 = -q^2 - i\epsilon$. Combine the result with the answer from part (d) to form the total cross section for $e^+e^- \to$ strongly interacting particles, to order α_g. Show that all infrareddivergent logarithms cancel out of this quantity, so that this total cross section is well-defined in the limit $\mu \to 0$.
(f) Finally, collect the terms of order 1 from the integrations of parts (d) and (e) and combine them. To evaluate certain of these terms, you may find the following formula useful:

$$\int_0^1 dx \frac{\log(1-x)}{x} = -\frac{\pi^2}{6}$$

(It is not hard to prove this.) Show that the total cross section is given, to this order in α_g, by

$$\sigma\left(e^+e^- \to \bar{q}q \text{ or } \bar{q}qg\right) = \frac{4\pi\alpha^2}{3s} \cdot 3Q_f^2 \cdot \left(1 + \frac{3\alpha_g}{4\pi}\right)$$

This formula gives a second way of measuring the strong-interaction coupling constant. The experimental results agree (within the current experimental errors) with the results obtained by the method of part (c). We will discuss the measurement of α_s more fully in Section 17.6.

17.6 Chapter 8 Invitation to Renormalization: Ultraviolet Cutoffs and Critical Fluctuations

The main purpose of Part II of this book is to develop a general theory of renormalization. This theory will explain the origin of ultraviolet divergences in field theory and will indicate when these divergences can be removed systematically. It will also give a way to convert the divergences of Feynman diagrams from a problem into a tool. We will apply this tool to study the asymptotic large- or small-momentum behavior of field theory amplitudes.

When we first encountered an ultraviolet divergence in the calculation of the one-loop vertex correction in Section 6.3, it seemed an aberration that ought to disappear before it caused us too much discomfort. In Chapter 7 we saw further examples of ultraviolet-divergent diagrams, enough to convince us that such divergences occur ubiquitously in Feynman diagram computations. Thus it is necessary for anyone studying field theory to develop a point of view toward these divergences. Most people begin with the belief that any theory that contains divergences must be nonsense. But this viewpoint is overly restrictive, since it excludes not only quantum field theory but even the classical electrodynamics of point particles.

With some experience, one might adopt a more permissive attitude of peaceful coexistence with the divergences: One can accept a theory with divergences, as long as they do not appear in physical predictions. In Chapter 7 we saw that all of the divergences that appear in the one-loop radiative corrections to electron scattering from a heavy target can be eliminated by consistently eliminating the bare values of the mass and charge of the electron in favor of their measured physical values. In Chapter 10, we will argue that all of the ultraviolet divergences of QED, in all orders of perturbation theory, can be eliminated in this way. Thus, as long as one is willing to consider the mass and charge of the electron as measured parameters, the predictions of QED perturbation theory will always be free of divergences. We will also show in Chapter 10 that QED

belongs to a well-defined class of field theories in which all ultraviolet divergences are removed after a fixed small number of physical parameters are taken from experiment. These theories, called renormalizable quantum field theories, are the only ones in which perturbation theory gives well-defined predictions.

Ideally, though, one should take the further step of trying to understand
physically why the divergences appear and why their effects are more severe in some theories than in others. This direct approach to the divergence problem was pioneered in the 1960s by Kenneth Wilson. The crucial insights needed to solve this problem emerged from a correspondence, discovered by Wilson and others, between quantum field theory and the statistical physics of magnets and fluids. Wilson's approach to renormalization is the subject of Chapter 12. The present chapter gives a brief introduction to the issues in condensed matter physics that have provided insight into the problem of ultraviolet divergences.

Formal and Physical Cutoffs

Ultraviolet divergences signal that quantities calculated in a quantum field theory depend on some very large momentum scale, the ultraviolet cutoff. Equivalently, in position space, divergent quantities depend on some very small distance scale.

The idea of a small-distance cutoff in the continuum description of a system occurs in classical field theories as well. Typically the cutoff is at the scale of atomic distances, where the continuum description no longer applies. However, the size of the cutoff manifests itself in certain parameters of the continuum theory. In fluid dynamics, for instance, parameters such as the viscosity and the speed of sound are of just the size one would expect by combining typical atomic radii and velocities. Similarly, in a magnet, the magnetic susceptibility can be estimated by assuming that the energy cost of flipping an electron spin is on the order of a tenth of an eV , as we would expect from atomic physics. Each of these systems possesses a natural ultraviolet cutoff at the scale of an atom; by understanding the physics at the atomic scale, we can compute the parameters that determine the physics on larger scales.

In quantum field theory, however, we have no precise knowledge of the fundamental physics at very short distance scales. Thus, we can only measure parameters such as the physical charge and mass of the electron, not compute them from first principles. The presence of ultraviolet divergences in the relations between these physical parameters and their bare values is a sign that these parameters are controlled by the unknown short-distance physics.

Whether we know the fundamental physics at small distance scales or not, we need two kinds of information in order to write an effective theory for large-distance phenomena. First, we must know how many parameters from the small distance scale are relevant to large-distance physics. Second, and more importantly, we must know what degrees of freedom from the underlying theory appear at large distances.

In fluid mechanics, it is something of a miracle, from the atomic point of view, that any large-distance degrees of freedom even exist. Nevertheless, the equations that express the transport of energy and mass over large distances do have smooth, coherent solutions. The large-distance degrees of freedom are
the flows that transport these conserved quantities, and sound waves of long wavelength.

In quantum field theory, the large-distance physics involves only those particles that have masses that are very small compared to the fundamental cutoff scale. These particles and their dynamics are the quantum analogues of the large-scale flows in fluid mechanics. The simplest way to naturally arrange for such particles to appear is to make use of particles that naturally have zero

mass. So far in this book, we have encountered two types of particles whose mass is precisely zero, the photon and the chiral fermion. (In Chapter 11 we will meet one further naturally massless particle, the Goldstone boson.) We might argue that QED exists as a theory on scales much larger than its cutoff because the photon is naturally massless and because the left- and right-handed clectrons are very close to being chiral fermions.

There is another way that particles of zero or almost zero mass can arise in quantum field theory: We can simply tune the parameters of a scalar field theory so that the scalar particles have masses small compared to the cutoff. This method of introducing particles with small mass seems arbitrary and unnatural. Nevertheless, it has an analogue in statistical mechanics that is genuinely interesting in that discipline and can teach us some important lessons.

Normally, in a condensed matter system, the thermal fluctuations are correlated only over atomic distances. Under special circumstances, however, they can have much longer range. The clearest example of this phenomenon occurs in a ferromagnet. At high temperature, the electron spins in a magnet are disorganized and fluctuating; but at low temperature, these spins align to a fixed direction.* Let us think about how this alignment builds up as the temperature of the magnet is lowered. As the magnet cools from high temperature, clusters of correlated spins become larger and larger. At a certain point the temperature of magnetization–the entire sample becomes a single large cluster with a well-defined macroscopic orientation. Just above this temperature, the magnet contains large clusters of spins with a common orientation, which in turn belong to still larger clusters, such that the orientations on the very largest scale are still randomized through the sample. This situation is illustrated in Fig. 8.1. Similar behavior occurs in the vicinity of any other second-order phase transition, for example, the order-disorder transition in binary alloys, the critical point in fluids, or the superfluid transition in Helium-4.

The natural description of these very long wavelength fluctuations is in terms of a fluctuating continuum field. At the lowest intuitive level, we might

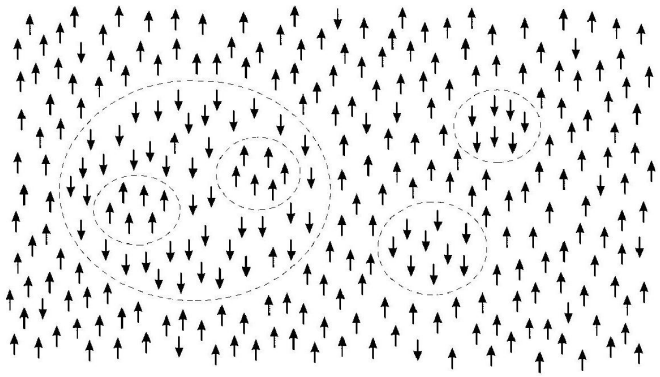


Figure 8.1. Clusters of oriented spins near the critical point of a ferromagnet.

substitute quantum for statistical fluctuations and try to describe this system as a quantum field theory. In Section 9.3 we will derive a somewhat more subtle relation that makes a precise connection between the statistical and the quantum systems. Through this connection, the behavior of any statistical system near a second-order phase transition can be translated into the behavior of a particular quantum field theory. This quantum field theory has a field with a mass that is very small compared to the basic atomic scale and that goes to zero precisely at the phase transition.

But this connection seems to compound the problem of ultraviolet divergences in quantum field

[6]*In a real ferromagnet, the long-range magnetic dipole-dipole interaction causes the state of uniform magnetization to break up into an array of magnetic domains. In this book, we will ignore this interaction and think of a magnetic spin as a pure orientation. It is this idealized system that is directly analogous to a quantum field theory.

theory: If the wealth of phase transitions observed in Nature generates a similar wealth of quantum field theories, how can we possibly define a quantum field theory without detailed reference to its origins in physics at the scale of its ultraviolet cutoff? Saying that a quantum field theory makes predictions independent of the cutoff would be equivalent to saying that the statistical fluctuations in the neighborhood of a critical point are independent of whether the system is a magnet, a fluid, or an alloy. But is this statement so obviously incorrect? By reversing the logic, we would find that quantum field theory makes a remarkably powerful prediction for condensed matter systems, a prediction of universality for the statistical fluctuations near a critical point. In fact, this prediction is verified experimentally.

A major theme of Part II of this book will be that these two ideas -cutoff independence in quantum field theory and universality in the theory of critical phenomena-are naturally the same idea, and that understanding either of these ideas gives insight into the other.

Landau Theory of Phase Transitions

To obtain a first notion of what could be universal in the phenomena of phase transitions, let us examine the simplest continuum theory of second-order phase transitions, due to Landau.

First we should review a little thermodynamics and clarify our nomenclature. In thermodynamics, a first-order phase transition is a point across which some thermodynamic variable (the density of a fluid, or the magnetization of a ferromagnet) changes discontinuously. At a phase transition point, two quite distinct thermodynamic states (liquid and gas, or magnetization parallel and antiparallel to a given axis) are in equilibrium. The thermodynamic quantity that changes discontinuously across the transition, and that characterizes the difference of the two competing phases, is called the order parameter. In most circumstances, it is possible to change a second thermodynamic parameter in such a way that the two competing states move closer together in the thermodynamic space, so that at some value of this parameter, these two states become identical and the discontinuity in the order parameter disappears. This endpoint of the line of first-order transitions is called a second-order phase transition, or, more properly, a critical point. Viewed from the other direction, a critical point is a point at which a single thermodynamic state bifurcates into two macroscopically distinct states. It is this bifurcation that leads to the long-ranged thermal fluctuations discussed in the previous section.

A concrete example of this behavior is exhibited by a ferromagnet. Let us assume for simplicity that the material we are discussing has a preferred axis of magnetization, so that at low temperature, the system will have its spins ordered either parallel or antiparallel to this axis. The total magnetization along this axis, M, is the order parameter. At low temperature, application of an external magnetic field H will favor one or the other of the two possible states. At $H = 0$, the two states will be in equilibrium; if H is changed from a small negative to a small positive value, the thermodynamic state and the value of M will change discontinuously. Thus, for any fixed (low) temperature, there is a first-order transition at $H = 0$. Now consider the effect of raising the temperature: The fluctuation of the spins increases and the value of $|M|$ decreases. At some temperature T_C the system ceases to be magnetized at $H = 0$. At this point, the first-order phase transition disappears and the two competing thermodynamic states coalesce. The system thus has a critical point at $T = T_C$. The location of these various transitions in the $H - T$ plane is shown in Fig. 8.2.

Landau described this behavior by the use of the Gibbs free energy G; this is the thermodynamic potential that depends on M and T, such that

$$\left.\frac{\partial G}{\partial M}\right|_T = H \tag{8.1}$$

He suggested that we concentrate our attention on the region of the critical

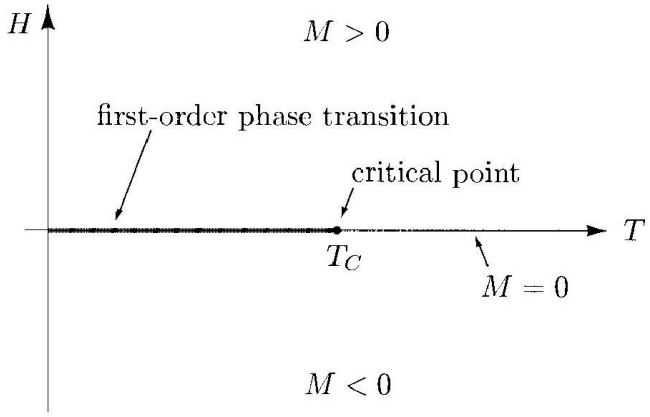


Figure 8.2. Phase diagram in the $H - T$ plane for a uniaxial ferromagnet.

point: $T \approx T_C, M \approx 0$. Then it is reasonable to expand $G(M)$ as a Taylor series in M. For $H = 0$, we can write

$$G(M) = A(T) + B(T)M^2 + C(T)M^4 + \cdots . \tag{8.2}$$

Because the system has a symmetry under $M \to -M, G(M)$ can contain only even powers of M. Since M is small, we will ignore the higher terms in the expansion. Given Eq. (8.2), we can find the possible values of M at $H = 0$ by solving

$$0 = \frac{\partial G}{\partial M} = 2B(T)M + 4C(T)M^3 \tag{8.3}$$

If B and C are positive, the only solution is $M = 0$. However, if $C > 0$ but B is negative below some temperature T_C, we have a nontrivial solution for $T < T_C$, as shown in Fig. 8.3. More concretely, approximate for $T \approx T_C$:

$$B(T) = b\,(T - T_C)\,, \quad C(T) = c \tag{8.4}$$

Then the solution to Eq. (8.3) is

$$M = \begin{cases} 0 & \text{for } T > T_C \\ \pm\left[(b/2c)\,(T_C - T)\right]^{1/2} & \text{for } T < T_C \end{cases} \tag{8.5}$$

This is just the qualitative behavior that we expect at a critical point.

To find the value of M at nonzero external field, we could solve Eq. (8.1) with the left-hand side given by (8.2). An equivalent procedure is to minimize a new function, related to (8.2). Define

$$G(M, H) = A(T) + B(T)M^2 + C(T)M^4 - HM \tag{8.6}$$

Then the minimum of $G(M, H)$ with respect to M at fixed H gives the value of M that satisfies Eq. (8.1). The minimum is unique except when $H = 0$ and $T < T_C$, where we find the double

minimum in the second line of (8.5). This is consistent with the phase diagram shown in Fig. 8.2.

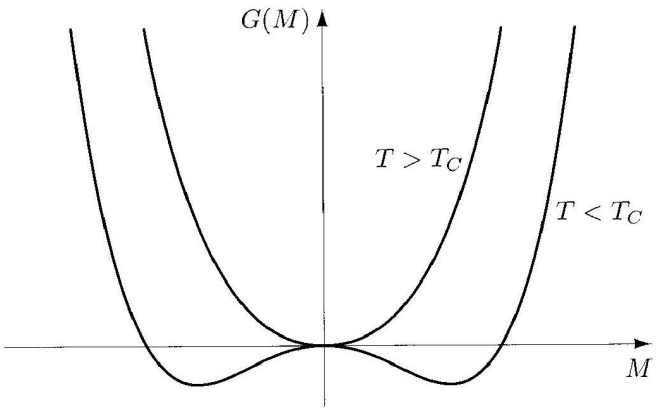


Figure 8.3. Behavior of the Gibbs free energy $G(M)$ in Landau theory, at temperatures above and below the critical temperature.

To study correlations in the vicinity of the phase transition, Landau generalized this description further by considering the magnetization M to be the integral of a local spin density:

$$M = \int d^3x s(\mathbf{x}) \tag{8.7}$$

Then the Gibbs free energy (8.6) becomes the integral of a local function of $s(\mathbf{x})$,

$$G = \int d^3x \left[\frac{1}{2} (\boldsymbol{\nabla} s)^2 + b\,(T - T_C)\, s^2 + c s^4 - H s \right] \tag{8.8}$$

which must be minimized with respect to the field configuration $s(\mathbf{x})$. The first term is the simplest possible way to introduce the tendency of nearby spins to align with one another. We have rescaled $s(\mathbf{x})$ so that the coefficient of this term is set to $1/2$. In writing this free energy integral, we could even consider H to vary as a function of position. In fact, it is useful to do that; we can turn on $H(\mathbf{x})$ near $x = 0$ and see what response we find at another point.

The minimum of the free energy expression (8.8) with respect to $s(\mathbf{x})$ is given by the solution to the variational equation

$$0 = \delta G[s(\mathbf{x})] = -\nabla^2 s + 2b\,(T - T_C)\, s + 4cs^3 - H(\mathbf{x}) \tag{8.9}$$

For $T > T_C$, where the macroscopic magnetization vanishes and so $s(\mathbf{x})$ should be small, we can find the qualitative behavior by ignoring the s^3 term. Then $s(\mathbf{x})$ obeys a linear equation,

$$\left(-\nabla^2 + 2b\,(T - T_C) \right) s(\mathbf{x}) = H(\mathbf{x}) \tag{8.10}$$

To study correlations of spins, we will set

$$H(\mathbf{x}) = H_0 \delta^{(3)}(\mathbf{x}) \tag{8.11}$$

The resulting configuration $s(\mathbf{x})$ is then the Green's function of the differential operator in Eq. (8.10), so we call it $D(\mathbf{x})$:

$$\left(-\nabla^2 + 2b\,(T - T_C) \right) D(\mathbf{x}) = H_0 \delta^{(3)}(\mathbf{x}) \tag{8.12}$$

This Green's function tells us the response at $\mathbf{x}$ when the spin at $\mathbf{x} = 0$ is forced into alignment with H. In Sections 9.2 and 9.3 we will see that $D(\mathbf{x})$ is also proportional to the zero-field spin-spin correlation function in the thermal ensemble,

$$D(\mathbf{x}) \propto \langle s(\mathbf{x})s(0)\rangle \equiv \sum_{\text{all } s(\mathbf{x})} s(\mathbf{x})s(0)e^{-\mathbf{H}/kT} \tag{8.13}$$

where $\mathbf{H}$ is the Hamiltonian of the magnetic system.
The solution to Eq. (8.12) can be found by Fourier transformation:

$$D(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{H_0 e^{i\mathbf{k}\cdot\mathbf{x}}}{|\mathbf{k}|^2 + 2b\,(T - T_C)} \tag{8.14}$$

This is just the integral we encountered in our discussion of the Yukawa potential, Eq. (4.126). Evaluating it in the same way, we find

$$D(\mathbf{x}) = \frac{H_0}{4\pi}\frac{1}{r}e^{-r/\xi} \tag{8.15}$$

where

$$\xi = [2b\,(T - T_C)]^{-1/2} \tag{8.16}$$

is the correlation length, the range of correlated spin fluctuations. Notice that this length diverges as $T \to T_C$.

The main results of this analysis, Eqs. (8.5) and (8.16), involve unknown constants b, c that depend on physics at the atomic scale. On the other hand, the power-law dependence in these formulae on $(T - T_C)$ follows simply from the structure of the Landau equations and is independent of any details of the microscopic physics. In fact, our derivation of this dependence did not even use the fact that G describes a ferromagnet; we assumed only that G can be expanded in powers of an order parameter and that G respects the reflection symmetry $M \to -M$. These assumptions apply equally well to many other types of systems: binary alloys, superfluids, and even (though the reflection symmetry is less obvious here) the liquid-gas transition. Landau theory predicts that, near the critical point, these systems show a universal behavior in the dependence of M, ξ, and other thermodynamic quantities on $(T - T_C)$.

Critical Exponents

The preceding treatment of the Landau theory of phase transitions emphasizes its similarity to classical field theory. We set up an appropriate free energy and found the thermodynamically preferred configuration by solving a classical variational equation. This gives only an approximation to the full statistical problem, analogous to the approximation of replacing quantum by classical dynamics in field theory. In Chapter 13, we will use methods of quantum field theory to account properly for the fluctuations about the preferred Landau thermodynamic state. These modifications turn out to be profound, and rather counterintuitive.

To describe the form of these modifications, let us write Eq. (8.15) more generally as

$$\langle s(\mathbf{x})s(0)\rangle = A\frac{1}{r^{1+\eta}}f(r/\xi) \tag{8.17}$$

where A is a constant and $f(y)$ is a function that satisfies $f(0) = 1$ and $f(y) \to 0$ as $y \to \infty$. Landau theory predicts that $\eta = 0$ and $f(y)$ is a simple exponential. This expression has a form

strongly analogous to that of a Green's function in quantum field theory. The constant A can be absorbed into the field-strength renormalization of the field $s(\mathbf{x})$. The correlation length ξ is, in general, a complicated function of the atomic parameters, but in the continuum description we can simply trade these parameters for ξ. It is appropriate to consider ξ as a cutoff-independent, physical parameter, since it controls the large-distance behavior of a physical correlation. In fact, the analogy between Eq. (8.15) and the Yukawa potential suggests that we should identify ξ^{-1} with the physical mass in the associated quantum field theory. Then Eq. (8.17) gives a cutoff-independent, continuum representation of the statistical system.

If we were working in quantum field theory, we would derive corrections to Eq. (8.17) as a perturbation series in the parameter c multiplying the nonlinear term in (8.9). This would generalize the Landau result to

$$\langle s(\mathbf{x})s(0)\rangle = \frac{1}{r}F(r/\xi, c). \tag{8.18}$$

The perturbative corrections would depend on the properties of the continuum field theory. For example, $F(y, c)$ would depend on the number of components of the field $s(\mathbf{x})$, and its series expansion would differ depending on whether the magnetization formed along a preferred axis, in a preferred plane, or isotropically. For order parameters with many components, the expansion would also depend on higher discrete symmetries of the problem. However, we expect that systems described by the same Landau free energy (for example, a single-axis ferromagnet and a liquid-gas system) should have the same perturbation expansion when this expansion is written in terms of the physical mass and coupling. The complete universality of Landau theory then becomes a more limited concept, in which systems have the same large-distance correlations if their order parameters have the same symmetry. We might say that statistical systems divide into distinct universality classes, each with a characteristic large-scale behavior.

If this were the true behavior of systems near second-order phase transitions, it would already be a wonderful confirmation of the ideas required to formulate cutoff-independent quantum field theories. However, the true behavior of statistical systems is still another level more subtle. What one finds experimentally is a dependence of the form of Eq. (8.17), where the function $F(y)$ is the same within each universality class. There is no need for an auxiliary parameter c. On the other hand, the exponent η takes a specific nonzero value in each universality class. Other power-law relations of Landau theory are also modified, in a specific manner for each universality class. For example, Eq. (8.5) is changed, for $T < T_C$, to

$$M \propto (T_C - T)^{\beta} \tag{8.19}$$

where the exponent β takes a fixed value for all systems in a given universality class. For three-dimensional single-axis magnets and for fluids, $\beta = 0.313$. The powers in these nontrivial scaling relations are called critical exponents.

The modification from Eq. (8.18) to Eq. (8.17) does not imperil the idea that a condensed matter system, in the vicinity of a second-order phase transition, has a well-defined, cutoff-independent, continuum behavior. However, we would like to understand why Eq. (8.17) should be expected as the correct representation. The answer to this question will come from a thorough analysis of the ultraviolet divergences of the corresponding quantum field theory. In Chapter 12, when we finally conclude our cxplication of the ultraviolet divergences, we will find that we have in hand the tools not only to justify Eq. (8.17), but also to calculate the values of the critical exponents using Feynman diagrams. In this way, we will uncover a bcautiful application of quantum field

theory to the domain of atomic physics. The success of this application will guide us, in Part III, to cven more powerful tools, which we will need in the relativistic domain of elementary particles.

17.7 L11 Electron and Photon Propagators at One-Loop

17.7.1 1. Mass Operator

В наинизшем приближении массовый оператор $-i\mathcal{N}(p)$ определяется диаграммой

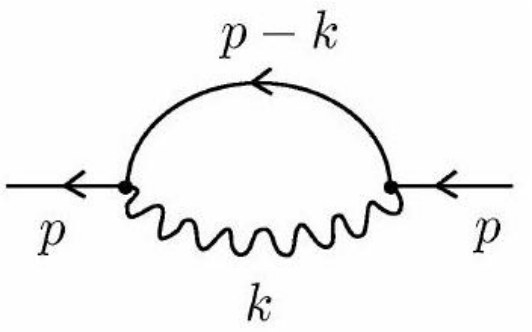


Рис. 23

аналитическое выражение которой даётся выражением

$$-i\mathcal{M}(p) = (-ie)^2 \int \frac{\mathrm{d}^{(d)}k}{(2\pi)^d} \gamma^\mu i S_c(p-k) \gamma^\nu \left(-iD^\alpha_{\mu\nu}(k)\right)$$

Проще всего вычисление проводится в калибровке Фейнмана $\alpha = 1$, в которой

$$-i\mathcal{M}(p) = -4\pi e^2 \int \frac{\mathrm{d}^{(d)}k}{(2\pi)^d} \frac{\gamma^\mu[(\not{p}-\not{k})+m]\gamma_\mu}{(p-k)^2-m^2} \frac{1}{k^2-\lambda^2}$$

Мы ввели массу фотона $\lambda^2 \ll m^2$, необходимую (см. ниже) для удовлетворения условий нормировки. Разумеется, масса фотона выпадает из физически наблюдаемых величин (см. лекцию 13). Пусть $d = 4-\varepsilon$. Тогда $\gamma^\mu \not{p} \gamma_\mu = -(2-\varepsilon)\not{p}$, $\quad \gamma^\mu\gamma_\mu = d = 4-\varepsilon$, и

$$\mathcal{M}(p) = -\frac{4\pi i e^2}{(2\pi)^d} \int \mathrm{d}^{(d)}k \frac{(4-\varepsilon)m - (2-\varepsilon)(\not{p}-\not{k})}{[(p-k)^2-m^2]\,(k^2-\lambda^2)}$$

Воспользуемся формулой (35 Л4), положив $A = (p-k)^2 - m^2, \quad B = k^2 - \lambda^2$:

$$\mathcal{M}(p) = -\frac{4\pi i e^2}{(2\pi)^d} \int \mathrm{d}^{(d)}k \int\limits_0^1 \mathrm{d}x \frac{(4-\varepsilon)m - (2-\varepsilon)(\not{p}-\not{k})}{\left[(k-xp)^2 - a^2\right]^2}$$

$$a^2 = -x(1-x)p^2 + xm^2 + (1-x)\lambda^2$$

После сдвига переменной интегрирования $k \to k + xp$ интеграл переписывается как

$$\mathcal{M}(p) = -\frac{4\pi i e^2}{(2\pi)^d} \int\limits_0^1 \mathrm{d}x \int \mathrm{d}^{(d)}k \frac{(4-\varepsilon)m - (2-\varepsilon)(1-x)\not{p}}{\left(k^2-a^2\right)^2}$$

причём линейное относительно $\not{\varepsilon}$слагаемое в числителе теперь может быть опущено, так как интегрирование по углам обращает его в нуль. Интеграл в импульсном пространстве берётся при помощи поворота Вика и формулы (3 Л4):

$$\mathcal{N}(p) = \frac{e^2\Gamma(\varepsilon/2)}{(4\pi)^{1-\varepsilon/2}} \int\limits_0^1 \mathrm{d}x[(4-\varepsilon)m - (2-\varepsilon)(1-x)\not{p}] \cdot \left[-x(1-x)p^2 + xm^2 + (1-x)\lambda^2\right]^{-\varepsilon/2}$$

Последнее выражение разложим по ε с точностью $O\left(\varepsilon^0\right)$:

$$\mathcal{N}(p) = \frac{e^2}{2\pi\varepsilon}(4m - \not{p}) + \frac{e^2}{2\pi}\left[\frac{1}{2}(\ln 4\pi + \psi(1))(4m - \not{p}) + \left(-m + \frac{1}{2}\not{p}\right)\right] -$$

$$-\frac{e^2}{2\pi}\int\limits_0^1 \mathrm{d}x[(1+x)m - (1-x)(\not{p} - m)] \cdot \ln\left[-x(1-x)p^2 + xm^2 + (1-x)\lambda^2\right] \tag{1}$$

Обратим внимание на то, что в точках нормировки оператор $\mathcal{M}$ используется в обкладках $\bar{u}_p\mathcal{M}u_p$, причём $p^2 = m^2$ и $\not{p}u_p = mu_p, \bar{u}_p\not{p} = m\bar{u}_p$ (см. (41 Л10) и (42 Л10)). Поэтому для приведения оператора $\mathcal{M}$ к виду (40 а Л10) следует, во первых, сделать вычитание

$$\mathcal{M}(\not{p}) \longrightarrow \mathcal{M}(\not{p}) - \mathcal{M}(\not{p})|_{\not{p}=m, p^2=m^2} \equiv (\not{p} - m)\mathcal{M}'(\not{p}) \tag{2}$$

Затем операцию вычитания следует повторить для величины $\mathcal{M}'^1$. В результате двух таких вычитаний первые два слагаемых в (1) исчезают. Последнее слагаемое в (1) переписываем как

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$$

$$\mathcal{M}_1 = \frac{e^2}{2\pi}(\not{p} - m)\int\limits_0^1 \mathrm{d}x(1-x)\ln\left[m^2(1-\rho)\,(x-x_1)\,(x-x_2)\right]$$

$$\mathcal{M}_2 = -\frac{e^2 m}{2\pi}\int\limits_0^1 \mathrm{d}x(1+x)\ln\left[m^2(1-\rho)\,(x-x_1)\,(x-x_2)\right] \tag{3}$$

где введены параметры

$$\rho = \frac{m^2 - p^2}{m^2}, \quad \beta = \frac{\lambda^2}{m^2}, \quad x_{1,2} = \frac{1}{2}\left[-\frac{\rho-\beta}{1-\rho} \pm \sqrt{\left(\frac{\rho-\beta}{1-\rho}\right)^2 - \frac{4\beta}{1-\rho}}\right] \tag{4}$$

Вычисления проводятся в предположении

$$\rho > 0, \longrightarrow x_{1,2} < 0 \tag{5}$$

Напомним, что m^2 имеет бесконечно малую отрицательную мнимую часть. Поэтому

$$\lim_{\rho\to+0} x_{1,2} \equiv \tilde{x}_{1,2} = \frac{1}{2}\beta \mp i\sqrt{\beta} \tag{6}$$

Проведём одно вычитание (2) для $\mathcal{M}_2$:

$$\mathcal{M}_2 = -\frac{e^2 m}{2\pi}\int\limits_0^1 \mathrm{d}x(1+x)\ln\frac{(1-\rho)(x-x_1)(x-x_2)}{(x-\tilde{x}_1)(x-\tilde{x}_2)} =$$

$$= \frac{e^2 m}{2\pi}\cdot\frac{1}{2}\int\limits_0^1 \mathrm{d}x\left(x^2+2x\right)\left[\frac{1}{x-x_1}+\frac{1}{x-x_2}-\frac{1}{x-\tilde{x}_1}-\frac{1}{x-\tilde{x}_2}\right] \tag{7}$$

При преобразовании интеграла (7) было учтено, что

$$(1-\rho)(1-x_1)(1-x_2) = 1 \longrightarrow (1-\tilde{x}_1)(1-\tilde{x}_2) = 1$$

Последний интеграл вычисляется при помощи формулы

$$\frac{x^2+ax}{x-x_i} = x + (x_i+a) + \frac{x_i(x_i+a)}{x-x_i} \tag{8}$$

Имеем:

$$\mathcal{M}_2 = \frac{e^2 m}{2\pi}\cdot\frac{1}{2}(-\rho)\left\{\frac{1-\beta}{1-\rho} - \frac{1}{\rho}\sum_{i=1,2}\left[x_i(x_i+2)\ln\frac{1-x_i}{-x_i} - \tilde{x}_i(\tilde{x}_i+2)\ln\frac{1-\tilde{x}_i}{-\tilde{x}_i}\right]\right\} \tag{9}$$

Так как

$$-\rho = \frac{1}{m^2}(\not{p}-m)(\not{p}+m) \tag{10}$$

то

10

$$\mathcal{M}_2'(\not{p}) = (\not{p}+m)\mathcal{M}_2''(\rho)$$

$$\mathcal{M}_2''(\rho) = \frac{e^2}{2\pi}\frac{1}{2m}\left\{\frac{1-\beta}{1-\rho} - \frac{1}{\rho}\sum_{i=1,2}\left[x_i(x_i+2)\ln\frac{1-x_i}{-x_i} - \tilde{x}_i(\tilde{x}_i+2)\ln\frac{1-\tilde{x}_i}{-\tilde{x}_i}\right]\right\} \tag{11}$$

Для приведения вклада в $\mathcal{M}$ от $\mathcal{M}_2$ к окончательному виду следует вместо $\mathcal{M}_2'(\not{p})$ использовать величину

$$(\not{p}+m)\mathcal{M}_2''(\rho) - (2m)\mathcal{M}_2''(0) + 2m\mathcal{M}_2''(\rho) - 2m\mathcal{M}_2''(\rho) = (\not{p}-m)\mathcal{M}_2''(\rho) + 2m\rho\left\{\frac{1}{\rho}\left[\mathcal{M}_2''(\rho) - \mathcal{M}_2''(0)\right]\right\} \tag{12}$$

Заметим, что в пределе $\lambda^2 \to 0, \rho > 0$

$$\tilde{x}_1 = \tilde{x}_2 = 0, \quad x_1 = 0, \quad x_2 = -\frac{\rho}{1-\rho}, \quad \mathcal{M}_2''(\rho) = \frac{e^2}{2\pi}\frac{1}{2m(1-\rho)}\left[1 - \frac{2-3\rho}{1-\rho}\ln\rho\right] \tag{13}$$

[61] Заметим, что согласно данным определениям $1 - \mathcal{M}'(\not{p})|_{p^2=m^2,\not{p}=m} = Z_2^{-1}$

Поэтому первое слагаемое в (12) определено корректно в пределе нулевой массы фотона. Для вычисления второго слагаемого в (12) следует вычислить $\mathcal{M}_2''(0)$, для чего необходимо в (11) перейти к пределу $\rho \to 0$ при фиксированном значении $0 < \lambda^2 \ll m^2$. В результате несложных вычислений находим:

$$\mathcal{M}_2''(0) = -\frac{e^2}{2\pi}\frac{1}{2m}\left(1 + 2\ln\frac{\lambda}{m}\right) \tag{14}$$

В (14) все коэффициенты, зависящие от β, берутся в точке $\beta = 0$, поскольку при этом не возникает неопределённостей и расходимостей. Собирая формулы (2), (12), (13) и (14), получаем для $\mathcal{M}_2$:

$$\mathcal{M}_2 = \frac{e^2}{2\pi m}(\not{p} - m)^2 \left\{ \frac{1}{2(1-\rho)}\left[1 - \frac{2-3\rho}{1-\rho}\ln\rho\right] - \frac{\not{x} + m}{m\rho}\left[\frac{1}{1-\rho} - \frac{(2-3\rho)}{(1-\rho)^2}\ln\rho + 1 + 2\ln\frac{\lambda}{m}\right]\right\} \tag{15}$$

Действуя аналогично и делая вычитание (2) для

$$\mathcal{M}_1'(\not{p}) = \frac{e^2}{2\pi}\cdot\frac{1}{2}\left\{ \text{ const } + \sum_{i=1,2}\left[x_i + x_i\,(x_i - 2)\ln\frac{1-x_i}{-x_i}\right]\right\} \tag{16}$$

находим:

$$\mathcal{M}_1 = \frac{e^2}{2\pi m}(\not{p} - m)^2\frac{\not{p} + m}{m}\frac{1}{2(1-\rho)}\left(1 + \frac{2-\rho}{1-\rho}\ln\rho\right) \tag{17}$$

Складывая (15) и (17), окончательно находим:

$$\mathcal{M} = \frac{e^2}{2\pi m}(\not{p} - m)^2\left\{\frac{1}{2(1-\rho)}\left(1 - \frac{2-3\rho}{1-\rho}\ln\rho\right) - \right.$$
$$\left. -\frac{\not{p} + m}{m\rho}\left[\frac{1}{2(1-\rho)}\left(2 - \rho + \frac{\rho^2 + 4\rho - 4}{1-\rho}\ln\rho\right) + 1 + 2\ln\frac{\lambda}{m}\right]\right\} \tag{18}$$

Оператор (18) вычислен в предположении $\rho > 0, \rho \gg \lambda/m$. При аналитическом продолжении в область $\rho < 0$ следует помнить, что точка ветвления $\rho = 0$ обходится снизу, так как m^2 содержит отрицательную бесконечно малую мнимую часть. Поэтому при $\rho < 0$

$$\ln\rho = \ln|\rho| - i\pi, \quad \rho < 0 \tag{19}$$

Из (18) видно, что при нулевой массе фотона массовый оператор не определён, если используются условия нормировки (40 а Л10).

При $p^2 \gg m^2$ из (18) получаем следующую оценку:

$$\mathcal{M} \approx \frac{e^2}{4\pi}\not{p}\ln\frac{p^2}{m^2} \tag{20}$$

Так как (20) есть поправка к \$\not\$10 (см. (39 Л10)), то для того чтобы ТВ была справедливой, должно быть

$$\frac{e^2}{4\pi}\ln\frac{p^2}{m^2} \ll 1 \tag{21}$$

Поскольку $e^2 = 1/137$, то условие (21) нарушается лишь при чрезвычайно больших импульсах.

Заметим, что массовый оператор калибровочно не инвариантен. В этом нет противоречий, поскольку массовый оператор не является непосредственно наблюдаемой величиной. В частности, в калибровке $\alpha = 3$ удовлетворение условиям нормировки (40 а Л10) не требует введения массы фотона

17.7.2 2. Polarization Operator

Можно существенно облегчить вычисления, если сначала найти мнимую часть диаграммы и затем восстановить всю диаграмму при помощи дисперсионного соотношения. Продемонстрируем этот метод на примере вычисления поляризационного оператора. В первом приближении ТВ он даётся петлёй в диаграмме

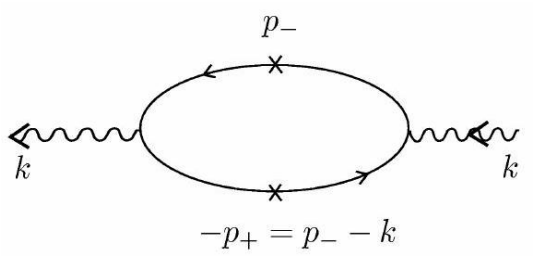


Рис. 24

Здесь внешняя линия виртуального фотона формально рассматривается как отвечающая реальной частице - векторному бозону массы $M^2 = k^2$, взаимодействующему с электрон-позитронным полем по тому же закону, что и фотон. При этом все соотношения условия унитарности остаются справедливыми. Диаграмма представляет диагональный элемент амплитуды iM_{ii}, дающей переход бозона самого в себя через распад на электрон-позитронную пару. Крестики на диаграмме показывают, по каким линиям она должна быть рассечена на две части так, чтобы дать реальное промежуточное состояние, фигурирующее при применении соотношения унитарности. Это состояние содержит электрон и позитрон с 4 -импульсами $p_- = p$ и $p_+ = (k - p)$, соответственно.

Для вычисления мнимой части M_{ii} при помощи условия унитарности в соотношении (34′5) в данном случае следует использовать меру

$$\sum_\lambda \int \frac{\mathrm{d}^3 p_-}{(2\pi)^3 2p_-^0} \frac{\mathrm{d}^3 p_+}{(2\pi)^3 2p_+^0}$$

причём далее $\sum\limits_\lambda$ означает сумму лишь по поляризациям промежуточных частиц, $p_-^2 = p_+^2 = m^2$ и в сумме по промежуточным состояниям 3 -вектора $\mathbf{p}_-$и $\mathbf{p}_+$считаются независимыми. Имеем:

$$2\,\mathrm{Im}\, M_{ii} = (2\pi)^4 \sum_\lambda \int \frac{\mathrm{d}^3 p_-}{(2\pi)^3 2p_-^0} \frac{\mathrm{d}^3 p_+}{(2\pi)^3 2p_+^0} \delta^{(4)}\left(k - p_- - p_+\right) |M_{\lambda i}|^2 \tag{22}$$

где амплитуда $M_{\lambda i}$ даётся частью диаграммы на рис. 24 правее крестиков:

$$iM_{\lambda i} = \bar{u}_{p_-} \left(-ie\gamma^\mu\right) u_{-p_+} \sqrt{4\pi} e_\mu \tag{23}$$

Введём параметр

$$t = k^2 = (p_- + p_+)^2 = 2\left(p_- p_+ + m^2\right)$$

В процессе вычисления будем предполагать, что $t > 4m^2$. Так как интеграл в (22) лоренцинвариантен, то вычислим его в системе отсчёта, где $\mathbf{k} = 0, k^0 = \sqrt{t}$. Сначала проинтегрируем по $\mathbf{p}_+$:

$$2\operatorname{Im} M_{ii} = \frac{1}{(4\pi)^2}\sum_\lambda \int \mathrm{d}o_{\mathbf{p}} \int\limits_0^\infty \frac{|\mathbf{p}|^2\ \mathrm{d}|\mathbf{p}|}{\left(p^0\right)^2}\delta\left(\sqrt{t} - 2p^0\right)|M_{\lambda i}|^2$$

Так как $|\mathbf{p}|\mathrm{d}|\mathbf{p}| = p^0\ \mathrm{d}p^0$, то

$$2\operatorname{Im} M_{ii} = \frac{1}{(4\pi)^2}\cdot\frac{|\mathbf{p}|}{2p^0}\int \mathrm{d}o\mathbf{p}\sum_\lambda |M_{\lambda i}|^2 \tag{24}$$

В (24) $p^0 = (1/2)\sqrt{t}, |\mathbf{p}| = (1/2)\sqrt{t-4m^2}$, d $o_{\mathbf{p}}$ - дифференциал телесного угла, внутри которого находится вектор $\mathbf{p} = \mathbf{p}_-$. С учётом (23)

$$\sum_\lambda |M_{\lambda i}|^2 = 4\pi e^2 e^*_\mu e_\nu \sum_\lambda \operatorname{tr}\gamma^\mu u_{-p_+}\bar{u}_{-p_+}\gamma^\nu u_{p_-}\bar{u}_{p_-} \tag{25}$$

В (25) $\sum\limits_\lambda$ означает независимое суммирование по поляризациям электрона и позитрона. Поэтому при помощи (56 Л8) находим:

$$\sum_\lambda |M_{\lambda i}|^2 = 4\pi e^2 e^*_\mu e_\nu \operatorname{tr}\gamma^\mu\left(\not{p}_+ - m\right)\gamma^\nu\left(\not{p}_- + m\right) = 16\pi e^2 e^*_\mu e_\nu\left[p^\mu_- p^\nu_+ + p^\nu_- p^\mu_+ - \left(p_+p_- + m^2\right)\eta^{\mu\nu}\right] \tag{26}$$

Здесь второе равенство получено при помощи формул (65 Л8) и (66 Л8). Так как $k = p_- + p_+$и

$$k_\mu\left[p^\mu_- p^\nu_+ + p^\nu_- p^\mu_+ - \left(p_+p_- + m^2\right)\eta^{\mu\nu}\right] \equiv 0$$

то

$$\int \mathrm{d}o_{\mathbf{p}}\left[p^\mu_- p^\nu_+ + p^\nu_- p^\mu_+ - \left(p_+p_- + m^2\right)\eta^{\mu\nu}\right] = 4\pi a\left(k^2\eta^{\mu\nu} - k^\mu k^\nu\right) \tag{27}$$

Коэффициент a в (27) находится путём вычисления свёртки по векторным индексам в (27):

$$a = -\frac{t + 2m^2}{3t} \tag{28}$$

С другой стороны,

$$iM_{ii} = \sqrt{4\pi}e^*_\mu\sqrt{4\pi}e_\mu\frac{i\mathcal{P}^{\mu\nu}}{4\pi} \tag{29}$$

Если считать вектора поляризации бозона вещественными, то сопоставляя формулы (24), (26), (27), (28) и (29), получаем:

$$2e_\mu e_\nu \operatorname{Im}\mathcal{P}^{\mu\nu} = -\frac{2e^2}{3} e_\mu e_\nu \sqrt{\frac{t-4m^2}{t}} \cdot \frac{t+2m^2}{t} \left(k^2\eta^{\mu\nu} - k^\mu k^\nu\right)$$

Теперь можно отбросить вектора поляризации бозона и использовать определение (18 Л10). Окончательно находим:

$$\operatorname{Im}\mathcal{P}(t) = -\frac{e^2}{3}\sqrt{\frac{t-4m^2}{t}} \cdot \frac{t+2m^2}{t} \tag{30}$$

Очевидно, $\operatorname{Im}\mathcal{P}(t) = 0$ при $t < 4m^2$. Поэтому по мнимой части (30) при помощи дисперсионного соотношения восстанавливается вся функция:

$$\mathcal{P}(t) = -\frac{e^2}{3\pi}\int\limits_{4m^2}^{\infty} \frac{\mathrm{d}t'}{t'-t-i\varepsilon}\sqrt{\frac{t'-4m^2}{t'}} \cdot \frac{t'+2m^2}{t'} \tag{31}$$

Этот интеграл является представлением Челлена-Лемана (см. (15 Л5)). Величина (30) является значением мнимой части поляризационного оператора на верхнем берегу разреза. Окончательное выражение для поляризационного оператора получается при учёте условия нормировки (40b Л10), согласно которому необходимо сделать замену $\mathcal{P}(t) \to \mathcal{P}(t) - \mathcal{P}(0)$:

$$\mathcal{P}(t) = -\frac{e^2}{3\pi}t\int\limits_{4m^2}^{\infty} \frac{\mathrm{d}t'}{t'-t-i\varepsilon}\sqrt{\frac{t'-4m^2}{t'}} \cdot \frac{t'+2m^2}{t'^2} \tag{32}$$

Будем считать, что $t < 0$. Тогда интеграл (32) берётся при помощи подстановки $t'/4m^2 = 1/\left(1-x^2\right), \quad 0 < x < 1$:

$$\mathcal{P}(t) = \frac{e^2}{3\pi}\left[-\frac{5}{3} + \frac{4\xi}{(1-\xi)^2} - \left(\xi^2 - 4\xi + 1\right)\frac{1+\xi}{(1-\xi)^3}\ln\xi\right] \tag{33}$$

Мы выразили $\mathcal{P}$ через переменную

$$t/m^2 = -(1-\xi)^2/\xi \tag{34}$$

Значение $\mathcal{P}(t)$ в предельном случае малых t легче всего ищется непосредственно из (32). Для этого следует положить под интегралом $t = 0$:

$$\mathcal{P}(t) = -\frac{e^2}{15\pi}\cdot\frac{t}{m^2}, \quad |t| \ll 4m^2 \tag{35}$$

Из (34) видно, что при стремлении переменной ξ к единице снизу t стремится к нулю. Чтобы продвинуться далее от $t = 0$ до $t = 4m^2$ следует положить $\xi = e^{i\varphi}$. Тогда согласно (34)

$$t = 4m^2\sin^2\frac{\varphi}{2}, \quad 0 \le \varphi \le \pi \tag{36}$$

В области $t > 4m^2$ переменная ξ заключена в интервале $-1 < \xi < 0$. Из (36) следует, что в этом интервале $\ln\xi = \ln|\xi| + i\pi$. Из (34) получаем также, что при больших $|t|$

$$t\xi = -m^2, \quad |t| \gg 4m^2 \tag{37}$$

Поэтому из (33) получаем асимптотику при больших $|t|$:

$$\mathcal{P}(t) = \begin{cases} \frac{e^2}{3\pi} \ln \frac{|t|}{m^2}, & \text{если } -t \gg 4m^2 \\ \frac{e^2}{3\pi} \left(\ln \frac{t}{m^2} - i\pi \right), & \text{если } t \gg 4m^2 \end{cases} \tag{38}$$

Мнимые части $\mathcal{P}(t)$, полученные при помощи (30) и (38), совпадают. Это подтверждает равенство $\ln \xi = \ln |\xi| + i\pi$ при $-1 < \xi < 0$.

По смыслу ТВ полученные формулы справедливы до тех пор пока $|\mathcal{P}| \ll 1$ (см. (20 Л10)). Поэтому ТВ справедлива до тех пор пока

$$\frac{e^2}{3\pi} \ln \frac{|t|}{m^2} \ll 1$$

Мы опять пришли к ограничению (21).

Константа перенормировки Z_3 получается, если в (20 Л10) подставить неперенормированный оператор (31) и устремить t к нулю. Обрезая интеграл в (31) масштабом $\Lambda \to \infty$, находим с точностью $O(e^2)$:

$$Z_3^{-1}(\Lambda) \equiv 1 - \mathcal{P}(t=0;\Lambda) \approx 1 + \frac{2e^2}{3\pi} \ln \frac{\Lambda}{m} \tag{39}$$

Теперь в соотношении (45 Л10) $e_0^2(\Lambda) = Z_3^{-1}(\Lambda) e^2$ сделаем подстановку $\Lambda \to M, e_0^2(\Lambda) \to e^2(M)$, где M - масштаб нормировки, $M \gg m$, а $e^2(M)$ - бегущая константа связи. Таким образом

$$e^2(M) = \left[1 + \frac{2e^2}{3\pi} \ln \frac{M}{m} \right] e^2$$

Отсюда с точностью $O(e^3)$ получаем:

$$M \frac{\mathrm{d}e(M)}{\mathrm{d}M} \equiv \beta(e(M)) = \frac{e^3(M)}{3\pi} \tag{40}$$

Решая это дифференциальное уравнение, находим:

$$e^2(M) = \frac{e^2}{1 - \frac{e^2}{3\pi} \ln \frac{M^2}{m^2}} \tag{41}$$

Формула (41) показывает, что эффективная константа связи растёт при продвижении в ультрафиолетовую область. Это свойство является общим для скалярной теории и КЭД.

17.7.3 3. Radiative Corrections to Coulomb's Law

Рассмотрим квантовые поправки к кулоновскому полю неподвижного сверхмассивного точечного заряда e_1, потенциал которого (без учёта поправок)

$$\varphi_0(r) \equiv A_0^{(e)}(r) = \frac{e_1}{r}, \quad \mathbf{A}^{(e)} = 0 \tag{42}$$

В импульсном пространстве

$$\varphi_0(\mathbf{k}) = \frac{4\pi e_1}{\mathbf{k}^2} \tag{43}$$

С учётом квантовых поправок потенциал (42) заменяется на

$$\varphi = A_0^{(e)} + \mathcal{D}_{0\mu}\frac{\mathcal{P}^{\mu\nu}}{4\pi}A_\nu^{(e)} \tag{44}$$

Получим первую поправку. Для этого поляризационный оператор возьмём в (32) и заменим точный фотонный пропагатор на свободный. С учётом того что $k^0 = 0$ получаем:

$$\varphi(\mathbf{k}) = \varphi_0(\mathbf{k}) + \mathcal{P}\left(-\mathbf{k}^2\right)\varphi_0(\mathbf{k}) \tag{45}$$

Отсюда видно, что возмущение потенциала есть

$$\delta\varphi(\mathbf{k}) = \frac{4\pi e_1}{\mathbf{k}^2}\mathcal{P}\left(-\mathbf{k}^2\right) = \frac{4\pi e_1}{y^2}\mathcal{P}\left(-y^2\right) \equiv \delta\varphi\left(-y^2\right)$$
$$\delta\varphi(r) = \int e^{i\mathbf{kr}}\delta\varphi(\mathbf{k})\frac{\mathrm{d}^{(3)}k}{(2\pi)^3} = \frac{1}{2\pi^2}\int\limits_0^\infty \delta\varphi\left(-y^2\right)\frac{\sin yr}{r}y\ \mathrm{d}y \tag{46}$$

Так как подынтегральное выражение чётно, то

$$\delta\varphi(r) = \frac{1}{4\pi^2 r}\operatorname{Im}\int\limits_{-\infty}^{+\infty}\delta\varphi\left(-y^2\right)e^{iry}y\ \mathrm{d}y \tag{47}$$

Теперь сместим контур интегрирования с вещественной оси переменной y в верхнюю полуплоскость так, показано на рис. 25.

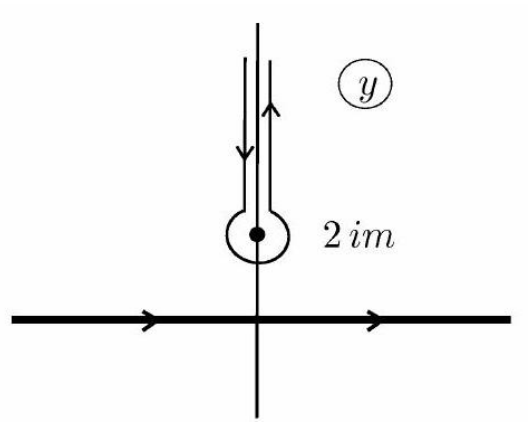


Рис. 25

На рис. 25 $\operatorname{Im}\mathcal{P}$ на левом берегу разреза задаётся при помощи (30). Положим $y = ix$. Тогда (47) переписывается как

$$\delta\varphi(r) = \frac{e_1}{r}\left[-\frac{2}{\pi}\int\limits_{2m}^\infty \operatorname{Im}\mathcal{P}\left(t = x^2\right)\frac{e^{-rx}}{x}\ \mathrm{d}x\right]$$

Теперь выпишем потенциал (45):

$$\varphi(r) = \frac{e_1}{r} + \delta\varphi(r) = \frac{e_1}{r}\left[1 + \frac{2e^2}{3\pi}\int\limits_1^\infty e^{-2mr\zeta}\left(1 + \frac{1}{2\zeta^2}\right)\frac{\sqrt{\zeta^2-1}}{\zeta^2}\ \mathrm{d}\zeta\right] \tag{48}$$

Оценим $\varphi(r)$ в двух предельных случаях.

а) $mr \gg 1$.

Интеграл (48) легко оценивается при помощи подстановки $\zeta = 1 + \xi$:

$$\varphi(r) = \frac{e_1}{r}\left[1 + \frac{e^2}{4\sqrt{\pi}} \cdot \frac{e^{-2mr}}{(mr)^{3/2}}\right], \quad r \gg \frac{1}{m} \tag{49}$$

б) $mr \ll 1$.

Опуская вычисления, приводим ответ (см. Л. Д. Ландау, Е. М. Лифшиц. Теоретическая физика. Т. 4, §114):

$$\varphi(r) = \frac{e_1}{r}\left[1 + \frac{2e^2}{3\pi}\left(\ln\frac{1}{mr} - C - \frac{5}{6}\right)\right], \quad r \ll \frac{1}{m} \tag{50}$$

где $C = 0,577...$ - постоянная Эйлера.

Мы видим, что поляризация вакуума искажает кулоново поле точечного заряда в области $r \lesssim 1/m (= \hbar/mc)$, где m - масса электрона. Вне этой области искажение поля убывает по экспоненциальному закону.

Задачи

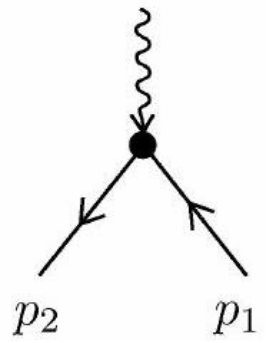
1. Вычислить в однопетлевом приближении константу перенормировки дираковского поля Z_2.

2*. Установить путём прямого вычисления что в калибровке $\alpha = 3$ для удовлетворения массовым оператором условий нормировки ($40a\Omega 10$) не требуется введение массы фотона.

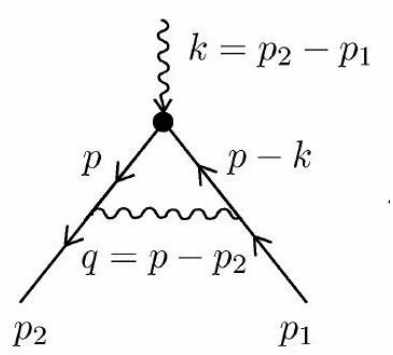
17.8 L12 Vertex Operator and Anomalous Magnetic Moment of the Electron

17.8.1 1. Electron Form Factors

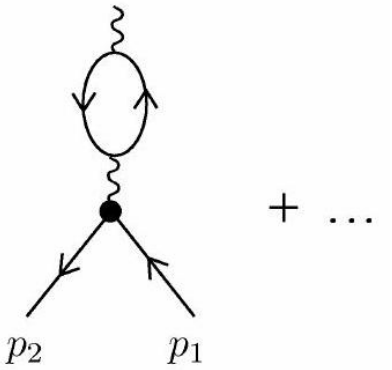
Рассмотрим вершинный оператор $\Gamma^\mu(p_2, p_1)$ в случае, когда две электронные линии являются внешними, а фотонная - внутренней с 4 -импульсом $k = p_2 - p_1$ (если рассматривать её как входящую). Электронным внешним линиям отвечают множители $u_1 = u_{p_1}$ и $\bar{u}_2 = \bar{u}_{p_2}$, так что Γ входит в выражение для диаграмм в виде $J^\mu_{fi} = \bar{u}_2 \Gamma^\mu u_1$. Амплитуда перехода, описывающая рас-



(a)



(b)



(c)
Рис. 26

сеяние электрона во внешнем поле в первом порядке по этому полю представляется как

$$M_{fi} = -eJ^{\mu}_{fi}\mathcal{A}^{(e)}_{\mu}(k) \tag{1}$$

где $\mathcal{A}^{(e)}_{\mu}$ - эффективное (с учётом поляризации вакуума) внешнее поле. Однопетлевой вклад в эффективное внешнее поле задаётся диаграммой (c) на рис. 26. Поскольку эта амплитуда является скаляром (а не псевдоскаляром), а $\mathcal{A}^{(e)}_{\mu}$ - вектором, то величина J^{μ}_{fi} является вектором во всех приближениях и она может быть выражена лишь через истинные векторы, которые могут быть составлены из имеющихся в нашем распоряжении 4 -векторов p_1 и $p_2\,(k = p_2 - p_1)$ и биспиноров u_1 и u_2. Всего имеется три таких вектора:

$$\bar{u}_2\gamma^{\mu}u_1, \quad (\bar{u}_2u_1)\,P^{\mu}, \quad (\bar{u}_2u_1)\,k^{\mu} \tag{2}$$

где $P^{\mu} = p^{\mu}_1 + p^{\mu}_2$. Вследствие тождества Уорда (10 Л10), формул (39 Л10), (40 а Л10) и равенства (38 а Л8) имеем:

$$k_{\mu}J^{\mu}_{fi} = 0 \tag{3}$$

Так как 4 -импульсы p_1 и p_2 находятся на массовой поверхности, то $k_{\mu}P^{\mu} = 0$, но $k_{\mu}k^{\mu} \neq 0$. Кроме того, вследствие (38 а Л8) $k_{\mu}\bar{u}_2\gamma^{\mu}u_1 = 0$. Поэтому

$$J^{\mu}_{fi} = f_1 \cdot (\bar{u}_2u_1)\,P^{\mu} + f_2 \cdot (\bar{u}_2\gamma^{\mu}u_1) \tag{4}$$

где f_1 и f_2 - инварианты, которые называются электромагнитными формфакторами электрона. Из двух векторов p_1 и $p_2\,(p_1^2 = p_2^2 = m^2)$ можно составить лишь одну инвариантную величину, в качестве которой возьмём k^2. Формфакторы являются функциями лишь k^2.

При помощи формул (5 Л8), (10 Л8) и (38 Л8) получаем:

$$(\bar{u}_2\sigma^{\mu\nu}u_1)\,k_\nu = -m\,(\bar{u}_2\gamma^\mu u_1) + \frac{1}{2}\,(\bar{u}_2 u_1)\,P^\mu \tag{5}$$

Поэтому представление (4) может быть переписано как

$$J^\mu_{fi} = f\left(k^2\right)(\bar{u}_2\gamma^\mu u_1) - \frac{1}{m}g\left(k^2\right)(\bar{u}_2\sigma^{\mu\nu}u_1)\,k_\nu \tag{6}$$

где f и g - другие формфакторы.

Член в амплитуде рассеяния (1), пропорциональный $g\,(k^2)$, имеет вид

$$\delta M_{fi} = \frac{e}{m}g\left(k^2\right)(\bar{u}_2\sigma^{\mu\nu}u_1)\,k_\nu\mathcal{A}^{(e)}_\mu \tag{7}$$

Рассмотрим чисто магнитное поле $\mathcal{A}^{(0)\mu} = (0, \mathbf{A})$ и пусть $k^\mu = (0, \mathbf{k}), \mathbf{k} \to 0$. Это значит, что поле не зависит от времени и оно медленно изменяется в пространстве. Поэтому далее в (7) можно заменить "одетое"поле $\mathcal{A}^{(e)}_\mu$ на "голое"поле $A^{(e)}_\mu 1$. Перепишем (7) с учётом того, что в данном случае имеются лишь пространственно-пространственные компоненты тензора электромагнитного поля и $H^i(\mathbf{k}) = i\varepsilon_{ijk}k_jA^{(e)}_k(\mathbf{k})$:

$$\delta M_{fi} = \frac{e}{m}g(0)\bar{u}_2\left(\frac{i}{2}\varepsilon_{ijk}\sigma^{jk}\right)u_1H^i$$

Такой же вклад в амплитуду даст добавка к гамильтониану

$$\delta\mathcal{H} = \int \mathrm{d}^{(3)}x\bar{\psi}\left[-\frac{e}{m}g(0)\left(\frac{i}{2}\varepsilon_{ijk}\sigma^{jk}\right)H^i\right]\psi \longrightarrow \int \mathrm{d}^{(3)}x\varphi^\dagger\left[-\frac{e}{m}g(0)\mathbf{sH}\right]\varphi \tag{8}$$

Последнее выражение в (8) получается из предыдущего в нерелятивистском пределе в стандартном представлении. Сравнение (8) с уравнениями (28 Л8) и (30 Л8) показывает, что радиационные поправки дают дополнительный вклад в магнитный момент электрона, так что его полный магнитный момент (в обычных единицах

$$\mu = \frac{e\hbar}{2mc}(1 + g(0)) \tag{9}$$

Очевидно, что первое слагаемое в (6) не может дать вклад в аномальный магнитный момент. Действительно, при $k \to 0$ для первого слагаемого в (6) имеем $f\,(k^2)\,(\bar{u}_2\gamma^\mu u_1) \to 2f(0)p^\mu$, где p^μ - импульс электрона.

17.8.2 2. Analytical Properties of Form Factors

Амплитуда (1) описывает два канала реакции. В канале рассеяния переменная

$$t \equiv k^2 = (p_2 - p_1)^2 \leq 0 \tag{10}$$

Путём замены $p_2 \to p_-, p_1 \to -p_+$мы приходим к аннигиляционному каналу, отвечающему рождению электрон-позитронной пары с 4 -импульсами p_-и p_+. В этом канале

$$t = (p_- + p_+)^2 \geq 4m^2 \tag{11}$$

Область значений $0 < t < 4m^2$ - нефизическая.

Применим условие унитарности (34 Л5). В канале рассеяния ($t < 0$) нет физических промежуточных состояний: один свободный электрон не может изменить свой импульс или родить какие либо другие реальные (не виртуальные) частицы; в нефизической области их также нет. Поэтому при $t < 4m^2$ правая сторона равенства (34 Л5) отсутствует, так что

$$M_{fi} = M_{if}^* \tag{12}$$

Равенство (12) справедливо, конечно, во всех приближениях. Перестановка начального и конечного состояний означает перестановку p_1 и p_2, а тем самым замену $k \to -k$. Из соотношения (12) для выражения (1) вытекает, что

$$J_{fi}^\mu \mathcal{A}_\mu^{(e)}(k) = J_{if}^{\mu *} \mathcal{A}_\mu^{(e)*}(-k)$$

Но вследствие вещественности электромагнитного поля $\mathcal{A}_\mu^{(e)}(k) = \mathcal{A}_\mu^{(e)*}(-k)$, и потому

$$J_{fi}^\mu = J_{if}^{\mu *}, \quad t < 4m^2 \tag{13}$$

Легко проверить, что (см. (8 Л8) и (11 Л8))

$$(\bar{u}_2 \gamma^\mu u_1) = (\bar{u}_1 \gamma^\mu u_2)^*, \quad (\bar{u}_2 \sigma^{\mu\nu} u_1) = -(\bar{u}_1 \sigma^{\mu\nu} u_2)^*$$

Поэтому f_{if}^* отличается от f_{fi} лишь заменой функций $f(t)$ и $g(t)$ их комплексно сопряжёнными. Тогда из (13) следует, что

$$\operatorname{Im} f(t) = \operatorname{Im} g(t) = 0, \quad t < 4m^2 \tag{14}$$

Из диаграмм (b) и (c) на рис. 26 видно, что в аннигиляционном канале ($t > 4m^2$) уже в однопетлевом приближении существуют реальные промежуточные частицы. Поэтому правая часть условия унитарности для амплитуды (1) отлична от нуля и формфакторы f и g комплексны. Аналитические свойства функций $f(t)$ и $g(t)$ аналогичны свойствам функций $\mathcal{P}(t)$. Эти функции аналитичны в комплексной плоскости переменной t, разрезанной вдоль положительной вещественной оси $t > 4m^2$, причём

$$f^*(t) = f(t^*), \quad g^*(t) = g(t^*) \tag{15}$$

Условие перенормируемости (46 Л10), применённое к (6), даёт

$$f(0) = 1 \tag{16}$$

Это условие учитывается автоматически, если в дисперсионном соотношении использовать не $f(t)$, а $(f(t) - 1)/t$. Таким образом

$$f(t) - 1 = \frac{t}{\pi} \int\limits_{4m^2}^{\infty} \frac{\operatorname{Im} f(t') \, \mathrm{d}t'}{t' (t' - t - i\varepsilon)} \tag{17}$$

[6] 1 1 Действительно, в конце лекции 11 было показано, что (в однопетлевом приближении) слабое медленно меняющееся "одетое"кулоново поле фактически совпадает с "голым". Это же утверждение во всех порядках ТВ имеет место в отношении любого слабого и медленно изменяющегося электромагнитного поля.

Для формфактора $g(t)$ условие перенормировки (46 Л10) не даёт ограничений аналогичная формула имеет вид

$$g(t) = \frac{1}{\pi} \int\limits_{4m^2}^{\infty} \frac{\operatorname{Im} g\left(t'\right) \mathrm{d}t'}{t' - t - i\varepsilon} \tag{18}$$

В (17), (18) мнимые части функций, как и обычно, берутся на верхних берегах разреза.

Рассмотрение, проведенное до сих пор, справедливо во всех порядках ТВ.

17.8.3 3. Calculation of Imaginary Parts Using the Feynman Integral

Выпишем амплитуду, соответствующую диаграмме (b) на рис. 26 в аннигиляционном канале $(p_2 \to p_-, p_1 \to -p_+, q \to p - p_-, k = p_- + p_+)$:

$$-ie\bar{u}_{p_-}\Gamma^{\mu}u_{-p_+} = (-ie)^3\bar{u}_{p_-}\gamma^{\nu}i\int \frac{\mathrm{d}^{(4)}p}{(2\pi)^4}S_c(p)\gamma^{\mu}S_c(p-k)\gamma^{\lambda}u_{-p_+}D^{\alpha}_{\lambda\nu}\left(p-p_-\right) \tag{19}$$

Далее обкладки $\bar{u}_{p_-}...u_{-p_+}$во всех формулах опускаются для сокращения записи, но равенства $\bar{u}_{p_-}\not{p}_- = m\bar{u}_{p_-}$и $\not{p}_+u_{-p_+} = -mu_{-p_+}$используются в вычислениях. В калибровке Фейнмана $\alpha = 1$ имеем:

$$f\left(k^2\right)\gamma^{\mu} - \frac{1}{m}g\left(k^2\right)\sigma^{\mu\nu}k_{\nu} = \int \frac{i\varphi^{\mu}\left(p^0,\mathbf{p}\right)\mathrm{d}^{(4)}p}{\left(p^2-m^2\right)\left[(p-k)^2-m^2\right]} \tag{20}$$

где

$$\varphi^{\mu}\left(p^0,\mathbf{p}\right) = -\frac{e^2}{4\pi^3}\cdot\frac{\gamma^{\nu}(\not{p}+m)\gamma^{\mu}(\not{p}-\not{k}+m)\gamma_{\nu}}{\left[\left(p-p_-\right)^2-\lambda^2\right]} \tag{21}$$

Здесь опять введена бесконечно малая масса фотона λ в фотонный пропагатор.

Разрез проходит в t-плоскости при $t > 4m^2$. При таких значениях $t = k^2$ существует система отсчёта K_0, в которой $k = (k^0, \mathbf{0})$. В системе K_0

$$(p-k)^2 = \left(p^0-k^0\right)^2 - \mathbf{p}^2, \quad p^0_- = p^0_+ = \frac{1}{2}k^0 = \frac{\sqrt{t}}{2}, \quad \mathbf{p}_- = -\mathbf{p}_+, \quad |\mathbf{p}_-| = \frac{1}{2}\sqrt{t-4m^2} \tag{22}$$

Введём обозначение $\omega = +\sqrt{|\mathbf{p}|^2+m^2}$. Тогда

$$\mathrm{d}^4p = |\mathbf{p}|\omega\mathrm{d}p^o\ \mathrm{d}\omega\mathrm{d}o_{\mathbf{p}} \tag{23}$$

В системе отсчёта K_0 знаменатель в интеграле (20) имеет полюса по p^0 в точках

$$\begin{array}{ll} \text{a)}\ p_0 = \omega - i\varepsilon, & a')\ p_0 = -\omega + i\varepsilon \\ \text{b)}\ p_0 = k_0 - \omega + i\varepsilon, & b')\ p_0 = k_0 + \omega - i\varepsilon \end{array} \tag{24}$$

Для определённости считаем $k^0 > 0$, что отвечает процессу рождения пары (ответ не зависит от этого предположения, поскольку является функцией $\left(k^0\right)^2$). Ясно, что лишь полюса a) и b) могут совпасть при интегрировании в (20), что происходит при $\omega = (1/2)k^0$ (это возможно лишь при $k^0 > 2m$). На рис. 27 показано расположение полюсов в комплексной плоскости p^0.

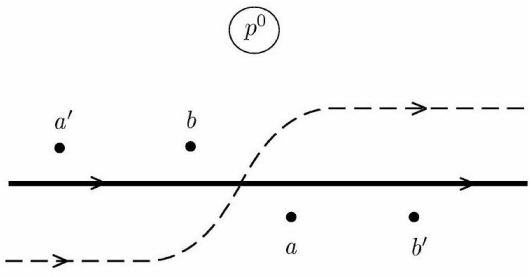


Рис. 27

При замене в интеграле (20) $k^2 \to k^2 - i\delta, \delta \to +0$, интеграл изменяется скачкообразно, поскольку при такой замене мы попадаем на нижний берег разреза. Однако из рис. 27 видно, что если полюса a) и b) не совпадают, то при указанной замене параметра скачка быть не может, поскольку контур интегрирования может быть плавно изменён (на рис. 27 новый контур интегрирования изображён пунктиром). Поэтому бесконечно малое смещение полюсов не переводит их на другую сторону контура интегрирования. Отсюда видно, что мнимая часть (или скачок интеграла) появляется лишь при условии совпадения полюсов a) и b). Такое совпадение происходит лишь при одновременном обращении в нуль обоих сомножителей в знаменателе интеграла (20). Это значит, что для нахождения скачка в интеграле (20) следует вставить под интеграл вместо знаменателя множитель $\delta\left(p^2 - m^2\right)\delta\left[(p-k)^2 - m^2\right]$. Для вычисления коэффициента при этом множителе заменим исходный контур интегрирования C на два других C' и C'' так, как показано на рис. 28.

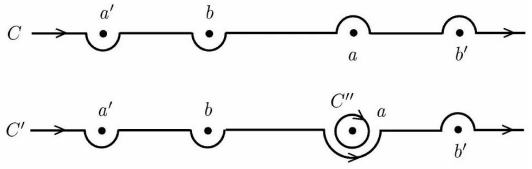


Рис. 28

Интеграл по контуру C' уводится от полюсов a) и b) и потому не даёт скачка. Интеграл по контуру C'' легко вычисляется взятием вычета, что эквивалентно замене под интегралом (20)

$$\frac{1}{p^2 - m^2} \to (-2\pi i)\delta\left(p^2 - m^2\right) \tag{25}$$

Знак "-" в (25) имеет место вследствие отрицательной ориентации контура C''. Теперь очевидно, что разность значений интеграла (20) на верхнем и нижнем берегах разреза даётся величиной

$$\int\limits_{p_0>0} \mathrm{d}^{(4)}pi\varphi^\mu\left(p^0, \mathbf{p}\right)\left[-2\pi i\delta\left(p^2 - m^2\right)\right]\left[\frac{1}{(p-k)^2 - m^2 + i\delta} - \frac{1}{(p-k)^2 - m^2 - i\delta}\right] =$$

$$= \int\limits_{p_0>0} \mathrm{d}^{(4)}pi\varphi^\mu\left(p^0, \mathbf{p}\right)\left[-2\pi i\delta\left(p^2 - m^2\right)\right]\left[-2\pi i\delta\left((p-k)^2 - m^2\right)\right] \tag{26}$$

Можно доказать общее правило Каткосского для отыскания скачка диаграмм:

[61] Совпадение полюсов внутренних фотонной и одной из фермионных линий невозможно, поскольку вследствие сохранения 4 -импульса электрон не может излучить частицу.

1. разобьём диаграмму на две всеми возможными способами путём разрезания её пропагаторов;

2. для каждого разрезанного пропагатора частицы массы m делаем замену $(p^2-m^2+i\varepsilon)^{-1}\to(-2\pi i)\delta(p^2-m^2)$, затем вычисляем петлевые интегралы;

3. суммируем вклады от всех возможных разрезаний.

В результате интегрирования в системе K_0 δ-функции в (26) легко снимаются. При помощи (22) и (23) и (26) находим для скачка интеграла (20):

$$\mathrm{Disc}\int\frac{i\varphi^\mu \mathrm{d}^4p}{(p^2-m^2)\left[(p-k)^2-m^2\right]}=\int\limits_{p_0>0}|\mathbf{p}|\omega\mathrm{d}p^o\ \mathrm{d}\omega\mathrm{d}o_{\mathbf{p}}i\varphi^\mu\times$$

$$\times\left[-2\pi i\delta\left(p_0^2-\omega^2\right)\right]\left[-2\pi i\delta\left(\left(p_0-k_0\right)^2-\omega^2\right)\right]=-i\frac{\pi^2}{2}\sqrt{\frac{t-4m^2}{t}}\int\mathrm{d}o_{\mathbf{p}}\varphi^\mu\left(\frac{\sqrt{t}}{2},\mathbf{p}\right)\tag{27}$$

$$p^0=\frac{\sqrt{t}}{2},\quad |\mathbf{p}|=\frac{1}{2}\sqrt{t-4m^2}\tag{28}$$

Вычислим интеграл в (27) в системе центра инерции K_0, в которой

$$p_-=\left(\frac{\sqrt{t}}{2},0,0,\frac{\sqrt{t-4m^2}}{2}\right),\quad p=\left(\frac{\sqrt{t}}{2},\mathbf{p}\right),\quad \mathbf{p}=\frac{\sqrt{t-4m^2}}{2}(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)\tag{29}$$

При помощи формул (62 Л8), (22), (28) и (29) величина (21) переписывается в виде

$$\varphi^\mu=\frac{e^2}{\pi^3\rho}\left(-2p^\mu\not{p}+\not{k}\gamma^\mu\not{p}+4mp^\mu-2mk^\mu\right),\quad \rho=\left(t-4m^2\right)(1-\cos\theta)+2\lambda^2\tag{30}$$

Введём обозначения

$$\varkappa^\mu=\int\mathrm{d}o_{\mathbf{p}}\frac{p^\mu}{\rho},\quad \varkappa^{\mu\nu}=\int\mathrm{d}o_{\mathbf{p}}\frac{p^\mu p^\nu}{\rho}\tag{31}$$

Приведём ненулевые значения компонент этих величин:

$$\begin{aligned}\varkappa^0&=\tfrac{\pi\sqrt{t}}{t-4m^2}\ln\tfrac{t-4m^2}{\lambda^2},&\qquad \varkappa^3&=\tfrac{\pi}{\sqrt{t-4m^2}}\left[-2+\ln\tfrac{t-4m^2}{\lambda^2}\right]\\ \varkappa^{00}&=\tfrac{\pi t}{2(t-4m^2)}\ln\tfrac{t-4m^2}{\lambda^2},&\qquad \varkappa^{03}&=\tfrac{\pi}{2}\sqrt{\tfrac{t}{t-4m^2}}\left[-2+\ln\tfrac{t-4m^2}{\lambda^2}\right]\end{aligned}\tag{32}$$

$$\varkappa^{33}=\tfrac{\pi}{2}\left[-2+\ln\tfrac{t-4m^2}{\lambda^2}\right],\quad \varkappa^{\alpha\beta}=\tfrac{\pi}{2}\delta^{\alpha\beta},\quad \alpha,\beta=1,2$$

Из (22), (28) и (30) следует, что

$$\varphi^0=0\tag{33}$$

Далее в этом пункте обозначаем символом $\approx$ те равенства, которые имеют место лишь в обкладках $\bar u_{p_-}\ldots u_{-p_+}$. Например,

$$\gamma^0\approx 0,\quad \not{k}\gamma^3\not{p}_-=-\not{p}_-\not{k}\gamma^3\approx m\gamma^3\not{k}\tag{34}$$

При помощи формул (32) и (34) находим:

$$\int \mathrm{d}o_{\mathbf{p}}\varphi^3 \approx \frac{2e^2}{\pi^2}\left\{\frac{m}{\sqrt{t-4m^2}}\left[-2+\ln\frac{t-4m^2}{\lambda^2}\right]+\frac{m}{t-4m^2}\left(\ln\frac{t-4m^2}{\lambda^2}\right)\gamma^3\not{K}\right\} \tag{35}$$

Теперь воспользуемся формулой (5), которая в аннигиляционном канале имеет вид

$$\sigma^{\mu\nu}k_\nu \approx -m\gamma^\mu+\frac{1}{2}\left(p_-^\mu-p_+^\mu\right) \tag{36}$$

При $\mu=3$ с учётом (22) отсюда следует, что

$$\frac{1}{2}\sqrt{t-4m^2}\approx\frac{1}{2}\gamma^3\not{K}+m\gamma^3 \tag{37}$$

и (35) переписывается как

$$\int \mathrm{d}o_{\mathbf{p}}\varphi^3 = \frac{2e^2}{\pi^2}\left\{\frac{2m}{t-4m^2}\left(-1+\ln\frac{t-4m^2}{\lambda^2}\right)\gamma^3\not{K}+\frac{2m^2}{t-4m^2}\left(-2+\ln\frac{t-4m^2}{\lambda^2}\right)\gamma^3\right\} \tag{38}$$

Так как $\gamma^0\approx 0$, то (см. (34), (22) и (29)):

$$m\gamma^3\not{K}\approx -\not{p}_-\not{K}\gamma^3\approx-\frac{1}{2}t\gamma^3 \tag{39}$$

Поэтому наиболее общий вид левой части в равенстве (38), допустимый структурой (20), таков:

$$\begin{aligned}\int \mathrm{d}o_{\mathbf{p}}\varphi^3 = \frac{e^2}{\pi^2\left(t-4m^2\right)}\Bigg\{&4m^2\left(-2+\ln\frac{t-4m^2}{\lambda^2}\right)\gamma^3+\\ &+4m\left(-1+\ln\frac{t-4m^2}{\lambda^2}\right)\gamma^3\not{K}+x\left(\frac{1}{2}t\gamma^3+m\gamma^3\not{K}\right)\Bigg\}\end{aligned} \tag{40}$$

где x - некое вещественное число, которое будет определено ниже.

Теперь займёмся вычислением величин $\left(\int \mathrm{d}o_{\mathbf{p}}\varphi^\alpha\right)$ с $\alpha=1,2$. При помощи формул (30) и (32) находим:

$$\int \mathrm{d}o_{\mathbf{p}}\varphi^\alpha=\frac{e^2}{\pi^2}\left\{\gamma^\alpha+\frac{2}{t-4m^2}\left(\ln\frac{t-4m^2}{\lambda^2}\right)\not{K}\gamma^\alpha\not{p}_-+\frac{2}{\sqrt{t-4m^2}}\not{K}\gamma^\alpha\gamma^3\right\} \tag{41}$$

Теперь учтём, что

$$\not{p}_-=\not{K}-\not{p}_+,\quad \gamma^3=\frac{2\not{p}_+-\not{K}}{\sqrt{t-4m^2}} \tag{42}$$

Поэтому

$$\int \mathrm{d}o_{\mathbf{p}}\varphi^\alpha\approx\frac{e^2}{\pi^2\left(t-4m^2\right)}\left\{\left[\left(3t-4m^2\right)-2t\left(\ln\frac{t-4m^2}{\lambda^2}\right)\right]\gamma^\alpha+\left[4m-2m\left(\ln\frac{t-4m^2}{\lambda^2}\right)\right]\gamma^\alpha\not{K}\right\} \tag{43}$$

В случае $\mu=1,2$ равенство (36) принимает вид

$$\frac{1}{2}\gamma^\alpha \not{k} \approx -m\gamma^\alpha \tag{44}$$

Поэтому имеем наиболее общий вид левой части равенства (43), совместимый со структурой (20):

$$\int \mathrm{d}o_{\mathbf{p}}\varphi^\alpha = \frac{e^2}{\pi^2\left(t-4m^2\right)}\left\{\left(3t-4m^2-2t\ln\frac{t-4m^2}{\lambda^2}\right)\gamma^\alpha + \right.$$
$$\left. + \left(4m-2m\ln\frac{t-4m^2}{\lambda^2}\right)\gamma^\alpha \not{k} + y\left(m\gamma^\alpha + \frac{1}{2}\gamma^\alpha \not{k}\right)\right\} \tag{45}$$

Здесь y - некое вещественное число. Чтобы все выражения (33), (40) и (45) собирались в выражение вида (6), коэффициенты при γ^3 и $\gamma^3\not{k}$ в (40) должны быть равны коэффициентам при γ^α и $\gamma^\alpha \not{1}$ в (45), соответственно. Это требование однозначно определяет числа x и y:

$$x = 6 - 4\ln\frac{t-4m^2}{\lambda^2}, \quad y = -4m + 4m\ln\frac{t-4m^2}{\lambda^2}$$

Подставляя эти числа в соответствующие величины, окончательно находим:

$$\int \mathrm{d}o_{\mathbf{p}}\varphi^\mu \approx -\frac{e^2}{\pi^2\left(t-4m^2\right)}\left\{\left[-3t+8m^2+2\left(t-2m^2\right)\ln\frac{t-4m^2}{\lambda^2}\right]\gamma^\mu - 4m\sigma^{\mu\nu}k_\nu\right\} \tag{46}$$

Теперь, сопоставляя формулы (20), (27) и (46), находим мнимые части формфакторов:

$$\mathrm{Im}\, g(t) = \frac{e^2m^2}{\sqrt{t\left(t-4m^2\right)}}$$
$$\mathrm{Im}\, f(t) = \frac{e^2}{4\sqrt{t\left(t-4m^2\right)}}\left[-3t+8m^2+2\left(t-2m^2\right)\ln\frac{t-4m^2}{\lambda^2}\right] \tag{47}$$

Инфракрасная расходимость имеется только в $f(t)$.

Формфакторы могут быть вычислены точно по их мнимым частям при помощи уже использовавшейся подстановки $t' = 4m^2/\left(1-x^2\right)$ (см. Л. Д. Ландау, Е. М. Лифшиц. Теоретическая физика. Т. 4, §117). Однако мы ограничимся их асимптотическими значениями.

Согласно (18) и (47)

$$g(0) = \frac{e^2m^2}{\pi}\int\limits_{4m^2}^{\infty}\frac{\mathrm{d}t}{t\sqrt{t\left(t-4m^2\right)}} = \frac{e^2}{2\pi} \tag{48}$$

При $|t| \ll 4m^2$ можно пренебречь параметром t под знаком интеграла в (17):

$$f(t)-1 = \frac{e^2t}{8\pi m^2}\int\limits_0^1 \mathrm{d}x\left[-1-2x^2+\left(1+x^2\right)\ln\frac{4x^2}{1-x^2}+\left(1+x^2\right)\ln\frac{m^2}{\lambda^2}\right] = \frac{e^2t}{3\pi m^2}\left(\ln\frac{m}{\lambda}-\frac{3}{8}\right)$$

Поэтому

$$f(t)-1=\frac{e^2t}{3\pi m^2}\left(\ln\frac{m}{\lambda}-\frac{3}{8}\right),\quad g(t)=\frac{e^2}{2\pi},\quad |t|\ll 4m^2 \tag{49}$$

При $t'\gg 4m^2$ из (47) получаем, что $\operatorname{Im} f\left(t'\right)=(e^2/2)\ln\left(t'/\lambda^2\right)$. Оценим сначала $f(t)$ при $-t\gg 4m^2$. Делая соответствующие пренебрежения, сводим интеграл в (17) к интегралу

$$f(t)-1=\frac{e^2t}{2\pi}\int\limits_{4m^2}^{\infty}\frac{\ln\frac{t'}{\lambda^2}\,\mathrm{d}t'}{t'\left(t'+|t|\right)}$$

который легко оценивается при помощи подстановки $t'=4m^2e^x, 0<x<\infty$:

$$f(t)-1=\frac{e^2t}{2\pi}\int\limits_{0}^{\infty}\frac{\left(x+\ln\frac{4m^2}{\lambda^2}\right)\mathrm{d}x}{\left(|t|+4m^2e^x\right)}$$

Знаменатель в подынтегральном выражении почти равен $|t|$ при $x<x_0$ и быстро падает при $x>x_0$, где $x_0=\ln\left[|t|/4m^2\right]$. Поэтому при $-t\gg 4m^2$

$$f(t)-1=-\frac{e^2}{2\pi}\left(\frac{1}{2}\ln^2\frac{|t|}{m^2}+2\ln\frac{m}{\lambda}\ln\frac{|t|}{m^2}\right) \tag{50}$$

При больших положительных $|t|$ этот формфактор приобретает мнимую часть, а вещественная остаётся прежней. Легко сделать аналогичную оценку для $g(t)$. Таким образом

$$f(t)-1=-\frac{e^2}{2\pi}\left(\frac{1}{2}\ln^2\frac{|t|}{m^2}+2\ln\frac{m}{\lambda}\ln\frac{|t|}{m^2}\right)+\begin{cases} i\frac{e^2}{2}\ln\frac{|t|}{\lambda^2}, & \text{если } t\gg 4m^2\\ 0, & \text{если } -t\gg 4m^2\end{cases} \tag{51}$$

$$g(t)=-\frac{e^2m^2}{\pi t}\ln\frac{|t|}{m^2}+\begin{cases} i\frac{e^2m^2}{t}, & \text{если } t\gg 4m^2\\ 0, & \text{если } -t\gg 4m^2\end{cases} \tag{52}$$

17.8.4 4. Anomalous Magnetic Moment of the Electron

Согласно (9) и (48) в однопетлевом приближении магнитный момент электрона

$$\mu=\frac{e\hbar}{2mc}\left(1+\frac{e^2}{2\pi\hbar c}\right) \tag{53}$$

Эта формула впервые была получена Ю. Швингером в 1949 году.

17.9 L13 Infrared Problem. Lamb Shift of Energy Levels

Мы видели, что некоторые величины в КЭД вследствие инфракрасных расходимостей не могут быть вычислены без конечной массы фотона. В этой лекции будет показано, что в физически наблюдаемых величинах (в отличие от ненаблюдаемых величин) нефизическая масса фотона может быть устремлена к нулю и при этом наблюдаемые величины имеют конечный предел.

17.9.1 1. Massive Vector Field

Поскольку массивное векторное поле имеет (для каждого импульса) три физических степени свободы (а не две - как в случае фотонного поля), то для корректного обращения с массивным векторным полем следует внести исправления в некоторые формулы

Действие массивного векторного поля имеет вид

$$\mathcal{S} = -\frac{1}{4\pi}\int \mathrm{d}^{(4)}x \left\{ \frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 - \frac{\lambda^2}{2}A_\mu^2 \right\} \tag{1}$$

Мы выбрали нормировку, совпадающую с нормировкой электромагнитного поля (сравни с (16 Л7)) для облегчения дальнейшего предельного перехода. Действие (1) приводит к следующему уравнению движения:

$$\partial_\nu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \lambda^2 A^\mu = 0 \tag{2}$$

Из (2) следует условие

$$\partial_\mu A^\mu = 0 \tag{3}$$

В результате уравнение движения (2) упрощается:

$$\left(\partial_\nu\partial^\nu + \lambda^2\right) A^\mu = 0 \tag{4}$$

При помощи уравнения движения (2) находим пропагатор (сравни с (53 Л7))

$$D_{\mu\nu}(k) = \frac{4\pi}{k^2-\lambda^2}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{\lambda^2}\right) \tag{5}$$

Сумма по поляризациям векторной частицы даёт правило (сравни с (33 Л7)):

$$\sum_{\alpha=1}^{3} e_\alpha^\mu e_\alpha^{\nu\star} = -\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{\lambda^2}\right) \tag{6}$$

Эта формула вытекает из равенств $k_\mu e_\alpha^\mu = 0$ и $k^2 = \lambda^2$.

Взаимодействие векторного поля с электрон-позитронным полем описывается тем же действием $\mathcal{S}_{\text{int}} = -e\int \mathrm{d}^{(4)}x A_\mu J^\mu$, что и в случае фотонного поля. Поэтому вследствие калибровочной инвариантности, то есть независимости физических величин от продольной части пропагатора, мы можем опустить в (5) продольную часть и пользоваться пропагатором

$$D_{\mu\nu}(k) = \frac{4\pi}{k^2-\lambda^2}\eta_{\mu\nu} \tag{7}$$

Именно пропагатор (7) использовался в предыдущих лекциях.

17.9.2 2. Emission of Soft Photons

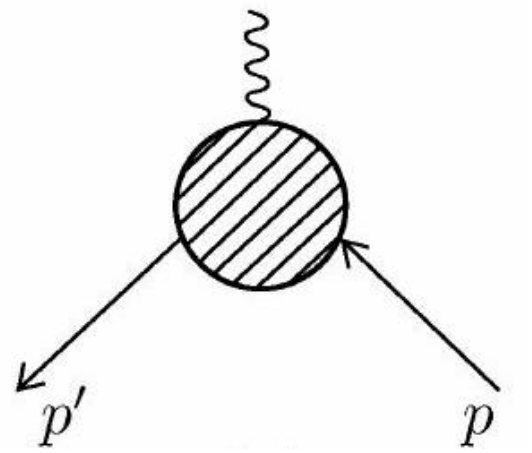
Для определённости рассмотрим процесс рассеяния электрона внешним электромагнитным полем. Например, это может быть неподвижное массивное ядро. На рис. 29 (а) изображена диаграмма упругого процесса рассеяния электрона, в результате которого электрон переходит из состояния с импульсом p в состояние с импульсом p'. Диаграммы на рис. 29$(b), (c)$ изображают процесс

PSfrag неупругого рассеяния, отличающийся от упругого процесса испусканием дополнительного мягкого фотона ($k \to 0$).

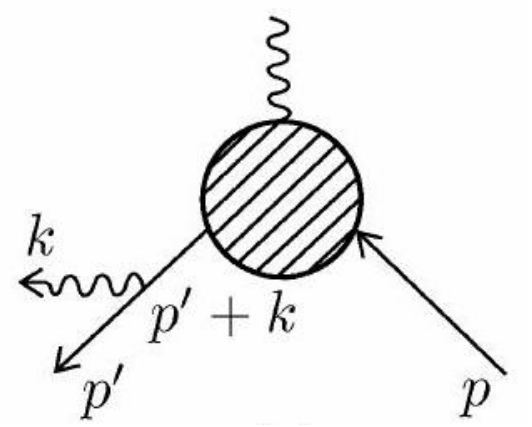
Диаграмма на рис. 29 (a) аналитически представляется в виде

$$M_{fi}^{(elas)} = \bar{u}_{p'} M u_p \tag{8}$$

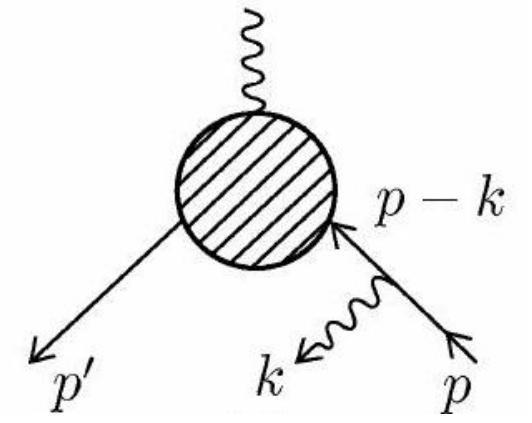
Рассмотрим амплитуду, изображённую на рис. 29 (c). Эта амплитуда получается из амплитуды,



(a)



(b)



(c)
Рис. 29
изображённой на рис. 29(a), путём замены

$$u_p \to \sqrt{4\pi}eS_c(p-k)\gamma^\mu e^*_\mu u_p = \sqrt{4\pi}e\frac{\not p - \not k + m}{(p-k)^2-m^2}^{\ *} u_p \approx -\sqrt{4\pi}e\frac{\not p + m}{2pk-\lambda^2}\phi^* u_p \tag{9}$$

Здесь было учтено, что $k^2=\lambda^2$ и $k\to 0$. Так как

$$\not p\phi^* = 2pe^* - \phi^*\not p, \quad \not p u_p = m u_p$$

то (9) переписывается как

$$u_p \to -\sqrt{4\pi}e u_p \frac{pe^*}{pk-\lambda^2/2} \tag{10}$$

Аналогичное рассмотрение показывает, что диаграмма, изображённая на рис. 29 (b), получается из диаграммы, изображённой на рис. 29 (a), путём замены

$$\bar u_{p'} \to \sqrt{4\pi}e\frac{p'e^*}{p'k+\lambda^2/2}\bar u_{p'} \tag{11}$$

Таким образом, сумма амплитуд (b) и (c) на рис. 29

$$M_{fi} = M_{fi}^{(elas)}\left[\sqrt{4\pi}e\left(\frac{p'e^*}{p'k+\lambda^2/2} - \frac{pe^*}{pk-\lambda^2/2}\right)\right] \tag{12}$$

а соответствующее неупругое сечение

$$\mathrm{d}\sigma_1 = \mathrm{d}\sigma^{(elas)} 4\pi e^2 \left|\frac{p'e}{p'k+\lambda^2/2} - \frac{pe}{pk-\lambda^2/2}\right|^2 \cdot \frac{\mathrm{d}^{(3)}k}{(2\pi)^3 2\omega_k}, \quad \omega_k = \sqrt{\mathbf{k}^2+\lambda^2} \tag{13}$$

Из формулы (13) видна причина, по которой следует учитывать лишь фотоны, излучённые внешними электронными линиями, но не учитывать мягкие фотоны, излучённые внутренними линиями: внешние электронные линии дают в амплитуде дополнительный множитель - электронный пропагатор, импульс которого при $k\to 0$ находится очень близко к массовой поверхности. Такой пропагатор находится вблизи своего полюса и он численно велик. Поэтому при $k\to 0$ дифференциальное сечение (13) стремится к бесконечности. Если же мягкий фотон излучается внутренней линией, то численно большого множителя не появляется, так как по импульсу виртуального электрона происходит интегрирование и потому можно считать, что он находится вдали от массовой поверхности.

Просуммируем в (13) по поляризациям векторной частицы при помощи (6). Легко увидеть, что второе слагаемое в (6) даёт относительный вклад в сечение $\sim O(\lambda^2)$. Так как $\lambda^2\to 0$, то вклад
в сечение от второго слагаемого в (6) можно опустить. По этой же причине может быть опущено слагаемое λ^2 в знаменателях под знаком модуля в (13). В результате получаем:

$$\mathrm{d}\sigma_1 = -e^2\left(\frac{p'}{p'k} - \frac{p}{pk}\right)^2 \frac{\mathrm{d}^{(3)}k}{4\pi^2\omega_k}\,\mathrm{d}\sigma^{(elas)} \tag{14}$$

Пусть $\mathbf{n}=\mathbf{k}/\omega_k$, а $\mathbf{v}$ и $\mathbf{v}'$ - начальная и конечная скорости электрона. Тогда в случае $\lambda=0$ выражение (14) в трёхмерных обозначениях принимает вид [1]

$$\mathrm{d}\sigma_1 = e^2 \left(\frac{[\mathbf{v}', \mathbf{n}]}{1 - \mathbf{n}\mathbf{v}'} - \frac{[\mathbf{v}, \mathbf{n}]}{1 - \mathbf{n}\mathbf{v}} \right)^2 \cdot \frac{\mathrm{d}|\mathbf{k}|\mathrm{d}o_{\mathbf{k}}}{4\pi^2 \omega_k} \, \mathrm{d}\sigma^{(elas)} \tag{15}$$

Точно к такому же выражению для сечения мы придём, если в (13) с $\lambda = 0$ просуммировать по двум поляризациям безмассовых фотонов при помощи правила (33 Л7), справедливого для реальных безмассовых фотонов. Мы видим, что формула (14) для неупругого сечения справедлива как в случае излучения реальных безмассовых мягких фотонов, так и в случае излучения мягких векторных частиц с бесконечно малой, но ненулевой массой, и тремя поляризациями.

Обратим внимание на то, что в случае реальных фотонов ($\lambda = 0$) амплитуда (12) калибровочно инвариантна, т.е. она не изменяется при преобразовании вектора поляризации согласно (56 Л7). Таким образом, в случае реальных фотонов можно суммировать в сечении (13) по трём поляризациям фотона, включая продольную поляризацию. Это не изменит результат вследствие калибровочной инвариантности амплитуды и сечения. Но это означает, что при суммировании по поляризациям реального фотона можно пользоваться формулой (6) с последующим переходом к пределу $\lambda \to 0$. Именно такой подход и был использован при получении формулы (14). Приведенное рассуждение объясняет переход формулы (14) в формулу (15) при $\lambda \to 0$. Сама же формула (15) легче всего получается из формулы (13) при $\lambda = 0$ при помощи правила (33 Л7).

Множитель, стоящий перед $\mathrm{d}\sigma^{(\mathrm{elas})}$ в (15), совпадает с (делённой на ω_k) классической интенсивностью $\mathrm{d}\, I$ тормозного излучения малых частот (см. Л. Д. Ландау, Е. М. Лифшиц. Теоретическая физика, Т. 2, §69). Таким образом, излучение длинноволновых фотонов является, по сути, классическим процессом.

Полученные формулы применимы, конечно, при условии

$$\omega \sim |\mathbf{k}| \ll p^0 \tag{16}$$

Необходимо также, чтобы передача импульса $\mathbf{q} = \mathbf{p}' - \mathbf{p}$ рассеивающему центру (ядру) была велика по сравнению с изменением этой величины вследствие испускания мягкого фотона. Так как

$$\delta|\mathbf{p}| \sim \frac{\partial|\mathbf{p}|}{\partial p^0}\omega_k \sim \frac{\omega_k}{v}$$

то имеем ограничение

$$\omega_k/|\mathbf{q}|v \ll 1 \tag{17}$$

Полученный результат легко обобщается на случай испускания нескольких фотонов. Например, в случае испускания двух мягких фотонов с импульсами и векторами поляризации, нумеруемыми индексами 1 и 2, амплитуда (12) содержала бы вместо одного множителя в квадратной скобке два аналогичных множителя:

$$\left[\sqrt{4\pi}e \left(\frac{p'e_1^*}{p'k_1} - \frac{pe_1^*}{pk_1} \right) \sqrt{4\pi}e \left(\frac{p'e_2^*}{p'k_2} - \frac{pe_2^*}{pk_2} \right) \right] \tag{18}$$

[61] Действительно, $p = p^0(1, \mathbf{v})$, и так как при $\lambda = 0 \quad \omega_k = |\mathbf{k}|$, то $pk = p^0|\mathbf{k}|(1 - \mathbf{n}\mathbf{v}), \quad pp' = p_0 p'_0 \, (1 - \mathbf{v}\mathbf{v}')$, и потому

Простое вычисление показывает, что квадратные скобки в (*) и (**) равны одному и тому же значению $(\mathbf{n}, \mathbf{v}-\mathbf{v}')^2 \left[(1-\mathbf{nv})^2 (1-\mathbf{nv}')^2\right]^{-1}$. Отсюда следует равенство выражений (14) и (15) в случае нулевой массы фотона.

В этом нетрудно убедиться прямым вычислением. Действительно, рассмотрим диаграмму, отличающуюся от диаграммы на рис. 29(с тем, что входящий электрон с импульсом p излучает сначала мягкий фотон с импульсом k_1 и затем мягкий фотон с импульсом k_2. Будем считать фотоны безмассовыми. Вычисление, аналогичное тому, которое привело к формуле (10), теперь показывает, что в данном случае вклад в амплитуду получается из амплитуды на рис. 29 (а), если в ней сделать замену (учитываются лишь главные вклады по k_1 и k_2)

$$u_p \to u_p \frac{\left[-\sqrt{4\pi} e\,(pe_1^*)\right]\left[-\sqrt{4\pi} e\,(pe_2^*)\right]}{(pk_1+pk_2)\,(pk_1)}$$

Если с правой частью этого равенства сложить такую же величину с перестановкой $k_1 \leftrightarrow k_2$, то мы получим

$$u_p \to u_p \left[-\frac{\sqrt{4\pi} e\,(pe_1^*)}{pk_1}\right]\left[-\frac{\sqrt{4\pi} e\,(pe_2^*)}{pk_2}\right]$$

Теперь очевидно, что если учесть все возможности излучения двух мягких фотонов внешними электронными линиями, то в результате амплитуда получит множитель (18), поскольку этот множитель калибровочно инвариантен.

В случае испускания n мягких фотонов для каждого фотона имеется свой множитель в амплитуде. Поэтому сечение процесса с испусканием n мягких фотонов представляется в виде

$$\mathrm{d}\sigma_n = \mathrm{d}\sigma^{(elas)} \mathrm{d}w_1 \ldots \mathrm{d}w_n \tag{19}$$

где $\mathrm{d}w_i (i = 1 \ldots n)$ - вероятность отдельного испускания фотона k_i. Поэтому полное неупругое сечение представляется в виде

$$\mathrm{d}\sigma = \mathrm{d}\sigma^{(elas)} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\int\limits_{\lambda}^{\omega_{\max}} \mathrm{d}w_\omega \right)^n \tag{20}$$

Здесь $\omega_{\max}$ - некое значение частоты, удовлетворяющее условиям (16), (17). Под знаком суммы в (20) множитель $1/n\,!$ учитывает тождественность фотонов.

Как было указано,

$$-|\mathbf{k}|^2 \left(\frac{p'}{p'k} - \frac{p}{pk}\right)^2 = -\frac{m^2}{(p_0')^2 (1-\mathbf{nv}')^2} - \frac{m^2}{p_0^2 (1-\mathbf{nv}')^2} + \frac{2(1-\mathbf{vv}')}{(1-\mathbf{nv})(1-\mathbf{nv}')} =$$

$$= \left(\frac{\mathbf{v}'^2}{(1-\mathbf{nv}')^2} + \frac{\mathbf{v}^2}{(1-\mathbf{nv})^2} - \frac{2\mathbf{vv}'}{(1-\mathbf{nv})(1-\mathbf{nv}')}\right) - \left[\frac{1}{(1-\mathbf{nv}')^2} + \frac{1}{(1-\mathbf{nv})^2} - \frac{2}{(1-\mathbf{nv})(1-\mathbf{nv}')}\right] \quad (*)$$

С другой стороны

$\left(\frac{p'}{p'k} - \frac{p}{pk}\right)^2 = \left(\frac{\mathbf{v}'^2}{(1-\mathbf{nv}')^2} + \frac{\mathbf{v}^2}{(1-\mathbf{nv})^2} - \frac{2\mathbf{vv}'}{(1-\mathbf{nv})(1-\mathbf{nv}')}\right) - \left[\frac{(\mathbf{nv}')^2}{(1-\mathbf{nv}')^2} + \frac{(\mathbf{nv})^2}{(1-\mathbf{nv})^2} - 2\frac{(\mathbf{nv})(\mathbf{nv}')}{(1-\mathbf{nv})(1-\mathbf{nv}')}\right]$. (**)

$$\mathrm{d}w_\omega = \frac{\mathrm{d}I}{\omega_k} \tag{21}$$

где d I - классическая интенсивность излучения мягких фотонов. Поэтому величина (21) может интерпретироваться как среднее число излучённых фотонов d$\bar{n}$ в заданном интервале частот. Тогда среднее число излучённых фотонов

$$\bar{n} = \int\limits_{\lambda}^{\omega_{\max}} \frac{\mathrm{d}I}{\omega_k} = \int\limits_{\lambda}^{\omega_{\max}} \mathrm{d}w_\omega \sim e^2 \ln \frac{\omega_{\max}}{\lambda} \tag{22}$$

Последняя оценка получена при помощи (15). Хотя $\omega_{\max} \ll p^0$, при $\lambda \to 0$ с логарифмической точностью имеем оценку

$$\bar{n} = e^2 \ln \frac{p^0}{\lambda} \tag{23}$$

Если $\bar{n} \sim 1$, то ТВ оказывается неприменимой, т.к. первая поправка (15) оказывается того же порядка, что и d $\sigma^{(\text{elas})}$. Однако сама формула (15) тем точнее, чем меньше энергия мягкого фотона.

Выход из этой противоречивой ситуации таков.

При $\lambda \to 0$ среднее число излучённых фотонов $\bar{n} \to \infty$. Мягкие фотоны излучаются статистически независимо. Этот факт подтверждается всеми порядками ТВ. Поэтому к процессу множественного излучения фотонов можно применить формулу Пуассона: вероятность w_n излучения n фотонов выражается через их среднее число $\bar{n}$ формулой

$$w_n = \frac{\bar{n}^n}{n!} e^{-\bar{n}} \tag{24}$$

Теперь формулу (20) можно модернизировать следующим образом:

$$\mathrm{d}\sigma = \sum_{n=0}^{\infty} \mathrm{d}\sigma_n, \qquad \mathrm{d}\sigma_n = \mathrm{d}\tilde{\sigma}^{(elas)} w_n \tag{25}$$

Так как $\sum\limits_n w_n = 1$, то согласно

$$\mathrm{d}\sigma = \mathrm{d}\tilde{\sigma}^{(elas)} \tag{25}$$

При этом, конечно, упругое сечение в формулах (25), (26) отличается от упругого сечения в (20) множителем:

$$e^{-\bar{n}}\ \mathrm{d}\tilde{\sigma}^{(elas)} = \mathrm{d}\sigma^{(elas)} \tag{27}$$

Таким образом мы видим, что сечение чисто упругого рассеяния равно нулю при $\lambda \to 0$ (или $\bar{n} \to \infty$), что видно из формулы (27). Вместе с этим обращается в нуль сечение рассеяния с излучением любого конечного числа мягких фотонов. Тем не менее те сечения, которые вычисляются при помощи ТВ, конечны во всех порядках по константе связи, если учесть также и излучение мягких фотонов. Более того, в каждом порядке по константе связи существует конечный предел для сечений при устремлении массы фотона к нулю, хотя в каждом порядке по константе связи может быть излучено лишь конечное число мягких фотонов. Этот факт демонстрируется в следующем пункте. Таким образом, сечение

рассеяния в борновском (без учёта радиационных поправок) приближении может интерпретироваться как полное сечение рассеяния с учётом излучения любого числа мягких фотонов, но в наинизшем приближении по константе связи (F. Bloch, A. Nordsieck, 1937).

17.9.3 3. Cancellation of Infrared Divergences

Для подтверждения сделанного в конце предыдущего пункта утверждения и проверки формулы (26) проследим за сокращением инфракрасной расходимости в низшем порядке ТВ.

Сначала продолжим вычисление сечения (14). Имеем:

$$\varkappa \equiv \left(\frac{p'}{p'k} - \frac{p}{pk}\right)^2 = \frac{m^2}{(p'k)^2} + \frac{m^2}{(pk)^2} - \frac{2pp'}{(pk)\,(p'k)}$$

Воспользуемся равенством

$$-2pp' = -2m^2 + (p-p')^2$$

при помощи которого находим:

$$\varkappa = \frac{m^2}{(pk)^2\,(p'k)^2}\left[(pk)^2 + (p'k)^2 - 2(pk)\,(p'k)\right] + \frac{(p-p')^2}{(pk)\,(p'k)} = \frac{m^2\left[(p'-p)\,k\right]^2}{(pk)^2\,(p'k)^2} + \frac{(p'-p)^2}{(pk)\,(p'k)} \tag{28}$$

В нерелятивистском пределе величина (28)

$$\varkappa \approx \frac{(\mathbf{qk})^2}{m^2\omega_k^4} - \frac{\mathbf{q}^2}{m^2\omega_k^2}, \quad \mathbf{q} = \mathbf{p}' - \mathbf{p} \tag{29}$$

Подставим (29) в (14) и выполним тривиальное интегрирование по углам:

$$\mathrm{d}\sigma_1 = \mathrm{d}\sigma_0 \frac{e^2\mathbf{q}^2}{\pi m^2} \int\limits_0^{\omega_{\max}} \left[1 - \frac{\mathbf{k}^2}{3\,(\mathbf{k}^2+\lambda^2)}\right] \cdot \frac{\mathbf{k}^2\,\mathrm{d}|\mathbf{k}|}{(\mathbf{k}^2+\lambda^2)^{3/2}} \tag{30}$$

где $\mathrm{d}\sigma_0$ - упругое сечение рассеяния в неком порядке по теории возмущений. В предположении $\omega_{\max}/\lambda \gg 1$ интеграл в (30) легко берётся:

$$\mathrm{d}\sigma_1 = \mathrm{d}\sigma_0 \frac{2e^2}{3\pi} \cdot \frac{\mathbf{q}^2}{m^2}\left(\ln\frac{2\omega_{\max}}{\lambda} - \frac{5}{6}\right), \quad \mathbf{q}^2 \ll m^2 \tag{31}$$

Все сечения $\mathrm{d}\sigma_n$ являются рядами относительно e^2:

$$\mathrm{d}\sigma_n = \sum_i \mathrm{d}\sigma_n^{(i)} \tag{32}$$

так что величина $\mathrm{d}\sigma_n^{(i)}$ пропорциональна e^{2i}.

Согласно формулам (1), (6) и (49) лекции 13 в однопетлевом приближении

$$\begin{aligned} iM_{fi} &= (-ieA_\mu \bar{u}_{p'}\gamma^\mu u_p)\, f\left(-\mathbf{q}^2\right) = iM_{fi}^{(1)} f\left(-\mathbf{q}^2\right) \\ f\left(-\mathbf{q}^2\right) &= 1 - \frac{e^2\mathbf{q}^2}{3\pi m^2}\left(\ln\frac{m}{\lambda} - \frac{3}{8}\right), \quad \mathbf{q}^2 \ll m^2 \end{aligned} \tag{33}$$

Нас интересуют лишь инфракрасно расходящиеся вклады. Поэтому мы не учитываем вклад в iM_{fi} от формфактора g, а также вклад от амплитуды (c) на рис. 26. В нашем приближении в (31) следует использовать $\mathrm{d}\sigma_0^{(2)}$. Это сечение строится по амплитуде первого приближения $iM_{fi}^{(1)} = -i\ \mathrm{e}\ A_\mu \bar{u}_{p'}\gamma^\mu u_p$. Согласно (33)

$$\mathrm{d}\sigma_0^{(2)} + \mathrm{d}\sigma_0^{(4)} = \mathrm{d}\sigma_0^{(2)} f^2 = \mathrm{d}\sigma_0^{(2)} \left[1 - \frac{2e^2\mathbf{q}^2}{3\pi m^2}\left(\ln\frac{m}{\lambda} - \frac{3}{8}\right)\right] \tag{34}$$

При помощи (31) и (34) находим:

$$\mathrm{d}\sigma_0^{(2)} + \mathrm{d}\sigma_0^{(4)} + \mathrm{d}\sigma_1^{(4)} = \mathrm{d}\sigma_0^{(2)} \left[1 - \frac{2e^2\mathbf{q}^2}{3\pi m^2}\left(\ln\frac{m}{2\omega_{\max}} + \frac{11}{24}\right)\right] \tag{35}$$

Теперь в (35) можно перейти к пределу $\lambda \to 0$. Однако появляется зависимость сечения от $\omega_{\max}$. Эта зависимость является следствием того, что величина $\omega_{\max}$ входит в определение рассеяния как процесса, в котором может быть испущено любое число мягких фотонов с энергиями меньше $\omega_{\max}$. Сечение такого процесса будет тем меньше, чем меньше $\omega_{\max}$. Из формулы (35) видно также, что радиационные поправки к сечению малы, если частота $\omega_{\max}$ соизмерима с энергией покоя электрона.

Сравнение (35) и (34) показывает, что в нерелятивистском пределе инфракрасно регуляризованный формфактор

$$f_{\omega_{\max}}(t) = 1 + \frac{e^2 t}{3\pi m^2}\left(\ln\frac{m}{2\omega_{\max}} + \frac{11}{24}\right), \quad |t| \ll m^2 \tag{36}$$

Замечание. Полученный результат означает следующее: в величине (35) взаимно сокращаются вклады виртуальных и реальных испущенных фотонов с энергиями, меньшими $\omega_{\max}$. Поэтому регуляризованный формфактор (36) содержит лишь вклады тех промежуточных виртуальных фотонов, энергии которых больше $\omega_{\max}$.

Доказательство сокращения инфракрасных расходимостей в сечениях в четвёртом порядке относительно константы связи при любых энергиях проводится аналогично, но при помощи более трудоёмких вычислений. Эти вычисления содержатся в (Л. Д. Ландау, Е. М. Лифшиц. Теоретическая физика, Т. 4).

Имеет место следующее утверждение: в суммах (20) и (25) инфракрасные расходимости сокращаются во всех порядках относительно константы связи.

17.9.4 4. Lamb Shift of Energy Levels

Рассмотрим многоуровневую нерелятивистскую систему, взаимодействующую с электромагнитным полем. Мы будем иметь ввиду водородоподобный атом с зарядом ядра Ze, причём

$$Ze^2 \ll 1 \tag{37}$$

Эта система имеет ряд дискретных одноэлектронных уровней. Электрон взаимодействует с квантованным фотонным полем и вследствие этого дискретные уровни сдвигаются. В этом пункте вычисляется сдвиг дискретных уровней, обусловленный этим взаимодействием.

Согласно (30 Л8) нерелятивистский гамильтониан электрона имеет вид (в рассматриваемой задаче релятивистскими поправками к гамильтониану можно пренебречь)

$$\mathcal{H} = \mathcal{H}_0 - \frac{e}{m}\mathbf{p}\mathbf{A}^{\perp}, \quad \mathcal{H}_0 = \frac{\mathbf{p}^2}{2m} + \Phi(r), \quad \mathbf{p} = -i\boldsymbol{\partial}, \quad \Phi(r) = -\frac{Ze^2}{r} \tag{38}$$

Здесь $\mathbf{A}^{\perp}(x)$ - квантованное фотонное поле (33 Л7). Спиновую составляющую гамильтониана взаимодействия с фотонным полем (см. гамильтониан Паули (30 Л8)) здесь можно опустить ввиду относительной малости ($\sim \mathbf{k}|a \sim \alpha$, где a - боровский радиус, а α - постоянная тонкой структуры) соответствующей поправки. Обозначим через $|N\rangle$ электронный уровень невозмущённого гамильтониана:

$$\mathcal{H}_0|N\rangle = E_N|N\rangle \tag{39}$$

При помощи (36 Л7) находим матричный элемент фотонного поля между фотонным вакуумом и однофотонным состоянием:

$$\left\langle \mathbf{k}, \alpha \left| \mathbf{A}^{\perp}(\mathbf{x}) \right| 0 \right\rangle = \sqrt{4\pi}\mathbf{e}^{\perp}_{\mathbf{k}\,\alpha} e^{-i\mathbf{k}\mathbf{x}} \approx \sqrt{4\pi}\mathbf{e}^{\perp}_{\mathbf{k}\,\alpha} \tag{40}$$

Последнее приближённое равенство означает, что задача решается в дипольном приближении, которое в данном случае является удовлетворительным. Воспринимая второе слагаемое в $\mathcal{H}$ как возмущение, выпишем первую неисчезающую поправку к уровням:

$$\delta E_N = \int \frac{\mathrm{d}^{(3)}k}{(2\pi)^3 2|\mathbf{k}|} \sum_{\alpha} \sum_{N'} \frac{4\pi e^2}{m^2} \cdot \frac{\left|\left\langle N|\mathbf{p}\mathbf{e}^{\perp}_{\mathbf{k}}\alpha N'\right\rangle\right|^2}{E_N - E_{N'} - |\mathbf{k}|} = \frac{2e^2}{3\pi m^2} \sum_{N'} \int_0^{\infty} \mathrm{d}|\mathbf{k}| \cdot |\mathbf{k}| \frac{|\mathbf{p}_{NN'}|^2}{E_N - E_{N'} - |\mathbf{k}|} \tag{41}$$

При переходе ко второму равенству в (41) мы воспользовались правилом суммирования по векторам поляризации (33 Л7) и затем проинтегрировали по углам виртуального фотона. Интеграл по энергии виртуального фотона $|\mathbf{k}|$ в (41) расходится линейно. Необходимо выделить логарифмическую расходимость на фоне линейной. Для этого воспользуемся тождеством

$$\frac{|\mathbf{k}|}{E_N - E_{N'} - |\mathbf{k}|} = \frac{E_N - E_{N'}}{E_N - E_{N'} - |\mathbf{k}|} - 1$$

Таким образом

$$\delta E_N = -\frac{2e^2}{3\pi m^2} \left(\mathbf{p}^2\right)_{NN} \int_0 \mathrm{d}|\mathbf{k}| + \frac{2e^2}{3\pi m^2} \sum_{N'} \int_0 \mathrm{d}|\mathbf{k}| \frac{|\mathbf{p}_{NN'}|^2 (E_N - E_{N'})}{E_N - E_{N'} - |\mathbf{k}|} \tag{42}$$

Предположим, что мы рассмотрели бы эту же задачу для свободного электрона с гамильтонианом $\mathcal{H}_0 = \mathbf{p}^2/2m$. В этом случае, если состояния $|N\rangle$ - плоские волны, то $\mathbf{p}_{NN'} \sim \delta_{NN'}$ и второе слагаемое в правой части (42) обращается в нуль. Первое, линейно расходящееся слагаемое, устраняется путём перенормировки массы [1]. Действительно, пусть затравочный гамильтониан

$$\mathcal{H}_0 = \frac{\mathbf{p}^2}{2m_0} = \frac{\mathbf{p}^2}{2(m + \delta m)} = \frac{\mathbf{p}^2}{2m} - \frac{\delta m}{2m^2}\mathbf{p}^2 \tag{43}$$

где m_0 - затравочная масса. В свободной теории (43) имеются лишь непрерывне уровни $|N\rangle$ - плоские волны. Подберём δm в (43) так чтобы линейные расходимости всех уровней энергии сокращались. Согласно (42) и (43) имеем:

$$\delta E_N = -\frac{2e^2}{3\pi m^2}\left(\mathbf{p}^2\right)_{NN}\int\limits_0 \mathrm{d}|\mathbf{k}| - \frac{\delta m}{2m^2}\left(\mathbf{p}^2\right)_{NN} = 0 \longrightarrow \delta m = -\frac{4e^2}{3\pi}\int\limits_0 \mathrm{d}|\mathbf{k}|$$

Так как к гамильтониану (38) должно быть добавлено второе слагаемое из правой части (43), то мы видим, что с точностью $O\left(e^2\right)$ линейно расходящиеся поправки ко всем уровням энергии сокращаются. Поэтому сдвиг дискретного уровня энергии вследствие взаимодействия с фотонным полем определяется вторым слагаемым в (42), в котором вычислим интеграл по фотонной энергии от нуля до $\omega_{\max}$. Так как наше вычисление - нерелятивистское, то неизвестная частота $\omega_{\max}$ должна удовлетворять следующим неравенствам:

$$\left(Ze^2\right)^2 m \ll \omega_{\max} \ll m \tag{44}$$

Здесь первое неравенство означает, что учитываемые энергии фотонов в (42) много больше разности энергий дискретных уровней: $|E_N - E_{N'}| \ll \omega_{\max}$. Таким образом

$$\delta E_N = \frac{2e^2}{3\pi m^2}\sum_{N'}\left|\mathbf{p}_{NN'}\right|^2\left(E_{N'} - E_N\right)\ln\frac{\omega_{\max}}{\left|E_{N'} - E_N\right|} \tag{45}$$

Вследствие первого неравенства (44) аргумент логарифма возможно заменить на $[\omega_{\max}/\left|E_{N'} - E_N\right|]$. Перепишем (45) в виде

$$\delta E_N = \frac{2e^2}{3\pi m^2}\sum_{N'}\left|\mathbf{p}_{NN'}\right|^2\left(E_{N'} - E_N\right)\ln\frac{2\omega_{\max}}{m} + \frac{2e^2}{3\pi m^2}\sum_{N'}\left|\mathbf{p}_{NN'}\right|^2\left(E_{N'} - E_N\right)\ln\frac{m}{2\left|E_{N'} - E_N\right|} \tag{46}$$

Для дальнейших преобразований воспользуемся следующими соотношениями:

$$\mathbf{p}_{NN'}\left(E_{N'} - E_N\right) = \left[\mathbf{p}, \mathcal{H}_0\right]_{NN'} = -i(\boldsymbol{\partial}\Phi)_{NN'} \tag{47}$$

$$\sum_{N'}\left|\mathbf{p}_{NN'}\right|^2\left(E_{N'} - E_N\right) \equiv \frac{1}{2}\sum_{N'}\left\{\left[\mathbf{p}_{NN'}\left(E_{N'} - E_N\right)\right]\mathbf{p}_{N'N} - \mathbf{p}_{NN'}\left[\left(E_N - E_{N'}\right)\mathbf{p}_{N'N}\right]\right\} =$$

$$= \frac{i}{2}\sum_{N'}\left\{-(\boldsymbol{\partial}\Phi)_{NN'}\mathbf{p}_{N'N} + \mathbf{p}_{NN'}(\boldsymbol{\partial}\Phi)_{N'N}\right\} = \frac{i}{2}[\mathbf{p}, \boldsymbol{\partial}\Phi]_{NN} = \frac{1}{2}(\Delta\Phi)_{NN} \tag{48}$$

$$\mathbf{p}_{NN'} = -im\left[\mathbf{r}, \mathcal{H}_0\right]_{NN'} = -im\mathbf{r}_{NN'}\left(E_{N'} - E_N\right) \tag{49}$$

Ещё раз перепишем первое и второе слагаемые в (46) при помощи (48) и (49), соответственно:

$$\delta E_N^{(I)} = \frac{e^2}{3\pi m^2}(\Delta\Phi)_{NN}\ln\frac{2\omega_{\max}}{m} + \frac{2e^2}{3\pi}\sum_{N'}\left|\mathbf{r}_{NN'}\right|^2\left(E_{N'} - E_N\right)^3\ln\frac{m}{2\left|E_{N'} - E_N\right|} \tag{50}$$

[61] В предыдущих лекциях мы выяснили, что в дираковской теории поправки к массе электрона расходятся логарифмически. Однако изучаемый здесь гамильтониан нерелятивистской заряженной частицы в равной степени может описывать как фермиевскую, так и бозевскую частицу. Поэтому линейная расходимость перенормировки её массы вследствие взаимодействия с электромагнитным полем не противоречит полученным результатам.

Здесь верхний индекс (I) указывает на то, что соответствующий вклад получен от фотонов с частотами $\omega < \omega_{\max}$.

Зависимость сдвига уровней (50) от нефизического параметра $\omega_{\max}$ указывает на то, что формула (50) учитывает лишь часть сдвига: в дополнение к (50) следует учесть вклад от жёстких фотонов, который содержится в диаграмме (b) на рис. 26, а также вклад от поправки потенциала Φ, описываемый диаграммой (c) на рис. 26. Диаграмма (b) на рис. 26 даёт квантовые поправки к движению электрона, которые можно рассматривать как поправки вследствие изменения эффективного электронного гамильтониана. Вклад в эффективный гамильтониан от диаграммы (b) на рис. 26 следует учесть, используя инфракрасно регуляризованный формфактор f (36). При этом частота $\omega_{\max}$ в (50) и (36) - одна и та же. Действительно, в регуляризованном формфакторе (36) фактически учитывается лишь вклад от виртуальных фотонов с частотами $\omega > \omega_{\max}$ (см. замечание после (36)). Вклад в сдвиг уровней именно от такого формфактора и следует добавить к (50), поскольку в (50) уже учтён вклад от фотонов с частотами $\omega < \omega_{\max}$. Формфактор g задаётся согласно (49 Л12) и он не содержит нефизических параметров, поскольку не требует инфракрасной регуляризации. Таким образом добавка к амплитуде, описываемая диаграммой (b) на рис. 26, имеет вид

$$\delta_{(b)} M_{fi}(\mathbf{q}) = -\left\{ \left[f_{\omega_{\max}}\left(-\mathbf{q}^2\right) - 1 \right] \bar{u}_f \gamma^0 u_i - \frac{g\left(-\mathbf{q}^2\right)}{m} \left(\bar{u}_f \sigma^{0\mu} u_i \right) q_\mu \right\} \Phi(\mathbf{q}) \tag{51}$$

Здесь $q-4$-импульс, приносимый потенциалом Φ, причём $q = (0, \mathbf{q})$. Добавка к амплитуде, вытекающая из радиационных поправок к потенциалу и описываемая диаграммой (c) на рис. 26, согласно (45 Л11) равна

$$\delta_{(c)} M_{fi}(\mathbf{q}) = -\mathcal{P}\left(-\mathbf{q}^2\right) \left(\bar{u}_f \gamma^0 u_i \right) \Phi(\mathbf{q}) \tag{52}$$

Представим сумму амплитуд (51) и (52) в виде

$$\begin{gathered} \delta M_{fi} = \delta_{(b)} M_{fi} + \delta_{(c)} M_{fi} = -u_f^\dagger \Lambda \Phi u_i \\ \Lambda = \left[f_{\omega_{\max}}\left(-\mathbf{q}^2\right) - 1 \right] + \mathcal{P}\left(-\mathbf{q}^2\right) + \frac{g}{2m} \gamma \mathbf{q} \end{gathered} \tag{53}$$

Величина $\delta\mathcal{H}^{(II)} \equiv \Lambda\Phi$ должна восприниматься как высокочастотная радиационная поправка к одноэлектронному гамильтониану. При помощи (53), (36), (35 Л11) и (49 Л12) находим значение этой поправки в импульсном представлении:

$$\delta\mathcal{H}^{(II)}(\mathbf{q}) = \left\{ -\frac{e^2 \mathbf{q}^2}{3\pi m^2} \left(\ln \frac{m}{2\omega_{\max}} + \frac{11}{24} - \frac{1}{5} \right) + \frac{e^2}{4\pi m} \gamma \mathbf{q} \right\} \Phi(\mathbf{q}) \tag{54}$$

Функция, соответствующая (54) в координатном представлении, имеет вид

$$\delta\mathcal{H}^{(II)}(\mathbf{x}) = \frac{e^2}{3\pi m^2} \left(\ln \frac{m}{2\omega_{\max}} + \frac{11}{24} - \frac{1}{5} \right) \Delta\Phi(\mathbf{x}) - \frac{e^2}{4\pi m} i\gamma \boldsymbol{\partial} \Phi(\mathbf{x}) \tag{55}$$

Соответствующее смещение уровней получается усреднением оператора (55) по волновой функции невозмущённого состояния электрона в атоме:

$$\delta E_N^{(II)} = \frac{e^2}{3\pi m^2} \left(\ln \frac{m}{2\omega_{\max}} + \frac{11}{24} - \frac{1}{5} \right) \langle N | \Delta\Phi | N \rangle - i \frac{e^2}{4\pi m} \langle N | \gamma \boldsymbol{\partial} \Phi | N \rangle \tag{56}$$

Замечание. Хотя формфакторы f и g были получены для электронов на массовой поверхности, их использование в (56) является законным, хотя усреднение идёт по связанным состояниям. Действительно, формфакторы в (56) получены от вклада виртуальных фотонов с энергиями $\omega > \omega_{\max}$. С другой стороны, так как энергия связанных состояний (включая энергию покоя) $\left[1-(Ze^2)^2\right]m$ почти совпадает с энергией покоя электрона и вследствие первого неравенства (44) использование формфакторов $f_{\omega_{\max}}$ и g в (53) законно.

В первом слагаемом в (56) можно пользоваться непосредственно нерелятивистскими волновыми функциями электрона. Так как диагональные матричные элементы от спиновых операторов равны нулю, то во втором слагаемом в (56) необходимо использовать в качестве невозмущённых собственных функций $|N\rangle$ собственные функции гамильтониана Дирака (см. (33 Л8)) $\mathcal{H}_0 = -i\boldsymbol{\alpha}\boldsymbol{\partial} + m\gamma^0 + \Phi(r)$. Тогда при помощи уравнения (39), в котором E_N включает энергию покоя электрона, в стандартном представлении выражаем спинор χ через спинор ϕ. Пренебрегая энергией связи и потенциалом Φ (их учёт был бы превышением точности), находим:

$$\chi = -\frac{i}{2m}\boldsymbol{\sigma}\boldsymbol{\partial}\phi \tag{57}$$

Спинор ϕ в (57) следует рассматривать как нерелятивистскую волновую функцию. Подставляя (57) в (56), после несложных вычислений получаем:

$$\delta E_N^{(II)} = \frac{e^2}{3\pi m^2}\left(\ln\frac{m}{2\omega_{\max}} + \frac{19}{30}\right)\langle N|\Delta\Phi|N\rangle + \frac{e^2}{4\pi m^2}\left\langle N\left|(\boldsymbol{\sigma}\mathbf{l})\frac{1}{r}\cdot\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right|N\right\rangle \tag{58}$$

В (58) и далее $|N\rangle$ - уже нерелятивистский вектор состояния электрона в центральном потенциале $\Phi(r)$ и $\mathbf{l}$ - оператор орбитального момента.

Складывая (50) и (58), находим полный сдвиг уровней:
$\delta E_N = \frac{2e^2}{3\pi}\sum\limits_{N'}|\mathbf{r}_{NN'}|^2\left(E_{N'}-E_N\right)^3\ln\frac{m}{2|E_{N'}-E_N|} + \frac{19}{30}\cdot\frac{e^2}{3\pi m^2}\langle N|\Delta\Phi|N\rangle + \frac{e^2}{4\pi m^2}\left\langle N\left|(\boldsymbol{\sigma}\mathbf{l})\frac{1}{r}\cdot\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right|N\right\rangle$.

В (59) сокращается промежуточная частота $\omega_{\max}$, что является необходимым условием корректности вычисления. Все матричные элементы в (59) берутся по отношению к нерелятивистским волновым функциям электрона в атоме. Для водородоподобного атома эти функции зависят только от четырёх квантовых чисел: главного квантового числа n, орбитального момента l, полного момента j и его проекции M, так что $|N\rangle = |nljM\rangle$. Это состояние может быть представлено как двухкомпонентный спинор (см. (74 Л8)) $|nljM\rangle = f_n(r)\Omega_{jlM}(\theta,\phi)$.

Второе слагаемое в (59) вычисляется при помощи уравнений

$$\Delta\Phi(\mathbf{x}) = 4\pi Ze^2\delta^{(3)}(\mathbf{x})$$
$$\langle nljM|\Delta\Phi|nljM\rangle = 4\pi Ze^2\left|\psi_N(0)\right|^2 = 4m^3\left(Ze^2\right)^4 n^{-3}\delta_{l0} \tag{60}$$

Полезно ввести обозначение

$$L_{nl} = \frac{n^3}{2m\left(Ze^2\right)^4}\sum_{N'}\left|\langle N'|\mathbf{r}|N\rangle\right|^2\cdot\left(E_{n'}-E_n\right)^3\ln\frac{m\left(Ze^2\right)^2}{2\left|E_{n'}-E_n\right|} \tag{61}$$

Поскольку невозмущённые уровни энергии пропорциональны $\left(Ze^2\right)^2$, а характерный размер атома пропорционален $\left(Ze^2\right)^{-1}$, то величины L_{nl} (61) не зависят от Z и они могут

быть найдены численно для атома водорода. При помощи обозначения (61) перепишем первое слагаемое в (59) как

$$\frac{4mZ^4e^{10}}{3\pi n^3}\left[L_{nl}+\frac{n^3}{2m\left(Ze^2\right)^4}\ln\frac{1}{\left(Ze^2\right)^2}\cdot\sum_{N'}\left|\mathbf{r}_{NN'}\right|^2\left(E_{N'}-E_N\right)^3\right] \tag{62}$$

Ещё раз используя (48) и (49) преобразуем второе слагаемое в квадратной скобке в (62) к виду

$$\frac{n^3}{4m^3\left(Ze^2\right)^4}\ln\frac{1}{\left(Ze^2\right)^2}\cdot\langle nljM|\Delta\Phi|nljM\rangle=\ln\frac{1}{\left(Ze^2\right)^2}\cdot\delta_{l0} \tag{63}$$

Последнее равенство здесь получается при помощи (60). Используя (60), (62), и (63), переписываем лэмбовский сдвиг в окончательном виде:

$$\delta E_{nlj}^{(L)}=\frac{4mZ^4e^{10}}{3\pi n^3}\left\{L_{nl}+\left[\ln\frac{1}{\left(Ze^2\right)^2}+\frac{19}{30}\right]\delta_{l0}\right\}+\frac{e^2}{4\pi m^2}\left\langle nljM\left|(\boldsymbol{\sigma}\mathbf{l})\frac{1}{r}\cdot\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right|nljM\right\rangle \tag{64}$$

Необходимо по отдельности рассмотреть случаи $l=0$ и $l\neq 0$.

При $l=0$ последнее слагаемое в (64) исчезает и мы имеем (в обычных единицах)

$$\delta E_{n0}^{(L)}=\frac{4mc^2Z^4\alpha^5}{3\pi n^3}\left[L_{n0}+\ln\frac{1}{(Z\alpha)^2}+\frac{19}{30}\right] \tag{65}$$

Приведём числовые значения нескольких величин L_{n0}:

n	=	1	2	3	4	∞
L_{n0}	=	$-2,984$	$-2,812$	$-2,768$	$-2,750$	$-2,721$

Так как невозмущённые уровни $E_n=-mc^2(Z\alpha)^2/\left(2n^2\right)$, то

$$\left|\frac{\delta E_{n0}}{E_{n0}}\right|\sim Z^2\alpha^3\ln\frac{1}{Z\alpha} \tag{66}$$

При $l\neq 0$ исчезает второе слагаемое в (64). Так как $(1/r)\mathrm{d}\Phi/\mathrm{d}r=Ze^2/r^3$ и

$$\begin{gathered}\langle nljM|\boldsymbol{\sigma}\mathbf{l}|nljM\rangle=j(j+1)-l(l+1)-\frac{3}{4}\\ \left\langle nljM\left|\frac{1}{r^3}\right|nljM\right\rangle=\frac{2\left(Ze^2m\right)^3}{n^3l(l+1)(2l+1)}\end{gathered} \tag{67}$$

то

$$\delta E_{nljM}^{(L)}=\frac{4mc^2Z^4\alpha^5}{3\pi n^3}\left[L_{nl}+\frac{3}{8}\cdot\frac{j(j+1)-l(l+1)-3/4}{l(l+1)(2l+1)}\right] \tag{68}$$

Согласно (110 Л8) релятивистская поправка к энергии водородоподобных атомов - энергия тонкой структуры:

$$\delta E_{nljM}=-\frac{mc^2(Z\alpha)^4}{2n^3}\left[\frac{1}{j+1/2}-\frac{3}{4n}\right] \tag{69}$$

Из сравнения формул (68) и (69) видно, что квантовая теория поля даёт вклад в сдвиг уровней по порядку величины в α раз меньше чем вклад от релятивистских поправок в одноэлектронной задаче. Однако радиационный сдвиг (68) снимает последнее вырождение, сохраняющееся в формуле (69): вырождение уровней с одинаковыми значениями n и j, но разными $l = j \pm 1/2$. Именно
поэтому лэмбовский сдвиг уровней легко отделим и потому измерим.

Задачи

1. Вычислить радиационный сдвиг уровней заряженного пространственного осциллятора.

Указание. Поскольку гармонический потенциал есть некая идеализация, то можно считать, что рассматривается движение нерелятивистского электрона в глубокой, но конечной потенциальной яме с потенциалом $\Phi(r) = m\omega^2 r^2/2, \quad r < L, \quad \Phi(r) = m\omega^2 L^2/2, \quad r > L$. Считать, что $\hbar\omega \ll m\omega^2 L^2/2 \ll mc^2$. В этом случае при вычислении радиационных поправок к нижним уровням учёт высокочастотного вклада при помощи формфакторов (51) является оправданным.

18 Radiative Corrections: Some Formal Developments 2

18.1 Renormalization and Gauge Invariance by Zi

18.1.1 Zi.III.2 Renormalizable vs. Nonrenormalizable

Старый взгляд против современной точки зрения

Теперь мы знаем, что представление амплитуды мезон-мезонного рассеяния в терминах физической константы связи λ_P ведет к исчезновению зависимости от обрезания Λ (по крайней мере до порядка λ_P^2). Неужели это простое везение?

Как оказалось, существуют квантовые теории поля, в которых это условие справедливо, и квантовые теории поля, в которых оно теряет свою силу. Таким образом, получаем два класса квантовых теорий поля. Исторически сложилось, что первый класс квантовых теорий называют «перенормируемыми теориями» и относят к «хорошим». Второй класс квантовых теорий, известных как «неперенормируемые теории», вызывает у теоретических физиков страх и отвращение.

Однако новый взгляд на теории поля как на эффективные низкоэнергетические теории, дополняющие некоторой основополагающей теории, позволил современным физикам взглянуть на неперенормируемые теории в более позитивном ключе по сравнению с предыдущим поколением. Надеюсь, я смогу пояснить это утверждение в этой и следующей главах.

Школьный анализ размерностей

Начнем с анализа размерностей, которому вас обучили в средней школе. Используя натуральные единицы, для которых $\hbar = 1$ и $c = 1$, размерность длины и времени будет одинаковой и равной обратной размерности массы (а также энергии и импульса). Специалисты в области физики элементарных частиц подсчитывают размерность в терминах массы, ибо привыкли думать в масштабах энергии. С другой стороны,

специалисты в области физики конденсированных сред обычно действуют исходя из масштаба длины. Значит, физики из упомянутых отраслей оперируют величинами с (равными и) противоположными размерностями. Мы будем пользоваться обозначениями из физики элементарных частиц.

Поскольку действие $S \equiv \int d^4x\mathcal{L}$ входит в интеграл по траекториям как e^{iS}, оно не имеет размерности и, следовательно, размерность лагранжиана (точнее, плотности лагранжиана) $\mathcal{L}$ совпадает с размерностью массы в четвертой степени. Тот факт, что размерность $\mathcal{L}$ равна 4, будем записывать как $[\mathcal{L}] = 4$. Аналогично имеем $[x] = -1$ и $[\partial] = 1$. Рассмотрим теорию скалярного поля $\mathcal{L} = \frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right] - \lambda\varphi^4$. Для 4-мерного параметра $(\partial\varphi)^2$ справедливо утверждение $[\varphi] = 1$ (поскольку $2(1 + [\varphi]) = 4$), вследствие чего $[\lambda] = 0$, т. е. параметр λ не имеет размерности. Существует простое правило, согласно которому для каждого члена $\mathcal{L}$ размерность различных составляющих, включая константу связи и массу, должна в сумме составлять 4 (т. е., к примеру, $[\lambda] + 4[\varphi] = 4$).

Как обстоит дело с фермионным полем ψ? Применяя это правило к лагранжиану $\mathcal{L} = \bar{\psi}i\gamma^\mu\partial_\mu\psi + ...$, видим, что $[\psi] = \frac{3}{2}$. (В дальнейшем будем опускать... ; и так понятно, что речь идет лишь о части лагранжиана. Более того, раз мы проводим анализ размерностей, то будем опускать некоторые несущественные коэффициенты, такие как численные коэффициенты и гамма-матрицы во взаимодействии Ферми, к которому мы вскоре придем.) Анализируя член взаимодействия $f\varphi\bar{\psi}\psi$, видим, что константа связи Юкавы f безразмерна. Однако в теории слабых взаимодействий с $\mathcal{L} = G\bar{\psi}\psi\bar{\psi}\psi$ константа связи Ферми G имеет размерность -2 (так как $-2 + 4\left(\frac{3}{2}\right) = 4$; это понятно?)

Из лагранжиана Максвелла $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ следует, что $[A_\mu] = 1$, т. е. размсрность A_μ совпадает с размерностью ∂_μ: размерности векторного и скалярного полей одинаковы. Из электромагнитного взаимодействия $eA_\mu\bar{\psi}\gamma^\mu\psi$ следует, что e не имеет размерности. Такой же вывод мы получим, если запишем закон Кулона в естественных единицах $V(r) = \alpha/r$ с коэффициентом α, равным $\alpha = e^2/4\pi$.

Амплитуда рассеяния расходится

Теперь мы готовы дать эвристическое объяснение неперенормируемости теории. Рассмотрим теорию слабых взаимодействий Ферми. Допустим, нам надо вычислить амплитуду $\mathcal{M}$ четырехфермионного взаимодействия, скажем, рассеяния нейтрино на нейтрино с энергией, намного меньшей Λ. В низшем порядке имеем $\mathcal{M} \sim G$. Попробуем записать амплитуду в следующем порядке: $\mathcal{M} \sim G + G^2(?)$ и догадаться, что должно быть на месте (?). I оо определению, все массы и энергии пренебрежительно малы, по сравнеIию с порогом Λ, так что можно принять их равными нулю. Поскольку $[G] = -2$, из анализа размерностей следует, что неизвестный коэффициент должен иметь размерность $+2$. Единственно возможным параметром для (?) является Λ^2. Значит, амплитуда следующего порядка записывается в виде $\mathcal{M} \sim G + G^2\Lambda^2$. Чтобы удостовериться в справедливости нашего заключения, рассмотрим фейнмановскую диаграмму на рис. III.2.1: видим, что действительно $G^2 \int^{\Lambda} d^4p(1/p)(1/p) \sim G^2\Lambda^2$.

Теория без обрезания или, что эквивалентно,

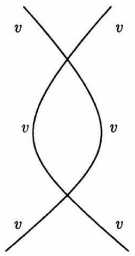


Рис. III.2.1 с Λ или принять $\Lambda = \infty$, оказывается несостоятельной: для физической величины предсказано бесконечное значение. Поэтому теорию слабых взаимодействий Ферми назвали неперенормируемой.

Поддавшись отчаянию, некоторые теоретики решили отказаться от квантовой теории поля. Другие предпринимали многочисленные попытки, чтобы «исцелить» теорию слабого взаимодействия. К примеру, они предлагали исходить из того, что ряд $\mathcal{M} \sim G\left[1 + G\Lambda^2 + \left(G\Lambda^2\right)^2 + \left(G\Lambda^2\right)^4 + \ +...\right] = Gf\left(G\Lambda^2\right)$, где неизвестная функция f обладает следующим свойством: значение $f(\infty)$ конечно. Теперь мы знаем, что данный подход нельзя было назвать плодотворным.

Зато к концу 1960-х гг. Глэшоу, Салам и Вайнберг, основываясь на результатах исследований других ученых, разработали теорию электрослабых взаимодействий, объединяющую электромагнитное и слабое взаимодействия (глава VII.2). Теория слабых взаимодействий Ферми входит в теорию электрослабых взаимодействий как эффективная низкоэнергетическая теория.

Теория Ферми оказалась несостоятельной

С современной точки зрения мы рассматриваем обрезание Λ как реально существующее, поэтому почти слышим, как жалуется теория на зависимость амплитуды четырехфермионного взаимодействия $\mathcal{M} \sim G + \ + G^2\Lambda^2$ от обрезания, ведь теория не может объяснить, что произойдет при $\Lambda \sim (1/G)^{\frac{1}{2}}$. Второй член в ряду возмущений становится сравним с первым, так что теория возмущений перестает быть применимой.

К такому выводу можно прийти иным путем. Предположим, нам ничего не известно об обрезании. Основываясь на анализе размерностей и учи- тывая тот факт, что размерность G равна -2, приходим к заключению, что амплитуда нейтрино-нейтринного рассеяния для энергии E центра масс равна $\mathcal{M} \sim G + G^2E^2 + ...$ Когда значение E достигает масштаба $\sim (1/G)^{\frac{1}{2}}$, амплитуда становится порядка единицы и должна появиться новая физика просто потому, что сечение достигает унитарного предела из обычной квантовой механики. (Помните про фазовый сдвиг и тому подобное?)

Все это восходит еще к Юкаве, который одновременно с разработкой мезонной теории для ядерных сил предложил также учитывать промежуточный векторный бозон в теории слабых взаимодействий Ферми. (В 1930-х гг. не было ясного различия между сильным и слабым взаимодействиями.) Рассмотрим кратко теорию векторного бозона массой M, взаимодействующего с фермионным полем с безразмерной константой связи g:

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + M^2A_\mu A^\mu + gA_\mu\bar{\psi}\gamma^\mu\psi$$

Вычислим фермион-фермионное рассеяние. Для фейнмановской диаграммы на рис. III.2.2 амплитуда равна $(-ig)^2\left(\bar{u}\gamma^\mu u\right)\left[i/\left(k^2 - M^2 + i\varepsilon\right)\right]\left(\bar{u}\gamma_\mu u\right)$. Если переданный импульс k много меньше M, амплитуда сводится к виду $i\left(g^2/M^2\right)\left(\bar{u}\gamma^\mu u\right)\left(\bar{u}\gamma_\mu u\right)$. Но это в точности

так, если бы фермионы взаимодействуют согласно теории Ферми $G\left(\bar{\psi}\gamma^{\mu}\psi\right)\left(\bar{\psi}\gamma_{\mu}\psi\right)$, где $G = g^2/M^2$.

Если мы беспечно будем проводить расчеты на основе эффективной низкоэнергетической теории $G\left(\bar{\psi}\gamma^{\mu}\psi\right)\left(\bar{\psi}\gamma_{\mu}\psi\right)$, то в конечном итоге потерпим неудачу. Да, действительно, при энергиях $(1/G)^{\frac{1}{2}} = M/g$ появляется векторный бозон. Возникает новая физика.

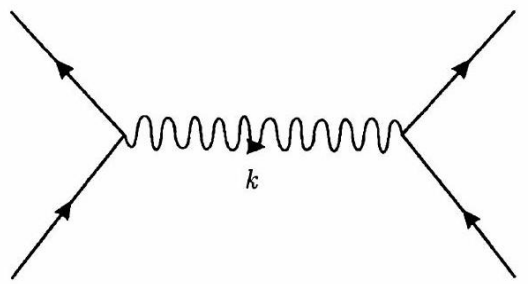


Рис. III. 2.2

Я считаю правильным, что физические теории, в отличие от других теорий человеческой мысли, в состоянии сообщить о своей области применимости, вне которой они становятся внутренне противоречивыми.

Теория Эйнштейна оказывается несостоятельной

Общеизвестно, что теория гравитации также неперенормируема. Просто сравнивая закон Ньютона $V(r) = G_N M_1 M_2/r$ с законом Кулона $V(r) = \alpha/r$, видим, то гравитационная постоянная Ньютона G_N имеет размерность массы -2. Этим все сказано. Приходим к печальному заключению, согласно которому теория гравитации, подобно теории слабых взаимодействий Ферми, является неперенормируемой. Чтобы подтвердить этот факт, попробуем вычислить гравитон-гравитонное рассеяние для энергии E. Получим ряд $\sim \left[1 + G_N E^2 + \left(G_N E^2\right)^2 + ...\right]$.

Как и для случая теории Ферми, неперенормируемость квантовой гравитации сказывается на том, что на масштабе массы Планка $(1/G_N)^{\frac{1}{2}} \equiv M_{\text{Planck}} \sim 10^{19} m_{\text{proton}}$ должна появиться новая физика. Когда обнаружилась несостоятельность теории Ферми, оказалось, что новая физика суть электрослабая теория. Окажется ли теория струн новой физикой для теории Эйнштейна? [1]

Упражнения

III.2.1. Рассмотрим теорию d-мерного скалярного поля $S = \int d^d x \left(\frac{1}{2}(\partial\varphi)^2 + \right.$ $\left. + \frac{1}{2}m^2\varphi^2 + \lambda\varphi^4 + ... + \lambda_n\varphi^n + ...\right)$. Покажите, что $[\varphi] = (d-2)/2$ и $[\lambda_n] = n(2-d)/2 + d$. Обратите внимание, что при $d = 2$ поле φ безразмерно.

'J. Polchinski, String Theory.

18.1.2 Zi.III.3 Counterterms and Physical Perturbation Theory

Перенормируемость

Из эвристических рассуждений предыдущей главы следует, что теория является неперенормируемой, если ее константа связи имеет размерность массы в отрицательной

степени. Как обстоит дело с теориями, где константа связи безразмерна, например, с квантовой электродинамикой и с φ^4-теорией? На самом деле уже доказано, что обе они являются перенормируемыми, хотя доказать, что теория перенормируема, гораздо труднее, чем неперенормируемость теории. Действительно, чтобы доказать перенормируемость неабелевой калибровочной теории (речь о которой пойдет позже), потребовались усилия многих выдающихся физиков, в частности, 't Хоофта, Вельтмана, Б. Ли, Зинн-Жюстена и многих других.

Снова рассмотрим простую φ^4-теорию. Сначала сделаем простое замечание: физическая константа связи λ_P есть функция от s_0, t_0 и u_0 [см. формулу (III.1.4)]. Для теоретических рассуждений удобнее положить s_0, t_0 и u_0 равными μ^2 и использовать вместо (III.1.4) более простое определение:

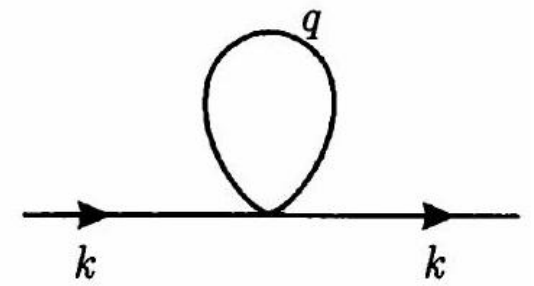
$$-i\lambda_P = -i\lambda + 3iC\lambda^2 \log\left(\frac{\Lambda^2}{\mu^2}\right) + O\left(\lambda^3\right)^1 .$$

Мы уже знаем, что до порядка λ^2 амплитуда мезон-мезонного рассеяния, выраженная через физическую константу связи λ_P, не зависит от обрезания Λ. Как доказать справедливость этого утверждения для любого порядка λ? Из анализа размерностей следует только, что до любого порядка λ зависимость амплитуды мезон-мезонного рассеяния от обрезания должна выражаться суммой членов $[\log(\Lambda/\mu)]^p$, где p - некоторая степень.

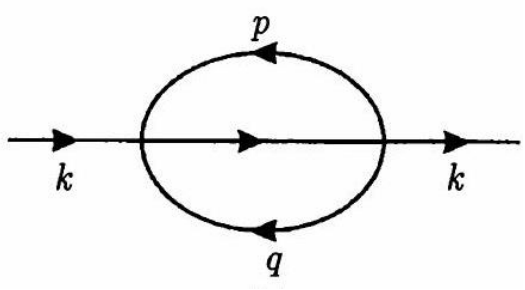
[1] В самом деле. кинематическую точку $s_0 = t_0 = u_0 = \mu^2$ нельзя достичь экспериментально. Конечно, не только амплитуда мезон-мезонного рассеяния зависит от обрезания. Рассмотрим обратный оператор к пропагатору φ до порядка λ^2 на рис. III.3.1. Для фейнмановской диаграммы на рис. III.3.1a имеем $-i\lambda \int^{\Lambda} [d^4q/(2\pi)^4]\,[i/\left(q^2 - m^2 + i\varepsilon\right)]$. Точное значение нас не интересует; заметим только, что оно квадратично зависит от Λ, а не от k^2. Для диаграммы на рис. III.3.1b справедлив двойной интеграл:

$$I(k, m, \Lambda; \lambda) \equiv (-i\lambda)^2 \int^{\Lambda}\int^{\Lambda} \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon}\frac{i}{q^2 - m^2 + i\varepsilon}\frac{i}{(p+q+k)^2 - m^2 + i\varepsilon}$$

После подсчета степеней p и q обнаружим квадратичную зависимость интеграла $\sim \int (d^8P/P^6)$ и, следовательно, функции I от обрезания Λ.



(a)



(b)

Рис. III.3.1

Из лоренцевской инвариантности следует, что I есть функция от k^2, которая раскладывается в ряд $D + Ek^2 + Fk^4 + ...$ Величина D соответствует функции I с нулевым внешним импульсом k и, значит, квадратично зависит от обрезания Λ. Теперь, дважды дифференцируя I по k и приравнивая затем k к нулю, находим E. При этом в два раза уменьшается значение степеней p и q в подынтегральном выражении, так что E начинает зависеть от порога Λ лишь логарифмически. Аналогично находим значение F четырехкратным дифференцированием I по k и приравниваем k к нулю. В результате значения степеней p и q уменьшатся в 4 раза и величину F можно будет найти как интеграл $\sim \int d^8P/P^{10}$ для больших значений P. Это сходящийся интеграл, следовательно, он не зависит от обрезания. Таким образом, F и члены в (...) не зависят от обрезания при его стремлении к бесконечности, и поэтому они не должны нас заботить.

С учетом всего вышесказанного приходим к обратному пропагатору $k^2 - m^2 + a + bk^2$ с точностью $O(k^2)$, в котором параметры a и b квадратично и логарифмически зависят от обрезания. Пропагатор сводится к виду:

$$\frac{1}{k^2 - m^2} \to \frac{1}{(1+b)k^2 - (m^2 - a)}$$

Полюс в k^2 сдвигается к $m_P^2 \equiv m^2 + \delta m^2 \equiv (m^2 - a)(1+b)^{-1}$, т. е. к физической массе. Этот сдвиг называют перенормировкой. С физической точки зрения довольно естественно, что квантовые флуктуации приводят к слвигу массы.

Что можно сказать по поводу вычета в полюсе пропагатора, который теперь равен не 1, а $(1+b)^{-1}$?

Для объяснения сдвига в вычете вспомним, как мы беспечно нормировали поле φ, так что $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + ...$ Коэффициент при k^2 в обратном пропагаторе низшего порядка $k^2 - m^2$ равен 1, следовательно, коэффициент при $\frac{1}{2}(\partial\varphi)^2$ в функции $\mathcal{L}$ также равен 1. Однако мы не можем гарантировать, что при включении поправок высшего порядка коэффициент при $\frac{1}{2}(\partial\varphi)^2$ в эффективном лагранжиане $\mathcal{L}$ останется равным 1. Действительно, он сдвигается к $(1 + b)$. Данный сдвиг получил название «перенормировки волновой функции», хотя ни о какой волновой функции речи не идет. Более современным названием мог бы послужить термин «перенормировка поля». (В нашем случае слово «перенормировка» употреблено кстати, ибо мы действительно нормируем поле, хотя сильно к этому и не стремимся.)

К тому же, гораздо проще сказать «логарифмически расходящийся», чем «логарифмически зависимый от обрезания Λ, поэтому довольно часто мы будем употреблять привычный, хотя и менее точный термин «расходящийся». В φ^4-теории перенормировка волновой функции и перенормировка константы связи являются логарифмически расходящимися, тогда как перенормировка массы расходится квадратично.

Затравочная теория возмучений против физической теории возмущений

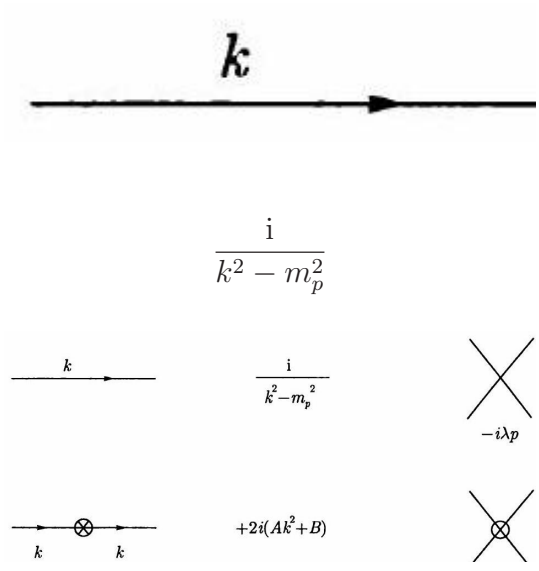
До сих пор мы занимались голой или затравочной теорией возмущений. Нам следовало добавить индекс 0 к величинам φ, m и λ. Поле φ_0 называется затравочным полем, параметры m_0 и λ_0 - затравочной массой и затравочной константой связи соответственно. Я не употреблял индекс 0 в первой части этой книги, поскольку не хотел вводить обозначения до того, как вы узнаете, что именно они обозначают.

С этой точки зрения применение затравочной теории возмущений кажется неоправданным, и это действительно так. Не лучше было бы начать с теории нулевого порядка, сразу записанной в терминах физической массы m_P и физической константы связи λ_P, которые измеряются экспериментаторами, и соответственно изучать теорию возмущений этой теории? В самом деле, это так и происходит, и такой способ вычислений называется перенормированной теорией возмущений или, как я предпочитаю ее называть, физической теорией возмущений.

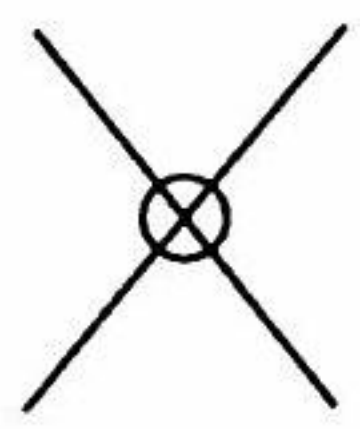
Запишем:

$$\mathcal{L} = \frac{1}{2}\left[(\partial\varphi)^2 - m_P^2\varphi^2\right] - \frac{\lambda_P}{4!}\varphi^4 + A(\partial\varphi)^2 + B\varphi^2 + C\varphi^4$$

(Пара слов об обозначениях: педант, возможно, пожелал бы употребить поле φ с индексом P, но давайте как можно реже менять обозначения.) Физическая теория возмущений работает следующим образом. Правила Фейнмана не меняются, за исключением того, что константа связи обозначается как λ_P, а пропагатор $i/\left(k^2 - m_P^2 + i\varepsilon\right)$ записывается уже с физической массой. Последние три члена в выражении (4) называют контрчленами. Коэффициенты A, B и C вычисляются итерационно (см. далее), по мере перехода к все более и более высоким порядкам теории возмущений. На фейнмановских диаграммах они обозначаются крестиками (рис. III.3.2) с соответствующими правилами Фейнмана. Все интегралы по импульсам обрезаны.



$$+2i\left(Ak^2 + B\right)$$



$4!iC$

Рис. III.3.2

Теперь объясню, почему коэффициенты A, B и C определяются итерационно. Предположим, мы вычислили их до порядка λ_P^N; обозначим их в этом порядке как A_N, B_N и C_N. Нарисуем все диаграммы для поряд- ка λ_P^{N+1}. Найдем значения A_{N+1}, B_{N+1} и C_{N+1} при условии, что вычисленный до порядка λ_P^{N+1} пропагатор имеет при m_P полюс с вычетом равным 1, и условии, что амплитуда мезон-мезонного рассеяния, для определенных значений кинематических переменных, равна $-i\lambda_P$. Иначе говоря, контрчлены фиксируются условием, согласно которому m_p и λ_P есть то, что мы о них думаем. Естественно, A, B и C будут зависимыми от обрезания. Обратите внимание, что существует точно три условия определения трех неизвестных A_{N+1}, B_{N+1} и C_{N+1}.

При таком рассуждении становится очевидным, почему работает физическая теория возмущений. Она справедлива тогда, когда все физические величины, которые мы вычисляем, не зависят от обрезания. Представьте, к примеру, что вы долго и упорно вычисляли амплитуду мезон-мезонного рассеяния до порядка λ_P^{17}. Она будет содержать как зависимые, так и независимые от обрезания члены. Вы просто добавляете в полученное выражение коэффициент C_{17} и подбираете его значение таким образом, чтобы зависимые от обрезания члены сократились.

И тут вы начинаете волноваться. Вы говорите: «Что, если я захочу вычислить амплитуду превращения двух мезонов в четыре мезона? Этой ситуации соответствуют диаграммы с шестью внешними концами. Если я получу зависимые от порога обрезания члены, что мне делать, ведь в выражении (4) нет контрчлена вида $D\varphi^6$ и нельзя устранить зависимость от обрезания?» Очень проницательно с вашей стороны, но не стоит беспокоиться по этому поводу. Проблему здесь решает теорема подсчета степеней.

Степень расходимости

Рассмотрим диаграмму с B_E внешними φ-линиями. Во-первых, определение: говорят, что диаграмма A имеет кажущуюся степень расходимости D, если она расходится как Λ^D. (Степень логарифмической расходимости $\log\Lambda$ равна $D = 0$.) Теорема гласит, что D принимает значение, равное:

$$D = 4 - B_E$$

Доказательство приведу позже, а пока объясню смысл формулы (5). Для пропагатора имеем $B_E = 2$ и, согласно формуле, $D = 2$. Действительно, расходимость получается

квадратичной. Для амплитуды мезон-мезонного рассеяния имеем $B_E = 4$ и, следовательно, $D = 0$, что как раз соответствует логарифмической расходимости.

Согласно теореме, для диаграммы с шестью внешними ветками (которую иногда формально называют шеститочечной функцией) имеем $B_E = 6$

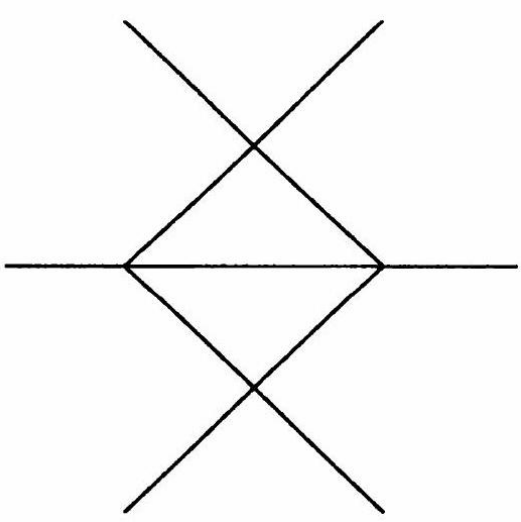


Рис. III.3.3

и $D = -2$. Получается, что по теореме диаграмма сходится или не зависит от обрезания (зависимость от обрезания исчезает при Λ^{-2}). Не сомневайтесь в этом, а просто постройте несколько диаграмм и проверьте данное утверждение. Чем больше на диаграмме внешних веток, тем лучше она сходится.

Доказательство теоремы основано на простом подсчете степеней. Кроме B_E и D, определим параметр B_I, характеризующий количество внутренних линий, параметр V - количество вершин и параметр L - количество петель. (Для удобства возьмем конкретную диаграмму на рис. III.3.3: $B_E = 6, D = -2, B_I = 5, V = 4$ и $L = 2$.)

Число петель соответствует числу интегрирований $\int [d^4k/(2\pi)^4]$, которые нужно выполнить. Каждой внутренней линии соответствует импульс, по которому надо провести интегрирование, так что получаем еще B_I интегралов. Реальное число интегрирований уменьшается за счет дельта-функций сохранения импульса, в вершинах (по одной на каждую вершину). В результате получаем V дельта-функций, одна из которых связана с сохранением общего импульса для всей диаграммы. Число петель равно

$$L = B_I - (V - 1).$$

[Если вы ставите это под сомнение, проанализируйте диаграмму на рис. III.3.3 и получите тождество $2 = 5 - (4 - 1)$.]

Из каждой вершины исходят четыре линии (или входят, в зависимости от того, как на это посмотреть). Каждая внешняя линия исходит (или входит) из одной вершины. Каждая внутренняя линия соединяет две вершины. Таким образом

$$4V = B_E + 2B_I$$

(Для рис. III. 3.3 имеем $4 \cdot 4 = 6 + 2 \cdot 5$.)

Наконец, каждой петле соответствует интеграл $\int d^4k$, а каждой внутренней линии - выражение $i/\left(k^2-m^2+i\varepsilon\right)$, что в два раза уменьшает значения степеней импульса. Получаем,

$$D = 4L - 2B_I$$

(Для рис. III. 3.3 имеем $-2 = 4 \cdot 2 - 2 \cdot 5$.)

Объединяя (6), (6) и (8), приходим к теореме (5)[2]. Теперь вы видите, что ее доказательство строится на подсчете степеней.

Степень расходимости с фермионами

Проверим, усвоили вы наши рассуждения, предшествующие доказательству теоремы (5), или нет. Для этого обратимся к теории Юкавы, с которой мы встречались в (II.5.19). (Для удобства обозначения опустим контрчлены.)

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m_P\right)\psi + \frac{1}{2}\left[(\partial\varphi)^2 - \mu_P^2\varphi^2\right] - \lambda_P\varphi^4 + f_P\varphi\bar{\psi}\psi$$

Теперь мы должны найти число внутренних F_I и внешних F_E фермионных линий и проследить за числом вершин V_f и V_λ, с константами связи f и λ соответственно.

У нас всего пять уравнений. Например, соотношение (7) можно разложить на два, поскольку теперь мы считаем отдельно фермионные линии и отдельно бозонные линии. Например, имеем:

$$V_f + 4V_\lambda = B_E + 2B_I$$

В итоге получаем:

$$D = 4 - B_E - \frac{3}{2}F_E$$

[2] Кажущаяся степень расходимости характеризует расходимость фейнмановской диаграммы при условии, что все внутренние импульсы масштабируются одинаково $k \to \lambda k$, где λ стремится к бесконечности. При более строгом подходе мы должны следить за импульсами в поддиаграммах (части полной диаграммы), которые стремятся к бесконечности при остальных фиксированных импульсах. Итак, для расходящихся амплитуд, т. е. класса диаграмм с $D \geqslant 0$, справедливо $(B_E, F_E) = (0,2), (2,0), (1,2)$ и $(4,0)$, что точно соответствует шести членам в лагранжиане (9). Значит, нам необходимы шесть контрчленов.

Обратите внимание, что подобный подсчет степеней расходимости доказывает факт появления всех членов с размерностью $\leqslant 4$. Например, записывая лагранжиан (9), мы забыли включить в него член $\lambda_P\varphi^4$. Согласно теореме мы должны будем включить этот член в виде контрчлена [член с $(B_E, F_E) = (4,0)$ в приведенном выше списке.]

Выражения (5) и (11) имеют одно общее свойство: они оба зависят только от числа внешних линий и не зависят от числа вершин V. Следовательно, для заданного числа внешних линий не имеет значения, до какого порядка теории возмущений мы двигаемся, поскольку кажущаяся степень расходимости остается одинаковой. Дальнейшие рассуждения приводят нас к мысли, что мы всего лишь формализуем процесс подсчета степеней из предыдущей главы. [Вспомним, что размерность поля Бозе $[\varphi]$ равна 1, а

размерность поля Ферми ψ равна $\frac{3}{2}$, что соответствует коэффициентам 1 и $\frac{3}{2}$ в уравнении (11).]

Наши рассуждения нельзя считать строгим доказательством перенормируемости таких теорий, как теория Юкавы. Для большей строгости вам придется обратиться к монографиям по теории перенормировок, в которых исчерпывающе обсуждаются такие мистические вопросы, как перекрывающиеся расходимости и «лесная» формула Циммермана.

Неперенормируемые теории поля

В контексте наших рассуждений полезно будет узнать, откуда неперенормируемые теории получили свои неприятные особенности. Рассмотрим теорию слабых взаимодействий Ферми, записанную в упрощенной форме:

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m_P\right)\psi + G(\bar{\psi}\psi)^2.$$

Аналогами выражений (6), (7) и (8) будут $L = F_I - (V-1), 4V = F_E + 2F_I$ и $D = 4L - F_I$. Выражая кажущуюся степень расходимости через число внешних фермионных линий, получаем:

$$D = 4 - \frac{3}{2}F_E + 2V.$$

Сравним это выражение с соответствующим уравнением (5) для перенормируемых теорий и с уравнением (11). Видим, что теперь D зависит от V. То есть при расчете, например, фермион-фермионного рассеяния ($F_E = 4$), характер расходимости будет ухудшаться по мере нашего продвижения к более высокому порядку ряда возмущений. Это подтверждает сделанные в предыдущей главе выводы. Действительно, неприятным моментом является условие, что для любого заданного F_E мы придем к расходящимся диаграммам с большим числом V, так что придется включать нескончаемый поток контрчленов $(\bar{\psi}\psi)^3, (\bar{\psi}\psi)^4, (\bar{\psi}\psi)^5$... с произвольными константами связи, которые следует определить из эксперимента. Предсказательные возможности теории сильно ограничиваются.

Одно время неперенормируемые теории считались безнадежными. С современной точки зрения, основанной на подходе эффективных теорий поля, речь о которой пойдет в главе VIII.3, они вполне приемлемы.

Зависимость от размерности

Кажущаяся степень расходимости зависит от размерности d пространства-времени, так как каждой петле сопоставляется интеграл $\int d^d k$. К примеру, рассмотрим взаимодействие Ферми $G(\bar{\psi}\psi)^2$ в $(1+1)$-мерном пространстве-времени. Из трех уравнений, приводящих к выражению (12), одно меняется на $D = 2L - F_I$, поэтому в итоге получаем:

$$D = 2 - \frac{1}{2}F_E$$

Полученное выражение есть аналог (12) для 2-мерного пространства-времени. В отличие от уравнения (12), сюда не входит параметр V, поэтому двумерные диаграммы содержат $F_E = 2$ и 4, которые сокращаются подходящими контрчленами. Взаимодействие Ферми

перенормируемо в (1+ +1)-мерном пространстве-времени. К этому вопросу мы еще вернемся в главе VII.4.

Явление Вайскопфа

В качестве заключения скажу, что поправка к массе поля Бозе и поля Ферми расходится неодинаково. Этот факт впервые обнаружил Вайскопф, поэтому я буду называть его явлением Вайскопфа. Вернемся к уравнению (11) и увидим, что B_E и F_E дают разные вклады в значение кажущейся степени расходимости D. Для $B_E = 2, F_E = 0$ имеем $D = 2$, так что поправка к массе поля Бозе расходится квадратично, что мы уже явно наблюдали [параметр a в (3)]. Однако для $F_E = 2, B_E = 0$ имеем $D = 1$. То есть масса фермиона расходится как бы линейно. В действительности в 4-мерной теории поля нельзя получить линейную зависимость от обрезания. Легче всего это увидеть на примере интеграла Фейнмана из упражнения II.5.1, соответствующего диаграмме на рис. II.5.1:

$$(if)^2 i^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-\mu^2} \frac{\not p + \not k + m}{(p+k)^2 - m^2} \equiv A\left(p^2\right) \not p + B\left(p^2\right)$$

где для удобства введены две неизвестные функции $A\left(p^2\right)$ и $B\left(p^2\right)$. (Для нас сейчас не имеет значения, придерживаемся мы затравочной или физической теории возмущений. В случае последней считайте, что я просто опустил подстрочный индекс P.)

Посмотрим на подынтегральное выражение при больших k. Видим, что интеграл будет иметь вид $\int^{\Lambda} d^4k \, (k/k^4)$ и выглядит линейно расходящимся, но, вследствие симметрии относительно отражений $k \to -k$, равен нулю в ведущем порядке. Интеграл в (14) расходится только логарифмически. Довольно часто кажущаяся степень расходимости D дает лишь оценку сверху, насколько сильны могут быть расходимости (отсюда определение «кажущаяся»). Действительно, из (14) можно доказать, что, например, $B\left(p^2\right)$ должно быть пропорционально m. В качестве упражнения попробуйте с помощью правил Фейнмана (см. главу II.5) доказать справедливость этого утверждения для квантовой электродинамики.

Для бозона квантовые флуктуации дают $\delta\mu \propto \Lambda^2/\mu$, а для фермиона (например, электрона) квантовая поправка к массе $\delta m \propto m \log(\Lambda/m)$ гораздо меньше. Интересно, что в начале двадцатого века физики рассматривали электрон как заряженный шарик радиуса a. Электростатическая энергия такого шарика, значение которой было порядка e^2/a, считалась массой электрона. Трактуя $1/a$ как Λ, мы могли бы сказать, что в классической физике масса электрона пропорциональна Λ и линейно расходится. Следовательно, приходим к одному из определений явления Вайскопфа: «Бозоны ведут себя хуже классического заряда, а фермионы - лучше».

В 1939 году Вайскопф пришел к выводу, что разницу в степенях расходимости можно объяснить эвристически на основе квантовой статистики. «Плохое» поведение бозонов связано с их стадным поведением. Фермион оттолкнул бы виртуальные фермионы, флуктуирующие в вакууме, и создал бы тем самым вокруг себя полость в вакуумном распределении заряда. Таким образом, собственная энергия фермиона менее сингулярна, чем могла бы быть без учета квантовой статистики. У бозона все наоборот.

«Плохое» поведение бозонов будет преследовать нас и дальше.

Упражнения

III.3.1. Покажите, что в $(1+1)$-мерном пространстве-времени дираковское поле ψ имеет размерность $\frac{1}{2}$, так что константа связи Ферми безразмерна.

III.3.2. Выведите (11) из (13).

III.3.3. Покажите, что $B\left(p^2\right)$ в выражении (14) равен нулю, если положить $m=0$. Покажите, что такое же явление справедливо и для квантовой электродинамики.

III.3.4. Мы показали, что вклад (14) в δm логарифмически расходится. Убедитесь, что это верно для любого конечного порядка теории возмущений.

18.1.3 Zi.III.4 Gauge Invariance: Photon Knows No Rest

Нарушение основного тождества

В главе I.7 говорилось, что интеграл по траекториям для произвольной теории поля можно формально вычислить с помощью соотношения, которое вполне заслуженно называют Центральным тождеством квантовой теории поля:

$$\int D\varphi e^{-\frac{1}{2}\varphi\cdot K\cdot\varphi - V(\varphi)+J\cdot\varphi} = e^{-V(\delta/\delta J)} e^{\frac{1}{2}J\cdot K^{-1}\cdot J}$$

Для любой теории поля мы всегда можем собрать все поля и поместить их в один гигантский вектор-столбец φ. Затем мы выделим из вектора φ квадратичный член и запишем его как $\frac{1}{2}\varphi\cdot K\cdot\varphi$; оставшийся вектор назовем $V(\varphi)$. Я пользуюсь компактным обозначением, при котором все координаты пространства-времени и все индексы полей, включая лоренцевские индексы, включены в индексы в формальной матрице K. Часто уравнение (1) будем записывать с $V=0$:

$$\int D\varphi e^{-\frac{1}{2}\varphi\cdot K\cdot\varphi + J\cdot\varphi} = e^{\frac{1}{2}J\cdot K^{-1}\cdot J}$$

Что делать, если K не имеет обратной матрицы?

Такой случай не является чем-то эзотерическим, присущим патологическим теориям поля. Он реализуется в одном из самых основных действий в физике, действии Максвелла:

$$S(A)=\int d^4x\mathcal{L}=\int d^4x\left[\frac{1}{2}A_\mu\left(\partial^2 g^{\mu\nu}-\partial^\mu\partial^\nu\right)A_\nu + A_\mu J^\mu\right]$$

Формальная матрица K в уравнении (2) пропорциональна дифференциальному оператору $\left(\partial^2 g^{\mu\nu}-\partial^\mu\partial^\nu\right)\equiv Q^{\mu\nu}$. Матрица не имеет обратной матрицы, если некоторые из ее собственных значений равны нулю, т. е. если при ее действии на вектор матрица зануляет этот вектор. Видим, что $Q^{\mu\nu}$ зануляет вектор $\partial_\nu\Lambda(x) : Q^{\mu\nu}\partial_\nu\Lambda(x)=0$. Значит, $Q^{\mu\nu}$ не имеет обратной.

В этом явлении нет ничего таинственного; мы уже встречались с ним в классической физике. Действительно, когда мы впервые изучали законы электричества, нам говорили, что только «падение напряжения» между двумя точками имеет физический смысл. На более сложном уровне мы узнали, что всегда можно добавить некоторую константу (или произвольную функцию времени) в электростатический потенциал (который, конечно, есть просто «напряжение»), поскольку по определению его градиентом является электрическое поле. На еще более сложном уровне мы видим, что, решая уравнение Максвелла (которое

получается из экстремума действия), мы находим обратную матрицу Q^{-1}. [Уравнение Максвелла $\partial_\mu F^{\mu\nu}$ я записал здесь в виде $Q_{\mu\nu}A^\nu = J_\mu$ с решением $A^\nu = (Q^{-1})^{\nu\mu}\, J_\mu$.]

Итак, Q^{-1} не существует! Что делать? Мы знаем, что следует наложить дополнительное условие на калибровочный потенциал A^μ, так называемую «фиксацию калибровки».

Простое объяснение

Чтобы подчеркнуть довольно земное происхождение фиксации калибровки (которую в старых книгах превращали во что-то таинственное и почти безнадежно сложное для понимания), рассмотрим обычный интеграл $\int\limits_{-\infty}^{+\infty} dAe^{-A\cdot K\cdot A}$, в котором $A = (a, b)$ есть двухкомпонентный вектор, а $K = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ есть матрица, не имеющая обратной. Конечно, вы понимаете, в чем состоит трудность: у нас есть $\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} dadbe^{-a^2}$, в то время как интеграла по b не существует. Для определения интеграла вставим в него дельта-функцию $\delta(b-\xi)$. Теперь интеграл определен и фактически не зависит от произвольного числа ξ. В более общем виде можно ввести в интеграл дельта-функцию $\delta[f(b)]$, где f - некоторая функция на наш выбор. В случае обычного интеграла эти действия выглядят нелепыми и лишними, но мы будем пользоваться аналогом этой процедуры в других случаях.

В калибровочных теориях необходимость наложения руками связи, фиксирующей калибровку, можно объяснить с физической точки зрения. В главе I.5 мы обошли вопрос фиксации калибровки за счет того, что рассматривали вместо фотона массивный векторный мезон. В результате мы заменили $Q^{\mu\nu}$ на $(\partial^2 + m^2)\, g^{\mu\nu} - \partial^\mu\partial^\nu$, для которой существует обратная матрица (на самом деле мы даже нашли ее явным вычислением).

Рассмотрим движущийся массивный векторный мезон. С помощью лоренцевских преобразований мы всегда можем привести его в покоящуюся систему координат и применить к ней все, что знаем о группе вращений, а именно то, что мезон со спином 1 имеет три спиновых состояния (в классической физике - состояния поляризации). Однако, если векторный мезон безмассов, мы больше не можем привести его в покоящуюся систему координат. Фотон не знает покоя!

Безмассовое поле со спином 1 отличается от массивного поля со спином 1 - в этом состоит корень проблемы. Теперь у нас есть лишь вращения вокруг направлсния движения фотона, т. е. $O(2)$. У фотона всего две поляризационные степени свободы. (Из классической электродинамики вы знаете, что электромагнитная волна обладает двумя поперечными степенями свободы.) Таково истинное физическое происхождение калибровочной инвариантности.

С этой точки зрения калибровочная инвариантность является, строго говоря, не настоящей симметрией, а лишь отражением того факта, что мы пользуемся избыточным описанием: лоренцевским [1] векторным полем для описания двух физических степеней свободы.

Ограничение функционального интеграла

Теперь обратимся к методу борьбы с этой избыточностью, изобретенному Фаддеевым и Поповым. Вы увидите, что он аналогичен методу, использованному нами в разобранном выше примере. Даже в случае электромагнетизма применение этого метода кажется

излишним, однако для неабелевых калибровочных теорий и теории гравитации он имеет большое значение (в главе VII.1 вы убедитесь в этом). Я изложу метод в наиболее общем виде. В следующем разделе мы рассмотрим конкретный пример. Если у вас возникнут трудности с пониманием этого раздела, возможно, полезно будет читать эти два раздела параллельно.

Предположим, нам необходимо вычислить интеграл $I \equiv \int DAe^{iS(A)}$; это может быть обычный интеграл или интеграл по траекториям. Допустим, вследствие преобразования $A \to A_g$ подынтегральное выражение и мера не меняются, т. е. $S(A) = S(A_g)$ и $DA = DA_g$. Очевидно, что преобразования образуют группу, поскольку при преобразовании с g' подынтегральное выражение и мера не изменятся при совместном преобразовании g и g' и $A_g \to (A_g)_{g'} = A_{gg'}$. Нам бы хотелось записать интеграл I в виде $I = \left(\int Dg\right) J$, где J не зависит от g, и отфакторизовать тем самым избыточное интегрирование по g. Обратите внимание, что Dg есть инвариантная мера для группы преобразований, а $\int Dg$ есть объем группы. Не забывайте, что

[1] Четырехкомпонентным. - Прим. ред. мы пользуемся компактным обозначением интеграла по траекториям: параметры A и g есть функции координат пространства-времени x.

Подчеркну, что в наших действиях нет ничего выдающегося и мистического. Для вычисления интеграла $I = \int dxdye^{iS(x,y)}$, где $S(x,y)$ - некоторая функция от x^2+y^2, удобно перейти к полярным координатам $I = \left(\int d\theta\right) J = (2\pi)J$, где $J = \int drre^{iS(r)}$ - интеграл по радиальной координате r. Множитель 2π есть в точности объем группы вращений в двух измерениях.

Фаддеев и Попов продемонстрировали универсальный и очень элегантный способ «перехода к полярным координатам». Они предложили записывать единицу как $1 = \Delta(A)\int Dg\delta\left[f\left(A_g\right)\right]$, т. е. равенством, определяющим $\Delta(A)$. Здесь f есть некоторая функция на наш выбор, а $\Delta(A)$ - детерминант Фаддеева-Попова, зависящий от f. Обратите внимание, что $\left[\Delta\left(A_{g'}\right)\right]^{-1} = \int Dg\delta\left[f\left(A_{g'g}\right)\right] = \int Dg''\delta\left[f\left(A_{g''}\right)\right] = [\Delta(A)]^{-1}$, где второе равенство следует из определения $g'' = g'g$ и условия $Dg'' = Dg$. Другими словами, мы показали, что $\Delta(A) = \Delta\left(A_g\right)$: детерминант Фаддеева-Попова является калибровочно-инвариантным. Теперь подставим 1 в интеграл I, который необходимо вычислить, и получим:

$$\begin{aligned} I &= \int DAe^{iS(A)} = \\ &= \int DAe^{iS(A)}\Delta(A)\int Dg\delta\left[f\left(A_g\right)\right] = \\ &= \int Dg\int DAe^{iS(A)}\Delta(A)\delta\left[f\left(A_g\right)\right]. \end{aligned}$$

Раз мы физики, а не математики, можем спокойно поменять порядок интегрирования.

С точки зрения физики, всегда можно заменить переменные интегрирования, пока не будет доказано обратное, поэтому заменим A на $A_{g^{-1}}$. Получим:

$$I = \left(\int Dg\right)\int DAe^{iS(A)}\Delta(A)\delta[f(A)]$$

где учитывается условие инвариантности $DA, S(A)$ и $\Delta(A)$ относительно преобразования $A \to A_{g^{-1}}$.

Вот так. Мы отфакторизовали интегрирование по группе $\left(\int Dg\right)$.

Объем компактной группы конечен, однако в калибровочных теориях для каждой точки пространства-времени существует отдельная группа, поэтому $\left(\int Dg\right)$ является бесконечным фактором. (Это также объясняет, почему в теориях с глобальными симметриями нет необходимости в фиксации калибровки.) К счастью, как говорилось в главе I.3, в рамках теории поля мы можем не беспокоиться по поводу общих множителей в интеграле по траекториям Z, поэтому множитель $\left(\int Dg\right)$ может быть просто отброшен.

Фиксация калибровки в электродинамике

Теперь применим метод Фаддеева-Попова к электродинамике. Преобразование, относительно которого действие инвариантно, конечно же $A_\mu \to A_\mu - \partial_\mu \Lambda$, где g обозначено как Λ, а $A_g \equiv A_\mu - \partial_\mu \Lambda$. Обратите внимание, что, поскольку начальный интеграл I не зависит от f, он не будет зависеть от f, несмотря на то, что f появляется в (5). Пусть $f(A) = \partial A - \sigma$, где σ - функция от x. В частности, интеграл I не зависит от σ, поэтому можно проинтегрировать I с произвольным функционалом от σ, например, с функционалом $e^{-(i/2\xi)\int d^4x \sigma(x)^2}$.

Теперь применим метод. Сначала вычислим

$$[\Delta(A)]^{-1} \equiv \int Dg \delta\left[f\left(A_g\right)\right] = \int D\Lambda \delta\left(\partial A - \partial^2 \Lambda - \sigma\right)$$

Видим, что в интеграле (5) параметр $\Delta(A)$ умножается на $\delta[f(A)]$, так что, вычисляя (6), можно принять $f(A) = \partial A - \sigma$ равной нулю. Из уравнения (6) получаем $\Delta(A) \ll= \left[\int D\Lambda \delta\left(\partial^2 \Lambda\right)\right]^{-1}$. Правая часть этого выражения не зависит от A, поэтому его можно отбросить. С точностью до несущественных общих множителей, которые можно отбросить, интеграл I равен просто $\int DA e^{iS(A)} \delta(\partial A - \sigma)$.

Интегрируя по $\sigma(x)$ мы окончательно получаем:

$$\begin{aligned} Z &= \int D\sigma e^{-(i/2\xi)\int d^4x \sigma(x)^2} \int DA e^{iS(A)} \delta(\partial A - \sigma) = \\ &= \int DA e^{iS(A) - (i/2\xi)\int d^4x (\partial A)^2}. \end{aligned}$$

Не правда ли, изящный трюк придумали Фаддеев и Попов?

Итак, $S(A)$ в (3) эффективно заменяется выражением:

$$\begin{aligned} S_{\ni\phi.}(A) &= S(A) - \frac{1}{2\xi} \int d^4x (\partial A)^2 = \\ &= \int d^4x \left\{ \frac{1}{2} A_\mu \left[\partial^2 g^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right] A_\nu + A_\mu J^\mu \right\} \end{aligned}$$

а $Q^{\mu\nu}$ заменяется на $Q^{\mu\nu}_{3.} = \partial^2 g^{\mu\nu} - (1 - 1/\xi)\partial^\mu \partial^\nu$. В импульсном пространстве матрица $Q^{\mu\nu}_{\text{эфф.}} = -k^2 g^{\mu\nu} + (1 - 1/\xi) k^\mu k^\nu$, в самом деле, имеет обратную матрицу. Можете проверить, что

$$Q^{\mu\nu}_{\exists\phi\phi.} \left[-g_{\nu\lambda} + (1 - \xi) \frac{k_\nu k_\lambda}{k^2} \right] \frac{1}{k^2} = \delta^\mu_\lambda.$$

Следовательно, фотонный пропагатор можно представить в виде:

$$\frac{(-i)}{k^2}\left[g_{\nu\lambda}-(1-\xi)\frac{k_\nu k_\lambda}{k^2}\right]$$

в согласии с выводами главы II.7.

В связи с тем что метод Фаддеева-Попова является во многом хитростью, многие физики, в том числе и я, предпочитают следовать явным аргументам Фейнмана, представленным в главе II.7. Однако, как я уже отмечал, в теории Янга-Миллса и теории Эйнштейна без метода Фаддеева-Попова не обойтись.

Размышления о калибровочной симметрии

Как мы увидим позже и как вы, возможно, слышали, многие явления в мире, выходящие за рамки электродинамики, объясняются калибровочными теориями. В некотором смысле калибровочные теории также оказываются неудовлетворительными, поскольку построены на избыточности описания. Электромагнитное калибровочное преобразование $A_\mu \to A_\mu - \partial_\mu\Lambda$ не является истинной симметрией, при которой два физических состояния наделены одинаковыми свойствами. Скорее, оно гласит, что два калибровочных потенциала A_μ и $A_\mu - \partial_\mu\Lambda$ описывают одно и то же физическое состояние. В главе IV.4 я покажу, что A_μ необходимо в первую очередь для уравнения Шредингера. В рамках классической физики можно спокойно обойтись лишь с помощью $\vec{E}$ и $\vec{B}$. Некоторые физики пытаются сформулировать квантовую электродинамику без использования A_μ, но пока не преуспели в поиске альтернативы тому, что мы имеем на данный момент. Возможно, что действительно глубокий прорыв в теоретической физике будет включать в себя формулировку квантовой электродинамики без использования A_μ.

18.1.4 Zi.III.5 Field Theory Without Relativistic Invariance

Замедляясь с возрастом

Когда появилась квантовая теория поля, она была релятивистской. С наступлением периода зрелости теория нашла свое применение в физике конденсированных сред. Можно долго говорить о роли квантовой теории поля в теории конденсированных сред, но у нас более скромная цель, заключающаяся в нахождении нерелятивистского предела квантовой теории поля.

Лоренц-инвариантная теория скалярного поля

$$\mathcal{L}=\left(\partial\Phi^\dagger\right)\left(\partial\Phi\right)-m^2\Phi^\dagger\Phi-\lambda\left(\Phi^\dagger\Phi\right)^2$$

(при условии $\lambda>0$) описывает систему взаимодействующих бозонов. Она обязательно включает в себя физику медленно движущихся бозонов. Для ясности рассмотрим сначала релятивистское уравнение Клейна-Гордона

$$\left(\partial^2+m^2\right)\Phi=0$$

для свободного скалярного поля. Мода с энергией $E=m+\varepsilon$ осциллировала бы во времени как $\Phi\propto e^{-iEt}$. В нерелятивистском пределе кинетическая энергия ε гораздо меньше массы m. Поэтому имеем право записать $\Phi(\vec{x},t)=e^{-imt}\varphi(\vec{x},t)$ при условии, что поле φ осциллирует во времени гораздо медленнее, чем e^{-imt}. Подставляя данное выражение в уравнение (2) и дважды применяя тождество $(\partial/\partial t)e^{-imt})\,(...)=e^{-imt}(-im+\partial/\partial t)(...)$, получаем $(-im+$

$\partial/\partial t)^2\varphi - \vec{\nabla}^2\varphi + m^2\varphi = 0$. Член $(\partial^2/\partial t^2)\,\varphi$ мал по сравнению со значением $-2im(\partial/\partial t)\varphi$, поэтому его можно отбросить. В результате придем к уравнению Шредингера:

$$i\frac{\partial}{\partial t}\varphi = -\frac{\vec{\nabla}^2}{2m}\varphi.$$

Кстати, уравнение Клейна-Гордона было получено раньше уравнения Шредингера.

С учетом всего вышесказанного можно легко найти нерелятивистский предел квантовой теории поля. Просто подставим в уравнение (1) выражение

$$\Phi(\vec{x},t) = \frac{1}{\sqrt{2m}}e^{-imt}\varphi(\vec{x},t)$$

(Множитель $1/\sqrt{2m}$ нужен для удобства дальнейших вычислений.) Например,

$$\frac{\partial\Phi^\dagger}{\partial t}\frac{\partial\Phi}{\partial t} - m^2\Phi^\dagger\Phi \to \frac{1}{2m}\left\{\left[\left(im + \frac{\partial}{\partial t}\right)\varphi^\dagger\right]\left[\left(-im + \frac{\partial}{\partial t}\right)\varphi\right] - m^2\varphi^\dagger\varphi\right\} \simeq$$
$$\simeq \frac{1}{2}i\left(\varphi^\dagger\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi^\dagger}{\partial t}\varphi\right)$$

После интегрирования по частям приходим к выражению

$$\mathcal{L} = i\varphi^\dagger\partial_0\varphi - \frac{1}{2m}\partial_i\varphi^\dagger\partial_i\varphi - g^2\left(\varphi^\dagger\varphi\right)^2$$

где $g^2 = \lambda/4m^2$

В главе I.9 мы видели, что в теории (1) есть сохраняющийся нетеровский ток $J_\mu = i\left(\Phi^\dagger\partial_\mu\Phi - \partial_\mu\Phi^\dagger\Phi\right)$. Как и можно было ожидать, плотность J_0 сводится к $\varphi^\dagger\varphi$, тогда как J_i сводится к виду $(i/2m)\left(\varphi^\dagger\partial_i\varphi - \partial_i\varphi^\dagger\varphi\right)$. Когда вы впервые изучали квантовую механику, не задавались ли вы вопросом, почему плотность $\rho \equiv \varphi^\dagger\varphi$ и ток $J_i = (i/2m)\left(\varphi^\dagger\partial_i\varphi - \partial_i\varphi^\dagger\varphi\right)$ выглядят так по-разному? Многие выражения при редукции из более симметричной в менее симметричную теорию неизбежно становятся менее изящными.

Число сопряжено фазовому углу

Позвольте обратить ваше внимание на некоторые отличия между релятивистским и нерелятивистским случаями.

Самое большое отличие состоит в том, что релятивистская теория квадратична по производным по времени, тогда как нерелятивистская теория линейна по производным по времени. Следовательно, в нерелятивистской теории плотность импульса, сопряженная полю φ, а именно $\partial\mathcal{L}/\delta\partial_0\varphi$, равна просто $i\varphi^\dagger$, так что $\left[\varphi^\dagger(\vec{x},t), \varphi\left(\vec{x}',t\right)\right] = -\delta^{(D)}\left(\vec{x}-\vec{x}'\right)$. В физике конденсированных сред часто полезно записать $\varphi = \sqrt{\rho}e^{i\theta}$, так что:

$$\mathcal{L} = \frac{i}{2}\partial_0\rho - \rho\partial_0\theta - \frac{1}{2m}\left[\rho\left(\partial_i\theta\right)^2 + \frac{1}{4\rho}\left(\partial_i\rho\right)^2\right] - g^2\rho^2$$

Первый член является полной производной. Второй член сообщает нечто очень важное [1] из физики конденсированных сред: в каноническом формализме (глава I.8) плотность импульса, сопряженная фазовому полю $\theta(x)$, равна $\delta\mathcal{L}/\delta\partial_0\theta = -\rho$, следовательно, согласно Гейзенбергу:

$$[\rho(\vec{x},t), \theta\left(\vec{x}',t\right)] = i\delta^{(D)}\left(\vec{x}-\vec{x}'\right)$$

Проинтегрировав и введя параметр $N \equiv \int d^D x \rho(\vec{x}, t)$, равный общему количеству бозонов, получим одно из важнейших соотношений в физике конденсированных сред:

$$[N, \theta] = i$$

Число оказалось сопряженным фазовому углу, подобно тому, как импульс сопряжен с координатой. Изящно, не правда ли? Из курса физики конденсированных сред вы, может быть, знаете, что это фундаментальное соотношение лежит в основе физики контакта Джозефсона.

Вам, должно быть, известно, что система бозонов с короткодействующим отталкиванием при нулевой температуре является сверхтекучей жидкостью. В частности, Боголюбов доказал, что система содержит элементарное возбуждение с линейной дисперсией [2]. О сверхтекучести речь пойдет в главе V.1.

Знак отталкивания

В нерелятивистской теории (уравнение (7)) бозоны отталкиваются друг от друга, и этот факт очевиден. Чтобы собрать частицы в области высокой плотности, потребовалась бы энергия $g^2\rho^2$. В релятивистской теории менее очевидно, что член $\lambda\left(\Phi^\dagger\Phi\right)^2$ с положительным λ соответствует отталкиванию. В упражнении III.5.3 описывается один из возможных методов, а пока прибегнем лишь к эвристическим соображениям. Гамильтониан (плотность) включает в себя лагранжиан со знаком минус, поэтому для больших значений Φ он записывается как $\lambda\left(\Phi^\dagger\Phi\right)^2$ и не ограничен снизу при $\lambda < 0$. Мы знаем, что физически свободный Бозе-газ склонен к конденсированию и сгущению, и при притягивающем взаимодействии он обязательно сколлапсирует. Естественным образом можно предположить, что условие $\lambda > 0$ соответствует отталкиванию.

Теперь рассмотрим более надежный способ. Если применить основное тождество квантовой теории поля, можно переписать интеграл по траекто-

[1] P. Anderson, Basic Notations of Condensed Matter Physics, p. 235.

[2] Например. Л. Д. Ландау, Е. М. Лифшиц. Статистическая физика, стр. 238. риям, фигурирующий в уравнении (1), в виде:

$$Z = \int D\Phi D\sigma e^{i\int d^4x\left[\left(\partial\Phi^\dagger\right)(\partial\Phi) - m^2\Phi^\dagger\Phi + 2\sigma\Phi^\dagger\Phi + (1/\lambda)\sigma^2\right]}$$

Физики-твердотельщики называют преобразование из (1) в лагранжиан $\mathcal{L} = \left(\partial\Phi^\dagger\right)(\partial\Phi) - m^2\Phi^\dagger\Phi + 2\sigma\Phi^\dagger\Phi + (1/\lambda)\sigma^2$ преобразованием ХаббардаСтратоновича. Поле, не обладающее кинетической энергией, например σ, называется в теории поля вспомогательным, его можно отынтегрировать в интеграле по траекториям. Когда мы будем рассматривать в главе VIII.4 сверхтекучие жидкости, вспомогательные поля будут играть важную роль.

Действительно, в главе III.2 уже говорилось, как теория с промежуточным векторным бозоном генерирует теорию слабых взаимодействий Ферми. Аналогичные физические законы действуют и в нашем случае: теория (10), в которой поле Φ взаимодействует с «промежуточным σ-бозоном», может генерировать теорию (1).

Если бы σ было «нормальным скалярным полем», которые нам известны, т. е. если бы члены квадратичные по σ, в лагранжиане, имели вид $\frac{1}{2}(\partial\sigma)^2 - \frac{1}{2}M^2\sigma^2$, то пропагатор был бы $i/\left(k^2 - M^2 + i\varepsilon\right)$. Амплитуда рассеяния двух Φ-бозонов была бы в этом случае пропорциональна пропагатору. Из главы I.4 нам известно, что обмен скалярным полем приводит к притяжению.

Однако, σ не есть нормальное поле, что следует из того факта, что лагранжиан содержит единственный квадратичный член $+(1/\lambda)\sigma^2$. Следовательно, его пропагатор равен просто $i/(1\lambda) = i\lambda$, его знак (при $\lambda > 0$) противоположен знаку нормального пропагатора, получаемого при малой передаче импульса $i/\left(k^2 - M^2 + i\varepsilon\right) \simeq -i/M^2$. Заключаем, что обмен σ-бозоном приводит к отталкиванию.

Это объяснение, к тому же, показывает, что сила отталкивания имеет бесконечно малый радиус действия, типа дельта-функционного взаимодействия. В главе I.4 говорилось, что расстояние определяется соотношением членов k^2 и M^2. В нашем случае член M^2 бесконечно большой. Можно также утверждать, что взаимодействие $\lambda\left(\Phi^\dagger\Phi\right)$ включает в себя рождение двух бозонов и их уничтожение в одной и той же точке пространства-времени.

Конечная плотность

Специалисты в области физики элементарных частиц иногда забывают, что специалистов в области конденсированных сред не интересует пустое пространство и что они хотели бы иметь конечную плотность $\bar{\rho}$-бозонов. Из статистической механики известен способ, при котором в лагранжиан (6) добавляется химический потенциал $\mu\varphi^\dagger\varphi$. С точностью до несущественных (в данном контексте!) аддитивных констант перепишем лагранжиан в виде:

$$\mathcal{L} = i\varphi^\dagger\partial_0\varphi - \frac{1}{2m}\partial_i\varphi^\dagger\partial_i\varphi - g^2\left(\varphi^\dagger\varphi - \bar{\rho}\right)^2$$

Удивительно, но в релятивистской и нерелятивистской теориях поля масса появляется в разных местах. Для дальнейшего мне потребуется понятие спонтанного нарушения симметрии, поэтому на этом пока все. К вопросу о сверхтекучести вернемся в свое время.

Упражнения

III.5.1. Получите уравнение Клейна-Гордона для частицы в электростатическом потенциале (таком, как потенциал ядра), основываясь на калибровочном принципе замены $(\partial/\partial t)$ в уравнении (2) на $\partial/\partial t - ieA_0$. Покажите, что в нерелятивистском пределе оно сводится к уравнению Шредингера для частицы во внешнем потенциале.

III.5.2. Найдите нерелятивистский предел дираковского лагранжиана.

III.5.3. Для конкрстной теории поля мы можем вычислить амплитуду рассеяния двух частиц в нерелятивистском пределе. Затем потребовать существование потенциала взаимодействия $U(\vec{x})$ между двумя частицами и с помощью нерелятивистской квантовой механики вычислить амплитуду рассеяния, например, в борновском приближении. Сравнивая две амплитуды рассеяния, можно найти значение $U(\vec{x})$. Вычислите таким способом юкавский и кулоновский потенциалы. Применение этого метода к взаимодействию $\lambda\left(\Phi^\dagger\Phi\right)^2$ несколько затрудняется из-за того, что дельта-функционное взаимодействие сингулярно. Тем не менее с помощью этого метода можно определить, является сила отталкивающей или притягивающей.

18.1.5 Zi.III.6 Magnetic Moment of the Electron

Триумф Дирака

Во введении я сказал, что не собираюсь в этой книге уделять много места вычислениям, но не могу удержаться и не рассказать сейчас о величайшем триумфе квантовой теории поля.

После того как Дирак написал свое уравнение, следующим шагом был вопрос о взаимодействии электрона с электромагнитным полем. Согласно калибровочному принципу, уже примененному к уравнению Шредингера в электромагнитном поле, чтобы получить уравнение Дирака для электрона во внешнем электромагнитном поле, мы должны просто заменить обычную производную ∂_μ ковариантной производной $D_\mu = \partial_\mu - ieA_\mu$:

$$(i\gamma^\mu D_\mu - m)\,\psi = 0$$

Сравните с уравнением (II.1.27).

Действуя на это уравнение $(i\gamma^\mu D_\mu + m)$, получаем $-(\gamma^\mu\gamma^\nu D_\mu D_\nu + \ +m^2)\,\psi = 0$. Имеем $\gamma^\mu\gamma^\nu D_\mu D_\nu = \frac{1}{2}\left(\{\gamma^\mu,\gamma^\nu\} + [\gamma^\mu,\gamma^\nu]\right) D_\mu D_\nu = D_\mu D^\mu$ - $-i\sigma^{\mu\nu} D_\mu D_\nu$ и $i\sigma^{\mu\nu} D_\mu D_\nu = (i/2)\sigma^{\mu\nu}\,[D_\mu, D_\nu] = (e/2)\sigma^{\mu\nu}F_{\mu\nu}$. Таким образом,

$$\left(D_\mu D^\mu - \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} + m^2\right)\psi = 0.$$

Теперь рассмотрим слабое постоянное магнитное поле, направленное для определенности вдоль третьей оси. Оно настолько слабо, что можно пренебречь членом $(A_i)^2$ в $(D_i)^2$. Согласно калибровочной инвариантности, мы можем выбрать $A_0 = 0, A_1 = -\frac{1}{2}Bx^2$ и $A_2 = \frac{1}{2}Bx^1$ (так что $F_{12} = \partial_1 A_2 - \partial_2 A_1 = B$). Как мы увидим, это одно из вычислений, в котором нам действительно нужно следить за множителями типа 2. Тогда

$$\begin{aligned}(D_i)^2 &= (\partial_i)^2 - ie\,(\partial_i A_i + A_i\partial_i) + O\left(A_i^2\right) = \\ &= (\partial_i)^2 - 2\frac{ie}{2}B\left(x^1\partial_2 - x^2\partial_1\right) + O\left(A_i^2\right) = \\ &= \vec{\nabla}^2 + e\vec{B}\cdot\vec{x}\times\vec{p} + O\left(A_i^2\right).\end{aligned}$$

Обратите внимание, что мы воспользовались условием $\partial_i A_i + A_i\partial_i = (\partial_i A_i) + 2A_i\partial_i$, где в $(\partial_i A_i)$ частные производные действуют только на A_i. В $\vec{L} \equiv \vec{x}\times\vec{p}$ вы, наверное, узнали оператор орбитального углового момента импульса. Таким образом, орбитальный угловой момент порождает орбитальный магнитный момент, взаимодействующий с магнитным полем.

В приведенном расчете есть определенный физический смысл. Если бы мы изучали взаимодействие заряженного скалярного поля Φ с внешним электромагнитным полем, мы бы начали с уравнения

$$\left(D_\mu D^\mu + m^2\right)\Phi = 0$$

получаемого путем замены обычной производной в уравнении КлейнаГордона ковариантными производными. Затем мы провели бы аналогичное уравнению (3) вычисление. Сравнивая выражения (4) и (2), видим, что спин электрона вносит вклад в виде $(e/2)\sigma^{\mu\nu}F_{\mu\nu}$.

Как и в главе II.1, запишем $\psi = \begin{pmatrix}\phi\\ \chi\end{pmatrix}$ в дираковском базисе и сконцентрируемся на параметре ϕ, поскольку в нерелятивистском пределе он много больше χ. Вспомним, что в этом базисе $\sigma^{ij} = \varepsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}$. Значит, член $(e/2)\sigma^{\mu\nu}F_{\mu\nu}$, действующий на ϕ, равен $(e/2)\sigma^3\,(F_{12} - F_{21}) = (e/2)2\sigma^3 B = 2e\vec{B}\cdot\vec{S}$, так как $\vec{S} = (\vec{\sigma}/2)$. Убедитесь, что вы понимаете происхождение всех множителей 2 ! Между тем, согласно тому, что я говорил в главе II.1, нам следует записать $\phi = e^{-imt}\Psi$, где Ψ осциллирует гораздо медленнее, чем e^{-imt}, так что

имеет место $(\partial_0^2 + m^2)\, e^{-imt}\Psi \simeq \simeq e^{-imt}[-2im(\partial/\partial t)\Psi]$. Объединяя все выражения, приходим к уравнению:

$$\left[-2im\frac{\partial}{\partial t} - \vec{\nabla}^2 - e\vec{B}\cdot(\vec{L}+2\vec{S})\right]\Psi = 0.$$

Полюбуйтесь на него! Уравнение Дирака как по волшебству говорит нам, что спиновый момент импульса взаимодействует с магнитным полем с коэффициентом 2, по сравнению с орбитальным моментом импульса. Этот экспериментальный факт в то время сильно озадачивал физиков. Вычисление, приводящее к уравнению (5), справедливо считается одним из важнейших в истории физики.

Известна история о том, что Дирак, открыв свое уравнение, не сделал этого вычисления. Вот насколько он был уверен в справедливости своего уравнения! По другой версии, он опасался, что магнитный момент окажется неверным и что Природа не воспользовалась преимуществом его красивейшего уравнения. Существует другой способ увидеть, что уравнение Дирака содержит магнитный момент, - применить разложение Гордона, доказательство которого приведено в упражнении:

$$\bar{u}\,(p')\,\gamma^\mu u(p) = \bar{u}\,(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}\,(p'-p)_\nu}{2m}\right]u(p).$$

Посмотрев на взаимодействие с электромагнитным полем $\bar{u}\,(p')\,\gamma^\mu u(p)\times\ \times A_\mu\,(p'-p)$, мы видим, что первый член в уравнении (6) зависит от импульса $(p'+p)^\mu$ и будет присутствовать в уравнении, даже при взаимодействии заряженной скалярной частицы с электромагнитным полем в первом порядке. Второй член содержит спин и дает магнитный момент. Иначе говоря, член $\bar{u}\,(p')\,\gamma^\mu u(p)$ содержит магнитный момент.

Аномальный магнитный момент

Совершенствование техники эксперимента позволило в конце 1940-х гг. установить, что магнитный момент электрона превышает значение, вычисленное Дираком, в $1,00118 \pm 0,00003$ раза. После этого целью любой теории квантовой электродинамики явился расчет данного аномального магнитного момента. Вы, должно быть, знаете, что впечатляющий успех, достигнутый Швингером при решении этой проблемы, позволил установить вне всяких сомнений, справедливость релятивистской квантовой теории поля, по крайней мере в том, что касается электромагнитных явлений.

Прежде чем погрузиться в вычисления, вспомним, что из лоренцевской инвариантности и сохранения тока следует (см. пример III.6.3), что матричный элемент электромагнитного тока должен иметь следующий вид (здесь $|p,s\rangle$ означает состояние электрона с импульсом p и поляризацией s):

$$\langle p',s'\,|J^\mu(0)|\,p,s\rangle = \bar{u}\,(p',s')\left[\gamma^m u F_1\left(q^2\right) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2\left(q^2\right)\right]u(p,s),$$

где $q \equiv (p'-p)$. Функции $F_1\,(q^2)$ и $F_2\,(q^2)$, о которых ничего не может нам сказать лоренцевская инвариантность, известны как форм-факторы. В главном порядке по переданному импульсу q выражение (7) с помощью разложения Гордона сводится к виду:

$$\bar{u}\,(p',s')\left\{\frac{(p'+p)^\mu}{2m}F_1(0) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\left[F_1(0)+F_2(0)\right]\right\}u(p,s).$$

Коэффициент в первом члене есть электрический заряд, наблюдаемый экспериментаторами и по определению равный 1. То есть $F_1(0) = 1$. Магнитный момент электрона отличается от дираковского значения на $1 + F_2(0)$.

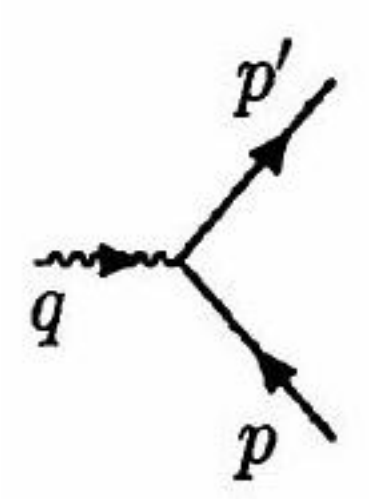
Триумф Швингера

Давайте теперь вычислим $F_2(0)$ до порядка $\alpha = e^2/4\pi$. Сначала нарисуем все соответствующие фейнмановские диаграммы в данном порядке (рис. III.6.1). За исключением диаграммы 1 b, все фейнмановские диаграммы, очевидно, пропорциональны $\bar{u}(p', s')\,\gamma^\mu u(p, s)$ и, значит, вносят свой вклад в $F_1(q^2)$, что, однако, не должно нас волновать. Как удачно! Нам необходимо будет вычислить лишь одну фейнмановскую диаграмму.

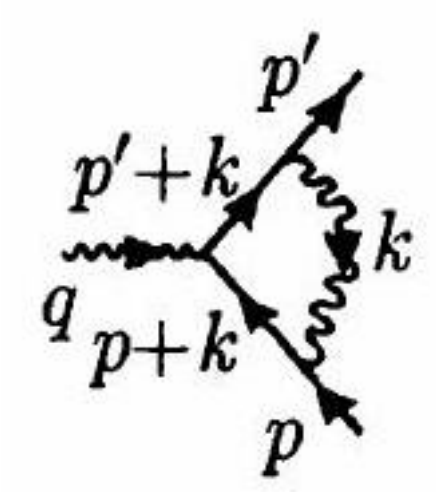
Для удобства нормируем вклад рисунка 1 b путем его сравнения со вкладом рисунка 1a низшего порядка и запишем сумму двух вкладов в виде $\bar{u}\,(\gamma^\mu + \Gamma^\mu)\,u$. Применяя правила Фейнмана, получаем:

$$\Gamma^\mu = \int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2}\left(ie\gamma^\nu \frac{i}{\not p' + \not k - m}\gamma^\mu \frac{i}{\not p + \not k - m} ie\gamma_\nu\right).$$

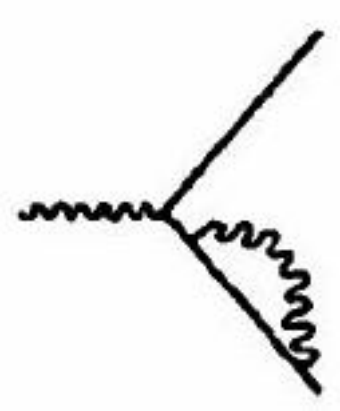
Сейчас я перехожу к довольно подробным вычислениям не только в связи с его важностью, но и потому, что мы воспользуемся несколькими изящными трюками, к которым впервые прибегли Швингер и Фейнман. Конечно, вам стоит самим проделать все выкладки.



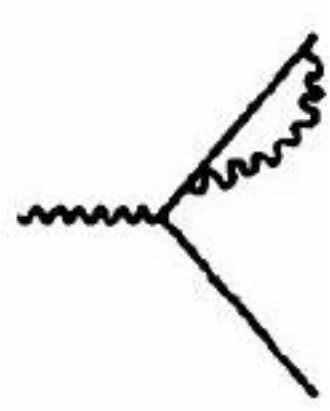
(a)



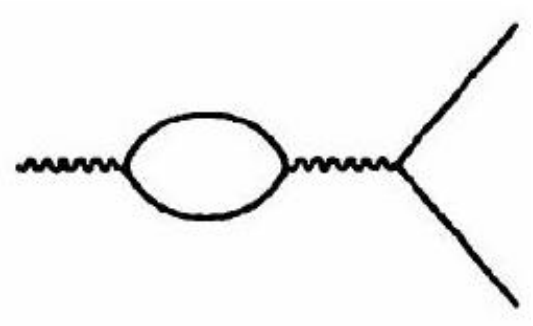
(b)



(c)



(d)



(e)
Рис. III.6.1
Сделав некоторые упрощения, получим $\Gamma^\mu = -ie^2 \int [d^4k/(2\pi)^4]\,(N^\mu/D)$, где

$$N^\mu = \gamma^\nu \left(\not{p}' + \not{k} + m\right) \gamma^\mu (\not{p} + \not{k} + m)\gamma_\nu$$

и

$$\frac{1}{D} = \frac{1}{(p'+k)^2 + m^2} \frac{1}{(p+k)^2 + m^2} \frac{1}{k^2} = 2 \int d\alpha d\beta \frac{1}{\mathcal{D}}.$$

Мы воспользовались тождеством (D.16). Интеграл вычисляется по треугольнику на плоскости (α, β), ограниченной прямыми $\alpha = 0, \beta = 0$ и $\alpha + +\beta = 1$. Имеем

$$\mathcal{D} = \left[k^2 + 2k\left(\alpha p' + \beta p\right)\right]^3 = \left[l^2 - (\alpha + \beta)^2 m^2\right]^3 + O\left(q^2\right)$$

где мы выделили полный квадрат, положив $k = l - (\alpha p' + \beta p)$.

Наша стратегия состоит в приведении N^μ к виду, состоящему из линейной комбинации γ^μ, p^μ и p'. Применяя разложение Гордона (6), запишем (7) в виде:

$$\bar{u}\left\{\gamma^\mu \left[F_1\left(q^2\right) + F_2\left(q^2\right)\right] - \frac{1}{2m}\left(p' + p\right)^\mu F_2\left(q^2\right)\right\} u$$

Следовательно, чтобы выделить $F_2(0)$, можно выкинуть все члены, пропорциональные γ^μ, которые мы учитывали, преобразуя N. Итак, продолжим. Выражая k через l в уравнении (9), получаем

$$N^\mu = \gamma^\nu \left[\not{P}' + m\right] \gamma^\mu [\not{P} + m]\gamma_\nu$$

где $P'\mu \equiv (1-\alpha)p'^\mu - \beta p^\mu$ и $P^\mu \equiv (1-\beta)p^\mu - \alpha p'$. Я буду неоднократно прибегать к тождествам из приложения D и не предупреждать вас каждый раз об этом. Удобно члены в N^μ сгруппировать по степеням m. (Далее я отказываюсь писать в виде завершенных грамматических предложений.)

1. Член m^2 : член γ^μ отбрасывается.

2. Члены m: выстраиваются по степеням l. Линейный по l член зануляется при интегрировании по соображениям симметрии. Остается член, который не зависит от l:

$$m\left(\gamma^\nu \not{P}'\gamma^\mu\gamma_\nu + \gamma^\nu\gamma^\mu \not{P}\gamma_\nu\right) = 4m\left[(1-2\alpha)p'^\mu + (1-2\beta)p^\mu\right] \to$$
$$\to 4m(1-\alpha-\beta)\left(p'+p\right)^\mu$$

На последнем шаге я воспользовался таким соображением: поскольку $\mathcal{D}$ симметрично относительно $\alpha \leftrightarrow \beta$, можно симметризовать члены в выражении для N^μ.

3. Наконец, самый сложный член m^0. Член второго порядка по l: обратите внимание на возможность эффективной замены $l^\sigma l^\tau$ внутри $\int d^4l/(2\pi)^4$ на $\frac{1}{4}\eta^{\sigma\tau}l^2$ в силу лоренцевской инвариантности (этот шаг возможен потому, что мы сдвинули переменную интегрирования; так что $\mathcal{D}$ есть лоренц-инвариантная функция от l^2.) Таким образом, член второго порядка по l приводит к появлению члена γ^μ. Отбрасываем его. Снова отбрасываем линейный по l член, оставляя в итоге [с учетом тождества (D.6)!]

$$\gamma^\nu \not{P}'\gamma^\mu\gamma_\nu = -2\not{P}\gamma^\mu\not{P}' \to$$
$$\to -2[(1-\beta)\not{p} - \alpha m]\gamma^\mu\left[(1-\alpha)\not{p}' - \beta m\right].$$

где на последнем шаге вспоминаем, что Γ^μ стоит в обкладках $\bar{u}\left(p'\right)$ и $u(p)$. Опять удобно члены в (14) сгруппировать по степеням m. Прибегая к различным, уже известным нам трюкам, находим, что член m^2 можно отбросить, член m дает $2m\left(p'+p\right)^\mu\left[\alpha(1-\alpha)+\beta(1-\beta)\right]$, а член m^0 дает нам $2m\left(p'+p\right)^\mu\left[-2(1-\alpha)(1-\beta)\right]$. Объединяя все вместе, приходим к тому, что $N^\mu \to 2m\left(p'+p\right)^\mu(\alpha+\beta)(1-\alpha-\beta)$.

Итак, теперь можно вычислить интеграл $\int\left[d^4l/(2\pi)^4\right](1/\mathcal{D})$ с учетом (D.11). В конечном итоге получим

$$\Gamma^\mu = -2ie^2\int d\alpha d\beta\left(\frac{-i}{32\pi^2}\right)\frac{1}{(\alpha+\beta)^2m^2}N^\mu =$$
$$= -\frac{e^2}{8\pi^2}\frac{1}{2m}\left(p'+p\right)^\mu$$

и, следовательно (аплодисменты, пожалуйста):

$$F_2(0) = \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi}$$

Когда в 1948 году Швингер объявил о полученном им результате, это произвело огромное впечатление на сообщество физиков-теоретиков.

В настоящей главе я рассказал не об одном, а о двух триумфах великих физиков двадцатого века, хотя первый из двух триумфов не является по своей сути результатом теории поля.

Упражнения

III.6.1. Найдите $\bar{u}\left(p'\right)\left(\not{p}'\gamma^\mu + \gamma^\mu\not{p}\right)u(p)$ двумя разными способами и докажите тем самым справедливость разложения Гордона.

III.6.2. Проверьте, что (7) согласовано с сохранением тока. [Указание: из трансляционной инвариантности (опустив спиновую переменную) следует:

$$\langle p'\left|J^\mu(x)\right|p\rangle = \langle p'\left|J^\mu(0)\right|p\rangle\, e^{i(p'-p)x}$$

и поэтому

$$\langle p' \left| \partial_\mu J^\mu(x) \right| p \rangle = i \left(p' - p\right)_\mu \langle p' \left| J^\mu(0) \right| p \rangle \, e^{i(p'-p)x}$$

Таким образом, сохранение тока означает, что $q^\mu \langle p' \left| J^\mu(0) \right| p \rangle = 0$.]

III.6.3. Согласно лоренцевской инвариантности правая часть (7) должна быть вектором. Возможны только $\bar{u}\gamma^\mu u, (p+p')^\mu \, \bar{u}u$ и $(p-p')^\mu \, \bar{u}u$. Последний член исключается, поскольку он противоречит сохранению тока. Покажите, что уравнение (7) действительно имеет самый общий вид.

18.1.6 Zi.III.7 Polarizing the Vacuum and Renormalizing the Charge

Фотон может флуктуировать в электрон и позитрон

Одним из ранних достижений квантовой электродинамики явилось понимание того, как квантовые флуктуации влияют на характер распространения фотона. Фотон всегда может превратиться в электрон и позитрон, которэе по истечении короткого промежутка времени, предписанного принципом неопределенности, аннигилируют друг с другом, снова образуя фотон. Этот процесс изображен на рис. III.7.1.

Квантовые флуктуации не ограничиваются теми процессами, что мы сейчас описали. Электрон и позитрон могут взаимодействовать друг с другом, обмениваясь фотоном, который в свою очередь может превратиться в электрон и позитрон и т. д. Этот процесс изображен на рис. III.7.2; здесь заштрихованный объект, обозначаемый как $i\Pi_{\mu\nu}(q)$ и называемый тензором поляризации вакуума, представляется бесконечным числом фейнмановских диаграмм (рис. III.7.3). Рисунок III.7.1 получается из рисунка III.7.2 приближением $i\Pi_{\mu\nu}(q)$ диаграммой низшего порядка.

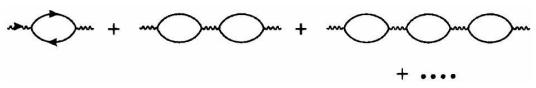


Рис. III.7.1

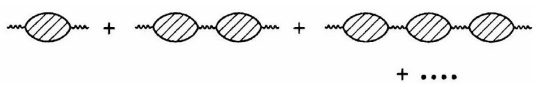


Рис. III.7.2

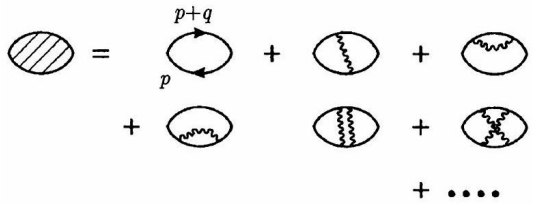


Рис. III.7.3

Удобно переписать лагранжиан $\mathcal{L} = \bar{\psi}\left[i\gamma^\mu\left(\partial_\mu - ieA_\mu\right) - m\right]\psi$ - $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, заменяя $A \to (1/e)A$, что мы всегда имеем право сделать, в виде:

$$\mathcal{L} = \bar{\psi}\left[i\gamma^\mu\left(\partial_\mu - iA_\mu\right) - m\right]\psi - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu}$$

Обратите внимание, что калибровочное преобразование, оставляющее $\mathcal{L}$ инвариантным, задается как $\psi \to e^{i\alpha}\psi$ и $A_\mu \to A_\mu + \partial_\mu \alpha$. Фотонный пропагатор (глава III.4), получаемый, грубо говоря, обращением $(1/4e^2)\, F_{\mu\nu}F^{\mu\nu}$, теперь пропорционален e^2:

$$iD_{\mu\nu}(q) = \frac{-ie^2}{q^2}\left[g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right]$$

Каждый раз после обмена фотоном амплитуда умножается на e^2. Подобное изменение хотя и тривиально, но очень удобно, поскольку ни в малейшей степени не влияет на физику. Например, для фейнмановской диаграммы, которую мы вычисляли в главе II.6 для электрон-электронного рассеяния, множитель e^2 рассматривался как обусловленный не вершинами взаимодействия, а скорее фотонным пропагатором. В данной интерпретации e^2 характеризует легкость, с которой фотон распространяется в пространстве-времени. Чем меньше e^2, тем больше требуется действия, чтобы фотон распространялся. Чем труднее фотону распространяться, тем слабее эффект электромагнетизма.

Диаграммное доказательство калибровочной инвариантности, которое мы приводили в главе II.7, означает, что $q^\mu \Pi_{\mu\nu}(q) = 0$. С учетом лоренцевской инвариантности из этого условия следует:

$$\Pi_{\mu\nu}(q) = \left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\Pi\left(q^2\right)$$

Физический, или ренормированный, фотонный пропагатор, изображенный на рис. III.7.2, дается в этом случае геометрической прогрессией:

$$\begin{aligned} iD^P_{\mu\nu}(q) &= iD_{\mu\nu}(q) + iD_{\mu\lambda}(q)i\Pi^{\lambda\rho}(q)iD_{\rho\nu}(q) + \\ &+ iD_{\mu\lambda}(q)i\Pi^{\lambda\rho}(q)iD^\sigma_\rho(q)i\Pi^\kappa_\sigma(q)iD_{\kappa\nu}(q) + \ldots = \\ &= \frac{-ie^2}{q^2}g_{\mu\nu}\left\{1 - e^2\Pi\left(q^2\right) + \left[e^2\Pi\left(q^2\right)\right]^2 + \ldots\right\} + q_\mu q_\nu \text{ член } = \\ &= \frac{-ie^2}{q^2}g_{\mu\nu}\frac{1}{1+e^2\Pi\left(q^2\right)} + q_\mu q_\nu \text{ член.} \end{aligned}$$

С учетом (3) член $(1-\xi)\,(q_\mu q_\lambda/q^2)$ в пропагаторе $D_{\mu\lambda}(q)$ зануляется при сворачивании с $\Pi^{\lambda\rho}(q)$. Следовательно, в пропагаторе $iD^P_{\mu\nu}$ калибровочный параметр ξ входит лишь в член $q_\mu q_\nu$ и выпадает из физических амплитуд, как было объяннено в главе II.7.

Вычет в полюсе в $iD^P_{\mu\nu}(q)$ представляет собой физический или перенормированный квадрат заряда:

$$e^2_R = e^2\frac{1}{1+e^2\Pi(0)}$$

Отдаем должное калибровочной инвариантности

Чтобы выразить e_R через e, проведем вычисление в низшем порядке

$$i\Pi_{\mu\nu}(q) = (-)\int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left(i\gamma_\nu\frac{i}{\not p + \not q - m}i\gamma_\mu\frac{i}{\not p - m}\right)$$

Для больших значений p подынтегральное выражение будет иметь вид $1/p^2$ с поправкой m^2/p^4, т. е. одна часть интеграла будет расходиться квадратично, а другая часть - логарифмически. (Видите, как легко перейти на плохой язык.) Это совсем не

концептуальная проблема, как я объяснил в главе III.1. Мы просто регуляризуем интеграл. Поскольку калибровочная инвариантность итрает ключевую роль мы должны проверить, что регуляризация ее не нарушает.

При регуляризации Паули-Вилларса (III.1.12) выражение (7) заменяется на:

$$i\Pi_{\mu\nu}(q) = (-)\int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left(i\gamma_\nu \frac{i}{\not p+\not q-m} i\gamma_\mu \frac{i}{\not p-m}\right) -$$

$$-\sum_a c_a\,\mathrm{tr}\left(i\gamma_\nu \frac{i}{\not p+\not q-m_a} i\gamma_\mu \frac{i}{\not p-m_a}\right).$$

Теперь подынтегральное выражение будет иметь вид $\left(1-\sum\limits_a c_a\right)(1/p^2)$ со вторым членом $\left(m^2-\sum\limits_a c_a m_a^2\right)(1/p^4)$. Поэтому интеграл будет сходиться, если мы выберем c_a и m_a так, что:

$$\sum_a c_a = 1$$

и

$$\sum_a c_a m_a^2 = m^2$$

Очевидно, что необходимо ввести как минимум две регуляторные массы. Мы верим, что можно пренебречь физикой выше масштаба m_a. Интеграл в (7) эффективно обрезается, когда импульс p превышает m_a.

Калибровочно-инвариантная форма (3) в действительности означает, что нам потребуется меньшее количество регуляторных членов, чем мы предполагали. Представим себе разложение (6) по степеням q. Поскольку

$$\Pi_{\mu\nu}(q) = \left(q_\mu q_\nu - g_{\mu\nu}q^2\right)[\Pi(0)+...]$$

в фейнмановском интеграле нас интересуют лишь члены $O\left(q^2\right)$ и старше. Если разложить подынтегральную функцию из уравнения (6), увидим, что член $O\left(q^2\right)$ для больших значений p ведет себя как $1/p^4$ и, следовательно, вносит логарифмически расходящийся (опять говоря на плохом языке) вклад. (Между прочим, аналогичное объяснение приводилось в главе III.3). Создается впечатление, что нам необходим только один регулятор. Этого нельзя сказать точно, ведь мы не доказали, что $\Pi\left(q^2\right)$ раскладывается в степенной ряд по q^2. Тем не менее продолжим вычисления.

Раз интеграл сходится, проходит доказательство калибровочной инвариантности из главы II.7. Вспомним схему доказательства. При вычислении $q^\mu\Pi_{\mu\nu}(q)$ мы пользовались тождеством

$$\frac{1}{\not p+\not q-m}\not q\frac{1}{\not p-m} = \frac{1}{\not p-m} - \frac{1}{\not p+\not q-m}$$

для разбиения подынтегрального выражения на две части, которые взаимно сокращаются при сдвиге переменной интегрирования $p \to p+q$. Из упражнения (II.7.2) вспомним, что в некоторых случаях сдвиг невозможен. Однако он возможен в случае, когда интеграл сходится. После регуляризации мы как раз получили сходящийся интеграл. В любом случае

не попробуешь не узнаешь, так что, проведя явное вычисление тензора поляризации вакуума $\Pi_{\mu\nu}(q)$, мы увидим, что он действительно имеет форму (3).

Выучив несколько вычислительных трюков из предыдущей главы, вы сейчас готовы взяться за вычисления. Я вам помогу их проделать. Чтобы не загромождать страницу, будем опускать регуляторные члены в (7) на промежуточных шагах. Восстановим их ближе к концу. После нескольких шагов вы должны получить

$$i\Pi_{\mu\nu}(q) = -\int \frac{d^4p}{(2\pi)^4}\frac{N_{\mu\nu}}{D}$$

где $N_{\mu\nu} = \operatorname{tr}\left[\gamma_\nu(\not{p}+\not{q}+m)\gamma_\mu(\not{p}+m)\right]$, и

$$\frac{1}{D} = \int\limits_0^1 d\alpha \frac{1}{\mathcal{D}}$$

где $\mathcal{D} = \left[l^2+\alpha(1-\alpha)q^2 - m^2 + i\varepsilon\right]^2$ и $l = p+\alpha q$. Выражая p через l, вы узнаете, что $N_{\mu\nu}$ фактически равно

$$-4\left(\frac{1}{2}g_{\mu\nu}l^2 + \alpha(1-\alpha)\left(2q_\mu q_\nu - g_{\mu\nu}q^2\right) - m^2 g_{\mu\nu}\right)$$

Интегрируя по l с учетом (D.12) и (D.13) и записывая явно вклады от регуляторов, получаем

$$\Pi_{\mu\nu}(q) = -\frac{1}{4\pi^2}\int\limits_0^1 d\alpha\left[F_{\mu\nu}(m) - \sum_a c_a F_{\mu\nu}\left(m_a\right)\right]$$

где

$$\begin{aligned}
&F_{\mu\nu}(m) = \\
=&\frac{1}{2}g_{\mu\nu}\left\{\Lambda^2 - 2\left[m^2-\alpha(1-\alpha)q^2\right]\log\frac{\Lambda^2}{m^2-\alpha(1-\alpha)q^2} + m^2 - \alpha(1-\alpha)q^2\right\} \\
&-\left[\alpha(1-\alpha)\left(2q_\mu q_\nu - g_{\mu\nu}q^2\right) - m^2 g_{\mu\nu}\right]\left[\log\frac{\Lambda^2}{m^2-\alpha(1-\alpha)q^2} - 1\right].
\end{aligned}$$

Не забывайте, что вы самостоятельно проводите вычисления, а я лишь направляю вас. В приложении D Λ вводится для придания смысла разным расходящимся интегралам. Поскольку наш интеграл сходится, нам не следует использовать параметр Λ. Действительно, благодаря (8) и (9) параметр Λ выпадает из формулы (10), так что в конечном итоге получаем формулу:

$$\begin{aligned}
\Pi_{\mu\nu}(q) = &-\frac{1}{2\pi^2}\left(q_\mu q_\nu - g_{\mu\nu}q^2\right)\int\limits_0^1 d\alpha\alpha(1-\alpha)\times \\
&\times\left\{\log\left[m^2-\alpha(1-\alpha)q^2\right] - \sum_a c_a \log\left[m_a^2-\alpha(1-\alpha)q^2\right]\right\}.
\end{aligned}$$

Вот так! Тензор поляризации вакуума действительно имеет форму $\Pi_{\mu\nu}(q) = (q_\mu q_\nu - g_{\mu\nu}q^2)\,\Pi\,(q^2)$. Наша схема регуляризации действительно уважает калибровочную инвариантность.

При $q^2 \ll m_a^2$ (желательно, чтобы интересующий нас кинематический режим был много меньше порога нашего неведения) определяем $\log M^2 \equiv \sum\limits_a c_a \log m_a^2$ в (12) и получаем:

$$\Pi\left(q^2\right)\frac{1}{2\pi^2}\int\limits_0^1 d\alpha\alpha(1-\alpha)\log\frac{M^2}{m^2-\alpha(1-\alpha)q^2}.$$

Заметим, что наши эвристические рассуждения в самом деле правильны. В конечном итоге нам нужен только один регулятор, хотя на промежуточных шагах их должно было быть два. По существу, спор о количестве регуляторов не столь важен.

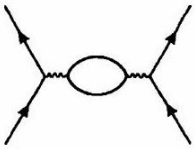
В главе III.1 я уже упоминал о размерной регуляризации как альтернативе регуляризации Паули-Вилларса. Размерную регуляризацию придумали для того, чтобы сохранить калибровочную инвариантность в неабелевых калибровочных теориях (о которых мы поговорим в следующих главах). Поучительно вычислить Π с помощью размерной регуляризации (упражнение III.7.1).

Электрический заряд

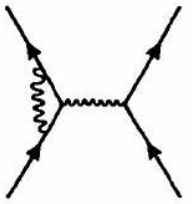
Окончательное выражение для $\Pi\,(q^2)$ содержит параметр M^2, характеризующий порог нашего неведения. Заключаем, что

$$e_R^2 = e^2\frac{1}{1+(e^2/12\pi^2)\log\left(M^2/m^2\right)} \simeq e^2\left(1-\frac{e^2}{12\pi^2}\log\frac{M^2}{m^2}\right).$$

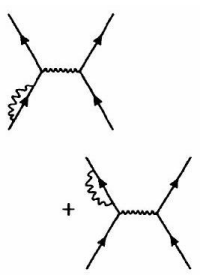
Квантовые флуктуации эффективно уменьшают заряд. Я объясню физический смысл этого явления позднее в главе, посвященной ренормализационной группе.



(a)



(b)



(c)

Рис. III.7.4

Вы скажете, что с физической точки зрения сила заряда характеризуется тем, как один электрон рассеивается на другом. До порядка e^4 строим диаграммы, как в главе II.7, и дополнительно другие диаграммы, как на рис. III.7.4a,b,c. Диаграмму на рисунке 4 a мы вычислили, но что же делать с диаграммами 4b и 4c? В книгах говорится, что вклады III.7.4b и III.7.4c в перенормировку заряда взаимно сокращаются. Заряд - это мера того, что фотон распространяется. Преимущество лагранжиана (1) в том, что это утверждение становится очевидным.

Чтобы проверить этот более или менее очевидный факт, прибегнем к физической, или ренормированной, теории возмущений (см. главу III.3). Перепишем лагранжиан в виде функции от физических, или ренормированных, полей (как и прежде, будем опускать подстрочный индекс P):

$$\begin{aligned}\mathcal{L} =&\bar{\psi}\left(i\gamma^{\mu}\left(\partial_{\mu}-iA_{\mu}\right)-m_{P}\right)\psi-\frac{1}{4e_{P}^{2}}F_{\mu\nu}F^{\mu\nu}+\\&+A\bar{\psi}i\gamma^{\mu}\left(\partial_{\mu}-iA_{\mu}\right)\psi+B\bar{\psi}\psi-CF_{\mu\nu}F^{\mu\nu}\end{aligned}$$

где коэффициенты A, B и C при контрчленах определяются итеративным методом. Дело в том, что калибровочная инвариантность гарантирует, что $\bar{\psi}i\gamma^{\mu}\partial_{\mu}\psi$ и $\bar{\psi}\gamma^{\mu}A_{\mu}\psi$ всегда идут в комбинации $\bar{\psi}i\gamma^{\mu}\left(\partial_{\mu}-iA_{\mu}\right)\psi$, т. е. силу взаимодействия A_{μ} с $\bar{\psi}\gamma^{\mu}\psi$ нельзя изменить. Мы можем изменить лишь ту легкость, с которой фотон распространяется в пространстве-времени.

Это утверждение имеет огромный физический смысл. Экспериментально установлено (с большой степенью точности), что заряды электрона и протона имеют противоположные знаки и точно равны между собой. Если бы они не равнялись друг другу, между объектами макроскопического мира действовала бы остаточная электростатическая сила. Предположим, мы открыли принцип, согласно которому голые заряды электрона и протона точно равны друг другу (в дальнейшем мы увидим, что в теориях великого объединения этот факт действительно вытекает из теории групп.) Как мы можем быть уверенными, что под воздействием квантовых флуктуаций заряды не станут несколько отличными друг от друга? Все же протон участвует в сильном взаимодействии, а электрон - нет, поэтому электромагнитное рассеяние двух протонов на больших расстояниях описывается гораздо большим количеством диаграмм. Теперь ясно, что равенство зарядов есть необходимое условие по той очевидной причине, что перенормировка заряда имеет отношение к фотону. В конце концов, все обуславливается калибровочной инвариантностью.

Аналитичность

Рассмотрим функцию П(s), определенную в (13). С помощью аналитического продолжсния сс можно задать на всей комплексной s-плоскости. Как правило, логарифм

$\log z$ задают так, чтобы он имел разрез на отрицательной действительной оси. Тогда подынтегральная функция в (13) будет иметь разрез на положительной действительной q^2-оси и простираться от $q^2 = m^2/\alpha(1-\alpha)$ до бесконечности. Поскольку максимальное значение $\alpha(1-\alpha)$ в области интегрирования равно $\frac{1}{4}$, функция $\Pi(q^2)$ есть аналитическая функция на комплексной q^2-плоскости, которая имеет разрез вдоль действительной оси, начинающийся от $q_c^2 = 4m^2$.

Представим себе виртуальный движущийся фотон с импульсом $q^\mu = \left(\sqrt{q^2}, \overrightarrow{0}\right)$. При $\sqrt{q^2} \geqslant 2m$ у него достаточно энергии для превращения в реальную пару электрон-позитрон, а не просто виртуальную пару, которая практически тут же перестает существовать. В этом состоит физический смысл разреза. (Если вы скрупулезно выполняете все упражнения, то, должно быть, уже заметили, что аналогичные выводы были получены в упражнениях I.7.5 и III.1.2.)

В связи с тем что в квантовой теории поля амплитуды вычисляются как интегралы от произведений пропагаторов, становится более или менее ясно, что амплитуды есть аналитические функции от внешних кинематических переменных. Например, амплитуда рассеяния $\mathcal{M}$ в главе III.1 представляет собой аналитическую функцию от s, t и u с несколькими разрезами. В период с конца 1950-х до начала 1960-х годов было приложено множество усилий, посвященных изучению аналитических свойств фейнмановских диаграмм 1.

[1] Техническое отступление: если восстановить в пропагаторах член $i\varepsilon$, который мы по ходу дела отбрасывали. увидим, что в интегральном представлении для $\Pi(q^2)$ мы должны написать $m^2 - i\varepsilon$. Отсюда следует, что мы должны вычислять $\Pi(q^2)$ чуть выше разреза для $q^2 \geqslant 4m^2$.

Модификация кулоновского потенциала

Основное внимание мы уделяли перенормировке заряда, которую полностью определяет $\Pi(0)$, однако в выражении (13) мы получили полную функцию $\Pi(q^2)$, которая описывает, как модифицируется зависимость фотонного пропагатора от q. Из обсуждения в главе I.5 мы знаем, что кулоновский потенциал есть фурье-преобразование фотонного пропагатора (см. также упражнение III.5.3). Следовательно, кулоновское взаимодействие модифицируется из закона $1/r$ на длине порядка $(2m)^{-1}$, обратной к характерному значению q в $\Pi(q^2)$. Эта модификация проверена экспериментально как часть лэмбовского сдвига в атомной спектроскопии, еще одного великого триумфа квантовой электродинамики.

Упражнения

III.7.1. Вычислите $\Pi_{\mu\nu}(q)$, используя размерную регуляризацию. Начните с уравнения (6), найдите след в $N_{\mu\nu}$, сдвиньте переменную интегрирования с p до l и т. д., действуя в точности как в тексте. Продолжайте до тех пор, пока не дойдете до вычисления интеграла по импульсу l в петле. На этом шаге вы «считаете», что живете в d-мерном пространстве-времени, что позволяет заменить, например, член типа $l_\mu l_\nu$ в $N_{\mu\nu}$ на $(1/d)g_{\mu\nu}l^2$. Интегрирование необходимо провести с учетом уравнения (III.1.15) и его обобщений. Покажите, что структура (3) автоматически возникает при продолжении в $d = 4$.

III.7.2. Изучите модифицированный закон Кулона, определяемый интегралом Фурье $\int d^3q \left\{1/\vec{q}^2\left[1 + e^2\Pi\left(\vec{q}^2\right)\right]\right\} e^{i\vec{q}\vec{x}}$.

18.2 K-l.II: Regularization

The divergences which appear in radiative corrections are not yet a catastrophe for a theory (remind, for example, the infinite self-energy of an electric charge in its own Coulomb field) but require a quantitative description. To get a finite difference of the two infinite quantities, one has to give them some meaning. This can be achieved by introducing a kind of regularization of divergent integrals. The most natural way of regularization is to cut off the integral on the upper or lower bound of integration. There are also different ways of regularization based on a modification of the integrand or of the measure of integration. Below we consider three most popular kinds of regularization: the ultraviolet cutoff in Euclidean space (Λ-regularization), the Pauli-Villars regularization, and the dimensional regularization.

18.2.1 Euclidean integral and the ultraviolet cutoff

All the integrals in quantum field theory are written in Minkowski space; however, the ultraviolet divergence appears for large values of modulus of momentum and it is useful to regularize it in Euclidean space. Transition to Euclidean space can be achieved by replacing the zeroth component of momentum $k_0 \to ik_4$, so that the squares of all momenta and the scalar products change the sign $k^2 = k_0^2 - \vec{k}^2 \to -k_4^2 - \vec{k}^2 = -k_E^2$ and the measure of integration becomes equal to $d^4k \to id^4k_E$, where the integration over the fourth component of momenta goes along the imaginary axis. To go to the integration along the real axis, one has to perform the (Wick) rotation of the integration contour by 90^o (see. Fig.18.1). This is possible since the integral over the big circle vanishes and during the transformation of the contour it does not cross the poles.

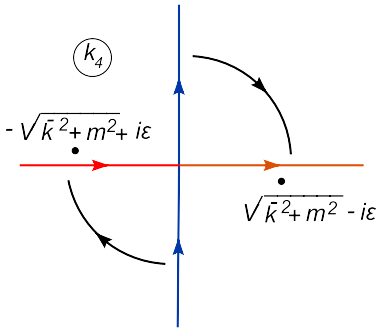


Figure 18.1: The Wick rotation of the integration contour

When transferring to Euclidean space the poles in all the propagators disappear. Now the integral in 4-dimensional Euclidean space can be evaluated in spherical coordinates and the integral over the modulus can be cut on the upper limit. Let us demonstrate how this method works in the case of the simplest scalar diagram shown in Fig.18.2. The corresponding pseudo-Euclidean integral has the form

$$I(p^2) = \frac{1}{(2\pi)^4}\int \frac{d^4k}{[k^2-m^2][(p-k)^2-m^2]}. \tag{18.1}$$

Transforming it to Euclidean space one gets

$$I(p_E^2) = \frac{i}{(2\pi)^4}\int \frac{d^4k}{[k_E^2+m^2][(p-k)_E^2+m^2]}. \tag{18.2}$$

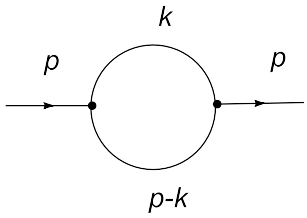


Figure 18.2: The simplest divergent diagram in a scalar theory

(in what follows the index will be omitted.)

For calculation of this kind of integrals we use the following approach. First, we transform the product of several brackets in the denominator into the single bracket with the help of the so-called Feynman parametrization. The following general formula is valid:

$$\begin{aligned}\frac{1}{A_1^{\alpha_1}A_2^{\alpha_2}\cdots A_n^{\alpha_n}} &= \frac{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_n)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)}\int\limits_0^1 dx_1dx_2\cdots dx_n \\ &\cdot \frac{\delta(1-x_1-x_2-\cdots-x_n)x_1^{\alpha_1-1}x_2^{\alpha_2-1}\cdots x_n^{\alpha_n-1}}{[A_1x_1+A_2x_2+\cdots A_nx_n]^{\alpha_1+\alpha_2+\cdots+\alpha_n}}.\end{aligned} \tag{18.3}$$

Here $\Gamma(\alpha)$ is the Euler Γ-function which has the following properties:

$$\Gamma(1)=1,\ \Gamma(n+1)=n!,\ x\Gamma(x)=\Gamma(x+1),\ \Gamma(1+x)=e^{[-x\gamma_E+\sum\limits_{n=2}^{\infty}\frac{(-x)^n}{n}\zeta(n)]},$$

where γ_E is the Euler constant and $\zeta(n)$ is the Riemann zeta-function. The Γ-function is finite for positive values of the argument and has simple poles at negative integer values and at zero.

In our case, $(n=2, \alpha_1=\alpha_2=1)$ and eq.(18.3) has the form:

$$\begin{aligned}\frac{1}{[k^2+m^2][(p-k)^2+m^2]} &= \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)}\int\limits_0^1 \frac{dx_1x_2\delta(1-x_1-x_2)}{[[k^2+m^2]x_1+[(p-k)^2+m^2]x_2]^2} \\ &= \int\limits_0^1 \frac{dx}{[k^2-2pkx+p^2x+m^2]^2}.\end{aligned} \tag{18.4}$$

Thus, integral (18.2) can be written as

$$I(p^2)=\frac{i}{(2\pi)^4}\int\limits_0^1 dx\int\frac{d^4k}{[k^2-2kpx+p^2x+m^2]^2}\overset{k\to k-px}{=}\frac{i}{(2\pi)^4}\int\limits_0^1 dx\int\frac{d^4k}{[k^2+p^2x(1-x)+m^2]^2} \tag{18.5}$$

Now the integral depends only on the modulus of k and one can use the spherical coordinates:

$$I(p^2)=\frac{i}{(2\pi)^4}\int\limits_0^1 dx\ \Omega_4\int\limits_0^\Lambda \frac{k^3dk}{[k^2+p^2x(1-x)+m^2]^2}, \tag{18.6}$$

where the volume of the 4-dimensional sphere equals $\Omega_4 = 2\pi^2$ (in general $\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$). The integral over the modulus can be easily calculated

$$\frac{1}{2}\int_0^{\Lambda^2} \frac{k^2 dk^2}{[k^2 + p^2 x(1-x) + m^2]^2} = \frac{1}{2}\log(\frac{\Lambda^2}{p^2 x(1-x) + m^2}) + 1, \tag{18.7}$$

and, as one can see, is logarithmically divergent at the upper limit. The full answer has the form

$$I(p^2) = \frac{i}{16\pi^2}\int_0^1 dx \left(\log(\frac{\Lambda^2}{p^2 x(1-x) + m^2}) + 1\right). \tag{18.8}$$

The last integral over x can also be evaluated and takes the simplest form in the limiting cases for $m = 0$ or $p = 0$. Now one can go back to Minkowski space $p_E^2 => -p^2$.

The regularization with the ultraviolet cut-off is quite natural and relatively simple. The drawback of this regularization is Euclidean rather than Lorentzian invariance and also the absence of the gauge invariance. Therefore, it is not useful in the gauge theories. However, one should notice that the noninvariance of a regularization is acceptable since the invariance is restored when removing the regularization. Still, this aspect complicates the calculation as one has to take care of the validity of all the identities.

18.2.2 Pauli-Villars Regularization

Another method of regularization which is called the Pauli-Villars regularization is based on the introduction of a set of additional heavy fields with a wrong sign of the kinetic term. These fields are not physical and are introduced essentially with the purpose of regularization of divergent integrals. The main trick is in the replacement

$$\frac{1}{p^2 - m^2} \to \frac{1}{p^2 - m^2} - \frac{1}{p^2 - M^2}, \tag{18.9}$$

where $M \to \infty$ is the mass of the Pauli-Villars fields. As a result, the propagator for large momenta decreases faster, which ensures the convergence of the integrals. The divergences manifest themselves as logs and powers of M^2 instead of the cutoff parameter Λ^2.

One uses sometimes the modifications of the Pauli-Villars regularization when the replacement (18.9) is performed not for each propagator but for the loop as a whole. This method of regularization is called the regularization over circles. It is used in Abelian gauge theories for the loops made of the matter fields. This way one can preserve the gauge invariance. However, in non-Abelian theories we face some problems related to the loops of the gauge fields which cannot become massive without violating the gauge invariance. This problem is often solved by introducing an additional regularization for the vector fields, for example, with the help of higher derivatives. Here we will not consider this regularization.

The positive property of the Pauli-Villars regularization is the explicit Lorentz and gauge (in abelian case) invariance, but it requires complicated calculations since one has to calculate massive diagrams, while massless integrals are much simpler.

18.2.3 Dimensional Regularization

The most popular in gauge theories is the so-called dimensional regularization. In this case, one modifies the integration measure.

The technique of dimensional regularization consists of analytical continuation from an integer to a noninteger number of dimensions. Basically one goes from some D to $D-2\epsilon$, where $\epsilon \to 0$. In particular, we will be interested in going from 4 to $4-2\epsilon$ dimensions. In this case, all the ultraviolet and infrared singularities manifest themselves as pole terms in ϵ. To perform this continuation to non-integer number of dimensions, one has to define all the objects such as the metric, the measure of integration, the γ matrices, the propagators, etc. Though this continuation is not unique, one can define a self-consistent set of rules, which allows one to perform the calculations.

The metric: $g_4^{\mu\nu} \to g_{4-2\epsilon}^{\mu\nu}$. Though it is rather tricky to define the metric in non-integer dimensions, one usually needs only one relation, namely $g^{\mu\nu}g_{\mu\nu} = \delta_\mu^\mu = D = 4-2\epsilon$.

The measure: $d^4q \to (\mu^2)^\epsilon d^{4-2\epsilon}q$, where μ is a parameter of dimensional regularization with dimension of a mass. The integration with this measure is defined by an analytical continuation from the integer dimensions.

The γ matrices: The usual anticommutation relation holds $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$; however, some relations involving the dimension are modified:

$$\gamma^\mu\gamma_\mu = D = 4-2\epsilon; \quad Tr\gamma^\mu\gamma^\nu = g^{\mu\nu}Tr1 = g^{\mu\nu}\begin{cases} 2^{[D/2]} \\ 4 \end{cases}.$$

Usually $Tr1 = 4$ is taken. Then the γ-algebra is straightforward:

$$Tr\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = Tr1[g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\nu\rho}g^{\mu\sigma}],$$

$$\gamma^\mu\gamma^\nu\gamma^\mu = -\gamma^\mu\gamma^\mu\gamma^\nu + 2g^{\mu\nu}\gamma^\mu = -(4-2\epsilon)\gamma^\nu + 2\gamma^\nu = -(2-2\epsilon)\gamma^\nu, \quad \text{etc.}$$

What is not well-defined is the γ^5 since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and cannot be continued to an arbitrary dimension. This creates a problem in dimensional regularization since there is no consistent way of definition of γ^5.

The propagator: In momentum space the continuation is simple

$$\frac{1}{p^2-m^2} \to \frac{1}{p^2-m^2}.$$

However, in coordinate space one has: (take $m=0$ for simplicity)

$$\int \frac{d^4p}{p^2}e^{ipx} \sim \frac{1}{x^2} \Rightarrow \int \frac{d^{4-2\epsilon}p}{p^2}e^{ipx} \sim \frac{1}{[x^2]^{1-\epsilon}}.$$

The basic integrals: The main idea is to calculate the integral in the space-time dimension where it is convergent and then analytically continue the answer to the needed dimension.

Consider the earlier discussed example (18.1) and use the Euclidean representation (18.5). Let us rewrite it formally in D-dimensional space

$$\int \frac{d^Dk}{[k^2+M^2]^2} = \frac{\Omega_D}{2}\int_0^\infty \frac{(k^2)^{D/2-1}dk^2}{[k^2+M^2]^2}, \qquad M^2 \equiv p^2x(1-x)+m^2. \tag{18.10}$$

The integral over k^2 is now the table one

$$\int_0^\infty \frac{(k^2)^{D/2-1}dk^2}{[k^2+M^2]^2} \overset{k^2\to k^2M^2}{=} (M^2)^{\frac{D}{2}-2}\int_0^\infty \frac{x^{D/2-1}dx}{(x+1)^2} = (M^2)^{\frac{D}{2}-2}\,\frac{\Gamma(\frac{D}{2})\Gamma(2-\frac{D}{2})}{\Gamma(2)}, \tag{18.11}$$

where we assume that the dimension D is such that the integral exists. In this case this is 2 and 3. The main formula (18.11) allows one to perform the analytical continuation over D into the region $D = 4-2\varepsilon$. For $\varepsilon = 0$, i.e., in 4 dimensions, the integral does not exist since the Γ-function has a pole at zero argument. However, in the vicinity of zero we get a regularized expression.

Collecting all together we get

$$I(p^2) = \frac{i}{(2\pi)^D}\frac{\Omega_D}{2}\int_0^1 dx \frac{\Gamma(D/2)\Gamma(2-D/2)}{[p^2x(1-x)+m^2]^{2-D/2}}. \tag{18.12}$$

Substituting now $D = 4-2\varepsilon$ and transforming back into the pseudo-Euclidean space one finds

$$I(p^2) = \frac{i(-\pi)^{2-\varepsilon}}{(2\pi)^{4-2\varepsilon}}\Gamma(\varepsilon)\int_0^1 \frac{dx(\mu^2)^\varepsilon}{[p^2x(1-x)-m^2]^\varepsilon} \tag{18.13}$$

Expanding the denominator into the series over ε, we finally arrive at

$$I(p^2) = \frac{i}{16\pi^2}\Gamma(1+\varepsilon)\left(\frac{1}{\varepsilon} - \int_0^1 dx\log[\frac{p^2x(1-x)-m^2}{-\mu^2}] + \log(4\pi)\right). \tag{18.14}$$

Comparing it with eq.(18.8) we see that the ultraviolet divergence now takes the form of the pole over ε instead of the logarithm of the cutoff. This is less visual but much simpler in the calculations and also is automatically gauge invariant.

We present below the main integrals needed for the one-loop calculations. They can be obtained via the analytical continuation from the integer values of D. We will write them down directly in the pseudo-Euclidean space.

$$\int \frac{d^Dp}{[p^2-2kp+m^2]^\alpha} = i\frac{\Gamma(\alpha-D/2)}{\Gamma)\alpha)}\frac{(-\pi)^{D/2}}{[m^2-k^2]^{\alpha-D/2}}, \tag{18.15}$$

$$\int \frac{d^{4-2\epsilon}p}{[p^2-2kp+m^2]^2} = i\frac{\Gamma)\epsilon)}{\Gamma(2)}\frac{(-\pi)^{2-\epsilon}}{[m^2-k^2]^\epsilon}, \quad \Gamma(\epsilon)\sim\frac{1}{\epsilon}\to\infty,$$

$$\int \frac{d^{4-2\epsilon}p\ \ p_\mu}{[p^2-2kp+m^2]^2} = i\frac{\Gamma(\epsilon)}{\Gamma(2)}\frac{(-\pi)^{2-\epsilon}k_\mu}{[m^2-k^2]^\epsilon}, \tag{18.16}$$

$$\int \frac{d^{4-2\epsilon}p\ \ p_\mu p_\nu}{[p^2-2kp+m^2]^2} = i(-\pi)^{2-\epsilon}\left[\frac{\Gamma(\epsilon)}{\Gamma(2)}\frac{k_\mu k_\nu}{[m^2-k^2]^\epsilon} + \frac{g^{\mu\nu}}{2}\frac{\Gamma(\epsilon-1)}{\Gamma(2)}\frac{1}{[m^2-k^2]^{\epsilon-1}}\right]$$

The key formula is (18.15). All the rest can be obtained from it by the differentiation. Notice the singularity in the r.h.s. of (18.15) for $\alpha = D/2-n$, $n = 0,1,...$ These integrals remain non-regularized. However, they usually do not appear in the real calculations.

Let us mention one important rule used in dimensional regularization and related to the massless theories. By definition it is accepted that zero to any power is zero. Thus, for example, the following integral is zero

$$\int \frac{d^Dk}{(k^2)^\alpha} = 0, \quad \forall\ \alpha. \tag{18.17}$$

In fact, here we have a cancellation of the ultraviolet and infrared divergences which both have the form of a pole over $1/\varepsilon$. There is no any inconsistency here and this way of doing is self-consistent in the calculations of dimensionally regularized integrals.

This rule leads, in particular, to the vanishing of all the diagrams of the tad-pole type in the massless case. However, in the massive case they survive and are important for the restoration of the gauge invariance. As it will be clear later, in the Standard Model the tad-poles give their contribution to the renormalization of the quark masses and provide the transversality of the vector propagator in a theory with spontaneous symmetry breaking.

18.3 K-l.III: Examples of Calculations. One-loop Integrals

All further calculations will be performed using dimensional regularization. Below we show how the rules described above can be applied to calculate in various models of quantum field theory.

18.3.1 The scalar theory

We start with the simplest scalar case and consider the theory described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4. \tag{18.18}$$

The Feynman rules in this case are:

$= \dfrac{i}{p^2-m^2},$

$= -i\lambda$

First, we find the one-loop divergent diagrams. As it follows from Fig.16.5, they are the propagator of the scalar field and the quartic vertex.

The propagator: In the first order there is only one diagram of the tad-pole type shown in Fig.18.3.

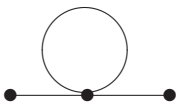


Figure 18.3: The one-loop propagator diagram

The corresponding integral is

$$J_1(p^2) = \frac{-i\lambda}{(2\pi)^{4-2\varepsilon}}\frac{i}{2}\int\frac{d^{4-2\varepsilon}k(\mu^2)^\varepsilon}{k^2-m^2}, \tag{18.19}$$

where $1/2$ is the combinatoric factor. Calculating the integral (18.19), according to (18.16), we find

$$J_1(p^2) = \frac{-i\lambda}{(4\pi)^{2-\varepsilon}}\frac{\Gamma(-1+\varepsilon)}{2\Gamma(1)}m^2(\frac{\mu^2}{m^2})^\varepsilon = \frac{i\lambda}{32\pi^2}m^2\left[\frac{1}{\varepsilon}+1-\gamma_E+\log(4\pi)-\log\frac{m^2}{\mu^2}\right] \tag{18.20}$$

The fact that the integral diverges quadratically manifests itself in the structure of the multiplier $\Gamma(-1+\varepsilon)$ which has a pole at $\varepsilon=0$ as well as at $\varepsilon=1$. However, since we are interested in the limit $\varepsilon\to 0$, we expand the answer in the Loran series in ε. As one can see, even in the case of

quadratically divergent integrals the divergence takes the form of a simple pole over ε, but the integral has the dimension equal to two. Notice, however, that for $m = 0$ the integral equals zero in accordance with the properties of dimensional regularization mentioned above.

The vertex: Here one also has only one diagram but the external momenta can be adjusted in several ways (see Fig.18.4). As a result the total contribution to the vertex function consists of

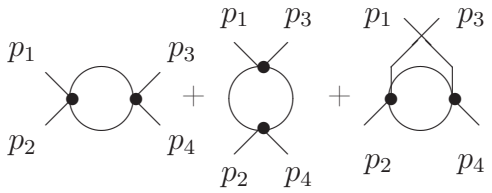


Figure 18.4: The one-loop vertex diagram

three parts

$$I_1 = I_1(s) + I_1(t) + I_1(u),$$

where we introduced the commonly accepted notation for the Mandelstam variables (we assume here that the momenta p_1 and p_2 are incoming and the momenta p_3 and p_4 are outgoing)

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2,\ t = (p_1 - p_3)^2 = (p_2 - p_4)^2,\ u = (p_1 - p_4)^2 = (p_2 - p_3)^2,$$

and the integral equals

$$I_1(s) = \frac{(-i\lambda)^2}{48} \frac{(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}} i^2 \int \frac{d^{4-2\varepsilon}k}{[k^2 - m^2][(p-k)^2 - m^2]} \tag{18.21}$$

(1/48 is the combinatoric coefficient). We have already calculated this integral and the answer has the form (18.14). Now we perform the calculation in a different and simpler way applicable to the massless integrals.

Two comments are in order. The first one concerns the evaluation of the combinatoric coefficient. It comes from the expansion of the S-matrix within the Wick theorem. In the case when all the particles are different like, for example, in QED, the combinatoric coefficient is usually 1. For identical particles their permutations are taken into account already in the Lagrangian (the factors 1/2 and /4! in (18.18)) and lead to nontrivial coefficients. There exists a simple method to calculate the combinatoric coefficient in these cases. The coefficient equals 1/Sym, where Sym is the symmetry factor of a diagram. Consider the diagram shown in Fig.18.4. If one does not distinguish the arrangement of momenta, then the diagram has the following symmetries: the permutation of external lines entering into the left vertex, the permutation of external lines entering into the right vertex, the permutation of the vertices, the permutation of internal lines. Altogether one has: $2 \times 2 \times 2 \times 2 = 16$. Hence, the combinatoric coefficient equals 1/16 but, since we distinguish three different momentum arrangements, one has 1/48. The same rule is valid for the multiloop diagrams and we will use it in the next section.

The second comment is related to the calculation of the massless integrals which are much simpler, and in some cases one can get the answer without any explicit integration. The method, which we will describe below, is applicable to a certain type of massless integrals and is based on conformal properties of the massless integrals depending on one external argument and uses the symmetry between the coordinate and momentum representations.

The key formula is the Fourier-transformation of the propagator of a massless particle

$$\int \frac{d^4p\ e^{ipx}}{p^2} = \frac{i\pi^2}{x^2}, \tag{18.22}$$

which can be generalized to an arbitrary dimension and any power of the propagator as follows:

$$\int \frac{d^D p\ e^{ipx}}{(p^2)^\alpha} = i(-\pi)^{D/2}\frac{\Gamma(D/2-\alpha)}{\Gamma(\alpha)}\frac{1}{(x^2)^{D/2-\alpha}}. \tag{18.23}$$

Obviously, this formula is also valid for the coordinate integration instead of momentum. This way the transition from momentum representation to the coordinate one and vice versa is performed with the help of (18.23) and is accompanied by the factor $\frac{\Gamma(D/2-\alpha)}{\Gamma(\alpha)}$.

Let us go back to the diagram Fig.18.4. In momentum space it corresponds to the integral over the momenta running along the loop. However, in coordinate space it is just the product of the two propagators and does not contain any integration. Therefore, the integral in momentum space can be replaced by the Fourier-transform of the square of the propagator. Since in the massless case all the propagators in both momentum and coordinate representation are just the powers of p^2 or x^2, all of them are easily calculated with the help of relation (18.23).

In the case of the integral (18.21) for $m = 0$ one first has to mentally transform both the propagators into coordinate space which, according to (18.23), gives the factor $(\frac{\Gamma(1-\varepsilon)}{\Gamma(1)})^2$, then multiply the obtained propagators (this gives $1/(x^2)^{2-2\varepsilon}$) and transform the obtained result back into momentum space that gives the factor $\frac{\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}$ and the power of momenta $1/(p^2)^\varepsilon$ (the same as in the argument of the last Γ-function). Besides this, each loop contains the factor $i(-\pi)^{2-\varepsilon}$. Collecting all together one gets

$$I_1(s) = \frac{(-i\lambda)^2}{48}\frac{(\mu^2)^\varepsilon i^2}{(2\pi)^{4-2\varepsilon}}\int\frac{d^{4-2\varepsilon}k}{k^2(p-k)^2} = \frac{\lambda^2}{48}\frac{i\pi^{2-\varepsilon}}{(2\pi)^{4-2\varepsilon}}\left(\frac{\mu^2}{-s}\right)^\varepsilon\frac{\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(1)\Gamma(1)\Gamma(2-2\varepsilon)}$$

$$= \frac{i}{48}\frac{\lambda^2}{(4\pi)^{2-\varepsilon}}\left[\frac{\mu^2}{-s}\right]^\varepsilon\frac{1}{\varepsilon(1-2\varepsilon)}\frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)} = \frac{i}{48}\frac{\lambda^2}{16\pi^2}[\frac{1}{\varepsilon}+2-\gamma_E+\log 4\pi+\ln\frac{\mu^2}{-s}],$$

which coincides with (18.14) at $m = 0$.

The described method for calculation of massless integrals is applicable to any integral depending on one external momentum (propagator type) and allows one to perform the calculations in any number of loops simply writing down the corresponding factors without explicit integration. In the case when the integral depends on more than one external momentum (like for a triangle or a box) and they cannot be put equal to zero the method is not directly applicable though some modifications are available. We do not consider them here.

The four-point vertex in the one-loop approximation thus equals (we take the common factor $1/4!\phi^4$ out of the brackets):

$$\Gamma_4 = -i\lambda\left\{1-\frac{\lambda}{16\pi^2}\left(\frac{3}{2\varepsilon}+3-\frac{3}{2}\gamma_E+\frac{3}{2}\log 4\pi+\frac{1}{2}\ln\frac{\mu^2}{-s}+\frac{1}{2}\ln\frac{\mu^2}{-t}+\frac{1}{2}\ln\frac{\mu^2}{-u}\right)\right\}. \tag{18.24}$$

As one can see, the Euler constant and the logarithm of 4π always accompany the pole term $1/\varepsilon$ and can be absorbed into the redefinition of μ^2.

18.3.2 Quantum electrodynamics

Consider now the calculation of the diagrams in the gauge theories. We start with quantum electrodynamics. The QED Lagrangian has the form

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}^2 + \bar\psi(i\gamma^\mu\partial_\mu - m)\psi + e\bar\psi\gamma^\mu A_\mu\psi - \frac{1}{2\xi}(\partial_\mu A_\mu)^2, \tag{18.25}$$

where the electromagnetic stress tensor is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the last term in (34.2) fixes the gauge. In what follows we choose the Feynman or the diagonal gauge ($\xi = 1$).

The Feynman rules corresponding to the Lagrangian (34.2) are shown in Fig.18.5.

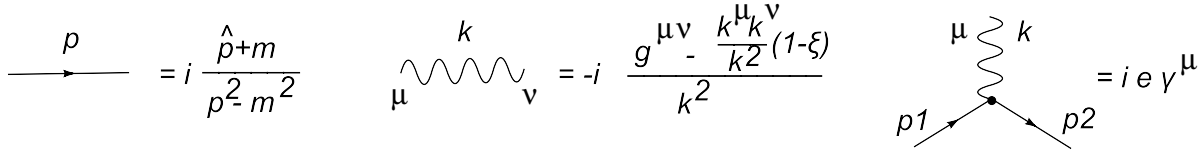


Figure 18.5: The Feynman rules for QED

In quantum electrodynamics the divergences appear only in the photon propagator, the electron propagator, and the triple vertex. The one-loop divergent diagrams are shown in Fig.18.6.

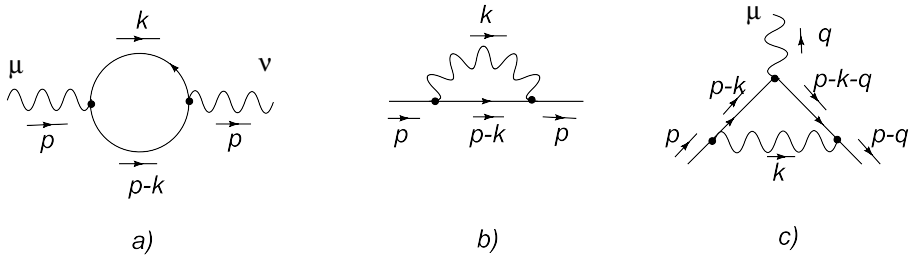


Figure 18.6: The one-loop divergent diagrams in QED

We begin with the vacuum polarization graph. It is given by the diagram shown in Fig. 18.6a). The corresponding expression looks like:

$$\Pi_{\mu\nu}(p) = (-)\frac{e^2}{(2\pi)^4}\int d^4k \frac{Tr[\gamma^\mu(m+\hat{k})\gamma^\nu(m+\hat{k}-\hat{p})]}{[m^2-k^2][m^2-(k-p)^2]}, \tag{18.26}$$

where the "-" sign comes from the fermion loop and $\hat{q} \equiv \gamma^\mu q_\mu$. We first go to dimension $4-2\epsilon$. Then the integral (18.26) becomes

$$\Pi^{Dim}_{\mu\nu}(p) = (-)\frac{e^2(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int d^{4-2\varepsilon}k \frac{Tr[\gamma^\mu(m+\hat{k})\gamma^\nu(m+\hat{k}-\hat{p})]}{[m^2-k^2][m^2-(k-p)^2]}, \tag{18.27}$$

Let us put $m = 0$ for simplicity. This will allow us to get a simple answer at the end. First, we calculate the trace of the γ-matrices:

$$Tr\gamma^\mu\hat{k}\gamma^\nu(\hat{k}-\hat{p}) = Tr(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)k^\rho(k-p)^\sigma = 4k^\rho(k-p)^\sigma[g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}-g^{\mu\nu}g^{\rho\sigma}].$$

So the integral now looks like

$$I^{Dim}_{\rho\sigma}(p) = (-)\frac{(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int \frac{d^{4-2\varepsilon}k k^\rho(k-p)^\sigma}{k^2(k-p)^2}.$$

Using the Feynman parametrization and performing the integration according to the formulae given above one finds

$$I^{Dim}_{\rho\sigma}(p) = (-)\frac{(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int_0^1 dx \int \frac{d^{4-2\varepsilon}k k^\rho(k-p)^\sigma}{[k^2-2pkx+p^2x]^2} \tag{18.28}$$

$$= (-)i\frac{(-\mu^2)^\varepsilon \pi^{2-\varepsilon}}{(2\pi)^{4-2\varepsilon}} \left\{ -\Gamma(\varepsilon)\int_0^1 \frac{dx p^\rho p^\sigma x(1-x)}{[p^2x(1-x)]^\varepsilon} + \Gamma(\varepsilon-1)\frac{g^{\rho\sigma}}{2}\int_0^1 \frac{dx}{[p^2x(1-x)]^{\varepsilon-1}} \right\}.$$

To evaluate the remaining integrals, we use the standard integral for the Euler beta-function

$$\int_0^1 dx x^{\alpha-1}(1-x)^{\beta-1} = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

which gives in our case

$$\int_0^1 dx x^{1-\varepsilon}(1-x)^{1-\varepsilon} = \frac{\Gamma(2-\varepsilon)\Gamma(2-\varepsilon)}{\Gamma(4-2\varepsilon)}.$$

Thus, the integral (18.28) becomes

$$I^{Dim}_{\rho\sigma}(p) = \frac{i}{16\pi^2}(4\pi)^\varepsilon \left(-\frac{\mu^2}{p^2}\right)^\varepsilon \frac{\Gamma^2(2-\varepsilon)\Gamma(\varepsilon)}{\Gamma(4-2\varepsilon)} \left[p^\rho p^\sigma + \frac{1}{2}\frac{g^{\rho\sigma}p^2}{1-\varepsilon}\right], \tag{18.29}$$

where we have used that $\Gamma(-1+\varepsilon) = -\frac{\Gamma(\varepsilon)}{1-\varepsilon}$. Multiplying eq.(18.29) by the trace

$$[g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}]p^\rho p^\sigma = p^\mu p^\nu + p^\nu p^\mu - g^{\mu\nu}p^2 = 2p^\mu p^\nu - g^{\mu\nu}p^2,$$

$$[g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} - g^{\mu\nu}g^{\rho\sigma}]g^{\rho\sigma}p^2 = g^{\mu\nu}p^2 + g^{\mu\nu}p^2 - g^{\mu\nu}(4-2\varepsilon)p^2 = -(2-2\varepsilon)p^2 g^{\mu\nu},$$

we find

$$\begin{aligned}\Pi^{Dim}_{\mu\nu}(p) &= i\frac{4e^2}{16\pi^2}(4\pi)^\varepsilon \left(-\frac{\mu^2}{p^2}\right)^\varepsilon \frac{\Gamma^2(2-\varepsilon)\Gamma(\varepsilon)}{\Gamma(4-2\varepsilon)}\left[2p^\mu p^\nu - g^{\mu\nu}p^2 - g^{\mu\nu}p^2\right] \\ &= -i\frac{8e^2}{16\pi^2}(4\pi)^\varepsilon \left(-\frac{\mu^2}{p^2}\right)^\varepsilon (g^{\mu\nu}p^2 - p^\mu p^\nu)\frac{\Gamma^2(2-\varepsilon)\Gamma(\varepsilon)}{\Gamma(4-2\varepsilon)}.\end{aligned} \tag{18.30}$$

Expanding now over ε with the help of

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon}\Gamma(1+\varepsilon),\ \Gamma(2-\varepsilon) = (1-\varepsilon)\Gamma(1-\varepsilon),\ \Gamma(4-2\varepsilon) = (3-2\varepsilon)(2-2\varepsilon)(1-2\varepsilon)\Gamma(1-2\varepsilon),$$

we finally get

$$\begin{aligned}\Pi^{Dim}_{\mu\nu}(p) &= -i\frac{e^2}{16\pi^2}(4\pi)^\varepsilon \left(-\frac{\mu^2}{p^2}\right)^\varepsilon (g^{\mu\nu}p^2 - p^\mu p^\nu)\frac{4(1+5/3\varepsilon)}{3\varepsilon}e^{-\gamma\varepsilon} \\ &= -ie^2\frac{g^{\mu\nu}p^2 - p^\mu p^\nu}{16\pi^2}\frac{4}{3}\left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \log\frac{-\mu^2}{p^2} + \frac{5}{3}\right], \\ &= i(g^{\mu\nu}p^2 - p^\mu p^\nu)\Pi^{Dim}(p^2),\end{aligned} \tag{18.31}$$

where

$$\Pi^{Dim}(p^2) = -\frac{e^2}{16\pi^2}\frac{4}{3}\left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \log\frac{-\mu^2}{p^2} + \frac{5}{3}\right]. \tag{18.32}$$

Given the expression for the vacuum polarization one can construct the photon propagator as shown in Fig.18.7.

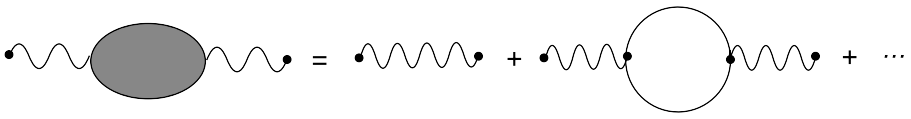


Figure 18.7: The photon propagator in QED

One has

$$\begin{aligned} G_{\mu\nu}(p) &= \frac{-i}{p^2}g^{\mu\nu} + \frac{-i}{p^2}g^{\mu\rho}\Pi_{\rho\sigma}\frac{-i}{p^2}g^{\sigma\nu} + \cdots \\ &= \frac{-i}{p^2}g^{\mu\nu} - \frac{\Pi^{\mu\nu}}{p^4} + \cdots = \frac{-i}{p^2}g^{\mu\nu} - \frac{i(g^{\mu\nu} - p^\mu p^\nu/p^2)}{p^2}\Pi(p^2) + \cdots \\ &= \frac{-i}{p^2}(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2})(1 + \Pi(p^2) + \cdots) - \frac{i}{p^2}\frac{p^\mu p^\nu}{p^2}, \end{aligned}$$

where $\Pi(p^2)$ is given by eq.(18.32). Notice that the radiative corrections are always proportional to the transverse tensor $P_{\mu\nu} = g_{\mu\nu} - p_\mu p_\nu/p^2$. This is a consequence of the gauge invariance and follows from the Ward identities.

Consider now the electron self-energy graph Fig.18.6b). The corresponding integral is

$$\Sigma(\hat{p}) = -\frac{e^2}{(2\pi)^4}\int d^4k \frac{\gamma^\mu(\hat{p} - \hat{k} + m)\gamma^\mu}{k^2[(p-k)^2 - m^2]}. \tag{18.33}$$

Acting in a usual way we go to dimension $4-2\varepsilon$, convert the indices of the γ-matrices and introduce the Feynman parametrization. The result is

$$\Sigma^{Dim}(\hat{p}) = -\frac{e^2(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int_0^1 dx \int \frac{d^{4-2\varepsilon}k[-2(1-\varepsilon)(\hat{p} - \hat{k}) + (4-2\varepsilon)m]}{[k^2 - 2kpx + p^2x - m^2x]^2}. \tag{18.34}$$

The integral over k can now be evaluated according to the standard formulas

$$\Sigma^{Dim}(\hat{p}) = -i\frac{e^2}{16\pi^2}\frac{(-\mu^2)^\varepsilon}{(4\pi)^{-\varepsilon}}\Gamma(\varepsilon)\int_0^1 dx \frac{-2(1-\varepsilon)\hat{p}(1-x) + (4-2\varepsilon)m}{[p^2x(1-x) - m^2x]^\varepsilon}. \tag{18.35}$$

This expression can be expanded in series in ε

$$\begin{aligned} \Sigma^{Dim}(\hat{p}) &= -i\frac{e^2}{16\pi^2}\left[-\frac{\hat{p} - 4m}{\varepsilon} + \hat{p} - 2m - (\hat{p} - 4m)(-\gamma_E + \log(4\pi))\right. \\ &+ \left.\int_0^1 dx[2\hat{p}(1-x) - 4m]\log\frac{p^2x(1-x) - m^2x}{-\mu^2}\right]. \end{aligned} \tag{18.36}$$

Notice that the linear divergence of the integral manifests itself as a simple pole in ε, and the coefficient has the dimension equal to 1 and is Lorentz invariant (this is either $\hat{p}$ or m).

At last, consider the vertex function Fig.18.6c). The corresponding integral is

$$\Gamma_1(p,q) = \frac{e^3}{(2\pi)^4}\int d^4k \frac{\gamma^\nu(\hat{p} - \hat{k} - \hat{q} + m)\gamma^\mu(\hat{p} - \hat{k} + m)\gamma^\nu}{[(p-k-q)^2 - m^2][(p-k)^2 - m^2]k^2}. \tag{18.37}$$

Transfer to dimension $4-2\varepsilon$ and introduce the Feynman parametrization. This gives

$$\Gamma_1^{Dim}(p,q) = \frac{e^3(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\Gamma(3)\int\limits_0^1 dx\int\limits_0^x dy \times \int \frac{d^{4-2\varepsilon}k[\gamma^\nu(\hat{p}-\hat{k}-\hat{q}+m)\gamma^\mu(\hat{p}-\hat{k}+m)\gamma^\nu]}{[((p-k-q)^2-m^2)y+((p-k)^2-m^2)(x-y)+k^2(1-x)]^3}. \tag{18.38}$$

The integral over k is straightforward and gives

$$\Gamma_1^{Dim}(p,q) = ie\frac{e^2}{16\pi^2}\frac{(-\mu^2)^\varepsilon}{(4\pi)^{-\varepsilon}}\int\limits_0^1 dx\int\limits_0^x dy \tag{18.39}$$
$$\left\{\Gamma(1+\varepsilon)\frac{[\gamma^\nu(\hat{p}(1-x)-\hat{q}(1-y)+m)\gamma^\mu(\hat{p}(1-x)+\hat{q}y+m)\gamma^\nu]}{[(p-q)^2y(1-x)+p^2(1-x)(x-y)+q^2y(x-y)-m^2x]^{1+\varepsilon}}\right.$$
$$\left.+\frac{\Gamma(\varepsilon)}{2}\frac{\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\rho\gamma^\nu}{[(p-q)^2y(1-x)+p^2(1-x)(x-y)+q^2y(x-y)-m^2x]^\varepsilon}\right\}.$$

As one can see, the first integral is finite and the second one is logarithmically divergent. Expanding in series in ε we find

$$\Gamma_1^{Dim}(p,q) = ie\frac{e^2}{16\pi^2}\left\{\frac{\gamma^\mu}{\varepsilon}-2\gamma^\mu-\gamma^\mu(\gamma_E-\log(4\pi))\right. \tag{18.40}$$
$$-\ 2\gamma^\mu\int\limits_0^1 dx\int\limits_0^x dy\ \log\left[\frac{(p-q)^2y(1-x)+p^2(1-x)(x-y)+q^2y(x-y)-m^2x}{-\mu^2}\right]$$
$$\left.+\int\limits_0^1 dx\int\limits_0^x dy\frac{\gamma^\nu(\hat{p}(1-x)-\hat{q}(1-y)+m)\gamma^\mu(\hat{p}(1-x)+\hat{q}y+m)\gamma^\nu}{(p-q)^2y(1-x)+p^2(1-x)(x-y)+q^2y(x-y)-m^2x}\right\}.$$

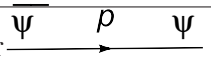
18.3.3 Quantum chromodynamics

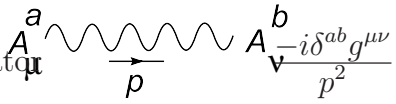
Consider now the non-Abelian gauge theories and, in particular, QCD. The Lagrangian of QCD has the form

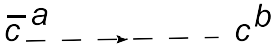
$$\mathcal{L}_{QD} = -\frac{1}{4}(F^a_{\mu\nu})^2+\bar{\psi}(i\gamma^\mu\partial_\mu-m)\psi+g\bar{\psi}\gamma^\mu A^a_\mu T^a\psi-\frac{1}{2\xi}(\partial_\mu A^a_\mu)^2$$
$$+\ \partial_\mu\bar{c}^a\partial_\mu c^2+gf^{abc}\partial_\mu\bar{c}^a A^b_\mu c^c, \tag{18.41}$$

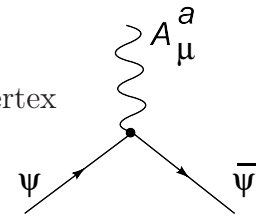
where the stress tensor of the gauge field is now $F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu$ and the last terms represent the Faddeev-Popov ghosts.

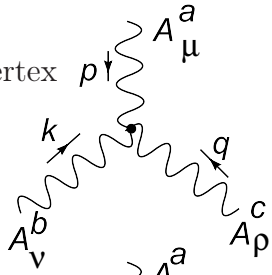
The Lagrangian (18.41) generates the following set of Feynman rules:

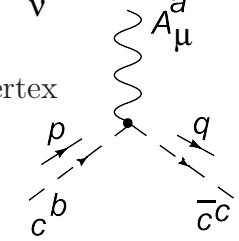
the spinor propagator  $= \frac{i}{\hat{p} - m}$

the vector propagator  $\frac{-i\delta^{ab} g^{\mu\nu}}{p^2}$

the ghost propagator  $= \frac{i\delta^{ab}}{p^2}$

the spino-gauge vertex  $-ig\gamma^{\mu} T^{a}$

the triple gauge vertex 
$= -gf^{abc}[(p-q)^{\rho} g^{\mu\nu}$
$+(q-k)^{\mu} g^{\rho\nu}$
$+(k-p)^{\nu} g^{\mu\rho}]$

the ghost-gauge vertex  $= -gf^{abc} q^{\mu}$

Consider the one-loop divergent diagrams. We start with the gluon propagator. Besides the diagram shown in Fig.18.6), one has additional contributions to the vacuum polarization from the diagrams shown in Fig.18.8. The first diagram takes into account the gluon self-interaction and the second one the contribution of the Faddeev-Popov ghosts. (As has already been mentioned, the tad-pole diagrams should not be included since they are automatically zero.) These diagrams depend on the choice of the gauge, and to evaluate them we have to fix the gauge. In what follows we choose the Feynman gauge (or the diagonal gauge) for the gluon field.

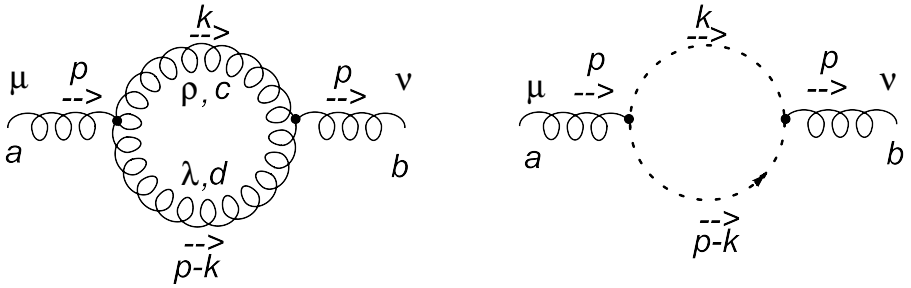


Figure 18.8: The vacuum polarization diagrams in the Yang-Mills theory

Then for the first diagram we have the expression

$$\Pi^{ab}_{\mu\nu}(p) = \frac{g^2 C_A \delta^{ab}}{2(2\pi)^4}\int \frac{d^4k}{k^2(k-p)^2}[(2p-k)^\rho g^{\mu\lambda}+(2k-p)^\mu g^{\rho\lambda}-(k+p)^\lambda g^{\mu\rho}] \times [(2p-k)^\rho g^{\lambda\nu}-(k+p)^\lambda g^{\nu\rho}+(2k-p)^\nu g^{\rho\lambda}], \quad (18.42)$$

where $1/2$ is a combinatorial factor and C_2 is the quadratic Casimir operator which for the SU(N) group equals N. It comes from the contraction of the gauge group structure constants f^{abc}

$$f^{abc}f^{dbc} = C_2\delta^{ad}.$$

Contracting the indices and going to $4-2\epsilon$ dimensions, one gets

$$\Pi^{Dim\ (ab)}_{\mu\nu}(p) = \delta^{ab}\frac{g^2C_A}{2}\frac{(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int\frac{d^{4-2\varepsilon}k}{k^2(k-p)^2}\{g^{\mu\nu}[4p^2+k^2+(k-p)^2] +(3-2\varepsilon)(2k-p)^\mu(2k-p)^\nu-(2p-k)^\mu(2p-k)^\nu-(k+p)^\mu(k+p)^\nu\}. \quad (18.43)$$

To calculate the integrals, one can use the formulas given above. The first step is the Feynman parametrization, eq.(18.4), and then the momentum integration is performed according to eqs.(18.16). Applying these rules we get for the integral (18.43)

$$\Pi^{Dim\ (ab)}_{\mu\nu}(p) = i\frac{g^2C_A\delta^{ab}}{(4\pi)^{2-\varepsilon}}\left[\frac{-\mu^2}{p^2}\right]^\varepsilon\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(2-\varepsilon)}{\Gamma(4-2\varepsilon)}[g^{\mu\nu}p^2(\frac{19}{2}-6\varepsilon)-p^\mu p^\nu(11-7\varepsilon)]. \quad (18.44)$$

The second diagram corresponds to the integral

$$\Pi^{Dim\ (ab)}_{\mu\nu}(p) = i\frac{g^2C_A\delta^{ab}}{(4\pi)^{2-\varepsilon}}\left[\frac{-\mu^2}{p^2}\right]^\varepsilon\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(2-\varepsilon)}{\Gamma(4-2\varepsilon)}[g^{\mu\nu}p^2(\frac{19}{2}-6\varepsilon)-p^\mu p^\nu(11-7\varepsilon)]. \quad (18.45)$$

here the "-" sign comes from the Fermi statistics of the ghost fields.

Calculation is now straightforward and gives

$$\Pi^{Dim\ (ab)}_{\mu\nu}(p) = i\frac{g^2C_A\delta^{ab}}{(4\pi)^{2-\varepsilon}}\left(-\frac{\mu^2}{p^2}\right)^\varepsilon\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(2-\varepsilon)}{\Gamma(4-2\varepsilon)}[g^{\mu\nu}p^2/2+p^\mu p^\nu(1-\varepsilon)]. \quad (18.46)$$

Adding up the two contributions together, one finally has

$$\Pi^{Dim\ (ab)}_{\mu\nu}(p) = iC_A\frac{2g^2\delta^{ab}}{16\pi^2}(4\pi)^\varepsilon\left[\frac{-\mu^2}{p^2}\right]^\varepsilon\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(2-\varepsilon)}{\Gamma(4-2\varepsilon)}(5-3\varepsilon)[g^{\mu\nu}p^2-p^\mu p^\nu] \quad (18.47)$$

or expanding in ε

$$\Pi^{Dim\ (ab)}_{\mu\nu}(p) = iC_A\delta^{ab}g^2\frac{g^{\mu\nu}p^2-p^\mu p^\nu}{16\pi^2}\frac{5}{3}\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi+\log\frac{-\mu^2}{p^2}+\frac{31}{15}\right]. \quad (18.48)$$

Acting the same way as in QED one can calculate the contribution to the gluon propagator.

Notice that the final result for the sum of the two diagrams is again proportional to the transverse tensor $P_{\mu\nu} = g_{\mu\nu} - p_\mu p_\nu/p^2$. This is not true, however, for the diagram with the gauge fields and is valid only if one takes into account the ghost contribution. Notice also the opposite sign of the resulting expression compared to that of eq.(18.31). This is due to a non-Abelian nature of the gauge fields and has very important consequences to be discussed later.

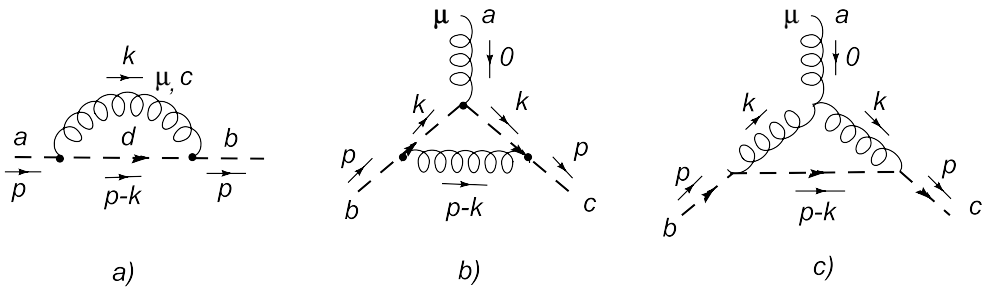


Figure 18.9: The ghost propagator and the ghost-gluon vertex diagrams in QCD

Consider also the ghost propagator. Here there is only one diagram shown in Fig.18.9a). It corresponds to the integral

$$\Pi^{Dim\ (ab)}(p) = -C_A\delta^{ab}\frac{g^2(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int d^{4-2\varepsilon}k\frac{k^\mu p^\mu}{k^2(k-p)^2}, \tag{18.49}$$

which equals

$$\begin{aligned}\Pi^{Dim\ (ab)}(p) &= -iC_A\delta^{ab}\frac{g^2}{2(4\pi)^{2-\varepsilon}}\left(\frac{-\mu^2}{p^2}\right)^\varepsilon p^2\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}\\ &= -iC_A\delta^{ab}\frac{g^2}{32\pi^2}p^2\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi+\log\frac{-\mu^2}{p^2}+2\right].\end{aligned} \tag{18.50}$$

Analogously one can calculate the vertex diagrams. We consider in more detail the calculation of the ghost-gluon vertex as a simpler one. The corresponding diagrams are shown in Fig.18.9. To simplify the evaluation, we put one of the momenta equal to zero. Then the first diagram gives the integral

$$V_{1\rho}^{Dim\ (abc)}(p) = i\frac{C_A}{2}f^{abc}\frac{g^3(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int d^{4-2\varepsilon}k\frac{k^\mu k^\rho p^\mu}{(k^2)^2(k-p)^2}. \tag{18.51}$$

Using the equality $kp = 1/2[k^2+p^2-(k-p)^2]$ and substituting it into (18.51) we find that the first two terms are reduced to the standard integrals and the last one leads to the tad-pole structure and is equal to zero. Adding up all together we get

$$\begin{aligned}V_{1\rho}^{Dim\ (abc)}(p) &= -C_A\frac{1}{4}f^{abc}\frac{g^3}{(4\pi)^{2-\varepsilon}}\left(-\frac{\mu^2}{p^2}\right)^\varepsilon p^\rho\frac{\Gamma(\varepsilon)\Gamma(2-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(3-2\varepsilon)}(1+2\varepsilon)\\ &= -C_A\frac{1}{8}f^{abc}\frac{g^3}{16\pi^2}p^\rho\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi+\log\frac{-\mu^2}{p^2}+4\right].\end{aligned} \tag{18.52}$$

The second diagram gives

$$V_{2\rho}^{Dim\ (abc)}(p) = -i\frac{C_A}{2}f^{abc}\frac{g^3(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}}\int d^{4-2\varepsilon}k\frac{(p-k)^\mu p^\nu[k^\nu g^{\mu\rho}+k^\mu g^{\nu\rho}-2k^\rho g^{\mu\nu}]}{(k^2)^2(k-p)^2}. \tag{18.53}$$

Contracting the indices in the numerator we have $(p-k)^\rho kp + p^\rho k(p-k) - 2k^\rho p(p-k)$, which after integration leads to

$$\begin{aligned}V_{2\rho}^{Dim\ (abc)}(p) &= -C_A\frac{3}{8}f^{abc}\frac{g^3}{(4\pi)^{2-\varepsilon}}\left(-\frac{\mu^2}{p^2}\right)^\varepsilon p^\rho\frac{\Gamma(\varepsilon)\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}(1-\frac{2}{3}\varepsilon)\\ &= -C_A\frac{3}{8}f^{abc}\frac{g^3}{16\pi^2}p^\rho\left[\frac{1}{\varepsilon}-\gamma_E+\log 4\pi+\log\frac{-\mu^2}{p^2}+\frac{4}{3}\right].\end{aligned} \tag{18.54}$$

Adding up the two contributions together we find

$$V_\rho^{Dim\ (abc)}(p) = -C_A \frac{1}{2} f^{abc} \frac{g^3}{16\pi^2} p^\rho \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \log \frac{-\mu^2}{p^2} + 2 \right]. \tag{18.55}$$

Having in mind that at the tree level the vertex has the form $V_\rho^{tree\ (abc)}(p) = -g f^{abc} p^\rho$ we get the vertex function in the one-loop approximation as

$$V_\rho^{(abc)}(p) = -g f^{abc} p^\rho \left\{ 1 + C_A \frac{1}{2} \frac{g^2}{16\pi^2} \left[\frac{1}{\varepsilon} - \gamma_E + \log 4\pi + \log \frac{-\mu^2}{p^2} + 2 \right] \right\}. \tag{18.56}$$

18.4 K-l.IV: Renormalization. General Idea

Thus, we have convinced ourselves that the integrals for the radiative corrections are indeed ultraviolet divergent in accordance with the naive power counting. The question then is: how to get a sensible result for the cross-sections of the scattering processes, decay widths, etc? To answer this question let us see what is the reason for divergences at large values of momenta. In coordinate space the large values of momenta correspond to the small distances. Hence, the ultraviolet divergences allow for the singularities at small distances. Indeed, the simplest divergent loop diagram (Fig.18.2) in coordinate space is the product of two propagators. Each propagator is uniquely defined in momentum as well as in coordinate space, but the square of the propagator has already an ill-defined Fourier-transform, it is ultraviolet divergent. The reason is that the square of the propagator is singular as $x^2 \to 0$ and behaves like $1/(x^2)^2$. In fact, the causal Green function (the propagator) is the so-called distribution which is defined on smooth functions. It has the δ-function like singularities and needs an additional definition for the product of several such functions at a single point. The discussed diagram is precisely this product.

The general approach to the elimination of the ultraviolet divergences known as the $\mathcal{R}$-operation was developed in the 1950s. It consists in the introduction to the initial Lagrangian of additional local (or quasi-local) terms, called the counter-terms, which serve the task of the definition of the product of distributions at the coinciding points. The counter-terms lead to additional diagrams which cancel the ultraviolet divergences. The peculiarity of this procedure, being the subject of the Bogoliubov-Parasiuk theorem, is in that the singularities are local in coordinate space, i.e., are the functions of a single point and can contain only a finite number of derivatives. In the theories belonging to the renormalizable class, where the number of divergent structures is finite, the number of types of the counter-terms is also finite, they repeat the terms of the original Lagrangian. This means that the introduction of the counter-terms in this case is equivalent to the modification of the coefficients of various terms., i.e. to the modification of the normalization of these terms. That is why this procedure was called the *renormalization* procedure.

It should be stressed that the parameters of the original Lagrangian like the masses, the coupling constants and the fields themselves are not, strictly speaking, observable. They can be infinite. It is important that the renormalized parameters which enter the final answers are meaningful.

Below we show by several examples of renormalizable theories how one introduces the counter-terms into the Lagrangian, how they lead to the renormalization of the original parameters and how the renormalization procedure allows one to get finite results for the Green functions.

18.4.1 The scalar theory. The one-loop approximation

We start with the one-loop approximation and consider for simplicity the scalar theory (18.18). It belongs to the renormalizable type and has a finite number of ultraviolet divergent structures.

The one-loop divergent diagrams in this theory were calculated in the third lecture. Here we are interested in the singular parts, i.e., the poles in ε. They are given by eqs. (18.20) and (18.24.

$$\text{The propagator}: \quad Sing\ J_1(p^2) = -im^2(\tfrac{\lambda}{16\pi^2})(-\tfrac{1}{2\varepsilon}),$$

$$\text{The vertex}: \quad Sing\ \Gamma_4(s,t,u) = -i\lambda(\tfrac{\lambda}{16\pi^2})(-\tfrac{3}{2\varepsilon}).$$

Note that the singular parts do not depend on momenta, i.e. their Fourier-transform has the form of the δ-function in coordinate space.

In order to remove the obtained singularities we add to the Lagrangian (18.18) extra terms, the counter-terms equal to the singular parts with the opposite sign (the factor i belongs to the S-matrix and does not enter into the Lagrangian), namely,

$$\Delta\mathcal{L} = \frac{1}{2\varepsilon}\frac{\lambda}{16\pi^2}\ (-\frac{m^2}{2}\phi^2) + \frac{\lambda}{16\pi^2}\frac{3}{2\varepsilon}\ (-\frac{\lambda}{4!}\phi^4). \tag{18.57}$$

These counter-terms correspond to additional vertices shown in Fig.18.10, where the cross denotes

Figure 18.10: The one-loop counter-terms in the scalar theory

the contribution corresponding to (18.57). With account taken of the new diagrams the expressions for the propagator (18.20) and the vertex (18.24) become

$$J_1(p^2) = \frac{i\lambda}{32\pi^2}m^2\left(1-\gamma_E+\log(4\pi)-\log(m^2/\mu^2)\right). \tag{18.58}$$

$$\Delta\Gamma_4 = i\lambda\left\{\frac{\lambda}{16\pi^2}\left(3-\frac{3}{2}\gamma_E+\frac{3}{2}\log(4\pi)+\frac{1}{2}\ln\frac{\mu^2}{-s}+\frac{1}{2}\ln\frac{\mu^2}{-t}+\frac{1}{2}\ln\frac{\mu^2}{-u}\right)\right\}. \tag{18.59}$$

Notice that the obtained expressions have no infinities but contain the dependence on the regularization parameter μ^2 which was absent in the initial theory. The appearance of this dependence on a dimensional parameter is inherent in any regularization and is called the dimensional transmutation, i.e., an appearance of a new scale in a theory.

What we have done is equivalent to *subtraction* of divergences from the diagrams. In doing this we have subtracted just the singular parts. This way of subtraction is called the *minimal subtraction scheme* or the MS-scheme. One can make the subtraction differently, for instance, subtract also the finite parts. It is useful to subtract the Euler constant and $\log 4\pi$ which accompany the pole terms. This subtraction scheme is called the *modified minimal subtraction scheme* or the $\overline{MS}$-scheme. It is equivalent to the redefinition of the parameter μ^2. Another popular scheme of subtraction is the so-called MOM-scheme when the subtractions are made for fixed values of momenta. For example, in the case of the vertex function one can make the subtraction at the point $s = t = u = l^2$. This subtraction is called the subtraction at a symmetric point.

The difference between various subtraction schemes is in the finite parts; in the one-loop approximation this is just the constant independent of momentum, however, in higher loops one already has momentum dependent terms. Therefore, the finite parts of the Green functions depend on a *subtraction scheme*. Note that this dependence in general is not reduced to the redefinition

of the parameter μ, since there are usually a few divergent Green functions and all of them are independent.

Thus, in the three subtraction schemes discussed above we have three different values for the vertex function

$$\begin{aligned}
\Gamma_4^{MS} &= -i\lambda\left\{1-\frac{\lambda}{16\pi^2}\left[3-\frac{3}{2}\gamma_E+\frac{3}{2}\log 4\pi+\frac{1}{2}\ln\frac{\mu^2}{-s}+\frac{1}{2}\ln\frac{\mu^2}{-t}+\frac{1}{2}\ln\frac{\mu^2}{-u}\right]\right\},\\
\Gamma_4^{\overline{MS}} &= -i\lambda\left\{1-\frac{\lambda}{16\pi^2}\left[3+\frac{1}{2}\ln\frac{\mu^2}{-s}+\frac{1}{2}\ln\frac{\mu^2}{-t}+\frac{1}{2}\ln\frac{\mu^2}{-u}\right]\right\},\\
\Gamma_4^{MOM} &= -i\lambda\left\{1-\frac{\lambda}{16\pi^2}\left[\frac{1}{2}\ln\frac{l^2}{-s}+\frac{1}{2}\ln\frac{l^2}{-t}+\frac{1}{2}\ln\frac{l^2}{-u}\right]\right\}.
\end{aligned}$$

The counter-terms are also different. It is useful to write them in the following way

$$\Delta\mathcal{L} = -(Z-1)\frac{m^2}{2}\phi^2-(Z_4-1)\frac{\lambda}{4!}\phi^4, \tag{18.60}$$

where for different subtraction schemes one has

$$\begin{aligned}
Z^{MS} &= 1+\frac{1}{2\varepsilon}\frac{\lambda}{16\pi^2},\\
Z^{\overline{MS}} &= 1+[\frac{1}{2\varepsilon}+1-\gamma_E+\log(4\pi)]\frac{\lambda}{16\pi^2},\\
Z_4^{MS} &= 1+\frac{3}{2\varepsilon}\frac{\lambda}{16\pi^2},\\
Z_4^{\overline{MS}} &= 1+[\frac{3}{2\varepsilon}-3\gamma_E+3\log(4\pi)]\frac{\lambda}{16\pi^2},\\
Z_4^{MOM} &= 1+[\frac{3}{2\varepsilon}+3-3\gamma_E+3\log(4\pi)+\frac{3}{2}\ln\frac{\mu^2}{l^2}]\frac{\lambda}{16\pi^2}.
\end{aligned} \tag{18.61}$$

The Lagrangian (18.18) together with the counter-terms (18.60) can be written as

$$\mathcal{L}+\Delta\mathcal{L} = Z_2\frac{1}{2}(\partial_\mu\varphi)^2-Z\frac{m^2}{2}\varphi^2-Z_4\frac{\lambda}{4!}\varphi^4 = \mathcal{L}_{Bare}, \tag{18.62}$$

where the renormalization constants Z and Z_4 are given by (18.61) and the renormalization constant Z_2 in the one-loop approximation equals 1.

Writing the "bare" Lagrangian in the same form as the initial one but in terms of the "bare" fields and couplings

$$\mathcal{L}_{Bare} = \frac{1}{2}(\partial_\mu\varphi_B)^2-\frac{m_B^2}{2}\varphi_B^2-\frac{\lambda_B}{4!}\varphi_B^4 \tag{18.63}$$

and comparing it with (18.62), we get the connection between the "bare" and renormalized quantities

$$\varphi_B = \sqrt{Z_2}\varphi,\quad m_B^2 = ZZ_2^{-1}m^2,\quad \lambda_B = Z_4Z_2^{-2}\lambda. \tag{18.64}$$

Equations (18.63) and (18.64) imply that the one-loop radiative corrections calculated from the Lagrangian (18.63) with parameters chosen according to (18.64,18.61) are finite.

18.4.2 The scalar theory. The two-loop approximation

Consider now the two-loop diagrams. For simplicity and in order to complete all the integrations we restrict ourselves to the massless case. Since we are going to calculate the diagrams off mass shell, no infrared divergences may appear.

The propagator: In this order of PT there is only one diagram shown in Fig.18.11.

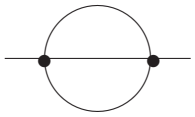


Figure 18.11: The two-loop propagator type diagram

The corresponding integral equals

$$J_2(p^2) = \frac{(-i\lambda)^2}{3!}\frac{i^3(\mu^2)^{2\varepsilon}}{(2\pi)^{8-4\varepsilon}}\int\frac{d^{4-2\varepsilon}kd^{4-2\varepsilon}q}{q^2(k-q)^2(p-k)^2},$$

(1/3! is a combinatorial coefficient). Let us use the method of evaluation of the massless diagrams described above. One has to transform each of the propagators into coordinate space, multiply them and transform back to momentum space. This reduces to writing down the corresponding transformation factors. One gets

$$J_2(p^2) = \frac{i\lambda^2}{6}\frac{(i\pi^2)^{2-\varepsilon}}{(2\pi)^{8-4\varepsilon}}p^2\left(\frac{\mu^2}{-p^2}\right)^{2\varepsilon}\frac{\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)\Gamma(-1+2\varepsilon)}{\Gamma(1)\Gamma(1)\Gamma(1)\Gamma(3-3\varepsilon)}$$

$$= \frac{i}{6}\frac{\lambda^2}{(16\pi^2)^2}\left[\frac{\mu^2}{-p^2}\right]^{2\varepsilon}\frac{p^2}{(2-3\varepsilon)(1-3\varepsilon)(1-2\varepsilon)2\varepsilon} = \frac{i}{24}\frac{\lambda^2}{(16\pi^2)^2}p^2\left[\frac{1}{\varepsilon}+\frac{13}{2}+2\ln\frac{\mu^2}{-p^2}\right],$$

where the Euler constant and $\log 4\pi$ are omitted.

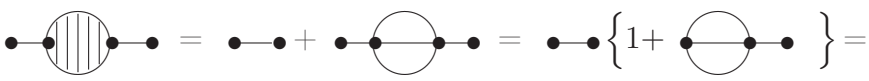
The appeared ultraviolet divergence, the pole in ε, can be removed via the introduction of the (quasi)local counter-term

$$\Delta\mathcal{L} = \frac{1}{2}(Z_2-1)(\partial\phi)^2, \tag{18.65}$$

where the wave function renormalization constant Z_2 in the $\overline{MS}$ scheme is obtained by taking the singular part of the integral with the opposite sign

$$Z_2 = 1 - \frac{1}{24\varepsilon}\left(\frac{\lambda}{16\pi^2}\right)^2. \tag{18.66}$$

After that the propagator in the massless case takes the form



$$= \frac{i}{p^2}\left\{1 - \frac{1}{24}\frac{\lambda^2}{(16\pi^2)^2}\left(\frac{13}{2}+2\ln\frac{\mu^2}{-p^2}\right)\right\}. \tag{18.67}$$

The vertex: In the given order there are two diagrams (remind that in the massless case the tad-poles equal to zero) shown in Fig.18.12.

Figure 18.12: The two-loop vertex diagrams

The first diagram by analogy with the one-loop case equals the sum of s, t and u channels

$$I_{21} = I_{21}(s) + I_{21}(t) + I_{21}(u),$$

where each integral is nothing else but the square of the one-loop integral

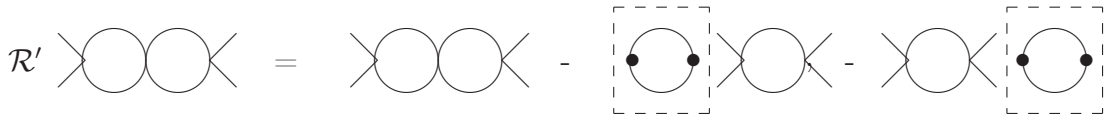
$$I_{21}(s) = \frac{(-i\lambda)^3}{96}\left(\frac{(\mu^2)^\varepsilon}{(2\pi)^{4-2\varepsilon}} i^2 \int \frac{d^{4-2\varepsilon}k}{k^2(p-k)^2}\right)^2 = -\frac{i}{96}\frac{\lambda^3}{(16\pi^2)^2}(\frac{1}{\varepsilon} + 2 + \ln\frac{\mu^2}{-s})^2. \qquad (18.68)$$

(1/96 is the combinatorial coefficient).

Opening the bracket we, for the first time here, come across the second order pole term $1/\varepsilon^2$ and the single pole $\log(-\mu^2/s)/\varepsilon$ accompanying it. This latter pole is not harmless since its Fourier-transform is not a local function of coordinates. This means that it can not be eliminated by a local counter-term. This would be an unremovable problem if it were not the one-loop counter-terms (18.57) which created the new vertices shown in Fig.18.10. In the same order of λ^3 one gets additional diagrams presented in Fig.18.13.

Figure 18.13: The diagrams with the counter-terms in the two-loop approximation

These diagrams lead to the subtraction of divergences in the subgraphs (left and right) in the first diagram of Fig.18.12. The subtraction of divergent subgraphs (the $\mathcal{R}$-operation without the last subtraction called the $\mathcal{R}'$-operation) looks like



where the subgraph surrounded with the dashed line means its singular part, and the rest of the graph is obtained by shrinking down the singular subgraph to a point. The result has the form

$$\mathcal{R}' I_{21}(s) = -\frac{i}{4}\frac{\lambda^3}{(16\pi^2)^2}\left\{(\frac{1}{\varepsilon} + 2 + \ln\frac{\mu^2}{-s})^2 - \frac{2}{\varepsilon}(\frac{1}{\varepsilon} + 2 + \ln\frac{\mu^2}{-s})\right\} =$$

$$= -\frac{i}{4}\frac{\lambda^3}{(16\pi^2)^2}\left(-\frac{1}{\varepsilon^2} + 4 + \ln^2\frac{\mu^2}{-s} + 4\ln\frac{\mu^2}{-s}\right).$$

Notice that after the subtractions of subgraphs the singular part is local, i.e. in momentum space does not contain $\ln p^2$. The terms with the single pole $1/\varepsilon$ are absent since the diagram can be factorized into two diagrams of the lower order.

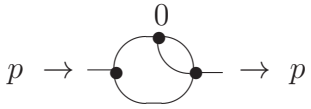
The contribution of a given diagram to the vertex function equals

$$\begin{aligned}\Delta\Gamma_4 &= -i\lambda\left\{\frac{1}{4}\frac{\lambda^2}{(16\pi^2)^2}\left(-\frac{3}{\varepsilon^2}+12\right.\right. \\ &\left.\left.+\ln^2\frac{\mu^2}{-s}+4\ln\frac{\mu^2}{-s}+\ln^2\frac{\mu^2}{-t}+4\ln\frac{\mu^2}{-t}+\ln^2\frac{\mu^2}{-u}+4\ln\frac{\mu^2}{-u}\right)\right\}\end{aligned} \tag{18.69}$$

The contribution to the renormalization constant of the four-point vertex in the $\overline{MS}$ scheme is equal to the singular part with the opposite sign

$$\Delta Z_4 = +\frac{3}{4\varepsilon^2}\left(\frac{\lambda}{16\pi^2}\right)^2. \tag{18.70}$$

The second diagram with the crossed terms contains 6 different cases. Consider one of them. Since we are interested here in the singular parts contributing to the renormalization constants, we perform some simplification of the original integral. We use a very important property of the minimal subtraction scheme that the renormalization constants depend only on dimensionless coupling constants and do not depend on the masses and the choice of external momenta. Therefore, we put all the masses equal to zero, and to avoid artificial infrared divergences, we also put equal to zero one of the external momenta. Then the diagram becomes the propagator type one:



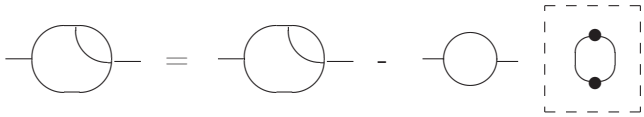
The corresponding integral is:

$$I_{22}(p^2) = \frac{(-i\lambda)^3}{48}\frac{(\mu^2)^{2\varepsilon}}{(2\pi)^{8-4\varepsilon}}i^4\int\frac{d^{4-2\varepsilon}qd^{4-2\varepsilon}k}{q^2(k-q)^2k^2(p-k)^2},$$

(1/48 is the combinatorial coefficient). Since putting one of the momenta equal to zero we reduced the diagram to the propagator type, we can again use the advocated method to calculate the massless integral. One has

$$\begin{aligned}I_{22}(p^2) &= \frac{i\lambda^3}{48}\frac{(\mu^2)^{2\varepsilon}}{(2\pi)^{8-4\varepsilon}}i\pi^2\frac{\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(1)\Gamma(1)\Gamma(2-2\varepsilon)}\int\frac{d^{4-2\varepsilon}k}{(k^2)^{1+\varepsilon}(p-k)^2} \\ &= -\frac{i}{48}\frac{\lambda^3}{(16\pi^2)^2}\left(\frac{\mu^2}{-p^2}\right)^{2\varepsilon}\frac{\Gamma(1-\varepsilon)\Gamma(1-\varepsilon)\Gamma(\varepsilon)\Gamma(1-2\varepsilon)\Gamma(1-\varepsilon)\Gamma(2\varepsilon)}{\Gamma(1)\Gamma(1)\Gamma(2-2\varepsilon)\Gamma(1+\varepsilon)\Gamma(1)\Gamma(2-3\varepsilon)} \\ &= -\frac{i}{48}\frac{\lambda^3}{(16\pi^2)^2}\left(\frac{\mu^2}{-p^2}\right)^{2\varepsilon}\frac{1}{2\varepsilon^2(1-2\varepsilon)(1-3\varepsilon)} \\ &= -\frac{i}{48}\frac{\lambda^3}{(16\pi^2)^2}\left\{\frac{1}{2\varepsilon^2}+\frac{5}{2\varepsilon}+2+\frac{\ln(-\mu^2/p^2)}{\varepsilon}+\ln^2\frac{\mu^2}{-p^2}+5\ln\frac{\mu^2}{-p^2}\right\}.\end{aligned}$$

As one can see, in this case we again have the second order pole in ε and, accordingly, the single pole with the logarithm of momentum. The reason of their appearance is the presence of the divergent subgraph. Here we again have to look at the counter-terms of the previous order which eliminate the divergence from the one-loop subgraph. The subtraction of divergent subgraphs (the $\mathcal{R}$-operation without the last subtraction) looks like

$\mathcal{R}'$ 

or

$$\mathcal{R}'I_2(s) = -\frac{i}{2}\frac{\lambda^3}{(16\pi^2)^2}\left\{\left(\frac{\mu^2}{-p^2}\right)^{2\varepsilon}\frac{1}{2\varepsilon^2(1-2\varepsilon)(1-3\varepsilon)} - \left(\frac{\mu^2}{-p^2}\right)^{\varepsilon}\frac{1}{\varepsilon^2(1-2\varepsilon)}\right\}$$

$$= -\frac{i}{2}\frac{\lambda^3}{(16\pi^2)^2}\left\{\left(\frac{1}{2\varepsilon^2}+\frac{5}{2\varepsilon}+2+\frac{\ln(-\mu^2/p^2)}{\varepsilon}+\ln^2\frac{\mu^2}{-p^2}+5\ln\frac{\mu^2}{-p^2}\right)\right.$$

$$\left. -\left(\frac{1}{\varepsilon^2}+\frac{2}{\varepsilon}+4+\frac{\ln(-\mu^2/p^2)}{\varepsilon}+\frac{1}{2}\ln^2\frac{\mu^2}{-p^2}+2\ln\frac{\mu^2}{-p^2}\right)\right\} =$$

$$= -\frac{i}{2}\frac{\lambda^3}{(16\pi^2)^2}\left\{-\frac{1}{2\varepsilon^2}+\frac{1}{2\varepsilon}-2+\frac{1}{2}\ln^2\frac{\mu^2}{-p^2}+3\ln\frac{\mu^2}{-p^2}\right\}.$$

Once again, after the subtraction of the divergent subgraph the singular part is local, i.e. in momentum space does not depend on $\ln p^2$.

The contribution to the vertex function from this diagram is:

$$\Delta\Gamma_4 = -i\lambda\left\{\frac{1}{2}\frac{\lambda^2}{(16\pi^2)^2}\left(-\frac{3}{\varepsilon^2}+\frac{3}{\varepsilon}-12+\frac{1}{2}\ln^2\frac{\mu^2}{-p^2}+3\ln\frac{\mu^2}{-p^2}+\ldots\right)\right\} \tag{18.71}$$

and, accordingly,

$$\Delta Z_4 = (\frac{3}{2\varepsilon^2}-\frac{3}{2\varepsilon})\left(\frac{\lambda}{16\pi^2}\right)^2. \tag{18.72}$$

Thus, due to (18.61) and (18.72) in the two-loop approximation the quartic vertex renormalization constant in the $\overline{MS}$ scheme looks like:

$$Z_4 = 1+\frac{3}{2\varepsilon}\frac{\lambda}{16\pi^2}+\left(\frac{\lambda}{16\pi^2}\right)^2(\frac{9}{4\varepsilon^2}-\frac{3}{2\varepsilon}). \tag{18.73}$$

With taking account of the two-loop renormalization of the propagator (18.66) one has:

$$Z_\lambda = Z_4 Z_2^{-2} = 1+\frac{3}{2\varepsilon}\frac{\lambda}{16\pi^2}+\left(\frac{\lambda}{16\pi^2}\right)^2(\frac{9}{4\varepsilon^2}-\frac{17}{12\varepsilon}). \tag{18.74}$$

The statement is that the counter-terms introduced this way eliminate all the ultraviolet divergences up to two-loop order and make the Green functions and hence the radiative corrections finite. In the case of nonzero mass, one should also add the mass counter-term.

18.4.3 The general structure of the R-operation

We are ready to formulate now the general procedure of getting finite expressions for the Green functions off mass shell in an arbitrary local quantum field theory. It consists of:

In any order of perturbation theory in the coupling constant one introduces to the Lagrangian the (quasi) local counter-terms. They perform the subtraction of divergences in the diagrams of a given order. The subtraction of divergences in the subgraphs is provided by the counter-terms of the lower order. After the subtraction of divergences in the subgraphs the rest of the divergences

are always local. The Green functions of the given order calculated on the basis of the initial Lagrangian with account of the counter-terms are ultraviolet finite.

The structure of the counter-terms as functions of the field operators depends on the type of a theory. According to the classification discussed in the first lecture, the theories are divided into three classes: *superrenormalizable* (a finite number of divergent diagrams), *renormalizable* (a finite number of types of divergent diagrams) and *non-renormalizable* (a infinite number of types of divergent diagrams). Accordingly, in the first case one has a finite number of counter-terms; in the second case, a infinite number of counter-terms but they repeat the structure of the initial Lagrangian, and in the last case, one has an infinite number of structures with an increasing number of the fields and derivatives.

In the case of renormalizable and superrenormalizable theories, since the counter-terms repeat the structure of the initial Lagrangian, the result of the introduction of counter-terms can be represented as

$$\mathcal{L} + \Delta\mathcal{L} = \mathcal{L}_{Bare} = \mathcal{L}(\phi_B, \{g_B\}, \{m_B\}), \tag{18.75}$$

i.e., $\mathcal{L}_{Bare}$ is the same Lagrangian $\mathcal{L}$ but with the fields, masses and coupling constants being the "bare" ones related to the renormalized quantities by the multiplicative equalities

$$\phi_i^{Bare} = Z_i^{1/2}(\{g\}, 1/\varepsilon)\phi, \quad g_i^{Bare} = Z_g^i(\{g\}, 1/\varepsilon)g_i, \quad m_i^{Bare} = Z_m^i(\{g\}, 1/\varepsilon)m_i, \tag{18.76}$$

where the renormalization constants Z_i depend on the renormalized parameters and the parameter of regularization (for definiteness we have chosen $1/\varepsilon$). In some cases the renormalization can be nondiagonal and the renormalization constants become matrices.

The renormalization constants are not unique and depend on the renormalization scheme. This arbitrariness, however, does not influence the observables expressed through the renormalized quantities. We will come back to this problem later when discussing the group of renormalization. In the gauge theories Z_i may depend on the choice of the gauge though in the minimal subtraction scheme the renormalizations of the masses and the couplings are gauge invariant.

In the minimal schemes the renormalization constants do not depend on dimensional parameters like masses and do not depend on the arrangement of external momenta in the diagrams. This property allows one to simplify the calculation of the counter-terms putting the masses and some external momenta to zero, as it was exemplified above by calculation of the two-loop diagrams. In making this trick, however, one has to be careful not to create artificial infrared divergences. Since in dimensional regularization they also have the form of poles in ε, this may lead to the wrong answers.

In renormalizable theory the finite Green function is obtained from the "bare" one, i.e., is calculated from the "bare" Lagrangian by multiplication on the corresponding renormalization constant

$$\Gamma(\{p^2\}, \mu^2, g_\mu) = Z_\Gamma(1/\varepsilon, g_\mu)\Gamma_{Bare}(\{p^2\}, 1/\varepsilon, g_{Bare}), \tag{18.77}$$

where in the n-th order of perturbation theory the "bare" parameters in the r.h.s. have to be expressed in terms of the renormalized ones with the help of relations (18.76) taken in the (n-1)-th order. The remaining constant Z_Γ creates the counter-term of the n-th order of the form $\Delta\mathcal{L} = (Z_\Gamma - 1)O_\Gamma$, where the operator O_Γ reflects the corresponding Green function. If the Green function is finite by itself (for instance, has many legs), then one has to remove the divergences only in the subgraphs and the corresponding renormalization constant $Z_\Gamma = 1$.

Note that since the propagator is inverse to the operator quadratic in fields in the Lagrangian, the renormalization of the propagator is also inverse to the renormalization of the 1-particle irre-

ducible two-point Green function

$$D(p^2, \mu^2, g_\mu) = Z_2^{-1}(1/\varepsilon, g_\mu) D_{Bare}(p^2, 1/\varepsilon, g_{Bare}). \tag{18.78}$$

The propagator renormalization constant is also the renormalization constant of the corresponding field, but the fields themselves, contrary to the masses and couplings, do not enter into the expressions for observables.

We would like to stress once more that the $\mathcal{R}$-operation works independently on the fact renormalizable or non-renormalizable the theory is. In local theory the counter-terms are local anyway. But only in renormalizable theory the counter-terms are reduced to the multiplicative renormalization of the finite number of fields and parameters.

One can perform the $\mathcal{R}$-operation for each diagram separately. For this purpose one has first of all to subtract the divergences in the subgraphs and then subtract the divergence in the diagram itself which has to be local. This serves as a good test that the divergences in the subgraphs are subtracted correctly. In this case the $\mathcal{R}$-operation can be symbolically written in a factorized form

$$\mathcal{R}G = \prod_{div.subgraphs} (1 - M_\gamma) G, \tag{18.79}$$

where G is the initial diagram, M is the subtraction operator (for instance, subtraction of the singular part of the regularized diagram) and the product goes over all divergent subgraphs including the diagram itself. By a subgraph we mean here the 1-particle irreducible diagram consisting of the vertices and lines of the diagram which is UV divergent. The 1-particle irreducible is called the diagram which can not be made disconnected by deleting of one line.

We have demonstrated above the application of the $\mathcal{R}$-operation to the two–loop diagrams in a scalar theory. Consider some other examples of diagrams with larger number of loops shown in Fig.18.14. They appear in the ϕ^4 theory in the three-loop approximation.

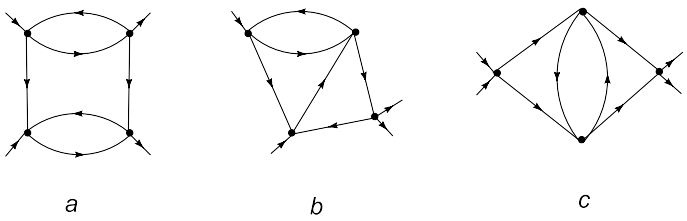


Figure 18.14: The multiloop diagrams in the ϕ^4 theory

In order to perform the $\mathcal{R}$-operation for these diagrams one first has to find out the divergent subgraphs. They are shown in Fig.18.15.

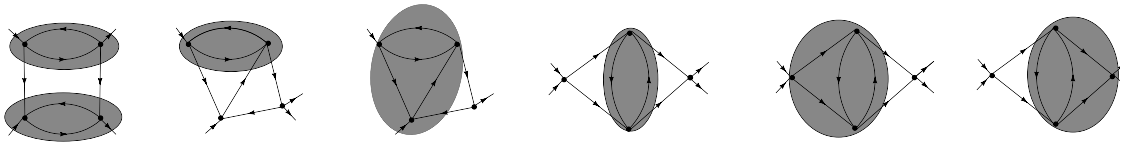


Figure 18.15: The divergent subgraphs in the diagrams of Fig.18.14

Let us use the factorized representation of the $\mathcal{R}$-operation in the form of (18.79). For the three chosen diagrams one has, respectively,

$$\begin{aligned} RG_a &= (1-M_G)(1-M_{\gamma_1})(1-M_{\gamma_1'})G_a, \\ RG_b &= (1-M_G)(1-M_{\gamma_2})(1-M_{\gamma_1})G, \\ RG_c &= (1-M_G)(1-M_{\gamma_2})(1-M_{\gamma_2'})(1-M_{\gamma_1})G, \end{aligned}$$

where γ_1 and γ_2 are the one- and two-loop divergent subgraphs shown in Fig.18.15.

The result of the application of the $\mathcal{R}$-operation without the last subtraction ($\mathcal{R}'$-operation) for the diagrams of interest graphically is as follows:

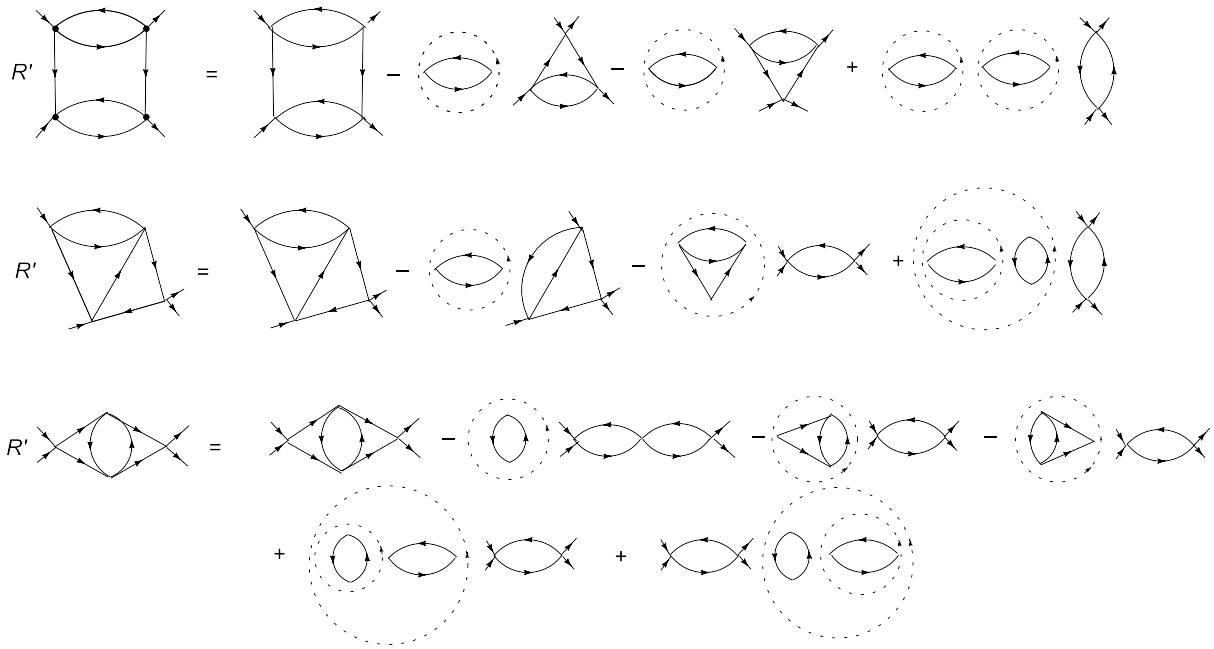
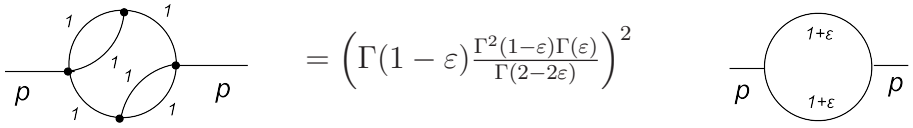


Figure 18.16: The $\mathcal{R}'$-operation for the multiloop diagrams

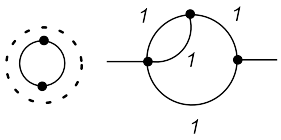
Here, as before, the graph surrounded with the dashed circle means its singular part and the remaining graph is obtained by shrinking the singular subgraph to a point.

Let us demonstrate how the $\mathcal{R}'$-operation works for the diagram Fig.18.14a). Since the result of the $\mathcal{R}'$-operation does not depend on external momenta, we put two momenta on the diagonal to be equal to zero so that the integral takes the propagator form. Then we can use the method based on Fourier-transform, as it was explained above. One has

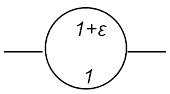


$$= \left(\Gamma(1-\varepsilon)\frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}\right)^2 \left(\Gamma(1-\varepsilon)\frac{\Gamma^2(1-2\varepsilon)\Gamma(3\varepsilon)}{\Gamma^2(1+\varepsilon)\Gamma(2-4\varepsilon)}\right)\left(\frac{\mu^2}{p^2}\right)^{3\varepsilon} \cong \frac{1}{\varepsilon^3(1-2\varepsilon)^2(1-4\varepsilon)}\left(\frac{\mu^2}{p^2}\right)^{3\varepsilon}.$$

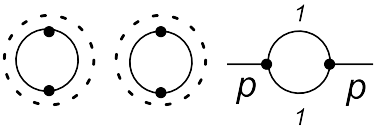
We use here the angular integration measure in the $4-2\varepsilon$ dimensional space accepted above, which results in the multiplication of the standard expression by $\Gamma(1-\varepsilon)$ in order to avoid the unwanted transcendental functions. Following the scheme shown in Fig.18.16 we get



$$= \tfrac{1}{\varepsilon}\Gamma(1-\varepsilon)\tfrac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}$$

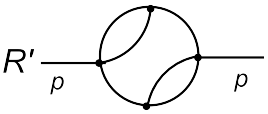


$$= \tfrac{1}{\varepsilon}\Gamma(1-\varepsilon)\tfrac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}\Gamma(1-\varepsilon)\tfrac{\Gamma(1-\varepsilon)\Gamma(1-2\varepsilon)\Gamma(2\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(2-3\varepsilon)}\left(\tfrac{\mu^2}{p^2}\right)^{2\varepsilon} \cong \tfrac{1}{\varepsilon^3(1-2\varepsilon)(1-3\varepsilon)}\left(\tfrac{\mu^2}{p^2}\right)^{2\varepsilon}.$$



$$= \tfrac{1}{\varepsilon^2}\Gamma(1-\varepsilon)\tfrac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)}{\Gamma(2-2\varepsilon)}\left(\tfrac{\mu^2}{p^2}\right)^{\varepsilon} \cong \tfrac{1}{\varepsilon^3(1-2\varepsilon)}\left(\tfrac{\mu^2}{p^2}\right)^{\varepsilon}.$$

Combining all together one finds



$$\cong \tfrac{1}{\varepsilon^3(1-2\varepsilon)^2(1-4\varepsilon)}\left(\tfrac{\mu^2}{p^2}\right)^{3\varepsilon} - 2\tfrac{1}{\varepsilon^3(1-2\varepsilon)}\left(\tfrac{\mu^2}{p^2}\right)^{\varepsilon} + \tfrac{1}{\varepsilon^3(1-2\varepsilon)}\left(\tfrac{\mu^2}{p^2}\right)^{\varepsilon}$$

$$= \frac{1-\varepsilon-\varepsilon^2}{\varepsilon^3}.$$

Note the cancellation of all nonlocal contributions. The singular part after the R'-operation is always local.

The realization of the $\mathcal{R}'$-operation for each diagram G allows one to find the contribution of a given diagram to the corresponding counter-term and, in the case of a renormalizable theory, to find the renormalization constant equal to

$$Z = 1 - \mathcal{K}\,\mathcal{R}'G, \tag{18.80}$$

where $\mathcal{K}$ means the extraction of the singular part. Adding the contribution of various diagrams we get the resulting counter-term of a given order and, accordingly, the renormalization constant.

18.5 K-l.V: Renormalization. Gauge Theories and the Standard Model

Consider now the gauge theories. The difference from the scalar case is in the relations between various renormalization constants which follow from the gauge invariance. If the regularization and the renormalization scheme do not break the symmetry these relations hold automatically. In the opposite case, this is an additional requirement imposed on the counter-terms.

18.5.1 Quantum electrodynamics

Quantum electrodynamics (34.2) is a renormalizable theory; hence, the counter-terms repeat the structure of the Lagrangian. They can be written as

$$\Delta\mathcal{L}_{QED} = -\frac{Z_3-1}{4}F_{\mu\nu}^2 + (Z_2-1)i\bar\psi\hat\partial\psi - m(Z-1)\bar\psi\psi + e(Z_1-1)\bar\psi\hat A\psi. \tag{18.81}$$

The term that fixes the gauge is not renormalized. In the leading order of perturbation theory we calculated the corresponding diagrams with the help of dimensional regularization (see

(18.32),(18.36),(18.40)). Their singular parts with the opposite sign give the proper renormalization constants. They are, respectively,

$$\begin{aligned} Z_1 &= 1-\frac{e^2}{16\pi^2}\frac{1}{\varepsilon}, \\ Z_2 &= 1-\frac{e^2}{16\pi^2}\frac{1}{\varepsilon}, \\ Z_3 &= 1-\frac{e^2}{16\pi^2}\frac{4}{3\varepsilon}, \\ Z &= 1-\frac{e^2}{16\pi^2}\frac{4}{\varepsilon}. \end{aligned} \tag{18.82}$$

Adding (18.81) with (34.2) we get

$$\begin{aligned} &\mathcal{L}_{QED}+\Delta\mathcal{L}_{QED} = -\frac{Z_3}{4}F_{\mu\nu}^2 + Z_2 i\bar\psi\hat\partial\psi - mZ\bar\psi\psi + eZ_1\bar\psi\hat A\psi - \frac{1}{2\xi}(\partial_\mu A_\mu)^2 \\ &= -\frac{1}{4}F_{\mu\nu B}^2 + i\bar\psi_B\hat\partial\psi_B - mZZ_2^{-1}\bar\psi_B\psi_B + eZ_1Z_2^{-1}Z_3^{-1/2}\bar\psi_B\hat A_B\psi_B \\ &\quad -\frac{Z_3^{-1}}{2\xi}(\partial_\mu A_{\mu B})^2, \end{aligned} \tag{18.83}$$

that gives

$$\psi_B = Z_2^{1/2}\psi, \quad A_B = Z_3^{1/2}A, \quad m_B = ZZ_2^{-1}m, \quad e_B = Z_1Z_2^{-1}Z_3^{-1/2}e, \quad \xi_B = Z_3\xi. \tag{18.84}$$

The gauge invariance here manifests itself in two places. First, the transversality of the radiative correction to the photon propagator means that the gauge fixing term is not renormalized and, hence, the gauge parameter ξ is renormalized as a gauge field. Second, the gauge invariance connects the vertex Green function and the fermion propagator (the Ward identity), which leads to the identity $Z_1 = Z_2$. Since the dimensional regularization which we use throughout the calculations does not break the gauge invariance, this identity is satisfied automatically (see (18.82)). This means that the renormalization of the coupling (18.84) is defined by the photon propagator only. Note, however, that this is not true in general in a non-Abelian theory.

18.5.2 Quantum chromodynamics

The complications which appear in non-Abelian theories are caused by the presence of many vertices with the same coupling as it follows from the gauge invariance. Hence, they have to renormalize the same way, i.e there appear new identities, called the Slavnov-Taylor identities. The full set of the counter-terms in QCD looks like

$$\begin{aligned} &\Delta\mathcal{L}_{QD} = -\frac{Z_3-1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - g(Z_1-1)f^{abc}A_\mu^a A_\nu^b\partial_\mu A_\nu^c \\ &-(Z_4-1)\frac{g^2}{4}f^{abc}f^{ade}A_\mu^b A_\nu^c A_\mu^d A_\nu^e + (\tilde Z_3-1)\partial_\mu\bar c^a\partial_\mu c^a + g(\tilde Z_1-1)f^{abc}\partial_\mu\bar c^a A_\mu^b c^c \\ &+i(Z_2-1)\bar\psi\hat\partial\psi - m(Z-1)\bar\psi\psi + g(Z_{1\psi}-1)\bar\psi\hat A^a T^a\psi, \end{aligned} \tag{18.85}$$

that being added to the initial Lagrangian gives

$$\begin{aligned}
\mathcal{L}_{QD}+\Delta\mathcal{L}_{QD} &= -\frac{Z_3}{4}(\partial_\mu A^a_\nu-\partial_\nu A^a_\mu)^2 - gZ_1 f^{abc}A^a_\mu A^b_\nu \partial_\mu A^c_\nu \\
&-Z_4\frac{g^2}{4}f^{abc}f^{ade}A^b_\mu A^c_\nu A^d_\mu A^e_\nu - \tilde{Z}_3\partial_\mu\bar{c}^a\partial c^a - g\tilde{Z}_1 f^{abc}\partial_\mu\bar{c}^a A^b_\mu c^c \\
&+iZ_2\bar{\psi}\hat{\partial}\psi - mZ\bar{\psi}\psi + gZ_{1\psi}\bar{\psi}\hat{A}^a T^a\psi - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 \\
&= -\frac{1}{4}(\partial_\mu A^a_{\nu B}-\partial_\nu A^a_{\mu B})^2 - gZ_1Z_3^{-3/2}f^{abc}A^a_{\mu B}A^b_{\nu B}\partial_\mu A^c_{\nu B} \\
&-Z_4Z_3^{-2}\frac{g^2}{4}f^{abc}f^{ade}A^b_{\mu B}A^c_{\nu B}A^d_{\mu B}A^e_{\nu B} + \partial_\mu\bar{c}^a_B\partial_\mu c^a_B + g\tilde{Z}_1\tilde{Z}_3^{-1}Z_3^{-1/2}f^{abc}\partial_\mu\bar{c}^a_B A^b_{\mu B}c^c_B \\
&+\frac{Z_3^{-1}}{2\xi}(\partial_\mu A^a_{\mu B})^2 + i\bar{\psi}_B\hat{\partial}\psi_B - mZZ_2^{-1}\bar{\psi}_B\psi_B + gZ_{1\psi}Z_2^{-1}Z_3^{-1/2}\bar{\psi}_B\hat{A}^a_B T^a\psi_B. \qquad (18.86)
\end{aligned}$$

This results in the relations between the renormalized and the "bare" fields and couplings

$$\begin{aligned}
&\psi_B = Z_2^{1/2}\psi, \quad A_B = Z_3^{1/2}A, \quad c_B = \tilde{Z}_3^{1/2}c, \\
&m_B = ZZ_2^{-1}m, \quad g_B = Z_1Z_3^{-3/2}g, \quad \xi_B = Z_3\xi, \qquad (18.87)\\
&Z_1Z_3^{-1} = \tilde{Z}_1\tilde{Z}_3^{-1}, \quad Z_4 = Z_1^2Z_3^{-1}, \quad Z_{1\psi}Z_2^{-1} = Z_1Z_3^{-1}.
\end{aligned}$$

The last line of equalities follows from the requirement of identical renormalization of the coupling in various vertices and represents the Slavnov-Taylor identities for the singular parts.

The explicit form of the renormalization constants in the lowest approximation follows from the one-loop diagrams calculated earlier (see (18.31), (18.36), (18.40), (18.48), (18.50), (18.56). Aa usual, one has to take the singular part with the opposite sign. One has in the $\overline{MS}$ scheme

$$\begin{aligned}
Z_2 &= 1-\frac{g^2}{16\pi^2}\frac{C_F}{\varepsilon}, \\
Z_3 &= 1+\frac{g^2}{16\pi^2}(\frac{5}{3\varepsilon}C_A-\frac{4}{3\varepsilon}T_f n_f), \\
Z &= 1-\frac{g^2}{16\pi^2}\frac{4C_F}{\varepsilon}, \\
\tilde{Z}_1 &= 1-\frac{g^2}{16\pi^2}\frac{C_A}{2\varepsilon}, \qquad (18.88)\\
\tilde{Z}_2 &= 1+\frac{g^2}{16\pi^2}\frac{C_A}{2\varepsilon}, \\
Z_g &= \tilde{Z}_1\tilde{Z}_2^{-1}Z_3^{-1/2} = 1-\frac{g^2}{16\pi^2}(\frac{11}{6\varepsilon}C_A-\frac{4}{3\varepsilon}T_f n_f),
\end{aligned}$$

where the following notation for the Casimir operators of the gauge group is used

$$f^{abc}f^{dbc} = C_A\delta^{ad}, \quad (T^aT^a)_{ij} = C_F\delta_{ij}, \quad Tr(T^aT^b) = T_F\delta^{ab}.$$

For the $SU(N)$ group and the fundamental representation of the fermion fields they are equal to

$$C_A = N, \quad C_F = \frac{N^2-1}{2N}, \quad T_F = \frac{1}{2}.$$

18.5.3 The Standard Model of fundamental interactions

In the Standard Model of fundamental interactions besides the gauge interactions and the quartic interaction of the Higgs fields there are also Yukawa type interactions of the fermion fields with the Higgs field. These interactions are also renormalizable and is characterized by the Yukawa coupling constants, one for each fermion field. The peculiarity of the SM is that the masses of the fields appear as a result of spontaneous symmetry breaking when the Higgs field develops a vacuum expectation value. As a result the masses are not independent but are expressed via the coupling constant multiplied by the vacuum expectation value. Here there are two possibilities: to treat the Yukawa couplings as independent quantities and to renormalize them in a usual way and then express the renormalized masses via the renormalized couplings or to start with the masses of particles and to treat the Yukawa couplings as secondary quantities. The first approach is usually used within the minimal subtraction scheme where the renormalizations do not depend on masses. On the contrary, in the scheme when the subtraction is carried out on mass shell (the so-called "on-shell" scheme), one usually takes masses of particles as the basis. Under this way of subtraction the pole of the propagator is not shifted and the renormalized mass coincides with the mass of a physical particle. Below we consider the renormalizations in the SM in the $\overline{MS}$ scheme and concentrate on the renormalization of the fields and the couplings.

Another property of the Standard Model is that it has the gauge group $SU_c(3)\times SU_L(2)\times U_Y(1)$ which is spontaneously broken to $SU_c(3)\times U_{EM}(1)$. In the theories with spontaneously broken symmetry, according to the Goldstone theorem there are massless particles, the goldstone bosons. These particles indeed are present in the SM but they are not the physical degrees of freedom and due to the Higgs effect are absorbed by vector bosons turning into longitudinal degrees of freedom of massive vector particles.

Thus, there are two possibilities to formulate the SM as a theory with spontaneous symmetry breaking: the *unitary* formulation in which nonphysical degrees of freedom are absent and vector bosons have three degrees of freedom, and the so-called *renormalizable* formulation in which goldstone bosons are present in the spectrum and vector fields have two degrees of freedom. These two formulations correspond to two different choices of the gauge in spontaneously broken theory.

In unitary gauge we have only physical degrees of freedom, i.e., the theory is automatically unitary, hence the name of this gauge. However, the propagator of the massive vector fields in this case has the form

$$G_{\mu\nu}(k) = -i\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{M^2}}{k^2 - M^2},$$

i.e., does nor decrease when momentum goes to infinity. This leads to the increase in the power of divergences and the theory happens to be formally nonrenormalizable despite the coupling constant being dimensionless. We have mentioned this fact in the first lecture.

On the other hand, in renormalizable gauge, where the vector fields have two degrees of freedom, the propagator behaves as

$$G_{\mu\nu}(k) = -i\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}}{k^2 - M^2},$$

which obviously leads to a renormalizable theory which explains the name of this gauge. However, the presence of the goldstone bosons calls into question the unitarity of the theory since transitions between the physical and unphysical states become possible.

Since all the gauges are equivalent, one can work in any of them but in the unitary gauge one has to prove the renormalizability while in the renormalizable gauge one has to prove unitarity.

The gauge invariance of observables preserved in a spontaneously broken theory should guarantee the fulfilment of both the requirements simultaneously. Note that in spontaneous symmetry breaking the symmetry of the Lagrangian is preserved, it is the boundary condition that breaks the symmetry.

The rigorous proof of that the theory is simultaneously renormalizable and unitary is not so obvious and eventually was awarded the Nobel prize, but can be seen by using some intermediate gauge called the R_ξ-gauge. The gauge fixing term in this case is chosen in the form

$$-\frac{1}{2\xi}(\partial_\mu A^a_\mu - \xi g F^a_i \chi_i)^2, \quad gF^a_i = \frac{v}{2}\begin{pmatrix} g & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & g \\ 0 & 0 & g' \end{pmatrix},$$

where v is the vacuum expectation value of the Higs field, and χ_i are the goldstone bosons. In this gauge the vector propagator has the form

$$G_{\mu\nu}(k) = -i\frac{g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2 - \xi M^2}(1-\xi)}{k^2 - M^2},$$

and at $\xi = 0$ corresponds to the renormalizable gauge while as $\xi \to \infty$ it corresponds to the unitary one. Since all the observables do not depend on ξ, we can choose $\xi = 0$ when investigating the renormalizability properties and choose $\xi = \infty$ in examining the unitarity. Since we are interested here in the renormalizability of the SM, in what follows we will work in a renormalizable gauge.

The Lagrangian of the Standard Model consists of the following three parts:

$$\mathcal{L} = \mathcal{L}_{gauge} + \mathcal{L}_{Yukawa} + \mathcal{L}_{Higgs}, \tag{18.89}$$

The gauge part is totally fixed by the requirement of the gauge invariance leaving only the values of the couplings as free parameters

$$\begin{aligned} \mathcal{L}_{gauge} &= -\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} - \frac{1}{4}W^i_{\mu\nu}W^i_{\mu\nu} - \frac{1}{4}B_{\mu\nu}B_{\mu\nu} \\ &+ i\overline{L}_\alpha \gamma^\mu D_\mu L_\alpha + i\overline{Q}_\alpha \gamma^\mu D_\mu Q_\alpha + i\overline{E}_\alpha \gamma^\mu D_\mu E_\alpha \\ &+ i\overline{U}_\alpha \gamma^\mu D_\mu U_\alpha + i\overline{D}_\alpha \gamma^\mu D_\mu D_\alpha + (D_\mu H)^\dagger (D_\mu H), \end{aligned} \tag{18.90}$$

where the following notation for the covariant derivatives is used

$$\begin{aligned} G^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu, \\ W^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\epsilon^{ijk} W^j_\mu W^k_\nu, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\ D_\mu L_\alpha &= (\partial_\mu - i\frac{g}{2}\tau^i W^i_\mu + i\frac{g'}{2}B_\mu)L_\alpha, \\ D_\mu E_\alpha &= (\partial_\mu + ig'B_\mu)E_\alpha, \\ D_\mu Q_\alpha &= (\partial_\mu - i\frac{g}{2}\tau^i W^i_\mu - i\frac{g'}{6}B_\mu - i\frac{g_s}{2}\lambda^a G^a_\mu)Q_\alpha, \\ D_\mu U_\alpha &= (\partial_\mu - i\frac{2}{3}g'B_\mu - i\frac{g_s}{2}\lambda^a G^a_\mu)U_\alpha, \\ D_\mu D_\alpha &= (\partial_\mu + i\frac{1}{3}g'B_\mu - i\frac{g_s}{2}\lambda^a G^a_\mu)D_\alpha. \end{aligned}$$

The Yukawa part of the Lagrangian which is needed for the generation of the quark and lepton masses is also chosen in the gauge invariant form and contains arbitrary Yukawa couplings (we ignore the neutrino masses, for simplicity)

$$\mathcal{L}_{Yukawa} = y^L_{\alpha\beta}\overline{L}_\alpha E_\beta H + y^D_{\alpha\beta}\overline{Q}_\alpha D_\beta H + y^U_{\alpha\beta}\overline{Q}_\alpha U_\beta \tilde{H} + h.c., \tag{18.91}$$

where $\tilde{H} = i\tau_2 H^\dagger$.

At last the Higgs part of the Lagrangian contains the Higgs potential which is chosen in such a way that the Higgs field acquires the vacuum expectation value and the potential itself is stable

$$\mathcal{L}_{Higgs} = -V = m^2 H^\dagger H - \frac{\lambda}{2}(H^\dagger H)^2. \tag{18.92}$$

Here there are two arbitrary parameters: m^2 λ. The ghost fields and the gauge fixing terms are omitted.

The Lagrangian of the SM contains the following set of free parameters:

- 3 gauge couplings g_s, g, g';
- 3 Yukawa matrices $y^L_{\alpha\beta}, y^D_{\alpha\beta}, y^U_{\alpha\beta}$;
- Higgs coupling constant λ;
- Higgs mass parameter m^2;
- the number of the matter fields (generations).

All particles obtain their masses due to spontaneous breaking of the $SU_{left}(2)$ symmetry group via a nonzero vacuum expectation value (v.e.v.) of the Higgs field

$$< H >= \begin{pmatrix} v \\ 0 \end{pmatrix}, \qquad v = m/\sqrt{\lambda}. \tag{18.93}$$

As a result, the gauge group of the SM is spontaneously broken down to

$$SU_c(3) \otimes SU_L(2) \otimes U_Y(1) \Rightarrow SU_c(3) \otimes U_{EM}(1).$$

The physical weak intermediate bosons are linear combinations of the gauge ones

$$W^\pm_\mu = \frac{W^1_\mu \mp iW^2_\mu}{\sqrt{2}}, \qquad Z_\mu = -\sin\theta_W B_\mu + \cos\theta_W W^3_\mu \tag{18.94}$$

with masses

$$m_W = \frac{1}{\sqrt{2}} gv, \qquad m_Z = m_W/\cos\theta_W, \qquad \tan\theta_W = g'/g, \tag{18.95}$$

while the photon field

$$\gamma_\mu = \cos\theta_W B_\mu + \sin\theta_W W^3_\mu \tag{18.96}$$

remains massless.

The matter fields acquire masses proportional to the corresponding Yukawa couplings:

$$M^u_{\alpha\beta} = y^u_{\alpha\beta} v, \; M^d_{\alpha\beta} = y^d_{\alpha\beta} v, \; M^l_{\alpha\beta} = y^l_{\alpha\beta} v, \; m_H = \sqrt{2}m. \tag{18.97}$$

The mass matrices have to be diagonalized to get the quark and lepton masses.

The explicit mass terms in the Lagrangian are forbidden because they are not $SU_{left}(2)$ symmetric. They would destroy the gauge invariance and, hence, the renormalizability of the Standard Model. To preserve the gauge invariance we use the mechanism of spontaneous symmetry breaking which, as was explained above, allows one to get the renormalizable theory with massive fields.

The Feynman rules in the SM include the ones for QED and QCD with additional new vertices corresponding to the $SU(2)$ group and the Yukawa interaction, as well as the vertices with goldstone particles if one works in the renormalizable gauge. We will not write them down due to their complexity, though the general form is obvious.

Consider the one-loop divergent diagrams in the SM. Besides the familiar diagrams in QED and QCD discussed above one has the diagrams presented in Fig.18.17. The diagrams containing the goldstone bosons are omitted. The calculation of these diagrams is similar to what we have

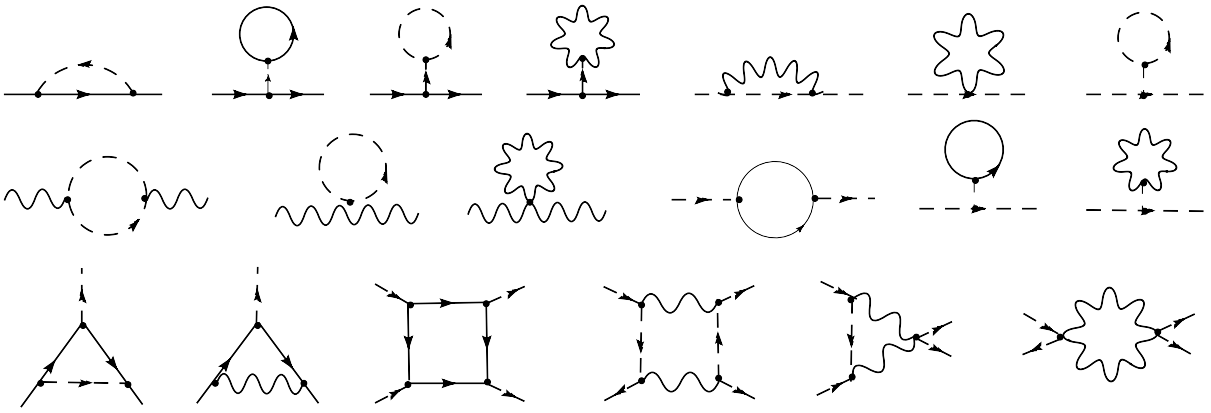


Figure 18.17: Some divergent one-loop diagrams in the SM. The dotted line denotes the Higgs field, the solid line - the quark and lepton fields, and the wavy line - the gauge fields

done above. Therefore, we show only the results for the renormalization constants of the fields

and the coupling constants. They have the form (for the gauge fields we use the Feynman gauge)

$$
\begin{aligned}
Z_{2Q_L} &= 1-\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{1}{36}g'^2+\frac{3}{4}g^2+\frac{4}{3}g_s^2+\frac{1}{2}y_U^2+\frac{1}{2}y_D^2],\\
Z_{2u_R} &= 1-\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{4}{9}g'^2+\frac{4}{3}g_s^2+y_U^2],\\
Z_{2d_R} &= 1-\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{1}{9}g'^2+\frac{4}{3}g_s^2+y_D^2],\\
Z_{2L_L} &= 1-\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{1}{4}g'^2+\frac{3}{4}g^2+\frac{1}{2}y_L^2],\\
Z_{2e_R} &= 1-\frac{1}{\varepsilon}\frac{1}{16\pi^2}[g'^2+y_L^2],\\
Z_{2H} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{1}{2}g'^2+\frac{3}{2}g^2-3y_U^2-3y_D^2-y_L^2],\\
Z_{3B} &= 1-\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{20}{9}N_F+\frac{1}{6}N_H]g'^2 \qquad \mathrm{U(1)_Y\ boson}\\
Z_{3A} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[3-\frac{32}{9}N_F]e^2 \qquad \mathrm{photon}\\
Z_{3W} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{10}{3}-\frac{1}{3}(N_F+3N_F)-\frac{1}{6}N_H]g^2,\\
Z_{3G} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[5-\frac{4}{3}N_F]g_s^2,\\
Z_{g_3^2} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[-11+\frac{4}{3}N_F]g_s^2,\\
Z_{g_2^2} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[-\frac{22}{3}+\frac{4}{3}N_F+\frac{1}{6}N_H]g^2,\\
Z_{g'^2} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\frac{20}{9}N_F+\frac{1}{6}N_H]g'^2,\\
Z_{y_U^2} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[-\frac{17}{12}g'^2-\frac{9}{4}g^2-8g_s^2+\frac{9}{2}y_U^2+\frac{3}{2}y_D^2+y_L^2],\\
Z_{y_D^2} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[-\frac{5}{12}g'^2-\frac{9}{4}g^2-8g_s^2+\frac{3}{2}y_U^2+\frac{9}{2}y_D^2+y_L^2],\\
Z_{y_L^2} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[-\frac{15}{4}g'^2-\frac{9}{4}g^2+\frac{9}{4}y_L^2+3y_U^2+3y_D^2],\\
Z_\lambda &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[-\frac{3}{2}g'^2-\frac{9}{2}g^2+2)3y_U^2+3y_D^2+y_L^2)+6\lambda\\
&\quad -2)3y_U^4+3y_D^4+y_L^4)/\lambda+(\frac{3}{8}g'^4+\frac{9}{8}g^4+\frac{3}{4}g^2g'^2)/\lambda],
\end{aligned}
$$

where, for simplicity, we ignored the mixing between the generations and assumed the Yukawa matrices to be diagonal.

The difference from the expressions considered above is that the renormalization constant of the scalar coupling contains the terms of the type g^4/λ and y^4/λ. This is because writing the counter-term for the quartic vertex we factorized λ. The counter-terms themselves are proportional to g^4 and y^4 and are not equal to zero. Thus, the quantum corrections generate a new interaction even if it is absent initially. Since the gauge and Yukawa interactions belong to the renormalizable type, the number of types of the counter-terms is finite and the only new interaction which is generated this way, if it was absent, is the quartic scalar one. With allowance for this interaction the model is renormalizable.

Since the masses of all the particles are equal to the product of the gauge or Yukawa couplings and the vacuum expectation value of the Higgs field, in the minimal subtraction scheme the mass ratios are renormalized the same way as the ratio of couplings. To find the renormalization of the mass itself, one should know how the v.e.v. is renormalized or find explicitly the mass counter-term from Feynman diagrams. In this case, one has also to take into account the tad-pole diagrams shown in Fig.18.17, including the diagrams with goldstone bosons.

For illustration we present the renormalization constant of the b-quark mass in the SM

$$\begin{aligned} Z_{m_b} &= 1+\frac{1}{\varepsilon}\frac{1}{16\pi^2}[\sum_l \frac{y_l^4}{\lambda}+3\sum_q \frac{y_q^4}{\lambda}-\frac{3}{2}\lambda+\frac{3}{4}(y_b^2-y_t^2) \\ &- \frac{3}{16}\frac{(g^2+g'^2)^2}{\lambda}-\frac{3}{8}\frac{g^4}{\lambda}-3Q_b(Q_b-T_b^3)g'^2-4g_s^2]. \end{aligned} \tag{18.98}$$

The result for the t-quark can be obtained by replacing b by t. For the light quarks the Yukawa constants are very small and can be ignored in eq.(34.268).

Note that here we again have the Higgs self-interaction coupling λ in the denominator. It appears from the tad-pole diagrams but, contrary to the previous case, the renormalization constant Z_{m_q} is not multiplied by λ and the denominator is not cancelled. This does not lead to any problems in perturbation theory since by order of magnitude $\lambda \sim g^2 \sim y^2$ and the loop expansion is still valid.

18.6 K-l.VI: Renormalization Group

The procedure formulated above allows one to eliminate the ultraviolet divergences and get the finite expression for any Green function in any local quantum field theory. In renormalizable theories this procedure is reduced to the multiplicative renormalization of parameters (masses and couplings) and multiplication of the Green function by its own renormalization constant. This is true for any regularization and subtraction scheme. Thus, for example, in the cutoff regularization and dimensional regularization the relation between the "bare" and renormalized Green functions looks like

$$\Gamma(\{p^2\},\mu^2,\{g_\mu\}) = Z_\Gamma(\Lambda^2/\mu^2,\{g_\mu\})\Gamma_{Bare}(\{p^2\},\Lambda,\{g_{Bare}\}) \tag{18.99}$$

$$\Gamma(\{p^2\},\mu^2,\{g_\mu\}) = Z_\Gamma(1/\varepsilon,\{g_\mu\})\Gamma_{Bare}(\{p^2\},1/\varepsilon,\{g_{Bare}\}), \tag{18.100}$$

where $\{p^2\}$ is the set of external momenta, $\{g\}$ is the set of masses and couplings, and

$$g_{Bare} = Z_g()\Lambda^2/\mu^2,\{g_\mu\})g \quad \text{or} \quad g_{Bare} = Z_g((1/\varepsilon,\{g_\mu\})g.$$

It is obvious that the operation of multiplication by the constant Z obeys the group property. Indeed, after the elimination of divergences one can multiply the couplings, masses and the Green functions by finite constants and this will be equivalent to the choice of another renormalization scheme. Since these finite constants can be changed continuously, we have a continuous Lie group which got the name of renormalization group. The group transformations of multiplication of the couplings and the Green functions are called the Dyson transformations.

18.6.1 The group equations and solutions via the method of characteristics

In what follows we stick to dimensional regularization and rewrite relation (18.100) in the form

$$\Gamma_{Bare}(\{p^2\},1/\varepsilon,\{g_{Bare}\}) = Z_\Gamma^{-1}(1/\varepsilon,\{g_\mu\})\Gamma(\{p^2\},\mu^2,\{g_\mu\}). \tag{18.101}$$

It is obvious that the l.h.s. of this equation does not depend on the parameter of dimensional transmutation μ and, hence, the r.h.s. should not also depend on it. This allows us to write the functional equation for the renormalized Green function. Differentiating it with respect to the continuous parameter μ one can get the differential equation which has a practical value: solving this equation one can get the improved expression for the Green function which corresponds to summation of an infinite series of Feynman diagrams.

Consider an arbitrary Green function Γ obeying equation (18.100) with the normalization condition

$$\Gamma(\{p^2\}, \mu^2, 0) = 1.$$

Differentiating (18.100) with respect to μ^2 one gets:

$$\mu^2 \frac{d}{d\mu^2}\Gamma = \left(\mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial g}{\partial \mu^2}\frac{\partial}{\partial g}\right)\Gamma = \mu^2 \frac{d \ln Z_\Gamma}{d\mu^2} Z_\Gamma \Gamma_{Bare},$$

or

$$\left(\mu^2 \frac{\partial}{\partial \mu^2} + \beta(g)\frac{\partial}{\partial g} + \gamma_\Gamma\right)\Gamma(\{p^2\}, \mu^2, g_\mu) = 0, \tag{18.102}$$

where we have introduced the so-called beta function $\beta(g)$ and the anomaly dimension of the Green function $\gamma_\Gamma(g)$ defined as

$$\beta = \mu^2 \frac{dg}{d\mu^2}|_{g_{bare}}, \tag{18.103}$$

$$\gamma_\Gamma = -\mu^2 \frac{d \ln Z_\Gamma}{d\mu^2}|_{g_{bare}}. \tag{18.104}$$

Equation (18.102) is called the *renormalization group equation* in partial derivatives (in Ovsyannikov form). In the western literature it is also called the Callan-Simanzik equation.

The solution of the renormalization group equation can be written in terms of characteristics:

$$\Gamma\left(e^t \frac{\{p^2\}}{\mu^2}, g\right) = \Gamma\left(\frac{\{p^2\}}{\mu^2}, \bar{g}(t, g)\right) \mathrm{e}^{\int\limits_0^t \gamma_\Gamma(\bar{g}(t,g))dt}, \tag{18.105}$$

where the characteristic equation is (for definiteness we restrict ourselves to a single coupling)

$$\frac{d}{dt}\bar{g}(t, g) = \beta(\bar{g}), \quad \bar{g}(0, g) = g. \tag{18.106}$$

The quantity $\bar{g}(t, g)$ is called the *effective charge* or *effective coupling*.

We will consider the useful properties of this solution (18.105) later and we first derive several other similar equations. Since the vertex function usually comes with the coupling, one can consider the product

$$g\Gamma\left(\frac{\{p^2\}}{\mu^2}, g\right). \tag{18.107}$$

If Γ is the n-point function, then the renormalization of the coupling g is given by

$$g_{Bare} = Z_\Gamma Z_2^{-n/2} g,$$

and the product (18.107) is renormalized as

$$g\Gamma = Z_2^{n/2} g_{Bare}\Gamma_{Bare}.$$

Hence, one has the same equation as (18.100) with solution (18.105) but with $Z_\Gamma = Z_2^{n/2}$ and $\gamma_\Gamma = -n/2\gamma_2$. (Recall that the anomalous dimension γ_2 is defined with respect to the renormalization constant Z_2^{-1}.)

Furthermore, one can construct the so-called *invariant charge* by multiplying the product (18.107) by the corresponding propagators

$$\xi = g\Gamma\left(\frac{\{p^2\}}{\mu^2}, g\right)\prod_i^n D^{1/2}\left(\frac{p_i^2}{\mu^2}, g\right). \tag{18.108}$$

The invariant charge ξ, being RG-invariant, obeys the RG equation without the anomalous dimension and plays an important role in the formulation of the renormalization group together with the effective charge. In some cases, for instance in the MOM subtraction scheme, the effective and invariant charges coincide.

The usefulness of solution (18.105) is that it allows one to sum up an infinite series of logs coming from the Feynman diagrams in the infrared ($t \to -\infty$) or ultraviolet ($t \to \infty$) regime and improve the usual perturbation theory expansions. This in its turn extends the applicability of perturbation theory and allows one to study the infrared or the ultraviolet asymptotics of the Green functions.

To demonstrate the power of the RG, let us consider the invariant charge in a theory with a single coupling and restrict ourselves to the massless case. Let the perturbative expansion be

$$\xi(\frac{p^2}{\mu^2}, g) = g(1 + bg\ln\frac{p^2}{\mu^2} + ...). \tag{18.109}$$

The β function in the one-loop approximation is given by

$$\beta(g) = bg^2. \tag{18.110}$$

Notice that the coefficient b of the logarithm in eq.(18.109) coincides with that of the β function. Alternatively the β function can be defined as the derivative of the invariant charge with respect to logarithm of momentum

$$\beta(g) = p^2\frac{d}{dp^2}\xi(\frac{p^2}{\mu^2}, g)|_{p^2=\mu^2}. \tag{18.111}$$

This definition is useful in the MOM scheme where the mass is not considered as a coupling but as a parameter and the renormalization constants depend on it. We will come back to the discussion of this question below when considering different definitions of the mass.

According to eq.(18.105) (with vanishing anomalous dimension) the RG-improved expression for the invariant charge corresponding to the perturbative expression (18.109) is:

$$\xi_{RG}(\frac{p^2}{\mu^2}, g) = \xi_{PT}(1, \bar{g}(\frac{p^2}{\mu^2}, g)) = \bar{g}(\frac{p^2}{\mu^2}, g), \tag{18.112}$$

where we have put in eq.(18.105) $p^2 = \mu^2$ and then replaced t by $t = \ln p^2/\mu^2$. The effective coupling is a solution of the characteristic equation

$$\frac{d}{dt}\bar{g}(t, g) = b\bar{g}^2, \quad \bar{g}(0, g) = g, \quad t \equiv \ln\frac{p^2}{\mu^2}. \tag{18.113}$$

The solution of this equation is

$$\bar{g}(t,g) = \frac{g}{1-bgt}. \tag{18.114}$$

Being expanded over t, the geometrical progression (18.114) reproduces the expansion (18.109); however, it sums the infinite series of terms of the form $g^n t^n$. This is called the leading log approximation (LLA) in QFT. To get the correction to the LLA, one has to consider the next term in the expansion of the β function. Then one can sum up the next series of terms of the form $g^n t^{n-1}$ which is called the next to leading log approximation (NLLA), etc. This procedure allows one to describe the leading asymptotics of the Green functions for $t \to \pm\infty$.

Consider now the Green function with non-zero anomalous dimension. Let its perturbative expansion be

$$\Gamma(\frac{p^2}{\mu^2}, g) = 1 + cg \ln \frac{p^2}{\mu^2} + ... \tag{18.115}$$

Then in the one-loop approximation the anomalous dimension is

$$\gamma(g) = cg. \tag{18.116}$$

Again the coefficient of the logarithm coincides with that of the anomalous dimension. In analogy with eq.(18.111) the anomalous dimension can be defined as a derivative with respect to the logarithm of momentum

$$\gamma(g) = p^2 \frac{d}{dp^2} \ln \Gamma(\frac{p^2}{\mu^2}, g)|_{p^2=\mu^2}. \tag{18.117}$$

Substituting (18.116) into eq.(18.105), one has in the exponent

$$\int_0^t \gamma(\bar{g}(t,g)dt = \int_g^{\bar{g}} \frac{\gamma(g)}{\beta(g)} dg = \int_g^{\bar{g}} \frac{cg}{bg^2} dg = \frac{c}{b} \ln \frac{\bar{g}}{g}.$$

This gives for the Green function the improved expression

$$\Gamma_{RG} = \left(\frac{\bar{g}}{g}\right)^{-c/b} = \left(\frac{1}{1-bgt}\right)^{c/b} \approx 1 + ct + ... \tag{18.118}$$

Thus, one again reproduces the perturbative expansion, but expression (18.118) again contains the whole infinite sum of the leading logs. To get the NLLA, one has to take into account the next term in eq.(18.116) together with the next term of expansion of the β function.

All the formulas can be easily generalized to the case of multiple couplings and masses.

18.6.2 The effective coupling

By virtue of the central role played by the effective coupling in RG formulas, consider it in more detail. The behaviour of the effective coupling is determined by the β function. Qualitatively, the β function can exhibit the behaviour shown in Fig.18.18. We restrict ourselves to the region of small couplings.

In the first case, the β-function is positive. Hence, with increasing momentum the effective coupling unboundedly increases. This situation is typical of most of the models of QFT in the one-loop approximation when $\beta(g) = bg^2$ and $b > 0$. The solution of the RG equation for the effective coupling in this case has the form of a geometric progression (18.114). It is characterized

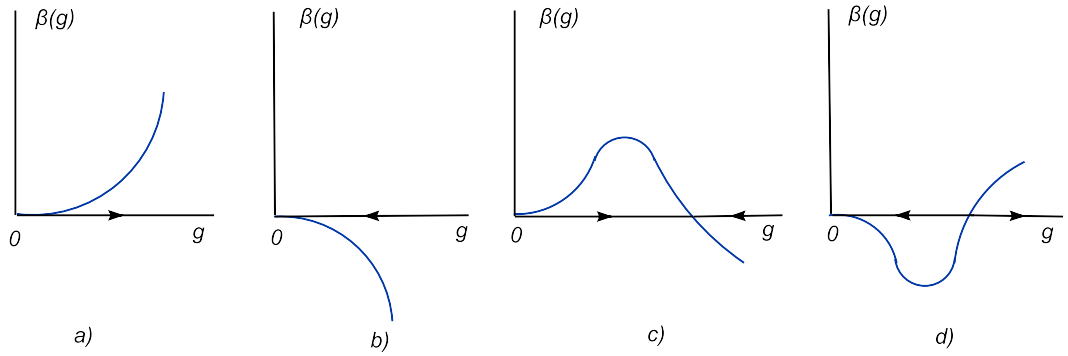


Figure 18.18: The possible form of the β-function. The arrows show the behaviour of the effective coupling in the ultraviolet regime ($t \to \infty$)

by the presence of a pole at high energies, called the Landau pole. We will consider this pole in detail later.

In the second case, the β-function is negative and, hence, the effective coupling decreases with increasing momentum. This situation appears in the one-loop approximation when $b < 0$, which takes place in the gauge theories. Here we also have a pole but in the infrared region.

In the third case, the β-function has zero: at first, it is positive and then is negative. This means that for small initial values the effective coupling increases; and for large ones, decreases. In both the cases, with increasing momentum it tends to the fixed value defined by the zero of the β-function. This is the so-called ultraviolet stable *fixed point*. It appears in some models in higher orders of perturbation theory.

Eventually, in the last case one also has the fixed point but now for the small initial coupling it decreases and for the large one it increases, i.e., with increasing momentum the effective coupling moves away from the fixed point, it is ultraviolet unstable. On the contrary, with decreasing momentum it tends to the fixed point, i.e., it is infrared stable. It appears in some models in lower dimensions, for instance, in statistical physics.

18.6.3 Dimensional regularization and the $\overline{MS}$ scheme

Consider now the calculation of the β function and the anomalous dimensions in some particular models within the dimensional regularization and the minimal subtraction scheme. Note that in transition from dimension 4 to $4 - 2\varepsilon$ the dimension of the coupling is changed and the "bare" coupling acquires the dimension $[g_B] = 2\varepsilon$. That is why the relation between the "bare" and renormalized coupling contains the factor $(\mu^2)^\varepsilon$

$$g_B = (\mu^2)^\varepsilon Z_g g. \tag{18.119}$$

Hence, even before the renormalization when $Z_g = 1$, in order to compensate this factor the dimensionless coupling g should depend on μ. Differentiating (18.119) with respect to μ^2 one gets

$$0 = \varepsilon Z_g g + \frac{d\log Z_g}{d\log\mu^2} Z_g g + Z_g \frac{dg}{d\log\mu^2},$$

i.e.,

$$\beta_{4-2\varepsilon}(g) \equiv \frac{dg}{d\log\mu^2} = -\varepsilon g + g\frac{d\log Z_g}{d\log\mu^2} = -\varepsilon g + \beta_4(g). \tag{18.120}$$

In the $\overline{MS}$ scheme the renormalization constants are given by the pole terms in $1/\varepsilon$ expansion and so does the bare coupling. They can be written as

$$Z_\Gamma = 1 + \sum_{n=1}^{\infty} \frac{c_n(g)}{\varepsilon^n} = 1 + \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{c_{nm} g^m}{\varepsilon^n}. \tag{18.121}$$

And similarly

$$g_{Bare} = (\mu^2)^\varepsilon \left[g + \sum_{n=1}^{\infty} \frac{a_n(g)}{\varepsilon^n} \right] = (\mu^2)^\varepsilon \left[g + \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} \frac{a_{nm} g^{m+1}}{\varepsilon^n} \right]. \tag{18.122}$$

Differentiating eq.(18.121) with respect to $\ln \mu^2$ and having in mind the definitions (18.103) and (18.104), one has:

$$-[1 + \sum_{n=1}^{\infty} \frac{c_n(g)}{\varepsilon^n}] \gamma_\Gamma(g) = [-\varepsilon g + \beta(g)] \frac{d}{dg} \sum_{n=1}^{\infty} \frac{c_n(g)}{\varepsilon^n}.$$

Equalizing the coefficients of equal powers of ε, one finds

$$\gamma_\Gamma(g) = g \frac{d}{dg} c_1(g), \tag{18.123}$$

$$g \frac{d}{dg} c_n(g) = [\gamma_\Gamma(g) + \beta(g) \frac{d}{dg}] c_{n-1}(g), \quad n \geq 2. \tag{18.124}$$

One sees that the coefficients of higher poles c_n, $n \geq 2$ are completely defined by that of the lowest pole c_1 and the β function. In its turn the β-function is also defined by the lowest pole. To see this, consider eq.(18.122). Differentiating it with respect to $\ln \mu^2$ one has

$$\varepsilon \left[g + \sum_{n=1}^{\infty} \frac{a_n(g)}{\varepsilon^n} \right] + [-\varepsilon g + \beta(g)] \left[1 + \frac{d}{dg} \sum_{n=1}^{\infty} \frac{a_n(g)}{\varepsilon^n} \right] = 0. \tag{18.125}$$

Equalizing the coefficients of equal powers of ε, one finds

$$\beta(g) = (g \frac{d}{dg} - 1) a_1(g), \tag{18.126}$$

$$(g \frac{d}{dg} - 1) a_n(g) = \beta(g) \frac{d}{dg} a_{n-1}(g), \quad n \geq 2. \tag{18.127}$$

Thus, knowing the coefficients of the lower poles one can reproduce all the higher order divergences. This means that they are not independent, all the information about them is connected in the lowest pole. In particular, substituting in (18.127) the perturbative expansion (18.122) one can solve the recurrent equation and find for the highest pole term

$$a_{nn} = a_{11}^n, \tag{18.128}$$

i.e. in the leading order one has the geometric progression

$$g_B = \mu^{2\varepsilon} \frac{g}{1 - g a_{11}/\varepsilon}, \tag{18.129}$$

which reflects the fact that the effective coupling in the LLA is also given by a geometric progression (18.114).

The pole equations are easily generalized for the multiple couplings case, the higher poles are also expressed through the lower ones though the solutions of the RG equations are more complicated.

Consider now some particular models and calculate the corresponding β-functions and the anomalous dimensions.

The ϕ^4 theory

The renormalization constants in the $\overline{MS}$ scheme up to two loops are given by eqs. (18.66,18.70,18.74). ($g \equiv \lambda/16\pi^2$)

$$Z_4 = 1 + \frac{3}{2\varepsilon}g + g^2\left(\frac{9}{4\varepsilon^2} - \frac{3}{2\varepsilon}\right), \tag{18.130}$$

$$Z_2^{-1} = 1 + \frac{g^2}{24\varepsilon}, \tag{18.131}$$

$$Z_g = 1 + \frac{3}{2\varepsilon}g + g^2\left(\frac{9}{4\varepsilon^2} - \frac{17}{12\varepsilon}\right). \tag{18.132}$$

Notice that the higher pole coefficient $a_{22} = 9/4$ in the last expression is the square of the lowest pole one $a_{11} = 3/2$ in accordance with eq.(18.128).

Applying now eqs.(18.123) and (18.126) one gets

$$\gamma_4(g) = \frac{3}{2}g - 3g^2, \tag{18.133}$$

$$\gamma_2(g) = \frac{1}{12}g^2, \tag{18.134}$$

$$\beta(g) = g(\gamma_4 + 2\gamma_2) = \frac{3}{2}g^2 - \frac{17}{6}g^2. \tag{18.135}$$

One can see from eq.(18.135) that the first coefficient of the β-function is 3/2, i.e., the ϕ^4 theory belongs to the type of theories shown in Fig.18.18a). In the leading log approximation (LLA) one has a Landau pole behaviour. In the two-loop approximation (NLLA) the β-function gets a non-trivial zero and the effective coupling possesses an UV fixed point like the one shown in Fig.18.18). However, this fixed point is unstable with respect to higher orders and is not reliable. Here we encounter the problem of divergence of perturbation series in quantum field theory, they are the so-called asymptotic series which have a zero radius of convergence.

QED

In QED in the one-loop approximation the renormalization constants in the Feynman gauge are given by eq.(18.82). Due to the Ward identities the renormalization of the coupling is defined by the photon wave function renormalization constant Z_3 and is gauge invariant. Equation (18.82) allows one to determine the anomalous dimensions and the β-function

$$\gamma_1(\alpha) = -\alpha, \tag{18.136}$$

$$\gamma_2(\alpha) = \alpha, \tag{18.137}$$

$$\gamma_3(\alpha) = \frac{4}{3}\alpha, \tag{18.138}$$

$$\gamma_m(\alpha) = -4\alpha, \tag{18.139}$$

$$\beta_\alpha(\alpha) = \frac{4}{3}\alpha^2, \tag{18.140}$$

where we use the notation $\alpha \equiv e^2/16\pi^2$.

Thus, in QED in the one-loop approximation the effective coupling behaves the same way a in the ϕ^4 theory and has a Landau pole in the LLA. In this theory, the next term of expansion of the β-function is also calculated. It has the same sign.

QCD

In QCD the calculation of the β function can be based on various vertices. The result should be the same due to the gauge invariance. To simplify the calculations, we choose the ghost-ghost-vector vertex. The renormalization constants in the one-loop approximation in the Feynman gauge are given by (18.88) and lead to the following anomalous dimensions and the β-function:

$$\tilde{\gamma}_1(\alpha) = -\frac{C_2}{2}\alpha, \tag{18.141}$$

$$\tilde{\gamma}_2(\alpha) = -\frac{C_2}{2}\alpha, \tag{18.142}$$

$$\gamma_3(\alpha) = -(\frac{5}{3}C_2 - \frac{2}{3}n_f)\alpha, \tag{18.143}$$

$$\beta_\alpha(\alpha) = \alpha(2\tilde{\gamma}_1 + 2\tilde{\gamma}_2 + \gamma_3) = -(\frac{11}{3}C_- \frac{2}{3}n_f)\alpha^2, \tag{18.144}$$

where like in QED we take $\alpha \equiv g^2/16\pi^2$, the Casimir operator C in the case of SU(3) groups is equal to 3, and n_f is the number of quark flavours.

One can see from eq.(18.144) that if the number of flavours is less than $\frac{11}{2}C_2 = \frac{33}{2}$, the β-function is negative and the effective coupling decreases and tends to zero with increasing momentum. This type of behaviour of the effective coupling is called the *asymptotic freedom*. It takes place only in gauge theories.

18.6.4 Λ_{QCD}

The solution of the characteristic equation for the effective coupling, which is a differential equation of the first order, depends on initial conditions. Therefore, the solution (18.114) depends on the choice of the initial point and the value of the coupling at this point. However, this choice is not unique and one can choose another initial point and another value of the coupling and still get the same solution, as it is shown in Fig.18.19.

In fact, every curve is not characterized by two numbers (the initial point and the coupling), but by one number and the transition from one curve to another is defined by the change of this number. To see this, consider the one-loop expression for the effective coupling in a gauge theory and rewrite it in equivalent form

$$\bar{g}(\frac{Q^2}{\mu^2}, g_\mu) = \frac{g_\mu}{1 - \beta_0 g_\mu \ln\frac{Q^2}{\mu^2}} = \frac{1}{\frac{1}{g_\mu} - \beta_0 \ln\frac{Q^2}{\mu^2}} \equiv -\frac{1}{\beta_0 \ln\frac{Q^2}{\Lambda^2}} = \bar{g}(\frac{Q^2}{\Lambda^2}), \tag{18.145}$$

where we have introduced the notation

$$\Lambda^2 = \mu^2 e^{-\frac{1}{\beta_0 \alpha_\mu}}. \tag{18.146}$$

This quantity is called Λ_{QCD} in quantum chromodynamics and can be introduced in any model. The numerical value of Λ is defined from experiment.

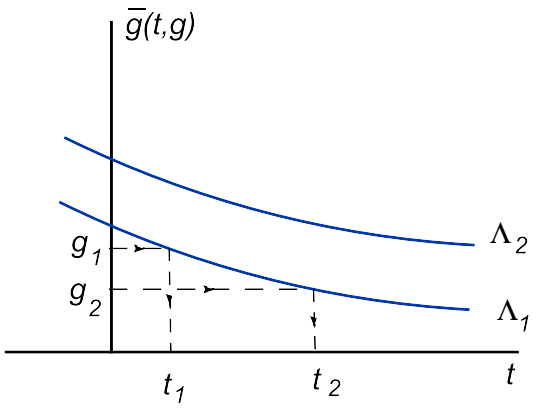


Figure 18.19: Different parametrizations of the effective coupling. Each curve is characterized by a single parameter Λ

Equation (18.146) can be generalized to any number of loops. For this purpose, let us rewrite the RG equation for the effective coupling in the Gell-Mann – Low form. One has

$$\ln \frac{Q^2}{\mu^2} = \int\limits_{g_\mu}^{g_Q} \frac{dg}{\beta_g(g)}. \tag{18.147}$$

Combining the lower limit with $\ln \mu^2$ one gets

$$\ln \frac{Q^2}{\Lambda^2} = \int^{g_Q} \frac{dg}{\beta_g(g)}, \tag{18.148}$$

where

$$\Lambda^2 = \mu^2 exp\left(\int^{g_\mu} \frac{dg}{\beta_g(g)} \right), \tag{18.149}$$

which is the generalization of eq.(18.146) for an arbitrary number of loops.

The quantity Λ, introduced this way, is μ-independent but depends on the renormalization scheme due to the scheme dependence of the β-function. However, the scheme dependence of Λ is given exactly (!) in one-loop order. Indeed, since Λ does not depend on μ, let us choose μ in such a way that $g_\mu \to 0$. Then for the β-function one can use the perturbative expansion

$$\beta_\alpha(\alpha) = \beta_0 \alpha^2 + \beta_1 \alpha^3 + ...$$

or

$$\int \frac{d\alpha}{\beta(\alpha)} = -\frac{1}{\beta_0 \alpha} + \ln \alpha + O(\alpha).$$

In this limit the ratio of two parameters Λ belonging to two different schemes is

$$\ln \frac{\Lambda_1^2}{\Lambda_2^2} = -\frac{1}{\beta_0}\left[\frac{1}{\alpha_1} - \frac{1}{\alpha_2}\right] = -\frac{1}{\beta_0}\left[c_1 - c_2\right], \tag{18.150}$$

where the coefficients c_1 and c_2 are calculated in the one-loop order. They can be found from perturbative expansion of any physical quantity in two different schemes

$$\begin{aligned} R &= g_1(1 + c_1 g_1 + ...) \\ &= g_2(1 + c_2 g_2 + ...). \end{aligned}$$

Since Λ does not depend on g, one can take any value of g, and eq.(18.150) is always valid. The difference $c_1 - c_2$ does not depend on a particular choice of R (though each of them depends) and is universal.

It should be noted that the quantities like the invariant or effective coupling, the β-function, etc. are not directly observable. Therefore, their dependence on the subtraction scheme does not contradict the independence of predictions of the method of calculations. We perform the perturbative expansion over the coupling which is scheme dependent, but the coefficients are also scheme dependent. As a result, within the given accuracy defined by the order of perturbation theory the answer is universal.

In the minimal subtraction schemes when the renormalizations depend only on dimensionless couplings, the one-loop renormalization constants and hence the anomalous dimensions and the β-function are the same in all schemes; the difference starts from two loops. The exception is the β-function in a theory with a single coupling like QED, QCD or the ϕ^4 theory, where the difference starts from three loops. Indeed, if one has two subtraction schemes M_1 and M_2 so that the couplings in two schemes are related by

$$g_2 = q(g_1) = g_1 + c g_1^2 + O(g_1^3),$$

then the β-functions $\beta_1(g_1)$ and $\beta_2(g_2)$ are connected by the relation

$$\beta_2(g_2) = \frac{dq(g_1)}{dg_1}\beta_1(g_1)$$

and their perturbative expansions are

$$\begin{aligned} \beta_1(g_1) &= \beta_0 g_1 + \beta_1 g_1^2 + \beta_2 g_1^3 + ..., \\ \beta_2(g_2) &= \beta_0 g_2 + \beta_1 g_2^2 + \beta_2' g_2^3 + ... \end{aligned}$$

so that the first two terms of the β-function are universal.

As for the further terms of expansion, they depend on the renormalization scheme and one can use this dependence as discretion, for instance, one can put all of them equal to zero. Then we would have an exact β-function. However, one should have in mind that it is not valuable by itself but rather in the aggregate with the PT expansion for the Green functions for which we construct the solution of the RG equation. This expansion in our "exact" scheme is unknown.

18.6.5 The running masses

In the minimal subtraction scheme the renormalization of the mass is performed the same way as the renormalization of the couplings, i.e., the mass is treated as an additional coupling and is renormalized multiplicatively, namely,

$$m_{Bare} = Z_m m,$$

where the mass renormalization constant Z_m is independent of the mass parameters and depends only on dimensionless couplings. Then, in full analogy with the effective coupling one can introduce the *effective* or the "running" mass

$$\frac{d}{dt}\bar{m}(t,g) = \bar{m}\gamma_m(\bar{g}), \quad \bar{m}(0,g) = m_0. \tag{18.151}$$

Solving this equation together with the equation for the effective coupling (18.106) one has

$$\bar{m}(t,g) = m_0 \mathrm{e}^{\int\limits_0^t \gamma_m(\bar{g}(t,g))dt} = m_0 \mathrm{e}^{\int\limits_g^{\bar{g}} \frac{\gamma(g)}{\beta(g)}dg}. \tag{18.152}$$

In the one-loop order

$$\beta(\alpha) = b\alpha^2, \quad \gamma_m(\alpha) = c\alpha$$

and the solution is

$$m(t) = m_0 \left(\frac{\alpha(t)}{\alpha_0}\right)^{c/b}.$$

This is the running mass!

The natural question arises: what is the physical mass measured in experiment and how is it related to the running mass and at what scale?

To answer this question, consider why the mass is running. This is due to the radiative corrections. If one considers the value of momentum which is bigger than the mass, i.e. $p^2 > m^2$, then the particles are created, they are running inside the loops and give the contribution to the running. On the contrary, if $p^2 < m^2$, particles are not created, they "decouple" and do not contribute to the running. In the MOM scheme this takes place automatically because for the momentum smaller than the mass the diagram simply disappears. In the minimal scheme, on the contrary, this does not happen. Hence, it is quite natural in this case to stop the running at the value of $p^2 = m^2$ and to identify the physical mass with the running mass at the scale of the mass, i.e

$$m^2 = \bar{m}^2(m^2).$$

However, this is true only up to finite corrections. Let us come back to the definition of the mass term in the Lagrangian. It is chosen in such a way that the propagator of a particle, which is the inverse to the quadratic form, has the pole at $p^2 = m^2$. Therefore, a more appropriate definition of the physical mass is the position of the pole of the propagator with allowance for the radiative corrections,.i.e.,

$$\text{physical mass} \equiv \text{pole mass}$$

This definition of a mass does not depend on a scale and it is also scheme independent and may have physical meaning. The pole mass can be expressed through the running mass at the scale of a mass with finite and calculable corrections.

Consider as an example the quark mass in QCD. The quark propagator is graphically presented in Fig.18.20.

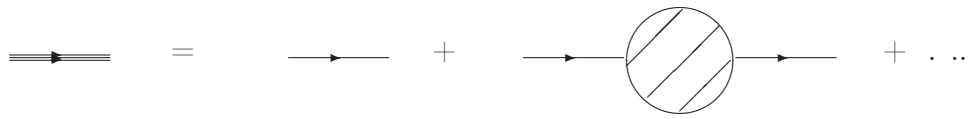


Figure 18.20: The quark propagator

The corresponding expression is

$$G(\hat{p},m) = \frac{i}{\hat{p}-m} + \frac{i}{\hat{p}-m}(iA\hat{p}+iBm)\frac{i}{\hat{p}-m} + ...$$
$$= \frac{i}{\hat{p}-m}\left[1 - \frac{A\hat{p}+Bm}{\hat{p}-m} + ...\right] = \frac{i}{\hat{p}-m}\,\frac{1}{1+\frac{A\hat{p}+Bm}{\hat{p}-m}} = \frac{i}{\hat{p}-m+A\hat{p}+Bm}.$$

The pole mass is now defined as a root of the equation

$$\hat{p}(1+A(p^2)) - m(1-B(p^2)) = 0, \tag{18.153}$$

which gives in the lowest order

$$m_{pole} = m\frac{1-B(m^2)}{1+A(m^2)} = m[1-A(m^2)-B(m^2)].$$

To calculate the functions A and B, consider the one-loop diagram shown in Fig.18.21.

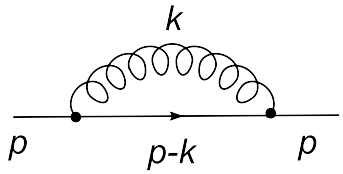


Figure 18.21: The quark propagator in one loop in QCD

The corresponding expression is

$$\Sigma = -\frac{g_s^2}{(2\pi)^4}C_F\int\frac{dk\;\gamma^\mu(\hat{p}-\hat{k}+m)\gamma^\nu}{[(p-k)^2-m^2]}\frac{g^{\mu\nu}}{k^2} \tag{18.154}$$

and was calculated earlier. The result has the form (18.36)

$$A(p^2,m^2) = \frac{g_s^2}{16\pi^2}C_F\left[\frac{1}{\varepsilon}-1-2\int\limits_0^1 dx(1-x)\log\frac{p^2x(1-x)-m^2}{-\mu^2}\right], \tag{18.155}$$

$$B(p^2,m^2) = \frac{g_s^2}{16\pi^2}C_F\left[-\frac{4}{\varepsilon}+2+4\int\limits_0^1 dx\log\frac{p^2x(1-x)-m^2}{-\mu^2}\right]. \tag{18.156}$$

After subtraction of divergences in the $\overline{MS}$-scheme one has

$$A^{\overline{MS}}(p^2,m^2) = -\frac{g_s^2}{16\pi^2}C_F\left[1+2\int\limits_0^1 dx(1-x)\log\frac{p^2x(1-x)-m^2}{-\mu^2}\right], \tag{18.157}$$

$$B^{\overline{MS}}(p^2,m^2) = \frac{g_s^2}{16\pi^2}C_F\left[2+4\int\limits_0^1 dx\log\frac{p^2x(1-x)-m^2}{-\mu^2}\right]. \tag{18.158}$$

Substituting $p^2=m^2$, one finds

$$A^{\overline{MS}}(m^2,m^2) = 2+\ln\frac{\mu^2}{m^2}, \quad B^{\overline{MS}}(m^2,m^2) = -6-4\ln\frac{\mu^2}{m^2}. \tag{18.159}$$

Thus, for the radiative correction to the pole mass we have

$$m_{pole} = m(\mu)\left[1+\frac{\alpha_s C_F}{4\pi})4+3\ln\frac{\mu^2}{m^2})\right]. \tag{18.160}$$

Substituting $C_F=4/3$ and $\mu^2=m^2$ one obtains the desired relation between the pole mass and the running mass at the mass scale

$$m_{pole} = m(m)\left[1+\frac{4}{3}\frac{\alpha_s}{\pi}\right]. \tag{18.161}$$

18.7 K-l.VII: Zero Charge and Asymptotic Freedom

Since the behaviour of the effective coupling has so essential consequences we consider two typical examples which are realized in quantum field theory in the one-loop approximation and presumably take place in a full theory. Usually, one speaks about the zero charge behaviour or the asymptotic freedom. We explain below what it means.

18.7.1 The zero charge

The notion of the zero charge appeared in QED in the leading log approximation. This is what takes place within the renormalization group method in the one-loop approximation. If one writes down the expression for the renormalized coupling as a function of the "bare" coupling, i.e. inverts eq.(18.129), one gets

$$g = \frac{g_B}{1+\beta_0 g_B/\varepsilon} = \frac{g_B}{1+\beta_0 g_B\log\Lambda^2}, \tag{18.162}$$

where the first coefficient of the β-function $\beta_0>0$. Then, removing the regularization, i.e., for $\varepsilon\to 0$ or $\Lambda\to\infty$, the renormalized coupling tends to zero independently of the value of the "bare" coupling. This is what is called the *zero charge*. For the effective coupling considered above the zero charge corresponds to the behaviour shown on the left panel of Fig.18.22 which is characterized by the Landau pole at high energies.

The zero charge behaviour is typical of QED, the ϕ^4 theory for positive quartic coupling and also the Yukawa type interactions, i.e., in those theories where the β-function is positive.

It is obvious that in the vicinity of the pole the perturbation theory does not work and, hence, the one-loop formula is not applicable. However, for small momenta transfer the one-loop

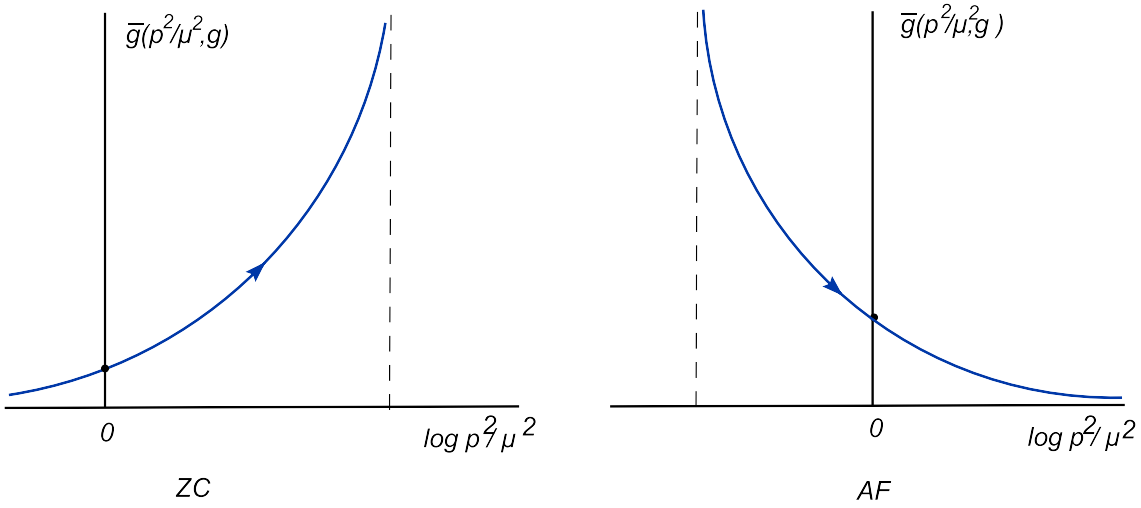


Figure 18.22: The behaviour of the effective coupling: the zero charge (left) and the asymptotic freedom (right)

approximation is reliable. For instance, in QED the effective expansion parameter is $e^2/16\pi^2 = \alpha/4\pi \approx 1/137/4\pi \approx 5.8 \cdot 10^{-4}$ and the next loop corrections (which have the same sign) do not play any essential role. The behaviour of the effective coupling in QED in the region up to 100 GeV has got the experimental confirmation in measuring the fine structure constant at the LEP accelerator. At the scale equal to the mass of the Z-boson M_Z the fine structure constant is not $1/137$ but $\alpha(M_Z) \approx 1/128$, which is in a good agreement with the one-loop formula.

The large momenta transfer in this case are limited by the pole provided the pole does not disappear in a full theory. It is still unclear how higher orders of perturbation theory influence this behaviour since the perturbation series is divergent and it is impossible to make definite conclusions without additional nonperturbative information.

The presence of the Landau pole indicates the presence of unphysical ghost states. To see this, consider the photon propagator in QED which due to the Ward identities coincides with the invariant charge and in the leading log approximation has the form of a geometric progression

$$G(p^2) = -i\frac{g^{\mu\nu} - p^\mu p^\nu/p^2}{p^2}\frac{1}{1 - \frac{4}{3}\sum Q^2 \frac{\alpha_0}{4\pi}\log(-p^2/m^2)}, \tag{18.163}$$

where Q is the electric charge of a particle (in the units of electron charge) running round the loop.

This expression has a pole in the Euclidean region at $p^2 = -m^2 exp(\frac{3\pi}{\alpha_0 Q^2 n_f})$. Substituting $m = m_e = 0.5$ MeV, $\alpha_0 \simeq 1/137$ and $\sum Q^2 = [(4/9 + 1/9)3 + 1)3] = 8$, one gets $p^2 \simeq -(5 \cdot 10^{31})^2$ GeV^2. That is the pole is very far off, even beyond the Planck scale, and at low energies one can ignore it. However, the presence of the pole indicates the presence of a new asymptotic state and the residue at the pole defines the norm of this state. In the case of the Landau pole the residue is negative, i.e., the new state is a ghost, it has the wrong sign of the kinetic term in the Lagrangian. This fact, in its turn, leads to negative probabilities, which indicates internal inconsistency of the theory.

Usually, it is assumed that there are two ways out of this trouble: either the higher order corrections improve the behaviour of the theory at high momenta so that the Landau pole disappears, or that the zero charge theory is contradictory by itself, but at high energies it is part of a more

general theory where the behaviour of the coupling is improved. The example of such a behaviour is given by the Grand Unified Theories where QED is one of the branches of a non-Abelian gauge theory with the asymptotically free behaviour. In both the cases the theory at high energies is modified. At the same time, the zero charge theory is infrared free, i.e. for small momenta transfer the coupling goes to zero.

18.7.2 The asymptotic freedom

The name *asymptotic freedom* originates from the non-Abelian gauge theories where it was found that the sign of the first coefficient of the β-function is negative. The effective coupling in this case behaves as is shown in the right panel of Fig.18.22 and tends to zero at high momenta transfer. This means that quarks in QCD are quasi-free particles, i.e., practically do not interact. This way one explains the success of the so-called parton model of the strong interactions at high energies, according to which the proton behaves as a set of free partons, and at high energies the interaction takes place with the individual partons and their interaction does not play any role.

The behaviour of the effective coupling in QCD at high energies was tested at various accelerators and in various experiments and the validity of the renormalization group formula was confirmed. The accuracy of modern measurements assumes the inclusion of the next terms of perturbative expansion. In QCD in the $\overline{MS}$ scheme the four terms of the β-function are known. Below we present the two-loop expression

$$\beta_\alpha(\alpha_s) = -\frac{1}{4\pi}[11 - \frac{2}{3}n_f]\alpha_s^2 - \frac{1}{(4\pi)^2}[102 - \frac{38}{3}n_f]\alpha_s^3 + O(\alpha_s^4). \tag{18.164}$$

As one can see, if the number of quarks in not too big, both the coefficients of the β-function are negative. All the experimental data fit a single curve for the effective coupling with the parameter $\Lambda_{QCD} \simeq 200$ MeV (see Fig.18.23)

In four-dimensional space the asymptotic freedom occurs only in non-Abelian gauge theories. But in the case when one has several interactions, like in the Standard Model, the non-Abelian coupling may draw other couplings into the asymptotically free region. Consider, for instance, the behaviour of the Yukawa couplings in the SM. For simplicity, let us take a single Yukawa coupling for the t-quark and a single gauge coupling. Then in the one-loop approximation the equations for the effective couplings look like

$$\begin{aligned} \frac{dg}{dt} &= -bg^2, \quad g \equiv \frac{g_s^2}{16\pi^2}, \\ \frac{dy}{dt} &= y(ay - cg), \quad y \equiv \frac{y_t^2}{16\pi^2}, \quad t \equiv \log\frac{q^2}{q_0^2}, \end{aligned} \tag{18.165}$$

where the coefficients b, a and c are always positive and for the SM are equal to $7, 9/2$ and 8, respectively. The solutions to these equations are

$$\begin{aligned} g &= \frac{g_0}{1 + bg_0 t}, \quad y = \frac{y_0 E}{1 - ay_0 F}, \\ E(t) &= (g/g_0)^{c/b}, \quad F(t) = \int_0^t E(t')dt'. \end{aligned} \tag{18.166}$$

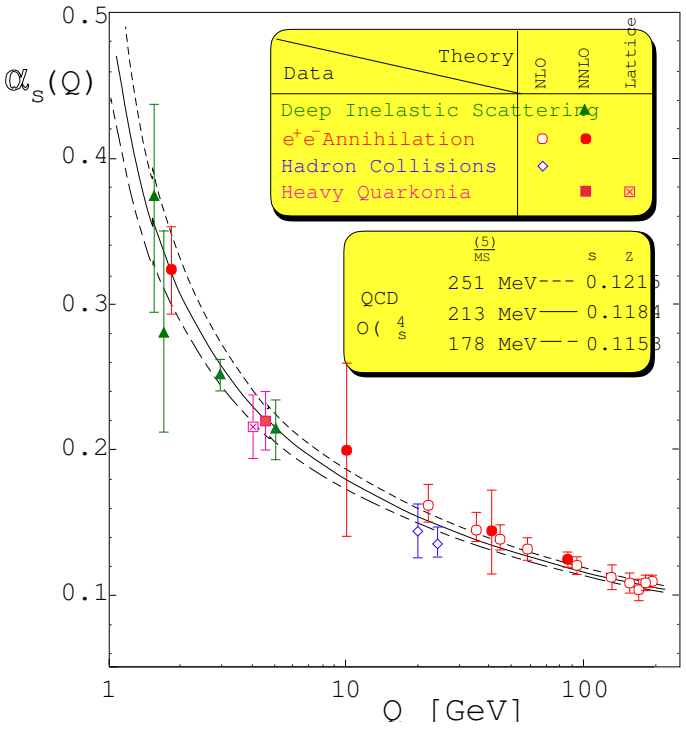


Figure 18.23: The variation of the effective coupling of the strong interactions α_s with energy

In the case of a single Yukawa coupling it can be written in an explicit form

$$y = \frac{y_0 (\frac{g}{g_0})^{c/b}}{1 + \frac{y_0}{g_0}\frac{a}{c-b}[(\frac{g}{g_0})^{c/b-1} - 1]}. \tag{18.167}$$

Graphically, it can be presented in a phase diagram shown in Fig.18.24. For the initial condition

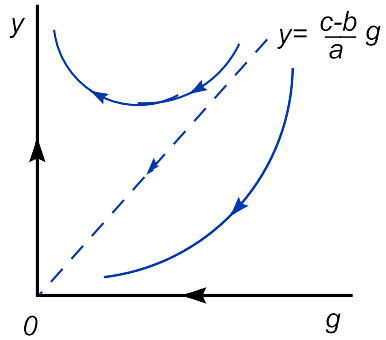


Figure 18.24: The behaviour of the Yukawa and gauge couplings for various initial conditions

such that $y_0 > (c-b)/a\ g_0$ the Yukawa coupling increases with momenta and has the Landau pole, while for $y_0 \leq (c-b)/a\ g_0$ it demonstrates the asymptotically free behaviour. In a similar way in the Grand Unified Theories one can reach the asymptotic freedom for all the couplings.

The back side of the asymptotic freedom at high energies is the presence of a pole at low energies or the infrared pole. In this region, we also go beyond the validity of perturbation theory since the coupling increases. To find the true behaviour of the coupling one has to attract independent

nonperturbative information. However, in QCD the region near the infrared pole $p \sim \Lambda_{QCD}$ is in the phase of hadronization, i.e., in this region the quark-gluon description is no more adequate. Therefore, the behaviour of the effective coupling in this region is not described by perturbative QCD.

18.7.3 The screening and anti-screening of the charge

The variation of the coupling with momenta transfer or with the scale, which is the characteristic feature of quantum field theory, has its analog in a classical theory. This analogy allows one to understand the qualitative reason for the variation of the coupling.

Indeed, let us consider the electromagnetic phenomena. Consider the dielectric medium and put the test electric charge in it. The medium will be polarized. The electric dipoles present in the medium will be rearranged in such a way as to screen the charge (see Fig.18.25). This is a consequence of the Coulomb law: the opposite charges are attracted and the same charges are repulsed. This is the essence of the electric screening phenomena.

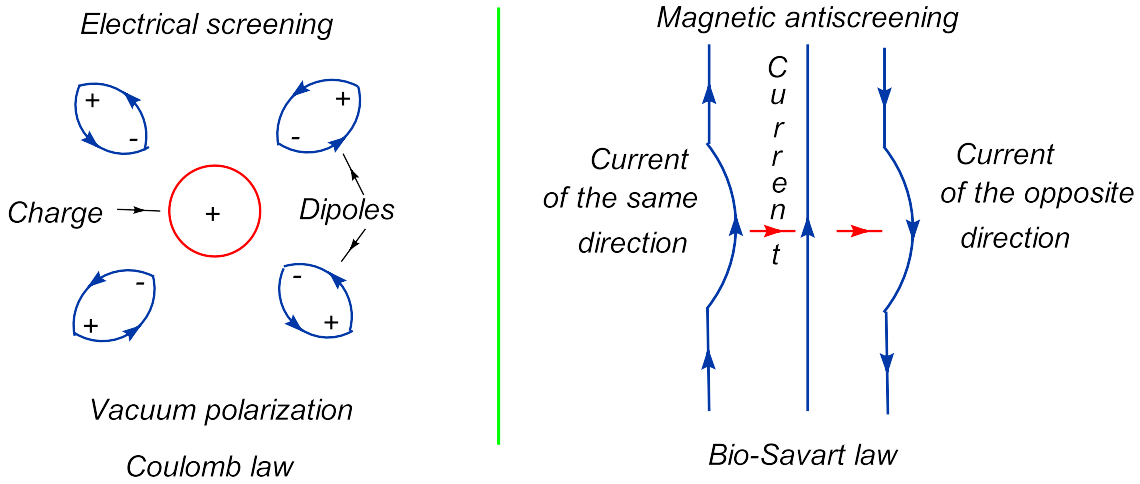


Figure 18.25: The electric screening and magnetic anti-screening

The opposite situation occurs in magnetic medium. According to the Bio-Savart law, the electric currents of the same direction are attracted and the opposite direction are repulsed (see Fig.18.25). This leads to the anti-screening in magnetic medium.

In quantum field theory the role of the medium is played by the vacuum. The vacuum is polarized in the presence of created virtual pairs. The matter particles as well as transversely polarized quanta of the gauge fields act like the electric dipoles in the dielectric and cause the screening of the charge. At the same time, the longitudinal quanta of the gauge fields behave like currents and cause the anti-screening. These two effects are in competition (see eq.(18.48) above) and, for instance, in QCD with a small number of quarks the effect of anti-screening prevails.

Thus, the couplings become the functions of the distance or momentum transfer described by the renormalization group equations.

18.7.4 Non-Abelian Gauge Transformations in QFT

Theory

Рассмотрим модель, содержащую безмассовые векторные поля B^a_μ (калибровочные поля или глюоны) и фермионы ψ_i (кварки).

Кварки преобразуются по некоторому представлению неабелевой группы с генераторами t^a и антисимметричными структурными константами f^{abc}:

$$\psi_i \to \psi_i + ig\omega^a t^a_{ij}\psi_j,$$

где g - константа связи (заряд), а $\omega^a(x)$ - локальные параметры калибровочных преобразований.

(???)

$$\left[t^a, t^b\right] = if^{abc}t^c,$$

тождество Якоби:

$$f^{\mathrm{abc}}\, f^{\mathrm{ade}} + f^{\mathrm{abd}}\, f^{\mathrm{aec}} + f^{\mathrm{abe}}\, f^{\mathrm{acd}} = 0$$

Глюоны преобразуются как:

$$B^a_\mu \to \left(B^a_\mu\right)^\omega = B^a_\mu + \partial_\mu\omega^a + gf^{abc}B^b_\mu\omega^c$$

Составленный из глюонов тензор $F^a_{\mu\nu}$ принадлежит присоединенному представлению,

$$F^a_{\mu\nu} \equiv \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + gf^{abc}B^b_\mu B^c_\nu,$$

(тут какое-то не сложное доказательство может дать вид преобразования:)

$$F^a_{\mu\nu} \to F^a_{\mu\nu} + gf^{abc}F^b_{\mu\nu}\omega^c$$

Лагранжиан векторного поля глюонов включает в себя свободную часть и слагаемые, отвечающие взаимодействию глюонов, строится из калибровочно инвариантной комбинации $F^a_{\mu\nu}F^a_{\mu\nu}$:

$$\begin{aligned}\mathscr{L}(B) &= -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \frac{1}{2\alpha}\left(\partial_\mu B^a_\mu\right)^2 = \\ &= -\frac{1}{4}\left(\partial_\mu B^a_\nu - \partial_\nu B^a_\mu\right)^2 - \frac{1}{2\alpha}\left(\partial_\mu B^a_\mu\right)^2 - \frac{g}{2}f^{abc}\left(\partial_\mu B^a_\nu - \partial_\nu B^a_\mu\right)B^b_\mu B^c_\nu - \frac{g^2}{4}f^{abc}f^{ade}B^b_\mu B^c_\nu B^d_\mu B^e_\nu\end{aligned}$$

Ковариантная производная имеет вид

$$\nabla_\mu\psi \equiv \left(\partial_\mu - igB^a_\mu t^a\right)\psi$$

Ее преобразование имеет вид:

$$\begin{aligned}\nabla_\mu\psi = \left(\partial_\mu - igB^a_\mu t^a\right)\psi &\to \left(\partial_\mu - ig\left(B^a_\mu\right)^\omega t^a\right)\left(1 + ig\omega^b t^b\right)\psi \\ &= \left(\partial_\mu - igB^a_\mu t^a - igt^a\partial_\mu\omega^a - ig^2 f^{adc}B^d_\mu t^a\omega^c\right)\left(1 + ig\omega^b t^b\right)\psi \\ &= \left(1 + ig\omega^b t^b\right)\left(\partial_\mu - igB^a_\mu t^a\right)\psi = \left(1 + ig\omega^b t^b\right)\nabla_\mu\psi\end{aligned}$$

Его усложнение по сравнению с КЭЛ обусловлено, конечно, некоммутативностью.

Коммутатор ковариантных производных, как всегда, связан с тензором $F^a_{\mu\nu}$:

$$[\nabla_\mu, \nabla_\nu] = -ig\left(\partial_\mu B^a_\nu - \partial_\nu B^a_\mu\right)t^a - g^2 B^a_\mu B^b_\nu\left[t^a, t^b\right] = -igF^a_{\mu\nu}t^a.$$

примеры

построение глюонов (????)

посмотрим, как построить глюонную теорию.
хз

применения

явное отличие неабелевых от абелевых

приведем такое сравнение, чтобы ориентроваться.

18.7.5 Faddeev–Popov Method

суть метода Фаддеева-Попова

Метод Фаддеева-Попова получает дополнительные поправки к лагранжиану??.
??? при чем тут петли??
Он основан на идее исключения “лишнего” интегрирования по полям, связанным друг с другом калибровочным преобразованием. (?????)
В квантовой электродинамике мы столкнулись с проблемой необратимости оператора

$$K_{\mu\nu} = g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu$$

в лагранжиане КЭД

$$\mathscr{L}_0(A) = -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 = \frac{1}{2}\left(\partial_\mu A_\nu \partial_\nu A_\mu - \partial_\mu A_\nu \partial_\mu A_\nu\right) = \frac{1}{2}A_\mu K_{\mu\nu} A_\nu$$

и решили ее, введя в лагранжиан неинвариантное слагаемое с параметром α.
(???)
Аналогичное слагаемое мы, с теми эке целями, сразу включили и в лагранжиан глюонов $\mathscr{L}(B)$.

$$\mathscr{L}(B) = -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} - \frac{1}{2\alpha}\left(\partial_\mu B^a_\mu\right)^2$$

Однако оказывается, что в неабелевом случае этого мало: потребуются и другие добавки к лагранжиану.
(!!!)
Метод получения дополнительных поправок, предложенный Фаллеевым и Поповым, основан на идее исключения "лишнего" интегрирования по полям, связанным друг с другом калибровочным преобразованием.

аналогия с чем-то простым, уже известным

Зи там привел ее, а я хз.
потом разберусь.

фиксация калибровки

(?? пример бы???)

Будем говорить, что условие $f(B) = 0$ фиксирует калибровку, если оно нарушается калибровочным преобразованием $B \to B^\omega$.

То есть если $\omega^a \neq 0$ влечет за собой $f\left(B^\omega\right) \neq 0$.

Реально мы ограничимся простейшими калибровочными условиями типа

$$f(B) = \partial_\mu B^a_\mu = 0.$$

?? тут примеры хочу записать

рецепт континуального интегрирования по полям

(?? то есть казалось бы, это тупо усложнение, зачем так делать?)

Имеется следующий рецепт континуального интегрирования по полям B^a_μ:

$$\int \mathscr{D}B\Phi(B) = \int \mathscr{D}B\delta(f(B))\Delta_f[B]\Phi(B)$$

(??? что за рецепт, как пользоваться???)

Предлагается запить

$$\int \mathscr{D}B \sim \int \mathscr{D}B\delta(f(B))\Delta_f[B]$$

Действительно, ведь более подробно:

$$\begin{aligned}
\int \mathscr{D}B &= \int \mathscr{D}B\mathscr{D}f\left(B^\omega\right)\delta\left(f\left(B^\omega\right)\right) = \\
&= \int \mathscr{D}B\mathscr{D}\omega \det\frac{\partial f\left(B^\omega\right)}{\partial\omega}\delta\left(f\left(B^\omega\right)\right) \\
&= \int \mathscr{D}\omega \int \mathscr{D}B\delta\left(f\left(B^\omega\right)\right)\Delta_f[B] \sim \\
&\sim \int \mathscr{D}B\delta(f(B))\Delta_f[B]
\end{aligned}$$

То есть сначала мы ввели дополнительное интегрирование по значениям функции f.

(???) Поскольку они скоррелированы со значениями ω, то на втором шаге сделана замена континуальной переменной $f \to \omega$.

Введенный на следующем шаге функционал

$$\Delta_f[B] \doteq \det\left[\frac{\partial f\left(B^\omega\right)}{\partial\omega}\right]_{f=0}$$

$$\Delta_f^{-1}[B] = \int \mathscr{D}\omega\delta\left(f\left(B^\omega\right)\right)$$

определяется видом f и по построению калибровочно инвариантен, то есть $\Delta_f\left[B^\omega\right] = \Delta_f[B]$.

Считая, что таким же свойством обладает и континуальная мера $\mathscr{D}B$, мы можем калибровочной заменой превратить B^ω в B.

Тогда интегрирование по $\mathscr{D}\omega$ даст просто числовой множитель (объем калибровочной группы), который мы, как всегда, опускаем.

В итоге получается следующий рецепт континуального интегрирования по полям B^a_μ:

$$\int \mathscr{D}B\Phi(B) = \int \mathscr{D}B\delta(f(B))\Delta_f[B]\Phi(B)$$

Здесь $\Phi(B)$ - любой калибровочно инвариантный объект.

Лишние поля из интеграла устранены условием $f(B) = 0$;

в виде платы за это пришлось внести под знак интеграла величину $\Delta_f[B]$.

Весь интеграл как целое, конечно же, от f не зависит.

вычисление детерминанта Фаддеева-Попова

Приведем рабочий метод вычисления детерминатнов.

Для нахождения $\Delta_f[B]$ на поверхности $f(B) = 0$ нужно уметь вычислять детерминант

$$\Delta_f[B] = \det\left[\frac{\partial f\left(B^\omega\right)}{\partial\omega}\right]_{f=0}$$

при $\omega = 0$.

Пусть $f(B) = \partial_\mu B^a_\mu = 0$.

Тогда

$$f\left(B^\omega\right) = \partial_\mu\left(B^a_\mu\right)^\omega = \partial^2\omega^a + gf^{acb}\partial_\mu\left(B^c_\mu\omega^b\right) = \partial^2\omega^a + gf^{acb}B^c_\mu\partial_\mu\omega^b = M^{ab}\omega^b$$

где $M^{ab} = \delta^{ab}\partial^2 + gf^{acb}B^c_\mu\partial_\mu$,

В итоге:

$$\Delta_f[B] = \det M = \int \mathscr{D}\bar{c}\mathscr{D}c\exp\left(i\bar{c}^a M^{ab}c^b\right)$$

То есть руководствуясь?????, мы представили искомый детерминант в виде континуального интеграла по фермионным полям $\bar{c}$, c.

лагранжиан гостов

Структура лагранжиана гостов $\mathscr{L}_{\text{gh}}$ (и даже сам факт их наличия) существенно зависит от вида калибровочного условия, задаваемого функцией f.

Введение функционала $\Delta_f[B]$ оказалось в данном случае эквивалентным добавлению к лагранжиану двух новых слагаемых,

$$\mathscr{L}_{\text{gh}} = -\partial_\mu\bar{c}^a\partial_\mu c^a + gf^{acb}\bar{c}^a B^c_\mu\partial_\mu c^b = -\partial_\mu\bar{c}^a\partial_\mu c^a - gf^{acb}\partial_\mu\bar{c}^a B^c_\mu c^b$$

отвечающих полям гостов.

пропагатор гостов

????

Сразу видно, что пропагатор гостов равен (с обратным знаком и множителем δ^{ab}) пропагатору скалярного поля.

и что?

(???)

Вершина взаимодействия связывает госты с глюонами.

Однако в in- и out-состояниях госты, по определению, присутствовать не должны, это частицы фиктивные и сугубо виртуальные, их единственная роль в диаграммах - образовывать замкнутые петли.

Кстати, можно было изначально свести $\Delta_f[B]$ к этим самым петлям, если раскрывать $\det M$ с помощью $\det M = \exp \operatorname{tr}(\ln M)$.

отсутствие гостов в КЭД

Отметим, что отсутствие гостов в КЭД объясняется тривиальностью оператора $M^{ab} = \delta^{ab}\partial^2$ в этом случае.

(?)

независимость интеграла от калибровочного параметра: доказательство

Нам осталось выполнить давнее обещание - доказать независимость интегралов вида (208) от калибровочного параметра α.

Для этого модифицируем калибровочное условие, положив

$$f(B) = \partial_\mu B^a_\mu - b^a.$$

Функционал $\Delta_f[B]$ удобно записать в форме

$$\Delta_f[B] = \det\left(\delta^{ab}\partial^2 + g f^{acb}\partial_\mu \cdot B^c_\mu\right),$$

одинаковой при любом выборе функции $b^a(x)$.

Формуле (209) это не противоречит, ибо $\partial_\mu \cdot B^a_\mu = B^a_\mu \partial_\mu + \partial_\mu B^a_\mu$.

Более того, оба представления для $\Delta_f[B]$ эквивалентны при любом B в силу очевидного равенства $\operatorname{tr}\ln(1+MN) = \operatorname{tr}\ln(1+NM)$, а соответствующие им выражения (210) для вклада гостов в лагранжиан - в силу $\partial^x_\mu D^c(x-y) = -\partial^y_\mu D^c(x-y)$.

Сам интеграл (208) по построению не может зависеть от f, а значит и от b, поэтому если мы подвергнем его дополнительному интегрированию по $\mathscr{D}b$ с весом $\exp\left(-ib^a b^a/2\alpha\right)$, это приведет лишь к домножению на несущественный числовой фактор:

$$\begin{aligned}\int \mathscr{D}B\Phi(B) &= \int \mathscr{D}B\delta\left(\partial_\mu B^a_\mu - b^a\right)\Delta_f[B]\Phi(B)\\ &\sim \int \mathscr{D}B\mathscr{D}b\delta\left(\partial_\mu B^a_\mu - b^a\right)\Delta_f[B]\Phi(B)\exp\left(-\frac{i}{2\alpha}b^a b^a\right)\\ &= \int \mathscr{D}B\mathscr{D}\bar{c}\mathscr{D}c\Phi(B)\exp i\left(\mathscr{L}_{\text{gh}} - \frac{1}{2\alpha}\left(\partial_\mu B^a_\mu\right)^2\right)\end{aligned}$$

мы воспроизвели слагаемое с калибровочным параметром α и показали, что континуальные интегралы вида (208) от него не зависят.

При этом интегралы от калибровочно неинвариантных величин (например, n -хвостки) вполне могут зависеть от α.

случай поперечной калибровки (???)

Отметим, что в случае поперечной калибровки $\alpha = 0$, когда $\exp\left(-\left(\partial_\mu B^a_\mu\right)^2/2\alpha\right)$ не существует, вместо него как раз появляется $\delta\left(\partial_\mu B^a_\mu\right)$ в континуальном интеграле: эту δ -функцию можно условно трактовать как предел экспоненты при $\alpha \to 0$.

калибровочная инвариантность S-матрицы (???)

Осталось сказать, что вместе с α-независимостью интегралов (208) нами установлена и калибровочная инвариантность S -матрицы.

???? я хз.

лагранжиан кварков и глюонов

Структура лагранжиана гостов $\mathscr{L}_{\text{gh}}$ (и даже сам факт их наличия) существенно зависит от вида калибровочного условия, задаваемого функцией f.

Лагранжиан не калибровочной теории кварков и глюонов имеет вид:

$$\begin{aligned}\mathscr{L}(\psi, B) =& -\frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \bar\psi\left(i\gamma_\mu\nabla_\mu - m\right)\psi + \mathscr{L}_{\text{gh}} - \frac{1}{2\alpha}\left(\partial_\mu B^a_\mu\right)^2 \\ =& -\frac{1}{4}\left(\partial_\mu B^a_\nu - \partial_\nu B^a_\mu\right)^2 + \bar\psi\left(i\gamma_\mu\partial_\mu - m\right)\psi - \partial_\mu\bar c^a\partial_\mu c^a - \frac{1}{2\alpha}\left(\partial_\mu B^a_\mu\right)^2 + g\bar\psi\gamma_\mu B^a_\mu t^a\psi \\ &+ g f^{acb}\bar c^a B^c_\mu\partial_\mu c^b - \frac{g}{2}f^{abc}\left(\partial_\mu B^a_\nu - \partial_\nu B^a_\mu\right)B^b_\mu B^c_\nu - \frac{g^2}{4}f^{abc}f^{ade}B^b_\mu B^c_\nu B^d_\mu B^e_\nu\end{aligned}$$

(как его использовать???)

теория от Киселева

На примере электромагнитного поля мы убедились, что его квантование в голоморфном представлении мало чем отличается от квантования скалярного поля.

А именно, две физические моды безмассового векторного поля полностью определяют гамильтониан поля и имеют соответствующие релятивистски инвариантные амплитуды, по которым можно провести интегрирование и в континуальном интеграле.

Амплитуды квантовых переходов строятся именно для этих двух поперечных мод.

Однако сами моды определены совершенно нековариантным образом в терминах векторов поляризации,

которые имеют нулевую временную компоненты и две пространственно поперечные компоненты.

Это, в частности, означает, что пропагатор таких мод имеет совершенно нековариантный вид, а значит,

доказательство релятивистской инвариантности амплитуд квантовых переходов становится настоящей проблемой.

С другой стороны, мы убедились, что уравнения движения для безмассового векторного поля могут быть получены из принципа экстремальности калибровочно инвариантного действия.

В этом действии, однако, присутствуют нефизические калибровочные степени свободы, которые мы можем исключить, налагая связи и фиксируя калибровку, а значит, исключая

степени излишние свободы. С точки зрения континуального интеграла

$$\tilde{Z} = \int \mathcal{D}\mathcal{A} e^{\mathrm{i}S}, \quad S = \int \mathrm{d}^4 x \mathcal{L}_{\mathrm{g.f.}}, \quad \mathcal{L}_{\mathrm{g.f.}} = -\frac{1}{2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu}$$

в релятивистски инвариантном интеграле по всем компонентам поля содер- жится лишнее интегрирование по параметрам локальной группы внутренней симметрии, от которых зависят компоненты поля. От этого излишнего нефи- зического фактора можно избавиться. Для этого заметим несколько фактов.

Первое, в мере интегрирования $\mathcal{D}\mathcal{A}$ можно делать замену переменных в виде калибрвочного преобразования инвариантным образом:

$$\mathcal{D}\mathcal{A} = \mathcal{D}\mathcal{A}^g$$

В самом деле, из вида калибровочного преобразования

$$\mathcal{A}^g_\mu = g\mathcal{A}_\mu g^{-1} - \frac{\mathrm{i}}{e} g \partial_\mu g^{-1}$$

ясно, сдвиг не влияет на интеграл, а якобиан замены переменных равен единице, так как $\det\left(g \cdot \mathbb{1} \cdot g^{-1}\right) = 1$. В силу калибровочной инвариантности действия такая замена переменных не меняет континуальный интеграл. Второе, на примере 2-мерного интегралая

$$\int \mathrm{d}x_1 dx_2 \delta^{(2)}\left(f^\alpha(x)\right) = \int \det\left(\frac{\partial x^\beta}{\partial f^\alpha}\right) \mathrm{d}f_1 df_2 \delta\left(f_1\right) \delta\left(f_2\right) = \det\left(\frac{\partial x^\beta}{\partial f^\alpha}\right)\Bigg|_{\boldsymbol{f}=0}$$

мы можем заключить, что использование обратной матрицы дает тождество

$$\det\left(\frac{\partial f^\alpha}{\partial x^\beta}\right)\Bigg|_{\boldsymbol{f}=0} \int \mathrm{d}x_1 dx_2 \delta^{(2)}\left(f^\alpha(x)\right) = 1$$

которое справедливо и для континуального интеграла

$$\operatorname{Det}\left(\mathcal{M}_X(\mathcal{A})\right)|_{X(\mathcal{A})=0} \int \mathcal{D}g \delta\left(X\left(\mathcal{A}^g\right)\right) = 1$$

В этом интеграле $\mathcal{D}g$ - инвариантная мера интегрирования по параметрам группы, скажем, интеграл по параметрам u, которые отсчитываются в экс-

поненциальном представлении возле единицы $\exp(\mathrm{i}eu)$, так что $\mathcal{D}g \cong \mathcal{D}u$, - так что замена в виде группового умножения $g \mapsto \tilde{g} = g \cdot g_1$ с фиксированным элементом g_1 никак не меняет параметров при разложении экспоненциального представления возле единицы, $\exp(\mathrm{i}eu) \mapsto \exp(\mathrm{i}eu)$. При заданном калибровочном поле $X\left(\mathcal{A}^g\right) = 0$ - это калибровочное условие, которое имеет единственное решение (по крайней мере в рамках теории возмущений). Матрица

$$\left(\mathcal{M}_X(\mathcal{A})\right)^a_b|_{X(\mathcal{A})=0} = \left.\frac{\partial X^a\left(\mathcal{A}^g\right)}{\partial u^b}\right|_{u=0}$$

Детерминант этой матрицы является калибровочно инвариантным в силу инвариантности метрики, поскольку при $\tilde{g} = gg_1$

$$\int \mathcal{D}g \delta\left(X\left(\mathcal{A}^{gg_1}\right)\right) = \int \mathcal{D}\tilde{g} \delta\left(X\left(\mathcal{A}^{\tilde{g}}\right)\right) = \int \mathcal{D}g \delta\left(X\left(\mathcal{A}^g\right)\right)$$

а значит,

$$\mathrm{Det}\left(\mathcal{M}_X\left(\mathcal{A}^{g_1}\right)\right)\big|_{X(\mathcal{A})=0} = \mathrm{Det}\left(\mathcal{M}_X(\mathcal{A})\right)\big|_{X(\mathcal{A})=0}$$

Тогда, вводя единицу в подынтегральное выражение

$$\tilde{Z} = \int \mathcal{D}\mathcal{A} e^{\mathrm{i}S}\,\mathrm{Det}\left(\mathcal{M}_X(\mathcal{A})\right)\Big|_{X(\mathcal{A})=0} \int \mathcal{D}g\,\delta\left(X\left(\mathcal{A}^g\right)\right)$$

мы можем выделить интеграл по параметрам калибровочной группы

$$\tilde{Z} = \int \mathcal{D}g \int \mathcal{D}\mathcal{A} e^{\mathrm{i}S}\,\mathrm{Det}\left(\mathcal{M}_X(\mathcal{A})\right)\Big|_{X(\mathcal{A})=0} \delta\left(X\left(\mathcal{A}^g\right)\right)$$

и воспользоваться заменой переменных $\mathcal{A}^g \mapsto \left(\mathcal{A}^g\right)^{g^{-1}} = \mathcal{A}$, так что калибровочно инвариантные факторы не изменятся, но изменится аргумент калибровочного условия, который теперь перестанет зависеть от параметров интегрирования по калибровочной группе, и

$$Z' = \tilde{Z}\left(\int \mathcal{D}g\right)^{-1} = \int \mathcal{D}\mathcal{A} e^{\mathrm{i}S}\,\mathrm{Det}\left(\mathcal{M}_X(\mathcal{A})\right)\Big|_{X(\mathcal{A})=0} \delta(X(\mathcal{A}))$$

Такая факторизация «объема калибровочной групыы» носит название трюка Фаддеева-Попова. Этот трюк позволяет нам избавиться в амплитуде от интегрирования по нефизическим степеням свободы, т. е. от интеграла по группе внутренней симметрии. При этом итоговая амплитуда не зависит от вида калибровки $X^a(\mathcal{A}) = 0$, поскольку $\tilde{Z}\left(\int \mathcal{D}g\right)^{-1}$ - это, как мы видели, калибровочно инвариантная величина по построению!

Третий шаг - переход от дельта-функции к более прагматическому выражению. Заменим калибровку

$$X(\mathcal{A}) \mapsto X(\mathcal{A}) - b(x)$$

При этом значение амплитуды Z' совершенно не зависит от функции $b(x)$, так что мы можем проинтегрировать по этой функции с любым весом. Например,

$$Z = \int \mathcal{D}b \exp\left(-\frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x b^2(x)\right) Z'$$

так что после поворота Вика и перехода к координатам в евклидова пространства $\mathrm{d}^4x = -\mathrm{i}\mathrm{d}^4x_E$ гауссов интеграл сходится при $\xi > 0$. Снимая интегрирование за счет континуальной дельта-функции, мы приходим к окончательному калибровочно инвариантному выражению для амплитуды

$$Z = \int \mathcal{D}\mathcal{A} e^{\mathrm{i}S_F}\,\mathrm{Det}\left(\mathcal{M}_X(\mathcal{A})\right)\Big|_{X(\mathcal{A})=0}$$

где действие с учетом калибровочного условия

$$S_F = S_{\text{g.f.}} - \frac{\mathrm{i}}{2\xi}\int \mathrm{d}^4x X^2(\mathcal{A})$$

Здесь фиксирующий калибровку вклад в лагранжиан

$$\mathcal{L}_{\text{g.fix.}} = -\frac{1}{2\xi}X^2(\mathcal{A})$$

можно теперь выбрать в явно релятивистски инвариантном виде. При введении источников j^μ для калибровочного поля необходимо сохранить калибровочную инвариантность исходного лагранжиана без $\mathcal{L}_{\text{g.fix.}}$, что конечно легко достигается требованием сохранения тока и соответствующим

законом преобразования при калибровочных преобразованиях $j^\mu \mapsto g j^\mu g^{-1}$, ТаК ЧТО

$$\mathcal{L}_{\text{int}} = \text{tr}\left(\mathcal{A}_\mu j^\mu\right)$$

строгое обоснование действия Фаддеева-Попова

В заключение укажем более строгий способ обоснования трюка Фаддева-Попова.

Калибровочно и релятивистски инвариантный лагранжиан калибровочного векторного поля с точки зрения гамильтоновой механики описывается физическими величинами - трехмерно поперечным полем и канонически сопряженным к этому полю импульсом,

а также трехмерно продольным полем и канонически сопряженным к нему импульсом,

но содержит также зависимость от временной компоненты поля $\mathcal{A}_0$, которая выделяется следующим образом: канонически сопряженный к ней импульс строится как производная лагранжиана по скорости изменения этой компоненты по времени,

т. е. по $\partial_0 \mathcal{A}_0$.

Однако, в лагранжиан не входит $\partial_0 \mathcal{A}_0$, поскольку напряженность поля не содержит этой производной, так что,

следуя формализму связи лагранжиана и гамильтониана,

мы приходим к выводу, что временная компонента калибровочного поля является множителем Лагранжа.

Обозначим в традициях механики трехмерно поперечное поле символом q, сопряженный ему импульс - p,

трехмерно продольное поле - x,

сопряженный ему импульс p_x, а временную компоненту поля $-\lambda$.

При этом в явном виде мы не пишем ни пространственные, ни групповые индексы внутренней симметрии.

Тогда лагранжиан калибровочного поля запишется в виде

$$\mathcal{L} = \dot{q}p + \dot{x}p_x - \mathcal{H}\left(q, p, x, p_x\right) - \lambda C\left(q, p, x, p_x\right)$$

где $C\left(q, p, x, p_x\right)$ - функция, которая оказывается связью: уравнение Эйлера-Лагранжа для величины λ сводятся к $C\left(q, p, x, p_x\right) = 0$.

По факту оказывается, что эта связь накладывает условие на значение продольной части канонического импульса p_x

Важно вычислить скобки Пуассона для физических величин $\{F, G\}_{\text{P}}$, которые при квантовании отвечают коммутаторам: скобка Пуассона связи с собой выражается через саму связь, а значит, связи не порождают новые связи в уравнениях движения; скобка Пуассона связи с функцией Гамильтона равна нулю, и следовательно, связь сохраняется со временем. Это также говорит о том, что связь является генератором симметрии: ее скобка Пауассона с обобщенными координатами (q, x) порождает преобразования поля, которые оставляют действие инвариантным. Это ничто иное как калибровочные преобразования, так что механическая система содержит степень свободы, от которой не зависят наблюдаемые величины. В нашем случае это трехмерно продольное поле x и соответсвующий ему импульс p_x: их можно исключить из рассмотрения и оставить только

физические моды. Это можно сделать, если ввести связи - калибровки $X\left(q,p,x,p_x\right)$, которые не инвариантны относительно действия генератора симметрии, а значит, скобки Пауассона связи и калибровки отличны от нуля, $\{C,X\}_{\mathrm{P}}\neq 0$. Сами калибровки также не должны порождать новые связи: $\{X,X\}\mathrm{p}=0$.

Допустиv, что существует каноническое преобразование переменных, при котором можно перейти от координат x к обобщенным координатам X. Тогда разрешая все связи, можно показать, что уравнения движения для физических степеней свободы - координат q и импульсов p - это уравнения гамильтоновой механики, в которой гамильтониан зависит только от физических степеней свободы (мы это знаем, поскольку для безмассового векторного поля мы как раз и построили

такой гамильтониан, исключив трехмерно продольное поле и временную компоненту поля).

Запишем амплитуду квантовых переходов в виде континуального интеграла от исходного действия с введением калиброки в виде

$$Z=\int \mathcal{D}q\mathcal{D}p\mathcal{D}x\mathcal{D}p_x\mathcal{D}\lambda\delta(X)\,\mathrm{Det}(\{C,X\}\mathrm{P})e^{\mathrm{i}\int \mathrm{d}^4x(\dot{q}p-\dot{x}p_x-\mathcal{H}-\lambda C)}$$

Интегрирвание по множителю Лагранжа выполняется элементарно и дает континуальную дельта-функцию $\delta(C)$, так что мы имеем полный аналог системы со связями: $C=X=0$ Остается рассмотреть выражение

$$\int \mathcal{D}x\mathcal{D}p_x\delta(C)\delta(X)\Big)\,\mathrm{Det}\left(\{C,X\}_{\mathrm{P}}\right)$$

которое следует сравнить с обычным интегрированием, рассмотренным ранее в (9.5): скобка Пуассона в точности соответствует якобиану перехода от одних координат к другим, так что этот интеграл следует положить равным единице и, при этом, использовать фиксированные значения x и p_x, которые в общем случае выражаются через физические переменный, но из можно просто считать равными нулю при $C=X=0$. В итоге, амплитуда сводится к интегралу по физическим степеням свободы, когда выполнены все калибровочные связи, а нединамическая компонента λ также не дает никакого вклада в действие,

$$Z=\int \mathcal{D}q\mathcal{D}pe^{\mathrm{i}\int \mathrm{d}^4x(\dot{q}p-\mathcal{H}(q,p))}$$

Именно, этого мы и добивались: построенный по физическим модам калибровочного поля континуальный интеграл эквивалентен калибровочно инвариантному континуальному интегралу,

где проводится интегрирование по всем компонентам поля с учетом фактора,

который отвечает скобке Пуассона генератора калибровочных преобразований и калировочного условия, которое фиксирует калибровку.

Эта

оператором, который вводится в трюке Фаддева-Попова.

Далее, чтобы рассматривать произвольные калибровочные условия, оставалось убедиться, что произведение дельта-функции от калибровки $\delta(X)$ на детерминант от оператора $\mathrm{Det}\left(\mathcal{M}_X\right)$, задающего производную калибровки по групповому параметру, является калибровочно инвариантным, что мы уже установили ранее.

примеры применений метода Фаддеева-Попова (!!!??)

самое интересное, я хз прямо.

примеры гостов

хз

примеры применений (!!!)

в гравитации, говорят, применяется.
как и где еще?

18.7.6 BRST Method

Более общий взгляд на происхождение и роль гостов возможен в рамках метода $BRST$, названного по именам авторов - Becchi, Rouet, Stora, Тютин и широко применяемого в теории калибровочных полей и струн.

мотивация и обзор методов

обзор применений (!!??)

вид преобразований

Соответствующие преобразования полей носят название $BRST$ -преобразований:

$$\delta\varphi = c_i t_i \varphi, \quad \delta c_i = -\frac{i}{2} f_{ijk} c_j c_k, \quad \delta \bar{c}_i = b_i, \quad \delta b_i = 0$$

объяснение метода BRST

Центральным объектом данного метода является нильпотентный оператор δ.

простейшая реализация

Рассмотрим сначала простейшую реализацию BRST метода.
Пусть нильпотентный оператор δ имеет вид:

$$\delta = c_i t_i - \frac{i}{2} f_{ijk} c_i c_j \frac{\partial}{\partial c_k}, \quad [t_i, t_j] = i f_{ijk} t_k, \quad c_i c_j = -c_j c_i$$

где t_i-генераторы некоторой группы, а c_i-грассмановы переменные (будущие госты).

Нильпотентность δ, то есть свойство $\delta^2 = 0$, доказывается прямым вычислением.
???

Например, если генераторы t_i действуют на φ, то достаточно проверить

$$\delta^2 \varphi = \delta\left(c_i t_i \varphi\right) = -\frac{i}{2} f_{jki} c_j c_k t_i \varphi + c_j c_i t_j t_i \varphi = -\frac{i}{2} f_{jki} c_j c_k t_i \varphi + \frac{1}{2} c_j c_i \left[t_j, t_i\right] \varphi = 0$$

$$\delta^2 c_i = -\frac{i}{2} f_{jki} \delta\left(c_j c_k\right) = \frac{1}{2} f_{jki} f_{mnk} c_j c_m c_n = 0$$

Последнее равенство есть следствие циклической структуры тождества Якоби.

Теперь запишем оператор δ в том виде, в котором он обычно используется в калибровочных теориях, добавив к (213), не нарушая нильпотентности, антигостые $\bar{c}_i$ (тоже грассманновы) и вспомогательные бозонные поля b_i :

$$\delta = c_i t_i - \frac{i}{2} f_{ijk} c_i c_j \frac{\partial}{\partial c_k} + b_i \frac{\partial}{\partial \bar{c}_i}, \quad \delta^2 = 0$$

Соответствующие преобразования полей носят название $BRST$ -преобразований:

$$\delta\varphi = c_i t_i \varphi, \quad \delta c_i = -\frac{i}{2} f_{ijk} c_j c_k, \quad \delta \bar{c}_i = b_i, \quad \delta b_i = 0$$

Теперь t_i -это генераторы калибровочной группы; они действуют на все поля (φ), из которых строилась исходная теория (глюоны, кварки и т.д.), производя над ними калибровочные преобразования согласно типу каждого поля.

В привычных нам обозначениях первые соотношения из BRST преобразований имели бы вид

$$\delta\psi = igc^a t^a \psi, \quad \delta B^a_\mu = \partial_\mu c^a + g f^{abc} B^b_\mu c^c, \quad \delta c^a = -\frac{g}{2} f^{abc} c^b c^c$$

Отметим, что госты и антигосты выступают в данном подходе как независимые поля, хотя в лагранжиане (210) это не бросалось в глаза; в частности, преобразования (215) у них совершенно различны.

мы ввели BRST-преобразования как калибровочные, но с гостами в качестве параметров.

метод Фаддеева-Попова на BRST языке

Переформулируем теперь метод Фаддеева-Попова на BRST языке.

Назовем $BRST$ -инвариантным объект Φ, удовлетворяющий условию $\delta\Phi = 0$.

Если выбрать полный лагранжиан квантовой неабелевой теории в виде суммы

$$\mathscr{L} = \mathscr{L}_{\text{inv}} + \delta\Lambda$$

исходного калибровочно инвариантного лагранжиана $\mathscr{L}_{\text{inv}}(\varphi)$, содержащего только поля φ, и добавки $\delta\Lambda$ с неинвариантным $\Lambda(\varphi, c, \bar{c}, b)$, то весь лагранжиан $\mathscr{L}$ будет BRST-инвариантен.

Пусть калибровка фиксируется условиями $f_i(\varphi) = 0$.

Возьмем

$$\Lambda = \left(f_i + \frac{\alpha}{2} b_i\right) \bar{c}_i \quad \Longrightarrow \quad \mathscr{L} = \mathscr{L}_{\text{inv}} + c_j t_j f_i \bar{c}_i + f_i b_i + \frac{\alpha}{2} b_i b_i$$

Тогда континуальное интегрирование по $c, \bar{c}$ выдает детерминант $\det(t_j f_i)$ матрицы калибровочного преобразования условий f_i, который в точности соответствует функционалу $\Delta_f[B]$(207).

Интегрирование по полям b порождает $\exp(-i f_i f_i / 2\alpha)$, а при $\alpha = 0$ -соответственно, $\delta(f)$, воспроизводя лагранжиан (212).

Особенно прозрачным становится в этом подходе доказательство независимост теории от неинвариантной (фиксирующей калибровку) добавки Λ.

Действительно, континуальный интеграл типа (211) с инвариантным Φ,

$$\int \mathscr{D}\phi \mathscr{D}\bar{c} \mathscr{D}c \mathscr{D}b \Phi \exp i\left(\mathscr{L}_{\text{inv}} + \delta\Lambda\right) \equiv \int \mathscr{D}\Psi \Phi \exp i\left(\mathscr{L}_{\text{inv}} + \delta\Lambda\right)$$

при малых изменениях Λ сам может измениться только на величину вида

$$\int \mathscr{D}\Psi\Phi\delta X \exp i\left(\mathscr{L}_{\text{inv}} + \delta\Lambda\right) = \int \delta\left(\mathscr{D}\Psi\Phi X \exp i\left(\mathscr{L}_{\text{inv}} + \delta\Lambda\right)\right) = 0$$

равную нулю, поскольку интеграл не меняется при заменах переменных.

пример применений???

хз, я даже сам метод не очень пока понял.

19 What If There Is Another Fundamental Effect?

(theoretical analysis of different effects)

19.0.1 Vacuum in a Uniform Gravitational Field, Excitation of a Uniformly Accelerated Detector by Ginzburg, Frolov

(see now something about it in GR, later I'll write also here from QFT perspective)

Вакуум в пространстве-времени Минковского, рассматриваемый в инерциальной и в равномерно ускоренной системах отсчета. Вакуум в статическом однородном гравитационном поле

Квантовая теория поля в инерциальной системе отсчета.

2.2. Равномерно ускоренная система отсчета и система отсчета, покоящаяся в однородном гравитационном поле.

2.3. Квантовая теория поля в равномерно ускоренной системе отсчета.

2.4. Вакуум в статическом однородном гравитационном поле.

3. Детекторы, покоящиеся в равномерно ускоренной системе отсчета и в статическом однородном гравитационном поле

3.1. Модельное описание детектора.

3.2. Взаимодействие ускоренного детектора с вакуумом.

Описание в равномерно ускоренной системе отсчета.

3.3. Поведение детектора, покоящегося в статическом однородном гравитационном поле. 3.4.

Элементарные частицы в роли детекторов.

4. Возбуждение и излучение равномерно ускоренного детектора. Описание в инерциальной системе отсчета

5. Возбуждение и излучение ускоренного детектора и аномальный эффект Допплера.

(??? я и не знаю такого эффекта)

Несколько дополнительных замечаний

6. Принцип эквивалентности в квантовой области и «тепловой характер» вакуума в равномерно ускоренной системе отсчета

Заключительные замечания

19.0.2 Effect on a Lattice or Condensate in Particle Scatterings

(people say that since incoming particles are fast, there is no impact form substrate, lattice or condensate. I'll show here, why it is so? there is a thermal motion and there is and exchange of a virtual particle, so it is very not obvious for me, that these effects are really negligible)

20 Other, Useful Methods

20.1 Other ways of Calculation Feynman Diagrams

20.1.1 Scattering Amplitudes For All Masses and Spins

(maybe I'll create several sections about it, but now here it is a reminder that this method is maybe very useful)

20.2 Other Feynman Rules

(?????)

20.2.1 Feynman Rules for Non-Abelian Theory and Ghost Fields

(?????)

20.2.2 Feynman Rules for Non-Abelian Theory with Fermion Fields

(?????)

Суть (????)

Правила Фейнмана для неабелевой теории с фермионными полями по Киселеву (!?!?)

Для определения правил Фейнмана запишем все вклады в лагранжиан неабелевых калибровочных полей, которые взаимодействую со спинорнми дираковскими полями.

Калибровочное поле с тензором напряженности

$$F^c_{\mu\nu} = \partial_\mu \mathcal{A}^c_\nu - \partial_\nu \mathcal{A}^c_\mu - e f_{abc} \mathcal{A}^a_\mu \mathcal{A}^b_\nu$$

задает

$$\mathcal{L}_{\text{g.f.}} = -\frac{1}{2} \operatorname{tr} F^2_{\mu\nu} = -\frac{1}{4} F^c_{\mu\nu} F^{\mu\nu}_c$$

где наряду с квадратичным членом свободного поля с добавлием вклада от лоренцевской калибровки

$$\mathcal{L}_0 + \mathcal{L}_{\text{g.fix.}} = -\frac{1}{4} \left(\partial_\mu \mathcal{A}^c_\nu - \partial_\nu \mathcal{A}^c_\mu\right) \left(\partial^\mu \mathcal{A}^\nu_c - \partial^\nu \mathcal{A}^\mu_c\right) - \frac{1}{2\xi} \left(\partial_\mu \mathcal{A}^\mu_c\right)^2$$

есть вклады от кубического самодействия поля, линейного по заряду e,

$$\mathcal{L}_{eA} = \frac{1}{2} e \left(\partial_\mu \mathcal{A}^c_\nu - \partial_\nu \mathcal{A}^c_\mu\right) f_{abc} \mathcal{A}^\mu_a \mathcal{A}^\nu_b$$

и самодействия четвертой степени по полю, квадратичного по заряду,

$$\mathcal{L}_{e^2A} = -\frac{1}{4}e^2 f_{abc}\mathcal{A}^a_\mu \mathcal{A}^b_\nu f_{a'b'c}\mathcal{A}^\mu_{a'}\mathcal{A}^\nu_{b'}$$

Значит, на внутренних линиях калибровочному полу соответствует волнистая линия с пропагатором, который сохраняет групповой индекс поля,

$$\frac{\mathrm{i}}{k^2+\mathrm{i}0}\delta^b_a\left(-g_{\mu\nu}+(1-\xi)\frac{k_\mu k_\nu}{k^2}\right) \quad a \quad \sim\sim\sim\sim \quad b$$

Трехточечная вершина вычисляется дифференцированием по трем полям, причем, $\frac{1}{2}$ сокращается поскольку поля без производных входят неотличимым образом, стандартный фактор перед лагранжианом i в сочетании, например, с ∂_μ дает импульс p_μ, а исходное слагаемое записывается по явному виду лагранжиана: сначала фактор с групповыми индексами, так что импульс отвечает полю с третим групповым индексом и лоренцевым индексом в соответствии с лагранжианом, а остальные вклады получаются циклической перестановкой групповых индексов в структурных константах и соответствующими заменами импульса и лоренцевых индексов,

$$\begin{array}{l} ef_{abc}\left(q_\mu g_{\lambda\nu}-q_\nu g_{\lambda\mu}\right)+ \\ ef_{bca}\left(p_\nu g_{\mu\lambda}-p_\lambda g_{\mu\nu}\right)+ \\ ef_{cab}\left(k_\lambda g_{\nu\mu}-k_\mu g_{\nu\lambda}\right) \end{array}$$

Четыреехточечная вершина имеет $4!/4 = 6$ вкладов, которые получаются перестановками лоренцеых и групповых индексов из их исходной прескрипции в явном виде лагражиана с учетом антисимметрии структурных констант f_{abc} по перестановке групповых индексов,

$$-\mathrm{i}e^2\left\{\begin{array}{l} f_{abc}f_{a'b'c}\left(g_{\mu\mu'}g_{\nu\nu'}-g_{\mu\nu'}g_{\nu\mu'}\right)+ \\ f_{aa'c}f_{bb'c}\left(g_{\mu\nu}g_{\mu'\nu'}-g_{\mu\nu'}g_{\mu'\nu}\right)+ \\ f_{ab'c}f_{a'bc}\left(g_{\mu\mu'}g_{\nu'\nu}-g_{\mu\nu}g_{\nu'\mu'}\right) \end{array}\right\}$$

Для внешних линий калибровочного полая в конечном состоянии в амплитуду входят отрицательно частотные моды ε^{*a}_μ, а в начальном состоянии − положительно частотные ε^a_μ. Лагранжиан калибровочного взаимодействия спинорного дираковского поля с групповым индексом A представления группы локальной внутренней

симметрии с эрмитово самосопряженными генераторами t^a_{AB} имеет вид

$$\mathcal{L}_D = \bar\psi_A\left(\delta_{AB}\left(\mathrm{i}\partial_\mu\gamma^\mu - m\right) - e\gamma^\mu\mathcal{A}^a_\mu t^a_{AB}\right)\psi_B$$

Отсюда элементарно воспроизводим пропагатор фермиона на внутренних линиях

(пара диаграмм)

Для внешних линий частиц в конечном состоянии в амплитуду входят отрицательно частотные моды $\bar u_{\lambda,A}$, в начальном состоянии - положительно частотные моды $u_{\lambda,A}$ для античастиц в конечном состоянии - положительно частотные $v+\lambda, A$, а в начальном - отрицательно частотные $\bar v_{\lambda,A}$

Правила Фейнмана для диаграмм с духами Фаддеева-Попова мы уже приводили в разделе 9.6.

примеры

применим эти правила к какому-то конкретному случаю.

Part VI
QFT Processes and Effects

21 Typical Phenomena of Quantum Electrodynamics (?!!!)

Обсудим подробно типичные явления КТП.

(!??!?!? Киселева еще раз переучу, там многие есть!!! ну или других, важно нарабатать классические КТП явления!!)

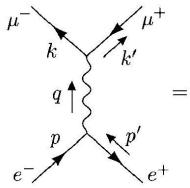
Finally, after three long chapters of formalism, we are ready to perform some real relativistic calculations, to begin working out the predictions of Quantum Electrodynamics. First we will return to the process considered in Chapter 1, the annihilation of an electron-positron pair into a pair of heavier fermions. We will study this paradigm process in extreme detail in the next three sections, then do a few more simple QED calculations in Sections 5.4 and 5.5. The problems at the end of the chapter treat several additional QED processes. More complete surveys of QED can be found in the books of Jauch and Rohrlich (1976) and of Berestetskii, Lifshitz, and Pitaevskii (1982).

21.1 5.1$e^+e^- \to \mu^+\mu^-$: Introduction

The reaction $e^+e^- \to \mu^+\mu^-$is the simplest of all QED processes, but also one of the most important in high-energy physics. It is fundamental to the understanding of all reactions in e^+e^-colliders, and is in fact used to calibrate such machines. The related process $e^+e^- \to q\bar{q}$ (a quark-antiquark pair) is extraordinarily useful in determining the properties of elementary particles.

In this section we will compute the unpolarized cross section for $e^+e^- \to \mu^+\mu^-$, to lowest order. In Chapter 1 we used elementary arguments to guess the answer (Eq. (1.8)) in the limit where all the fermions are massless. We now relax that restriction and retain the muon mass in the calculation. Retaining the electron mass as well would be easy but pointless, since the ratio $m_e/m_\mu \approx 1/200$ is much smaller than the fractional error introduced by neglecting higher-order terms in the perturbation series.

Using the Feynman rules from Section 4.8, we can at once draw the diagram and write down the amplitude for our process:



$$= \bar{v}^{s'}(p')(-ie\gamma^\mu)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^r(k)(-ie\gamma^\nu)v^{r'}(k').$$

Rearranging this slightly and leaving the spin superscripts implicit, we have

$$i\mathcal{M}\left(e^-(p)e^+(p') \to \mu^-(k)\mu^+(k')\right) = \frac{ie^2}{q^2}\left(\bar{v}(p')\gamma^\mu u(p)\right)\left(\bar{u}(k)\gamma_\mu v(k')\right) \tag{5.1}$$

This answer for the amplitude $\mathcal{M}$ is simple, but not yet very illuminating.
To compute the differential cross section, we need an expression for $|\mathcal{M}|^2$, so we must find the complex conjugate of $\mathcal{M}$. A bi-spinor product such as $\bar{v}\gamma^\mu u$ can be complex-conjugated as follows:

$$(\bar{v}\gamma^\mu u)^* = u^\dagger (\gamma^\mu)^\dagger \left(\gamma^0\right)^\dagger v = u^\dagger (\gamma^\mu)^\dagger \gamma^0 v = u^\dagger \gamma^0 \gamma^\mu v = \bar{u}\gamma^\mu v$$

(This is another advantage of the 'bar' notation.) Thus the squared matrix element is

$$|\mathcal{M}|^2 = \frac{e^4}{q^4} \left(\bar{v}\left(p'\right)\gamma^\mu u(p)\bar{u}(p)\gamma^\nu v\left(p'\right)\right)\left(\bar{u}(k)\gamma_\mu v\left(k'\right)\bar{v}\left(k'\right)\gamma_\nu u(k)\right) \tag{5.2}$$

At this point we are still free to specify any particular spinors $u^s(p)$, $\bar{v}^{s'}(p')$, and so on, corresponding to any desired spin states of the fermions. In actual experiments, however, it is difficult (though not impossible) to retain control over spin states; one would have to prepare the initial state from polarized materials and/or analyze the final state using spin-dependent multiple scattering. In most experiments the electron and positron beams are unpolarized, so the measured cross section is an average over the electron and positron spins s and s'. Muon detectors are normally blind to polarization, so the measured cross section is a sum over the muon spins r and r'.

The expression for $|\mathcal{M}|^2$ simplifies considerably when we throw away the spin information. We want to compute

$$\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'}\sum_r\sum_{r'} \left|\mathcal{M}\left(s,s'\to r,r'\right)\right|^2$$

The spin sums can be performed using the completeness relations from Section 3.3:

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m; \quad \sum_s v^s(p)\bar{v}^s(p) = \not{p} - m \tag{5.3}$$

Working with the first half of (5.2), and writing in spinor indices so we can freely move the v next to the $\bar{v}$, we have

$$\begin{aligned}\sum_{s,s'} \bar{v}_a^{s'}\left(p'\right)\gamma^\mu_{ab}u_b^s(p)\bar{u}_c^s(p)\gamma^\nu_{cd}v_d^{s'}\left(p'\right) &= \left(\not{p}'-m\right)_{da}\gamma^\mu_{ab}\left(\not{p}+m\right)_{bc}\gamma^\nu_{cd} \\ &= \operatorname{trace}\left[\left(\not{p}'-m\right)\gamma^\mu\left(\not{p}+m\right)\gamma^\nu\right]\end{aligned}$$

Evaluating the second half of (5.2) in the same way, we arrive at the desired simplification:

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\operatorname{tr}\left[\left(\not{p}'-m_e\right)\gamma^\mu\left(\not{p}+m_e\right)\gamma^\nu\right]\operatorname{tr}\left[\left(\not{k}+m_\mu\right)\gamma_\mu\left(\not{p}'-m_\mu\right)\gamma_\nu\right] \tag{5.4}$$

The spinors u and v have disappeared, leaving us with a much cleaner expression in terms of γ matrices. This trick is very general: Any QED amplitude involving external fermions, when squared and summed or averaged over spins, can be converted in this way to traces of products of Dirac matrices.

Trace Technology

This last step would hardly be an improvement if the traces had to be laboriously computed by brute force. But Feynman found that they could be worked out easily by appealing to the algebraic properties of the γ matrices. Since the evaluation of such traces occurs so often in QED calculations, it is worthwhile to pause and attack the problem systematically, once and for all.

We would like to evaluate traces of products of n gamma matrices, where $n = 0, 1, 2, \ldots$ (For the present problem we need $n = 2, 3, 4$.) The $n = 0$ case is fairly easy: $\operatorname{tr}\mathbf{1} = 4$. The trace of one γ matrix is also easy. From the explicit form of the matrices in the chiral representation, we have

$$\operatorname{tr}\gamma^\mu = \operatorname{tr}\begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix} = 0$$

It is useful to prove this result in a more abstract way, which generalizes to an arbitrary odd number of γ matrices:

$$\begin{aligned}
\operatorname{tr}\gamma^\mu &= \operatorname{tr}\gamma^5\gamma^5\gamma^\mu && \text{since } \left(\gamma^5\right)^2 = 1 \\
&= -\operatorname{tr}\gamma^5\gamma^\mu\gamma^5 && \text{since } \left\{\gamma^\mu,\gamma^5\right\} = 0 \\
&= -\operatorname{tr}\gamma^5\gamma^5\gamma^\mu && \text{using cyclic property of trace} \\
&= -\operatorname{tr}\gamma^\mu.
\end{aligned}$$

Since the trace of γ^μ is equal to minus itself, it must vanish. For $n\gamma$-matrices we would get n minus signs in the second step (as we move the second γ^5 all the way to the right), so the trace must vanish if n is odd.

To evaluate the trace of two γ matrices, we again use the anticommutation properties and the cyclic property of the trace:

$$\begin{aligned}
\operatorname{tr}\gamma^\mu\gamma^\nu &= \operatorname{tr}\left(2g^{\mu\nu}\cdot 1 - \gamma^\nu\gamma^\mu\right) && \text{(anticommuatation)} \\
&= 8g^{\mu\nu} - \operatorname{tr}\gamma^\mu\gamma^\nu && \text{(cyclicity)}
\end{aligned}$$

Thus $\operatorname{tr}\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$. The trace of any even number of γ matrices can be evaluated in the same way: Anticommute the first γ matrix all the way to the right, then cycle it back to the left. Thus for the trace of four γ matrices, we have

$$\begin{aligned}
\operatorname{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= \operatorname{tr}\left(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\right) \\
&= \operatorname{tr}\left(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu 2g^{\mu\rho}\gamma^\sigma + \gamma^\nu\gamma^\rho 2g^{\mu\sigma} - \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu\right)
\end{aligned}$$

Using the cyclic property on the last term and bringing it to the left-hand side, we find

$$\begin{aligned}
\operatorname{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= g^{\mu\nu}\operatorname{tr}\gamma^\rho\gamma^\sigma - g^{\mu\rho}\operatorname{tr}\gamma^\nu\gamma^\sigma + g^{\mu\sigma}\operatorname{tr}\gamma^\nu\gamma^\rho \\
&= 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)
\end{aligned}$$

In this manner one can always reduce a trace of $n\gamma$-matrices to a sum of traces of $(n-2)\gamma$-matrices. The case $n = 6$ is easy to work out, but has fifteen terms (the number of ways of grouping the six indices in pairs to make terms of the form $g^{\mu\nu}g^{\rho\sigma}g^{\alpha\beta}$). Fortunately, we will not need it in this book. (If you ever do need to evaluate such complicated traces, it may be easier to learn to use one of the several computer programs that can perform symbolic manipulations on Dirac matrices.)

Starting in Section 5.2, we will often need to evaluate traces involving γ^5. Since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, the trace of γ^5 times any odd number of other γ matrices is zero. It is also easy to show that the trace of γ^5 itself is zero:

$$\operatorname{tr}\gamma^5 = \operatorname{tr}\left(\gamma^0\gamma^0\gamma^5\right) = -\operatorname{tr}\left(\gamma^0\gamma^5\gamma^0\right) = -\operatorname{tr}\left(\gamma^0\gamma^0\gamma^5\right) = -\operatorname{tr}\gamma^5$$

The same trick works for $\operatorname{tr}\left(\gamma^\mu\gamma^\nu\gamma^5\right)$, if we insert two factors of γ^α for some α different from both μ and ν. The first nonvanishing trace involving γ^5 contains four other γ matrices. In this case the trick still works unless every γ matrix appears, so $\operatorname{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right) = 0$ unless $(\mu\nu\rho\sigma)$ is some permutation of (0123). From the anticommutation rules it also follows that interchanging any two

of the indices simply changes the sign of the trace, so tr $(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5)$ must be proportional to $\epsilon^{\mu\nu\rho\sigma}$. The overall constant turns out to be $-4i$, as you can easily check by plugging in $(\mu\nu\rho\sigma) = (0123)$.

Here is a summary of the trace theorems, for convenient reference:

$$\begin{aligned}
\mathrm{tr}(\mathbf{1}) &= 4 \\
\mathrm{tr}\,(\text{ any odd \# of } \gamma'\mathrm{s}) &= 0 \\
\mathrm{tr}\,(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu} \\
\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) &= 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) \\
\mathrm{tr}\left(\gamma^5\right) &= 0 \\
\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^5\right) &= 0 \\
\mathrm{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right) &= -4i\epsilon^{\mu\nu\rho\sigma}
\end{aligned} \tag{5.5}$$

Expressions resulting from use of the last formula can be simplified by means of the identities

$$\begin{aligned}
\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} &= -24 \\
\epsilon^{\alpha\beta\gamma\mu}\epsilon_{\alpha\beta\gamma\nu} &= -6\delta^\mu{}_\nu \\
\epsilon^{\alpha\beta\mu\nu}\epsilon_{\alpha\beta\rho\sigma} &= -2\left(\delta^\mu{}_\rho\delta^\nu{}_\sigma - \delta^\mu{}_\sigma\delta^\nu{}_\rho\right)
\end{aligned} \tag{5.6}$$

All of these can be derived by first appealing to symmetry arguments, then evaluating one special case to determine the overall constant.

Another useful identity allows one to reverse the order of all the γ matrices inside a trace:

$$\mathrm{tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\cdots) = \mathrm{tr}\,(\cdots\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu) \tag{5.7}$$

To prove this relation, consider the matrix $C \equiv \gamma^0\gamma^2$ (essentially the chargeconjugation operator). This matrix satisfies $C^2 = 1$ and $C\gamma^\mu C = -\left(\gamma^\mu\right)^T$. Thus if there are $n\gamma$-matrices inside the trace,

$$\begin{aligned}
\mathrm{tr}\,(\gamma^\mu\gamma^\nu\cdots) &= \mathrm{tr}\,(C\gamma^\mu CC\gamma^\nu C\cdots) \\
&= (-1)^n\,\mathrm{tr}\left[\left(\gamma^\mu\right)^T\left(\gamma^\nu\right)^T\cdots\right] \\
&= \mathrm{tr}\,(\cdots\gamma^\nu\gamma^\mu)
\end{aligned}$$

since the trace vanishes unless n is even. It is easy to show that the reversal identity (5.7) is also valid when the trace contains one or more factors of γ^5.

When two γ matrices inside a trace are dotted together, it is easiest to eliminate them before evaluating the trace. For example,

$$\gamma^\mu\gamma_\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{2}g_{\mu\nu}\left\{\gamma^\mu,\gamma^\nu\right\} = g_{\mu\nu}g^{\mu\nu} = 4 \tag{5.8}$$

The following contraction identities, all easy to prove using the anticommutation relations, can be used when other γ matrices lie in between:

$$\begin{aligned}
\gamma^\mu\gamma^\nu\gamma_\mu &= -2\gamma^\nu \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4g^{\nu\rho} \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu
\end{aligned} \tag{5.9}$$

Note the reversal of order in the last identity.
All of the γ matrix identities proved in this section are collected for reference in the Appendix.

Unpolarized Cross Section

We now return to the evaluation of the squared matrix element, Eq. (5.4). The electron trace is

$$\mathrm{tr}\left[(\not{p}' - m_e)\,\gamma^\mu\,(\not{p} + m_e)\,\gamma^\nu\right] = 4\left[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}\left(p\cdot p' + m_e^2\right)\right]$$

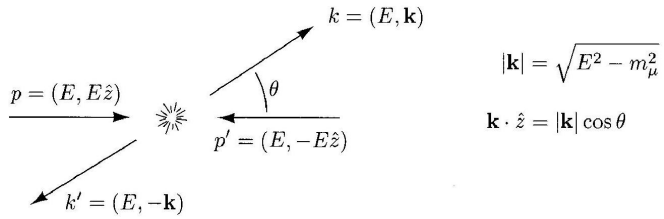
The terms with only one factor of m vanish, since they contain an odd number of γ matrices. Similarly, the muon trace is

$$\mathrm{tr}\left[(\not{k}+m_\mu)\,\gamma_\mu\,(\not{k}' - m_\mu)\,\gamma_\nu\right] = 4\left[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}\left(k\cdot k' + m_\mu^2\right)\right]$$

From now on we will set $m_e = 0$, as discussed at the beginning of this section. Dotting these expressions together and collecting terms, we get the simple result

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{q^4}\left[(p\cdot k)\,(p'\cdot k') + (p\cdot k')\,(p'\cdot k) + m_\mu^2\,(p\cdot p')\right] \tag{5.10}$$

To obtain a more explicit formula we must specialize to a particular frame of reference and express the vectors p, p', k, k', and q in terms of the basic kinematic variables - energies and angles- in that frame. In practice, the choice of frame will be dictated by the experimental conditions. In this book, we will usually make the simplest choice of evaluating cross sections in the center-ofmass frame. For this choice, the initial and final 4-momenta for $e^+e^- \to \mu^+\mu^-$ can be written as follows:



To compute the squared matrix element we need

$$q^2 = (p+p')^2 = 4E^2; \qquad p\cdot p' = 2E^2$$
$$p\cdot k = p'\cdot k' = E^2 - E|\mathbf{k}|\cos\theta; \quad p\cdot k' = p'\cdot k = E^2 + E|\mathbf{k}|\cos\theta$$

We can now rewrite Eq. (5.10) in terms of E and θ:

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{8e^4}{16E^4}\left[E^2(E-|\mathbf{k}|\cos\theta)^2 + E^2(E+|\mathbf{k}|\cos\theta)^2 + 2m_\mu^2E^2\right]\\ &= e^4\left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]\end{aligned} \tag{5.11}$$

All that remains is to plug this expression into the cross-section formula derived in Section 4.5. Since there are only two particles in the final state and we are working in the center-of-mass frame, we can use the simplified formula (4.84). For our problem $|v_{\mathcal{A}} - v_{\mathcal{B}}| = 2$ and $E_{\mathcal{A}} = E_{\mathcal{B}} = E_{\mathrm{cm}}/2$, so we have

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_{\rm cm}^2}\frac{|\mathbf{k}|}{16\pi^2 E_{\rm cm}} \cdot \frac{1}{4}\sum_{\rm spins}|\mathcal{M}|^2 \tag{5.12}$$
$$= \frac{\alpha^2}{4E_{\rm cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left[\left(1+\frac{m_\mu^2}{E^2}\right)+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right].$$

Integrating over $d\Omega$, we find the total cross section:

$$\sigma_{\rm total} = \frac{4\pi\alpha^2}{3E_{\rm cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right) \tag{5.13}$$

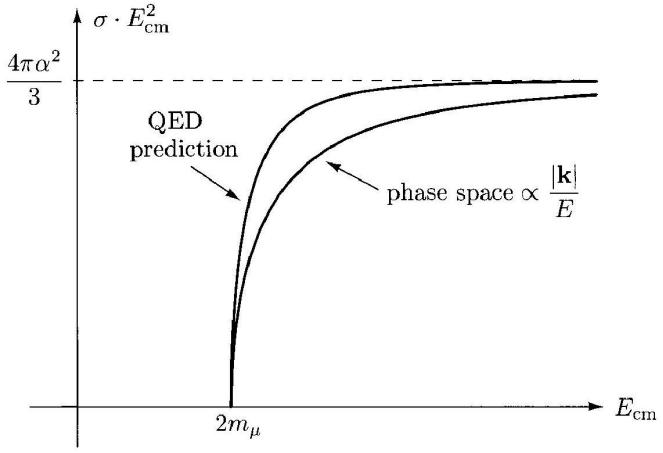


Figure 5.1. Energy dependence of the total cross section for $e^+e^- \to \mu^+\mu^-$, compared to "phase space" energy dependence.

In the high-energy limit where $E \gg m_\mu$, these formulae reduce to those given in Chapter 1:

$$\frac{d\sigma}{d\Omega} \underset{E\gg m_\mu}{\longrightarrow} \frac{\alpha^2}{4E_{\rm cm}^2}\left(1+\cos^2\theta\right)$$
$$\sigma_{\rm total} \underset{E\gg m_\mu}{\longrightarrow} \frac{4\pi\alpha^2}{3E_{\rm cm}^2}\left(1-\frac{3}{8}\left(\frac{m_\mu}{E}\right)^4-\cdots\right) \tag{5.14}$$

Note that these expressions have the correct dimensions of cross sections. In the high-energy limit, $E_{\rm cm}$ is the only dimensionful quantity in the problem, so dimensional analysis dictates that $\sigma_{\rm total} \propto E_{\rm cm}^{-2}$. Since we knew from the beginning that $\sigma_{\rm total} \propto \alpha^2$, we only had to work to get the factor of $4\pi/3$.

The energy dependence of the total cross-section formula (5.13) near threshold is shown in Fig. 5.1. Of course the cross section is zero for $E_{\rm cm} < 2m_\mu$. It is interesting to compare the shape of the actual curve to the shape one would obtain if $|\mathcal{M}|^2$ did not depend on energy, that is, if all the energy dependence came from the phase-space factor $|\mathbf{k}|/E$. To test Quantum Electrodynamics, an experiment must be able to resolve deviations from the naive phase-space prediction. Experimental results from pair production of both μ and τ leptons confirm that these particles behave as QED predicts. Figure 5.2 compares formula (5.13) to experimental measurements of the $\tau^+\tau^-$ threshold.

Before discussing our result further, let us pause to summarize how we obtained it. The method extends in a straightforward way to the calculation of unpolarized cross sections for other QED

processes. The general procedure is as follows:

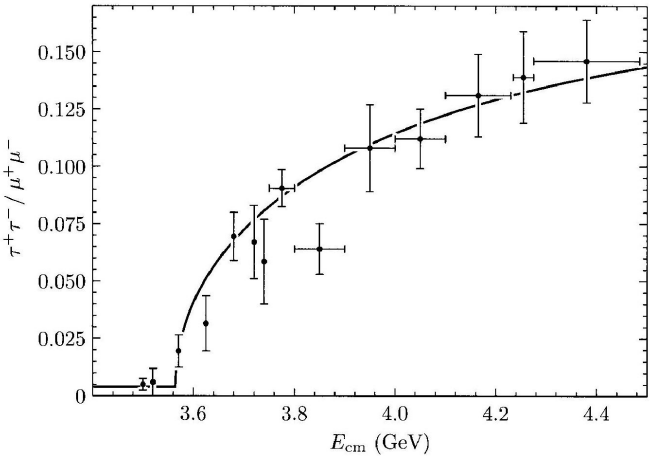


Figure 5.2. The ratio $\sigma\left(e^+e^- \to \tau^+\tau^-\right)/\sigma\left(e^+e^- \to \mu^+\mu^-\right)$of measured cross sections near the threshold for $\tau^+\tau^-$pair-production, as measured by the DELCO collaboration, W. Bacino, et. al., Phys. Rev. Lett. 41, 13 (1978). Only a fraction of τ decays are included, hence the small overall scale. The curve shows a fit to the theoretical formula (5.13), with a small energy-independent background added. The fit yields $m_\tau = 1782^{+2}_{-7}$MeV.

1. Draw the diagram(s) for the desired process.

2. Use the Feynman rules to write down the amplitude $\mathcal{M}$.

3. Square the amplitude and average or sum over spins, using the completeness relations (5.3). (For processes involving photons in the final state there is an analogous completeness relation, derived in Section 5.5.)

4. Evaluate traces using the trace theorems (5.5); collect terms and simplify the answer as much as possible.

5. Specialize to a particular frame of reference, and draw a picture of the kinematic variables in that frame. Express all 4-momentum vectors in terms of a suitably chosen set of variables such as E and θ.

6. Plug the resulting expression for $|\mathcal{M}|^2$ into the cross-section formula (4.79), and integrate over phase-space variables that are not measured to obtain a differential cross section in the desired form. (In our case these integrations were over the constrained momenta $\mathbf{k}'$ and $|\mathbf{k}|$, and were performed in the derivation of Eq. (4.84).)
   While other calculations (especially those involving loop diagrams) often require additional tricks, nearly every QED calculation will involve the basic procedures outlined here.

Production of Quark-Antiquark Pairs

The asymptotic energy dependence of the $e^+e^- \to \mu^+\mu^-$cross-section formula sets the scale for all e^+e^-annihilation cross sections. A particularly important example is the cross section for

$$e^+e^- \to \text{ hadrons },$$

that is, the total cross section for production of any number of strongly interacting particles.

In our current understanding of the strong interactions, given by the theory called Quantum Chromodynamics (QCD), all hadrons are composed of Dirac fermions called quarks. Quarks appear in a variety of types, called flavors, each with its own mass and electric charge. A quark also carries an additional quantum number, color, which takes one of three values. Color serves as the "charge" of QCD, as we will discuss in Chapter 17.

According to QCD , the simplest e^+e^-process that ends in hadrons is

$$e^+e^- \to q\bar{q}$$

the annihilation of an electron and a positron, through a virtual photon, into a quark-antiquark pair. After they are created, the quarks interact with one another through their strong forces, producing more quark pairs. Eventually the quarks and antiquarks combine to form some number of mesons and baryons.

To adapt our results for muon production to handle the case of quarks, we must make three modifications:

1. Replace the muon charge e with the quark charge $Q|e|$.

2. Count each quark three times, one for each color.

3. Include the effects of the strong interactions of the produced quark and antiquark.
   The first two changes are easy to make. For the first, it is simply necessary to know the masses and charges of each flavor of quark. For u, c, and t quarks we have $Q = 2/3$, while for d, s, and b quarks we have $Q = -1/3$. The crosssection formulae are proportional to the square of the charge of the final-state particle, so we can simply insert a factor of Q^2 into any of these formulae to obtain the cross section for production of any particular variety of quark. Counting colors is necessary because experiments measure only the total cross section for production of all three colors. (The hadrons that are actually detected are colorless.) In any case, this counting is easy: Just multiply the answer by 3 .

If you know a little about the strong interaction, however, you might think this is all a big joke. Surely the third modification is extremely difficult to make, and will drastically alter the predictions of QED. The amazing truth is that in the high-energy limit, the effect of the strong interaction on the quark production process can be completely neglected. As we will discuss in Part III, the only effect of the strong interaction (in this limit) is to dress
up the final-state quarks into bunches of hadrons. This simplification is due to a phenomenon called asymptotic freedom; it played a crucial role in the identification of Quantum Chromodynamics as the correct theory of the strong force.

Thus in the high-energy limit, we expect the cross section for the reaction $e^+e^- \to q\bar{q}$ to approach $3 \cdot Q^2 \cdot 4\pi\alpha^2/3E_{\rm cm}^2$. It is conventional to define

$$1 \text{ unit of R} \equiv \frac{4\pi\alpha^2}{3E_{\rm cm}^2} = \frac{86.8\text{nbarns}}{(E_{\rm cm} \text{ in GeV})^2} \tag{5.15}$$

The value of a cross section in units of R is therefore its ratio to the asymptotic value of the $e^+e^- \to \mu^+\mu^-$cross section predicted by Eq. (5.14). Experimentally, the easiest quantity to measure is the total rate for production of all hadrons. Asymptotically, we expect

$$\sigma\left(e^+e^- \to \text{ hadrons }\right) \underset{E_{\rm cm}\to\infty}{\longrightarrow} 3 \cdot \left(\sum_i Q_i^2\right) \text{R} \tag{5.16}$$

where the sum runs over all quarks whose masses are smaller than $E_{\rm cm}/2$. When $E_{\rm cm}/2$ is in the vicinity of one of the quark masses, the strong interactions cause large deviations from this formula. The most dramatic such effect is the appearance of bound states just below $E_{\rm cm} = 2m_q$, manifested as very sharp spikes in the cross section.

Experimental measurements of the cross section for e^+e^-annihilation to hadrons between 2.5 and 40 GeV are shown in Fig. 5.3. The data shows three distinct regions: a low-energy region in which u, d, and s quark pairs are produced; a region above the threshold for production of c quark pairs; and a region also above the threshold for b quark pairs. The prediction (5.16) is shown as a set of solid lines; it agrees quite well with the data in each region, as long as the energy is well away from the thresholds where the high-energy approximation breaks down. The dotted curves show an improved theoretical prediction, including higher-order corrections from QCD, which we will discuss in Section 17.2. This explanation of the e^+e^-annihilation cross section is a remarkable success of QCD. In particular, experimental verification of the factor of 3 in (5.16) is one piece of evidence for the existence of color.

The angular dependence of the differential cross section is also observed experimentally.* At high energy the hadrons appear in jets, clusters of several hadrons all moving in approximately the same direction. In most cases there are two jets, with back-to-back momenta, and these indeed have the angular dependence $(1 + \cos^2\theta)$.

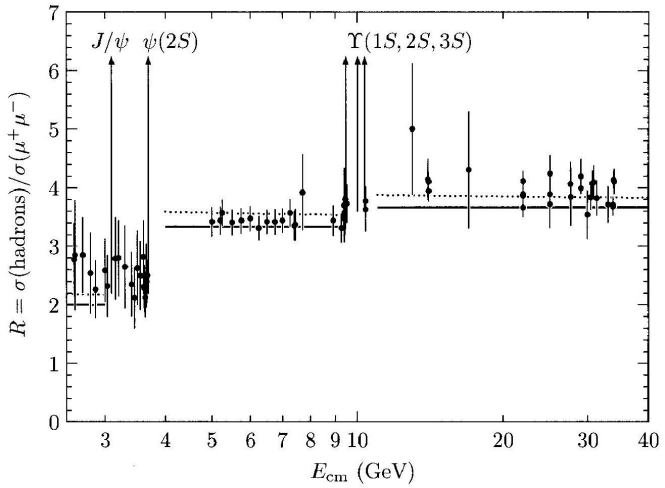


Figure 5.3. Experimental measurements of the total cross section for the reaction $e^+e^- \to$ hadrons, from the data compilation of M. Swartz, Phys. Rev. D53, 5268 (1996). Complete references to the various experiments are given there. The measurements are compared to theoretical predictions from Quantum Chromodynamics, as explained in the text. The solid line is the simple prediction (5.16).

21.2 5.2$e^+e^- \to \mu^+\mu^-$:Helicity Structure

The unpolarized cross section for a reaction is generally easy to calculate (and to measure) but hard to understand. Where does the $(1 + \cos^2\theta)$ angular dependence come from? We can answer this question by computing the $e^+e^- \to \mu^+\mu^-$cross section for each set of spin orientations separately.

First we must choose a basis of polarization states. To get a simple answer in the high-energy

[6]*The basic features of hadron production in high-energy e^+e^-annihilation are reviewed by P. Duinker, Rev. Mod. Phys. 54, 325 (1982).

limit, the best choice is to quantize each spin along the direction of the particle's motion, that is, to use states of definite helicity. Recall that in the massless limit, the left- and right-handed helicity states of a Dirac particle live in different representations of the Lorentz group. We might therefore expect them to behave independently, and in fact they do.

In this section we will compute the polarized $e^+e^- \to \mu^+\mu^-$cross sections, using the helicity basis, in two different ways: first, by using trace technology but with the addition of helicity projection operators to project out the desired left- or right-handed spinors; and second, by plugging explicit expressions for these spinors directly into our formula for the amplitude $\mathcal{M}$. Throughout this section we work in the high-energy limit where all fermions are effectively
massless. (The calculation can be done for lower energy, but it is much more difficult and no more instructive.) †

Our starting point for both methods of calculating the polarized cross section is the amplitude

$$i\mathcal{M}\left(e^-(p)e^+\left(p'\right) \to \mu^-(k)\mu^+\left(k'\right)\right) = \frac{ie^2}{q^2}\left(\bar{v}\left(p'\right)\gamma^\mu u(p)\right)\left(\bar{u}(k)\gamma_\mu v\left(k'\right)\right) \tag{5.1}$$

We would like to use the spin sum identities to write the squared amplitude in terms of traces as before, even though we now want to consider only one set of polarizations at a time. To do this, we note that for massless fermions, the matrices

$$\frac{1+\gamma^5}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \frac{1-\gamma^5}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.17}$$

are projection operators onto right- and left-handed spinors, respectively. Thus if in (5.1) we make the replacement

$$\bar{v}\left(p'\right)\gamma^\mu u(p) \longrightarrow \bar{v}\left(p'\right)\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u(p)$$

the amplitude for a right-handed electron is unchanged while that for a lefthanded electron becomes zero. Note that since

$$\bar{v}\left(p'\right)\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u(p) = v^\dagger\left(p'\right)\left(\frac{1+\gamma^5}{2}\right)\gamma^0\gamma^\mu u(p) \tag{5.18}$$

this same replacement imposes the requirement that $v\left(p'\right)$ also be a righthanded spinor. Recall from Section 3.5, however, that the right-handed spinor $v\left(p'\right)$ corresponds to a left-handed positron. Thus we see that the annihilation amplitude vanishes when both the electron and the positron are right-handed. In general, the amplitude vanishes (in the massless limit) unless the electron and positron have opposite helicity, or equivalently, unless their spinors have the same helicity.

Having inserted this projection operator, we are now free to sum over the electron and positron spins in the squared amplitude; of the four terms in the sum, only one (the one we want) is nonzero. The electron half of $|\mathcal{M}|^2$, for a right-handed electron and a left-handed positron, is then

$$\begin{aligned}\sum_{\text{spins}}\left|\bar{v}\left(p'\right)\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u(p)\right|^2 &= \sum_{\text{spins}}\bar{v}\left(p'\right)\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)u(p)\bar{u}(p)\gamma^\nu\left(\frac{1+\gamma^5}{2}\right)v\left(p'\right) \\ &= \operatorname{tr}\left[\not{p}'\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)\not{p}\gamma^\nu\left(\frac{1+\gamma^5}{2}\right)\right] \\ &= \operatorname{tr}\left[\not{p}''\gamma^\mu\not{p}\gamma^\nu\left(\frac{1+\gamma^5}{2}\right)\right]\end{aligned}$$

† The general formalism for S-matrix elements between states of definite helicity is presented in a beautiful paper of M. Jacob and G. C. Wick, Ann. Phys. 7, 404 (1959).

$$= 2\left(p'^{\mu}p^{\nu} + p'^{\nu}p^{\mu} - g^{\mu\nu}p\cdot p' - i\epsilon^{\alpha\mu\beta\nu}p'_{\alpha}p_{\beta}\right) \tag{5.19}$$

The indices in this expression are to be dotted into those of the muon half of the squared amplitude. For a right-handed $\mu^-$and a left-handed μ^+, an identical calculation yields

$$\sum_{\text{spins}} \left|\bar{u}(k)\gamma_{\mu}\left(\frac{1+\gamma^5}{2}\right)v\left(k'\right)\right|^2 = 2\left(k_{\mu}k'_{\nu} + k_{\nu}k'_{\mu} - g_{\mu\nu}k\cdot k' - i\epsilon_{\rho\mu\sigma\nu}k^{\rho}k'^{\sigma}\right) \tag{5.20}$$

Dotting (5.19) into (5.20), we find that the squared matrix element for $e^-_R e^+_L \to \mu^-_R\mu^+_L$in the center-of-mass frame is

$$\begin{aligned}
|\mathcal{M}|^2 &= \frac{4e^4}{q^4}\left[2(p\cdot k)\left(p'\cdot k'\right) + 2\left(p\cdot k'\right)\left(p'\cdot k\right) - \epsilon^{\alpha\mu\beta\nu}\epsilon_{\rho\mu\sigma\nu}p'_{\alpha}p_{\beta}k^{\rho}k'^{\sigma}\right] \\
&= \frac{8e^4}{q^4}\left[(p\cdot k)\left(p'\cdot k'\right) + \left(p\cdot k'\right)\left(p'\cdot k\right) - (p\cdot k)\left(p'\cdot k'\right) + \left(p\cdot k'\right)\left(p'\cdot k\right)\right] \\
&= \frac{16e^4}{q^4}\left(p\cdot k'\right)\left(p'\cdot k\right) \\
&= e^4(1+\cos\theta)^2
\end{aligned} \tag{5.21}$$

Plugging this result into (4.85) gives the differential cross section,

$$\frac{d\sigma}{d\Omega}\left(e^-_R e^+_L \to \mu^-_R\mu^+_L\right) = \frac{\alpha^2}{4E^2_{\text{cm}}}(1+\cos\theta)^2 \tag{5.22}$$

There is no need to repeat the entire calculation to obtain the other three nonvanishing helicity amplitudes. For example, the squared amplitude for $e^-_R e^+_L \to \mu^-_L\mu^+_R$is identical to (5.20) but with γ^5 replaced by $-\gamma^5$ on the left-hand side, and thus $\epsilon_{\rho\mu\sigma\nu}$ replaced by $-\epsilon_{\rho\mu\sigma\nu}$ on the right-hand side. Propagating this sign though (5.21), we easily see that

$$\frac{d\sigma}{d\Omega}\left(e^-_R e^+_L \to \mu^-_L\mu^+_R\right) = \frac{\alpha^2}{4E^2_{\text{cm}}}(1-\cos\theta)^2 \tag{5.23}$$

Similarly,

$$\begin{aligned}
\frac{d\sigma}{d\Omega}\left(e^-_L e^+_R \to \mu^-_R\mu^+_L\right) &= \frac{\alpha^2}{4E^2_{\text{cm}}}(1-\cos\theta)^2 \\
\frac{d\sigma}{d\Omega}\left(e^-_L e^+_R \to \mu^-_L\mu^+_R\right) &= \frac{\alpha^2}{4E^2_{\text{cm}}}(1+\cos\theta)^2
\end{aligned} \tag{5.24}$$

(These two results actually follow from the previous two by parity invariance.) The other twelve helicity cross sections (for instance, $e^-_L e^+_R \to \mu^-_L\mu^+_L$) are zero, as we saw from Eq. (5.18). Adding up all sixteen contributions, and dividing by 4 to average over the electron and positron spins, we

recover the unpolarized cross section in the massless limit, Eq. (5.14).

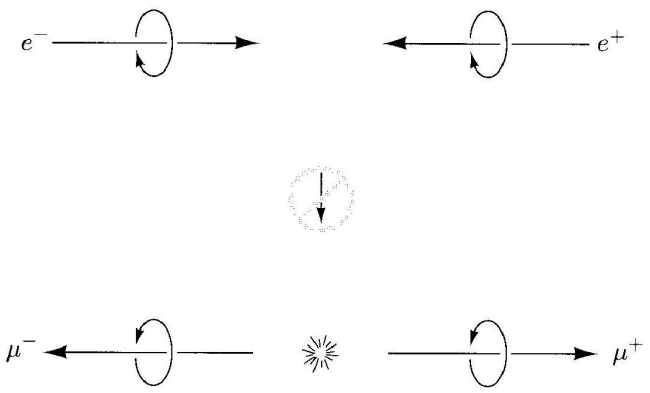


Figure 5.4. Conservation of angular momentum requires that if the z component of angular momentum is measured, it must have the same value as initially.

Note that the cross section (5.22) for $e_R^- e_L^+ \to \mu_R^- \mu_L^+$vanishes at $\theta = 180°$. This is just what we would expect, since for $\theta = 180°$, the total angular momentum of the final state is opposite to that of the initial state (see Figure 5.4).

This completes our first calculation of the polarized $e^+e^- \to \mu^+\mu^-$cross sections. We will now redo the calculation in a manner that is more straightforward, more enlightening, and no more difficult. We will calculate the amplitude $\mathcal{M}$ (rather than the squared amplitude) directly, using explicit values for the spinors and γ matrices. This method does have its drawbacks: It forces us to specialize to a particular frame of reference much sooner, so manifest Lorentz invariance is lost. More pragmatically, it is very cumbersome except in the nonrelativistic and ultra-relativistic limits.

Consider again the amplitude

$$\mathcal{M} = \frac{e^2}{q^2} \left(\bar{v}\left(p'\right) \gamma^\mu u(p)\right) \left(\bar{u}(k) \gamma_\mu v\left(k'\right)\right) \tag{5.25}$$

In the high-energy limit, our general expressions for Dirac spinors become

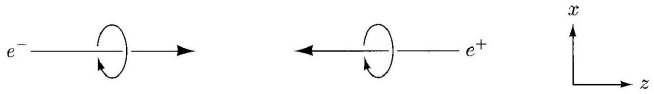
$$\begin{aligned} u(p) &= \begin{pmatrix} \sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \bar{\sigma}}\xi \end{pmatrix} \underset{E\to\infty}{\longrightarrow} \sqrt{2E} \begin{pmatrix} \frac{1}{2}(1 - \hat{p} \cdot \sigma)\xi \\ \frac{1}{2}(1 + \hat{p} \cdot \sigma)\xi \end{pmatrix} \\ v(p) &= \begin{pmatrix} \sqrt{p \cdot \sigma}\xi \\ -\sqrt{p \cdot \bar{\sigma}}\xi \end{pmatrix} \underset{E\to\infty}{\longrightarrow} \sqrt{2E} \begin{pmatrix} \frac{1}{2}(1 - \hat{p} \cdot \sigma)\xi \\ -\frac{1}{2}(1 + \hat{p} \cdot \sigma)\xi \end{pmatrix} \end{aligned} \tag{5.26}$$

A right-handed spinor satisfies $(\hat{p} \cdot \boldsymbol{\sigma})\xi = +\xi$, while a left-handed spinor has $(\hat{p} \cdot \boldsymbol{\sigma})\xi = -\xi$. (Remember once again that for antiparticles, the handedness of the spinor is the opposite of the handedness of the particle.) We must evaluate expressions of the form $\bar{v}\gamma^\mu u$, so we need

$$\gamma^0\gamma^\mu = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} = \begin{pmatrix} \bar{\sigma}^\mu & 0 \\ 0 & \sigma^\mu \end{pmatrix} \tag{5.27}$$

Thus we see explicitly that the amplitude is zero when one of the spinors is left-handed and the other is right-handed. In the language of Chapter 1, the Clebsch-Gordan coefficients that couple the vector photon to the product of such spinors are zero; those coefficients are just the off-block-diagonal elements of the matrix $\gamma^0\gamma^\mu$ (in the chiral representation).

Let us choose p and p' to be in the $\pm z$-directions, and first consider the case where the electron is right-handed and the positron is left-handed:



Thus for the electron we have $\xi = \binom{1}{0}$, corresponding to spin up in the z direction, while for the positron we have $\xi = \binom{0}{1}$, also corresponding to (physical) spin up in the z-directon. Both particles have $(\hat{p}\cdot\sigma)\xi = +\xi$, so the spinors are

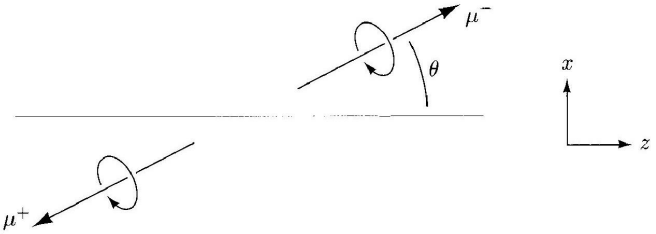
$$u(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \qquad v\,(p') = \sqrt{2E}\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \tag{5.28}$$

The electron half of the matrix element is therefore

$$\bar{v}\,(p')\,\gamma^\mu u(p) = 2E(0,-1)\sigma^\mu\begin{pmatrix}1\\0\end{pmatrix} = -2E(0,1,i,0) \tag{5.29}$$

We can interpret this expression by saying that the virtual photon has circular polarization in the $+z$-direction; its polarization vector is $\epsilon_+ = (1/\sqrt{2})(\hat{x} + i\hat{y})$.

Next we must calculate the muon half of the matrix element. Let the μ^- be emitted at an angle θ to the z-axis, and consider first the case where it is right-handed (and the $\mu^+$is therefore left-handed):



To calculate $\bar{u}(k)\gamma^\mu v\,(k')$ we could go back to expressions (5.26), but then it would be necessary to find the correct spinors ξ corresponding to polarization along the muon momentum. It is much easier to use a trick: Since any expression of the form $\bar{\psi}\gamma^\mu\psi$ transforms like a 4 -vector, we can just rotate the result
(5.29). Rotating that vector by an angle θ in the xz-plane, we find

$$\begin{aligned}\bar{u}(k)\gamma^\mu v\,(k') &= [\bar{v}\,(k')\,\gamma^\mu u(k)]^* \\ &= [-2E(0,\cos\theta,i,-\sin\theta)]^* \\ &= -2E(0,\cos\theta,-i,-\sin\theta).\end{aligned} \tag{5.30}$$

This vector can also be interpreted as the polarization of the virtual photon; when it has a nonzero overlap with (5.29), we get a nonzero amplitude. Plugging (5.29) and (5.30) into (5.25), we see that the amplitude is

$$\mathcal{M}\left(e_R^- e_L^+ \to \mu_R^- \mu_L^+\right) = \frac{e^2}{q^2}(2E)^2(-\cos\theta - 1) = -e^2(1+\cos\theta), \tag{5.31}$$

in agreement with (1.6), and also with (5.21). The differential cross section for this set of helicities can now be obtained in the same way as above, yielding (5.22).

We can calculate the other three nonvanishing helicity amplitudes in an analogous manner. For a left-handed electron and a right-handed positron, we easily find

$$\bar{v}\left(p'\right)\gamma^\mu u(p) = -2E(0,1,-i,0) \equiv -2E\cdot\sqrt{2}\epsilon_-^\mu$$

Perform a rotation to get the vector corresponding to a left-handed $\mu^-$and a right-handed μ^+:

$$\bar{u}(k)\gamma^\mu v\left(k'\right) = -2E(0,\cos\theta,i,\sin\theta)$$

Putting the pieces together in various ways yields the remaining amplitudes,

$$\begin{aligned}\mathcal{M}\left(e_L^- e_R^+ \to \mu_L^- \mu_R^+\right) &= -e^2(1+\cos\theta)\\ \mathcal{M}\left(e_R^- e_L^+ \to \mu_L^- \mu_R^+\right) = \mathcal{M}\left(e_L^- e_R^+ \to \mu_R^- \mu_L^+\right) &= -e^2(1-\cos\theta)\end{aligned} \tag{5.32}$$

21.3 5.3$e^+e^- \to \mu^+\mu^-$: Nonrelativistic Limit

Now let us go to the other end of the energy spectrum, and discuss the reaction $e^+e^- \to \mu^+\mu^-$in the extreme nonrelativistic limit. When E is barely larger than m_μ, our previous result (5.12) for the unpolarized differential cross section becomes

$$\frac{d\sigma}{d\Omega} \underset{|\mathbf{k}|\to 0}{\longrightarrow} \frac{\alpha^2}{2E_{\rm cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}} = \frac{\alpha^2}{2E_{\rm cm}^2}\frac{|\mathbf{k}|}{E} \tag{5.33}$$

We can recover this result, and also learn something about the spin dependence of the reaction, by evaluating the amplitude with explicit spinors. Once again we begin with the matrix element

$$\mathcal{M} = \frac{e^2}{q^2}\left(\bar{v}\left(p'\right)\gamma^\mu u(p)\right)\left(\bar{u}(k)\gamma_\mu v\left(k'\right)\right)$$

Before:

After:

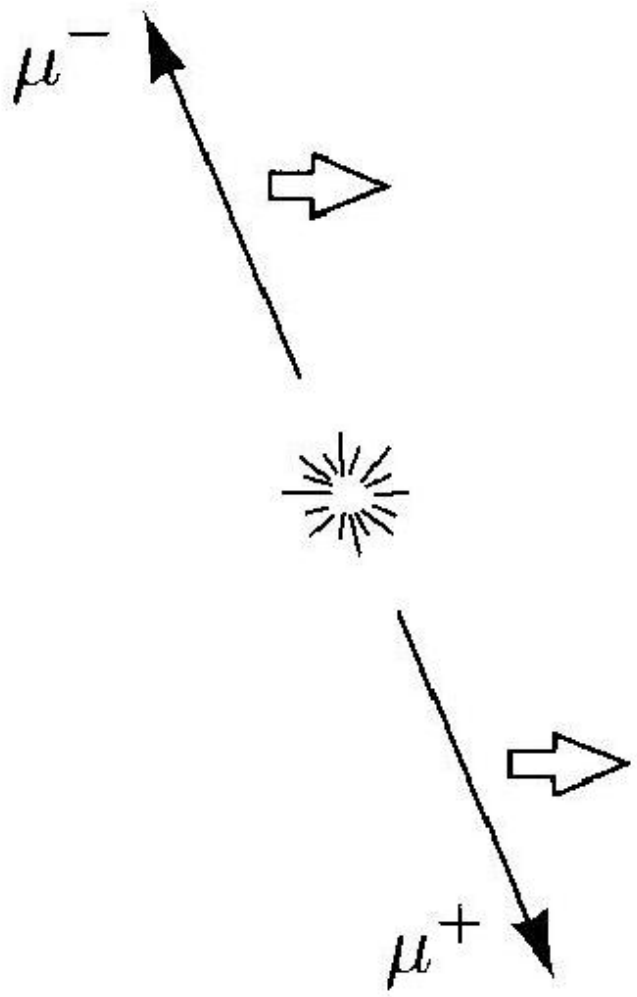


Figure 5.5. In the nonrelativistic limit the total spin of the system is conserved, and thus the muons are produced with both spins up along the z-axis.

The electron and positron are still very relativistic, so this expression will be simplest if we choose them to have definite helicity. Let the electron be righthanded, moving in the $+z$-direction, and the positron be left-handed, moving in the $-z$-direction. Then from Eq. (5.29) we have

$$\bar{v}\,(p')\,\gamma^\mu u(p) = -2E(0,1,i,0) \tag{5.34}$$

In the other half of the matrix element we should use the nonrelativistic expressions

$$u(k) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}, \quad v\,(k') = \sqrt{m}\begin{pmatrix}\xi'\\ -\xi'\end{pmatrix}. \tag{5.35}$$

Keep in mind, in the discussion of this section, that the spinor ξ' gives the flipped spin of the antiparticle. Leaving the muon spinors ξ and ξ' undetermined for now, we can easily compute

$$\begin{aligned}\bar{u}(k)\gamma^\mu v\,(k') &= m\,(\xi^\dagger, \xi^\dagger)\begin{pmatrix}\bar{\sigma}^\mu & 0\\ 0 & \sigma^\mu\end{pmatrix}\begin{pmatrix}\xi'\\ -\xi'\end{pmatrix}\\ &= \begin{cases}0 & \text{for } \mu = 0,\\ -2m\xi^\dagger\sigma^i\xi' & \text{for } \mu = i\end{cases}\end{aligned} \tag{5.36}$$

To evaluate $\mathcal{M}$, we simply dot (5.34) into (5.36) and multiply by $e^2/q^2 = e^2/4m^2$. The result is

$$\mathcal{M}\left(e_R^- e_L^+ \to \mu^+\mu^-\right) = -2e^2 \xi^\dagger \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \xi' \tag{5.37}$$

Since there is no angular dependence in this expression, the muons are equally likely to come out in any direction. More precisely, they are emitted in an s-wave; their orbital angular momentum is zero. Angular momentum conservation therefore requires that the total spin of the final state equal 1 , and indeed the matrix product gives zero unless both the muon and the antimuon have spin up along the z-axis (see Fig. 5.5).

To find the total rate for this process, we sum over muon spins to obtain $\mathcal{M}^2 = 4e^4$, which yields the cross section

$$\frac{d\sigma}{d\Omega}\left(e_R^- e_L^+ \to \mu^+\mu^-\right) = \frac{\alpha^2}{E_{\rm cm}^2}\frac{|\mathbf{k}|}{E} \tag{5.38}$$

The same expression holds for a left-handed electron and a right-handed positron. Thus the spin-averaged cross section is just $2 \cdot (1/4)$ times this expression, in agreement with (5.33).

Bound States

Until now we have considered the initial and final states of scattering processes to be states of isolated single particles. Very close to threshold, however, the Coulomb attraction of the muons should become an important effect. Just below threshold, we can still form $\mu^+\mu^-$pairs in electromagnetic bound states.

The treatment of bound states in quantum field theory is a rich and complex subject, but one that lies mainly beyond the scope of this book. ‡ Fortunately, many of the familiar bound systems in Nature can be treated (at least to a good first approximation) as nonrelativistic systems, in which the internal motions are slow. The process of creating the constituent particles out of the vacuum is still a relativistic effect, requiring quantum field theory for its proper description. In this section we will develop a formalism for computing the amplitudes for creation and annihilation of two-particle, nonrelativistic bound states. We begin with a computation of the cross section for producing a $\mu^+\mu^-$bound state in e^+e^-annihilation.

Consider first the case where the spins of the electron and positron both point up along the z-axis. From the preceding discussion we know that the resulting muons both have spin up, so the only type of bound state we can produce will have total spin 1, also pointing up. The amplitude for producing free muons in this configuration is

$$\mathcal{M}\left(\uparrow\uparrow \to \mathbf{k}_1 \uparrow, \mathbf{k}_2 \uparrow\right) = -2e^2 \tag{5.39}$$

independent of the momenta (which we now call $\mathbf{k}_1$ and $\mathbf{k}_2$) of the muons.
Next we need to know how to write a bound state in terms of free-particle states. For a general two-body system with equal constituent masses, the center-of-mass and relative coordinates are

$$\mathbf{R} = \frac{1}{2}\left(\mathbf{r}_1 + \mathbf{r}_2\right), \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{5.40}$$

These have conjugate momenta

$$\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2, \quad \mathbf{k} = \frac{1}{2}\left(\mathbf{k}_1 - \mathbf{k}_2\right) \tag{5.41}$$

The total momentum $\mathbf{K}$ is zero in the center-of-mass frame. If we know the force between the particles (for $\mu^+\mu^-$, it is just the Coulomb force), we can

solve the nonrelativistic Schrdinger equation to find the Schrdinger wavefunction, $\psi(\mathbf{r})$. The bound state is just a linear superposition of free states of definite $\mathbf{r}$ or $\mathbf{k}$, weighted by this wavefunction. For our purposes it is more convenient to build this superposition in momentum space, using the Fourier transform of $\psi(\mathbf{r})$:
$$

$$\widetilde{\psi}(\mathbf{k}) = \int d^3x e^{i\mathbf{k}\cdot\mathbf{r}}\psi(\mathbf{r}); \quad \int \frac{d^3k}{(2\pi)^3}|\widetilde{\psi}(\mathbf{k})|^2 = 1 \tag{5.42}$$

$$

If $\psi(\mathbf{r})$ is normalized conventionally, $\widetilde{\psi}(\mathbf{k})$ gives the amplitude for finding a particular value of $\mathbf{k}$. An explicit expression for a bound state with mass $M \approx 2m$, momentum $\mathbf{K} = 0$, and spin 1 oriented up is then

$$|B\rangle = \sqrt{2M}\int \frac{d^3k}{(2\pi)^3}\widetilde{\psi}(\mathbf{k})\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}|\mathbf{k}\uparrow, -\mathbf{k}\uparrow\rangle \tag{5.43}$$

The factors of $(1/\sqrt{2m})$ convert our relativistically normalized free-particle states so that their integral with $\widetilde{\psi}(\mathbf{k})$ is a state of norm 1. (The factors should involve $\sqrt{2E_{\pm\mathbf{k}}}$, but for a nonrelativistic bound state, $|\mathbf{k}| \ll m$.) The outside factor of $\sqrt{2M}$ converts back to the relativistic normalization assumed by our formula for cross sections. These normalization factors could easily be modified to describe a bound state with nonzero total momentum $\mathbf{K}$.

Given this expression for the bound state, we can immediately write down the amplitude for its production:

$$\mathcal{M}(\uparrow\uparrow\to B) = \sqrt{2M}\int \frac{d^3k}{(2\pi)^3}\widetilde{\psi}^*(\mathbf{k})\frac{1}{\sqrt{2m}}\frac{1}{\sqrt{2m}}\mathcal{M}(\uparrow\uparrow\to \mathbf{k}\uparrow, -\mathbf{k}\uparrow) \tag{5.44}$$

Since the free-state amplitude from (5.39) is independent of the momenta of the muons, the integral over $\mathbf{k}$ gives $\psi^*(0)$, the position-space wavefunction evaluated at the origin. It is quite natural that the amplitude for creation of a two-particle state from a pointlike virtual photon should be proportional to the value of the wavefunction at zero separation. Assembling the pieces, we find that the amplitude is simply

$$\mathcal{M}(\uparrow\uparrow\to B) = \sqrt{\frac{2}{M}}\left(-2e^2\right)\psi^*(0) \tag{5.45}$$

In a moment we will compute the cross section from this amplitude. First, however, let us generalize this discussion to treat bound states with more general spin configurations. The analysis leading up to (5.37) will cast any S matrix element for the production of nonrelativistic fermions with momenta $\mathbf{k}$ and $-\mathbf{k}$ into the form of a spin matrix element

$$i\mathcal{M}\left(\text{ something } \to \mathbf{k}, \mathbf{k}'\right) = \xi^\dagger[\Gamma(\mathbf{k})]\xi' \tag{5.46}$$

where $\Gamma(\mathbf{k})$ is some 2×2 matrix. We now must replace the spinors with a normalized spin wavefunction for the bound state. In the example just completed,
we replaced

[6]‡ Reviews of this subject can be found in Bodwin, Yennie, and Gregorio, Rev. Mod. Phys. 57, 723 (1985), and in Sapirstein and Yennie, in Kinoshita (1990).

$$\xi'\xi^\dagger \to \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} \tag{5.47}$$

More generally, a spin-1 state is obtained by the replacement

$$\xi'\xi^\dagger \to \frac{1}{\sqrt{2}}\mathbf{n}^*\cdot\boldsymbol{\sigma} \tag{5.48}$$

where $\mathbf{n}$ is a unit vector. Choosing $\mathbf{n} = (\hat{x}+i\hat{y})/\sqrt{2}$ gives back (5.47), while the choices $\mathbf{n} = (\hat{x}-i\hat{y})/\sqrt{2}$ and $\mathbf{n} = \hat{z}$ give the other two spin- 1 states $\downarrow\downarrow$ and $(\uparrow\downarrow+\downarrow\uparrow)/\sqrt{2}$. (The relative minus sign in (5.48) for this last case comes from the rule (3.135) for the flipped spin.) Similarly, the spin-zero state $(\uparrow\downarrow-\downarrow\uparrow)/\sqrt{2}$ is given by the replacement

$$\xi'\xi^\dagger \to \frac{1}{\sqrt{2}}\mathbf{1} \tag{5.49}$$

involving the 2×2 unit matrix. With these rules, we can convert an S-matrix element of the form (5.46) quite generally into an S-matrix element for production of a bound state at rest:

$$i\mathcal{M}(\text{ something } \to B) = \sqrt{\frac{2}{M}}\int\frac{d^3k}{(2\pi)^3}\widetilde{\psi}^*(\mathbf{k})\,\mathrm{tr}\left(\frac{\mathbf{n}^*\cdot\boldsymbol{\sigma}}{\sqrt{2}}\Gamma(\mathbf{k})\right) \tag{5.50}$$

where the trace is taken over 2-component spinor indices. For a spin-0 bound state, replace $\mathbf{n}\cdot\boldsymbol{\sigma}$ by the unit matrix.

Vector Meson Production and Decay

Equation (5.45) can be straightforwardly converted into a cross section for production of $\mu^+\mu^-$bound states in e^+e^-annihilation. To make it easier to extract all the physics in this equation, let us introduce polarization vectors for the initial and final spin configurations: $\epsilon_+ = (\hat{x}+i\hat{y})/\sqrt{2}$, from Eq. (5.29), and $\mathbf{n}$, from Eq. (5.48). Then (5.45) can be rewritten in a more invariant form as

$$\mathcal{M}\left(e_R^-e_L^+ \to B\right) = \sqrt{\frac{2}{M}}\left(-2e^2\right)\left(\mathbf{n}^*\cdot\boldsymbol{\epsilon}_+\right)\psi^*(0) \tag{5.51}$$

The bound state spin polarization $\mathbf{n}$ is projected parallel to $\boldsymbol{\epsilon}_+$. Note that if the electrons are initially unpolarized, the cross section for production of B will involve the polarization average

$$\frac{1}{4}\left(|\mathbf{n}^*\cdot\boldsymbol{\epsilon}_+|^2 + |\mathbf{n}^*\cdot\boldsymbol{\epsilon}_-|^2\right) = \frac{1}{4}\left((n^x)^2+(n^y)^2\right) \tag{5.52}$$

Thus, the bound states produced will still be preferentially polarized along the e^+e^-collision axis.

Assuming an unpolarized electron beam, and summing (5.52) over the three possible directions of $\mathbf{n}$, we find the following expression for the total cross section for production of the bound state: $\sigma\left(e^+e^-\to B\right) = \frac{1}{2}\frac{1}{2m}\frac{1}{2m}\int\frac{d^3K}{(2\pi)^3}\frac{1}{2E_{\mathbf{K}}}(2\pi)^4\delta^{(4)}\left(p+p'-K\right)\cdot\frac{2}{M}\left(4e^4\right)\frac{1}{2}|\psi(0)|^2$.
Notice that the 1-body phase space integral can remove only three of the four delta functions. It is conventional to rewrite the last delta function using $\delta\left(P^0-K^0\right) = 2K^0\delta\left(P^2-K^2\right)$. Then

$$\sigma\left(e^+e^-\to B\right) = 64\pi^3\alpha^2\frac{|\psi(0)|^2}{M^3}\delta\left(E_{\mathrm{cm}}^2-M^2\right) \tag{5.54}$$

The last delta function enforces the constraint that the total center-of-mass energy must equal the bound-state mass; thus, the bound state is produced as a resonance in e^+e^-annihilation. If the bound state has a finite lifetime, this delta function will be broadened into a resonance peak. In practice, the intrinsic spread of the e^+e^-beam energy is often a more important broadening mechanism. In either case, (5.54) correctly predicts the area under the resonance peak.

If the bound state B can be produced from e^+e^-, it can also annihilate back to e^+e^-, or to any other sufficiently light lepton pair. According to (4.86), the total width for this decay mode is given by

$$\Gamma\left(B \to e^+e^-\right) = \frac{1}{2M}\int d\Pi_2 |\mathcal{M}|^2 \tag{5.55}$$

where $\mathcal{M}$ is just the complex conjugate of the matrix element (5.51) we used to compute B production. Thus

$$\Gamma = \frac{1}{2M}\int \left(\frac{1}{2\pi}\frac{d\cos\theta}{2}\right)\frac{8e^4}{M}|\psi(0)|^2\left(|\mathbf{n}\cdot\boldsymbol{\epsilon}|^2 + |\mathbf{n}\cdot\boldsymbol{\epsilon}^*|^2\right) \tag{5.56}$$

Now we must sum over electron polarization states and average over the three possible values of $\mathbf{n}$. We thus obtain

$$\Gamma\left(B \to e^+e^-\right) = \frac{16\pi\alpha^2}{3}\frac{|\psi(0)|^2}{M^2} \tag{5.57}$$

The formula for the decay width of B is very similar to that for the production cross section, and this is no surprise: Both calculations involve the square of the same matrix element, summed over initial and final polarizations. The two calculations differed only in how we formed the polarization averages, and in the phase-space factors. By this logic, the relation we have found between the two quantities,

$$\sigma\left(e^+e^- \to B\right) = 4\pi^2 \cdot \frac{3\Gamma\left(B \to e^+e^-\right)}{M} \cdot \delta\left(E_{\rm cm}^2 - M^2\right) \tag{5.58}$$

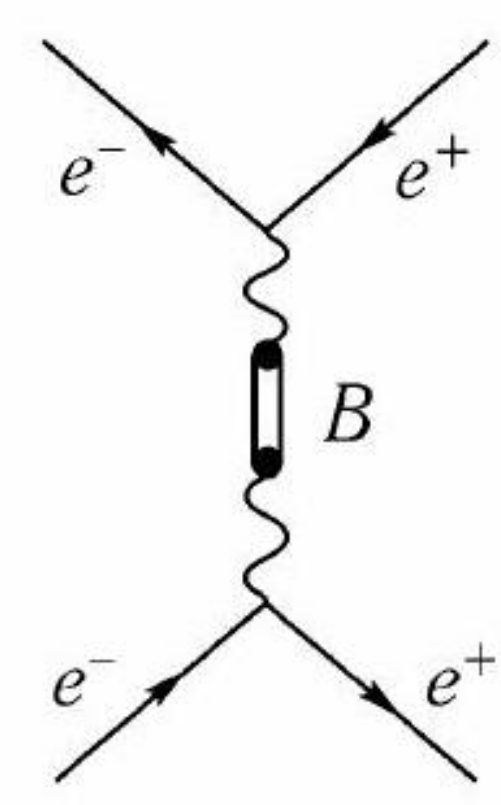
is very general and completely independent of the details of the matrix element computation. The factor 3 in (5.58) came from the orientation average for $\mathbf{n}$; for a spin- J bound state, this factor would be $(2J+1)$.

The most famous application of this formalism is to bound states not of muons but of quarks: quarkonium. We saw the experimental evidence for $q\bar{q}$ bound states (the J/ψ and Υ, for example) in Fig. 5.3. (The resonance peaks are much too high and too narrow to show in the figure, but their sizes have been carefully measured.) Equations (5.54) and (5.57) must be multiplied by a color factor of 3 to give the production cross section and decay width for a spin-1 $q\bar{q}$ bound state. The value $\psi(0)$ of the $q\bar{q}$ wavefunction at the origin cannot be computed from first principles, but can be estimated from a nonrelativistic model of the $q\bar{q}$ spectrum with a phenomenologically chosen potential. Alternatively, we can use the formula

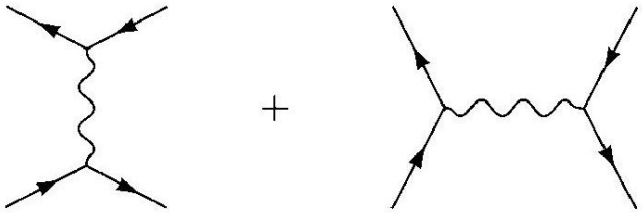
$$\Gamma\left(B(q\bar{q}) \to e^+e^-\right) = 16\pi\alpha^2 Q^2\frac{|\psi(0)|^2}{M^2} \tag{5.59}$$

to measure $\psi(0)$ for a $q\bar{q}$ bound state. For example, the $1S$ spin- 1 state of $s\bar{s}$, the ϕ meson, has an e^+e^-partial width of 1.4 keV and a mass of 1.02 GeV . From this we can infer $|\psi(0)|^2 = (1.2\text{fm})^{-3}$. This result is physically reasonable, since hadronic dimensions are typically $\sim$ 1fm.

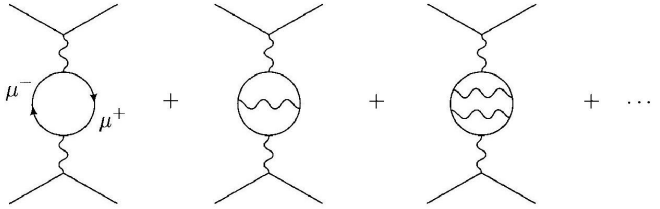
Our viewpoint in this section has been quite different from that of earlier sections: Instead of computing everything from first principles, we have pieced together an approximate formula using a bit of quantum field theory and a bit of nonrelativistic quantum mechanics. In principle, however, we could treat bound states entirely in the relativistic formalism. Consider the annihilation of an e^+e^-pair to form a $\mu^+\mu^-$bound state, which subsequently decays back into e^+e^-. In our present formalism we might represent this process by the diagram



The net process is simply $e^+e^- \to e^+e^-$(Bhabha scattering). What would happen if we tried to compute the Bhabha scattering cross section directly in QED perturbation theory? Obviously there is no $\mu^+\mu^-$contribution in the tree-level diagrams:



As we go to higher orders in the perturbation series, however, we find (among others) the following set of diagrams:

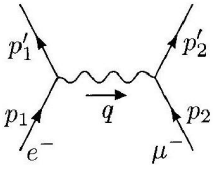


At most values of $E_{\rm cm}$, these diagrams give only a small correction to the tree-level expression. But when $E_{\rm cm}$ is near the $\mu^+\mu^-$threshold, the diagrams involving the exchange of photons within the muon loop contain the Coulomb interaction between the muons, and therefore become quite large. One must sum over all such diagrams, and it can be shown that this summation is equivalent to solving the nonrelativistic Schrdinger equation.* The final prediction is that the cross section contains a resonance peak, whose area is given by (5.54) and whose width is given by (5.57).

21.4 5.4 Crossing Symmetry

Electron-Muon Scattering

Now that we have completed our discussion of the process $e^+e^- \to \mu^+\mu^-$, let us consider a different but closely related QED process: electron-muon scattering, or $e^-\mu^- \to e^-\mu^-$. The lowest-order Feynman diagram is just the previous one turned on its side:



$$= \frac{ie^2}{q^2}\,\bar{u}(p_1')\gamma^\mu u(p_1)\,\bar{u}(p_2')\gamma_\mu u(p_2).$$

The relation between the processes $e^+e^- \to \mu^+\mu^-$and $e^-\mu^- \to e^-\mu^-$becomes clear when we compute the squared amplitude, averaged and summed over spins:
$\frac{1}{4}\sum_{\rm spins}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\,\mathrm{tr}\left[(\not{p}\mathrm{l}_1 + m_e)\,\gamma^\mu\,(\not{p}_1 + m_e)\,\gamma^\nu\right]\mathrm{tr}\left[(\not{p}\mathrm{l}_2' + m_\mu)\,\gamma_\mu\,(\not{p}_2 + m_\mu)\,\gamma_\nu\right]$.
This is exactly the same as our result (5.4) for $e^+e^- \to \mu^+\mu^-$, with the replacements

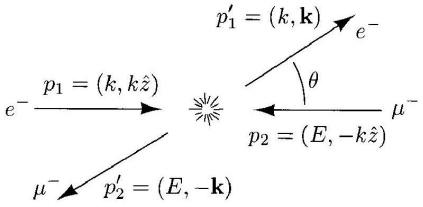
$$p \to p_1, \quad p' \to -p_1', \quad k \to p_2', \quad k' \to -p_2 \tag{5.60}$$

*This analysis is carried out in Berestetskii, Lifshitz, and Pitaevskii (1982).

So instead of evaluating the traces from scratch, we can just make the same replacements in our previous result, Eq. (5.10). Setting $m_e = 0$, we find

$$\frac{1}{4}\sum_{\rm spins}|\mathcal{M}|^2 = \frac{8e^4}{q^4}\left[(p_1\cdot p_2')\,(p_1'\cdot p_2) + (p_1\cdot p_2)\,(p_1'\cdot p_2') - m_\mu^2\,(p_1\cdot p_1')\right] \tag{5.61}$$

To evaluate this expression, we must work out the kinematics, which will be completely different. Working in the center-of-mass frame, we make the following assignments:



$$E^2 = k^2 + m_\mu^2$$
$$\mathbf{k}\cdot\hat{z} = k\cos\theta$$
$$E + k = E_{\rm cm}$$

The combinations we need are

$$p_1\cdot p_2 = p_1'\cdot p_2' = k(E+k); \quad p_1'\cdot p_2 = p_1\cdot p_2' = k(E + k\cos\theta)$$
$$p_1\cdot p_1' = k^2(1-\cos\theta); \quad q^2 = -2p_1\cdot p_1' = -2k^2(1-\cos\theta)$$

Our expression for the squared matrix element now becomes

$$\frac{1}{4}\sum_{\rm spins}|\mathcal{M}|^2 = \frac{2e^4}{k^2(1-\cos\theta)^2}\left((E+k)^2 + (E+k\cos\theta)^2 - m_\mu^2(1-\cos\theta)\right) \tag{5.62}$$

To find the cross section from this expression, we use Eq. (4.84), which in the case where one particle is massless takes the simple form

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{CM}} = \frac{|\mathcal{M}|^2}{64\pi^2(E+k)^2} \tag{5.63}$$

Thus we have our result for unpolarized electron-muon scattering in the center-of-mass frame:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2k^2(E+k)^2(1-\cos\theta)^2}\left((E+k)^2 + (E+k\cos\theta)^2 - m_\mu^2(1-\cos\theta)\right) \tag{5.64}$$

where $k = \sqrt{E^2 - m_\mu^2}$. In the high-energy limit where we can set $m_\mu = 0$, the differential cross section becomes

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2E_{\text{cm}}^2(1-\cos\theta)^2}\left(4 + (1+\cos\theta)^2\right) \tag{5.65}$$

Note the singular behavior

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{\theta^4} \quad \text{as } \theta \to 0 \tag{5.66}$$

of formulae (5.64) and (5.65). This singularity is the same as in the Rutherford formula (Problem 4.4). Such behavior is always present in Coulomb scattering; it arises from the nearly on-shell (that is, $q^2 \approx 0$) virtual photon.

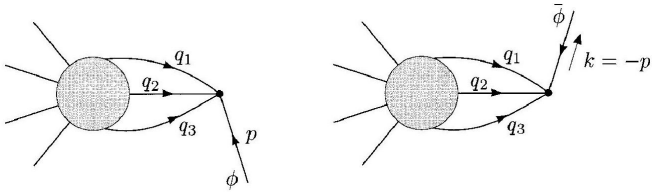
Crossing Symmetry

The trick we made use of here, namely the relation between the two processes $e^+e^- \to \mu^+\mu^-$and $e^-\mu^- \to e^-\mu^-$, is our first example of a type of relation known as crossing symmetry. In general, the S-matrix for any process involving a particle with momentum p in the initial state is equal to the S-matrix for an otherwise identical process but with an antiparticle of momentum $k = -p$ in the final state. That is,

$$\mathcal{M}(\phi(p) + \cdots \to \cdots) = \mathcal{M}(\cdots \to \cdots + \bar{\phi}(k)) \tag{5.67}$$

where $\bar{\phi}$ is the antiparticle of ϕ and $k = -p$. (Note that there is no value of p for which p and k are both physically allowed, since the particle must have $p^0 > 0$ and the antiparticle must have $k^0 > 0$. So technically, we should say that either amplitude can be obtained from the other by analytic continuation.)

Relation (5.67) follows directly from the Feynman rules. The diagrams that contribute to the two amplitudes fall into a natural one-to-one correspondence, where corresponding diagrams differ only by changing the incoming ϕ into the outgoing $\bar{\phi}$. A typical pair of diagrams looks like this:



In the first diagram, the momenta q_i coming into the vertex from the rest of the diagram must add up to $-p$, while in the second diagram they must add up to k. Thus the two diagrams are equal, except for any possible difference in the external leg factors, if $p = -k$. If ϕ is a spin-zero

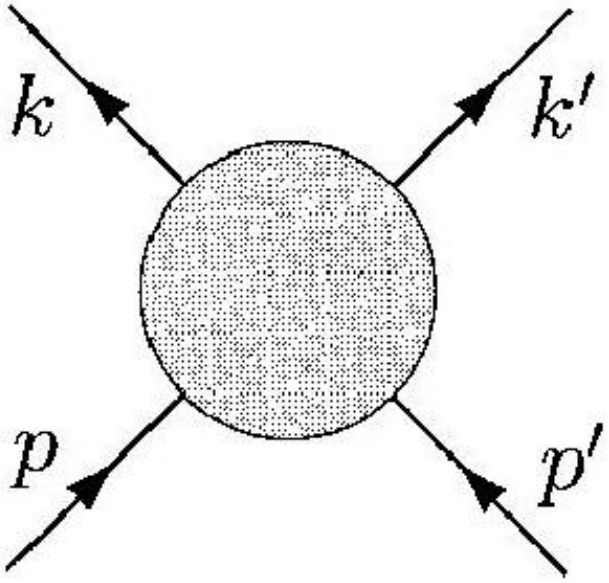
boson, there is no external leg factor, so the identity is proved. If ϕ is a fermion, the analysis becomes more subtle, since the relation depends on the relative phase convention for the external spinors u and v. If we simply replace p by $-k$ in the fermion polarization sum, we find

$$\sum u(p)\bar{u}(p) = \not{p} + m = -(\not{k} - m) = -\sum v(k)\bar{v}(k). \tag{5.68}$$

The minus sign can be compensated by changing our phase convention for $v(k)$. In practice, it is easiest to cancel by hand one minus sign for each crossed fermion. With appropriate conventions for the spinors $u(p)$ and $v(k)$, it is possible to prove the identity (5.67) without spin-averaging.

Mandelstam Variables

It is often useful to express scattering amplitudes in terms of variables that make it easy to apply crossing relations. For 2-body $\to$ 2-body processes, this can be done as follows. Label the four external momenta as

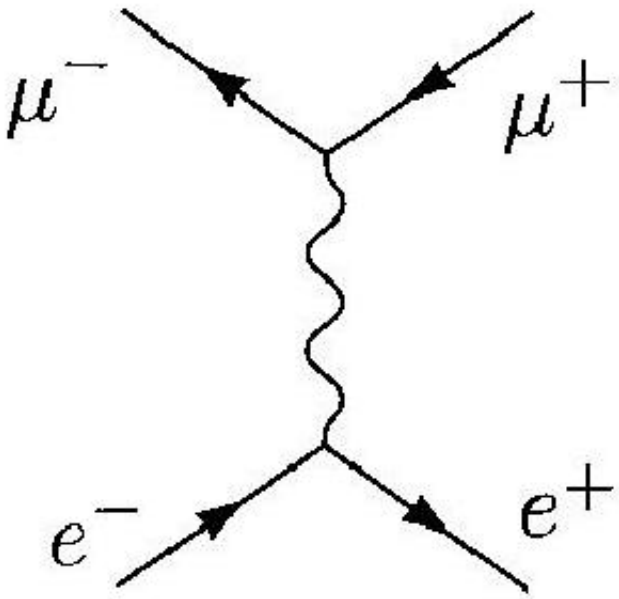


We now define three new quantities, the Mandelstam variables:

$$\begin{aligned} s &= (p + p')^2 = (k + k')^2 \\ t &= (k - p)^2 = (k' - p')^2 \\ u &= (k' - p)^2 = (k - p')^2 \end{aligned} \tag{5.69}$$

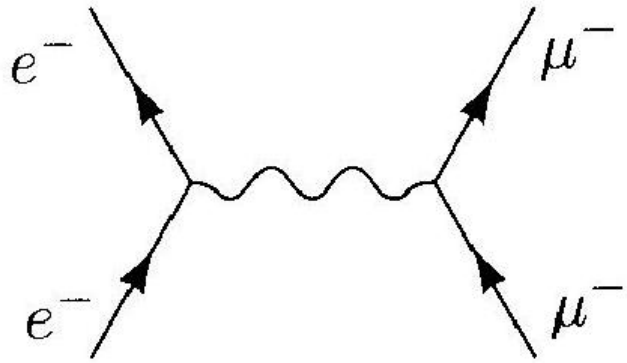
The definitions of t and u appear to be interchangeable (by renaming $k \to k'$); it is conventional to define t as the squared difference of the initial and final momenta of the most similar particles. For any process, s is the square of the total initial 4-momentum. Note that if we had defined all four momenta to be ingoing, all signs in these definitions would be $+$.

To illustrate the use of the Mandelstam variables, let us first consider the squared amplitude for $e^+e^- \to \mu^+\mu^-$, working in the massless limit for simplicity. In this limit we have $t = -2p \cdot k = -2p' \cdot k'$ and $u = -2p \cdot k' = -2p' \cdot k$, while of course $s = (p + p')^2 = q^2$. Referring to our previous result (5.10), we find



$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{s^2}\left[\left(\frac{t}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right] \tag{5.70}$$

To convert to the process $e^-\mu^- \to e^-\mu^-$, we turn the diagram on its side and make use of the crossing relations, which become quite simple in terms of Mandelstam variables. For example, the crossing relations tell us to change the sign of p', the positron momentum, and reinterpret it as the momentum of the outgoing electron. Therefore $s = (p + p')^2$ becomes what we would now call t, the difference of the outgoing and incoming electron momenta. Similarly, t becomes s, while u remains unchanged. Thus for $e^{--}\mu^- \to e^-\mu^-$,
we can immediately write down

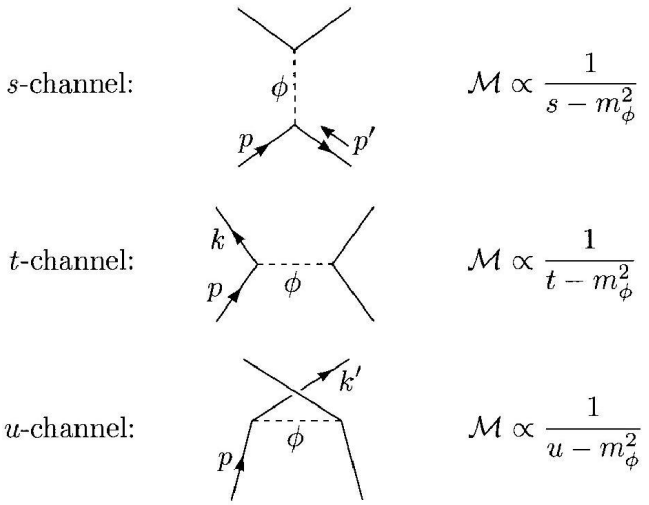


$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{t^2}\left[\left(\frac{s}{2}\right)^2 + \left(\frac{u}{2}\right)^2\right] \tag{5.71}$$

You can easily check that this agrees with (5.61) in the massless limit. Note that while (5.70) and (5.71) look quite similar, they are physically very different: The denominator of the first is just $s^2 = E_{\text{cm}}^4$, but that of the second involves t, which depends on angles and goes to zero as $\theta \to 0$.

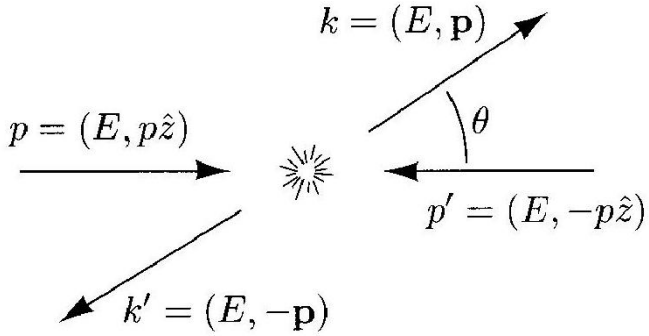
When a 2-body $\to$ 2-body diagram contains only one virtual particle, it is conventional to

describe that particle as being in a certain "channel". The channel can be read from the form of the Feynman diagram, and each channel leads to a characteristic angular dependence of the cross section:



In many cases, a single process will receive contributions from more than one channel; these must be added coherently. For example, the amplitude for Bhabha scattering, $e^+e^- \to e^+e^-$, is the sum of $s^-$and t-channel diagrams; Mller scattering, $e^-e^- \to e^-e^-$, involves t - and u-channel diagrams.

To get a better feel for s, t, and u, let us evaluate them explicitly in the center-of-mass frame for particles all of mass m. The kinematics is as usual:



Thus the Mandelstam variables are

$$\begin{aligned} s &= (p + p')^2 = (2E)^2 = E_{\text{cm}}^2 \\ t &= (k - p)^2 = -p^2 \sin^2\theta - p^2(\cos\theta - 1)^2 = -2p^2(1 - \cos\theta) \\ u &= (k' - p)^2 = -p^2 \sin^2\theta - p^2(\cos\theta + 1)^2 = -2p^2(1 + \cos\theta) \end{aligned} \tag{5.72}$$

Thus we see that $t \to 0$ as $\theta \to 0$, while $u \to 0$ as $\theta \to \pi$. (When the masses are not all equal, the limiting values of t and u will shift slightly.)

Note from (5.72) that when all four particles have mass m, the sum of the Mandelstam variables is $s + t + u = 4E^2 - 4p^2 = 4m^2$. This is a special case of a more general relation, which is often quite useful:

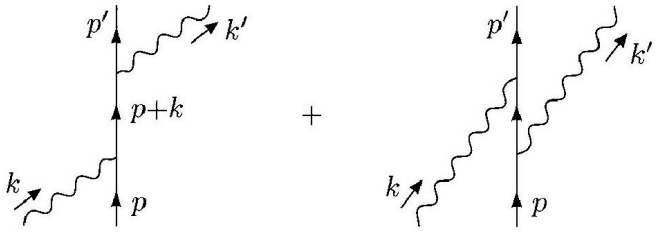
$$s + t + u = \sum_{i=1}^{4} m_i^2 \tag{5.73}$$

where the sum runs over the four external particles. This identity is easy to prove by adding up the terms on the right-hand side of Eqs. (5.69), and applying momentum conservation in the form $(p+p'-k-k')^2 = 0$.

21.5 5.5 Compton Scattering

We now move on to consider a somewhat different QED process: Compton scattering, or $e^-\gamma \to e^-\gamma$. We will calculate the unpolarized cross section for this reaction, to lowest order in α. The calculation will employ all the machinery we have developed so far, including the Mandelstam variables of the previous section. We will also develop some new technology for dealing with external photons.

This is our first example of a calculation involving two diagrams:



As usual, the Feynman rules tell us exactly how to write down an expression for $\mathcal{M}$. Note that since the fermion portions of the two diagrams are identical, there is no relative minus sign between the two terms. Using $\epsilon_\nu(k)$ and $\epsilon^*_\mu(k')$ to denote the polarization vectors of the initial and final photons, we have

$$\begin{aligned} i\mathcal{M} =& \bar{u}(p')(-ie\gamma^\mu)\epsilon^*_\mu(k')\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2}(-ie\gamma^\nu)\epsilon_\nu(k)u(p) \\ &+ \bar{u}(p')(-ie\gamma^\nu)\epsilon_\nu(k)\frac{i(\not{p}-\not{k}'+m)}{(p-k')^2-m^2}(-ie\gamma^\mu)\epsilon^*_\mu(k')u(p) \\ = -ie^2\epsilon^*_\mu(k')\epsilon_\nu(k)\bar{u}(p')&\left[\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2}+\frac{\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu}{(p-k')^2-m^2}\right]u(p). \end{aligned}$$

We can make a few simplifications before squaring this expression. Since $p^2 = m^2$ and $k^2 = 0$, the denominators of the propagators are

$$(p+k)^2 - m^2 = 2p\cdot k \quad \text{and} \quad (p-k')^2 - m^2 = -2p\cdot k'$$

To simplify the numerators, we use a bit of Dirac algebra:

$$\begin{aligned} (\not{p}+m)\gamma^\nu u(p) &= (2p^\nu - \gamma^\nu\not{p} + \gamma^\nu m)u(p) \\ &= 2p^\nu u(p) - \gamma^\nu(\not{p}-m)u(p) \\ &= 2p^\nu u(p). \end{aligned}$$

Using this trick on the numerator of each propagator, we obtain

$$i\mathcal{M} = -ie^2\epsilon^*_\mu(k')\epsilon_\nu(k)\bar{u}(p')\left[\frac{\gamma^\mu\not{p}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu\not{k}'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u(p). \tag{5.74}$$

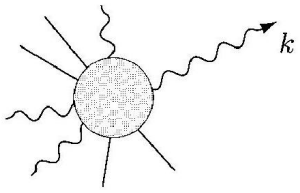
Photon Polarization Sums

The next step in the calculation will be to square this expression for $\mathcal{M}$ and sum (or average) over electron and photon polarization states. The sum over electron polarizations can be performed as before, using the identity $\Sigma u(p)\bar{u}(p) = \not{p} + m$. Fortunately, there is a similar trick for summing over photon polarization vectors. The correct prescription is to make the replacement

$$\sum_{\text{polarizations}} \epsilon_\mu^* \epsilon_\nu \longrightarrow -g_{\mu\nu} \tag{5.75}$$

The arrow indicates that this is not an actual equality. Nevertheless, the replacement is valid as long as both sides are dotted into the rest of the expression for a QED amplitude $\mathcal{M}$.

To derive this formula, let us consider an arbitrary QED process involving an external photon with momentum k:



$$= i\mathcal{M}(k) \equiv i\mathcal{M}^\mu(k)\epsilon_\mu^*(k).$$

Since the amplitude always contains $\epsilon_\mu^*(k)$, we have extracted this factor and defined $\mathcal{M}^\mu(k)$ to be the rest of the amplitude $\mathcal{M}$. The cross section will be proportional to

$$\sum_\epsilon \left|\epsilon_\mu^*(k)\mathcal{M}^\mu(k)\right|^2 = \sum_\epsilon \epsilon_\mu^* \epsilon_\nu \mathcal{M}^\mu(k)\mathcal{M}^{\nu *}(k)$$

For simplicity, we orient k in the 3 -direction: $k^\mu = (k, 0, 0, k)$. Then the two transverse polarization vectors, over which we are summing, can be chosen to be

$$\epsilon_1^\mu = (0, 1, 0, 0); \quad \epsilon_2^\mu = (0, 0, 1, 0)$$

With these conventions, we have

$$\sum_\epsilon \left|\epsilon_\mu^*(k)\mathcal{M}^\mu(k)\right|^2 = \left|\mathcal{M}^1(k)\right|^2 + \left|\mathcal{M}^2(k)\right|^2 \tag{5.77}$$

Now recall from Chapter 4 that external photons are created by the interaction term $\int d^4x e j^\mu A_\mu$, where $j^\mu = \bar{\psi}\gamma^\mu\psi$ is the Dirac vector current. Therefore we expect $\mathcal{M}^\mu(k)$ to be given by a matrix element of the Heisenberg field j^μ:

$$\mathcal{M}^\mu(k) = \int d^4x e^{ik\cdot x} \langle f|j^\mu(x)|i\rangle \tag{5.78}$$

where the initial and final states include all particles except the photon in question.

From the classical equations of motion, we know that the current j^μ is conserved: $\partial_\mu j^\mu(x) = 0$. Provided that this property still holds in the quantum theory, we can dot k_μ into (5.78) to obtain

$$k_\mu \mathcal{M}^\mu(k) = 0. \tag{5.79}$$

The amplitude $\mathcal{M}$ vanishes when the polarization vector $\epsilon_\mu(k)$ is replaced by k_μ. This famous relation is known as the Ward identity. It is essentially a statement of current conservation, which is a consequence of the gauge symmetry (4.6) of QED. We will give a formal proof of the

Ward identity in Section 7.4, and discuss a number of subtle points skimmed over in this quick "derivation".

It is useful to check explicitly that the Compton amplitude given in (5.74) obeys the Ward identity. To do this, replace $\epsilon_\nu(k)$ by k_ν or $\epsilon^*_\mu(k')$ by k'_μ, and manipulate the Dirac matrix products. In either case (after a bit of algebra) the terms from the two diagrams cancel each other to give zero.

Returning to our derivation of the polarization sum formula (5.75), we note that for $k^\mu = (k, 0, 0, k)$, the Ward identity takes the form

$$k\mathcal{M}^0(k) - k\mathcal{M}^3(k) = 0 \tag{5.80}$$

Thus $\mathcal{M}^0 = \mathcal{M}^3$, and we have

$$\begin{aligned}\sum_\epsilon \epsilon^*_\mu \epsilon_\nu \mathcal{M}^\mu(k)\mathcal{M}^{\nu*}(k) &= \left|\mathcal{M}^1\right|^2 + \left|\mathcal{M}^2\right|^2 \\ &= \left|\mathcal{M}^1\right|^2 + \left|\mathcal{M}^2\right|^2 + \left|\mathcal{M}^3\right|^2 - \left|\mathcal{M}^0\right|^2 \\ &= -g_{\mu\nu}\mathcal{M}^\mu(k)\mathcal{M}^{\nu*}(k)\end{aligned}$$

That is, we may sum over external photon polarizations by replacing $\sum \epsilon^*_\mu \epsilon_\nu$ with $-g_{\mu\nu}$.

Note that this proves (pending our general proof of the Ward identity) that the unphysical timelike and longitudinal photons can be consistently omitted from QED calculations, since in any event the squared amplitudes for producing these states cancel to give zero total probability. The negative norm of the timelike photon state, a property that troubled us in the discussion after Eq. (4.132), plays an essential role in this cancellation.

The Klein-Nishina Formula

The rest of the computation of the Compton scattering cross section is straightforward, although it helps to be somewhat organized. We want to average the squared amplitude over the initial electron and photon polarizations, and sum over the final electron and photon polarizations. Starting with expression (5.74) for $\mathcal{M}$, we find

$$\begin{aligned}\tfrac{1}{4}\textstyle\sum_{\text{spins}} |\mathcal{M}|^2 = \tfrac{e^4}{4} g_{\mu\rho}g_{\nu\sigma} \cdot \operatorname{tr}\Big\{ (\not{p}' + m)\left[\tfrac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \tfrac{\gamma^\nu \not{k}'\gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}\right] \\ \cdot(\not{p} + m)\left[\tfrac{\gamma^\sigma \not{k}\gamma^\rho + 2\gamma^\rho p^\sigma}{2p\cdot k} + \tfrac{\gamma^\rho \not{k}'\gamma^\sigma - 2\gamma^\sigma p^\rho}{2p\cdot k'}\right]\Big\} \\ \equiv \tfrac{e^4}{4}\left[\tfrac{\mathbf{I}}{(2p\cdot k)^2} + \tfrac{\mathbf{II}}{(2p\cdot k)(2p\cdot k')} + \tfrac{\mathbf{III}}{(2p\cdot k')(2p\cdot k)} + \tfrac{\mathbf{IV}}{(2p\cdot k')^2}\right],\end{aligned} \tag{5.81}$$

where I, II, III, and IV are complicated traces. Note that IV is the same as I if we replace k with $-k'$. Also, since we can reverse the order of the γ matrices inside a trace (Eq. (5.7)), we see that II =III. Thus we must work only to compute I and II.

The first of the traces is

$$\mathbf{I} = \operatorname{tr}\left[(\not{p}' + m)\left(\gamma^\mu \not{k} \not{\gamma}^\nu + 2\gamma^\mu p^\nu\right)(\not{p} + m)\left(\gamma_\nu \not{k}\gamma_\mu + 2\gamma_\mu p_\nu\right)\right].$$

There are 16 terms inside the trace, but half contain an odd number of γ matrices and therefore vanish. We must now evaluate the other eight terms, one at a time. For example,

$$\begin{aligned}\mathrm{tr}\left[\not{p}'\gamma^\mu\ \not{k}\gamma^\nu\ \not{p}\gamma_\nu\ \not{k}\gamma_\mu\right] &= \mathrm{tr}\left[(-2\ \not{p}')\ \not{k}(-2\ \not{p})\ \not{\mathbb{1}}\ c\right] \\ &= \mathrm{tr}\left[4\ \not{p}'\ \not{k}(2p\cdot k - \not{k}\ \not{p})\right] \\ &= 8p\cdot k\,\mathrm{tr}\left[\not{p}'\ \not{k}x\right] \\ &= 32(p\cdot k)\,(p'\cdot k)\,.\end{aligned}$$

By similar use of the contraction identities (5.8) and (5.9), and other Dirac algebra such as $\not{p}\ \not{p} = p^2 = m^2$, each term in $\mathbf{I}$ can be reduced to a trace of no more than two γ matrices. When the smoke clears, we find

$$\mathbf{I} = 16\left(4m^4 - 2m^2 p\cdot p' + 4m^2 p\cdot k - 2m^2 p'\cdot k + 2(p\cdot k)\,(p'\cdot k)\right). \tag{5.82}$$

Although it is not obvious, this expression can be simplified further. To see how, introduce the Mandelstam variables:

$$\begin{aligned}s &= (p+k)^2 = 2p\cdot k + m^2 = 2p'\cdot k' + m^2 \\ t &= (p'-p)^2 = -2p\cdot p' + 2m^2 = -2k\cdot k' \\ u &= (k'-p)^2 = -2k'\cdot p + m^2 = -2k\cdot p' + m^2\end{aligned} \tag{5.83}$$

Recall from (5.73) that momentum conservation implies $s+t+u = 2m^2$. Writing everything in terms of s, t, and u, and using this identity, we eventually obtain

$$\mathbf{I} = 16\left(2m^4 + m^2\left(s-m^2\right) - \frac{1}{2}\left(s-m^2\right)\left(u-m^2\right)\right) \tag{5.84}$$

Sending $k \leftrightarrow -k'$, we can immediately write

$$\mathbf{IV} = 16\left(2m^4 + m^2\left(u-m^2\right) - \frac{1}{2}\left(s-m^2\right)\left(u-m^2\right)\right) \tag{5.85}$$

Evaluating the traces in the numerators II and III requires about the same amount of work as we have just done. The answer is

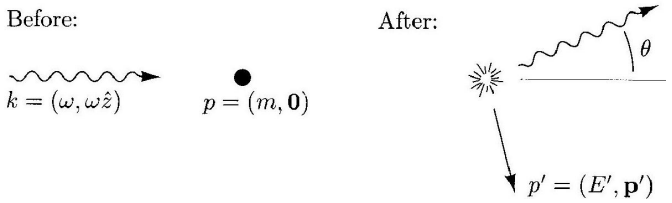
$$\mathbf{II} = \mathbf{III} = -8\left(4m^4 + m^2\left(s-m^2\right) + m^2\left(u-m^2\right)\right) \tag{5.86}$$

Putting together the pieces of the squared matrix element (5.81), and rewriting s and u in terms of $p\cdot k$ and $p\cdot k'$, we finally obtain

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = 2e^4\left[\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right] \tag{5.87}$$

To turn this expression into a cross section we must decide on a frame of reference and draw a picture of the kinematics. Compton scattering is most often analyzed in the "lab" frame, in which the electron is initially at rest:

Before:



We will express the cross section in terms of ω and θ. We can find ω', the energy of the final photon, using the following trick:

$$\begin{aligned} m^2 = (p')^2 &= (p+k-k')^2 = p^2 + 2p\cdot(k-k') - 2k\cdot k' \\ &= m^2 + 2m(\omega-\omega') - 2\omega\omega'(1-\cos\theta) \end{aligned} \tag{5.88}$$

hence, $\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1-\cos\theta)$.

The last line is Compton's formula for the shift in the photon wavelength. For our purposes, however, it is more useful to solve for ω':

$$\omega' = \frac{\omega}{1+\frac{\omega}{m}(1-\cos\theta)} \tag{5.89}$$

The phase space integral in this frame is

$$\begin{aligned} \int d\Pi_2 &= \int \frac{d^3k'}{(2\pi)^3}\frac{1}{2\omega'}\frac{d^3p'}{(2\pi)^3}\frac{1}{2E'}(2\pi)^4\delta^{(4)}(k'+p'-k-p) \\ &= \int \frac{(\omega')^2\,d\omega' d\Omega}{(2\pi)^3}\frac{1}{4\omega' E'} \\ &\quad \times 2\pi\delta\left(\omega' + \sqrt{m^2+\omega^2+(\omega')^2-2\omega\omega'\cos\theta} - \omega - m\right) \\ &= \int \frac{d\cos\theta}{2\pi}\frac{\omega'}{4E'}\frac{1}{\left|1+\frac{\omega'-\omega\cos\theta}{E'}\right|} \\ &= \frac{1}{8\pi}\int d\cos\theta \frac{\omega'}{m+\omega(1-\cos\theta)} \\ &= \frac{1}{8\pi}\int d\cos\theta \frac{(\omega')^2}{\omega m} \end{aligned} \tag{5.90}$$

Plugging everything into our general cross-section formula (4.79) and setting $|v_{\mathcal{A}} - v_{\mathcal{B}}| = 1$, we find

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{2\omega}\frac{1}{2m}\cdot\frac{1}{8\pi}\frac{(\omega')^2}{\omega m}\cdot\left(\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\right)$$

To evaluate $|\mathcal{M}|^2$, we replace $p\cdot k = m\omega$ and $p\cdot k' = m\omega'$ in (5.87). The shortest way to write the final result is

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right] \tag{5.91}$$

where ω'/ω is given by (5.89). This is the (spin-averaged) Klein-Nishina formula, first derived in 1929. †

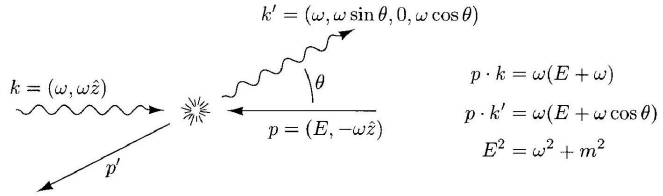
In the limit $\omega \to 0$ we see from (5.89) that $\omega'/\omega \to 1$, so the cross section becomes

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(1+\cos^2\theta\right); \quad \sigma_{\text{total}} = \frac{8\pi\alpha^2}{3m^2} \tag{5.92}$$

This is the familiar Thomson cross section for scattering of classical electromagnetic radiation by a free electron.

High-Energy Behavior

To analyze the high-energy behavior of the Compton scattering cross section, it is easiest to work in the center-of-mass frame. We can easily construct the differential cross section in this frame from the invariant expression (5.87). The kinematics of the reaction now looks like this:



Plugging these values into (5.87), we see that for $\theta \approx \pi$, the term $p \cdot k / p \cdot k'$ becomes very large, while the other terms are all of $\mathcal{O}(1)$ or smaller. Thus for $E \gg m$ and $\theta \approx \pi$, we have

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 \approx 2e^4 \cdot \frac{p \cdot k}{p \cdot k'} = 2e^4 \cdot \frac{E + \omega}{E + \omega\cos\theta} \tag{5.93}$$

The cross section in the CM frame is given by (4.84):

$$\begin{aligned} \frac{d\sigma}{d\cos\theta} &= \frac{1}{2} \cdot \frac{1}{2E} \cdot \frac{1}{2\omega} \cdot \frac{\omega}{(2\pi)4(E+\omega)} \cdot \frac{2e^4(E+\omega)}{E + \omega\cos\theta} \\ &\approx \frac{2\pi\alpha^2}{2m^2 + s(1+\cos\theta)} \end{aligned} \tag{5.94}$$

Notice that, since $s \gg m^2$, the denominator of (5.94) almost vanishes when the photon is emitted in the backward direction ($\theta \approx \pi$). In fact, the electron mass m could be neglected completely in this formula if it were not necessary to cut off this singularity. To integrate over $\cos\theta$, we can drop the electron mass term if we supply an equivalent cutoff near $\theta = \pi$. In this way, we can approximate the total Compton scattering cross section by

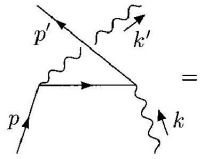
$$\int_{-1}^{1} d(\cos\theta) \frac{d\sigma}{d\cos\theta} \approx \frac{2\pi\alpha^2}{s} \int_{-1+2m^2/s}^{1} d(\cos\theta) \frac{1}{(1+\cos\theta)} \tag{5.95}$$

Thus, we find that the total cross section behaves at high energy as

$$\sigma_{\text{total}} = \frac{2\pi\alpha^2}{s} \log\left(\frac{s}{m^2}\right) \tag{5.96}$$

The main dependence α^2/s follows from dimensional analysis. But the singularity associated with backward scattering of photons leads to an enhancement by an extra logarithm of the energy.

Let us try to understand the physics of this singularity. The singular term comes from the square of the u-channel diagram,



$$= -ie^2\, \epsilon_\mu(k)\epsilon^*_\nu(k')\bar{u}(p')\gamma^\mu \frac{\not{p} - \not{k}' + m}{(p-k')^2 - m^2}\gamma^\nu u(p).$$

[6†] O. Klein and Y. Nishina, Z. Physik, 52, 853 (1929).

The amplitude is large at $\theta \approx \pi$ because the denominator of the propagator is then small ($\sim m^2$) compared to s. To be more precise, define $\chi \equiv \pi - \theta$. We will be interested in values of χ that are somewhat larger than m/ω, but still small enough that we can approximate $1 - \cos\chi \approx \chi^2/2$. For χ in this range, the denominator is

$$(p - k')^2 - m^2 = -2p \cdot k' \approx -2\omega^2 \left(\frac{m^2}{2\omega^2} + 1 - \cos\chi \right) \approx -\left(\omega^2\chi^2 + m^2\right) \tag{5.98}$$

This is small compared to s over a wide range of values for χ, hence the enhancement in the total cross section.

Looking back at (5.93), we see that for χ such that $m/\omega \ll \chi \ll 1$, the squared amplitude is proportional to $1/\chi^2$, and hence we expect $\mathcal{M} \propto 1/\chi$. But we have just seen that the denominator of $\mathcal{M}$ is proportional to χ^2, so there must be a compensating factor of χ in the numerator. We can understand the physical origin of that factor by looking at the amplitude for a particular set of electron and photon polarizations.

Suppose that the initial electron is right-handed. The dominant term of (5.97) comes from the term that involves $(\not{p} - \not{k}')$ in the numerator of the propagator. Since this term contains three γ-matrices in (5.97) between the $\bar{u}$ and the u, the final electron must also be right-handed. The amplitude is therefore

$$i\mathcal{M} = -ie^2 \epsilon_\mu(k) \epsilon^*_\nu(k')\, u^\dagger_R(p')\, \sigma^\mu \frac{\bar{\sigma} \cdot (p - k')}{-(\omega^2\chi^2 + m^2)} \sigma^\nu u_R(p) \tag{5.99}$$

where

$$u_R(p) = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad u_R(p') = \sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.100}$$

If the initial photon is left-handed, with $\epsilon_\mu(k) = (1/\sqrt{2})(0, 1, -i, 0)$, then

$$\sigma^\mu \epsilon_\mu(k) = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}$$

and the combination $u^\dagger_R(p')\, \sigma^\mu \epsilon_\mu(k)$ vanishes. The initial photon must therefore be right-handed. Similarly, the amplitude vanishes unless the final photon is right-handed. The kinematic situation for this set of polarizations is shown

Before:

After:

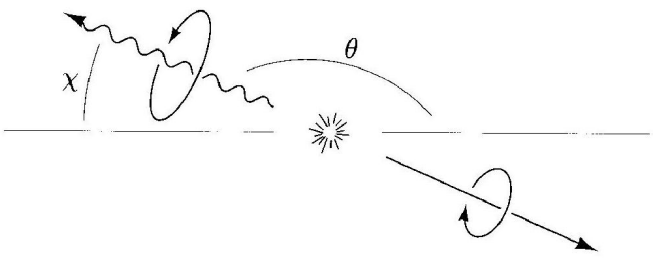


Figure 5.6. In the high-energy limit, the final photon is most likely to be emitted at backward angles. Since helicity is conserved, a unit of spin angular momentum is converted to orbital angular momentum.

in Fig. 5.6. Note that the total spin angular momentum of the final state is one unit less than that of the initial state.

Continuing with our calculation, let us consider the numerator of the propagator in (5.99). For χ in the range of interest, the dominant term is

$$-\sigma^1 \left(p - k'\right)^1 = \sigma^1 \cdot \omega\chi$$

This is the factor of χ anticipated above. It indicates that the final state is a p-wave, as required by angular momentum conservaton. Assembling all the pieces, we obtain

$$\mathcal{M}\left(e_R^- \gamma_R \to e_R^- \gamma_R\right) \approx e^2\sqrt{2E}\sqrt{2}\frac{\omega\chi}{(\omega^2\chi^2 + m^2)}\sqrt{2E}\sqrt{2} \approx \frac{4e^2\chi}{\chi^2 + m^2/\omega^2}. \tag{5.101}$$

We would find the same result in the case where all initial and final particles are left-handed.

Notice that for directly backward scattering, $\chi = 0$, the matrix element (5.101) vanishes due to the angular momentum zero in the numerator. Thus, at angles very close to backward, we should also take into account the mass term in the numerator of the propagator in (5.97). This term contains only two gamma matrices and so converts a right-handed electron into a left-handed electron. By an analysis similar to the one that led to Eq. (5.101), we can see that this amplitude is nonvanishing only when the initial photon is lefthanded and the final photon is right-handed. Following this analysis in more detail, we find

$$\mathcal{M}\left(e_R^- \gamma_L \to e_L^- \gamma_R\right) \approx \frac{4e^2 m/\omega}{\chi^2 + m^2/\omega^2} \tag{5.102}$$

The reaction with all four helicities reversed gives the same matrix element.

To compare this result to our previous calculations, we should add the contributions to the cross section from (5.101) and (5.102) and equal contributions for the reactions involving initial left-handed electrons, and divide by 4 to average over initial spins. The unpolarized differential cross section should then be

$$\begin{aligned}\frac{d\sigma}{d\cos\theta} &= \frac{1}{2}\frac{1}{2E}\frac{1}{2\omega}\frac{\omega}{(2\pi)4(E+\omega)}\left[\frac{8e^4\chi^2}{\left(\chi^2 + m^2/\omega^2\right)^2} + \frac{8e^4 m^2/\omega^2}{\left(\chi^2 + m^2/\omega^2\right)^2}\right] \\ &= \frac{4\pi\alpha^2}{s\left(\chi^2 + 4m^2/s\right)},\end{aligned} \tag{5.103}$$

which agrees precisely with Eq. (5.94).

The importance of the helicity-flip process (5.102) just at the kinematic endpoint has an interesting experimental consequence. Consider the process of inverse Compton scattering, a high-energy electron beam colliding with a low-energy photon beam (for example, a laser beam) to produce a highenergy photon beam. Let the electrons have energy E and the laser photons have energy ϖ, let the energy of the scattered photon be $E' = yE$, and assume for simplicity that $s = 4E\varpi \gg m^2$. Then the computation we have just done applies to this situation, with the highest energy photons resulting from scattering that is precisely backward in the center-of-mass frame. By computing $2k \cdot k'$ in the center-of-mass frame and in the lab frame, it is easy to show that the final photon energy is related to the center-of-mass scattering angle through

$$y \approx \frac{1}{2}(1-\cos\theta) \approx 1 - \frac{\chi^2}{4}$$

Then Eq. (5.103) can be rewritten as a formula for the energy distribution of backscattered photons near the endpoint:

$$\frac{d\sigma}{dy} = \frac{2\pi\alpha^2}{s\left((1-y)+m^2/s\right)^2}\left[(1-y)+\frac{m^2}{s}\right] \tag{5.104}$$

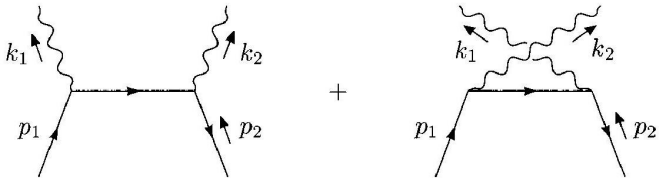
where the first term in brackets corresponds to the helicity-conserving process and the second term to the helicity-flip process. Thus, for example, if a right-handed polarized laser bearn is scattered from an unpolarized highenergy electron beam, most of the backscattered photons will be right-handed but the highest-energy photons will be left-handed. This effect can be used experimentally to measure the polarization of an electron beam or to create high-energy photon sources with adjustable energy distribution and polarization.

Pair Annihilation into Photons

We can still obtain one more result from the Compton-scattering amplitude. Consider the annihilation process

$$e^+e^- \to 2\gamma$$

given to lowest order by the diagrams



This process is related to Compton scattering by crossing symmetry; we can obtain the correct amplitude from the Compton amplitude by making the replacements

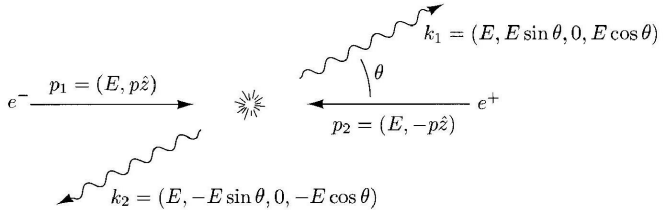
$$p \to p_1 \quad p' \to -p_2 \quad k \to -k_1 \quad k' \to k_2$$

Making these substitutions in (5.87), we find

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = -2e^4\left[\frac{p_1\cdot k_2}{p_1\cdot k_1}+\frac{p_1\cdot k_1}{p_1\cdot k_2}+2m^2\left(\frac{1}{p_1\cdot k_1}+\frac{1}{p_1\cdot k_2}\right)\right. \tag{5.105}$$
$$\left. -m^4\left(\frac{1}{p_1\cdot k_1}+\frac{1}{p_1\cdot k_2}\right)^2\right]$$

The overall minus sign is the result of the crossing relation (5.68) and should be removed.

Now specialize to the center-of-mass frame. The kinematics is



A routine calculation yields the differential cross section,

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha^2}{s}\left(\frac{E}{p}\right)\left[\frac{E^2+p^2\cos^2\theta}{m^2+p^2\sin^2\theta} + \frac{2m^2}{m^2+p^2\sin^2\theta} - \frac{2m^4}{\left(m^2+p^2\sin^2\theta\right)^2}\right] \tag{5.106}$$

In the high-energy limit, this becomes

$$\frac{d\sigma}{d\cos\theta} \underset{E\gg m}{\longrightarrow} \frac{2\pi\alpha^2}{s}\left(\frac{1+\cos^2\theta}{\sin^2\theta}\right) \tag{5.107}$$

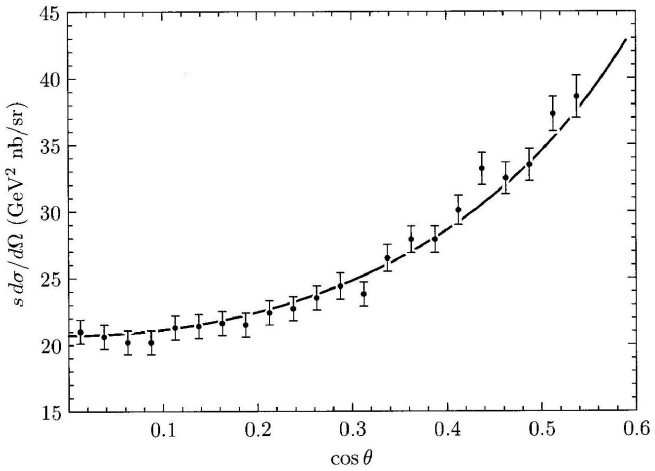


Figure 5.7. Angular dependence of the cross section for $e^+e^- \to 2\gamma$ at $E_{\rm cm} = 29$GeV, as measured by the HRS collaboration, M. Derrick, et. al., Phys. Rev. D34, 3286 (1986). The solid line is the lowest-order theoretical prediction, Eq. (5.107).

except when $\sin\theta$ is of order m/p or smaller. Note that since the two photons are identical, we count all possible final states by integrating only over $0 \le \theta \le \pi/2$. Thus the total cross section is computed as

$$\sigma_{\rm total} = \int_0^1 d(\cos\theta)\frac{d\sigma}{d\cos\theta} \tag{5.108}$$

Figure 5.7 compares the asymptotic formula (5.107) for the differential cross section to measurements of e^+e^-annihilation into two photons at very high energy.

Problems

5.1 Coulomb scattering. Repeat the computation of Problem 4.4, part (c), this time using the full relativistic expression for the matrix element. You should find, for the spin-averaged cross section,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4|\mathbf{p}|^2\beta^2\sin^4(\theta/2)}\left(1-\beta^2\sin^2\frac{\theta}{2}\right)$$

where $\mathbf{p}$ is the electron's 3 -momentum and β is its velocity. This is the Mott formula for Coulomb scattering of relativistic electrons. Now derive it in a second way, by working out the cross section for electron-muon scattering, in the muon rest frame, retaining the electron

mass but sending $m_\mu \to \infty$.

5.2 Bhabha scattering. Compute the differential cross section $d\sigma/d\cos\theta$ for Bhabha scattering, $e^+e^- \to e^+e^-$. You may work in the limit $E_{\rm cm} \gg m_e$, in which it is permissible to ignore the electron mass. There are two Feynman diagrams; these must be added in the invariant matrix element before squaring. Be sure that you have the correct relative sign between these diagrams. The intermediate steps are complicated, but the final result is quite simple. In particular, you may find it useful to introduce the Mandelstam variables s, t, and u. Note that, if we ignore the electron mass, $s+t+u=0$. You should be able to cast the differential cross section into the form

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right]$$

Rewrite this formula in terms of $\cos\theta$ and graph it. What feature of the diagrams causes the differential cross section to diverge as $\theta \to 0$?

5.3 The spinor product formalism introduced in Problem 3.3 provides an efficient way to compute tree diagrams involving massless particles. Recall that in Problem 3.3 we defined spinor products as follows: Let u_{L0}, u_{R0} be the left- and right-handed spinors at some fixed lightlike momentum k_0. These satisfy

$$u_{L0}\bar{u}_{L0} = \left(\frac{1-\gamma^5}{2}\right)\not{0}, \quad u_{R0}\bar{u}_{R0} = \left(\frac{1+\gamma^5}{2}\right)\not{0}. \tag{1}$$

(These relations are just the projections onto definite helicity of the more standard formula $\sum u_0\bar{u}_0 = \not{p}_0$.) Then define spinors for any other lightlike momentum p by

$$u_L(p) = \frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{R0}, \quad u_R(p) = \frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{L0} \tag{2}$$

We showed that these spinors satisfy $\not{p}u(p) = 0$; because there is no m around, they can be used as spinors for either fermions or antifermions. We defined

$$s(p_1,p_2) = \bar{u}_R(p_1)\,u_L(p_2), \quad t(p_1,p_2) = \bar{u}_L(p_1)\,u_R(p_2)$$

and, in a special frame, we proved the properties

$$t(p_1,p_2) = (s(p_2,p_1))^*, \quad s(p_1,p_2) = -s(p_2,p_1), \quad |s(p_1,p_2)|^2 = 2p_1\cdot p_2 \tag{3}$$

Now let us apply these results.

(a) To warm up, give another proof of the last relation in Eq. (3) by using (1) to rewrite $|s(p_1,p_2)|^2$ as a trace of Dirac matrices, and then applying the trace calculus.

(b) Show that, for any string of Dirac matrices,

$$\mathrm{tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\cdots\right] = \mathrm{tr}\left[\cdots\gamma^\rho\gamma^\nu\gamma^\mu\right]$$

where $\mu,\nu,\rho,\ldots = 0,1,2,3$, or 5 . Use this identity to show that

$$\bar{u}_L(p_1)\,\gamma^\mu u_L(p_2) = \bar{u}_R(p_2)\,\gamma^\mu u_R(p_1)$$

(c) Prove the Fierz identity

$$\bar{u}_L(p_1)\,\gamma^\mu u_L(p_2)\,[\gamma_\mu]_{ab} = 2\left[u_L(p_2)\,\bar{u}_L(p_1) + u_R(p_1)\,\bar{u}_R(p_2)\right]_{ab}$$

where $a, b = 1, 2, 3, 4$ are Dirac indices. This can be done by justifying the following statements: The right-hand side of this equation is a Dirac matrix; thus, it can be written as a linear combination of the 16Γ matrices discussed in Section 3.4. It satisfies

$$\gamma^5[M] = -[M]\gamma^5,$$

thus, it must have the form

$$[M] = \left(\frac{1-\gamma^5}{2}\right)\gamma_\mu V^\mu + \left(\frac{1+\gamma^5}{2}\right)\gamma_\mu W^\mu$$

where V^μ and W^μ are 4 -vectors. These 4 -vectors can be computed by trace technology; for example,

$$V^\nu = \frac{1}{2}\,\mathrm{tr}\left[\gamma^\nu\left(\frac{1-\gamma^5}{2}\right)M\right].$$

(d) Consider the process $e^+e^- \to \mu^+\mu^-$, to the leading order in α, ignoring the masses of both the electron and the muon. Consider first the case in which the electron and the final muon are both right-handed and the positron and the final antimuon are both left-handed. (Use the spinor v_R for the antimuon and $\bar{u}_R$ for the positron.) Apply the Fierz identity to show that the amplitude can be evaluated directly in terms of spinor products. Square the amplitude and reproduce the result for

$$\frac{d\sigma}{d\cos\theta}\left(e_R^- e_L^+ \to \mu_R^- \mu_L^+\right)$$

given in Eq. (5.22). Compute the other helicity cross sections for this process and show that they also reproduce the results found in Section 5.2.
(e) Compute the differential cross section for Bhabha scattering of massless electrons, helicity state by helicity state, using the spinor product formalism. The average over initial helicities, summed over final helicities, should reproduce the result of Problem 5.2. In the process, you should see how this result arises as the sum of definite-helicity contributions.

5.4 Positronium lifetimes.

(a) Compute the amplitude $\mathcal{M}$ for e^+e^-annihilation into 2 photons in the extreme nonrelativistic limit (i.e., keep only the term proportional to zero powers of the electron and positron 3 -momentum). Use this result, together with our formalism for fermion-antifermion bound states, to compute the rate of annihilation of the $1S$ states of positronium into 2 photons. You should find that the spin-1 states of positronium do not annihilate into 2 photons, confirming the symmetry argument of Problem 3.8. For the spin-0 state of positronium, you should find a result proportional to the square of the $1S$ wavefunction at the origin. Inserting the value of this wavefunction from nonrelativistic quantum mechanics, you should find

$$\frac{1}{\tau} = \Gamma = \frac{\alpha^5 m_e}{2} \approx 8.03 \times 10^9 \mathrm{sec}^{-1}$$

A recent measurement [‡] gives $\Gamma = 7.994 \pm .011 \mathrm{nsec}^{-1}$; the 0.5% discrepancy is accounted for by radiative corrections.
(b) Computing the decay rates of higher-l positronium states is somewhat more difficult; in the rest of this problem, we will consider the case $l = 1$. First, work out the terms in the $e^+e^- \to 2\gamma$

amplitude proportional to one power of the 3 -momentum. (For simplicity, work in the center-of-mass frame.) Since

$$\int \frac{d^3p}{(2\pi)^3} p^i \psi(\mathbf{p}) = i \frac{\partial}{\partial x^i} \psi(\mathbf{x}) \Big|_{\mathbf{x}=0}$$

this piece of the amplitude has overlap with P-wave bound states. Show that the $S = 1$, but not the $S = 0$ states, can decay to 2 photons. Again, this is a consequence of C.
(c) To compute the decay rates of these P-wave states, we need properly normalized state vectors. Denote the three P-state wavefunctions by

$$\psi_i = x^i f(|\mathbf{x}|), \quad \text{normalized to } \int d^3x \psi_i^*(\mathbf{x}) \psi_j(\mathbf{x}) = \delta_{ij}$$

and their Fourier transforms by $\psi_i(\mathbf{p})$. Show that

$$|B(\mathbf{k})\rangle = \sqrt{2M} \int \frac{d^3p}{(2\pi)^3} \psi_i(\mathbf{p}) a^\dagger_{\mathbf{p}+\mathbf{k}/2} \Sigma^i b^\dagger_{-\mathbf{p}+\mathbf{k}/2} |0\rangle$$

is a properly normalized bound-state vector if Σ^i denotes a set of three 2×2 matrices normalized to

$$\sum_i \mathrm{tr}\left(\Sigma^{i\dagger}\Sigma^i\right) = 1$$

To build $S = 1$ states, we should take each Σ^i to contain a Pauli sigma matrix. In general, spin-orbit coupling will split the multiplet of $S = 1, L = 1$ states according to the total angular momentum J. The states of definite J are given by

$$\begin{aligned} J = 0: &\quad \Sigma^i = \tfrac{1}{\sqrt{6}} \sigma^i \\ J = 1: &\quad \Sigma^i = \tfrac{1}{2} \epsilon^{ijk} n^j \sigma^k \\ J = 2: &\quad \Sigma^i = \tfrac{1}{\sqrt{3}} h^{ij} \sigma^j \end{aligned}$$

where $\mathbf{n}$ is a polarization vector satisfying $|\mathbf{n}|^2 = 1$ and h^{ij} is a traceless tensor, for which a typical value might be $h^{12} = 1$ and all other components zero.
(d) Using the expanded form for the $e^+e^- \to 2\gamma$ amplitude derived in part (b) and the explicit form of the $S = 1, L = 1$, definite- J positronium states found in part (c), compute, for each J, the decay rate of the state into two photons.

5.5 Physics of a massive vector boson. Add to QED a massive photon field B_μ of mass M, which couples to electrons via

$$\Delta H = \int d^3x \left(g \bar{\psi} \gamma^\mu \psi B_\mu\right)$$

A massive photon in the initial or final state has three possible physical polarizations, corresponding to the three spacelike unit vectors in the boson's rest frame. These can be characterized invariantly, in terms of the boson's 4 -momentum k^μ, as the three vectors $\epsilon^{(i)}_\mu$ satisfying

$$\epsilon^{(i)} \cdot \epsilon^{(j)} = -\delta^{ij}, \quad k \cdot \epsilon^{(i)} = 0$$

[6‡] D. W. Gidley et. al., Phys. Rev. Lett. 49, 525 (1982).

The four vectors $(k_\mu/M, \epsilon_\mu^{(i)})$ form a complete orthonormal basis. Because B_μ couples to the conserved current $\bar{\psi}\gamma^\mu\psi$, the Ward identity implies that k_μ dotted into the amplitude for B production gives zero; thus we can replace:

$$\sum_i \epsilon_\mu^{(i)} \epsilon_\nu^{(i)*} \to -g_{\mu\nu}$$

This gives a generalization to massive bosons of the Feynman trick for photon polarization vectors and simplifies the calculation of B production cross sections. (Warning: This trick does not work (so simply) for "non-Abelian gauge fields".) Let's do a few of these computations, using always the approximation of ignoring the mass of the electron.
(a) Compute the cross section for the process $e^+e^- \to B$. Compute the lifetime of the B, assuming that it decays only to electrons. Verify the relation

$$\sigma\left(e^+e^- \to B\right) = \frac{12\pi^2}{M}\Gamma\left(B \to e^+e^-\right)\delta\left(M^2 - s\right)$$

discussed in Section 5.3.
(b) Compute the differential cross section, in the center-of-mass system, for the process $e^+e^- \to \gamma + B$. (This calculation goes over almost unchanged to the realistic process $e^+e^- \to \gamma + Z^0$; this allows one to measure the number of decays of the Z^0 into unobserved final states, which is in turn proportional to the number of neutrino species.)
(c) Notice that the cross section of part (b) diverges as $\theta \to 0$ or π. Let us analyze the region near $\theta = 0$. In this region, the dominant contribution comes from the t-channel diagram and corresponds intuitively to the emission of a photon from the electron line before e^+e^-annihilation into a B. Let us rearrange the formula in such a way as to support this interpretation. First, note that the divergence as $\theta \to 0$ is cut off by the electron mass: Let the electron momentum be $p^\mu = (E, 0, 0, k)$, with $k = (E^2 - m_e^2)^{1/2}$, and let the photon momentum be $k^\mu = (xE, xE\sin\theta, 0, xE\cos\theta)$. Show that the denominator of the propagator then never becomes smaller than $\mathcal{O}(m_e^2/s)$. Now integrate the cross section of part (b) over forward angles, cutting off the θ integral at $\theta^2 \sim (m_e^2/s)$ and keeping only the leading logarithmic term, proportional to $\log(s/m_e^2)$. Show that, in this approximation, the cross section for forward photon emission can be written

$$\sigma\left(e^+e^- \to \gamma + B\right) \approx \int dx f(x) \cdot \sigma\left(e^+e^- \to B \text{ at } E_{\text{cm}}^2 = (1-x)s\right)$$

where the annihilation cross section is evaluated for the collision of a positron of energy E and an electron of energy $(1-x)E$, and the function $f(x)$, the Weiszacker-Williams distribution function, is given by

$$f(x) = \frac{\alpha}{2\pi}\frac{1+(1-x)^2}{x} \cdot \log\left(\frac{s}{m_e^2}\right)$$

This function arises universally in processes in which a photon is emitted collinearly from an electron line, independent of the subsequent dynamics. We will meet it again, in another context, in Problem 6.2.
5.6 This problem extends the spinor product technology of Problem 5.3 to external photons.
(a) Let k be the momentum of a photon, and let p be another lightlike vector, chosen so that $p \cdot k \neq 0$. Let $u_R(p), u_L(p)$ be spinors of definite helicity for fermions with the lightlike momentum p, defined according to the conventions of Problem 5.3. Define photon polarization vectors as follows:

$$\epsilon_+^\mu(k) = \frac{1}{\sqrt{4p\cdot k}}\bar{u}_R(k)\gamma^\mu u_R(p), \quad \epsilon_-^\mu(k) = \frac{1}{\sqrt{4p\cdot k}}\bar{u}_L(k)\gamma^\mu u_L(p).$$

Use the identity

$$u_L(p)\bar{u}_L(p) + u_R(p)\bar{u}_R(p) = \not{p}$$

to compute the polarization sum

$$\epsilon_+^\mu\epsilon_+^{\nu*} + \epsilon_-^\mu\epsilon_-^{\nu*} = -g^{\mu\nu} + \frac{k^\mu p^\nu + k^\nu p^\mu}{p\cdot k}$$

The second term on the right gives zero when dotted with any photon emission amplitude $\mathcal{M}^\mu$, so we have

$$|\epsilon_+\cdot\mathcal{M}|^2 + |\epsilon_-\cdot\mathcal{M}|^2 = \mathcal{M}^\mu\mathcal{M}^{\nu*}(-g_{\mu\nu});$$

thus, we can use the vectors $\epsilon_+, \epsilon_-$to compute photon polarization sums.
(b) Using the polarization vectors just defined, and the spinor products and the Fierz identity from Problem 5.3, compute the differential cross section for a massless electron and positron to annihilate into 2 photons. Show that the result agrees with the massless limit derived in (5.107):

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha^2}{s}\left(\frac{1+\cos^2\theta}{\sin^2\theta}\right)$$

in the center-of-mass frame. It follows from the result of part (a) that this answer is independent of the particular vector p used to define the polarization vectors; however, the calculation is greatly simplified by taking this vector to be the initial electron 4 -vector.

21.6 Anomalous Magnetic Moment

(там много всего вообще, вот и теория про это.)

21.6.1 On higher order terms

Tenth-Order QED Contribution to the Muon $g-2$

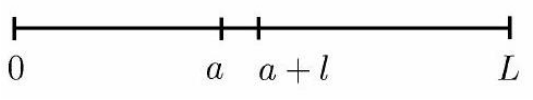
см. Complete Tenth-Order QED Contribution to the Muon Tatsumi Aoyama,1, 2 Masashi Hayakawa,

21.7 Well-known Vacuum Effects

(other will be in the latter part of the note)

21.7.1 Casimir Effect by Vergeles

Рассмотреть свободное безмассовое скалярное поле $\phi(x)$ на отрезке длины L с граничными условиями $\phi(0) = \phi(a) = \phi(a+l) = \phi(L) = 0$, где $a \sim (1/2)L, l \ll L$ (см. рисунок).



Вычислить силу $(-\partial\mathcal{E}_0/\partial l)$, где $\mathcal{E}_0$ - энергия нулевых колебаний вакуума.

Замечание. Рассмотренная ситуация даёт одномерную модель взаимодействия двух близко расположенных параллельных проводящих пластин, устанавливающих соответствующие граничные условия для электромагнитного поля.

Указание. Для решения задачи необходимо пройти следующие этапы:

а) На отрезке $[a, a+l]$ проквантовать скалярное поле с лагранжианом

$$L = \frac{1}{2}\int\limits_0^l \mathrm{d}x \left(\dot{\phi}^2 - (\phi')^2\right)$$

и граничными условиями $\phi(0) = \phi(l)$.

б) Выразить гамильтониан через бозонные операторы рождения и уничтожения и вычислить энергию вакуума $\mathcal{E}_0$.

в) Регуляризовать энергию вакуума двумя методами:

(i)

$$\mathcal{E}_{0(reg)} = \sum_n \frac{\omega_n}{2} e^{-\varepsilon\omega_n/2}, \quad \varepsilon \to +0$$

(ii) При помощи дзета-функции Римана

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

формально полагая

$$\sum_{n=1}^{\infty} n = \zeta(-1) = -\frac{1}{12}$$

Сравнить оба метода.

г) Вычислить регуляризованную энергию системы $\mathcal{E}_{0(reg)}$, определённой на рис. 1, и вычислить силу

$$f = -\frac{\partial\mathcal{E}_0(reg)}{\partial l}$$

21.7.2 Casimir Effect in Different Configurations 1

(I'll add many articles shortly about it here, see special my note about it for now)

21.7.3 Casimir Effect in Different Configurations 2

(I'll add many articles shortly about it here, see special my note about it for now)

21.7.4 Vacuum polarization по Matthew Schwartz

Теория Vacuum polarization по Matthew Schwartz

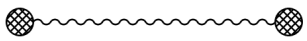
In the previous lecture, we calculated the Casimir effect. We found that the energy of a system involving two plates was infinite; however the observable, namely the force on the plates, was finite. At an intermediate step in calculating the force we needed to model the inability of the plates to restrict ultra-high frequency radiation. We found that the force was independent of the model and only determined by radiation with wavelengths of the plate separation, exactly as physical intuition would suggest. More precisely, we proved the force was independent of how we modeled the interactions of the fields with the plates as long as the very short wavelength modes were effectively removed and the longest wavelength modes were not affected. Some of our models were inspired by physical arguments, as in a step-function cutoff representing an atomic spacing; others, such as the ζ-function regulator, were not. That the calculated force is independent of the model is very satisfying: macroscopic physics (the force) is independent of microscopic physics (the atoms). Indeed, for the Casimir calculation, it doesn't matter if the plates are made of atoms, aether, phlogiston or little green aliens.

The program of systematically making testable predictions about long-distance physics in spite of formally infinite short distance fluctuations is known as renormalization. Because physics at short- and long-distance decouples, we can deform the theory at short distance any way we like to get finite answers - we are unconstrained by physically justifiable models. In fact, our most calculationally efficient deformation will be analytic continuation to $d = 4 - \varepsilon$ dimensions with $\varepsilon \to 0$. In some quantum field theory applications, such as condensed matter systems or string theory, there is a real physical cutoff. The beauty of renormalization, however, is that the existence of a physical cutoff is totally irrelevant: quantitative predictions about long distance physics do not care what the short-distance cutoff really is, or even whether or not it exists. The basic principle of renormalization in quantum field theory is - Observables, such as S-matrix elements, are finite and (in principle) calculable functions of other observables.

One can think of general correlation functions $\langle \Omega | T \{\phi(x_1) \cdots \phi(x_n)\} | \Omega \rangle$ as a useful proxy for observables. Most of the conceptual confusion, both historically and among students learning the subject, stems from trying to express observables in terms of non-observable quantities, such as coupling constants in a Lagrangian. In practice: - Infinite results associated with high-energy divergences may appear in intermediate steps of calculations, such as in loop graphs. - Infinities are tamed by a deformation procedure called regularization. The regulator dependence must drop out of physical predictions. - Coefficients of terms in the Lagrangian, such as coupling constants, are not observable. They can be solved for in terms of the regulator and will drop out of physical predictions. We will find that loops can produce behavior different from anything possible at tree-level. In particular, - Non-analytic behavior, such as $\ln \frac{s}{s_0}$, is characteristic of loop effects. Tree-level amplitudes are always rational polynomials in external momenta and never involve logarithms. In many cases the non-analytic behavior will comprise the entire physical prediction associated with the loop.

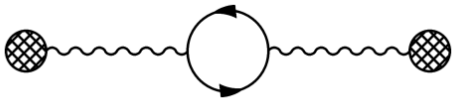
In the previous lecture we gave an example of renormalization in ϕ^4 theory. We found that the expression for a correlation function in terms of the coupling constant λ was infinite: $\langle \phi^4 \rangle_s = \lambda - \frac{\lambda^2}{16\pi^2} \ln\left(\frac{s}{\Lambda^2}\right) + \cdots = \infty$. However, expressing the correlation function at the scale s in terms of the correlation function at a different scale s_0 gave a finite prediction: $\langle \phi^4 \rangle_s = \langle \phi^4 \rangle_{s_0} - \frac{1}{16\pi^2} \langle \phi^4 \rangle_{s_0}^2 \ln\left(\frac{s}{s_0}\right) + \cdots$. Although ϕ^4 theory was just a toy example, renormalization in QED, which we begin in this lecture, is conceptually identical. Recall that the Coulomb potential $V(r) = \frac{e^2}{4\pi r}$

is given by the exchange of a single photon



$$= \frac{e^2}{p^2}$$

Indeed $\frac{1}{4\pi r}$ is just the Fourier transform of the propagator (cf. Lecture I-3). A one-loop correction to Coulomb's law comes from a e^+e^-loop inside the photon line:



This will give us a correction to $V(r)$ proportional to e^4. We will show that while the charge e is infinite it can be replaced by a finite renormalized charge order-by-order in perturbation theory. The physical effect will be a measurable correction to Coulomb's law predicted by quantum field theory with logarithmic scale dependence, as in the ϕ^4 toy model.

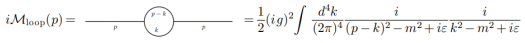
The process represented by the Feynman diagram in Eq. (2) is known as vacuum polarization. The diagram shows the creation of virtual e^+e^-pairs which act like a virtual dipole. In the same way that a dielectric material such as water would get polarized if we put it in a electric field, this vacuum polarization tells us how the vacuum itself gets polarized when it interacts with electromagnetic radiation.

Since the renormalization of the graph is no different than it was in ϕ^4 theory, the only difficult part of calculating vacuum polarization is in the evaluation of the loop. Indeed, the loop in Eq. (2) is complicated, involving photons and spinors, but we can do it exploiting some tricks developed through the hard work of our predecessors. Our approach will be to build up the

QED vacuum polarization graph in pieces, starting with ϕ^3 theory, then scalar QED, and finally real QED. For convenience, some of the more mathematical aspects of regularization are combined into one place in Appendix B, which is meant to provide a general reference. In the following, we assume familiarity with the results from that appendix. 2 Scalar ϕ^3 theory As a warm-up for the vacuum polarization calculation in QED, we will start with scalar ϕ^3 theory with Lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi\left(\Box + m^2\right)\phi + \frac{g}{3!}\phi^3$$

Now we want to compute



$$i\mathcal{M}_{\text{loop}}(p) = \quad = \frac{1}{2}(ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(p-k)^2 - m^2 + i\varepsilon} \frac{i}{k^2 - m^2 + i\varepsilon}$$

which will tell us the 1-loop correction to the Yukawa potential. We will allow the initial and final line to be off-shell: $p^2 \neq m^2$, since we are calculating the correction to the initial ϕ propagator $\frac{i}{p^2-m^2}$ which also must have $p^2 \neq m^2$ to make any sense, and since we will be embedding this graph into a correction to Coulomb's law (see Eq. (14) below). First, we can use the Feynman parameter trick from Appendix B

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[A + (B-A)x]^2}$$

with $A - (p-k)^2 - m^2 + i\varepsilon$ and $B - k^2 - m^2 + i\varepsilon$. Then we complete the square

$$\begin{aligned} A + [B-A]x &= (p-k)^2 - m^2 + i\varepsilon + \left[k^2 - (p-k)^2\right] x \\ &= (k - p(1-x))^2 + p^2 x(1-x) - m^2 + i\varepsilon \end{aligned}$$

which gives

$$i\mathcal{M}_{\text{loop}}(p) = \frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4}\int_0^1 dx \frac{1}{\left[(k-p(1-x))^2 + p^2x(1-x) - m^2 + i\varepsilon\right]^2}$$

Now shift $k^\mu \to k^\mu + p^\mu(1-x)$ in the integral. The measure is unchanged, and we get

$$i\mathcal{M}_{\text{loop}}(p) = \frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4}\int_0^1 dx \frac{1}{\left[k^2 - (m^2 - p^2x(1-x)) + i\varepsilon\right]^2}$$

At this point, we need to introduce a regulator. We will use Pauli-Villars regularization (see Appendix B), which adds a fictitious scalar of mass Λ with fermionic statistics. This particle is an unphysical ghost particle. We can use the Pauli-Villars formula from Appendix B:

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{\left(k^2 - \Delta + i\varepsilon\right)^2} = -\frac{i}{16\pi^2}\ln\frac{\Delta}{\Lambda^2}$$

Comparing to Eq. (8), we have $\Delta = m^2 - p^2x(1-x)$ so that

$$i\mathcal{M}_{\text{loop}}(p) = -\frac{ig^2}{32\pi^2}\int_0^1 dx \ln\left(\frac{m^2 - p^2x(1-x)}{\Lambda^2}\right)$$

This integral can be done - the integrand is perfectly well behaved between $x = 0$ and $x = 1$. For $m = 0$ it has the simple form

$$\mathcal{M}_{\text{loop}}(p) = \frac{g^2}{32\pi^2}\left[2 - \ln\frac{-p^2}{\Lambda^2}\right]$$

Note that the 2 cannot be physical, because we can remove it by redefining $\Lambda^2 \to \Lambda^2 e^{-2}$. Also note that when this diagram contributes to the Coulomb potential (as in Eq. (14) below), the virtual momentum p^μ is spacelike ($p^2 < 0$) so $\ln\frac{-p^2}{\Lambda^2}$ is real. Then,

$$\mathcal{M}_{\text{loop}}(p) = -\frac{g^2}{32\pi^2}\ln\frac{Q^2}{\Lambda^2}$$

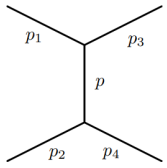
An important point is that the regulator scale Λ has to be just a number, independent of any external momenta. With the Pauli-Villars regulator we are using here, Λ is the mass of some heavy fictitious particle. It corresponds to a deformation of the theory at very high energies/short distances, like the modeling of the wall in the Casimir force. On the other hand, Q is a physical scale like the plate separation in the Casimir force. Thus Λ cannot depend on Q. In particular, the $\ln Q^2$ dependence cannot be removed by a redefinition of Λ like the 2 in Eq. (11) was. This point is so important it's worth repeating: the short distance deformation (Λ) can't depend on long-distance physical quantities (Q). This separation of scales is critical to being able to take $\Lambda \to \infty$ to make predictions by relating observables at different long-distance scales such as Q and Q_0. The coefficient of $\ln Q^2$ is in fact regulator independent and will generate the physical prediction from the loop.

2.1

Renormalization

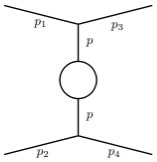
The diagram we computed is a correction to the tree level ϕ propagator. To see this, observe that the propagator is essentially the same as the t-channel scattering diagram

$$i\mathcal{M}^0(p) = \quad = (ig)^2\frac{i}{p^2}$$



If we insert our scalar bubble in the middle, we get

$$i\mathcal{M}^1(p) = \quad = (ig)^2\frac{i}{p^2}i\mathcal{M}_{\text{loop}}(p)\frac{i}{p^2} = ig^2\frac{1}{p^2}\left[-\frac{g^2}{32\pi^2}\ln\frac{-p^2}{\Lambda^2}\right]\frac{1}{p^2}$$



Since $p^2 < 0$, let us write $Q^2 = -p^2$ with $Q > 0$. Then,

$$\mathcal{M}(Q) = \mathcal{M}^0(Q) + \mathcal{M}^1(Q) = \frac{g^2}{Q^2}\left(1 - \frac{1}{32\pi^2}\frac{g^2}{Q^2}\ln\frac{Q^2}{\Lambda^2} + \mathcal{O}\left(g^4\right)\right)$$

Note that g is not a number in ϕ^3 theory but has dimensions of mass. This actually makes ϕ^3 a little more confusing than QED, but not insurmountably so. Let's just simplify things by writing the whole thing in terms of a new Q-dependent variable $\tilde{g}^2 \equiv \frac{g^2}{Q^2}$ which is dimensionless. Then

$$\mathcal{M}(Q) = \tilde{g}^2 - \frac{1}{32\pi^2}\tilde{g}^4\ln\frac{Q^2}{\Lambda^2} + \mathcal{O}\left(\tilde{g}^6\right)$$

Then we can define a renormalized coupling $\hat{g}_R$ at some fixed scale Q_0 by

$$\tilde{g}_R \equiv \mathcal{M}\left(Q_0\right)$$

This is called a renormalization condition. It is a definition, and by definition, it holds to all orders in perturbation theory. The renormalization condition defines the coupling in terms of an observable. Therefore, you can only have one renormalization condition for each parameter in the theory. This is critical to the predictive power of quantum field theory. It follows that $\tilde{g}_R^2$ is a formal power series in $\tilde{g}$

$$\tilde{g}_R = \mathcal{M}\left(Q_0\right) = \tilde{g}^2 - \frac{1}{32\pi^2}\tilde{g}^4\ln\frac{Q_0^2}{\Lambda^2} + \mathcal{O}\left(\tilde{g}^6\right)$$

which can be inverted to give $\tilde{g}$ as a power series in $\tilde{g}_R$

$$\bar{g}^2 = \tilde{g}_R^2 + \frac{1}{32\pi^2}\tilde{g}_R^4\ln\frac{Q_0^2}{\Lambda^2} + \mathcal{O}\left(\tilde{g}_R^6\right)$$

Substituting into Eq.(16) produces a prediction for the matrix element at the scale Q in terms of the matrix element at the scale Q_0

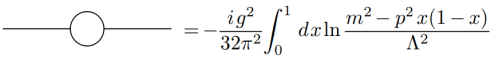
$$\mathcal{M}(Q) = \tilde{g}^2 - \frac{1}{32\pi^2}\tilde{g}^4\ln\frac{Q^2}{\Lambda^2} + \mathcal{O}\left(\tilde{g}^6\right) = \tilde{g}_R^2 + \frac{1}{32\pi^2}\tilde{g}_R^4\ln\frac{Q_0^2}{Q^2} + \mathcal{O}\left(\tilde{g}_R^6\right)$$

Thus we can measure $\mathcal{M}$ at one Q and then make a nontrivial prediction at another value of Q.

3

Vacuum polarization in QED

In ϕ^3 theory, we found

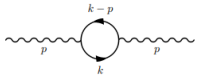


$$= -\frac{ig^2}{32\pi^2}\int_0^1 dx \ln\frac{m^2 - p^2x(1-x)}{\Lambda^2}$$

The integral in QED is quite similar. We will first evaluate the vacuum polarization graph in scalar QED, and then in spinor QED.

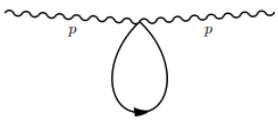
Scalar QED

In scalar QED the vacuum polarization diagram is



$$= (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(p-k)^2 - m^2 + i\varepsilon} \frac{i}{k^2 - m^2 + i\varepsilon} (2k^\mu - p^\mu)(2k^\nu - p^\nu)$$

For external photons, we could contract the μ and ν indices with polarization vectors, but instead we keep them free so that this diagram can be embedded in a Coulomb exchange diagram as in Eq. (14). This integral is the same as in ϕ^3 theory, except for the numerator factors. In scalar QED there is also another diagram



$$= 2ie^2 g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon}$$

So we can add them together to get

$$i\Pi_2^{\mu\nu} = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{-4k^\mu k^\nu + 2p^\mu k^\nu + 2p^\nu k^\mu - p^\mu p^\nu + 2g^{\mu\nu}\left((p-k)^2 - m^2\right)}{\left[(p-k)^2 - m^2 + i\varepsilon\right]\left[k^2 - m^2 + i\varepsilon\right]}$$

Fortunately, we do not need to evaluate the entire integral. By looking at what possible form it could have, we can isolate the part which will contribute to a correction to Coulomb's law and just calculate that part. By Lorentz invariance, the most general form that $\Pi_2^{\mu\nu}$ could have is

$$\Pi_2^{\mu\nu} = \Delta_1\left(p^2, m^2\right) p^2 g^{\mu\nu} + \Delta_2\left(p^2, m^2\right) p^\mu p^\nu$$

for some form factors Δ_1 and Δ_2. Note that $\Pi_2^{\mu\nu}$ cannot depend on k^μ, since k^μ is integrated over.

As a correction to Coulomb's law, this vacuum polarization graph will contribute to the same process that the photon propagator does. Let us define the photon propagator in momentum space by

$$\langle\Omega\left|T\left\{A^\mu(x)A^\nu(y)\right\}\right|\Omega\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} iG^{\mu\nu}(p)$$

Note that this expression only depends on $x-y$ by translation invariance. This is the all-orders non-perturbative definition of the propagator $G^{\mu\nu}(p)$, which is sometimes called the dressed propagator. At leading order, in Feynman gauge, the dressed propagator reduces to the free propagator

$$iG^{\mu\nu}(p) = \frac{-ig^{\mu\nu}}{p^2 + i\varepsilon} + \mathcal{O}\left(e^2\right)$$

Including the 1-loop correction, with the parametrization in Eq. (24), the propagator is (suppressing the is pieces)

$$\begin{aligned} iG^{\mu\nu}(p) &= \frac{-ig^{\mu\nu}}{p^2} + \frac{-ig^{\mu\alpha}}{p^2} i\Pi^2_{\alpha\beta} \frac{-ig^{\beta\nu}}{p^2} + \mathcal{O}\left(e^4\right) \\ &= \frac{-ig^{\mu\nu}}{p^2} + \frac{-i}{p^2}\left(\Delta_1 g^{\mu\nu} + \Delta_2 \frac{p^\mu p^\nu}{p^2}\right) + \mathcal{O}\left(e^4\right) \\ &= -i\frac{(1+\Delta_1)\, g^{\mu\nu} + \Delta_2 \frac{p^\mu p^\nu}{p^2}}{p^2 + i\varepsilon} \end{aligned}$$

Note that we are calculating loop corrections to a Green's function, not an S-matrix element, so we do not truncate the external propagators and add polarization vectors. One point of using a dressed propagator is that once we calculate Δ_1 and Δ_2 we can just use $G^{\mu\nu}(p)$ instead of the tree level propagator in QED calculations to include the loop effect.

Next note that the Δ_2 term is just a change of gauge - it gives a correction to the unphysical gauge parameter ξ in covariant gauges. Since ξ drops out of any physical process, by gauge invariance, so will Δ_2. Thus we only need to compute Δ_1. This means extracting the term proportional to $g^{\mu\nu}$ in $\Pi^{\mu\nu}$.

Most of the terms in the amplitude in Eq. (23) cannot give $g^{\mu\nu}$. For example, the $p^\mu p^\nu$ term must be proportional to $p^\mu p^\nu$ and can therefore only contribute to Δ_2, so we can ignore it. For the $p^\mu k^\nu$ term, we can pull p^μ out of the integral, so whatever the remaining integral gives, it must provide a p^ν by Lorentz invariance. So these terms can be ignored too. The $k^\mu k^\nu$ term is important - it may give a $p^\mu p^\mu$ piece, but may also give a $g^{\mu\nu}$ piece, which is what we're looking for. So we only need to consider

$$\Pi_2^{\mu\nu} = ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{-4k^\mu k^\nu + 2g^{\mu\nu}\left((p-k)^2 - m^2\right)}{(p-k)^2 - m^2 + i\varepsilon]\,[k^2 - m^2 + i\varepsilon]}$$

Now we need to compute the integral. The denominator can be manipulated using Feynman parameters just as with the ϕ^3 theory

$$\Pi_2^{\mu\nu} = ie^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{-4k^\mu k^\nu + 2g^{\mu\nu}\left((p-k)^2 - m^2\right)}{\left[(k - p(1-x))^2 + p^2 x(1-x) - m^2 + i\varepsilon\right]^2}$$

However, now when we shift $k^\mu \to k^\mu + p^\mu(1-x)$ we get a correction to the numerator. We get

$$\Pi_2^{\mu\nu} = ie^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{-4\left(k^\mu + p^\mu(1-x)\right)\left(k^\nu + p^\nu(1-x)\right) + 2g^{\mu\nu}\left((xp-k)^2 - m^2\right)}{\left[k^2 + p^2 x(1-x) - m^2 + i\varepsilon\right]^2}$$

As we have said, we don't care about $p^\mu p^\nu$ pieces, or pieces linear in p^ν. Also, pieces like $p \cdot k$ are odd under $k \to -k$ while the rest of the integrand, including the measure, is even. So these terms must give zero by symmetry. All that is left is

$$\Pi_2^{\mu\nu} = 2ie^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{-2k^\mu k^\nu + g^{\mu\nu}\left(k^2 + x^2p^2 - m^2\right)}{\left[k^2 + p^2 x(1-x) - m^2 + i\varepsilon\right]^2}$$

It looks like this integral is much more badly divergent than the ϕ^3 theory - it is now quadratically instead of logarithmically divergent. That is, if we cut off at $k = \Lambda$ we will get something proportional to Λ^2 due to the $k^\mu k^\nu$ and k^2 terms. Quadratic divergences are not technically a problem for

renormalization. However, in Lecture III-8 we will see, on very general grounds, that in gauge theories like scalar QED, all divergences should be logarithmic. In this case, the quadratic divergence from the $k^\mu k^\nu$ term and the k^2 term precisely cancel due to gauge invariance. This cancellation can only be seen using a regulator which respects gauge invariance, such as dimensional regularization. In d dimensions (using $k^\mu k^\nu \to \frac{1}{d}k^2 g^{\mu\nu}$ from Appendix B), the integral becomes

$$\Pi_2^{\mu\nu} = 2ie^2\mu^{4-d}g^{\mu\nu}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d\left[\frac{\left(1-\frac{2}{d}\right)k^2+x^2p^2-m^2}{[k^2+p^2x(1-x)-m^2+i\varepsilon]^2}\right.}$$

Using the formulas from Appendix B:

$$\int \frac{d^dk}{(2\pi)^d}\frac{k^2}{\left(k^2-\Delta+i\varepsilon\right)^2} = -\frac{d}{2}\frac{i}{(4\pi)^{d/2}}\frac{1}{\Delta^{1-\frac{d}{2}}}\Gamma\left(\frac{2-d}{2}\right)$$
$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2-\Delta+i\varepsilon\right)^2} = \frac{i}{(4\pi)^{d/2}}\frac{1}{\Delta^{2-\frac{\pi}{2}}}\Gamma\left(\frac{4-d}{2}\right)$$

with $\Delta = m^2 - p^2x(1-x)$ we find

$$\Pi_2^{\mu\nu} = -2\frac{e^2}{(4\pi)^{d/2}}g^{\mu\nu}\mu^{4-d}\int_0^1 dx\left[\left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}} + \left(x^2p^2-m^2\right)\Gamma\left(2-\frac{d}{2}\right)\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}\right]$$

Using $\Gamma\left(2-\frac{d}{2}\right) = \left(1-\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)$ this simplifies to

$$\Pi_2^{\mu\nu} = -2\frac{e^2}{(4\pi)^{d/2}}p^2g^{\mu\nu}\Gamma\left(2-\frac{d}{2}\right)\mu^{4-d}\int_0^1 dx x(2x-1)\left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}}$$

For completeness, we also give the result including also the $p^{\mu i}p^\nu$ terms

$$\Pi_2^{\mu\nu} = \frac{-2e^2}{(4\pi)^{d/2}}\left(p^2g^{\mu\nu}-p^\mu p^\nu\right)\Gamma\left(2-\frac{d}{2}\right)\mu^{4-d}\int_0^1 dx x(2x-1)\left(\frac{1}{m^2-p^2x(1-x)}\right)^{2-\frac{d}{2}}$$

You should verify this through direct calculation (see Problem 1), but it is the unique result consistent with Eq. (35) which satisfies the Ward identity $p_\mu\Pi_2^{m=n} = 0$. Expanding $d = 4-\varepsilon$ we get, in the $\epsilon \to 0$ limit,

$$\Pi_2^{\mu\nu} = -\frac{e^2}{8\pi^2}\left(p^2g^{\mu\nu}-p^{\mu k}p^\nu\right)\int_0^1 dx x(2x-1)\left[\frac{2}{\varepsilon}+\ln\left(\frac{4\pi e^{-\gamma\mu^2}}{m^2-p^2x(1-x)}\right)+\mathcal{O}(\varepsilon)\right]$$

The $\frac{1}{c}$ gives the infinite, regulator-dependent constant. It is also standard to define $\tilde{\mu}^2 = 4\pi e^{-\gamma_r}\mu^2$ which removes the $\ln(4\pi)$ and $e^{-\gamma x}$ factors. For $Q^2 = -p^2 > 0$ and $m\&Q$, the integral over x is easy to do and we find

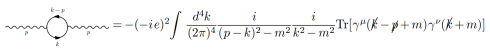
$$\Pi_2^{\mu\nu} = -\frac{e^2}{48\pi^2}\left(p^2g^{\mu\nu}-p^\mu p^\nu\right)\left(\frac{2}{\varepsilon}+\ln\frac{\tilde{\mu}^2}{Q^2}+\frac{8}{3}\right), \quad m\&Q$$

At this point, rather than continue with the scalar QED calculation, let's calculate the loop in QED, as it's almost exactly the same.

3.2

Spinor QED

In spinor QED, the loop is



where the -1 in front comes from the fermion loop. Note that there is only 1 diagram in this case. Using our trace formulas (see Lecture II-6), we find

$$\mathrm{Tr}\left[\gamma^\mu(\not{k}-\not{p}+m)\gamma^\nu(\not{k}+m)\right] = 4\left[-p^\mu k^\nu - k^\mu p^\nu + 2k^\mu k^\nu + g^{\mu\nu}\left(-k^2 + p\cdot k + m^2\right)\right]$$

We can drop the p^μ and p^ν terms as before giving

$$i\Pi_2^{\mu\nu} = -4e^2\int\frac{d^4k}{(2\pi)^4}\frac{2k^\mu k^\nu + g^{\mu\nu}\left(-k^2+p\cdot k+m^2\right)}{\left[(p-k)^2-m^2+i\varepsilon\right]\left[k^2-m^2+i\varepsilon\right]}$$

Introducing Feynman parameters and changing $k^\mu \to k^\mu + p^\mu(1-x)$ and again dropping the $p^{\mu t}$ and p^ν terms we get

$$\Pi_2^{\mu\nu} = 4ie^2\int\frac{d^4k}{(2\pi)^4}\int_0^1 dx\frac{2k^\mu k^\nu - g^{\mu\nu}\left(k^2 - x(1-x)p^2 - m^2\right)}{\left[k^2+p^2x(1-x)-m^2\right]^2}$$

This integral is quite similar to the one for scalar QED, Eq. (31). The result is

$$\begin{aligned}\Pi_2^{\mu\nu} &= -8p^2g^{\mu\nu}\frac{e^2}{(4\pi)^{d/2}}\Gamma\left(2-\frac{d}{2}\right)\mu^{4-d}\int_0^1 dx x(1-x)\left(\frac{1}{m^2-p^2x(1-x)}\right)^{2-\frac{d}{2}}\\ &= -\frac{e^2}{2\pi^2}p^2g^{\mu\nu}\int_0^1 dx x(1-x)\left[\frac{2}{\varepsilon}+\ln\left(\frac{\tilde{\mu}^2}{m^2-p^2x(1-x)}\right)+\mathcal{O}(\varepsilon)\right]\end{aligned}$$

So we find (for large $Q^2 = -p^2 \gg m^2$)

$$\Pi_2^{\mu\nu} = -\frac{e^2}{12\pi^2}p^2g^{\mu\nu}\left(\frac{2}{\varepsilon}+\ln\frac{\tilde{\mu}^2}{Q^2}+\frac{5}{3}+\mathcal{O}(\varepsilon)\right),\quad m\leqslant Q$$

We see that the electron loop gives the same pole and $\ln\frac{\mu^2}{Q^2}$ terms as a scalar loop, multiplied by a factor of 4.

It is not hard to compute the $p^\mu p^\nu$ pieces as well (see Problem 1). The full result is

$$\Pi_2^{\mu\nu} = \frac{-8e^2}{(4\pi)^{d/2}}\left(p^2g^{\mu\nu}-p^\mu p^\nu\right)\Gamma\left(2-\frac{d}{2}\right)\mu^{4-d}\int_0^1 dx x(1-x)\left(\frac{1}{m^2-p^2x(1-x)}\right)^{2-\frac{d}{2}}$$

which, as in the scalar QED case, automatically satisfies the Ward identity.

4

Physics of vacuum polarization

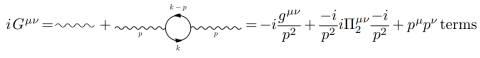
We have found that vacuum polarization loop in QED gives

$$i\Pi_2^{\mu\nu} = i\left(-p^2 g^{\mu\nu} + p^\mu p^\nu\right) e^2 \Pi_2\left(p^2\right)$$

where

$$\Pi_2\left(p^2\right) = \frac{1}{2\pi^2}\int_0^1 dx x(1-x)\left[\frac{2}{\varepsilon} + \ln\left(\frac{\tilde{\mu}^2}{m^2 - p^2 x(1-x)}\right)\right]$$

Thus the dressed photon propagator at 1-loop in Feynman gauge is



$$= -i\frac{\left(1 - e^2\Pi_2\left(p^2\right)\right) g^{\mu\nu}}{p^2} + p^\mu p^\nu \text{ terms}$$

This directly gives the Fourier transform of the corrected Coulomb potential:

$$\tilde{V}(p) = e^2 \frac{1 - e^2\Pi_2\left(p^2\right)}{p^2}$$

Now we need to renormalize. A natural renormalization condition is that the potential between two particles at some reference scale r_0 should be $V\left(r_0\right) \equiv -\frac{e_R^2}{4\pi r_0}$ which would define a renormalized e_R. It is easier to continue working in momentum space and to define the renormalized charge as $\tilde{V}\left(p_0^2\right) \equiv e_R^2 p_0^{-2}$ exactly. So

$$e_R^2 \equiv p_0^2 \hat{V}\left(p_0^2\right) = e^2 - e^4\Pi_2\left(p_0^2\right) + \cdots$$

Solving for the bare coupling e as a function of e_R to order e_R gives

$$e^2 = e_R^2 + e_R\Pi_2\left(p_0^2\right) + \cdots$$

Since $\Pi_2\left(p_0^2\right)$ is infinite, e is infinite as well, but that is ok since e is not observable. The potential at another scale p, which is measurable, is

$$p^2\tilde{V}(p) = e^2 - e^4\Pi_2\left(p^2\right) + \cdots = e_R^2 - e_R^4\left[\Pi_2\left(p^2\right) - \Pi_2\left(p_0^2\right)\right] + \cdots$$

For concreteness, let us take $p_0 = 0$, corresponding to $r = \infty$ so that the renormalized electric charge agrees with the macroscopically measured electric charge. Then

$$\Pi_2\left(p^2\right) - \Pi_2(0) = -\frac{1}{2\pi^2}\int_0^1 dx x(1-x)\ln\left[1 - \frac{p^2}{m^2}x(1-x)\right]$$

Thus, we have

$$\tilde{V}\left(p^2\right) = \frac{e_R^2}{p^2}\left\{1 + \frac{e_R^2}{2\pi^2}\int_0^1 dx x(1-x)\ln\left[1 - \frac{p^2}{m^2}x(1-x)\right] + \mathcal{O}\left(e_R^4\right)\right\}$$

which is a totally finite correction to the Coulomb potential. It is also a well-defined perturbation expansion in terms of a small parameter e_R which is also finite. We will now study some of the physical implications of this potential

Small momentum - Lamb shift

(?? мб больший раздел про Лембовский сдвиг сделаю, пока так вот.)

Суть Лэмбовского сдвига (????)

The Lamb shift is a difference in energy between two energy levels ${}^2S_{1/2}$ and ${}^2P_{1/2}$ (in term symbol notation) of the hydrogen atom which was not predicted by the Dirac equation, according to which these states should have the same energy. Interaction between vacuum energy fluctuations and the hydrogen electron in these different orbitals is the cause of the Lamb shift, as was shown subsequent to its discovery. The Lamb shift has since played a significant role through vacuum energy fluctuations in theoretical prediction of Hawking radiation from black holes.

This effect was first measured in 1947 in the Lamb-Retherford experiment on the hydrogen microwave spectrum [1] and this measurement provided the stimulus for renormalization theory to handle the divergences. It was the harbinger of modern quantum electrodynamics developed by Julian Schwinger, Richard Feynman, Ernst Stueckelberg, Sin-ltiro Tomonaga and Freeman Dyson. Lamb won the Nobel Prize in Physics in 1955 for his discoveries related to the Lamb shift.

In 1978, on Lamb's 65th birthday, Freeman Dyson addressed him as follows: "Those years, when the Lamb shift was the central theme of physics, were golden years for all the physicists of my generation. You were the first to see that this tiny shift, so elusive and hard to measure, would clarify our thinking about particles and fields."

Теория по Schwartz-y

First, let's look at the small-momentum, large-distance limit. For $|p^2| < m$

$$\int_0^1 dx x(1-x) \ln\left[1 - \frac{p^2}{m^2}x(1-x)\right] \approx \int_0^1 dx x(1-x)\left[-\frac{p^2}{m^2}x(1-x)\right] = -\frac{p^2}{30m^2}$$

$$\text{implying } \tilde{V}(p) = \frac{e_R^2}{p^2} - \frac{e_R^4}{60\pi^2 m^2} + \cdots$$

The Fourier transform of a 1 is $\delta(r)$, so we find

$$V(r) = -\frac{e_R^2}{4\pi r} - \frac{e_R}{60\pi^2 m^2}\delta(r)$$

This agrees with the Coulomb potential up to a correction known the Uehling term [Uehling Phys. Rev. 48, 55-63 (1935)]

What is the physical effect of this extra term? One way to find out is to plug this potential into the Schrödinger equation and see how the states of the Hydrogen atom change. Equivalently, we can evaluate the effect in time-independent perturbation theory by evaluating the leading order energy shift $\Delta E = \langle\psi_i|\Delta V|\psi_i\rangle$ using $\Delta V = -\frac{e^4}{60\pi^4 m^2}\delta(r)$. Since only the $L = 0$ atomic orbitals have support at $r = 0$ this extra term will only affect the S states of the Hydrogen atom. The energy is negative, so their energies will be lowered. You might recall that, at leading order, the energy spectrum of the Hydrogen atom is determined only by the principal atomic number n, so the $2P_{1/2}$ and $2S_{1/2}$ levels (for example) are degenerate. Thus the Uehling term contributes to the splitting of these levels, known as the Lamb shift. It changes the $2S_{1/2}$ state by -27MHz which is

a measurable contribution to the −1028MHz Lamb shift. More carefully, you can show in Problem 2 that the 1-loop potential is

$$V(r) = -\frac{e^2}{4\pi r}\left(1 + \frac{e^2}{6\pi^2}\int_1^\infty dx e^{-2mrx}\frac{2x^2+1}{2x^4}\sqrt{x^2-1}\right)$$

This is known as the Uehling potential [Uehling 1935]. For $r \gg \frac{1}{m}$,

$$V(r) = -\frac{\alpha}{r}\left[1 + \frac{\alpha}{4\sqrt{\pi}}\frac{1}{(mr)^{3/2}}e^{-2mr}\right], \quad r \gg \frac{1}{m}$$

This shows that the finite correction has extent $1/m = r_e$, the Compton radius of the electron. Since r_e is much smaller than the characteristic size of the L modes, the Bohr radius $a_0 \sim \frac{m}{\alpha}$, our δ-function approximation is valid.

By the way, the measurement of the Lamb shift in 1947 by Wallis Lamb [Ref] was one of the key experiments that convinced people to take quantum field theory seriously. Measurements of the hyperfine splitting between the $2S_{1/2}$ and $2P_{1/2}$ states of the Hydrogen atom had been attempted for many years, but it was only by using microwave technology developed during the Second World War that Lamb was able to provide an accurate measurement. He found $\Delta E \simeq 1000$MHz. Shortly after his measurement, Hans Bethe calculated the dominant theoretical contribution. His calculation was of a vertex correction which was infrared divergent. Now we know that the infrared divergence is canceled when all the relevant contributions are included, but Bethe simply cut off the divergence by hand at what he argued was a natural physical scale, the electron mass. His result was that $\Delta E = -\frac{Z^4\alpha^5\ \mathrm{m}_e}{12\pi}\ln(\alpha^4 Z^4) \approx -1000$MHz in excellent agreement with Lamb's value. The next year, Feynman, Schwinger and Tomonaga all independently provided the complete calculation, including the Uehling term and the spin-orbit coupling. Due to a subtlety regarding gauge invariance, only Tomonaga got it right the first time. The full 1-loop result gives $E\left(2S_{1/2}\right) - E\left(2P_{1/2}\right) = 1051$MHz. The current best measurement of this shift is 1054 MHz. 4.2

Large momentum - logarithms of p

In the small distance limit $r \& \frac{1}{m}$ it is easier to consider the potential in momentum space. Then we have from Eq. (51)

$$\begin{aligned}
\tilde{V}\left(p^2\right) &= \frac{e_R^2}{p^2} + \frac{e_R^4}{p^2}\frac{1}{2\pi^2}\int_0^1 dx x(1-x)\ln\left[1 - \frac{p^2}{m^2}x(1-x)\right] + \mathcal{O}\left(e_R^6\right) \\
&\approx \frac{e_R^2}{p^2} + \frac{e_R^4}{p^2}\frac{1}{2\pi^2}\ln\left(-\frac{p^2}{m^2}\right)\int_0^1 dx x(1-x) + \mathcal{O}\left(e_R^6\right) \\
&= \frac{e_R^2}{p^2}\left(1 + \frac{e_R^2}{12\pi^2}\ln\frac{-p^2}{m^2}\right) + \mathcal{O}\left(e_R^6\right)
\end{aligned}$$

Recall that for t-channel exchange, $Q^2 = -p^2 > 0$ so this logarithm is real. If we compare the potential at two high energy scales $Q \gg m$ and $Q_0 > m$ we find

$$Q^2\tilde{V}\left(Q^2\right) - Q_0^2 V\left(Q_0^2\right) = \frac{e_R^4}{12\pi^2}\ln\frac{Q_0^2}{Q^2}$$

which is independent of m. Note however, that setting $m = 0$ directly in Eq. (57) results in a divergence. This kind of divergence is known as a mass singularity, which is a type of infrared divergence. In this case, the divergence is naturally regulated by $m \neq 0$. On other occasions we will have to introduce an artificial infrared regulator (like a photon mass) to produce finite answers. Infrared divergences are the subject of Lecture III-6. One way to write the radiative correction to the potential is where

$$\tilde{V}\left(Q^2\right) = \frac{e_{\text{eff}}^2(Q)}{p^2}$$

In this case, for simplicity, we have defined the renormalized charge $e_R \equiv e_{\text{eff}}\left(m\right)$ at $Q = m$ rather than at $Q = 0$. (One could also define ε_R at $Q = 0$ as with the Uehling potential; however then one would need to include the full m dependence to regulate the $m = 0$ singularity.)

Eq. (62) is to be interpreted as an effective charge in QED which grows as the distance gets smaller (momentum gets larger). Near any particular fixed value of the momentum transfer p^μ, the potential looks like a Coulomb potential with a charge $e_{\text{eff}}\left(p^2\right)$ instead of e_R. This is a useful concept because the charge depends only weakly on p^2, through a logarithm. Thus, for small variations of p around a reference scale, the same effective charge can be used. Eq. (62) only comes into play when one compares the charge at very different momentum transfers.

The sign of the coefficient of the $\ln \frac{Q}{m}$ term is very important; this sign implies the effective charge gets larger at short distances. At large distances, the charge is increasingly screened by the virtual electron-positron dipole pairs. At smaller distances, there is less room for the screening and the effective charge increases. However, the effective charge only increases at small distances very slowly. In fact taking $\alpha_R = \frac{e_R^2}{4\text{x}} = \frac{1}{137}$ so that $e_R = 0.303$ we get an effective fine structure constant of the form

$$\alpha_{e\text{eff}}\left(-p^2\right) = \frac{1}{137}\left[1 + 0.00077 \ln\left(\frac{-p^2}{m^2}\right)\right]$$

Because the coefficient of the logarithm is numerically small, one has to measure the potential at extremely high energies to see its effect. In fact, only very few high precision measurements are sensitive to this logarithm.

Despite the difficulty of probing extremely high energies in QED experimentally, one can at least ask what would happen if we attempted scattering at $t = -p^2 \gg m^2$. From Eq. (63) we can see that at some extraordinarily high energies $p \sim 10^{286}$eV, the loop correction, the logarithm, is as important as the tree-level value, the 1. Thus perturbation theory is breaking down. At these scales, the 2-loop value will also be as large as the 1-loop and tree level values, and so on. The scale where this happens is known as a Landau pole. So, - QED has a Landau Pole: perturbation theory breaks down at short distances. This means that QED is not a complete theory in the sense that it does not tell us how to compute scattering amplitudes at all energies.

4.3

Running coupling

It is not difficult to include certain higher-order corrections to the effective electric charge. Adding more loops in series, we can sum a set of graphs to all orders in the coupling constant:

These corrections to the propagator immediately translate into corrections to the momentumspace potential:

$$\tilde{V}(Q) = -\frac{e_R^2}{Q^2}\left[1 + \frac{e_R^2}{12\pi^2}\ln\frac{Q^2}{m^2} + \left(\frac{e_R^2}{12\pi^2}\ln\frac{Q^2}{m^2}\right)^2 + \cdots\right]$$
$$= -\frac{1}{Q^2}\left[\frac{e_R^2}{1 - \frac{c\hbar}{12\pi^2}\ln\frac{Q^2}{m^2}}\right]$$

So now the momentum-dependent electric charge becomes

$$e_{\text{eff}}^2(Q) = \frac{e_R^2}{1 - \frac{c_R^2}{12\pi^2}\ln\frac{Q^2}{m^2}}$$

which is known as a running coupling. Note we have defined this running coupling to have the same renormalization condition as the one-loop effective charge: $e_{\text{eff}} = e_R$ at $p^2 = -m^2$. Although the running coupling includes contributions from all orders in perturbation theory, it still has a Landau pole at $p \sim 10^{286}$eV.

Running couplings will play an increasingly important role as we study more complicated problems in quantum field theory. They are best understood through the renormalization group. As a preview of how the renormalization group works, note that Eq. (66) can be written as

$$\frac{1}{e_{eff}^2(Q)} = \frac{1}{e_R^2} - \frac{1}{12\pi^2}\ln\frac{Q^2}{m^2}$$

The renormalization group comes from the simple observation that there is nothing special about the renormalization point. Here we have defined $e_R = e_{eff}(m)$, but we could have renormalized at any other point μ^2 instead of m^2, and the results would be the same. Then we would have

$$\frac{1}{e_{eff}^2(Q)} = \frac{1}{e_{eff}^2(\mu)} - \frac{1}{12\pi^2}\ln\frac{Q^2}{\mu^2}$$

The left hand side is independent of μ. So taking the μ derivative gives Or

$$0 = -\frac{2}{e_{\text{eff}}^3}\frac{d}{d\mu}e_{\text{eff}} + \frac{1}{12\pi^2}\frac{2}{\mu}$$

$$\mu\frac{de_{\text{eff}}}{d\mu} = \frac{e_{\text{eff}}^3}{12\pi^2}$$

This is known as a renormalization group equation. We even have a special name for the right hand side of this particular equation, the $\boldsymbol{\beta}$-function. In general,

$$\mu\frac{de}{d\mu} \equiv \beta(e)$$

and we have derived that $\beta(e) = \frac{e^3}{12\pi^2}$ at 1-loop. The renormalization group is the subject of Lecture Lecture III-9.

21.8 Finding Main Fundamental Constants Using QFT

21.8.1 Testing Electron Charge

(??? how to do it???)

21.8.2 Testing Fine Structure Constant

(??? how to do it???)

21.8.3 Testing Ridberg's Constant

21.9 Finding Some Other Other Constants Using QFT

(there are many constants, so I'll also write about how some of them can be determined)

21.10 Determining Fundamental Quantities of Nature

21.10.1 Determining Proton Radius

(main theory about it, some experiments, some main discussions. I heard more than 1 hour talk about it, but have no time to write what is going on there)

21.10.2 Determining Proton Radius: Extra Understanding

(a lot of things were done about it, so there will be a lot of theory which I can write here)

22 Radiation and Scattering of Light

22.1 Radiation of Light by Landau 4

22.1.1 Electromagnetic Interaction Operator

(???)

22.1.2 Emission and Absorption

22.1.3 Dipole Radiation

(see field theory and QM (??))

22.1.4 Electric Multipole Radiation

(see field theory and QM (??))

22.1.5 Magnetic Multipole Radiation

22.1.6 Angular Distribution and Polarization of Radiation

22.1.7 Introduction to Quantum Optics (?!??!)

(тут связка с кв оптикой, потом мб больше будет теор разделов. some very important sections, actually)

22.2 Fundamentals of Scattering

(??? see book of Landau??)

22.2.1 Scattering Tensor

(??? see book of Landau??)

22.2.2 Scattering by Freely Oriented Systems

(??? see book of Landau??)

22.2.3 Scattering on Molecules

(??? see book of Landau??)

22.2.4 Natural Width of Spectral Lines

22.2.5 Resonance Fluorescence

23 Phenomena in Particle Physics

(see special note about particle physics)

24 Other Models of Quantum Electrodynamics

24.1 Electron Interaction by Landau (??)

80.Рассеяние электрона во внешнем поле 360 81.Рассеяние электронов и позитронов на электроне364 82.Ионизационные потери быстрых частиц 374 83.Уравнение Брейта381 84.Позитроний388 85.Взаимодействие атомов на далеких расстояниях 393

24.2 Electron-Photon Interaction by Landau 1

24.2.1 86. Photon Scattering by an Electron

Сохранение 4-импульса при рассеянии фотона свободным электроном (эффект Комптона) выражается равенством

$$p + k = p' + k'$$

где p и k – 4-импульсы электрона и фотона до столкновения, а p' и k' - их 4-импульсы после столкновения. Введенные в \$66 кинематические инварианты:

$$s = (p+k)^2 = (p'+k')^2 = m^2 + 2pk = m^2 + 2p'k',$$
$$t = (p-p')^2 = (k'-k)^2 = 2\left(m^2 - pp'\right) = -2kk',$$
$$u = (p-k')^2 = (p'-k)^2 = m^2 - 2pk' = m^2 - 2p'k,$$
$$s + t + u = 2m^2.$$

Рассматриваемый процесс изображается двумя диаграммами Фейнмана $(74, 14)$, и его амплитуда

$$M_{fl} = -4\pi e^2 e'^*_\mu e_v \left(\bar{u}' Q^{\mu v} u\right),$$

$$Q^{\mu\nu} = \frac{1}{s-m^2}\gamma^\mu(\gamma p + \gamma k + m)\gamma^\nu + \frac{1}{u-m^2}\gamma^\nu\left(\gamma p - \gamma k' + m\right)\gamma^\mu.$$

$$d\sigma = \frac{\pi e^4}{4}\frac{dt}{(s-m^2)^2}\,\mathrm{Sp}\left\{\left(\gamma p' + m\right)Q^{\lambda\mu}(\gamma p + m)\bar{Q}_{\lambda\mu}\right\}$$

Аналогичным образом вычисляется g:

$$g(s,u) = \frac{2m^2}{(s-m^2)(u-m^2)}\left\{4m^2 + \left(s-m^2\right) + \left(u-m^2\right)\right\}.$$

В результате для сечения получим

$$d\sigma = 8\pi r_e^2 \frac{m^2 dt}{(s-m^2)^2}\left\{\left(\frac{m^2}{s-m^2} + \frac{m^2}{u-m^2}\right)^2 + \right.$$
$$\left. + \left(\frac{m^2}{s-m^2} + \frac{m^2}{u-m^2}\right) - \frac{1}{4}\left(\frac{s-m^2}{u-m^2} + \frac{u-m^2}{s-m^2}\right)\right\}.$$

где $r_e = e^2/m$.

(вывод потом понимать буду)

$$d\sigma = \frac{r_e^2}{2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\vartheta\right)do'$$

Наибольший член в сечении $(86, 6)$

$$d\sigma \approx 8\pi r_e^2 \frac{m^2 dt}{4(s-m^2)(m^2-u)},$$

откуда

$$d\sigma = \frac{r_B^2}{2}\frac{do'}{1 + (\pi-\theta)^2\omega^2/m^2}.$$

Сечение $d\sigma/do' \sim r_e^2$ в узком конусе $\pi - \theta \leqslant m/\omega$, а вне его уменьшается по порядку величины в ω^2/m^2 раз.

24.2.2 87. Photon Scattering by an Electron: Polarization Effects

Общие формулы

$$e^{(1)} = \frac{N}{\sqrt{-N^2}}, \quad e^{(2)} = \frac{P}{\sqrt{-P^2}},$$

где

$$P^\lambda = \left(p^\lambda + p'^\lambda\right) - K^\lambda \frac{pK + p'K}{K^2}, \quad N^\lambda = e^{\lambda\mu\nu\rho} P_\mu q_v K_\rho$$
$$K^\lambda = k^\lambda + k'^\lambda, \quad q^\lambda = k'^\lambda - k^\lambda = p^\lambda - p'^\lambda.$$

Непосредственным проецированием тензора $Q_{\mu\nu}$ находим

$$Q_0 = \frac{1}{2} Q^{\mu\nu} \left(e^{(1)}_\mu e^{(1)}_v + e^{(2)}_u e^{(2)}_v\right)$$

и т. д. Для вычисления удобно сначала выразить $Q_{\mu v}$ через взаимно ортогональные 4-векторы P, N, q, K:

$$Q^{\mu\nu} = \gamma^\mu \frac{\gamma P/2 + m}{s - m^2} \gamma^\nu + \gamma^\nu \frac{\gamma P/2 + m}{u - m^2} \gamma^\mu - \frac{1}{t} \left[\gamma^\mu (\gamma K) \gamma^\nu - \gamma^\nu (\gamma K) \gamma^\mu\right].$$

$$\bar{u}' \left(\gamma p + \gamma p'\right) u = 2m \bar{u}' u,$$
$$\bar{u}' \gamma^5 (\gamma q) u = \bar{u}' \left[\gamma^5 (\gamma p) + (\gamma p') \gamma^5\right] u = 2m \bar{u}' \gamma^5 u$$

то можно заменить в $Q^{\mu\nu}$ слагаемые

$$\gamma p + \gamma p' \to 2m, \quad \gamma^5 (\gamma q) \to 2m\gamma^5$$

Поэтому матрицы плотности начального и конечного фотонов, отнесенные к 4-ортам $e^{(1)}, e^{(2)}$, будут

$$\rho^{(\gamma)} = \tfrac{1}{2}(1 + \xi\sigma), \qquad \xi = (\xi_1, \xi_2, \xi_3);$$
$$\rho^{(\gamma)'} = \tfrac{1}{2}(1 + \xi'\sigma), \quad \xi' = (-\xi'_1, -\xi'_2, \xi'_3).$$

Теперь тензорный след

$$\rho^{(\nu')}_{\lambda\mu} Q^{\mu\nu} \rho^{(\nu)}_{\nu\rho} \tilde{\tilde{Q}}^{\rho\lambda}$$

Рассеяние на неполяризованных электронах

$$\rho^{(e)} = \frac{1}{2}(\gamma p + m), \quad \rho^{(e)'} = \frac{1}{2}\left(\gamma p' + m\right)$$

и удвоить результат, который должен быть подставлен вместо $|M_{fi}|^2$ в формулу для сечения $(64, 22)$

$$d\sigma = \frac{1}{32\pi^2} \frac{dt d\varphi}{\left(s - m^2\right)^2} |M_{fi}|^2$$

В лабораторной системе формула $(87, 11)$ принимает вид

$$d\sigma = \tfrac{r_e^2}{4} \left(\tfrac{\omega'}{\omega}\right)^2 do' \left\{F_0 + F_3 (\xi_3 + \xi'_3) + F_{11}\xi_1\xi'_1 + F_{22}\xi_2\xi'_2 + F_{33}\xi_3\xi'_3\right\}$$
$$do' = \sin\theta d\theta d\varphi, \quad (87, 12$$

где

$$F_0 = \frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\vartheta, \quad F_3 = \sin^2\vartheta$$

$$F_{11} = 2\cos\vartheta, \quad F_{22} = \left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega}\right)\cos\vartheta, \quad F_{33} = 1 + \cos^2\vartheta$$

(U. Fano, 1949).

Согласно изложенным в §65 правилам величины ξ_i'' равны отношениям коэффициентов при $\xi_i'\dot{\kappa}$ члену, не содержащему ξ':

$$\xi_1^{(f)} = \frac{F_{11}}{F}\xi_1, \quad \xi_2^{(f)} = \frac{F_{22}}{F}\xi_2, \quad \xi_3^{(f)} = \frac{F_3 + F_{33}\xi_3}{F}.$$

В частности, при рассеянии неполяризованного фотона

$$\xi_1^{(f)} = \xi_2^{(f)} = 0, \quad \xi_3^{(f)} = \frac{\sin^2\theta}{\omega/\omega' + \omega'/\omega - \sin^2\theta}$$

$$d\sigma = \frac{r_e^2}{4}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2 + 4\cos^2\Theta\right)do',$$

Рассеяние на поляризованных электронах

$$d\sigma = \frac{1}{2}d\sigma\left(\xi,\xi'\right) +$$

$$+ \frac{r_e^2}{8}\left(\frac{\omega'}{\omega}\right)^2 do'\left\{\mathbf{f}\xi\xi_2 + \mathbf{f}'\xi_2' + \mathbf{g}\xi'\xi_2 + \mathbf{g}'\xi'\xi_2' + G_{ik}\xi_i\xi_k' + ...\right\}.$$

фотона поляризованным электроном

$$d\sigma(\xi,\xi) = d\sigma(\xi) + \frac{1}{2}r_e^2\left(\frac{\omega'}{\omega}\right)^2\xi_2\mathrm{f}\xi do'$$

отличается от $d\sigma(\xi)$ только при наличии у фотона круговой поляризации, а у электронов-отличной от нуля проекции среднего спина на плоскость рассеяния. По той же причине электрон отдачи поляризуется только в случае, если фотон обладает круговой поляризацией; вектор же возникающей поляризации электрона лежит при этом в плоскости рассеяния:

$$\xi^{(f)} = \frac{1}{F}\xi_2\mathbf{g}.$$

Соотношения симметрии

Качественные свойства поляризационных эффектов при рассеянии фотонов на электронах следуют уже из общих требований симметрин.

Параметр ξ_2 циркулярной поляризации - псевдоскаляр (см. §8). Поэтому в силу требования P-инвариантности члены вида $\sim \xi_2$ (или $\propto \xi_2'$) в сечении рассеяния могли бы возникнуть лишь как произведение ξ_2 на какой-либо псевдоскаляр, составленный из имеющихся в нашем распоряжении векторов $\mathbf{k}$ и $\mathbf{k}'^1$). Но из двух полярных векторов нельзя составить псевдоскаляр. Отсюда и следует, что указанных членов в сечении не может быть.

Параметры линейной поляризации ξ_1 и ξ_3 связаны с компонентами двумерного (в плоскости, перпендикулярной k) симметричного тензора

$$S_{\alpha\beta} = \frac{1}{2}\left(\rho_{\alpha\beta}^{(\nu)} + \rho_{\beta\alpha}^{(\nu)}\right) = \frac{1}{2}\begin{pmatrix} 1+\xi_3 & \xi_1 \\ \xi_1 & 1-\xi_3 \end{pmatrix}.$$

24.2.3 88. Two-Photon Annihilation of an Electron Pair

Они отличаются от диаграмм рассеяния фотона на электроне заменой

$$p \to p_-, \quad p' \to -p_+, \quad k \to -k_1, \quad k' \to k_2.$$

Оба процесса - два перекрестных канала одной и той же (обобщенной) реакции. После замены (88, 2) кинематические инварианты (86, 2) приобретают следующий смысл:

$$\begin{aligned} s &= (p_- - k_1)^2, \\ t &= (p_- + p_+)^2 = (k_1 + k_2)^2, \\ u &= (p_- - k_2)^2. \end{aligned}$$

Если рассеяние фотона было s-каналом, то аннигиляция есть t-канал.

Физическая область аниигиляционного канала есть область II на рис. 7 (см. с. 300). При заданном t (заданной энергии в системе центра инерцин) интервал изменения s определяется уравнением границы $su = m^4$. Вместе с соотношением $s + t + u = 2m^2$ это дает $-\frac{t}{2} - \frac{1}{2}\sqrt{t\,(t-4m^2)} \leqslant s - m^2 \leqslant -\frac{t}{2} + \frac{1}{2}\sqrt{t\,(t-4m^2)}$. Интегрирование выражения (88, 4) элементарно; результат надо еше разделить на 2, учитывая тождественность двух конечных частиц (фотонов). Таким образом, получим

$$\sigma = \frac{\pi r_e^2}{2\tau^2(\tau-1)}\left[\left(\tau^2+\tau-\frac{1}{2}\right)\ln\frac{\sqrt{\tau}+\sqrt{\tau-1}}{\sqrt{\tau}-\sqrt{\tau-1}} - (\tau+1)\sqrt{\tau(\tau-1)}\right],$$

(88, 6) где $\tau = t/4m^2$ (P. A. M. Dirac, 1930).

В нерелятивистском пределе ($\tau \to 1$) находим отсюда

$$\sigma = \frac{\pi r_e^2}{2\sqrt{\tau-1}} \text{ (н. р.)}.$$

В системе центра инерции электрон, позитрон и оба фотона имеют одинаковые энергии $\varepsilon = \omega$. Инварианты:

$$m^2 - s = 2\varepsilon(\varepsilon - |\mathbf{p}|\cos\theta), \quad m^2 - u = 2\varepsilon(\varepsilon + |\mathbf{p}|\cos\theta),$$
$$t = 4\varepsilon^2,$$

где θ - угол между импульсами электрона и одного из фотонов. Подставив (88, 12) в (88, 4), получим угловое распределение аннигиляционных фотонов

$$d\sigma = \frac{r_e^2 m^2}{4\varepsilon|\mathrm{p}|}\left[\frac{\varepsilon^2 + \mathrm{p}^2\left(1+\sin^2\theta\right)}{\varepsilon^2 - \mathrm{p}^2\cos^2\theta} - \frac{2\mathrm{p}^4\sin^4\theta}{\left(\varepsilon^2 - \mathrm{p}^2\cos^2\theta\right)^2}\right]do.$$

В ультрарелятивистском случае оно имеет симметричные максимумы в направлениях $\theta = 0$ и $\theta = \pi$. Вблизи $\theta = 0$

$$d\sigma \approx \frac{r_e^2 m^2 do}{2\varepsilon^2\left(\theta^2 + m^2/\varepsilon^2\right)} \text{ (у. р.)}.$$

Подставив (88, 12) в (88, 4), получй угловое распределение аннигиляционных фотонов

$$d\sigma = \frac{r_e^2 m^2}{4\varepsilon|\mathrm{p}|}\left[\frac{\varepsilon^2 + \mathrm{p}^2\left(1+\sin^2\theta\right)}{\varepsilon^2 - \mathrm{p}^2\cos^2\theta} - \frac{2\mathrm{p}^4\sin^4\theta}{\left(\varepsilon^2 - \mathrm{p}^2\cos^2\theta\right)^2}\right]do.$$

В ультрарелятивистском случае оно имеет симметричные максимумы в направлениях $\theta = 0$ и $\theta = \pi$. Вблизи $\theta = 0$

$$d\sigma \approx \frac{r_e^2 m^2 do}{2\varepsilon^2 \left(\theta^2 + m^2/\varepsilon^2\right)} \text{ (у. р.).}$$

Полное сечение получается из $(88, 6)$:

$$\sigma = \pi r_e^2 \frac{1 - v^2}{4v} \left[\frac{3 - v^4}{v} \ln \frac{1 + v}{1 - v} - 2\left(2 - v^2\right)\right],$$

где $v = |\mathbf{p}|/\varepsilon = \sqrt{\varepsilon^2 - m^2}/\varepsilon$ - скорость сталкивающихся частин.

24.2.4 89. Positronium Annihilation

В нерелятивистском пределе согласно $(88, 11)$[1])

$$\overrightarrow{\sigma}_{2\gamma} = \pi \left(\frac{e^2}{mc^2}\right)^2 \frac{c}{v},$$

где v-относительная скорость частиц. Мы получим вероятность аннигиляции $\overline{\boldsymbol{\omega}}_{2\gamma}$, умножив $\bar{\sigma}_{2\gamma}$ на плотность потока, равную $v|\psi(0)|^2$. Здесь $\psi(r)$ - нормированная на 1 волновая функция основного состояния позитрония:

$$\psi(r) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}, \quad a = \frac{2\hbar^2}{me^2}$$

(боровский радиус позитрония a в два раза больше радиуса атома водорода из-за вдвое меньшей приведенной массы). Эта

Аналогичным образом найдем, что вероятность распада ортопозитрония связана с усредненным по спинам сечением трехфотонной аннигиляции свободной пары соотноением

$$w_1 = \frac{4}{3} \bar{w}_{3\gamma} = \frac{4}{3} |\psi(0)|^2 \left(v \bar{\sigma}_{3\gamma}\right)_{0 \to 0}$$

(3/4 - относительный статистический вес состояния со спином 1). Забегая вперед, укажем, что

$$\bar{\sigma}_{3\nu} = \frac{4\left(\pi^2 - 9\right) c}{3v} \alpha \left(\frac{e^2}{mc^2}\right)^2.$$

Поэтому продолжительность жизни ортопозитрония

$$\tau_1 = \frac{9\pi}{2\left(\pi^2 - 9\right)} \frac{\hbar}{mc^2 \alpha^5} = 1,4 \cdot 10^{-7} \text{с}.$$

Неравенство $\Gamma_1 \ll |E_{\text{осн}}|$ в этом случае, разумеется, выполняется еще в большей степени, чем для парапозитрония.

Вычислим сечение трехфотонной аннигиляции свободной пары (A. Ore, J. L. Powell, 1949).

Соответствующую амплитуду запишем в виде

$$M_{fi} = (4\pi)^{s/2} e_\lambda^{(3)^*} e_\mu^{(2)^*} e_v^{(1^{**})} \bar{u}\left(-p_+\right) Q^{\lambda\mu v} u\left(p_-\right),$$

где

$$Q^{\lambda\mu\nu} = \sum_{\text{пер}} \gamma^\lambda G\left(k_3 - p_+\right)\gamma^\mu G\left(p_- - k_1\right)\gamma^\nu,$$

причем сумма берется по всем перестановкам номеров фотонов $1, 2, 3$ вместе с одновременными такими же перестановками соответствующих тензорных индексов $\lambda\mu v$.

Квадрат модуля амплитуды, усредненный по поляризациям электрона и позитрона и просуммированный по поляризациям фотонов:

$$\frac{1}{4}\sum_{\text{поляр}} |M_{fi}|^2 = (4\pi)^3 \,\mathrm{Sp}\left\{\rho_+ Q^{\lambda\mu\nu}\rho_- \bar{Q}_{\lambda\mu\nu}\right\},$$

где

$$\rho_- = \frac{1}{2}\left(\gamma p_- + m\right), \quad \rho_+ = \frac{1}{2}\left(\gamma p_+ - m\right).$$

В результате все же довольно длинного вычисления пояучается

$$\frac{1}{4}\sum_{\text{поляр}} |M_{fi}|^2 = (4\pi)^3 e^6 \cdot 16\left[\left(\frac{m-\omega_1}{\omega_2\omega_3}\right)^2 + \left(\frac{m-\omega_2}{\omega_1\omega_3}\right)^2 + \left(\frac{m-\omega_3}{\omega_1\omega_2}\right)^2\right].$$

Подставив это выражение в $(89, 8)$, найдем дифференциальное сечение трехфотонной аннигиляции:

$$\begin{aligned} d\tilde{\sigma}_{3\gamma} =& \frac{e^6}{\pi^2 m^2 v}\left[\left(\frac{m-\omega_1}{\omega_2\omega_3}\right)^2 + \left(\frac{m-\omega_2}{\omega_1\omega_3}\right)^2 + \left(\frac{m-\omega_3}{\omega_1\omega_2}\right)^2\right] \times \\ & \times \delta\left(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3\right)\delta\left(\omega_1 + \omega_2 + \omega_3 - 2m\right)\frac{d^3k_1 d^3k_2 d^3k_3}{\omega_1\omega_2\omega_3}. \end{aligned}$$

Здесь надо еще исключить δ-функции. Первая нз них устраняется интегрированием по d^3k_3, после чего заменяем остальные дифференциалы:

$$d^3k_1 d^3k_2 \to 4\pi\omega_1^2 d\omega_1 \cdot 2\pi\omega_2^2 d\left(\cos\theta_{12}\right) d\omega_2$$

где θ_{12} - угол между $\mathbf{k}_1$ и $\mathbf{k}_2$; подразумевается, что уже произведено интегрирование по направлениям $\mathbf{k}_1$ и азимуту $\mathbf{k}_2$ относительно k_1. Дифференцируя равенство

$$\omega_3 = \sqrt{\omega_1^2 + \omega_2^2 + 2\omega_1\omega_2\cos\theta_{12}},$$

находим

$$d\cos\theta_{12} = \frac{\omega_3}{\omega_1\omega_2}d\omega_3$$

Полнсе сечение аннигиляции получается интегрированием $(89, 14)$ по обеим частотам:

$$\bar{\sigma}_{3y} = \frac{4\varepsilon^6}{3vm^2}3\int_0^m \int_{m-\omega_1}^m \frac{\left(\omega_1 + \omega_2 - m\right)^2}{\omega_1^2\omega_2^2}d\omega_1 d\omega_2.$$

Стояший здесь двойной интеграл равен $\left(\pi^2 - 9\right)/3$, и мы приходим к приведенной выше формуле $(89, 6)$.

24.2.5 90. Synchrotron radiation

квантовая отдача при испускании фотона. Последняя определяется отношением $\hbar\omega/\varepsilon$, и условие применимости классической теории требует его малостн. В этой связи удобно ввести параметр

$$\chi = \frac{H}{H_0}\frac{|p|}{m} \approx \frac{H\varepsilon}{H_0 m} \approx \frac{\hbar\omega_0}{\varepsilon}\left(\frac{\varepsilon}{m}\right)^3,$$

где $H_0 = m^2/|e|\hbar\,(= m^2c^3/|e|\hbar) = 4,4\cdot 10^{13}\Gamma$ с. В классической области $\chi \sim \hbar\omega/\varepsilon \ll 1$. В случае $\chi \geqslant 1$ энергия излученного фотона $\hbar\omega \sim \varepsilon$, причем при $\chi \gg 1$ (как мы увидим в дальнейшем) существенная область спектра простирается до частот, при которых энергия электрона после испускания

$$\varepsilon' \sim m\frac{H_0}{H} \ll \varepsilon.$$

Для того чтобы электрон оставался ультрарелятивистским, поле должно удовлетворять условию

$$\frac{H}{H_0} \ll 1$$

символическом виде

$$\psi = (2\hat{H})^{-1/2}u(\hat{p})\exp\left(-\frac{i}{\hbar}\hat{H}t\right)\varphi(\mathbf{r}),$$

где $\varphi(\mathbf{r}) \sim \exp(iS/\hbar)$ - квазиклассические волновые функцин бесспиновой частицы ($S(\mathbf{r})$ - ее классическое действие); $u(\hat{p})$ - операторный биспинор

$$u(\hat{p}) = \begin{pmatrix} (\hat{H}+m)^{1/2}w \\ (\hat{H}+m)^{-1/2}(\sigma\hat{p})\omega \end{pmatrix},$$

получающийся из биспинорной амплитуды плоской волны $u(p)$ $(23,9)$ заменой $\mathbf{p}$ и ε операторами [2])

$$\hat{\mathbf{p}} = \hat{\mathbf{p}} - e\mathbf{A} = -i\hbar\nabla - e\mathbf{A}, \quad \hat{H} = \left(\hat{\mathbf{p}}^2 + m^2\right)^{1/2},$$

$$V_{fi}(t) = e\sqrt{\frac{2\pi}{\hbar\omega}}\langle f|Q(t)|i\rangle e^{i\omega t},$$

где $\hat{Q}(t)$ обозначает гейзенберговский оператор

$$\widehat{Q}(t) = \frac{u_f^+(\hat{p})}{(2\hat{H})^{1/2}}\,(\alpha\mathrm{e}^*)\,e^{-i\hat{\mathrm{k}}(t)}\frac{u_i(\hat{p})}{(2\hat{H})^{1/2}},$$

$$dI = \frac{e^2}{4\pi^2}d^3k\int e^{-i\omega\tau}\left\langle i\left|Q^+\left(t+\frac{\tau}{2}\right)Q\left(t-\frac{\tau}{2}\right)\right|i\right\rangle d\tau.$$

$$\hat{\mathbf{p}}\exp(-i\mathbf{k}\hat{\mathbf{r}}) = \exp(-i\mathbf{k}\hat{\mathbf{r}})(\hat{\mathbf{p}} - \hbar\mathbf{k})$$
$$H(\hat{\mathbf{p}})\exp(-i\mathbf{k}\hat{\mathbf{r}}) = \exp(-i\mathbf{k}\hat{\mathbf{r}})H(\hat{\mathbf{p}} - \hbar\mathbf{k})$$

сдвига в импульсном пространстве. С помощью $(90,11)$. выносим в $(90,8)$ оператор $\exp(-i\mathrm{k}\hat{\mathbf{r}}(t))$ налево и записываем $\widehat{Q}(t)$ в виде

$$\hat{Q}(t) = \exp[-i\mathbf{k}\hat{\mathbf{r}}(t)]\hat{R}(t), \quad \hat{R}(t) = \frac{u_f^+\,(\hat{p}')}{\left(2\widehat{H}'\right)^{1/2}}\,(\alpha\mathrm{e}^*)\,\frac{u_i(\hat{p})}{(2\widehat{H})^{1/2}},$$

где $\hat{H}' = \hat{H} - \hbar\omega, \hat{\mathbf{p}}' = \hat{\mathbf{p}} - \hbar\mathbf{k}$. Теперь

$$\hat{Q}_2^+ \hat{Q}_1 = \hat{R}_2 \exp\left(i\mathbf{k}\hat{\mathbf{r}}_2\right) \exp\left(-i\mathbf{k}\hat{\mathbf{r}}_1\right) \hat{R}_1$$

$$L(\tau) = \exp\left\{ i \frac{\boldsymbol{\varepsilon}}{\boldsymbol{\varepsilon}'} \left(\mathbf{k}\mathbf{r}_2 - \mathbf{k}\mathbf{r}_1 - \omega\tau\right) \right\}$$

До сих пор мы не использовали конкретного вида траектории электрона. Выразим теперь $\boldsymbol{r}_2 - \mathbf{r}_1$ в $(90, 18)$ через $\mathbf{p}_1$ с помощью уравнения движения электрона в плоскости, перпендикулярной полю H (см. II, §21):

$$\mathbf{r}_2 - \mathbf{r}_1 = \frac{\mathbf{p}_1}{e\ddot{H}} \sin\frac{eH\tau}{\varepsilon} + \frac{[\mathbf{p}_1\mathbf{H}]}{eH^2} \left(1 - \cos\frac{eH\tau}{\varepsilon}\right).$$

Разлагая по степеням τ, имеем отсюда

$$\mathbf{k}\left(\mathbf{r}_2 - \mathbf{r}_1\right) - \omega\tau \approx \omega\tau \left\{ \left(\mathbf{v}_1\mathbf{n} - 1\right) + \tau \frac{e\mathbf{n}\left[\mathbf{p}_1\mathbf{H}\right]}{2\varepsilon^2} - \tau^2 \frac{e^2H^2}{6\varepsilon^2} \right\}$$

Таким образом, окончательно имеем

$$\exp(-i\omega t) \left\langle i \left| Q_2^+ Q_1 \right| i \right\rangle = R_2^* R_1 L(\tau)$$
$$R_2^* R_1 = \mathrm{Sp} \frac{1 + \xi_i \boldsymbol{\sigma}}{2} \left(\left(\mathrm{A}_2 - i\left[\mathbf{B}_2\boldsymbol{\sigma}\right]\right)\mathrm{e}\right) \frac{1 + \xi_i \boldsymbol{\sigma}}{2} \left(\left(\mathbf{A}_1 + i\left[\mathbf{B}_1\boldsymbol{\sigma}\right]\right)\mathrm{e}^*\right).$$

Множители $(1 + \xi\sigma)/2$ - двухрядные поляризационные матрицы плотности начального и конечного электронов.

$$\frac{dI}{d\omega} = \frac{ie^2\omega}{2\pi} \int\limits_{-\infty}^{\infty} \left(\frac{m^2}{\varepsilon^2\tau} + \frac{\varepsilon^2 + \varepsilon'^2}{4\varepsilon\varepsilon'} \omega_0^2 \tau \right) \exp\left\{ -\frac{i\omega\tau\varepsilon}{\boldsymbol{\varepsilon}'} \left(1 - v + \frac{\tau^2}{24}\omega_0^2\right) \right\} d\tau,$$

причем контур интегрирования выбирается указанным выше способом. Используя интегральное представление, функции Эйри Φ (см. III § b), нетрудно показать, что первый член сводится к интегралу от функции Эйри, а второй-к производной от нее. Окончательно находим

$$\frac{dI}{d\omega} = -\frac{e^2m^2\omega}{\sqrt{\pi}\varepsilon^2} \left\{ \int\limits_x^{\infty} \Phi(\xi) d\xi + \left(\frac{2}{x} + \frac{\hbar\omega}{\varepsilon}\chi x^{1/2}\right) \tilde{\Phi}'(x) \right\},$$
$$x = \left(\frac{\hbar\omega}{\varepsilon'\chi}\right)^{2/3} = \frac{m\iota^2}{\varepsilon^2} \left(\frac{\varepsilon\omega}{\varepsilon'\omega_0}\right)^{2/s}$$

(А. И. Никишов, В. И. Ритус, 1967).

Магнитотормозное излучение ириводит к возникновению поляризации движущихся электронов (А. А. Соколов, И. М. Тернов, 1963). Для рассмотрения этого вопроса надо найти всроятность радиационного перехода с обращением направления спина. Положнв в $(90, 21)\xi_i = -\xi_f \equiv \xi, |\zeta| = 1$, получим

$$R_2^* R_1 = \left(\mathbf{B}_1\mathbf{B}_2\right) - \left(\mathrm{e}^*\mathbf{B}_1\right)\left(\mathbf{e}_2\right) - \left(\mathbf{e}^*\left[\mathbf{B}_1\zeta\right]\right)\left(\mathbf{e}\left[\mathbf{B}_2\zeta\right]\right) - i\left(\zeta\mathrm{e}^*\right)\left(\mathbf{e}\left[\mathbf{B}_1\mathbf{B}_2\right]\right).$$

окончательно полную вероятность радиационного перехода с обращением спина:

$$w = \frac{5\sqrt{3}\alpha}{16}\frac{\hbar^2}{m^2}\left(\frac{\varepsilon}{m}\right)^5 \omega_0^3 \left(1 - \frac{2}{9}\zeta_\parallel^2 - \frac{8\sqrt{3}}{15}\frac{e}{|e|}\zeta_\perp\right),$$

где $\zeta_\parallel = \zeta\mathbf{v}, \zeta_\perp = \zeta\mathbf{H}/H$. Эта формула пригодна как для электронов ($e < 0$), так и для позитронов ($e > 0$).

Вероятность $(90, 30)$ не зависит от знака продольной поляризации $\zeta_\parallel$, но зависит от знака $\zeta_\perp$.

24.2.6 91. Pair Production by a Photon in a Magnetic Field

Образование электрон-позитронной пары фотоном в магнитном поле и магнитотормозное излучение - два 'перекрестных канала одной и той же реакции. Поэтому амплитуда M_{fi} процесса образования пары получается из амплитуды тормозного излучения просто путем замены

$$\varepsilon, p \to -\varepsilon_+, -p_+; \quad \varepsilon', p' \to \varepsilon_-, p_-; \quad \omega, k \to -\omega, -k \quad (91, 1)$$

(здесь ε_-, p_-и ε_+, p_+−энергии и импульсы электрона и пози трона в паре; $\boldsymbol{\varepsilon}, \mathbf{p}$ и $\boldsymbol{\varepsilon}', \mathbf{p}'$ - начальные и конечные энергии и импульсы электрона при тормозном излучении). В терминах углов и абсолютных значений преобразование импульсов есть

$$|\mathbf{p}| \to |\mathbf{p}_+|, |\mathbf{p}'| \to |\mathbf{p}_-|, \theta \to \pi - \theta_+, \theta' \to \theta_-, \varphi \to \varphi - \pi,$$

где $\theta_\pm$−углы между $\mathbf{p}_\pm$и $\mathbf{k}, \varphi$-угол между плоскостями $\mathbf{k}, \mathbf{p}_+$и $\mathbf{k}, \mathbf{p}_-$.

В случае же образования пары фотоном сечение выражается через амплитуду согласно

$$d\sigma = |M_{fi}|^2 \frac{1}{8\omega\varepsilon_-\varepsilon_+}\delta\left(\omega - \varepsilon_+ - \varepsilon_-\right)\frac{d^3p_+ d^3p_-}{(2\pi)^5}$$

или, после исключения δ-функции:

$$d\sigma = |M_{fi}|^2 \frac{|\mathrm{p}_+|\,|\mathbf{p}_-|}{8(2\pi)^5\omega} do_+ do_- d\varepsilon_+.$$

$$dw = \frac{m^2e^2}{\sqrt{\pi}}\frac{d\varepsilon_+}{\omega^2}\left\{\int_x^\infty \Phi(\xi)d\xi + \left(\frac{2}{x} - xx^{1/2}\right)\Phi'(x)\right\}.$$

где теперь

$$x = \left(\frac{m^3\omega}{|e|H{\varepsilon_+}^2-}\right)^{2/3}, \quad x = \frac{|e|H\omega}{m^3}\left(= \frac{\hbar^2|e|H\omega}{m^3c^5}\right).$$

Полная вероятность рождения пары в единицу времени получается интегрированием $(91, 8)$ по ε_+(причем, ввиду очевидной симметрии по отношению к ε_+и $\varepsilon_- = \omega - \varepsilon_+$, достаточно интегрировать от 0 до $\omega/2$ и затем удвонть результат). Производя замену переменной интегрирования ε_+на x и интегрируя первый член в $(91, 8)$ дважды по частям, получаем

$$w = -\frac{|e|^3H}{mx\sqrt{\pi}}\int_{(4/x)^{2/6}}^\infty \frac{2\left(x^{3/2} + 1/x\right)\Phi'(x)}{x^{1/4}\left(x^{3/2} - 4/x\right)^{1/2}}dx$$

(А. И. Никищов, В. И. Ритус, 1967).

24.2.7 92. Electron-nucleus bremsstrahlung. The non-relativistic case

24.2.8 93. Electron-nucleus bremsstrahlung. The relativistic case

24.2.9 94. Pair Production by a Photon in the Field of a Nucleus

24.3 Electron-Photon Interaction by Landau 2

24.3.1 95. Exact Theory of Pair Production in the Ultrarelativistic Case

(????)

24.3.2 96. Exact Theory of Bremsstrahlung in the Ultrarelativistic Case

(????)

24.3.3 97. Electron-electron bremsstrahlung in the ultra-relativistic case

(????)

24.3.4 98. Emission of soft photons in collisions

24.3.5 99. The method of equivalent photons

24.3.6 100. Pair production in collisions between particles

24.3.7 101. Photon Emission by an Electron in the Field of an Intense Electromagnetic Wave

24.4 Radiative Corrections in QED by Landau

(пока скорее всего это в предыдущую часть перенесу, пока тут зависла, все равно не так много это читал еще.)

113.Вычисление поляризационного оператора
114.Радиационные поправки к закону Кулона
115.Вычисление мнимой части поляризационного оператора по интегралу Фейнмана
116.Электромагнитные формфакторы электрона
117.Вычисление формфакторов электрона
118.Аномальный магнитный момент электрона
119.Вычисление массового оператора
120.Испускание мягких фотонов с ненулевой массой
121.Рассеяние электрона во внешнем поле во втором борновском приближении
122.Радиационные поправки к рассеянию электрона во внешнем поле
123.Радиационное смещение атомных уровней
124.Радиационное смещение уровней мезоатомов

125.Релятивистское уравнение для связанных состояний
126.Двойное дисперсионное соотношение
127.Рассеяние фотона на фотоне
128.Когерентное рассеяние фотона в поле ядра
129.Радиационные поправки к уравнениям электромагнитного поля

Расщепление фотона в магнитном поле

Вычисление интегралов по четырехмерным областям

24.5 Asymptotic Formulas of Quantum Electrodynamics

132.Асимптотическое поведение фотонного пропагатора при больших импульсах
133.Связь между «затравочным» и истинным зарядами
134.Асимптотическое поведение амплитуд рассеяния при высоких энергиях
135.Выделение дважды логарифмических членов в вершинном операторе
136.Дважды логарифмическая асимптотика вершинного оператора
137.Дважды логарифмическая асимптотика амплитуды рассеяния электрона на мюоне

24.6 Other Effects (????)

24.6.1 Schwinger Effect (!?!?)

это рождение частиц в сильном электромагнитном поле.
(?? потом поизучаю, очень крутая тренировка!)

25 Other about Quantum Field Phenomena

25.1 Overview of Phenomena in QFT

Подробно опишем, что мы рассмотрели, что происходит вообще?

25.1.1 Comparison of Scattering Cross-Sections for Different Processes in QFT (?)

(сперва все прорешаю - потом напишу)

25.1.2 Comparison of Scattering Cross-Sections in QFT and in Relativistic Mechanics (?)

(давно вопрос был, как конкретно они отличаются? тут сравню, сперва их нарешаю много)

25.2 Some Phenomena of the Standard Model

(I'll write about some the most important, for details see note on SM)

Part VII
Problems in QFT

26 Problems about messing with typical QFT formalism

26.1 Problems about fundamentals of QFT

26.1.1 Problems about classical fields and mechanics for preparation to QFT

Stanford.exam-2022.2/5. Forces on external currents. (5/40p)

In problem set 7, you considered the "improved" electromagnetic stress-energy tensor,

$$\hat{T}^{\mu\nu} = -\eta_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

and showed that it was conserved, $\partial_\mu \hat{T}^{\mu\nu} = 0$, when no current was present.

a) When there is a nonzero current $J^\mu = (\rho, \mathbf{J})$, the stress-energy tensor defined in (3) has divergence $\partial_\mu \hat{T}^{\mu\nu} = -K^\nu$. Find K^ν.

Solution

: The only difference is that the equation of motion is now $\partial_\mu F^{\mu\nu} = J^\nu$. We already know that when $\partial_\mu F^{\mu\nu} = 0$, the stress-energy is conserved, so to compute K^ν we only need to consider what terms appear which are proportional to J^μ. This can only occur when the derivative ∂_μ hits the factor of $F^{\mu\rho}$ in the first term, so

$$\partial_\mu \hat{T}^{\mu\nu} = -\eta_{\rho\sigma}\left(\partial_\mu F^{\mu\rho}\right)F^{\nu\sigma} = -\eta_{\rho\sigma}J^\rho F^{\nu\sigma}.$$

Thus, the answer is $K^\nu = J_\sigma F^{\nu\sigma}$.

You might be wondering why we are allowed to use the original expression for the stress-energy tensor, as rerunning the Noether/Belinfante procedure will give additional terms when J^μ is nonzero. The point is that we're trying to find the change in the energy-momentum of the electromagnetic field alone, so we shouldn't include those extra terms, which represent something like potential energy-momentum.

b) The vector $K^\mu = \left(K^0, \mathbf{K}\right)$ is the four-momentum delivered to the current by the field, per unit space and unit time. Explicitly write K^0 and $\mathbf{K}$ in terms of $\rho, \mathbf{J}, \mathbf{E}$, and $\mathbf{B}$.

Solution

Expanding the components out, we have

$$K^i = J_\sigma F^{i\sigma} = J_0 F^{i0} + J_j F^{ij} = J_0 E^i - J_j \epsilon^{ijk} B^k.$$

Noting that $J_0 = J^0 = \rho$ and $J_i = -J^i$, we conclude

$$\mathbf{K} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B}$$

which of course is just the continuous version of the Lorentz force law. Similarly, we have

$$K^0 = J_i F^{0i} = \left(-J^i\right)\left(-F^{i0}\right) = \mathbf{J} \cdot \mathbf{E}$$

which is the rate at which the electric field does work on the charges.

Асимптотическое состояние теории с простым взаимодействием полей

Пусть у нас

$$L = (Lkg1) + (Lkg2) - \tilde{m}\phi_1\phi_2.$$

Какое его асимптотическое состояние?

Solution

(нужно записать через матрицы и повращать матрицей поворота, там потом новое поле - примерно сумма и разность двух полей до. задача по сути по линалу на диагонализацию симм матрицы.)
(не очень крутая задача, выдуманная теория взаимодействия полей, по сути на линал.)

Задачи 1 на Интегралы движения как генераторы симметрии

По аналогии с классической механикой, обобщенный импульс в теории поля можно записать, как

$$\pi_i(\vec{x},t)=\frac{\delta\int d^3\vec{y}\mathcal{L}\left(\phi_k(\vec{y},t),\partial_\mu\phi_k(\vec{y},t)\right)}{\delta\dot{\phi}_i(\vec{x},t)} \tag{0.1}$$

где мы используем, что

$$\frac{\delta\dot{\phi}_k(\vec{x},t)}{\delta\dot{\phi}_i(\vec{y},t)}=\delta_{ik}\delta(\vec{x}-\vec{y}) \tag{0.2}$$

Далее, по опять же по аналогии с классической механикой, мы вводим скобку Пуассона

$$\left\{F\left(\phi_k(\vec{x},t),\pi_k(\vec{x},t)\right),G\left(\phi_k(\vec{y},t),\pi_k(\vec{y},t)\right)\right\}=\int d^3z\sum_i\left(\frac{\delta F}{\delta\phi_i(\vec{z},t)}\frac{\delta G}{\delta\pi_i(\vec{z},t)}-\frac{\delta F}{\delta\pi_i(\vec{z},t)}\frac{\delta G}{\delta\phi_i(\vec{z},t)}\right) \tag{0.3}$$

и подразумеваем, что

$$\frac{\delta\phi_k(\vec{x},t)}{\delta\phi_i(\vec{y},t)}=\delta_{ik}\delta(\vec{x}-\vec{y}),\quad \frac{\delta\pi_k(\vec{x},t)}{\delta\pi_i(\vec{y},t)}=\delta_{ik}\delta(\vec{x}-\vec{y}),\quad \frac{\delta\phi_k(\vec{x},t)}{\delta\pi_i(\vec{y},t)}=\frac{\delta\pi_k(\vec{x},t)}{\delta\phi_i(\vec{y},t)}=0. \tag{0.4}$$

Легко проверить, что

$$\{\phi_i(\vec{x},t),\phi_j(\vec{y},t)\}=\{\pi_i(\vec{x},t),\pi_j(\vec{y},t)\}=0,\quad \{\phi_i(\vec{x},t),\pi_j(\vec{y},t)\}=\delta_{ij}\delta(\vec{x}-\vec{y}). \tag{0.5}$$

Общее утверждение заключается в том, что интегралы движения (заряды)

$$Q=\int d^3\vec{x}J^0(\vec{x},t),\quad \frac{dQ}{dt}=0,\quad \partial_\mu J^\mu=0 \tag{0.6}$$

генерируют симметрию (из которой они по теореме Нетер были вычисленны), а именно

$$\{\phi_i(\vec{x},t),Q\}=\delta\phi_i(\vec{x},t), \tag{0.7}$$

где инфинитезимальное преобразование симметрии есть $\phi_i'(x)=\phi_i(x)+\epsilon\delta\phi_i(x)$.
Упражнение 1. Проверьте (0.7) для комплексного скалярного поля ($U(1)$ симметрия).
Решение 1. Как мы уже знаем $U(1)$ сохраняющийся ток

$$j^\mu=i\left(\phi\partial^\mu\phi^*-\phi^*\partial^\mu\phi\right) \tag{0.8}$$

соответствует симметрии $\phi'(x)=e^{i\alpha}\phi(x)$, что при инфинитезимальном α запишется как

$$\delta\phi=i\phi,\quad \delta\phi^*=-i\phi^* \tag{0.9}$$

С другой стороны сохраняющийся заряд равен

$$Q=i\int d^3\vec{x}\left(\pi\phi-\phi^*\pi^*\right) \tag{0.10}$$

Тогда находим,

$$\begin{aligned}\{\phi(\vec{x},t),Q\}&=\int d^3z\left(\frac{\delta\phi(\vec{x},t)}{\delta\phi(\vec{z},t)}\frac{\delta Q}{\delta\pi(\vec{z},t)}-\frac{\delta\phi(\vec{x},t)}{\delta\pi(\vec{z},t)}\frac{\delta Q}{\delta\phi(\vec{z},t)}+\frac{\delta\phi(\vec{x},t)}{\delta\phi^*(\vec{z},t)}\frac{\delta Q}{\delta\pi^*(\vec{z},t)}-\frac{\delta\phi(\vec{x},t)}{\delta\pi^*(\vec{z},t)}\frac{\delta Q}{\delta\phi^*(\vec{z},t)}\right)=\\&=\int d^3z\left(\frac{\delta Q}{\delta\pi(\vec{z},t)}\frac{\delta\phi(\vec{x},t)}{\delta\phi(\vec{z},t)}\right)=\int d^3z\left(\frac{\delta Q}{\delta\pi(\vec{z},t)}\delta(\vec{x}-\vec{z})\right)=\frac{\delta Q}{\delta\pi(\vec{x},t)}=\\&=\frac{\delta\left(i\int d^3\vec{y}(\pi(\vec{y},t)\phi(\vec{y},t))-\phi^*(\vec{y},t)\pi^*(\vec{y},t))\right)}{\delta\pi(\vec{x},t)}=i\phi(\vec{x},t).\end{aligned} \tag{0.11}$$

Аналогично найдем, что $\{\phi^*(\vec{x},t),Q\} = -i\phi^*(\vec{x},t)$.

Упражнение 2. Проверьте, на примере скалярного поля, что Гамильтониан генерирует сдвиги по времени: $\{\phi(\vec{x},t),H\} = \dot{\phi}(\vec{x},t)$

Решение 2. Для Гамильтониана имеем

$$H = \int d^3\vec{x}\left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right), \tag{0.12}$$

тогда

$$\{\phi(\vec{x},t),H\} = \int d^3z\left(\frac{\delta\phi(\vec{x},t)}{\delta\phi(\vec{z},t)}\frac{\delta H}{\delta\pi(\vec{z},t)} - \frac{\delta\phi(\vec{x},t)}{\delta\pi(\vec{z},t)}\frac{\delta H}{\delta\phi(\vec{z},t)}\right) = \frac{\delta H}{\delta\pi(\vec{x},t)} = \pi(\vec{x},t) = \dot{\phi}(\vec{x},t) \tag{0.13}$$

Откуда мы получили, что Гамильтониан генерирует сдвиги по времени. Это не удивительно, так как Гамилтониан это по сути заряд

$$H = \int d^3\vec{x}T^{00}(\vec{x},t) \tag{0.14}$$

получающийся из компоненты тензора энергии импульса $T^{\mu 0}$.

Упражнение 3. Проверьте, на примере скалярного поля, что импульс генерирует пространственные сдвиги: $\{\phi(\vec{x},t),\vec{P}\} = -\nabla\phi(\vec{x},t)$

Решение 3. Для импульса имеем

$$P^i = \int d^3xT^{0i} = -\int d^3\vec{x}\pi(\vec{x},t)\partial_i\phi(x,t), \tag{0.15}$$

тогда

$$\{\phi(\vec{x},t),\vec{P}\} = \int d^3z\left(\frac{\delta\phi(\vec{x},t)}{\delta\phi(\vec{z},t)}\frac{\delta\vec{P}}{\delta\pi(\vec{z},t)} - \frac{\delta\phi(\vec{x},t)}{\delta\pi(\vec{z},t)}\frac{\delta\vec{P}}{\delta\phi(\vec{z},t)}\right) = \frac{\delta\vec{P}}{\delta\pi(\vec{x},t)} = -\nabla\phi(\vec{x},t) \tag{0.16}$$

Упражнение 4. Проверьте, на примере скалярного поля, что момент импульса генерирует пространственные повороты: $\{\phi(\vec{x},t),L_k\} = -\epsilon_{ijk}x^i\partial_j\phi(\vec{x},t)$

Решение 4. Для момента импульса имеем (см. Листок 1)

$$L_k = \frac{1}{2}\epsilon_{ijk}\int d^3\vec{x}\left(x^iT^{0j} - x^jT^{0i}\right) = \frac{1}{2}\epsilon_{ijk}\int d^3\vec{x}\pi(\vec{x},t)\left(x^i\partial^j\phi(\vec{x},t) - x^j\partial^i\phi(\vec{x},t)\right) \tag{0.17}$$

тогда

$$\{\phi(\vec{x},t),L_k\} = \int d^3z\left(\frac{\delta\phi(\vec{x},t)}{\delta\phi(\vec{z},t)}\frac{\delta L_k}{\delta\pi(\vec{z},t)} - \frac{\delta\phi(\vec{x},t)}{\delta\pi(\vec{z},t)}\frac{\delta L_k}{\delta\phi(\vec{z},t)}\right) = \frac{\delta L_k}{\delta\pi(\vec{x},t)} = \frac{1}{2}\epsilon_{ijk}\left(x^i\partial^j - x^j\partial^i\right)\phi(\vec{x},t) =$$
$$= \epsilon_{ijk}x^i\partial^j\phi(\vec{x},t). \tag{0.18}$$

Белавин-Пархоменко**.1. Классическая теория поля**

1 Классическая Теория Поля и Теорема Нётер (100 баллов)

В этой задаче, мы будем изучать классическую Теорему Нётер, которая утверждает, что любой непрерывной симметрии действия соответствует сохраняющийся ток.

(a). В качестве разминки, рассмотрим двухмерный изотропный гармонический осциллятор с координатами x, y. В терминах комплексной координаты $\phi = x + iy$ действие имеет вид:

$$S[\phi(t)] = \int dt\left(\frac{1}{2}\dot{\phi}^*(t)\dot{\phi}(t) - \frac{1}{2}m^2\phi^*(t)\phi(t)\right) \tag{0.1}$$

Покажите, что это действие инвариантно при преобразовании $\phi(t) \to \phi'(t) = e^{i\alpha}\phi(t)$, где α действительная постоянная. Как это преобразование действует на координаты x и y? Теперь рассмотрим такое же преобразование, только с зависимой от времени функцией $\alpha(t)$,

$$\phi(t) \to \phi'(t) = e^{i\alpha(t)}\phi(t) \tag{0.2}$$

Это больше не симметрия действия. Для инфинитезимального (бесконечно маленького) $\alpha(t)$ покажите, что изменение действия есть

$$\delta S = S\left[\phi'(t)\right] - S[\phi(t)] = \int dt Q(t)\dot{\alpha}(t) + \mathcal{O}\left(\alpha^2\right) \tag{0.3}$$

и найдите $Q(t)$ в терминах $\phi(t)$. Используйте (0.3) для вывода того, что из уравнений движения следует, что $\dot{Q}(t) = 0$, т.е. Q является сохраняющимся зарядом. Какой физической величине соответствует заряд Q в данном примере?

(b). Рассмотрим классическую теорию поля с действием

$$S = \int d^4x \mathcal{L}\left(\phi_i(x), \partial_\mu \phi_i(x)\right) \tag{0.4}$$

Предположим, что S инвариантно относительно инфинитезимальных преобразований вида

$$\phi_i(x) \to \phi_i'(x) = \phi_i(x) + \alpha F_i\left(\phi_j(x), \partial_\mu \phi_j(x)\right), \quad \alpha \ll 1. \tag{0.5}$$

Если S инвариантно, то как $\mathcal{L}$ изменятся при таких преобразованиях. Теперь объявляя α функцией координаты x и действуя аналогично части (a) выше, покажите, что существует ток j^μ, такой что при учете уравнений движения можно получить $\partial_\mu j^\mu = 0$, то есть ток сохраняется (сохраняющийся ток). Найдите явную формулу для j^μ.

(c). Рассмотрим комплексное скалярное поле $\phi(x)$ с действием

$$S = \int d^4x \left(\partial^\mu \phi^* \partial_\mu \phi - V(|\phi|)\right) \tag{0.6}$$

Покажите, что действие инвариантно относительно глобальных $U(1)$ фазовых поворотов $\phi'(x) = e^{i\alpha}\phi(x)$, и найдите соответствующий Нётеровский ток. Повторите тоже самое, но для сдвигов (трансляций):

$$x^\mu \to x'^\mu = x^\mu + a^\mu, \quad \phi(x) \to \phi'\left(x'\right) = \phi(x). \tag{0.7}$$

Получите соответствующие Нётеровские токи и следовательно Тензор энергии-импульса $T_{\mu\nu}$. (Обратите внимание, что Лагранжева плотность не является инвариантной при сдвигах, в том смысле, что: $\delta\mathcal{L}(x) = \mathcal{L}\left(\phi'(x), \partial_\mu \phi'(x)\right) - \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right) = a^\mu \partial_\mu \mathcal{L}(x)$. В данном месте может возникнуть путаница, так как иногда говорят, что Лагранжева плотность как раз инвариантна, подразумевая, что: $\mathcal{L}\left(\phi'\left(x'\right), \partial'_\mu \phi'\left(x'\right)\right) - \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right) = 0\big)$. Теперь рассмотрите масштабное преобразование

$$x'^\mu = \lambda x^\mu, \quad \phi'\left(x'\right) = \lambda^{-1}\phi(x). \tag{0.8}$$

При каком виде потенциала $V(|\phi|)$ данное преобразование является симметрией действия? Найдите соответствующий Нётеровский ток и выразите его в терминах $T_{\mu\nu}$.

(d). Утверждение, что теория является Лоренц инвариантной, является утверждением о наличии симметрии нашей теории (Лоренц-симметрии), а следовательно мы можем в полном праве воспользоваться теоремой Нётер и найти соответствующие данной симметрии сохраняющиеся токи. Рассмотрим инфинитезимальное преобразование Лоренца для теории одного скалярного поля

$$x'^\mu = \Lambda^\mu_\nu x^\nu = x^\mu + \omega^\mu{}_\nu x^\nu, \quad \phi(x) \to \phi'\left(x'\right) = \phi(x), \quad \omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{0.9}$$

Покажите, что такому преобразованию соответствуют сохраняющиеся токи вида

$$M^{\lambda\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}. \tag{0.10}$$

Соответственно независящий от времени тензор получается по формуле $L^{\mu\nu} = \int d^3x M^{0\mu\nu}$. А величина $L_k = \frac{1}{2}\epsilon_{ijk}L^{ij}$ в физике называется моментом импульса.

(e). Найдите сохраняющиеся токи соответствующие Лоренц симметрии для теории векторного поля A_μ с Лагранжианом $\mathcal{L}\left(A_\mu(x), \partial_\nu A_\mu(x)\right)$. В данном случае

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \quad A_\mu(x) \to A'_\mu\left(x'\right) = \Lambda^\nu_\mu A_\nu(x). \tag{0.11}$$

Аналогично задаче (d) рассмотрите инфинитезимальное преобразование Лоренца и покажите, что у сохраняющихся токов $M^{\lambda\mu\nu}$ в данном случае появляется еще и спиновая часть

$$M^{\lambda\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} + S^{\lambda\mu\nu}. \tag{0.12}$$

Найдите $S^{\lambda\mu\nu}$.

(f). Покажите, что в Лоренц инвариантной теории одного скалярного поля $\phi\left(x^\mu\right)$ с Лагранжевой плотностью $\mathcal{L}\left(\phi, \partial_\mu \phi\right)$ Тензор энергии-импулься $T_{\mu\nu}$, полученный как Нётеровский ток, всегда симметричен. (В данном случае как раз удобно рассмотреть разность: $0 = \mathcal{L}\left(\phi'\left(x'\right), \partial'_\mu \phi'\left(x'\right)\right) - \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right)\big)$.

Белавин-Пархоменко**.1. Классическая теория поля Решения**

1 Классическая Теория Поля и Теорема Нётер (100 баллов)

(a). - Преобразование имеет вид

$$\phi'(t) = e^{i\alpha}\phi(t) \quad \Rightarrow \quad \dot{\phi}'(t) = e^{i\alpha}\dot{\phi}(t). \tag{0.1}$$

Таким образом получаем для действия

$$S\left[\phi'(t)\right] = \int dt \left(\frac{1}{2}\left(e^{-i\alpha}\dot{\phi}^*(t)\right)\left(e^{i\alpha}\dot{\phi}(t)\right) - \frac{1}{2}m^2\left(e^{-i\alpha}\phi^*(t)\right)\left(e^{i\alpha}\phi(t)\right)\right) = S[\phi(t)]. \tag{0.2}$$

Далее имеем

$$\phi = x + iy \quad \Rightarrow \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \tag{0.3}$$

это матрица поворота на угол α (из группы $SO(2)$).
Далее теперь имеем $\alpha(t) \ll 1$, следовательно

$$\begin{aligned} \phi'(t) &= e^{i\alpha(t)}\phi(t) = \phi(t) + i\alpha(t)\phi(t) + \mathcal{O}\left(\alpha^2\right), \\ \dot{\phi}'(t) &= \dot{\phi}(t) + i\dot{\alpha}(t)\phi(t) + i\alpha(t)\dot{\phi}(t) + \mathcal{O}\left(\alpha^2\right). \end{aligned} \tag{0.4}$$

Находим для действия

$$\begin{aligned} S\left[\phi'(t)\right] = \int dt \Bigg[\frac{1}{2}\left(\dot{\phi}^*(t) - i\dot{\alpha}(t)\phi^*(t) - i\alpha(t)\dot{\phi}^*(t)\right)\left(\dot{\phi}(t) + i\dot{\alpha}(t)\phi(t) + i\alpha(t)\dot{\phi}(t)\right) - \\ -\frac{1}{2}m^2\left(\phi^*(t) - i\alpha(t)\phi^*(t)\right)\left(\phi(t) + i\alpha(t)\phi(t)\right) + \mathcal{O}\left(\alpha^2\right)\Bigg] = \\ = S[\phi(t)] + \int dt \left[-\frac{i\dot{\alpha}(t)}{2}\left(\phi^*(t)\dot{\phi}(t) - \dot{\phi}^*(t)\phi(t)\right)\right] + \mathcal{O}\left(\alpha^2\right). \end{aligned} \tag{0.5}$$

Таким образом мы находим

$$Q(t) = -\frac{i}{2}\left(\phi^*(t)\dot{\phi}(t) - \dot{\phi}^*(t)\phi(t)\right) \tag{0.6}$$

На уравнениях движения ("on-shell") вариация действия равна нулю $\delta S = 0$ для любого $\alpha(t)$, следовательно получаем:

$$\int dt\alpha(t)\dot{Q}(t) = 0, \forall\alpha(t), \quad \Rightarrow \quad \dot{Q}(t) = 0 \quad \text{и поэтому} \quad Q(t) = Q \text{ сохраняющий заряд "on-shell".} \tag{0.7}$$

В обычных координатах x, y имеем для заряда

$$Q = -\frac{i}{2}((x - iy)(\dot{x} + i\dot{y}) - \text{ к.с. }) = x\dot{y} - y\dot{x} \tag{0.8}$$

данное выражение называется в физике моментом импульса. (аббревиатура к.с. означает комплексно сопряженная часть)

(b). - Мы предполагаем, что мера d^4x инвариантна. Из инвариантности действия S мы получаем (здесь $\alpha = \text{const}$, не зависит от x)

$$\mathcal{L}\left(\phi'(x), \partial_\mu\phi'(x)\right) = \mathcal{L}\left(\phi(x), \partial_\mu\phi(x)\right) + \alpha\partial_\mu G^\mu. \tag{0.9}$$

Последнее слагаемое есть полная производная, поэтому при интегрировании по d^4x оно исчезает. Теперь пусть (здесь уже мы ввели зависящую от x функцию $\alpha(x)$)

$$\phi_i'(x) = \phi_i(x) + \alpha(x)F_i\left(\phi(x), \partial_\mu\phi(x)\right), \tag{0.10}$$

тогда получим

$$\mathcal{L}\left(\phi_i\ +\alpha(x)F_i\left(\phi,\partial_\mu\phi\right),\partial_\mu\phi_i(x)+\partial_\mu\alpha(x)F_i+\alpha(x)\partial_\mu F_i\right)=$$
$$=\mathcal{L}\left(\phi_i,\partial_\mu\phi_i\right)+\frac{\partial\mathcal{L}}{\partial\phi_i}\alpha(x)F_i+\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_i\right)}\left(\partial_\mu\alpha F_i+\alpha\partial_\mu F_i\right) \tag{0.11}$$

но как мы знаем (это мы бы получили если бы α не зависело от x)

$$\frac{\partial\mathcal{L}}{\partial\phi_i}F_i+\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_i\right)}\partial_\mu F_i=\partial_\mu G^\mu \tag{0.12}$$

Поэтому можно получить, что

$$\delta S=-\int d^4x\alpha(x)\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_i\right)}F_i-G^\mu\right) \tag{0.13}$$

На уравнениях движения ("on-shell") $\delta S=0$, следовательно

$$j^\mu=\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_i\right)}F_i-G^\mu,\quad \partial_\mu j^\mu=0. \tag{0.14}$$

(c). · При $U(1)$ преобразованиях мы имеем (по аналогии с задачей (b))

$$\delta\phi=i\alpha\phi,\quad \delta\phi^*=-i\alpha\phi^*,\quad G^\mu=0 \tag{0.15}$$

откуда получаем

$$F_\phi=i\phi,\quad F_{\phi^*}=-i\phi^* \tag{0.16}$$

и следовательно сохраняющийся ток дается формулой

$$j^\mu=\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}F_\phi+\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi^*\right)}F_{\phi^*}=\left(-\partial^\mu\phi^*\right)\left(i\phi\right)+\left(-\partial^\mu\phi\right)\left(-i\phi^*\right)=i\left(\phi^*\partial^\mu\phi-\phi\partial^\mu\phi^*\right). \tag{0.17}$$

При сдвигах (трансляциях) мы имеем

$$\phi'\left(x'\right)=\phi(x)\quad\Rightarrow\quad\phi'(x)=\phi(x-a), \tag{0.18}$$

следовательно

$$\delta_{a^\mu}\phi=-a^\mu\partial_\mu\phi(x),\quad \delta_{a^\mu}\phi^*=-a^\mu\partial_\mu\phi^*(x), \tag{0.19}$$

а также

$$\delta_{a^\mu}\mathcal{L}=-a^\mu\partial_\mu\mathcal{L}=-a^\mu\partial_\nu\left(\delta^\nu_\mu\mathcal{L}\right) \tag{0.20}$$

откуда получаем, что

$$G^\nu_{a^\mu}=-a^\mu\delta^\nu_\mu\mathcal{L} \tag{0.21}$$

Таким образом для сохраняющегося тока находим (см. пункт (b))

$$j^\nu_{a^\mu}=\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi\right)}\delta_{a^\mu}\phi+\frac{\partial\mathcal{L}}{\partial\left(\partial_\nu\phi^*\right)}\delta_{a^\mu}\phi^*-G^\nu_{a^\mu}=$$
$$=\partial^\nu\phi^*\left(-a^\mu\partial_\mu\phi\right)+\partial^\nu\phi\left(-a^\mu\partial_\mu\phi\right)+a^\mu\delta^\nu_\mu\mathcal{L}=$$
$$=-a^\mu\left(\partial^\nu\phi^*\partial_\mu\phi+\partial_\mu\phi^*\partial^\nu\phi-\delta^\nu_\mu\mathcal{L}\right). \tag{0.22}$$

Эти сохраняющиеся токи называются тензором энергии-импульса

$$T_{\mu\nu}=\partial_\nu\phi^*\partial_\mu\phi+\partial_\mu\phi^*\partial_\nu\phi-\eta_{\mu\nu}\left(\partial^\lambda\phi^*\partial_\lambda\phi-V(|\phi|)\right). \tag{0.23}$$

Заметим, что

$$T^{00}=\dot\phi^*\dot\phi+\partial_i\phi^*\partial_i\phi+V(|\phi|)=\mathcal{H}=\ \left(\text{Гамильтонова плотность}\ \right). \tag{0.24}$$

При масштабных преобразованиях мы имеем

$$x' = \lambda x, \quad \phi'(x') = \lambda^{-1}\phi(x), \tag{0.25}$$

следовательно $d^4x' = \lambda^4 d^4x$, следовательно для инвариантности действия нам нужно, чтобы $\mathcal{L}' = \lambda^{-4}\mathcal{L}$. С другой стороны мы имеем

$$\mathcal{L}' = \lambda^{-4}\partial_\mu\phi^*\partial^\mu\phi - V\left(\lambda^{-1}|\phi|\right), \tag{0.26}$$

следовательно мы должны потребовать

$$V\left(\lambda^{-1}|\phi|\right) = \lambda^{-4}V(|\phi|) \tag{0.27}$$

откуда получаем, что

$$V(|\phi|) \sim |\phi|^4 \tag{0.28}$$

и в нашей теории не должно быть никакого массово члена $\sim |\phi|^2$. Для того чтобы найти Нетеровский ток рассмотрим вариацию

$$\delta\phi(x) = \phi'(x) - \phi(x) = \lambda^{-1}\phi\left(\lambda^{-1}x\right) - \phi(x). \tag{0.29}$$

Если $\lambda = 1 + \delta\lambda$, то мы получим

$$\delta\phi(x) = -\delta\lambda\left(\phi(x) + x^\mu\partial_\mu\phi(x)\right). \tag{0.30}$$

И аналогично

$$\partial'_\mu\phi'(x') = \lambda^{-1}\frac{\partial}{\partial(\lambda x)^\mu}\phi(x) = \lambda^{-2}\partial_\mu\phi(x) \tag{0.31}$$

следовательно

$$\delta\left(\partial_\mu\phi(x)\right) = \lambda^{-2}\partial_\mu\phi\left(\lambda^{-1}x\right) - \phi(x) = -\delta\lambda\left(2\partial_\mu\phi(x) + x^\nu\partial_\mu\partial_\nu\phi(x)\right). \tag{0.32}$$

Таким образом мы получаем

$$\begin{aligned}\delta\mathcal{L} &= \partial^\mu\phi^*(-\delta\lambda)\left(2\partial_\mu\phi + x^\nu\partial_\mu\partial_\nu\phi\right) + \partial^\mu\phi(-\delta\lambda)\left(2\partial^\mu\phi^* + x^\nu\partial_\mu\partial_\nu\phi^*\right) - 2V(|\phi|)(-\delta\lambda)\left(2 + x^\nu\partial_\nu\log|\phi|^2\right) = \\ &= -4\delta\lambda\left(\partial^\mu\phi^*\partial_\mu\phi - V(|\phi|)\right) - \delta\lambda x^\nu\partial_\nu\left(\partial^\mu\phi^*\partial_\mu\phi\right)\left(\partial^\mu\phi^*\partial_\mu\phi - V(|\phi|)\right) = -\delta\lambda\partial_\nu\left(x^\nu\left(\partial^\mu\phi^*\partial_\mu\phi - V(|\phi|)\right)\right)\end{aligned} \tag{0.33}$$

Откуда мы получаем

$$G^\mu = -x^\mu\left(\partial^\nu\phi^*\partial_\nu\phi - V(|\phi|)\right). \tag{0.34}$$

И для тока имеем

$$\begin{aligned}j^\mu &= -\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\left(\phi + x^\nu\partial_\nu\phi\right) - \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi^*\right)}\left(\phi^* + x^\nu\partial_\nu\phi^*\right) - G^\mu = \\ &= -\partial^\mu\phi^*\left(\phi + x^\nu\partial_\nu\phi\right) - \partial^\mu\phi\left(\phi^* + x^\nu\partial_\nu\phi^*\right) + x^\mu\left(\partial^\nu\phi^*\partial_\nu\phi - V(|\phi|)\right).\end{aligned} \tag{0.35}$$

В терминах $T^{\mu\nu}$ находим

$$j^\mu = -x_\nu T^{\mu\nu} - \partial^\mu|\phi|^2. \tag{0.36}$$

Используя тот факт, что $\partial_\mu|\phi|^2$ можно записать как

$$\partial_\mu|\phi|^2 = -\frac{1}{3}x^\nu\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2\right)|\phi|^2 + \frac{1}{3}\partial^\rho\left(\left(x_\rho\partial_\mu - x_\mu\partial_\rho\right)|\phi|^2\right), \tag{0.37}$$

мы можем переписать ток в виде

$$j^\mu = -x_\nu\hat{T}^{\mu\nu} - \frac{1}{3}\partial_\rho\left(\left(x^\rho\partial^\mu - x^\mu\partial^\rho\right)|\phi|^2\right), \quad \hat{T}^{\mu\nu} = T^{\mu\nu} - \frac{1}{3}\left(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^2\right)|\phi|^2. \tag{0.38}$$

Добавки которые получили ток и тензор энергии-импульса называются улучшающими членами ("improvement terms").

(d). В данном случае мы имеем

$$\phi'(x) = \phi\left(x^\mu - \omega^\mu{}_\nu x^\nu\right) = \phi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x),$$
$$\partial_\mu \phi'(x) = \partial_\mu \phi(x) - \omega^\lambda_\nu x^\nu \partial_\mu \partial_\lambda \phi(x) - \omega^\lambda{}_\mu \partial_\lambda \phi(x). \tag{0.39}$$

Таким образом с одной стороны для Лагранжевой плотности мы получаем

$$\mathcal{L}\left(\phi'(x), \partial_\mu \phi'(x)\right) = \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right) - \frac{\partial \mathcal{L}}{\partial \phi}\omega^\mu{}_\nu x^\nu \partial_\mu \phi - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\omega^\lambda{}_\nu x^\nu \partial_\mu \partial_\lambda \phi - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\omega^\lambda{}_\mu \partial_\lambda \phi. \tag{0.40}$$

С другой стороны

$$\mathcal{L}\left(x^\mu - \omega^\mu{}_\nu x^\nu\right) = \mathcal{L}(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L}. \tag{0.41}$$

Откуда получим

$$\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\omega^\mu{}_\nu x^\nu \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\omega^\lambda{}_\nu x^\nu \partial_\mu \partial_\lambda \phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}\omega^\lambda{}_\mu \partial_\lambda \phi \tag{0.42}$$

Далее используя уравнение движения (Эйлера-Лагранжа):

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\lambda \frac{\partial \mathcal{L}}{\partial\left(\partial_\lambda \phi\right)} \tag{0.43}$$

можно получить

$$\omega^\mu{}_\nu \left(\partial_\lambda \frac{\partial \mathcal{L}}{\partial\left(\partial_\lambda \phi\right)} x^\nu \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\lambda \phi\right)} x^\nu \partial_\mu \partial_\lambda \phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \phi\right)} \partial_\mu \phi - x^\nu \partial_\mu \mathcal{L}\right) = 0. \tag{0.44}$$

Далее заметим, что полученное выражение можно записать в виде

$$\omega^\mu{}_\nu \partial_\lambda \left(x^\nu \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\lambda \phi\right)} \partial_\mu \phi - \delta^\lambda_\mu \mathcal{L}\right)\right) = \omega_{\mu\nu} \partial_\lambda \left(x^\nu T^{\lambda\mu}\right) = 0 \tag{0.45}$$

Так как $\omega_{\mu\nu} = -\omega_{\nu\mu}$ мы можем окончательно записать

$$\omega_{\mu\nu} \partial_\lambda \left(x^\nu T^{\lambda\mu} - x^\mu T^{\lambda\nu}\right) = 0. \tag{0.46}$$

Это равенство выполняется для произвольного антисимметричного $\omega_{\mu\nu}$, следовательно мы получаем сохраняющийся Нетеровский ток

$$M^{\lambda\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu} \tag{0.47}$$

Заметим, что то, что данный ток сохраняется сразу дает ответ на пункт (f), так как

$$0 = \partial_\lambda M^{\lambda\mu\nu} = T^{\mu\nu} - T^{\nu\mu} \tag{0.48}$$

но мы решим пункт (f) похожим, но немного другим способом.
(e). В данном случае для векторного поля A_μ мы получаем

$$A'_\mu(x) = \left(\delta^\nu_\mu + \omega^\nu_\mu\right) A_\nu\left(x^\lambda - \omega^\lambda_\rho x^\rho\right) = A_\mu(x) - \omega^\lambda_\rho x^\rho \partial_\lambda A_\mu(x) + \omega_\mu{}^\nu A_\nu(x),$$
$$\partial_\nu A'_\mu(x) = \partial_\nu A_\mu(x) - \omega^\lambda_\nu \partial_\lambda A_\mu(x) - \omega^\lambda_\rho x^\rho \partial_\lambda \partial_\nu A_\mu(x) + \omega_\mu{}^\lambda \partial_\nu A_\lambda(x). \tag{0.49}$$

Ход решения аналогичен пункту (d), за исключением того, что векторное поле приобретает дополнительные поправки, связанные с "векторностью", или можно сказать, связанные со спином $s = 1$. Лагранжева плотность, являясь скаляром, преобразуется как и раньше:

$$\mathcal{L}\left(x^\mu - \omega^\mu{}_\nu x^\nu\right) = \mathcal{L}(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L}. \tag{0.50}$$

Далее следуя ходу решения задачи (d) получаем

$$0 = -\omega_{\mu\nu} \partial_\lambda \left(x^\nu T^{\lambda\mu} - x^\mu T^{\lambda\nu}\right) + \frac{\partial \mathcal{L}}{\partial A_\mu}\omega_\mu{}^\lambda A_\lambda + \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu A_\mu\right)}\omega_\mu{}^\lambda \partial_\nu A_\lambda \tag{0.51}$$

Используя уравнения движения

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = \partial_\sigma \frac{\partial \mathcal{L}}{\partial \left(\partial_\sigma A_\mu\right)} \tag{0.52}$$

находим $\left(\omega_\mu{}^\lambda = -\omega^\lambda{}_\mu\right)$

$$\begin{aligned} 0 &= -\omega_{\mu\nu}\partial_\lambda \left(x^\nu T^{\lambda\mu} - x^\mu T^{\lambda\nu}\right) - \omega^\lambda{}_\mu \left(\partial_\sigma \frac{\partial \mathcal{L}}{\partial \left(\partial_\sigma A_\mu\right)} A_\lambda + \frac{\partial \mathcal{L}}{\partial \left(\partial_\nu A_\mu\right)} \partial_\nu A_\lambda\right) = \\ &= -\omega_{\mu\nu}\partial_\lambda \left(x^\nu T^{\lambda\mu} - x^\mu T^{\lambda\nu} + \frac{\partial \mathcal{L}}{\partial \left(\partial_\lambda A_\mu\right)} A^\nu - \frac{\partial \mathcal{L}}{\partial \left(\partial_\lambda A_\nu\right)} A^\mu\right). \end{aligned} \tag{0.53}$$

Откуда получаем

$$M^{\lambda\mu\nu} = x^\mu T^{\lambda\mu} - x^\nu T^{\lambda\mu} + S^{\lambda\mu\nu} = x^\mu T^{\lambda\mu} - x^\nu T^{\lambda\mu} + \frac{\partial \mathcal{L}}{\partial \left(\partial_\lambda A_\nu\right)} A^\mu - \frac{\partial \mathcal{L}}{\partial \left(\partial_\lambda A_\mu\right)} A^\nu \tag{0.54}$$

Изначально задача была сформулирована в очень замысловатом варианте: $M^{\lambda\mu\nu} = x^\mu \Theta^{\lambda\nu} - x^\nu \Theta^{\lambda\mu}$, где $\Theta^{\mu\nu} = T^{\mu\nu} + S^{\mu\nu}$ тензор Белинфанте. Такой подход осложняет понимание материала и может быть найден в книге Вайнберга 1 том п. 7.

(f). В данном случае мы имеем

$$\begin{aligned} &\phi'\left(x'\right) = \phi(x), \\ &\partial'_\mu \phi'\left(x'\right) = \partial_\mu \phi(x) - \omega^\nu{}_\mu \partial_\nu \phi(x). \end{aligned} \tag{0.55}$$

Тогда получим

$$0 = \mathcal{L}\left(\phi'\left(x'\right), \partial'_\mu \phi'\left(x'\right)\right) - \mathcal{L}\left(\phi(x), \partial_\mu \phi(x)\right) = -\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \phi\right)} \omega^\nu{}_\mu \partial_\nu \phi = -\omega_{\nu\mu}\left(T^{\mu\nu} - T^{\nu\mu}\right), \tag{0.56}$$

откуда

$$T^{\mu\nu} = T^{\nu\mu} \tag{0.57}$$

МФТИ-Киселев-2020.**3**.^{C}

Найти поток частиц с релятивистской нормировкой состояний

$$\left\langle \boldsymbol{k} | \boldsymbol{k}' \right\rangle = 2\epsilon(\boldsymbol{k})(2\pi)^3 \delta\left(\boldsymbol{k} - \boldsymbol{k}'\right)$$

Solution

Релятивистская нормировка задает число частиц, которое равно

$$\mathcal{N} = \langle \boldsymbol{k} | \boldsymbol{k} \rangle = 2\varepsilon(\boldsymbol{k})(2\pi)^3 \delta(\boldsymbol{k})\big|_{k=0} = 2\varepsilon(\boldsymbol{k}) V_{[3]}$$

Для свободных частиц вектор потока направлен по волновому вектору $j = j \cdot n$, где $n = \boldsymbol{k}/k$. Тогда за время $T \to \infty$ через площадь $S = S \cdot n$ с нормалью, параллельной потоку, пройдет число частиц

$$\mathcal{N} = j \cdot ST = jST$$

Но если v - скорость движения частицы, то

$$T = \frac{L}{v}, \quad V_{[3]} = SL$$

и, следовательно,

$$\mathcal{N} = \frac{1}{v} j V_{[3]}$$

а значит,

$$j = 2\varepsilon(k) \cdot v$$

26.1.2 Problems about free quantum fields 1

Stanford.exam-2022.1/5. Field uncertainties. (10/40p)

Quantum field theories have states with definite numbers of particles, in analogy with the number states $|n\rangle$ of a quantum harmonic oscillator. However, there also exist states with definite field profiles, in analogy with the position states $|x\rangle$. A general state is a superposition of such field eigenstates, and thus can have an uncertain field value.

a) Consider a free real scalar field ϕ of mass m in the vacuum state. The variance of the field value at $\mathbf{x}$ is

$$\sigma^2_{\phi(\mathbf{x})} = \left\langle 0 \left| \phi(\mathbf{x})^2 \right| 0 \right\rangle - (\langle 0|\phi(\mathbf{x})|0\rangle)^2.$$

Show that this quantity is infinite.

b) There's nothing wrong with the result in part (a), because in practice we can never measure the field value at a point. We only measure "smeared" field values averaged over some length scale a, such as

$$\phi_a(\mathbf{x}) = \frac{1}{a^3\pi^{3/2}} \int d^3y \phi(\mathbf{y}) e^{-|\mathbf{x}-\mathbf{y}|^2/a^2}.$$

Write $\sigma^2_{\phi_a(\mathbf{x})}$ in the form of an integral over a single variable.

c) The standard deviation of the smeared field is finite. Find expressions for $\sigma^2_{\phi_a(\mathbf{x})}$ in the limits $a \ll 1/m$ and $a \gg 1/m$.

(Hint: in each regime, your answer should have the form αa^β for some values of α and β. If you're having trouble doing the final integral in one of these cases, you can start from the result $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.)

d) Consider a cubical copper microwave cavity of volume $V = 1\ \mathrm{m}^3$. The electric field in the cavity does not have a definite value, due to the quantization of the electromagnetic field. Find a rough, order of magnitude estimate for the standard deviation of the average electric field in the cavity in its ground state, and evaluate it in volts per meter. (Hint: you will need to go from natural units back to SI units using $\hbar = 10^{-34}\mathrm{Js}, c = 3 \times 10^8$ m/s, and $e = 1.6 \times 10^{-19}\mathrm{C}$.)

Solution a)

The expectation value is

$$\langle 0|\phi(\mathbf{x})|0\rangle = 0$$

while the expectation value of the square is

$$\begin{aligned}
\left\langle 0 \left| \phi(\mathbf{x})^2 \right| 0 \right\rangle &= \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\sqrt{2E(\mathbf{p})}\sqrt{2E(\mathbf{q})}} \\
&\quad \times \langle 0| \left(a(\mathbf{p}) e^{i\mathbf{x}\cdot\mathbf{p}} + a^\dagger(\mathbf{p}) e^{-i\mathbf{x}\cdot\mathbf{p}} \right) \left(a(\mathbf{q}) e^{i\mathbf{x}\cdot\mathbf{q}} + a^\dagger(\mathbf{q}) e^{-i\mathbf{x}\cdot\mathbf{q}} \right) \\
&= \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\sqrt{2E(\mathbf{p})}\sqrt{2E(\mathbf{q})}} \left\langle 0 \left| a(\mathbf{p}) a^\dagger(\mathbf{q}) e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{q})} \right| 0 \right\rangle \\
&= \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\sqrt{2E(\mathbf{p})}\sqrt{2E(\mathbf{q})}} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}) e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{q})} \\
&= \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{1}{2E(\mathbf{p})} \\
&= \frac{1}{2\pi^2} \int_0^\infty d|\mathbf{p}| \frac{|\mathbf{p}|^2}{\sqrt{|\mathbf{p}|^2 + m^2}}
\end{aligned}$$

which is infinite.

Solution b)

The expectation value still vanishes, but now we have

$$\begin{aligned}\left\langle 0\left|\phi_a(\mathbf{x})^2\right|0\right\rangle &= \frac{1}{a^6\pi^3}\int d\mathbf{y}d\mathbf{z}e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{a^2}}e^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{a^2}}\langle 0|\phi(\mathbf{y})\phi(\mathbf{z})|0\rangle \\ &= \frac{1}{a^6\pi^3}\int d\mathbf{y}d\mathbf{z}\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{2E(\vec{p})}e^{i\mathbf{p}(\mathbf{y}-\mathbf{z})}e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{a^2}}e^{-\frac{|\mathbf{x}-\mathbf{z}|^2}{a^2}}.\end{aligned}$$

Next, we perform the shifts $\mathbf{z}\to \mathrm{z}+\mathrm{x}$ and $\mathbf{y}\to\mathbf{y}+\mathrm{x}$. Then the integral

$$\begin{aligned}\int d\mathbf{y}e^{i\mathbf{p}(\mathbf{y})}e^{-\frac{|\mathbf{y}|^2}{a^2}} &= \int d\mathbf{y}e^{i|\mathbf{p}||\mathbf{y}|\cos\theta}e^{-\frac{|\mathbf{y}|^2}{a^2}} \\ &= (2\pi)\int_0^\infty d|\mathbf{y}||\mathbf{y}|^2\int_{-1}^1 d\cos\theta|\mathbf{y}|e^{i|\mathbf{p}|\cos\theta}e^{-\frac{|\mathbf{y}|^2}{a^2}} \\ &= \frac{(4\pi)}{|\mathbf{p}|}\int_0^\infty d|\mathbf{y}||\mathbf{y}|\sin(|\mathbf{y}||\mathbf{p}|)e^{-\frac{|\mathbf{y}|^2}{a^2}} \\ &= a^3\pi^{3/2}e^{-\frac{|\mathbf{p}|^2a^2}{4}}\end{aligned}$$

Inserting this above, we find

$$\left\langle 0\left|\phi_a(\mathbf{x})^2\right|0\right\rangle = \int\frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{2E(\vec{p})}e^{-\frac{|\mathbf{p}|^2a^2}{2}}$$

Simplifying, we thus conclude that

$$\sigma^2_{\phi_a(\mathbf{x})} = \left\langle 0\left|\phi_a(\mathbf{x})^2\right|0\right\rangle = \frac{1}{4\pi^2}\int_0^\infty d|\mathbf{p}|\frac{|\mathbf{p}|^2}{\sqrt{|\mathbf{p}|^2+m^2}}e^{-\frac{|\mathbf{p}|^2a^2}{2}}$$

Solution c)

Changing to the variable $t=a|\mathbf{p}|$, we have

$$\sigma^2_{\phi_a(\mathbf{x})} = \frac{1}{4\pi^2a^2}\int_0^\infty dt\frac{t^2}{\sqrt{t^2+a^2m^2}}e^{-\frac{t^2}{2}}$$

Expanding around $ma\ll 1$, we find

$$\lim_{ma\to 0}\sigma^2_{\phi_a(\mathbf{x})} = \frac{1}{4\pi^2a^2}\int_0^\infty dtte^{-\frac{t^2}{2}} = \frac{1}{4\pi^2a^2}.$$

Expanding around $ma\gg 1$, we find

$$\lim_{1/(ma)\to 0}\sigma^2_{\phi_a(\mathbf{x})} = \frac{1}{4\pi^2a^3m}\int_0^\infty dtt^2e^{-\frac{t^2}{2}} = \frac{1}{4\pi^{3/2}a^3m\sqrt{2}}.$$

Solution d)

In part (c), we showed that for a massless vector field, the standard deviation of the field average over a region of size a is $\sigma\sim 1/a$. The same logic applies to a massless vector field A_μ. The electric field in the cavity is related to A_μ by one spatial or temporal derivative, which by dimensional analysis also gives a factor of $1/a$. Thus, we expect $\sigma_E\sim 1/a^2\sim 1/V^{2/3}$. (We could also have jumped directly to this result. We already know $[E]=2$ from problem set 1, and part (**c**) tells us the answer is neither zero nor infinity, so by dimensional analysis, $1/a^2$ is the only possible answer.)

Going from natural units back to SI units, we have

$$\sigma_E \sim \frac{\hbar c}{e a^2} \sim 10^{-7}\ \text{V/m}.$$

This might be surprisingly large if you haven't seen it before, but it lines up with typical "quantum noise" scales in precision electronics. Any answer within a few orders of magnitude of this is acceptable.

Белавин-Пархоменко.2. Квантование поля Клейна-Гордона и эффект Казимира

Квантование комплексного поля Клейна-Гордона (задача 2.2. П-Ш) (50 баллов)

Рассмотрим комплексное скалярное поле $\phi(x)$ с действием

$$S = \int d^4x \left(\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi\right) \tag{0.1}$$

(a). Найдите сопряженные импульсы к $\phi(x)$ и $\phi^*(x)$ и канонические коммутационные соотношения. Покажите, что Гамильтониан дается формулой

$$H = \int d^3x \left(\pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi\right). \tag{0.2}$$

Вычислите Гейзенберговское уравнение движения для поля $\phi(x)$ и покажите, что это действительно уравнение движения для поля Клейна-Гордона.

(b). Диагонализуйте H введя операторы рождения и уничтожения:

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} \left(a_p e^{-ipx} + b_p^\dagger e^{ipx}\right), \quad E_p = \sqrt{p^2 + m^2}. \tag{0.3}$$

Покажите, что теория содержит два набора частиц массой m.

(c). Перепишите сохраняющийся заряд

$$Q = \int d^3x \frac{i}{2} \left(\phi^* \pi^* - \pi \phi\right) \tag{0.4}$$

в терминах операторов рождения и уничтожения и вычислите заряд частиц каждого типа.

(d). Рассмотрите случай двух комплексных полей Клейна-Гордона одинаковой массы. Обозначим поля ϕ_a, где $a = 1, 2$. Покажите, что теперь есть четыре сохраняющихся заряда. Один дается обобщением части (c), а другие три даются формулой

$$Q^i = \int d^3x \frac{i}{2} \left(\phi_a^* \left(\sigma^i\right)_{ab} \pi_b^* - \pi_a \left(\sigma^i\right)_{ab} \phi_b\right), \tag{0.5}$$

где σ^i матрицы Паули. Обобщите результат на случай n комплексных скалярных полей. Для решения этой задачи, нужно вспомнить из первой лекции, что сохраняющиеся заряды получаются из сохраняющихся токов, а те в свою очередь по теореме Нетер возникают вследствие некоторой симметрии. В данном случае мы имеем дело с симметрией $U(n) = U(1) \times SU(n)$:

$$\phi_a(x) \to \phi'_a(x) = U_{ab} \phi_b(x). \tag{0.6}$$

Сохраняющийся заряд для $U(1)$ это заряд в пункте (c). Для группы $SU(n)$ введем генераторы симметрии $\left(T^i\right)_{ab}$, тогда

$$\delta \phi_a = i \alpha^i \left(T^i\right)_{ab} \phi_b, \quad \delta \phi_a^* = -i \alpha^i \phi_b^* \left(T^i\right)_{ba}, \tag{0.7}$$

Дальше постарайтесь продолжить сами.

2 Эффект Казимира (50 баллов)

(a). В данной задаче мы изучим немного иной подход (чем который был на лекции) к эффекту Казимира. Рассмотрите безмассовое скалярное поле в пространстве размерности $d+1$. Пусть само пространство представляет из себя огромный d-мерный куб с длиной стороны D (+ время). В этом пространстве находятся две пластины на расстоянии L друг от друга. Выберем координаты так, чтобы одна пластина находилась в $x_1 = 0$, а вторая в $x_1 = L$. Мы подразумеваем отражающие граничные условия для поля

$$\phi(\vec{x}, t) = 0, \quad \text{при} \quad x_1 = 0, L. \tag{0.8}$$

Также будем считать, что $\phi(\vec{x}, t) = \phi(\vec{x} + D\vec{n}, t)$, где $\vec{n} = (1, 0, ...0)$, то есть у нас переодические граничные условия на гранях огромного ($D \gg L$, в силу этого условия считайте, что спектр вне пластин и вдоль пластин непрерывный) пространства-куба. Таким образом полная энергия вакуума складывается из энергии вакуума между пластинами $E(L)$ и из энергии вакуума вне пластин $E(D-L)$:

$$E = E(L) + E(D - L). \tag{0.9}$$

Покажите, что энергия Казимира на единицу площади дается формулой (здесь A это площадь)

$$\frac{E(L)}{A} = -\frac{\Gamma(-d/2)\zeta(-d)}{2(4\pi)^{d/2}} \left(\frac{\pi}{L}\right)^d \tag{0.10}$$

При вычислении используйте регуляризацию дзета-функцией:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{0.11}$$

Например: $\zeta(-1) = \sum\limits_{n=1}^{\infty} n = -\frac{1}{12}$, (не удивляйтесь, это формальный математический метод регуляризации, часто оказывающийся весьма удобным). Также Г-функция определяется как

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \tag{0.12}$$

Объясните почему наличие квантового поля приводит к силе между пластинами.
(b). Изучите эффект Казимира в трех пространственных измерениях для поля Клейна-Гордона в случае когда масса m не равна нулю. Граничные условия для ϕ и ход решения точно такие же как на лекции. Используйте, выведенную на лекции, формулу Абеля-Плана:

$$\sum_{n=1}^{\infty} f(n) - \int_0^{\infty} f(\tau) d\tau = -\frac{1}{2} f(0) + i \int_0^{\infty} dt \frac{f(it) - f(-it)}{e^{2\pi t} - 1} \tag{0.13}$$

(из которой в свою очередь получалась формула Эйлера-Маклорена). Обратите внимание на то, что у подынтегральной функции есть точки ветвления и поэтому $f(it)$ не на всей области интегрирования равна $f(-it)$. Найдите силу на единицу площади $F(a, m)$, где a - растояние между пластинами (не ожидайте, что ответ будет элементарной функцией!) Найдите асимптотики $F(a, m)$ для $am \ll 1$ и $am \gg 1$. (Обратите внимание, что если в этой задаче наивно использовать формулу Эйлера-Маклорена, а не Абеля-Плана, то мы получим для энергии Казимира ноль, что очевидно физически неверно!)

Белавин-Пархоменко.2. Квантование поля Клейна-Гордона и эффект Казимира Решения

Квантование комплексного поля Клейна-Гордона (задача 2.2. П-Ш) (50 баллов)

$$S = \int d^4x \left(\partial^\mu \phi^* \partial_\mu \phi + m^2 |\phi|^2\right) \tag{0.1}$$

получаем

$$\pi = \frac{\delta S}{\delta \dot{\phi}} = \dot{\phi}^*, \quad \pi^* = \frac{\delta S}{\delta \dot{\phi}^*} = \dot{\phi} \tag{0.2}$$

Для Гамильтониана находим

$$\mathcal{H}(x)=\int d^3x\left(\pi^*\cdot\pi+\nabla\phi^*\cdot\nabla\phi+m^2|\phi|^2\right) \tag{0.3}$$

Используем

$$[\phi(x),\pi(y)]|_{x^0=y^0}=i\delta^{(3)}(\vec{x}-\vec{y}) \tag{0.4}$$

и получаем

$$\begin{aligned}\dot{\pi}\left(x^0,\vec{x}\right)&=-i\left[\pi\left(x^0,\vec{x}\right),\int d^3y\left(\pi^*\left(x^0,\vec{y}\right)\pi\left(x^0,\vec{y}\right)+\nabla\phi^*\left(x^0,\vec{y}\right)\nabla\phi\left(x^0,\vec{y}\right)+m^2\left|\phi\left(x^0,\vec{y}\right)\right|^2\right)\right]=\\&=-i\int d^3y\left(-\nabla^2\phi^*\left(x^0,\vec{y}\right)(-i)\delta^{(3)}(\vec{x}-\vec{y})-im^2\phi^*\left(x^0,\vec{y}\right)\delta^{(3)}(\vec{x}-\vec{y})\right)=\\&=\left(\nabla^2-m^2\right)\phi^*\left(x^0,\vec{x}\right).\end{aligned} \tag{0.5}$$

В итоге находим $\left(\dot{\pi}=\ddot{\phi}^*\right)$

$$\ddot{\phi}^*-\nabla^2\phi^*+m^2\phi^*=\left(-\partial^2+m^2\right)\phi^*=0. \tag{0.6}$$

(b). Так как $\phi(x)$ комплексное поле, то в его разложении по модам участвуют два разных типа оператора рождения и уничтожения a_p,b_p:

$$\phi(x)=\int\frac{d^3p}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}e^{i\vec{p}\vec{x}}+b^\dagger_{\vec{p}}e^{-i\vec{p}\vec{x}}\right) \tag{0.7}$$

Для импульса получаем

$$\pi(x)=i\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\vec{p}}}{2}}\left(a^\dagger_{\vec{p}}e^{-i\vec{p}\vec{x}}-b_{\vec{p}}e^{i\vec{p}\vec{x}}\right) \tag{0.8}$$

где коммутационные соотношения для операторов рождения и уничтожения даются формулами:

$$\left[a_{\vec{p}},a^\dagger_{\vec{q}}\right]=\left[b_{\vec{p}},b^\dagger_{\vec{q}}\right]=(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}) \tag{0.9}$$

Далее получаем (пусть $x^0=0$)

$$\begin{aligned}\int d^3x\pi^*(x)\pi(x)&=\int\frac{d^3xd^3pd^3q}{(2\pi)^6}\frac{\sqrt{E_{\vec{p}}E_{\vec{q}}}}{2}\left(a_{\vec{p}}e^{i\vec{p}\vec{x}}-b^\dagger_{\vec{p}}e^{-i\vec{p}\vec{x}}\right)\left(a^\dagger_{\vec{q}}e^{-i\vec{q}\vec{x}}-b_{\vec{q}}e^{i\vec{q}\vec{x}}\right)=\\&=\int\frac{d^3p}{(2\pi)^3}\frac{E_{\vec{p}}}{2}\left(a_{\vec{p}}a^\dagger_{\vec{p}}-a_{\vec{p}}b_{-\vec{p}}-b^\dagger_{\vec{p}}a^\dagger_{-\vec{p}}+b^\dagger_{\vec{p}}b_{\vec{p}}\right),\\\int d^3x\nabla\phi^*(x)\nabla\phi(x)&=\int\frac{d^3xd^3pd^3q}{(2\pi)^6}\frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}}\left(-ia^\dagger_{\vec{p}}\vec{p}e^{-i\vec{p}\vec{x}}+i\vec{p}b_{\vec{p}}e^{i\vec{p}\vec{x}}\right)\left(i\vec{q}a_{\vec{q}}e^{i\vec{q}\vec{x}}-i\vec{q}b^\dagger_{\vec{q}}e^{-i\vec{q}\vec{x}}\right)=\\&=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\vec{p}^2a^\dagger_{\vec{p}}a_{\vec{p}}+\vec{p}^2a^\dagger_{\vec{p}}b^\dagger_{-\vec{p}}+\vec{p}^2b_{\vec{p}}a_{-\vec{p}}+\vec{p}^2b_{\vec{p}}b^\dagger_{\vec{p}}\right),\\\int d^3xm^2|\phi|^2&=\int\frac{d^3xd^3pd^3q}{(2\pi)^6}\frac{m^2}{2\sqrt{E_{\vec{p}}E_{\vec{p}}}}\left(a^\dagger_{\vec{p}}a_{\vec{p}}+a^\dagger_{\vec{p}}b^\dagger_{-\vec{p}}+b_{\vec{p}}a_{-\vec{p}}+b_{\vec{p}}b^\dagger_{\vec{p}}\right).\end{aligned} \tag{0.10}$$

В итоге получаем для Гамильтониана

$$\begin{aligned}H&=\int\frac{d^3p}{(2\pi)^3}\left[\frac{E_{\vec{p}}}{2}\left(a_{\vec{p}}a^\dagger_{\vec{p}}+a^\dagger_{\vec{p}}a_{\vec{p}}+b^\dagger_{\vec{p}}b_{\vec{p}}+b_{\vec{p}}b^\dagger_{\vec{p}}\right)+\frac{E_{\vec{p}}}{2}\left(a^\dagger_{\vec{p}}b^\dagger_{-\vec{p}}+b_{\vec{p}}a_{-\vec{p}}-a_{-\vec{p}}b_{\vec{p}}-b^\dagger_{-\vec{p}}a^\dagger_{\vec{p}}\right)\right]=\\&=\int\frac{d^3p}{(2\pi)^3}E_{\vec{p}}\left(a^\dagger_{\vec{p}}a_{\vec{p}}+b^\dagger_{\vec{p}}b_{\vec{p}}\right)+(\text{ бесконечная константа }),\end{aligned} \tag{0.11}$$

где $E_{\vec{p}}=\sqrt{p^2+m^2}$. Таким образом теория описывает две частицы (a и b) массой m.

(c). Для заряда получаем

$$\begin{aligned}
Q &= \frac{i}{2}\int d^3x\phi^*\pi^* + (\text{ к.с. }) = \\
&= \frac{i}{2}\int \frac{d^3xd^3pd^3q}{(2\pi)^6}\frac{(-i)}{2}\sqrt{\frac{E_{\vec{q}t}}{E_{\vec{p}}}}\left(a^\dagger_{\vec{p}}e^{-i\vec{p}\vec{x}}+b_{\vec{p}}e^{i\vec{p}\vec{x}}\right)\left(a_{\vec{q}}e^{i\vec{q}\vec{x}}-b^\dagger_{\vec{q}}e^{-i\vec{q}\vec{x}}\right)+(\text{ к.с. }) = \\
&= \frac{1}{4}\int \frac{d^3p}{(2\pi)^3}\left(a^\dagger_{\vec{p}}a_{\vec{p}}-a^\dagger_{\vec{p}}b^\dagger_{-\vec{p}}+b_{\vec{p}}a_{-\vec{p}}-b_{\vec{p}}b^\dagger_{\vec{p}}+a^\dagger_{\vec{p}}a_{\vec{p}}-b_{-\vec{p}}a_{\vec{p}}+a^\dagger_{-\vec{p}}b^\dagger_{\vec{p}}-b_{\vec{p}}b^\dagger_{\vec{p}}\right) = \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left(a^\dagger_{\vec{p}}a_{\vec{p}}-b^\dagger_{\vec{p}}b_{\vec{p}}\right)+(\text{ бесконечная константа }).
\end{aligned} \tag{0.12}$$

В итоге получаем, что $Q(a)=\frac{1}{2}$ и $Q(b)=-\frac{1}{2}$.
(d). Итак вариации полей даются формулами

$$\delta\phi_a = i\alpha^i\left(T^i\right)_{ab}\phi_b,\quad \delta\phi^*_a = -i\alpha^i\phi^*_b\left(T^i\right)_{ba}. \tag{0.13}$$

Откуда для сохраняющегося тока получаем (см. решение Листка 1)

$$\begin{aligned}
j^\mu_i &= \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi_a\right)}\left(-i\left(T_i\right)_{ab}\phi_b\right)+(\text{ к.с. }) = \\
&= i\partial^\mu\phi^*_a\left(T_i\right)_{ab}\phi_b - i\phi^*_b\left(T_i\right)_{ba}\partial^\mu\phi_a.
\end{aligned} \tag{0.14}$$

В итоге получаем

$$Q_i = \int d^3xj^0_i = -i\int d^3x\dot{\phi}^*_a\left(T_i\right)_{ab}\phi_b + i\int d^3x\phi^*_b\left(T_i\right)_{ba}\dot{\phi}_a. \tag{0.15}$$

Далее замечая, что импульсы равны $\pi_a=\dot{\phi}^*_a, \pi^*_a=\dot{\phi}_a$, получаем для зарядов

$$Q_i = i\int d^3x\left[\phi^*_a\left(T_i\right)_{ab}\pi^*_b - \pi_a\left(T_i\right)_{ab}\phi_b\right] \tag{0.16}$$

В частном случае группы $SU(2)$ мы имеем

$$\left(T^i\right)_{ab} = \frac{1}{2}\left(\sigma^i\right)_{ab} \tag{0.17}$$

2 Эффект Казимира (50 баллов)

(a). Итак спектр поля между пластинами

$$\omega_{n,p_2,...,p_d} = \left[\left(\frac{\pi n}{L}\right)^2+p_2^2+...+p_d^2\right]^{1/2} = \left[\left(\frac{\pi n}{L}\right)^2+\vec{p}^{\,2}_{\|}\right]^{1/2},\quad n=1,2,3,... \tag{0.18}$$

Для энергии на единицу площади получаем

$$\begin{aligned}
\frac{E(L)}{A} &= \sum_{n=1}^{\infty}\int\frac{d^{d-1}p_{\|}}{(2\pi)^{d-1}}\frac{1}{2}\left[\left(\frac{\pi n}{L}\right)^2+\vec{p}^{\,2}_{\|}\right]^{1/2} = \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{\Omega_{d-1}}{(2\pi)^{d-1}}\int\limits_0^\infty p_{\|}^{d-2}dp_{\|}\left[\left(\frac{\pi n}{L}\right)^2+\vec{p}^{\,2}_{\|}\right]^{1/2}
\end{aligned} \tag{0.19}$$

где $\Omega_{d-1}=\frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}$ - площадь $d-2$-мерного шара. Далее делая замену переменных в интеграле $p_{\|}=\left(\frac{\pi n}{L}\right)t$ получаем

$$\frac{E(L)}{A}=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{\pi n}{L}\right)^{d}\frac{\Omega_{d-1}}{(2\pi)^{d-1}}\int\limits_{0}^{\infty}dtt^{d-2}\sqrt{1+t^2}=\frac{1}{2}\frac{\Omega_{d-1}}{(2\pi)^{d-1}}\frac{\pi^d}{L^d}\zeta(-d)\int\limits_{0}^{\infty}dtt^{d-2}\sqrt{1+t^2}\tag{0.20}$$

Далее для интеграла находим

$$\int\limits_{0}^{\infty}dtt^{d-2}\sqrt{1+t^2}\overset{u=1+t^2}{=}\frac{1}{2}\int\limits_{1}^{\infty}duu^{1/2}(u-1)^{\frac{d-3}{2}}\overset{v=\frac{1}{u}}{=}\frac{1}{2}\int\limits_{0}^{1}dvv^{-\frac{d}{2}-1}(1-v)^{\frac{d}{2}-\frac{3}{2}}=$$
$$=\frac{1}{2}B\left(-\frac{d}{2},\frac{d-1}{2}\right)=\frac{\Gamma(-d/2)\Gamma\left(\frac{d-1}{2}\right)}{2\Gamma(-1/2)}.\tag{0.21}$$

В итоге получаем

$$\frac{E(L)}{A}=-\frac{1}{2^{d+1}}\zeta(-d)\frac{\pi^{d/2}}{L^d}\Gamma\left(-\frac{d}{2}\right)\tag{0.22}$$

И для силы на единицу площади находим

$$\frac{F}{A}=-\frac{\partial}{\partial L}\left(\frac{E(L)}{A}+\frac{E(D-L)}{A}\right)\overset{D\to\infty}{=}\left(-\frac{d}{2}\right)\Gamma\left(-\frac{d}{2}\right)\frac{\zeta(-d)\pi^{d/2}}{2^dL^{d+1}}=\frac{\pi^{d/2}\Gamma(1-d/2)\zeta(-d)}{2^dL^{d+1}}\tag{0.23}$$

Мы видим, что пластины притягиваются.
(b). Итак в данном случае мы имеем

$$\frac{E-E_0}{L^2}=\sum_{n=1}^{\infty}f(n)-\int\limits_{0}^{\infty}f(\tau)d\tau\tag{0.24}$$

где

$$f(\tau)=\frac{1}{2}\int\frac{d^2p_{\|}}{(2\pi)^2}\sqrt{\vec{p}_{\|}^2+\frac{\pi^2\tau^2}{a^2}+m^2}\Phi\left(\frac{\vec{p}_{\|}^2+\frac{\pi^2\tau^2}{a^2}+m^2}{\Lambda^2}\right).\tag{0.25}$$

Используя формулу Абеля-Плана находим

$$\sum_{n=1}^{\infty}f(n)-\int\limits_{0}^{\infty}f(\tau)d\tau=-\frac{1}{2}f(0)+i\int\limits_{0}^{\infty}dt\frac{f(it)-f(-it)}{e^{2\pi t}-1}.\tag{0.26}$$

Для функции $F(z)=\sqrt{z^2+a^2}$, мы имеем

$$F(iy)=\begin{cases}\sqrt{a^2-y^2}, & |y|<a\\ i\sqrt{y^2-a^2}, & y>a\\ -i\sqrt{y^2-a^2}, & y<a\end{cases}\tag{0.27}$$

В итоге получаем

$$\frac{E-E_0}{L^2}=-\frac{1}{2}f(0)+i\frac{1}{2}\int\frac{d^2p_{\|}}{(2\pi)^2}\int\limits_{\sqrt{\frac{a^2}{\pi^2}\left(p_{\|}^2+m^2\right)}}^{\infty}dt\frac{2i\sqrt{\frac{\pi^2}{a^2}t^2-p_{\|}^2-m^2}}{e^{2\pi t}-1}\tag{0.28}$$

где мы уже убрали ультрафиолетовый регулятор $\Phi\left(p^2/\Lambda^2\right)$, так как получившееся выражение сходится. Далее делая замену координат $t=\frac{a}{\pi}\sqrt{p_{\|}^2+m^2}x$, получаем

$$\frac{E-E_0}{L^2}=-\frac{1}{2}f(0)-\frac{a}{\pi}\int\frac{d^2p_{\|}}{(2\pi)^2}\left(p_{\|}^2+m^2\right)\int\limits_{1}^{\infty}dx\frac{\sqrt{x^2-1}}{e^{2ax\sqrt{p_{\|}^2+m^2}}-1}\tag{0.29}$$

Далее для интеграла в правой части формулы делая замену $p_{\|}^2=ym^2$ имеем

$$I(am) = \frac{am^4}{4\pi^2}\int\limits_0^\infty dy(y+1)\int\limits_1^\infty dx\frac{\sqrt{x^2-1}}{e^{2(am)x\sqrt{y+1}-1}} = \frac{am^4}{2\pi^2}\iint\limits_1^\infty dydx\frac{y^3\sqrt{x^2-1}}{e^{2(am)xy}-1} \tag{0.30}$$

Таким образом для силы на единицу площади получаем получаем

$$\frac{F(am)}{L^2} = mI'(am) \tag{0.31}$$

Теперь если $am \ll 1$, мы находим

$$\begin{aligned} I(am) &= \frac{am^4}{2\pi^2}\iint\limits_1^\infty dydx\frac{y^3\sqrt{x^2-1}}{e^{2(am)xy}-1} \overset{t=amy}{=} \frac{am^4}{2\pi^2}\frac{1}{(am)^4}\int\limits_1^\infty dx\sqrt{x^2-1}\int\limits_{am}^\infty dt\frac{t^3}{e^{2tx}-1} \approx \\ &\approx \frac{1}{2\pi^2a^3}\int\limits_1^\infty dx\sqrt{x^2-1}\int\limits_0^\infty dt\frac{t^3}{e^{2tx}-1} = \frac{1}{2\pi^2a^3}\int\limits_1^\infty dx\sqrt{x^2-1}\frac{\pi^4}{240x^4} = \frac{1}{2\pi^2a^3}\frac{\pi^4}{240}\cdot\frac{1}{3} = \\ &= \frac{\pi^2}{1440a^3}. \end{aligned} \tag{0.32}$$

Откуда для энергии находим в случаем $am \ll 1$:

$$\frac{E-E_0}{L^2} = -\frac{1}{2}f(0) - \frac{\pi^2}{1440a^3}, \quad \text{откуда} \quad \frac{F}{L^2} = \frac{\pi^2}{480a^4} \tag{0.33}$$

что как раз является ответом для безмассового случая, который был разобран на лекции.
Теперь если $am \gg 1$, мы получаем

$$\begin{aligned} I(am) &= \frac{am^4}{2\pi^2}\iint\limits_1^\infty dydx\frac{y^3\sqrt{x^2-1}}{e^{2(am)xy}-1} \approx \frac{am^4}{2\pi^2}\iint\limits_1^\infty dydxy^3\sqrt{x^2-1}e^{-2(am)xy} = \frac{am^4}{2\pi^2}\int\limits_1^\infty dyy^3\frac{K_1(2yam)}{2yam} = \\ &= \frac{m^3}{4\pi^2}\int\limits_1^\infty dyy^2K_1(2yam). \end{aligned} \tag{0.34}$$

Далее используя асимптотику для функции бесселя $K_1(x)$ при $x \gg 1$:

$$K_1(x) \approx \sqrt{\frac{\pi}{2x}}e^{-x} \tag{0.35}$$

получаем

$$I(am) \approx \frac{m^3}{4\pi^2}\int\limits_1^\infty dyy^2K_1(2yam) = \frac{m^{5/2}}{8\pi^{3/2}a^{1/2}}\int\limits_1^\infty dyy^{3/2}e^{-2yam} \approx \frac{m^3}{16(\pi am)^{3/2}}e^{-2am} \tag{0.36}$$

Откуда для энергии находим в случаем $am \gg 1$:

$$\frac{E-E_0}{L^2} = -\frac{1}{2}f(0) - \frac{m^3}{16(\pi am)^{3/2}}e^{-2am}, \quad \text{откуда} \quad \frac{F}{L^2} = \frac{m^4}{8(\pi am)^{3/2}}e^{-2am} \tag{0.37}$$

То есть взаимодействие (притяжение) пластин экспоненциально подавленно.

Белавин-Пархоменко.3. Функции Грина

1 Функции Грина для Простого Гармонического Осциллятора (ПГО)

В этом упражнении мы будем изучать несколько разных функций Грина для одномерного ПГО. Обозначим координату ϕ, тогда Лагранжиан дается формулой

$$L = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2. \tag{0.1}$$

(a). Используйте выражение для Гейзенберговского оператора $\phi(t)$ в терминах операторов рождения и уничтожения, чтобы вычислить следующие функции Грина

$$\begin{aligned} G_R(t) &= \theta(t)\langle 0|[\phi(t),\phi(0)]|0\rangle, \\ G_A(t) &= -\theta(-t)\langle 0|[\phi(t),\phi(0)]|0\rangle, \\ G_F(t) &= \theta(t)\langle 0|\phi(t)\phi(0)|0\rangle + \theta(-t)\langle 0|\phi(0)\phi(t)|0\rangle. \end{aligned} \tag{0.2}$$

Индексы R, A, F означают запаздывающая ("retarted"), опережающая ("advanced") и Фейнмановская ("Feynman") функции Грина, соответственно.

(b). Покажите непосредственно из формул (0.2), что все функции G_R, G_A и G_F удовлетворяют уравнению

$$\left(\partial_t^2 + m^2\right) G_{R,A,F}(t) = -i\delta(t). \tag{0.3}$$

Поэтому они являются действительно функциями Грина для дифференциального оператора $\partial_t^2 +$ m^2. Покажите, что все эти три функции Грина отличаются друг от друга решением однородного уравнения и что они могут быть классифицированы по их поведению при $t \to \pm\infty$.

(c). Используйте преобразование Фурье $\tilde{G}(\omega) = \int\limits_{-\infty}^{\infty} dt e^{i\omega t} G(t)$, чтобы решить (0.3). Заметьте, что у $\tilde{G}(\omega)$ много разных неопределенностей. Проанализируйте их и покажите, что они соответствуют добавлению решений однородного уравнения для $G(t)$. Используя это знание, запишите явное Фурье представление для функций Грина G_R, G_A и G_F.

Пропагатор Фейнмана Применяя дифференциальный оператор $\partial_\mu\partial^\mu +$ m^2 к определению пропагатора Фейнмана $D_F(x)$:

$$D_F(\vec{x},t) = \theta(t)\langle 0|\phi(\vec{x},t)\phi(0,0)|0\rangle + \theta(-t)\langle 0|\phi(0,0)\phi(\vec{x},t)|0\rangle \tag{0.4}$$

и используя коммутационные соотношения, получите уравнение

$$\left(\partial_\mu\partial^\mu + m^2\right) D_F(x) = -i\delta^{(4)}(x). \tag{0.5}$$

(30 баллов) Амплитуда для частицы поля Клейна-Гордона (П-Ш, з. 2.3) Вычислите явно в терминах функции Бесселя, амплитуду перехода из точки y в точку x:

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)}, \quad E_{\vec{p}} = \sqrt{p^2+m^2} \tag{0.6}$$

для $(x-y)$ пространственно подобного, т.е. $(x-y)^2 = -r^2$.

Белавин-Пархоменко.исток3. Функции Грина Решения

1 Функции Грина для Простого Гармонического Осциллятора (ПГО)

(a). - Координата $\phi(t)$ в данном случае записывается через операторы рождения и уничтожения как

$$\phi(t) = \frac{1}{\sqrt{2m}}\left(ae^{-imt} + a^\dagger e^{imt}\right) \tag{0.1}$$

Тогда для коммутатора полей находим $\left(\left[a, a^\dagger\right] = 1\right)$:

$$[\phi(t),\phi(0)] = \frac{1}{2m}\left[ae^{-imt} + a^\dagger e^{imt}, a + a^\dagger\right] = \frac{1}{2m}\left(e^{-imt} - e^{imt}\right) = -\frac{i}{m}\sin(mt) \tag{0.2}$$

Откуда получаем для $G_R(t)$ и $G_A(t)$:

$$G_R(t) = -\frac{i}{m}\theta(t)\sin(mt),$$
$$G_A(t) = \frac{i}{m}\theta(-t)\sin(mt). \tag{0.3}$$

И для пропагатора $G_F(t)$ получаем

$$G_F(t) = \theta(t)\frac{1}{2m}\left\langle 0\left|\left(ae^{-imt}+a^\dagger e^{imt}\right)\left(a+a^\dagger\right)\right|0\right\rangle + \theta(-t)\frac{1}{2m}\left\langle 0\left|\left(a+a^\dagger\right)\left(ae^{-imt}+a^\dagger e^{imt}\right)\right|0\right\rangle =$$
$$= \theta(t)\frac{1}{2m}e^{-imt} + \theta(-t)\frac{1}{2m}e^{imt} = \frac{1}{2m}e^{-im|t|}. \tag{0.4}$$

(b). Мы покажем для G_F (для $G_{R,A}$ аналогично):

$$\left(\partial_t^2+m^2\right)G_F(t) =$$
$$\left(\partial_t^2+m^2\right)\left[\theta(t)\langle 0|\phi(t)\phi(0)|0\rangle + \theta(-t)\langle 0|\phi(0)\phi(t)|0\rangle\right] = \delta'(t)\langle 0|\phi(t)\phi(0)|0\rangle - \delta'(t)\langle 0|\phi(0)\phi(t)|0\rangle +$$
$$+ 2\delta(t)\langle 0|\dot\phi(t)\phi(0)|0\rangle - 2\delta(t)\langle 0|\phi(0)\dot\phi(t)|0\rangle + \theta(t)\left\langle 0\left|\left(\partial_t^2+m^2\right)\phi(t)\phi(0)\right|0\right\rangle +$$
$$+ \theta(-t)\left\langle 0\left|\phi(0)\left(\partial_t^2+m^2\right)\phi(t)\right|0\right\rangle. \tag{0.5}$$

Используя то, что $\delta'(t)f(t) = -\delta(t)f'(t)$, а также то, что $\left(\partial_t^2+m^2\right)\phi(t) = 0$, получаем

$$\left(\partial_t^2+m^2\right)G_F(t) = \delta(t)\langle 0|[\dot\phi(t),\phi(0)]|0\rangle = -i\delta(t) \tag{0.6}$$

Далее, например, имеем

$$G_F(t) - G_R(t) = \frac{e^{imt}}{2m} = f(t) \tag{0.7}$$

что является однородным решением уравнения: $\left(\partial_t^2+m^2\right)f(t) = 0$. Аналогично для других разностей. Далее имеем при $t\to\pm\infty$:

$$G_R(t) \to \begin{cases} \sim\left(e^{imt}-e^{-imt}\right), & t\to+\infty \\ 0, & t\to-\infty \end{cases},$$
$$G_A(t) \to \begin{cases} 0, & t\to+\infty \\ \sim\left(e^{imt}-e^{-imt}\right), & t\to-\infty \end{cases},$$
$$G_F(t) \to \begin{cases} \sim e^{-imt}, & t\to+\infty \\ \sim e^{imt}, & t\to-\infty \end{cases}, \tag{0.8}$$

и мы видим, что все функции имееют разное поведение при $t\to\pm\infty$.
(c). Для фурье образа получаем

$$\left(-\omega^2+m^2\right)\tilde G(\omega) = -i \tag{0.9}$$

откуда

$$\tilde G(\omega) = \frac{i}{\omega^2-m^2} \tag{0.10}$$

Для обратного преобразования имеем:

$$G(t) = i\int\limits_C \frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega^2-m^2} \tag{0.11}$$

Мы видим, что на действительной оси есть два полюса в точках $\omega_1 = -m$ и $\omega_2 = m$. Ответ для функции Грина завит от выбора контура интегрирования C. Если контур C идет вдоль действительной оси, чуть выше полюсов, то при интегрировании мы получим, что $G(t) = G_R(t)$. Если контур C идет вдоль действительной

оси, чуть ниже полюсов, то при интегрировании мы получим, что $G(t) = G_A(t)$. Если контур C идет ниже полюса $\omega_1 = -m$, но выше полюса $\omega_2 = m$, то при интегрировании мы получим, что $G(t) = G_F(t)$.

Пропагатор Фейнмана По аналогии с пунктом (b) задачи 1, получаем

$$\begin{aligned}\left(\partial_\mu\partial^\mu + m^2\right) D_F(\vec{x},t) =& \delta'(t)\langle 0|\phi(\vec{x},t)\phi(0,0)|0\rangle - \delta'(t)\langle 0|\phi(0,0)\phi(\vec{x},t)|0\rangle + \\ &+ 2\delta(t)\langle 0|\dot{\phi}(\vec{x},t)\phi(0,0)|0\rangle - 2\delta(t)\langle 0|\phi(0,0)\dot{\phi}(\vec{x},t)|0\rangle = \\ =&\delta(t)\langle 0|[\dot{\phi}(\vec{x},0),\phi(0,0)]|0\rangle = i\delta(t)\langle 0|[\pi(\vec{x},0),\phi(0,0)]|0\rangle = -i\delta(t)\delta^{(3)}(\vec{x}) = \\ =&-i\delta^{(4)}(x). \end{aligned} \tag{0.12}$$

(30 баллов) Амплитуда для частицы поля Клейна-Гордона (П-Ш, з. 2.3) Пусть $\left(x^0 - y^0\right) = t$ и $\vec{x} - \vec{y} = \vec{r}$, и $(x-y)^2 = t^2 - r^2 < 0$. Так как амплитуда Лоренц инвариантна (докажите как упражнение), то мы можем перейти в систему координат в которой $x^0 - y^0 = 0$ и $\vec{x} - \vec{y} = \vec{r}$. Тогда для амплитуды получаем

$$\begin{aligned} D(x-y) =& \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\sqrt{p^2+m^2}}e^{i\vec{p}\vec{r}} = \frac{2\pi}{(2\pi)^3}\int\limits_0^\infty dp\frac{p^2}{2\sqrt{p^2+m^2}}\int\limits_{-1}^{1} d\cos\theta e^{ipr\cos\theta} = \\ =& \frac{1}{8\pi^2}\int\limits_0^\infty dp\frac{p^2}{\sqrt{m^2+p^2}}\frac{e^{ipr}-e^{-ipr}}{ipr} = -\frac{i}{8\pi^2 r}\int\limits_{-\infty}^{\infty} dp\frac{pe^{ipr}}{\sqrt{p^2+m^2}} \end{aligned} \tag{0.13}$$

У интеграла есть две точки ветвления $p = \pm im$. Изогнем контур так, чтобы он сел на вертикальный разрез, идущий от im до $+i\infty$, тогда получим для амплитуды, делая замену $\rho = -ip$:

$$\begin{aligned} D(x-y) =& \frac{1}{4\pi^2 r}\int\limits_m^\infty d\rho\frac{\rho e^{-\rho r}}{\sqrt{\rho^2-m^2}} \overset{\rho = m\,\mathrm{ch}\,\xi}{=} \frac{m}{4\pi^2 r}\int\limits_0^\infty d\xi\,\mathrm{ch}\,\xi e^{-mr\,\mathrm{ch}\,\xi} = \frac{mK_1(mr)}{4\pi^2 r} = \\ =& \frac{m}{4\pi^2\sqrt{-(x-y)^2}}K_1\left(m\sqrt{-(x-y)^2}\right) \end{aligned} \tag{0.14}$$

где $K_1(x)$ функция Макдональда.

Белавин-Пархоменко.4. Континуальный интеграл в Квантовой Механике

Евклидова корреляционная функция

(a). Как мы знаем, если сделать поворот контура интегрирования по времени (поворот Вика) для пропагатора Фейнмана в пространстве Минковского: $t = -ix_4$, где x_4 действительная переменная, то мы получим выражение для Евклидового пропагатора:

$$D\left(x_E\right) = \int \frac{d^4p_E}{(2\pi)^4}\frac{e^{ip_E x_E}}{p_E^2+m^2} \tag{0.1}$$

где $p_E^2 = \vec{p}^2 + p_4^2$ и $x_E^2 = \vec{x}^2 + x_4^2$. Иногда удобно воспользоваться следующим приемом (Швингеровское представление), а именно записать $D\left(x_E\right)$, как

$$D\left(x_E\right) = \int\limits_0^\infty d\tau \int \frac{d^4p_E}{(2\pi)^4}e^{-\tau\left(p_E^2+m^2\right)}e^{ip_E x_E} \tag{0.2}$$

Покажите, что при $|x_E| \gg m^{-1}$:

$$D\left(x_E\right) \sim \frac{m^{1/2}}{|x_E|^{3/2}}e^{-m|x_E|} \tag{0.3}$$

(b). Используя представление Швингера вычислите $D\left(x_E\right)$ явно в терминах функций Бесселя.

Оператор эволюции (в мнимом времени $\tau = it$) По определению, оператор эволюции $\hat{U}(\tau, \tau_0)$ удовлетворяет уравнению

$$-\frac{\partial}{\partial\tau}\hat{U}(\tau,\tau_0) = \hat{H}(\tau)U(\tau,\tau_0), \quad \hat{U}(\tau_0,\tau_0) = \hat{I} \tag{0.4}$$

где $\hat{H}(\hat{p},\hat{q},\tau)$ - Гамильтониан нашей квантово-механической системы. Мы предполагаем, что спектр Гамильтониана ограничен снизу. Оператор эволюции существует только при $\tau \geqslant \tau_0$.

(a). Проверьте, что оператор эволюции удовлетворяет свойству композиции:

$$\hat{U}(\tau,\tau_0) = \hat{U}(\tau,\tau_1)\hat{U}(\tau_1,\tau_0), \quad \tau \geqslant \tau_1 \geqslant \tau_0 \tag{0.5}$$

(b). В координатном представлении $|q\rangle$ и для Гамильтониана вида $\hat{H} = \frac{1}{2}p^2 + V(q,\tau)$, уравнение для оператора эволюции запишется как

$$-\frac{\partial}{\partial\tau}\left\langle q\left|\hat{U}(\tau,\tau_0)\right|q'\right\rangle = \left(-\frac{1}{2}\frac{\partial^2}{\partial q^2} + V(q,\tau)\right)\left\langle q\left|\hat{U}(\tau,\tau_0)\right|q'\right\rangle, \quad \left\langle q\left|\hat{U}(\tau_0,\tau_0)\right|q'\right\rangle = \delta(q-q') \tag{0.6}$$

Рассмотрим $\tau - \tau_0 = \Delta \to 0$, и записывая $\left\langle q\left|\hat{U}(\tau,\tau_0)\right|q'\right\rangle = e^{-\sigma(q,q'|\tau_0)}$, будем искать $\sigma(q,q'|\tau_0)$ в виде

$$\sigma(q,q'|\tau_0) = \frac{1}{\Delta}\sigma_0(q,q'|\tau_0) + a\log\Delta + b + \Delta\sigma_1(q,q'|\tau_0) + O\left(\Delta^2\right) \tag{0.7}$$

где a, b - некоторые константы. Используя уравнения для оператора эволюции, покажите, что

$$\sigma_0(q,q'|\tau_0) = \frac{1}{2}(q-q')^2, \quad a = 1/2, \quad \sigma_1(q,q'|\tau_0) = \frac{1}{q-q'}\int_{q'}^{q} V(y|\tau_0)\,dy \tag{0.8}$$

а также, что $b = \frac{1}{2}\log 2\pi$, которая определяется из условия $\left\langle q\left|\hat{U}(\tau_0,\tau_0)\right|q'\right\rangle = \delta(q-q')$.

Stanford.Zhou.2022.1.1. natural units

In this course, we will work in "natural" units, where $\hbar = k_B = c = \mu_0 = \epsilon_0 = 1$. As a result, any physical quantity $\mathcal{A}$ has the same dimensions as $(1\text{eV})^n$ for some n, which we write as $[\mathcal{A}] = n$. For example, we have

$$[\text{ energy }] = [\text{ momentum }] = [\text{ mass }] = [\text{ temperature }] = 1$$

and

$$[\text{ length }] = [\text{ time }] = -1.$$

These results immediately imply, e.g., [frequency] = 1, [speed] = 0, and [volume] = −3. If the mass of a particle in natural units is $m = 1\text{eV}$, that means its mass in SI units is

$$m = \frac{1\text{eV}}{c^2} = 1.8 \times 10^{-36}\ \text{kg}$$

One physical interpretation is that a particle of this mass has rest energy 1eV.

a) Find the wavelength and period of a photon of energy 1eV in SI units.

Solution:

Since $E = \hbar\omega$, the period is

$$T = \frac{2\pi\hbar}{1\text{eV}} = 4 \times 10^{-15}\ \text{s}$$

Since $\lambda = cT$, the wavelength is

$$\lambda = \frac{2\pi\hbar c}{1\text{eV}} = 1.2 \times 10^{-6}\ \text{m}$$

b) Express the temperature $T = 1\text{eV}$ in SI units.

Solution:

Since energy has dimensions of $k_B T$, the temperature is

$$T = \frac{1\text{eV}}{k_B} = 12000 \text{ K}$$

This is the temperature where the typical thermal energy of a degree of freedom is $\sim$ 1eV, which would be sufficient to break apart molecules and ionize most atoms.

If you commit the above results to memory, you should always be able to recover numeric values in SI units. The other skill you need is going from SI units to natural units.

c) Find $[G]$, where G is Newton's constant.

Solution:

You can do this part, and all the parts below, by brute force. But it's faster and more fun to get the answers by thinking about equations you already know.

For example, the gravitational potential energy formula implies $[E] = \left[GM^2/R\right]$, so $[G] = [R/M] = -2$. (In natural units, $G \sim 1/M_{\text{pl}}^2$ where M_{pl} is the Planck mass.)

d) Find $[n], [P]$, and $[\rho]$, where n is number density, P is pressure, and ρ is mass density.

Solution:

Since n is a dimensionless number per volume, $[n] = [1/\text{ volume }] = 3$. Noting that $dU = PdV$ for thermodynamic work, $[P] = [\text{ energy / volume }] = 4$. Similarly, $[\rho] = [\text{ mass / volume }] = 4$.

e) Find $[\phi]$, where ϕ is a real scalar field.

Solution:

Note that the action S is the integral of a Lagrangian (with units of energy) over time, so

$$[S] = [\text{ energy } \cdot \text{ time }] = 0$$

In field theory, the action is the integral over 4-dimensional spacetime of a Lagrangian density $\mathcal{L}$, which implies $[\mathcal{L}] = 4$. Examining the kinetic term, we have

$$\left[(\partial_\mu \phi)^2\right] = 4$$

and we have $[\partial_\mu] = 1$, which implies $[\phi] = 1$.

f) Find $[q], [A], [E]$ and $[B]$, where q is electric charge, A is vector potential, and E and B are electric and magnetic fields.

Solution:

In natural units, the force between charges is $F = q^2/4\pi r^2$. Since $[F] = [dp/dt] = 2$, we conclude $[q] = 0$, i.e. charge is dimensionless. (In case you were wondering, the specific value of the electron charge is about -0.3. This result is more commonly expressed in terms of the "fine structure constant" $\alpha = e^2/4\pi \approx 1/137$, and later, the fact that $\alpha \ll 1$ will tell us that electromagnetic interactions in QFT can be treated in perturbation theory.)

For the vector potential, one could recall the Lorenz gauge equation of motion $\partial^2 A^\mu = J^\mu$ where J^μ is the current density. We have $\left[\partial^2\right] = 2$ and $[J] = [\text{ charge/volume }] = 3$, so we conclude $[A] = 1$.

The electric and magnetic fields are spatial or time derivatives of the vector potential, so $[E] = [B] = 2$.

Once you're comfortable with natural units, they'll be an incredibly convenient tool for making rough estimates. For example, the mass of the proton is $m_p \sim$ GeV, and everything in nuclear physics is roughly governed by this scale. From this, we can immediately conclude that the radius of the proton is roughly $r \sim \text{GeV}^{-1}$, in natural units.

g) Write down rough expressions for the density and electric field within a nucleus, and the temperature above which nuclei melt into quark-gluon plasma, in natural units.

Solution:

It's just $\rho \sim \text{GeV}^4, E \sim \text{GeV}^2$, and $T \sim \text{GeV}$. These are all very rough estimates, dropping dimensionless factors like $\alpha \sim 1/137$ or dependence on the pion mass, but they're all within a few orders of magnitude.

Technically, all of the estimates here will be a bit off, because some of these quantities are actually determined by the pion mass $m_\pi \sim 10^{-1}\text{GeV}$, as the pion governs the forces between nucleons. For a more careful treatment, see Astronomical reach of fundamental physics by Burrows and Ostriker.

Stanford.Zhou.2022.1. The harmonic oscillator in quantum mechanics. (15/45p)

This exercise reviews the quantum harmonic oscillator, which has Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}.$$

a) Write the Hamiltonian in terms of the ladder operators a and $a^\dagger$, where

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}x + i\frac{p}{\sqrt{m\omega}}\right).$$

Solution:

Of course, we have

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}x - i\frac{p}{\sqrt{m\omega}}\right).$$

Following the usual textbook steps, you should find

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right)$$

Equivalent forms, such as having $a^\dagger a + aa^\dagger$ in parentheses, are also acceptable.

b) The normalized vacuum state $|0\rangle$ is defined to satisfy $a|0\rangle = 0$. The number states are then defined by $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ for any integer n. Show that $\left[a, a^\dagger\right] = 1$, and use this fact alone to show that the number states are properly normalized.

Solution:

Using the fact that $[x, p] = i$, we have

$$\begin{aligned}\left[a, a^\dagger\right] &= \frac{1}{2}\left[\sqrt{m\omega}x + i\frac{p}{\sqrt{m\omega}}, \sqrt{m\omega}x - i\frac{p}{\sqrt{m\omega}}\right] \\ &= \frac{1}{2}([x, -ip] + [ip, x]) = 1\end{aligned}$$

as desired.

Now, by repeatedly using the definition of the number states, we have

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n|0\rangle$$

Note that we have

$$\left[a, \left(a^\dagger\right)^n\right] = n\left(a^\dagger\right)^{n-1}.$$

The intuition is that to convert $a\left(a^\dagger\right)^n$ into $\left(a^\dagger\right)^n a$, we need to move an a through an $a^\dagger$ a total of n times. For each move, we use the identity $aa^\dagger = a^\dagger a + \left[a, a^\dagger\right] = a^\dagger a + 1$, and therefore pick up a factor of $\left(a^\dagger\right)^{n-1}$. The factor of n in the final result is because we need to do n such moves.

To check the normalization, we just compute

$$\begin{aligned}\langle n|n\rangle &= \frac{1}{n!}\left\langle 0\left|a^n\left(a^\dagger\right)^n\right|0\right\rangle \\ &= \frac{1}{n!}\left\langle 0\left|a^{n-1}a\left(a^\dagger\right)^n\right|0\right\rangle \\ &= \frac{1}{n!}\left\langle 0\left|a^{n-1}\left(\left(a^\dagger\right)^n a + n\left(a^\dagger\right)^{n-1}\right)\right|0\right\rangle \\ &= \frac{1}{(n-1)!}\left\langle 0\left|a^{n-1}\left(a^\dagger\right)^{n-1}\right|0\right\rangle \\ &= \langle n-1|n-1\rangle.\end{aligned}$$

This shows that the norms of all the number states are equal, so they're all normalized because $|0\rangle$ is.

c) Calculate the expectation values of x, p, and the number operator $N = a^\dagger a$ in the number state $|n\rangle$.

Solution:

We can write x and p in terms of ladder operators as

$$x = \frac{1}{\sqrt{2m\omega}}\left(a^\dagger + a\right), \quad p = i\sqrt{\frac{m\omega}{2}}\left(a^\dagger - a\right).$$

Then $\langle n|x|n\rangle$ has two terms, one proportional to $\langle n|n+1\rangle$ and the other proportional to $\langle n|n-1\rangle$, which both vanish. Thus, $\langle n|x|n\rangle = 0$, and by similar reasoning $\langle n|p|n\rangle = 0$. As for the number operator,

$$\langle n|N|n\rangle = \left\langle n\left|a^\dagger a\right|n\right\rangle = \langle n-1|\sqrt{n}\sqrt{n}|n-1\rangle = n.$$

d) Calculate the standard deviations $\Delta x, \Delta p$ and ΔN in the number state $|n\rangle$. For what n is the Heisenberg uncertainty product $\Delta x \Delta p$ minimal?

Solution:

For a given state, the standard deviation of any operator A is defined by

$$\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$$

where the expectation values are taken with respect to the state. Now first, we note that

$$\begin{aligned}\left\langle n\left|x^2\right|n\right\rangle &= \frac{1}{2m\omega}\left\langle n\left|\left(a + a^\dagger\right)\left(a + a^\dagger\right)\right|n\right\rangle \\ &= \frac{1}{2m\omega}\left\langle n\left|a^\dagger a + aa^\dagger\right|n\right\rangle \\ &= \frac{1}{2m\omega}(n + (n+1)) \\ &= \frac{2n+1}{2m\omega}.\end{aligned}$$

From this and the previous part, we conclude

$$\Delta x = \sqrt{\frac{2n+1}{2m\omega}}.$$

A very similar computation for p gives

$$\Delta p = \sqrt{\frac{m\omega(2n+1)}{2}}.$$

Clearly the Heisenberg uncertainty product is minimal for $n = 0$, in which case $\Delta x \Delta p = 1/2$, saturating the uncertainty relation.

Finally, we have

$$\left\langle n\left|N^2\right|n\right\rangle = n\langle n|N|n\rangle = n^2$$

from which we conclude $\Delta N = 0$, as expected.

e) Suppose the particle begins in the vacuum state $|0\rangle$, and at time $t = 0$, we apply an impulse α. This can be modeled by a Hamiltonian term $-\alpha x \delta(t)$, and the state immediately after the impulse is

$$|\alpha\rangle = e^{i\alpha x}|0\rangle.$$

Show that $|\alpha\rangle$ is an eigenvector of a, and find the eigenvalue.

Solution:

For simplicity, let's define $\beta = \alpha/\sqrt{2m\omega}$, so that

$$|\alpha\rangle = e^{i\beta\left(a+a^\dagger\right)}|0\rangle.$$

We'll show a direct solution. First, note that $\left[a, a + a^\dagger\right] = 1$. This implies that

$$\left[a, \left(a + a^\dagger\right)^n\right] = n\left(a + a^\dagger\right)^{n-1}$$

where the factor of n comes from commuting the a past each copy of $a + a^\dagger$. Now we have

$$\begin{aligned}
a|\alpha\rangle &= a\sum_{n=0}^{\infty}\frac{(i\beta)^n}{n!}\left(a + a^\dagger\right)^n|0\rangle \\
&= \sum_{n=0}^{\infty}\frac{(i\beta)^n}{n!}\left(\left[a, \left(a + a^\dagger\right)^n\right] + \left(a + a^\dagger\right)^n a\right)|0\rangle \\
&= \sum_{n=0}^{\infty}\frac{(i\beta)^n}{n!}\left(n\left(a + a^\dagger\right)^{n-1}\right)|0\rangle \\
&= \sum_{n=0}^{\infty}\frac{(i\beta)^n}{(n-1)!}\left(a + a^\dagger\right)^{n-1}|0\rangle \\
&= \sum_{n=0}^{\infty}\frac{(i\beta)^{n+1}}{n!}\left(a + a^\dagger\right)^n|0\rangle \\
&= i\beta|\alpha\rangle
\end{aligned}$$

where we Taylor expanded the exponential, used the definition of the commutator, used $a|0\rangle = 0$ and Eq. (S28), and shifted the variable n. Thus, $|\alpha\rangle$ is an eigenvector of a with eigenvalue $i\beta = i\alpha/\sqrt{2m\omega}$.

f) Find the expectation values of x, p, and N, and their standard deviations, for all $t > 0$. (Hint: after you find the answers for the initial state $|\alpha\rangle$, it is easiest to generalize to arbitrary t using Heisenberg picture.)

Solution:

First, let's calculate the expectation values in the initial state $|\alpha\rangle$. Since x commutes with $e^{i\alpha x}$, we have

$$\langle\alpha|x|\alpha\rangle = \left\langle 0\left|e^{-i\alpha x}xe^{i\alpha x}\right|0\right\rangle = \langle 0|x|0\rangle = 0.$$

Similarly, we have

$$\left\langle\alpha\left|x^2\right|\alpha\right\rangle = \left\langle 0\left|x^2\right|0\right\rangle = \frac{1}{2m\omega}$$

where we used Eq. (S24). This gives a standard deviation

$$\Delta x = \frac{1}{\sqrt{2m\omega}}$$

Next, for p it's useful to use the decomposition into ladder operators, as

$$\langle\alpha|a|\alpha\rangle = \langle\alpha|i\beta|\alpha\rangle = i\beta, \quad \left\langle\alpha\left|a^\dagger\right|\alpha\right\rangle = (a|\alpha\rangle)^\dagger|\alpha\rangle = -i\beta$$

where we used the result of part (e). We therefore have

$$\langle\alpha|p|\alpha\rangle = i\sqrt{\frac{m\omega}{2}}\left\langle\alpha\left|\left(a^\dagger - a\right)\right|\alpha\right\rangle = (-2i\beta)i\sqrt{\frac{m\omega}{2}} = \alpha$$

which is unsurprising, as the state $|\alpha\rangle$ as defined by exerting an impulse α on the ground state. Next,

$$\begin{aligned}\left\langle\alpha\left|p^2\right|\alpha\right\rangle &= -\frac{m\omega}{2}\left\langle\alpha\left|\left(a^\dagger - a\right)^2\right|\alpha\right\rangle \\ &= -\frac{m\omega}{2}\left\langle\alpha\left|\left(a^\dagger a^\dagger + aa - aa^\dagger - a^\dagger a\right)\right|\alpha\right\rangle \\ &= \frac{m\omega}{2}\left\langle\alpha\left|\left(-a^\dagger a^\dagger - aa + 2a^\dagger a + 1\right)\right|\alpha\right\rangle \\ &= \frac{m\omega}{2}\left(4\beta^2 + 1\right) \\ &= \frac{m\omega}{2} + \alpha^2.\end{aligned}$$

Therefore, the standard deviation is

$$\Delta p = \sqrt{\frac{m\omega}{2}}$$

which is unchanged from the ground state. Finally, for the number states,

$$\langle\alpha|N|\alpha\rangle = \left\langle\alpha\left|a^\dagger a\right|\alpha\right\rangle = \beta^2 = \frac{\alpha^2}{2m\omega}$$

and

$$\begin{aligned}\left\langle\alpha\left|N^2\right|\alpha\right\rangle &= \left\langle\alpha\left|a^\dagger a a^\dagger a\right|\alpha\right\rangle \\ &= \beta^2\left\langle\alpha\left|aa^\dagger\right|\alpha\right\rangle \\ &= \beta^2\left\langle\alpha\left|\left(1 + a^\dagger a\right)\right|\alpha\right\rangle \\ &= \beta^2 + \beta^4\end{aligned}$$

from which we conclude

$$\Delta N = \beta = \frac{\alpha}{\sqrt{2m\omega}}$$

You could also derive all of the above results using the Baker-Campbell-Hausdorff theorem, but such machinery isn't necessary. We simply leaned on the result of part **(e)** to get all of the results, and the solution to part **(e)** is effectively a proof of that theorem in the simple case we need.

Now let's consider how these expectation values and uncertainties evolve over time. The evolution of an expectation value of any operator $\mathcal{O}$ is

$$\langle\mathcal{O}\rangle(t) = \left\langle\psi\left|e^{iHt}\mathcal{O}e^{-iHt}\right|\psi\right\rangle$$

Since N commutes with H, this implies that N and ΔN just stay constant over time. As for x and p, the easiest way is to compute the "Heisenberg" operators $\mathcal{O}_H(t) = e^{iHt}\mathcal{O}e^{-iHt}$. The basic result is that Heisenberg operators satisfy Hamilton's equations, which here imply

$$\frac{dp_H}{dt} = -m\omega^2 x_H, \quad \frac{dx_H}{dt} = \frac{p_H}{m}$$

The solution of this pair of equations is

$$p_H(t) = p\cos\omega t - m\omega x\sin\omega t, \quad x_H(t) = x\cos\omega t + \frac{p}{m\omega}\sin\omega t$$

This means the expectation values we're after follow immediately from those we already calculated. We know that $\langle\alpha|x|\alpha\rangle = 0$ and $\langle\alpha|p|\alpha\rangle = \alpha$, from which we conclude

$$\langle x\rangle(t) = \frac{\alpha}{m\omega}\sin\omega t, \quad \langle p\rangle(t) = \alpha\cos\omega t$$

Similarly, we have

$$\begin{aligned}\left\langle x^2\right\rangle(t) &= \left\langle \alpha \left| x^2\cos^2\omega t + \frac{(xp+px)}{m\omega}\cos\omega t\sin\omega t + \frac{p^2}{m^2\omega^2}\sin^2\omega t \right| \alpha \right\rangle \\ &= \cos^2\omega t\left\langle \alpha\left|x^2\right|\alpha\right\rangle + \frac{\sin^2\omega t}{m^2\omega^2}\left\langle \alpha\left|p^2\right|\alpha\right\rangle \\ &= \left(\frac{\alpha\sin\omega t}{m\omega}\right)^2 + \frac{1}{2m\omega}\end{aligned}$$

which implies that the position uncertainty does not change over time,

$$\Delta x(t) = \frac{1}{\sqrt{2m\omega}}$$

By an extremely similar computation, we find the momentum uncertainty doesn't change over time either,

$$\Delta p(t) = \sqrt{\frac{m\omega}{2}}$$

Your result in part (**e**) shows that $|\alpha\rangle$ is a so-called coherent state. You might have heard that they are important because they are the "most classical" states. A more important reason is that they are the states you automatically get when you drive a quantum system. As you can see from your results, in the limit of strong driving, Δx and Δp become negligible compared to x and p, and we recover classical physics. Later we will see how a similar result allows quantum fields to behave like classical fields.

26.1.3 Problems about free quantum fields 2

Stanford.Zhou.2022.1. The relativistic classical point particle. (12/45p)

The spacetime trajectory of a relativistic point particle is $x^\mu(\tau) = \left(x^0(\tau), \mathbf{x}(\tau)\right)$, where τ is an arbitrary parameter. The corresponding action is proportional to the relativistic "length" of the trajectory, where the relativistic line element is

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = dt^2 - dx^2 - dy^2 - dz^2$$

The action is therefore

$$S = -\alpha\int_{\mathcal{P}} ds = -\alpha\int_{\tau_1}^{\tau_2} d\tau\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}$$

where α is a constant, and τ_1 and τ_2 are the initial and final values of the parameter.

a) The easiest way to understand the nonrelativistic limit $\left|\partial_t x^i\right| \ll 1$ is to set $\tau = t$. By demanding that the action reduces to that of a free nonrelativistic particle of mass m (plus a constant), determine the value of the constant α.

Solution:

Setting $\tau = t$, we have

$$S = -\alpha\int dt\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}} = -\alpha\int dt\sqrt{1-v(t)^2}$$

where $v(t)$ is the ordinary speed. In the nonrelativistic limit $v \ll 1$,

$$S \approx -\alpha\int dt\left(1 - \frac{v^2}{2} + O\left(v^4\right)\right)$$

The first term is just an irrelevant constant, while the next gives a Lagrangian $\alpha v^2/2$. On the other hand, we know the Lagrangian for a free nonrelativistic particle is $mv^2/2$, so we conclude $\alpha = m$.

If we don't just set $\tau = t$, there are four Euler-Lagrange equations and canonical momenta,

$$\frac{dp^\mu}{d\tau} = \frac{\partial L}{\partial x_\mu}, \quad p^\mu = \frac{\partial L}{\partial\left(dx_\mu/d\tau\right)}$$

b) Find the Euler-Lagrange equations for a general parameter τ, then show that they are equivalent to the conservation of the physical four-momentum of the particle.

Solution:

For simplicity, let a dot denote a derivative with respect to τ, so that

$$L = -m\sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} = -m\sqrt{\dot{x}^2}.$$

The canonical momenta are

$$p^\mu = \frac{\partial L}{\partial \dot{x}_\mu} = -\frac{m}{2\sqrt{\dot{x}^2}}\left(2\dot{x}^\mu\right) = -\frac{m\dot{x}^\mu}{\sqrt{\dot{x}^2}}.$$

The Lagrangian has no direct dependence on x^μ, so the Euler-Lagrange equations just state that p^μ is conserved, $dp^\mu/d\tau = 0$.

This isn't the end of the problem, though; we still need to show how the canonical momentum p^μ is related to physical four-momentum. One way to do this is to let τ be the proper time s of the particle. Then $\dot{x}^2 = 1$, and $p^\mu = -m dx^\mu/ds$-momentum. Alternatively, we can let τ be the coordinate time t, in which case $p^\mu = -m\left(dx^\mu/dt\right)/\sqrt{1-v^2}$. In both cases, it's clear that p^μ is the negative of the physical four-momentum. (This negative sign is weird, but harmless, since the physical results are the same. It occurs because we used a mostly negative metric, which will pay off later in the course.)

c) A simple local, Lorentz invariant way to include a force on the particle is to add

$$S_{\text{int}} = -q\int_{\mathcal{P}} A_\mu\left(x^\mu\right) dx^\mu = -q\int_{\tau_1}^{\tau_2} A_\mu\left(x^\mu\right)\frac{dx^\mu}{d\tau}d\tau$$

to the action, where $A^\mu\left(x^\mu\right)$ is a given four-vector field. Calculate p^μ and $\partial L/\partial x^\mu$, continuing to assume general τ.

Solution:

There are a lot of μ 's floating around, so to avoid confusion, let's use the fact that we can always rename a dummy index to rewrite the interaction as

$$S_{\text{int}} = -q\int A_\nu\left(x^\mu\right)\dot{x}^\nu d\tau$$

First, the canonical momenta are changed to

$$p^\mu = \frac{\partial L}{\partial \dot{x}_\mu} = -\frac{m\dot{x}^\mu}{\sqrt{\dot{x}^2}} - qA^\mu\left(x^\mu\right)$$

Next, the right-hand side of the Euler-Lagrange equations is now nontrivial,

$$\frac{\partial L}{\partial x_\mu} = -q\dot{x}^\nu \partial^\mu A_\nu\left(x^\mu\right)$$

where the partial derivative on the right is with respect to x^μ.

A common issue with this part was mismatched indices (having μ up on one side and down on the other side), which can be avoided by taking care to make the index positions consistent at each step. Another issue was duplicated indices (having more than two copies of μ, which is meaningless), which can be avoided by renaming dummy indices when required.

d) Now set τ to be the proper time s experienced by the particle (so that $ds = d\tau$) and evaluate the Euler-Lagrange equations, simplifying as much as possible.

Solution:

From this point on, we'll leave the x^μ argument of the vector potential implicit. Specializing the proper time, the canonical momenta are

$$p^\mu = -m\dot{x}^\mu - qA^\mu$$

We therefore have

$$\frac{dp^\mu}{ds} = -m\ddot{x}^\mu - q\frac{dA^\mu}{ds} = -m\ddot{x}^\mu - q\dot{x}^\nu \partial_\nu A^\mu$$

Equating this with Eq. (S67) gives

$$m\ddot{x}^\mu = -q\dot{x}^\nu \left(\partial_\nu A^\mu - \partial^\mu A_\nu\right) = q\dot{x}_\nu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)$$

This is an acceptable final answer; the right-hand side can also be written as $q\dot{x}_\nu F^{\mu\nu}$ where F is the electromagnetic field strength tensor. If you treat A^μ as a vector potential and plug in the definitions of the electric and magnetic fields, you'll recover the Lorentz force law.

A warning: if you set τ to proper time before doing part (c), and apply the Euler-Lagrange equations anyway, you'll get nonsense. The reason is that the derivation of the EulerLagrange equation assumes all the variables $x^\mu(\tau)$ can be varied independently, but when $d\tau = ds$ we automatically have the constraint $\sqrt{\eta_{\mu\nu}\left(dx^\mu/d\tau\right)\left(dx^\nu/d\tau\right)} = 1$. Lagrangians with constraints are subtle and important, but beyond the scope of this course. For much more about them, see Quantization of Gauge Systems by Henneaux and Teitelboim.

Stanford.Zhou.2022.1. The complex scalar field. (5/45p)

The Lagrangian density for a canonically normalized free real scalar field of mass m is

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{1}{2}m^2\phi^2.$$

Now consider a theory of two free real scalar fields ϕ_1 and ϕ_2, both with mass m.

a) Write their Lagrangian density in terms of the complex scalar field $\Phi = \left(\phi_1 + i\phi_2\right)/\sqrt{2}$ and its complex conjugate Φ^*.

Solution:

The mass terms involve $\phi_1^2 + \phi_2^2$, which clearly indicates the answer involves the complex norm $\Phi^*\Phi$. Treating the kinetic term the same way, and fixing the numeric constants, gives

$$\mathcal{L} = \left(\partial_\mu\Phi^*\right)\left(\partial^\mu\Phi\right) - m^2\Phi^*\Phi$$

Complex fields are always equivalent to a pair of equal mass real fields, and are useful because such pairs occur frequently in nature, for reasons we'll see later. (At low energies, we actually don't know of any complex scalar fields, but the electron is described by a Dirac field, which is a complex fermion field built from two equal mass real fermion fields.)

Complex fields are convenient once you get to know them, but they come with an annoying problem: it is not obvious how to vary the action with respect to Φ, because any change in Φ also changes Φ^*. It turns out that you will always get the right results (i.e. results that are equivalent to what you'd get working in terms of the two real fields) by treating Φ and Φ^* as if they were independent real fields, even though they clearly aren't. (For an explanation why, see page 56 of Sidney Coleman's lecture notes.)

b) Compute the conjugate momenta Π and Π^* of Φ and Φ^*, and the Euler-Lagrange equations for Φ and Φ^*.

Solution:

Note that the kinetic term contains $\dot{\Phi}^*\dot{\Phi}$, where a dot denotes a time derivative. Thus, the canonical momenta are

$$\Pi = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}} = \dot{\Phi}^*, \quad \Pi^* = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}^*} = \dot{\Phi}$$

The Euler-Lagrange equations are

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \Phi\right)}-\frac{\partial \mathcal{L}}{\partial \Phi}=0, \quad \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial^\mu \Phi^*\right)}-\frac{\partial \mathcal{L}}{\partial \Phi^*}=0$$

from which we read off

$$\left(\partial^2+m^2\right) \Phi^*=0, \quad \left(\partial^2+m^2\right) \Phi=0$$

which is just the Klein-Gordan equation.

c) Show that the action is invariant under the transformation $\Phi \rightarrow e^{i\alpha}\Phi$, for any real α. What is the equivalent symmetry in terms of the real scalar fields ϕ_1 and ϕ_2?

Solution:

Each term in the Lagrangian has one power of Φ and one power of Φ^*, and $\Phi^* \rightarrow e^{-i\alpha}\Phi$, so the phase cancels out. In terms of the real scalar fields, an equivalent symmetry is a rotation,

$$\phi_1 \rightarrow \phi_1 \cos\alpha - \phi_2 \sin\alpha, \quad \phi_2 \rightarrow \phi_2 \cos\alpha + \phi_1 \sin\alpha$$

Stanford.Zhou.2022.2. Quantization of the complex scalar field. (30/45p)

In problem set 1 we considered a complex scalar field Φ with Lagrangian density

$$\mathcal{L}=\left(\partial_\mu \Phi^*\right)\left(\partial^\mu \Phi\right)-m^2 \Phi^* \Phi$$

and found the canonical fields Φ and Φ^* and momenta Π and Π^*. Letting $\omega_{\mathbf{k}}=\sqrt{|\mathbf{k}|^2+m^2}$, introduce the following mode expansion for the canonical fields and momenta:

$$\begin{aligned}
\Phi(\mathbf{x}) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}+b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \\
\Phi^*(\mathbf{x}) &= \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left[b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}+a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \\
\Pi(\mathbf{x}) &= -i\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{k}}}{2}}\left[b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}-a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right], \\
\Pi^*(\mathbf{x}) &= -i\int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{k}}}{2}}\left[a(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}-b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right]
\end{aligned}$$

We quantize the fields by imposing the canonical commutation relations

$$[\Phi(\mathbf{x}),\Pi(\mathbf{y})]=[\Phi^*(\mathbf{x}),\Pi^*(\mathbf{y})]=i\delta^{(3)}(\mathbf{x}-\mathbf{y})$$

with all other commutators between these fields vanishing. (Technically, we should write Hermitian conjugates $\Phi^\dagger$ and $\Pi^\dagger$ here, since Φ and Π are now operators, but we'll continue to use stars to emphasize the links between the classical and quantum theories.)

a) Show that the operators $a(\mathbf{k}), a^\dagger(\mathbf{k}), b(\mathbf{k})$ and $b^\dagger(\mathbf{k})$ obey

$$\left[a(\mathbf{k}),a^\dagger(\mathbf{p})\right]=\left[b(\mathbf{k}),b^\dagger(\mathbf{p})\right]=(2\pi)^3\delta^{(3)}(\mathbf{k}-\mathbf{p})$$

with all other commutators between these operators vanishing. This implies that we have two sets of independent creation and annihilation operators for each $\mathbf{k}$.

Solution:

In order to keep the expressions as simple as possible, we'll define

$$d\mathbf{x}=d^3x, \quad d\mathbf{p}=\frac{d^3p}{(2\pi)^3}, \quad \not{\delta}(\mathbf{p})=(2\pi)^3\delta^{(3)}(\mathbf{p})$$

As a simple of this notation, we have

$$\int d\mathbf{x} e^{-i\mathbf{p}\cdot\mathbf{x}} = \phi(\mathbf{p}), \quad \int d\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{x}} = \delta(\mathbf{x})$$

If you just remember that every momentum delta function comes with a slash, and every momentum integral comes with a bar, all the factors of 2π will take care of themselves.

Now, we note that

$$\begin{aligned}\int d\mathbf{x} e^{i\mathbf{k}\cdot\mathbf{x}}\Phi(\mathbf{x}) &= \int \frac{d\mathbf{x}d\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\left[a(\mathbf{k})e^{i\left(\mathbf{k}+\mathbf{k}'\right)\cdot\mathbf{x}} + b^\dagger(\mathbf{k})e^{-i\left(\mathbf{k}'-\mathbf{k}\right)\mathbf{x}}\right] \\ &= \int \frac{d\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}}\left[a(\mathbf{k})\phi\left(\mathbf{k}+\mathbf{k}'\right) + b^\dagger(\mathbf{k})\phi\left(\mathbf{k}'-\mathbf{k}\right)\right] \\ &= \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}\left(a(-\mathbf{k}) + b^\dagger(\mathbf{k})\right).\end{aligned}$$

By a similar argument, we have

$$\int d\mathbf{x} e^{i\mathbf{k}\cdot\mathbf{x}}(i\Pi(\mathbf{x})) = \sqrt{\frac{\omega_{\mathbf{k}}}{2}}\left(b(-\mathbf{k}) - a^\dagger(\mathbf{k})\right).$$

Combining these equations to solve for the creation and annihilation operators yields

$$\begin{aligned}a(\mathbf{k}) &= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}} e^{-i\mathbf{k}\cdot\mathbf{x}}\left[\omega_{\mathbf{k}}\Phi(\mathbf{x}) + i\Pi^*(\mathbf{x})\right] \\ b(\mathbf{k}) &= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}} e^{-i\mathbf{k}\cdot\mathbf{x}}\left[\omega_{\mathbf{k}}\Phi^*(\mathbf{x}) + i\Pi(\mathbf{x})\right] \\ a^\dagger(\mathbf{k}) &= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}} e^{i\mathbf{k}\cdot\mathbf{x}}\left[\omega_{\mathbf{k}}\Phi^*(\mathbf{x}) - i\Pi(\mathbf{x})\right] \\ b^\dagger(\mathbf{k}) &= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}} e^{i\mathbf{k}\cdot\mathbf{x}}\left[\omega_{\mathbf{k}}\Phi(\mathbf{x}) - i\Pi^*(\mathbf{x})\right]\end{aligned}$$

Let's start by considering commutators involving $a(\mathbf{k})$. We only have nonzero commutators between Φ and Π, and between Φ^* and Π^*. This automatically implies that $[a(\mathbf{k}), a(\mathbf{p})] = \left[a(\mathbf{k}), b^\dagger(\mathbf{p})\right] = 0$, leaving just two to manually check. First, we have

$$\begin{aligned}[a(\mathbf{k}), b(\mathbf{p})] &= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}}\frac{d\mathbf{y}}{\sqrt{2\omega_{\mathbf{p}}}} e^{-i\mathbf{k}\cdot\mathbf{x}}e^{-i\mathbf{p}\cdot\mathbf{y}}\left[\omega_{\mathbf{k}}\Phi(\mathbf{x}) + i\Pi^*(\mathbf{x}), \omega_{\mathbf{p}}\Phi^*(\mathbf{y}) + i\Pi(\mathbf{y})\right] \\ &= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}}\frac{d\mathbf{y}}{\sqrt{2\omega_{\mathbf{p}}}} e^{-i\mathbf{k}\cdot\mathbf{x}}e^{-i\mathbf{p}\cdot\mathbf{y}}\left(\delta(\mathbf{y}-\mathbf{x})\left(\omega_{\mathbf{p}} - \omega_{\mathbf{k}}\right)\right) \\ &= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{p}}}} e^{-i\mathbf{k}\cdot\mathbf{x}}e^{-i\mathbf{p}\cdot\mathbf{x}}\left(\omega_{\mathbf{p}} - \omega_{\mathbf{k}}\right) \\ &= \not{\delta}(\mathbf{k}+\mathbf{p})\frac{\omega_{\mathbf{p}} - \omega_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{p}}}} = 0\end{aligned}$$

where the final step follows from $\omega_{\mathbf{k}} = \omega_{-\mathbf{k}}$. Next, we have

$$\begin{aligned}\left[a(\mathbf{k}), a^\dagger(\mathbf{p})\right] &= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}}\frac{d\mathbf{y}}{\sqrt{2\omega_{\mathbf{p}}}} e^{-i\mathbf{k}\cdot\mathbf{x}}e^{i\mathbf{p}\cdot\mathbf{y}}\left[\omega_{\mathbf{k}}\Phi(\mathbf{x}) + i\Pi^*(\mathbf{x}), \omega_{\mathbf{p}}\Phi^*(\mathbf{y}) - i\Pi(\mathbf{y})\right] \\ &= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}}\frac{d\mathbf{y}}{\sqrt{2\omega_{\mathbf{p}}}} e^{-i\mathbf{k}\cdot\mathbf{x}}e^{i\mathbf{p}\cdot\mathbf{y}}\left(\delta(\mathbf{y}-\mathbf{x})\left(\omega_{\mathbf{p}} + \omega_{\mathbf{k}}\right)\right) \\ &= \int \frac{d\mathbf{x}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{p}}}} e^{-i\mathbf{k}\cdot\mathbf{x}}e^{i\mathbf{p}\cdot\mathbf{x}}\left(\omega_{\mathbf{p}} + \omega_{\mathbf{k}}\right) \\ &= \not{\delta}(\mathbf{k}-\mathbf{p})\frac{\omega_{\mathbf{p}} + \omega_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}\sqrt{2\omega_{\mathbf{p}}}} = \not{\delta}(\mathbf{k}-\mathbf{p})\text{`}\end{aligned}$$

as desired. All of the other commutators can be derived similarly.

The vacuum state $|0\rangle$ is defined to be the unique state where $a(\mathbf{k})|0\rangle = b(\mathbf{k})|0\rangle = 0$ for all $\mathbf{k}$. The states $a^\dagger(\mathbf{k})|0\rangle$ and $b^\dagger(\mathbf{k})|0\rangle$ each contain one particle, while the state $a^\dagger\left(\mathbf{k}_1\right)a^\dagger\left(\mathbf{k}_2\right)|0\rangle$ contains two particles, and so on.

b) In problem set 1, you showed that the complex scalar field Lagrangian had the symmetry $\Phi \to e^{i\alpha}\Phi, \Phi^* \to e^{-i\alpha}\Phi^*$. Compute the Noether current J^μ and conserved charge Q associated with the symmetry.

Solution:

The infinitesimal transformation is $\delta\Phi = i\Phi$ and $\delta\Phi^* = -i\Phi$, so Noether's theorem gives

$$\begin{aligned} J^\mu &= \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi\right)}\delta\Phi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi^*\right)}\delta\Phi^* \\ &= i\left(\left(\partial^\mu\Phi^*\right)\Phi - \left(\partial^\mu\Phi\right)\Phi^*\right) \end{aligned}$$

The charge is just the integral of the charge density,

$$Q = i\int d\mathbf{x}\left(\left(\partial^0\Phi^*\right)\Phi - \left(\partial^0\Phi\right)\Phi^*\right)$$

Of course, if we want to evaluate this in the canonically quantized theory, we should express it in terms of the fields and canonical momenta, giving the final answer

$$Q = i\int d\mathbf{x}\left(\Pi(\mathbf{x})\Phi(\mathbf{x}) - \Pi^*(\mathbf{x})\Phi^*(\mathbf{x})\right)$$

c) Write Q in terms of the creation and annihilation operators. You should find the result is indeterminate up to a constant; resolve this by defining the vacuum to have zero charge, $Q|0\rangle = 0$. What is the charge in the three other states mentioned above?

Solution:

The fundamental reason the answer will be undetermined up to a constant is that Noether's theorem, which applies to classical fields, tells us (S30), but not the order that $\Pi(\mathbf{x})$ and $\Phi(\mathbf{x})$ are multiplied together. In the classical theory, that simply doesn't matter, but in the quantum theory it does because the fields are noncommuting operators. So without further information, the "right" answer in the quantum theory could contain $\Pi(\mathbf{x})\Phi(\mathbf{x})$, or $\Phi(\mathbf{x})\Pi(\mathbf{x})$, or even a mixture like $(\Pi(\mathbf{x})\Phi(\mathbf{x}) + \Phi(\mathbf{x})\Pi(\mathbf{x}))/2$.

This is called an ordering ambiguity, and in general there is no way to resolve it. (Ordering information is intrinsically quantum; it simply doesn't exist in the classical theory!) But in simple cases like this, all of the possible orderings give you the same operator Q, just shifted by a constant. So we can fix the ambiguity by demanding the reasonable assumption $Q|0\rangle = 0$, which is known as normal ordering.

Anyway, let's not worry about this for now, and just directly evaluate the result. First,

$$\begin{aligned} \int d\mathbf{x} i\Pi(\mathbf{x})\Phi(\mathbf{x}) &= \int \frac{d\mathbf{x}d\mathbf{k}d\mathbf{k}'}{2}\sqrt{\frac{\omega_\mathbf{k}}{\omega_{\mathbf{k}'}}}\left(b(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right)\left(a\left(\mathbf{k}'\right)e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger\left(\mathbf{k}'\right)e^{-i\mathbf{k}'\cdot\mathbf{x}}\right) \\ &= \int \frac{d\mathbf{k}d\mathbf{k}'}{2}\sqrt{\frac{\omega_\mathbf{k}}{\omega_{\mathbf{k}'}}}\left[\left(b(\mathbf{k})a\left(\mathbf{k}'\right) - a^\dagger(\mathbf{k})b^\dagger\left(\mathbf{k}'\right)\right)\phi\left(\mathbf{k}+\mathbf{k}'\right)\right. \\ &\qquad \left. + \left(b(\mathbf{k})b^\dagger\left(\mathbf{k}'\right) - a^\dagger(\mathbf{k})a\left(\mathbf{k}'\right)\right)\phi\left(\mathbf{k}-\mathbf{k}'\right)\right] \\ &= \int \frac{d\mathbf{k}}{2}\left(b(\mathbf{k})a(-\mathbf{k}) - a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k}) + b(\mathbf{k})b^\dagger(\mathbf{k}) - a^\dagger(\mathbf{k})a(\mathbf{k})\right). \end{aligned}$$

By a very similar computation, the second term is

$$\int d\mathbf{x}\left(-i\Pi^*(\mathbf{x})\right)\Phi^*(\mathbf{x}) = \int \frac{d\mathbf{k}}{2}\left(-a(\mathbf{k})b(-\mathbf{k}) + b(\mathbf{k})^\dagger a(-\mathbf{k})^\dagger - a(\mathbf{k})a(\mathbf{k})^\dagger + b(\mathbf{k})^\dagger b(\mathbf{k})\right)$$

The first term of (S33) cancels with the first term of (S34). To see this very explicitly:

$$\int d\mathbf{k}b(\mathbf{k})a(-\mathbf{k}) = \int d\mathbf{k}'a\left(\mathbf{k}'\right)b\left(-\mathbf{k}'\right) = \int d\mathbf{k}a(\mathbf{k})b\left(-\mathbf{k}'\right)$$

where we changed the integration variable to $\mathbf{k}' = -\mathbf{k}$, used the commutation relation, and then renamed $\mathbf{k}'$ to $\mathbf{k}$. Similarly, the second term of (S33) cancels with the second term of S34. This leaves

$$Q = \int \frac{d\mathbf{k}}{2}\left(b(\mathbf{k})b^\dagger(\mathbf{k}) + b(\mathbf{k})^\dagger b(\mathbf{k}) - a^\dagger(\mathbf{k})a(\mathbf{k}) - a(\mathbf{k})a(\mathbf{k})^\dagger\right)$$

At this point, we need to impose normal ordering. We will have $Q|0\rangle = 0$ if all the annihilation operators are on the right, so we move them to the right and throw out the constants this produces, giving

$$Q = \int d\mathbf{k}\left(b^\dagger(\mathbf{k})b(\mathbf{k}) - a^\dagger(\mathbf{k})a(\mathbf{k})\right)$$

Evidently, the conserved charge is the number of type b particles minus the number of type a particles. So the charges in the three states mentioned above are $-1, 1$, and -2, respectively.

d) To check that this is the same symmetry operation that we started out with, we can see how it acts on the operators of the theory. Quantum mechanically, symmetries act on operators by conjugation, and we expect to have

$$e^{i\alpha Q}\Phi(\mathbf{x})e^{-i\alpha Q} = e^{i\alpha}\Phi(\mathbf{x}), \quad e^{i\alpha Q}\Phi^*(\mathbf{x})e^{-i\alpha Q} = e^{-i\alpha}\Phi^*(\mathbf{x})$$

Show that this implies the commutation relations

$$[Q, \Phi(\mathbf{x})] = \Phi(\mathbf{x}), \quad [Q, \Phi^*(\mathbf{x})] = -\Phi^*(\mathbf{x})$$

and show that these relations hold. Thus, Q generates phase rotations of the field.

Solution:

Taking α infinitesimal, we have

$$(1 + i\alpha Q)\Phi(\mathbf{x})(1 - i\alpha Q) = (1 + i\alpha)\Phi(\mathbf{x})$$

and matching up the order α terms gives

$$iQ\Phi - i\Phi Q = i\Phi$$

which is precisely the desired result. The proof for Φ^* is similar.
To show the first commutation relation, we have

$$\begin{aligned}
[Q, \Phi(\mathbf{x})] &= \int \frac{d\mathbf{k}d\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}} \left[b^\dagger(\mathbf{k})b(\mathbf{k}) - a^\dagger(\mathbf{k})a(\mathbf{k}), a\left(\mathbf{k}'\right)e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger\left(\mathbf{k}'\right)e^{-i\mathbf{k}'\cdot\mathbf{x}}\right] \\
&= \int \frac{d\mathbf{k}d\mathbf{k}'}{\sqrt{2\omega_{\mathbf{k}'}}} \left(\left[b^\dagger(\mathbf{k})b(\mathbf{k}), b^\dagger\left(\mathbf{k}'\right)\right]e^{-i\mathbf{k}'\cdot\mathbf{x}} - \left[a^\dagger(\mathbf{k})a(\mathbf{k}), a\left(\mathbf{k}'\right)\right]e^{i\mathbf{k}'\cdot\mathbf{x}}\right) \\
&= \int \frac{d\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \left(b^\dagger(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}\right) = \Phi(\mathbf{x})
\end{aligned}$$

as desired. The proof of the other commutation relation is similar, but we don't have to write it because it follows immediately from the result we just proved. Taking the Hermitian conjugate of both sides,

$$\Phi^*(\mathbf{x}) = [Q, \Phi(\mathbf{x})]^\dagger = (Q\Phi(\mathbf{x}) - \Phi(\mathbf{x})Q)^\dagger = \Phi^*(\mathbf{x})Q - Q\Phi^*(\mathbf{x}) = -[Q, \Phi^*(\mathbf{x})]$$

which is the desired result. (Here we used the fact that Q is Hermitian.)

e) Use Noether's theorem to find the stress-energy tensor $T^{\mu\nu}$ and the associated conserved total four-momentum P^μ.

Solution:

Using the same procedure as in class, and suppressing x arguments, we have

$$\begin{aligned}
T^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi\right)}\partial^\nu\Phi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Phi^*\right)}\partial^\nu\Phi^* - \eta^{\mu\nu}\mathcal{L} \\
&= \partial^\mu\Phi^*\partial^\nu\Phi + \partial^\mu\Phi\partial^\nu\Phi^* - \eta^{\mu\nu}\left(\left(\partial_\rho\Phi^*\right)\left(\partial^\rho\Phi\right) - m^2\Phi^*\Phi\right)
\end{aligned}$$

The total energy is

$$\begin{aligned}
P^0 &= \int d\mathbf{x}T^{00} \\
&= \int d\mathbf{x}\partial^0\Phi\partial^0\Phi^* + \partial^0\Phi^*\partial^0\Phi - \left(\partial_\rho\Phi^*\right)\left(\partial^\rho\Phi\right) + m^2\Phi^*\Phi \\
&= \int d\mathbf{x}\Pi^*\Pi + (\nabla\Phi^*)\cdot(\nabla\Phi) + m^2\Phi^*\Phi
\end{aligned}$$

The total momentum is

$$\begin{aligned} P^i &= \int d\mathbf{x} T^{0i} \\ &= \int d\mathbf{x} \partial^0 \Phi^* \partial^i \Phi + \partial^0 \Phi \partial^i \Phi^* \\ &= \int d\mathbf{x} \Pi \partial^i \Phi + \Pi^* \partial^i \Phi^* \end{aligned}$$

f) Write the Hamiltonian $H = P^0$ and the spatial momenta $\mathbf{P}$ in terms of the creation and annihilation operators, again defining the vacuum to have zero four-momentum. What is the four-momentum in the three other states mentioned above?

Solution:

Starting with the Hamiltonian, we have three terms:

$$\begin{aligned} \int d\mathbf{x} \Pi^* \Pi &= -\int d\mathbf{x} d\mathbf{k} d\mathbf{k}' \frac{\sqrt{\omega_{\mathbf{k}} \omega_{\mathbf{k}'}}}{2} \left(a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - b^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \left(b\left(\mathbf{k}'\right) e^{i\mathbf{k}'\cdot\mathbf{x}} - a^\dagger\left(\mathbf{k}'\right) e^{-i\mathbf{k}'\cdot\mathbf{x}} \right) \\ \int d\mathbf{x} \left(\nabla \Phi^*\right) \cdot (\nabla \Phi) &= -\int \frac{d\mathbf{x} d\mathbf{k} d\mathbf{k}'}{2\sqrt{\omega_{\mathbf{k}} \omega_{\mathbf{k}'}}} \left(\mathbf{k} b(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - \mathbf{k} a^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \cdot \left(\mathbf{k}' a\left(\mathbf{k}'\right) e^{i\mathbf{k}'\cdot\mathbf{x}} - \mathbf{k}' b^\dagger\left(\mathbf{k}'\right) e^{-i\mathbf{k}'\cdot\mathbf{x}} \right) \\ \int d\mathbf{x} m^2 \Phi^* \Phi &= \int \frac{d\mathbf{x} d\mathbf{k} d\mathbf{k}'}{2\sqrt{\omega_{\mathbf{k}} \omega_{\mathbf{k}'}}} m^2 \left(b(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \left(a\left(\mathbf{k}'\right) e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger\left(\mathbf{k}'\right) e^{-i\mathbf{k}'\cdot\mathbf{x}} \right) \end{aligned}$$

When we do the $d\mathbf{x}$ integrations, each line gives four terms, along with appropriate delta functions,

$$\begin{aligned} \int d\mathbf{x} \Pi^* \Pi &= \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} \omega_{\mathbf{k}}^2 \left(-a(\mathbf{k}) b(-\mathbf{k}) + b^\dagger(\mathbf{k}) b(\mathbf{k}) + a(\mathbf{k}) a^\dagger(\mathbf{k}) - b^\dagger(\mathbf{k}) a^\dagger(-\mathbf{k}) \right) \\ \int d\mathbf{x} \left(\nabla \Phi^*\right) \cdot (\nabla \Phi) &= \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} |\mathbf{k}|^2 \left(b(\mathbf{k}) a(-\mathbf{k}) + a^\dagger(\mathbf{k}) a(\mathbf{k}) + b(\mathbf{k}) b^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k}) b^\dagger(-\mathbf{k}) \right) \\ \int d\mathbf{x} m^2 \Phi^* \Phi &= \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} m^2 \left(b(\mathbf{k}) a(-\mathbf{k}) + b(\mathbf{k}) b^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k}) a(\mathbf{k}) + a^\dagger(\mathbf{k}) b^\dagger(-\mathbf{k}) \right). \end{aligned}$$

Summing the terms, all the cross terms (those with one a and one b) cancel, and normal ordering leaves

$$\begin{aligned} H &= \int \frac{d\mathbf{k}}{2\omega_{\mathbf{k}}} \left(\omega_{\mathbf{k}}^2 + |\mathbf{k}|^2 + m^2 \right) \left(a^\dagger(\mathbf{k}) a(\mathbf{k}) + b^\dagger(\mathbf{k}) b(\mathbf{k}) \right) \\ &= \int d\mathbf{k} \omega_{\mathbf{k}} \left(a^\dagger(\mathbf{k}) a(\mathbf{k}) + b^\dagger(\mathbf{k}) b(\mathbf{k}) \right) \end{aligned}$$

This tells us that each one of the a-type and b-type particles has energy $\omega_{\mathbf{k}}$. Thankfully, the momentum is a little easier to deal with. We have

$$\begin{aligned} \int d\mathbf{x} \Pi \partial^i \Phi &= -i \int \frac{d\mathbf{x} d\mathbf{k} d\mathbf{k}'}{2} \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}} \left(b(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \partial^i \left(a\left(\mathbf{k}'\right) e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger\left(\mathbf{k}'\right) e^{-i\mathbf{k}'\cdot\mathbf{x}} \right) \\ &= \int \frac{d\mathbf{x} d\mathbf{k} d\mathbf{k}'}{2} \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}} k^i \left(b(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \left(-a\left(\mathbf{k}'\right) e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger\left(\mathbf{k}'\right) e^{-i\mathbf{k}'\cdot\mathbf{x}} \right) \\ &= \int \frac{d\mathbf{k}}{2} k^i \left(b(\mathbf{k}) a(-\mathbf{k}) + a^\dagger(\mathbf{k}) a(\mathbf{k}) + b^\dagger(\mathbf{k}) b(\mathbf{k}) + a^\dagger(\mathbf{k}) b^\dagger(-\mathbf{k}) \right) \end{aligned}$$

The easy way to handle the other term in P^i is to notice that when you conjugate this term, you almost get the other term, but the operators are in the reverse order. But of course, we're going to fix the ordering ambiguity with normal ordering later, so this doesn't matter. So we can treat the other term as

$$\left(\int d\mathbf{x} \Pi \partial^i \Phi \right)^\dagger = \int \frac{d\mathbf{k}}{2} k^i \left(a^\dagger(-\mathbf{k}) b^\dagger(\mathbf{k}) + a(\mathbf{k}) a^\dagger(\mathbf{k}) + b(\mathbf{k}) b^\dagger(\mathbf{k}) + b(-\mathbf{k}) a(\mathbf{k}) \right)$$

By substituting $\mathbf{k} = -\mathbf{k}'$ and then renaming $\mathbf{k}'$ back to $\mathbf{k}$, the first and last terms here cancel with the first and last terms of S62. The second and third terms add, and normal ordering gives

$$P^i = \int d\mathbf{k} k^i \left(a^\dagger(\mathbf{k}) a(\mathbf{k}) + b^\dagger(\mathbf{k}) b(\mathbf{k}) \right)$$

which tells us that a-type and b-type particles both carry momentum $\mathbf{k}$. Therefore, the states mentioned above have four-momentum $(\omega_{\mathbf{k}}, \mathbf{k})$, $(\omega_{\mathbf{k}}, \mathbf{k})$, and $(\omega_{\mathbf{k}_1} + \omega_{\mathbf{k}_2}, \mathbf{k}_1 + \mathbf{k}_2)$.

g) By definition, the operator H generates time translations of the field - this is the content of the Schrodinger equation, $i\partial_t |\Psi\rangle = H|\Psi\rangle$, which holds unchanged in quantum field theory. As for the momenta $\mathbf{P}$, we expect

$$e^{i\mathbf{a}\cdot\mathbf{P}} \Phi(\mathbf{x}) e^{-i\mathbf{a}\cdot\mathbf{P}} = \Phi(\mathbf{x} - \mathbf{a})$$

for any vector a, with a similar result for all the other fields. Show that this implies

$$\left[P^i, \Phi(\mathbf{x}) \right] = -i\partial^i \Phi(\mathbf{x})$$

and show that this relation holds. Thus, $\mathbf{P}$ generates spatial translations.

Solution:

We again take infinitesimal a and expand, giving

$$(1 + i\mathbf{a}\cdot\mathbf{P})\Phi(\mathbf{x})(1 - i\mathbf{a}\cdot\mathbf{P}) = \Phi(\mathbf{x}) - \mathbf{a}\cdot\nabla\Phi(\mathbf{x})$$

Equating the order a components and taking care to remember that $\partial^i = -\partial_i$ gives the desired result.

By the way, you might wonder why we're treating the Hamiltonian and momentum slightly differently. The reason is that in this problem set, we're still working in Schrodinger picture, where the operators $\Phi(\mathbf{x})$ depend on space but not time. In Heisenberg picture, the Heisenberg equation of motion for operators gives the analogous result $[H, \Phi(\mathbf{x}, t)] = -i\partial^0 \Phi(\mathbf{x}, t)$, so that in general we have $[P^\mu, \Phi(x)] = -i\partial^\mu \Phi(x)$.

Now, evaluating the commutator, we have

$$\begin{aligned}
\left[P^i, \Phi(\mathbf{x}) \right] &= \int d\mathbf{k} d\mathbf{k}' \frac{k^i}{\sqrt{2\omega_{\mathbf{k}'}}} \left[a^\dagger(\mathbf{k}) a(\mathbf{k}) + b^\dagger(\mathbf{k}) b(\mathbf{k}), a\left(\mathbf{k}'\right) e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger\left(\mathbf{k}'\right) e^{-i\mathbf{k}'\cdot\mathbf{x}} \right] \\
&= \int d\mathbf{k} d\mathbf{k}' \frac{k^i}{\sqrt{2\omega_{\mathbf{k}'}}} \left[a^\dagger(\mathbf{k}) a(\mathbf{k}) + b^\dagger(\mathbf{k}) b(\mathbf{k}), a\left(\mathbf{k}'\right) e^{i\mathbf{k}'\cdot\mathbf{x}} + b^\dagger\left(\mathbf{k}'\right) e^{-i\mathbf{k}'\cdot\mathbf{x}} \right] \\
&= \int d\mathbf{k} \frac{k^i}{\sqrt{2\omega_{\mathbf{k}}}} \left(a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - b^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \\
&= -i\partial^i \int \frac{d\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}} \left(a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} + b^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right) = -i\partial^i \Phi(\mathbf{x}).
\end{aligned}$$

Stanford.Zhou.2022.3.1. Locality in relativistic quantum field theory. (20/45 p)

The vacuum two-point correlation function of a real scalar field is

$$D(x - y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle.$$

It only depends on the difference of positions $z = x - y$, by translational invariance, and it quantifies correlations between the field values at x and y in the vacuum state.

a) Show that

$$D(z) = \int \frac{d^4 p}{(2\pi)^4} (2\pi) \delta\left(p^2 - m^2\right) \theta\left(p^0\right) e^{-ip\cdot z}$$

where the Heaviside step function θ is 1 if the argument is positive and 0 otherwise.

Solution:

Using the same d and ϕ notation as in the solutions to the second problem set,

$$\begin{aligned}\langle 0|\phi(x)\phi(y)|0\rangle &= \int \frac{d\mathbf{p}d\mathbf{q}}{\sqrt{2\omega_\mathbf{p}}\sqrt{2\omega_\mathbf{q}}} e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\omega_p x^0} e^{-i\mathbf{q}\cdot\mathbf{y}} e^{i\omega_q y^0} \left\langle 0 \left| a(\mathbf{q}) a^\dagger(\mathbf{p}) \right| 0 \right\rangle \\ &= \int \frac{d\mathbf{p}d\mathbf{q}}{\sqrt{2\omega_\mathbf{p}}\sqrt{2\omega_\mathbf{q}}} e^{i\mathbf{p}\cdot\mathbf{x}} e^{-i\omega_p x^0} e^{-i\mathbf{q}\cdot\mathbf{y}} e^{i\omega_q y^0} \phi(\mathbf{p}-\mathbf{q}) \\ &= \int \frac{d\mathbf{p}}{2\omega_\mathbf{p}} e^{-i\omega_p z^0} e^{i\mathbf{p}\cdot\mathbf{z}}.\end{aligned}$$

On the other hand, starting from the desired expression, we have

$$\begin{aligned}D(z) &= \int d\mathbf{p} e^{i\mathbf{p}\cdot\mathbf{z}} \int_0^\infty \frac{dp^0}{2\pi} (2\pi)\delta\left(p^2 - m^2\right) e^{-ip^0 z^0} \\ &= \int \frac{d\mathbf{p}}{2\omega_\mathbf{p}} e^{-i\omega_p z^0} e^{i\mathbf{p}\cdot\mathbf{z}}\end{aligned}$$

which matches.

b) Show that if $z'^\mu = \Lambda^\mu{}_\nu z^\nu$ for a proper orthochronous Lorentz transformation Λ, then $D\left(z'\right) = D(z)$. (Hint: show that d^4p and $\delta\left(p^2 - m^2\right)\theta\left(p^0\right)$ are each Lorentz invariant.)

Solution:

For $D\left(z'\right)$, let's change the integration variable from p to $p'^\mu = \Lambda^\mu_\nu p^\nu$. Then

$$\begin{aligned}D\left(z'\right) &= \int \frac{d^4p'}{(2\pi)^4} (2\pi)\delta\left(p'^2 - m^2\right)\theta\left(p'^0\right) e^{-ip'\cdot z'} \\ &= \int \frac{d^4p'}{(2\pi)^4} (2\pi)\delta\left(p^2 - m^2\right)\theta\left(p'^0\right) e^{-ip\cdot z}\end{aligned}$$

which is precisely the same thing as $D(z)$ provided that

$$d^4p'\delta\left(p'^2 - m^2\right)\theta\left(p'^0\right) = d^4p\delta\left(p^2 - m^2\right)\theta\left(p^0\right)$$

Lorentz transformations always preserve inner products, $p^2 = p^2$, so the two delta functions are the same. Furthermore, for timelike momenta (which are the only ones that contribute to the integral, because of the presence of the delta functions), orthochronous Lorentz transformations always preserve the sign of the time component, by definition. Physically this corresponds to the fact that you can't boost a particle from positive energy to negative energy, and you can check it by explicitly computing the effects of Lorentz boosts. (Of course, Lorentz boosts certainly can change the sign of the time component of spacelike vectors.)

The tricky part is to show that proper orthochronous Lorentz transformations preserve spacetime volume, $d^4p' = d^4p$. There are a few different ways to show this. The most formal way is to just compute the Jacobian determinant. The Jacobian matrix for the transformations between the variables is just

$$\frac{\partial p'^\mu}{\partial p^\nu} = \Lambda^\mu_\nu$$

and its determinant is

$$J = -\frac{1}{24}\epsilon_{\mu\nu\rho\sigma}\Lambda^\mu_{\mu'}\Lambda^\nu_{\nu'}\Lambda^\rho_{\rho'}\Lambda^\sigma_{\sigma'}\epsilon^{\mu'\nu'\rho'\sigma'}$$

as you can show by expanding out the definition of the determinant and using the antisymmetry of ϵ. On the other hand, we know the Levi-Civita symbol is an invariant tensor, so we just get

$$J = -\frac{1}{24}\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = 1$$

Another method is to note that in terms of matrix multiplications, $\eta = \Lambda^T\eta\Lambda$, which implies

$$\det\eta = \left(\det\Lambda^T\right)(\det\eta)(\det\Lambda)$$

and therefore $J = \det\Lambda = \pm 1$. Since the proper orthochronous Lorentz transformations are connected to the identity, we must have $J = 1$. Finally, you could just manually show that $J = 1$ when Λ^μ_ν is a rotation or a boost, i.e. by showing that the matrices

$$\Lambda^\mu_\nu = \begin{pmatrix} \cosh\gamma & \sinh\gamma & & \\ \sinh\gamma & \cosh\gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \Lambda^\mu_\nu = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & \cos\theta & \sin\theta & \\ & & -\sin\theta & \cos\theta \end{pmatrix}$$

have unit determinant. This is sufficient, since any proper orthochronous Lorentz transformation can be built out of rotations and boosts.

c) Show that for spacelike separation, $z^2 < 0$, the correlation function is exponentially decaying but nonzero. You can do this in two ways. First, you can show that

$$D(z) = \frac{m}{4\pi^2\sqrt{-z^2}} K_1\left(\sqrt{-z^2}m\right)$$

where K_1 is a modified Bessel function of the second kind. Alternatively, you may numerically integrate $D(z)$ and graph the result. In both cases you may use any books or computer programs needed, such as Mathematica or Abramowitz and Stegun.

Solution:

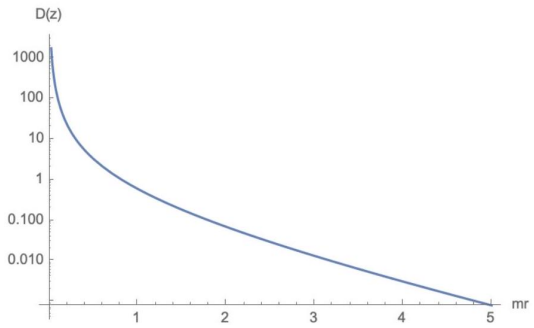
By our result from part (b), we can choose a frame where $z^\mu = (0, \mathbf{z})$, and set up spherical coordinates with the axis aligned with $\mathbf{z}$. Defining $r = \sqrt{-z^2} = |\mathbf{z}|$ and $p = |\mathbf{p}|$, we have

$$\begin{aligned} D(z) &= \int \frac{d\mathbf{p}}{2\omega_\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{z}} \\ &= \frac{1}{(2\pi)^3} \int_0^\infty dp \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi \frac{p^2}{2\sqrt{p^2+m^2}} e^{ipr\cos\theta} \\ &= \frac{1}{(2\pi)^2} \int_0^\infty dp \frac{p}{\sqrt{p^2+m^2}} \frac{\sin(pr)}{r} \end{aligned}$$

At this point we can straightforwardly evaluate the integral numerically, e.g. in Mathematica.

```
Dz[r_, m_] := NIntegrate[p Sin[p r] / (r Sqrt[p^2 + m^2]),
  {p, 0, Infinity}]

LogPlot[Dz[r, 1], {r, 0, 5}, AxesLabel → {"mr", "D(z)"}]
```



Alternatively, if we want to stay analytic to the end, a little more work is necessary. First, we replace the integral with an equivalent complex integral over the entire real axis,

$$D(z) = -\frac{i}{8\pi^2 r} \int_{-\infty}^\infty dp \frac{pe^{ipr}}{\sqrt{p^2+m^2}}$$

To evaluate the integral, we deform the contour so that it wraps around the vertical branch cut that starts at $p = im$. Across this branch cut, the square root flips sign, so the contour integral going down the left end of the branch cut is equal to the contour integral going up the right end, giving

$$D(z) = \frac{1}{4\pi^2 r}\int_m^\infty dx \frac{xe^{-xr}}{\sqrt{x^2-m^2}}$$

where $p = ix$. Finally, looking up this integral in a book or Mathematica yields the final result,

$$D(z) = \frac{m}{4\pi^2 r}K_1(mr)$$

A key requirement for a relativistic theory is that it is local, meaning that effects don't propagate faster than the speed of light. That means any change applied to the field at x should only affect observable results at y if x and y aren't spacelike separated, $(y-x)^2 \geq 0$.

You might thus be concerned about part (c), which implies a field in the vacuum "knows" about the values of the field at spacelike separation. But there's nothing wrong with this. For example, to define a reference frame in special relativity, one synchronizes the clocks by sending light pulses throughout all space. After synchronization, all the clocks "know" about the values on the other clocks, even at spacelike separation. But that doesn't mean that changes propagate faster than light; it's just a consequence of how we set up the system. Similarly, ensuring a field is in the vacuum state requires absorbing all particles throughout all space, and this process sets up correlations between field values.

To test if our theory is local, we must see whether changes in the state propagate faster than light; that is the subject of the rest of the question.

d) One way to interact with a quantum field is to measure its value at a point, a process which changes the state. From quantum mechanics, we know that measurements of two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ do not affect each other if $[\mathcal{O}_1, \mathcal{O}_2] = 0$. Show that

$$\langle 0|[\phi(x), \phi(y)]|0\rangle = 0$$

for spacelike separation, $(x-y)^2 < 0$, as expected from locality.

Solution:

We have

$$\langle 0|[\phi(x), \phi(y)]|0\rangle = D(z) - D(-z).$$

On the other hand, when z is spacelike, it can always be transformed into $-z$ by a Lorentz transformation. (A simple way to see this is to note that both z and $-z$ can be transformed into a standard four-vector like $\left(0, 0, 0, \sqrt{-z^2}\right)$, by first boosting to get rid of the time component, and rotating to get the spatial component pointing the right way.) Therefore, $D(z) = D(-z)$, so the commutator vanishes.

Note that when z is timelike, you can't Lorentz transform it into $-z$, so the argument that $D(z) = D(-z)$ fails. That makes sense because the commutator should be nonzero for timelike separations; if it was zero for all separations, then the theory would be trivial, as no measurements would ever affect anything.

e) Another way to interact with a field is to couple it to a classical source $J(x)$, which for a real scalar field corresponds to taking the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{1}{2}m^2\phi^2 + \phi(x)J(x)$$

This is the scalar analogue of driving the electromagnetic field with a current $J^\mu(x)$. For simplicity, we'll treat $\phi(x)$ as a classical field for now, though similar conclusions will hold when it is a quantum field. Show that the classical equation of motion is

$$\left(\partial_\mu\partial^\mu + m^2\right)\phi(x) = J(x).$$

Solution:

The Euler-Lagrange equations are

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

where we now have

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)} = \partial_\mu\partial^\mu\phi, \quad \frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi + J$$

which together give the desired result.
f) Show that the equation of motion is solved by

$$\phi(x) = i \int d^4y G(x-y) J(y)$$

where G is a Green's function of the Klein-Gordan operator, which means

$$\left(\partial_\mu \partial^\mu + m^2\right) G(z) = -i\delta^{(4)}(z)$$

The factors of $-i$ here are purely conventional, and will simplify results later.

Solution:

We have

$$\begin{aligned}\left(\partial^2 + m^2\right)\phi(x) &= i\left(\partial^2 + m^2\right)\int d^4y G(x-y)J(y) \\ &= i\int d^4y \left(-i\delta^{(4)}(x-y)\right) J(y) \\ &= J(x)\end{aligned}$$

as desired. Note that there is no term from the derivatives acting on the current because the derivatives are with respect to x, while J is a function of y. g) By taking Fourier transforms, we may heuristically write the Green's function as

$$G(z) = \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot z}}{p^2 - m^2}$$

which formally obeys Eq. (8). However, this expression is not mathematically welldefined because the integral blows up when $p^2 = m^2$. To get a definite result, we must add " $i\epsilon$ " terms to the denominator to keep it from vanishing. There are multiple ways to do this, which physically corresponds to the fact that there are multiple possible Green's functions, depending on the field's boundary conditions. Three key examples are the retarded, advanced, and Feynman Green's functions,

$$\begin{aligned}G_R(z) &= \lim_{\epsilon\to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot z}}{\left(p^0 + i\epsilon\right)^2 - |\mathbf{p}|^2 - m^2} \\ G_A(z) &= \lim_{\epsilon\to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot z}}{\left(p^0 - i\epsilon\right)^2 - |\mathbf{p}|^2 - m^2} \\ G_F(z) &= \lim_{\epsilon\to 0^+} \int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot z}}{p^2 - m^2 + i\epsilon}\end{aligned}$$

where the limit notation means ϵ approaches zero from the positive end. By using the residue theorem to perform the integral over p^0, show that

$$\begin{aligned}G_R(z) &= \theta\left(z^0\right)\left(D(z) - D(-z)\right) \\ G_A(z) &= \theta\left(-z^0\right)\left(D(-z) - D(z)\right) \\ G_F(z) &= \theta\left(z^0\right) D(z) + \theta\left(-z^0\right) D(-z)\end{aligned}$$

Solution:

Applying partial fractions, we have

$$G_R(z) = i\int \frac{d^3p dp^0}{(2\pi)^4} \frac{1}{2\omega_p}\left[\frac{1}{\left(p^0 - \omega_p + i\epsilon\right)} - \frac{1}{\left(p^0 + \omega_p + i\epsilon\right)}\right] e^{-ip\cdot z}$$

which implies both poles are shifted below the real p^0 axis. Now we perform the integral over p^0. When $z^0 > 0$, the integrand is exponentially damped below the real p^0 axis, so it makes no difference if we replace the contour over the real line with a closed contour which additionally includes a semicircle (of infinite radius) below the real axis, as shown.

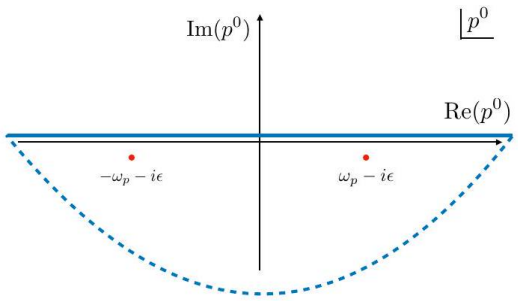


Figure 1: Contour for the retarded propagator for $z^0 > 0$.

This closed contour integral can be calculated using the residue theorem. Each pole contributes $-2\pi i$ times its residue (where the minus sign is because the contour goes around the poles clockwise), giving

$$G_R(z) = \int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{2\omega_p}e^{-i\omega_p z^0 + i\mathbf{p}\cdot\mathbf{z}} - \frac{1}{2\omega_p}e^{i\omega_p z^0 + i\mathbf{p}\cdot\mathbf{z}}\right]$$

Reindexing $\mathbf{p} \to -\mathbf{p}$ in the second term gives

$$\begin{aligned} G_R(z) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left[e^{-i\omega_p z^0 + i\mathbf{p}\cdot\mathbf{z}} - e^{i\omega_p z^0 - i\mathbf{p}\cdot\mathbf{z}}\right] \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left[e^{-ip\cdot z} - e^{ip\cdot z}\right] = D(z) - D(-z) \end{aligned}$$

where here we are defining $p = (\omega_p, \mathbf{p})$. On the other hand, when $z^0 < 0$, the integral is exponentially damped above the real p^0 axis. We can thus close the contour above the real axis, and it encircles no poles, so the integral vanishes. Thus,

$$G_R(z) = \theta\left(z^0\right)\left(D(z) - D(-z)\right)$$

as desired. The proof for $G_A(z)$ is extremely similar, but now with both poles above the real axis.

Finally, the Feynman propagator is

$$\begin{aligned} G_F(z) &= i\int \frac{d^4p}{(2\pi)^4}\left[\frac{1}{(p^0 - \omega_p + i\epsilon)(p^0 + \omega_p - i\epsilon)}\right]e^{-ip\cdot z} \\ &= i\int \frac{d^3p dp^0}{(2\pi)^4}\frac{1}{2\omega_p}\left[\frac{1}{(p^0 - \omega_p + i\epsilon)} - \frac{1}{(p^0 + \omega_p - i\epsilon)}\right]e^{-ip\cdot z} \end{aligned}$$

which has one pole above the real axis, and one pole below. In figure 2 we show the contours of integration

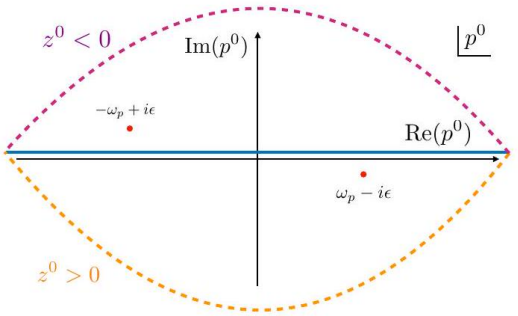


Figure 2: Contour for the Feynman propagator.

for the p^0 integral. If $z^0 > 0$ the dashed, orange line in fig. 2 represents a contour of a semi-circle with infinite radius in the lower half of the complex plane and does not contribute to the integral. The integral along the the real p^0 axis consequently yields minus one times the residue at $p^0 = \omega_p + i\epsilon$ and the other residue does not contribute. Conversely, if $z^0 < 0$, the purple dashed contour in the positive imaginary half-plane contributes zero to the integral and the integral along the real axis yields the residue at $p^0 = -\omega_p + i\epsilon$. We thus conclude

$$G_F(z) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_p} \left[\theta\left(z^0\right) e^{-ipz} + \theta\left(-z^0\right) e^{ipz}\right]$$
$$= \theta\left(z^0\right) D(z) + \theta\left(-z^0\right) D(-z)$$

h) The retarded Greens function applies when the field is zero before the source acts. Show that $G_R(x-y)$ vanishes for spacelike separation, as expected from locality.

Solution:

When z^0 is negative, $G_R(z)$ is just zero by the definition of the step function. When z^0 is positive, $G_R(z)$ is just the same thing as $\langle 0|[\phi(x), \phi(y)]|0\rangle$, which we showed was zero for spacelike separation in part (d).

i) The Feynman propagator will play a crucial role when we introduce Feynman diagrams because it is equal to $\langle 0|T\phi(x)\phi(y)|0\rangle$. Does it vanish for spacelike separation?

Solution:

It doesn't vanish for spacelike separation. For example, when z^0 is positive, the Feynman propagator is just $D(z)$, and we already showed in part **(c)** that this can be nonzero for spacelike separations. For this reason, the Feynman propagator has little use in classical field theory (which is why it doesn't show up in electromagnetism classes, while the retarded and advanced propagators do). But it will be useful when we set up perturbation theory.

Some students tried to solve this problem by setting $z^0 = 0$, but this is a bit tricky because the step functions are discontinuous there. There's no need to do this, because we already showed that $D(z)$ can be nonzero for general spacelike separation, not just when $z^0 = 0$.

Stanford.Zhou.2022.3.2 Recovering classical field theory. (10/45p)

If an operator $\mathcal{O}$ is time-independent in Schrodinger picture, then in Heisenberg picture,

$$\frac{d\mathcal{O}(t)}{dt} = i[H(t), \mathcal{O}(t)]$$

In quantum field theory, working in the Heisenberg picture allows fields to depend on spacetime, making results look more Lorentz invariant. For example, we saw that in Heisenberg picture, a real scalar field obeys the Klein-Gordan equation $\left(\partial^2 + m^2\right)\phi = 0$. Now consider the Lagrangian of Eq. (5), which additionally includes a source term $J(x)$. In this case, the Hamiltonian is explicitly time-dependent,

$$H(t) = H_0 - \int d^3\mathbf{x}\phi(\mathbf{x}, t)J(\mathbf{x}, t)$$

where H_0 is the free Hamiltonian.

a) Show that in Heisenberg picture, the field $\phi(x)$ obeys Eq. (6). (Thus, by the logic of problem 1, expectation values of a quantum field $\phi(x)$ respond locally to sources.)

b) The impulse causes operators to instantaneously shift in value at $t = 0$. Show that

$$\phi\left(\mathbf{x}, 0^+\right) = \phi\left(\mathbf{x}, 0^-\right), \quad \pi\left(\mathbf{x}, 0^+\right) = \pi\left(\mathbf{x}, 0^-\right) + j(\mathbf{x}).$$

Here, 0^+means a time right after $t = 0$, and 0^-means a time right before, i.e.

$$f\left(0^+\right) - f\left(0^-\right) = \lim_{\epsilon \to 0^+} \int_{-\epsilon}^{\epsilon} \frac{df(t)}{dt} dt$$

for any function of time.

c) Show that this is equivalent to the annihilation operators shifting by

$$a\left(\mathbf{p}, 0^+\right) = a\left(\mathbf{p}, 0^-\right) + \frac{i}{\sqrt{2\omega_{\mathbf{p}}}} \tilde{j}(\mathbf{p})$$

where $\tilde{j}$ is the Fourier transform of j.

d) Show that the field is in a coherent state after the impulse, in the sense that the state is an eigenvector of $a\left(\mathbf{p}, 0^+\right)$. Then evaluate $\left\langle 0 \left| a^\dagger\left(\mathbf{p}, 0^+\right) a\left(\mathbf{p}, 0^+\right) \right| 0 \right\rangle$, which gives the number of particles of momentum $\mathbf{p}$ produced, per unit volume of momentum space.

Solution:

Note that the new term in the Hamiltonian commutes with $\phi(\mathbf{y},t)$ for any $\mathbf{y}$. Therefore, the Heisenberg equation of motion for ϕ is exactly the same as before,

$$\dot{\phi}(x) = \pi(x)$$

As for the canonical momentum, we have

$$\begin{aligned}\dot{\pi}(x) &= \nabla^2\phi(x) - m^2\phi(x) + i\left[-\int d^3\mathbf{y}\phi(\mathbf{y},t)J(\mathbf{y},t), \pi(\mathbf{x},t)\right] \\ &= \nabla^2\phi(x) - m^2\phi(x) + (-i)(i)\int d^3\mathbf{y}\delta^{(3)}(\mathbf{y}-\mathbf{x})J(\mathbf{y},t) \\ &= \nabla^2\phi(x) - m^2\phi(x) + J(x).\end{aligned}$$

Combining these two equations gives the result.

With the source, quantum fields can evolve in time nontrivially. Suppose we start in the vacuum state $|0\rangle$, and at time $t = 0$ apply an impulse to the field via

$$J(x) = \delta(t)j(\mathbf{x})$$

This is the scalar analogue of suddenly turning an electric current on and off in electromagnetism. It is also closely related to the last half of problem 2 of problem set 1.

Solution:

Integrating the Heisenberg equation of motion for an infinitesimal time centered at $t = 0$,

$$\phi\left(\mathbf{x},0^+\right) - \phi\left(\mathbf{x},0^-\right) = -i\left[\int d^3\mathbf{y}\phi(\mathbf{y},0)j(\mathbf{y}), \phi(\mathbf{x},0)\right] = 0$$

On the other hand, we have

$$\begin{aligned}\pi\left(\mathbf{x},0^+\right) - \pi\left(\mathbf{x},0^-\right) &= -i\left[\int d^3\mathbf{y}\phi(\mathbf{y},0)j(\mathbf{y}), \pi(\mathbf{x},0)\right] \\ &= -i\int d^3\mathbf{y}i\delta(\mathbf{y}-\mathbf{x})j(\mathbf{y}) \\ &= j(\mathbf{x})\end{aligned}$$

Solution to c:

First, note that

$$a^\dagger\left(\mathbf{p},0^+\right) = a^\dagger\left(\mathbf{p},0^-\right) - \frac{i}{\sqrt{2\omega_\mathbf{p}}}\tilde{j}^*(\mathbf{p})$$

Furthermore, because $j(\mathbf{x})$ is real, we have

$$\tilde{j}^*(\mathbf{p}) = \left(\int d\mathbf{x}j(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{x}}\right)^* = \int d\mathbf{x}j(\mathbf{x})e^{i\mathbf{p}\cdot\mathbf{x}} = \tilde{j}(-\mathbf{p})$$

The change in the field operator is

$$\phi\left(\mathbf{x},0^+\right) - \phi\left(\mathbf{x},0^-\right) = \int \frac{d\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\left(\frac{i\tilde{j}(\mathbf{k})}{\sqrt{2\omega_\mathbf{k}}}e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{i\tilde{j}^*(\mathbf{k})}{\sqrt{2\omega_\mathbf{k}}}e^{-i\mathbf{k}\cdot\mathbf{x}}\right)$$

Substituting $\mathbf{k}' = -\mathbf{k}$ in the second term and renaming $\mathbf{k}'$ to $\mathbf{k}$ gives

$$\phi\left(\mathbf{x},0^+\right) - \phi\left(\mathbf{x},0^-\right) = i\int \frac{d\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}}\left(\frac{\tilde{j}(\mathbf{k})}{\sqrt{2\omega_\mathbf{k}}}e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{\tilde{j}^*(-\mathbf{k})}{\sqrt{2\omega_\mathbf{k}}}e^{i\mathbf{k}\cdot\mathbf{x}}\right) = 0.$$

On the other hand, we have

$$\begin{aligned}\pi\left(\mathbf{x},0^{+}\right)-\pi\left(\mathbf{x},0^{-}\right) &= -i\int d\mathbf{k}\sqrt{\frac{\omega_{\mathbf{k}}}{2}}\left(\frac{i\tilde{j}(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}e^{i\mathbf{k}\cdot\mathbf{x}}-\frac{-i\tilde{j}^{*}(\mathbf{k})}{\sqrt{2\omega_{\mathbf{k}}}}e^{-i\mathbf{k}\cdot\mathbf{x}}\right)\\ &= \frac{1}{2}\int d\mathbf{k}\left(\tilde{j}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}+\tilde{j}^{*}(\mathbf{k})e^{-i\mathbf{k}\cdot\mathbf{x}}\right)\\ &= \frac{1}{2}\int d\mathbf{k}\left(\tilde{j}(\mathbf{k})+\tilde{j}^{*}(-\mathbf{k})\right)e^{i\mathbf{k}\cdot\mathbf{x}}\\ &= j(\mathbf{x})\end{aligned}$$

as desired.

Solution to d:

Because we are in the vacuum state before the impulse, $a\left(\mathbf{p},0^{-}\right)|0\rangle=0$, we have

$$a\left(\mathbf{p},0^{+}\right)|0\rangle=\frac{i}{\sqrt{2\omega_{\mathbf{p}}}}\tilde{j}(\mathbf{p})|0\rangle$$

which implies the field is in a coherent state after the impulse. Then we have

$$\left\langle 0\left|a^{\dagger}\left(\mathbf{p},0^{+}\right)a\left(\mathbf{p},0^{+}\right)\right|0\right\rangle=\left\langle 0\left|\frac{(-i)\tilde{j}^{*}(\mathbf{p})}{\sqrt{2\omega_{\mathbf{p}}}}\frac{i\tilde{j}(\mathbf{p})}{\sqrt{2\omega_{\mathbf{p}}}}\right|0\right\rangle=\frac{|\tilde{j}(\mathbf{p})|^{2}}{2\omega_{\mathbf{p}}}$$

Incidentally, the fact that the field is in a coherent state implies that the number of particles in a given range of momenta is Poisson distributed.

As you saw in problem set 1, coherent states of the quantum harmonic oscillator have the same position and momentum uncertainties as the vacuum state. Similarly, in quantum field theory, coherent states of the field have the same ϕ and π uncertainties as the vacuum, while their expectation values behave classically. When the driving is strong, the uncertainties become negligible compared to the expectation values, and the number of particles becomes large so that we can no longer see their discreteness. We therefore recover a classical field.

Stanford.Zhou.2022.3.3 Practice with time-ordered exponentials. (10/45p)

Suppose the Hamiltonian is $H=H_0+H_{\text{int}}$, where H_0 is a time-independent free Hamiltonian, and H_{int} is an interaction which could be time-dependent. In interaction picture, operators evolve under the free Hamiltonian H_0 alone, and the states are $|\psi(t)\rangle_I$. Let's first review some results derived in lecture.

a) Show that

$$i\partial_t|\psi(t)\rangle_I=H_{\text{int},I}(t)|\psi(t)\rangle_I$$

where $H_{\text{int},I}(t)$ is the interaction Hamiltonian in the interaction picture.

Solution:

By definition, matrix elements of operators are the same in all pictures, so for any states $|\psi\rangle$ and $|\psi'\rangle$, and any operator $\mathcal{O}$, we have

$${}_I\left\langle\psi'(t)\left|\mathcal{O}_I(t)\right|\psi(t)\right\rangle_I={}_S\left\langle\psi'(t)\left|\mathcal{O}_S\right|\psi(t)\right\rangle_S$$

On the other hand, we know that

$$\mathcal{O}_I(t)=e^{iH_0t}\mathcal{O}_Se^{-iH_0t}$$

so comparing the expressions gives

$$|\psi(t)\rangle_I=e^{iH_0t}|\psi(t)\rangle_S$$

Applying $i\partial_t$ to both sides gives

$$\begin{aligned}i\partial_t|\psi(t)\rangle_I &= -H_0e^{iH_0t}|\psi(t)\rangle_S+ie^{iH_0t}\partial_t|\psi(t)\rangle_S\\ &= -H_0|\psi(t)\rangle_I+e^{iH_0t}H_S|\psi(t)\rangle_S\\ &= -e^{iH_0t}H_0e^{-iH_0t}|\psi(t)\rangle_I+e^{iH_0t}H_Se^{-iH_0t}|\psi(t)\rangle_I\\ &= H_{\text{int},I}(t)|\psi(t)\rangle_I\end{aligned}$$

as desired.
b) Show that if the interaction picture time evolution operator is defined as

$$|\psi(t_f)\rangle_I = U(t_f, t_i)\,|\psi(t_i)\rangle_I$$

for $t_f > t_i$, then it is given by Dyson's formula,

$$U(t_f, t_i) = T\exp\left(-i\int_{t_i}^{t_f} dt H_{\text{int},I}(t)\right)$$

where T denotes time ordering.

Solution:

Abbreviating $H_{\text{int},I}(t)$ as $H(t)$, note that the time-ordered exponential is

$$U(t_f, t_i) = \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{t_i}^{t_f} dt_n \int_{t_i}^{t_f} dt_{n-1}\cdots\int_{t_i}^{t_f} dt_1 T\left(H(t_n)\,H(t_{n-1})\cdots H(t_1)\right).$$

We can incorporate the effect of the time ordering by changing the integration bounds,

$$U(t_f, t_i) = \sum_{n=0}^{\infty}(-i)^n\int_{t_i}^{t_f} dt_n \int_{t_i}^{t_n} dt_{n-1}\cdots\int_{t_i}^{t_2} dt_1 H(t_n)\,H(t_{n-1})\cdots H(t_1)$$

where the factor of $1/n\,!$ cancelled because the new integration region has $1/n\,!$ times the volume of the original one. Now, part **(a)** tells us that we want to prove

$$i\partial_{t_f} U(t_f, t_i) = H(t_f)\,U(t_f, t_i).$$

When we differentiating Eq. (S61) with respect to t_f, we only pick up one term from the upper bound of the outer integral, giving

$$i\partial_{t_f} U(t_f, t_i) = H(t_f)\sum_{n=1}^{\infty}(-i)^{n-1}\int_{t_i}^{t_f} dt_{n-1}\cdots\int_{t_i}^{t_2} dt_1 H(t_{n-1})\cdots H(t_1)$$

Shifting the sum in n downward by one recovers $U(t_f, t_i)$, as desired.
Now let's do some concrete calculations with the time evolution operator.
c) Suppose the interaction is only turned on for two moments, i.e. it has the form

$$H_{\text{int},I}(t) = g\left(h_1\delta(t - t_1) + h_2\delta(t - t_2)\right)$$

Write out $U(t_f, t_i)$ up to and including terms of order g^2, assuming $t_i < t_1 < t_2 < t_f$.

Solution:

We have

$$\int_{t_i}^{t_f} dt H_{\text{int},I}(t) = g(h_1 + h_2)$$

so the exponential without the time-ordering would be

$$1 - ig(h_1 + h_2) - \frac{g^2}{2}(h_1 + h_2)^2 + O\left(g^3\right).$$

The time-ordering puts contributions that occur later on the left, which means all the h_2 terms are to the left of the h_1 terms, giving

$$1 - ig\left(h_1 + h_2\right) - \frac{g^2}{2}\left(h_1^2 + 2h_2h_1 + h_2^2\right) + O\left(g^3\right)$$

d) Now consider a general $H_{\text{int},I}(t)$ which is proportional to a coupling g. It is generally true that $U\left(t_3,t_2\right)U\left(t_2,t_1\right) = U\left(t_3,t_1\right)$. Explicitly show that this result is true, up to and including terms of order g^2, in the case $t_1 < t_2 < t_3$.

Solution:

The desired right-hand side is

$$U\left(t_3,t_1\right) = 1 - ig\int_{t_1}^{t_3} dt H(t) - g^2\int_{t_1}^{t_3} dt\int_{t_1}^{t} dt' H(t)H\left(t'\right) + O\left(g^3\right)$$

On the left-hand side, expanding the product of the two terms gives

$$\begin{aligned} U\left(t_3,t_2\right)U\left(t_2,t_1\right) &= 1 - ig\int_{t_1}^{t_2} dt H(t) - ig\int_{t_2}^{t_3} dt H(t) \\ &- g^2\int_{t_1}^{t_2} dt\int_{t_2}^{t_3} dt' H(t)H\left(t'\right) - g^2\int_{t_1}^{t_2} dt\int_{t_1}^{t} dt' H(t)H\left(t'\right) - g^2\int_{t_2}^{t_3} dt\int_{t_2}^{t} dt' H(t)H\left(t'\right) + O\left(g^3\right). \end{aligned}$$

Clearly, the constant and $O(g)$ terms agree, so the tricky part is showing the equality of the $O\left(g^2\right)$ terms,

$$\int_{t_1}^{t_3} dt\int_{t_1}^{t} dt' H(t)H\left(t'\right) = \int_{t_1}^{t_2} dt\int_{t_2}^{t_3} dt' H(t)H\left(t'\right) + \int_{t_1}^{t_2} dt\int_{t_1}^{t} dt' H(t)H\left(t'\right) + \int_{t_2}^{t_3} dt\int_{t_2}^{t} dt' H(t)H\left(t'\right)$$

The left-hand side integrates over all $t_1 < t' < t < t_3$, and t_2 is between t_1 and t_3. All terms on the right-hand side enforce $t' < t$. The second term includes the region where both t and t' are less than t_2, the third term includes the regions where both are greater than t_2, and the first term corresponds to when t_2 is between t and t'.

26.1.4 Problems about free quantum fields 3

Stanford.Zhou.2022 4.1. Wick's theorem. (7/45 p)

For both parts of this problem, you can use Wick's theorem for two real scalar fields.
a) Show that $T\left(\phi\left(x_1\right)\phi\left(x_2\right)\right)$ and: $\phi\left(x_1\right)\phi\left(x_2\right)$: both remain the same when x_1 and x_2 are exchanged. Using these results, explain why $D_F\left(x_1 - x_2\right) = D_F\left(x_2 - x_1\right)$.

Solution:

For the time ordered product this is true by definition; it puts the later operator on the left no matter what their original order was. As for the normal ordered product, we can split each field into parts with only creation and annihilation operators,

$$\phi\left(x_i\right) = \phi_i^+ + \phi_i^-, \quad 0 = \phi_i^-|0\rangle = \langle 0|\phi_i^+.$$

The normal ordered product is defined by

$$:\phi_1\phi_2: = \phi_1^+\phi_2^+ + \phi_1^+\phi_2^- + \phi_2^+\phi_1^- + \phi_1^-\phi_2^-.$$

Since $\left[\phi_i^+,\phi_j^+\right] = \left[\phi_i^-,\phi_j^-\right] = 0$, this expression is identical if we exchange x_1 and x_2. Finally, Wick's theorem tells us that the Feynman propagator is the difference of a time ordered product and a normal ordered product, so it is also symmetric in its arguments.
b) Prove Wick's theorem for a time ordered product of three scalar fields,

$$\begin{aligned} T\left(\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\right) =: \phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right): &+ \phi\left(x_1\right)D_F\left(x_2 - x_3\right) \\ &+ \phi\left(x_2\right)D_F\left(x_3 - x_1\right) + \phi\left(x_3\right)D_F\left(x_1 - x_2\right). \end{aligned}$$

Solution:

We simply work through the cases for the time ordering of x_1, x_2, and x_3. By definition,

$$T\left(\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\right) = \phi_1 T\left\{\phi_2\phi_3\right\}\big|_{t_1>t_2,t_3} + \phi_2 T\left\{\phi_1\phi_3\right\}\big|_{t_2>t_1,t_3} + \phi_3 T\left\{\phi_1\phi_2\right\}\big|_{t_3>t_2,t_1}$$

Now let's apply Wick's theorem to the last term,

$$\phi_3 T\left\{\phi_1\phi_2\right\}\big|_{t_3>t_2,t_1} = \left(\phi_3^+ + \phi_3^-\right)\left(\phi_1^+\phi_2^+ + \phi_1^+\phi_2^- + \phi_2^+\phi_1^- + \phi_1^-\phi_2^- + D_F\left(x_2 - x_3\right)\right)\big|_{t_3>t_2,t_1}$$

A term is normal ordered if all the ϕ_i^+'s are to the left of all the ϕ_i^-'s, so

$$:\phi_1\phi_2\phi_3: = \left(\phi_3^+ + \phi_3^-\right)\left(\phi_1^+\phi_2^+ + \phi_1^+\phi_2^- + \phi_2^+\phi_1^- + \phi_1^-\phi_2^-\right)\big|_{t_3>t_2,t_1} + \left[\phi_3^-, \phi_2^+\right]\phi_1 + \left[\phi_3^-, \phi_1^+\right]\phi_2\big|_{t_3>t_2,t_1}$$

We have therefore shown that

$$\phi_3 T\left\{\phi_1\phi_2\right\}\big|_{t_3>t_2,t_1} = \left(:\phi_1\phi_2\phi_3: + \phi_1\left[\phi_3^-, \phi_2^+\right] + \phi_2\left[\phi_3^-, \phi_1^+\right] + \phi_3 D_F\left(x_1 - x_2\right)\right) \times \theta\left(x_3^0 - x_2^0\right)\theta\left(x_3^0 - x_1^0\right).$$

When we add the other two terms, the first terms of each one combine to give: $\phi_1\phi_2\phi_3$: with no step functions. In addition, pairs of commutator terms combine into Feynman propagators by

$$D_F(x,y) = \theta\left(x^0 - y^0\right)\left[\phi^-(x), \phi^+(y)\right] + \theta\left(y^0 - x^0\right)\left[\phi^-(y), \phi^+(x)\right]$$

To see how this works, let's consider the terms that eventually combine to yield $\phi\left(x_3\right) D_F\left(x_1 - x_2\right)$,

$$\phi_3\left(D_F\left(x_1 - x_2\right)\theta\left(t_3 - t_2\right)\theta\left(t_3 - t_1\right) + \left[\phi_1^-, \phi_2^+\right]\theta\left(t_1 - t_3\right)\theta\left(t_1 - t_2\right) + \left[\phi_2^-, \phi_1^+\right]\theta\left(t_2 - t_1\right)\theta\left(t_2 - t_3\right)\right).$$

The first term here is the desired term when t_3 is the latest time. The second and third terms combine to give the Feynman propagator for any ordering of t_1 and t_2, under the condition that t_3 is not the latest. So the sum is $\phi\left(x_3\right) D_F\left(x_1 - x_2\right)$, and the logic for the other terms is similar.

Stanford.Zhou.2022 4.2 Diagrams in ϕ^3 theory. (18/45p)

Consider a real scalar field ϕ of mass m with a cubic self-interaction, so that

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - \frac{1}{2}m^2\phi^2, \quad \mathcal{L}_{\text{int}} = \frac{\lambda}{3!}\phi^3.$$

Let $|0\rangle$ denote the free vacuum, the interaction Hamiltonian density is $\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$, and H_I is the interaction Hamiltonian in interaction picture.

a) Using Wick's theorem, evaluate the vacuum correlation function

$$\left\langle 0\left|T\exp\left(-i\int dt H_I(t)\right)\right|0\right\rangle$$

up to and including terms of order λ^2. Your result should be an explicit expression in terms of integrals of the position-space Feynman propagator. (You will run into terms proportional to $D_F(0)$, which is infinite. Don't worry about this for now; we'll come back to this issue when we discuss renormalization.)

b) Draw all distinct diagrams that contribute to the vacuum correlation function at order λ^3 and λ^4. For this part, you do not need to evaluate the diagrams or compute any symmetry factors, but you should neatly display all of your diagrams in one box.

c) Now consider the two-point correlation function

$$\langle\Omega\left|T\left(\phi_H(x)\phi_H(y)\right)\right|\Omega\rangle = \frac{\left\langle 0\left|T\left(\phi_I(x)\phi_I(y)\exp\left(-i\int dt H_I\right)\right)\right|0\right\rangle}{\left\langle 0\left|T\exp\left(-i\int dt H_I\right)\right|0\right\rangle}$$

where ϕ_H and ϕ_I are Heisenberg and interaction picture fields. As discussed in class, only connected diagrams contribute to the left-hand side, i.e. diagrams where all fields are connected to at least one of the external points. Find all diagrams that contribute up to and including order λ^4, along with their symmetry factors, which you can compute with any method. Again, give your answer by drawing everything neatly inside one box. (Hint: there are 13 fully connected diagrams, i.e. diagrams where all fields are connected to both of the external points.)

Solution to a:

Let the desired correlation function be $c_0 + \lambda c_1 + \lambda^2 c_2 + \mathcal{O}\left(\lambda^3\right)$. Then

$$c_0 = \langle 0|0\rangle = 1$$

and

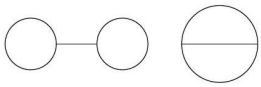
$$c_1 = \frac{i}{3!}\left\langle 0\left|T\left(\int d^4x\phi(x)\phi(x)\phi(x)\right)\right|0\right\rangle = 0$$

because we have an odd number of copies of ϕ.

Then,

$$\begin{aligned} c_2 &= \frac{1}{2!}\left(\frac{i}{3!}\right)^2\left\langle 0\left|T\left(\int d^4xd^4y\phi(x)\phi(x)\phi(x)\phi(y)\phi(y)\phi(y)\right)\right|0\right\rangle \\ &= -\frac{1}{72}\int d^4xd^4y\left[3\cdot 3\cdot D_F(x-x)D_F(x-y)D_F(y-y) + 3!\cdot D_F(x-y)^3\right] \\ &= -\int d^4xd^4y\left(\frac{1}{8}D_F(0)^2D_F(x-y) + \frac{1}{12}D_F(x-y)^3\right). \end{aligned}$$

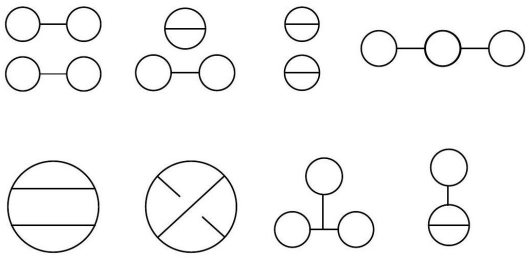
The terms in c_2 correspond to the diagrams shown below:



(??? интересно, тут как раз эти 8 и 12 это перестановки вершин и пара других правил симметрий? вроде да, так и получается, впишу эти рассуждения потом)

Solution to b:

There are no diagrams at any odd order. The 8 diagrams at order λ^4 are shown below.



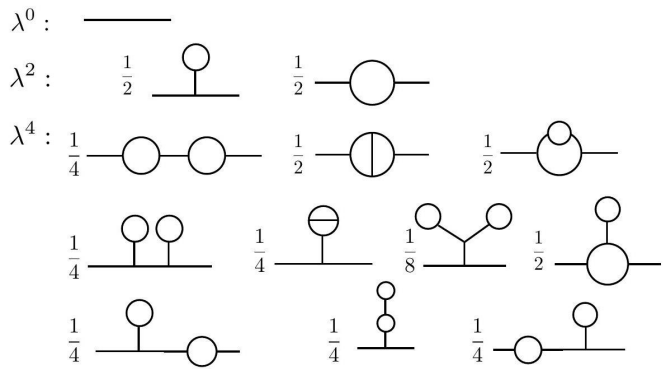
Solution c: (??????????????)

(??? пока вообще не понял, почему такие симм. факторы???? научусь выводить еще это потом!!!)

Again, odd orders in λ always yield correlation functions on the right-hand side with an odd number of fields, which vanish since $\langle 0|\phi|0\rangle = 0$. The fully connected diagrams that contribute at orders λ^0, λ^2, and λ^4 are shown below, along with their symmetry factors.

These symmetry factors can all be found with the following rules:

- $1/2$ for every vertex connected to itself.
- $1/k!$ for k propagators connecting the same two vertices.
- $1/2$ for two interchangeable vertices.



(??? почему именно такое их количество????)

For λ^2:

1) vertex connected to itself, 2) 2 propagators connecting the same two vertices

For the first line in λ^4:

1) Two elements of "two propagators connecting the same two vertices", 2) two interchangeable vertices, 3) 2 propagators connecting the same two vertices.

For the second line in λ^4:

1) Two elements of "vertex connected to itself", 2) two interchangeable vertices and 2 propagators connecting the same two vertices, 3) two elements of "vertex connected to itself" and two interchangeable vertices, 4) vertex connected to itself.

For the third line in λ^4:

1) Vertex connected to itself and two propagators connecting the same two vertices, 2) vertex connected to itself and two propagators connecting the same two vertices, 3) vertex connected to itself and two propagators connecting the same two vertices.

Here is some computation by hand if one wants to make sure:

(????!!! сделаю это, а то пока не уверен!!!)

(to do: also include the four non-fully connected diagrams) (?????)

Stanford.Zhou.2022 4.3 Recovering nonrelativistic quantum mechanics. (15/45p)

A free complex scalar field Φ of mass m satisfies the Klein-Gordan equation, so its plane wave modes $e^{-ip\cdot x}$ satisfy $\omega^2 = |\mathbf{k}|^2 + m^2$. This implies we have plane waves with both positive and negative ω, which is universal in relativistic theories. Upon quantization, excitations of these modes correspond to matter and antimatter, with opposite charges.

The nonrelativistic limit is $|\mathbf{k}| \ll m$, but since antimatter is inherently a feature of relativistic theories, we should also throw away the negative ω modes. To do this, take

$$\Phi(x) = e^{-imt}\chi(x)/\sqrt{2m}$$

and assume $|\dot{\chi}| \ll m\chi$, so that only nonrelativistic positive ω modes are excited. The $\sqrt{2m}$ factor transfers us back to nonrelativistic normalization.

a) Simplify the complex scalar field action to

$$S = \int d^4x \left(i\chi^*\dot{\chi} - \frac{1}{2m}\nabla\chi^* \cdot \nabla\chi \right)$$

by dropping a term that is small for $|\dot{\chi}| \ll m\chi$.

b) Find the Euler-Lagrange equations for χ and χ^*, and the conserved current $J^\mu = (\rho, \mathbf{J})$ corresponding to the symmetry $\chi \to e^{-i\alpha}\chi$ and $\chi^* \to e^{i\alpha}\chi^*$. (Be careful to account for minus signs from the relativistic metric.)

For the relativistic complex scalar field, we had to canonically quantize Φ, Π, Φ^*, and Π^* as two pairs of phase space variables. But for this nonrelativistic field, the canonical momentum for χ is just $i\chi^*$, so we only have one pair. This is because we threw away the negative ω solutions, and it implies the canonical commutation relations are

$$[\chi(\mathbf{x}), \chi^*(\mathbf{y})] = \delta^{(3)}(\mathbf{x} - \mathbf{y})$$

with all other commutators vanishing. These can be satisfied with the mode expansion

$$\chi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} a(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}}, \quad \chi^*(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} a^\dagger(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}}$$

c) Show that $\left[a(\mathbf{p}), a^\dagger(\mathbf{q})\right] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})$. (Of course, the other commutators vanish.) Then show that the single-particle states $|\mathbf{x}\rangle = \chi^*(\mathbf{x})|0\rangle$ and $|\mathbf{p}\rangle = a^\dagger(\mathbf{p})|0\rangle$ behave like nonrelativistic position and momentum states, obeying

$$\langle\mathbf{y}|\mathbf{x}\rangle = \delta^{(3)}(\mathbf{y}-\mathbf{x}), \quad \langle\mathbf{q}|\mathbf{p}\rangle = (2\pi)^3\delta^{(3)}(\mathbf{q}-\mathbf{p}), \quad \langle\mathbf{p}|\mathbf{x}\rangle = e^{i\mathbf{p}\cdot\mathbf{x}}$$

The factors of 2π are just due to our Fourier transform convention.

d) A general single-particle state takes the form

$$|\psi(t)\rangle = \int d^3\mathbf{x}\psi(\mathbf{x},t)|\mathbf{x}\rangle$$

where $\psi(\mathbf{x},t)$ is the particle's position-space wavefunction at time t. As always in quantum theories, the state evolves over time as $i\partial_t|\psi\rangle = H|\psi\rangle$, where H is the Hamiltonian. Write this equation as a partial differential equation involving $\psi(\mathbf{x},t)$.

e) Take the term in the action you dropped in part (**a)** and simplify it by applying the Euler-Lagrange equations you found in part (b). Find how the Hamiltonian in part (**d)** changes when this term is added. What is its physical interpretation?

Solution to a:

The Lagrangian is

$$\begin{aligned}\mathcal{L} &= (\partial_\mu\Phi)(\partial^\mu\Phi)^* - m^2\Phi^*\Phi \\ &= \frac{1}{2m}\left(\partial_\mu\left(e^{-imt}\chi\right)\right)\left(\partial^\mu\left(e^{imt}\chi^*\right)\right) - \frac{m}{2}\chi^*\chi \\ &= -\frac{1}{2m}\nabla\chi\cdot\nabla\chi^* + \frac{1}{2m}\left(\partial_t\left(e^{-imt}\chi\right)\right)\left(\partial_t\left(e^{imt}\chi^*\right)\right) - \frac{m}{2}\chi^*\chi\end{aligned}$$

The middle term is

$$\frac{1}{2m}(\dot\chi - im\chi)(\dot\chi^* + im\chi^*) = \frac{\dot\chi^*\dot\chi}{2m} + \frac{i}{2}(\dot\chi\chi^* - \dot\chi^*\chi) + \frac{m}{2}\chi^*\chi$$

Of these terms, the first is negligible in the nonrelativistic limit, and the last term cancels with the last term of Eq. S16). To get the second term in the desired form, we integrate the action by parts,

$$\frac{i}{2}\int d^4x\,(\dot\chi\chi^* - \dot\chi^*\chi) = i\int d^4x\chi^*\dot\chi$$

Solution to b:

As usual, we treat χ and χ^* as independent variables. For both parts of this question, we need to compute the derivatives of $\mathcal{L}$ with respect to $\partial_\mu\chi$ and $\partial_\mu\chi^*$. These are

$$\frac{\partial\mathcal{L}}{\partial\dot\chi} = i\chi^*, \quad \frac{\partial\mathcal{L}}{\partial(\nabla\chi)} = -\frac{1}{2m}\nabla\chi^*$$

and

$$\frac{\partial\mathcal{L}}{\partial\dot\chi^*} = 0, \quad \frac{\partial\mathcal{L}}{\partial(\nabla\chi^*)} = -\frac{1}{2m}\nabla\chi$$

The Euler-Lagrange equation for χ is

$$\frac{\partial\mathcal{L}}{\partial\chi} = \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi)} = \partial_t\frac{\partial\mathcal{L}}{\partial\dot\chi} + \nabla\cdot\frac{\partial\mathcal{L}}{\partial(\nabla\chi)}$$

The left-hand side is zero, and plugging in the results for the right-hand side gives

$$i\dot\chi^* = \frac{1}{2m}\nabla^2\chi^*$$

Similarly, the Euler-Lagrange equation for χ^* is

$$\frac{\partial\mathcal{L}}{\partial\chi^*} = \partial_t\frac{\partial\mathcal{L}}{\partial\dot\chi^*} + \nabla\cdot\frac{\partial\mathcal{L}}{\partial(\nabla\chi^*)}$$

Now the left-hand side is nonzero, but the first term on the right-hand side is zero, giving

$$i\dot{\chi} = -\frac{1}{2m}\nabla^2\chi$$

As expected, the two Euler-Lagrange equations are conjugates of each other, and the equation of motion for χ is just the Schrodinger equation.

For infinitesimal α, the change in the fields is $\delta\chi = -i\alpha\chi$ and $\delta\chi^* = i\alpha\chi^*$, so Noether's theorem gives

$$J^\mu = -i\chi\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\chi\right)} + i\chi^*\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\chi^*\right)}$$

The density of the conserved charge is

$$\rho = -i\chi\frac{\partial\mathcal{L}}{\partial\dot{\chi}} + i\chi^*\frac{\partial\mathcal{L}}{\partial\dot{\chi}^*} = \chi^*\chi$$

The current density is

$$\mathbf{J} = -i\chi\frac{\partial\mathcal{L}}{\partial(\nabla\chi)} + i\chi^*\frac{\partial\mathcal{L}}{\partial\left(\nabla\chi^*\right)} = \frac{i}{2m}\left(\chi\nabla\chi^* - \chi^*\nabla\chi\right)$$

These are the same form as the probability density and probability current in nonrelativistic quantum mechanics. (But here they don't mean the same thing: we're still talking about classical field theory, not quantum mechanics.)

Solution to c:

We use the same d and ϕ notation as in the solutions to the second problem set. Inverting the Fourier transform gives

$$a(\mathbf{p}) = \int d\mathbf{x}\chi(\mathbf{x})e^{-i\mathbf{p}\cdot\mathbf{x}}$$

which implies

$$\left[a(\mathbf{p}), a^\dagger(\mathbf{q})\right] = \int d\mathbf{x}d\mathbf{y}\left[\chi(\mathbf{x}), \chi^*(\mathbf{y})\right]e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{q}\cdot\mathbf{y}} = \int d\mathbf{x}e^{-i\mathbf{p}\cdot\mathbf{x}}e^{i\mathbf{q}\cdot\mathbf{x}} = \not{\delta}(\mathbf{p}-\mathbf{q})$$

Now, we have

$$\langle\mathbf{y}|\mathbf{x}\rangle = \langle 0\left|\chi^*(\mathbf{y})\chi(\mathbf{x})\right|0\rangle = \delta(\mathbf{y}-\mathbf{x})$$

where we applied the commutation relation and used $\chi^*(\mathbf{y})|0\rangle = 0$. Next,

$$\langle\mathbf{q}|\mathbf{p}\rangle = \left\langle 0\left|a(\mathbf{q})a^\dagger(\mathbf{p})\right|0\right\rangle = \not{\delta}(\mathbf{q}-\mathbf{p})$$

where we applied the commutation relation and used $a(\mathbf{q})|0\rangle = 0$. Finally,

$$\langle\mathbf{p}|\mathbf{x}\rangle = \left\langle 0\left|a(\mathbf{p})\int d\mathbf{q}a^\dagger(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}\right|0\right\rangle = e^{i\mathbf{p}\cdot\mathbf{x}}$$

Solution to d:

First, we need to find the Hamiltonian. Applying the Legendre transform (or equivalently considering the element T^{00} of the stress-energy tensor),

$$\mathcal{H} = \dot{\chi}\frac{\partial\mathcal{L}}{\partial\dot{\chi}} - \mathcal{L} = \frac{1}{2m}\nabla\chi^*\cdot\nabla\chi$$

Therefore, in terms of creation and annihilation operators, the Hamiltonian is

$$\begin{aligned} H &= \frac{1}{2m}\int d\mathbf{x}\nabla\chi^*\cdot\nabla\chi \\ &= \frac{1}{2m}\int d\mathbf{x}\int d\mathbf{p}d\mathbf{q}a^\dagger(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}(i\mathbf{p}\cdot(-i\mathbf{q}))a(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}} \\ &= \frac{1}{2m}\int d\mathbf{x}\int d\mathbf{p}d\mathbf{q}(\mathbf{p}\cdot\mathbf{q})e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}a^\dagger(\mathbf{p})a(\mathbf{q}) \\ &= \frac{1}{2m}\int d\mathbf{p}|\mathbf{p}|^2a^\dagger(\mathbf{p})a(\mathbf{p}). \end{aligned}$$

The Hamiltonian acts on position eigenstates as

$$\begin{aligned} H|\mathbf{x}\rangle &= \frac{1}{2m}\int d\mathbf{p}|\mathbf{p}|^2a^\dagger(\mathbf{p})a(\mathbf{p})\int d\mathbf{q}a^\dagger(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}|0\rangle \\ &= \frac{1}{2m}\int d\mathbf{p}|\mathbf{p}|^2a^\dagger(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}|0\rangle \\ &= -\frac{\nabla^2}{2m}\int d\mathbf{p}a^\dagger(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}}|0\rangle \\ &= -\frac{\nabla^2}{2m}|\mathbf{x}\rangle \end{aligned}$$

Finally, the time evolution equation is

$$\int d\mathbf{x}i\dot{\psi}|\mathbf{x}\rangle = \int d\mathbf{x}\psi H|\mathbf{x}\rangle = -\frac{1}{2m}\int d\mathbf{x}\psi\nabla^2|\mathbf{x}\rangle = -\frac{1}{2m}\int d\mathbf{x}\left(\nabla^2\psi\right)|\mathbf{x}\rangle$$

where we integrated by parts in the last step. Matching the coefficients of $|\mathbf{x}\rangle$ gives

$$i\dot{\psi} = -\frac{\nabla^2\psi}{2m}$$

which is of course the nonrelativistic single-particle Schrodinger equation. Thus, our nonrelativistic quantum scalar field theory contains single-particle nonrelativistic quantum mechanics. In addition, it naturally contains states with arbitrarily many particles.

Notice that the single-particle quantum wavefunction ψ obeys the same equation of motion as the classical field χ, and both have a conserved current of the same form. This formal correspondence is a general structural feature of quantum field theory, and one of the original clues used to construct it. It is also the reason the pioneers of quantum field theory called it "second quantization," under the mistaken impression they were quantizing ψ a second time rather than χ for a first time.

Solution to e:

We had dropped the term $\dot{\chi}^*\dot{\chi}/2m$, and plugging in the Euler-Lagrange equations gives

$$\mathcal{L} \supset \frac{1}{8m^3}\nabla^2\chi^*\nabla^2\chi$$

This contributes an additional term to the Hamiltonian,

$$H \supset -\frac{1}{8m^3}\int d\mathbf{x}\nabla^2\chi^*\nabla^2\chi$$

and following the same steps as in part (d) yields

$$H \supset -\frac{1}{8m^3}\int d\mathbf{p}|\mathbf{p}|^4a^\dagger(\mathbf{p})a(\mathbf{p})$$

This is simply the leading relativistic correction to the kinetic energy, which you often see, e.g. when computing the fine structure of the hydrogen atom. Applying a similar (but more technically complex) procedure to the Dirac equation, known as the Foldy-Wouthuysen transformation, one can derive the fine structure corrections for the hydrogen atom. You can continue turning the crank to get relativistic corrections at all orders.

Stanford.Zhou.2022 4.4.⋆ The force between sources. (5/45p)

Correlation functions often do not have a direct physical interpretation. For example, the vacuum correlation function $\langle 0 \left| T \exp\left(-i \int dt H_I(t)\right) \right| 0 \rangle$ can be used to find the energy of the interacting vacuum relative to the free vacuum, but this isn't directly measurable because we can't turn off the interaction in real life.

Instead, let's again consider a free real scalar field with source $J(x)$,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - \frac{1}{2}m^2\phi^2 + \phi(x)J(x).$$

If we change the vacuum by including a static source $J(x)$, the resulting shift in vacuum energy can be directly interpreted as the energy of the source, which is measurable.

a) Using Wick's theorem, show that

$$\left\langle 0 \left| T \exp\left(i \int d^4x \phi(x) J(x)\right) \right| 0 \right\rangle = \exp\left(-\frac{1}{2}\int d^4x d^4y J(x) D_F(x-y) J(y)\right).$$

b) Now consider the case where two point sources are turned on for a long time T,

$$J(x) = g f(t)\left(\delta^{(3)}(\mathbf{x}) + \delta^{(3)}(\mathbf{x}-\mathbf{R})\right), \quad f(t) = \begin{cases} 1 & -T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}.$$

If you plug this into Eq. (12), you'll get infinite terms involving $D_F(0)$, reflecting the fact that point sources have infinite energy. Discarding these terms so we can focus on the sources' interaction energy, evaluate Eq. (12) assuming $T \gg R \gg 1/m$.

c) Relate your answer to the interaction energy $V(R)$ between the sources, and find it. Is the force between them attractive or repulsive?

Solution to a:

Again, odd-order terms vanish, so the left-hand side is

$$\sum_n \frac{i^{2n}}{(2n)!}\int d^4x_1 \cdots d^4x_{2n} J\left(x_1\right) \cdots J\left(x_{2n}\right) \left\langle 0 \left| T\phi\left(x_1\right) \cdots \phi\left(x_{2n}\right) \right| 0 \right\rangle$$

There are $(2n-1)(2n-3)\cdots(3)(1)$ possible Wick contractions, all of which yield the same thing, giving

$$\sum_n \frac{i^{2n}}{(2n)(2n-2)\cdots(2)}\left(\int d^4x d^4y J(x) D_F(x-y) J(y)\right)^n$$

Simplifying the numeric factors gives

$$\sum_n \frac{1}{n!}\left(-\frac{1}{2}\int d^4x d^4y J(x) D_F(x-y) J(y)\right)^n$$

which is indeed the Taylor series of the desired right-hand side.

Solution to b:

It's helpful to go to momentum space, where we get

$$\exp\left(-\frac{i}{2}\int d^4x d^4y J(x) J(y) \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon}\right)$$

Taking only the two interaction terms and imposing the position-space delta functions gives

$$\exp\left(-ig^2 \int dx^0 dy^0 f\left(x^0\right) f\left(y^0\right) \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip^0\left(x^0-y^0\right)}e^{-i\mathbf{p}\cdot\mathbf{R}}}{p^2 - m^2 + i\epsilon}\right)$$

At this point, we can perform the temporal integrals; they are a representation of the delta function,

$$\int_{-T/2}^{T/2} dx^0 \int_{-T/2}^{T/2} dy^0 e^{-ip^0\left(x^0-y^0\right)} = 2\pi T\delta\left(p^0\right)$$

for large T, where the normalization can be found by integrating both sides against p^0. We thus get

$$\exp\left(ig^2T\int\frac{d^3p}{(2\pi)^3}\frac{e^{-i\mathbf{p}\cdot\mathbf{R}}}{|\mathbf{p}|^2+m^2}\right)$$

where we dropped the $i\epsilon$ since the denominator won't vanish. This is a standard integral you can look up or do with spherical coordinates and contour integration (we'll see it again later in the course), so

$$\exp\left(ig^2T\frac{e^{-mr}}{4\pi r}\right)$$

Solution to c:

If the interaction energy is V, then the correlation function should be e^{-iVT}, reflecting the change in energy of the vacuum. Comparing with above expression gives

$$V(r) = -\frac{g^2e^{-mr}}{4\pi r}$$

The force falls off exponentially on the scale $1/m$, and it is attractive. (Incidentally, since we just have a free field theory and classical sources, we could have done this all much more easily in classical field theory. But the approach we've just taken is much more powerful and general.)

Stanford.Zhou.2022 5.1. Mandelstam variables. (4/45 p)

Consider any $2 \to 2$ scattering process, where the two incoming particles have momenta p_1 and p_2, and the two outgoing particles have momenta p_3 and p_4. By momentum conservation, $p_1 + p_2 = p_3 + p_4$, and $p_i^2 = m_i^2$ where m_i is the mass of particle i. In this situation, it is often useful to work in terms of the Mandelstam variables

$$s = (p_1+p_2)^2, \quad t = (p_1-p_3)^2, \quad u = (p_1-p_4)^2$$

a) Show that $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$.
b) In the center of mass frame, the total energy is E_{cm} and the angle of $\mathbf{p}_1$ to $\mathbf{p}_3$ is θ. Write s, t, and u in terms of E_{cm} and θ, assuming all four particles are massless.

Solution:

Just directly using the definitions,

$$\begin{aligned} s+t+u &= p_1^2 + 2p_1\cdot p_2 + p_2^2 + p_1^2 - 2p_1\cdot p_3 + p_3^2 + p_1^4 - 2p_1\cdot p_4 + p_4^2 \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 + 2p_1\cdot(p_1+p_2-p_3-p_4) \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 \end{aligned}$$

Solution:

This is easiest if we use concrete coordinate expressions. Aligning the z-axis with $\mathbf{p}_1$,

$$p_1^\mu = \frac{E_{\text{cm}}}{2}(1,0,0,1), \quad p_2^\mu = \frac{E_{\text{cm}}}{2}(1,0,0,-1)$$

where the spatial momenta are opposite because we're working in the center of mass frame, and equal in magnitude to the energy because the particles are massless. After the scattering, we have

$$p_3^\mu = \frac{E_{\text{cm}}}{2}(1,0,\sin\theta,\cos\theta), \quad p_4^\mu = \frac{E_{\text{cm}}}{2}(1,0,-\sin\theta,-\cos\theta).$$

At this point we can just compute the Mandelstam variables explicitly, giving

$$s = E_{\text{cm}}^2, \quad t = -E_{\text{cm}}^2 \frac{1-\cos\theta}{2} = -E_{\text{cm}}^2 \sin^2\frac{\theta}{2}, \quad u = -E_{\text{cm}}^2 \frac{1+\cos\theta}{2} = -E_{\text{cm}}^2 \cos^2\frac{\theta}{2}.$$

Note that they sum to zero, as expected from part (a).

МФТИ-Киселев-2020.4.^{C}

Показать, что для свободного комплексного скалярного поля электрический заряд выражается через лоренц-инвариантные амплитуды $a(\boldsymbol{k})$ и $a_c(\boldsymbol{k})$ в виде

$$Q = \int \mathrm{d}^3 r j^0 = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} e \left\{ a^*(\boldsymbol{k}) a(\boldsymbol{k}) - a_c^*(\boldsymbol{k}) a_c(\boldsymbol{k}) \right\}$$

Solution

Как известно, из теоремы Нетер для действия скалярного комплексного поля, при глобальных калибровочных преобразованиях, сохраняется ток:

$$j^\mu = -ie(\partial_\mu \varphi^* \varphi - \varphi^* \partial_\mu \varphi)$$

где поле, как известно, имеет вид:

$$\varphi(k) = \frac{c\hbar}{\sqrt{\hbar\omega}} \hat{\mathfrak{q}}(k) = \frac{e}{2\omega} \left\{ e^{\mathrm{i}kx} \hat{\mathfrak{a}}^\dagger(k) + e^{-\mathrm{i}kx} \hat{\mathfrak{a}}(k) \right\}$$

Всё, осталось подставить, и просто посчитать.
В итоге у нас слагаемые преобразуются и

$$Q = \int \mathrm{d}^3 r j^0 = \int \frac{\mathrm{d}^3 k}{(2\pi)^3 2k_0} e \left\{ a^*(\boldsymbol{k}) a(\boldsymbol{k}) - a_c^*(\boldsymbol{k}) a_c(\boldsymbol{k}) \right\}$$

МФТИ-Киселев-2020.5 Energy-momentum tensor for a scalar field

Для решения в виде плоской монохроматической волны для скалярного поля $\phi \mapsto \frac{1}{\sqrt{2k_0}} e^{\mp \mathrm{i}kx}$ proof that компоненты тензора энергии-импульса

$$T_0^0 \mapsto k_0, \quad T_0^\alpha \mapsto \boldsymbol{k}$$

Solution

$$T_\nu^\mu := \frac{\partial L}{\partial \partial_\mu \varphi} \partial_\nu \varphi + \frac{\partial L}{\partial \partial_\mu \varphi^*} \partial_\nu \varphi^* - \delta_\nu^\mu L; \qquad L = \partial_\mu \varphi \partial^\mu \varphi^* - m^2 \varphi \varphi^*$$

$$T_\mu^\nu = \partial^\nu \varphi^* \partial_\mu \varphi + \partial_\mu \varphi^* \partial^\nu \varphi - \delta_\mu^\nu (\partial_\mu \varphi \partial^\mu \varphi^* - m^2 \varphi \varphi^*)$$

$\phi \mapsto \frac{1}{\sqrt{2k_0}} e^{\mp \mathrm{i}kx}$.
(?? I'll check again later)
s

$$T_0^0 = \frac{2}{2k_0} (\pm ik_0)(\mp ik_0) = k_0$$

$$T_0^\alpha = \frac{1}{2k_0} (\pm ik_0^\alpha (-ik_0) + (\pm ik_0)(\mp ik^\alpha) = k^\alpha$$

26.1.5 Problems about free quantum fields 4

МФТИ-Киселев-2020.6 Current for a scalar field

Для решения в виде плоской монохроматической волны для скалярного поля $\phi \mapsto \frac{1}{\sqrt{2k_0}} e^{\mp \mathrm{i}kx}$ Найти, что компоненты тока

$$j_0 \mapsto \pm e, \quad j_\alpha \mapsto \pm e\boldsymbol{k}$$

Solution

$j^\mu = -ie(\partial_\mu \varphi^* \varphi - \varphi^* \partial_\mu \varphi)$
$\phi \mapsto \frac{1}{\sqrt{2k_0}} e^{\mp \mathrm{i} kx}$

$$j_0 = ie\left(\frac{1}{\sqrt{2k_0}}\right)^2 (\pm ik_0 - (\mp ik_0)) = \pm e$$

$$j_\alpha = ie\left(\frac{1}{\sqrt{2k_0}}\right)^2 (\mp ik_\alpha - (\pm ik_0)) = \pm ek_\alpha$$

МФТИ-Киселев-2020.7.^{C} Invariant energy-momentum tensor

Какой вид имеет тензор энергии-импульса релятивистски инвариантного вакуума?

Solution

Этот объект удовлетворяет:
Для баланса энергии-импульса должно выполняться $\partial_\mu T^\mu_\nu = 0$.
симметричность.
во всех системах отсчета он должен иметь один и тот же вид.
А вообще тензор энергии-импульса в теории поля в пространстве Минковского: $T_{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}$ где лагранжиан $\mathcal{L}$ является плотностью функции Лагранжа

$$\frac{1}{c}L = \int \mathrm{d}^3 x \mathcal{L}$$

Тогда для вещественного скалярного поля лагранжиан

$$c\mathcal{L} = \frac{1}{2}\left\{ g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \frac{m^2c^2}{\hbar^2}\varphi^2 \right\}$$

И тензор энергии-импульса

$$cT_{\mu\nu} = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\frac{1}{2}\left\{ g^{\mu'\nu'}\partial_{\mu'}\varphi\partial_{\nu'}\varphi - \frac{m^2c^2}{\hbar^2}\varphi^2 \right\}$$

МФТИ-Мусаев-2023.3. Оператор заряда

Найти оператор заряда для свободного комплексного скалярного поля.

МФТИ-Мусаев-2023.4 Оператор момента

Найти оператор момента $M^{\mu,\rho\sigma}$ для свободного действительного векторного поля. Интерпретировать интеграл от компонент $M^{0,0i}$, где $i = 1, 2, 3$.

МФТИ-Мусаев-2023.5. Симметрии модели трех комплексных полей

Рассмотрим теорию трех комплексных скалярных полей ϕ^a, где $a = 1, 2, 3$, сохраняющиеся заряды которой заданы выражениями

$$Q^i = \int d^3x \left(\phi^{*a}\left(\sigma^i\right)_{ab}\pi^{*b} - \pi^a\left(\sigma^i\right)_{ab}\phi^b\right),$$

где σ^i - три сигма-матрицы Паули, π^a - канонический импульс. Вычислить коммутационные соотношения для зарядов. Какой симметрией обладает лагранжиан с такими зарядами?

МФТИ-Мусаев-2023.6 Калибровочное преобразование

Как выглядит калибровочное преобразование, которое реализует переход от фотонного пропагатора в калибровке Фейнмана $D^F{}_{\text{K}}$ пропагатору в поперечной калибровке Ландау D^L?

а) Чему равна функция $D^F(x-y)-D^L(x-y)$ в координатном представлении?

b) Как меняется оператор вектор-потенциала при смене калибровки $\mathrm{A}^F_\mu(x) \longrightarrow \mathrm{A}^L_\mu(x)$?

26.1.6 Problems about relativity and Poincare group 1

МФТИ-Киселев-2020.1 4-вектор как спинор (?!??!)

Рассмотреть вещественный 4-вектор в представлении группы Лоренца $\left(\frac{1}{2},\frac{1}{2}\right)$

Solution

Рассмотрим произвольную эрмитово сопряженную величину $\hat{V}$ в представлении $\left(\frac{1}{2},\frac{1}{2}\right)$.

$\hat{V}$ несет пару индексов $\{\alpha\dot{\alpha}\}$, может быть разложена по базису матриц σ^ν :

$$(\hat{V})_{\alpha\dot{\alpha}} = \sigma^\nu_{\alpha\dot{\alpha}} V_\nu$$

причем

$$V^\mu = \frac{1}{2}\operatorname{tr}\left\{\bar{\sigma}^\mu \hat{V}\right\} = \frac{1}{2}\operatorname{tr}\left\{\hat{V}\bar{\sigma}^\mu\right\}$$

так как по определению в (1.58) и (1.59), очевидно,

что антисимметричная по пространственным индексам часть произведения матриц не дает вклада в след, ввиду бесследовости обычных 3 -мерных матриц Паули,

$$\operatorname{tr}\{\bar{\sigma}^\mu\sigma^\nu\} = 2g^{\mu\nu}$$

Согласно установленным нами законам преобразования верхних и нижних спинорных индексов,

преобразования группы $\mathrm{SL}(2,\mathbb{C})$ переводят $\hat{V}$ в эрмитову величину

$$\hat{V}' = \Lambda_- \hat{V} \Lambda_-^\dagger$$

которая опять может быть разложена по исходному базису релятивистских матриц Паули:

$$V'^\mu = \frac{1}{2}\operatorname{tr}\left\{\bar{\sigma}^\mu \hat{V}'\right\} = \frac{1}{2}\operatorname{tr}\left\{\hat{V}'\bar{\sigma}^\mu\right\}$$

Заметим, что

$$\det \hat{V}' = \det\left\{\Lambda_- \hat{V} \Lambda_-^\dagger\right\} = |\det \Lambda_-|^2 \det \hat{V} = \det \hat{V}$$

При этом

$$\hat{V} = \begin{pmatrix} V_0+V_3 & V_1-\mathrm{i}V_2 \\ V_1+\mathrm{i}V_2 & V_0-V_3 \end{pmatrix}$$

так что

$$\det \hat{V} = V_0^2 - \boldsymbol{V}^2$$

а равенство детерминантов $\det \hat{V}' = \det \hat{V}$ означает, что преобразование сохраняет лоренц-инвариантную длину 4-вектора,

т. е. представляет собой элемент группы Лоренца на 4-векторах.

Это представление является двузначным, так как матрицы Λ_- и $-\Lambda_-$ приводят к идентичным преобразованиям 4-вектора.

В случае инфинитезимальных преобразований $\omega^{\lambda\rho} \to 0$,

$$\Lambda_- = \mathbb{1} - \frac{\mathrm{i}}{2}\sigma_{\lambda\rho}\omega^{\lambda\rho}, \quad \Lambda_-^\dagger = \mathbb{1} + \frac{\mathrm{i}}{2}\bar{\sigma}_{\lambda\rho}\omega^{\lambda\rho}$$

находим, что

$$V'^{\mu} = V^{\mu} + \frac{\mathrm{i}}{4}\,\mathrm{tr}\left\{\bar{\sigma}_{\lambda\rho}\bar{\sigma}^{\mu}\sigma^{\nu} - \sigma_{\lambda\rho}\sigma^{\nu}\bar{\sigma}^{\mu}\right\}\omega^{\lambda\rho}V_{\nu}$$

где, например, разложение на симметричную и антисимметричную по перестановке индексов части

$$\bar{\sigma}^{\mu}\sigma^{\nu} = g^{\mu\nu} - 2\mathrm{i}\bar{\sigma}^{\mu\nu}$$

приводит к

$$\mathrm{tr}\left\{\bar{\sigma}_{\lambda\rho}\bar{\sigma}^{\mu}\sigma^{\nu}\right\} = -2\mathrm{i}\,\mathrm{tr}\left\{\bar{\sigma}_{\lambda\rho}\bar{\sigma}^{\mu\nu}\right\}$$

и затем, как легко заметить, в силу симметрийных свойств по перестановке пространственных индексов выражение для следа матриц должно иметь определенную тензорную структуру,

$$\mathrm{tr}\left\{\bar{\sigma}_{\lambda\rho}\bar{\sigma}^{\mu\nu}\right\} = A\left\{\delta^{\mu}_{\lambda}\delta^{\nu}_{\rho} - \delta^{\nu}_{\lambda}\delta^{\mu}_{\rho}\right\} + B\hat{\varepsilon}^{\mu\nu}_{\lambda\rho}$$

а коэффициенты A, B можно определить, используя явный вид генераторов при определенном выборе пространственных индексов, так что в итоге полу- чаем

$$\mathrm{tr}\left\{\bar{\sigma}_{\lambda\rho}\bar{\sigma}^{\mu\nu}\right\} = \frac{1}{2}\left\{\delta^{\mu}_{\lambda}\delta^{\nu}_{\rho} - \delta^{\nu}_{\lambda}\delta^{\mu}_{\rho}\right\} - \frac{\mathrm{i}}{2}\hat{\varepsilon}^{\mu\nu}_{\lambda\rho}$$

Совершенно аналогично

$$\mathrm{tr}\left\{\sigma_{\lambda\rho}\sigma^{\mu\nu}\right\} = \frac{1}{2}\left\{\delta^{\mu}_{\lambda}\delta^{\nu}_{\rho} - \delta^{\nu}_{\lambda}\delta^{\mu}_{\rho}\right\} + \frac{\mathrm{i}}{2}\hat{\varepsilon}^{\mu\nu}_{\lambda\rho}$$

В итоге приведение подобных членов дает

$$V'^{\mu} = V^{\mu} + \omega^{\mu\nu}V_{\nu}$$

т. е. инфинитезимальное преобразование 4-вектора.

МФТИ-Киселев-2020.**2**

Доказать равенства

$$(\sigma^{\mu}\bar{\sigma}^{\nu} + \sigma^{\nu}\bar{\sigma}^{\mu})^{\alpha}_{\beta} = 2g^{\mu\nu}\delta^{\alpha}_{\beta}$$
$$(\bar{\sigma}^{\mu}\sigma^{\nu} + \bar{\sigma}^{\nu}\sigma^{\mu})^{\dot{\alpha}}_{\dot{\beta}} = 2g^{\mu\nu}\delta^{\dot{\alpha}}_{\dot{\beta}}$$

Solution

По определению $\sigma^{\mu} = (1, \sigma^{i}), \bar{\sigma}^{\mu} = (1, -\sigma^{i})$.
Посчитаем отдельно

$$\left(\sigma^{0}\bar{\sigma}^{i} + \sigma^{i}\bar{\sigma}^{0}\right) = (-\sigma^{i} + \sigma^{i}) = 0$$
$$\left(\sigma^{i}\bar{\sigma}^{j} + \sigma^{j}\bar{\sigma}^{i}\right) = -(\sigma^{i}\sigma^{j} + \sigma^{j}\sigma^{i}) = -2\delta^{ij}$$

Поэтому

$$(\sigma^{\mu}\bar{\sigma}^{\nu} + \sigma^{\nu}\bar{\sigma}^{\mu})^{\alpha}_{\beta} = 2g^{\mu\nu}\delta^{\alpha}_{\beta}$$

Аналогично доказывается второе.

МФТИ-Киселев-2020.**3**

Доказать равенства

$$\mathrm{tr}\left\{\bar{\sigma}_{\lambda\rho}\bar{\sigma}^{\mu\nu}\right\} = \tfrac{1}{2}\left\{\delta^{\mu}_{\lambda}\delta^{\nu}_{\rho} - \delta^{\nu}_{\lambda}\delta^{\mu}_{\rho}\right\} - \tfrac{\mathrm{i}}{2}\hat{\epsilon}^{\mu\nu}_{\lambda\rho}$$
$$\mathrm{tr}\left\{\sigma_{\lambda\rho}\sigma^{\mu\nu}\right\} = \tfrac{1}{2}\left\{\delta^{\mu}_{\lambda}\delta^{\nu}_{\rho} - \delta^{\nu}_{\lambda}\delta^{\mu}_{\rho}\right\} + \tfrac{\mathrm{i}}{2}\hat{\epsilon}^{\mu\nu}_{\lambda\rho}$$

Solution

По определению

$$\sigma^{\mu\nu} = -\sigma^{\nu\mu} = \frac{i}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$$
$$\bar{\sigma}^{\mu\nu} = -\bar{\sigma}^{\nu\mu} = \frac{i}{4}(\bar{\sigma^\mu}\sigma^\nu - \bar{\sigma^\nu}\sigma^\mu)$$

Посмотрим, как можно их расписать через компоненты.

$$\sigma^{00} = 0$$
$$\sigma^{0i} = -\sigma^{\nu\mu} = \frac{i}{4}(\sigma^0(-\sigma^i) - \sigma^i\sigma^0) = -\frac{i}{2}\sigma^i$$
$$\bar{\sigma^{0i}} = \frac{i}{2}\sigma^i$$
$$\bar{\sigma}^{ij} = \frac{i}{4}(-\sigma^i\sigma^j - (-\sigma^j)\sigma^i) = -\frac{i}{4}\cdot 2i\varepsilon_{ijk}\sigma^k = \frac{1}{2}\varepsilon_{ijk}\sigma^k$$
$$\bar{\sigma}^{ij} = \frac{1}{2}\varepsilon_{ijk}\sigma^k$$

В последних равенствах использовалось соотношение на матрицы Паули:

$$\sigma^i\sigma^j = i\varepsilon_{ijk}\sigma^k + \delta_{ij}\sigma^0$$

Таким образом, исходное уравнение для пространственных индексов $\operatorname{tr}\{\sigma_{\lambda\rho}\sigma^{\mu\nu}\}$ можно переписать как:

$$\operatorname{tr}\left\{\sigma_{ij}\sigma^{kl}\right\} = \operatorname{tr}\left\{\frac{1}{2}\varepsilon_{ijm}\sigma^m\frac{1}{2}\varepsilon_{kln}\sigma^n\right\} = \frac{1}{4}\varepsilon_{ijm}\varepsilon_{kln}\operatorname{tr}\{\sigma_m\sigma^n\} = \frac{1}{4}\varepsilon_{ijm}\varepsilon_{kln}\operatorname{tr}\{2\delta^{mn}\}$$

И дальше просто преобразуем до конца:

$$\operatorname{tr}\left\{\sigma_{ij}\sigma^{kl}\right\} = \frac{1}{2}\varepsilon_{ijm}\varepsilon_{kln} = 1\frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$$

А если есть временной индекс, то

$$\operatorname{tr}\left\{\sigma_{ij}\sigma^{k0}\right\} = \operatorname{tr}\left\{\frac{1}{2}\varepsilon_{ijk}\sigma^n\frac{i}{2}\sigma^k\right\} = \frac{i}{2}\varepsilon_{ijk} = -\frac{i}{2}\varepsilon_{ijk0} = \frac{i}{2}\varepsilon_{ij}^{k0}$$

В последнем переходе показано как от трехмерного символа Леви-Чевиты перейти к четырехмерному. Также при подъеме пространственной части метрика домножилась на (-1), а при подъеме временной - на 1.

Осталось разобрать случай

$$\operatorname{tr}\left\{\sigma_{i0}\sigma^{k0}\right\} = -\left(\frac{i}{2}\right)^2\operatorname{tr}\left\{\sigma^i\sigma^k\right\} = \frac{1}{2}\delta_{ik} \equiv \frac{1}{2}[\delta_i^k\delta_0^0 - \delta_i^0\delta_0^k]$$

Собирая все вместе, получаем

$$\operatorname{tr}\{\sigma_{\lambda\rho}\sigma^{\mu\nu}\} = \frac{1}{2}\left\{\delta_\lambda^\mu\delta_\rho^\nu - \delta_\lambda^\nu\delta_\rho^\mu\right\} + \frac{\mathrm{i}}{2}\hat{\epsilon}_{\lambda\rho}^{\mu\nu}$$

Теперь то же самое для $\operatorname{tr}\{\bar{\sigma}_{\lambda\rho}\bar{\sigma}^{\mu\nu}\}$
действуя аналогично, получаем:

$$\operatorname{tr}\{\sigma_{\lambda\rho}\sigma^{\mu\nu}\} = \frac{1}{2}\left\{\delta_\lambda^\mu\delta_\rho^\nu - \delta_\lambda^\nu\delta_\rho^\mu\right\} + \frac{\mathrm{i}}{2}\hat{\epsilon}_{\lambda\rho}^{\mu\nu}$$

МФТИ-Киселев-2020.**4**

Показать, что величины

$$\theta\sigma^\mu\bar{\chi} = \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \quad \text{и} \quad \bar{\theta}\bar{\sigma}^\mu\chi = \bar{\theta}_{\dot{\alpha}}\left(\bar{\sigma}^\mu\right)^{\dot{\alpha}\alpha}\chi_\alpha$$

ведут себя так же, как 4-векторы.

Solution

То есть нужно доказать, что
В самом деле, воспользуемся стандартным способом: построим, например, величину

$$V_\mu \theta \sigma^\mu \bar{\chi} = V_\mu \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}$$

и покажем, что она является инвариантом.
Действительно, преобразование из группы $SL(2,\mathbb{C})$ дает

$$\theta^\alpha \mapsto \theta^{\alpha_1} \left(\Lambda^{-1}\right)^\alpha_{\alpha_1}, \quad \bar{\chi}^{\dot{\alpha}} \mapsto \bar{\chi}^{\dot{\alpha}_1} \left(\Lambda_+\right)^{\dot{\alpha}}_{\dot{\alpha}_1}, \quad V_\mu \mapsto V_\nu \left(\Lambda^{-1}\right)^\nu_\mu$$

где Λ^ν_μ - матрица преобразований координат, так что ковектор преобразуется обратной матрицей, а также, как мы показали,

$$\hat{V}_{\alpha\dot{\alpha}} = V_\mu \left(\sigma^\mu\right)_{\alpha\dot{\alpha}} \mapsto \left(\Lambda_-\right)^{\alpha_1}_\alpha \hat{V}_{\alpha_1\dot{\alpha}_1} \left(\Lambda^{-1}_+\right)^{\dot{\alpha}_1}_{\dot{\alpha}}$$

Поэтому величина в явном виде

$$V_\mu \theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \bar{\chi}^{\dot{\alpha}}$$

в явном виде переходит сама в себя, а значит, является инвариантом.
С другой стороны, этот инвариант - скалярное линейное отображение 4-ковектора на числа, и, стало быть, по определению оно представляет собой 4-вектор, что и доказывает наше утверждение о характере преобразований билинейной по киральным полям форме $\theta\sigma^\mu\bar{\chi}$.

МФТИ-Киселев-2020.5

Доказать, что $(\theta_\alpha)^\dagger = \bar{\theta}_{\dot{\alpha}}$ и $\left(\bar{\chi}^{\dot{\alpha}}\right)^\dagger = \chi^\alpha$.

Solution

Напомним, что по определению $(\theta_\alpha)^\dagger = (\theta_{\dot{\alpha}})^*$,
то есть мы совершаем комплексное сопряжение над спинором, а также заменяем точечный индекс на неточечный. (????)
Совершим преобразования поднятия индексов и перехода из сопряженного спинора к обычному над $\bar{\theta}_{\dot{\alpha}}$:

$$\bar{\theta}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\theta}^{\dot{\beta}} = \varepsilon_{\dot{\alpha}\dot{\beta}} [i\sigma_2^{\alpha\dot{\beta}} \theta_\alpha]^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \theta^*_\alpha = \delta^\alpha_{\dot{\alpha}} \theta^*_\alpha = (\theta_\alpha)^\dagger$$

Однако то, что компоненты равны не значит, что это один и тот же объект, потому что они могут преобразовываться по-разному. Поэтому проверим, что они преобразуются одинаково:

$$(\theta')^\dagger_\alpha = ((\Lambda_-)^\beta_\alpha \theta_\beta)^\dagger = \exp\left(\frac{i}{2}\sigma^{\mu\nu}\omega_{\mu\nu}\right)^\beta_\alpha \theta^\dagger_\beta$$

а кстати легко показать, записывая в явном виде, что $(\sigma^{\mu\nu})^\dagger = \bar{\sigma}^{\mu\nu}$, поэтому

$$\bar{\theta}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\theta}^{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}} (\Lambda_+)^{\dot{\beta}}_{\dot{\gamma}} \bar{\theta}^{\dot{\gamma}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \exp\left(\frac{i}{2}\sigma^{\mu\nu}\omega_{\mu\nu}\right)^{\dot{\beta}}_{\dot{\alpha}} \bar{\theta}_{\dot{\beta}}$$

В итоге $(\theta_\alpha)^\dagger$ и $\bar{\theta}_{\dot{\alpha}}$ преобразуются одинаково, так что это один и тот же объект.
Аналогично доказывается, что $\left(\bar{\chi}^{\dot{\alpha}}\right)^\dagger = \chi^\alpha$

МФТИ-Киселев-2020.6*

Покажите, что представления группы Лоренца со спином $s = 1$: $(1,0)$ и $(0,1)$ отвечают самодуальным и антисамодуальным тензорным полям второго ранга в пространстве-времени Минковского, т.е. при определении поля, дуального к $B_{\mu\nu}$, как

$$\tilde{B}^{\mu\nu} = \frac{1}{2}\hat{\epsilon}^{\mu\nu\mu'\nu'} B_{\mu'\nu'}$$

имеют место соотношения самодуальности и антисамодуальности в пространстве-времени Минковского:

$$\tilde{B}^{\mu\nu} = \pm \mathrm{i} B^{\mu\nu}$$

[Hint: При выводе учесть, что представления (1,0) и $(0,1)$ - это бесследовые матрицы в индексах с точками и без точек.]

Solution

МФТИ-Киселев-2020.**7**

Доказать, что квадрат псевдовектора Паули-Любанского имеет вид

$$W^2 = -\frac{1}{2}\left\{p^2 S^2 - 2p_\nu p^\mu S_{\mu\lambda} S^{\nu\lambda}\right\}$$

Solution

Просто запишем W^2 квадрат в тензорном виде, вспомнив, что произведение двух символов Леви-Чивиты запишутся через определитель матрицы ниже:

$$W^2 = -\frac{1}{4}\begin{vmatrix} \delta^\nu_{\nu'} & \delta^\nu_{\lambda'} & \delta^\nu_{\rho'} \\ \delta^\lambda_{\nu'} & \delta^\lambda_{\lambda'} & \delta^\lambda_{\rho'} \\ \delta^\rho_{\nu'} & \delta^\rho_{\lambda'} & \delta^\rho_{\rho'} \end{vmatrix} p_\nu p^{\nu'} S_{\lambda\rho} S^{\lambda'\rho'}$$

что можно представить в виде

$$W^2 = -\frac{1}{4}\left\{p_\nu p^\nu S_{\lambda\rho} S^{\lambda\rho} + p_\nu p^\rho S_{\lambda\rho} S^{\nu\lambda} + p_\nu p^\lambda S_{\lambda\rho} S^{\rho\nu} - -p_\nu p^\rho S_{\lambda\rho} S^{\lambda\nu} - p_\nu p^\lambda S_{\lambda\rho} S^{\nu\rho} - p_\nu p^\nu S_{\lambda\rho} S^{\rho\lambda}\right\}$$

и после использования антисимметрии тензора спина получаем

$$W^2 = -\frac{1}{2}\left\{p^2 S^2 - 2p_\nu p^\lambda S_{\lambda\rho} S^{\nu\rho}\right\}$$

МФТИ-Киселев-2020.**1**.^{C}

Доказать, что компоненты псевдовектора Паули–Любанского для безмассовых полей равны

$$W_0 = \hbar \boldsymbol{p}\cdot\boldsymbol{s}, \quad W^\alpha = \hbar\left\{p_0 \boldsymbol{s}^\alpha \mp \mathrm{i}(\boldsymbol{p}\times\boldsymbol{s})^\alpha\right\}$$

Solution

В искомом базисе поля удовлетворяют уравнению на собственные значения оператора W^2. Но у повышающего оператора в группе SU(2) существуют только два собственных вектора: во-первых, это «вакуум» с нулевым моментом

$$\mathcal{J}^\pm|0\rangle = 0$$

а во-вторых, это старший вектор с нулевым собственным значением

$$\mathcal{J}^+_+ |\lambda_+, \lambda_+\rangle = 0$$

Точно так же для понижающего оператора есть два собственных вектора: вакуум (2.8) и младший вектор с нулевым собственным значением

$$\mathcal{J}^-_- |\lambda_-, -\lambda_-\rangle = 0$$

При этом, конечно, на этих полях (т. е. при действии операторов на поля) проекция спина на ось импульса или, как говорят, спиральность имеет значения

$$\frac{p\cdot\mathcal{J}^\pm}{p_0} = \mathcal{J}^\pm_3 = \pm\lambda_\pm$$

причем на физических полях $W^2 = 0$.

Таким образом, среди безмассовых полей со спином базис составляют так называемые киральные поля полуцелого спина и поляризованные поля целого спина: правые поля с положительной киральностью и спиральностью $\mathfrak{s} = \lambda_+$:

$$\mathcal{J}^- \equiv 0 \to \mathcal{K} = -\mathrm{i}s, \quad \mathcal{J}^+ = s$$

левые поля с отрицательной киральностью и спиральностью $\mathfrak{s} = -\lambda_-$:

$$\mathcal{J}^+ \equiv 0 \to \mathcal{K} = \mathrm{i}s, \quad \mathcal{J}^- = s$$

а также их произведения и вакуум. При этом произведения полей, конечно, могут оказаться приводимыми представлениями.

Рассмотрим компоненты псевдовектора Паули- -Любанского для киральных и поляризованных полей в представлениях, отвечающим полям группы Лоренца $(\lambda_+, 0)$ и $(0, \lambda_-)$, в которых генераторы $\mathcal{J}^{(\pm)}$, отличные от нуля конечно, совпадают со спином s. С учетом

$$S_{\beta\gamma} = \hbar\varepsilon_{\beta\gamma\rho}s^\rho$$

нулевая компонента

$$W_0 = -\frac{1}{2}\hat{\varepsilon}^{0\alpha\beta\gamma}p_\alpha S_{\beta\gamma} = \hbar p\cdot s = \pm p_0\hbar\lambda_\pm$$

Половина задачи решена!

При вычислении пространственной компоненты необходимо использовать то, Что

$$\frac{1}{\hbar}S_{0\gamma} = \mathcal{K}^\gamma = \mp\mathrm{i}s^\gamma$$

откуда

$$W^\alpha = -\frac{1}{2}\hat{\varepsilon}^{\alpha 0\beta\gamma}p_0 S_{\beta\gamma} - \frac{1}{2}2\hat{\varepsilon}^{\alpha\beta 0\gamma}p_\beta S_{0\gamma} = \hbar\left\{p_0 s^\alpha \mp \mathrm{i}(p\times s)^\alpha\right\}$$

Вот и вся задача решена.

26.1.7 Problems about relativity and Poincare group 2

МФТИ-Киселев-2020.**2**.^{C}

Доказать, что квадрат псевдовектора Паули-Любанского для безмассовых полей имеет вид

$$W^2 = -4p_0^2\hbar^2\left\{\partial^+\cdot\partial^- - \frac{1}{p_0^2}\left(\boldsymbol{p}\cdot\boldsymbol{\jmath}^+\right)\left(\boldsymbol{p}\cdot\boldsymbol{\jmath}^-\right) - \frac{\mathrm{i}}{p_0}\boldsymbol{p}\cdot\left(\boldsymbol{J}^+\times\boldsymbol{\jmath}^-\right)\right\}$$

Solution

Псевдовектор Паули-Любанского определяется как

$$W^m = -\frac{1}{2}\hat{\epsilon}^{mnkl}p_n S_{kl}$$

При $p^2 = 0$

$$\begin{aligned}W^2 &= p_n p^k S_{kl}S^{nl} = p_n p^k S_{k0}S^{n0} + p_n p^k S_{k\alpha}S^{n\alpha}\\ &= -p^\alpha p^\beta S_{0\alpha}S^{0\beta} + p_0^2 S^{0\alpha}S_{0\alpha} - p^\gamma p^\beta S^{\gamma\alpha}S_{\beta\alpha} + p_0 p^\beta\left(S_{\beta\alpha}S^{0\alpha} - S_{0\alpha}S^{\beta\alpha}\right)\\ &= \hbar^2\left\{(\boldsymbol{p}\cdot\mathcal{K})^2 - p_0^2\mathcal{K}^2 - (\boldsymbol{p}\times s)^2 + p_0\boldsymbol{p}\cdot\left\{(\boldsymbol{s}\times\mathcal{K}) - (\mathcal{K}\times\boldsymbol{s})\right\}\right\}\end{aligned}$$

Раскрывая квадрат векторного произведения $(p\times s)^2 = p^2s^2 - (p\cdot s)^2$, находим

$$\frac{1}{\hbar^2}W^2 = -p_0^2\left(s^2 + \mathcal{K}^2\right) + (\boldsymbol{p}\cdot\boldsymbol{s})^2 + (\boldsymbol{p}\cdot\boldsymbol{\mathcal{K}})^2 + p_0\boldsymbol{p}\cdot\left\{(\boldsymbol{s}\times\mathcal{K}) - (\mathcal{K}\times\boldsymbol{s})\right\}$$

Выражая генераторы спина и бустов через эрмитовы векторы $\mathcal{J}^+$ и $\mathcal{J}^-$, $s = \mathcal{J}^+ + \mathcal{J}^-$, $\quad \mathrm{i}\mathcal{K} = \mathcal{J}^+ - \mathcal{J}^-$, получаем

$$W^2 = -4p_0^2\hbar^2\left\{\boldsymbol{\jmath}^+\cdot\boldsymbol{\jmath}^- - \frac{1}{p_0^2}\left(\boldsymbol{p}\cdot\boldsymbol{\jmath}^+\right)\left(\boldsymbol{p}\cdot\boldsymbol{\jmath}^-\right) - \frac{\mathrm{i}}{p_0}\boldsymbol{p}\cdot\left(\boldsymbol{\jmath}^+\times\boldsymbol{\jmath}^-\right)\right\}$$

задача решена.

Упражнения к лекции 3. Лоренц инвариантность

(Параграф 2 в книжке Средницкого)
Итак преобразование Лоренца это линейная однородная замена координат

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} \tag{0.1}$$

которая сохраняет интервал $x^2 = x^{\mu} x_{\mu} = \eta_{\mu\nu} x^{\mu} x^{\nu}$, где $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Это означает, что

$$\eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}_{\sigma} = \eta_{\rho\sigma}. \tag{0.2}$$

Заметим, преобразования Лоренца включают в себя обычные пространственные вращения: для это возьмем $\Lambda^0_0 = 1, \Lambda^0{}_i = \Lambda^i{}_0 = 0$, и $\Lambda^i{}_j = R_{ij}$, где R_{ij} ортогональная матрица поворотов трехмерного пространства.
Для инфинитезимальных преобразования Лоренца мы можем записать

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}, \tag{0.3}$$

где

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \tag{0.4}$$

То есть есть 6 независимых инфинитезимальных преобразований Лоренца. Это три поворота $\omega_{ij} = -\epsilon_{ijk} \hat{n}_k \delta\theta$ на угол $\delta\theta$ относительно оси $\hat{n}$ и три буста $\omega_{i0} = \hat{n}_i \delta\eta$ в направлении $\hat{n}$ с быстротой $\delta\eta$.
Упражнение 1. Проверьте, что (0.4) следует из (0.2).
Решение: Подставляем (0.3) в (0.2) и получаем

$$\eta_{\mu\nu} \left(\delta^{\mu}{}_{\rho} + \omega^{\mu}{}_{\rho}\right) \left(\delta^{\nu}{}_{\sigma} + \omega^{\nu}{}_{\sigma}\right) = \eta_{\rho\sigma}, \tag{0.5}$$

откуда

$$\eta_{\sigma\rho} + \omega_{\sigma\rho} + \omega_{\rho\sigma} + O\left(\omega^2\right) = \eta_{\sigma\rho} \tag{0.6}$$

и находим $\omega_{\sigma\rho} = -\omega_{\rho\sigma}$. $\bigcirc$
В квантовой теории поля симметрии представляются унитарными (или антиунитарными) операторами. Это значит, что мы сопоставляем оператор $U(\Lambda)$ каждому правильному ортохроному преобразованию Лоренца $\Lambda \left(\Lambda^0_0 \geqslant 1, \det \Lambda = 1\right)$. Эти операторы должны удовлетворять свойству

$$U\left(\Lambda' \Lambda\right) = U\left(\Lambda'\right) U(\Lambda) \tag{0.7}$$

Для инфинитезимальных преобразований Лоренца мы можем записать

$$U(1 + \omega) = I + \frac{i}{2} \omega_{\mu\nu} M^{\mu\nu} \tag{0.8}$$

где $M^{\mu\nu} = -M^{\nu\mu}$ набор эрмитовых операторов, которые называются генераторами группы Лоренца. Теперь рассмотрим $U(\Lambda)^{-1} U\left(\Lambda'\right) U(\Lambda) = U\left(\Lambda^{-1} \Lambda' \Lambda\right)$, и пусть $\Lambda' = 1 + \omega'$, и разложим обе части этого равенства до линейных по ω членов. Мы получим

$$\omega_{\mu\nu} U(\Lambda)^{-1} M^{\mu\nu} U(\Lambda) = \omega_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} M^{\rho\sigma}. \tag{0.9}$$

Далее, так как $\omega_{\mu\nu}$ произвольно, то анитисимметричная часть ее сомножителей на каждой стороне уравнения должна быть одинаковой. В данном случае, так как $M^{\mu\nu}$ уже само антисимметрично мы получаем

$$U(\Lambda)^{-1} M^{\mu\nu} U(\Lambda) = \Lambda^{\mu}_{\rho} \Lambda^{\nu}_{\sigma} M^{\rho\sigma}. \tag{0.10}$$

Упражнение 2. Проверьте, что (0.10) следует из $U(\Lambda)^{-1} U\left(\Lambda'\right) U(\Lambda) = U\left(\Lambda^{-1} \Lambda' \Lambda\right)$.
Решение: Имеем

1. $U(\Lambda)^{-1} U\left(1 + \omega'\right) U(\Lambda) = U(\Lambda)^{-1} \left(I + \frac{i}{2} \omega_{\mu\nu} M^{\mu\nu}\right) U(\Lambda) = I + \frac{i}{2} \omega_{\mu\nu} U(\Lambda)^{-1} M^{\mu\nu} U(\Lambda)$
2. $U\left(\Lambda^{-1}(1 + \omega)\Lambda\right) = U\left(\delta^{\rho}{}_{\sigma} + \left(\Lambda^{-1}\right)^{\rho}{}_{\mu} \omega^{\mu}{}_{\nu} \Lambda^{\nu}{}_{\sigma}\right) = U\left(\delta^{\rho}{}_{\sigma} + \omega^{\mu}{}_{\nu} \Lambda_{\mu}{}^{\rho} \Lambda^{\nu}{}_{\sigma}\right) = U\left(\delta^{\rho}{}_{\sigma} + \omega_{\mu\nu} \Lambda^{\mu\rho} \Lambda^{\nu}{}_{\sigma}\right) =$

$$= I + \frac{i}{2} \omega_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} M^{\rho\sigma}, \tag{0.11}$$

откуда получаем сначала (0.9), а потом уже (0.10).

То есть мы видим, что каждый векторный индекс у $M^{\mu\nu}$ преобразуется по его собственному преобразованию Лоренца. Это общий результат: любой оператор который несет один или более векторных индексов должен преобразовываться подобным образом. Например, рассмотрим оператор момента $P^\mu = \int d^3x T^{0\mu}$, где $P^0 = H$ - гамильтониан, а P^i компоненты оператора полного импульса. Мы ожидаем, что

$$U(\Lambda)^{-1} P^\mu U(\Lambda) = \Lambda^\mu{}_\nu P^\nu. \tag{0.12}$$

Теперь пусть $\Lambda = 1 + \omega$ в уравнении (0.10), тогда раскладывая его до линейных по ω членов мы получим коммутационное соотношения на операторы $M^{\mu\nu}$:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(\eta^{\mu\sigma} M^{\nu\rho} - \eta^{\mu\rho} M^{\nu\sigma} + \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\nu\sigma} M^{\mu\rho}\right) \tag{0.13}$$

Эти коммутационные соотношения определяют алгебру Ли группы Лоренца.
Упражнение 3. Проверьте, что (0.13) следует из (0.10).
Решение: Имеем

$U(1+\omega)^{-1} M^{\mu\nu} U(1+\omega) = \left(I - \frac{i}{2}\omega_{\sigma\rho} M^{\sigma\rho}\right) M^{\mu\nu} \left(I + \frac{i}{2}\omega_{\lambda\delta} M^{\lambda\delta}\right) = M^{\mu\nu} + \frac{i}{2}\omega_{\sigma\rho}\left[M^{\mu\nu}, M^{\sigma\rho}\right] + O\left(\omega^2\right)$,
$\left(\delta^\mu{}_\rho + \omega^\mu{}_\rho\right)\left(\delta^\nu{}_\sigma + \omega^\nu{}_\sigma\right) M^{\rho\sigma} = M^{\mu\nu} + \omega^\mu{}_\rho M^{\rho\nu} + \omega^\nu{}_\sigma M^{\mu\sigma} + O\left(\omega^2\right) = M^{\mu\nu} + \omega_{\sigma\rho}\eta^{\mu\sigma} M^{\rho\nu} - \omega_{\sigma\rho}\eta^{\nu\rho} M^{\mu\sigma}$.
После антисимметризации части 2. получаем (0.13).
Введем компоненты оператора момента импульса $\boldsymbol{J}$ по формуле $J_i = \frac{1}{2}\epsilon_{ijk} M^{jk}$ и компоненты оператора буста $\boldsymbol{K}$ по формуле $K_i = M^{i0}$. Тогда мы можем найти используя (0.13):

$$\begin{aligned} &[J_i, J_j] = i\epsilon_{ijk} J_k \\ &[J_i, K_j] = i\epsilon_{ijk} K_k \\ &[K_i, K_j] = -i\epsilon_{ijk} J_k \end{aligned} \tag{0.15}$$

Упражнение 4. Выведите уравнения (0.15).
Решение: Покажем, что $[J_i, J_j] = i\epsilon_{ijk} J_k$ $\left(\epsilon_{123} = -\epsilon^{123} = 1\right)$. Имеем

$$\begin{aligned} [J_i, J_j] &= \frac{1}{4}\epsilon_{imn}\epsilon_{jst}\left[M^{mn}, M^{st}\right] = \frac{i}{4}\epsilon_{imn}\epsilon_{jst}\left(\eta^{mt} M^{ns} - \eta^{ms} M^{nt} + \eta^{ns} M^{mt} - \eta^{nt} M^{ms}\right) = \\ &= -\frac{i}{4}\epsilon_{imn}\epsilon_{jst}\left(\delta^{mt} M^{ns} - \delta^{ms} M^{nt} + \delta^{ns} M^{mt} - \delta^{nt} M^{ms}\right) \end{aligned} \tag{0.16}$$

Далее заметим, что $M^{ij} = -\epsilon^{ijk} J_k$ (мы поднимаем и опускаем индексы с помощью $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, поэтому так как здесь мы работаем с индексами $i = 1, 2, 3$, с каждым поднятием индекса мы зарабатываем фактор)-1)), откуда получаем

$$[J_i, J_j] = \frac{i}{4}\left(\epsilon_{imn}\epsilon_{jsm}\epsilon^{nsk} J_k - \epsilon_{imn}\epsilon_{jmt}\epsilon^{ntk} J_k + \epsilon_{imn}\epsilon_{jnt}\epsilon^{mtk} J_k - \epsilon_{imn}\epsilon_{jsn}\epsilon^{msk} J_k\right) \tag{0.17}$$

Далее используя, что $\epsilon_{ijs}\epsilon_{mns} = \delta_{im}\delta_{jn} - \delta_{im}\delta_{jm}$, находим

$$\begin{aligned} &[J_i, J_j] = \\ &= \frac{i}{4}\left(\left(\delta_{is}\delta_{nj} - \delta_{ij}\delta_{ns}\right)\epsilon^{nsk} J_k - \left(\delta_{ij}\delta_{nt} - \delta_{it}\delta_{nj}\right)\epsilon^{ntk} J_k + \left(\delta_{it}\delta_{mj} - \delta_{ij}\delta_{mt}\right)\epsilon^{mtk} J_k - \left(\delta_{ij}\delta_{ms} - \delta_{is}\delta_{mj}\right)\epsilon^{msk} J_k\right) = \\ &= \frac{i}{4}\left(\epsilon^{jik} J_k + \epsilon^{jik} J_k + \epsilon^{jik} J_k + \epsilon^{jik} J_k\right) = -i\epsilon^{ijk} J_k = i\epsilon_{ijk} J_k. \end{aligned} \tag{0.18}$$

В итоге мы получили, что $[J_i, J_j] = i\epsilon_{ijk} J_k$.
Далее из уравнения (0.12), мы можем получить при $\Lambda = 1 + \omega$:

$$[P^\mu, M^{\rho\sigma}] = i\left(\eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho\right) \tag{0.19}$$

Упражнение 5. Выведите (0.19) из (0.12).
Решение: Имеем

$$1.\ \left(I-\frac{i}{2}\omega_{\sigma\rho}M^{\sigma\rho}\right)P^{\mu}\left(I+\frac{i}{2}\omega_{\sigma\rho}M^{\sigma\rho}\right)=P^{\mu}+\frac{i}{2}\omega_{\sigma\rho}\left[P^{\mu},M^{\sigma\rho}\right]+O\left(\omega^{2}\right)$$

$$2.\ \left(\delta^{\mu}{}_{\nu}+\omega^{\mu}{}_{\nu}\right)P^{\nu}=P^{\mu}+\omega^{\mu}{}_{\nu}P^{\nu}=P^{\mu}+\omega_{\sigma\rho}\eta^{\mu\sigma}P^{\rho}=P^{\mu}+\frac{1}{2}\omega_{\sigma\rho}\left(\eta^{\mu\sigma}P^{\rho}-\eta^{\mu\rho}P^{\sigma}\right) \tag{0.20}$$

Сравнивая выражения 1. и 2. получаем (0.19).
Из формулы (0.19) можно получить

$$\begin{aligned}&[J_i,H]=0\\&[J_i,P_j]=i\epsilon_{ijk}P_k\\&[K_i,H]=iP_i\\&[K_i,P_j]=i\delta_{ij}H\end{aligned} \tag{0.21}$$

Упражнение 6. Выведите уравнения (0.21) из (0.19).
Решение: Получим, что $[K_i,H]=iP_i$. Имеем:

$$[K_i,H]=\left[M^{i0},P^{0}\right]=-i\left(\eta^{0i}P^{0}-\eta^{00}P^{i}\right)=iP^{i} \tag{0.22}$$

Остальные выражения получаются аналогично.
Также компоненты P^{μ} должны коммутировать с друг другом

$$\begin{aligned}&[P_i,P_i]=0\\&[P_i,H]=0\end{aligned} \tag{0.23}$$

Уравнения (0.15), (0.21), и (0.23) образуют алгебру Ли группы Пуанкаре.
Упражнение 7. Покажите, что мера

$$d\mu(p)=\frac{d^{3}\vec{p}}{(2\pi)^{3}2E_{\vec{p}}},\quad E_{\vec{p}}=\sqrt{\vec{p}^{2}+m^{2}} \tag{0.24}$$

Лоренц инвариантна
Решение:

$$d\mu(p)=\frac{d^{3}p}{(2\pi)^{3}2E_{p}}=\int\limits_{p_0>0}dp^{0}\delta\left(\left(p^{0}\right)^{2}-E_{p}^{2}\right)\frac{d^{3}\vec{p}}{(2\pi)^{3}}=\int\limits_{p_0>0}\frac{d^{4}p}{(2\pi)^{3}}\delta\left(p^{2}-m^{2}\right) \tag{0.25}$$

далее очевидно, что если мы перейдем к $p^{\mu}=\Lambda^{\mu}{}_{\nu}p^{\nu}$, то $p^{2}=p^{2}$ и $d^{4}p=d^{4}p'$ и соответственно $d\mu(p)=d\mu\left(p'\right)$.

Stanford.Zhou.2022.2.1. Lorentz transformations. (10/45 p)

Lorentz transformations $x^{\mu}\rightarrow x^{\mu}=\Lambda^{\mu}{}_{\nu}x^{\nu}$ are linear transformations that leave inner products invariant, meaning that $x^{\mu}y_{\mu}=x'^{\mu}y'_{\mu}$ for any four-vectors x and y.

a) Show that this implies

$$\eta_{\mu\nu}=\eta_{\rho\sigma}\Lambda^{\rho}_{\mu}\Lambda^{\sigma}_{\nu}$$

Solution:

Since the inner product stays the same, we have

$$\eta_{\mu\nu}x^{\mu}y^{\nu}=\eta_{\rho\sigma}x'^{\rho}y'^{\sigma}=\eta_{\rho\sigma}\Lambda^{\rho}_{\mu}\Lambda^{\sigma}_{\nu}x^{\mu}y^{\nu}$$

In other words, we have

$$\left(\eta_{\mu\nu}-\eta_{\rho\sigma}\Lambda^{\rho}_{\mu}\Lambda^{\sigma}_{\nu}\right)x^{\mu}y^{\nu}=0$$

for any four-vectors x and y, which implies the quantity in parentheses vanishes. (Concretely, we could take $x^\mu = \delta_0^\mu$ and $y^\nu = \delta_0^\nu$ to show the $\mu = \nu = 0$ component vanishes, and so on for all 16 pairs of values.)

b) All proper, orthochronous Lorentz transformations (i.e. all those which preserve the orientation of space and the direction of time) can be decomposed into infinitesimal Lorentz transformations. These take the form

$$x^\mu \to x'^\mu = x^\mu + \epsilon \omega^\mu{}_\nu x^\nu$$

where ϵ is infinitesimal. Show that $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

Solution:

Plugging $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon\omega^\mu{}_\nu$ into Eq. (1) yields

$$\eta_{\mu\nu} = \eta_{\rho\sigma}\left(\delta^\rho_\mu\delta^\sigma_\nu + \epsilon\omega^\rho_\mu\delta^\sigma_\nu + \epsilon\delta^\rho_\mu\omega^\sigma_\nu\right)$$

where we dropped a term of order ϵ^2. Contracting the indices on the right-hand side,

$$\eta_{\mu\nu} = \eta_{\mu\nu} + \epsilon\omega^\rho_\mu\eta_{\rho\nu} + \epsilon\eta_{\mu\sigma}\omega^\sigma_\nu$$

from which we conclude $\omega_{\nu\mu} + \omega_{\mu\nu} = 0$, as desired.

c) The elements of any infinitesimal Lorentz transformation $\omega^\mu{}_\nu$ can be written as a 4×4 matrix, where μ and ν index the row and column, respectively. For an infinitesimal rotation $\epsilon = d\theta$ about the z-axis, write out this matrix, and denote it by iJ^3 for later. What exponential of J^3 corresponds to a finite rotation by an angle θ?

Solution:

An infinitesimal rotation keeps t and z the same, and maps $x \to x + \epsilon y$ and $y \to y - \epsilon x$, so we can immediately write down

$$iJ^3 = \begin{pmatrix} & 1 & \\ -1 & & \end{pmatrix}$$

The appropriate matrix exponential is $e^{i\theta J^3}$. The purpose of the factor of i in the definition is to ensure J^3 is a Hermitian matrix, in accordance with how it's treated in quantum mechanics.

d) Write down the matrix iK^1 corresponding to an infinitesimal boost by $\epsilon = dv$ about the x-axis. What exponential of K^1 corresponds to a finite boost by a velocity v?

Solution:

An infinitesimal boost keeps y and z the same, and maps $x \to x + \epsilon t$ and $t \to t + \epsilon x$, so

$$iK^1 = \begin{pmatrix} 1 & 1 & \\ & & \\ & & \end{pmatrix}$$

Note that this is not an antisymmetric matrix. The quantity that is antisymmetric is $\omega_{\mu\nu}$, with two indices down, but raising an index to get $\omega^\mu{}_\nu$ produces some signs. The appropriate exponential for a finite boost is $e^{i\alpha K^1}$, where $\alpha = \tanh^{-1} v$ is the rapidity. (The most common mistake was instead writing e^{ivK^1}.)

There's a minor subtlety: the usual taught form of the Lorentz transformation has $x' = \gamma(x - vt)$, which suggests the matrix should have minus signs. But in that form, we're keeping the situation the same, and x' denotes the coordinate seen by an observer moving to the right. In this problem, we're considering the case where there's just one observer but physical objects are boosted to the right. Nonetheless, if you used the "passive" convention instead, it's no big deal since none of the other results in this problem will be affected.

Note that the factor of i in the definition of K^1 is a common convention, but it implies K^1 is not Hermitian. This is also sensible, as K^1 doesn't actually correspond to a physical observable. At a deeper level, it is related to the fact that non-compact non-Abelian Lie groups don't have finite-dimensional unitary representations, which will be discussed in section.

e) Defining J^1, J^2, K^2, and K^3 similarly, the generators obey commutation relations

$$\left[J^i, J^j\right] = i\epsilon^{ijk}J^k, \quad \left[J^i, K^j\right] = i\epsilon^{ijk}K^k, \quad \left[K^i, K^j\right] = -i\epsilon^{ijk}J^k$$

This is the Lie algebra of the Lorentz group. Physically, these results tells us that infinitesimal rotations and boosts are vectors (i.e. angular velocity and velocity are vectors), and that composing boosts can yield a rotation. Prove these results for the cases $(i, j) = (1, 1)$ and $(i, j) = (1, 2)$. (The proofs for other cases are similar.)

Solution:

The full set of generators is, up to a factor of i,

$$iJ^1 = \begin{pmatrix} & & \\ & -1 & 1 \\ & & \end{pmatrix} \quad iJ^2 = \begin{pmatrix} & & \\ & & -1 \\ & & \end{pmatrix} \quad iJ^3 = \begin{pmatrix} & & \\ & -1 & \\ & & 1 \end{pmatrix}$$
$$iK^1 = \begin{pmatrix} & & \\ & & 1 \\ & 1 & \end{pmatrix} \quad iK^2 = \begin{pmatrix} & & \\ & & 1 \\ & 1 & \end{pmatrix} \quad iK^3 = \begin{pmatrix} & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

For the case $(i, j) = (1, 1)$, the first and third relations are clearly satisfied, because the right-hand side is just zero, and $\left[J^1, K^1\right] = 0$ because these matrices are nonzero only in independent blocks. For the case $(i, j) = (1, 2)$, we need to check $\left[J^1, J^2\right] = iJ^3$, $\left[J^1, K^2\right] = iK^3$, and $\left[K^1, K^2\right] = -iJ^3$, which is straightforward to do manually.

Stanford.Zhou.2022.2.3.* Conserved currents of Lorentz transformations. (10/45p)

This somewhat tricky problem combines the ideas of the first two. It is completely optional: the problem set will be graded out of 40 points, so that you will receive up to 100% credit if you don't do this problem, and up to 125% credit if you do.

Under a Lorentz transformation, a scalar field profile ϕ gets mapped to ϕ', so that $\phi'(x') = \phi(x)$. This implies that

$$\phi'(x) = \phi\left(\Lambda^{-1}x\right).$$

For an infinitesimal Lorentz transformation (2), this corresponds to

$$\phi'(x) = \phi(x) - \epsilon\omega^{\mu\nu}x_\nu\partial_\mu\phi(x)$$

to first order in ϵ. Because an infinitesimal Lorentz transformation is parametrized by a rank 2 tensor $\omega^{\mu\nu}$, the corresponding Noether current will be a rank 3 tensor $J^{\mu\nu\rho}$, where the first index is the usual index that comes from Noether's theorem, and the last two describe the Lorentz transformation. For simplicity, you can do the entire problem for a real scalar field. (This corresponds to a complex scalar field with $a(\mathbf{k}) = b(\mathbf{k})$, which in turn implies $\Phi = \Phi^*$ and $\Pi = \Pi^*$.)

a) Show that for a scalar field,

$$J^{\mu\nu\rho} = x^\nu T^{\mu\rho} - x^\rho T^{\mu\nu}$$

where $T^{\mu\nu}$ is the stress-energy tensor. Let the associated conserved charges be $M^{\mu\nu}$.

Solution:

This is just like the derivation of the stress-energy tensor, except that the constant displacement a^μ is replaced with $\omega^{\mu\nu}x_\nu$. Thus, the current for a given ω is

$$J^\mu = \omega^{\rho\nu}x_\nu T^\mu_\rho = \frac{1}{2}\omega_{\rho\nu}\left(T^{\mu\rho}x^\nu - T^{\mu\nu}x^\rho\right)$$

where we used the antisymmetry of ω in the second step. The general rank-three conserved tensor can be found by "stripping off" the $\omega_{\rho\nu}$, giving

$$J^{\mu\nu\rho} = T^{\mu\rho}x^\nu - T^{\mu\nu}x^\rho$$

as desired.

b) Since $M^{\mu\nu}$ is antisymmetric, there are six independent conserved charges. The three independent M^{ij} physically correspond to the angular momentum of the field. What is the physical meaning of the other three conserved quantities M^{0i}?

Solution:

We have

$$M^{0i} = \int d\mathbf{x}J^{00i} = \int d\mathbf{x}T^{0i}x^0 - T^{00}x^i$$

The first term is just the total momentum P^i times the time t. The second term is an integral of the energy density times x^i, so it is proportional to the location of the "center of energy",

$$X^i_{\rm CE} = \frac{\int d\mathbf{x} T^{00} x^i}{\int d\mathbf{x} T^{00}} = \frac{\int d\mathbf{x} T^{00} x^i}{E}$$

where E is the total energy. Plugging in these definitions, we have

$$M^{0i} = P^i t - E X^i_{\rm CE}$$

Since this is conserved, its time derivative vanishes, giving

$$\frac{P^i}{E} = \frac{dX^i_{\rm CE}}{dt}$$

In other words, the conservation of M^{0i} means that the center of energy moves at constant speed. This is a very general principle in relativistic theories.

c) Show that after normal ordering, the angular momentum is

$$M^{ij} = i \int \frac{d^3k}{(2\pi)^3} a(\mathbf{k})^\dagger \left(k^j \frac{\partial}{\partial k_i} - k^i \frac{\partial}{\partial k_j} \right) a(\mathbf{k})$$

The form of this answer implies that the particles created and annihilated by scalar fields do not carry any intrinsic angular momentum (i.e. spin).

Solution:

We have

$$M^{ij} = \int d\mathbf{x} J^{0ij} = \int d\mathbf{x} x^i T^{0j} - x^j T^{0i}$$

The resulting integrals look very similar to those we already did in 2(f), but the factor of x^i complicates things: it means we can no longer do the $d\mathbf{x}$ integral trivially. The trick is to convert the factor of x^i into a momentum derivative. From a starting point much like (S60), we have

$$\begin{aligned}
\int d\mathbf{x} x^i \pi \partial^j \phi &= -\frac{i}{2} \int d\mathbf{x} d\mathbf{k} d\mathbf{k}' \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}} x^i \left(a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \partial^j \left(a\left(\mathbf{k}'\right) e^{i\mathbf{k}'\cdot\mathbf{x}} + a^\dagger\left(\mathbf{k}'\right) e^{-i\mathbf{k}'\cdot\mathbf{x}} \right) \\
&= \frac{1}{2} \int d\mathbf{x} d\mathbf{k} d\mathbf{k}' \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}} x^i k'^j \left(a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} - a^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \left(-a\left(\mathbf{k}'\right) e^{i\mathbf{k}'\cdot\mathbf{x}} + a^\dagger\left(\mathbf{k}'\right) e^{-i\mathbf{k}'\cdot\mathbf{x}} \right) \\
&= -\frac{i}{2} \int d\mathbf{x} d\mathbf{k} d\mathbf{k}' \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}} k'^j \left(a(\mathbf{k}) \partial_{k_i} e^{i\mathbf{k}\cdot\mathbf{x}} + a^\dagger(\mathbf{k}) \partial_{k_i} e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \\
&\quad \times \left(-a\left(\mathbf{k}'\right) e^{i\mathbf{k}'\cdot\mathbf{x}} + a^\dagger\left(\mathbf{k}'\right) e^{-i\mathbf{k}'\cdot\mathbf{x}} \right) \\
&= -\frac{i}{2} \int d\mathbf{k} d\mathbf{k}' \sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{k}'}}} k'^j \left(\left(a(\mathbf{k}) a\left(\mathbf{k}'\right) - a^\dagger(\mathbf{k}) a^\dagger\left(\mathbf{k}'\right) \right) \partial_{k_i} \phi\left(\mathbf{k} + \mathbf{k}'\right) \right. \\
&\left. + \left(a(\mathbf{k}) a^\dagger\left(\mathbf{k}'\right) - a^\dagger(\mathbf{k}) a\left(\mathbf{k}'\right) \right) \partial_{k_i} \phi\left(\mathbf{k} - \mathbf{k}'\right) \right).
\end{aligned}$$

To evaluate the delta function, we need to integrate by parts, which causes the ∂_{k_i}'s to hit the rest of the integrand. Now, the ∂_{k_i} can either act on $\sqrt{\omega_{\mathbf{k}}}$ or the ladder operators. If it hits the $\sqrt{\omega_{\mathbf{k}}}$, it produces a factor of k^i which makes the integrand proportional to $k^i k'^j$. However, since M^{ij} involves an antisymmetrization in i and j, these terms don't contribute to the final answer. We only care about the terms where ∂_{k_i} acts on the ladder operators, giving

$$\begin{aligned}
M^{ij} = \frac{i}{2} \int d\mathbf{k} k^j \left(-\left(\partial_{k_i} a(\mathbf{k})\right) a(-\mathbf{k}) + \left(\partial_{k_i} a^\dagger(\mathbf{k})\right) a^\dagger(-\mathbf{k}) \right. \\
\left. + \left(\partial_{k_i} a(\mathbf{k})\right) a^\dagger(\mathbf{k}) - \left(\partial_{k_i} a^\dagger(\mathbf{k})\right) a(\mathbf{k}) \right) - (i \leftrightarrow j \text{ term }).
\end{aligned}$$

Let's first show that the first term vanishes. For convenience we define

$$I^{ij} = \int d\mathbf{k} k^j a(-\mathbf{k}) \partial_{k_i} a(\mathbf{k})$$

Integrating by parts again, we have

$$\begin{aligned} I^{ij} &= -\int d\mathbf{k}\partial_{k_i}\left(k^j a(-\mathbf{k})\right) a(\mathbf{k}) \\ &= -\int d\mathbf{k}\delta^{ij} a(-\mathbf{k})a(\mathbf{k}) + k^j\left(\partial_{k_i} a(-\mathbf{k})\right) a(\mathbf{k}) \\ &= -\int d\mathbf{k}\delta^{ij} a(-\mathbf{k})a(\mathbf{k}) + k^j\left(\partial_{k_i} a(\mathbf{k})\right) a(-\mathbf{k}) \\ &= -I^{ij} - \int d\mathbf{k}\delta^{ij} a(-\mathbf{k})a(\mathbf{k}) \end{aligned}$$

where we did the usual $\mathbf{k} \to -\mathbf{k}$ trick in the third step. This shows that I^{ij} is proportional to δ^{ij}. Since the final answer M^{ij} involves an antisymmetrization in i and j, this shows the first term does not contribute to M^{ij}. Similarly, the second term doesn't contribute either.

You might be worried about what happens to the boundary terms when we integrate by parts. Usually, we justify throwing away boundary terms by saying the integrand goes to zero at infinity. But here, it doesn't: the integrand is always a nonzero operator, for any $\mathbf{k}$. However, as the integrand goes to infinity, we get creation and annihilation operators for extremely high momentum particles. These operators have zero matrix elements if you sandwich them between any two physically accessible states, so they might as well just be zero.

We are now left with the last two terms in S81. Normal ordering now gives

$$\begin{aligned} M^{ij} &= \frac{i}{2}\int d\mathbf{k}k^j\left(a^\dagger(\mathbf{k})\partial_{k_i} a(\mathbf{k}) - \left(\partial_{k_i} a^\dagger(\mathbf{k})\right) a(\mathbf{k})\right) - (i \leftrightarrow j \text{ term }) \\ &= \frac{i}{2}\int d\mathbf{k}k^j\left(a^\dagger(\mathbf{k})\partial_{k_i} a(\mathbf{k}) + a^\dagger(\mathbf{k})\partial_{k_i} a(\mathbf{k})\right) - (i \leftrightarrow j \text{ term }) \\ &= i\int d\mathbf{k}a^\dagger(\mathbf{k})\left(k^j\partial_{k_i} - k^i\partial_{k_j}\right) a(\mathbf{k}) \end{aligned}$$

where we integrated by parts again. This is the final result.

The only reason this result looks more complicated than our result for the spatial momentum is that we expanded the field in plane wave modes, which are eigenfunctions of translations but not rotations. Alternatively, we could have expanded the field in spherical waves (i.e. equal to a spherical Bessel function in r times a spherical harmonic), in which case the angular momentum expansion looks simple and the momentum expansion looks complicated. Plane waves are useful for describing the initial states in a particle physics experiment, but spherical waves can be useful in other situations, such as the emission of photons from a excited atom.

Stanford.Zhou.2022 6.1. The Lorentz algebra. (10/45 p)

You previously showed the rotation and boost generators obey commutation relations

$$\left[J^i, J^j\right] = i\epsilon^{ijk}J^k, \quad \left[J^i, K^j\right] = i\epsilon^{ijk}K^k, \quad \left[K^i, K^j\right] = -i\epsilon^{ijk}J^k$$

These are Euclidean indices, which is why pairs of upstairs indices can be contracted.

a) Defining $J^i_\pm = \left(J^i \pm iK^i\right)/2$, show that

$$\left[J^i_+, J^j_-\right] = 0, \quad \left[J^i_\pm, J^j_\pm\right] = i\epsilon^{ijk}J^k_\pm$$

so that the Lorentz algebra is just two copies of the algebra of the rotation group.

Solution:

In general, for any numbers a and b,

$$\begin{aligned} \frac{1}{4}\left[J^i + iaK^i, J^j + ibK^j\right] &= \frac{1}{4}\left(\left[J^i, J^j\right] + ia\left[K^i, J^j\right] + ib\left[J^i, K^j\right] - ab\left[K^i, K^j\right]\right) \\ &= \frac{i}{4}\epsilon^{ijk}\left((1 + a\mathbf{b})J^k + i(a + \mathbf{b})K^k\right) \end{aligned}$$

For $a = -b = 1$ we find

$$\left[J^i_+, J^j_-\right] = 0$$

as desired. For $a = b = \pm 1$ we find

$$\left[J_{\pm}^{i}, J_{\pm}^{j}\right] = i\epsilon^{ijk} J_{\pm}^{k}$$

also as desired.

The commutation relations can be expressed more concisely if we collect all six generators into the antisymmetric Lorentz tensor $J^{\mu\nu}$, where

$$J^{ij} = \epsilon^{ijk} J^{k}, \quad J^{0i} = K^{i}$$

The commutation relations (1) can then all be written together as

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}\right).$$

By definition, a representation of the Lorentz algebra is a choice of $J^{\mu\nu}$ that satisfies these commutation relations. The simplest example is the trivial representation $J^{\mu\nu} = 0$, which describes the action of Lorentz transformations on scalars.

b) In the vector representation, each generator $J^{\mu\nu}$ is a 4×4 matrix, with elements

$$\left(J^{\mu\nu}\right)_{\alpha\beta} = i\left(\delta_{\alpha}^{\mu}\delta_{\beta}^{\nu} - \delta_{\alpha}^{\nu}\delta_{\beta}^{\mu}\right)$$

This describes the action of Lorentz transformations on four-vectors V^{β}, as you've already seen. Starting from (5), verify the generators satisfy the Lorentz algebra (4). (Hint: upper and lower indices should be contracted in pairs to perform matrix multiplication. For example, $\left(J^{01}J^{23}\right)_{\alpha\beta} = \left(J^{01}\right)_{\alpha}^{\gamma}\left(J^{23}\right)_{\gamma\beta}$. If you get confused, start by plugging in specific values for the indices and doing the sums explicitly.)

Solution:

We just directly evaluate the commutator,

$$[J^{\mu\nu}, J^{\rho\sigma}]_{\alpha\delta} = \eta^{\beta\gamma}\left[-\left(\delta_{\alpha}^{\mu}\delta_{\beta}^{\nu} - \delta_{\alpha}^{\nu}\delta_{\beta}^{\mu}\right)\left(\delta_{\gamma}^{\rho}\delta_{\delta}^{\sigma} - \delta_{\gamma}^{\sigma}\delta_{\delta}^{\rho}\right) + \delta_{\alpha}^{\rho}\delta_{\beta}^{\sigma} - \delta_{\alpha}^{\sigma}\delta_{\beta}^{\rho}\right)\left(\delta_{\gamma}^{\mu}\delta_{\delta}^{\nu} - \delta_{\gamma}^{\nu}\delta_{\delta}^{\mu}\right)\Big]$$

There are 8 terms in total, each of which has the form $\eta^{\beta\gamma}$ times a product of four delta functions. For each one, we can apply the identity $\delta_{\alpha}^{\mu}\delta_{\beta}^{\nu}\eta^{\alpha\beta} = \eta^{\mu\nu}$ to reduce the term to a product of two delta functions times a metric tensor. Collecting the terms in four pairs yields the desired result.

This is tedious to write out. An easier way is to note that $\left(J^{\mu\nu}\right)_{\alpha\beta} = i\left(\delta_{\alpha}^{\mu}\delta_{\beta}^{\nu} - \delta_{\alpha}^{\nu}\delta_{\beta}^{\mu}\right)$ is antisymmetric in μ and ν, and in ρ and σ, which means the result also has to be antisymmetric in these pairs of indices. In this way, you can reduce the amount of algebra needed by a factor of 4, though you have to be a bit careful with signs.

c) The gamma matrices are 4×4 matrices defined to satisfy

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}$$

Show that

$$[\gamma^{\mu}\gamma^{\nu}, \gamma^{\rho}\gamma^{\sigma}] = 2\left(\eta^{\nu\rho}\gamma^{\mu}\gamma^{\sigma} - \eta^{\mu\rho}\gamma^{\nu}\gamma^{\sigma} + \eta^{\nu\sigma}\gamma^{\rho}\gamma^{\mu} - \eta^{\mu\sigma}\gamma^{\rho}\gamma^{\nu}\right)$$

Solution:

We just directly use the definition of the commutator, and then use (6) repeatedly to get the two terms in the same form, picking up factors of the metric with each anticommutation:

$$\begin{aligned}
[\gamma^{\mu}\gamma^{\nu}, \gamma^{\rho}\gamma^{\sigma}] &= \gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma} - \gamma^{\rho}\gamma^{\sigma}\gamma^{\mu}\gamma^{\nu} \\
&= \gamma^{\mu}\left\{\gamma^{\nu}, \gamma^{\rho}\right\}\gamma^{\sigma} - \gamma^{\mu}\gamma^{\rho}\gamma^{\nu}\gamma^{\sigma} - \gamma^{\rho}\gamma^{\sigma}\gamma^{\mu}\gamma^{\nu} \\
&= 2\eta^{\nu\rho}\gamma^{\mu}\gamma^{\sigma} - \gamma^{\mu}\gamma^{\rho}\left\{\gamma^{\nu}, \gamma^{\sigma}\right\} + \gamma^{\mu}\gamma^{\rho}\gamma^{\sigma}\gamma^{\nu} - \gamma^{\rho}\gamma^{\sigma}\gamma^{\mu}\gamma^{\nu} \\
&= 2\eta^{\nu\rho}\gamma^{\mu}\gamma^{\sigma} - 2\eta^{\nu\sigma}\gamma^{\mu}\gamma^{\rho} + \left\{\gamma^{\mu}, \gamma^{\rho}\right\}\gamma^{\sigma}\gamma^{\nu} - \gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\gamma^{\nu} - \gamma^{\rho}\gamma^{\sigma}\gamma^{\mu}\gamma^{\nu} \\
&= 2\eta^{\nu\rho}\gamma^{\mu}\gamma^{\sigma} - 2\eta^{\nu\sigma}\gamma^{\mu}\gamma^{\rho} + 2\eta^{\mu\rho}\gamma^{\sigma}\gamma^{\nu} - \gamma^{\rho}\left\{\gamma^{\mu}, \gamma^{\sigma}\right\}\gamma^{\nu} \\
&= 2\eta^{\nu\rho}\gamma^{\mu}\gamma^{\sigma} - 2\eta^{\nu\sigma}\gamma^{\mu}\gamma^{\rho} + 2\eta^{\mu\rho}\gamma^{\sigma}\gamma^{\nu} - 2\eta^{\mu\sigma}\gamma^{\rho}\gamma^{\nu} \\
&= 2\eta^{\nu\rho}\gamma^{\mu}\gamma^{\sigma} + 2\eta^{\nu\sigma}\gamma^{\rho}\gamma^{\mu} - 2\eta^{\mu\rho}\gamma^{\nu}\gamma^{\sigma} - 2\eta^{\mu\sigma}\gamma^{\rho}\gamma^{\nu}
\end{aligned}$$

Note that in the last step, we used the anticommutation relation on each of the middle two terms; the factors of $\eta^{\nu\sigma}\eta^{\mu\rho}$ cancel between the two terms.

d) In the Dirac spinor representation, the generators are the 4×4 matrices

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]$$

This describes the action of Lorentz transformations on Dirac spinors. Show that these generators satisfy the Lorentz algebra (4).

Solution:

Plugging in the definitions, we have

$$\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] &= -\frac{1}{16}\left[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu, \gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho\right] \\
&= -\frac{1}{16}\left([\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma] - [\gamma^\nu\gamma^\mu, \gamma^\rho\gamma^\sigma] - [\gamma^\mu\gamma^\nu, \gamma^\sigma\gamma^\rho] + [\gamma^\nu\gamma^\mu, \gamma^\sigma\gamma^\rho]\right) \\
&= -\frac{1}{4}\left[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma\right] \\
&= -\frac{1}{2}\left(\eta^{\nu\rho}\gamma^\mu\gamma^\sigma + \eta^{\nu\sigma}\gamma^\rho\gamma^\mu - \eta^{\mu\rho}\gamma^\nu\gamma^\sigma - \eta^{\mu\sigma}\gamma^\rho\gamma^\nu\right)
\end{aligned}$$

In the third step, we used the anticommutation relations on the last three terms, and in the fourth step we used the result of part (c). To finish the problem, we note that

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right] + \frac{1}{2}\left\{\gamma^\mu, \gamma^\nu\right\} = -2iS^{\mu\nu} + \eta^{\mu\nu}$$

Applying this to all four terms, the factors of the metric all cancel out, leaving the desired result

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\eta^{\nu\rho}S^{\mu\sigma} + i\eta^{\nu\sigma}S^{\rho\mu} - i\eta^{\mu\rho}S^{\nu}\sigma - i\eta^{\mu\sigma}S^{\rho\nu}$$

Stanford.Zhou.2022 6.2 Properties of gamma matrices. (15/45p)

In this problem, you should only use the defining property (6) of the gamma matrices. Below, $\mathbb{1}_4$ denotes a 4×4 identity matrix.

a) Show that contractions of gamma matrices satisfy

$$\begin{aligned}
\gamma^\mu\gamma_\mu &= 4\mathbb{1}_4 \\
\gamma^\mu\gamma^\nu\gamma_\mu &= -2\gamma^\nu \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4\eta^{\nu\rho}\mathbb{1}_4 \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu
\end{aligned}$$

Solution:

Since γ^μ and γ_μ commute, we have

$$\gamma^\mu\gamma_\mu = \frac{1}{2}\left(\gamma^\mu\gamma_\mu + \gamma_\mu\gamma^\mu\right) = \frac{1}{2}\left\{\gamma^\mu, \gamma_\mu\right\} = \eta^\mu_\mu = 4$$

where the right-hand side has an implicit 4×4 spinor identity matrix. Now, the proofs of the other three statements are very similar, but we use the defining anticommutation relation (6) to get the γ^μ and γ_μ next to each other, so we can use (9). The results are:

$$\begin{aligned}
\gamma^\mu\gamma^\nu\gamma_\mu &= \gamma^\mu\left\{\gamma^\nu,\gamma_\mu\right\} - \gamma^\mu\gamma_\mu\gamma^\nu \\
&= 2\gamma^\mu\eta_\mu^\nu - 4\gamma^\nu \\
&= -2\gamma^\nu \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= \gamma^\mu\gamma^\nu\left\{\gamma^\rho,\gamma_\mu\right\} - \gamma^\mu\gamma^\nu\gamma_\mu\gamma^\rho \\
&= 2\eta_\mu^\rho\gamma^\mu\gamma^\nu - \gamma^\mu\left\{\gamma^\nu,\gamma_\mu\right\}\gamma^\rho + \gamma^\mu\gamma_\mu,\gamma^\nu\gamma^\rho \\
&= 2\gamma^\rho\gamma^\nu - 2\gamma^\nu\gamma^\rho + 4\gamma^\nu\gamma^\rho \\
&= 4\eta^{\rho\nu} \\
\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= 2\eta_\mu^\sigma\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu\gamma^\sigma \\
&= 2\gamma^\sigma\gamma^\nu\gamma^\rho - 2\eta_\mu^\rho\gamma^\mu\gamma^\nu\gamma^\sigma + \gamma^\mu\gamma^\nu\gamma_\mu\gamma^\rho\gamma^\sigma \\
&= 2\gamma^\sigma\gamma^\nu\gamma^\rho - 2\gamma^\rho\gamma^\nu\gamma^\sigma + 2\eta_\mu^\nu\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma_\mu\gamma^\nu\gamma^\rho\gamma^\sigma \\
&= 2\gamma^\sigma\gamma^\nu\gamma^\rho - 2\gamma^\rho\gamma^\nu\gamma^\sigma - 2\gamma^\nu\gamma^\rho\gamma^\sigma \\
&= 2\gamma^\sigma\left\{\gamma^\nu,\gamma^\rho\right\} - 2\gamma^\sigma\gamma^\rho\gamma^\nu - 2\gamma^\rho\gamma^\nu\gamma^\sigma - 2\left\{\gamma^\nu,\gamma^\rho\right\}\gamma^\sigma \\
&= 2\gamma^\sigma\left\{\gamma^\nu,\gamma^\rho\right\} - 2\gamma^\sigma\gamma^\rho\gamma^\nu - 2\gamma^\rho\gamma^\nu\gamma^\sigma - 2\left\{\gamma^\nu,\gamma^\rho\right\}\gamma^\sigma + 2\gamma^\rho\gamma^\nu\gamma^\sigma \\
&= -2\gamma^\sigma\gamma^\rho\gamma^\nu
\end{aligned}$$

b) Show that the traces of products of gamma matrices obey

$$\begin{aligned}
\operatorname{tr}\gamma^\mu &= 0 \\
\operatorname{tr}\gamma^\mu\gamma^\nu &= 4\eta^{\mu\nu} \\
\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho &= 0 \\
\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma &= 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)
\end{aligned}$$

Solution:

We use the cyclic property of the trace,

$$\operatorname{tr} ABC...XYZ = \operatorname{tr} BC...XYZA$$

as well as the results we proved in part (a). For example, we have

$$\operatorname{tr}\gamma^\mu = \operatorname{tr}\left[\frac{1}{4}\gamma^\mu\gamma^\nu\gamma_\nu\right] = \operatorname{tr}\left[\frac{1}{4}\gamma_\nu\gamma^\mu\gamma^\nu\right] = -\frac{1}{2}\operatorname{tr}\gamma^\mu$$

where we used (9), the cyclic property of the trace, and (10). The only way this equation can be satisfied is if $\operatorname{tr}\gamma^\mu = 0$, as desired. A similar idea works for the trace of three gamma matrices,

$$\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho = \frac{1}{4}\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\sigma = \frac{1}{4}\operatorname{tr}\gamma_\sigma\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = -\frac{1}{2}\operatorname{tr}\gamma^\rho\gamma^\nu\gamma^\mu = -\frac{1}{2}\operatorname{tr}\gamma^\mu\gamma^\rho\gamma^\nu$$

We can bring the right-hand side to the same form as the left-hand side using the anticommutation relation,

$$-\frac{1}{2}\operatorname{tr}\gamma^\mu\gamma^\rho\gamma^\nu = -\frac{1}{2}\operatorname{tr}\gamma^\mu\left\{\gamma^\rho,\gamma^\nu\right\} + \frac{1}{2}\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho = \frac{1}{2}\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho$$

where the second step uses (13). We thus conclude $\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho = 0$. One can generalize this argument to show that the trace of an odd number of gamma matrices always vanishes. (Of course, there are multiple ways to show this; you could also have done it by inserting factors of $\left(\gamma^5\right)^2 = 1$, as in Peskin.)

Now let's consider the equations with an even number of gamma matrices. For two gamma matrices, we can use the cyclic property of the trace to produce an anticommutator, giving

$$\operatorname{tr}\gamma^\mu\gamma^\nu = \frac{1}{2}\operatorname{tr}\left[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right] = \operatorname{tr}\left[\eta^{\mu\nu}\right] = 4\eta^{\mu\nu}$$

For four gamma matrices, we can use a similar idea, repeatedly using the anticommutation relations to bring γ^μ to the right and then moving it back to the left using the cyclic property of the trace,

$$\begin{aligned}
\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma &= \operatorname{tr}\left\{\gamma^\mu,\gamma^\nu\right\}\gamma^\rho\gamma^\sigma - \operatorname{tr}\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma \\
&= 8\eta^{\mu\nu}\eta^{\rho\sigma} - \operatorname{tr}\gamma^\nu\left\{\gamma^\mu,\gamma^\rho\right\}\gamma^\sigma - \operatorname{tr}\gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma \\
&= 8\eta^{\mu\nu}\eta^{\rho\sigma} - 8\eta^{\mu\rho}\eta^{\nu\sigma} + \operatorname{tr}\gamma^\nu\gamma^\rho\left\{\gamma^\mu,\gamma^\sigma\right\} - \operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \\
&= 8\eta^{\mu\nu}\eta^{\rho\sigma} - 8\eta^{\mu\rho}\eta^{\nu\sigma} + 8\eta^{\mu\sigma}\eta^{\rho\nu} - \operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma
\end{aligned}$$

Adding $\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ to both sides and dividing by two gives the result.

While these relations might seem random, there's a very simple way to summarize what they mean. Consider an arbitrarily long product of arbitrary gamma matrices,

$$\operatorname{tr}\left(\gamma^1\gamma^3\gamma^0\gamma^0\gamma^2\gamma^3\gamma^2\gamma^1\ldots\right)$$

The defining anticommutation relations tells us that we pick up a minus sign when we swap the order of any two adjacent, distinct gamma matrices. So such a trace can always be rewritten in the form

$$\pm\operatorname{tr}\left(\left(\gamma^0\right)^{n_0}\left(\gamma^1\right)^{n_1}\left(\gamma^2\right)^{n_2}\left(\gamma^3\right)^{n_3}\right)$$

where n_0 through n_3 are nonnegative integers. The anticommutation relations also tell us that $\gamma^0\gamma^0 = 1$ and $\gamma^1\gamma^1 = \gamma^2\gamma^2 = \gamma^3\gamma^3 = -1$, which reduces the trace above (up to a sign) to

$$\pm\operatorname{tr}\left(\left(\gamma^0\right)^{m_0}\left(\gamma^1\right)^{m_1}\left(\gamma^2\right)^{m_2}\left(\gamma^3\right)^{m_3}\right)$$

where the m_i are 0 or 1, depending on the parity of the n_i. Looking back at what we've just proven, (13) says the trace of any single gamma matrix is zero, (14) says the trace of the product of any two distinct gamma matrices is zero, and (15) and (16) say the analogous results for three and four gamma matrices. Therefore, the trace S45 is automatically zero unless all of the m_i are equal to zero. The punchline is that the only way to get a nonzero trace is to multiply an even number of copies of each type of gamma matrix.

c) It will be useful to introduce the matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Show that

$$\begin{aligned}\left\{\gamma^5,\gamma^\mu\right\} &= 0\\ \left(\gamma^5\right)^2 &= \mathbb{1}_4\\ \operatorname{tr}\gamma^5 &= 0\\ \operatorname{tr}\gamma^\mu\gamma^\nu\gamma^5 &= 0\\ \operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5 &= -4i\epsilon^{\mu\nu\rho\sigma}.\end{aligned}$$

Solution:

We can check the first statement for each value of μ. For example, for $\mu = 0$ we have

$$\left\{\gamma^5,\gamma^0\right\} = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 + i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3$$

The second term can be brought to the first term by performing three anticommutations, each of which flips the sign. Thus, the sum of the two terms is zero. The logic for $\mu = 1, 2, 3$ is similar.

To prove the second statement, we note that

$$\begin{aligned}\left(\gamma^5\right)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3\\ &= \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3\\ &= -\gamma^2\gamma^3\gamma^2\gamma^3\\ &= -\gamma^3\gamma^3\\ &= 1\end{aligned}$$

where we just used the anticommutation relations, which imply $\gamma^0\gamma^0 = 1$ and $\gamma^1\gamma^1 = \gamma^2\gamma^2 = \gamma^3\gamma^3 = -1$.

To prove the third statement, we can simply use (16),

$$\operatorname{tr}\gamma^5 = i\operatorname{tr}\gamma^0\gamma^1\gamma^2\gamma^3 = 4i\left(\eta^{01}\eta^{23} - \eta^{02}\eta^{13} + \eta^{03}\eta^{12}\right) = 0$$

For the fourth statement, we can do some casework on the values of μ and ν. If $\mu = \nu$, then the γ^μ and γ^ν multiply to give \pm 1, leaving us with $\operatorname{tr}\gamma^5 = 0$. On the other hand, suppose $\mu \neq \nu$, then we can bring the factors of γ^μ and γ^ν within γ^5 to the left, by performing anticommutations. Multiplying them with $\gamma^\mu\gamma^\nu$ yields \pm 1 and leaves the trace of a product of two different gamma matrices, which is zero.

Finally, for the last statement, let's first show that the left-hand side is totally antisymmetric in its four indices. For the μ and ν indices, note that

$$\operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5 = 2\eta^{\mu\nu}\operatorname{tr}\gamma^\rho\gamma^\sigma\gamma^5 - \operatorname{tr}\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^5 = -\operatorname{tr}\gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^5$$

By similar reasoning, we get a sign flip when we exchange the ν and ρ indices, and the ρ and σ indices, which suffices to show that the left-hand side is antisymmetric in all four indices. Thus, it must be equal to $A\epsilon^{\mu\nu\rho\sigma}$ for some constant A. To find the constant, we just consider a special case,

$$\operatorname{tr}\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5 = A\epsilon^{0123} = A$$

The left-hand side is just

$$-i\operatorname{tr}\left(\gamma^5\right)^2 = -4i$$

which yields the desired result.

Stanford.Zhou.2022 6.3 Invariance of the Dirac Lagrangian. (15/45p)

The Dirac Lagrangian is

$$\mathcal{L} = \bar{\Psi}\left(i\not{\partial} - m\mathbb{1}_4\right)\Psi$$

where $\bar{\Psi} = \Psi^\dagger\gamma^0$ and $\not{\partial} = \gamma^\mu\partial_\mu$. The four-component spinor Ψ is acted on by the gamma matrices. In general, a Lorentz transformation Λ will change a spinor according to

$$\Psi(x) \to \Psi'(x') = U(\Lambda)\Psi(x)$$

where $U(\Lambda)$ is some 4×4 matrix, not necessarily unitary.
a) Show that the Dirac Lagrangian is invariant under Lorentz transformations if

$$U^{-1}(\Lambda) = \gamma^0 U^\dagger(\Lambda)\gamma^0, \quad U^{-1}(\Lambda)\gamma^\mu U(\Lambda) = \Lambda^\mu_\nu\gamma^\nu$$

Solution:

From basic special relativity, we know that under a Lorentz transformation,

$$x^\mu \to x^\mu = \Lambda^\mu_\nu x^\nu, \quad \frac{\partial}{\partial x_\mu} \to \frac{\partial}{\partial x'_\mu} = \Lambda^\mu_\nu\frac{\partial}{\partial x^\nu}$$

Now let's show that each term in the Dirac Lagrangian is invariant. The mass term is invariant because

$$\bar{\Psi}\Psi = \Psi^\dagger\gamma^0\Psi \to \Psi^\dagger U^\dagger\gamma^0 U\Psi = \Psi^\dagger\gamma^0 U^{-1}U\Psi = \bar{\Psi}\Psi$$

To show the kinetic term is invariant, note that

$$\begin{aligned}
\bar{\Psi}\not{\partial}\Psi &\to \Psi^\dagger U^\dagger\gamma^0\gamma_\mu\Lambda^\mu_\nu\partial^\nu U\Psi \\
&= \Psi^\dagger\gamma^0 U^{-1}\gamma_\mu\Lambda^\mu_\nu U\partial^\nu\Psi \\
&= \bar{\Psi}\Lambda_{\mu\rho}\gamma^\rho\Lambda^\mu_\nu\partial^\nu\Psi \\
&= \bar{\Psi}\eta_{\nu\rho}\gamma^\rho\partial^\nu\Psi \\
&= \bar{\Psi}\not{\partial}\Psi.
\end{aligned}$$

b) Show that for an infinitesimal Lorentz transformation $\Lambda^{\mu\nu} = \eta^{\mu\nu} + \omega^{\mu\nu}$ the above relations are satisfied for

$$U(\Lambda) = \mathbb{1}_4 - \frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}$$

where $S^{\mu\nu}$ is the Lorentz generator defined by (8). (Hint: use $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$, which holds in every representation of the gamma matrices.)

Solution:

Expanding out the definitions, we have

$$U = \mathbb{1} + \frac{1}{8}\left[\gamma^\mu,\gamma^\nu\right]\omega_{\mu\nu}$$

which implies

$$U^{-1} = \mathbb{1} - \frac{1}{8}\left[\gamma^\mu,\gamma^\nu\right]\omega_{\mu\nu}$$

since we're considering infinitesimal translations. To prove the first result note that

$$\begin{aligned}\gamma^0 U^\dagger \gamma^0 &= \mathbb{1} + \frac{1}{8}\gamma^0 \left[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger\right] \gamma^0 \omega_{\mu\nu} \\ &= \mathbb{1} + \frac{1}{8}\gamma^0 \left[\gamma^0\gamma^\nu\gamma^0, \gamma^0\gamma^\mu\gamma^0\right] \gamma^0 \omega_{\mu\nu} \\ &= \mathbb{1} - \frac{1}{8}\left[\gamma^\mu, \gamma^\nu\right] \omega_{\mu\nu} \\ &= U^{-1}\end{aligned}$$

In the first step, we used the fact that $[A,B]^\dagger = \left[B^\dagger, A^\dagger\right]$, in the second step we used $\gamma_\mu^\dagger = \gamma^0\gamma_\mu\gamma^0$, and in the third step we used $\gamma^0\gamma^0 = 1$.

For the second relation, note that

$$\gamma_\mu \left[\gamma_\rho, \gamma_\sigma\right] = 4\left(\eta_{\mu\rho}\gamma_\sigma - \eta_{\mu\sigma}\gamma_\rho\right) + \left[\gamma_\rho, \gamma_\sigma\right]\gamma_\mu$$

Next,

$$\gamma^\mu U = \gamma^\mu \left(\mathbb{1} + \frac{1}{8}\left[\gamma_\rho, \gamma_\sigma\right]\omega^{\rho\sigma}\right) = U\gamma^\mu + \frac{1}{2}\left(\omega^{\mu\sigma}\gamma_\sigma - \omega^{\sigma\mu}\gamma_\sigma\right) = U\gamma^\mu + \omega^{\mu\sigma}\gamma_\sigma$$

Finally, we find

$$U^{-1}\gamma_\mu U = U^{-1}\left(U\gamma_\mu + \omega_{\mu\sigma}\gamma^\sigma\right) = \left(\eta_{\mu\nu} + \omega_{\mu\nu}\right)\gamma^\nu = \Lambda_{\mu\nu}\gamma^\nu$$

as desired. Note that in the penultimate step, we dropped a term of order ω^2 since the Lorentz transformation is infinitesimal. **c)** A finite Lorentz transformation is given by exponentiating a generator,

$$U(\Lambda) = \exp\left(-\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}\right)$$

Explicitly write down the 4×4 matrix $U(\Lambda)$ for a rotation about the x-axis by an angle θ, and a boost along the z axis with rapidity ϕ. Use the Dirac representation of the gamma matrices, as this will yield simple results in the nonrelativistic limit.

Solution:

For rotations around the x-axis by an angle θ we have $\omega^{23} = -\omega^{32} = \theta$ and all other entries zero. In the Dirac representation, we find

$$\left[\gamma^2, \gamma^3\right] = \left[\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\right] = -\begin{pmatrix} [\sigma_2,\sigma_3] & 0 \\ 0 & [\sigma_2,\sigma_3] \end{pmatrix} = 2i\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

Consequently,

$$U(\Lambda(\theta)) = \exp\left\{\frac{i\theta}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}\right\} = \cos\frac{\theta}{2}\mathbb{1} + i\sin\frac{\theta}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

This makes sense, because it just says that both of the two-component spinors that make up a Dirac spinor transform like spin 1/2 particles in ordinary quantum mechanics.

For boosts along the z direction we have $\omega^{03} = -\omega^{30} = \phi$ and all other entries zero, and

$$\left[\gamma^0, \gamma^3\right] = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}\right] = 2\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

Consequently,

$$U(\Lambda(\phi)) = \exp\left\{\frac{\phi}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}\right\} = \cosh\frac{\phi}{2}\mathbb{1} + \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}\sinh\frac{\phi}{2}$$

In other words, boosts mix the two two-component spinors, while keeping the spin state the same.

d) Show that the Dirac Lagrangian is invariant under $\Psi \to e^{-i\alpha}\Psi$, and find the associated conserved current J_V^μ. Then show explicitly that $\partial_\mu J_V^\mu = 0$ using the Dirac equation.

Solution:

The Lagrangian is invariant because we also have $\bar{\Psi} \to e^{i\alpha}\bar{\Psi}$, so the phases cancel out. Now, to avoid confusion when applying Noether's theorem, let's explicitly write out the spinor indices. The changes in the spinors are

$$(\delta\Psi)_a = -i\Psi_a, \quad (\delta\bar{\Psi})_a = (i\bar{\Psi})_a$$

and Noether's theorem states

$$J_V^\mu = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\Psi\right)_a}(\delta\Psi)_a + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\bar{\Psi}\right)_a}(\delta\bar{\Psi})_a$$

where there is an implicit sum over the spinor index a. (It ranges from 0 to 3, covering the four elements of the spinor, but it is not a Lorentz index.) The second term is just zero, so we get

$$J_V^\mu = \left(i\bar{\Psi}\gamma^\mu\right)_a(-i\Psi)_a$$

We can rewrite this without explicit spinor indices as

$$J_V^\mu = \bar{\Psi}\gamma^\mu\Psi$$

where there are now implicit spinor matrix multiplications. To check this is conserved, note that

$$\partial_\mu J_V^\mu = \left(\partial_\mu\bar{\Psi}\right)\gamma^\mu\Psi + \bar{\Psi}\gamma^\mu\partial_\mu\Psi = im\bar{\Psi}\Psi - im\bar{\Psi}\Psi = 0$$

e) Show that when $m = 0$, the Dirac Lagrangian is also invariant under $\Psi \to e^{-i\alpha\gamma^5}\Psi$, and find the associated conserved current J_A^μ. What is $\partial_\mu J_A^\mu$ when m is nonzero?

Solution:

Using the fact that γ^5 anticommutes with all gamma matrices, we have

$$\bar{\Psi} = \Psi^\dagger\gamma^0 \to \Psi^\dagger e^{i\alpha\gamma^5}\gamma^0 = \Psi^\dagger\gamma^0 e^{-i\alpha\gamma^5} = \bar{\Psi}e^{-i\alpha\gamma^5}$$

Thus, the Lagrangian becomes

$$\Psi^\dagger e^{-i\alpha\gamma^5}\not{\partial}e^{-i\alpha\gamma^5}\Psi.$$

Moving the $e^{-i\alpha\gamma^5}$ past the $\not{\partial}$ flips the sign of the exponential, so we end up with a factor of $e^{i\alpha\gamma^5}e^{-i\alpha\gamma^5} = 1$. Now, the changes in the spinors per α, for infinitesimal α, are

$$\delta\Psi = -i\gamma^5\Psi, \quad \delta\bar{\Psi} = -i\bar{\Psi}\gamma^5$$

which means the Noether current is

$$J_A^\mu = \bar{\Psi}\gamma^\mu\gamma^5\Psi$$

For nonzero m, the divergence of the current is

$$\begin{aligned}\partial_\mu J_A^\mu &= \left(\partial_\mu\bar{\Psi}\right)\gamma^\mu\gamma^5\Psi + \bar{\Psi}\gamma^\mu\gamma^5\partial_\mu\Psi \\ &= \left(\partial_\mu\bar{\Psi}\right)\gamma^\mu\gamma^5\Psi - \bar{\Psi}\gamma^5\gamma^\mu\partial_\mu\Psi \\ &= im\bar{\Psi}\gamma^5\Psi + im\bar{\Psi}\gamma^5\Psi \\ &= 2im\bar{\Psi}\gamma^5\Psi\end{aligned}$$

Stanford.Zhou.2022 6.4.* Spinors in three dimensions. (5/45p)

In this course, we focus on spinors in four dimensions for good reason. In this optional problem, you'll see how the same mathematical structures appear in three dimensions.

a) Consider spinors in three spacetime dimensions. What are the smallest nonzero matrices that can satisfy (6)? Write down three such matrices γ^0, γ^1, and γ^2 explicitly.

Solution:

The minimum size is 2×2, and one example set is

$$\gamma^0 = \sigma^2, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^3$$

where the σ^i are the Pauli matrices.

b) We define the spinor Lorentz generators by (8) in any dimension. Since there are now only two spatial dimensions, there is only one rotation generator S^{12}. What phase does a spinor pick up after a 2π rotation?

Solution:

We have
Then a 2π rotation yields

$$S^{12} = \frac{i}{4}\left[i\sigma^1, i\sigma^3\right] = -\frac{\sigma^2}{2}$$

$$\exp\left(-\pi\sigma^2\right) = -I$$

and therefore a phase of π, just like in 4 dimensions.

c) How does the tensor product of two spinor representations decompose into irreducible representations of the Lorentz group?

Solution:

By the exact same logic as in 4 dimensions, we can extract a Lorentz scalar by $\bar{\psi}\psi$, and a Lorentz vector (which only has three components) by $\bar{\psi}\gamma^\mu\psi$. Since spinors have 2 components, the tensor product has 4 components, so this decomposition is complete: there isn't anything else. For instance, you might consider putting $[\gamma^\mu, \gamma^\nu]$ in the middle, but in three dimensions the commutator of two gamma matrices is just another gamma matrix, so you don't get anything new. Similarly, you can't use γ^5 to get anything new, as its analogue here, $i\gamma^0\gamma^1\gamma^2$, is just proportional to the identity.

In two spatial dimensions there are exotic particles called anyons, which can pick up an arbitrary phase after a 2π rotation. However, they can't be described by the conventional quantum fields covered in this course.

26.1.8 Problems about spinors and Dirac equation

33 Решения уравнения Дирака для свободных частиц.

11

Покажите, что $\exp\left[\begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}\right] = \begin{pmatrix} \operatorname{ch}\eta & \operatorname{sh}\eta \\ \operatorname{sh}\eta & \operatorname{ch}\eta \end{pmatrix}$, и

$$\exp\left[\begin{pmatrix} -\eta\sigma^3/2 & 0 \\ 0 & \eta\sigma^3/2 \end{pmatrix}\right] = \begin{pmatrix} \operatorname{ch}(\eta/2) - \sigma^3\operatorname{sh}(\eta/2) & 0 \\ 0 & \operatorname{ch}(\eta/2) + \sigma^3\operatorname{sh}(\eta/2) \end{pmatrix}$$

Решение: Имеем

$$\begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}^{2n} = \begin{pmatrix} \eta^{2n} & 0 \\ 0 & \eta^{2n} \end{pmatrix}, \quad \begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}^{2n+1} = \begin{pmatrix} 0 & \eta^{2n+1} \\ \eta^{2n+1} & 0 \end{pmatrix} \tag{0.14}$$

В итоге получим

$$\exp\left[\begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}\right] = \sum_{k=0}^{\infty}\frac{1}{k!}\begin{pmatrix} 0 & \eta \\ \eta & 0 \end{pmatrix}^k = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\begin{pmatrix} \eta^{2n} & 0 \\ 0 & \eta^{2n} \end{pmatrix} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\begin{pmatrix} 0 & \eta^{2n+1} \\ \eta^{2n+1} & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} \operatorname{ch}\eta & \operatorname{sh}\eta \\ \operatorname{sh}\eta & \operatorname{ch}\eta \end{pmatrix}. \tag{0.15}$$

Также имеем

$$\begin{pmatrix} -\eta\sigma^3/2 & 0 \\ 0 & \eta\sigma^3/2 \end{pmatrix}^{2n} = \begin{pmatrix} \left(\frac{\eta}{2}\right)^{2n} & 0 \\ 0 & \left(\frac{\eta}{2}\right)^{2n} \end{pmatrix}, \quad \begin{pmatrix} -\eta\sigma^3/2 & 0 \\ 0 & \eta\sigma^3/2 \end{pmatrix}^{2n+1} = \begin{pmatrix} -\left(\frac{\eta}{2}\right)^{2n+1}\sigma^3 & 0 \\ 0 & \left(\frac{\eta}{2}\right)^{2n+1}\sigma^3 \end{pmatrix}. \tag{0.16}$$

И аналогично получаем

$$\exp\left[\begin{pmatrix} -\eta\sigma^3/2 & 0 \\ 0 & \eta\sigma^3/2 \end{pmatrix}\right] = \begin{pmatrix} \operatorname{ch}(\eta/2) - \sigma^3\operatorname{sh}(\eta/2) & 0 \\ 0 & \operatorname{ch}(\eta/2) + \sigma^3\operatorname{sh}(\eta/2) \end{pmatrix} \tag{0.17}$$

что и требовалось показать.

12

Докажите, что $(p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2 = m^2$, где $\sigma = (1, \boldsymbol{\sigma})$ и $\bar{\sigma} = (1, -\boldsymbol{\sigma})$.
Решение: Имеем

$$(p\cdot\sigma)(p\cdot\bar{\sigma}) = p_\mu p_\nu \sigma^\mu\bar{\sigma}^\nu = p_\mu p_\nu \frac{1}{2}\left(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu\right) = p_\mu p_\nu g^{\mu\nu} = p^2 = m^2 \tag{0.18}$$

где мы использовали, что $\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu = 2g^{\mu\nu}$, что легко проверить прямым вычислением и учитывая, что $\left\{\sigma^i, \sigma^j\right\} = 2\delta^{ij}$.$\bigcirc$

13*.

Найдите собственные числа и общий вид матрицы $\sqrt{p\cdot\sigma}$.
Решение: Имеем

$$p\cdot\sigma = g_{\mu\nu}p^\mu\sigma^\nu = p^0\sigma^0 - \boldsymbol{p}\boldsymbol{\sigma} = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix}. \tag{0.19}$$

Находя собственные значения $\lambda_{1,2} = p^0 \pm |\boldsymbol{p}|$, легко переписать данную матрицу в виде

$$p\cdot\sigma = S\Lambda S^{-1} = \begin{pmatrix} -p^3 - |\boldsymbol{p}| & -p^3 + |\boldsymbol{p}| \\ -p^1 - ip^2 & -p^1 - ip^2 \end{pmatrix}\begin{pmatrix} p^0 - |\boldsymbol{p}| & 0 \\ 0 & p^0 + |\boldsymbol{p}| \end{pmatrix}\begin{pmatrix} -\frac{1}{2|\boldsymbol{p}|} & \frac{p^3 - |\boldsymbol{p}|}{2|\boldsymbol{p}|(p^1+ip^2)} \\ \frac{1}{2|\boldsymbol{p}|} & -\frac{p^3+|\boldsymbol{p}|}{2|\boldsymbol{p}|(p^1+ip^2)} \end{pmatrix}. \tag{0.20}$$

Тогда для $\sqrt{p\cdot\sigma}$ получим

$$\begin{aligned}\sqrt{p\cdot\sigma} = S\sqrt{\Lambda}S^{-1} &= \begin{pmatrix} -p^3 - |\boldsymbol{p}| & -p^3 + |\boldsymbol{p}| \\ -p^1 - ip^2 & -p^1 - ip^2 \end{pmatrix}\begin{pmatrix} \sqrt{p^0 - |\boldsymbol{p}|} & 0 \\ 0 & \sqrt{p^0 + |\boldsymbol{p}|} \end{pmatrix}\begin{pmatrix} -\frac{1}{2|\boldsymbol{p}|} & \frac{p^3 - |\boldsymbol{p}|}{2|\boldsymbol{p}|(p^1+ip^2)} \\ \frac{1}{2|\boldsymbol{p}|} & -\frac{p^3+|\boldsymbol{p}|}{2|\boldsymbol{p}|(p^1+ip^2)} \end{pmatrix} = \\ &= \begin{pmatrix} \frac{|\boldsymbol{p}|\mathfrak{s} - p^3\mathfrak{r}}{2|\boldsymbol{p}|} & -\frac{\mathfrak{r}}{2|\boldsymbol{p}|}\left(p^1 - ip^2\right) \\ -\frac{\mathfrak{r}}{2|\boldsymbol{p}|}\left(p^1 + ip^2\right) & \frac{|\boldsymbol{p}|\mathfrak{s} + \mathfrak{r}p^3}{2|\boldsymbol{p}|} \end{pmatrix} = g_{\mu\nu}\tilde{p}^\mu\sigma^\mu = \tilde{p}\cdot\sigma, \quad \tilde{p} = \left(\frac{\mathfrak{s}}{2}, \frac{\mathfrak{r}}{2}\frac{\boldsymbol{p}}{|\boldsymbol{p}|}\right),\end{aligned} \tag{0.21}$$

где $\mathfrak{s} = \sqrt{p^0 + |\boldsymbol{p}|} + \sqrt{p^0 - |\boldsymbol{p}|}$, а $\mathfrak{r} = \sqrt{p^0 + |\boldsymbol{p}|} - \sqrt{p^0 - |\boldsymbol{p}|}$.

14

Проверить, что $\psi(x) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix} e^{-ipx}$ есть решение уравнения Дирака.
Решение: Мы должны проверить, что

$$\left(\gamma^\mu p_\mu - m\right)u(p) = \left(g_{\mu\nu}\gamma^\mu p^\nu - m\right)u(p) = \left(\gamma^0 p^0 - \gamma^i p^i - m\right)u(p) = 0 \tag{0.22}$$

где спинор $u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix}$, а γ-матрицы взяты в киральном (Вейлевском) представлении:

$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\cdot$ Имеем

$$\begin{aligned}(\gamma^\mu p_\mu - m)\, u(p) &= \begin{pmatrix} -m & p^0\sigma^0 - p^i\sigma^i \\ p^0\sigma^0 + p^i\sigma^i & -m \end{pmatrix} u(p) = \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar\sigma & -m \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar\sigma}\xi \end{pmatrix} = \\ &= \begin{pmatrix} (-m\sqrt{p\cdot\sigma} + p\cdot\sigma\sqrt{p\cdot\bar\sigma})\xi \\ (-m\sqrt{p\cdot\bar\sigma} + p\cdot\bar\sigma\sqrt{p\cdot\sigma})\xi \end{pmatrix} = \begin{pmatrix} (-m\sqrt{p\cdot\sigma} + \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma})\xi \\ (-m\sqrt{p\cdot\bar\sigma} + \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma})\xi \end{pmatrix} = \\ &= \begin{pmatrix} (-m\sqrt{p\cdot\sigma} + \sqrt{p\cdot\sigma}m)\xi \\ (-m\sqrt{p\cdot\bar\sigma} + \sqrt{p\cdot\bar\sigma}m)\xi \end{pmatrix} = 0, \end{aligned} \tag{0.23}$$

что и требовалось показать.

15

Покажите, что $\bar u u = 2m\xi^\dagger\xi$, где $u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar\sigma}\xi \end{pmatrix}$, а $\bar u(p) = u^\dagger(p)\gamma^0$.

Решение: Имеем

$$\bar u(p) = u^\dagger(p)\gamma^0 = \begin{pmatrix} \xi^\dagger\sqrt{p\cdot\sigma} & \xi^\dagger\sqrt{p\cdot\bar\sigma} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \xi^\dagger\sqrt{p\cdot\bar\sigma} & \xi^\dagger\sqrt{p\cdot\sigma} \end{pmatrix} \tag{0.24}$$

где мы использовали, что $(\sqrt{p\cdot\sigma})^\dagger = \sqrt{p\cdot\sigma}$ и $(\sqrt{p\cdot\bar\sigma})^\dagger = \sqrt{p\cdot\bar\sigma}$. Далее получаем

$$\bar u(p)u(p) = \begin{pmatrix} \xi^\dagger\sqrt{p\cdot\bar\sigma}\xi^\dagger\sqrt{p\cdot\sigma} \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar\sigma}\xi \end{pmatrix} = 2\xi^\dagger\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma}\xi = 2m\xi^\dagger\xi \tag{0.25}$$

что и требовалось показать.

16

Покажите, что $\bar v^r(p)v^s(p) = -2m\delta^{rs}$, $v^{r\dagger}(p)v^s(p) = 2E_{\mathbf{p}}\delta^{rs}$, где $v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar\sigma}\eta^s \end{pmatrix}$ и $\eta^{r\dagger}\eta^s = \delta^{rs}, s, r = 1, 2$.

Решение: Имеем

$$\bar v^r(p)v^s(p) = \begin{pmatrix} -\eta^{r\dagger}\sqrt{p\cdot\bar\sigma} & \eta^{r\dagger}\sqrt{p\cdot\sigma} \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar\sigma}\eta^s \end{pmatrix} = -2m\eta^{r\dagger}\eta^s = -2m\delta^{rs}. \tag{0.26}$$

Далее

$$\begin{aligned} v^{r\dagger}(p)v^s(p) &= \begin{pmatrix} \eta^{r\dagger}\sqrt{p\cdot\sigma} - \eta^{r\dagger}\sqrt{p\cdot\bar\sigma} \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar\sigma}\eta^s \end{pmatrix} = \eta^{r\dagger}(p\cdot\sigma + p\cdot\bar\sigma)\eta^s = \eta^{r\dagger}\left(2p^0\sigma^0\right)\eta^s = \\ &= 2p^0\eta^{r\dagger}\eta^s = 2E_{\boldsymbol{p}}\delta^{rs} \end{aligned} \tag{0.27}$$

что и требовалось показать.

17

Проверьте, что $\bar u^r(p)v^s(p) = \bar v^r(p)u^s(p) = 0$ и $u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p}) = v^{r\dagger}(-\mathbf{p})u^s(\mathbf{p}) = 0$.

Решение: Имеем

$$\bar{u}^r(p)v^s(p) = \begin{pmatrix} \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}} & \xi^{r\dagger}\sqrt{p\cdot\sigma} \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix} = \xi^{r\dagger}(\sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma} - \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}})\eta^s =$$
$$= \xi^{r\dagger}(m-m)\eta^s = 0,$$
$$\bar{v}^r(p)u^s(p) = \begin{pmatrix} -\eta^{r\dagger}\sqrt{p\cdot\bar{\sigma}} & \eta^{r\dagger}\sqrt{p\cdot\sigma} \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} = \eta^{r\dagger}(-\sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma} + \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}})\xi^s =$$
$$= \eta^{r\dagger}(-m+m)\xi^s = 0. \tag{0.28}$$

Далее имеем

$$u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p}) = \begin{pmatrix} \xi^{r\dagger}\sqrt{p\cdot\sigma} & \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}} \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\eta^s \\ -\sqrt{p\cdot\sigma}\eta^s \end{pmatrix} = 0 \tag{0.29}$$

для $v^{r\dagger}(-\mathbf{p})u^s(\mathbf{p}) = 0$ проверяется аналогично.

18

Докажите формулу $\sum\limits_{s=1,2}\xi^s\xi^{s\dagger} = \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, где $\xi^{r\dagger}\xi^s = \delta^{rs}$ и покажите, что $\sum\limits_s u^s(p)\bar{u}^s(p) = \gamma\cdot p + m$, а $\sum\limits_s v^s(p)\bar{v}^s(p) = \gamma\cdot p - m$

Решение: Так как $\xi^s, s = 1,2$ образуют ортонормированный базис: $\xi^{r\dagger}\xi^s = \delta^{rs}$, то взяв произвольный вектор $\boldsymbol{e} = e_1\xi^1 + e_2\xi^2$, имеем

$$\left(\sum_{s=1,2}\xi^s\xi^{s\dagger}\right)\boldsymbol{e} = e_1\sum_{s=1,2}\xi^s\xi^{s\dagger}\xi^1 + e_2\sum_{s=1,2}\xi^s\xi^{s\dagger}\xi^2 = e_1\sum_{s=1,2}\xi^s\delta^{s1} + e_2\sum_{s=1,2}\xi^s\delta^{s2} = e_1\xi^1 + e_2\xi^2 = \boldsymbol{e}, \tag{0.30}$$

откуда следует, что $\sum\limits_{s=1,2}\xi^s\xi^{s\dagger} = \mathbf{1}$. Далее получаем

$$\sum_s u^s(p)\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} \begin{pmatrix} \xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} & \xi^{s\dagger}\sqrt{p\cdot\sigma} \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} = \gamma\cdot p + m$$
$$\sum_s v^s(p)\bar{v}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix} \begin{pmatrix} -\eta^{s\dagger}\sqrt{p\cdot\bar{\sigma}} & \eta^{s\dagger}\sqrt{p\cdot\sigma} \end{pmatrix} = \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} = \gamma\cdot p - m$$

что и требовалось показать.

34 Матрицы Дирака и билинейные формы Дирака.

Проверьте, что $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$, и $\varepsilon^{0123} = -\varepsilon_{0123} = 1$.

Решение: Так как $\gamma_\mu\gamma_\nu = 2g_{\mu\nu} - \gamma_\nu\gamma_\nu$, и $g_{\mu\nu}$ симметричный тензор, и при свертке с $\varepsilon^{\mu\nu\rho\sigma}$ он равен нулю, то при свертке с $\varepsilon^{\mu\nu\rho\sigma}$ можно считать, что $\gamma_\mu\gamma_\nu \sim -\gamma_\nu\gamma_\mu$. Тогда $\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma \sim \varepsilon_{\mu\nu\rho\sigma}\gamma^0\gamma^1\gamma^2\gamma^3$. Таким образом получаем

$$-\frac{i}{4!}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = -\frac{i}{4!}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma}\gamma^0\gamma^1\gamma^2\gamma^3 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{0.32}$$

где мы воспользовались тем, что $\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = -4$!.

Проверьте, что $\gamma^{\mu\nu\rho} = i\varepsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5$, где $\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]} = \frac{1}{3!}\left(\gamma^\mu\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\mu\gamma^\rho\gamma^\nu + \gamma^\nu\gamma^\rho\gamma^\mu + \gamma^\rho\gamma^\mu\gamma^\nu - \gamma^\rho\gamma^\nu\gamma^\mu\right)$.

Решение: Очевидно, что если два каких-либо индекса совпадают, то $\gamma^{\mu\nu\rho} = 0$. Рассмотрим случай, когда все индексы μ, ν, ρ разные. Тогда гамма-матрицы антикоммутируют между собой, откуда $\gamma^{\mu\nu\rho} = \gamma^\mu\gamma^\nu\gamma^\rho$. Далее пусть σ есть индекс, который не равен индексам μ, ν, ρ (например $\mu = 2, \nu = 3, \rho = 0$, тогда $\sigma = 1$), тогда мы можем записать, используя, что $\gamma_\sigma\gamma^\sigma = 1$ (здесь нет суммирования по индексу σ)

$$\gamma^{\mu\nu\rho} = \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\sigma\gamma^\sigma = -\gamma_\sigma\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = i\varepsilon^{\mu\nu\rho\sigma}\gamma_\sigma\left(i\gamma^0\gamma^1\gamma^3\gamma^4\right) = i\varepsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5 \tag{0.33}$$

где мы использовали, что $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = \varepsilon^{\mu\nu\rho\sigma}\gamma^0\gamma^1\gamma^3\gamma^4$, так как все индексы разные.

Проверьте, что $\left(\gamma^5\right)^\dagger = \gamma^5, \quad \left(\gamma^5\right)^2 = 1, \quad \left\{\gamma^5, \gamma^\mu\right\} = 0$.

Решение: Используя, что $\left(\gamma^0\right)^\dagger = \gamma^0$, а $\left(\gamma^i\right)^\dagger = -\gamma^i, i = 1, 2, 3$, имеем:

$$\left(\gamma^5\right)^\dagger = \left(i\gamma^0\gamma^1\gamma^2\gamma^3\right)^\dagger = -i\left(\gamma^3\right)^\dagger\left(\gamma^2\right)^\dagger\left(\gamma^1\right)^\dagger\left(\gamma^0\right)^\dagger = i\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5 \tag{0.34}$$

Также

$$\left(\gamma^5\right)^2 = i\gamma^0\gamma^1\gamma^2\gamma^3 i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = -\gamma^2\gamma^3\gamma^2\gamma^3 = -\gamma^3\gamma^3 = 1 \tag{0.35}$$

И в итоге

$$\left\{\gamma^5, \gamma^\mu\right\} = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 + i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 = 0 \tag{0.36}$$

так как при переносе γ^μ через произведение $\gamma^0\gamma^1\gamma^2\gamma^3$, она антикоммутирует с тремя гамма-матрицами и коммутирует с собой, откуда и появляется знак минус.

Покажите, что $\partial_\mu j^\mu = 0, \partial_\mu j^{\mu 5} = 2im\bar{\psi}\gamma^5\psi$, где $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x), \quad j^{\mu 5}(x) = \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$, учитывая уравнения движения.

Решение: Уравнения движения есть

$$\begin{aligned}&\left(i\gamma^\mu\partial_\mu - m\right)\psi = 0\\ &\left(i\left(\partial_\mu\bar{\psi}\right)\gamma^\mu + m\bar{\psi}\right) = 0\end{aligned} \tag{0.37}$$

далее получаем

$$\partial_\mu j^\mu = \partial_\mu\left(\bar{\psi}(x)\gamma^\mu\psi(x)\right) = \left(\partial_\mu\bar{\psi}(x)\right)\gamma^\mu\psi(x) + \bar{\psi}(x)\gamma^\mu\partial_\mu\psi(x) = im\bar{\psi}(x)\psi(x) - m\bar{\psi}(x)\psi(x) = 0 \tag{0.38}$$

аналогично имеем

$$\partial_\mu j^{\mu 5} = \partial_\mu\left(\bar{\psi}\gamma^\mu\gamma^5\psi\right) = \left(\partial_\mu\bar{\psi}\right)\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5\partial_\mu\psi = im\bar{\psi}\gamma^5\psi + im\bar{\psi}\gamma^5\psi = 2im\bar{\psi}\gamma^5\psi \tag{0.39}$$

что и требовалось показать.

Покажите, что $j^\mu(x)$ и $j^{\mu 5}(x)$ являются нетеровскими токами, соответствующими преобразованиям $\psi(x) \to e^{i\alpha}\psi(x)$ и $\psi(x) \to e^{i\alpha\gamma^5}\psi(x)$.

Решение: Имеем $\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi$ и $\psi \to (1+i\alpha)\psi$, откуда $\delta\psi = i\psi$, и получим

$$j^\mu = -\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\delta\psi = \bar{\psi}\gamma^\mu\psi \tag{0.40}$$

далее $\psi \to \left(1+i\alpha\gamma^5\right)\psi$, откуда $\delta\psi = i\gamma^5\psi$ и получим

$$j^{\mu 5} = -\frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\psi\right)}\delta\psi = \bar{\psi}\gamma^\mu\gamma^5\psi \tag{0.41}$$

что и требовалось показать.

Используя тождество Фирца $\left(\sigma^\mu\right)_{\alpha\beta}\left(\sigma_\mu\right)_{\gamma\delta} = 2\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta}$, покажите, что

$$\left(\bar{u}_{1R}\sigma^\mu u_{2R}\right)\left(\bar{u}_{3R}\sigma_\mu u_{4R}\right) = -\left(\bar{u}_{1R}\sigma^\mu u_{4R}\right)\left(\bar{u}_{3R}\sigma_\mu u_{2R}\right)$$
$$\left(\bar{u}_{1L}\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\lambda u_{2L}\right)\left(\bar{u}_{3L}\bar{\sigma}_\mu\sigma_\nu\bar{\sigma}_\lambda u_{4L}\right) = 16\left(\bar{u}_{1L}\bar{\sigma}^\mu u_{2L}\right)\left(\bar{u}_{3L}\bar{\sigma}_\mu u_{4L}\right)$$

Решение: Используем, что $\left(\sigma^\mu\right)_{\alpha\beta}\left(\sigma_\mu\right)_{\gamma\delta} = 2\varepsilon_{\alpha\gamma}\varepsilon_{\beta\delta} = -2\varepsilon_{\alpha\gamma}\varepsilon_{\delta\beta} = -\left(\sigma^\mu\right)_{\alpha\delta}\left(\sigma_\mu\right)_{\gamma\beta}$, откуда

$$\begin{aligned}\left(\bar{u}_{1R}\sigma^\mu u_{2R}\right)\left(\bar{u}_{3R}\sigma_\mu u_{4R}\right) &= \bar{u}_{1R\alpha}u_{2R\beta}\bar{u}_{3R\gamma}u_{4R\delta}\left(\sigma^\mu\right)_{\alpha\beta}\left(\sigma_\mu\right)_{\gamma\delta} = -\bar{u}_{1R\alpha}u_{2R\beta}\bar{u}_{3R\gamma}u_{4R\delta}\left(\sigma^\mu\right)_{\alpha\delta}\left(\sigma_\mu\right)_{\gamma\beta} = \\ &= -\left(\bar{u}_{1R}\sigma^\mu u_{4R}\right)\left(\bar{u}_{3R}\sigma_\mu u_{2R}\right).\end{aligned} \tag{0.42}$$

Далее имеем

$$\left(\bar{u}_{1L}\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\lambda u_{2L}\right)\left(\bar{u}_{3L}\bar{\sigma}_\mu\sigma_\nu\bar{\sigma}_\lambda u_{4L}\right) = 2\epsilon_{\alpha\gamma}\bar{u}_{1L\alpha}\bar{u}_{3L\gamma}\epsilon_{\beta\delta}\left(\sigma^\nu\bar{\sigma}^\lambda u_{2L}\right)_\beta\left(\sigma_\nu\bar{\sigma}_\lambda u_{4L}\right)_\delta. \tag{0.43}$$

Теперь рассмотрим отдельно

$$\begin{aligned}\epsilon_{\beta\delta}\left(\sigma^\nu\bar{\sigma}^\lambda u_{2L}\right)_\beta\left(\sigma_\nu\bar{\sigma}_\lambda u_{4L}\right)_\delta &= \epsilon_{\beta\delta}\left(\sigma^\nu\right)_{\beta\rho}\left(\bar{\sigma}^\lambda\right)_{\rho\sigma}u_{2L\sigma}\left(\sigma_\nu\bar{\sigma}_\lambda u_{4L}\right)_\delta = -\epsilon_{\delta\beta}\left(\sigma^\nu\right)_{\beta\rho}\left(\bar{\sigma}^\lambda\right)_{\rho\sigma}u_{2L\sigma}\left(\sigma_\nu\bar{\sigma}_\lambda u_{4L}\right)_\delta = \\ &= -\left(\bar{\sigma}^\nu\right)_{\beta\delta}\epsilon_{\beta\rho}\left(\bar{\sigma}^\lambda\right)_{\rho\sigma}u_{2L\sigma}\left(\sigma_\nu\bar{\sigma}_\lambda u_{4L}\right)_\delta = -\left(\bar{\sigma}^\nu\right)_{\beta\delta}\left(\sigma^\lambda\right)_{\rho\beta}\epsilon_{\rho\sigma}u_{2L\sigma}\left(\sigma_\nu\bar{\sigma}_\lambda u_{4L}\right)_\delta = \\ &= -\epsilon_{\rho\sigma}u_{2L\sigma}\left(\sigma^\lambda\right)_{\rho\beta}\left(\bar{\sigma}^\nu\right)_{\beta\delta}\left(\sigma_\nu\bar{\sigma}_\lambda u_{4L}\right)_\delta = \epsilon_{\sigma\rho}u_{2L\sigma}\left(\sigma^\lambda\bar{\sigma}^\nu\sigma_\nu\bar{\sigma}_\lambda u_{4L}\right)_\rho = \\ &= (4)^2\epsilon_{\sigma\rho}u_{2L\sigma}u_{4L\rho},\end{aligned} \tag{0.44}$$

где мы использовали, что $\epsilon_{\alpha\beta}\left(\sigma^\mu\right)_{\beta\gamma} = \left(\bar{\sigma}^\mu\right)_{\beta\alpha}\epsilon_{\beta\gamma}$ и $\sigma^\mu\bar{\sigma}_\mu = \bar{\sigma}^\mu\sigma_\mu = 4$. Теперь получаем

$$\begin{aligned}\left(\bar{u}_{1L}\bar{\sigma}^\mu\sigma^\nu\bar{\sigma}^\lambda u_{2L}\right)\left(\bar{u}_{3L}\bar{\sigma}_\mu\sigma_\nu\bar{\sigma}_\lambda u_{4L}\right) &= 2\epsilon_{\alpha\gamma}\bar{u}_{1L\alpha}\bar{u}_{3L\gamma}\epsilon_{\beta\delta}\left(\sigma^\nu\bar{\sigma}^\lambda u_{2L}\right)_\beta\left(\sigma_\nu\bar{\sigma}_\lambda u_{4L}\right)_\delta = \\ &= 2\cdot(4)^2\epsilon_{\alpha\gamma}\bar{u}_{1L\alpha}\bar{u}_{3L\gamma}\epsilon_{\sigma\rho}u_{2L\sigma}u_{4L\rho} = \\ &= 16\bar{u}_{1L\alpha}\bar{u}_{3L\gamma}u_{2L\sigma}u_{4L\rho}\left(\bar{\sigma}^\mu\right)_{\alpha\sigma}\left(\bar{\sigma}_\mu\right)_{\gamma\rho} = \\ &= 16\left(\bar{u}_{1L}\bar{\sigma}^\mu u_{2L}\right)\left(\bar{u}_{3L}\bar{\sigma}_\mu u_{4L}\right),\end{aligned} \tag{0.45}$$

что и требовалось показать.

Белавин-Пархоменко.6. Поле Дирака

Все задачи в этом листке взяты из книги Пескина Шредера, глава 3.

Белавин-Пархоменко.6.1 Представления группы Лоренца

Итак коммутационные соотношения для генераторов алгебры Лоренца:

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right). \tag{0.1}$$

(a). Определим генераторы поворотов и бустов по формулам

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk}, \quad K^i = J^{0i} \tag{0.2}$$

где $i, j, k = 1, 2, 3$.

Инфинитезимальное преобразование Лоренца может быть записано как

$$\Phi \to (1 - i\boldsymbol{\theta}\cdot\boldsymbol{L} - i\boldsymbol{\beta}\cdot\boldsymbol{K})\Phi. \tag{0.3}$$

Запишите коммутационные соотношения для L^i и K^i явно. Покажите, что комбинации

$$\boldsymbol{J}_+ = \frac{1}{2}(\boldsymbol{L} + i\boldsymbol{K}), \quad \boldsymbol{J}_- = \frac{1}{2}(\boldsymbol{L} - i\boldsymbol{K}) \tag{0.4}$$

коммутируют друг с другом, и по отдельности удовлетворяют коммутационным соотношениям момента импульса.

(b). Конечномерные представления группы вращений нумеруются допустимыми значениям орбитального момента: целыми и полуцелыми числами. Из результата части (a) следуют, что все конечномерные представления группы Лоренца соответствуют парам целых или полуцелых чисел, (j_+, j_-), которые в свою очередь соответствуют паре представлений группы поворотов. Используя то, что $\boldsymbol{J} = \boldsymbol{\sigma}/2$ для представления спина- 1/2, запишите явно закон преобразования для 2 -компонентных объектов преобразующихся по представлениям $\left(\frac{1}{2}, 0\right)$ и $\left(0, \frac{1}{2}\right)$ группы Лоренца. Покажите, что это есть в точности преобразования для ψ_L и ψ_R:

$$\psi_L \to \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_L, \quad \psi_R \to \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_R. \tag{0.5}$$

(c). Тождество $\boldsymbol{\sigma}^T = -\sigma^2\boldsymbol{\sigma}\sigma^2$ позволяет переписать преобразование для ψ_L в эквивалентной форме

$$\psi' \to \psi'\left(1 + i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right), \tag{0.6}$$

где $\psi' = \psi_L^T\sigma^2$. Используя этот закон, мы можем записать объект, который преобразуется по представлению $\left(\frac{1}{2}, \frac{1}{2}\right)$ как 2×2 матрица, которая преобразуется по закону ψ_R с левой стороны и одновременно преобразуется по закону транспонированного ψ_L с правой. Запишите эту матрицу в виде

$$\begin{pmatrix} V^0 + V^3 & V^1 - iV^2 \\ V^1 + iV^2 & V^0 - V^3 \end{pmatrix}. \tag{0.7}$$

Покажите, что объект V^μ преобразуется как 4-вектор.

Белавин-Пархоменко.6.2 Тождество Гордона

Выведите тождество

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) \tag{0.8}$$

где $q = (p' - p)$.

Белавин-Пархоменко.6.3 Произведение спиноров

Пусть k_0^μ, k_1^μ фиксированные 4-векторы, удовлетворяющие $k_0^2 = 0, k_1^2 = -1, k_0 \cdot k_1 = 0$. Определим базис спиноров следующим способом: Пусть u_{L0} будет левополяризованным спинором для фермиона с импульсом k_0. Пусть $u_{R0} = k_1 u_{L0}$. Тогда для любого светоподобного импульса $p\,(p^2 = 0)$, определим:

$$u_L(p) = \frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{R0}, \quad u_R(p) = \frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{L0} \tag{0.9}$$

Такой набор условий определяет спиноры однозначно (кроме случая, когда p параллелен k_0).
(a). Покажите, что $\ell_0 u_{R0} = 0$. Покажите, что для любого светоподобного p $\not{u}_L(p) = \not{u}_R(p) = 0$.
(b). Для $k_0 = (E, 0, 0, -E), k_1 = (0, 1, 0, 0)$, постройте $u_{L0}, u_{R0}, u_L(p)$ и $u_R(p)$ явно.
(c). Определим спинорные произведения $s\left(p_1, p_2\right)$ и $t\left(p_1, p_2\right)$, для светоподоных p_1, p_2, по формулам

$$s\left(p_1, p_2\right) = \bar{u}_R\left(p_1\right) u_L\left(p_2\right), \quad t\left(p_1, p_2\right) = \bar{u}_L\left(p_1\right) u_R\left(p_2\right) \tag{0.10}$$

Используя явные выражения для u_λ в части (b), вычислите спинорные произведения явно и покажите, что $t\left(p_1, p_2\right) = \left(s\left(p_2, p_1\right)\right)^*$ и $s\left(p_1, p_2\right) = -s\left(p_2, p_1\right)$. Также покажите, что

$$\left|s\left(p_1, p_2\right)\right|^2 = 2 p_1 \cdot p_2 \tag{0.11}$$

Откуда видно, что спиноры это квадратные корни из скалярного произведения 4-векторов.

Белавин-Пархоменко.6.4 Майорановские фермионы

Релятивистское уравнение для безмассового 2-компонентного фермионного поля, которое преобразутеся как верхние две компоненты Дираковского спинора (ψ_L) можно записать как:

$$i\left(\partial_0 - \boldsymbol{\sigma} \cdot \boldsymbol{\nabla}\right) \psi_L = 0 \tag{0.12}$$

Обозначим такое двухкомпонентное поле $\chi_a(x), a = 1, 2$.
(a). Покажите, что можно записать уравнение для $\chi(x)$, как для массивного поля в следующем виде

$$i\bar{\sigma} \cdot \partial \chi - im\sigma^2 \chi^* = 0 \tag{0.13}$$

То есть, покажите вначале, что это уравнение является релятивистски инвариантным, а затем, что из него можно получить уравнение Клейна-Гордона $\left(\partial^2 + m^2\right) \chi = 0$. Такая форма фермионной массы называется Майорановское массовое слагаемое.
(b). Следует ли уравнение Майорана из лагранжиана? Казалось бы, массовое слагаемое, получается варьированием выражения $\left(\sigma^2\right)_{ab} \chi_a^* \chi_b^*$; Однако, так как σ^2 - антисимметричная матрица, это выражение обратилось бы в нуль, если бы $\chi(x)$ было бы обычным c-числовым полем. Известно, что при переходе к квантовой теории поля $\chi(x)$ становится антикоммутирующим квантовым полем. Следовательно, имеет смысл развивать классическую теорию поля, рассматривая $\chi(x)$ как классическое антикоммутирующее поле, то есть как поле, которое принимает значения в грассмановых числах, удовлетворяющих условиям:

$$\alpha\beta = -\beta\alpha, \quad \text{для любых } \alpha, \beta. \tag{0.14}$$

Заметим, что из этих соотношений следует, что $\alpha^2 = 0$. Грассманово поле $\xi(x)$ может быть разложено по базису функций как

$$\xi(x) = \sum_n \alpha_n \phi_n(x) \tag{0.15}$$

где $\phi_n(x)$ - ортогональные c-числовые функции и α_n - набор независимых грассмановых чисел. Определим комплексное сопряжение произведения грассмановых чисел как обращающее их порядок:

$$(\alpha\beta)^* \equiv \beta^* \alpha^* = -\alpha^* \beta^*. \tag{0.16}$$

Это правило по форме совпадает с эрмитовым сопряжением квантовых полей. Покажите, что классическое действие

$$S = \int d^4x \left[\chi^\dagger i\bar{\sigma} \cdot \partial \chi + \frac{im}{2}\left(\chi^T \sigma^2 \chi - \chi^\dagger \sigma^2 \chi^*\right)\right] \tag{0.17}$$

(где $\chi^\dagger = \left(\chi^*\right)^T$) является вещественным ($S^* = S$) и варьирование S относительно χ и χ^* приводит к уравнению Майорана.
(c). Запишем 4-компонентное дираковское поле

$$\psi(x) = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{0.18}$$

и напомним, что нижняя компонента ψ преобразуется по представлению, унитарно эквивалентному комплексно-сопряженному представлению ψ_L. Таким образом, можно переписать 4 -компонетное дираковское поле в терминах двух 2-компонентных спиноров:

$$\psi_L(x) = \chi_1(x), \quad \psi_R(x) = i\sigma^2 \chi_2^*(x). \tag{0.19}$$

Перепешите лагранжиан Дирака в терминах χ_1 и χ_2 и обратите внимание на форму массового слагаемого.
(d). Покажите, что действие пункта (c) имеет глобальную симметрию. Вычислите дивергенции токов

$$J^\mu = \chi^\dagger \bar{\sigma}^\mu \chi, \quad J^\mu = \chi_1^\dagger \bar{\sigma}^\mu \chi_1 - \chi_2^\dagger \bar{\sigma}^\mu \chi_2 \tag{0.20}$$

для теорий пунктов (b) и (c) соответственно, и свяжите результаты с симметриями этих теорий. Постройте теорию из N свободных массивных 2-компонентных фермионных полей с $O(N)$ симметрией (то есть с вращательной симметрией в N-мерном пространстве).
(e). Проквантуйте теорию Майорана пунктов (a) и (b). Иными словами, рассмотрите $\chi(x)$ как квантовое поле, удовлетворяющее каноническому антикоммутационному соотношению

$$\left\{\chi_a(\boldsymbol{x}), \chi_b^\dagger(\boldsymbol{y})\right\} = \delta_{ab}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{0.21}$$

Постройте эрмитовый гамильтониан и найдите представление канонических коммутационных соотношений, которое диагонализует гамильтониан в терминах набора операторов рождения и уничтожения. (Подсказка: сравните поле $\chi(x)$ с двумя верхними компонентами проквантованного дираковского поля)

Упражнения к Лекции 6. Поле Дирака

Упражнения идут по ходу главы 3 Пескина Шредера, каждое упражнение оценивается в 2.5 балла.

31 Лоренц инвариантность волновых уравнений.

31

Исходя из формулы для генераторов вращения $J^{\mu\nu} = i\left(x^\mu \partial^\nu - x^\nu \partial^\mu\right)$, показать, что для них выполняются следующие коммутационные соотношения алгебры Лоренца

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right).$$

Решение: Используем, что $\partial_\mu x^\nu = \delta_\mu^\nu$, откуда $\partial^\mu x^\nu = g^{\mu\nu}$, имеем

$$\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] &= -\left[\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right), \left(x^\rho\partial^\sigma - x^\sigma\partial^\rho\right)\right] = -\left[x^\mu\partial^\nu, x^\rho\partial^\sigma\right] + \ldots = \\
&= -g^{\nu\rho}x^\mu\partial^\sigma + g^{\mu\sigma}x^\rho\partial^\nu + \ldots = ig^{\nu\rho}\left(ix^\mu\partial^\rho + \ldots\right) + ig^{\mu\sigma}\left(\ldots - ix^\rho\partial^\nu\right) + \ldots = \\
&= i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right),
\end{aligned} \tag{0.1}$$

что и тербовалось показать.

26.1.9 Problems about spinors and Dirac equation 2

32 Уравнение Дирака.

2

Показать, что генераторы алгебры Лоренца в представлении гамма-матриц: $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$, удовлетворяет коммутационным соотношениям алгебры Лоренца:

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\left(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}\right).$$

Решение: Используем, что $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, имеем

$$\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] &= -\frac{1}{16}\left[\gamma^\mu\gamma^\nu, \gamma^\rho\gamma^\sigma\right] + \ldots = -\frac{1}{16}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu\gamma^\nu\right) + \ldots = \\
&= -\frac{1}{16}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma + \gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu - 2g^{\mu\sigma}\gamma^\rho\gamma^\nu\right) + \ldots = \\
&= -\frac{1}{16}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma - \gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu + 2g^{\mu\rho}\gamma^\sigma\gamma^\nu - 2g^{\mu\sigma}\gamma^\rho\gamma^\nu\right) + \ldots = \\
&= -\frac{1}{16}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma + \gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma - 2g^{\nu\sigma}\gamma^\mu\gamma^\rho + 2g^{\mu\rho}\gamma^\sigma\gamma^\nu - 2g^{\mu\sigma}\gamma^\rho\gamma^\nu\right) + \ldots = \\
&= -\frac{1}{16}\left(2g^{\nu\rho}\gamma^\mu\gamma^\sigma - 2g^{\nu\sigma}\gamma^\mu\gamma^\rho + 2g^{\mu\rho}\gamma^\sigma\gamma^\nu - 2g^{\mu\sigma}\gamma^\rho\gamma^\nu\right) + \ldots = \\
&= i\left(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}\right)
\end{aligned} \tag{0.2}$$

3

Покажите, что в киральном представлении гамма-матриц: $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$, генераторы алгебры Лоренца $S^{\mu\nu}$, даются выражениями

$$S^{0i} = \frac{i}{4}\left[\gamma^0, \gamma^i\right] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \quad S^{ij} = \frac{i}{4}\left[\gamma^i, \gamma^j\right] = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\varepsilon^{ijk}\Sigma^k.$$

Решение: Имеем

$$\begin{aligned} S^{0i} = \frac{i}{4}\left[\gamma^0, \gamma^i\right] = \frac{i}{4}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \\ = \frac{i}{4}\left(\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} - \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}\right) = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \end{aligned} \tag{0.3}$$

И также находим

$$\begin{aligned} S^{ij} = \frac{i}{4}\left[\gamma^i, \gamma^j\right] = \frac{i}{4}\left(\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\right) = \\ = \frac{i}{4}\left(\begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} - \begin{pmatrix} -\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i \end{pmatrix}\right) = -\frac{i}{4}\begin{pmatrix} \left[\sigma^i, \sigma^j\right] & 0 \\ 0 & \left[\sigma^i, \sigma^j\right] \end{pmatrix} = \\ = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\varepsilon^{ijk}\Sigma^k, \end{aligned} \tag{0.4}$$

где мы использовали то, что $\left[\sigma^i, \sigma^j\right] = 2i\varepsilon^{ijk}\sigma^k$ и $\varepsilon^{123} = 1$.◯

4

Покажите, что $[\gamma^\mu, S^{\rho\sigma}] = (\mathcal{J}^{\rho\sigma})^\mu_\nu \gamma^\nu$, где $(\mathcal{J}^{\rho\sigma})_{\alpha\beta} = i\left(\delta^\rho_\alpha\delta^\sigma_\beta - \delta^\rho_\beta\delta^\sigma_\alpha\right)$.
Решение: Имеем

$$\begin{aligned} [\gamma^\mu, S^{\rho\sigma}] = \frac{i}{4}\left[\gamma^\mu, [\gamma^\rho, \gamma^\sigma]\right] = \frac{i}{4}\left[\gamma^\mu, \gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho\right] = \frac{i}{4}\left([\gamma^\mu, \gamma^\rho\gamma^\sigma] - [\gamma^\mu, \gamma^\sigma\gamma^\rho]\right) = \\ = \frac{i}{4}\left(\gamma^\mu\gamma^\rho\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu - \gamma^\mu\gamma^\sigma\gamma^\rho + \gamma^\sigma\gamma^\rho\gamma^\mu\right) = \\ = \frac{i}{4}\left(-\gamma^\rho\gamma^\mu\gamma^\sigma + 2g^{\mu\rho}\gamma^\sigma - \gamma^\rho\gamma^\sigma\gamma^\mu + \gamma^\sigma\gamma^\mu\gamma^\rho - 2g^{\mu\sigma}\gamma^\rho + \gamma^\sigma\gamma^\rho\gamma^\mu\right) = \\ = \frac{i}{4}\left(-2g^{\mu\sigma}\gamma^\rho + 2g^{\mu\rho}\gamma^\sigma + 2g^{\mu\rho}\gamma^\sigma - 2g^{\mu\sigma}\gamma^\rho\right) = i\left(g^{\mu\rho}\gamma^\sigma - g^{\mu\sigma}\gamma^\rho\right) = \\ = i\left(g^{\rho\mu}\delta^\sigma_\nu - g^{\sigma\mu}\delta^\rho_\nu\right)\gamma^\nu = (\mathcal{J}^{\rho\sigma})^\mu_\nu\gamma^\nu. \end{aligned} \tag{0.5}$$

5

Покажите, что верно равенство

$$\left(1 + \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^\mu\left(1 - \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right) = \left(1 - \frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}\right)^\mu_\nu\gamma^\nu$$

Решение: Имеем
$\left(1 + \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right)\gamma^\mu\left(1 - \frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\right) = \gamma^\mu + \frac{i}{2}\omega_{\rho\sigma}\left[S^{\rho\sigma}, \gamma^\mu\right] + O\left(\omega^2\right) = \left(1 - \frac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma}\right)^\mu_\nu\gamma^\nu + O\left(\omega^2\right)$,
где мы использовали результат предыдущего упражнения.

6

Проверьте, что $\left(S^{ij}\right)^{\dagger}=S^{ij},\left(S^{0i}\right)^{\dagger}=-S^{0i},\quad \left(S^{ij}\right)^{\dagger}\gamma^{0}=\gamma^{0}S^{ij},\quad \left(S^{0i}\right)^{\dagger}\gamma^{0}=-\gamma^{0}S^{0i}$.

Решение: Используя результат упражнения 3, а также $\left(\sigma^{i}\right)^{\dagger}=\sigma^{i}$, очевидно, что $\left(S^{ij}\right)^{\dagger}=S^{ij},\left(S^{0i}\right)^{\dagger}=-S^{0i}$. Далее, используя, что $\left(\gamma^{0}\right)^{\dagger}=\gamma^{0}$, можно получить остальные равенства. Также все можно было получить непостредственно из определения $S^{\mu\nu}$ и из того, что $\left(\gamma^{0}\right)^{\dagger}=\gamma^{0}$ и $\left(\gamma^{i}\right)^{\dagger}=-\gamma^{i}$.

7

Покажите, что $\bar{\psi}\psi$ есть лоренцевский скаляр, а $\bar{\psi}\gamma^{\mu}\psi$ лоренцевский вектор, где $\bar{\psi}\equiv\psi^{\dagger}\gamma^{0}$.
Решение: При преобразовании Лоренца $x^{\mu}\to\Lambda_{\nu}^{\mu}x^{\nu}$, спинор преобразуется как

$$\psi(x)\to\Lambda_{\frac{1}{2}}\psi(x),\qquad \text{где}\qquad \Lambda_{\frac{1}{2}}=\exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right) \tag{0.7}$$

тогда имеем для $\bar{\psi}$

$$\bar{\psi}=\psi^{\dagger}\gamma^{0}\to\psi^{\dagger}\exp\left(\frac{i}{2}\omega_{\mu\nu}\left(S^{\mu\nu}\right)^{\dagger}\right)\gamma^{0}=\psi^{\dagger}\gamma^{0}\exp\left(\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)=\bar{\psi}\Lambda_{\frac{1}{2}}^{-1} \tag{0.8}$$

где мы использовали результаты упражнения 6. Далее имеем

$$\bar{\psi}\psi\to\bar{\psi}\Lambda_{\frac{1}{2}}^{-1}\Lambda_{\frac{1}{2}}\psi=\bar{\psi}\psi \tag{0.9}$$

и также

$$\bar{\psi}\gamma^{\mu}\psi\to\bar{\psi}\Lambda_{\frac{1}{2}}^{-1}\gamma^{\mu}\Lambda_{\frac{1}{2}}\psi=\Lambda_{\nu}^{\mu}\bar{\psi}\gamma^{\nu}\psi \tag{0.10}$$

где мы использовали результат упражнения 5.

8

Покажите, что уравнения Эйлера-Лагранжа для $\bar{\psi}$ (или $\psi^{\dagger}$) для лагранжиана $\mathcal{L}_{\text{Dirac}}=\bar{\psi}\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi$ приводят к уравнению Дирака $\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi(x)=0$.
Решение: Получаем из $\frac{\partial\mathcal{L}}{\partial\bar{\psi}}=0$:

$$\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi(x)=0 \tag{0.11}$$

32 Вейлевские спиноры.

9

Проверьте, что для левого ψ_{L} и правого ψ_{R} вейлевских спиноров, при бесконечно мылых вращениях $\boldsymbol{\theta}\left(\theta_{i}=\frac{1}{2}\varepsilon^{ijk}\omega_{jk}\right)$ и бустах $\boldsymbol{\beta}\left(\beta_{i}=\omega_{0i}\right)$, преобразования имеют вид

$$\psi_{\text{L}}\to\left(1-i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}-\boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_{L}$$
$$\psi_{\text{R}}\to\left(1-i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}+\boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_{R}.$$

Решение: Имеем

$$\begin{aligned}\begin{pmatrix}\psi_{L}\\ \psi_{R}\end{pmatrix}=\psi\to\left(1-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\psi&=\left(1-i\omega_{0i}S^{0i}-\frac{i}{2}\omega_{ij}S^{ij}\right)\begin{pmatrix}\psi_{L}\\ \psi_{R}\end{pmatrix}=\\ &=\left(1-\frac{1}{2}\begin{pmatrix}\omega_{0i}\sigma^{i} & 0\\ 0 & -\omega_{0i}\sigma^{i}\end{pmatrix}-\frac{i}{4}\omega_{ij}\varepsilon^{ijk}\begin{pmatrix}\sigma^{k} & 0\\ 0 & \sigma^{k}\end{pmatrix}\right)\begin{pmatrix}\psi_{L}\\ \psi_{R}\end{pmatrix}=\\ &=\left(1-\begin{pmatrix}\beta_{i}\frac{\sigma^{i}}{2}+i\theta_{i}\frac{\sigma^{i}}{2} & 0\\ 0 & -\beta_{i}\frac{\sigma^{i}}{2}+i\theta_{i}\frac{\sigma^{i}}{2}\end{pmatrix}\right)\begin{pmatrix}\psi_{L}\\ \psi_{R}\end{pmatrix}.\end{aligned} \tag{0.12}$$

В итоге получим преобразование $\psi_{\text{L}}\to\left(1-i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}-\boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_{L}$, и $\psi_{\text{R}}\to\left(1-i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2}+\boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_{R}\cdot\bigcirc$

10

Докажите тождество $\sigma^2\boldsymbol{\sigma}^* = -\boldsymbol{\sigma}\sigma^2$, и покажите, что величина $\sigma^2\psi_{\mathrm{L}}^*$ преобразуется подобно правому спинору.

Решение: Используя, что $\left(\sigma^1\right)^* = \sigma^1, \left(\sigma^2\right)^* = -\sigma^2, \left(\sigma^3\right)^* = \sigma^3$, а также $\left\{\sigma^i, \sigma^j\right\} = 2\delta^{ij}$, легко проверить, что $\sigma^2\boldsymbol{\sigma}^* = -\boldsymbol{\sigma}\sigma^2$. Далее получаем

$$\sigma^2\psi_{\mathrm{L}}^* \to \sigma^2\left(1 + i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_L^* = \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\sigma^2\psi_L^* \tag{0.13}$$

В итоге видим, что $\sigma^2\psi_{\mathrm{L}}^* \to \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\sigma^2\psi_L^*$, как и $\psi_{\mathrm{R}} \to \left(1 - i\boldsymbol{\theta}\cdot\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\beta}\cdot\frac{\boldsymbol{\sigma}}{2}\right)\psi_R$.

35 Квантование Дираковского поля.

25

Покажите, что плотность Дираковского гамильтониана дается выражением $\mathcal{H} = \bar{\psi}(-i\gamma.\ \nabla + m)\psi$.
Решение: Для канонического момента p имеем

$$p = \frac{\partial\mathcal{L}}{\partial_0\psi} = \bar{\psi}i\gamma^0 = i\psi^\dagger\gamma^0\gamma^0 = i\psi^\dagger \tag{0.46}$$

Далее для Гамильтоновой плотности получаем

$$\mathcal{H} = p\partial_0\psi - \mathcal{L} = i\bar{\psi}\gamma^0\partial_0\psi - \bar{\psi}\left(i\gamma^0\partial_0 + i\gamma^i\partial_i - m\right)\psi = \bar{\psi}(-i\gamma\cdot\nabla + m)\psi, \tag{0.47}$$

что и требовалось показать.

Покажите, что для данного "неправильного"дираковского поля

$$\psi(\mathbf{x}) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}e^{i\mathbf{px}}\sum_{s=1,2}\left(a_{\mathbf{p}}^s u^s(\mathbf{p}) + b_{-\mathbf{p}}^s v^s(-\mathbf{p})\right)$$

выполняются соотношения $\left[\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})\right] = \delta^{(3)}(\mathbf{x}-\mathbf{y})\times\mathbf{1}_{4\times4}$. Используйте коммутационные соотношения $\left[a_{\boldsymbol{p}}^r, a_{\boldsymbol{q}}^{s\dagger}\right] = \left[b_{\boldsymbol{p}}^r, b_{\boldsymbol{q}}^{s^\dagger}\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\delta^{rs}$.
Решение: Имеем

$$\begin{aligned}\left[\psi(\mathbf{x}), \psi^\dagger(\mathbf{y})\right] &= \int\frac{d^3pd^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}e^{i(\boldsymbol{p}\cdot\boldsymbol{x}-\boldsymbol{q}\cdot\boldsymbol{y})}\times\sum_{r,s}\left(\left[a_{\boldsymbol{p}}^r, a_{\boldsymbol{q}}^{s\dagger}\right]u^r(\boldsymbol{p})\bar{u}^s(\boldsymbol{q}) + \left[b_{-\boldsymbol{p}}^r, b_{-\boldsymbol{q}}^{s\dagger}\right]v^r(-\boldsymbol{p})\bar{v}^s(-\boldsymbol{q})\right)\gamma^0 = \\ &= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\left[\left(\gamma^0E_{\boldsymbol{p}} - \gamma\cdot\boldsymbol{p} + m\right) + \left(\gamma^0E_{\boldsymbol{p}} + \gamma\cdot\boldsymbol{p} - m\right)\right]\gamma^0 = \\ &= \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\times\mathbf{1}_{4\times4},\end{aligned} \tag{0.48}$$

что и требовалось показать.

27

Покажите, что для волновой функции из пункта 26, гамильтониан дается выражением

$$H = \int\frac{d^3p}{(2\pi)^3}\sum_s\left(E_{\mathbf{p}}a_{\mathbf{p}}^{s\dagger}a_{\mathbf{p}}^s - E_{\mathbf{p}}b_{\mathbf{p}}^{s\dagger}b_{\mathbf{p}}^s\right).$$

Решение: Имеем

$$H = \int d^3x\bar{\psi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\psi \tag{0.49}$$

где

$$\psi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}e^{i\mathbf{px}}\sum_{s=1,2}\left(a^s_{\mathbf{p}}u^s(\mathbf{p}) + b^s_{-\mathbf{p}}v^s(-\mathbf{p})\right),$$
$$\bar{\psi}(\boldsymbol{y}) = \psi^\dagger(\mathbf{y})\gamma^0 = \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}e^{-i\mathbf{qy}}\sum_{r=1,2}\left(a^{r\dagger}_{\mathbf{q}}\bar{u}^r(\mathbf{q}) + b^{r\dagger}_{-\mathbf{q}}\bar{v}^r(-\mathbf{q})\right). \tag{0.50}$$

Далее получаем

$$H = \int d^3x\bar{\psi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\psi =$$
$$= \int \frac{d^3pd^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}(2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{q})\sum_{r,s=1}^{2}\left(a^{r\dagger}_{\mathbf{q}}\bar{u}^r(\mathbf{q}) + b^{r\dagger}_{-\mathbf{q}}\bar{v}^r(-\mathbf{q})\right)(\boldsymbol{\gamma}\cdot\boldsymbol{p} + m)\left(a^s_{\mathbf{p}}u^s(\mathbf{p}) + b^s_{-\mathbf{p}}v^s(-\mathbf{p})\right) =$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_{r,s=1}^{2}\left(a^{r\dagger}_{\mathbf{p}}\bar{u}^r(\mathbf{p}) + b^{r\dagger}_{-\mathbf{p}}\bar{v}^r(-\mathbf{p})\right)(\boldsymbol{\gamma}\cdot\boldsymbol{p} + m)\left(a^s_{\mathbf{p}}u^s(\mathbf{p}) + b^s_{-\mathbf{p}}v^s(-\mathbf{p})\right) \tag{0.51}$$

далее используем из уравнений движения, что $(\boldsymbol{\gamma}\cdot\boldsymbol{p} + m)u^s(\boldsymbol{p}) = \gamma^0E_{\boldsymbol{p}}u^s(\boldsymbol{p})$ и $(\boldsymbol{\gamma}\cdot\boldsymbol{p} + m)v^s(-\boldsymbol{p}) = -\gamma^0E_{\boldsymbol{p}}v^s(-\boldsymbol{p})$ получим

$$H = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\sum_{r,s=1}^{2}\left(a^{r\dagger}_{\mathbf{p}}u^{r\dagger}(\mathbf{p}) + b^{r\dagger}_{-\mathbf{p}}v^{r\dagger}(-\mathbf{p})\right)\left(a^s_{\mathbf{p}}u^s(\mathbf{p}) - b^s_{-\mathbf{p}}v^s(-\mathbf{p})\right) \tag{0.52}$$

теперь используем, что $u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p}) = v^{r\dagger}(-\mathbf{p})u^s(\mathbf{p}) = 0$ и получим

$$H = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\sum_{r,s=1}^{2}\left(a^{r\dagger}_{\mathbf{p}}a^s_{\mathbf{p}}u^{r\dagger}(\mathbf{p})u^s(\mathbf{p}) - b^{r\dagger}_{-\mathbf{p}}b^s_{-\mathbf{p}}v^{r\dagger}(-\mathbf{p})v^s(-\mathbf{p})\right) \tag{0.53}$$

далее используя, что $u^{r\dagger}(\boldsymbol{p})u^s(\boldsymbol{p}) = v^{r\dagger}(\boldsymbol{p})v^s(\boldsymbol{p}) = 2E_{\boldsymbol{p}}\delta^{rs}$, в итоге находим

$$H = \int \frac{d^3p}{(2\pi)^3}\sum_s\left(E_{\mathbf{p}}a^{s\dagger}_{\mathbf{p}}a^s_{\mathbf{p}} - E_{\mathbf{p}}b^{s\dagger}_{\mathbf{p}}b^s_{\mathbf{p}}\right) \tag{0.54}$$

что и требовалось показать.

Покажите, что $\left[\psi_a(x), \bar{\psi}_b(y)\right] = \left(i\partial\!\!\!/_x + m\right)_{ab}\left[\phi(x), \phi(y)\right]$, где $\phi(x), \phi(y)$ - поля Клейна-Гордона, а ψ_a и $\bar{\psi}_b$ поля из пункта 26 в Гейзенберговском представлении.

Решение: Имеем

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1,2}\left(a^s_{\boldsymbol{p}}u^s(p)e^{-ip\cdot x} + b^s_{\boldsymbol{p}}v^s(p)e^{ip\cdot x}\right)$$
$$\bar{\psi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1,2}\left(a^{s\dagger}_{\boldsymbol{p}}\bar{u}^s(p)e^{ip\cdot x} + b^{s\dagger}_{\boldsymbol{p}}\bar{v}^s(p)e^{-ip\cdot x}\right) \tag{0.55}$$

тогда получим

$$\begin{aligned}\left[\psi_a(x),\bar{\psi}_b(y)\right] &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_{s=1,2}\left(u_a^s(p)\bar{u}_b^s(p)e^{-ip\cdot(x-y)}+v_a^s(p)\bar{v}_b^s(p)e^{ip\cdot(x-y)}\right)= \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\left((\not{p}+m)_{ab}e^{-ip\cdot(x-y)}+(\not{p}-m)_{ab}e^{ip\cdot(x-y)}\right)= \\ &= \left(i\not{\partial}_x+m\right)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\left(e^{-ip\cdot(x-y)}-e^{ip\cdot(x-y)}\right)= \\ &= \left(i\not{\partial}_x+m\right)_{ab}[\phi(x),\phi(y)], \end{aligned} \tag{0.56}$$

что и требовалось показать.

29

Покажите, что для "правильно"квантованного Дираковского поля

$$\psi(x)=\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s\left(a_{\mathbf{p}}^s u^s(p)e^{-ip\cdot x}+b_{\mathbf{p}}^{s\dagger}v^s(p)e^{ip\cdot x}\right)$$
$$\bar{\psi}(x)=\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s\left(b_{\mathbf{p}}^s \bar{v}^s(p)e^{-ip\cdot x}+a_{\mathbf{p}}^{s\dagger}\bar{u}^s(p)e^{ip\cdot x}\right)$$

гамильтониан дается выражением

$$H=\int \frac{d^3p}{(2\pi)^3}\sum_s\left(E_{\mathbf{p}}a_{\mathbf{p}}^{s\dagger}a_{\mathbf{p}}^s+E_{\mathbf{p}}b_{\mathbf{p}}^{s\dagger}b_{\mathbf{p}}^s\right)$$

где операторы антикоммутируют как $\left\{a_{\boldsymbol{p}}^r,a_{\boldsymbol{q}}^{s\dagger}\right\}=\left\{b_{\boldsymbol{p}}^r,b_{\boldsymbol{q}}^{s\dagger}\right\}=(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\delta^{rs}$.
Решение: Имеем

$$H=\int d^3x\bar{\psi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\psi \tag{0.57}$$

далее получаем

$$\begin{aligned} H &= \int d^3x\bar{\psi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+m)\psi= \\ &= \int \frac{d^3pd^3q}{(2\pi)^6}\frac{1}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}(2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{q})\sum_{r,s=1}^{2}\left(a_{\mathbf{q}}^{r\dagger}\bar{u}^r(\mathbf{q})+b_{-\mathbf{q}}^r\bar{v}^r(-\mathbf{q})\right)(\boldsymbol{\gamma}\cdot\boldsymbol{p}+m)\left(a_{\mathbf{p}}^s u^s(\mathbf{p})+b_{-\mathbf{p}}^{s\dagger}v^s(-\mathbf{p})\right)= \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_{r,s=1}^{2}\left(a_{\mathbf{p}}^{r\dagger}\bar{u}^r(\mathbf{p})+b_{-\mathbf{p}}^r\bar{v}^r(-\mathbf{p})\right)(\boldsymbol{\gamma}\cdot\boldsymbol{p}+m)\left(a_{\mathbf{p}}^s u^s(\mathbf{p})+b_{-\mathbf{p}}^{s\dagger}v^s(-\mathbf{p})\right) \end{aligned} \tag{0.58}$$

далее используем из уравнений движения, что $(\boldsymbol{\gamma}\cdot\boldsymbol{p}+m)u^s(\boldsymbol{p})=\gamma^0E_{\boldsymbol{p}}u^s(\boldsymbol{p})$ и $(\boldsymbol{\gamma}\cdot\boldsymbol{p}+m)v^s(-\boldsymbol{p})=-\gamma^0E_{\boldsymbol{p}}v^s(-\boldsymbol{p})$ получим

$$H=\int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\sum_{r,s=1}^{2}\left(a_{\mathbf{p}}^{r\dagger}u^{r\dagger}(\mathbf{p})+b_{-\mathbf{p}}^r v^{r\dagger}(-\mathbf{p})\right)\left(a_{\mathbf{p}}^s u^s(\mathbf{p})-b_{-\mathbf{p}}^{s\dagger}v^s(-\mathbf{p})\right) \tag{0.59}$$

теперь используем, что $u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p})=v^{r\dagger}(-\mathbf{p})u^s(\mathbf{p})=0$ и получим

$$H=\int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\sum_{r,s=1}^{2}\left(a_{\mathbf{p}}^{r\dagger}a_{\mathbf{p}}^s u^{r\dagger}(\mathbf{p})u^s(\mathbf{p})-b_{-\mathbf{p}}^r b_{-\mathbf{p}}^{s\dagger}v^{r\dagger}(-\mathbf{p})v^s(-\mathbf{p})\right) \tag{0.60}$$

далее используя, что $u^{r\dagger}(\boldsymbol{p})u^s(\boldsymbol{p})=v^{r\dagger}(\boldsymbol{p})v^s(\boldsymbol{p})=2E_{\boldsymbol{p}}\delta^{rs}$, в итоге находим
$H=\int\frac{d^3p}{(2\pi)^3}\sum_s\left(E_{\mathbf{p}}a_{\mathbf{p}}^{s\dagger}a_{\mathbf{p}}^s-E_{\mathbf{p}}b_{\mathbf{p}}^s b_{\mathbf{p}}^{s\dagger}\right)=\int\frac{d^3p}{(2\pi)^3}\sum_s\left(E_{\mathbf{p}}a_{\mathbf{p}}^{s\dagger}a_{\mathbf{p}}^s+E_{\mathbf{p}}b_{\mathbf{p}}^{s\dagger}b_{\mathbf{p}}^s\right)+$ (бесконечная константа),
что и требовалось показать.

30

Покажите, что унитарный оператор $U(\Lambda)$, преобразует $a^s_{\mathbf{p}}$ по правилу $U(\Lambda)a^s_pU^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda p}}{E_p}}a^s_{\Lambda p}$.

Решение: Имеем для одночастичного состояния $|\boldsymbol{p},s\rangle = \sqrt{2E_{\boldsymbol{p}}}a^{s\dagger}_{\boldsymbol{p}}|0\rangle$:

$$U(\Lambda)|\boldsymbol{p},s\rangle = |\Lambda\boldsymbol{p},s\rangle = \sqrt{2E_{\Lambda\boldsymbol{p}}}a^{s\dagger}_{\Lambda\boldsymbol{p}}|0\rangle \tag{0.62}$$

откуда получаем

$$U(\Lambda)|\boldsymbol{p},s\rangle = \sqrt{2E_{\boldsymbol{p}}}U(\Lambda)a^{s\dagger}_{\boldsymbol{p}}|0\rangle = \sqrt{2E_{\boldsymbol{p}}}U(\Lambda)a^{s\dagger}_{\boldsymbol{p}}U^{-1}(\Lambda)U(\Lambda)|0\rangle = \sqrt{2E_{\Lambda\boldsymbol{p}}}a^{s\dagger}_{\Lambda\boldsymbol{p}}|0\rangle, \tag{0.63}$$

учитывая, что $U(\Lambda)|0\rangle = |0\rangle$, получаем, что

$$\sqrt{2E_{\boldsymbol{p}}}U(\Lambda)a^{s\dagger}_{\boldsymbol{p}}U^{-1}(\Lambda) = \sqrt{2E_{\Lambda\boldsymbol{p}}}a^{s\dagger}_{\Lambda\boldsymbol{p}}, \tag{0.64}$$

откуда находим $U(\Lambda)a^{s\dagger}_pU^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda p}}{E_p}}a^{s\dagger}_{\Lambda\boldsymbol{p}}$.

31

Рассматривая маленький поворот вокруг оси z, покажите, что $\Lambda_{\frac12} \approx 1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu} = 1 - \frac{i}{2}\theta\Sigma^3$.

Решение: При повороте вокруг оси $z, \omega_{12} = -\omega_{21} = \theta$, остальные компоненты $\omega_{\mu\nu}$ равны нулю. Также $S^{ij} = \frac12\varepsilon^{ijk}\Sigma^k$, следовательно

$$\Lambda_{\frac12} \approx 1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu} = 1 - \frac{i}{2}\left(\omega_{12}S^{12} + \omega_{21}S^{21}\right) = 1 - \frac{i}{2}\left(\theta\frac12\Sigma^3 - \theta\left(-\frac12\Sigma^3\right)\right) = 1 - \frac{i}{2}\theta\Sigma^3, \tag{0.65}$$

что и требовалось показать.

32*.

Оператор углового момента для поля Дирака равен $\boldsymbol{J} = \int d^3x\psi^\dagger\left(-i\boldsymbol{x}\times\boldsymbol{\nabla} + \frac12\boldsymbol{\Sigma}\right)\psi$. Упростите отдельно выражение для спиновой i компоненты $J^i_{\text{spin}} = \int d^3x\psi^\dagger\left(\frac12\Sigma^i\right)\psi$ оператора углового момента и затем для орбитальной $J^i_{\text{orbit}} = \int d^3x\psi^\dagger\left(-i(\boldsymbol{x}\times\boldsymbol{\nabla})^i\right)\psi$. Покажите, что оператор $\boldsymbol{J} = \boldsymbol{J}_{\text{orbit}} + \boldsymbol{J}_{\text{spin}}$ уничтожает вакуум, то есть $\boldsymbol{J}|0\rangle = 0$. Покажите, что $\boldsymbol{J}a^{s\dagger}_0|0\rangle = \sum\limits_r\left(\xi^{r\dagger}\frac{\boldsymbol{\sigma}}{2}\xi^s\right)a^{r\dagger}_0|0\rangle$.

Решение: Используем, что

$$\psi(\boldsymbol{x}) = \int\frac{d^3p}{(2\pi)^3}\frac{e^{i\boldsymbol{p}\boldsymbol{x}}}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1,2}\left(a^s_{\boldsymbol{p}}u^s(\boldsymbol{p}) + b^{s\dagger}_{-\boldsymbol{p}}v^s(-\boldsymbol{p})\right),$$
$$\psi^\dagger(\boldsymbol{x}) = \int\frac{d^3p}{(2\pi)^3}\frac{e^{-i\boldsymbol{p}\boldsymbol{x}}}{\sqrt{2E_{\boldsymbol{p}}}}\sum_{s=1,2}\left(a^{s\dagger}_{\boldsymbol{p}}u^{s\dagger}(\boldsymbol{p}) + b^s_{-\boldsymbol{p}}v^{s\dagger}(-\boldsymbol{p})\right), \tag{0.66}$$

а также, что $u^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\xi^s\\ \sqrt{p\cdot\bar\sigma}\xi^s\end{pmatrix}$ и $v^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\eta^s\\ -\sqrt{p\cdot\bar\sigma}\eta^s\end{pmatrix}$, и $u^{r\dagger}(\boldsymbol{p})v^s(-\boldsymbol{p}) = v^{r\dagger}(-\boldsymbol{p})u^s(\boldsymbol{p}) = 0$, и $\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} = m$, и $\sqrt{p\cdot\sigma} = \begin{pmatrix}\frac{|\boldsymbol{p}|\mathfrak{s}-p^3\mathfrak{r}}{2|\boldsymbol{p}|} & -\frac{\mathfrak{r}}{2|\boldsymbol{p}|}\left(p^1-ip^2\right)\\ -\frac{\mathfrak{r}|\boldsymbol{p}|}{2|p^1+ip^2)} & \frac{|\boldsymbol{p}|\mathfrak{s}+\mathfrak{r}p^3}{2|\boldsymbol{p}|}\end{pmatrix} = g_{\mu\nu}\tilde p^\mu\sigma^\mu = \tilde p\cdot\sigma,\quad \tilde p = \left(\frac{\mathfrak{s}}{2},\frac{\mathfrak{r}}{2}\frac{\boldsymbol{p}}{|\boldsymbol{p}|}\right)$, где $\mathfrak{s} = \sqrt{p^0+|\boldsymbol{p}|}+\sqrt{p^0-|\boldsymbol{p}|}$, а $\mathfrak{r} = \sqrt{p^0+|\boldsymbol{p}|}-\sqrt{p^0-|\boldsymbol{p}|}$ (отметим, что $\mathfrak{s},\mathfrak{r}$ скаляры, а не векторы!).

Спиновая часть: Для спиновой части имеем

$$\begin{aligned}J^i_{\text{spin}} &= \int d^3x\frac{d^3pd^3q}{(2\pi)^6}\frac{e^{-ix(\boldsymbol{q}-\boldsymbol{p})}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\sum_{r,s=1,2}\left(a^{s\dagger}_{\mathbf{q}}u^{s\dagger}(\mathbf{q}) + b^s_{-\mathbf{q}}v^{s\dagger}(-\mathbf{q})\right)\frac{\Sigma^i}{2}\left(a^r_{\mathbf{p}}u^r(\mathbf{p}) + b^{r\dagger}_{-\mathbf{p}}v^r(-\mathbf{p})\right) =\\ &= \int d^3x\frac{d^3pd^3q}{(2\pi)^6}\frac{e^{-ix(\boldsymbol{q}-\boldsymbol{p})}}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}}\sum_{r,s=1,2}\Big(u^{s\dagger}(\boldsymbol{q})\tfrac{\Sigma^i}{2}u^r(\boldsymbol{p})a^{s\dagger}_{\boldsymbol{q}}a^r_{\boldsymbol{p}} + u^{s\dagger}(\boldsymbol{q})\tfrac{\Sigma^i}{2}v^r(-\boldsymbol{p})a^{s\dagger}_{\boldsymbol{q}}b^{r\dagger}_{-\boldsymbol{p}} +\\ &\qquad + v^{s\dagger}(-\boldsymbol{q})\tfrac{\Sigma^i}{2}u^r(\boldsymbol{p})b^s_{-\boldsymbol{q}}a^r_{\boldsymbol{p}} + v^{s\dagger}(-\boldsymbol{q})\tfrac{\Sigma^i}{2}v^r(-\boldsymbol{p})b^s_{-\boldsymbol{q}}b^{r\dagger}_{-\boldsymbol{p}}\Big).\end{aligned} \tag{0.67}$$

Далее используя, что $\int d^3x e^{-ix(\boldsymbol{q}-\boldsymbol{p})} = (2\pi)^3\delta(\boldsymbol{q}-\boldsymbol{p})$, получим

$$\begin{aligned} J^i_{\text{spin}} = \int \tfrac{d^3p d^3q}{(2\pi)^3}\tfrac{\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \sum_{r,s=1,2} \Big(& u^{s\dagger}(\boldsymbol{q})\tfrac{\Sigma^i}{2}u^r(\boldsymbol{p})a^{s\dagger}_{\boldsymbol{q}}a^r_{\boldsymbol{p}} + u^{s^\dagger}(\boldsymbol{q})\tfrac{\Sigma^i}{2}v^r(-\boldsymbol{p})a^{s\dagger}_{\boldsymbol{q}}b^{r\dagger}_{-\boldsymbol{p}} + \\ & + v^{s\dagger}(-\boldsymbol{q})\tfrac{\Sigma^i}{2}u^r(\boldsymbol{p})b^s_{-\boldsymbol{q}}a^r_{\boldsymbol{p}} + v^{s\dagger}(-\boldsymbol{q})\tfrac{\Sigma^i}{2}v^r(-\boldsymbol{p})b^s_{-\boldsymbol{q}}b^{r\dagger}_{-\boldsymbol{p}}\Big) = \\ = \int \tfrac{d^3p}{(2\pi)^3}\tfrac{1}{2E_{\mathbf{p}}} \sum_{r,s=1,2} \Big(& u^{s\dagger}(\boldsymbol{p})\tfrac{\Sigma^i}{2}u^r(\boldsymbol{p})a^{s\dagger}_{\boldsymbol{p}}a^r_{\boldsymbol{p}} + u^{s^\dagger}(\boldsymbol{p})\tfrac{\Sigma^i}{2}v^r(-\boldsymbol{p})a^{s\dagger}_{\boldsymbol{p}}b^{r\dagger}_{-\boldsymbol{p}} + \\ & + v^{s\dagger}(-\boldsymbol{p})\tfrac{\Sigma^i}{2}u^r(\boldsymbol{p})b^s_{-\boldsymbol{p}}a^r_{\boldsymbol{p}} + v^{s\dagger}(-\boldsymbol{p})\tfrac{\Sigma^i}{2}v^r(-\boldsymbol{p})b^s_{-\boldsymbol{p}}b^{r\dagger}_{-\boldsymbol{p}}\Big). \end{aligned} \tag{0.68}$$

Теперь вычисляем (используем, что $\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$ и $\left[\sigma^i,\sigma^j\right] = 2i\varepsilon^{ijk}\sigma^k$ и $\mathfrak{sr} = 2|\boldsymbol{p}|, \varepsilon^{123} = 1$):

$$\begin{aligned} & u^{s^\dagger}(\boldsymbol{p})\frac{\Sigma^i}{2}v^r(-\boldsymbol{p}) = \frac{1}{2}\begin{pmatrix} \xi^{s\dagger}\sqrt{p\cdot\sigma} & \xi^{s\dagger}\sqrt{p\cdot\bar\sigma}\end{pmatrix}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\bar\sigma}\eta^r\\ -\sqrt{p\cdot\sigma}\eta^r\end{pmatrix} = \\ & = \frac{1}{2}\xi^{s\dagger}\left(\sqrt{p\cdot\sigma}\sigma^i\sqrt{p\cdot\bar\sigma} - \sqrt{p\cdot\bar\sigma}\sigma^i\sqrt{p\cdot\sigma}\right)\eta^r = \\ & = \frac{1}{2}\xi^{s\dagger}\left(\left(\frac{\mathfrak{s}}{2}\sigma^0 - \frac{\mathfrak{r}}{2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}\right)\sigma^i\left(\frac{\mathfrak{s}}{2}\sigma^0 + \frac{\mathfrak{r}}{2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}\right) - \left(\frac{\mathfrak{s}}{2}\sigma^0 + \frac{\mathfrak{r}}{2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}\right)\sigma^i\left(\frac{\mathfrak{s}}{2}\sigma^0 - \frac{\mathfrak{r}}{2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}\right)\right)\eta^r = \\ & = \frac{1}{2}\xi^{s\dagger}\left(2\frac{\mathfrak{sr}}{4|\boldsymbol{p}|}\left[\sigma^i,\boldsymbol{\sigma}\boldsymbol{p}\right]\right)\eta^r = \frac{1}{2}\xi^{s\dagger}\left(2i\epsilon_{ijk}\sigma^k p^j\right)\eta^r = i\xi^{s\dagger}(\boldsymbol{p}\times\boldsymbol{\sigma})^i\eta^r, \\ & v^{s^\dagger}(-\boldsymbol{p})\frac{\Sigma^i}{2}u^r(\boldsymbol{p}) = \frac{1}{2}\begin{pmatrix}\eta^{s\dagger}\sqrt{p\cdot\bar\sigma} & -\xi^{s\dagger}\sqrt{p\cdot\sigma}\end{pmatrix}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^r\\ \sqrt{p\cdot\bar\sigma}\xi^r\end{pmatrix} = \\ & = \frac{1}{2}\eta^{s\dagger}\left(\sqrt{p\cdot\bar\sigma}\sigma^i\sqrt{p\cdot\sigma} - \sqrt{p\cdot\sigma}\sigma^i\sqrt{p\cdot\bar\sigma}\right)\xi^r = -i\eta^{s\dagger}(\boldsymbol{p}\times\boldsymbol{\sigma})^i\xi^r. \end{aligned}$$

Далее имеем

$$\begin{aligned} & u^{s^\dagger}(\boldsymbol{p})\frac{\Sigma^i}{2}u^r(\boldsymbol{p}) = \frac{1}{2}\begin{pmatrix} \xi^{s\dagger}\sqrt{p\cdot\sigma} & \xi^{s\dagger}\sqrt{p\cdot\bar\sigma}\end{pmatrix}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\xi^r\\ \sqrt{p\cdot\bar\sigma}\xi^r\end{pmatrix} = \\ & = \frac{1}{2}\xi^{s\dagger}\left(\sqrt{p\cdot\sigma}\sigma^i\sqrt{p\cdot\sigma} + \sqrt{p\cdot\bar\sigma}\sigma^i\sqrt{p\cdot\bar\sigma}\right)\xi^r = \\ & = \frac{1}{2}\xi^{s\dagger}\left(\left(\frac{\mathfrak{s}}{2}\sigma^0 - \frac{\mathfrak{r}}{2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}\right)\sigma^i\left(\frac{\mathfrak{s}}{2}\sigma^0 - \frac{\mathfrak{r}}{2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}\right) + \left(\frac{\mathfrak{s}}{2}\sigma^0 + \frac{\mathfrak{r}}{2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}\right)\sigma^i\left(\frac{\mathfrak{s}}{2}\sigma^0 + \frac{\mathfrak{r}}{2}\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{|\boldsymbol{p}|}\right)\right)\xi^r = \\ & = \frac{1}{2}\xi^{s\dagger}\left(\frac{\mathfrak{s}^2}{2}\sigma^i + \frac{\mathfrak{r}^2}{2|\boldsymbol{p}|^2}\boldsymbol{\sigma}\boldsymbol{p}\sigma^i\boldsymbol{\sigma}\boldsymbol{p}\right)\xi^r = \xi^{s\dagger}\left(m\sigma^i + (E_{\boldsymbol{p}} - m)\frac{\boldsymbol{\sigma}\boldsymbol{p}}{|\boldsymbol{p}|^2}p^i\right)\xi^r, \\ & v^{s^\dagger}(-\boldsymbol{p})\frac{\Sigma^i}{2}v^r(-\boldsymbol{p}) = \frac{1}{2}\begin{pmatrix}\eta^{s\dagger}\sqrt{p\cdot\bar\sigma} & -\eta^{s\dagger}\sqrt{p\cdot\sigma}\end{pmatrix}\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\bar\sigma}\eta^r\\ -\sqrt{p\cdot\sigma}\eta^r\end{pmatrix} = \\ & = \frac{1}{2}\eta^{s\dagger}\left(\sqrt{p\cdot\bar\sigma}\sigma^i\sqrt{p\cdot\bar\sigma} + \sqrt{p\cdot\sigma}\sigma^i\sqrt{p\cdot\sigma}\right)\eta^r = \eta^{s\dagger}\left(m\sigma^i + (E_{\boldsymbol{p}} - m)\frac{\boldsymbol{\sigma}\boldsymbol{p}}{|\boldsymbol{p}|^2}p^i\right)\eta^r \end{aligned}$$

где мы использовали, что $\mathfrak{s}^2 = 2\left(E_{\boldsymbol{p}} + m\right)$ и $\mathfrak{r}^2 = 2\left(E_{\boldsymbol{p}} - m\right)$, а также

$$\begin{aligned} \boldsymbol{\sigma}\boldsymbol{p}\sigma^i\boldsymbol{\sigma}\boldsymbol{p} &= \sigma^j\sigma^i\sigma^k p^j p^k = \left(\delta^{ji} + i\varepsilon^{jil}\sigma^l\right)\sigma^k p^j p^k = p^i\boldsymbol{\sigma}\boldsymbol{p} + i\varepsilon^{jil}\sigma^l\sigma^k p^j p^k = \\ &= p^i\boldsymbol{\sigma}\boldsymbol{p} + i\varepsilon^{jil}\left(\delta^{lk} + i\varepsilon^{lkm}\sigma^m\right)p^j p^k = p^i\boldsymbol{\sigma}\boldsymbol{p} + i\varepsilon^{jil}p^j p^l - \varepsilon^{jil}\varepsilon^{lkm}\sigma^m p^j p^k = \\ &= p^i\boldsymbol{\sigma}\boldsymbol{p} - \left(\delta^{jk}\delta^{im} - \delta^{jm}\delta^{ik}\right)\sigma^m p^j p^k = 2p^i\boldsymbol{\sigma}\boldsymbol{p} - |\boldsymbol{p}|^2\sigma^i. \end{aligned} \tag{0.71}$$

В итоге получаем

$$\begin{aligned} J^i_{\text{spin}} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\sum_{r,s=1,2}\Big(& \xi^{s\dagger}\left(m\sigma^i + (E_{\boldsymbol{p}} - m)\frac{\boldsymbol{\sigma}\boldsymbol{p}}{|\boldsymbol{p}|^2}p^i\right)\xi^r a^{s\dagger}_{\boldsymbol{p}}a^r_{\boldsymbol{p}} + i\xi^{s\dagger}(\boldsymbol{p}\times\boldsymbol{\sigma})^i\eta^r a^{s\dagger}_{\boldsymbol{p}}b^{r\dagger}_{-\boldsymbol{p}} - \\ & - i\eta^{s\dagger}(\boldsymbol{p}\times\boldsymbol{\sigma})^i\xi^r b^s_{-\boldsymbol{p}}a^r_{\boldsymbol{p}} + \eta^{s\dagger}\left(m\sigma^i + (E_{\boldsymbol{p}} - m)\frac{\boldsymbol{\sigma}\boldsymbol{p}}{|\boldsymbol{p}|^2}p^i\right)\eta^r b^s_{-\boldsymbol{p}}b^{r\dagger}_{-\boldsymbol{p}}\Big) \end{aligned} \tag{0.72}$$

Орбитальная часть: Для орбитальной компоненты имеем

$$
\begin{aligned}
J^i_{\text{orbit}} &= \int d^3x \psi^\dagger(x) \left(-i(\boldsymbol{x}\times\boldsymbol{\nabla})^i\right)\psi(x) = \\
&= \int d^3x \frac{d^3q d^3p}{(2\pi)^6}\frac{e^{-i\boldsymbol{xq}}}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}\sum_{s,r=1,2}\left(a^{s\dagger}_{\boldsymbol{q}}u^{s\dagger}(\boldsymbol{q})+b^s_{-\boldsymbol{q}}v^{s\dagger}(-\boldsymbol{q})\right)\left(-i\varepsilon^{ijk}x^j\frac{\partial}{\partial x^k}\right)e^{i\boldsymbol{px}}\left(a^r_{\boldsymbol{p}}u^r(\boldsymbol{p})+b^{r\dagger}_{-\boldsymbol{p}}v^r(-\boldsymbol{p})\right) = \\
&= \int \frac{d^3q d^3p}{(2\pi)^6}d^3x\frac{\varepsilon^{ijk}x^j e^{i\boldsymbol{x}(\boldsymbol{p}-\boldsymbol{q})}}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}\sum_{s,r=1,2}\left(a^{s\dagger}_{\boldsymbol{q}}u^{s\dagger}(\boldsymbol{q})+b^s_{-\boldsymbol{q}}v^{s\dagger}(-\boldsymbol{q})\right)p^k\left(a^r_{\boldsymbol{p}}u^r(\boldsymbol{p})+b^{r\dagger}_{-\boldsymbol{p}}v^r(-\boldsymbol{p})\right) = \\
&= \int \frac{d^3q d^3p}{(2\pi)^6}\frac{-i(2\pi)^3\varepsilon^{ijk}\frac{\partial\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})}{\partial p^j}}{\sqrt{2E_{\boldsymbol{p}}2E_{\boldsymbol{q}}}}\sum_{s,r=1,2}\left(a^{s\dagger}_{\boldsymbol{q}}u^{s\dagger}(\boldsymbol{q})+b^s_{-\boldsymbol{q}}v^{s\dagger}(-\boldsymbol{q})\right)p^k\left(a^r_{\boldsymbol{p}}u^r(\boldsymbol{p})+b^{r\dagger}_{-\boldsymbol{p}}v^r(-\boldsymbol{p})\right) = \\
&= i\int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\boldsymbol{q}}}}\sum_{s,r=1,2}\left(a^{s\dagger}_{\boldsymbol{q}}u^{s\dagger}(\boldsymbol{q})+b^s_{-\boldsymbol{q}}v^{s\dagger}(-\boldsymbol{q})\right)\varepsilon^{ijk}\frac{\partial}{\partial q^j}\left(q^k\frac{1}{\sqrt{2E_{\boldsymbol{q}}}}\left(a^r_{\boldsymbol{q}}u^r(\boldsymbol{q})+b^{r\dagger}_{-\boldsymbol{q}}v^r(-\boldsymbol{q})\right)\right), \quad (0.73)
\end{aligned}
$$

далее мы используем, что $\varepsilon^{ijk}\frac{\partial}{\partial q^j}q^k = \varepsilon^{ijk}\delta^{jk} = 0$ и также $\varepsilon^{ijk}q^k\frac{\partial}{\partial q^j}f(|\boldsymbol{q}|) = \varepsilon^{ijk}q^k\frac{q^j}{|\boldsymbol{q}|}f'(|\boldsymbol{q}|) = 0$, где f - произвольная функция. Отсюда в частности следует, что $\varepsilon^{ijk}q^k\frac{\partial}{\partial q^j}\frac{1}{\sqrt{2E_q}} = 0$, так как $E_{\boldsymbol{q}} = \sqrt{m^2+|\boldsymbol{q}|^2}$. В итоге мы получаем (заменяя $\boldsymbol{q}$ на $\boldsymbol{p}$)

$$
J^i_{\text{orbit}} = -i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_{s,r=1,2}\left(a^{s\dagger}_{\boldsymbol{p}}u^{s\dagger}(\boldsymbol{p})+b^s_{-\boldsymbol{p}}v^{s\dagger}(-\boldsymbol{p})\right)\varepsilon^{ijk}p^j\frac{\partial}{\partial p^k}\left(a^r_{\boldsymbol{p}}u^r(\boldsymbol{p})+b^{r\dagger}_{-\boldsymbol{p}}v^r(-\boldsymbol{p})\right), \quad (0.74)
$$

далее используя, что $u^{s\dagger}(\boldsymbol{p})u^r(\boldsymbol{p}) = v^{s\dagger}(-\boldsymbol{p})v^r(-\boldsymbol{p}) = 2E_{\boldsymbol{p}}\delta^{rs}$, и $u^{s\dagger}(\boldsymbol{p})v^r(-\boldsymbol{p}) = v^{s\dagger}(-\boldsymbol{p})u^r(\boldsymbol{p}) = 0$, получаем

$$
\begin{aligned}
J^i_{\text{orbit}} &= -i\int \frac{d^3p}{(2\pi)^3}\sum_{s=1,2}\varepsilon^{ijk}\left(a^{s\dagger}_{\boldsymbol{p}}p^j\frac{\partial a^s_{\boldsymbol{p}}}{\partial p^k}+b^s_{-\boldsymbol{p}}p^j\frac{\partial b^{s\dagger}_{-\boldsymbol{p}}}{\partial p^k}\right) - \\
&- i\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_{s,r=1,2}\varepsilon^{ijk}\left(u^{s\dagger}(\boldsymbol{p})p^j\frac{\partial u^r(\boldsymbol{p})}{\partial p^k}a^{s\dagger}_{\boldsymbol{p}}a^r_{\boldsymbol{p}}+u^{s\dagger}(\boldsymbol{p})p^j\frac{\partial v^r(-\boldsymbol{p})}{\partial p^k}a^{s\dagger}_{\boldsymbol{p}}b^{r\dagger}_{-\boldsymbol{p}}+\right. \\
&\left.+v^{s\dagger}(-\boldsymbol{p})p^j\frac{\partial u^r(\boldsymbol{p})}{\partial p^k}b^s_{-\boldsymbol{p}}a^r_{\boldsymbol{p}}+v^{s\dagger}(-\boldsymbol{p})p^j\frac{\partial v^r(-\boldsymbol{p})}{\partial p^k}b^s_{-\boldsymbol{p}}b^{r\dagger}_{-\boldsymbol{p}}\right). \quad (0.75)
\end{aligned}
$$

Используя, что $\varepsilon^{ijk}p^j\frac{\partial}{\partial p^k}\mathfrak{s} = \varepsilon^{ijk}p^j\frac{\partial}{\partial p^k}\frac{\mathfrak{r}}{2|\boldsymbol{p}|} = 0$, где $\mathfrak{s} = \sqrt{p^0+|\boldsymbol{p}|}+\sqrt{p^0-|\boldsymbol{p}|}$, а $\mathfrak{r} = \sqrt{p^0+|\boldsymbol{p}|}$ - $\sqrt{p^0-|\boldsymbol{p}|}$, получим

$$
\begin{aligned}
\varepsilon^{ijk}p^j\frac{\partial}{\partial p^k}\sqrt{p\cdot\sigma} &= \varepsilon^{ijk}p^j\frac{\partial}{\partial p^k}\left(\frac{\mathfrak{s}}{2}\sigma^0-\frac{\mathfrak{r}}{2|\boldsymbol{p}|}p^l\sigma^l\right) = -\frac{\mathfrak{r}}{2|\boldsymbol{p}|}\varepsilon^{ijk}p^j\sigma^k = -\frac{\mathfrak{r}}{2|\boldsymbol{p}|}(\boldsymbol{p}\times\boldsymbol{\sigma})^i, \\
\varepsilon^{ijk}p^j\frac{\partial}{\partial p^k}\sqrt{p\cdot\bar{\sigma}} &= \frac{\mathfrak{r}}{2|\boldsymbol{p}|}\varepsilon^{ijk}p^j\sigma^k = \frac{\mathfrak{r}}{2|\boldsymbol{p}|}(\boldsymbol{p}\times\boldsymbol{\sigma})^i, \quad (0.76)
\end{aligned}
$$

откуда находим

$$
\begin{aligned}
u^{s\dagger}(\boldsymbol{p})\varepsilon^{ijk}p^j\frac{\partial v^r(-\boldsymbol{p})}{\partial p^k} &= \xi^{s\dagger}\left(\sqrt{p\cdot\sigma}\varepsilon^{ijk}p^j\frac{\partial}{\partial p^k}\sqrt{p\cdot\bar{\sigma}}-\sqrt{p\cdot\bar{\sigma}}\varepsilon^{ijk}p^j\frac{\partial}{\partial p^k}\sqrt{p\cdot\sigma}\right)\eta^r = \\
&= \xi^{s\dagger}\left(\frac{\mathfrak{s}\mathfrak{r}}{2|\boldsymbol{p}|}\varepsilon^{ijk}p^j\sigma^k\right)\eta^r = \xi^{s\dagger}\left(\varepsilon^{ijk}p^j\sigma^k\right)\eta^r = \xi^{s\dagger}(\boldsymbol{p}\times\boldsymbol{\sigma})^i\eta^r, \\
v^{s\dagger}(-\boldsymbol{p})\varepsilon^{ijk}p^j\frac{\partial u^r(\boldsymbol{p})}{\partial p^k} &= \eta^{s\dagger}\left(\sqrt{p\cdot\bar{\sigma}}\varepsilon^{ijk}p^j\frac{\partial}{\partial p^k}\sqrt{p\cdot\sigma}-\sqrt{p\cdot\sigma}\varepsilon^{ijk}p^j\frac{\partial}{\partial p^k}\sqrt{p\cdot\bar{\sigma}}\right)\xi^r = \\
&= -\eta^{s\dagger}\left(\frac{\mathfrak{s}\mathfrak{r}}{2|\boldsymbol{p}|}\varepsilon^{ijk}p^j\sigma^k\right)\xi^r = -\eta^{s\dagger}\left(\varepsilon^{ijk}p^j\sigma^k\right)\xi^r = -\eta^{s\dagger}(\boldsymbol{p}\times\boldsymbol{\sigma})^i\xi^r. \quad (0.77)
\end{aligned}
$$

Далее находим

$$
\begin{aligned}
u^{s\dagger}(\boldsymbol{p})\varepsilon^{ijk}p^j\frac{\partial u^r(\boldsymbol{p})}{\partial p^k} &= \xi^{s\dagger}\left(\sqrt{p\cdot\sigma}\varepsilon^{ijk}p^j\frac{\partial}{\partial p^k}\sqrt{p\cdot\sigma}+\sqrt{p\cdot\bar{\sigma}}\varepsilon^{ijk}p^j\frac{\partial}{\partial p^k}\sqrt{p\cdot\bar{\sigma}}\right)\xi^r = \\
&= \xi^{s\dagger}\left(\frac{\mathfrak{r}^2}{2|\boldsymbol{p}|^2}(\boldsymbol{\sigma p})(\boldsymbol{p}\times\boldsymbol{\sigma})^i\right)\xi^r = i\xi^{s\dagger}\left((E_{\boldsymbol{p}}-m)\,\sigma^i-(E_{\boldsymbol{p}}-m)\,\frac{\boldsymbol{\sigma p}}{|\boldsymbol{p}|^2}p^i\right)\xi^r, \\
v^{s\dagger}(-\boldsymbol{p})\varepsilon^{ijk}p^j\frac{\partial v^r(-\boldsymbol{p})}{\partial p^k} &= \eta^{s\dagger}\left(\sqrt{p\cdot\bar{\sigma}}\varepsilon^{ijk}p^j\frac{\partial}{\partial p^k}\sqrt{p\cdot\bar{\sigma}}+\sqrt{p\cdot\sigma}\varepsilon^{ijk}p^j\frac{\partial}{\partial p^k}\sqrt{p\cdot\sigma}\right)\eta^r = \\
&= \eta^{s\dagger}\left(\frac{\mathfrak{r}^2}{2|\boldsymbol{p}|^2}(\boldsymbol{\sigma p})(\boldsymbol{p}\times\boldsymbol{\sigma})^i\right)\eta^r = i\eta^{s\dagger}\left((E_{\boldsymbol{p}}-m)\,\sigma^i-(E_{\boldsymbol{p}}-m)\,\frac{\boldsymbol{\sigma p}}{|\boldsymbol{p}|^2}p^i\right)\eta^r, \qquad (0.78)
\end{aligned}
$$

где мы использовали, что $\mathfrak{r}^2 = 2\,(E_{\boldsymbol{p}}-m)$, а также

$$
\begin{aligned}
(\boldsymbol{\sigma p})(\boldsymbol{p}\times\boldsymbol{\sigma})^i &= \varepsilon^{ijk}\sigma^l\sigma^k p^l p^j = \varepsilon^{ijk}\left(\delta^{lk}+i\varepsilon^{lkm}\sigma^m\right)p^l p^j = i\varepsilon^{ijk}\varepsilon^{mlk}\sigma^m p^l p^j = \\
&= i\left(\delta^{im}\delta^{jl}-\delta^{il}\delta^{jm}\right)\sigma^m p^l p^j = i\left(|\boldsymbol{p}|^2\sigma^i - p^i\boldsymbol{\sigma p}\right). \qquad (0.79)
\end{aligned}
$$

В итоге получаем

$$
\begin{aligned}
J^i_{\text{orbit}} &= -\,i\int\frac{d^3p}{(2\pi)^3}\sum_{s=1,2}\varepsilon^{ijk}\left(a^{s\dagger}_{\boldsymbol{p}}p^j\frac{\partial a^s_{\boldsymbol{p}}}{\partial p^k}+b^s_{-\boldsymbol{p}}p^j\frac{\partial b^{s\dagger}_{-\boldsymbol{p}}}{\partial p^k}\right)+ \\
&+\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}}\sum_{s,r=1,2}\left(\xi^{s\dagger}\left((E_{\boldsymbol{p}}-m)\,\sigma^i-(E_{\boldsymbol{p}}-m)\,\frac{\boldsymbol{\sigma p}}{|\boldsymbol{p}|^2}p^i\right)\xi^r a^{s\dagger}_{\boldsymbol{p}}a^r_{\boldsymbol{p}} - i\xi^{s\dagger}(\boldsymbol{p}\times\boldsymbol{\sigma})^i\eta^r a^{s\dagger}_{\boldsymbol{p}}b^{r\dagger}_{-\boldsymbol{p}}+\right. \\
&\left.+i\eta^{s\dagger}(\boldsymbol{p}\times\boldsymbol{\sigma})^i\xi^r b^s_{-\boldsymbol{p}}a^r_{\boldsymbol{p}}+\eta^{s\dagger}\left((E_{\boldsymbol{p}}-m)\,\sigma^i-(E_{\boldsymbol{p}}-m)\,\frac{\boldsymbol{\sigma p}}{|\boldsymbol{p}|^2}p^i\right)\eta^r b^s_{-\boldsymbol{p}}b^{r\dagger}_{-\boldsymbol{p}}\right). \qquad (0.80)
\end{aligned}
$$

В итоге находи для J^i:

$$
\begin{aligned}
J^i = J^i_{\text{spin}} + J^i_{\text{orbit}} &= -\,i\int\frac{d^3p}{(2\pi)^3}\sum_{s=1,2}\varepsilon^{ijk}\left(a^{s\dagger}_{\boldsymbol{p}}p^j\frac{\partial a^s_{\boldsymbol{p}}}{\partial p^k}+b^s_{-\boldsymbol{p}}p^j\frac{\partial b^{s\dagger}_{-\boldsymbol{p}}}{\partial p^k}\right)+ \\
&+\int\frac{d^3p}{(2\pi)^3}\sum_{s,r=1,2}\left(\xi^{s\dagger}\frac{\sigma^i}{2}\xi^r a^{s\dagger}_{\boldsymbol{p}}a^r_{\boldsymbol{p}}+\eta^{s\dagger}\frac{\sigma^i}{2}\eta^r b^s_{-\boldsymbol{p}}b^{r\dagger}_{-\boldsymbol{p}}\right). \qquad (0.81)
\end{aligned}
$$

Антикоммутируя $b^s_{-p}b^{r\dagger}_{-p}$ и выкидывая бесконечные константы, а также заменяя в данных членах $\boldsymbol{p}\to-\boldsymbol{p}$, получаем

$$
J^i = -i\int\frac{d^3p}{(2\pi)^3}\sum_{s=1,2}\varepsilon^{ijk}\left(a^{s\dagger}_{\boldsymbol{p}}p^j\frac{\partial a^s_{\boldsymbol{p}}}{\partial p^k}+b^{s\dagger}_{\boldsymbol{p}}p^j\frac{\partial b^s_{\boldsymbol{p}}}{\partial p^k}\right)+\int\frac{d^3p}{(2\pi)^3}\sum_{s,r=1,2}\left(\xi^{s\dagger}\frac{\sigma^i}{2}\xi^r a^{s\dagger}_{\boldsymbol{p}}a^r_{\boldsymbol{p}}-\eta^{s\dagger}\frac{\sigma^i}{2}\eta^r b^{r\dagger}_{\boldsymbol{p}}b^s_{\boldsymbol{p}}\right). \qquad (0.82)
$$

Это выражение можно переписать как

$$
\boldsymbol{J} = \int\frac{d^3p}{(2\pi)^3}\sum_{s,r=1,2}\left(a^{s\dagger}_{\boldsymbol{p}}\xi^{s\dagger}\left(\frac{\boldsymbol{\sigma}}{2}-i\delta^{rs}\boldsymbol{p}\times\frac{\partial}{\partial\boldsymbol{p}}\right)\xi^r a^r_{\boldsymbol{p}} - b^{r\dagger}_{\boldsymbol{p}}\eta^{s\dagger}\left(\frac{\boldsymbol{\sigma}}{2}+i\delta^{rs}\boldsymbol{p}\times\frac{\partial}{\partial\boldsymbol{p}}\right)\eta^r b^s_{\boldsymbol{p}}\right). \qquad (0.83)
$$

Далее, очевидно, что $\boldsymbol{J}|0\rangle = 0$. Теперь, так как $\boldsymbol{J}|0\rangle = 0$, имеем $\boldsymbol{J}a^{s\dagger}_0|0\rangle = \left[\boldsymbol{J}, a^{s\dagger}_0\right]|0\rangle$. Вычисляя коммутатор $\left[a^{r'\dagger}_{\boldsymbol{p}}a^r_{\boldsymbol{p}}, a^{s\dagger}_0\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{p})a^{r'\dagger}_0\delta^{rs}$, мы видим, что должны взять все в точке $\boldsymbol{p}=0$, откуда находим, что $\boldsymbol{J}a^{s\dagger}_0|0\rangle = \sum\limits_{r'}\left(\xi^{r'\dagger}\frac{\boldsymbol{\sigma}}{2}\xi^s\right)a^{r'\dagger}_0|0\rangle$.

33

Покажите, что запаздывающая функция Грина для уравнения Дирака есть $S_R^{ab}(x-y) = \left(i\partial\!\!\!/_x + m\right)_{ab} D_R(x-y)$, где $D_R(x-y)$ запаздывающая функция Грина для уравнения Клейна-Гордона.

Решение: С одной стороны мы имеем

$$S_R^{ab}(x-y) \equiv \theta\left(x^0-y^0\right)\right) \left\langle 0 \left| \left\{ \psi_a(x), \bar{\psi}_b(y) \right\} \right| 0 \right\rangle, \tag{0.84}$$

с другой стороны можно вычислить

$$\begin{aligned}
\left\langle 0 \left| \psi_a(x)\bar{\psi}_b(y) \right| 0 \right\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}} \sum_{s=1,2} u_a^s(p)\bar{u}_b^s(p)e^{-ip\cdot(x-y)} = \left(i\partial_x + m\right)_{ab} \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}} e^{-ip\cdot(x-y)}, \\
\left\langle 0 \left| \bar{\psi}_b(y)\psi_a(x) \right| 0 \right\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}} \sum_{s=1,2} v_a^s(p)\bar{v}_b^s(p)e^{-ip\cdot(y-x)} = -\left(i\partial\!\!\!/_x + m\right)_{ab} \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}} e^{-ip\cdot(y-x)},
\end{aligned} \tag{0.85}$$

откуда

$$\left\langle 0 \left| \left\{ \psi_a(x), \bar{\psi}_b(y) \right\} \right| 0 \right\rangle = \left(i\partial\!\!\!/_x + m\right)_{ab} \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\boldsymbol{p}}} \left(e^{-ip\cdot(x-y)} - e^{-ip\cdot(y-x)} \right) = \left(i\partial\!\!\!/_x + m\right)_{ab} \langle 0|[\phi(x),\phi(y)]|0\rangle. \tag{0.86}$$

В итоге получаем

$$\begin{aligned}
S_R^{ab}(x-y) &\equiv \theta\left(x^0-y^0\right)\right) \left\langle 0 \left| \left\{ \psi_a(x), \bar{\psi}_b(y) \right\} \right| 0 \right\rangle = \left(i\partial\!\!\!/_x + m\right)_{ab} \theta\left(x^0-y^0\right)\right) \langle 0|[\phi(x),\phi(y)]|0\rangle = \\
&= \left(i\partial\!\!\!/_x + m\right)_{ab} D_R(x-y),
\end{aligned} \tag{0.87}$$

что и требовалось показать.

Покажите, что $\partial\!\!\!/\partial\!\!\!/ = \partial^2$.

Решение: Имеем

$$\partial\!\!\!/\partial\!\!\!/ = \gamma^\mu\partial_\mu\gamma^\nu\partial_\nu = \gamma^\mu\gamma^\nu\partial_\mu\partial_\nu = \frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\partial_\mu\partial_\nu = g^{\mu\nu}\partial_\mu\partial_\nu = \partial^2 \tag{0.88}$$

что и требовалось показать.

36 Дискретные симметрии в теории Дирака.

Проверьте, что все величины: $\bar{\psi}\psi, \bar{\psi}\gamma^\mu\psi, i\bar{\psi}\left[\gamma^\mu,\gamma^\nu\right]\psi, \bar{\psi}\gamma^\mu\gamma^5\psi, i\bar{\psi}\gamma^5\psi$ являются эрмитовыми.

Решение: Используя то, что $\left(\gamma^0\right)^\dagger = \gamma^0, \left(\gamma^5\right)^\dagger = \gamma^5$, а $\left(\gamma^i\right)^\dagger = -\gamma^i, i = 1,2,3$, а также, что $\left\{\gamma^5,\gamma^\mu\right\} = 0, \left\{\gamma^\mu,\gamma^\nu\right\} = 2g^{\mu\nu}$ имеем:

$$\begin{aligned}
&(\bar{\psi}\psi)^\dagger = \left(\psi^\dagger\gamma^0\psi\right)^\dagger = \psi^\dagger\left(\gamma^0\right)^\dagger\psi = \psi^\dagger\gamma^0\psi = \bar{\psi}\psi \\
&\left(\bar{\psi}\gamma^\mu\psi\right)^\dagger = \left(\psi^\dagger\gamma^0\gamma^\mu\psi\right)^\dagger = \psi^\dagger\left(\gamma^\mu\right)^\dagger\left(\gamma^0\right)^\dagger\psi = \psi^\dagger\gamma^0\gamma^\mu\psi = \bar{\psi}\gamma^\mu\psi, \\
&\left(i\bar{\psi}\left[\gamma^\mu,\gamma^\nu\right]\psi\right)^\dagger = \left(i\psi^\dagger\gamma^0\left[\gamma^\mu,\gamma^\nu\right]\psi\right)^\dagger = -i\psi^\dagger\left(\left(\gamma^\nu\right)^\dagger\left(\gamma^\mu\right)^\dagger - \left(\gamma^\mu\right)^\dagger\left(\gamma^\nu\right)^\dagger\right)\gamma^0\psi = i\bar{\psi}\left[\gamma^\mu,\gamma^\nu\right]\psi, \\
&\left(\bar{\psi}\gamma^\mu\gamma^5\psi\right)^\dagger = \psi^\dagger\left(\gamma^5\right)^\dagger\left(\gamma^\mu\right)^\dagger\gamma^0\psi = \psi^\dagger\gamma^5\gamma^0\gamma^\mu\psi = \bar{\psi}\gamma^\mu\gamma^5\psi, \\
&\left(i\bar{\psi}\gamma^5\psi\right)^\dagger = -i\psi^\dagger\left(\gamma^5\right)^\dagger\left(\gamma^0\right)^\dagger\psi = -i\psi^\dagger\gamma^5\gamma^0\psi = i\bar{\psi}\gamma^5\psi,
\end{aligned} \tag{0.89}$$

что и требовалось проверить.

35

Покажите, что

$$\gamma^0\gamma^\mu\gamma^0 = \begin{cases} +\gamma^\mu, & \mu = 0 \\ -\gamma^\mu, & \mu = 1,2,3 \end{cases}$$

Решение: Используя то, что $\left(\gamma^0\right)^2 = 1$, а также $\left\{\gamma^0, \gamma^i\right\} = 0, i = 1,2,3$, получаем

$$\gamma^0\gamma^0\gamma^0 = \gamma^0, \quad \gamma^0\gamma^i\gamma^0 = -\gamma^i\gamma^0\gamma^0 = -\gamma^i \tag{0.90}$$

что и требовалось показать.

36

Проверьте, что

$$T\bar{\psi}\gamma^\mu\psi T = \begin{cases} +\bar{\psi}\gamma^\mu\psi(-t,\mathbf{x}), & \mu = 0 \\ -\bar{\psi}\gamma^\mu\psi(-t,\mathbf{x}), & \mu = 1,2,3 \end{cases}$$

Решение: Используя, что $T(c \text{ - число }) = (c - \text{ число })^* T$, и также $T\psi(t,\boldsymbol{x})T = \gamma^1\gamma^3\psi(-t,\boldsymbol{x})$, получим

$$\begin{aligned} T\bar{\psi}\gamma^\mu\psi T = T\bar{\psi}T\left(\gamma^\mu\right)^* T\psi T = \psi^\dagger(-t,\boldsymbol{x})\left[\gamma^1\gamma^3\right]^\dagger(\gamma)^0\left(\gamma^\mu\right)^*\gamma^1\gamma^3\psi(-t,\boldsymbol{x}) = \\ = \bar{\psi}(-t,\boldsymbol{x})\left(-\gamma^1\gamma^3\right)\left(\gamma^\mu\right)^*\gamma^1\gamma^3\psi(-t,\boldsymbol{x}) \end{aligned} \tag{0.91}$$

Теперь используя, что $\left(\gamma^0\right)^* = \gamma^0, \left(\gamma^1\right)^* = \gamma^1, \left(\gamma^3\right)^* = \gamma^3$ и $\left(\gamma^2\right)^* = -\gamma^2, \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, получим

$$\begin{aligned} T\bar{\psi}\gamma^0\psi T &= \bar{\psi}(-t,\boldsymbol{x})\left(-\gamma^1\gamma^3\right)\gamma^0\gamma^1\gamma^3\psi(-t,\boldsymbol{x}) = \bar{\psi}\gamma^0\psi(-t,\boldsymbol{x}), \\ T\bar{\psi}\gamma^1\psi T &= \bar{\psi}(-t,\boldsymbol{x})\left(-\gamma^1\gamma^3\right)\gamma^1\gamma^1\gamma^3\psi(-t,\boldsymbol{x}) = -\bar{\psi}\gamma^1\psi(-t,\boldsymbol{x}), \\ T\bar{\psi}\gamma^2\psi T &= -\bar{\psi}(-t,\boldsymbol{x})\left(-\gamma^1\gamma^3\right)\gamma^2\gamma^1\gamma^3\psi(-t,\boldsymbol{x}) = -\bar{\psi}\gamma^2\psi(-t,\boldsymbol{x}), \\ T\bar{\psi}\gamma^3\psi T &= \bar{\psi}(-t,\boldsymbol{x})\left(-\gamma^1\gamma^3\right)\gamma^3\gamma^1\gamma^3\psi(-t,\boldsymbol{x}) = -\bar{\psi}\gamma^3\psi(-t,\boldsymbol{x}), \end{aligned} \tag{0.92}$$

что и требовалось проверить.

37

Найдите чему равно преобразование зарядового сопряжения для биллинейных форм:
$C\bar{\psi}\psi C, C\bar{\psi}\gamma^\mu\psi C, Ci\bar{\psi}\left[\gamma^\mu,\gamma^\nu\right]\psi C, C\bar{\psi}\gamma^\mu\gamma^5\psi C, Ci\bar{\psi}\gamma^5\psi C$.
Решение: Используя, что $C\psi(x)C = -i\left(\bar{\psi}\gamma^0\gamma^2\right)^T$, и также $C\bar{\psi}C = \left(-i\gamma^0\gamma^2\psi\right)^T$ имеем:
$C\bar{\psi}\psi C = \left(-i\gamma^0\gamma^2\psi\right)^T\left(-i\bar{\psi}\gamma^0\gamma^2\right)^T = -\gamma^0_{ab}\gamma^2_{bc}\psi_c\bar{\psi}_d\gamma^0_{de}\gamma^2_{ea} = \bar{\psi}_d\gamma^0_{de}\gamma^2_{ea}\gamma^0_{ab}\gamma^2_{bc}\psi_c = -\bar{\psi}\gamma^2\gamma^0\gamma^0\gamma^2\psi = \bar{\psi}\psi$,
$Ci\bar{\psi}\gamma^5\psi C = i\left(-i\gamma^0\gamma^2\psi\right)^T\gamma^5\left(-i\bar{\psi}\gamma^0\gamma^2\right)^T = i\bar{\psi}\gamma^5\psi$,
аналогично можно проверить для остальных выражений.

26.1.10 Problems about spinors and Dirac equation 3

Stanford.Zhou.2022.7.1. Plane wave solutions of the Dirac equation. (10/45 p)

In the Weyl representation the gamma matrices are given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \sigma^\mu = \left(\mathbb{1}_2, \sigma^1, \sigma^2, \sigma^3\right), \quad \bar{\sigma}^\mu = \left(\mathbb{1}_2, -\sigma^1, -\sigma^2, -\sigma^3\right)$$

Here, $\mathbb{1}_n$ denotes an $n \times n$ identity matrix.
a) Show that $(p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2\mathbb{1}_2$.

Solution:

Inserting the definitions and suppressing the factors of the identity, the expression is

$$\left(p^0 - \mathbf{p}\cdot\boldsymbol{\sigma}\right)\left(p^0 + \mathbf{p}\cdot\boldsymbol{\sigma}\right) = \left(p^0\right)^2 - \frac{1}{2}p^i p^j\left\{\sigma^i,\sigma^j\right\} = \left(p^0\right)^2 - |\mathbf{p}|^2 = p^2$$

as desired, where we used $\left\{\sigma^i,\sigma^j\right\} = 2\delta^{ij}$.

b) Show that the Dirac equation has positive-frequency plane wave solutions

$$\psi(x) = u_s(p)e^{-ip\cdot x}, \quad u_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix}$$

where ξ_s is a two-component spinor, i.e. a complex vector with two elements. (Formally, given a matrix A with a complete basis of eigenvectors, with eigenvalues λ_i, we can define a matrix square root $\sqrt{A}$ to have the same eigenvectors, with eigenvalues $\sqrt{\lambda_i}$. But the only thing you need to know to do this problem is $\sqrt{A}\sqrt{A} = A$.)

Solution:

By the definition of the matrix square root and the result of part (a), we have

$$\sigma\cdot p = \sqrt{\sigma\cdot p}\cdot\sqrt{\sigma\cdot p}, \quad \bar{\sigma}\cdot p = \sqrt{\bar{\sigma}\cdot p}\cdot\sqrt{\bar{\sigma}\cdot p}, \quad m\mathbb{1}_2 = \sqrt{\sigma\cdot p}\cdot\sqrt{\bar{\sigma}\cdot p} = \sqrt{\bar{\sigma}\cdot p}\cdot\sqrt{\sigma\cdot p}$$

where we used $p^2 = m^2$. To show the Dirac equation is satisfied, note that the derivative ∂_μ just pulls down a factor of $-ip^\mu$. Thus, it suffices to show that

$$\left(p_\mu\gamma^\mu - m\mathbb{1}_4\right)u_s(p) = 0$$

To show this, we just plug in the definitions, giving

$$\begin{aligned}
\left(p_\mu\gamma^\mu - m\mathbb{1}_4\right)u_s(p) &= \begin{pmatrix} -m\mathbb{1}_2 & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} & -m\mathbb{1}_2 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{p\cdot\sigma}\left(-m\mathbb{1}_2 + \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}\right)\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\left(\sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma} - m\mathbb{1}_2\right)\xi_s \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{p\cdot\sigma}\left(-m\mathbb{1}_2 + \sqrt{p^2}\mathbb{1}_2\right)\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\left(\sqrt{p^2}\mathbb{1}_2 - m\mathbb{1}_2\right)\xi_s \end{pmatrix} \\
&= 0
\end{aligned}$$

where we used $p^2 = m^2$ in the last step.

c) Show that if we pick an orthonormal basis of two-component spinors ξ_s with $s\in\{1,2\}$, satisfying $\left(\xi_r\right)^\dagger\xi_s = \delta_{rs}$, then the Dirac spinors satisfy the orthogonality relations

$$u_r(p)^\dagger u_s(p) = 2p_0\delta_{rs}, \quad \bar{u}_r(p)u_s(p) = 2m\delta_{rs}$$

Solution:

To take the Hermitian conjugate, note that $\left(\sigma^\mu\right)^\dagger = \sigma^\mu$. Then we simply have

$$\begin{aligned}
u_s^\dagger(p)u_r(p) &= \begin{pmatrix} \xi_s^\dagger\sqrt{\sigma\cdot p} & \xi_s^\dagger\sqrt{\bar{\sigma}\cdot p} \end{pmatrix}\cdot\begin{pmatrix} \sqrt{\sigma\cdot p}\xi_r \\ \sqrt{\bar{\sigma}\cdot p}\xi_r \end{pmatrix} \\
&= \xi_s^\dagger(p\cdot\sigma + p\cdot\bar{\sigma})\xi_r = 2p^0\xi_s^\dagger\xi_r = 2p^0\delta_{rs}
\end{aligned}$$

The proof of the second orthogonality relation is similar:

$$\begin{aligned}
\bar{u}_s(p)u_r(p) = u_s^\dagger(p)\gamma^0u_r(p) &= \begin{pmatrix} \xi_s^\dagger\sqrt{\bar{\sigma}\cdot p} & \xi_s^\dagger\sqrt{\sigma\cdot p} \end{pmatrix}\cdot\begin{pmatrix} \sqrt{\sigma\cdot p}\xi_r \\ \sqrt{\bar{\sigma}\cdot p}\xi_r \end{pmatrix} \\
&= 2\xi_s^\dagger(\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}})\xi_r = 2m\xi_s^\dagger\xi_r = 2m\delta_{rs}.
\end{aligned}$$

d) Similarly, the Dirac equation has negative-frequency plane wave solutions

$$\psi(x) = v_s(p)e^{ip\cdot x}, \quad v_s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ -\sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix}$$

Show that these solve the Dirac equation, and satisfy the orthogonality relations

$$v_r(p)^\dagger v_s(p) = 2p_0\delta_{rs}, \quad \bar{v}_r(p)v_s(p) = -2m\delta_{rs}$$

Solution:

The proof that these plane waves solve the Dirac equation is very similar to that of part (b), except that now the derivative ∂_μ pulls down a factor of ip^μ, so we want to show

$$(p_\mu\gamma^\mu + m\mathbb{1}_4)\, v_s(p) = 0$$

This holds because

$$\begin{aligned}(p_\mu\gamma^\mu + m\mathbb{1}_4)\, v_s(p) &= \begin{pmatrix} m\mathbb{1}_2 & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} & m\mathbb{1}_2 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ -\sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{p\cdot\sigma}\left(m\mathbb{1}_2 - \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}\right)\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\left(\sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma} - m\mathbb{1}_2\right)\xi_s \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{p\cdot\sigma}\left(m\mathbb{1}_2 - \sqrt{p^2}\mathbb{1}_2\right)\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\left(\sqrt{p^2}\mathbb{1}_2 - m\mathbb{1}_2\right)\xi_s \end{pmatrix} \\ &= 0.\end{aligned}$$

The proofs of the orthogonality relations are very similar to part (c),

$$\begin{aligned}v_s^\dagger(p)v_r(p) &= \begin{pmatrix} \xi_s^\dagger\sqrt{\sigma\cdot p} & -\xi_s^\dagger\sqrt{\bar{\sigma}\cdot p} \end{pmatrix}\cdot\begin{pmatrix} \sqrt{\sigma\cdot p}\xi_r \\ -\sqrt{\bar{\sigma}\cdot p}\xi_r \end{pmatrix} \\ &= \xi_s^\dagger(p\cdot\sigma + p\cdot\bar{\sigma})\xi_r = 2p^0\xi_s^\dagger\xi_r = 2p^0\delta_{rs}\end{aligned}$$

and

$$\begin{aligned}\bar{v}_s(p)v_r(p) = v_s(p)\gamma^0 v_r(p) &= -\begin{pmatrix} \xi_s^\dagger\sqrt{\bar{\sigma}\cdot p} & \xi_s^\dagger\sqrt{\sigma\cdot p} \end{pmatrix}\cdot\begin{pmatrix} \sqrt{\sigma\cdot p}\xi_r \\ \sqrt{\bar{\sigma}\cdot p}\xi_r \end{pmatrix} \\ &= -2\xi_s^\dagger(\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}})\xi_r = -2m\xi_s^\dagger\xi_r = -2m\delta_{rs}.\end{aligned}$$

e) Show the completeness relations for the Dirac spinors,

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = \not{p} + m\mathbb{1}_4, \quad \sum_{s=1}^{2} v_s(p)\bar{v}_s(p) = \not{p} - m\mathbb{1}_4$$

Solution:

This follows straightforwardly from plugging in the definitions,

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = \sum_{s=1}^{2}\begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix}\begin{pmatrix} \xi_s^\dagger\sqrt{p\cdot\bar{\sigma}} & \xi_s^\dagger\sqrt{p\cdot\sigma} \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} = p_\mu\gamma^\mu + m\mathbb{1}_4$$

$$\sum_{s=1}^{2} v_s(p)\bar{v}_s(p) = \sum_{s=1}^{2}\begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ -\sqrt{p\cdot\bar{\sigma}}\xi_s \end{pmatrix}\begin{pmatrix} -\xi_s^\dagger\sqrt{p\cdot\bar{\sigma}} & \xi_s^\dagger\sqrt{p\cdot\sigma} \end{pmatrix} = \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} = p_\mu\gamma^\mu - m\mathbb{1}_4$$

In both cases we used the completeness relation for the two-component spinors, $\sum_{s=1}^{2}\xi_s\xi_s^\dagger = \mathbb{1}_2$.

Stanford.Zhou.2022.7.2 Useful spinor identities. (10/45p)

a) Prove the Gordon identity,

$$\bar{u}_r\left(p'\right)\gamma^\mu u_s(p)=\bar{u}_r\left(p'\right)\left(\frac{p'^\mu+p^\mu}{2m}+\frac{i\sigma^{\mu\nu}\left(p'_\nu-p_\nu\right)}{2m}\right)u_s(p)$$

where $\sigma^{\mu\nu}=\frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right]$.

(Hint: this can be done using only the Dirac equation and the defining property of the gamma matrices.)

b) Show the completeness relation for the Γ matrices,

$$\delta_{ac}\delta_{db}=\frac{1}{4}\sum_I\tilde{\Gamma}^I_{dc}\Gamma^I_{ab}$$

(Hint: it suffices to show that both sides give the same result when multiplied by a general matrix M_{cd}.)

c) Using (10), show the Fierz identity

$$\tilde{\Gamma}^I_{ab}\Gamma^J_{cd}=\frac{1}{16}\sum_{KL}\mathrm{tr}\left[\tilde{\Gamma}^I\tilde{\Gamma}^K\Gamma^J\Gamma^L\right]\tilde{\Gamma}^L_{ad}\Gamma^K_{cb}$$

d) Find the Fierz transformations for $\left(\bar{u}_1u_2\right)\left(\bar{u}_3u_4\right)$ and $\left(\bar{u}_1\gamma^\mu u_2\right)\left(\bar{u}_3\gamma_\mu u_4\right)$ explicitly. (Your final result should only contain the spinors and gamma matrices, not the Γ^I.)

Solution to (a)

It's easiest to start from the second term on the right-hand side. Note that

$$\begin{aligned}\bar{u}_r\left(p'\right)\left[\gamma^\mu,\gamma^\nu\right]\left(p'-p\right)_\nu u_s(p)&=\bar{u}_r\left(p'\right)\left(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu\right)\left(p'-p\right)_\nu u_s(p)\\&=\bar{u}_r\left(p'\right)\left[2\left(p'+p\right)^\mu-2\left(p'\gamma^\mu+\gamma^\mu\not{p}\right)\right]u_s(p)\\&=\bar{u}_r\left(p'\right)\left(-4m\gamma^\mu+2\left(p'^\mu+p^\mu\right)\right)u_s(p)\end{aligned}$$

where we used the Dirac equation (S33)(??) in the last step. Plugging this into the right hand side of (7) gives the desired result.

Combinations of gamma matrices can be used to produce a basis Γ^I for the space of 4×4 matrices, where I ranges from 1 to 16. Concretely, we have

$$\begin{aligned}&\Gamma^{11}=(i/2)\left[\gamma^0,\gamma^1\right]\\&\Gamma^3=\gamma^0\quad\Gamma^7=i\gamma^5\gamma^0\quad\Gamma^{12}=(i/2)\left[\gamma^0,\gamma^2\right]\\&\Gamma^1=\mathbb{1}_4\quad\Gamma^4=\gamma^1\quad\Gamma^8=i\gamma^5\gamma^1\quad\Gamma^{13}=(i/2)\left[\gamma^0,\gamma^3\right]\\&\Gamma^2=\gamma^5\quad\Gamma^5=\gamma^2\quad\Gamma^9=i\gamma^5\gamma^2\quad\Gamma^{14}=(i/2)\left[\gamma^1,\gamma^2\right]\\&\Gamma^6=\gamma^3\quad\Gamma^{10}=i\gamma^5\gamma^3\quad\Gamma^{15}=(i/2)\left[\gamma^1,\gamma^3\right]\\&\Gamma^{16}=(i/2)\left[\gamma^2,\gamma^3\right]\end{aligned}$$

We define the matrices $\tilde{\Gamma}^I$ similarly, but with lowered Lorentz indices; for example, $\tilde{\Gamma}^4=\gamma_1=-\gamma^1=-\Gamma^4$. (This ensures the equations below won't have annoying extra signs due to the signs in the metric.) An inner product for 4×4 matrices can be defined by

$$\mathrm{tr}(AB)=\sum_{ab}A_{ab}B_{ba}$$

Under this inner product, the matrices given above are orthogonal, in the sense that

$$\mathrm{tr}\left(\tilde{\Gamma}^I\Gamma^J\right)=4\delta^{IJ}$$

which implies they are linearly independent, and thus indeed form a basis. (You don't have to check (9), but it follows directly from the results you proved in problem set 6.)

Solution to (b)

We consider a general matrix M, where

$$M_{cd} = \sum_I m^I \Gamma^I_{cd}$$

Multiplying both sides with M_{cd}, the left-hand side is just M_{ab}, while the right-hand side is

$$\frac{1}{4}\sum_I \tilde{\Gamma}^I_{dc}\Gamma^I_{ab}\sum_J m^J\Gamma^J_{cd} = \sum_{IJ}\Gamma^I_{ab}m^J \operatorname{tr}\left(\tilde{\Gamma}^I\Gamma^J\right) = \sum_I \Gamma^I_{ab}m^I = M_{ab}$$

Since the two sides match for any matrix M, the completeness relation holds.

Solution to (c)

We note that

$$\begin{aligned}\tilde{\Gamma}^I_{ab}\Gamma^J_{cd} &= \tilde{\Gamma}^I_{ef}\Gamma^J_{gh}\delta_{ae}\delta_{fb}\delta_{cg}\delta_{hd}\\ &= \frac{1}{16}\tilde{\Gamma}^I_{ef}\Gamma^J_{gh}\sum_{KL}\tilde{\Gamma}^L_{ad}\Gamma^L_{he}\tilde{\Gamma}^K_{fg}\Gamma^K_{cb}\\ &= \frac{1}{16}\sum_{KL}\operatorname{tr}\left[\tilde{\Gamma}^I\tilde{\Gamma}^K\Gamma^J\Gamma^L\right]\tilde{\Gamma}^L_{ad}\Gamma^K_{cb}\end{aligned}$$

where we applied the completeness relation twice, and then used the definition of the trace.
Contracting both sides of the Fierz identity with four spinors $\bar{u}_{1a}u_{2b}\bar{u}_{3c}u_{4d}$ yields

$$\left(\bar{u}_1\tilde{\Gamma}^I u_2\right)\left(\bar{u}_3\Gamma^J u_4\right) = \frac{1}{16}\sum_{KL}\operatorname{tr}\left[\tilde{\Gamma}^I\tilde{\Gamma}^K\Gamma^J\Gamma^L\right]\left(\bar{u}_1\tilde{\Gamma}^L u_4\right)\left(\bar{u}_3\Gamma^K u_2\right)$$

In other words, the Fierz identity relates the product of a contraction of $\bar{u}_1$ with u_2 and a contraction of $\bar{u}_3$ and u_4 (with arbitrary gamma matrices in the middle) to a combination of contractions of $\bar{u}_1$ with u_4 and $\bar{u}_3$ with u_2. This "Fierz transformation" rearranges how the spinors are contracted with each other, which can be useful in calculations.

Solution to (d)

For the first case, we can apply the Fierz identity with $\tilde{\Gamma}^I = \Gamma^J = \mathbb{1}_4$. Then the right-hand side becomes

$$\begin{aligned}\frac{1}{16}\sum_{KL}\operatorname{tr}\left[\tilde{\Gamma}^I\tilde{\Gamma}^K\Gamma^J\Gamma^L\right]\tilde{\Gamma}^L_{ad}\Gamma^K_{cb} &= \frac{1}{16}\sum_{KL}\operatorname{tr}\left[\tilde{\Gamma}^K\Gamma^L\right]\tilde{\Gamma}^L_{ad}\Gamma^K_{cb}\\ &= \frac{1}{4}\sum_{KL}\delta^{KL}\tilde{\Gamma}^L_{ad}\Gamma^K_{cb}\\ &= \frac{1}{4}\sum_K \tilde{\Gamma}^K_{ad}\Gamma^K_{cb}.\end{aligned}$$

Contracting both sides with the spinors, we conclude

$$\begin{aligned}\left(\bar{u}_1u_2\right)\left(\bar{u}_3u_4\right) =& \frac{1}{4}\Big[\left(\bar{u}_1u_4\right)\left(\bar{u}_3u_2\right) + \left(\bar{u}_1\gamma^5u_4\right)\left(\bar{u}_3\gamma^5u_2\right) + \left(\bar{u}_1\gamma^\mu u_4\right)\left(\bar{u}_3\gamma_\mu u_2\right)\\ &- \left(\bar{u}_1\gamma^5\gamma^\mu u_4\right)\left(\bar{u}_3\gamma^5\gamma_\mu u_2\right) - \frac{1}{8}\left(\bar{u}_1\left[\gamma^\mu,\gamma^\nu\right]u_4\right)\left(\bar{u}_3\left[\gamma_\mu,\gamma_\nu\right]u_2\right)\Big].\end{aligned}$$

Note that there is an extra factor of $1/2$ in the final term to avoid double counting. This result is admittedly rather complex, but it's indeed rearranged as desired.

For the second case, we take $\tilde{\Gamma}^I = \gamma_\mu$ and $\Gamma^J = \gamma^\mu$. Computing the right-hand side requires some casework, and use of the trace identities derived in problem set 6. The results are

$$\begin{aligned}
\tilde{\Gamma}^K = \mathbb{1}_4 : &\sum_L \operatorname{tr}\left[\gamma_\mu \gamma^\mu \Gamma^L\right] \tilde{\Gamma}^L_{ad} \delta_{cb} = 16 \delta_{cb} \delta_{ad}, \\
\tilde{\Gamma}^K = \gamma_\nu : &\sum_L \operatorname{tr}\left[\gamma_\mu \gamma_\nu \gamma^\mu \Gamma^L\right] \tilde{\Gamma}^L_{ad} \gamma^\nu_{cb} = -8 \gamma_{\nu ad} \gamma^\nu_{cb}, \\
\tilde{\Gamma}^K = \gamma^5 : &\sum_L \operatorname{tr}\left[\gamma_\mu \gamma^5 \gamma^\mu \Gamma^L\right] \tilde{\Gamma}^L_{ad} \gamma_{5cb} = -16 \gamma_{5ad} \gamma_{5cb}, \\
\tilde{\Gamma}^K = i\gamma^5 \gamma_\nu : &- \sum_L \operatorname{tr}\left[\gamma_\mu \gamma^5 \gamma_\nu \gamma^\mu \Gamma^L\right] \tilde{\Gamma}^L_{ad} \left(\gamma_5 \gamma^\nu\right)_{cb} = 8 \left(i\gamma_5 \gamma_\nu\right)_{ad} \left(i\gamma_5 \gamma^\nu\right)_{cb}, \\
\tilde{\Gamma}^K = \frac{i}{2} \sum_L \left[\gamma_\nu, \gamma_\rho\right] : &\quad \frac{i}{2} \operatorname{tr}\left[\gamma_\mu \left[\gamma_\nu, \gamma_\rho\right] \gamma^\mu \Gamma^L\right] \tilde{\Gamma}^L_{ad} \left(\frac{i}{2}\left[\gamma_\nu, \gamma_\rho\right]\right)_{cb} = 0.
\end{aligned}$$

Putting this all together yields

$$\left(\bar{u}_1 \gamma_\mu u_2\right)\left(\bar{u}_3 \gamma^\mu u_4\right) = \left(\bar{u}_1 u_4\right)\left(\bar{u}_3 u_2\right) - \frac{1}{2}\left(\bar{u}_1 \gamma_\mu u_4\right)\left(\bar{u}_3 \gamma^\mu u_2\right) - \left(\bar{u}_1 \gamma^5 u_4\right)\left(\bar{u}_3 \gamma^5 u_2\right) - \frac{1}{2}\left(\bar{u}_1 \gamma^5 \gamma_\mu u_4\right)\left(\bar{u}_3 \gamma^5 \gamma^\mu u_2\right)$$

Stanford.Zhou.2022.7.3 Electromagnetism in relativistic notation. (20/45p)

In electromagnetism, the Lagrangian is a function of the four-potential $A^\mu = (\phi, \mathbf{A})$,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu, \quad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and $J^\mu = (\rho, \mathbf{J})$ is a classical current density.

a) Show that if the current is conserved, $\partial_\mu J^\mu = 0$, then the action remains the same under the gauge symmetry $A_\mu \to A_\mu + \partial_\mu \alpha$, for any smooth function α.

Solution:

The kinetic term is gauge invariant because $F_{\mu\nu}$ is,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu + \partial_\nu \partial_\mu \alpha - \partial_\nu A_\mu - \partial_\nu \partial_\mu \alpha = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The interaction term is gauge invariant when the current is conserved because

$$J^\mu A_\mu \to J^\mu A_\mu + J^\mu \partial_\mu \alpha = J^\mu A_\mu + \partial_\mu \left(J^\mu \alpha\right)$$

where the change is a total derivative term, which does not affect the action.

b) Show that the Euler-Lagrange equation for A^μ is

$$\partial_\mu F^{\mu\nu} = J^\nu$$

Solution:

The Euler-Lagrange equation is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} = \frac{\partial \mathcal{L}}{\partial A_\nu}$$

The right-hand side is clearly $-J^\nu$. As for the left-hand side, we use the product rule, giving

$$\begin{aligned}
\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu A_\nu\right)} &= -\frac{1}{2} \partial_\mu \left(F^{\rho\sigma} \frac{\partial F_{\rho\sigma}}{\partial\left(\partial_\mu A_\nu\right)}\right) \\
&= -\frac{1}{2} \partial_\mu \left(F^{\rho\sigma}\left(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho\right)\right) \\
&= -\frac{1}{2} \partial_\mu \left(F^{\mu\nu} - F^{\nu\mu}\right) \\
&= -\partial_\mu F^{\mu\nu}
\end{aligned}$$

This gives the desired result.

c) Defining the electric and magnetic fields by $E^i = F^{i0}$ and $\epsilon^{ijk}B^k = -F^{ij}$, show that (14) is equivalent to two of Maxwell's equations.

Solution:

First set $\nu = 0$. Since $F^{00} = 0$, we have $\partial_i F^{i0} = J^0$, which in vector calculus notation is Gauss's law, $\nabla \cdot \mathbf{E} = \rho$. Next set $\nu = j$. Here we have

$$J^j = \partial_0 F^{0j} + \partial_i F^{ij} = -\dot{E}^j - \partial_i \epsilon^{ijk} B^k$$

which in vector calculus notation is

$$\mathbf{J} = -\dot{\mathbf{E}} + \nabla \times \mathbf{B}$$

which is Ampere's law.

d) The other two of Maxwell's equations are

$$\epsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\nu\rho} = 0$$

which follows directly from the definition of $F_{\mu\nu}$. Interestingly, all four of Maxwell's equations can be written as a single spinor equation. Show that the equation

$$\gamma^\nu\gamma^\rho\gamma^\sigma\partial_\nu F_{\rho\sigma} = 2\gamma^\nu J_\nu$$

contains both (14) and (15). (This is just a mathematical trick with no physical meaning, but it's a nice application of the properties of gamma matrices.)

Solution:

To recover (14), we want to get the current by itself on the right-hand side. Thus, multiply both sides by γ^μ and take the trace, giving

$$\operatorname{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right)\partial_\nu F_{\rho\sigma} = 2\operatorname{tr}\left(\gamma^\mu\gamma^\nu\right)J_\nu = 8J^\mu$$

Simplifying the left-hand side, we have

$$\begin{aligned}\operatorname{tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right)\partial_\nu F_{\rho\sigma} &= 4\left(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)\partial_\nu F_{\rho\sigma} \\ &= 4\left(-\partial_\nu F^{\mu\nu} + \partial_\nu F^{\nu\mu}\right) \\ &= 8\partial_\nu F^{\nu\mu}\end{aligned}$$

where we used the fact that F is antisymmetric, so $\eta^{\rho\sigma}F_{\rho\sigma} = 0$. Equating the two sides and renaming the indices recovers 14.

To recover (15), we want to get a factor of $\epsilon^{\mu\nu\rho\sigma}$ on the left-hand side. Therefore, multiply both sides by $\gamma^5\gamma^\mu$ and take the trace. By the results we've derived previously, the right-hand side vanishes and the trace on the left-hand side is proportional to $\epsilon^{\mu\nu\rho\sigma}$, as desired.

e) In the Lagrangian (13), we could have also included a term of the form $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$, which is Lorentz invariant and gauge invariant. What is it in terms of $\mathbf{E}$ and $\mathbf{B}$? Show that this term is a total derivative, and thus does not contribute to the action.

Solution:

To write this term in terms of $\mathbf{E}$ and $\mathbf{B}$, we just expand out the contractions. There are 24 terms, corresponding to the cases where μ, ν, ρ, and σ are all distinct. However, the antisymmetry of F means that groups of eight of them all give the same thing, e.g.

$$\begin{aligned}&\epsilon^{0123}F_{01}F_{23} = \epsilon^{1023}F_{10}F_{23} = \epsilon^{0132}F_{01}F_{32} = \epsilon^{1032}F_{10}F_{32} \\ &= \epsilon^{2301}F_{23}F_{01} = \epsilon^{2310}F_{23}F_{10} = \epsilon^{3201}F_{32}F_{01} = \epsilon^{3210}F_{32}F_{10}\end{aligned}$$

There are thus only three distinct terms to write down. Adjusting the index position for convenience,

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} &= 8\left(\epsilon_{0123}F^{01}F^{23}+\epsilon_{0213}F^{02}F^{13}+\epsilon_{0312}F^{03}F^{12}\right)\\ &= 8\left(-F^{01}F^{23}+F^{02}F^{13}-F^{03}F^{12}\right)\\ &= 8\left(F^{10}F^{23}+F^{20}F^{31}+F^{30}F^{12}\right)\\ &= 8\left(F^{10}F^{23}+F^{20}F^{31}+F^{30}F^{12}\right)\\ &= -8\mathbf{E}\cdot\mathbf{B}.\end{aligned}$$

Now, to show the term is a total derivative, we note that

$$\begin{aligned}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} &= \epsilon^{\mu\nu\rho\sigma}\left[\partial_\mu A_\nu\partial_\rho A_\sigma-\partial_\nu A_\mu\partial_\rho A_\sigma-\partial_\mu A_\nu\partial_\sigma A_\rho+\partial_\nu A_\mu\partial_\sigma A_\rho\right]\\ &= 4\epsilon^{\mu\nu\rho\sigma}\partial_\mu A_\nu\partial_\rho A_\sigma\\ &= 4\epsilon^{\mu\nu\rho\sigma}\partial_\mu\left(A_\nu\partial_\rho A_\sigma\right)-4\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\mu\left(\partial_\rho A_\sigma\right)\\ &= \partial_\mu\left(4\epsilon^{\mu\nu\rho\sigma}A_\nu\partial_\rho A_\sigma\right)\end{aligned}$$

which is a total derivative as desired. Note that the second and third steps in this derivation just follow from the antisymmetry of $\epsilon^{\mu\nu\rho\sigma}$.

For the rest of this problem suppose there is no current, $J^\mu = 0$.

f) Construct the stress-energy tensor by directly applying Noether's theorem.

Solution:

Applying Noether's theorem as usual, we have

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu A_\rho\right)}\partial^\nu A_\rho - \eta^{\mu\nu}\mathcal{L} = -F^{\mu\rho}\partial^\nu A_\rho + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

g) The stress-energy tensor you found in part (**e**) is conserved, but neither symmetric nor gauge invariant. However, we can define an "improved" stress-energy tensor,

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\rho\left(F^{\mu\rho}A^\nu\right)$$

Assuming the equations of motion hold, show that $\hat{T}^{\mu\nu}$ is symmetric, gauge invariant, conserved $\left(\partial_\mu\hat{T}^{\mu\nu}=0\right)$, and traceless $\left(\eta_{\mu\nu}\hat{T}^{\mu\nu}=0\right)$. Furthermore, show that it contains the familiar electromagnetic energy and momentum densities,

$$\hat{T}^{00} = \frac{1}{2}\left(|\mathbf{E}|^2+|\mathbf{B}|^2\right), \quad \hat{T}^{0i} = (\mathbf{E}\times\mathbf{B})^i$$

Solution:

Adding on the new term gives

$$\hat{T}^{\mu\nu} = \left(\partial_\rho F^{\mu\rho}\right)A^\nu + F^{\mu\rho}F^\nu_\rho + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

When the equations of motion hold, $\partial_\rho F^{\mu\rho}=0$, so we may simply drop the first term, giving

$$\hat{T}^{\mu\nu} = -\eta_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} + \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

where we cleaned up the indices a bit. In this form, the stress-energy tensor is clearly symmetric, and it is gauge invariant because it is written in terms of the gauge invariant F alone. To show conservation, it is easiest to note that

$$\partial_\mu\hat{T}^{\mu\nu} = \partial_\mu T^{\mu\nu} + \partial_\mu\partial_\rho\left(F^{\mu\rho}A^\nu\right) = 0$$

where the first term vanishes by Noether's theorem, and the second term vanishes by the antisymmetry of F. Finally, to show tracelessness, we note that

$$\eta_{\mu\nu}\hat{T}^{\mu\nu} = -\eta_{\mu\nu}\eta_{\rho\sigma}F^{\mu\rho}F^{\nu\sigma} + \frac{1}{4}\eta_{\mu\nu}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = -F_{\nu\sigma}F^{\nu\sigma} + F_{\rho\sigma}F^{\rho\sigma} = 0$$

Now let's consider the components. For the energy density, we have

$$\begin{aligned}\hat{T}^{00} &= -\eta_{\rho\sigma}F^{0\rho}F^{0\sigma} + \frac{1}{4}\eta^{00}F_{\rho\sigma}F^{\rho\sigma} \\ &= F^{01}F^{01} + F^{02}F^{02} + F^{03}F^{03} + \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma} \\ &= |\mathbf{E}|^2 + \frac{1}{2}\left(F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} + F_{12}F^{12} + F_{23}F^{23} + F_{31}F^{31}\right) \\ &= |\mathbf{E}|^2 + \frac{1}{2}\left(-|\mathbf{E}|^2 + |\mathbf{B}|^2\right) \\ &= \frac{1}{2}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right)\end{aligned}$$

as desired. For the momentum density, we have

$$\hat{T}^{0i} = -\eta_{\rho\sigma}F^{0\rho}F^{i\sigma} = F^{0j}F^{ij} = E^j\epsilon^{ijk}B^k = (\mathbf{E}\times\mathbf{B})^i.$$

In general, the improved stress-energy tensor considered here is called the Belinfante tensor; it is this stress-energy tensor that sources gravity in general relativity. We didn't see it earlier because the improvement terms are related to the spin angular momentum of the field, which is zero for scalar fields.

36 Дискретные симметрии в теории Дирака.

Проверьте, что все величины: $\bar{\psi}\psi, \bar{\psi}\gamma^\mu\psi, i\bar{\psi}\left[\gamma^\mu,\gamma^\nu\right]\psi, \bar{\psi}\gamma^\mu\gamma^5\psi, i\bar{\psi}\gamma^5\psi$ являются эрмитовыми.

Решение: Используя то, что $\left(\gamma^0\right)^\dagger = \gamma^0, \left(\gamma^5\right)^\dagger = \gamma^5$, а $\left(\gamma^i\right)^\dagger = -\gamma^i, i = 1,2,3$, а также, что $\left\{\gamma^5,\gamma^\mu\right\} = 0, \left\{\gamma^\mu,\gamma^\nu\right\} = 2g^{\mu\nu}$ имеем:

$$\begin{aligned}&(\bar{\psi}\psi)^\dagger = \left(\psi^\dagger\gamma^0\psi\right)^\dagger = \psi^\dagger\left(\gamma^0\right)^\dagger\psi = \psi^\dagger\gamma^0\psi = \bar{\psi}\psi \\ &\left(\bar{\psi}\gamma^\mu\psi\right)^\dagger = \left(\psi^\dagger\gamma^0\gamma^\mu\psi\right)^\dagger = \psi^\dagger\left(\gamma^\mu\right)^\dagger\left(\gamma^0\right)^\dagger\psi = \psi^\dagger\gamma^0\gamma^\mu\psi = \bar{\psi}\gamma^\mu\psi, \\ &\left(i\bar{\psi}\left[\gamma^\mu,\gamma^\nu\right]\psi\right)^\dagger = \left(i\psi^\dagger\gamma^0\left[\gamma^\mu,\gamma^\nu\right]\psi\right)^\dagger = -i\psi^\dagger\left(\left(\gamma^\nu\right)^\dagger\left(\gamma^\mu\right)^\dagger - \left(\gamma^\mu\right)^\dagger\left(\gamma^\nu\right)^\dagger\right)\gamma^0\psi = i\bar{\psi}\left[\gamma^\mu,\gamma^\nu\right]\psi, \\ &\left(\bar{\psi}\gamma^\mu\gamma^5\psi\right)^\dagger = \psi^\dagger\left(\gamma^5\right)^\dagger\left(\gamma^\mu\right)^\dagger\gamma^0\psi = \psi^\dagger\gamma^5\gamma^0\gamma^\mu\psi = \bar{\psi}\gamma^\mu\gamma^5\psi, \\ &\left(i\bar{\psi}\gamma^5\psi\right)^\dagger = -i\psi^\dagger\left(\gamma^5\right)^\dagger\left(\gamma^0\right)^\dagger\psi = -i\psi^\dagger\gamma^5\gamma^0\psi = i\bar{\psi}\gamma^5\psi,\end{aligned} \tag{0.89}$$

что и требовалось проверить.

35

Покажите, что

$$\gamma^0\gamma^\mu\gamma^0 = \begin{cases}+\gamma^\mu, & \mu = 0 \\ -\gamma^\mu, & \mu = 1,2,3\end{cases}$$

Решение: Используя то, что $\left(\gamma^0\right)^2 = 1$, а также $\left\{\gamma^0,\gamma^i\right\} = 0, i = 1,2,3$, получаем

$$\gamma^0\gamma^0\gamma^0 = \gamma^0, \quad \gamma^0\gamma^i\gamma^0 = -\gamma^i\gamma^0\gamma^0 = -\gamma^i \tag{0.90}$$

что и требовалось показать.

36

Проверьте, что

$$T\bar{\psi}\gamma^\mu\psi T = \begin{cases}+\bar{\psi}\gamma^\mu\psi(-t,\mathbf{x}), & \mu = 0 \\ -\bar{\psi}\gamma^\mu\psi(-t,\mathbf{x}), & \mu = 1,2,3\end{cases}$$

Решение: Используя, что $T(c\text{ - число }) = (c-\text{ число })^*T$, и также $T\psi(t,\boldsymbol{x})T = \gamma^1\gamma^3\psi(-t,\boldsymbol{x})$, получим

$$T\bar{\psi}\gamma^{\mu}\psi T = T\bar{\psi}T\left(\gamma^{\mu}\right)^{*}T\psi T = \psi^{\dagger}(-t,\boldsymbol{x})\left[\gamma^{1}\gamma^{3}\right]^{\dagger}(\gamma)^{0}\left(\gamma^{\mu}\right)^{*}\gamma^{1}\gamma^{3}\psi(-t,\boldsymbol{x}) = \\ = \bar{\psi}(-t,\boldsymbol{x})\left(-\gamma^{1}\gamma^{3}\right)\left(\gamma^{\mu}\right)^{*}\gamma^{1}\gamma^{3}\psi(-t,\boldsymbol{x}) \tag{0.91}$$

Теперь используя, что $\left(\gamma^{0}\right)^{*} = \gamma^{0}, \left(\gamma^{1}\right)^{*} = \gamma^{1}, \left(\gamma^{3}\right)^{*} = \gamma^{3}$ и $\left(\gamma^{2}\right)^{*} = -\gamma^{2}, \{\gamma^{\mu},\gamma^{\nu}\} = 2g^{\mu\nu}$, получим

$$T\bar{\psi}\gamma^{0}\psi T = \bar{\psi}(-t,\boldsymbol{x})\left(-\gamma^{1}\gamma^{3}\right)\gamma^{0}\gamma^{1}\gamma^{3}\psi(-t,\boldsymbol{x}) = \bar{\psi}\gamma^{0}\psi(-t,\boldsymbol{x}), \\ T\bar{\psi}\gamma^{1}\psi T = \bar{\psi}(-t,\boldsymbol{x})\left(-\gamma^{1}\gamma^{3}\right)\gamma^{1}\gamma^{1}\gamma^{3}\psi(-t,\boldsymbol{x}) = -\bar{\psi}\gamma^{1}\psi(-t,\boldsymbol{x}), \\ T\bar{\psi}\gamma^{2}\psi T = -\bar{\psi}(-t,\boldsymbol{x})\left(-\gamma^{1}\gamma^{3}\right)\gamma^{2}\gamma^{1}\gamma^{3}\psi(-t,\boldsymbol{x}) = -\bar{\psi}\gamma^{2}\psi(-t,\boldsymbol{x}), \\ T\bar{\psi}\gamma^{3}\psi T = \bar{\psi}(-t,\boldsymbol{x})\left(-\gamma^{1}\gamma^{3}\right)\gamma^{3}\gamma^{1}\gamma^{3}\psi(-t,\boldsymbol{x}) = -\bar{\psi}\gamma^{3}\psi(-t,\boldsymbol{x}), \tag{0.92}$$

что и требовалось проверить.

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Найдите чему равно преобразование зарядового сопряжения для биллинейных форм:
$C\bar{\psi}\psi C, C\bar{\psi}\gamma^{\mu}\psi C, Ci\bar{\psi}\left[\gamma^{\mu},\gamma^{\nu}\right]\psi C, C\bar{\psi}\gamma^{\mu}\gamma^{5}\psi C, Ci\bar{\psi}\gamma^{5}\psi C$.
Решение: Используя, что $C\psi(x)C = -i\left(\bar{\psi}\gamma^{0}\gamma^{2}\right)^{T}$, и также $C\bar{\psi}C = \left(-i\gamma^{0}\gamma^{2}\psi\right)^{T}$ имеем:
$C\bar{\psi}\psi C = \left(-i\gamma^{0}\gamma^{2}\psi\right)^{T}\left(-i\bar{\psi}\gamma^{0}\gamma^{2}\right)^{T} = -\gamma^{0}_{ab}\gamma^{2}_{bc}\psi_{c}\bar{\psi}_{d}\gamma^{0}_{de}\gamma^{2}_{ea} = \bar{\psi}_{d}\gamma^{0}_{de}\gamma^{2}_{ea}\gamma^{0}_{ab}\gamma^{2}_{bc}\psi_{c} = -\bar{\psi}\gamma^{2}\gamma^{0}\gamma^{0}\gamma^{2}\psi = \bar{\psi}\psi$,
$Ci\bar{\psi}\gamma^{5}\psi C = i\left(-i\gamma^{0}\gamma^{2}\psi\right)^{T}\gamma^{5}\left(-i\bar{\psi}\gamma^{0}\gamma^{2}\right)^{T} = i\bar{\psi}\gamma^{5}\psi$,
аналогично можно проверить для остальных выражений.

МФТИ-Киселев-2020.8 Тождество вейлевского спинора

Для правого вейлевского спинора покажите, что из уравнения движения следует тождество

$$\frac{1}{\hbar}\boldsymbol{W}\bar{\chi} = \frac{1}{2}\boldsymbol{p}\bar{\chi}$$

Solution

(?? вроде типичная по теории задача? напишу теорию точнее!!!)
Правое киральное безмассовое поле $\bar{\chi}^{\dot{\alpha}}(x)$, преобразующееся по представлению $\left(\frac{1}{2},0\right)$ группы $\mathrm{SL}(2,\mathbb{C})$, согласно общему формализму удовлетворяет уравнению

$$\mathfrak{s}p^{\mu}\bar{\chi} = \frac{1}{\hbar}W^{\mu}\bar{\chi},$$

где спиральность $\mathfrak{s} = \frac{1}{2}$, а оператор W^{k} задается покомпонентно, так что для временно́й компоненты

$$\frac{1}{\hbar}W^{0} = \frac{1}{2}p\cdot\sigma$$

а уравнения для пространственных компонент, как мы показали, являются следствием заданного представления для безмассового поля. В явном виде уравнение для p_0 принимает вид

$$p_{0}\bar{\chi} = p\cdot\sigma\bar{\chi}$$

или с учетом определения 4-вектора сигма-матриц

$$p_{\mu}\sigma^{\mu}\bar{\chi} = 0 \leftrightarrow p_{\mu}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = 0$$

МФТИ-Киселев-2020.9 Отраженный спинор (???)

Показать, что если χ удовлетворяет уравнению

$$\boldsymbol{p}\cdot\boldsymbol{\sigma}\bar{\chi}(\boldsymbol{p}) = |\boldsymbol{p}|\bar{\chi}(\boldsymbol{p}),$$

то спинор $\chi_{cp}(-\boldsymbol{p}) = -\mathrm{i}\sigma_2\bar{\chi}^*(\boldsymbol{p})$ удовлетворяет уравнению

$$-\boldsymbol{p}\cdot\boldsymbol{\sigma}\chi_{cp}(-\boldsymbol{p}) = |\boldsymbol{p}|\chi_{cp}(-\boldsymbol{p})$$

Solution

(???? доучу скоро. суть - что решаем ур движения, да? Тут как раз то самое сопряжение, напишу в 1й части тоже про это!)

Найдем решения уравнения для спинора (6.1) в импульсном представлении, т. е. в виде плоской волны

$$\bar{\chi}(x) = e^{-\frac{\mathrm{i}}{\hbar}p\cdot x}\sum_{\lambda}\bar{\chi}_{(\lambda)}(\boldsymbol{p})$$

где $p\cdot x = p_0x_0 - \boldsymbol{p}\cdot\boldsymbol{x}$. Как нам известно, матрица $p\cdot\sigma = \sigma_p$ имеет пару собственных векторов-столбцов, так что в их базисе

$$\boldsymbol{p}\cdot\boldsymbol{\sigma} = |\boldsymbol{p}|\sigma_3 = |\boldsymbol{p}|\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

и, принимая условие нормировки $\bar{\chi}^\dagger\bar{\chi} = 2|p|$

$$\bar{\chi}(\boldsymbol{p}) = \sqrt{2|\boldsymbol{p}|}\begin{pmatrix}1\\0\end{pmatrix},\quad \boldsymbol{p}\cdot\boldsymbol{\sigma}\bar{\chi}(\boldsymbol{p}) = |\boldsymbol{p}|\bar{\chi}(\boldsymbol{p})$$
$$\chi_{cp}(\boldsymbol{p}) = \sqrt{2|p|}\begin{pmatrix}0\\1\end{pmatrix},\quad -\boldsymbol{p}\cdot\boldsymbol{\sigma}\chi_{cp}(-\boldsymbol{p}) = |\boldsymbol{p}|\chi_{cp}(-\boldsymbol{p})$$

МФТИ-Киселев-2020. Гамильтониан спиноров

Вычислить гамильтониан правого вейлевского спинора в терминах амплитуд релятивистски нормированных мод.

Solution

Обобщенный импульс равен $\frac{\delta L_{\mathrm{R}}}{\delta\dot{\bar{\chi}}(x)} = \mathrm{i}\hbar\chi\sigma^0$, поэтому гамильтониан вейлевского классического поля: (??? проверю!)

$$H_{\mathrm{R}} = \int \mathrm{d}^3x\left\{\mathrm{i}\hbar\chi\sigma^0\dot{\bar{\chi}} - c\chi p_\mu\sigma^\mu\bar{\chi}\right\} = c\int \mathrm{d}^3x\{\chi\boldsymbol{p}\cdot\sigma\bar{\chi}\}$$

В итоге разложение классического правого спинорного поля по модам принимает вид

$$\bar{\chi}(x) = \int\frac{\mathrm{d}^3p}{2p_0(2\pi\hbar)^3}\left\{e^{-\frac{1}{\hbar}p\cdot x}\bar{\chi}(p)\vartheta(p) + e^{\frac{1}{\hbar}p\cdot x}\chi_{cp}(p)\bar{v}(p)\right\}$$

где уже энергия p_0 принимает только положительные значения $p_0 = |p|, p\cdot x = p_0x_0 - p\cdot\boldsymbol{x}$, а $\vartheta(p)$ и $v(p)$ - комплекснозначные параметры. С учетом ортогональности базисных спиноров находим для гамильтониана

$$H_{\mathrm{R}} = \int\varepsilon(p)\frac{\mathrm{d}^3p}{2p_0(2\pi\hbar)^3}\{\bar{\vartheta}(p)\vartheta(p) - v(p)\bar{v}(p)\}$$

где $\varepsilon(p) = c|p| = cp_0 > 0$ - положительно определенная энергия спинорной моды.

МФТИ-Киселев-2020.11^C Заряд

Вычислить заряд правого вейлевского спинора в терминах амплитуд релятивистски нормированных мод.

Solution

Аналогично предыдущему пункту, заряд классического поля

$$Q = e\int \frac{\mathrm{d}^3 p}{2p_0(2\pi\hbar)^3}\{\bar{\vartheta}(\boldsymbol{p})\vartheta(\boldsymbol{p}) + v(\boldsymbol{p})\bar{v}(\boldsymbol{p})\}$$

Просто подставили разложение спинорного поля по модам в формулу для тока.

МФТИ-Киселев-2020.12 Проекторы

Показать, что проекторы на состояния с заданной проекцией спина частицы на вектор поляризации имеют вид

$$P_\pm = \frac{1}{2}\left(1 + \lambda\gamma_5 \not{\epsilon}\right)$$

а для античастиц -

$$P^c_\pm = \frac{1}{2}\left(1 - \lambda\gamma_5 \not{\epsilon}\right)$$

где λ — направление спина вдоль вектора поляризации ε^μ, ортогонального 4-импульсу p:

$$\lambda = \pm 1, \quad \epsilon^2 = -1, \quad \epsilon \cdot p = 0$$

Solution

(?? очень известная из теории тема, напишу потом)

МФТИ-13 Сумма по поляризациям дираковских частиц и античастиц

Вычислить сумму по поляризациям дираковских частиц и античастиц:

$$\Pi(\boldsymbol{p}) = \sum_\lambda u_\lambda(\boldsymbol{p})\bar{u}_\lambda(\boldsymbol{p}) = \not{p} + mc, \quad \Pi^c(\boldsymbol{p}) = \sum_\lambda v_\lambda(\boldsymbol{p})\bar{v}_\lambda(\boldsymbol{p}) = \not{p} - mc$$

Solution

Для покоящихся спиноров сумму по поляризациям или матрицу

$$\Pi(0) := \sum_\lambda u_\lambda(0)\bar{u}_\lambda(0)$$

можно вычислить в явном виде как прямое произведение строк и столбцов

$$\Pi(\mathbf{0}) = mc(1,0,1,0)\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + mc(0,1,0,1)\begin{pmatrix}0\\1\\0\\1\end{pmatrix} = mc\begin{pmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{pmatrix}$$

или в ковариантном виде

$$\Pi(0) = mc\left(\gamma_0 + 1\right) = \left(p' + mc\right), \qquad p' = (mc, 0)$$

Поскольку преобразование Лоренца для матрицы A сводятся к преобразованию 4-вектора A^ν в движущуюся систему отсчета:

$$\Lambda \not{A} \Lambda^{-1} \to \not{A}$$

сумма по поляризациям для движущихся частиц

$$\Pi(p) = (\not{p} + mc)$$

и аналогично для античастиц

$$\Pi^c(p) = \sum_\lambda v_\lambda(p)\bar{v}_\lambda(p) = (\not{p} - mc)$$

(?? нам что-то нужно подставить, для этого пишем то, что ниже, я забыл, что?)
Запишем уравнение Дирака

$$(\mathrm{i}\gamma^\mu\partial_\mu - m)\,\psi = 0$$

Рассмотрим для начала положительно частотные решения уравнения (16.1). Для этого запипем ψ в виде

$$\psi = u(\mathbf{p})e^{-ip\cdot x}$$

где $u(\mathbf{p})$ - четырёхкомпонентных спинор, зависящий только от $\mathbf{p}$. С учётом (16.2), уравнение Дирака примет вид

$$(\gamma^\mu p_\mu - m)\,u(\mathbf{p}) = \begin{pmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\bar\sigma^\mu & -m \end{pmatrix} u(\mathbf{p}) = 0$$

Здесь мы использовали следующие обозначения: $\sigma^\mu = \left(1, \sigma^i\right), \bar\sigma^\mu = \left(1, -\sigma^i\right)$. Найдём общий вид $u(\mathbf{p})$. Для этого запишем спинор $u(\mathbf{p})$ как

$$u(\mathbf{p}) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Тогда имеем

$$(p\cdot\sigma)u_2 = mu_1, \quad (p\cdot\bar\sigma)u_1 = mu_2$$

Заметим, что $(p\cdot\sigma)(p\cdot\bar\sigma) = m^2$. Действительно,

$$(p\cdot\sigma)(p\cdot\bar\sigma) = p_0^2 - p_ip_j\sigma^i\sigma^j = p_0^2 - p_ip_j\delta^{ij} = p_0^2 - \mathbf{p}^2 = p^2 = m^2$$

Всвязи с этим, представим двухкомпонентный спинор u_1 в виде $u_1 = (p\cdot\sigma)\xi'$. В таком случае спинор u_2 примет вид: $u_2 = m\xi'$. Значит,

$$u(\mathbf{p}) = A\begin{pmatrix} (p\cdot\sigma)\xi' \\ m\xi' \end{pmatrix}$$

Выбирая $A = \frac{1}{m}, \xi' = \sqrt{p\cdot\bar\sigma}\xi\ \left(\xi^\dagger\xi = 1\right)$, получаем

$$u(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar\sigma}\xi \end{pmatrix}$$

Аналогично можно рассмотреть и отрицательно частотные решения уравнения Дирака:

$$\psi = v(\mathbf{p})e^{\mathrm{i}p\cdot x}$$
$$(\gamma^\mu p_\mu + m)\,v(\mathbf{p}) = \begin{pmatrix} m & p_\mu\sigma^\mu \\ p_\mu\bar\sigma^\mu & m \end{pmatrix} v(\mathbf{p}) = 0$$

Решение имеет вид:

$$v(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta \\ -\sqrt{p\cdot\bar\sigma}\eta \end{pmatrix}$$

где η - некоторый постоянный двухкомпонентный спинор с нормировкой $\eta^\dagger\eta = 1$. Вычислим теперь суммы по поляризациям $\sum\limits_s u_s(\mathbf{p})\bar u_s(\mathbf{p}), \sum\limits_s v_s(\mathbf{p})\bar v_s(\mathbf{p})$:

$$\sum_s u_s(\mathbf{p})\bar u_s(\mathbf{p}) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\xi_s \\ \sqrt{p\cdot\bar\sigma}\xi_s \end{pmatrix}\left(\xi_s^\dagger\sqrt{p\cdot\bar\sigma}, \xi_s^\dagger\sqrt{p\cdot\sigma}\right) = \begin{pmatrix} m & p_\mu\sigma^\mu \\ p_\mu\bar\sigma^\mu & m \end{pmatrix} = p\gamma + m = \not{p} + m$$

Здесь учтено, что $\sum\limits_s \xi_s\xi_s^\dagger = 1$. Аналогично находим, что

$$\sum_s v_s(\mathbf{p})\bar v_s(\mathbf{p}) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\eta_s \\ -\sqrt{p\cdot\bar\sigma}\eta_s \end{pmatrix}\left(-\eta_s^\dagger\sqrt{p\cdot\bar\sigma}, \eta_s^\dagger\sqrt{p\cdot\sigma}\right) = \begin{pmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\bar\sigma^\mu & -m \end{pmatrix} = p\gamma - m = \not{p} - m$$

МФТИ-Мусаев-2023. 7. Оператор спиральности

Доказать, что оператор спиральности $\chi = \frac{1}{2}\vec\Sigma\vec n$ (где $\vec\Sigma = \gamma_0\gamma_5\vec\gamma$, $\vec n$ - единичный вектор вдоль импульса частицы, $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$) коммутирует со свободным дираковским гамильтонианом. Найти собственные значения оператора спиральности.

МФТИ-Мусаев-2023**.8. Суммы произведений биспиноров**

Вычислить суммы произведений биспиноров

$$\sum_{\sigma} u(\sigma, \pm E, \vec{p})\bar{u}(\sigma, \pm E, \vec{p}), \qquad \text{где} \qquad E = \sqrt{p^2 + m^2} > 0.$$

по спиральности частиц $\sigma = \pm\frac{1}{2}$.

МФТИ-Мусаев-2023**.9**/19 **Гамма матрицы**

Пусть $A = A_\mu\gamma_\mu$, а R - произвольное матричное выражение. Доказать равенство

$$\gamma_\mu A R \gamma_\mu + A\gamma_\mu R\gamma_\mu = 2RA$$

МФТИ-Мусаев-2023**.10**/19

Найти сохраняющийся оператор, соответствующий инвариантности лагранжиана дираковского поля относительно лоренцевых бустов и трехмерных поворотов. Выразить его в терминах операторов рождения и уничтожения электронов и позитронов. Вычислить спин возбуждений дираковского поля, при условии, что спин безмассового векторного поля равен 1.

МФТИ-Мусаев-2023**.11**/19

Показать, что при унитарных преобразованиях

$$UU^+ = U^+U = 1$$

не меняются нормировка состояний, средние значения операторов, коммутационные соотношения, свойство эрмитовости операторов, собственные значения операторов.

26.2 Problems about Useful QFT Formalism

26.2.1 Problems about operators in QFT

26.2.2 Problems about propagators in QFT

МФТИ-Мусаев-2023**.1**/19 **Причинность**

Показать, что фейнмановская функция Грина свободного действительного скалярного поля является причинной. В каком смысле следует понимать причинность?

МФТИ-Мусаев-2023**.2**/19 **Смысл пропагатора**

Показать, что пропагатор скалярной частицы

$$D(x-y) = i\langle 0|\mathbf{T}\{\phi(x)\phi(y)\}|0\rangle$$

описывает следующие последовательности событий: **a)** при $t_x > t_y$ в точке y рождается частица, которая затем уничтожается в точке x; **b)** при $t_x < t_y$ частица рождается в точке x и уничтожается в точке y;
Проверить в x - и p-представлениях, что D пропорциональна функции Грина оператора д'Аламбера $\Box$.

26.2.3 Problems about path integral formalism in QFT 1

Белавин-Пархоменко**.7.1 Действие мировой линии и пропагатор Фейнмана**

(a).

Классическое действие для релятивистской частицы массы m дается произведением m с собственным временем вдоль мировой линии частицы

$$S[x] = -m \int d\tau = -m \int_0^1 ds \sqrt{-\dot{x}^\mu \dot{x}_\mu}, \tag{0.1}$$

где $\dot{A} = \frac{\partial A}{\partial s}$. Такое релятивисткое действие мировой линии инвариантно относительно репараметризаций $s \to s'(s)$, где $s'(0) = 0, s'(1) = 1$, and $ds'/ds > 0$. Введем множитель Лагранжа $\lambda(s) > 0$. Покажите, что следующее действие

$$S[x, \lambda] = \frac{1}{2} \int_0^1 ds \left(\frac{\dot{x}^2(s)}{\lambda(s)} - m^2 \lambda(s) \right) \tag{0.2}$$

классически эквивалентно (в итоге дает тоже самое уравнение движения для x^μ) действию (0.1). (Подсказка: Исключите $\lambda(s)$ из уравнений движения.) Теперь в добавок, мы введем импульс p_μ. Покажите, что

$$S[x, \lambda, p] = - \int_0^1 ds \left(\dot{x}^\mu p_\mu + \frac{\lambda}{2} \left(p^2 + m^2 \right) \right) \tag{0.3}$$

классически эквивалентно действию (0.1).

(b).

Покажите, что действие (0.3) инвариантно относительно репараметризаций, если $\lambda(s)$ преобразуется определенным образом. Используя эту инвариантность, покажите, что можно фиксировать $\lambda(s) = \tau$, где $\tau > 0$ и не зависит от s.

Теперь мы определим пропагатор Фейнмана $G_F(x, y)$ как интеграл по путям из одной точки пространства x в другую y:

$$G_F(x, y) = \int_x^y [dx] \int [d\lambda][dp] e^{\frac{i}{\hbar} S[x, \lambda, p]} \tag{0.4}$$

(c).

Используя эквивалентность между инегралом по путям и Шредингеровской формулировкой квантовой механики, покажите, что

$$G_F(x, y) = \int_0^\infty d\tau \left\langle y \left| e^{-\frac{i}{\hbar} \hat{H} \tau} \right| x \right\rangle \tag{0.5}$$

с $\hat{H} = \hat{p}^2 + m^2$.

Интеграл по τ, как мы уже знаем, называется интегралом Швингера по собственному времени. Чтобы сделать этот интеграл формально хорошо определенным - сходящимся, мы добаляем маленькую мнимую часть к Гамильтониану $\hat{H} \to \hat{H} - i\epsilon$.

(d). Покажите, что это требование сходимости, по сути эквивалентно $i\epsilon$ прескрипции для пропагатора Фейнмана:

$$G_F(x, y) = \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2 + m^2 - i\epsilon} \tag{0.6}$$

Белавин-Пархоменко.7.1 Действие мировой линии и пропагатор Фейнмана

(a). Классическое действие для релятивистской частицы массы m дается произведением m с собственным временем вдоль мировой линии частицы

$$S[x] = -m \int d\tau = -m \int_0^1 ds \sqrt{-\dot{x}^\mu \dot{x}_\mu} \tag{0.1}$$

где $\dot{A} = \frac{\partial A}{\partial s}$. Такое релятивисткое действие мировой линии инвариантно относительно репараметризаций $s \to s'(s)$, где $s'(0) = 0, s'(1) = 1$, and $ds'/ds > 0$. Введем множитель Лагранжа $\lambda(s) > 0$. Покажем, что следующее действие

$$S[x, \lambda] = \frac{1}{2} \int\limits_0^1 ds \left(\frac{\dot{x}^2(s)}{\lambda(s)} - m^2 \lambda(s) \right) \tag{0.2}$$

классически эквивалентно)- в итоге дает тоже самое уравнение движения для x^μ) действию (0.1). С одной стороны для уравнения (0.1) получаем следующие уравнения движения

$$\delta_x S[x] = 0, \quad \text{откуда:} \quad \frac{d}{ds} \left(\frac{m \dot{x}^\mu}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}} \right) = 0 \tag{0.3}$$

С другой стороны уравнения движения для уравнения (0.2) следующие

$$\delta_\lambda S[x, \lambda] = 0, \quad \text{откуда:} \quad \frac{\dot{x}^2}{\lambda^2} + m^2 = 0, \tag{0.4}$$

и мы находим $\lambda = \frac{1}{m}\sqrt{-\dot{x}^2}$. Теперь подставляя это в действие (0.2) мы получим

$$S[x, \lambda] = \frac{1}{2} \int\limits_0^1 ds \left(-m \frac{-\dot{x}^2}{\sqrt{-\dot{x}^2}} - m^2 \frac{1}{m} \sqrt{-\dot{x}^2} \right) = -m \int\limits_0^1 ds \sqrt{-\dot{x}^\mu \dot{x}_\mu} = S[x] \tag{0.5}$$

Альтернативно мы можем также найти уравнения движения для x у уравнения (0.2), имеем

$$\delta_x S[x, \lambda] = 0, \quad \text{откуда:} \quad \frac{d}{ds} \left(\frac{\dot{x}^\mu}{\lambda} \right) = 0 \tag{0.6}$$

подставляя в него $\lambda = \frac{1}{m}\sqrt{-\dot{x}^2}$, получим

$$\frac{d}{ds} \left(\frac{m \dot{x}^\mu}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}} \right) = 0 \tag{0.7}$$

что совпадает с уравнениями движения для исходного действия $S[x]$. Теперь в добавок, мы введем импульс p_μ. Покажем, что

$$S[x, \lambda, p] = - \int\limits_0^1 ds \left(\dot{x}^\mu p_\mu + \frac{\lambda}{2} \left(p^2 + m^2 \right) \right) \tag{0.8}$$

классически эквивалентно действию (0.1). Аналогично находим уравнения движения, из вариации действия по импульсу:

$$\delta_p S[x, \lambda, p] = 0, \quad \text{откуда:} \quad \dot{x}^\mu + \lambda p^\mu = 0 \tag{0.9}$$

откуда мы находим, что $p^\mu = -\frac{1}{\lambda}\dot{x}^\mu$ и подставляя это в действие (0.8) имеем

$$S[x, \lambda, p] = - \int\limits_0^1 ds \left(-\dot{x}^\mu \frac{1}{\lambda} \dot{x}_\mu + \frac{\lambda}{2} \left(\frac{1}{\lambda^2} \dot{x}^2 + m^2 \right) \right) = \frac{1}{2} \int\limits_0^1 ds \left(\frac{\dot{x}^2(s)}{\lambda(s)} - m^2 \lambda(s) \right) = S[x, \lambda]. \tag{0.10}$$

То есть мы пришли к действию $S[x, \lambda]$, которое как мы уже показали эквивалентно действию $S[x]$.
(b). Покажем, что действие (0.8) инвариантно относительно репараметризаций, если $\lambda(s)$ преобразуется определенным образом. Имеем, при преобразовании $s' = s'(s)$:

$$ds' = \frac{ds'}{ds} ds, \quad \frac{dx^\mu}{ds} = \frac{dx^\mu}{ds'} \frac{ds'}{ds} \tag{0.11}$$

и теперь если мы определим преобразование для $\lambda(s)$ как $\lambda(s) = \lambda'(s') \frac{ds'}{ds}$, то мы получим

$$S[x,\lambda,p] = -\int ds\left(\dot{x}^\mu p_\mu + \frac{\lambda}{2}\left(p^2+m^2\right)\right) = -\int\limits_0^1 ds'\frac{ds}{ds'}\left(\frac{ds'}{ds}\frac{dx^\mu}{ds'}p_\mu + \frac{ds'}{ds}\frac{\lambda'(s')}{2}\left(p^2+m^2\right)\right) =$$

$$= -\int\limits_0^1 ds'\left(\frac{dx^\mu}{ds'}p_\mu + \frac{\lambda'(s')}{2}\left(p^2+m^2\right)\right) = S[x,\lambda,p]. \tag{0.12}$$

Используя эту инвариантность, покажем, что можно фиксировать $\lambda(s)=\tau$, где $\tau>0$ и не зависит от s. Имеем

$$\lambda'(s') = \frac{ds}{ds'}\lambda(s) \tag{0.13}$$

и пусть $\lambda'(s')=\tau>0$, тогда мы получаем уравнение для $s'(s)$:

$$\frac{ds}{ds'}\lambda(s)=\tau, \quad \text{откуда} \quad \frac{ds'}{ds}=\frac{1}{\tau}\lambda(s) \tag{0.14}$$

и решение данного уравнения

$$s'(s) = \frac{1}{\tau}\int\limits_0^s \lambda(x)dx, \quad \text{где} \quad \tau = \int\limits_0^1 \lambda(s)ds \tag{0.15}$$

То есть мы явно построили преобразование $s'(s)$ при котором $\lambda'(s')=\tau$.
Теперь мы определим пропагатор Фейнмана $G_F(x,y)$ как интеграл по путям из одной точки пространства x в другую y:

$$G_F(x,y) = \int\limits_x^y [dx]\int [d\lambda][dp]e^{\frac{i}{\hbar}S[x,\lambda,p]} \tag{0.16}$$

(c). Используя эквивалентность между инегралом по путям и Шредингеровской формулировкой квантовой механики, покажем, что

$$G_F(x,y) = \int\limits_0^\infty d\tau\left\langle y\left|e^{-\frac{i}{\hbar}\hat{H}\tau}\right|x\right\rangle \tag{0.17}$$

с $\hat{H}=\hat{p}^2+m^2$. Имеем

$$\int [d\lambda] = \text{Vol(diff)}\times\int\limits_0^\infty d\tau \tag{0.18}$$

где Vol(diff) - объем группы репараметризаций. (На самом деле данная формула требует уточнения и более детального вывода, но здесь мы опустим эти детали). В итоге получаем

$$G_F(x,y) = \int\limits_x^y [dx]\int [d\lambda][dp]e^{\frac{i}{\hbar}S[x,\lambda,p]} = \int\limits_0^\infty d\tau\int\limits_x^y [dx][dp]e^{-\frac{i}{\hbar}\int\limits_0^1 ds\left(\dot{x}^\mu p_\mu+\frac{\tau}{2}\left(p^2+m^2\right)\right)}. \tag{0.19}$$

С другой стороны мы знаем, что

$$\left\langle y\left|e^{-\frac{iHT}{\hbar}}\right|x\right\rangle = \int [dx][dp]e^{\frac{i}{\hbar}\int\limits_0^T dt(p_\mu\dot{x}^\mu - H(x,p))} \tag{0.20}$$

откуда получим

$$\int\limits_x^y [dx][dp]e^{-\frac{i}{\hbar}\int\limits_0^1 ds\left(\dot{x}^\mu p_\mu+\frac{\tau}{2}\left(p^2+m^2\right)\right)} = \left\langle y\left|e^{\frac{i}{\hbar}\left(-\frac{\tau}{2}\left(p^2+m^2\right)\right)}\right|x\right\rangle = \left\langle y\left|e^{-\frac{i}{\hbar}\hat{H}\frac{\tau}{2}}\right|x\right\rangle. \tag{0.21}$$

Далее переобозначим τ, как $\tau \to 2\tau$, чтобы избавиться от лишнего фактора $1/2$. Такой скейлинг приведет к коэффициенту 2 перед всем интегралом, но у нас уже и так есть численный коэффициент, равный объему группы репараметризаций, так что все вычисления которые мы делаем, определены с точностью до константы. В итоге находим

$$G_F(x,y) = \int\limits_0^\infty d\tau \left\langle y \left| e^{-\frac{i}{\hbar}\hat{H}\tau} \right| x \right\rangle \tag{0.22}$$

Интеграл по τ, как мы уже знаем, называется интегралом Швингера по собственному времени. Чтобы сделать этот интеграл формально хорошо определенным - сходящимся, мы добаляем маленькую мнимую часть к Гамильтониану $\hat{H} \to \hat{H} - i\epsilon$.

(d). Покажем, что это требование сходимости, по сути эквивалентно $i\epsilon$ прескрипции для пропагатора Фейнмана:

$$G_F(x,y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2 + m^2 - i\epsilon}. \tag{0.23}$$

Вставим два раза полный набор состояний $1 = \int \frac{d^4p}{(2\pi)^4} |p\rangle\langle p|$ в формулу для $G_F(x,y)$, получим

$$\begin{aligned} G_F(x,y) &= \int\limits_0^\infty d\tau \left\langle y \left| e^{-\frac{i}{\hbar}(\hat{H}-i\epsilon)\tau} \right| x \right\rangle = \int\limits_0^\infty d\tau \int \frac{d^4p d^4q}{(2\pi)^8} \langle y|q\rangle \left\langle q \left| e^{-\frac{i}{\hbar}(\hat{H}-i\epsilon)\tau} \right| p \right\rangle \langle p|x\rangle = \\ &= \int\limits_0^\infty d\tau \int \frac{d^4p d^4q}{(2\pi)^8} e^{-iqy} (2\pi)^4 \delta(p-q) e^{-\frac{i}{\hbar}\left(p^2+m^2-i\epsilon\right)\tau} e^{ipx} = \\ &= \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \int\limits_0^\infty d\tau e^{-\frac{i}{\hbar}\left(p^2+m^2-i\epsilon\right)\tau} = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{\hbar}{i\left(p^2+m^2-i\epsilon\right)} \end{aligned} \tag{0.24}$$

Но, так как мы вычисляем все с точностью до константы, то мы действительно можем написать, что

$$G_F(x,y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2 + m^2 - i\epsilon}. \tag{0.25}$$

Белавин-Пархоменко.7.2 Сходимость теории возмущений в 0 -мерной КТП

Большое понимание КТП может быть получено из изучения интеграла

$$Z(j) = \int\limits_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 + jx} \tag{0.7}$$

где $\lambda \geqslant 0$ и $j \in \mathbb{R}$. Этот интеграл может рассматриваться как производящий функционал для Евклидовой КТП в $d = 0$ пространстве времени. Заметим, что данный интеграл быстро сходится.

(a). Положите $\lambda = 0$ и вычислите $Z(j)$ в этом случае. Используйте это, чтобы вычислить свободные "функции Грина":

$$\langle x^n \rangle = \frac{\int\limits_{-\infty}^{\infty} dx x^n e^{-\frac{1}{2}x^2}}{\int dx e^{-\frac{1}{2}x^2}} \tag{0.8}$$

где $n \geqslant 1$. Дайте интерпретацию результата в терминах диаграмм Фейнмана и покажите, что для $n = 2, 4, 6, 8$ функции Грина корректно даются их симметрийным фактором.

(b). Теперь рассмотрим $Z(0)$ для $\lambda > 0$. Предполагая, что λ достаточно мало, разложите $Z(0)$ по теории возмущений $Z(0) = \sum\limits_{n=0}^{\infty} \lambda^n Z_n$ и вычислите Z_n. Покажите, что Z_n растет факториально с $n \to \infty$ и по этому теория возмущений (ряд Тейлора) имеет нулевой радиус сходимости. Объясните этот результат в терминах интегрального представления для $Z(0)$.

Казалось бы результат пункта (b) является катастрофой для нашей уверенности в теории возмущений для КТП. На самом деле, такая ситуация закономерна и в Квантовой механике и в КТП, по причинам подобным

тем, что мы наблюдали на нашем примере. Эта ситуация не безнадежна по двум причинам. Первая причина состоит в том, что теория возмущений все равно имеет смысл как асимптотическое разложение: покуда мы ограничиваем наше разложение до порядка $n \lesssim \lambda^{-1}$ в теории возмущений, пертурбативный ответ дает нам хорошее приближение к полному ответу. Вторая причина в том, что существуют разные методы (такие как преобразование Бореля), которые позволяют получить полный ответ из знания коэффициентов Z_n теории возмущений, даже если такой ряд формально расходится.

Белавин-Пархоменко .7.2 Сходимость теории возмущений в 0 -мерной Квантовой теории поля (КТП)

Большое понимание КТП может быть получено из изучения интеграла

$$Z(j) = \int\limits_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4 + jx} \tag{0.26}$$

где $\lambda \geqslant 0$ и $j \in \mathbb{R}$. Этот интеграл может рассматриваться как производящий функционал для Евклидовой КТП в $d = 0$ пространстве времени. Заметим, что данный интеграл быстро сходится.

(a). Положим $\lambda = 0$ и вычислим $Z(j)$ в этом случае. Имеем

$$Z(j) = \int\limits_{-\infty}^{\infty} dx e^{-\frac{1}{2}x^2 + jx} = \int dx e^{-\frac{1}{2}(x-j)^2 + \frac{j^2}{2}} = \sqrt{2\pi} e^{\frac{1}{2}j^2} \tag{0.27}$$

Используем это, чтобы вычислить свободные "функции Грина":

$$\langle x^n \rangle = \frac{\int\limits_{-\infty}^{\infty} dx x^n e^{-\frac{1}{2}x^2}}{\int dx e^{-\frac{1}{2}x^2}} \tag{0.28}$$

где $n \geqslant 1$. Очевидно, что если $n = 2k+1$ то $\left\langle x^{2k+1} \right\rangle = 0$. Для $n = 2k$ получим

$$\left\langle x^{2k} \right\rangle = \left(\frac{d}{dj} \right)^{2k} e^{\frac{j^2}{2}} \Bigg|_{j=0} = (2k-1)!! \tag{0.29}$$

В итоге получаем

$$\langle x^n \rangle = \begin{cases} 0, & n \text{ нечетное} \\ (n-1)!!, & n \text{ четное.} \end{cases} \tag{0.30}$$

В терминах диаграмм Фейнмана, $\left\langle x^{2k} \right\rangle$ есть число способов соединить $2k$ точек попарно:

(b). Теперь рассмотрим $Z(0)$ для $\lambda > 0$. Предполагая, что λ достаточно мало, разложим $Z(0)$ по теории возмущений $Z(0) = \sum\limits_{n=0}^{\infty} \lambda^n Z_n$ и вычислим Z_n. Имеем

$$\begin{aligned} Z(0) &= \int dx e^{-\frac{1}{2}x^2 - \frac{\lambda}{4!}x^4} = \int dx \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{4!} x^4 \right)^n e^{-\frac{1}{2}x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{4!} \right)^n \int dx x^{4n} e^{-\frac{1}{2}x^2} = \\ &= \sqrt{2\pi} \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{4!} \right)^n \left\langle x^{4n} \right\rangle . \end{aligned} \tag{0.31}$$

В итоге получаем для Z_n:

$$Z_n = \sqrt{2\pi} \frac{(-1)^n}{n!(4!)^n} \left\langle x^{4n} \right\rangle = \sqrt{2\pi}(-1)^n \frac{(4n-1)!!}{n!(4!)^n} = \sqrt{2\pi}(-1)^n \frac{(4n)!}{2^{2n}(2n)!n!(4!)^n} = \frac{\sqrt{2\pi}(-1)^n}{2^{5n}3^n} \frac{(4n)!}{n!(2n)!} \tag{0.32}$$

Покажем, что Z_n растет факториально с $n \to \infty$, для этого используем формулу Стирлинга:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{0.33}$$

откуда получаем

$$Z_n = \frac{\sqrt{2\pi}(-1)^n}{2^{5n}3^n}\frac{(4n)!}{n!(2n)!} \sim \frac{1}{2^{5n}3^n}\frac{(4n)^{4n}}{n^n(2n)^{2n}}\frac{e^n e^{2n}}{e^{4n}} = \frac{2^n}{3^n}\frac{n^n}{e^n} \sim \left(\frac{2}{3}n\right)! \tag{0.34}$$

и по этому теория возмущений (ряд Тейлора) имеет нулевой радиус сходимости. Если бы данный ряд имел ненулевой радиус сходимости $R > 0$, то мы бы получили

$$Z(0) = \sum_{n=0}^{\infty} \lambda^n Z_n \leqslant \sum_{n=0}^{\infty} R^n Z_n \tag{0.35}$$

для $|\lambda| < R$, но тогда $Z(0)$ было бы конечным при $\lambda < 0$, но очевидно, что при $\lambda < 0$ наш интеграл $Z(0)$ расходится.

Казалось бы результат пункта (b) является катастрофой для нашей уверенности в теории возмущений для КТП. На самом деле, такая ситуация закономерна и в Квантовой механике и в КТП, по причинам подобным тем, что мы наблюдали на нашем примере. Эта ситуация не безнадежна по двум причинам. Первая причина состоит в том, что теория возмущений все равно имеет смысл как асимптотическое разложение: покуда мы ограничиваем наше разложение до порядка $n \lesssim \lambda^{-1}$ в теории возмущений, пертурбативный ответ дает нам хорошее приближение к полному ответу. Вторая причина в том, что существуют разные методы (такие как преобразование Бореля), которые позволяют получить полный ответ из знания коэффициентов Z_n теории возмущений, даже если такой ряд формально расходится.

Белавин-Пархоменко.7.3 Уравнение Швингера-Дайсона и петлевое разложение

В интеграле по путям мы тоже можем интегрировать по частям. В этой задаче вы покажите, что это позволяет получить так называемые уравнения Швингера-Дайсона, которые утверждают, что классические уравнения движения выполняются внутри корреляционных функций, с точностью до контактных членов.

(a).

Для разминки, рассмотрите интеграл $I[J] = \int\limits_{-\infty}^{\infty} dx e^{-s(x)-Jx}$ с $s(x) = \frac{1}{2}x^2 + \frac{\lambda}{4!}x^4$. Покажите, что $I[J]$ удовлетворяет дифференциальному уравнению $\left(J + s'\left[-\frac{\partial}{\partial J}\right]\right) I[J] = 0$. Это уравнение Швингера-Дайсона для интеграла $I[J]$ - игрушечной модели производящего функционала в КТП.

(b).

Теперь, рассмотрите статистическую сумму $Z[J] = \int [d\phi] e^{\frac{i}{\hbar}S[\phi(x)] + \frac{i}{\hbar}\int d^d x J(x)\phi(x)}$, для взаимодействующей теории поля

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3, \tag{0.9}$$

с источником $J(x)$. Выведите уравнение Швингера-Дайсона для $Z[J]$.

Теперь возьмем оператор, среднее которого мы хотим найти, например произведение полей. Обозначим такой оператор $f[\phi]$. Определим среднее такого оператора $f[\phi]$ как

$$\langle f[\phi] \rangle = \frac{\int [d\phi] f[\phi] e^{\frac{i}{\hbar}S[\phi]}}{\int [d\phi] e^{\frac{i}{\hbar}S[\phi]}} \tag{0.10}$$

(c).

Выведите следующее уравнение:

$$\left\langle \frac{\delta f[\phi]}{\delta \phi(x)} + \frac{i}{\hbar} f[\phi] \frac{\delta S[\phi]}{\delta \phi(x)} \right\rangle = 0 \tag{0.11}$$

(d).

Рассмотрите разложение $Z[J]$ в терминах статистической суммы свободной теории $Z_0[J]$. Покажите, что такое разложение может быть записанно как ряд по $\hbar$. Также рассмотрите Z[0] как разложение по диаграммам Фейнмана. Покажите, что каждая связная диаграмма Фейнмана пропорциональна $\hbar^{L-1}$, где L - количество петель в диаграмме (количество незафиксированных импульсов у пропагаторов). Чтобы показать это, изучите как зависят пропагаторы и вершины от $\hbar$. А затем покажите, что число петель L в связной диаграмме Фейнмана связанно с количеством пропагаторов P и вершин V по формуле $L = P - V + 1$. (Подсказка: используйте то, что в каждой вершине должен выполняться закон сохранения импульса)

Белавин-Пархоменко.7.3 Уравнение Швингера-Дайсона и петлевое разложение

В интеграле по путям мы тоже можем интегрировать по частям. В этой задаче мы покажем, что это позволяет получить так называемые уравнения Швингера-Дайсона, которые утверждают, что классические уравнения движения выполняются внутри корреляционных функций, с точностью до контактных членов.

(a). Для разминки, рассмотрим интеграл $I[J] = \int\limits_{-\infty}^{\infty} dx e^{-s(x)-Jx}$ с $s(x) = \frac{1}{2}x^2 + \frac{\lambda}{4!}x^4$. Покажем, что $I[J]$ удовлетворяет дифференциальному уравнению $\left(J + s'\left[-\frac{\partial}{\partial J}\right]\right) I[J] = 0$. Это уравнение Швингера-Дайсона для интеграла $I[J]$ - игрушечной модели производящего функционала в КТП. Имеем

$$\int\limits_{-\infty}^{\infty} dx \frac{d}{dx}\left(e^{-s(x)-Jx}\right) = e^{-s(x)-Jx}\Big|_{-\infty}^{\infty} = 0 \tag{0.36}$$

с другой стороны

$$0 = \int\limits_{-\infty}^{\infty} dx \frac{d}{dx}\left(e^{-s(x)-Jx}\right) = \int\limits_{-\infty}^{\infty} dx \left(-J - s'(x)\right) e^{-s(x)-Jx} = -\left(J + s'\left[-\frac{\partial}{\partial J}\right]\right) I[J] \tag{0.37}$$

Так как $s'(x) = x + \frac{\lambda}{6}x^3$, получим

$$\left(\frac{\lambda}{6}\frac{\partial^3}{\partial J^3} + \frac{\partial}{\partial J} - J\right) I[J] = 0 \tag{0.38}$$

(b). Теперь, рассмотрим статистическую сумму $Z[J] = \int [d\phi] e^{\frac{i}{\hbar}S[\phi(x)] + \frac{i}{\hbar}\int d^d x J(x)\phi(x)}$, для взаимодействующей теории поля

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3, \tag{0.39}$$

с источником $J(x)$. Выведим уравнение Швингера-Дайсона для $Z[J]$. Имеем

$$\begin{aligned} 0 = \int [d\phi] \frac{\delta}{\delta\phi(y)}\left(e^{\frac{i}{\hbar}S[\phi(x)] + \frac{i}{\hbar}\int d^d x J(x)\phi(x)}\right) = \int [d\phi]\left(\frac{i}{\hbar}\frac{\delta S[\phi]}{\delta\phi(y)} + \frac{i}{\hbar}J(y)\right) e^{\frac{i}{\hbar}S[\phi(x)] + \frac{i}{\hbar}\int d^d x J(x)\phi(x)} = \\ = \frac{i}{\hbar}\left(J(y) + \frac{\delta S\left[\frac{\hbar}{i}\frac{\delta}{\delta J(x)}\right]}{\delta\phi(y)}\right) Z[J] = 0 \end{aligned} \tag{0.40}$$

Далее находим $\frac{\delta S}{\delta\phi(y)} = -\left(\partial^2 + m^2\right)\phi(y) - \frac{g}{2}\phi^2(y)$. И мы получим

$$\left(\frac{g}{2}\left(\frac{\hbar}{i}\frac{\delta}{\delta J(y)}\right)^2 + \left(\partial^2 + m^2\right)\left(\frac{\hbar}{i}\frac{\delta}{\delta J(y)}\right) - J(y)\right) Z[J] = 0 \tag{0.41}$$

Теперь возьмем оператор, среднее которого мы хотим найти, например произведение полей. Обозначим такой оператор $f[\phi]$. Определим среднее такого оператора $f[\phi]$ как

$$\langle f[\phi]\rangle = \frac{\int [d\phi] f[\phi] e^{\frac{i}{\hbar}S[\phi]}}{\int [d\phi] e^{\frac{i}{\hbar}S[\phi]}} \tag{0.42}$$

(c). Выведим следующее уравнение:

$$\left\langle \frac{\delta f[\phi]}{\delta\phi(x)} + \frac{i}{\hbar} f[\phi] \frac{\delta S[\phi]}{\delta\phi(x)} \right\rangle = 0 \tag{0.43}$$

Имеем

$$0 = \frac{\int [d\phi] \frac{\delta}{\delta\phi(x)} \left(f[\phi] e^{\frac{i}{\hbar} S[\phi]} \right)}{\int [d\phi] e^{\frac{i}{\hbar} S[\phi]}} = \left\langle \frac{\delta f[\phi]}{\delta\phi(x)} + \frac{i}{\hbar} f[\phi] \frac{\delta S[\phi]}{\delta\phi(x)} \right\rangle \tag{0.44}$$

(d). Рассмотрим разложение $Z[J]$ в терминах статистической суммы свободной теории $Z_0[J]$. Покажем, что такое разложение может быть записанно как ряд по $\hbar$. Имеем

$$Z[J] = \int [d\phi] e^{\frac{i}{\hbar} \int d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{g}{3!} \phi^3 \right) + \frac{i}{\hbar} \int d^d x J(x) \phi(x)} \tag{0.45}$$

Сделаем замену $\phi \to \sqrt{\hbar}\phi$ и $J \to \sqrt{h} J$, тогда получим

$$\begin{aligned} Z[J] &= \int [d\phi] e^{i \int d^d x \left(\frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{g\sqrt{\hbar}}{3!} \phi^3 \right) + i \int d^d x J(x) \phi(x)} = \\ &= \int [d\phi] \sum_{n=0}^{\infty} \left(\int d^d x \frac{-ig\sqrt{\hbar}\phi^3}{3!} \right)^n e^{iS_0[\phi] + i \int d^d x J(x) \phi(x)} = \sum_{n=0}^{\infty} \hbar^{n/2} \left(\int d^d x \frac{g}{3!} \frac{\delta^3}{\delta J(x)^3} \right)^n Z_0[J] \end{aligned} \tag{0.46}$$

Также рассмотрим $Z[0]$ как разложение по диаграммам Фейнмана. Покажем, что каждая связная диаграмма Фейнмана пропорциональна $\hbar^{L-1}$, где L - количество петель в диаграмме (количество незафиксированных импульсов у пропагаторов). Имеем

$$\text{Пропагатор } = \langle \phi(x)\phi(y) \rangle = \hbar \int \frac{d^d p}{(2\pi)^d} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}, \quad \text{Вершина } = \frac{1}{\hbar} \int d^d x \left(-\frac{ig}{3!} \phi^3 \right). \tag{0.47}$$

В фурье пространстве, каждая вершина дает фактор $\sim \frac{1}{\hbar} \delta^{(d)} \left(\sum k_i \right)$, где дельта функция накладывает условия сохранения энергии и импульса в вершине. Каждый пропагатор дает фактор $\sim \hbar \frac{i}{k^2 - m^2}$. Таким образом выражение для диаграммы общего вида с P пропагаторами и V вершинами будет иметь вид

$$\int \frac{d^d k_1 ... d^d k_P}{(2\pi)^{dP}} \frac{\hbar^P i^P}{(k_1^2 - m^2) ... (k_P^2 - m^2)} \frac{1}{\hbar^V} \delta^{(d)} \left(\sum k_{i_1} \right) ... \delta^{(d)} \left(\sum k_{i_V} \right) \sim \hbar^{P-V} \tag{0.48}$$

С другой стороны при интегрировании, дельта функции вершин сократят число интегралов до $L = P - V + 1$. Дополнительная единица в этой формуле происходит из-за того, что остается одна дельта функция, обеспечивающая сохранение входящих и выходящих из диаграммы энергии и импульса. В итоге мы видим, что разложение по петлям идет как $\hbar^{L-1}$, где L - число петель.

Белавин-Пархоменко.8. Функциональный интеграл для фермионных полей

1. Интегралы по Грассмановым переменным

(a). Show that

$$\left(\prod_i \int d\theta_i^* d\theta_i \right) e^{-\theta_i^* B_{ij} \theta_j} = \det B \tag{0.1}$$

где B - Эрмитова матрица, θ_i - комплексные Грассмановые переменные.

b) Compute the Grassman integral

$$\int d\psi_n ... d\psi_1 e^{\frac{1}{2} \sum\limits_{i,j=1}^{n} \psi_i M_{ij} \psi_j} \tag{0.3}$$

где M антисимметричная комплексная матрица коммутирующих чисел (обычная антисимметричная комплексная матрица), а ψ_i - Грассмановы числа.

Solution to a)

Имеем

$$\left(\prod_i \int d\theta_i^* d\theta_i\right) e^{-\theta_i^* B_{ij} \theta_j} = \left(\prod_i \int d\theta_i'^* d\theta_i'\right) e^{-\theta_i'^* b_i \theta_j'} = \prod_i b_i = \det B \tag{0.2}$$

Solution to b)

Данная матрица M может быть приведена к виду

$$M = \begin{pmatrix} 0 & m_1 & & & \\ -m_1 & 0 & & & \\ & & 0 & m_2 & \\ & & -m_2 & 0 & \\ & & & & \ddots \end{pmatrix} \tag{0.4}$$

причем если матрица имеет нечетный размер, то на диагонали вместо последнего блока 2×2 стоит ноль. Таким образом в случае матрицы нечетного размера данный интеграл равен нулю. Для матрицы четного размера $n = 2k$ находим

$$\int d\psi_n ... d\psi_1 e^{\frac{1}{2} \sum\limits_{i,j=1}^{n} \psi_i M_{ij} \psi_j} = \int d\psi'_{2k} d\psi'_{2k-1} ... d\psi'_2 d\psi'_1 e^{m_1 \psi'_1 \psi'_2 + ... m_k \psi'_{2k-1} \psi'_{2k}} = \prod_{i=1}^{k} m_i = \sqrt{\det M} \tag{0.5}$$

2. Теорема Вика для фермионных полей

Выведим формулу для свободного Дираковского поля:

$$\left\langle \psi(x_1) ... \psi(x_n) \bar{\psi}(y_1) ... \bar{\psi}(y_n) \right\rangle = \sum_{i=1}^{n} (-1)^{n+i} S(x_1 - y_i) \left\langle \psi(x_2) ... \psi(x_n) \bar{\psi}(y_1) ... \widehat{\bar{\psi}(y_i)} ... \bar{\psi}(y_n) \right\rangle \tag{0.6}$$

где $\langle \psi(x) \bar{\psi}(y) \rangle = S(x-y)$, а шляпка $\widehat{...}$ над полем означает, что этого поля нет в корреляционной функции.

(Подсказка: Рассмотрите функциональный интеграл: $\langle ... \rangle = \frac{1}{Z} \int D\psi' D\bar{\psi}' ... e^{i \int d^4x \bar{\psi}' (i\not{\partial} - m) \psi'}$. Сделайте сдвиг полей: $\psi'(x) = \psi(x), \bar{\psi}'(x) = \bar{\psi}(x) + \bar{\epsilon}(x)$, где $\bar{\epsilon}(x)$ - инфинитезимальная Грассманова функция. Такой сдвиг не изменит значение функционального интеграла, поэтому можно получить дифференциальное уравнение на корреляционную функцию, решение которого и есть формула (0.3).)

Solution

Рассмотрим функциональный интеграл:

$$\frac{1}{Z} \int D\psi' D\bar{\psi}' \psi'(x_2) ... \psi'(x_n) \bar{\psi}'(y_1) ... \bar{\psi}'(y_n) e^{i \int d^4x \bar{\psi}' (i\not{\partial} - m) \psi'} \tag{0.7}$$

Теперь сделаем замену переменных

$$\psi'(x) = \psi(x), \quad \bar{\psi}'(x) = \bar{\psi}(x) + \bar{\epsilon}(x) \tag{0.8}$$

где $\bar{\epsilon}(x)$ - произвольная инфинитезимальная Грассманова функция, быстро затухающая при $x \to \infty$. Имеем

$$\begin{aligned} &\frac{1}{Z} \int D\psi' D\bar{\psi}' \ \psi'(x_2) ... \psi'(x_n) \bar{\psi}'(y_1) ... \bar{\psi}'(y_n) e^{i \int d^4x \bar{\psi}' (i\not{\partial} - m) \psi'} = \\ &= \frac{1}{Z} \int D\psi D\bar{\psi} \ \psi(x_2) ... \psi(x_n) \left(\bar{\psi}(y_1) ... \bar{\psi}(y_n) + \sum_{i=1}^{n} (-1)^{i-1} \bar{\epsilon}(y_i) \bar{\psi}(y_1) ... \widehat{\bar{\psi}(y_i)} ... \bar{\psi}(y_n) \right) e^{i \int d^4x (\bar{\psi} + \bar{\epsilon}) (i\not{\partial} - m) \psi}, \end{aligned} \tag{0.9}$$

где мы использовали то, что $\bar{\epsilon}$ антикоммутирует со всеми полями. Теперь раскладывая данное выражение до линейных членов по $\bar{\epsilon}$ и учитывая очевидное равенство

$$\frac{1}{Z}\int D\psi' D\bar{\psi}'\psi'\left(x_2\right)...\psi'\left(x_n\right)\bar{\psi}'\left(y_1\right)...\bar{\psi}'\left(y_n\right)e^{i\int d^4x\bar{\psi}'(i\not{\partial}-m)\psi'}= \\ =\frac{1}{Z}\int D\psi D\bar{\psi}\psi\left(x_2\right)...\psi\left(x_n\right)\bar{\psi}\left(y_1\right)...\bar{\psi}\left(y_n\right)e^{i\int d^4x\bar{\psi}(i\not{\partial}-m)\psi} \tag{0.10}$$

получим

$$0=\sum_{i=1}^{n}(-1)^{n+i}\bar{\epsilon}\left(y_i\right)\left\langle\psi\left(x_2\right)...\psi\left(x_n\right)\bar{\psi}\left(y_1\right)...\widehat{\bar{\psi}\left(y_i\right)}...\bar{\psi}\left(y_n\right)\right\rangle+ \\ +i\int d^4x\left(i\not{\partial}_x-m\right)\left\langle\psi\left(x_2\right)...\psi\left(x_n\right)\bar{\psi}\left(y_1\right)...\bar{\psi}_n\left(y_n\right)\bar{\epsilon}(x)\psi(x)\right\rangle, \tag{0.11}$$

откуда, из произвольности $\bar{\epsilon}(x)$, следует, что

$$0=\sum_{i=1}^{n}(-1)^{n+i}\delta^{(4)}\left(x-y_i\right)\left\langle\psi\left(x_2\right)...\psi\left(x_n\right)\bar{\psi}\left(y_1\right)...\widehat{\bar{\psi}\left(y_i\right)}...\bar{\psi}\left(y_n\right)\right\rangle+ \\ +i\left(i\not{\partial}_x-m\right)\left\langle\psi(x)\psi\left(x_2\right)...\psi\left(x_n\right)\bar{\psi}\left(y_1\right)...\bar{\psi}_n\left(y_n\right)\right\rangle \tag{0.12}$$

и мы в итоге получаем уравнение

$$\left(i\not{\partial}_x-m\right)\left\langle\psi(x)\psi\left(x_2\right)...\psi\left(x_n\right)\ \bar{\psi}\left(y_1\right)...\bar{\psi}_n\left(y_n\right)\right\rangle= \\ =\sum_{i=1}^{n}(-1)^{n+i}i\delta^{(4)}\left(x-y_i\right)\left\langle\psi\left(x_2\right)...\psi\left(x_n\right)\bar{\psi}\left(y_1\right)...\widehat{\bar{\psi}\left(y_i\right)}...\bar{\psi}\left(y_n\right)\right\rangle. \tag{0.13}$$

Используя то, что $\left(i\not{\partial}_x-m\right)S(x-y)=\left(i\not{\partial}_x-m\right)\langle\psi(x)\bar{\psi}(y)\rangle=i\delta^{(4)}(x-y)$, мы видим, что решение данного уравнения есть

$$\left\langle\psi(x)...\psi\left(x_n\right)\bar{\psi}\left(y_1\right)...\bar{\psi}\left(y_n\right)\right\rangle=\sum_{i=1}^{n}(-1)^{n+i}S\left(x-y_i\right)\left\langle\psi\left(x_2\right)...\psi\left(x_n\right)\bar{\psi}\left(y_1\right)...\widehat{\bar{\psi}\left(y_i\right)}...\bar{\psi}\left(y_n\right)\right\rangle$$

что и требовалось показать.

3. Задача из Пескина Шредера 9.2 пункт (d)

Пусть $\psi(t)$ и $\bar{\psi}(t)$ два Грассмановых (фермионных) поля. Определим фермионный осциллятор следующим Лагранжианом:

$$L_E=\bar{\psi}\dot{\psi}+\omega\bar{\psi}\psi \tag{0.15}$$

Данный Лагранжиан соответствует Гамильтониану

$$H=\omega\bar{\psi}\psi,\qquad\text{где}\qquad\{\bar{\psi},\psi\}=1 \tag{0.16}$$

то есть это простая двух-уровневая система. Вычислим функциональный интеграл, предполагая что фермионные (Грассмановы) поля удовлетворяют следующим граничным условиям $\psi(t+\beta)=-\psi(t)$

$$Z=\int\limits_{\psi(0)=-\psi(\beta)}D\psi D\bar{\psi}\exp\left(-\int\limits_0^{\beta}dtL_E(\psi(t),\bar{\psi}(t))\right) \tag{0.17}$$

(Подсказка: Используйте разложение в ряд Фурье). Покажите, что ваш результат воспроизводит статистическую сумму квантово-механической двух-уровневой системы - то есть частица может сидеть либо на одном уровне, либо на другом, а расстояние между уровнями ω и температура $T=1/\beta$.

Раскладывая $\psi(t)$ в ряд Фурье

$$\psi(t) = \sum_{n=-\infty}^{\infty} \theta_n \frac{e^{\frac{\pi i t}{\beta}(2n+1)}}{\sqrt{\beta}} \tag{0.18}$$

где θ_n - Грассмановы числа, находим для статистической суммы:

$$Z = \prod_n \int d\bar{\theta}_n d\theta_n e^{-\sum\limits_n \bar{\theta}_n \theta_n \left(\frac{\pi i(2n+1)}{\beta}+\omega\right)} = \prod_{n=-\infty}^{\infty} \left(\frac{\pi i(2n+1)}{\beta}+\omega\right) = \prod_{n=0}^{\infty} \left(\omega^2 + \frac{\pi^2(2n+1)^2}{\beta^2}\right), \tag{0.19}$$

затем мы, беря логарифм, получаем:

$$\ln Z = \sum_{n=0}^{\infty} \ln\left(\omega^2 + \frac{\pi^2(2n+1)^2}{\beta^2}\right) \tag{0.20}$$

Дифференцируя данное выражение по ω, находим

$$\frac{\partial \ln Z}{\partial \omega} = \sum_{n=0}^{\infty} \frac{2\omega}{\omega^2 + \frac{\pi^2(2n+1)^2}{\beta^2}} = \frac{1}{2}\beta \tanh\left(\frac{\beta\omega}{2}\right). \tag{0.21}$$

Теперь находим

$$\ln Z[\omega] = \ln Z[0] + \ln\cosh\frac{\beta\omega}{2} \tag{0.22}$$

и в конечном итоге

$$Z[\omega] = Z[0]\cosh\frac{\beta\omega}{2} \tag{0.23}$$

В свою очередь для двух-уровневой системы с щелью ω мы получаем

$$Z[\omega] = e^{-\beta\omega/2} + e^{\beta\omega/2} = 2\cosh\frac{\beta\omega}{2}. \tag{0.24}$$

Теперь разберемся почему в данном случае мы использовали антипериодические граничные условия. Наша цель записать подходящее выражение для следа некоторого оператора:

$$\mathrm{tr}[\hat{O}] \stackrel{\mathrm{def}}{=} O_{\downarrow\downarrow} + O_{\uparrow\uparrow} \tag{0.25}$$

Во первых, вспомним определение и свойства Грассмановых состояний и операторов. Для Грассманового понижающего оператора $\hat{\psi}$ мы имеем

$$\hat{\psi}|\downarrow\rangle = 0, \quad \hat{\psi}|\uparrow\rangle = |\downarrow\rangle \tag{0.26}$$

где состояние системы $|\uparrow\rangle$ и $|\downarrow\rangle$ удовлетворяют

$$\langle\downarrow|\downarrow\rangle = \langle\uparrow|\uparrow\rangle = 1, \quad \langle\uparrow|\downarrow\rangle = 0. \tag{0.27}$$

Общий вид собственного вектора для такого оператора есть:

$$|\psi\rangle = |\downarrow\rangle + \psi|\uparrow\rangle, \quad \langle\psi| = -\langle\uparrow| + \langle\downarrow|\psi \tag{0.28}$$

и можно проверить, что

$$\hat{\psi}|\psi\rangle = \psi|\psi\rangle, \quad \langle\psi|\hat{\psi} = -\langle\psi|\psi, \quad \langle\psi|\psi'\rangle = \psi - \psi' \tag{0.29}$$

Следует заметить, что у $\langle\psi| = -\langle\uparrow| + \langle\downarrow|\psi$, есть знак минус перед $\langle\uparrow|$! Такое определение для бра состояния нужно, чтобы сделать Грассмановы переменные подобными обычным бозонным, т.е. чтобы иметь

$$1 = \int |\psi\rangle d\psi\langle\psi|, \quad \langle\psi|\psi'\rangle = \delta(\psi - \psi') = \psi - \psi' \tag{0.30}$$

Тогда для любого оператора $\hat{O}$ мы получаем

$$\hat{O}|\uparrow\rangle = |\uparrow\rangle O_{\uparrow\uparrow} + |\downarrow\rangle O_{\downarrow\uparrow}, \quad \hat{O}|\downarrow\rangle = |\downarrow\rangle O_{\downarrow\downarrow} + |\uparrow\rangle O_{\uparrow\downarrow} \tag{0.31}$$

и мы моголи бы ожидать, что формула $\int d\psi \langle\psi|\hat{O}|\psi\rangle$ и есть выражение для следа оператора $\hat{O}$, но

$$\int d\psi \langle\psi|\hat{O}|\psi\rangle = \int d\psi \left(\psi \left(O_{\downarrow\downarrow} - O_{\uparrow\uparrow}\right) - O_{\uparrow\downarrow}\right) = O_{\downarrow\downarrow} - O_{\uparrow\uparrow} = \mathrm{tr}\left[(-1)^{\hat{F}}\hat{O}\right] \tag{0.32}$$

где $(-1)^F$ действует как

$$\langle\uparrow|(-1)^F = -\langle\uparrow|, \quad \langle\downarrow|(-1)^F = \langle\downarrow|. \tag{0.33}$$

То есть мы видим, что такое наивное определение следа не приводит нас к желаемому результату, поэтому, чтобы получить след поератора $\hat{O} : O_{\downarrow\downarrow} + O_{\uparrow\uparrow}$, нам нужно записать выражение как

$$\mathrm{tr}[\hat{O}] = O_{\downarrow\downarrow} + O_{\uparrow\uparrow} = \int d\psi \left\langle \psi \left| (-1)^{\hat{F}}\hat{O} \right| \psi \right\rangle \tag{0.34}$$

где мы использовали, что $(-1)^{\hat{F}} \cdot (-1)^{\hat{F}} = 1$. Теперь, используя, что $\langle\psi|(-1)^{\hat{F}} = -\langle -\psi|$ мы имеем

$$\mathrm{tr}[\hat{O}] = -\int d\psi \langle -\psi|\hat{O}|\psi\rangle \tag{0.35}$$

С другой стороны, используя тот же самый метод вывода континуального интеграла, что и для бозонных полей, т.е. вставляя полные наборы из Грассмановым переменных в матричный элемент через каждый небольшой промежуток времени ϵ, мы получаем

$$\left\langle \psi_f \left| e^{-\beta H} \right| \psi_i \right\rangle = - \int\limits_{\substack{\psi(0)=\psi_i \\ \psi(\beta)=\psi_f}} D\bar{\psi}D\psi \exp\left(- \int\limits_0^\beta dt L_E(\psi(t), \bar{\psi}(t)) \right) \tag{0.36}$$

и поэтому для статистической суммы находим

$$Z = \mathrm{tr}\left[e^{-\beta H}\right] = -\int d\psi \left\langle -\psi \left| e^{-\beta H} \right| \psi \right\rangle = \int\limits_{\psi(\beta)=-\psi(0)} D\bar{\psi}D\psi \exp\left(- \int\limits_0^\beta dt L_E(\psi(t), \bar{\psi}(t)) \right) \tag{0.37}$$

что и требовалось показать.

Также сделаем небольшое замечание о фермионных переменных. По существу мы имеем два разных разложения фермионного поля $\psi(t)$:

$$\begin{aligned} &1.\ \psi(t) = \left\{ \psi_n, \quad \psi_n^2 = 0, \quad n = 0, ..., N, \quad t_n = \beta\frac{n}{N}, \quad N \to \infty \right\}, \\ &2.\ \psi(t) = \left\{ \sum_{k=-\infty}^{\infty} \theta_k e^{\frac{i\pi k t}{\beta}}, \quad \theta_k^2 = 0 \right\}. \end{aligned} \tag{0.38}$$

Причем, как мы видим для вывода континуального интеграла из матричного элемента (мы не проделали данный вывод, а лишь указали путь к нему, оставив это как упражнение) мы использовали первое разложение, а для вычисления континуального интеграла для нашего конкретного Лагранжиана мы пользовались вторым разложением. На первый взгляд два данных разложени выглядят по-разному, но если мы сделаем время дискретным во втором разложении, то мы получим

$$\psi_n = \sum_{k=0}^{N} \theta_k e^{i\pi k \frac{n}{N}} \tag{0.39}$$

Из этого выражения мы можем видеть, что ψ_n и θ_k связанны просто унитарной матрицей, поэтому это всего лишь унитарное преобразование, которое обычно называют преобразованием Фурье. В нашем конкретном случае

$$\psi_n = \sum_{k=0}^{N} \theta_k e^{i\pi(2k+1)\frac{n}{N}} \tag{0.40}$$

и можно получить

$$\psi_0 = \sum_{k=0}^{N} \theta_k, \quad \psi_N = -\sum_{k=0}^{N} \theta_k \tag{0.41}$$

откуда можно видеть, что $\psi_0 = -\psi_N$.

Белавин-Пархоменко.9.1. Диаграммы Фейнмана для теории ϕ^4

Интегралы вычислять не надо, нужно только их уметь записать, используя теорему Вика для свободного поля, а также, что

$$\langle\phi(x)\phi(y)\rangle_0 := \frac{\int D\phi\phi(x)\phi(y)e^{iS_0}}{\int D\phi e^{iS_0}} = G(x-y) \tag{0.1}$$

где $S_0 = \frac{1}{2}\int d^dx\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right)$. Обратите внимание на индекс $\langle ...\rangle_0$, который означает, что усреднение идет по свободному полю! Каждый такой пропагатор $G(x-y)$ представляется графически как линия между точками x и y:

$$G(x-y) = \quad x \bullet\!\!\!

Что можно сказать про $\langle\phi(x)\rangle, \langle\phi(x)\phi(y)\phi(z)\rangle$ и $\langle\phi(x_1)...\phi(x_{2n+1})\rangle$?

(d). Нарисуйте диаграммы Фейнмана и напишите выражения для них до λ^2 включительно для корреляционной функции

$$\langle\phi(x)\phi(y)\phi(z)\phi(w)\rangle. \tag{0.7}$$

Solution

Рассмотрим теорию ϕ^4 с взаимодействием $S_I = -\int dd^dx\frac{\lambda}{4!}\phi^4(x)$. Если мы хотим найти среднее какого-то числа полей по теории возмущений

$$\langle\phi(x_1)\phi(x_2)...\phi(x_n)\rangle := \frac{\int D\phi\phi(x_1)\phi(x_2)...\phi(x_n)e^{iS_0+iS_I}}{\int D\phi e^{iS_0+iS_I}} \tag{0.2}$$

мы раскладываем возмущение в ряд Тейлора $e^{iS_I} = e^{-i\int d^4x\frac{\lambda}{4!}\phi^4(x)} = \sum\limits_{k=0}^{\infty}\frac{(iS_I)^k}{k!}$, и дальше усредняем все (берем функциональный интеграл) по свободному полю. То есть на деле мы просто применяем уже доказанную нами теорему Вика, то есть находим все возможные попарные свертки полей. Каждый множитель iS_I генерирует на языке диаграмм Фейнмана вершину - узел из которого в теории ϕ^4 торчат 4 отрезка:

$$\frac{\lambda}{4!}\phi^4(x) =>$$

Выражение вида $\frac{(iS_I)^k}{k!}$ пропорционально λ^k, и мы говорим о нем, как о k порядке в теории возмущений.

(a). Начнем с простого рассмотрения вакуумных диаграмм. Рассмотрим функциональный интеграл

$$\int D\phi e^{iS_0+iS_I} \tag{0.3}$$

и найдем выражения для диаграмм Фейнмана до второго порядка включительно $\left(\lambda^2\right)$. Имеем

$$\int D\phi e^{iS_0+iS_I} = \int D\phi\sum_{k=0}^{\infty}\frac{(iS_I)^k}{k!}e^{iS_0} \tag{0.4}$$

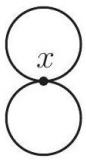
В нулевом порядке по λ получаем просто свободную статсумму

$$\int D\phi e^{iS_0} = Z_0 \tag{0.5}$$

В первом пордяке по λ мы имеем

$$\begin{aligned}\int D\phi(iS_I)e^{iS_0} = -i\int D\phi\int d^dx\frac{\lambda}{4!}\phi^4(x)e^{iS_0} = -\frac{i\lambda}{4!}Z_0\int d^dx\langle\phi(x)\phi(x)\phi(x)\phi(x)\rangle_0 = \\ = -\frac{i\lambda}{4!}Z_0\int d^dx3G^2(0) = -\frac{i\lambda}{8}Z_0G^2(0)\cdot V_d,\end{aligned} \tag{0.6}$$

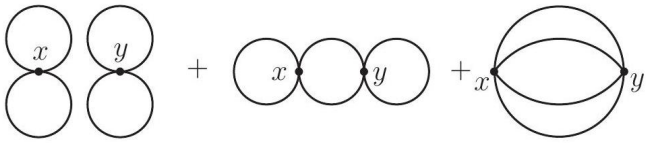
где $V_d = \int d^dx$. В данном случае мы имеем только одну диаграмму вида:



Мы не включаем в выражение для диагрммы множитель Z_0. Во втором порядке мы имеем

$$\int D\phi \frac{1}{2!}\left(iS_I\right)^2 e^{iS_0} = -\frac{1}{2!}\int D\phi \int d^dx \frac{\lambda}{4!}\phi^4(x) \int d^dy \frac{\lambda}{4!}\phi^4(y) e^{iS_0} = -\frac{\lambda^2}{2!(4!)^2} Z_0 \int d^dx d^dy \left\langle \phi^4(x)\phi^4(y)\right\rangle_0 =$$
$$= -\frac{\lambda^2}{2!(4!)^2} Z_0 \int d^dx d^dy \left(\left(3G^2(0)\right)^2 + 72G(0)G^2(x-y)G(0) + 4!G^4(x-y)\right) =$$
$$= \frac{1}{2!}\left(-\frac{i\lambda}{8}\right)^2 Z_0 G^4(0)\cdot V_d^2 - \frac{\lambda^2}{16} Z_0 G^2(0) \int d^dx d^dy G^2(x-y) - \frac{\lambda^2 Z_0}{2!(4!)} \int d^dx d^dy G^4(x-y) \tag{0.7}$$

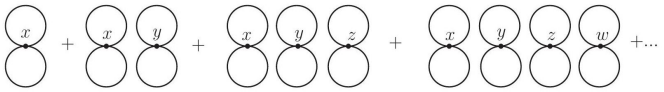
В данном случае мы имеем три диаграммы вида:



(b). Покажем, что вакуумные диаграммы "экспоненцируются", то есть можно написать

$$\int D\phi e^{iS_0+iS_I} = \exp(\text{ сумма по всем связным вакуумным диаграммам }), \tag{0.8}$$

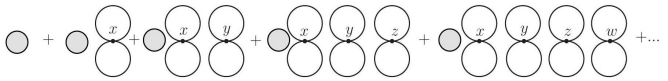
где связными называются диаграммы, которые представляют собой один связный граф. Из предыдущего пункта уже видно, что в разложении возмущения в ряд Тейлора мы можем выделить из бесконечной суммы диаграмм, например, ряд диаграмм вида:



Данный ряд равен

$$-\frac{i\lambda}{8}G^2(0)\cdot V_d + \frac{1}{2!}\left(-\frac{i\lambda}{8}\right)^2 \left(G^2(0)\right)^2 \cdot V_d^2 + + \frac{1}{3!}\left(-\frac{i\lambda}{8}\right)^3 \left(G^2(0)\right)^3 \cdot V_d^3 + ... =$$
$$= \exp\left(-\frac{i\lambda}{8}G^2(0)\cdot V_d\right) - 1 \tag{0.9}$$

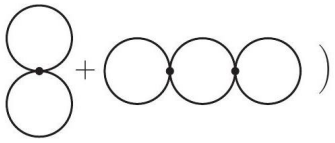
Далее можно заметить, что $\int D\phi e^{iS_I} e^{iS_0}$ можно представить как следующий ряд



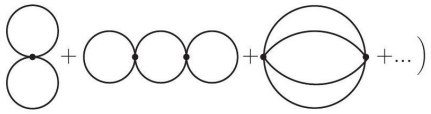
где заштрихованный кружочек означает сумму все остальных диграмм (отличных по структуре, то есть не равных "двум склееным кружочкам"). Как мы видим, выделенные диаграммы могут быть "экспоненциированы", то есть в итоге мы можем написать:

$$\int D\phi\, e^{iS_I} e^{iS_0} = \exp\,($$ $$)$$

Далее рассуждая алогично, мы можем “экспоненциировать" диаграммы следующего типа и записать

$$\int D\phi\, e^{iS_I} e^{iS_0} = \exp\Big($$  $$\Big)$$

где более маленький заштрихованный кружочек означает сумму все остальных диграмм, отличных уже от тех двух которые стоят в экспоненте. В итоге повторяя процедуру экспоненциирования диаграмм каждого типа, мы получим

$$\int D\phi\, e^{iS_I} e^{iS_0} = Z_0 \exp\Big($$  $$+\ldots\Big)$$

(c). Следствием пункта (b) является то, что мы можем написать

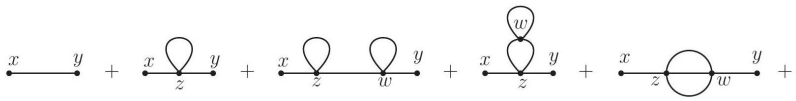
$$\langle \phi(x_1)\,\phi(x_2)\ldots\phi(x_n)\rangle = \frac{\int D\phi\phi(x_1)\,\phi(x_2)\ldots\phi(x_n)\,e^{iS_0+iS_I}}{\int D\phi e^{iS_0+iS_I}} =$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\left\langle \phi(x_1)\ldots\phi(x_n)\,(iS_I)^k\right\rangle_0\right)_{\text{все диаграммы без вакуумных пузырьков}}, \qquad (0.10)$$

то есть это означает, что мы должны раскладывать взаимодействие в ряд тейлора по λ, находить все возможные свертки полученных корреляторов по теореме Вика, но из полученного множества всех Фейнмановских диаграм, мы должны взять только те диаграммы, которые не содержат отдельных кусков (вакуумных пузырьков), которые не связанны с внешними точками $x_1,\ldots,x_n$. Далее, пользуясь этим следствием, нарисуем диаграммы Фейнмана и напишим выражения для них до λ^2 включительно для корреляционной функции $\langle\phi(x)\phi(y)\rangle$. Имеем

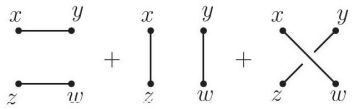
$$\begin{aligned}\langle\phi(x)\phi(y)\rangle =& \langle\phi(x)\phi(y)\rangle_0 + \frac{(-i\lambda)}{4!}\left\langle\phi(x)\phi(y)\int d^dz\phi^4(z)\right\rangle_{0,\text{ связ}} + \\ &+\frac{(-i\lambda)^2}{2!(4!)^2}\left\langle\phi(x)\phi(y)\int d^dz\phi^4(z)\int d^dw\phi^4(w)\right\rangle_{0,\text{ связ}} + O\left(\lambda^3\right) = \\ =& G(x-y) - \frac{i\lambda}{2}\int d^dz G(x-z)G(0)G(z-y) + \\ &+\left(\frac{-i\lambda}{2}\right)^2\int d^dz d^dw G(x-z)G(0)G(z-w)G(0)G(w-y) - \\ &-\frac{\lambda^2}{4}\int d^dz d^dw G(x-z)G^2(w-z)G(0)G(z-y) - \\ &-\frac{\lambda^2}{6}\int d^dz d^dw G(x-z)G^3(z-w)G(w-y) + O\left(\lambda^3\right). \end{aligned} \qquad (0.11)$$

На диаграммном языке мы имеем:

$\langle\phi(x)\phi(y)\rangle =$ 

Коррелятор $\langle\phi(x_1)\ldots\phi(x_{2n+1})\rangle = 0$ в силу симметрии действия: $S[\phi] = S[-\phi]$, также это можно заметить из того факта, что в каждом порядке теории возмущений будет возникать усреднение по свободному полю (по S_0) нечетного числа полей, а такие корреляторы в свободной теории поля равны нулю.

(d). Нарисуем диаграммы Фейнмана и напишим выражения для них до λ^2 включительно для корреляционной функции $\langle\phi(x)\phi(y)\phi(z)\phi(w)\rangle$. Имеем

$\lambda^0:\ \langle\phi(x)\phi(y)\phi(z)\phi(w)\rangle_0 =$ 

Все остальные диаграммы выглядят как диаграммы первого порядка по λ у которых пропагаторы перенормированны еще одной дополнительной петлей.

Белавин-Пархоменко.9.2. Диаграммы Фейнмана для теории ϕ^3

Рассмотрим теперь теорию скалярного поля с взаимодействие $S_I = -\int d^dx\lambda\frac{\phi^3(x)}{3!}$. Проделайте пункты (a)-(d) из Задачи 1.

Данная задача разобрана в параграфе 9 книги Средницкого.

Белавин-Пархоменко.**4. Континуальный интеграл в Квантовой Механик Решения**

1 Евклидова корреляционная функция

(a). Имеем для $D\left(x_E\right)$

$$D\left(x_E\right)=\int\limits_0^\infty d\tau\int\frac{d^4p_E}{(2\pi)^4}e^{-\tau\left(p_E^2+m^2\right)}e^{ip_Ex_E}=\int\limits_0^\infty d\tau\frac{1}{16\pi^2\tau^2}e^{-\frac{x_E^2}{4\tau}-\tau m^2\tau=\frac{|x_E|}{2m}t}$$
$$=\frac{m}{8\pi^2\left|x_E\right|}\int\limits_0^\infty\frac{dt}{t^2}e^{-\frac{1}{2}m|x_E|\left(t+\frac{1}{t}\right)}. \tag{0.1}$$

Далее, так как $m\left|x_E\right|\gg1$, мы можем воспользоваться методом перевала. Для этого находим для седловой точки:

$$\left(t+\frac{1}{t}\right)'=1-\frac{1}{t^2}=0,\quad \text{откуда}\quad t_0=1 \tag{0.2}$$

Далее подставляя $t=t_0+\delta t$, получаем

$$D\left(x_E\right)=\frac{m}{8\pi^2\left|x_E\right|}\int\limits_0^\infty\frac{dt}{t^2}e^{-\frac{1}{2}m|x_E|\left(t+\frac{1}{t}\right)}\approx\frac{m}{8\pi^2\left|x_E\right|}e^{-m|x_E|}\int\limits_{-\infty}^\infty d(\delta t)e^{-\frac{1}{2}m|x_E|(\delta t)^2}=$$
$$=\frac{m}{8\pi^2\left|x_E\right|}e^{-m|x_E|}\sqrt{\frac{2\pi}{m\left|x_E\right|}}\sim\frac{m^{1/2}}{\left|x_E\right|^{3/2}}e^{-m|x_E|}. \tag{0.3}$$

(b). Имеем

$$D\left(x_E\right)=\int\limits_0^\infty d\tau\frac{1}{16\pi^2\tau^2}e^{-\frac{x_E^2}{4\tau}-\tau m^2} \tag{0.4}$$

Используя формулу

$$K_\nu(2\sqrt{pq})=\frac{1}{2}\left(\frac{p}{q}\right)^{\nu/2}\int\limits_0^\infty x^{\nu-1}e^{-px-q/x}dx \tag{0.5}$$

находим $\left(p=m^2,q=x_E^2/4,\nu=-1\right)$

$$D\left(x_E\right)=\frac{m}{4\pi^2\left|x_E\right|}K_{-1}\left(m\left|x_E\right|\right)=\frac{m}{4\pi^2\left|x_E\right|}K_1\left(m\left|x_E\right|\right). \tag{0.6}$$

2 Оператор эволюции (в мнимом времени $\tau=it$)

(a). Мы можем записать решение уравнения

$$-\frac{\partial}{\partial\tau}\hat{U}\left(\tau,\tau_0\right)=\hat{H}(\tau)U\left(\tau,\tau_0\right),\quad\hat{U}\left(\tau_0,\tau_0\right)=\hat{I} \tag{0.7}$$

в форме (см. Пескин Шредер, глава 4)

$$\hat{U}\left(\tau,\tau_0\right)=\hat{I}-\int\limits_{\tau_0}^\tau d\tau_1\hat{H}\left(\tau_1\right)+\int\limits_{\tau_0}^\tau d\tau_1\int\limits_{\tau_0}^{\tau_1}d\tau_2\hat{H}\left(\tau_1\right)\hat{H}\left(\tau_2\right)-...=$$
$$=\hat{I}-\int\limits_{\tau_0}^\tau d\tau_1\hat{H}\left(\tau_1\right)+\frac{1}{2!}\int\limits_{\tau_0}^\tau d\tau_1d\tau_2T\left\{\hat{H}\left(\tau_1\right)\hat{H}\left(\tau_2\right)\right\}-...\equiv T\left\{e^{-\int\limits_{\tau_0}^\tau d\tau'\hat{H}\left(\tau'\right)}\right\}, \tag{0.8}$$

где символ T означает временное упорядочение, т.е. оператор с большим временем ставится слева от оператора с меньшим. Далее имеем

$$T\left\{\left(\int\limits_{\tau_0}^{\tau} d\tau' \hat{H}\left(\tau'\right)\right)^n\right\} = C_n^k T\left\{\left(\int\limits_{\tau_1}^{\tau} d\tau' \hat{H}\left(\tau'\right)\right)^k \left(\int\limits_{\tau_0}^{\tau_1} d\tau' \hat{H}\left(\tau'\right)\right)^{n-k}\right\} = \\ = C_n^k T\left\{\left(\int\limits_{\tau_1}^{\tau} d\tau' \hat{H}\left(\tau'\right)\right)^k\right\} T\left\{\left(\int\limits_{\tau_0}^{\tau_1} d\tau' \hat{H}\left(\tau'\right)\right)^{n-k}\right\}, \tag{0.9}$$

где $C_n^k = \frac{n!}{k!(n-k)!}$. В итоге получаем, что

$$\hat{U}\left(\tau,\tau_0\right) = T\left\{e^{-\int\limits_{\tau_0}^{\tau} d\tau' \hat{H}(\tau')}\right\} = T\left\{e^{-\int\limits_{\tau_1}^{\tau} d\tau' \hat{H}(\tau')}\right\} T\left\{e^{-\int\limits_{\tau_0}^{\tau_1} d\tau' \hat{H}(\tau')}\right\} = \hat{U}\left(\tau,\tau_1\right)\hat{U}\left(\tau_1,\tau_0\right) \tag{0.10}$$

(b). Имеем $\left\langle q\left|\hat{U}\left(\tau,\tau_0\right)\right|q'\right\rangle = e^{-\sigma\left(q,q'|\tau_0\right)}$, где

$$\sigma\left(q,q'|\tau_0\right) = \frac{1}{\Delta}\sigma_0\left(q,q'|\tau_0\right) + a\log\Delta + b + \Delta\sigma_1\left(q,q'|\tau_0\right) + O\left(\Delta^2\right) \tag{0.11}$$

тогда ($\Delta = \tau - \tau_0$):

$$-\frac{\partial}{\partial\tau}e^{-\sigma\left(q,q'|\tau_0\right)} = -\frac{\partial}{\partial\Delta}e^{-\sigma\left(q,q'|\tau_0\right)} = \left(-\frac{1}{\Delta^2}\sigma_0\left(q,q'|\tau_0\right) + \frac{a}{\Delta} + \sigma_1\left(q,q'|\tau_0\right) + O(\Delta)\right)e^{-\sigma\left(q,q'|\tau_0\right)}, \\ -\frac{1}{2}\partial_q^2 e^{-\sigma\left(q,q'|\tau_0\right)} = \left(-\frac{1}{2\Delta^2}\left(\partial_q\sigma_0\left(q,q'|\tau_0\right)\right)^2 + \frac{1}{2\Delta}\partial_q^2\sigma_0\left(q,q'|\tau_0\right) - \partial_q\sigma_0\left(q,q'|\tau_0\right)\partial_q\sigma_1\left(q,q'|\tau_0\right)\right)e^{-\sigma\left(q,q'|\tau_0\right)} \tag{0.12}$$

Получаем в порядке $1/\Delta^2$

$$\left(\partial_q\sigma_0\left(q,q'|\tau_0\right)\right)^2 = 2\sigma_0\left(q,q'|\tau_0\right) \tag{0.13}$$

откуда

$$\sigma_0\left(q,q'|\tau_0\right) = \frac{1}{2}\left(q-q_0\right)^2 \tag{0.14}$$

Далее в порядке $1/\Delta$ находим

$$a = \frac{1}{2} \tag{0.15}$$

Далее имеем

$$e^{-\sigma\left(q,q'|\tau_0\right)} = \frac{1}{e^b\sqrt{\Delta}}e^{-\frac{(q-q_0)^2}{2\Delta}} \xrightarrow{\Delta} \frac{\sqrt{2\pi}}{e^b}\delta\left(q-q_0\right) \tag{0.16}$$

и используя граничное условие $\left\langle q\left|\hat{U}\left(\tau_0,\tau_0\right)\right|q'\right\rangle = \delta\left(q-q'\right)$ мы находим

$$q_0 = q', \quad b = \frac{1}{2}\log 2\pi. \tag{0.17}$$

Далее в порядке Δ^0 получаем уравнение

$$\sigma_1\left(q,q'|\tau_0\right) = -\partial_q\sigma_0\left(q,q'|\tau_0\right)\partial_q\sigma_1\left(q,q'|\tau_0\right) + V(q,\tau) \tag{0.18}$$

откуда

$$\sigma_1\left(q,q'|\tau_0\right) + \left(q-q'\right)\partial_q\sigma_1\left(q,q'|\tau_0\right) = V(q,\tau) \tag{0.19}$$

Далее получаем

$$\partial_q\left(\left(q-q'\right)\sigma_1\left(q,q'|\tau_0\right)\right)=V(q,\tau),\qquad \text{откуда}\qquad \sigma_1\left(q,q'|\tau_0\right)=\frac{1}{q-q'}\int\limits_{q'}^{q}V\left(y|\tau_0\right)dy \tag{0.20}$$

В конечном итоге мы получили

$$\left\langle q\left|\hat{U}\left(\tau,\tau_0\right)\right|q'\right\rangle=\frac{1}{\sqrt{2\pi\Delta}}e^{-\frac{(q-q')^2}{2\Delta}-\frac{\Delta}{q-q'}\int\limits_{q'}^{q}V(y|\tau_0)dy} \tag{0.21}$$

26.2.4 Problems about path integral formalism in QFT 3

Белавин-Пархоменко.5. Связь континуального интеграла и классической статфизики

1 Полубесконечная струна

Рассмотрим полубесконечную струну в некотором потенциале V, положения точек которой задается функцией $q(\tau)$, где $\tau\in[0,+\infty)$. Граничное условие для левого конца струны $q(0)=q_0$. А статистический вес конфигурации задается формулой

$$[Dq(\tau)]\exp(-\beta W[q(\tau)])=[Dq(\tau)]\exp\left(-\beta\int d\tau\left(V(q(\tau))+\frac{1}{2}\left(\frac{dq(\tau)}{d\tau}\right)^2\right)\right) \tag{0.1}$$

где член $\frac{1}{2}\left(\frac{dq(\tau)}{d\tau}\right)^2$ представляет упругую энергию деформации. Покажите, что q_0-зависимость для статистической суммы

$$Z\left(q_0\right)=\ \text{const}\ \cdot\Psi_0\left(q_0\right), \tag{0.2}$$

где $\Psi_0(q)$ - волновая функция основного состояния Гамильтониана

$$\hat{H}=\frac{1}{2}\hat{p}^2+V(\hat{q}). \tag{0.3}$$

2. Временное упорядочение

На лекции мы нашли, что корреляционная функция координат струны ($A[q(\tau)]=\beta W[q(\tau)]$):

$$G\left(\tau_1,...,\tau_N\right)\overset{\text{def}}{=}\left\langle q\left(\tau_1\right)...q\left(\tau_N\right)\right\rangle=\frac{1}{Z}\int[Dq(\tau)]q\left(\tau_1\right)...q\left(\tau_N\right)e^{-A[q(\tau)]} \tag{0.4}$$

может быть представлена как

$$\left\langle q\left(\tau_1\right)...q\left(\tau_N\right)\right\rangle=\left\langle 0\left|\hat{q}_E\left(\tau_N\right)\hat{q}_E\left(\tau_{N-1}\right)...\hat{q}_E\left(\tau_1\right)\right|0\right\rangle, \tag{0.5}$$

где $\hat{q}_E(\tau)=e^{\hat{H}\tau}\hat{q}e^{-\hat{H}\tau}$ - оператор координаты в представлении Гейзенберга в мнимом времени, и $\hat{q}\left|q_0\right\rangle=q_0\left|q_0\right\rangle$ (Вопрос на засыпку (5 баллов): как выглядит собственная волновая функция $\psi_{q_0}(q)=\langle q|q_0\rangle$ оператора координаты q, в координатном представлении, с собственным значением q_0 : $\hat{q}\psi_{q_0}(q)=q_0\psi_{q_0}(q)$?). Важно, что все операторы поставлены в порядке убывания времени: $\tau_N\geqslant\tau_{N-1}\geqslant...\geqslant\tau_1$. Далее, если мы хотим аналитически продолжить функцию $G\left(\tau_1,...,\tau_N\right)$, то есть рассмотреть ее значения для комплексных чисел $\tau_1,...,\tau_N$, то это имеет смысл только если выполняется неравенство

$$\operatorname{Re}\tau_N\geqslant\operatorname{Re}\tau_{N-1}\geqslant...\geqslant\operatorname{Re}\tau_1 \tag{0.6}$$

то есть $G\left(\tau_1,...,\tau_N\right)$ аналитична только в этой области. Покажите, что делая Виковский поворот: $\tau_k=e^{i\alpha}t_k$, где $\alpha\to\frac{\pi}{2}-0$, мы получим соотношение

$$\left\langle q\left(\tau_1\right)...q\left(\tau_N\right)\right\rangle=\left\langle 0\left|T\left(\hat{q}\left(t_1\right)...\hat{q}\left(t_N\right)\right)\right|0\right\rangle, \tag{0.7}$$

где $\tau_k=(i-0)t_k$ и $\hat{q}\left(t_k\right)$ обычный оператор Гейзенберга в реальном времени и символ T означает временное упорядочение - то есть операторы с большим временем ставятся слева от операторов с меньшим.

3 Теорема Вика для ПГО (Простого Гармонического Осциллятора)

(a). Пусть $\phi(t)$ координата обычного гармонического осциллятора, то есть

$$\phi(t)=\frac{1}{\sqrt{2m}}\left(ae^{-imt}+a^{\dagger}e^{imt}\right) \tag{0.8}$$

Докажите, что

$$T\left(\phi(t)\phi\left(t'\right)\right)=G_F\left(t-t'\right)+:\phi(t)\phi\left(t'\right): \tag{0.9}$$

где:: обозначает нормальное упорядочение - то есть мы должны все операторы уничтожения поставить справа от операторов рождения (например: $aa^{\dagger}aa^{\dagger}:=\left(a^{\dagger}\right)^2a^2$), и где пропагатор Фейнмана дается формулой

$$G_F(t)=\langle 0|T(\phi(t)\phi(0))|0\rangle \tag{0.10}$$

Обобщите (0.9) на произведение $2n$ полей и используйте это для доказательства теоремы Вика:

$$\langle 0\left|T\left(\phi\left(t_1\right)\phi\left(t_2\right)...\phi\left(t_{2n}\right)\right)\right|0\rangle=\sum_{\text{по всем парам}}G_F\left(t_{i_1}-t_{i_2}\right)...G_F\left(t_{i_{2n-1}}-t_{i_{2n}}\right). \tag{0.11}$$

(b). Вычислите явно

$$\langle 1\left|T\left(\phi\left(t_1\right)\phi\left(t_2\right)\phi\left(t_3\right)\phi\left(t_4\right)\right)\right|1\rangle \tag{0.12}$$

где $|1\rangle$ - первое возбужденное состояние ПГО, и проверьте, что теорема Вика не работает. Что не так в этом случае?

Белавин-Пархоменко .5. Связь континуального интеграла и классической статфизики Решения

1 Полубесконечная струна

Данную статистическую сумму можно записать как:

$$Z\left(q_0\right)=\lim_{T\to\infty}\int dq_f\int\limits_{q(0)=q_0}^{q(T)=q_f}[Dq(\tau)]e^{-\beta\int\limits_0^T d\tau\left(\frac{1}{2}\left(\frac{dq}{d\tau}\right)^2+V(q)\right)} \tag{0.1}$$

Далее, как мы знаем, интеграл по путям равен матричному элементу от оператора эволюции в квантовой механике $\left(\beta=\frac{1}{\hbar}=1\right)$:

$$Z\left(q_0\right)=\lim_{T\to\infty}\int dq_f\left\langle q_f|U(T,0)|q_0\right\rangle=\lim_{T\to\infty}\int dq_f\left\langle q_f\left|e^{-TH}\right|q_0\right\rangle \tag{0.2}$$

где $\hat{H}=\frac{1}{2}\hat{p}^2+V(q)$. Вставляя полный набор собственных состояний Гамильтониана: $1=\sum\limits_{n=0}^{\infty}|n\rangle\langle n|$, мы получаем

$$Z\left(q_0\right)=\lim_{T\to\infty}\int dq_f\sum_{n=0}^{\infty}\left\langle q_f|n\right\rangle e^{-TE_n}\left\langle n|q_0\right\rangle\to\left(\lim_{T\to\infty}\int dq_f\Psi_0\left(q_f\right)e^{-E_0T}\right)\Psi_0^*\left(q_0\right)=\text{const}\cdot\Psi_0^*\left(q_0\right). \tag{0.3}$$

Так как статсумма является вещественной функцией, можно записать

$$Z\left(q_0\right)=\text{ const }\cdot\Psi_0\left(q_0\right). \tag{0.4}$$

С одной стороны мы видим, что константа при $E_0>0$ стремится к нулю, но мы всегда можем нормировать нашу статистическую сумму так, чтобы сократить эту константу. В общем случае мы также не придаем значения бесконечным константам перед функциональным интегралом или интегралом по путям, так как обычно при подсчете физических величин такие бесконечные константы сокращаются.

2. Временное упорядочение

Ответ на вопрос на засыпку: $\psi_{q_0}(q) = \langle q|q_0\rangle = \delta(q-q_0)$.
Сама задача была по сути очевидной, так как мы уже получили равенство на лекции:

$$\langle q(\tau_1)...q(\tau_N)\rangle = \langle 0|T\left(\hat{q}_E(\tau_1)\hat{q}_E(\tau_2)...\hat{q}_E(\tau_N)\right)|0\rangle \tag{0.5}$$

Далее, делая Виковский поворот $\left(\tau_k = e^{i\alpha}t_k, \alpha \to \frac{\pi}{2}-0\right) : \tau_k = (i-0)t_k$ мы не нарушаем T упорядочение (т. е. если изначально $\tau_i > \tau_j$, то и $t_i > t_j$), поэтому получим

$$\langle q(\tau_1)...q(\tau_N)\rangle = \langle 0|T\left(\hat{q}(t_1)\hat{q}(t_2)...\hat{q}(t_N)\right)|0\rangle \tag{0.6}$$

4. Теорема Вика для ПГО (Простого Гармонического Осциллятора)

(a). Итак мы имеем

$$\phi(t) = \frac{1}{\sqrt{2m}}\left(ae^{-imt} + a^\dagger e^{imt}\right) \tag{0.7}$$

Пусть теперь $t > t'$ откуда получаем

$$\begin{aligned}
&T(\phi(t)\phi(t')) = \phi(t)\phi(t') = \frac{1}{2m}\left(ae^{-imt} + a^\dagger e^{imt}\right)\left(ae^{-imt'} + a^\dagger e^{imt'}\right) = \\
&= \frac{1}{2m}\left(a^2e^{-imt-imt'} + a^\dagger a e^{imt}e^{-imt'} + \left(a^\dagger\right)^2 e^{imt+imt'} + aa^\dagger e^{-imt+imt'}\right) = \\
&= \frac{1}{2m}\left(a^2e^{-imt-imt'} + a^\dagger a e^{imt}e^{-imt'} + \left(a^\dagger\right)^2 e^{imt+imt'} + \left(1+a^\dagger a\right)e^{-imt+imt'}\right) = \\
&=: \phi(t)\phi(t') : + \frac{1}{2m}e^{-im\left(t-t'\right)}.
\end{aligned} \tag{0.8}$$

Аналогично, если $t < t'$, то

$$T(\phi(t)\phi(t')) =: \phi(t)\phi(t') : + \frac{1}{2m}e^{im\left(t-t'\right)}. \tag{0.9}$$

Откуда получаем

$$T(\phi(t)\phi(t')) =: \phi(t)\phi(t') : + \frac{1}{2m}e^{-im|t-t'|} =: \phi(t)\phi(t') : + G_F(t-t'). \tag{0.10}$$

Обобщение на случай $2n$ полей тривиально следует из метода индукции примененного к утверждению (см. Пескин Шредер параграф 4.3):

$$T(\phi(t_1)\phi(t_2)...\phi(t_m)) =: \phi(t_1)\phi(t_2)...\phi(t_m) + \text{ все возможные свертки:}, \tag{0.11}$$

где каждая свертка дает пропагатор G_F.
(b). В данном случае имеем

$$\begin{aligned}
T(\phi_1\phi_2\phi_3\phi_4) =& G_{12}G_{34} + G_{13}G_{24} + G_{14}G_{23} + \\
&+ G_{12} : \phi_3\phi_4 : + G_{14} : \phi_2\phi_2 : + G_{13} : \phi_2\phi_4 : + G_{23} : \phi_1\phi_4 : + \\
&+ G_{24} : \phi_1\phi_3 : + G_{34} : \phi_1\phi_2 : + :: \phi_1\phi_2\phi_3\phi_4 :
\end{aligned} \tag{0.12}$$

Теорема Вика работает, когда все нормально-упорядоченные слагаемые исчезают. Очевидно это происходит, когда мы берем матричный элемент по вакуумным состояниям $\langle 0|...|0\rangle$. Но в нашем случае имеем, например

$$\langle 1|: \phi_1\phi_2 :|1\rangle = \frac{1}{2m}\left\langle 1\left|a^\dagger a\left(e^{-imt_1}e^{imt_2} + e^{-imt_2}e^{imt_1}\right)\right|1\right\rangle = \frac{1}{2m}\cos\left(m\left(t_1-t_2\right)\right) \neq 0, \tag{0.13}$$

так как $\left\langle 1\left|a^\dagger a\right|1\right\rangle = 1$, в случае вакуума мы бы получили $\left\langle 0\left|aa^\dagger\right|0\right\rangle = 0$.

МФТИ-Киселев-2020.14 Уравнения Швингера–Дайсона для Юкавской теории

Вывести уравнения Швингера–Дайсона и графическое представление для двухточечной вершинной функции для биспинора Дирака с юкавским взаимодействием с вещественным скалярным полем.

Записать правила Фейнмана.

Solution

МФТИ-Киселев-2020.15^C Уравнения Швингера–Дайсона для скалярной электродинамики

Вывести уравнения Швингера–Дайсона и графическое представление для двухточечной вершинной функции для скалярной электродинамики. Записать правила Фейнмана.

Solution

(???????)

Правила Фейнмана для вычисления амплитуд квантовых переходов в теории со скалярными полями при заданных начальных и конечных состояниях.

Эти правила зависят, конечно, от исходного (bare) лагранжиана, что подразумевает определение состава полей и их взаимодействий. Для (нейтрального) вещественного поля φ и заряженного (комплексного) поля Φ, лагранжиан свободных полей имеет стандартный вид

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu\varphi\partial^\mu\varphi - M^2\varphi^2\right) + \partial_\mu\Phi^\dagger\partial^\mu\Phi - m^2\Phi^\dagger\Phi$$

где M -масса нейтрального поля, а m - масса заряженного поля, а лагранжиан взаимодействия имеет вид

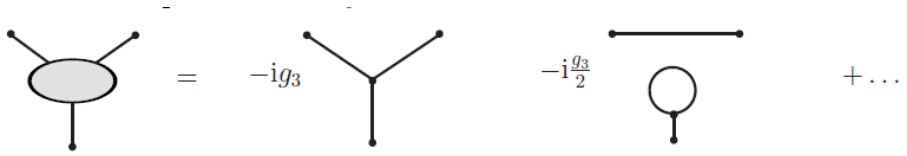
$$\mathcal{L}_{\text{int}} = -\Lambda\varphi\Phi^\dagger\Phi - \frac{\lambda}{4}\left(\Phi^\dagger\Phi\right)^2$$

где Λ - размерная постоянная юкавского взаимодействия полей, а λ константа самодействия заряженного поля.

В импульсном представлении амплитуда квантового перехода строится по диаграмме с учетом закона сохранения 4-импульса, которые приписываются всем линиям,

$$\mathrm{i}\mathcal{M} = (2\pi)^4\delta\left(\sum p\right)\mathrm{i}\mathfrak{M}$$

причем в $\mathfrak{M}$ внутренним линиям отвечают следующие пропагаторы: вещественное поля φ - штриховая линия -



Вершина, как мы убедились, получаются из затравочного действия путем дифференцирования лагранжиана $\mathcal{L}_{\text{int}}$ по полям и домножением на мнимую единицу. Внешним линиям в конечном состоянии отвечают следующие моды:

заряженная частица - отрицательно частотная мода поля $\hat{\Phi}^\dagger$,

заряженная античастица - отрицательно частотная мода поля $\hat{\Phi}$,

нейтральная частица - отрицательно частотная мода поля $\hat{\varphi}$.

Внешним линиям в начальном состоянии отвечают следующие моды: заряженная античастица - положительно частотная мода поля $\hat{\Phi}^\dagger$, заряженная частица - положительно частотная мода поля $\hat{\Phi}$, нейтральная частица - положительно частотная мода поля $\hat{\varphi}$. Для скалярных частиц в принятой релятивистски ковариантной нормировке состояний все эти моды в координатном представлении отвечают волнам де Бройля с соответствующей прескрипцией для импульсов и энергий в начальных и конечных состояниях,

чтобы выполнялся закон сохранения 4 импульса. В импульсном же представлении эти моды оказываются равными 1. Однако понимание того, как они были получены, упростит нам задачу в случае полей с ненулевым спином, когда моды - это многокомпонентные строки или столбцы, а не единицы.

При наличии в диаграмме петель закон сохранения 4-импульса приводит к дополнительному интегрированию по 4-импульсу, который может принимать произвольные значения,

$$\int \frac{\mathrm{d}^4 p}{(2\pi)^4}$$

для каждой петли. Подчеркнем, что при наличии тождественных полей необходимо учесть симметрийный фактор: для внешних полей провести симметризацию по тождественным частицам и сосчитать варианты прескрипции 4-импульсов этим полям, для внутренних линий сосчитать число графических вариантов возникновения одних и тех же диаграмм по заданному набору линий и вершин. Пример расчета симметрийного фактора мы видели в теории кубического самодействия нейтрального поля.

И наконец, повторим, как связана амплитуда с дифференциальной шириной распада частицы массой M с 4 -импульсм p

$$\mathrm{d}\Gamma = \frac{|\mathfrak{M}|^2}{2M} d\mathfrak{p}_f$$

где релятивистски инвариантный фазовый объем частиц в конечном состоя- нии

$$\mathrm{d}\mathfrak{p}_f = (2\pi)^4 \delta\left(p - \sum_{n=1}^{n_f} k_{(n)}\right) \prod_{n=1}^{n_f} \frac{\mathrm{d}^3 k_{(n)}}{2E_{(n)}}$$

и дифференциальным сечением рассеяния частиц с 4-импульсами p и $\tilde{p}$

$$\mathrm{d}\sigma = \frac{|\mathfrak{M}|^2}{4\sqrt{(p\cdot\tilde{p})^2 - m^2\tilde{m}^2}} d\mathfrak{p}_f$$

где фазовый объем в конечном состоянии $\mathrm{d}\mathfrak{p}_f = (2\pi)^4 \delta\left(p + \tilde{p} - \sum_{n=1}^{n_f} k_{(n)}\right) \prod_{n=1}^{n_f} \frac{\mathrm{d}^3 k_{(n)}}{2E_{(n)}}$. Отметим, что эти правила относятся к случаю, когда одноточечная функция Грина поля равна нулю. Однако, легко сообразить, что ненулевое

вакуумное значение поля $v = \langle 0|\hat{\varphi}(x)|0\rangle$ отвечает просто случаю внешнего поля с нулевым импульсом. Такую внешнюю линию обычно изображают с символом $\otimes$ на конце.

При этом вершины взаимодействия внешнего вакуумного поля легко определить, если в исходном лагранжиане произвести замену поля, скажем, $\varphi \mapsto v + \varphi$, так что новое поле уже не имеет вакуумного среднего.

МФТИ-Киселев-2020.16.C Уравнение Швингера-Дайсона для $\lambda\phi^4/4!$ (????)

Вывести уравнения Швингера–Дайсона и графическое представление для двухточечной вершинной функции для массивного скалярного поля с самодействием $\lambda\phi^4/4!$. Записать правила Фейнмана.

Solution

(???????)
Правила Фейнмана:
1. точный пропагатор дается суммой всех связных диаграмм
2. каждой вершине отвечает $-i\lambda\int d^4x$
3. Свободный пропагатор $D_0^F(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2-m^2+i\varepsilon}$
4. Комбинаторные множители.

МФТИ-Киселев-2020.17 Количество петель, внутренних линий и связных компонент

Доказать, что число петель N_L в диаграмме с N_V степенями действия взаимодействия V, числом связных компонент диаграммы N_c и числом внутренних линий N_I связано соотношением

$$N_L = N_I + N_c - N_V$$

Привести примеры одно- и двухпетлевых диаграмм с одно- и двухсвязными компонентами в теории с взаимодействием $V \sim \lambda\phi^4$

Solution

(?? в чем идея - тут напишу)
Внутренняя линия добавляет $\int d^4x$, а петля снимает.
Для одной связной петли

$$N_V = N_I - N_L + 1$$

Поэтому для нескольких:

$$\sum N_v = \sum N_I - \sum N_L + N_C = N_v = N_I - N_L + N_C$$

Амплитуду квантовых переходов $\hat{Z}(j) = \int \mathcal{D}\varphi \exp\left\{\frac{\mathrm{i}}{\hbar}\left(S_0(\varphi) + j_{x'}\varphi_{x'}\right) + \frac{\mathrm{i}}{\hbar}S_{\mathrm{int}}(\varphi)\right\}$ можно выразить с помощью действия оператора поля на функционал, построенный для свободных полей:

$$\hat{Z}(j) = \exp\left\{\frac{\mathrm{i}}{\hbar}S_{\mathrm{int}}\left(-\mathrm{i}\hbar\frac{\delta}{\delta j}\right)\right\}\int \mathcal{D}\varphi \exp\left\{\frac{\mathrm{i}}{\hbar}\left(S_0(\varphi) + j_{x'}\varphi_{x'}\right)\right\}$$

Этот функционал позволяет найти общее соотношение для числа петель, степени возмущения и числа внутренних линий в связной диаграмме.

Действительно, если N_V - степень вклада возмущения в функцию Грина, и значит, N_V - это число внутренних вершин в диаграмме, то вклад вершин в число интегрирований по координатам равен N_V (vertex, number of vertecies).

Если в диаграмме есть внутренняя линия, то она соединяет 2 точки, т. е. каждой внутренней линии отвечает одно интегрирование по координатам на конце внутренней линии, так что при наличии N_I внутренних линий (internal lines) N_I интегрирований относятся именно к этим линиям, и нужно добавить еще одну начальную точку, так что при последовательном счете, если рассматривать связную цепочку внутренних линий, имеется $N_I + 1$ число интегрирований в цепочке.

При этом, поскольку внутренняя линия соединяет именно вершины,

по сути мы полагаем, что в диаграмме столько же и вершин.

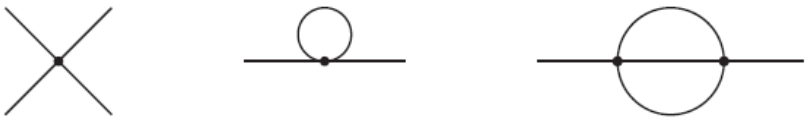
Однако если диаграмма связная, то некоторые начальные и конечные точки в цепочке внутренних линий могут совпасть.

Каждое такое совпадение означает наличие замкнутой петли.

Число петель N_L (number of loops) определяет число точек посчитанных дважды, так что общее число интегрирований по внутренним вершинам или просто число вершин в действительности равно

$$N_V = N_I + 1 - N_L, \quad N_L = N_I + 1 - N_V$$

В справедливости данной формулы читатель может наглядно убедиться на примере четырех- и двухточечной функции Грина в теории «фи-в-четвертой», как это показано на рис. ниже



МФТИ-Киселев-2020.18^C Разложение диаграмм по петлям

Докажите, что разложение связных диаграмм по петлям совпадает с разложением по постоянной Планка $\hbar$.

Solution

(?? в чем идея - тут напишу. ничего сейчас не понимаю!!!)
Заметим, что связные функции Грина $\mathrm{i}G(j)/\hbar$ с фиксированным числом петель получаются суммированием диаграмм с N_V степенями возмущения и N_I внутренними линиями, причем каждое возмущение дает фактор $1/\hbar$, а каждая внутренняя линия отвечает обратному оператору на квадратичном по траектории свободном действии, т. е. вносит фактор $\hbar$(cm. связь связной двухточечной функции Грина со связной частью двухточечной функции Грина для амплитуды вероятности $Z(j)$).

В итоге связные функции Грина с петлями имеют следующий масштаб по $\hbar$:

$$\frac{1}{\hbar}G^{(L)} \sim \sum_{N_V,N_I} \left(\frac{1}{\hbar}\right)^{N_V} \hbar^{N_I}\Bigg|_{N_L \text{ fixed}} \sim \hbar^{N_L-1}$$

Таким образом, разложение связных функций Грина по петлям совпадает с разложением по степеням $\hbar$. Это означает, что в классическом пределе $\hbar \to 0$ среди связных диаграмм остаются только диаграммы без петель или, как говорят, древесные диаграммы.

Example (???)

(хочется еще пример, потом приведу.)

26.2.5 Problems about path integral formalism in QFT 4

KUL-aqft-2023-hw2.1/3 Identity for Grassmann integral

Assume that $\left\{\theta_i, \bar{\theta}_i, \eta_i, \bar{\eta}_i\right\}$ is a collection of independent Grassmann variables with $i = 1, ..., n$, and B_{ij} is an invertible $n \times n$ matrix. Show that the following identity is true

$$Z[B, \eta, \bar{\eta}] \equiv \int \prod_{i=1}^{n} d\bar{\theta}_i d\theta_i \exp\left(-\bar{\theta}_i B_{ij} \theta_j + \bar{\eta}_i \theta_i + \bar{\theta}_i \eta_i\right) = \det(B) \exp\left(\bar{\eta}_i \left(B^{-1}\right)_{ij} \eta_j\right).$$

Using this result compute the "two-point function" $\left\langle \theta_i \bar{\theta}_j \right\rangle$. Express your result in two alternative ways, one explicit expression in terms of the matrix B_{ij} and one in terms of derivatives of the "path integral" $Z[B, \eta, \bar{\eta}]$.

Solution

KUL-aqft-2023-hw2.2/3 Value of Grassmann integral (?????)

Assume that $w_i(x)$ are functions of the n real variables x_i. Take θ_i and $\bar{\theta}_i$ to be n independent Grassmann variables. Compute the integral

$$Z[w] \equiv \frac{1}{(2\pi)^{n/2}} \int d^n x \prod_{i=1}^{n} d\bar{\theta}_i d\theta_i \exp\left(-\bar{\theta}_i \partial_j w_i(x) \theta_j - \frac{1}{2} w_i(x) w_i(x)\right).$$

Assume that $\det\left[\partial_i w_j(x)\right] > 0$.

Now take

$$w_i(x) := x_i + \frac{1}{2} g_{ijk} x_j x_k, \qquad \text{with} \qquad g_{ijk} = g_{ikj} = g_{kij}.$$

Assume that all g_{ijk} are small and find the Feynman rules for this perturbative expansion. Evaluate the one- and two-loop contributions to the "path integral" $Z[g]$.

Solution

(??? вроде хорошая задача, подумаю потом!!! не до этого все еще.)

KUL-aqft-2023-hw1.1 Typical integral

Consider the integral

$$Z(\lambda) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} dx e^{-\frac{x^2}{2!} - \frac{\lambda}{4!} x^4}$$

This is a simple toy model for the path integral in quantum field theory.

1. What is the range of λ such that $Z(\lambda)$ converges?

2. Find a compact expression for the series expansion of $Z(\lambda)$ for small λ of the form

$$Z_N(\lambda) = \sum_{n=0}^{N} c_n \lambda^n$$

Is your series convergent? What is the radius of convergence?

3. Use your favorite computer program or www.wolframalpha.com to compute $Z(0.1)$. Evaluate your series expression $Z_N(\lambda)$ for $\lambda = 0.1$ and for $N = 1, 5, 10, 50$. Discuss and interpret your results.

4. Find a series expansion for $Z(\lambda)$ for $\lambda \gg 1$ of the form

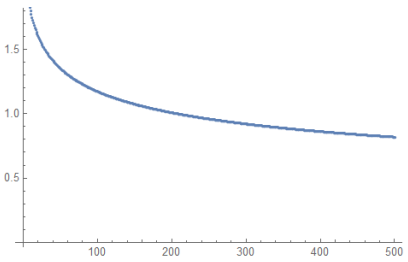
$$\hat{Z}_N(\lambda) = \sum_{n=0}^{N} d_n \lambda^{-(2n+1)/4}$$

Is this series convergent? For what value of N will you obtain a value for $\hat{Z}_N(0.1)$ which is close to the exact value $Z(0.1)$.

Solution

I'll check with Wolfram:

```
f[l_] = Integrate[ Exp[-x^2/2 - l x^4/24], {x, - \ [Infinity], \ [Infinity]}]
ListPlot[Table[f[n], {n, 500}]]
```



So indeed there is no reason to worry about it.

1.

What is the range of λ such that $Z(\lambda)$ converges?

It converges for $\lambda \geq 0$, since for it it behaves like $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2}}$, which converges. For $\lambda < 0$, integral behaves like $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx e^{|\lambda| \frac{x^4}{4!}}$ and it diverges.

1. Another solution

A Series Use Eq. 1.131 from Osborn to express Z with two exponential Taylor series.

$$\begin{aligned} Z &= \exp\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}\right) \exp\left(-\frac{\lambda}{4!}x^4\right)\Big|_{x=0} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{m!n!}\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)^m \left(\frac{x^4}{4!}\right)^n\Big|_{x=0} \\ &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-\lambda)^n}{m!n!} 2^{-m}(4!)^{-n}\left(\frac{\partial^2}{\partial x^2}\right)^m x^{4n}\Big|_{x=0} \end{aligned}$$

Only the $4n$-th derivative of x^{4n} is nonvanishing, so restrict to $m = 2n$. This derivative equals $(4n)!$.

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{(2n)!n!} 2^{-2n} (4!)^{-n} (4n)! \\ &= \sum_{n=0}^{\infty} (-\lambda)^n \frac{(4!4)^{-n}(4n)!}{n!(2n)!} \\ &= \sum_{n=0}^{\infty} (-\lambda)^n \frac{6^{-n}\Gamma\left(2n+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(n+1)} \end{aligned}$$

This is of that power series form with $c_n = (-1)^n \frac{6^{-n}\Gamma(2n+1/2)}{\sqrt{\pi}\Gamma(n+1)}$. This is divergent because the $(4n)$! grows faster than everything else. This can be seen from the Stirling formula, $\mathcal{O}(n!) = \mathcal{O}\left(\sqrt{n}\left(\frac{n}{e}\right)^n\right)$. The radius of convergence is 0. But that's normal.

2.

Well, I have no idea how to implement mathematical analysis for this problem and it will take me a lot of time to revise mathematical tools for it. But the problem is about the results, so I can just use a numerical method from paragraph below. The result is that if diverges at around $N \approx 42$.

3.

I write:

```
g[numb_]:= Series[f[1] // Normal, {1, 0, numb}, Assumptions->{1 \[Element] Reals, 1 > 0}]//Normal
g[1] /. 1 -> 0.1
g[5] /. 1 -> 0.1
g[10] /. 1 -> 0.1
g[50] /. 1 -> 0.1
```

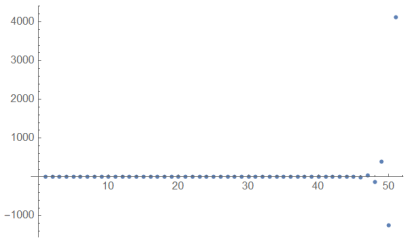
I get:

$$Z(0.1) = 2.47732 \quad Z_1(0.1) = 2.4753 \quad Z_5(0.1) = 2.47731 \quad Z_{10}(0.1) = 2.47732 \quad Z_{50}(0.1) = 4112.56$$

Wow, it diverges at big N...

Let's analyse more:

```
TableForm[N[Table[g[numb], {numb, 0, 5}] ] ]
ListPlot[%, PlotRange -> All]
```



4.

I will use Watson's lemma from Mathematical Analysis II by Vladimir A. Zorich (page 741 in russian version). This lemma is as follows.

Let's $\alpha > 0, \beta > 0, 0 < a \leqslant \infty$, $f \in C([0,a],\mathbb{R})$. Then for $W(\lambda) := \int\limits_0^a x^{\beta-1} f(x) e^{-\lambda x^\alpha} dx$

$$\lim_{\lambda \to +\infty} W(\lambda) \sim \frac{1}{\alpha} f(0) \Gamma(\beta/\alpha) \lambda^{-\frac{\beta}{\alpha}} + O\left(\lambda^{-\frac{\beta+1}{\alpha}}\right),$$

if $f(x) = f(0) + O(x)$ with $x \to 0$.

b) If $f(x) = a_0 + a_1 x + ... + a_n x^n + O\left(x^{n+1}\right)$ with $x \to 0$, then

$$W(\lambda) = \frac{1}{\alpha} \sum_{k=0}^{n} a_k \Gamma\left(\frac{k+\beta}{\alpha}\right) \lambda^{-\frac{k+\beta}{\alpha}} + O\left(\lambda^{-\frac{n+\beta+1}{\alpha}}\right)$$

c) If f infinitely differentiable in $x = 0$, then

$$W(\lambda) \simeq \frac{1}{\alpha} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \Gamma\left(\frac{k+\beta}{\alpha}\right) \lambda^{-\frac{k+\beta}{\alpha}}.$$

So in our case we will have only terms for even terms, because for odd ones $f^{(k)}(0) = 0$. And in notation above we have $\beta = 1, \alpha = 4, \lambda \to \frac{\lambda}{4!}, f(x) = e^{-x^2/2}$, so we will get:

I checked with Wolfram, $f^{(2k)}(0) = (-1)^{k/2}(2k)!!$, so

$$Z_N = \frac{1}{4} \sum_{2k=0}^{N} \frac{(-1)^{k/2}(2k)!!}{(2k)!} \Gamma\left(\frac{2k+1}{4}\right) \lambda^{-\frac{2k+1}{4}}$$

I suppose that this series are convergent and better then above ones.

О смысле этого интеграла и результатов (?????)

(напишу потом, оставался вопрос, потом подумаю.)

МФТИ-Мусаев-2023.18/19 Функциональный интеграл фермионов

Рассмотрим квантовую механику, заданную лагранжианом

$$L_E = \bar{\psi}\dot{\psi} + \omega\bar{\psi}\psi$$

где $\bar{\psi}, \psi$ - грассмановы переменные и $\{\bar{\psi}, \psi\} = 1$. Считая, что переменные удовлетворяют антипериодическим граничным условиям $\psi(t + \beta) = -\psi(t)$, вычислить функциональный интеграл теории. Почему осмысленны именно антипериодические условия?

26.3 Other Problems about Messing with Typical QFT Formalism

26.3.1 Problems about understanding diagrams

KUL-aqft-2023-hw1.3

Consider the quantum field theory of a scalar with the interaction potential

$$V(\phi) = \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4$$

Find all connected one- and two-loop graphs (with their appropriate symmetry factors) which contribute to $\langle \phi(x_1)\, \phi(x_2) \rangle$ and $\langle \phi(x_1)\, \phi(x_2)\, \phi(x_3) \rangle$.

(тут много проблем и вопросов, пока так и не могу вывести ничего, дня 3 тренироваться нужно, чтобы еще дойти до этой задачи.)

Solution

(??)

me. Integral for (?)

(put nice diagrams here)

Answer

me. Integral for (?)

(put nice diagrams here)

Answer

me. Integral for (?)

(put nice diagrams here)

Answer

me. Integral for (?)

(put nice diagrams here)

Answer

me. Integral for (?)

(put nice diagrams here)

Answer

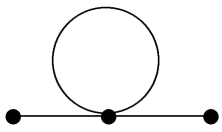
me. Integral for (?)

(put nice diagrams here)

Answer

me. Integral for "головастика" ϕ^4

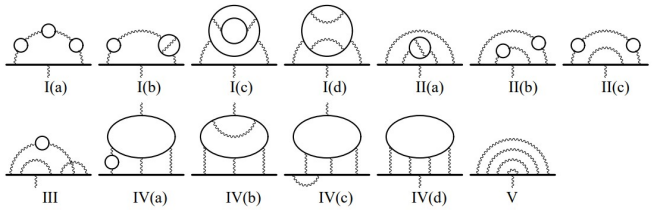
Записать интеграл для диаграммы "головастик":



Solution (????)

(в теории уже это написано, там напишу лучше еще, сюда потом буду писать.)

Integrals for muon



(типа выберите тут 3 диаграммы, напишите для них интеграл)

Solution (????)

(в теории уже это написано, там напишу лучше еще, сюда потом буду писать.)

Integrals for muon

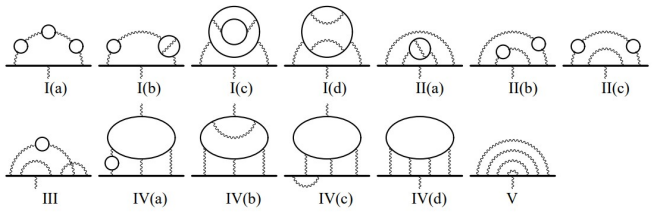


FIG. 1. Vertex diagrams representing 13 gauge-invariant subsets contributing to the lepton $g-2$ at the eighth-order. Solid and wavy lines represent lepton and photon lines, respectively.

(типа выберите тут 3 диаграммы, напишите для них интеграл. а вот руками их никто не считает, все слишком сложно, численно считают...)

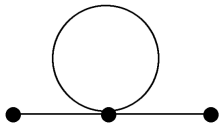
Solution

(очень крутая задача, очень хочется подумать про это, но пока нужно теорию доучивать.)

26.3.2 Problems about computation of diagrams

Интеграл для "головастика" ϕ^4

Вычислить интеграл для диаграммы "головастик":



Solution (????)

(в теории уже это написано, там напишу лучше еще, сюда потом буду писать.)

27 Problems about physics in QFT

27.1 Problems about decays and scatterings

27.1.1 Problems about typical scatterings in QED

Stanford.Zhou.2022.8.1. Magnetic moments in QED. (15/45 p)

The interaction Hamiltonian in quantum electrodynamics for a Dirac field of charge q is

$$H_I = q \int d^3x \bar{\Psi} \gamma^\mu \Psi A_\mu$$

To understand the physical meaning of this expression, we can evaluate its matrix elements in states $|p,s\rangle = \sqrt{2E_p} a_s^\dagger(p)|0\rangle$ with a single fermion. We assume the electromagnetic field is in a quantum state with negligible field uncertainty, so that its field operator A_μ can be replaced with a time-independent classical expectation value $A_\mu^{\mathrm{cl}}(\mathbf{x})$.

a) Show that the matrix elements of the Schrodinger picture Hamiltonian are

$$\langle p,s\,|H_I|\,p',s'\rangle = q\int d^3x e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \bar{u}_s(p)\gamma^\mu u_{s'}(p')\, A_\mu^{\mathrm{cl}}(\mathbf{x})$$

when $|p',s'\rangle \neq |p,s\rangle$.

b) Evaluate both sides of (4) in a static electric field, corresponding to general $A^0(\mathbf{x})$ and $\mathbf{A}=0$. Show that they agree when $q'=q$, as one would expect.

c) Show that if the fermion was a classical spinning ball with uniform mass and charge density, then its g-factor would be 1. This strongly disagrees with the measured value.

d) Evaluate both sides of (4) in a static magnetic field, corresponding to $A^0=0$ and $\nabla\times\mathbf{A}=\mathbf{B}(\mathbf{x})$, and infer the value of g. (Hint: to relate spin in the Dirac theory to spin in the nonrelativistic theory, recall that in the nonrelativistic theory, spinors ξ_s have two components and the spin operator is $\mathbf{S}=\boldsymbol{\sigma}/2$. You already showed in problem set 7 how the Dirac spinors $u_s(p)$ are built from ξ_s. You will have to integrate by parts, so assume $\mathbf{A}$ and $\mathbf{B}$ vanish at infinity.)

Solution:

Plugging in the Schrodinger picture mode expansion of the Dirac field, and using the abbreviated notation introduced in the solutions to problem set 2,

$$\begin{aligned}\langle p,s\,|H_I|\,p',s'\rangle &= q\sum_{r,r'}\int d\mathbf{x}\frac{d\mathbf{q}}{\sqrt{2E_q}}\frac{d\mathbf{q}'}{\sqrt{2E_{q'}}}\sqrt{2E_p 2E_{p'}}\\ &\times\left\langle 0\left|a_s(\mathbf{p})\left(a_r^\dagger(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}}\bar{u}_r(q)+b_r(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}\bar{v}_r(q)\right)\mathcal{A}\left(a_{r'}(\mathbf{q}')\,e^{i\mathbf{q}'\cdot\mathbf{x}}u_{r'}(q)+b_{r'}^\dagger(\mathbf{q}')\,e^{-i\mathbf{q}'\cdot\mathbf{x}}v_{r'}(q')\right)a_{s'}^\dagger(\mathbf{p}')\right|0\right\rangle.\end{aligned}$$

Now let's think about the structure of this matrix element. We can only get nonzero contributions from the terms $a_r^\dagger(\mathbf{q})a_{r'}(\mathbf{q}')$ and $b_r(\mathbf{q})b_{r'}^\dagger(\mathbf{q}')$. On the other hand, the latter term yields

$$\left\langle 0\left|a_s(\mathbf{p})b_r(\mathbf{q})b_{r'}^\dagger(\mathbf{q}')\,a_{s'}^\dagger(\mathbf{p}')\right|0\right\rangle = \delta_{ss'}\delta_{rr'}\phi(\mathbf{p}-\mathbf{p}')\,\phi(\mathbf{q}-\mathbf{q}')$$

and therefore cannot contribute matrix elements with $|p',s'\rangle\neq|p,s\rangle$. Discarding this term, we're left with

$$\begin{aligned}\langle p,s\,|H_I|\,p',s'\rangle &= q\int d\mathbf{x}d\mathbf{q}d\mathbf{q}'\frac{\sqrt{2E_p2E_{p'}}}{\sqrt{2E_q2E_{q'}}}e^{-i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{x}}\sum_{r,r'}\bar{u}_r(q)\not{A}u_{r'}(q')\left\langle 0\left|a_s(\mathbf{p})a_r^\dagger(\mathbf{q})a_{r'}(\mathbf{q}')\,a_{s'}^\dagger(\mathbf{p}')\right|0\right\rangle\\ &= q\int d\mathbf{x}d\mathbf{q}'\frac{\sqrt{2E_{p'}}}{\sqrt{2E_{q'}}}e^{-i(\mathbf{p}-\mathbf{q}')\cdot\mathbf{x}}\sum_{r'}\bar{u}_s(p)\not{A}u_{r'}(q')\left\langle 0\left|a_{r'}(\mathbf{q}')\,a_{s'}^\dagger(\mathbf{p}')\right|0\right\rangle\\ &= q\int d\mathbf{x}e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)\not{A}u_{s'}(p').\end{aligned}$$

This is the desired result. Using the Gordon identity we find

$$\langle p,s\left|H_I\right|p',s'\rangle = \frac{q}{2m}\int d\mathbf{x}e^{-i\left(\mathbf{p}-\mathbf{p}'\right)\cdot\mathbf{x}}\bar{u}_s(p)\left(p'_\mu+p_\mu+i\sigma_{\mu\nu}\left(p'^\nu-p^\nu\right)\right)u_{s'}\left(p'\right)A^\mu(x)$$

In nonrelativistic quantum mechanics, the single-particle states are $|\mathbf{p},s\rangle$. If the particle has charge q' and g-factor g, then its magnetic moment is $\mu=(gq'/2m)\,\mathbf{S}$ where $\mathbf{S}$ is the spin. In terms of the particle's position $\mathbf{x}$, the Hamiltonian contains the terms

$$H_I^{\text{nr}} = q'A^0(\mathbf{x}) - \boldsymbol{\mu}\cdot\mathbf{B}(\mathbf{x})$$

To find the values of q' and g, we equate matrix elements in the nonrelativistic limit,

$$\langle p,s\left|H_I\right|p',s'\rangle = 2m\left\langle\mathbf{p},s\left|H_I^{\text{nr}}\right|\mathbf{p}',s'\right\rangle \text{ when } |\mathbf{p}|,|\mathbf{p}'|\ll m$$

where the factor of $2m$ converts between relativistic and nonrelativistic normalization. We could evaluate both sides for general $A_\mu^{\text{cl}}(\mathbf{x})$, but it is easier to consider two special cases. In both cases it will be helpful to use the Gordon identity from set 7.

Solution:

If we set $\mathbf{A}=0$, then the matrix element above reduces to

$$\langle p,s\left|H_I\right|p',s'\rangle = \frac{q}{2m}\int d\mathbf{x}e^{-i\left(\mathbf{p}-\mathbf{p}'\right)\cdot\mathbf{x}}\bar{u}_s(p)\left(p'_0+p_0+i\sigma_{0i}\left(p^i-p^i\right)\right)u_{s'}\left(p'\right)A^0(x)$$

since σ_{00} vanishes. In the nonrelativistic limit, $p_0,p'_0\approx m$ are much greater than p^i,p^i, so we may simply drop the latter terms, giving

$$\begin{aligned}\langle p,s\left|H_I\right|p',s'\rangle &= q\int d\mathbf{x}e^{-i\left(\mathbf{p}-\mathbf{p}'\right)\cdot\mathbf{x}}\bar{u}_s(p)u_{s'}\left(p'\right)A^0(x)\\ &= 2mq\delta_{ss'}\int d\mathbf{x}e^{-i\left(\mathbf{p}-\mathbf{p}'\right)\cdot\mathbf{x}}A^0(x)\end{aligned}$$

On the other hand, in nonrelativistic quantum mechanics we have

$$\begin{aligned}2m\left\langle\mathbf{p},s\left|q'A^0(\hat{\mathbf{x}})\right|\mathbf{p}',s'\right\rangle &= 2mq'\delta_{ss'}\left\langle\mathbf{p}\left|A^0(\hat{\mathbf{x}})\right|\mathbf{p}'\right\rangle\\ &= 2mq'\delta_{ss'}\int d\mathbf{y}d\mathbf{z}\langle\mathbf{p}|\mathbf{y}\rangle\left\langle\mathbf{y}\left|A^0(\hat{\mathbf{x}})\right|\mathbf{z}\right\rangle\langle\mathbf{z}|\mathbf{p}'\rangle\\ &= 2mq'\delta_{ss'}\int d\mathbf{y}d\mathbf{z}e^{-i\mathbf{p}\cdot\mathbf{y}}e^{i\mathbf{p}'\cdot\mathbf{z}}\delta(\mathbf{y}-\mathbf{z})A^0(\mathbf{z})\\ &= 2mq'\delta_{ss'}\int d\mathbf{y}e^{-i\left(\mathbf{p}-\mathbf{p}'\right)\cdot\mathbf{y}}A^0(\mathbf{y})\\ &= 2mq'\delta_{ss'}\int d\mathbf{x}e^{-i\left(\mathbf{p}-\mathbf{p}'\right)\cdot\mathbf{x}}A^0(\mathbf{x})\end{aligned}$$

The two matrix elements match if $q=q'$, as desired.

Solution:

For simplicity, let's consider a thin ring of radius r, angular velocity ω, mass m, and charge q. Then the angular momentum is

$$S = mvr = m\omega r^2$$

and the magnetic moment is

$$\mu = IA = \frac{q}{2\pi/\omega}\left(\pi r^2\right) = \frac{q\omega r^2}{2}$$

Therefore, the ring has $\mu/S=q/2m$, corresponding to a g-factor of 1. Since a ball is just a superposition of such rings, this shows that $g=1$ for a ball, or more generally any spinning object with axial symmetry.

Solution:

In this case, since A^0 vanishes, we have

$$\langle p,s\,|H_I|\,p',s'\rangle = \frac{q}{2m}\int d\mathbf{x}e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)\left(p'_i+p_i+i\sigma_{i\nu}\left(p'^{\nu}-p^{\nu}\right)\right)u_{s'}\left(p'\right)A^i(x)$$

This might look more complicated than what we had before using the Gordon identity, but this form is useful because it lets us isolate the physically distinct effects of different terms, and figure out which to keep in the nonrelativistic limit. (Though of course you could also do the problem without using the Gordon identity too.) Now, we need to choose which term to keep within the parentheses.

- The p_i and p'_i terms are order mv.
- The term $\sigma_{i0}\left(p'^0-p^0\right)$ is not order m, since the m 's cancel. Instead, it's order mv^2.
- The term $\sigma_{ij}\left(p^j-p^j\right)$ is order mv.

The second contribution is therefore negligible in the nonrelativistic limit. The first contribution is not negligible, but it also has nothing to do with spin: it will just yield the $\mathbf{p}\cdot\mathbf{A}$ term in the nonrelativistic Hamiltonian that physically corresponds to the magnetic force on a charged particle. (Considering it won't tell us anything new, because we already know that $q'=q$ from part (b).) We therefore focus on the third term, giving

$$\begin{aligned}\langle p,s\,|H_I|\,p',s'\rangle &\supset \frac{q}{2m}\int d\mathbf{x}e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\bar{u}_s(p)\left(i\sigma_{ij}\left(p'^j-p^j\right)\right)u_{s'}\left(p'\right)A^i(x)\\ &= \frac{q}{2m}\int d\mathbf{x}e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\left(p^j-p'^j\right)\left(\bar{u}_s(p)\frac{\left[\gamma^i,\gamma^j\right]}{2}u_{s'}\left(p'\right)\right)A^i(x)\end{aligned}$$

To make further progress, we use the results we derived in the Weyl representation in problem set 7. First, because we're only considering the leading term in the nonrelativistic limit, it suffices to expand the spinor solutions to lowest order in m,

$$u_s(p)=\begin{pmatrix}\sqrt{p\cdot\sigma}\xi_s\\ \sqrt{p\cdot\bar{\sigma}}\xi_s\end{pmatrix}\approx\sqrt{m}\begin{pmatrix}\xi_s\\ \xi_s\end{pmatrix}$$

We can also explicitly evaluate the commutators. For example, we have

$$\left[\gamma^1,\gamma^2\right]=-2i\begin{pmatrix}\sigma^3 & \\ & \sigma^3\end{pmatrix}$$

from which we conclude in general that

$$\left[\gamma^i,\gamma^j\right]=-2i\epsilon^{ijk}\begin{pmatrix}\sigma^k & \\ & \sigma^k\end{pmatrix}$$

Plugging these results in and simplifying, we find

$$\langle p,s\,|H_I|\,p',s'\rangle \supset q\int d\mathbf{x}e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}(-i)\left(p^j-p'^j\right)\left(\epsilon^{ijk}\xi_s^\dagger\sigma^k\xi_{s'}\right)A^i(x)$$

To handle the $p^j-p'^j$, we write it as a derivative and integrate by parts, dropping a boundary term,

$$\begin{aligned}\langle p,s\,|H_I|\,p',s'\rangle &\supset q\int d\mathbf{x}\frac{\partial}{\partial x_j}\left(e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\right)\left(\epsilon^{ijk}\xi_s^\dagger\sigma^k\xi_{s'}\right)A^i(x)\\ &= q\xi_s^\dagger\sigma^k\xi_{s'}\int d\mathbf{x}e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}\epsilon^{ijk}\partial^j A^i(x)\\ &= -q\xi_s^\dagger\sigma^k\xi_{s'}\int d\mathbf{x}e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}B^k.\end{aligned}$$

We can now compare this to the nonrelativistic result,

$$2m\left\langle\mathbf{p},s\right|(-\boldsymbol{\mu}\cdot\mathbf{B})\left|\mathbf{p}',s'\right\rangle=-gq'\xi_s^\dagger S^k\xi_{s'}\int d\mathbf{x}e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}B^k$$

by the same logic as in part (b). Since $q'=q$ and $\mathbf{S}=\boldsymbol{\sigma}/2$, we conclude that $g=2$.

The agreement of this value you found in part **(d)** with the experimentally measured value for the electron was one of the early triumphs of the Dirac equation.

Stanford.Zhou.2022.8.3. Electron-positron annihilation to muons. (15/45p)

One of the key successes of quantum electrodynamics is its description of particle creation and annihilation processes at relativistic energies. In this exercise we will consider the process $e^+e^- \to \mu^+\mu^-$, where the electron and muon are Dirac fields of charge e, and mass m_e and m_μ respectively.

a) Let the initial momenta be $p^\mu_{e^+}$ and $p^\mu_{e^-}$ and the final momenta be $p^\mu_{\mu^+}$ and $p^\mu_{\mu^-}$. Find the scattering matrix element $\mathcal{M}$ for this process, to leading nonvanishing order in e.

b) Let $|\overline{\mathcal{M}}|^2$ be the square of the matrix element, summed over final spin states and averaged over initial spin states. Compute $|\overline{\mathcal{M}}|^2$ in terms of e, m_e, m_μ, and the Mandelstam variables s and t, where

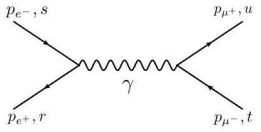
$$s = (p_{e^+} + p_{e^-})^2, \quad t = \left(p_{e^-} - p_{\mu^-}\right)^2, \quad u = \left(p_{e^-} - p_{\mu^+}\right)^2$$

c) Rewrite $|\overline{\mathcal{M}}|^2$ in terms of e, m_e, m_μ, s, and the angle θ between $\mathbf{p}_{e^-}$and $\mathbf{p}_{\mu^-}$.

d) Starting from equation (4.84) of Peskin and Schroeder, compute the differential cross section $d\sigma/d\Omega$ and the total cross section.

Solution:

There is one relevant Feynman diagram, which gives

$\mathcal{M}_{e^+e^-\to\mu^+\mu^-} =$ 

$$= \bar{u}_t(p_\mu^-)(-ie\gamma^\mu_{tu})v_u(p_{\mu^+})\left(\frac{-i\eta_{\mu\nu}}{(p_{e^+}+p_{e^-})^2}\right)\bar{v}_r(p_{e^+})(-ie\gamma^\nu_{rs})u_s(p_{e^-}).$$

Solution:

Before starting, we note that the Mandelstam variables are simply related to inner products,

$$\frac{s}{2} = m_e^2 + p_{e^+}\cdot p_{e^-} = m_{\mu^2} + p_{\mu^+}\cdot p_{\mu^-}$$

and

$$t = m_e^2 + m_\mu^2 - p_{e^-}\cdot p_{\mu^-} = m_e^2 + m_\mu^2 - p_{e^+}\cdot p_{\mu^+}$$
$$u = m_e^2 + m_\mu^2 - p_{e^-}\cdot p_{\mu^+} = m_e^2 + m_\mu^2 - p_{e^+}\cdot p_{\mu^-}$$

Now, squaring, summing over final spins and averaging over initial spins yields

$$|\overline{\mathcal{M}}|^2 = \frac{e^4}{s^2}\frac{1}{4}\sum_{\text{spins}} \bar{u}_t\left(p_\mu^-\right)\gamma^\mu_{tu}v_u\left(p_{\mu^+}\right)\bar{v}_{u'}\left(p_\mu^+\right)\gamma^\nu_{u't'}u_{t'}\left(p_{\mu^-}\right)\bar{v}_r\left(p_{e^+}\right)\left(\gamma_\mu\right)_{rs}u_s\left(p_{e^-}\right)\bar{u}_{s'}\left(p_{e^-}\right)\left(\gamma_\nu\right)_{s'r'}v_{r'}\left(p_{e^+}\right)$$
$$= \frac{e^4}{s^2}\frac{1}{4}\operatorname{tr}\left[\left(\not{p}_{\mu^-} + m_\mu\right)\gamma^\mu\left(\not{p}_{\mu^+} - m_\mu\right)\gamma^\nu\right]\operatorname{tr}\left[\left(\not{p}_{e^-} + m_e\right)\gamma_\mu\left(\not{p}_{e^+} - m_e\right)\gamma_\nu\right].$$

Both traces have the same form, and can be evaluated using the results from problem set 6,

$$\operatorname{tr}\left[(\not{p} + m)\gamma^\mu\left(\not{p}_2 - m\right)\gamma^\nu\right] = \operatorname{tr}\left[\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu\right] - m^2\operatorname{tr}\left[\gamma^\mu\gamma^\nu\right]$$
$$= 4\left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - (p_1\cdot p_2)\,\eta^{\mu\nu}\right) - 4m^2\eta^{\mu\nu}$$
$$= 4\left(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{s}{2}\eta^{\mu\nu}\right)$$

where in the last step we used (S44). Inserting this result twice above, we have

$$|\overline{\mathcal{M}}|^2 = \frac{4e^4}{s^2}\left(p^\mu_{e^+}p^\nu_{e^-} + p^\mu_{e^-}p^\nu_{e^+} - \frac{s}{2}\eta^{\mu\nu}\right)\left(p_{\mu^+,\mu}p_{\mu^-,\nu} + p_{\mu^-,\mu}p_{\mu^+,\nu} - \frac{s}{2}\eta_{\mu\nu}\right)$$
$$= \frac{4e^4}{s^2}\left(s^2 - s\left(p_{\mu^+}\cdot p_{\mu^-} + p_{e^+}\cdot p_{e^-}\right) + 2\left(p_{e^+}\cdot p_{\mu^+}\right)\left(p_{e^-}\cdot p_{\mu^-}\right) + 2\left(p_{e^+}\cdot p_{\mu^-}\right)\left(p_{e^-}\cdot p_{\mu^+}\right)\right)$$

We can then write the inner products in terms of s, t, and u using (S44) and (S45). There are many possible forms for the answer, since

$$s + t + u = 2m_e^2 + 2m_\mu^2.$$

If we eliminate u, then we get

$$|\overline{\mathcal{M}}|^2 = \frac{2e^4}{s^2}\left[(s+t)^2 + t^2 - 4t\left(m_e^2 + m_\mu^2\right) + 2\left(m_e^2 + m_\mu^2\right)^2\right]$$

Alternatively, if we eliminate s inside the brackets, then we get

$$|\overline{\mathcal{M}}|^2 = \frac{2e^4}{s^2}\left[\left(u - 2\left(m_e^2 + m_\mu^2\right)\right)^2 + \left(t - 2\left(m_e^2 + m_\mu^2\right)\right)^2 - 2\left(m_e^2 + m_\mu^2\right)^2\right]$$

For the rest of this problem, we specialize to the center of mass frame.

Solution:

In the centre-of-mass frame we have

$$E_{e^-} = E_{e^+}, \quad E_{\mu^-} = E_{\mu^+}, \quad \mathbf{p}_{e^-} = -\mathbf{p}_{e^+}, \quad \mathbf{p}_{\mu^-} = -\mathbf{p}_{\mu^+}$$

which implies that

$$s = 4E_{e^-}^2 = 4E_{\mu^-}^2.$$

Furthermore, we have $E_{e^\pm}^2 = |\mathbf{p}_{e^\pm}|^2 + m_e^2$ and $E_{\mu^\pm}^2 = |\mathbf{p}_{\mu^\pm}|^2 + m_\mu^2$.
Starting from S55, we need to eliminate t in favor of θ. Starting from the definition of t,

$$\begin{aligned} t &= m_e^2 + m_\mu^2 - 2E_{e^-}E_{\mu^-} + 2\,|\mathbf{p}_{e^-}|\,|\mathbf{p}_{\mu^-}|\cos\theta \\ &= \frac{1}{2}\left(2m_e^2 + 2m_\mu^2 - s + \sqrt{s - 4m_e^2}\sqrt{s - 4m_\mu^2}\cos\theta\right) \end{aligned}$$

Plugging this into S55 and simplifying gives

$$|\overline{\mathcal{M}}|^2 = e^4\left[1 + \frac{4\left(m_e^2 + m_\mu^2\right)}{s} + \left(1 - 4m_e^2/s\right)\left(1 - 4m_\mu^2/s\right)\cos^2\theta\right]$$

Also note that solving (S60) for $\cos\theta$ gives

$$\cos\theta = 2\frac{t - m_e^2 - m_\mu^2 + \frac{s}{2}}{\sqrt{s - 4m_e^2}\sqrt{s - 4m_\mu^2}}$$

Solution:

Equation (4.84) of Peskin and Schroeder is

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_{e^-}2E_{e^+}\,|\mathbf{v}_{e^-} - \mathbf{v}_{e^+}|}\frac{|\mathbf{p}_{\mu^-}|}{(2\pi)^2 4E_{\mathrm{cm}}}|\overline{\mathcal{M}}|^2$$

Our first task is to write the phase space factors in the same variables we've been using above. We note that $\sqrt{s} = E_{\mathrm{cm}} = 2E_{e^-} = 2E_{e^+}$in the center of mass frame. Furthermore,

$$|\mathbf{v}_{e^-} - \mathbf{v}_{e^+}| = 2\,|\mathbf{v}_{e^-}| = 2\frac{|\mathbf{p}_{e^-}|}{E_{e^-}} = 2\frac{\sqrt{E_{e^-}^2 - m_{e^-}^2}}{E_{e^-}} = 2\sqrt{1 - 4m_e^2/s}$$

and similarly

$$|\mathbf{p}_{\mu^-}| = \frac{\sqrt{s}}{2}\sqrt{1 - 4m_\mu^2/s}$$

Plugging these results in and simplifying gives

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{\sqrt{1-4m_\mu^2/s}}{\sqrt{1-4m_e^2/s}}|\overline{\mathcal{M}}|^2$$

To find the total cross section, we integrate over solid angle, noting that

$$\int d\Omega = 4\pi, \quad \int \cos^2\theta d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\cos^2\theta = \frac{4\pi}{3}$$

The final result is

$$\begin{aligned}\sigma &= \frac{e^4}{16\pi s}\frac{\sqrt{1-4m_\mu^2/s}}{\sqrt{1-4m_e^2/s}}\left[1+\frac{4\left(m_e^2+m_\mu^2\right)}{s}+\frac{\left(1-4m_e^2/s\right)\left(1-4m_\mu^2/s\right)}{3}\right] \\ &= \frac{e^4}{12\pi s}\frac{\sqrt{1-4m_\mu^2/s}}{\sqrt{1-4m_e^2/s}}\left(1+2m_e^2/s\right)\left(1+2m_\mu^2/s\right).\end{aligned}$$

As a check, we recover the expected result in the ultrarelativistic limit,

$$\sigma \approx \frac{e^4}{12\pi s} = \frac{4\pi\alpha^2}{3s}$$

which matches the result in Peskin and Schroeder.

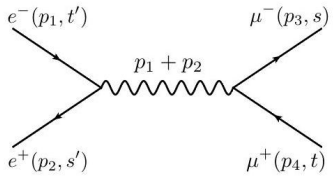
You can see section 5.1 of Peskin and Schroeder to get started or check your answer. But note that the book neglects the mass of the electron, while here we account for it.

Stanford.Zhou.2022.9.1. $e^+e^- \to \mu^+\mu^-$ at one loop. (8/45p)

This problem set guides you through the calculation of loop diagrams and their application to the renormalization of QED. Unlike previous problem sets, it is ordered sequentially: each problem is easiest to approach after completing the previous problems. For reference, you may find it useful to consult sections 6.3 and 7.5 of Peskin and Schroeder, but this problem set is self-contained and its results are somewhat more general.

We will use dimensional regularization with $d = 4 - 2\epsilon$. To avoid confusion, we will rename the $i\epsilon$ in the Feynman propagator to $i0$ in this problem set.

In the previous problem set, you considered the leading order (or "tree level") contribution to the scattering matrix element for $e^+e^- \to \mu^+\mu^-$,

$\mathcal{M}^{\mathrm{LO}} =$ 

$$= \bar{u}_s(p_3)(-ie\gamma^\nu)v_t(p_4)\frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2+i0}\bar{v}_{s'}(p_2)(-ie\gamma^\mu)u_{t'}(p_1)$$

which is proportional to e^2. The next-to-leading order contribution $\mathcal{M}^{\mathrm{NLO}}$ is of order e^4, and contains Feynman diagrams with one loop.

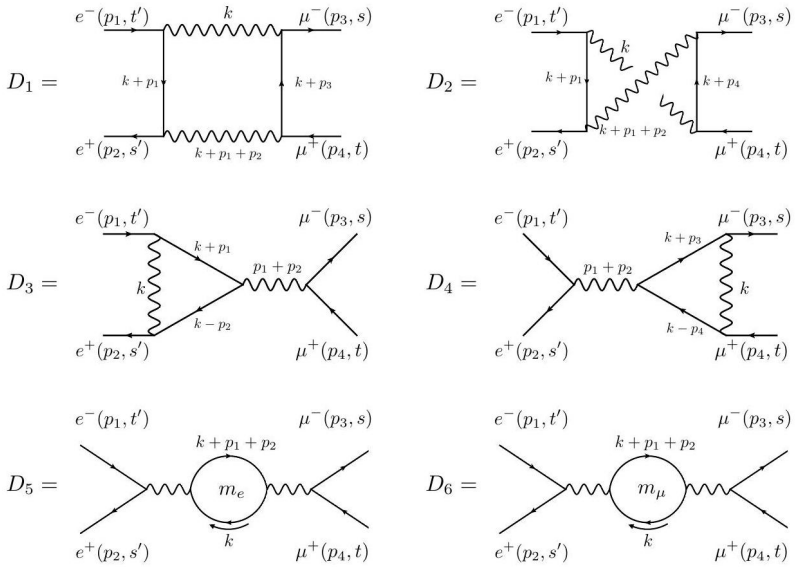
a) Draw all Feynman diagrams that contribute to $\mathcal{M}^{\mathrm{NLO}}$.

b) Write down $\mathcal{M}^{\mathrm{NLO}}$ using the Feynman rules. You don't have to evaluate the loop integrals; this is just to show you examples of their typical form.

Solution to a:

The matrix element at next-to-leading order (NLO) is given by

$$M^{\mathrm{NLO}} = D_1 + D_2 + D_3 + D_4 + D_5 + D_6$$



Solution to b:

(!???!! проделаю это еще раз и перепишу)

$$D_1 = \int \frac{d^dk}{(2\pi)^d} \bar{u}_s(p_3)(-ie\gamma^\mu) \frac{i\left(\not{k}+\not{p}_3+m_\mu\right)}{(k+p_3)^2-m_\mu^2+i0}(-ie\gamma^\nu) v_t(p_4) \frac{-i\eta_{\mu\sigma}}{k^2+i0}$$
$$\times \bar{v}_{s'}(p_2)(-ie\gamma^\rho) \frac{i\left(\not{k}+\not{p}_1+m_e\right)}{(k+p_1)^2-m_e^2+i0}(-ie\gamma^\sigma) u_{t'}(p_1) \frac{-i\eta_{\nu\rho}}{(k+p_1+p_2)^2+i0}.$$

$$D_2 = \int \frac{d^dk}{(2\pi)^d} \bar{u}_s(p_3)(-ie\gamma^\mu) \frac{i\left(\not{k}+\not{p}_4+m_\mu\right)}{(k+p_4)^2-m_\mu^2+i0}(-ie\gamma^\nu) v_t(p_4) \frac{-i\eta_{\mu\sigma}}{k^2+i0}$$
$$\times \bar{v}_{s'}(p_2)(-ie\gamma^\sigma) \frac{i\left(k+\not{p}_1+m_e\right)}{(k+p_1)^2-m_e^2+i0}(-ie\gamma^\rho) u_{t'}(p_1) \frac{-i\eta_{\nu\rho}}{(k+p_1+p_2)^2+i0}.$$

$$D_3 = \int \frac{d^dk}{(2\pi)^d} \bar{u}_s(p_3)(-ie\gamma^\mu) v_t(p_4) \frac{-i\eta_{\mu\nu}}{(p_3+p_4)^2+i0} \frac{-i\eta_{\sigma\rho}}{k^2+i0}$$
$$\times \bar{v}_{s'}(p_2)(-ie\gamma^\rho) \frac{i\left(\not{k}-\not{p}_2+m_e\right)}{(k-p_2)^2-m_e^2+i0}(-ie\gamma^\nu) \frac{i\left(\not{k}+\not{p}_1+m_e\right)}{(k+p_1)^2-m_e^2+i0}(-ie\gamma^\sigma) u_{t'}(p_1).$$

$$D_4 = \int \frac{d^dk}{(2\pi)^d} \bar{v}_{s'}(p_2)(-ie\gamma^\mu) u_{t'}(p_1) \frac{-i\eta_{\mu\nu}}{(p_1+p_2)^2+i0} \frac{-i\eta_{\sigma\rho}}{k^2+i0}$$
$$\times \bar{u}_s(p_3)(-ie\gamma^\rho) \frac{i\left(k+\not{p}_3+m_\mu\right)}{(k+p_3)^2-m_\mu^2+i0}(-ie\gamma^\nu) \frac{i\left(k\not{k}-\not{p}_4+m_\mu\right)}{(k-p_4)^2-m_\mu^2+i0}(-ie\gamma^\sigma) v_t(p_4).$$

$$D_5 = \int \frac{d^dk}{(2\pi)^d} \bar{u}_s(p_3)(-ie\gamma^\nu) v_t(p_4) \bar{v}_{s'}(p_2)(-ie\gamma^\mu) u_{t'}(p_1)$$
$$\times \frac{-i\eta_{\mu\rho}}{(p_1+p_2)^2+i0} \frac{\operatorname{tr}\left[(-ie\gamma^\rho) i\left(\not{k}+m_\mu\right)(-ie\gamma^\sigma) i\left(\not{k}+\not{p}_1+\not{p}_2+m_\mu\right)\right]}{\left(k_1+m_\mu^2+i0\right)(k+p_1+p_2)^2-m_\mu^2+i0} \frac{-i\eta_{\sigma\nu}}{(p_1+p_2)^2+i0}.$$

$$D_6 = \int \frac{d^dk}{(2\pi)^d} \bar{u}_s(p_3)(-ie\gamma^\nu) v_t(p_4) \bar{v}_{s'}(p_2)(-ie\gamma^\mu) u_{t'}(p_1)$$
$$\times \frac{-i\eta_{\mu\rho}}{(p_1+p_2)^2+i0} \frac{\operatorname{tr}\left[(-ie\gamma^\rho) i\left(\not{k}+m_\mu\right)(-ie\gamma^\sigma) i\left(\not{k}+\not{p}_1+\not{p}_2+m_\mu\right)\right]}{\left(k^2-m_\mu^2+i0\right)(k+p_1+p_2)^2-m_\mu^2+i0} \frac{-i\eta_{\sigma\nu}}{(p_1+p_2)^2+i0}$$

If you collect all factors of i, e and (-1), the common pre-factor is $i^8(-1)^6e^4 = (4\pi\alpha)^2$.

Stanford.Zhou.2022.9.5.* The scalar triangle integral. (5/45p)

a) Evaluate the scalar triangle integral $I_{\triangle}(a,b,c;p_1,p_2)$ when $p_1^2=p_2^2=0$, but for general a,b,c, and d, and give your answer in terms of $s=(p_1+p_2)^2$.

Solution:

(!! write again, from what we start?)

Introducing $s=(p_1+p_2)^2$, Feynman parameters and shifting the loop momentum $k\to k+(1-x_1-x_2)\,p_2-x_2p_1$ we find

$$I_T(a,b,c;s)=\frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)}\int\frac{d^dk}{(2\pi)^d}\int\limits_0^1 dx_1x_1^{a-1}dx_2x_2^{b-1}\left(1-x_1-x_2\right)^{c-1}\frac{1}{\left[k^2+sx_2\left(1-x_1-x_2\right)\right]^{a+b+c}}.$$

Next, we perform the Wick rotation $k^0=ik_E^0$ and integrate out k_E. We find

$$I_T(a,b,c;s)=i(-1)^{a+b+c}\pi^{d/2}(-s-i0)^{-a-b-c+\frac{d}{2}}\frac{\Gamma\left(a+b+c-\frac{d}{2}\right)}{(2\pi)^d\Gamma(a)\Gamma(b)\Gamma(c)}\int\limits_0^1 dx_1\int\limits_0^{1-x_1}dx_2$$
$$\times x_1^{a-1}x_2^{-a-c+\frac{d}{2}-1}\left(1-x_1-x_2\right)^{-a-b+\frac{d}{2}-1}$$

Next, we perform the transformation $x_2=(1-x_1)\,x_2$ and subsequently integrate out the parameter integrals over x_1 and x_2. We find

$$I_T(a,b,c;s)=i(4\pi)^{-\frac{d}{2}}(-1)^{a+b+c}(-s-i0)^{-a-b-c+\frac{d}{2}}\frac{\Gamma\left(-a-b+\frac{d}{2}\right)\Gamma\left(-a-c+\frac{d}{2}\right)\Gamma\left(a+b+c-\frac{d}{2}\right)}{\Gamma(b)\Gamma(c)\Gamma(-a-b-c+d)}.$$

b) It turns out there is a simple relation between the triangle and bubble integrals,

$$I_{\triangle}(1,1,1;p_1,p_2)|_{p_1^2=p_2^2=0}=CI_B\left(1,1;(p_1+p_2)^2\right)$$

Find the coefficient C in terms of d and s.

Solution:

By comparing our previous results, we conclude

$$C=-\frac{2(d-3)}{(d-4)s}$$

Рассеяние $e^+e^-\to\mu^+\mu^-$

(решу скоро!!)

Рассеяние $e^+e^-\to\pi^+\pi^-$

В ведущем порядке теории возмущений квантовой электродинамики вычислить дифференциальное и полное сечения элеткрон-позитронной аннигиляции в пион-антипион,

считая пионы точечными скалярными частицами: $e^+e^-\to\pi^+\pi^-$. Сравнить распределение по углами в системе центра масс с распределением в случае образования мюонов.

сечение рождения очарованных кварков

(задача их ктп киселева)

в глюон-глюоннном слиянии: g g cc. Рассмотреть синглетный и октетный по цвету вклады в сечение.

распад мюона $\mu^- \to e^- \bar{\nu}_e \nu_\mu$

Вычислить ширину трёхчастичного распада мюона на электрон и нейтрино: $\mu^- \to e^- \bar{\nu}_e \nu_\mu$

время распада заряженного пиона $\pi^- \to \mu^- \bar{\nu}_\mu$.

Вычислить время двухчастичного распада заряженного пиона: $\pi^- \to \mu^- \bar{\nu}_\mu$. Сравнить ширину распада пиона на электрон и мюон.

время распада нейтрона $n \to p e^- \bar{\nu}_e$

Распад Z-бозона на нейтрино

Вычислить ширину двухчастичного распада Z -бозона на нейтрино $Z \to \nu\bar{\nu}$

сечение $\bar{q}q \to \bar{c}c$

В ведушем порядке теории возмушений КХД вычислить сечение $\bar{q}q \to \bar{c}c$

ПШ-5.1. Кулоновское рассеяние.

Повторите решение задачи 4.4, часть (c), на этот раз используя для матричного элемента полное релятивистское выражение. Необходимо показать, что усредненное по спинам сечение равно

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4|\mathbf{p}|^2\beta^2 \sin^4(\theta/2)} \left(1 - \beta^2 \sin^2 \frac{\theta}{2}\right)$$

где $\mathbf{p}$ - 3-импульс электрона, а β - его скорость. Это формула получила название формулы Мотта для кулоновского рассеяния релятивистских электронов. Выведите эту формулу другим способом, используя сечение рассеяния электрона на мюоне, в системе покоя мюона и в пределе $m_\mu \to \infty$.

Solution

(пишу коротко, потому что есть как раз теор часть для всяких деталей этого)

ПШ-5.2. Рассеяние Баба.

Вычислите дифференциальное сечение $d\sigma/d\cos\theta$ процесса $e^+e^- \to e^+e^-$. Вычисления можно сделать в пределе $E_{\text{cm}} \gg m_e$, то есть пренебречь массой электрона. Процесс описывается двумя фейнмановскими диаграммами. Вклады от них следует сложить, прежде чем находить квадрат инвариантного матричного элемента. Убедитесь, что вы получили правильный относительный знак между вкладами диаграмм. Вычисления довольно сложны, но окончательный результат очень прост. В частности, его удобно записать, используя мандельстамовские переменные s, t, u. Заметим, что в пределе нулевой массы электрона $u + s + t = 0$. Попробуйте привести дифференциальное сечение к виду:

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{s}\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right]$$

Перепишите эту формулу через $\cos\theta$ и постройте соответствующий график. Почему сечение расходится при $\theta \to 0$?

Solution

(пишу коротко, потому что есть как раз теор часть для всяких деталей этого)

ПШ-5.3.Куча вопросов о времени жизни позитрония

(мб много сделаю параграфов из этой задачи)

(a) Вычислите амплитуду $\mathscr{M}$ аннигиляции электрон-позитронной пары с образованием двух фотонов в нерелятивистском пределе (т. е. оставьте только слагаемые, пропорциональные нулевым степеням 3-импульсов электрона и позитрона). Используя этот результат и формализм связанных состояний фермион-антифермион, исследуйте аннигиляцию 1S-состояния позитрония с образованием двух фотонов. Покажите, что состояния позитрония со спином 1 не могут аннигилировать в два фотона (что подтверждает соображения симметрии, высказанные в задаче 3.8). Для состояния позитронов со спином 0 покажите, что результат пропорционален квадрату волновой функции 1 S - состояния в нуле. Подставив волновую функцию, известную из нерелятивистской квантовой механики, нужно получить следующий ответ:

$$\frac{1}{\tau} = \Gamma = \frac{\alpha^5 m_e}{2} \approx 8.03 \times 10^9 \mathrm{c}^{-1}.$$

Последние измерения* дают $\Gamma = 7.994 \pm 0.011$ нс $^{-1}$; расхождение в 0.5% говорит о необходимости учета радиационных поправок.

(b) Вычисление вероятностей распада состояний с большими значениями l - несколько более сложная задача. Мы рассмотрим случай $l = 1$. Прежде всего, выделим слагаемые в амплитуде $e^+e^- \to 2\gamma$, которые пропорциональны первой степени 3-импульса (для простоты ведем расчет в системе центра инерции). Так как

$$\int \frac{d^3p}{(2\pi)^3} p^i \psi(\mathbf{p}) = i \frac{\partial}{\partial x^i} \psi(\mathbf{x}) \bigg|_{\mathbf{x}=0}$$

эта часть амплитуды будет «перекрываться» со связанными P-состояниями. Покажите, что состояния со спином 1 (а не со спином 0) могут аннигилировать в два фотона. Это снова есть следствие C-инвариантности.

(c) Чтобы вычислить вероятности распада P-состояний, нам нужны соответствующим образом отнормированные волновые функции. Обозначим три волновые функции P-состояний как

$$\psi_i = x^i f(|\mathbf{x}|), \quad \text{причем они нормированы условием } \int d^3x \psi_i^*(\mathbf{x}) \psi_j(\mathbf{x}) = \delta_{ij},$$

а их фурье-образы как $\psi_i(\mathbf{p})$. Покажите, что

$$|B(\mathbf{k})\rangle = \sqrt{2M} \int \frac{d^3p}{(2\pi)^3} \psi_i(\mathbf{p}) a^\dagger_{\mathbf{p}+\mathbf{k}/2} \Sigma^i b^\dagger_{-\mathbf{p}+\mathbf{k}/2} |0\rangle$$

правильно нормированный вектор связанного состояния, если Σ^i обозначает набор трех матриц 2×2, нормированных условием:

$$\sum_i \mathrm{tr}\left(\Sigma^{i\dagger}\Sigma^i\right) = 1$$

* D. W. Gidley et. al., Phys. Rev. Lett. 49, 525 (1982).

Чтобы построить состояния с $S = 1$, нужно сделать так, чтобы каждая Σ^i содержала матрицу Паули. Вообще говоря, спин-орбитальное взаимодействие расщепит мультиплет с $S = 1, L = 1$ в соответствии с полным моментом J. Состояния с определенным J таковы:

$$\begin{aligned} J = 0: &\quad \Sigma^i = \tfrac{1}{\sqrt{6}} \sigma^i \\ J = 1: &\quad \Sigma^i = \tfrac{1}{2} \varepsilon^{ijk} n^j \sigma^k, \\ J = 2: &\quad \Sigma^i = \tfrac{1}{\sqrt{3}} h^{ij} \sigma^j \end{aligned}$$

где $\mathbf{n}$ - вектор поляризации, удовлетворяющий $|\mathbf{n}|^2 = 1$, а h^{ij} - тензор с нулевым следом, типичное значение которого $h^{12} = 1$, а все остальные компоненты равняются нулю.

(d) Используя выражение для амплитуды аннигиляции, выведенное в (b), и явное выражение в (c) для состояний позитрона с $S = 1, L = 1$ и определенным J, вычислите для каждого J вероятность аннигиляции состояния с образованием двух фотонов.

ПШ-5.5. Физика массивного векторного бозона.

Введем в КЭД массивное фотонное поле B_μ массой M, которое взаимодействует с электроном по закону

$$\Delta H = \int d^3x \left(g \bar{\psi} \gamma^\mu \psi B_\mu\right)$$

Массивный фотон в начальном и конечном состоянии имеет три возможные физические поляризации, соответствующие трем пространственноподобным единичным векторам в системе покоя бозона. Они могут быть инвариантно описаны (через 4-импульс бозона k^μ) как три вектора $\varepsilon_\mu^{(i)}$, Удовлетворяющие условиям:

$$\varepsilon^{(i)} \cdot \varepsilon^{(j)} = -\delta^{ij}, \quad k \cdot \varepsilon^{(i)} = 0.$$

Четыре вектора $\left(k_\mu/M, \varepsilon_\mu^{(i)}\right)$ образуют ортогональный базис. Так как B_μ связано с сохраняющемся током $\bar{\psi}\gamma^\mu\psi$, то из тождества Уорда вытекает, что произведение k_μ на амплитуду рождения B равно нулю. Поэтому мы можем заменить

$$\sum_i \varepsilon_\mu^{(i)} \varepsilon_\nu^{(i)*} \to -g_{\mu\nu}$$

Это обобщает фейнмановский прием с векторами поляризации фотона на случай массивных бозонов и упрощает вычисление сечений процессов рождения B. (Внимание! Этот прием не работает (по крайней мере, столь просто) в случае неабелевых калибровочных полей.) Выполните некоторые вычисления, во всех случаях пренебрегая массой электрона.

(a) Вычислите сечение процесса $e^+e^- \to B$. Найдите время жизни B, предполагая, что он распадается только на электроны. Проверьте соотношение

$$\sigma\left(e^+e^- \to B\right) = \frac{12\pi^2}{M}\Gamma\left(B \to e^+e^-\right)\delta\left(M^2 - s\right)$$

(оно обсуждалось в разделе 5.3).

(b) Вычислите дифференциальное сечение процесса $e^+e^- \to \gamma + B$ в системе центра масс. (Аналогичный процесс, действительно имеющий место, - это $e^+e^- \to \gamma + Z^0$. Полученный результат позволяет измерить число распадов Z_0 в ненаблюдаемые конечные состояния, которое в свою очередь оказывается пропорциональным числу типов нейтрино.)

(c) Заметьте, что сечение из (b) расходится при $\theta \to 0$ или π. Проанализируем область вблизи $\theta = 0$. Здесь главный вклад дает t-канал. Интуитивно это соответствует тому, что фотон излучается с электронной линии до момента аннигиляции. Преобразуйте формулу так, чтобы показать, что такая интерпретация действительно правильна. Прежде всего, заметьте, что расходимость при $\theta \to 0$ обрезается электронной массой. Положите импульс электрона равным $(E, 0, 0, k)$, где $k = \left(E^2 - m_e^2\right)^{1/2}$, а импульс фотона - $k^\mu = (xE, xE\sin\theta, 0, xE\cos\theta)$. Покажите, что знаменатель пропагатора никогда не становится меньше, чем $\mathcal{O}\left(m_e^2/s\right)$. Теперь проинтегрируйте сечение из (b) по углам, обрезая интеграл по θ при $\theta^2 \sim \left(m_e^2/s\right)$ и оставляя только ведущий логарифмический вклад, пропорциональный $\ln\left(s/m_e^2\right)$. Покажите, что в этом приближении сечение для излучения фотона в направлении вперед может быть записано как

$$\sigma\left(e^+e^- \to \gamma + B\right) \approx \int dx f(x) \cdot \sigma\left(e^+e^- \to B \text{ при } E_{\text{ци}}^2 = (1-x)s\right),$$

где сечение аннигиляции вычислено для случая столкновения позитрона с энергией E и электрона с энергией $(1-x)E$, а $f(x)$, а функция распределения Вайцзеккера -Вильямса $f(x)$ равна

$$f(x) = \frac{\alpha}{2\pi}\frac{1+(1-x)^2}{x} \cdot \ln\left(\frac{s}{m_e^2}\right)$$

Эта функция всегда возникает в тех процессах, где фотон излучается коллинеарно с электронной линии. Мы встретимся с ней снова в задаче 6.2.

ПШ-5.6 О спинорах и фотонах (???)

5.6. В этой задаче формализм спинорного произведения распространяется на внешние фотоны.

(a) Пусть k - импульс фотона, а p - другой светоподобный вектор, выбранный так, что $p \cdot k \neq 0$. Введем $u_R(p)$ и $u_L(p)$ - спиноры с определенной спиральностью, соответствующие фермионам со светоподобным импульсом p, определенные в соответствии с соглашением из задачи 5.3. Определим векторы поляризации фотона:

$$\varepsilon_+^\mu(k) = \frac{1}{\sqrt{4p\cdot k}}\bar{u}_R(k)\gamma^\mu u_R(p), \quad \varepsilon_-^\mu(k) = \frac{1}{\sqrt{4p\cdot k}}\bar{u}_L(k)\gamma^\mu u_L(p)$$

Воспользуйтесь тождеством

$$u_L(p)\bar{u}_L(p) + u_R(p)\bar{u}_R(p) = \not{p}$$

и вычислите сумму по поляризациям:

$$\varepsilon_+^\mu \varepsilon_+^{\nu *} + \varepsilon_-^\mu \varepsilon_-^{\nu *} = -g^{\mu\nu} + \frac{k^\mu p^\nu + k^\nu p^\mu}{p \cdot k}.$$

Второе слагаемое справа даст нуль, если его свернуть с любой амплитудой излучения фотона $\mathscr{M}^\mu$, так что

$$|\varepsilon_+ \cdot \mathscr{M}|^2 + |\varepsilon_- \cdot \mathscr{M}|^2 = \mathscr{M}^\mu \mathscr{M}^{\nu *} \left(-g_{\mu\nu}\right);$$

Таким образом, векторы $\varepsilon_+, \varepsilon_-$можно использовать для вычисления сумм по поляризациям фотонов.
(b) Используя только что введенные поляризации и тождество Фирца из задачи 5.3, вычислите дифференциальное сечение аннигиляции безмассовых электрона и позитрона с образованием двух фотонов. Покажите, что в системе центра инерции результат совпадает с безмассовым пределом (5.107):

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha^2}{s}\left(\frac{1+\cos^2\theta}{\sin^2\theta}\right)$$

Из результатов (a) следует, что ответ не зависит от вектора p, который использовался для определения поляризационных векторов; однако вычисление сильно упрощается, если выбрать этот вектор равным первоначальному 4-вектору электрона.

27.1.2 Problems about other, special decays and scatterings

Stanford.Zhou.2022.8.2. Decays of the Higgs boson. (10/45p)

The Standard Model contains three charged leptons, the electron e, muon μ, and tau τ, which are described by Dirac fields and differ only by their mass. It also contains a spinless particle called the Higgs boson, described by a real scalar field h. The free Lagrangian is

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu h\right)\left(\partial^\mu h\right) - \frac{1}{2}m_h^2 h^2 + \sum_i \bar{\Psi}_i \left(i\partial\!\!\!/ - m_i\right)\Psi_i$$

where $i \in \{e, \mu, \tau\}$, and the numeric values of the masses are

$$m_h = 125\text{GeV}, \quad m_e = 511\text{keV}, \quad m_\mu = 105.7\text{MeV}, \quad m_\tau = 1777\text{MeV}$$

The Higgs field couples to the charged leptons by a Yukawa coupling proportional to mass,

$$\mathcal{L}_{\text{int}} = -\sum_i \frac{m_i}{v} h \bar{\Psi}_i \Psi_i$$

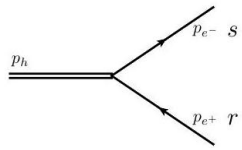
Here, v is a constant associated with the breaking of electroweak symmetry, but for this problem you will only need its value, $v = 246\text{GeV}$. For this problem, all of your final answers should be numeric, and given to at least two significant figures.
a) Compute the partial decay rate for a Higgs boson to an electron-positron pair $\Gamma_{H\to e^+e^-}$ at leading non-vanishing order in perturbation theory, giving your answer in eV. (Hint: sum over final spin states, and reuse results from problem set 5.)
b) Find the ratios $\Gamma_{H\to\mu^+\mu^-}/\Gamma_{H\to e^+e^-}$and $\Gamma_{H\to\tau^+\tau^-}/\Gamma_{H\to e^+e^-}$.
c) Find the probability a Higgs boson decays to $\tau^+\tau^-$, also known as the branching ratio

$$\text{BR}_{H\to\tau^+\tau^-} = \frac{\Gamma_{H\to\tau^+\tau^-}}{\sum_X \Gamma_{H\to X}}$$

The denominator is the total decay rate of the Higgs boson, equal to 4.1 MeV. (To check your answer, you can consult the so-called Yellow Report.)

Solution:

There is one Feynman diagram, giving matrix element

$$\mathcal{M}_{h\to e^+e^-} = \quad \text{  } \quad = \bar{u}_s(p_{e^-})\left(-i\frac{m_e}{v}\right)v_r(p_{e^+}).$$

Squaring the matrix element and summing over final spin states gives

$$\begin{aligned}
\left|\overline{\mathcal{M}}_{h\to e^+e^-}\right|^2 &= \frac{m_e^2}{v^2}\sum_{r,s}\bar{u}_s\left(p_{e^-}\right)v_r\left(p_{e^+}\right)\bar{v}_r\left(p_{e^+}\right)u_s\left(p_{e^-}\right) \\
&= \frac{m_e^2}{v^2}\,\mathrm{tr}\left[\left(\not{p}_{e^-}+m_e\right)\left(\not{p}_{e^+}-m_e\right)\right] \\
&= \frac{m_e^2}{v^2}\left(\mathrm{tr}\left[\not{p}_{e^-}\not{p}_{e^+}\right]-4m_e^2\right) \\
&= 2\frac{m_e^2}{v^2}\left(2p_{e^-}\cdot p_{e^+}-2m_e^2\right) \\
&= 2\frac{m_e^2}{v^2}\left(m_h^2-4m_e^2\right).
\end{aligned}$$

In the last line we used the fact that

$$m_h^2 = p_h^2 = \left(p_{e^-}+p_{e^+}\right)^2 = 2m_e^2+2p_{e^+}\cdot p_{e^-}.$$

The decay width of a Higgs to leptons $l^+l^-$is defined by

$$\Gamma_{h\to l^+l^-} = \frac{1}{2m_h}\int\frac{d^4p_{l^-}}{(2\pi)^4}(2\pi)\delta_+\left(p_{l^-}^2-m_l^2\right)\frac{d^4p_{l^+}}{(2\pi)^4}(2\pi)\delta_+\left(p_{l^+}^2-m_l^2\right)(2\pi)^4\delta^{(4)}\left(p_h-p_{l^-}-p_{l^+}\right)\left|\overline{\mathcal{M}}_{h\to e^+e^-}\right|^2$$

By reusing results from problem set 5, we immediately find

$$\Gamma_{h\to l^+l^-} = \frac{1}{16\pi m_h^2}\sqrt{m_h^2-4m_e^2}\left|\overline{\mathcal{M}}_{h\to e^+e^-}\right|^2 = \frac{m_e^2m_h}{8\pi v^2}\left(1-4m_e^2/m_h^2\right)^{3/2} = 0.021\text{eV}$$

Solution:

Note that since m_e, m_μ, and m_τ are all much less than m_h, we have

$$\Gamma_{h\to l^+l^-} \approx \frac{m_l^2m_h}{8\pi v^2}$$

for each charged lepton. That is, the decay rate is proportional to the mass squared, so the Higgs boson is much more likely to decay to heavier leptons,

$$\frac{\Gamma_{H\to\mu^+\mu^-}}{\Gamma_{H\to e^+e^-}} \approx \frac{m_\mu^2}{m_e^2} = 4.3\times10^4, \quad \frac{\Gamma_{H\to\tau^+\tau^-}}{\Gamma_{H\to e^+e^-}} \approx \frac{m_\tau^2}{m_e^2} = 1.2\times10^7$$

Solution:

Plugging in the numbers gives $\mathrm{BR}_{H\to\tau^+\tau^-} = 6.3\%$. (The Yellow Report gives a slightly different value because it accounts for higher-order corrections.)

The decay of Higgs bosons to taus was confirmed experimentally only recently [1][2], and first evidence for the Higgs coupling to muons has been detected [34]. Establishing the coupling of the Higgs to electrons remains a monumental experimental challenge.

Stanford.exam-2022.3/5. Pseudoscalar and vector decays. (10/40p)

In this problem we will consider decays of particles to pairs of electrons. As usual, the electron is described by a Dirac field Ψ of mass m_e. All decay rates should be computed in the rest frame of the decaying particle, and you can reuse results from problem sets.

a) Consider a pseudoscalar particle, described by a real scalar field ϕ of mass M, where $M > 2m_e$. Write down the Feynman rule for the interaction

$$\mathcal{L}_{\text{int}} = ig\phi\bar{\Psi}\gamma^5\Psi.$$

Then find the decay rate for $\phi \to e^+e^-$to leading nontrivial order in g.

Solution

: As usual, the Feynman rule comes from $-iH_{\text{int}} = iL_{\text{int}}$, so the vertex factor is $-g\gamma^5$. (This looks unusual, but it's correct: there had to be an i in L_{int} to ensure the action is real when g is.) Then

$$\mathcal{M} = -g\bar{u}_r\left(p_1\right)\gamma^5 v_s\left(p_2\right).$$

Squaring and summing over final spin states gives

$$\begin{aligned}
|\overline{\mathcal{M}}|^2 &= g^2\sum_{r,s}\bar{u}_r\left(p_1\right)\gamma^5 v_s\left(p_2\right)v_s^\dagger\left(p_2\right)\left(\gamma^5\right)^\dagger\left(\gamma^0\right)^\dagger u_r\left(p_1\right) \\
&= -g^2\,\mathrm{tr}\left(\left(\not{p}_1 + m_e\right)\gamma^5\left(\not{p}_2 - m_e\right)\gamma^5\right) \\
&= g^2\,\mathrm{tr}\left(\left(\not{p}_1 + m_2\right)\left(\not{p}_2 + m_e\right)\gamma^5\gamma^5\right) \\
&= 4g^2\left(p_1\cdot p_2 + m_e^2\right) \\
&= 2g^2M^2.
\end{aligned}$$

Plugging this into the decay rate result derived in problem set 5 gives

$$\Gamma = \frac{g^2}{8\pi}\sqrt{M^2 - 4m_e^2}.$$

b) Now consider a vector particle, described by a massive vector field A_μ of mass M, where $M > 2m_e$. Write down the Feynman rule for the interaction

$$\mathcal{L}_{\text{int}} = g\bar{\Psi}\gamma^\mu A_\mu\Psi.$$

Then find the decay rate for $A_\mu \to e^+e^-$to leading nontrivial order in g. (Hint: the Feynman rule for an incoming massive vector is $\epsilon_\mu(p)$, where ϵ_μ is its polarization. The polarization satisfies $\epsilon\cdot\epsilon = -1$ and $p\cdot\epsilon = 0$, since $\partial_\mu A^\mu = 0$. By rotational symmetry, the decay rate doesn't depend on the vector's polarization, so you can compute it by either averaging over possible polarizations using

$$\sum_\lambda \epsilon^\mu(p,\lambda)\epsilon^{*\nu}(p,\lambda) = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{M^2}$$

or by picking any specific polarization.)

Solution

The Feynman rule for the vertex is $ig\gamma^\mu$, as in ordinary QED, so

$$\mathcal{M} = ig\epsilon_\mu^\lambda\bar{u}_r\left(p_1\right)\gamma^\mu v_s\left(p_2\right).$$

Summing over final spin states and averaging over the initial three spin states, we have

$$|\overline{\mathcal{M}}|^2 = \frac{g^2}{3}\left(-\eta_{\mu\nu} + \frac{p_\mu p_\nu}{M^2}\right)\sum_{rs}\bar{u}_r\left(p_1\right)\gamma^\mu v_s\left(p_2\right)\bar{v}_2\left(p_2\right)\gamma^\nu u_r\left(p_1\right).$$

Note that the $p_\mu p_\nu$ terms won't contribute anything, as by the Dirac equation,

$$\bar{u}_r\left(p_1\right)pv_s\left(p_2\right) = \bar{u}_r\left(p_1\right)\left(\not{p}_1 + \not{p}_2\right)v_s\left(p_2\right) = \bar{u}_r\left(p_1\right)\left(m_e - m_e\right)v_s\left(p_2\right) = 0.$$

Thus, keeping just the metric term and rewriting the spinor sum as a trace,

$$\begin{aligned}|\overline{\mathcal{M}}|^2 &= -\frac{g^2}{3}\eta_{\mu\nu}\,\mathrm{tr}\left(\left(p_1+m_e\right)\gamma^\mu\left(p_2-m_e\right)\gamma^\nu\right)\\ &= -\frac{4g^2}{3}\eta_{\mu\nu}\left(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - \left(p_1\cdot p_2 - m_e^2\right)\eta^{\mu\nu}\right)\\ &= \frac{8g^2}{3}\left(p_1\cdot p_2 + 2m_e^2\right)\\ &= \frac{4g^2M^2}{3}\left(1+\frac{2m_e^2}{M^2}\right).\end{aligned}$$

Plugging this into our decay rate result gives

$$\Gamma = \frac{g^2}{12\pi}\left(1+\frac{2m_e^2}{M^2}\right)\sqrt{M^2-4m_e^2}.$$

Stanford.exam-2022.4/5. Higgs boson production at a muon collider. (15/40p)

The high energy physics community is currently debating how to best study the Higgs boson with future particle accelerators. One idea is to build a machine that collides muons and anti-muons at very high energies and produces Higgs bosons in their annihilation. You should treat the muon, Higgs, and photon in the same way as in problem set 8, and neglect any other particles or interactions. That is, consider the free Lagrangian

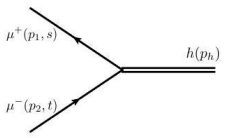
$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu h\right)^2 - \frac{1}{2}m_h^2h^2 + \bar{\Psi}\left(i\not{\partial} - m_\mu\right)\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

with interactions

$$\mathcal{L}_{\text{int}} = -\frac{m_\mu}{v}h\bar{\Psi}\Psi - e\bar{\Psi}A\Psi.$$

We first consider the case where a muon and anti-muon annihilate to produce a Higgs boson, $\mu^+\left(p_1\right)\mu^-\left(p_2\right)\to h\left(p_h\right)$.

a) Compute the scattering amplitude $\mathcal{M}_{p_1p_2\to p_h}$ for this process at lowest nontrivial order in perturbation theory, by explicitly using Wick's theorem. Then show that you get the same answer if you simply apply the Feynman rules, where the vertex is



$$= -\frac{im_\mu}{v}.$$

Solution a)

The initial and final states are

$$\begin{aligned}\left|\mu^+\left(p_1,s\right),\mu^-\left(p_2,t\right)\right\rangle &= \sqrt{2E\left(\overrightarrow{p_1}\right)}\sqrt{2E\left(\overrightarrow{p_2}\right)}b_s^\dagger\left(p_1\right)a_t^\dagger\left(p_2\right)|0\rangle,\\ |h\rangle &= \sqrt{2E\left(\overrightarrow{p_h}\right)}a^\dagger\left(p_h\right)|0\rangle.\end{aligned}$$

We can then compute for example the S-matrix element

$$\begin{aligned}\left\langle\mu^+\left(p_1,s\right),\mu^-\left(p_2,t\right)\left|\int d^4x i\mathcal{L}_{int}\right|h\right\rangle &= \left\langle\mu^+\left(p_1,s\right),\mu^-\left(p_2,t\right)\left|\int d^4x i\frac{m_h}{v}h(x)\bar{\Psi}_i(x)\Psi_i(x)\right|h\right\rangle\\ &= \frac{-im_\mu}{v}\sqrt{8E\left(p_h\right)E\left(p_1\right)E\left(p_2\right)}\left\langle 0\left|\int d^4x b_s\left(p_1\right)a_t\left(p_2\right)h(x)\bar{\Psi}_i(x)\Psi_i(x)a^\dagger\left(p_h\right)\right|0\right\rangle\end{aligned}$$

We find for example for

$$
\begin{aligned}
h(x)a^{\dagger}\left(p_h\right) &= \int \frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E(p)}}\left[a(p)e^{-ipx}+a^{\dagger}(p)e^{ipx}\right]a^{\dagger}\left(p_h\right)\\
&= a^{\dagger}\left(p_h\right)h(x)+\int \frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E(p)}}\left[a(p),a^{\dagger}\left(p_h\right)\right]e^{-ipx}\\
&= a^{\dagger}\left(p_h\right)h(x)+\int \frac{d\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E(p)}}(2\pi)^3\delta^{(3)}\left(\vec{p}-\vec{p_h}\right)e^{-ipx}\\
&= a^{\dagger}\left(p_h\right)h(x)+\frac{1}{\sqrt{2E\left(p_h\right)}}e^{-ip_hx}
\end{aligned}
$$

In our vacuum expectation value of S40, the creation operator of the scalar field then acts directly on the vacuum state on the left and annihilates it and only the commutator term remains. Similarly, we find

$$
\begin{aligned}
a_t\left(p_2\right)\Psi_i(x) &= u_i^t\left(p_2\right)\frac{e^{ip_2x}}{\sqrt{2E\left(p_2\right)}}-\Psi_i(x)a_t\left(p_2\right).\\
b_s\left(p_1\right)\bar{\Psi}_i(x) &= \bar{v}_i^s\left(p_1\right)\frac{e^{ip_1x}}{\sqrt{2E\left(p_1\right)}}-\bar{\Psi}_i(x)b_s\left(p_1\right).
\end{aligned}
$$

As

$$
\left\{b_s^{\dagger}\left(p_1\right),\Psi(x)\right\}=\left\{\bar{\Psi}_i(x),a_t^{\dagger}\left(p_2\right)\right\}=0,
$$

only the anti-commutator terms survive. We find

$$
\begin{aligned}
&\left\langle \mu^{+}\left(p_1,s\right),\mu^{-}\left(p_2,t\right)\left|\int d^4x i\mathcal{L}_{int}\right|h\right\rangle = -\frac{im_{\mu}}{v}\int d^4x e^{-ix\left(p_h-p_1-p_2\right)}\bar{v}_i^s\left(p_1\right)\delta_{ij}u_j^t\left(p_2\right)\\
&=(2\pi)^4\delta^{(4)}\left(p_h-p_1-p_2\right)\bar{v}_i^s\left(p_1\right)\left(-\frac{im_{\mu}}{v}\delta_{ij}\right)u_j^t\left(p_2\right).
\end{aligned}
$$

The scattering matrix elements are defined by

$$
S_{fi}=\delta_{fi}+(2\pi)^4\delta\left(p_f-p_i\right)\mathcal{M},
$$

and we associate $u_i^t\left(p_2\right)$ with an ingoing muon and $\bar{v}_j^s\left(p_1\right)$ with an ingoing anti-muon. We thus conclude

$$
\mathcal{M}_{p_1p_2\to p_h}=\bar{v}_i^s\left(p_1\right)\left(-i\frac{m_{\mu}}{v}\right)u_i^t\left(p_2\right)
$$

which is clearly also what we get from the Feynman rules.
The scattering cross section of particles a and b with masses m_a and m_b to produce N final state particles is

$$
\begin{aligned}
\sigma_{p_ap_b\to q_1\ldots q_N} &= \frac{1}{2\sqrt{\left(2p_ap_b\right)^2-4m_a^2m_b^2}}\int\prod_{i=1}^{N}\left(\frac{d^4q_i}{(2\pi)^4}(2\pi)\delta\left(q_i^2-m_i^2\right)\theta\left(q_i^0-m_i\right)\right)\\
&\times(2\pi)^4\delta^{(4)}\left(p_a+p_b-q_1-\cdots-q_N\right)\left|\overline{\mathcal{M}}_{p_ap_b\to q_1\ldots q_N}\right|^2
\end{aligned}
$$

where $\left|\overline{\mathcal{M}}_{p_ap_b\to q_1\ldots q_N}\right|^2$ is the modulus squared of the scattering amplitude, summed over final state spins and averaged over initial state spins.

b) Compute $\left|\overline{\mathcal{M}}_{p_1p_2\to p_h}\right|^2$ and write it in terms of the constants defined in (7) and (8).

Solution

: Using the result of part (a), we have

$$\begin{aligned}\left|\overline{\mathcal{M}}_{p_1p_2\to p_h}\right|^2 &= \frac{1}{4}\sum_{s,t}\bar{u}_i^t\left(p_2\right)\left(i\frac{m_\mu}{v}\right)v_i^s\left(p_1\right)\bar{v}_j^s\left(p_1\right)\left(-i\frac{m_\mu}{v}\right)u_j^t\left(p_2\right)\\ &= \frac{m_\mu^2}{4v^2}\,\mathrm{tr}\left[\left(\not{p}_1-m_\mu\right)\left(\not{p}_2+m_\mu\right)\right]\\ &= \frac{m_\mu^2}{4v^2}\,\mathrm{tr}\left[\not{p}_1\not{p}_2\right]-4m_\mu^2\\ &= \frac{m_\mu^2}{2v^2}\left(m_h^2-4m_\mu^2\right).\end{aligned}$$

The factor of 1/4 arises due to averaging over 2 spin states per initial state fermion.

c) Using this result and (10, compute the total cross section for this process. Your result should contain distributions - what is their interpretation?

Solution

Introducing $s=\left(p_1+p_2\right)^2$, we find

$$\begin{aligned}\sigma_{p_1p_2\to p_h} &= \frac{\pi}{\sqrt{s\left(s-4m_\mu^2\right)}}\int d^4p_h\delta\left(p_h^2-m_h^2\right)\theta\left(p_h^0-m_h\right)\delta^{(4)}\left(p_1+p_2-p_h\right)\left|\overline{\mathcal{M}}_{p_1p_2\to p_h}\right|^2\\ &= \frac{\pi}{\sqrt{s\left(s-4m_\mu^2\right)}}\delta\left(s-m_h^2\right)\theta\left(p_1^0+p_2^0-m_h\right)\left|\overline{\mathcal{M}}_{p_1p_2\to p_h}\right|^2\\ &= \frac{\pi m_\mu^2}{2v^2}\sqrt{1-\frac{4m_\mu^2}{m_h^2}}\delta\left(s-m_h^2\right)\theta\left(p_1^0+p_2^0-m_h\right).\end{aligned}$$

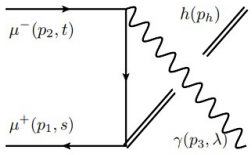
The distributions forces the center-of-mass energy s to be exactly equal to the Higgs boson mass, which makes sense by energy-momentum conservation. (There is something weird about this result, though: in physics you can never have two things be exactly equal to each other, so this would suggest Higgs bosons are never produced this way at all. The resolution is that all unstable particles actually have a slightly indefinite mass (heuristically due to the energy-time uncertainty principle applied to their finite decay time) so the delta function is replaced with a narrow peak.)

It is also possible to produce additional particles in the final state. For the rest of this problem, we consider the process $\mu^+\left(p_1\right)\mu^-\left(p_2\right)\to\gamma\left(p_3\right)h\left(p_h\right)$ at leading nontrivial order in perturbation theory.

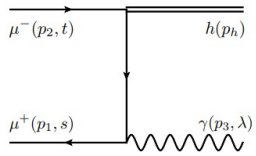
d) Draw all the relevant Feynman diagrams and use the Feynman rules to write down the corresponding amplitude $\mathcal{M}_{p_1p_2\to p_3p_h}$.

Solution

: There are two relevant diagrams:

$D_1 =$ 

$$= \bar{v}_s(p_1)\left(-i\frac{m_\mu}{v}\right)\frac{i(\not{p}_1 - \not{p}_h + m_\mu)}{(p_1-p_h)^2 - m_\mu^2}(-ie\gamma^\mu)u_t(p_2)\epsilon_\mu(p_3,\lambda).$$

$D_2 =$ 

$$= \bar{v}_s(p_1)(-ie\gamma^\mu)\frac{i(\not{p}_1 - \not{p}_3 + m_\mu)}{(p_1-p_3)^2 - m_\mu^2}\left(-i\frac{m_\mu}{v}\right)u_t(p_2)\epsilon_\mu(p_3,\lambda).$$

The amplitude is $D_1 + D_2$. All other diagrams are suppressed by more powers of m_μ/v, or more powers of e, or both.

e) Since a muon collider would operate at a center-of-mass energy much higher than the muon mass, we can treat m_μ as a small quantity. Compute $\left|\overline{\mathcal{M}}_{p_1p_2\to p_3p_h}\right|^2$ to leading nontrivial order in m_μ, and write it in terms of the Mandelstam invariants

$$s = (p_1+p_2)^2, \quad t = (p_1-p_h)^2, \quad u = (p_1-p_3)^2$$

and the constants defined in (7) and (8). (Hint: your expression will contain a sum over photon polarizations of the form

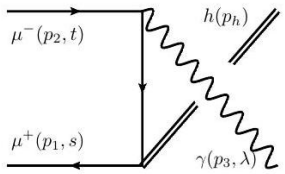
$$\sum_\lambda \epsilon^\mu(p_3,\lambda)\,\epsilon^{*\nu}(p_3,\lambda)$$

which you can simply replace with $-\eta^{\mu\nu}$.)

Solution

: First, we find the leading term in m_μ for our diagrams:

$$D_1 = \frac{-iem_\mu}{vt}\bar{s}_s(p_1)(p_1 - \not{p}_h)\gamma^\mu u_t(p_2)\epsilon_\mu(p_3,\lambda),$$
$$D_2 = \frac{-iem_\mu}{vu}\bar{v}_s(p_1)\gamma^\mu(p_h - \not{p}_2)u_t(p_2)\epsilon_\mu(p_3,\lambda).$$

$D_1 =$ 

Since these quantities already have a factor of m_μ in front, and we only want the lowest order in m_μ, we can set m_μ to zero when evaluating what's left. The Dirac equation for the spinors becomes $\bar{v}_s(p_1)\not{p}_1 = 0 = \not{p}_2 u_t(p_2)$, from which we find

$$D_1 = \frac{iem_\mu}{vt}\bar{v}_s(p_1)\not{p}_h\gamma^\mu u_t(p_2)\epsilon_\mu(p_3,\lambda),$$
$$D_2 = \frac{-iem_\mu}{vu}\bar{v}_s(p_1)\gamma^\mu p_h u_t(p_2)\epsilon_\mu(p_3,\lambda).$$

Next, we compute the terms in the square of the total amplitude,

$$\frac{1}{4}\sum_{s,t,\lambda} D_1 D_1^\dagger = -\frac{e^2 m_\mu^2}{4v^2 t^2}\,\mathrm{tr}\left[p_1 \not{p}_4 \gamma_\mu \not{p}_2 \gamma^\mu \not{p}_4\right] = \frac{e^2 m_\mu^2}{2v^2 t^2}\,\mathrm{tr}\left[\not{p}_1 \not{p}_4 \not{p}_2 \not{p}_4\right] = \frac{e^2 m_\mu^2 u}{v^2 t}.$$

$$\frac{1}{4}\sum_{s,t,\lambda} D_1 D_2^\dagger = \frac{e^2 m_\mu^2}{4v^2 tu}\,\mathrm{tr}\left[\not{p}_1 \not{p}_4 \gamma_\mu \not{p}_2 \not{p}_4 \gamma^\mu\right] = \frac{4e^2 m_\mu^2}{v^2 tu}\left(p_1 p_4\right)\left(p_2 p_4\right) = \frac{e^2 m_\mu^2}{v^2 tu}\left(m_h^2 - t\right)\left(m_h^2 - u\right).$$

$$\frac{1}{4}\sum_{s,t,\lambda} D_2 D_2^\dagger = -\frac{e^2 m_\mu^2}{4v^2 u^2}\,\mathrm{tr}\left[\not{p}_1 \gamma_\mu \not{p}_4 \not{p}_2 \not{p}_4 \gamma^\mu\right] = \frac{e^2 m_\mu^2}{2v^2 u^2}\,\mathrm{tr}\left[p_1 \not{p}_4 \not{p}_2 \not{p}_4\right] = \frac{e^2 m_\mu^2 t}{v^2 u}.$$

Above, we used the identities

$$\begin{aligned}
\gamma^\mu \not{p} \gamma_\mu &= -2\not{p} \\
\gamma^\mu \not{p} \not{q} \gamma_\mu &= 4pq \\
\mathrm{tr}[\not{p}\not{q}] &= 4pq \\
p_1 p_4 &= \frac{m_h^2 - t}{2} \\
p_2 p_4 &= \frac{m_h^2 - u}{2}, \\
m_h^2 &= s + t + u, \\
\mathrm{tr}\left[p_1 \not{p}_4 \not{p}_2 \not{p}_4\right] &= 4\left[2\left(p_1 p_4\right)\left(p_2 p_4\right) - \left(p_1 p_2\right) p_4^2\right] = 2tu
\end{aligned}$$

several of which follow from neglecting m_μ. After some algebra, we find

$$\left|\overline{\mathcal{M}}_{p_1 p_2 \to p_3 p_h}\right|^2 = \frac{1}{4}\sum_{s,t,\lambda}\left[D_1 D_1^\dagger + 2\,\mathrm{Re}\left(D_1 D_2^\dagger\right) + D_2 D_2^\dagger\right] = \frac{e^2 m_\mu^2}{v^2 tu}\left(s^2 + m_h^4\right).$$

f) Using this result and (10, compute the differential cross section $d\sigma/d\cos\theta$ for this process in the center-of-mass frame, where θ is the angle between the three-momenta of the muon and the photon. You may again work to leading nontrivial order in m_μ. Express your result in terms of s, θ, and the constants defined in (7) and (8).

Solution

Starting from 10, we have

$$\begin{aligned}
\sigma_{p_1 p_2 \to p_3 p_h} = \frac{1}{2(2\pi)^2 s}\int d^4 p_3 d^4 p_h \delta\left(p_3^2\right)\delta\left(p_h^2 - m_h^2\right)\theta\left(p_3^0\right)\theta\left(p_h^0 - m_h\right) \\
\times \delta^{(4)}\left(p_1 + p_2 - p_3 - p_h\right)\left|\overline{\mathcal{M}}_{p_1 p_2 \to p_3 p_h}\right|^2.
\end{aligned}$$

We work in the center-of-mass frame where $s = E_{CM}^2$ and do the p_h integral, giving
$\sigma_{p_1 p_2 \to p_3 p_h} = \frac{1}{2(2\pi)^2 s}\int d^4 p_3 \delta\left(p_3^2\right)\delta\left(E_{CM}^2 - 2E_{CM} p_3^0 - m_h^2\right)\theta\left(p_3^0\right)\theta\left(E_{CM} - p_3^0 - m_h\right)\left|\overline{\mathcal{M}}_{p_1 p_2 \to p_3 p_h}\right|^2$.
We parametrize the p_3 integral by writing

$$\begin{aligned}
\int d^4 p_3 \delta\left(p_3^2\right) &= \int_0^\infty dp_3^0 \frac{d\left|\vec{p}_3\right|^2}{2}\left|\overrightarrow{p_3}\right|\int d\cos\theta d\phi \delta\left(\left(p_3^0\right)^2 - \left|\vec{p}_3\right|^2\right) \\
&= \pi\int_0^\infty dp_3^0 p_3^0 \int d\cos\theta
\end{aligned}$$

We can integrate over p_3^0 using the remaining delta function,

$$\delta\left(E_{CM}^2 - 2E_{CM} p_3^0 - m_h^2\right) = \frac{1}{2E_{CM}}\delta\left(p_3^0 - \frac{1}{2}\left(E_{CM} - \frac{m_h^2}{E_{CM}}\right)\right).$$

Since we want a differential cross section, we simply drop the last integral over $\cos\theta$. Using

$$\begin{aligned}
t &= \left(p_2 - p_3\right)^2 = -2p_2 p_3 = -E_{CM} p_3^0 (1 - \cos\theta) \\
u &= \left(p_1 - p_3\right)^2 = -2p_1 p_3 = -E_{CM} p_3^0 (1 + \cos\theta)
\end{aligned}$$

we find

$$\frac{d\sigma_{p_1p_2\to p_3p_h}}{d\cos\theta} = \frac{s-m_h^2}{32s^2\pi}\left|\overline{\mathcal{M}}_{p_1p_2\to p_3p_h}\right|^2 = \frac{e^2m_\mu^2}{8\pi v^2 s^2}\frac{s^2+m_h^4}{(s-m_h^2)\left(1-\cos^2\theta\right)}.$$

Рассеяние электронов на мюонном нейтрино

Вычислить сечение рассеяния электронов на мюонном нейтрино в модели с четырёхфермионном взаимодействием: $e^-\nu_\mu \to \nu_e\mu^-$

МФТИ-Мусаев-2023.**12**/19

Найти дифференциальное сечение рассеяния электрона в кулоновском поле точечного ядра с зарядом Z.

МФТИ-Мусаев-2023.**13**/19

Показать, что амплитуды комптон-эффекта, соответствующие каждой из двух возможных фейнмановских диаграмм, порознь не являются калибровочно-инвариантными. Проверить калибровочную инвариантность суммарной амплитуды процесса.

МФТИ-Мусаев-2023.**14**/19

Показать, что при малых энергиях позитрона сечение аннигиляции пары электрон-позитрон в два фотона обратно пропорционально относительной скорости частиц, а вероятность аннигиляции не зависит от скорости.

МФТИ-Мусаев-2023.**15**/19 **Эффективный потенциал**

Получить из амплитуды рассеяния эффективный потенциал взаимодействия в борновском приближении для электронов и позитронов при обмене **a)** безмассовыми векторными бозонами (КЭД) **b)** скалярными частицами (потенциал Юкавы). Отдельно рассмотреть случаи массивных и безмассовых скалярных частиц.

МФТИ-Мусаев-2023.**16**/19 **Ограниченин на массу бозона Хиггса**

Экспериментальное значение аномального магнитного момента отличается от вычисления КЭД

$$|a_{\text{exp}} - a_{QED}| < 10^{-10}.$$

Вычислить ограничения, которые задает это значение на массу бозона Хиггса m_h и константу взаимодействия λ, если

$$H_{int} = \frac{\lambda}{\sqrt{2}}\int d^3x h\bar{\psi}\psi.$$

МФТИ-Мусаев-2023.**17**/19 **Эффект Швингера**

Получить выражение для плотности числа рожденных электрон-позитронных пар постоянным во времени однородным электрическим полем (эффект Швингера). Показать, что результат непертурбативен по напряженности поля.

27.2 Problems about Pure QED

(in spirit of LL4. there are a lot of problems in that book, so all of them will be collected here)

27.3 Problems about important methods and relations in QFT

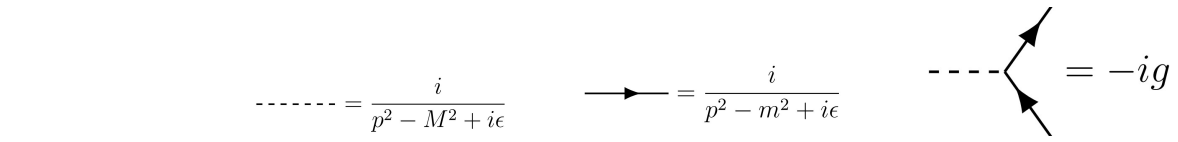
27.3.1 Problems about properties of QED (?!?!)

Stanford.Zhou.2022**5.2 Scalar Yukawa amplitudes. (7/45p)**

Let ϕ be a real scalar field of mass M, and ψ be a complex scalar field of mass m, with

$$\mathcal{L}_{\text{int}} = -g\psi^*\psi\phi$$

The field ψ can annihilate a particle, or create a particle with opposite $U(1)$ charge; these particles are conventionally called the ψ and ψ^*, respectively. Similarly, the field ϕ can create or annihilate a ϕ particle. (That is, particles are named after the field that annihilates them.) As discussed in section, the Feynman rules are



where a dashed line stands for ϕ and a solid line stands for ψ.

a) Assuming $M > 2m$, calculate the decay rate of a ϕ particle to leading order in g.

Start from equation

$$d\Gamma = \frac{1}{2m_{\mathscr{A}}}\left(\prod_f \frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\left|\mathscr{M}\left(m_{\mathscr{A}} \to \{p_f\}\right)\right|^2 (2\pi)^4\delta^{(4)}\left(p_{\mathscr{A}} - \sum p_f\right)$$

and do the phase space integrals.

b) Find the amplitude for $\psi(p_1)\,\psi^*(p_2) \to \psi(p_3)\,\psi^*(p_4)$ scattering to leading order in g, in terms of Mandelstam variables.

c) Suppose the energies we can reach in an experiment are higher than m, but much lower than M. In this case, we might not know that ϕ particles exist, since we can't produce them, so we would have to describe the scattering process in part (**b)** using an "effective" field theory in terms of ψ alone. Show that at leading order,

$$\mathcal{L}_{\text{int}} = -\lambda\psi^*\psi\psi^*\psi$$

will yield the same answer for part (**b)** for some value of λ, and find that value in terms of g and M. Assume all elements of the momenta p_i are much less than M.

Solution to a:

The Feynman diagram is just one copy of the interaction vertex, so $i\mathcal{M} = -ig$. Let the initial momentum be $k^\mu = (M, \mathbf{0})$ and the final momenta be p_1^μ and p_2^μ. Then

$$\begin{aligned}
\Gamma &= \frac{1}{2M}\int \frac{d^3\mathbf{p}_1}{(2\pi)^3 2E_1}\frac{d^3\mathbf{p}_2}{(2\pi)^3 2E_2}|-ig|^2(2\pi)^4\delta^{(4)}(k - p_1 - p_2) \\
&= \frac{g^2}{32\pi^2 M}\int \frac{d^3\mathbf{p}_1}{E_1}\frac{d^3\mathbf{p}_2}{E_2}\delta^{(4)}(k - p_1 - p_2) \\
&= \frac{g^2}{32\pi^2 M}\int \frac{d^3\mathbf{p}_1}{E_1^2}\delta(M - 2E_1) \\
&= \frac{g^2}{8\pi M}\int_0^\infty d|\mathbf{p}_1|\,\frac{|\mathbf{p}_1|^2}{E_1^2}\delta(M - 2E_1) \\
&= \frac{g^2}{8\pi M}\int_m^\infty dE_1\,\frac{|\mathbf{p}_1|}{E_1}\delta(M - 2E_1) \\
&= \frac{g^2}{16\pi M}\sqrt{1 - (2m/M)^2}.
\end{aligned}$$

Solution to b:

There are two Feynman diagrams, giving amplitude

$$i\mathcal{M} = \quad + \quad$$

$$= (-ig)^2 \left(\frac{i}{(p_1+p_2)^2 - M^2 + i\epsilon} + \frac{i}{(p_1-p_3)^2 - M^2 + i\epsilon} \right)$$

$$= -g^2 \left(\frac{i}{s - M^2 + i\epsilon} + \frac{i}{t - M^2 + i\epsilon} \right).$$

Solution to c:

The Feynman rule for the interaction in the effective theory is just

$$= -4i\lambda.$$

The factor of 4 comes about because there are 2 ways to choose which copy of ψ contracts with what, and similarly 2 ways for ψ^*.

This is also the amplitude for the scattering process in the effective theory. On the other hand, we have $s, t \ll M$, so the answer to part **(b)** becomes $2ig^2/M^2$. Equating these results gives $\lambda = -g^2/2M^2$. (The fact that λ is suppressed by powers of M is a generic feature of effective field theory.)

Stanford.Zhou.2022.5.3 Solving a trivial theory. (4/45p)

Consider a free real scalar field of mass m, but treat the mass term as the perturbation,

$$\mathcal{L}_0 = \frac{1}{2} \left(\partial_\mu \phi\right) \left(\partial^\mu \phi\right), \qquad \mathcal{L}_{\text{int}} = -\frac{1}{2} m^2 \phi^2$$

a) Write down the Feynman rules for this theory.
b) Evaluate the momentum-space propagator exactly, summing all of the (infinitely many) connected Feynman diagrams.

Solution to a:

The Feynman rules are:

propagator $\qquad$ $= \dfrac{i}{p^2 + i\epsilon}$

interaction vertex $\qquad$ $= -im^2.$

Solution to b:

The diagrams are

$$+ \quad + \quad + \ldots$$

which correspond to

$$\frac{i}{p^2+i\epsilon}+\frac{i}{p^2+i\epsilon}\left(-im^2\right)\frac{i}{p^2+i\epsilon}+\frac{i}{p^2+i\epsilon}\left(-im^2\right)\frac{i}{p^2+i\epsilon}\left(-im^2\right)\frac{i}{p^2+i\epsilon}+\ldots$$

This is a geometric series where the ratio between terms is $m^2/\left(p^2+i\epsilon\right)$, so the sum is

$$\frac{i}{p^2+i\epsilon}\frac{1}{1-\frac{m^2}{p^2+i\epsilon}}=\frac{i}{p^2-m^2+i\epsilon}$$

Of course, this is just the familiar propagator for a free massive scalar field. But still this problem is a good illustration.

Stanford.Zhou.2022**5.4 Pion scattering in the linear sigma model. (25/45p)**

Consider a theory with N free real scalar fields $\Phi_i(x)$ of equal mass m,

$$\mathcal{L}_0=\sum_{i=1}^{N}\frac{1}{2}\partial_\mu\Phi_i\partial^\mu\Phi_i-\frac{m^2}{2}\Phi_i\Phi_i$$

This Lagrangian is symmetric under rotations of the scalar fields among themselves. We can show this more clearly by defining an N-element vector $\mathbf{\Phi}$ with elements of Φ_i, so

$$\mathcal{L}_0=\frac{1}{2}\left(\partial_\mu\mathbf{\Phi}\right)^2-\frac{m^2}{2}\mathbf{\Phi}\cdot\mathbf{\Phi}$$

This is just shorthand for Eq. (6). Note that the index on Φ_i is not a spatial index (such as on ∂_i). It is a "flavor" index, meaning it just identifies which field we're talking about.

a) We quantize the theory by imposing the equal time commutation relations

$$[\Pi_i(\mathbf{x},0),\Phi_j(\mathbf{y},0)]=i\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad \Pi_i(x)=\partial_t\Phi_i(x)$$

with all other commutators vanishing. Show that the propagator is

$$\langle 0\left|T\left\{\Phi_i(x)\Phi_j(y)\right\}\right|0\rangle=\text{W. contr.}(\Phi_i(x)\Phi_j(y))=\delta_{ij}D_F(x-y).$$

where δ_{ij} is 1 if $i=j$, and 0 otherwise. The momentum space Feynman rule is

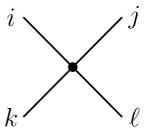
$$i\text{————}j=\frac{i\delta_{ij}}{p^2-m^2+i\epsilon}$$

where the i and j are flavor indices.

b) The linear sigma model additionally contains the interaction

$$\mathcal{L}_{\text{int}}\ =-\frac{\lambda}{4}(\mathbf{\Phi}\cdot\mathbf{\Phi})^2$$

which is also symmetric under rotations of $\mathbf{\Phi}$. Show that this interaction corresponds to the momentum space Feynman rule



$$=-2i\lambda\left(\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}\right).$$

(Hint: consider the cases where all four fields have the same flavor, and when two pairs of fields have the same flavor.)

e) Show that in terms of these fields, the Lagrangian is

$$\begin{aligned}\mathcal{L}=\frac{1}{2})\ \partial_\mu\boldsymbol{\pi})^2+\frac{1}{2}\left(\partial_\mu\sigma\right)^2-\frac{1}{2}m_\sigma^2\sigma^2\\ -\frac{\lambda_{4\pi}}{4}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})^2-\lambda_{3\sigma}\sigma^3-\frac{\lambda_{4\sigma}}{4}\sigma^4-\lambda_{\pi\sigma1}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma-\lambda_{\pi\sigma2}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma^2+C.\end{aligned}$$

and give expressions for $m_\sigma, \lambda_{4\pi}, \lambda_{3\sigma}, \lambda_{4\sigma}, \lambda_{\pi\sigma 1}, \lambda_{\pi\sigma 2}$ and C in terms of μ and λ. We see that σ is massive, but we have $N-1$ massless pion fields π_i.

f) Write the momentum-space Feynman rules in terms of μ and λ. Draw the pions with a solid line and flavor index i, and the σ particle with a double solid line. (Hint: if you've seen the pattern from the other Feynman rules in the problem set, it should be possible to write down the answer with some thought, but no explicit calculation. There is no Feynman rule corresponding to the constant C, which has no effect here.)

g) Compute the decay rate of a σ particle to leading order in λ, and give the corresponding lifetime in seconds if $\lambda = 0.1, \mu = 10\text{GeV}$ and $N = 3$.

h) Find the amplitude for $\pi_i(p_1)\,\pi_j(p_2) \to \pi_k(p_3)\,\pi_\ell(p_4)$ scattering to leading order in λ, in terms of Mandelstam variables. (Hint: there are four Feynman diagrams.)

i) Show that the amplitude in part (h) vanishes when all spatial momenta $\mathbf{p}_i$ go to zero.

j) We can see a similar phenomenon in the linear sigma model. Show that when we add a small term $a\Phi_N$ to the Lagrangian, the pion fields get a mass term m_π^2 proportional to a, and find this term. (Hint: to keep things from getting messy, work to lowest order in a as much as possible.)

Solution to a:

The commutation relations imply that the mode expansion of the field Φ_i involves a set of creation and annihilation operators $a_i^\dagger(\mathbf{k})$ and $a_i(\mathbf{k})$, whose commutators vanish for $i \neq j$. Therefore, when $i \neq j$, the contraction of two fields vanishes because Φ_i^+ and Φ_j^- automatically commute. When $i = j$, the derivation of the propagator is just the same as for a single real scalar field.

Solution to b:

Expanding out the interaction explicitly gives

$$\mathcal{L}_{\text{int}} = -\frac{\lambda}{4}\left(\sum_i \Phi_i\Phi_i\right)^2 = -\frac{\lambda}{4}\sum_{i,j}\Phi_i\Phi_i\Phi_j\Phi_j = -\frac{\lambda}{4}\sum_i \Phi_i^4 - \frac{\lambda}{2}\sum_{i<j}\Phi_i^2\Phi_j^2$$

The first term yields a vertex for four fields of the same flavor. Whenever we introduce a copy of this vertex, we get a factor of $4! = 24$ because of the ways to choose how to contract the copies of Φ_i with other things, so the overall vertex factor is $-6i\lambda$.

The second term yields a vertex for four fields where two pairs have the same flavor. These come with an extra factor of $2 \cdot 2 = 4$ because of the two copies of Φ_i and two copies of Φ_j, and thus the overall vertex factor is $-2i\lambda$.

Now let's check if this is compatible with the provided Feynman rule. When $i = j = k = \ell$, all the Kronecker deltas are equal to 1, so we get $-6i\lambda$. When $i = j$ and $k = \ell$, but $i \neq k$, then only the first Kronecker delta is equal to 1, and the others are equal to 0, so we get $-2i\lambda$.

c) Find the total cross section in the centre of mass frame for the processes

$$\Phi_1\Phi_2 \to \Phi_1\Phi_2, \quad \Phi_1\Phi_1 \to \Phi_2\Phi_2, \quad \Phi_1\Phi_1 \to \Phi_1\Phi_1$$

to leading order in λ, in terms of E_{cm}. (Hint: start from equation (4.85) of Peskin and Schroeder, and be careful with factors of 2.)

Solution to c:

The Feynman diagrams for all three of these processes are just one copy of the interaction vertex, and the amplitudes are just $-2i\lambda$, $-2i\lambda$, and $-6i\lambda$ respectively. Because all four masses are identical, we may use equation (4.85), which states

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{cm}}^2}$$

The scattering is isotropic, so the angular integrals are trivial.

For the first process, the angular integral just gives a factor of 4π, so

$$\sigma(\Phi_1\Phi_2 \to \Phi_1\Phi_2) = \frac{4\pi\lambda^2}{16\pi^2 E_{\text{cm}}^2} = \frac{\lambda^2}{4\pi E_{\text{cm}}^2}$$

For the second process, the angular integral only gives a factor of 2π because the particles in the final state are identical. (The state where the first Φ_2 exits left and the second exits right is exactly the same as the state where

the first Φ_2 exits right and the second exits left. So integrating over all possible directions for the momentum of the first Φ_2 would be double counting.) Thus,

$$\sigma\left(\Phi_1\Phi_1 \to \Phi_2\Phi_2\right) = \frac{\lambda^2}{8\pi E_{\text{cm}}^2}$$

Finally, the last cross section is 9 times bigger, giving

$$\sigma\left(\Phi_1\Phi_1 \to \Phi_1\Phi_1\right) = \frac{9\lambda^2}{8\pi E_{\text{cm}}^2}$$

In the linear sigma model, the potential energy of a uniform classical field is

$$V(\Phi) = \frac{m^2}{2}\mathbf{\Phi}\cdot\mathbf{\Phi} + \frac{\lambda}{4}(\mathbf{\Phi}\cdot\mathbf{\Phi})^2.$$

When we quantize the harmonic oscillator, the usual definition of the creation and annihilation operators in terms of x and p only makes sense if the potential's minimum is at $x = 0$. If the minimum is somewhere else, then many things go wrong. For instance, $a|0\rangle$ won't be zero, and more generally $a^\dagger$ and a won't have simple commutation relations with H, so won't properly raise and lower the energy. Similarly, it only makes sense to quantize fields in the usual way about minima of the potential $V(\Phi)$.

If $m^2 > 0$ and $\lambda > 0$, there is a unique minimum of the potential at $\mathbf{\Phi} = \mathbf{0}$, so our treatment above makes sense. Now suppose that $m^2 < 0$ and $\lambda > 0$.

d) Defining $\mu^2 = -m^2$, show that the minima of the potential are at

$$|\Phi| = v = \sqrt{\frac{\mu^2}{\lambda}}$$

The quantum theory thus has multiple vacuum states, but all have nonvanishing $\langle\mathbf{\Phi}\rangle$.

Solution to d:

Setting $dV/d|\mathbf{\Phi}|$ to zero gives $m^2|\mathbf{\Phi}| + \lambda|\mathbf{\Phi}|^3 = 0$, which yields the desired result.

By symmetry under rotations of $\mathbf{\Phi}$, we can suppose without loss of generality that we are in a vacuum state where $\langle\Phi_N\rangle = v$, with all others vanishing. Define the new fields

$$\sigma(x) = \Phi_N(x) - v, \quad \pi_i(x) = \Phi_i(x), \quad i \in \{1, 2, ..., N-1\}$$

which all have vanishing expectation values in this vacuum.

Solution to e:

This is straightforward algebra, and the answers are

$$m_\sigma = \sqrt{2}\mu, \quad \lambda_{4\pi} = \lambda, \quad \lambda_{3\sigma} = \mu\sqrt{\lambda}, \quad \lambda_{4\sigma} = \lambda$$

and

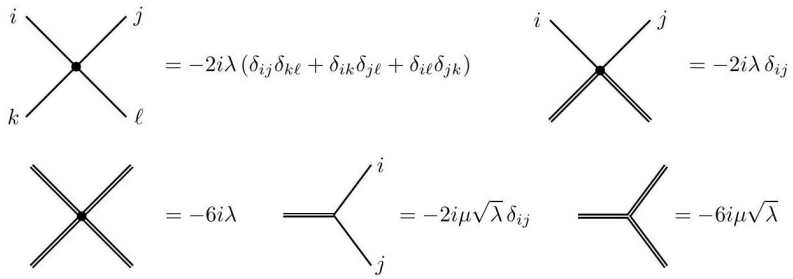
$$\lambda_{\pi\sigma 1} = \mu\sqrt{\lambda}, \quad \lambda_{\pi\sigma 2} = \frac{\lambda}{2}, \quad C = -\frac{\mu^4}{4\lambda}$$

Solution to f:

The propagators are

$$i \text{———} j = \frac{i\delta_{ij}}{p^2 + i\epsilon} \qquad \text{═══} = \frac{i}{p^2 - m_\sigma^2 + i\epsilon}$$

and the interaction vertices are



Solution to g:

The amplitude to decay to a pair of some species of pion is $i\mathcal{M} = -2i\mu\sqrt{\lambda}$, so recycling the result to problem 2(**a**) gives the decay rate

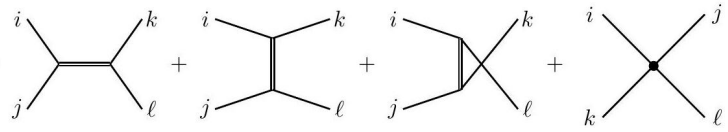
$$\Gamma = \frac{N-1}{2}\frac{(2\sqrt{\lambda}\mu)^2}{8\pi m_\sigma} = \frac{(N-1)\lambda\mu}{4\sqrt{2}\pi}.$$

where the factor of $1/2$ accounts for the two identical particles in the final state, and $N-1$ is the number of species of pion. Inverting this and converting back from natural units gives a lifetime 6×10^{-24} s.

Solution to h:

The amplitude is

$$i\mathcal{M} =$$



$$= (-2i\sqrt{\lambda}\mu)^2\left(\frac{i\delta_{ij}\delta_{k\ell}}{s-m_\sigma^2+i\epsilon} + \frac{i\delta_{ik}\delta_{j\ell}}{t-m_\sigma^2+i\epsilon} + \frac{i\delta_{i\ell}\delta_{jk}}{u-m_\sigma^2+i\epsilon}\right) - 2i\lambda\left(\delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\right)$$
$$= (-2i\lambda)\left(\frac{2\mu^2\delta_{ij}\delta_{k\ell}}{s-2\mu^2+i\epsilon} + \frac{2\mu^2\delta_{ik}\delta_{j\ell}}{t-2\mu^2+i\epsilon} + \frac{2\mu^2\delta_{i\ell}\delta_{jk}}{u-2\mu^2+i\epsilon} + \delta_{ij}\delta_{k\ell} + \delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}\right).$$

Solution to i:

In this limit, we have $s = t = u = 0$, so the denominators all become $-2\mu^2$. Then each of the first three diagrams cancels with one term from the last diagram.

This is another manifestation of Goldstone's theorem, discussed below. The general statement is that in the effective theory of pions alone (i.e. without σ), sometimes called the nonlinear sigma model, the Lagrangian only depends on the spatial derivatives $\partial_\mu\pi$ of the pion field. Since acting with a derivative on a field produces a factor of the particle momentum p^μ, the Feynman rules for the interactions always come with factors of momenta, so scattering amplitudes like the one we computed must vanish at zero momentum.

The theory above is an example of "spontaneous" symmetry breaking. The original theory has an $SO(N)$ rotational symmetry among the fields $\mathbf{\Phi}$. After fixing a vacuum state, we only have an $SO(N-1)$ rotational symmetry among the fields $\boldsymbol{\pi}$. The symmetries that rotate Φ_N into the π_i are still there, but operating with them moves us between different vacuum states; they tell us that the vacua all have the same energy. Thus, there is no energy cost for shifting a pion field π_i by a small constant, corresponding to the fact that there is no mass term for the pion fields. This is an example of a Goldstone's theorem, which states that each spontaneously broken continuous symmetry corresponds to a massless boson, called a Goldstone boson.

Pions are the lightest mesons, and mediate interactions between protons and neutrons. The linear sigma model was a phenomenological model which, among other things, explained why the pion was so light. The complete picture we have today is that pions are the Goldstone bosons corresponding to the spontaneous breaking of chiral symmetry, a symmetry of quantum chromodynamics which appears for massless up and down quarks.

In reality, the up and down quarks have small masses. This additional, "explicit" breaking of chiral symmetry explains why real pions have nonzero mass. In addition, the quarks differ in electric charge, which explains why the pions have different masses.

Solution to j:

The potential now includes $-a\Phi_N$, which tilts it in the Φ_N direction. There is now a unique vacuum state where only $\langle\Phi_N\rangle = v$ is nonzero. Setting the derivative of the potential to zero gives

$$-\mu^2 v + \lambda v^3 - a = 0$$

which rearranges to

$$\lambda v^2 = \mu^2 + \frac{a}{v}$$

To solve this at leading order in a, we replace the v on the right-hand side with the unperturbed expectation value $v_0 = \sqrt{\mu^2/\lambda}$. Solving for v then yields

$$v = \sqrt{\frac{\mu^2}{\lambda} + \frac{a}{\mu\sqrt{\lambda}}} = v_0 + \frac{a}{2\mu^2} + O\left(a^2\right)$$

To find the mass term for the pion, it's easiest to start with Eq. (17) and do a second shift, defining

$$\sigma' = \sigma - \frac{a}{2\mu^2}$$

The only term that will produce a pion mass term linear in a is $-\lambda_{\pi\sigma 1}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\sigma$, giving

$$\mathcal{L} \supset -\mu\sqrt{\lambda}(\boldsymbol{\pi}\cdot\boldsymbol{\pi})\frac{a}{2\mu^2} = -\frac{a\sqrt{\lambda}}{2\mu}\boldsymbol{\pi}\cdot\boldsymbol{\pi}$$

In other words, the pion mass is $\sqrt{a\sqrt{\lambda}/\mu}$.

Stanford.Zhou.2022 5.5.⋆ A contrived calculation. (5/45 p) (??????????????????)

In scalar ϕ^3 theory there is a diagram that contributes to the vacuum correlation function at order λ^8, shaped like a cube. Find its symmetry factor.
(This is somewhat involved, and the heuristic rules given in lecture will not be enough.)

Solution:

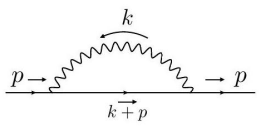
(???? вообще не знаю)
The most reliable way to do this is to simply count all the Wick contractions that yield a cube, building up the diagram one vertex at a time.
• Consider what the three copies of ϕ in the first vertex can contract with. The first has $7\cdot 3$ choices, the second has $6\cdot 3$ choices, and the third has $5\cdot 3$ choices.
• Now consider the opposite corner of the cube. There are 4 choices for which remaining uncontracted vertex is the opposite corner. Its three copies of ϕ have $3\cdot 3, 2\cdot 3$, and 3 choices for contractions.
• Finally, we attach the two halves of the cube together. There are 6 ways of choosing which vertices attach to which, and 2^6 choices for how ϕ fields are contracted to make the joining edges.
Therefore, the symmetry factor is

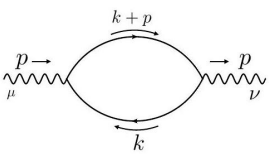
$$\frac{7\cdot 3\cdot 6\cdot 3\cdot 5\cdot 3\cdot 4\cdot 3\cdot 3\cdot 2\cdot 3\cdot 3\cdot 6\cdot 2^6}{8!\cdot(3!)^8} = \frac{1}{48}$$

This makes sense, as the symmetry group of the cube has 48 elements.

Stanford.Zhou.2022.9.4. Self-energy corrections in QED. (10/45p)

Loop corrections to the two-point correlation function play a key role in renormalization, and are often referred to as self-energy corrections. The one-loop self-energy corrections for the electron and photon in QED are given by the diagrams below.

$\Sigma(p^2) =$ 

$\Pi^{\mu\nu}(p^2) =$ 

Throughout this problem we will work in massless QED, $m_e = 0$.

a) Using the Feynman rules, write down $\Sigma\left(p^2\right)$ and $\Pi^{\mu\nu}\left(p^2\right)$. Since we are viewing these quantities as corrections to the electron and photon propagators, you should not include factors for the external legs, such as $u_s(p)$ or ϵ_μ, to get a scalar matrix element. Instead, Σ should be a 4×4 spinor matrix and $\Pi^{\mu\nu}$ a rank 2 Lorentz tensor.

Solution:

Using the Feynman rules, we simply read off the answers,

$$\Sigma\left(p^2\right) = \int \frac{d^dk}{(2\pi)^d}(-ie)\gamma^\mu \frac{i(\not{k}+\not{p})}{(k+p)^2+i0}(-ie)\gamma^\nu \frac{-ig_{\mu\nu}}{k^2+i0} = -4\pi\alpha \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu(\not{k}+\not{k})\gamma_\mu}{(k^2+i0)\left((k+p)^2+i0\right)}$$

and

$$\Pi^{\mu\nu}\left(p^2\right) = -\int \frac{d^dk}{(2\pi)^d} \frac{\operatorname{tr}\left[(-ie\gamma^\mu)\, i(\not{k}+\not{p})\,(-ie\gamma^\nu)\, i\not{k}\right]}{(k^2+i0)\left((k+p)^2+i0\right)} = -4\pi\alpha \int \frac{d^dk}{(2\pi)^d} \frac{\operatorname{tr}\left[\gamma^\mu(\not{k}+\not{p})\gamma^\nu\not{k}\right]}{(k^2+i0)\left((k+p)^2+i0\right)}$$

b) Evaluate the resulting loop integrals, expressing your final result in terms of p^2 and ϵ. (Hint: once you handle the gamma matrices, all of the integrals you get will be ones evaluated earlier in the problem set.)

Solution:

To compute Σ, we first note that $\gamma^\mu(\not{k}+\not{p})\gamma_\mu = (2-d)(\not{k}+\not{p})$. We then find that

$$\begin{aligned}
\Sigma\left(p^2\right) &= -4\pi\alpha(2-d)\left[\not{p} I_B(1,1) + \gamma_\mu I_B^\mu(1,1)\right] \\
&= -2\pi\alpha(2-d)\not{p} I_B(1,1) \\
&= \frac{\alpha}{\pi}\left(\frac{-p^2-i0}{4\pi}\right)^{-\epsilon} \not{p}\frac{i(1-\epsilon)\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{4(1-2\epsilon)\epsilon\Gamma(1-2\epsilon)}.
\end{aligned}$$

To compute $\Pi^{\mu\nu}$ we first compute the trace in the numerator.

$$\begin{aligned}
\operatorname{tr}\left[\gamma^\mu(\not{k}+\not{p})\gamma^\nu\not{k}\right] &= 4\left(2k^\mu k^\nu + p^\mu k^\nu + k^\mu p^\nu - \eta^{\mu\nu}\left(k^2 + k\cdot p\right)\right) \\
&= 4\left(2k^\mu k^\nu + p^\mu k^\nu + k^\mu p^\nu - \eta^{\mu\nu}\left(k^2 + \frac{1}{2}\left((k+p)^2 - k^2 - p^2\right)\right)\right).
\end{aligned}$$

With this we find

$$\begin{aligned}
\Pi^{\mu\nu}\left(p^2\right) &= -16\pi\alpha\left[2I_B^{\mu\nu}\left(1,1;p^2\right) + p^\mu I_B^\nu\left(1,1;p^2\right) + p^\nu I_B^\mu\left(1,1;p^2\right) + \eta^{\mu\nu}\frac{p^2}{2} I_B\left(1,1;p^2\right)\right] \\
&= -8\pi\alpha p^2 \frac{d-2}{d-1} I_B\left(1,1;p^2\right)\left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) \\
&= -\frac{\alpha}{\pi}\left(\frac{-p^2-i0}{4\pi}\right)^{-\epsilon} \frac{ip^2(1-\epsilon)\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{(1-2\epsilon)(3-2\epsilon)\epsilon\Gamma(1-2\epsilon)}\left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right).
\end{aligned}$$

c) The Euler-Mascheroni constant is defined as

$$\gamma_E = -\int_0^\infty dx e^{-x}\log x \approx 0.577$$

Show that

$$\Gamma(1-\epsilon) = 1 + \gamma_E\epsilon + O\left(\epsilon^2\right)$$

Solution:

Using the definition,

$$\begin{aligned}\Gamma(1-\epsilon) &= \int_0^\infty dx x^{-\epsilon} e^{-x} \\ &= \int_0^\infty dx e^{-\epsilon \log x} e^{-x} \\ &= \int_0^\infty dx \left(1 - \epsilon \log x + O\left(\epsilon^2\right)\right) e^{-x} \\ &= 1 + \gamma_E \epsilon + O\left(\epsilon^2\right)\end{aligned}$$

as desired.
d) Using (22), expand your results from part (b), dropping terms that vanish as $\epsilon \to 0$.

Solution:

Plugging in the above result and simplifying gives

$$\Sigma = i\frac{\alpha}{\pi}\left(\frac{-p^2 - i0}{4\pi}\right)^{-\epsilon} e^{-\epsilon\gamma_E} \not{p}\left(\frac{1}{4\epsilon} + \frac{1}{4} + \mathcal{O}(\epsilon)\right)$$

and

$$\Pi^{\mu\nu}\left(p^2\right) = i\frac{\alpha}{\pi}\left(\frac{-p^2 - i0}{4\pi}\right)^{-\epsilon} e^{-\epsilon\gamma_E}\left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\left(-\frac{1}{3\epsilon} - \frac{5}{9} + \mathcal{O}(\epsilon)\right)$$

27.3.2 Basic problems about renormalization

Экзамен. Перенормировка безмассовой теории ϕ^4

Решение:
В этой задаче мы будем изучать перенормировку безмассовой теории ϕ^4 в 4 мерном Евклидовом пространстве используя метод размерной регуляризации. Статистическая сумма нашей теории записывается как

$$Z = \int D\phi_0 \exp\left(-\int d^4x \left(\frac{1}{2}\left(\partial_\mu \phi_0\right)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4\right)\right) \tag{0.1}$$

где m_0 называется "затравочной" ("голой") массой и λ_0 затравочная константа связи. Мы вводим новое поле ϕ и константу связи $\lambda(\mu)$ по формулам

$$\begin{aligned}&\phi_0 = Z^{1/2}(\mu)\phi, \\ &m_0^2 = Z^{-1}(\mu)\delta m^2, \\ &\lambda_0 = Z^{-2}(\mu)(\lambda(\mu) + \delta\lambda(\mu)),\end{aligned} \tag{0.2}$$

где $\delta Z(\mu) = Z(\mu) - 1, \delta m^2 = Z(\mu)m_0^2$ и $\delta\lambda(\mu) = Z^2(\mu)\lambda_0 - \lambda(\mu)$ контрчлены, а μ вспомогательный масштаб. Статистическая сумма запишется как

$$Z = \int D\phi \exp\left(-\int d^4x \left(\frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{\lambda(\mu)}{4!}\phi^4 + \frac{\delta Z(\mu)}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}\delta m^2(\mu)\phi^2 + \frac{\delta\lambda(\mu)}{4!}\phi^4\right)\right). \tag{0.3}$$

Ренормализационные условия в данном случае записываются как

$$\Gamma^{(2)}(0) = 0,$$
$$\left.\frac{d\Gamma^{(2)}}{dp^2}\right|_{p^2=\mu^2} = 1,$$
$$\left.\Gamma^{(4)}\right|_{p_i=\mu e_i} = \lambda(\mu), \tag{0.4}$$

где $e_1, e_2, .., e_4$4-вектора, такие, что $e_1+e_2+e_3+e_4 = 0$ и $p_i^2 = 3\mu^2, p_i \cdot p_j = -\mu^{21}$. Мы опять предполагаем, что все контрчлены могут быть вычисленны шаг за шагом по степеням $\lambda(\mu)$:

$$\delta Z(\mu) = z^{(2)}(\mu)\lambda^2(\mu) + ...,$$
$$\delta m^2(\mu) = b^{(1)}(\mu)\lambda(\mu) + ...$$
$$\delta\lambda(\mu) = a^{(2)}(\mu)\lambda^2(\mu) + ..., \tag{0.5}$$

с коэффициентами, которые определяются из наших ренормализационных условий.
В первой петле для $\Sigma\left(p^2\right)$ мы имеем:

Найдем коэффициенты $z^{(1)}$ и $b^{(1)}$ для контчленов δZ и δm^2, исходя из ренормализационных условий $\Sigma(0) = \Sigma'(0) = 0$. Получаем

$$-\Sigma\left(p^2\right) = -\frac{\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2} - \lambda\left(p^2z^{(1)} + b^{(1)}\right) = -\lambda\left(p^2z^{(1)} + b^{(1)}\right) \tag{0.6}$$

(Например можно взять $e_1 = (\sqrt{3}, 0, 0, 0), e_2 = \left(-\frac{1}{\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}}, 0, 0\right), e_3 = \left(-\frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}, \sqrt{2}, 0\right), e_4 = \left(-\frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}, -\sqrt{2}, 0\right)$.)
Из уравнений $\Sigma(0) = 0$ и $\Sigma'\left(\mu^2\right) = 0$ мы получаем

$$b^{(1)}(\mu) = 0, \quad z^{(1)}(\mu) = 0. \tag{0.7}$$

Во втором порядке по λ^2, для $\Gamma^{(2)}$ мы рассматриваем три диаграммы:
$-\Sigma\left(p^2\right) =$

+
Первая диаграмма равна

$$\Sigma_a\left(p^2\right) = -\frac{\lambda^2(\mu)}{3!}\int\frac{d^dkd^dq}{(2\pi)^{2d}}\frac{1}{k^2q^2(p+k+q)^2} = -\frac{\lambda^2(\mu)}{3!}\int\frac{d^dkd^dq}{(2\pi)^{2d}}\int\limits_0^\infty dt_1dt_2dt_3e^{-k^2t_1-q^2t_2-(p+k+q)^2t_3} =$$

$$= -\frac{\lambda^2(\mu)}{3!}\int\limits_0^\infty dt_1dt_2dt_3\int\frac{d^dkd^dq}{(2\pi)^{2d}}e^{-k^2(t_1+t_3)-2k(q+p)t_3-q^2(t_2+t_3)-2pqt_3-p^2t_3} =$$

$$= -\frac{\lambda^2(\mu)}{3!}\int\limits_0^\infty dt_1dt_2dt_3\int\frac{d^dq}{(2\pi)^{2d}}e^{-q^2(t_2+t_3)-2pqt_3-p^2t_3+\frac{(q+p)^2t_3^2}{t_1+t_3}}\int d^dke^{-(t_1+t_3)\left(k+\frac{(q+p)t_3}{t_1+t_3}\right)^2} =$$

$$= -\frac{\lambda^2(\mu)}{3!}\int\limits_0^\infty dt_1dt_2dt_3\int\frac{d^dq}{(2\pi)^{2d}}e^{-\frac{q^2(t_1t_2+t_2t_3+t_1t_3)+2pqt_1t_3+p^2t_1t_3}{t_1+t_3}}\left(\frac{\pi}{t_1+t_3}\right)^{d/2} =$$

$$= -\frac{\lambda^2(\mu)}{3!}\int\limits_0^\infty dt_1dt_2dt_3\frac{1}{(2\pi)^{2d}e^{-p^2\frac{t_1t_2t_3}{t_1t_2+t_2t_3+t_1t_3}}\left(\frac{\pi(t_1+t_3)}{t_1t_2+t_2t_3+t_1t_3}\right)^{d/2}\left(\frac{\pi}{t_1+t_3}\right)^{d/2t_i\to t_i/p^2}} =$$

$$= -\frac{\lambda^2(\mu)\left(p^2\right)^{d-3}}{3!(4\pi)^d}\int\limits_0^\infty\frac{dt_1dt_2dt_3}{\left(t_1t_2+t_2t_3+t_1t_3\right)^{d/2}}e^{-\frac{t_1t_2t_3}{t_1t_2+t_2t_3+t_1t_3}} =$$

$$= -\frac{\lambda^2(\mu)\left(p^2\right)^{d-3}}{3!(4\pi)^d}\frac{\Gamma-d)\Gamma^3(d/2-1)}{\Gamma(3d/2-3)}. \tag{0.8}$$

Также для второй и третьей диаграммы мы получаем

$$\Sigma_b\left(p^2\right) = -\frac{\delta\lambda(\mu)}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2} = 0$$
$$\Sigma_c\left(p^2\right) = -\left(p^2\delta Z(\mu)+\delta m^2(\mu)\right) = -\lambda^2(\mu)\left(z^{(2)}(\mu)p^2+b^{(2)}(\mu)\right) \tag{0.9}$$

Из первого ренормализационного условия $\Gamma^{(2)}(0)=0$ получаем $\left(\Gamma^{(2)}=p^2+\Sigma\left(p^2\right)\right)$:

$$b^{(2)}(\mu) = 0 \tag{0.10}$$

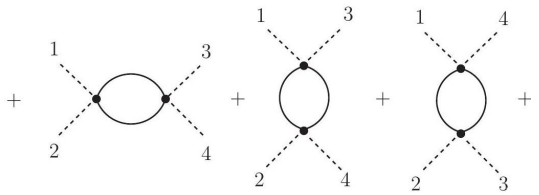
Из второго ренормализационного условия $\left.\frac{d\Gamma^{(2)}}{dp^2}\right|_{p^2=\mu^2}=1$ получаем:

$$z^{(2)}(\mu) = \frac{\left(\mu^2\right)^{d-4}\Gamma(4-d)\Gamma^3(d/2-1)}{3!(4\pi)^d\Gamma(3d/2-3)}, \tag{0.11}$$

откуда получим при $d=4-\epsilon$ и $\epsilon\to 0$:

$$\Gamma^{(2)}\left(p^2\right) = p^2+\Sigma_a\left(p^2\right)+\Sigma_b\left(p^2\right)+\Sigma_c\left(p^2\right) = p^2+\frac{1}{12}\frac{\lambda^2}{(4\pi)^4}p^2\left(\log\left(\mu^2/p^2\right)+1\right)+O\left(\lambda^3\right) \tag{0.12}$$

$-\Gamma^{(4)}\left(p_1,p_2,p_3,p_4\right) =$



Для $\Gamma^{(4)}\left(p_1,p_2,p_3,p_4\right)$ имеем получаем

$$\Gamma^{(4)}\left(p_1,p_2,p_3,p_4\right) = \lambda(\mu)-\frac{\lambda^2(\mu)}{2}\left(I\left(p_{12}^2\right)+I\left(p_{13}^2\right)+I\left(p_{14}^2\right)\right)+\lambda^2(\mu)a^{(2)}(\mu)+O\left(\lambda^3\right), \tag{0.13}$$

где $p_{12}=p_1+p_2, p_{13}=p_1+p_3, p_{14}=p_1+p_4$ и

$$I\left(q^2\right)=\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2(q+k)^2}=\int\frac{d^dk}{(2\pi)^d}\int\limits_0^1dx\frac{1}{\left(x(q+k)^2+(1-x)k^2\right)^2}=$$

$$=\int\limits_0^1dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2+2xqk+xq^2\right)^2}=\int\limits_0^1dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left((k+xq)^2+x(1-x)q^2\right)^2}=$$

$$=\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\int\limits_0^1dx\left(x(1-x)q^2\right)^{\frac{d}{2}-2}=\frac{\left(q^2\right)^{\frac{d}{2}-2}}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(2-\frac{d}{2}\right)\Gamma^2\left(\frac{d}{2}-1\right)}{\Gamma(d-2)}. \tag{0.14}$$

Далее из условия $\Gamma^{(4)}\big|_{p_i=\mu e_i}=\lambda(\mu)$, где $e_1,e_2,..,e_4$4-вектора, такие, что $e_1+e_2+e_3+e_4=0$ и $p_i^2=3\mu^2, p_i\cdot p_j=-\mu^2$, находим $\left(p_{12}^2=p_{13}^2=p_{14}^2=4\mu^2\right)$:

$$a^{(2)}(\mu)=\frac{3}{2}I\left(4\mu^2\right) \tag{0.15}$$

Затем находим, что

$$I\left(q^2\right)-I\left(\mu^2\right)\overset{d\to4}{=}-\frac{1}{(4\pi)^2}\log\frac{q^2}{4\mu^2} \tag{0.16}$$

И таким образом мы получаем

$$\Gamma^{(2)}\left(p^2\right)=p^2+\frac{1}{12}\frac{\lambda^2}{(4\pi)^4}p^2\left(\log\left(\mu^2/p^2\right)+1\right)+O\left(\lambda^3\right),$$
$$\Gamma^{(4)}\left(p^2\right)=\lambda+\frac{1}{2}\frac{\lambda^2}{(4\pi)^2}\left(\log\left(p_{12}^2/4\mu^2\right)+\log\left(p_{13}^2/4\mu^2\right)+\log\left(p_{14}^2/4\mu^2\right)\right)+O\left(\lambda^3\right), \tag{0.17}$$

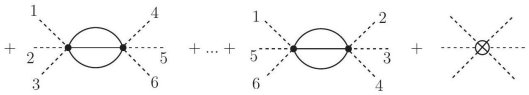
где $p_{12}^2=\left(p_1+p_2\right)^2$ и т.д.

2 Бета-функция безмассовой теории ϕ^6

Рассмотрим безмассовую теорию ϕ^6 в $d=3$ Евклидовом пространстве. Действие теории

$$S=\int d^3x\left(\frac{1}{2}\left(\partial_\mu\phi\right)^2+\frac{\lambda}{6!}\phi^6+\text{ контрчлены }\right). \tag{0.18}$$

Вычислим ведущий член в бета-функции $\beta(\lambda)$. Для $\Gamma^{(6)}\left(p_1,...,p_6\right)$ имеем
$-\Gamma^{(6)}\left(p_1,...,p_6\right)=$



И мы получаем

$$\Gamma^{(6)}\left(p_1,...,p_6\right)=\lambda-\frac{(-\lambda)^2}{6}\left(J\left(p_{123}^2\right)+...+J\left(p_{156}^2\right)\right)+\lambda^2a^{(2)}+O\left(\lambda^3\right) \tag{0.19}$$

где $p_{ijk}=p_i+p_j+p_k$ и в сумме имеется 10 слагаемых вида $J\left(p^2\right)$ и

$$J\left(p^2\right)=\int\frac{d^dqd^dk}{(2\pi)^{2d}}\frac{1}{k^2q^2(p+k+q)^2}=\frac{\left(p^2\right)^{d-3}}{(4\pi)^d}\frac{\Gamma(3-d)\Gamma^3(d/2-1)}{\Gamma(3d/2-3)}. \tag{0.20}$$

Теперь используя ренормальзационное условие

$$\Gamma^{(6)}\left(p_1,...,p_6\right)\Big|_{p_i=\mu e_i}=\lambda(\mu), \tag{0.21}$$

где $e_1, ..., e_6$ - какие-то произвольные 3 -вектора с условием $\sum_{i=1}^{6} e_i = 0$, мы найдем, что

$$a^{(2)}(\mu) = \frac{1}{6}\left(J\left(\mu^2 e_{123}^2\right) + ... + J\left(\mu^2 e_{156}^2\right)\right) \tag{0.22}$$

Далее замечая, что

$$J\left(p^2\right) - J\left(\mu^2\right) \overset{d\to 3}{=} -\frac{1}{2(4\pi)^2}\log\frac{p^2}{\mu^2} \tag{0.23}$$

получим

$$\Gamma^{(6)}\left(p_1, ..., p_6\right) = \lambda + \frac{\lambda^2}{12(4\pi)^2}\left(\log\frac{p_{123}^2}{\mu^2 e_{123}^2} + ... + \log\frac{p_{156}^2}{\mu^2 e_{156}^2}\right) + O\left(\lambda^3\right) \tag{0.24}$$

Далее подставляя $\Gamma^{(6)}\left(p_1, ..., p_6\right)$ в уравнение Каллана-Симанчика:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} - 6\gamma(\lambda)\right)\Gamma^{(6)}\left(p_1, ..., p_6\right) = 0 \tag{0.25}$$

находим

$$-\frac{10\lambda^2}{12(4\pi)^2}\left(\mu\frac{\partial}{\partial\mu}\log\mu^2\right) + \beta(\lambda) - 6\gamma(\lambda)\lambda + O\left(\lambda^3\right) = 0 \tag{0.26}$$

затем, используем, что $\gamma(\lambda) = O\left(\lambda^2\right)^2$, а следовательно слагаемое $\gamma(\lambda)\lambda = O\left(\lambda^3\right)$ и может не рассматриваться во втором порядке по λ, поэтому мы в итоге получаем:

$$\beta(\lambda) = \frac{10\lambda^2}{12(4\pi)^2}\left(\mu\frac{\partial}{\partial\mu}\log\mu^2\right) = \frac{5\lambda^2}{48\pi^2}. \tag{0.27}$$

(Строго говоря, это нужно было бы проверить, но в данном случае мы оставим это как упражнение.)

3 Задача 11.1 из Пескина-Шредера.

(a). Итак у нас есть свободное скалярное поле $\phi(x)$ с пропагатором

$$\langle T\phi(x)\phi(0)\rangle = D(x). \tag{0.28}$$

и конечно мы можем использовать теорему Вика. Можно использовать комбинаторные методы, но наипростейший путь использовать функциональный интеграл

$$\begin{aligned}
&\left\langle Te^{i\phi(x_0)}e^{-i\phi(0)}\right\rangle = \int D\phi e^{i(\phi(x_0)-\phi(0))}e^{\int d^dxd^dy\frac{1}{2}\phi(x)D^{-1}(x-y)\phi(y)} = \\
&= \int D\phi e^{\int d^dxd^dy\frac{1}{2}\phi(x)D^{-1}(x-y)\phi(y)+i\int d^dx\phi(x)(\delta(x-x_0)-\delta(x))} = \\
&= e^{-\frac{1}{2}\int d^dxd^dy(\delta(x-x_0)-\delta(x))D(x-y)(\delta(y-x_0)-\delta(y))} = e^{-\frac{1}{2}\int d^dx(\delta(x-x_0)-\delta(x))(D(x-x_0)-D(x))} = \\
&= e^{-\frac{1}{2}((D(0)-D(x_0)-(D(-x_0)-D(0)))} = e^{D(x_0)-D(0)}.
\end{aligned} \tag{0.29}$$

(b). У нас есть только производные ϕ, т.е. члены вида $\lambda_{n_1...n_m}^{k_1...k_m}\left(\nabla^{n_1}\phi\right)^{k_1}...\left(\nabla^{n_m}\phi\right)^{k_m}$. И мы находим ($\phi$ есть фаза и поэтому $[\phi] = 1$)

$$[\phi] = 1, \quad [\rho] = L^{2-d}, \quad \left[\lambda_{n_1...n_m}^{k_1...k_m}\right] = L^{\sum_i k_i n_i - d} \tag{0.30}$$

Принимая во внимание то, что $n_i, k_i \geqslant 1$ можно показать, что

$$P = \sum_{i=1}^{m} k_i n_i - d \geqslant m - d \tag{0.31}$$

Если $m = 1$, мы получим линейный член, что есть просто источник и это не интересно. Для $m = 2$ мы имеем $P = 2 - d$, только когда $k_1 = k_2 = n_1 = n_2 = 1$ и в этом случае мы получим $\rho(\nabla\phi)^2$. В остальных случаях мы имеем $P > 2 - d$, поэтому эти члены ростут быстрее, чем член с ρ, и они будут релевантными.

(c). Мы имеем

$$\langle s(x)s^*(0)\rangle = \left\langle Ae^{i\phi(x)}Ae^{-i\phi(0)}\right\rangle = A^2e^{D(x)-D(0)} \tag{0.32}$$

и для Лагранжиана $L = \frac{1}{2}\rho(\nabla\phi)^2$ получим

$$D(x) = \frac{1}{\rho}\int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2}e^{ikx} = \frac{1}{\rho}\int\frac{d^dk_E}{(2\pi)^d}\frac{1}{k_E^2}e^{-ik_Ex} = \frac{1}{\rho}\int\frac{d^dk_E}{(2\pi)^d}\int\limits_0^\infty dte^{-tk_E^2-ik_Ex} =$$

$$= \frac{1}{\rho}\int\limits_0^\infty dt\frac{t^{-d/2}}{(4\pi)^{d/2}}e^{-\frac{x^2}{4t}} \overset{t=1/s}{=} \frac{1}{(4\pi)^{d/2}\rho}\int\limits_0^\infty dss^{d/2-2}e^{-s\frac{x^2}{4}} = \frac{\Gamma(d/2-1)}{4\rho\pi^{d/2}|x|^{d-2}}. \tag{0.33}$$

Мы предполагаем, что $D(0) = \frac{\Gamma(d/2-1)}{4\rho\pi^{d/2}|a|^{d-2}}$, где a - размер решетки. Поэтому получаем

$$\langle s(x)s^*(0)\rangle = \left\langle Ae^{i\phi(x)}Ae^{-i\phi(0)}\right\rangle = A^2\exp\left(\frac{\Gamma(d/2-1)}{4\rho\pi^{d/2}}\left(\frac{1}{|x|^{d-2}} - \frac{1}{|a|^{d-2}}\right)\right). \tag{0.34}$$

В итоге мы можем написать для $d > 2$:

$$\langle s(x)s^*(0)\rangle = A^2e^{-\frac{\Gamma(d/2-1)}{4\rho\pi^{d/2}}\frac{1}{|a|^{d-2}}}\exp\left(\frac{\Gamma(d/2-1)}{4\rho\pi^{d/2}}\frac{1}{|x|^{d-2}}\right). \tag{0.35}$$

Для $d = 2$ мы имеем $D(x) = -\frac{1}{2\pi\rho}\log|x|$, поэтому:

$$\langle s(x)s^*(0)\rangle = \frac{A^2|a|^{2\pi\rho}}{|x|^{2\pi\rho}}. \tag{0.36}$$

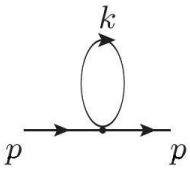
Белавин-Пархоменко.10. Кактусные диаграммы Фейнмана

В этой задаче мы будем изучать перенормировку массы в теории ϕ^4 в 3 мерном Евклидовом пространстве, на примере кактусных диаграмм (также они часто называются "daisy" диаграммы в англоязычной литературе). Статистическая сумма нашей теории записывается как

$$Z = \int D\phi\exp\left(-\int d^3x\left(\frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}m_0^2\phi^2 + \frac{\lambda}{4!}\phi^4\right)\right) \tag{0.1}$$

где m_0 называется "затравочной")"голой") массой.

(a). (30 баллов) Рассмотрим поправку первого порядка к пропагатору поля ϕ. А именно рассмотрим первую диаграмму, которая дает вклад в собственную энергетическую часть $\tilde{\Sigma}(p)$:

$-\tilde{\Sigma}(p) =$ 

$= -\frac{\lambda}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2+m_0^2}$

Как мы видим, данный интеграл расходится. Устранить данную расходимость мы можем многими способами, например введя регулятор $\Phi_{\Lambda^2}\left(k^2\right)$:

$$\tilde{\Sigma}(p) = \frac{\lambda}{2}\int\frac{d^3k}{(2\pi)^3}\frac{1}{k^2+m_0^2}\Phi_{\Lambda^2}\left(k^2\right) \tag{0.2}$$

Другой способ устранения расходимости, есть по сути выбор регулятора в виде $\Phi_{\Lambda^2}\left(k^2\right) = \theta\left(\Lambda^2-k^2\right)$, где $\theta(x)$ - тета-функция Хевисайда ("ступенька"). Тогда мы схематически можем записать

$$\tilde{\Sigma}(p) = \frac{\lambda}{2}\int\limits^\Lambda\frac{d^3k}{(2\pi)^3}\frac{1}{k^2+m_0^2} \tag{0.3}$$

Второй оригинальный способ устранить расходимость называется регуляризацией Паули-Вилларса, а именно мы записываем

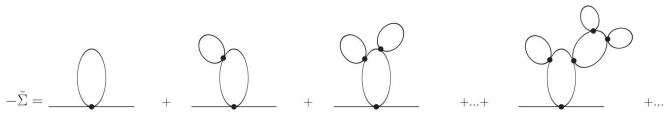
$$\frac{1}{k^2+m_0^2} \to \frac{1}{k^2+m_0^2} - \frac{1}{k^2+M^2} \tag{0.4}$$

где $M \gg m$. И последний способ, который мы рассмотрим в данном примере называется размерной регуляризацией: идея состоит в том, что мы можем рассмотреть интеграл по k в пространстве произвольной размерности d:

$$I(d) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2+m_0^2} \tag{0.5}$$

Очевидно, что при $d = 1$, данный интеграл сходится. Далее оказывается, что для данного интеграла $I(d)$ можно найти общий ответ в виде аналитической функции при любом действительном
d[1]. Замечательным свойством является то, что мы можем вычислить эту функцию при $d \to 3$ и получить конечный "регуляризованный" ответ. Вычислите собственную энергетическую часть $\tilde{\Sigma}(p)$ для всех трех регуляризаций.

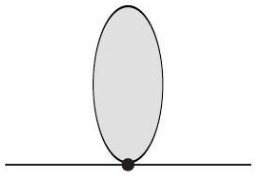
(b). Теперь рассмотрим бесконечный ряд "кактусных" диаграмм, которые дают вклад в $\tilde{\Sigma}(p)$:



Очевидно, что это далеко не все диаграммы которые дают вклад в $\tilde{\Sigma}(p)$, а только лишь их некоторое ничтожно малое подмножество [2]. Найдите сумму всех таких диаграмм. Вы должны получить самосогласованное уравнение (уравнение Дайсона) на $\tilde{\Sigma}(p)$ вида (как мы видим в данном случае $\tilde{\Sigma}$ не зависит от внешнего импульса p):

$$\tilde{\Sigma} = \frac{\lambda}{2} \int \frac{d^3 k}{k^2 + m_0^2 + \tilde{\Sigma}} \tag{0.6}$$

Нарисуйте данное самосогласованное уравнение на диаграммном языке, введя понятие перенормированного "кактуса", который по сути и есть $-\tilde{\Sigma}(p)$:



(c). Найдите $m^2 - m_0^2 = \tilde{\Sigma}(p)$ из уравнения (0.6), вычисляя интеграл с применением регуляризаций из пункта (a). Как отличаются ответы при разных регуляризациях?

Solution кактусных диаграмм Фейнмана

В этой задаче мы будем изучать перенормировку массы в теории ϕ^4 в 3 мерном Евклидовом пространстве, на примере кактусных диаграмм (также они часто называются "daisy" диаграммы в англоязычной литературе). Статистическая сумма нашей теории записывается как

$$Z = \int D\phi \exp\left(-\int d^3x \left(\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m_0^2 \phi^2 + \frac{\lambda}{4!}\phi^4\right)\right) \tag{0.1}$$

где m_0 называется "затравочной")"голой") массой.

(a). (30 баллов) Рассмотрим поправку первого порядка к пропагатору поля ϕ. А именно рассмотрим первую диаграмму, которая дает вклад в собственную энергетическую часть $\tilde{\Sigma}(p)$:

[1] Мы это уже делали в Листке 2, когда вычисляли эффект Казимира: там был интеграл, который сводился к произведению Гамма функций, на самом деле формально данный интеграл не сходился, но тем не менее мы получили для него ответ в виде аналитичекой фунции.

[2] Существуют теории, в которых данное подмножество "кактусных" диаграмм дает основной вклад в $\tilde{\Sigma}(p)$, но в нашем случае ϕ^4 это не так.

$$-\tilde{\Sigma}(p) = \quad \underset{p}{\longrightarrow} \overset{k}{\bigcirc} \underset{p}{\longrightarrow} \quad = -\frac{\lambda}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2+m_0^2}$$

Как мы видим, данный интеграл расходится. Устранить данную расходимость мы можем многими способами, например введя регулятор $\Phi_{\Lambda^2}\left(k^2\right)$:

$$\tilde{\Sigma}(p) = \frac{\lambda}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2+m_0^2}\Phi_{\Lambda^2}\left(k^2\right) \tag{0.2}$$

Другой способ устранения расходимости, есть по сути выбор регулятора в виде $\Phi_{\Lambda^2}\left(k^2\right) = \theta\left(\Lambda^2 - k^2\right)$, где $\theta(x)$ - тета-функция Хевисайда ("ступенька"). Тогда мы схематически можем записать

$$\tilde{\Sigma}(p) = \frac{\lambda}{2}\int^{\Lambda} \frac{d^3k}{(2\pi)^3}\frac{1}{k^2+m_0^2} \tag{0.3}$$

Второй оригинальный способ устранить расходимость называется регуляризацией Паули-Вилларса, а именно мы записываем

$$\frac{1}{k^2+m_0^2} \to \frac{1}{k^2+m_0^2} - \frac{1}{k^2+M^2} \tag{0.4}$$

где $M \gg m$. И последний способ, который мы рассмотрим в данном примере называется размерной регуляризацией: идея состоит в том, что мы можем рассмотреть интеграл по k в пространстве произвольной размерности d:

$$I(d) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m_0^2} \tag{0.5}$$

Очевидно, что при $d = 1$, данный интеграл сходится. Далее оказывается, что для данного интеграла $I(d)$ можно найти общий ответ в виде аналитической функции при любом действительном
d[1]. Замечательным свойством является то, что мы можем вычислить эту функцию при $d \to 3$ и получить конечный "регуляризованный" ответ. Вычислим собственную энергетическую часть $\tilde{\Sigma}(p)$ для всех трех регуляризаций. Имеем

$$\begin{aligned}
\tilde{\Sigma}(p) &= \frac{\lambda}{2}\frac{4\pi}{8\pi^3}\int_0^{\Lambda}\frac{k^2dk}{k^2+m_0^2} = \frac{\lambda}{4\pi^2}\left(\int_0^{\Lambda}dk - \int_0^{\Lambda}\frac{m_0^2}{k^2+m_0^2}dk\right) = \\
&= \frac{\lambda}{4\pi^2}\left(\Lambda - m_0\arctan\left(\frac{k}{m_0}\right)\Big|_0^{\Lambda}\right) = \frac{\lambda}{4\pi^2}\left(\Lambda - m_0\arctan\left(\frac{\Lambda}{m_0}\right)\right) \approx \\
&\approx \frac{\lambda}{4\pi^2}\left(\Lambda - \frac{\pi}{2}m_0 + \frac{m_0^2}{\Lambda} - \frac{m_0^4}{3\Lambda^3}\right) = \frac{\lambda}{4\pi^2}\Lambda - \frac{\lambda}{8\pi}m_0 + \frac{\lambda}{4\pi^2}\frac{m_0^2}{\Lambda} - \frac{\lambda}{12\pi^2}\frac{m_0^4}{\Lambda^3} + O\left(1/\Lambda^5\right).
\end{aligned} \tag{0.6}$$

Далее имеем:

$$\begin{aligned}
\tilde{\Sigma}(p) &= \frac{\lambda}{4\pi^2}\int_0^{\Lambda}k^2dk\left(\frac{1}{k^2+m_0^2} - \frac{1}{k^2+M^2}\right) = \frac{\lambda}{4\pi^2}\left(M\arctan\frac{k}{M} - m_0\arctan\frac{k}{m_0}\right)\Big|_0^{\infty} = \\
&= \frac{\lambda}{4\pi^2}\left(\frac{\pi}{2}M - \frac{\pi}{2}m_0\right) = \frac{\lambda}{8\pi}M - \frac{\lambda}{8\pi}m_0.
\end{aligned} \tag{0.7}$$

И в последнем случае получаем:

$$\tilde{\Sigma}(p) = \frac{\lambda}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m_0^2} = \frac{\lambda}{2}\frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\frac{1}{(2\pi)^d}\int\limits_0^\infty \frac{k^{d-1}dk}{k^2+m_0^2} = \frac{\lambda}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\int\limits_0^\infty dt\int\limits_0^\infty dk k^{d-1}e^{-t\left(k^2+m_0^2\right)} =$$

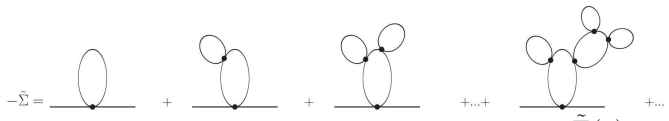
$$= \frac{\lambda}{(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\int\limits_0^\infty dte^{-tm_0^2}\int\limits_0^\infty \frac{dx}{2t}(x/t)^{\frac{d-2}{2}}e^{-x} = \frac{\lambda}{2(4\pi)^{d/2}\Gamma\left(\frac{d}{2}\right)}\int\limits_0^\infty dte^{-tm_0^2}\frac{\Gamma\left(\frac{d}{2}\right)}{t^{d/2}} =$$

$$= \frac{\lambda}{2(4\pi)^{d/2}}\int\limits_0^\infty dtt^{-d/2}e^{-tm_0^2} = \frac{\lambda}{2(4\pi)^{d/2}}\left(m_0^2\right)^{\frac{d}{2}-1}\Gamma\left(1-\frac{d}{2}\right)\overset{d\to 3}{=} -\frac{\lambda}{8\pi}m_0$$

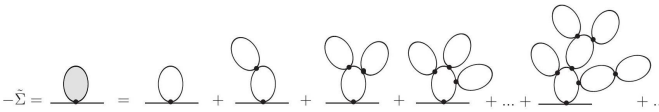
Мы видим, что размерная регуляризиация дает нам первую инвариантную константу и не содержит в данном случае расходящихся членов.

(b).

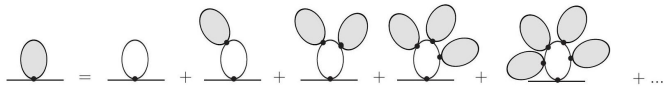
Теперь рассмотрим бесконечный ряд "кактусных" диаграмм, которые дают вклад в $\tilde{\Sigma}(p)$:



Очевидно, что это далеко не все диаграммы которые дают вклад в $\tilde{\Sigma}(p)$, а только лишь их некоторое ничтожно малое подмножество [2]. Найдем сумму всех таких диаграмм. Получим самосогласованное уравнение (уравнение Дайсона) на $\tilde{\Sigma}(p)$ вида (как мы видим в данном случае $\tilde{\Sigma}$ не зависит от внешнего импульса p). Введем понятие перенормированного "кактуса":



Тогда на такой перенормированный "кактус" можно написать самосогласованное уравнение, которое диаграммно рисуется как:



Откуда получается самосогласованное уравнение

$$-\tilde{\Sigma} = -\frac{\lambda}{2}\int d^3k\left(\frac{1}{k^2+m_0^2}+\frac{1}{k^2+m_0^2}(-\tilde{\Sigma})\frac{1}{k^2+m_0^2}+\frac{1}{k^2+m_0^2}(-\tilde{\Sigma})\frac{1}{k^2+m_0^2}(-\tilde{\Sigma})\frac{1}{k^2+m_0^2}+...\right) =$$
$$= -\frac{\lambda}{2}\int d^3k\frac{1}{k^2+m_0^2+\tilde{\Sigma}}, \tag{0.9}$$

и в итоге

$$\tilde{\Sigma} = \frac{\lambda}{2}\int d^3k\frac{1}{k^2+m_0^2+\tilde{\Sigma}} \tag{0.10}$$

(c). Найдем $m^2 - m_0^2 = \tilde{\Sigma}(p)$ из уравнения (0.10), вычисляя интеграл с применением регуляризаций из пункта (a). Имеем в первом случае

$$\tilde{\Sigma} = \frac{\lambda}{4\pi^2}\left(\Lambda - \sqrt{m_0^2+\tilde{\Sigma}}\arctan\left(\frac{\Lambda}{\sqrt{m_0^2+\tilde{\Sigma}}}\right)\right) \tag{0.11}$$

[61] Мы это уже делали в Листке 2, когда вычисляли эффект Казимира: там был интеграл, который сводился к произведению Гамма функций, на самом деле формально данный интеграл не сходился, но тем не менее мы получили для него ответ в виде аналитичекой фунции.

[2] Существуют теории, в которых данное подмножество "кактусных" диаграмм дает основной вклад в $\tilde{\Sigma}(p)$, но в нашем случае ϕ^4 это не так.

Или для перенормированной массы m^2 получим уравнение

$$m^2 = m_0^2 + \frac{\lambda}{4\pi^2}\left(\Lambda - m\arctan\left(\frac{\Lambda}{m}\right)\right) \tag{0.12}$$

В итоге получим в первом порядке ($\lambda\Lambda \gg m_0$):

$$m^2 \approx \frac{\lambda}{4\pi^2}\Lambda - \frac{\lambda^{3/2}}{16\pi^2}\Lambda^{1/2} \tag{0.13}$$

Во втором случаем получаем:

$$\tilde{\Sigma} = \frac{\lambda}{8\pi}M - \frac{\lambda}{8\pi}\sqrt{\tilde{\Sigma} + m_0^2} \tag{0.14}$$

откуда

$$\tilde{\Sigma} = \frac{\lambda}{8\pi}\left(M + \frac{\lambda}{16\pi} - \sqrt{\left(\frac{\lambda}{16\pi}\right)^2 + 2M\left(\frac{\lambda}{16\pi}\right) + m_0^2}\right). \tag{0.15}$$

Мы выбрали знак минус в решении для $\tilde{\Sigma}$, чтобы при $\lambda \to 0$ получить выражение для $\tilde{\Sigma}$ из пункта (a). Если опять $M \gg \lambda, m_0^2/\lambda$, мы получим

$$\tilde{\Sigma} \approx \frac{\lambda M}{8\pi} - \frac{\lambda^{3/2}}{16\sqrt{2}\pi^{3/2}}M^{1/2} + \frac{\lambda^2}{128\pi^2}. \tag{0.16}$$

В последнем случаем мы получаем

$$\tilde{\Sigma} = -\frac{\lambda}{8\pi}\sqrt{\tilde{\Sigma} + m_0^2} \tag{0.17}$$

откуда

$$\tilde{\Sigma} = \frac{\lambda}{8\pi}\left(\frac{\lambda}{16\pi} - \sqrt{\left(\frac{\lambda}{16\pi}\right)^2 + m_0^2}\right). \tag{0.18}$$

Белавин-Пархоменко.11 Перенормировки в теории ϕ^3

В этой задаче мы будем изучать перенормировки в теории ϕ^3 в 6 мерном Евклидовом пространстве. Статистическая сумма нашей теории записывается как

$$Z = \int D\phi_0 \exp\left(-\int d^6x\left(\frac{1}{2}\left(\partial_\mu\phi_0\right)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{3!}\phi_0^3\right)\right) \tag{0.1}$$

где m_0 называется "затравочной" ("голой") массой и λ_0 затравочная константа связи.

Применяя размерную регуляризацию, то есть работая в $d = 6 - \epsilon$ размерности, вычислите одно-петлевые контрчлены $\delta Z, \delta m^2$ и $\delta\lambda$, где $\delta Z = Z - 1, \delta m^2 = Zm_0^2 - m^2$ и $\delta\lambda = \lambda_0 Z^{3/2} - \lambda$. Используйте следующие ренормализационные условия: трех-точечная функция $\Gamma^{(3)}(0,0,0) = \lambda$ в точке, где $p_1 = p_2 = p_3 = 0$. И для функции $\Sigma(p)$ используйте условия: $\Sigma(0) = 0$ и $\Sigma'(0) = 0$, что предполагает, что $\Gamma^{(2)}(p) = p^2 + m^2 + O\left(p^4\right)$. По сути нужно повторить все, что было сделано в лекции, для одной петли, но на примере теории ϕ^3, а не ϕ^4.

Solution 1. (100 баллов)

В этой задаче мы будем изучать перенормировки в теории ϕ^3 в 6 мерном Евклидовом пространстве. Статистическая сумма нашей теории записывается как

$$Z = \int D\phi_0 \exp\left(-\int d^6x\left(\frac{1}{2}\left(\partial_\mu\phi_0\right)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{3!}\phi_0^3\right)\right) \tag{0.1}$$

где m_0 называется "затравочной" ("голой") массой и λ_0 затравочная константа связи.

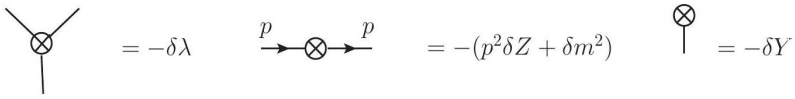
Применяя размерную регуляризацию, то есть работая в $d = 6 - \epsilon$ размерности, мы вычислим однопетлевые контрчлены $\delta Z, \delta m^2$ и $\delta\lambda$, где $\delta Z = Z - 1, \delta m^2 = Zm_0^2 - m^2$ и $\delta\lambda = \lambda_0 Z^{3/2} - \lambda$, используя следующие ренормализационные условия:

$$\begin{aligned}\Gamma^{(3)}(0,0,0) &= \lambda \\ \Gamma^{(2)}(p) &= p^2 + m^2 + O\left(p^4\right)\end{aligned} \tag{0.2}$$

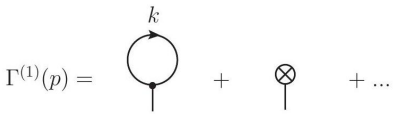
Итак мы имеем следующее действие с контр-членами

$$S = \int d^6x \left(\frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{3!}\phi^3 + \frac{1}{2}\delta Z\left(\partial_\mu\phi\right)^2 + \frac{1}{2}\delta m^2\phi^2 + \frac{\delta\lambda}{3!}\phi^3 + \delta Y\phi\right), \tag{0.3}$$

где $\phi_0 = Z^{1/2}\phi$. Здесь произошел небольшой обман, а именно мы руками ввели контр-член δY, который нужен для того, чтобы сделать одноточку $\langle\phi\rangle = 0$ во всех порядках по λ и как следствие сократить все диаграммы содержащие "головастиков". Мы поясним ниже, что это значит [1]. Итак у нас есть три контр-члена:



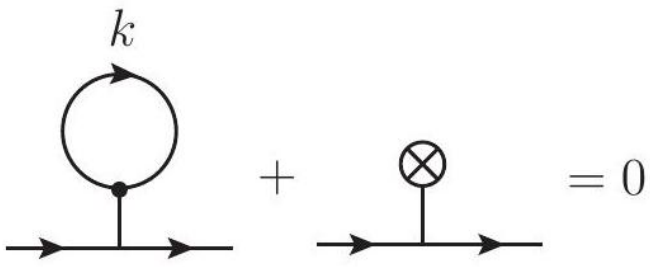
Тогда для одноточечной функции $\langle\phi\rangle$ в первом порядке по λ мы получим



И на языке формул:

$$\Gamma^{(1)}(p) = -\lambda\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2} - \delta Y = -\lambda\frac{\left(m^2\right)^{\frac{d}{2}-1}}{(4\pi)^{\frac{d}{2}}}\Gamma\left(1-\frac{d}{2}\right) - \delta Y = 0 \tag{0.4}$$

Тогда в первом порядке мы для контр-члена получим: $\delta Y = -\lambda\frac{\left(m^2\right)^{\frac{d}{2}-1}}{(4\pi)^{\frac{d}{2}}}\Gamma\left(1-\frac{d}{2}\right) + O\left(\lambda^2\right)$. Для более высоких порядков по λ мы опять выбираем контр-член δY так, чтобы в итоге получить $\langle\phi\rangle = 0$. Данный выбор контр-члена δY приводит к тому, что мы можем выкинуть все диаграммы "головастики"("tadpoles") из рассмотрения, так как их всегда можно будет сократить диаграммами с контр-членами δY. Например:



[61] Вообще говоря теория ϕ^3 является плохо определенной теорией, так как можно легко увидеть, что Гамильтониан данной теории неограничен снизу (в отличие от теории ϕ^4). Тем не менее теория возмущений не "видит" данной ужасной проблемы, и мы можем безнаказанно вычислять порядок за порядком контр-члены и перенормировки наших велечин.

Более строгое утверждение можно сформулировать так: мы ингорируем любую диаграмму, которую можно разрезать на две части, разрезав только одну линию. Причем одна из этих частей не будет содержать внешнии линии.

Теперь для собственной энергетической части мы имеем:

$$-\Sigma\left(p^2\right) = \xrightarrow{p+k} + \xrightarrow{p} \circlearrowright \xrightarrow{p} + \ldots$$

То есть на языке формул мы получаем

$$\begin{aligned}
-\Sigma\left(p^2\right) &= \frac{(-\lambda)^2}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{(p+k)^2+m^2}\frac{1}{k^2+m^2} - \left(p^2\delta Z + \delta m^2\right) = \\
&= \frac{\lambda^2}{2}\int\limits_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{1}{\left[(k+xp)^2+(m^2+x(1-x)p^2)\right]^2} - \left(p^2\delta Z + \delta m^2\right) = \\
&= \frac{\lambda^2\Gamma(2-d/2)}{2(4\pi)^{d/2}}\int\limits_0^1 dx \frac{1}{\left[m^2+x(1-x)p^2\right]^{2-d/2}} - \left(p^2\delta Z + \delta m^2\right), \qquad (0.5)
\end{aligned}$$

где мы использовали формулу

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2+\Delta\right)^n} = \frac{\Gamma\left(n-\frac{d}{2}\right)}{(4\pi)^{d/2}\Gamma(n)}\frac{1}{\Delta^{n-\frac{d}{2}}}. \qquad (0.6)$$

Далее из условий

$$\Sigma\left(p^2\right)\big|_{p^2=0} = 0, \quad \frac{d}{dp^2}\Sigma\left(p^2\right)\Big|_{p^2=0} = 0, \qquad (0.7)$$

мы находим

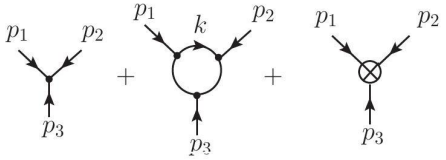
$$\begin{aligned}
\delta Z - \frac{\lambda^2\Gamma(2-d/2)}{2(4\pi)^{d/2}}\int\limits_0^1 dx \frac{(d/2-2)x(1-x)}{\left(m^2\right)^{3-d/2}} = 0, \\
\delta m^2 - \frac{\lambda^2\Gamma(2-d/2)}{2(4\pi)^{d/2}}\int\limits_0^1 dx \frac{1}{\left(m^2\right)^{2-d/2}} = 0, \qquad (0.8)
\end{aligned}$$

и в итоге получаем

$\delta Z = -\frac{\lambda^2\Gamma\left(2-\frac{d}{2}\right)\left(2-\frac{d}{2}\right)}{12(4\pi)^{d/2}}\frac{1}{(m^2)^{3-d/2}} = -\frac{\lambda^2\Gamma\left(3-\frac{d}{2}\right)}{12(4\pi)^{d/2}}\frac{1}{(m^2)^{3-d/2}} \overset{d=6-\epsilon}{=} \frac{\lambda^2}{6(4\pi)^3}\left(\frac{-1}{\epsilon}+\frac{1}{2}\log\frac{m^2}{4\pi}+\frac{\gamma}{2}+O(\epsilon)\right)$,

$\delta m^2 = \frac{\lambda^2\Gamma(2-d/2)}{2(4\pi)^{d/2}}\frac{1}{(m^2)^{2-d/2}} \overset{d=6-\epsilon}{=} \frac{\lambda^2 m^2}{(4\pi)^3}\left(\frac{-1}{\epsilon}+\frac{1}{2}\log\frac{m^2}{4\pi}+\frac{\gamma}{2}-\frac{1}{2}+O(\epsilon)\right)$.

Для трехточечной функции $\Gamma^{(3)}\left(p_1,p_2,p_3\right)$ мы получим

$-\Gamma^{(3)}(p_1,p_2,p_3) =$ 

И на языке формул имеем:

$$-\Gamma^{(3)}\left(p_1,p_2,p_3\right) = -\lambda + (-\lambda)^3\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2+m^2\right)\left(\left(k+p_2\right)^2+m^2\right)\left(\left(k+p_2+p_3\right)^2+m^2\right)} - \delta\lambda \qquad (0.10)$$

И из условия $\Gamma^{(3)}(0,0,0) = \lambda$, получаем

$$-\lambda^3 \int \frac{d^d k}{(2\pi)^d} \frac{1}{\left(k^2+m^2\right)^3} - \delta\lambda = 0 \tag{0.11}$$

откуда

$$\delta\lambda = -\frac{\lambda^3 \Gamma(3-d/2)}{2(4\pi)^{d/2} m^{6-d}} \overset{d=6-\epsilon}{=} -\frac{\lambda^3 \Gamma(\epsilon/2)}{2(4\pi)^{3-\epsilon/2} m^{\epsilon}} = \frac{\lambda^3}{(4\pi)^3}\left(\frac{-1}{\epsilon} + \frac{1}{2}\log\frac{m^2}{4\pi} + \frac{\gamma}{2} + O(\epsilon)\right). \tag{0.12}$$

Белавин-Пархоменко.**12. Перенормировка пропагатора в теории ϕ^4**

1. (100 баллов)

В этой задаче мы будем изучать перенормировку пропагатора в теории ϕ^4 в 4 мерном Евклидовом пространстве используя метод размерной регуляризации. Статистическая сумма нашей теории записывается как

$$Z = \int D\phi_0 \exp\left(-\int d^4x \left(\frac{1}{2}\left(\partial_\mu \phi_0\right)^2 + \frac{1}{2} m_0^2 \phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4\right)\right) \tag{0.1}$$

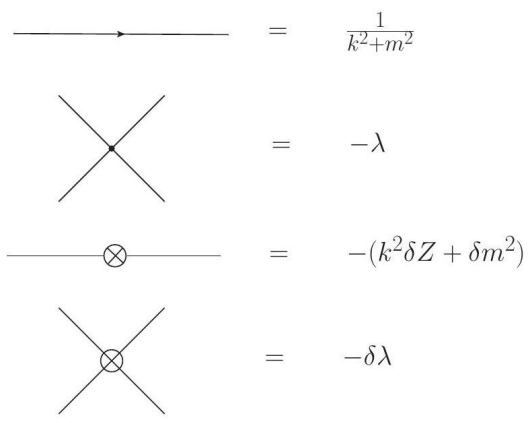
где m_0 называется "затравочной" ("голой") массой и λ_0 затравочная константа связи. Мы вводим новое поле ϕ и константы связи λ и m^2 по формулам

$$\begin{aligned} &\phi_0 = Z^{1/2}\phi, \\ &m_0^2 = Z^{-1}\left(m^2 + \delta m^2\right), \\ &\lambda_0 = Z^{-2}(\lambda + \delta\lambda) \end{aligned} \tag{0.2}$$

где $\delta Z = Z - 1, \delta m^2 = Z m_0^2 - m^2$ и $\delta\lambda = Z^2\lambda_0 - \lambda$ контрчлены. И статистическая сумма запишется как

$$Z = \int D\phi \exp\left(-\int d^4x \left(\frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\delta Z}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}\delta m^2\phi^2 + \frac{\delta\lambda}{4!}\phi^4\right)\right). \tag{0.3}$$

Нашу диаграммную технику в данном случае можно кратко записать как:



Контрчлены расскладываются по степеням взаимодействия λ:

$$\begin{aligned} &\delta Z = z^{(1)}\lambda + z^{(2)}\lambda^2 + ... \\ &\delta m^2 = b^{(1)}\lambda + b^{(2)}\lambda^2 + ... \\ &\delta\lambda = a^{(1)}\lambda + a^{(2)}\lambda^2 + ... \end{aligned} \tag{0.4}$$

с коэффициентами зависящими от $\epsilon = 4 - d$, таким образом, что корреляционные функции ренормированных полей ϕ имеют конечный предел при $\epsilon \to 0$ в каждом порядке по λ.

В этой задаче нужно будет найти перенормировку пропагатора, то есть двухточечной функции $\Gamma^{(2)}\left(p^2\right)$ до второй петли:

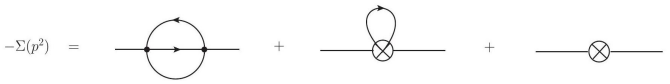
$$\Gamma^{(2)}\left(p^2\right) = p^2 + m^2 + \Sigma\left(p^2\right) \tag{0.5}$$

с ренормализационными условиями $\Sigma\left(p^2=0\right)=0$ и $\frac{d}{dp^2}\Sigma\left(p^2\right)\Big|_{p^2=0} = 0$ (кратко: $\Sigma(0)=\Sigma'(0)=0$).

(a). Однопетлевое вычисление $\Gamma^{(2)}\left(p^2\right)$ (30 баллов) В первой петле для $\Sigma\left(p^2\right)$ мы имеем:

Найдите коэффициенты $z^{(1)}$ и $b^{(1)}$ для контчленов δZ и δm^2, исходя из ренормализационных условий $\Sigma(0)=\Sigma'(0)=0$.

(b). Двухпетлевое вычисление $\Gamma^{(2)}\left(p^2\right)$ (70 баллов) Во второй петле для $\Sigma\left(p^2\right)$ мы имеем:



Вычислите первую диаграмму (I). Она дается довольно громоздким двойным интегралом. Вычислите сначала первый, а затем второй интегралы используя Фейнмановские параметры, а именно пользуясь формулой:

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int\limits_0^1 \frac{dv v^{\alpha-1}(1-v)^{\beta-1}}{[vA+(1-v)B]^{\alpha+\beta}}. \tag{0.6}$$

В итоге вы должны получить результат вида

$$\Sigma_I\left(p^2\right) = -\frac{\lambda^2}{3!}\frac{\Gamma(3-d)}{(4\pi)^d}\times$$
$$\int\limits_0^1 dwduw^{1-\frac{d}{2}}\left((1-w)+wu(1-u)\right)^{3-\frac{3d}{2}}\left[u(1-u)w(1-w)p^2+\left((1-w)+wu(1-u)\right)m^2\right]^{d-3}. \tag{0.7}$$

Данный интеграл расходится как $1/\epsilon^2$. Вычитая из него члены вида $\Sigma_I(0)$ и $p^2\Sigma_I'(0)$, получите выражение для

$$\Sigma_r\left(p^2\right) = \Sigma_I\left(p^2\right) - \Sigma_I(0) - p^2\Sigma_I'(0). \tag{0.8}$$

Покажите, что $\Sigma_r\left(p^2\right)$ при $d\to 4$ не имеет расходимостей. С другой стороны мы имеем

$$\begin{aligned}\Sigma\left(p^2\right) &= \Sigma_I\left(p^2\right) + \Sigma_{II}\left(p^2\right) + \Sigma_{III}\left(p^2\right)\\ &= \Sigma_r\left(p^2\right) + \Sigma_I(0) + p^2\Sigma_I'(0) + \Sigma_{II}\left(p^2\right) + \Sigma_{III}\left(p^2\right)\end{aligned} \tag{0.9}$$

Найдити коэффициенты в контлченах: $z^{(2)}, a^{(2)}$ и $b^{(2)}$ исходя из условия

$$\Sigma_I(0) + p^2\Sigma_I'(0) + \Sigma_{II}\left(p^2\right) + \Sigma_{III}\left(p^2\right) = 0. \tag{0.10}$$

Откуда в конечно итоге мы получим, что

$$\Sigma\left(p^2\right) = \Sigma_r\left(p^2\right) \tag{0.11}$$

Мы не включили сюда диаграммы вида:

Покажите, что их сумма равна нулю, если использовать коэффициенты $z^{(1)}$ и $b^{(1)}$ которые мы нашли в пункте (a).

Solution Белавин-Пархоменко.12. Перенормировка пропагатора в теории ϕ^4 Решения

В этой задаче мы будем изучать перенормировку пропагатора в теории ϕ^4 в 4 мерном Евклидовом пространстве используя метод размерной регуляризации. Статистическая сумма нашей теории записывается как

$$Z = \int D\phi_0 \exp\left(-\int d^4x \left(\frac{1}{2}\left(\partial_\mu \phi_0\right)^2 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4\right)\right) \tag{0.1}$$

где m_0 называется "затравочной" ("голой") массой и λ_0 затравочная константа связи. Мы вводим новое поле ϕ и константы связи λ и m^2 по формулам

$$\begin{aligned} &\phi_0 = Z^{1/2}\phi \\ &m_0^2 = Z^{-1}\left(m^2 + \delta m^2\right), \\ &\lambda_0 = Z^{-2}(\lambda + \delta\lambda) \end{aligned} \tag{0.2}$$

где $\delta Z = Z - 1, \delta m^2 = Zm_0^2 - m^2$ и $\delta\lambda = Z^2\lambda_0 - \lambda$ контрчлены. И статистическая сумма запишется как

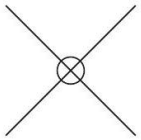
$$Z = \int D\phi \exp\left(-\int d^4x \left(\frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\delta Z}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}\delta m^2\phi^2 + \frac{\delta\lambda}{4!}\phi^4\right)\right). \tag{0.3}$$

Нашу диаграммную технику в данном случае можно кратко записать как:

$= \quad \frac{1}{k^2+m^2}$

$= \quad -\lambda$

$= \quad -(k^2\delta Z + \delta m^2)$


$= \quad -\delta\lambda$

Контрчлены расскладываются по степеням взаимодействия λ:

$$\begin{aligned} &\delta Z = z^{(1)}\lambda + z^{(2)}\lambda^2 + ... \\ &\delta m^2 = b^{(1)}\lambda + b^{(2)}\lambda^2 + ... \\ &\delta\lambda = a^{(1)}\lambda + a^{(2)}\lambda^2 + ... \end{aligned} \tag{0.4}$$

с коэффициентами зависящими от $\epsilon = 4 - d$, таким образом, что корреляционные функции ренормированных полей ϕ имеют конечный предел при $\epsilon \to 0$ в каждом порядке по λ.

В этой задаче мы найдем перенормировку пропагатора, то есть двухточечной функции $\Gamma^{(2)}\left(p^2\right)$ до второй петли:

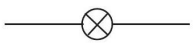
$$\Gamma^{(2)}\left(p^2\right) = p^2 + m^2 + \Sigma\left(p^2\right) \tag{0.5}$$

с ренормализационными условиями $\Sigma\left(p^2\right)\big|_{p^2=0} = 0$ и $\frac{d}{dp^2}\Sigma\left(p^2\right)\Big|_{p^2=0} = 0$ (кратко: $\Sigma(0) = \Sigma'(0) = 0$).

(a). Однопетлевое вычисление $\Gamma^{(2)}\left(p^2\right)$ (30 баллов) В первой петле для $\Sigma\left(p^2\right)$ мы имеем:

$-\Sigma(p^2)$ =

+



Найдем коэффициенты $z^{(1)}$ и $b^{(1)}$ для контчленов δZ и δm^2, исходя из ренормализационных условий $\Sigma(0) = \Sigma'(0) = 0$. Получаем

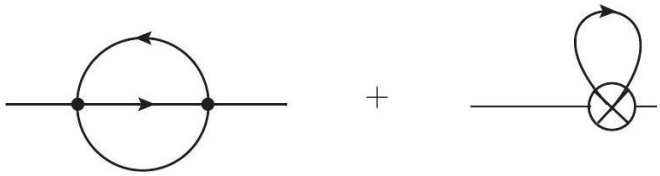
$$-\Sigma\left(p^2\right) = -\frac{\lambda}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2} - \lambda\left(p^2 z^{(1)} + b^{(1)}\right) = -\frac{\lambda}{2(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(1-\frac{d}{2}\right)}{\left(m^2\right)^{1-\frac{d}{2}}} - \lambda\left(p^2 z^{(1)} + b^{(1)}\right). \tag{0.6}$$

Из уравнений $\Sigma(0) = \Sigma'(0) = 0$ мы получаем

$$b^{(1)} = -\frac{1}{2(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(1-\frac{d}{2}\right)}{\left(m^2\right)^{1-\frac{d}{2}}} \overset{d=4-\epsilon}{=} \frac{1}{(4\pi)^2}\frac{m^2}{\epsilon} - \frac{m^2}{2(4\pi)^2}\left(\log\frac{m^2}{4\pi} + \gamma - 1\right) + O(\epsilon), \quad z^{(1)} = 0. \tag{0.7}$$

(b). Двухпетлевое вычисление $\Gamma^{(2)}\left(p^2\right)$ (70 баллов) Во второй петле для $\Sigma\left(p^2\right)$ мы имеем:

$$-\Sigma\left(p^2\right) =$$



Вычислим первую диаграмму (I). Имеем

$$\begin{aligned}
-\Sigma_I\left(p^2\right) &= \frac{(-\lambda)^2}{3!}\int \frac{d^dk d^dq}{(2\pi)^{2d}}\frac{1}{k^2+m^2}\frac{1}{q^2+m^2}\frac{1}{(p+q+k)^2+m^2} = \\
&= \frac{\lambda^2}{3!}\int \frac{d^dk d^dq}{(2\pi)^{2d}}\frac{1}{k^2+m^2}\int\limits_0^1 du \frac{1}{\left[u\left(q^2+2q(p+k)+(p+k)^2\right)+(1-u)q^2+m^2\right]^2} = \\
&= \frac{\lambda^2}{3!}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}\int\limits_0^1 du\int \frac{d^dq}{(2\pi)^d}\frac{1}{\left[(q+u(p+k))^2+m^2+u(1-u)(p+k)^2\right]^2} = \\
&= \frac{\lambda^2}{3!}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}\int\limits_0^1 du \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}\Gamma(2)}\frac{1}{\left[m^2+u(1-u)(p+k)^2\right]^{2-\frac{d}{2}}} = \\
&= \frac{\lambda^2}{3!}\frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{d/2}\Gamma(2)}\int \frac{d^dk}{(2\pi)^d}\int\limits_0^1 dudw w^{1-\frac{d}{2}}\frac{\Gamma^2)}{\Gamma\left(2-\frac{d}{2}\right)\Gamma(2)\left[m^2+wu(1-u)(p+k)^2+k^2(1-w)\right]^{3-\frac{d}{2}}} = \\
&= \frac{\lambda^2}{3!}\frac{\Gamma\left(3-\frac{d}{2}\right)}{(4\pi)^{d/2}}\int dudw w^{1-\frac{d}{2}}\int \frac{d^dk}{(2\pi)^d}\frac{1}{\left[k^2A+2pkwu(1-u)+m^2+p^2wu(1-u)\right]^{3-\frac{d}{2}}},
\end{aligned} \tag{0.8}$$

где $A = 1 - w + wu(1-u)$. Далее получаем

$$\begin{aligned}
-\Sigma_I\left(p^2\right) &= \frac{\lambda^2}{3!}\frac{\Gamma\left(3-\frac{d}{2}\right)}{(4\pi)^{d/2}}\int \frac{dudw w^{1-\frac{d}{2}}}{A^{3-\frac{d}{2}}}\int \frac{d^dk}{(2\pi)^d}\frac{1}{\left[\left(k+p\frac{wu(1-u)}{A}\right)^2+\frac{m^2+p^2wu(1-u)}{A}-p^2\frac{w^2u^2(1-u)^2}{A^2}\right]^{3-\frac{d}{2}}} = \\
&= \frac{\lambda^2}{3!}\frac{\Gamma\left(3-\frac{d}{2}\right)}{(4\pi)^{d/2}}\int \frac{dudw w^{1-\frac{d}{2}}}{A^{3-\frac{d}{2}}}\frac{\Gamma(3-d)}{(4\pi)^{d/2}\Gamma\left(3-\frac{d}{2}\right)}\left(\frac{A^2}{u(1-u)w(1-w)p^2+Am^2}\right)^{3-d}.
\end{aligned} \tag{0.9}$$

В итоге мы получим

$$\Sigma_I\left(p^2\right) = -\frac{\lambda^2}{3!}\frac{\Gamma(3-d)}{(4\pi)^d}\int dudww^{1-\frac{d}{2}}A^{3-\frac{3d}{2}}\left(u(1-u)w(1-w)p^2+Am^2\right)^{d-3} =$$
$$= -\frac{\lambda^2}{3!}\frac{\Gamma(3-d)}{(4\pi)^d}\left(m^2\right)^{d-3}\int dudww^{1-\frac{d}{2}}A^{-\frac{d}{2}}\left(1+Bp^2\right)^{d-3}, \tag{0.10}$$

где $B = \frac{u(1-u)w(1-w)}{Am^2}$. Данный интеграл расходится как $1/\epsilon^2$. Вычитая из него члены вида $\Sigma_I(0)$ и $p^2\Sigma_I'(0)$, получим

$$\Sigma_r\left(p^2\right) = \Sigma_I\left(p^2\right) - \Sigma_I(0) - p^2\Sigma_I'(0) =$$
$$= -\frac{\lambda^2}{3!}\frac{\Gamma(3-d)}{(4\pi)^d}\left(m^2\right)^{d-3}\int dudww^{1-\frac{d}{2}}A^{-\frac{d}{2}}\left(\left(1+Bp^2\right)^{d-3}-1-(d-3)p^2B\right) =$$
$$= -\frac{\lambda^2}{3!}\frac{\Gamma(-1+\epsilon)}{(4\pi)^{4-\epsilon}}\left(m^2\right)^{1-\epsilon}\int dudww^{1-\frac{d}{2}}A^{-\frac{d}{2}}\left(\left(1+Bp^2\right)e^{-\epsilon\log\left(1+Bp^2\right)}-1-(1-\epsilon)p^2B\right) =$$
$$= -\frac{\lambda^2}{3!}\frac{\epsilon(-1+\epsilon)}{(4\pi)^{4-\epsilon}}\left(m^2\right)^{1-\epsilon}\int dudww^{1-\frac{d}{2}}A^{-\frac{d}{2}}\left(p^2B-\left(1+p^2B\right)\log\left(1+p^2B\right)+O(\epsilon)\right)\overset{\epsilon\to 0}{=}$$
$$= \frac{\lambda^2}{3!}\frac{m^2}{(4\pi)^4}\int\limits_0^1\frac{dudw}{wA^2}\left(p^2B-\left(1+p^2B\right)\log\left(1+p^2B\right)\right). \tag{0.11}$$

Покажем, что $\Sigma_r\left(p^2\right)$ при $d\to 4$ не имеет расходимостей. Имеем при $w\to 0$

$$A\to 1,\quad B\to\frac{wu(1-u)}{m^2}\to 0 \tag{0.12}$$

поэтому мы получаем для подынтегрального выражения при $w\to 0$:

$$\frac{1}{wA^2}\left(p^2B-\left(1+p^2B\right)\log\left(1+p^2B\right)\right)\to -\frac{\left(p^2B\right)^2}{2wA^2}\to -\frac{p^4}{2m^4}wu^2(1-u)^2\to 0 \tag{0.13}$$

Откуда следует, что двойной инеграл по w и по u не имеет расходимостей при $w\to 0$. Теперь при $w\to 1$ мы имеем:

$$A\to 1-w+u(1-u)=\alpha+\beta,\quad B\to\frac{(1-w)u(1-u)}{m^2(1-w+u(1-u))}=\frac{1}{m^2}\frac{\alpha\beta}{\alpha+\beta} \tag{0.14}$$

где $\alpha = 1-w\to +0$ и $\beta = u(1-u) > 0$. Для A и B имеем следующие неравенства:

$$A\geqslant\max(\alpha,\beta),\quad B\leqslant\frac{1}{m^2}\min(\alpha,\beta)\to 0 \tag{0.15}$$

И для подынтегрального выражения получаем при $w\to 1$:

$$\left|\frac{1}{wA^2}\left(p^2B-\left(1+p^2B\right)\log\left(1+p^2B\right)\right)\right|\to\frac{\left(p^2B\right)^2}{2wA^2}\to\frac{p^4}{2}\frac{B^2}{A^2}\leqslant\frac{p^4}{2m^4}\frac{\min(\alpha,\beta)^2}{\max(\alpha,\beta)^2}\leqslant\frac{p^4}{2m^4}, \tag{0.16}$$

то есть мы опять видим, что подынтегральное выражение конечно и не содержит сингулярностей, таким образом мы видим, что $\Sigma_r\left(p^2\right)$ не имеет расходимостей.

С другой стороны мы имеем

$$\Sigma\left(p^2\right) = \Sigma_I\left(p^2\right)+\Sigma_{II}\left(p^2\right)+\Sigma_{III}\left(p^2\right)$$
$$= \Sigma_r\left(p^2\right)+\left(\Sigma_I(0)+p^2\Sigma_I'(0)+\Sigma_{II}\left(p^2\right)+\Sigma_{III}\left(p^2\right)\right) \tag{0.17}$$

Найдем коэффициенты в контлченах: $z^{(2)}, a^{(2)}$ и $b^{(2)}$ исходя из условия

$$\Sigma_I(0)+p^2\Sigma_I'(0)+\Sigma_{II}\left(p^2\right)+\Sigma_{III}\left(p^2\right)=0 \tag{0.18}$$

Итак имеем:

$$\Sigma_{II}\left(p^2\right) = -\frac{\lambda^2 a^{(2)}}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}, \quad \Sigma_{III}\left(p^2\right) = -\lambda^2\left(z^{(2)}p^2 + b^{(2)}\right), \tag{0.19}$$

откуда находим, что

$$z^{(2)} = \frac{\Sigma_I'(0)}{\lambda^2} = \frac{1}{3!}\frac{\Gamma(4-d)}{(4\pi)^d}\left(m^2\right)^{d-4}\int\limits_0^1 dudw\frac{w^{2-\frac{d}{2}}(1-w)u(1-u)}{(1-w+wu(1-u))^{1+\frac{d}{2}}} \overset{d=4-\epsilon}{=} \frac{1}{12(4\pi)^4}\frac{1}{\epsilon}+O(1). \tag{0.20}$$

и также

$$a^{(2)}\frac{\Gamma\left(1-\frac{d}{2}\right)}{2(4\pi)^{\frac{d}{2}}\left(m^2\right)^{1-\frac{d}{2}}} + b^{(2)} = -\frac{\Gamma(3-d)\left(m^2\right)^{d-3}}{3!(4\pi)^d}\int\limits_0^1\frac{dudww^{1-\frac{d}{2}}}{(1-w+wu(1-u))^{\frac{d}{2}}} \tag{0.21}$$

и в лидирующем порядке по ϵ получаем:

$$a^{(2)}\left(-\frac{m^2}{(4\pi)^2\epsilon}+O(1)\right)+b^{(2)} = \frac{m^2}{3(4\pi)^4\epsilon^2}+O(1/\epsilon) \tag{0.22}$$

Откуда в конечно итоге мы получим, что

$$\Sigma\left(p^2\right) = \Sigma_r\left(p^2\right) = \frac{\lambda^2}{3!}\frac{m^2}{(4\pi)^4}\int\limits_0^1\frac{dudw}{wA^2}\left(p^2B-\left(1+p^2B\right)\log\left(1+p^2B\right)\right), \tag{0.23}$$

где $A = 1-w+wu(1-u)$ и $B = \frac{u(1-u)w(1-w)}{Am^2}$.
Мы не включили сюда диаграммы вида:

Покажем, что их сумма равна нулю, если использовать коэффициенты $z^{(1)}$ и $b^{(1)}$, которые мы нашли в пункте (a). Имеем

$$\begin{aligned}
&-\frac{\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}\left(-\frac{\lambda}{2}\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+m^2}\right)\frac{1}{k^2+m^2} - \frac{\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}\left(-\lambda\left(k^2z^{(1)}+b^{(1)}\right)\right)\frac{1}{k^2+m^2} = \\
&= -\frac{\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}\left(-\frac{\lambda}{2}\int\frac{d^dq}{(2\pi)^d}\frac{1}{q^2+m^2}-\lambda b^{(1)}\right)\frac{1}{k^2+m^2} = -\frac{\lambda}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}(0)\frac{1}{k^2+m^2} = 0.
\end{aligned} \tag{0.24}$$

Белавин-Пархоменко.13. Перенормировка оператора $\phi^2(x)$ в теории ϕ^4

1. (100 баллов)

В этой задаче мы будем изучать перенормировку оператора $O(x) = \phi^2(x)$ в теории ϕ^4 в 4 мерном Евклидовом пространстве используя метод размерной регуляризации. Статистическая сумма нашей теории записывается как

$$Z = \int D\phi_0 \exp\left(-\int d^4x\left(\frac{1}{2}\left(\partial_\mu\phi_0\right)^2+\frac{1}{2}m_0^2\phi_0^2+\frac{\lambda_0}{4!}\phi_0^4\right)\right) \tag{0.1}$$

где m_0 называется "затравочной" ("голой") массой и λ_0 затравочная константа связи. Мы вводим новое поле ϕ и константы связи λ и m^2 по формулам

$$\begin{aligned}
&\phi_0 = Z^{1/2}\phi, \\
&m_0^2 = Z^{-1}\left(m^2+\delta m^2\right), \\
&\lambda_0 = Z^{-2}(\lambda+\delta\lambda)
\end{aligned} \tag{0.2}$$

где $\delta Z = Z - 1, \delta m^2 = Z m_0^2 - m^2$ и $\delta\lambda = Z^2\lambda_0 - \lambda$ контрчлены. И статистическая сумма запишется как

$$Z = \int D\phi \exp\left(-\int d^4x \left(\frac{1}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{\delta Z}{2}\left(\partial_\mu\phi\right)^2 + \frac{1}{2}\delta m^2\phi^2 + \frac{\delta\lambda}{4!}\phi^4\right)\right). \tag{0.3}$$

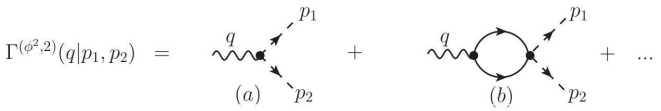
Определим $W^{\left(\phi^2,2\right)}\left(x|x_1,x_2\right)$ по формуле:

$$W^{\left(\phi^2,2\right)}\left(x|x_1,x_2\right) = \left\langle\phi^2(x)\phi\left(x_1\right)\phi\left(x_2\right)\right\rangle_{\text{связн}} = \left\langle\phi^2(x)\phi\left(x_1\right)\phi\left(x_2\right)\right\rangle - \left\langle\phi^2(x)\right\rangle\left\langle\phi\left(x_1\right)\phi\left(x_2\right)\right\rangle. \tag{0.4}$$

Удобно перейти в импульсное пространство и определить $\Gamma^{\left(\phi^2,2\right)}\left(q|p_1,p_2\right)$:

$$\Gamma^{\left(\phi^2,2\right)}\left(q|p_1,p_2\right) = \frac{W^{\left(\phi^2,2\right)}\left(q|p_1,p_2\right)}{W\left(p_1\right)W\left(p_2\right)}, \quad q = p_1 + p_2 \tag{0.5}$$

где $\Gamma^{\left(\phi^2,2\right)}\left(q|p_1,p_2\right)$ уже дается суммой всех связанных одночастично неприводимых диаграмм:



Таким обозом мы находим, что

$$\Gamma^{\left(\phi^2,2\right)}\left(q|p_1,p_2\right) = 1 - \frac{\lambda}{2}I\left(p_{12}^2\right) + \ldots \tag{0.6}$$

где $I\left(p^2\right)$ интеграл, который мы вычисляли на предыдущих лекциях. Мы знаем, что $I\left(p_{12}^2\right) = I_r\left(p_{12}^2\right) + I(0)$, где I_r конечен при $\epsilon \to 0$, а $I(0) = \left(m^2\right)^{d/2-2}\Gamma(2-d/2)/(4\pi)^{d/2}$ сингулярная часть, то есть мы видим, что в данных диаграммах есть расходимости. Мы можем определить перенормированный оператор $\left[\phi^2\right]_R$ и $\Gamma^{\left(\left[\phi^2\right]_R,2\right)}$ по формулам:

$$\phi^2 = Z_2^{-1}\left[\phi^2\right]_R, \quad \Gamma^{\left(\phi^2,2\right)} = Z_2^{-1}\Gamma^{\left(\left[\phi^2\right]_R,2\right)}, \tag{0.7}$$

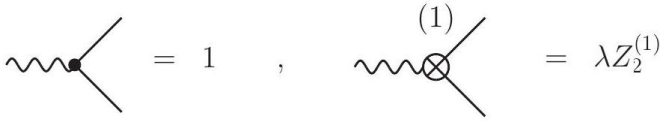
так, что $\Gamma^{\left(\left[\phi^2\right]_R,2\right)}$ имеет уже конечный предел при $\epsilon \to 0$. Удобно написать

$$\left[\phi^2\right]_R = Z_2\phi^2 \tag{0.8}$$

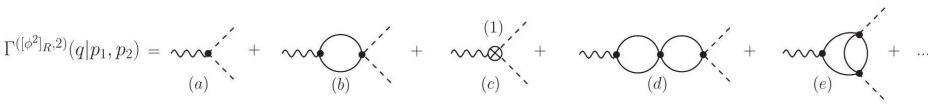
где коэффициент Z_2 находится порядок за порядком по степени λ:

$$Z_2 = 1 + \lambda Z_2^{(1)} + \lambda^2 Z_2^{(2)} + \ldots \tag{0.9}$$

Поэтому вычисление корреляционной функции с вставкой ренормированного оператора $\left[\phi^2\right]_R$, эквивалентно добавлению к нашей диаграммной технике серии контрчленных вершин:



Поэтому для корреляционной функции $\Gamma^{\left(\left[\phi^2\right]_R,2\right)}$ мы получим следующие диаграммы:



[61] Внешнии линии обозначаются здесь пунктиром, а волнистая линия просто обозначает, что в корреляционную функцию вставленна внешняя вершина-оператор с полным импульсом q.

И коэффициенты контрчленов $Z_2^{(1)}, Z_2^{(2)}, ...$ подбираются таким образом, чтобы сократить расходимости. Мы выбираем ренормализационное условие в виде [2]

$$\Gamma^{\left([\phi^2]_R,2\right)}(0|0,0)=1 \tag{0.10}$$

И мы можем легко найти, что $Z_2^{(1)}=\frac{1}{2}I(0)$ и $\Gamma^{\left([\phi^2]_R,2\right)}=1-\frac{\lambda}{2}I_r\left(p_{12}^2\right)+O\left(\lambda^2\right)$.

Покажите, что во втором порядке по λ, можно найти такое $Z_2^{(2)}$, что $\Gamma^{\left([\phi^2]_R,2\right)}$ не будет иметь сингулярностей. Помимо диаграмм (d) и (e) вам нужно найти еще три дополнительные диаграммы, которые дадут вклад в λ^2 порядке, одна из этих диаграмм есть просто операторный контрчлен $Z_2^{(2)}$. Также не забывайте, что у нас есть и обычные контрчлены. (Подсказка: Вспомните анализ двухпетлевой перенормировки для $\Gamma^{(4)}$).

Solution Белавин-Пархоменко.13. Перенормировка оператора $\phi^2(x)$ в теории ϕ^4

В этой задаче мы будем изучать перенормировку оператора $O(x)=\phi^2(x)$ в теории ϕ^4 в 4 мерном Евклидовом пространстве используя метод размерной регуляризации. Статистическая сумма нашей теории записывается как

$$Z=\int D\phi_0 \exp\left(-\int d^4x\left(\frac{1}{2}\left(\partial_\mu\phi_0\right)^2+\frac{1}{2}m_0^2\phi_0^2+\frac{\lambda_0}{4!}\phi_0^4\right)\right) \tag{0.1}$$

где m_0 называется "затравочной" ("голой") массой и λ_0 затравочная константа связи. Мы вводим новое поле ϕ и константы связи λ и m^2 по формулам

$$\begin{aligned}\phi_0&=Z^{1/2}\phi\\ m_0^2&=Z^{-1}\left(m^2+\delta m^2\right)\\ \lambda_0&=Z^{-2}(\lambda+\delta\lambda)\end{aligned} \tag{0.2}$$

где $\delta Z=Z-1, \delta m^2=Zm_0^2-m^2$ и $\delta\lambda=Z^2\lambda_0-\lambda$ контрчлены. И статистическая сумма запишется как

$$Z=\int D\phi \exp\left(-\int d^4x\left(\frac{1}{2}\left(\partial_\mu\phi\right)^2+\frac{1}{2}m^2\phi^2+\frac{\lambda}{4!}\phi^4+\frac{\delta Z}{2}\left(\partial_\mu\phi\right)^2+\frac{1}{2}\delta m^2\phi^2+\frac{\delta\lambda}{4!}\phi^4\right)\right). \tag{0.3}$$

Определим $W^{\left(\phi^2,2\right)}\left(x|x_1,x_2\right)$ по формуле:

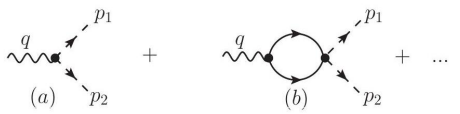
$$W^{\left(\phi^2,2\right)}\left(x|x_1,x_2\right)=\left\langle\phi^2(x)\phi\left(x_1\right)\phi\left(x_2\right)\right\rangle_{\text{связн}}=\left\langle\phi^2(x)\phi\left(x_1\right)\phi\left(x_2\right)\right\rangle-\left\langle\phi^2(x)\right\rangle\left\langle\phi\left(x_1\right)\phi\left(x_2\right)\right\rangle. \tag{0.4}$$

Удобно перейти в импульсное пространство и определить $\Gamma^{\left(\phi^2,2\right)}\left(q|p_1,p_2\right)$:

$$\Gamma^{\left(\phi^2,2\right)}\left(q|p_1,p_2\right)=\frac{W^{\left(\phi^2,2\right)}\left(q|p_1,p_2\right)}{W\left(p_1\right)W\left(p_2\right)},\quad q=p_1+p_2 \tag{0.5}$$

где $\Gamma^{\left(\phi^2,2\right)}\left(q|p_1,p_2\right)$ уже дается суммой всех связанных одночастично неприводимых диаграмм:

$\Gamma^{(\phi^2,2)}(q|p_1,p_2)$ =



Таким обозом мы находим, что

$$\Gamma^{\left(\phi^2,2\right)}\left(q|p_1,p_2\right)=1-\frac{\lambda}{2}I\left(p_{12}^2\right)+... \tag{0.6}$$

[62] Как всегда у нас есть свобода в выборе конечной части контрчленных коэффициентов. Мы фиксируем эту свободу выбором ренормализационного условия.

где $I\left(p^2\right)$ интеграл, который мы вычисляли на предыдущих лекциях. Мы знаем, что $I\left(p_{12}^2\right) = I_r\left(p_{12}^2\right) + I(0)$, где I_r конечен при $\epsilon \to 0$, а $I(0) = \left(m^2\right)^{d/2-2}\Gamma(2-d/2)/(4\pi)^{d/2}$ сингулярная часть, то есть мы видим, что в данных диаграммах есть расходимости. Мы можем определить перенормированные операторы $\left[\phi^2\right]_R$ и $\Gamma^{\left(\left[\phi^2\right]_R,2\right)}$ по формулам:

$$\phi^2 = Z_2^{-1}\left[\phi^2\right]_R, \quad \Gamma^{\left(\phi^2,2\right)} = Z_2^{-1}\Gamma^{\left(\left[\phi^2\right]_R,2\right)} \tag{0.7}$$

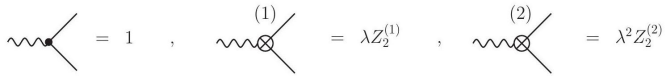
так, что $\Gamma^{\left(\left[\phi^2\right]_R,2\right)}$ имеет уже конечный предел при $\epsilon \to 0$. Удобно написать

$$\left[\phi^2\right]_R = Z_2\phi^2 \tag{0.8}$$

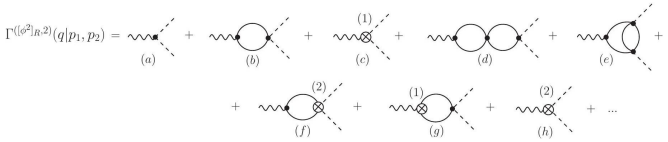
где коэффициент Z_2 находится порядок за порядком по степени λ:

$$Z_2 = 1 + \lambda Z_2^{(1)} + \lambda^2 Z_2^{(2)} + \ldots \tag{0.9}$$

Поэтому вычисление корреляционной функции с вставкой ренормированного оператора $\left[\phi^2\right]_R$, эквивалентно добавлению к нашей диаграммной технике серии контрчленных вершин:



Поэтому для корреляционной функции $\Gamma^{\left(\left[\phi^2\right]_R,2\right)}$ мы получим следующие диаграммы:



(g)
(h) ,

И коэффициенты контрчленов $Z_2^{(1)}, Z_2^{(2)}, \ldots$ подбираются таким образом, чтобы сократить расходимости. Мы выбираем ренормализационное условие в виде [2]

$$\Gamma^{\left(\left[\phi^2\right]_R,2\right)}(0|0,0) = 1 \tag{0.10}$$

Далее имеем

$$(a) = 1, \quad (b) = -\frac{\lambda}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2+m^2\right)\left((q+k)^2+m^2\right)} = -\frac{\lambda}{2}I\left(q^2\right), \quad (c) = \lambda Z_2^{(1)} \tag{0.11}$$

Вычисляем интеграл $I\left(q^2\right)$ используя Фейнмановские параметры:

$$\begin{aligned} I\left(q^2\right) &= \int \frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2+m^2\right)\left((q+k)^2+m^2\right)} = \int \frac{d^dk}{(2\pi)^d}\int\limits_0^1 dx \frac{1}{\left(x(q+k)^2+(1-x)k^2+m^2\right)^2} = \\ &= \int\limits_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2+2xqk+m^2+xq^2\right)^2} = \int\limits_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{1}{\left((k+xq)^2+m^2+x(1-x)q^2\right)^2} = \\ &= \frac{\Gamma\left(2-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}}\int\limits_0^1 dx\left(m^2+x(1-x)q^2\right)^{\frac{d}{2}-2}. \end{aligned} \tag{0.12}$$

При $d = 4$, Гамма-функция имеет полюс: $\Gamma\left(2-\frac{d}{2}\right) = \frac{2}{\epsilon} - \gamma + O(\epsilon)$, где $d = 4-\epsilon$, чтобы устранить данную сингулярность мы вычитаем из $I\left(q^2\right)$ интеграл на нулевом импульсе $I(0)$:

[61] Внешнии линии обозначаются здесь пунктиром, а волнистая линия просто обозначает, что в корреляционную функцию вставленна внешняя вершина-оператор с полным импульсом q.

[62] Как всегда у нас есть свобода в выборе конечной части контрчленных коэффициентов. Мы фиксируем эту свободу выбором ренормализационного условия.

$$I_r\left(q^2\right)=I\left(q^2\right)-I(0)=-\frac{1}{(4\pi)^2}\int\limits_0^1 dx\log\left(1+x(1-x)\frac{q^2}{m^2}\right)=$$
$$=-\frac{1}{16\pi^2}\left(\sqrt{\frac{q^2+4m^2}{q^2}}\log\frac{\sqrt{q^2+4m^2}+\sqrt{q^2}}{\sqrt{q^2+4m^2}-\sqrt{q^2}}-2\right), \tag{0.13}$$

также замечая, что очевидно $I_r(0)=0$. В итоге получаем ($q=p_1+p_2=p_{12}$):

$$\Gamma^{\left(\left[\phi^2\right]_R,2\right)}\left(q|p_1,p_2\right)=1-\frac{\lambda}{2}I_r\left(q^2\right)-\lambda\left(\frac{1}{2}I(0)-Z_2^{(1)}\right)+O\left(\lambda^2\right). \tag{0.14}$$

и из условия (0.10) следует, что $Z_2^{(1)}=\frac{1}{2}I(0)$ и $\Gamma^{\left(\left[\phi^2\right]_R,2\right)}=1-\frac{\lambda}{2}I_r\left(p_{12}^2\right)+O\left(\lambda^2\right)$.

Покажем, что во втором порядке по λ, можно найти такое $Z_2^{(2)}$, что $\Gamma^{\left(\left[\phi^2\right]_R,2\right)}$ не будет иметь сингулярностей. Имеем

$$(d)=\frac{(-\lambda)^2}{4}\left(I\left(q^2\right)\right)^2,\quad (f)=-\frac{\delta\lambda^{(2)}}{2}I\left(q^2\right)=-\frac{a^{(2)}\lambda^2}{2}I\left(q^2\right),\quad (g)=-\frac{\lambda^2Z_2^{(1)}}{2}I\left(q^2\right),\quad (h)=\lambda^2Z_2^{(2)}, \tag{0.15}$$

где $\delta\lambda=a^{(1)}\lambda+a^{(2)}\lambda^2+...$, и контрчлен $a^{(2)}=\frac{3}{2}I(0)$, находился из однопетлевой перенормировки вершины ϕ^4. Диаграмма () более сложная и дается интегралом

$$(e)=\frac{(-\lambda)^2}{2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}\frac{1}{(q+k)^2+m^2}I\left(\left(k+p_1\right)^2\right) \tag{0.16}$$

Вычислим его опять используя Фейнмановские параметры

$$(e)=\frac{\lambda^2\Gamma\left(2-\frac{d}{2}\right)}{2(4\pi)^{d/2}}\int\limits_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}\frac{1}{(q+k)^2+m^2}\frac{1}{\left(m^2+x(1-x)\left(k+p_1\right)^2\right)^{2-\frac{d}{2}}}=$$
$$=\frac{\lambda^2\Gamma\left(2-\frac{d}{2}\right)}{2(4\pi)^{d/2}}\int\limits_0^1 dxdy\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2+2ykq+yq^2+m^2\right)^2}\frac{z^{1-\frac{d}{2}}(1-z)}{\left(m^2+x(1-x)\left(k+p_1\right)^2\right)^{2-\frac{d}{2}}}=$$
$$=\frac{\lambda^2\Gamma\left(4-\frac{d}{2}\right)}{2(4\pi)^{d/2}}\int\limits_0^1 dxdydz\int\frac{d^dk}{(2\pi)^d}\frac{z^{1-\frac{d}{2}}(1-z)}{\left[z\left(m^2+x(1-x)\left(k+p_1\right)^2\right)+(1-z)\left(k^2+2ykq+yq^2+m^2\right)\right]^{4-\frac{d}{2}}}=$$
$$=\frac{\lambda^2\Gamma\left(4-\frac{d}{2}\right)}{2(4\pi)^{d/2}}\int\limits_0^1 dxdydz\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left[A\left(k+\frac{zx(1-x)p_1+y(1-z)q}{1-z+xx(1-x)}\right)^2+P^2+m^2\right]^{4-\frac{d}{2}}}, \tag{0.17}$$

где $A=1-z+zx(1-x)$ и

$$P^2=\frac{z(1-z)x(1-x)p_1^2+y(1-z)((1-y)(1-z)+zx(1-x))q^2-2xyz(1-x)(1-z)p_1q}{1-z+zx(1-x)}. \tag{0.18}$$

В итоге получаем

$$(e)=\frac{\lambda^2\Gamma(4-d)}{2(4\pi)^d}\int\limits_0^1 dxdydz\frac{z^{1-\frac{d}{2}}(1-z)}{A^{\frac{d}{2}}}\left(m^2+P^2\right)^{d-4}. \tag{0.19}$$

Данная диаграмма расходится как $1/\epsilon^2$, одна степень ϵ появляется из Гамма-функции:

$$\Gamma(4-d)=\Gamma(\epsilon)=\frac{1}{\epsilon}-\gamma+O(\epsilon) \tag{0.20}$$

а другая степень из интеграла по z, при интегрировании вблизи $z=0$:

$$\int\limits_0^1 z^{1-\frac{d}{2}}dz=\int\limits_0^1 dzz^{-1+\frac{\epsilon}{2}}=\frac{2}{\epsilon}z^{\frac{\epsilon}{2}}\Big|_0^1=\frac{2}{\epsilon} \tag{0.21}$$

Чтобы изолировать данную сингулярность мы запишем диаграмму (e), как

$$\begin{aligned}(e)=&\frac{\lambda^2\Gamma(4-d)}{2(4\pi)^d}\int\limits_0^1 dxdydzz^{1-\frac{d}{2}}\left(m^2+y(1-y)q^2\right)^{d-4}+\\&+\frac{\lambda^2\Gamma(4-d)}{2(4\pi)^d}\int\limits_0^1 dxdydzz^{1-\frac{d}{2}}\left(\frac{(1-z)}{A^{\frac{d}{2}}}\left(m^2+P^2\right)^{d-4}-\left(m^2+y(1-y)q^2\right)^{d-4}\right)\end{aligned} \tag{0.22}$$

где мы использовали то, что $P^2(z=0)=y(1-y)q^2$ и $A(z=0)=1$. Итак, вычисляя первое слагаемое имеем:

$$\begin{aligned}&\frac{\lambda^2\Gamma(4-d)}{2(4\pi)^{d/2}}\int\limits_0^1 dxdydzz^{1-\frac{d}{2}}\left(m^2+y(1-y)q^2\right)^{d-4}=\frac{\lambda^2\Gamma(\epsilon)}{\epsilon(4\pi)^{4-\epsilon}}\int\limits_0^1 dy\left(m^2+y(1-y)q^2\right)^{-\epsilon}=\\&=\frac{\lambda^2}{\epsilon(4\pi)^4}\left(\frac{1}{\epsilon}-\gamma\right)e^{\epsilon\log 4\pi}\int\limits_0^1 dye^{-\epsilon\log\left(m^2+y(1-y)q^2\right)}=\\&=\frac{\lambda^2}{\epsilon(4\pi)^4}\left(\frac{1}{\epsilon}-\gamma\right)(1+\epsilon\log 4\pi)\int\limits_0^1 dy\left(1-\epsilon\log\left(m^2+y(1-y)q^2\right)\right)=\\&=\frac{\lambda}{\epsilon(4\pi)^4}\left(\frac{1}{\epsilon}-\gamma\right)(1+\epsilon\log 4\pi)\left(1-\epsilon\left(\log m^2-(4\pi)^2I_r\left(q^2\right)\right)\right)=\\&=\frac{\lambda^2}{(4\pi)^4}\left(\frac{1}{\epsilon^2}+\frac{1}{\epsilon}\left(\log\frac{4\pi}{m^2}-\gamma+(4\pi)^2I_r\left(q^2\right)\right)\right)+\text{ конечные члены.}\end{aligned} \tag{0.23}$$

Для того, чтобы во втором слагаемом найти член $\sim\frac{1}{\epsilon}$, интеграл достаточно вычислить при $d=4$:

$$\begin{aligned}&\frac{\lambda^2\Gamma(4-d)}{2(4\pi)^d}\int\limits_0^1 dxdydzz^{1-\frac{d}{2}}\left(\frac{(1-z)}{A^{\frac{d}{2}}}\left(m^2+P^2\right)^{d-4}-\left(m^2+y(1-y)q^2\right)^{d-4}\right)=\\&=\frac{\lambda^2\Gamma(\epsilon)}{2(4\pi)^4}\int\limits_0^1 dxdydzz^{-1}\left(\frac{(1-z)}{A^2}-1\right)+\text{ конечные члены }=\\&=\frac{\lambda^2}{2(4\pi)^4}\frac{1}{\epsilon}+\text{ конечные члены.}\end{aligned} \tag{0.24}$$

Теперь замечая, что $I(0)=\frac{2}{\epsilon(4\pi)^2}+\frac{1}{(4\pi)^2}\left(\log\frac{4\pi}{m^2}-\gamma\right)$, получаем для диграммы (e):

$$(e)=\frac{\lambda^2}{4}I^2(0)+\frac{\lambda^2}{2}I(0)I_r\left(q^2\right)+\text{ конечные члены.} \tag{0.25}$$

В итоге получаем

$$(d)+(e)+(f)+(g)+(h)=\frac{\lambda^2}{4}I^2\left(q^2\right)+\frac{\lambda^2}{4}I^2(0)+\frac{\lambda^2}{2}I(0)I_r\left(q^2\right)-\frac{3\lambda^2}{4}I(0)I\left(q^2\right)-$$
$$-\frac{\lambda^2}{4}I(0)I\left(q^2\right)+\lambda^2 Z_2^{(2)}+\text{ конечные члены }=$$
$$=\lambda^2\left(\frac{1}{4}\left(I_r\left(q^2\right)+I(0)\right)^2+\frac{1}{4}I(0)^2+\frac{1}{2}I(0)I_r\left(q^2\right)-I(0)\left(I_r\left(q^2\right)+I(0)\right)+Z_2^{(2)}+\text{ к.ч. }\right)=$$
$$=\lambda^2\left(\frac{1}{4}I_r^2\left(q^2\right)-\frac{1}{2}I^2(0)+Z_2^{(2)}+\text{ к.ч. }(p_1,p_2)\right)$$

откуда мы получаем, что $Z_2^{(2)}=\frac{1}{2}I^2(0)$ - к.ч. $(0,0)$, и в итоге

$$\Gamma^{\left([\phi^2]_R,2\right)}(q|p_1,p_2)=1-\frac{\lambda}{2}I_r\left(q^2\right)+\frac{\lambda^2}{4}\left(I_r^2\left(q^2\right)+\text{ к.ч. }(p_1,p_2)-\text{ к.ч. }(0,0)\right),\tag{0.27}$$

отметим, что интеграл $I_r\left(q^2\right)$ уже тоже входит во множество конечных членов, но мы оставили здесь его в явном виде.

28 Other problems in QFT

28.1 Problems for checking knowledge of QFT

28.1.1 Questions on Understanding the Essence of QFT

KUL-aqft-2023-hw2.3/3 О виде лагранжиана

Assume that you have a theory of a real scalar field $\phi(x)$ in d space-time dimensions. Write down the most general Lorentz invariant Lagrangian with a canonical (i.e. quadratic in the fields and the derivatives) kinetic term and polynomial interactions of degree less than or equal to 6. Analyze the mass dimensions of all terms in this Lagrangian and find the mass dimensions of all couplings. What is the maximal value of d for which this theory is renormalizable? Find out a polynomial potential (with integer powers) for this model which will render the theory renormalizable in $d=5,6,7,8$. What can you conclude about interacting scalar theories in general space-time dimensions?

Solution

В чем проблемы с уравнением КГ и как их решаем?

Answer:

(это в теории написано, еще раз вопрос про это. потому что у нас нет основного состояния в гамильтониане КГ (????), потом с причинностью разговоры, тоже в теории их напишу.)

Критерии перенормируемости

Какая теория является перенормируемой, а какая нет?

Answer:

28.1.2 Typical Problems for Checking Knowledge of Fundamentals

(! тут будет хороший список задач, которые не такие уж громоздкие, но очень типичные. как и в других всех записях.)

Количество поляризаций фотона?

Сколько у фотона поляризаций и почему?

Solution

Две. Потому что у нас изначально 4 (??), потом одна калибровкой Лоренца уходит, вторая (????) гармоническими функциями (?? как??).

Бесконечная отрицательная энергия дираковского поля?

(типичный вопрос, оформлю потом его.)

28.2 Problems about mathematical preparation

28.2.1 Problems about integrals and mathematical features for QFT

KUL-aqft-2023-hw1.2 Derivative of a function

Prove the identity

$$G\left(\frac{\partial}{\partial b}\right)F(b) = F\left(\frac{\partial}{\partial x}\right)\left[G(x)e^{xb}\right]\Big|_{x=0},$$

by assuming that the functions F and G can be expanded in a convergent power series around 0. Extend the result to the case of many variables, i.e. to the case where $(\vec{b},\vec{x})$ are vectors in $\mathbb{R}^n$.

Solution

I will show this for a special case: Let $G(\underline{x}) = e^{\underline{x}\cdot\underline{\alpha}}, F(\underline{b}) = e^{\underline{\beta}\underline{b}}$. Then the left-hand side is

$$G\left(\frac{\partial}{\partial \underline{b}}\right)F(\underline{b}) = e^{\frac{\partial}{\partial \underline{b}}\cdot\underline{\alpha}}F(\underline{b}) = F(\underline{b}+\underline{\alpha}) = e^{\underline{\beta}\cdot(\underline{b}+\underline{\alpha})}.$$

Let us attempt to consider the right-hand side:

$$F\left(\frac{\partial}{\partial \underline{x}}\right)G(\underline{x})e^{\underline{x}\cdot\underline{b}}\Big|_{\underline{x}=\underline{0}} = e^{\underline{\beta}\cdot\frac{\partial}{\partial \underline{x}}}e^{\underline{x}\cdot(\underline{\alpha}+\underline{b})}\Big|_{\underline{x}=\underline{0}} = e^{(\underline{x}+\underline{\beta})\cdot(\underline{\alpha}+\underline{b})}\Big|_{\underline{x}=0} = e^{\underline{\beta}\cdot(\underline{\alpha}+\underline{b})}.$$

The result is then true for any F and G as one may express F and G as a Fourier series.

Another Solution, for $d = 1$ Dimensional

Say

$$G(x) = \sum_{m=0}^{\infty} g_m x^m, \text{ and}$$
$$F(x) = \sum_{n=0}^{\infty} f_n x^n.$$

On one hand, we have

$$G\left(\frac{\partial}{\partial b}\right)F(b) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g_m f_n \frac{\partial^m}{\partial b^m} b^n$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} g_m f_n \frac{n!}{(n-m)!} b^{n-m}.$$

On the other hand, we have

$$F\left(\frac{\partial}{\partial x}\right)\left\{G(x)e^{bx}\right\}\Big|_{x=0}=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}f_n g_m\frac{\partial^n}{\partial x^n}\left\{x^m e^{bx}\right\}\Big|_{x=0}.$$

To get $\frac{\partial^n}{\partial x^n}\left\{x^m e^{bx}\right\}\big|_{x=0}\neq 0$, we need m derivatives to fall on x^m, and thus $n-m$ on e^{bx}, in the Leibniz rule. This is assuming there are sufficient derivatives to share, i.e. $n\geq m$. If $n<m$, there are insufficient derivatives to get any nonzero contribution, so we may reduce the sum to $\sum_{m=0}^{n}$. There are $\left(\begin{array}{c} n \\ m \end{array}\right)$ ways to get in such a situation where the derivative does not vanish. [1] Using also $e^{bx}\big|_{x=0}=1$, we see that

$$\begin{aligned}\frac{\partial^n}{\partial x^n}\left\{x^m e^{bx}\right\}\Big|_{x=0}&=m!\left(\begin{array}{c} n \\ m \end{array}\right)b^{n-m}\\&=\frac{n!}{(n-m)!}b^{n-m}\end{aligned}$$

and thus

$$F\left(\frac{\partial}{\partial x}\right)\left\{G(x)e^{bx}\right\}\Big|_{x=0}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}f_n g_m\frac{n!}{(n-m)!}b^{n-m}$$

which is identical to (1).

Another Solution, for d Dimensional

We will just write the calculation here, it's the same idea as before. Say

$$\begin{aligned}G(\underline{x})&=\sum_{m=0}^{\infty}\underline{g_m}\cdot\underline{x}^m,\ \text{and}\\F(\underline{x})&=\sum_{n=0}^{\infty}\underline{f_n}\cdot\underline{x}^n\cdot\\G\left(\frac{\partial}{\partial\underline{b}}\right)F(\underline{b})&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\underline{g_m}\cdot\frac{\partial^m}{\partial\underline{b}^m}\underline{f_n}\cdot\underline{b}^n\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{n!}{(n-m)!}\underline{g_m}\cdot\underline{f_n}b^{n-m}\\F\left(\frac{\partial}{\partial\underline{x}}\right)\left\{G(\underline{x})e^{\underline{b}\cdot\underline{x}}\right\}\Big|_{\underline{x}=\underline{0}}&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\underline{f_n}\cdot\frac{\partial^n}{\partial\underline{x}^n}\left\{\underline{g_m}\cdot\underline{x}^m e^{\underline{b}\cdot\underline{x}}\right\}\Big|_{\underline{x}=\underline{0}}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\underline{f_n}\cdot\underline{g_m}m!\left(\begin{array}{c} n \\ m \end{array}\right)b^{n-m}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\underline{f_n}\cdot\underline{g_m}\frac{n!}{(n-m)!}b^{n-m}\end{aligned}$$

Stanford.Zhou.2022.9.2. Feynman parameters. (12/45p)

As you saw in problem 1, loop integrals take a few generic forms. For simplicity, let's consider a scalar theory, which doesn't have nontrivial factors in the numerator of the loop integrand. Then some generic loop integrals are the "tadpole", "bubble", and "triangle",

$$I_T\left(a;m^2\right)=\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2-m^2\right)^a}$$
$$I_B\left(a,b;p^2\right)=\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2\right)^a\left((k+p)^2\right)^b}$$
$$I_\triangle\left(a,b,c;p_1,p_2\right)=\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2\right)^a\left(\left(k+p_1\right)^2\right)^b\left(\left(k-p_2\right)^2\right)^c}$$

named after the kinds of Feynman diagrams in which they appear. Here we have suppressed $i0$ terms, but you should keep in mind that they're implicitly there, which will be important in the last subpart. We regard a, b, and c as general real exponents, though in practice they will usually be positive integers. The dimension d is also a real parameter.

To compute the bubble and triangle integrals, it is helpful to introduce Feynman parameters, which combine the factors into the denominator into a power of a single quantity, like the tadpole integral. We will first have to build up some mathematical machinery.

a) The gamma function is defined by

$$\Gamma(z)=\int_0^\infty dx x^{z-1}e^{-x}$$

and for positive integer n, satisfies $\Gamma(n)=(n-1)\,!$. Show that for real ν,

$$\frac{1}{A^\nu}=\frac{1}{\Gamma(\nu)}\int_0^\infty dx x^{\nu-1}e^{-xA}$$

b) Show that for general ν_i,

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}}=\frac{\Gamma\left(\sum_{i=1}^n\nu_i\right)}{\prod_{i=1}^n\Gamma\left(\nu_i\right)}\int_0^1\left(\prod_{i=1}^n dx_i\right)\delta\left(1-\sum_{i=1}^n x_i\right)\frac{\prod_{i=1}^n x_i^{\nu_i-1}}{\left[\sum_{i=1}^n x_iA_i\right]^{\sum_{i=1}^n\nu_i}}$$

Here, the x_i are called Feynman parameters.

Solution to a:

Using the result of part (a) n times yields

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}}=\frac{1}{\prod_{i=1}^n\Gamma\left(\nu_i\right)}\int_0^\infty\prod_{i=1}^n dx_i x_i^{\nu_i-1}e^{-\sum_{i=1}^n x_iA_i}$$

This is already pretty close to the desired answer, so we now introduce an identity

$$1=\int_0^\infty dy\delta\left(y-\sum_{i=1}^n x_i\right)$$

which converts our result to

$$\frac{1}{\prod_{i=1}^n A_i^{\nu_i}}=\frac{1}{\prod_{i=1}^n\Gamma\left(\nu_i\right)}\int_0^\infty dy\prod_{i=1}^n dx_i x_i^{\nu_i-1}\delta\left(y-\sum_{i=1}^n x_i\right)e^{-\sum_{i=1}^n x_iA_i}$$

Let us rescale all the x_i by y so they sum up to 1, as required in the final answer. The result is

$$\begin{aligned}\frac{1}{\prod_{i=1}^{n} A_i^{\nu_i}} &= \frac{1}{\prod_{i=1}^{n}\Gamma(\nu_i)} \int_0^\infty dy y^{\sum_{i=1}^{n}\nu_i} \prod_{i=1}^{n} dx_i x_i^{\nu_i-1} \delta\left(y - y\sum_{i=1}^{n} x_i\right) e^{-y\sum_{i=1}^{n} x_i A_i} \\ &= \frac{1}{\prod_{i=1}^{n}\Gamma(\nu_i)} \int_0^\infty dy y^{-1+\sum_{i=1}^{n}\nu_i} \prod_{i=1}^{n} dx_i x_i^{\nu_i-1} \delta\left(1 - \sum_{i=1}^{n} x_i\right) e^{-y\sum_{i=1}^{n} x_i A_i} \\ &= \frac{\Gamma\left(\sum_{i=1}^{n}\nu_i\right)}{\prod_{i=1}^{n}\Gamma(\nu_i)} \int_0^1 \prod_{i=1}^{n} dx_i x_i^{\nu_i-1} \delta\left(1 - \sum_{i=1}^{n} x_i\right) \frac{1}{\left[\sum_{i=1}^{n} x_i A_i\right]^{\sum_{i=1}^{n}\nu_i}}\end{aligned}$$

as desired, where we performed the y integral using (7). **c)** Apply (8) to $I_B\left(a,b;p^2\right)$ to get an integrand whose denominator contains a single quantity raised to the $a+b$ power.

Solution:

$$\begin{aligned}I_B\left(a,b;p^2\right) &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int \frac{d^dk}{(2\pi)^d} \int_0^\infty dx_1 x_1^{a-1} dx_2 x_2^{b-1} \frac{\delta(1-x_1-x_2)}{\left[k^2x_1 + (k+p)^2 x_2\right]^{a+b}} \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int \frac{d^dk}{(2\pi)^d} \int_0^\infty dx_1 x_1^{a-1} dx_2 x_2^{b-1} \frac{\delta(1-x_1-x_2)}{\left[k^2(x_1+x_2) + (2kp+p^2)x_2\right]^{a+b}} \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int \frac{d^dk}{(2\pi)^d} \int_0^\infty dx_1 x_1^{a-1}(1-x_1)^{b-1} \frac{1}{\left[k^2 + (2kp+p^2)(1-x_1)\right]^{a+b}}.\end{aligned}$$

d) Complete the evaluation of $I_B\left(a,b;p^2\right)$ by Wick rotating to Euclidean signature and performing all the remaining integrals. As a hint, the Euclidean integration measure is $d^dk_E = |k_E|^{d-1}\, d|k_E|\, d\Omega_d$, where the surface area of a unit sphere in d dimensions is

$$\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$

As a second hint, you can use the result

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 dx x^{a-1}(1-x)^{b-1}$$

To check your answer, if you set $d = 4-2\epsilon$ then you should find

$$I_B\left(1,1;p^2\right) = \left(\frac{-p^2-i0}{4\pi}\right)^{-\epsilon} \frac{i\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{(4\pi)^2(1-2\epsilon)\epsilon\Gamma(1-2\epsilon)}$$

Solution:

Next, we perform the linear shift $k \to k - p(1-x_1)$ and find

$$I_B\left(a,b;p^2\right) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int \frac{d^dk}{(2\pi)^d} \int_0^\infty dx_1 x_1^{a-1}(1-x_1)^{b-1} \frac{1}{\left[k^2 + p^2x_1(1-x_1)\right]^{a+b}}$$

Notice, that we dropped the " $+i0$ " Feynman prescription from the integral at the very beginning. We could have kept an infinitesimal mass term throughout. To track the infinitesimal imaginary part, we can simply associate it with the momentum, $p^2 + i0$. Next, we perform a Wick rotation $k^0 = ik_E^0$ and $\vec{k} = \vec{k}_E$.

$$
\begin{aligned}
I_B\left(a,b;p^2\right) &= i\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int\frac{d^dk_E}{(2\pi)^d}\int\limits_0^\infty dx_1x_1^{a-1}\left(1-x_1\right)^{b-1}\frac{1}{\left[-k_E^2+p^2x_1\left(1-x_1\right)\right]^{a+b}}\\
&= \frac{i(-1)^{a+b}}{2(2\pi)^d}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int\limits_0^\infty d\left(k_E^2\right)\int d\Omega_d\left(k_E^2\right)^{\frac{d}{2}-1}\int\limits_0^\infty dx_1x_1^{a-1}\left(1-x_1\right)^{b-1}\frac{1}{\left[k_E^2-p^2x_1\left(1-x_1\right)\right]^{a+b}}\\
&= \frac{i(-1)^{a+b}}{2(2\pi)^d}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(-a-b-\frac{d}{2}\right)}{\Gamma(a)\Gamma(b)}\Omega_d\int\limits_0^1 dx_1x_1^{a-1}\left(1-x_1\right)^{b-1}\left(-\left(p^2+i0\right)x_1\left(1-x_1\right)\right)^{\frac{d}{2}-a-b}\\
&= \frac{i(-1)^{a+b}\left(-\left(p^2+i0\right)\right)^{\frac{d}{2}-a-b}}{2(2\pi)^d}\frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(a+b-\frac{d}{2}\right)}{\Gamma(a)\Gamma(b)}\Omega_d\int\limits_0^1 dx_1x_1^{\frac{d}{2}-b-1}\left(1-x_1\right)^{\frac{d}{2}-a-1}\\
&= \frac{i(-1)^{a+b}\left(-\left(p^2+i0\right)\right)^{\frac{d}{2}-a-b}}{(4\pi)^{\frac{d}{2}}}\frac{\Gamma\left(a+b-\frac{d}{2}\right)}{\Gamma(a)\Gamma(b)}\frac{\Gamma\left(\frac{d}{2}-a\right)\Gamma\left(\frac{d}{2}-b\right)}{\Gamma(d-a-b)}.
\end{aligned}
$$

Above we used the Euler β-function integral

$$
\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}=\int\limits_0^1 dxx^{a-1}(1-x)^{b-1}
$$

Stanford.Zhou.2022.9.3. Passarino-Veltman reduction. (10/45p)

More generally, loop integrands will have momenta and other factors in the numerator. However, we can often use symmetry properties and a technique called Passarino-Veltman reduction to reduce them to the simpler loop integrands considered in problem 2. For example, consider the rank 2 tadpole integral

$$
I_T^{\mu\nu}\left(a;m^2\right)=\int\frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{\left(k^2-m^2\right)^a}
$$

Because the right-hand side is a Lorentz tensor, the left-hand side must be as well. As there are no momenta in the integral, the only option is the metric, so we must have

$$
I_T^{\mu\nu}\left(a;m^2\right)=\eta^{\mu\nu}A
$$

for some Lorentz scalar A. Contracting both sides with the metric, we find

$$
\begin{aligned}
A &= \frac{1}{d}\int\frac{d^dk}{(2\pi)^d}\frac{k^2}{\left(k^2-m^2\right)^a}\\
&= \frac{1}{d}\int\frac{d^dk}{(2\pi)^d}\frac{\left(k^2-m^2\right)+m^2}{\left(k^2-m^2\right)^a}
\end{aligned}
$$

This is just two "scalar" tadpole integrals, as defined in (3), so we conclude

$$
I_T^{\mu\nu}\left(a;m^2\right)=\frac{\eta^{\mu\nu}}{d}\left[I_T\left(a-1;m^2\right)+m^2I_T\left(a;m^2\right)\right]
$$

a) Apply the same reasoning to the rank 3 and rank 4 tadpole integrals,

$$
\begin{aligned}
I_T^{\mu\nu\rho}\left(a;m^2\right) &= \int\frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu k^\rho}{\left(k^2-m^2\right)^a}\\
I_T^{\mu\nu\rho\sigma}\left(a;m^2\right) &= \int\frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu k^\rho k^\sigma}{\left(k^2-m^2\right)^a}.
\end{aligned}
$$

Solution:

Again, only combinations of the metric tensor are available to us. There is no combination of metric tensors that would allow us to write down a tensor with 5 indices, so the first integral vanishes. (Another way of seeing this is that the integrand is odd under the transformation $k \to -k$ and the integral is consequently equal to minus itself and thus zero.)

The integrand with four Lorentz indices is symmetric under the exchange of any pair of indices, so we must have

$$I_T^{\mu\nu\rho\sigma}(a) = A\left(\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right)$$

Contracting with $\eta_{\mu\nu}\eta_{\rho\sigma}$ we find

$$\begin{aligned} I_T^{\mu\nu\rho\sigma}\eta_{\mu\nu}\eta_{\rho\sigma}(a) &= A\left(d^2+2d\right) \\ &= \int \frac{d^dk}{(2\pi)^d}\frac{\left(k^2\right)^2}{\left(k^2-m^2\right)^a} \\ &= \int \frac{d^dk}{(2\pi)^d}\frac{\left(k^2-m^2+m^2\right)^2}{\left(k^2-m^2\right)^a} \\ &= \int \frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2-m^2\right)^{a-2}} + \int \frac{d^dk}{(2\pi)^d}\frac{2m^2}{\left(k^2-m^2\right)^{a-1}} + \int \frac{d^dk}{(2\pi)^d}\frac{m^4}{\left(k^2-m^2\right)^a} \end{aligned}$$

from which we conclude

$$I_T^{\mu\nu\rho\sigma}\left(a;m^2\right) = \frac{\eta^{\mu\nu}\eta^{\rho\sigma}+\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}}{d^2+2d}\left(I_T\left(a-2;m^2\right)+2m^2I_T\left(a-1;m^2\right)+m^4I_T\left(a;m^2\right)\right)$$

b) Similarly, the rank 1 and 2 bubble integrals can be written as

$$\begin{aligned} I_B^{\mu}\left(a,b;p^2\right) &= \int \frac{d^dk}{(2\pi)^d}\frac{k^\mu}{\left(k^2\right)^a\left((k+p)^2\right)^b} = p^\mu C. \\ I_B^{\mu\nu}\left(a,b;p^2\right) &= \int \frac{d^dk}{(2\pi)^d}\frac{k^\mu k^\nu}{\left(k^2\right)^a\left((k+p)^2\right)^b} = \left(\eta^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right)A_\perp + \frac{p^\mu p^\nu}{p^2}B_\parallel. \end{aligned}$$

Express $A_\perp$ and $B_\parallel$and C in terms of p^2 and the scalar bubble integral (4).

Solution:

$$\begin{aligned} I_B^{\mu}\left(a,b;p^2\right)p_\mu = p^2C &= \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\frac{2kp}{\left(k^2\right)^a\left((k+p)^2\right)^b} \\ C &= \frac{1}{2p^2}\int \frac{d^dk}{(2\pi)^d}\frac{(k+p)^2-k^2-p^2}{\left(k^2\right)^a\left((k+p)^2\right)^b} \\ &= \frac{1}{2p^2}\left[I_B\left(a,b-1;p^2\right)-I_B\left(a-1,b;p^2\right)-p^2I_B\left(a,b;p^2\right)\right]. \\ I_B^{\mu\nu}\left(a,b;p^2\right)\eta_{\mu\nu} &= (d-1)A_\perp + B_\parallel = I_B(a-1,b). \\ I_B^{\mu\nu}\left(a,b;p^2\right)\frac{p^\mu p^\nu}{p^2} &= B_\parallel = \frac{1}{4p^2}\int \frac{d^dk}{(2\pi)^d}\frac{(2kp)^2}{\left(k^2\right)^a\left((k+p)^2\right)^b} \\ &= \frac{1}{4p^2}\int \frac{d^dk}{(2\pi)^d}\frac{\left((k+p)^2-k^2-p^2\right)^2}{\left(k^2\right)^a\left((k+p)^2\right)^b} \\ &= \frac{1}{4p^2}\Big[I_B\left(a-2,b;p^2\right)+I_B\left(a,b-2;p^2\right)+\left(p^2\right)^2I_B\left(a,b;p^2\right) \\ &\quad -2I_B\left(a-1,b-1;p^2\right)-2p^2I_B\left(a-1,b;p^2\right)-2p^2I_B\left(a,b-1;p^2\right)\Big] \\ A_\perp &= \frac{1}{d-1}\left[I_B\left(a-1,b;p^2\right)-B_\parallel\right]. \end{aligned}$$

28.2.2 Other Problems about γ-Matrices and Other Mathematics for Spinors

(всякие особые задачи пока в отдельной записи по модиф теориям поля. простые и скучные - тоже в подготовительной записи!)

МФТИ-Киселев-2020.9.^{C} Следы

Доказать, что след нечетного числа гамма-матриц Дирака равен нулю, а для четного n имеет место соотношение редукции

$$\mathrm{tr}\left(\gamma^{\mu_1}...\gamma^{\mu_n}\right)=g^{\mu_1\mu_2}\,\mathrm{tr}\left(\gamma^{\mu_3}...\gamma^{\mu_n}\right)+g^{\mu_1\mu_3}\,\mathrm{tr}\left(\gamma^{\mu_2}\gamma^{\mu_4}...\gamma^{\mu_n}\right)+...$$

Solution

(пока такое не научился доказывать)

ПШ-5.3.Куча вопросов о спинорах

(мб много сделаю параграфов из этой задачи)
Спинорные произведения, введенные в задаче 3.3, позволяют эффективно вычислять древесные диаграммы, включающие безмассовые частицы. Вспомним, как мы определяли спинорные произведения в задаче 3.3. Пусть u_{L0} и u_{R0} являются соответственно левым и правым спинорами при некотором фиксированном светоподобном импульсе k_0. Они удовлетворяют соотношениям:

$$u_{L0}\bar{u}_{L0}=\left(\frac{1-\gamma^5}{2}\right)\not{k}_0,\qquad u_{R0}\bar{u}_{R0}=\left(\frac{1+\gamma^5}{2}\right)\not{k}_0.$$

(Эти соотношения есть просто проекции более стандартной формулы $\sum u_0\bar{u}_0=\not{k}_0$ на состояние с определенной спиральностью.) Затем определим спиноры для любого другого светоподобного импульса p как

$$u_L(p)=\frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{R0},\quad u_R(p)=\frac{1}{\sqrt{2p\cdot k_0}}\not{p}u_{L0}.$$

Мы показали, что эти спиноры удовлетворяют условию $pu(p)=0$. Так как в задачу не входит масса m, то эти спиноры можно использовать как для фермионов, так и для антифермионов. Определим

$$s\left(p_1,p_2\right)=\bar{u}_R\left(p_1\right)u_L\left(p_2\right),\quad t\left(p_1,p_2\right)=\bar{u}_L\left(p_1\right)u_R\left(p_2\right).$$

В выделенной системе отсчета доказаны тождества:

$$t\left(p_1,p_2\right)=\left(s\left(p_2,p_1\right)\right)^*,\quad s\left(p_1,p_2\right)=-s\left(p_2,p_1\right),\quad \left|s\left(p_1,p_2\right)\right|^2=2p_1\cdot p_2.$$

Теперь воспользуемся этими результатами. (a) Для разминки дайте другое доказательство последнего соотношения в (3): воспользуйтесь (1), чтобы переписать $\left|s\left(p_1,p_2\right)\right|^2$ как след от матриц Дирака, а затем вычислите след.
(b) Покажите, что для любого набора дираковских матриц

$$\mathrm{tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\cdots\right]=\mathrm{tr}\left[\cdots\gamma^\rho\gamma^\nu\gamma^\mu\right]$$

где $\mu,\nu,\rho,...=0,1,2,3$, и 5. Воспользуйтесь этим тождеством для доказательства соотношения

$$\bar{u}_L\left(p_1\right)\gamma^\mu u_L\left(p_2\right)=\bar{u}_R\left(p_2\right)\gamma^\mu u_R\left(p_1\right)$$

(c) Докажите тождество Фирца:

$$\bar{u}_L\left(p_1\right)\gamma^\mu u_L\left(p_2\right)\left[\gamma_\mu\right]_{ab}=2\left[u_L\left(p_2\right)\bar{u}_L\left(p_1\right)+u_R\left(p_1\right)\bar{u}_R\left(p_2\right)\right]_{ab},$$

где $a,b=1,2,3,4$ - дираковские индексы. Доказательство может быть проведено путем проверки следующих утверждений. Правая часть этого соотношения является матрицей Дирака; следовательно, она может быть записана как линейная комбинация -матриц, которые обсуждались в разделе 3.4. Эта правая часть удовлетворяет равенству

$$\gamma^5[M]=-[M]\gamma^5,$$

и, следовательно, должна иметь вид:

$$[M] = \left(\frac{1-\gamma^5}{2}\right)\gamma_\mu V^\mu + \left(\frac{1+\gamma^5}{2}\right)\gamma_\mu W^\mu$$

где V^μ и W^μ – 4-векторы. Их можно найти с помощью техники вычисления следов, например:

$$V^\nu = \frac{1}{2}\,\mathrm{tr}\left[\gamma^\nu\left(\frac{1-\gamma^5}{2}\right)M\right]$$

(d) Рассмотрим процесс $e^+e^- \to \mu^+\mu^-$в главном порядке по α, пренебрегая массами электрона и мюона. Сначала рассмотрим случаи, когда электрон и конечный мюон правополяризованы, а позитрон и антимюон - левополяризованы*. Используя тождество Фирца, покажите, что амплитуду можно выразить через спинорные произведения. Возведите ее в квадрат и воспроизведите выражение (5.22) для

$$\frac{d\sigma}{d\cos\theta}\left(e_R^- e_L^+ \to \mu_R^- \mu_L^+\right)$$

Вычислите сечения для других спиральностей частиц и покажите, что они идентичны приведенным в разделе 5.2.

(e) Используя формализм спинорного произведения, вычислите дифференциальное сечение рассеяния Баба для состояний с заданными спиральностями частиц в случае безмассовых электронов. Сечение, усредненное по спиральностям исходных и просуммированное по спиральностям конечных частиц, должно воспроизводить ответ задачи 5.2. Нужно обратить внимание на то, как суммируются вклады от определенных спиральностей.

* Используйте спинор v_R для антимюона и $\bar{u}_R$ для позитрона.

Solution

(пишу коротко, потому что есть как раз теор часть для всяких деталей этого)

28.3 Problems about special quantum theories

28.3.1 Problems about Non-Abelian QED

МФТИ-Мусаев-2023.3.19/19. Духи и амплитуды

Показать, что в абелевой калибровочной теории духи не дают вклада в амплитуды.

28.3.2 Problems of Typical Modifications of QFT

Stanford.Zhou.2022.7.4.* Chern-Simons theory. (5/45p)

This optional problem presents a way to produce massive gauge fields.
a) The Proca Lagrangian for a massive vector field is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu$$

Find the equation of motion for A^μ and show that for $m \neq 0$, it implies

$$\left(\partial^2 + m^2\right)A^\mu = 0, \quad \partial_\mu A^\mu = 0$$

However, this action does not have a gauge symmetry.

Solution:

The derivation of the equation of motion is similar to problem 3(b), but now a new term appears on the right-hand side, giving

$$-\partial_\mu F^{\mu\nu} = m^2 A^\nu.$$

This is the equation of motion. Applying ∂_ν to both sides yields

$$m^2 \partial_\nu A^\nu = -\partial_\nu \partial_\mu F^{\mu\nu} = 0$$

by the antisymmetry of F, which shows $\partial_\mu A^\mu = 0$ when m is nonzero. Plugging this result back into the original equation of motion gives

$$-\partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = -\partial^2 A^\nu = m^2 A^\nu$$

as desired.

b) On the other hand, in two spatial dimensions it is possible to have massive gauge fields. We consider a Lagrangian with a "Chern-Simons" term,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{\alpha}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} A_\rho$$

where all indices take the values 0,1, and 2, and $\epsilon^{012} = 1$. (This new term is a relative of the term we considered in 3(e), but it is not a total derivative.) Show that the action is gauge invariant, and find the Euler-Lagrange equation for A^μ.

Solution:

The proof that the action is gauge invariant is similar to 3(e). Note that the change in the Lagrangian under a gauge transformation is

$$\delta\mathcal{L} = \frac{\alpha}{4} \epsilon^{\mu\nu\rho} F_{\mu\nu} \partial_\rho \alpha = \frac{\alpha}{2} \epsilon^{\mu\nu\rho} \left(\partial_\mu A_\nu\right) \partial_\rho \alpha = \partial_\rho \left(\frac{\alpha}{2} \epsilon^{\mu\nu\rho} \alpha \partial_\mu A_\nu\right)$$

which is a total derivative, as desired. As for the Euler-Lagrange equation, starting from our solution to 3(b), we pick up an extra term on both the left-hand and right-hand sides, giving

$$-\partial_\mu F^{\mu\nu} + \frac{\alpha}{2} \epsilon^{\mu\nu\rho} \partial_\mu A_\rho = \frac{\alpha}{4} \epsilon^{\mu\rho\nu} F_{\mu\rho}$$

We can clean this up using the antisymmetry of ϵ to give

$$\partial_\mu F^{\mu\nu} = \frac{\alpha}{2} \epsilon^{\mu\nu\rho} F_{\mu\rho}$$

c) The Chern-Simons term is sometimes called a "topological" mass term, because it doesn't depend on the metric. Show that the equation of motion implies

$$\left(\partial^2 + m^2\right) F^{\mu\nu} = 0$$

for some m you should find. Thus the field is massive, yet still has a gauge symmetry.

Solution:

This is a bit fiddly. The idea is that we want to use the equation of motion twice. But in its current form, we can't really do much to it, because of the factor of ϵ on the right-hand side which already contracts with most of the indices. However, we can clear factors of ϵ out of the way by contracting with additional copies of ϵ, and using

$$\epsilon^{\mu\nu\rho} \epsilon_{\mu\alpha\beta} = \delta^\nu_\alpha \delta^\rho_\beta - \delta^\nu_\beta \delta^\rho_\alpha.$$

To get started, we contract both sides of S82 with $\epsilon_{\nu\alpha\beta}$, giving

$$\epsilon_{\nu\alpha\beta} \partial_\mu F^{\mu\nu} = \alpha F_{\beta\alpha}$$

Now we can apply ∂^β to both sides and use the equation of motion again, giving

$$\epsilon_{\nu\alpha\beta} \partial^\beta \partial_\mu F^{\mu\nu} = \frac{\alpha^2}{2} \epsilon_{\mu\alpha\nu} F^{\mu\nu}$$

To get rid of the ϵ' s again, we contract both sides with $\epsilon^{\alpha\delta\sigma}$, giving

$$\partial^\delta \partial_\mu F^{\mu\sigma} - \partial^\sigma \partial_\mu F^{\mu\delta} = \alpha^2 F^{\sigma\delta}$$

Finally, to get the factor of ∂^2 we want, we can apply ∂_δ to both sides, giving

$$\partial^2 \partial_\mu F^{\mu\sigma} = \alpha^2 \partial_\delta F^{\sigma\delta}$$

where a term vanished due to the antisymmetry of F. Cleaning this up, we have

$$0 = \left(\partial^2 + \alpha^2\right)\left(\partial_\mu F^{\mu\nu}\right) = \left(\partial^2 + \alpha^2\right)\left(\frac{\alpha}{2}\epsilon^{\mu\nu\rho}F_{\mu\rho}\right)$$

where we used the equation of motion a final time. Now we just clear away the factor of ϵ by contracting with $\epsilon_{\nu\alpha\beta}$, which finally gives the result with $m = \alpha$.

Since there is no Chern-Simons term in three spatial dimensions, it has little role in particle physics, but it is important in the description of topological effects in two-dimensional condensed matter systems.

Stanford.Zhou.2022.8.4.* Nonminimal couplings. (5/45p)

In problem 1, you found the charge and g-factor for a Dirac fermion minimally coupled to the electromagnetic field, i.e. via the simplest possible interaction (1). However, if the fermion is composite, or interacts with heavier particles, we might need additional terms to describe the coupling. For all parts of this problem, you should adopt the formalism of problem 1 and work in the nonrelativistic limit. Detailed calculations are not needed; qualitative final answers (with justification) are sufficient.

a) The g-factors of the proton and neutron are not given by the result you found in problem 1. When the Dirac equation was invented, physicists explained this by adding an additional term to $\mathcal{H}_I$, proportional to $i\bar{\Psi}\left[\gamma^\mu, \gamma^\nu\right]\Psi F_{\mu\nu}$. Show that the physical effect of such a term is indeed to shift the magnetic dipole moment.

Solution:

If we follow the exact same logic as in problem 1, we'll end up with a term in $\langle p, s\,|H_I|\,p', s'\rangle$ proportional to

$$\int d\mathbf{x} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \bar{u}_s(p)\left[\gamma^\mu, \gamma^\nu\right] u_{s'}(p') F_{\mu\nu}(x)$$

In the presence of a uniform magnetic field, this becomes

$$\begin{aligned}\int d\mathbf{x} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \bar{u}_s(p)\left[\gamma^i, \gamma^j\right] u_{s'}(p') F_{ij}(x) &\propto \int d\mathbf{x} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \epsilon^{ijk}\xi_s^\dagger \sigma^k \xi_{s'} F_{ij} \\ &\propto \not{\delta}\left(\mathbf{p}-\mathbf{p}'\right)\xi_s^\dagger(\boldsymbol{\sigma}\cdot\mathbf{B})\xi_{s'}\end{aligned}$$

which, as we already saw, is precisely what you would get from a magnetic dipole moment.

You might be concerned that this term also causes an interaction with electric fields, as it contains

$$\int d\mathbf{x} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \bar{u}_s(p)\left[\gamma^0, \gamma^i\right] u_{s'}(p') F_{0i}(x).$$

However, we have

$$\left[\gamma^0, \gamma^i\right] = -2\begin{pmatrix} \sigma^i & \\ & -\sigma^i \end{pmatrix}$$

so that when we do the spinor contractions in the nonrelativistic limit, we get two terms that cancel in the nonrelativistic limit. So the only effect of this term is a magnetic dipole moment.

b) What is the physical effect of a term proportional to $\bar{\Psi}\left[\gamma^\mu, \gamma^\nu\right]\gamma^5\Psi F_{\mu\nu}$?

Solution:

By the same logic as before, we'll get a term in the matrix element like

$$\int d\mathbf{x} e^{-i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}} \bar{u}_s(p) \left[\gamma^\mu, \gamma^\nu\right] \gamma^5 u_{s'}(p') F_{\mu\nu}(x)$$

where

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & \\ & \mathbb{1}_2 \end{pmatrix}$$

in the Weyl representation. Recall that both $\left[\gamma^i, \gamma^j\right]$ and $\left[\gamma^i, \gamma^0\right]$ have Pauli matrices on the diagonal, but the latter has them with opposite signs. The presence of the γ^5 essentially exchanges the two. It is now the $\left[\gamma^i, \gamma^0\right]$ that contribute in the nonrelativistic limit, giving a term proportional to $\not{\delta}(\mathbf{p}-\mathbf{p}')\,\xi_s^\dagger(\boldsymbol{\sigma}\cdot\mathbf{E})\xi_{s'}$. Physically, this corresponds to giving the particle an electric dipole moment.

The presence of an electric dipole moment implies the violation of CP symmetry. Many past and ongoing experiments search for the electric dipole moment of the electron, which could arise from physics beyond the Standard Model, but so far none has been detected. The Standard Model generically predicts a sizable electric dipole moment for the neutron, which should have been easily detected decades ago, but mysteriously none has been found.

The above two are the only terms with dimension 5, and are therefore the simplest nonminimal couplings one could consider. Next, let's consider some dimension 6 terms.

c) What is the physical effect of a term proportional to $\bar{\Psi}\gamma^\mu\Psi\partial^\nu F_{\mu\nu}$?

Solution:

This is just like the coupling in problem 1, but we have replaced A_μ with $\partial^\nu F_{\mu\nu} = -J_\mu$. In the nonrelativistic limit, the leading effect is the one we found in problem 1(b), as this is the only one not suppressed by powers of v. By the same logic, the result is a term in H_I^{nr} proportional to $J^0(\mathbf{x}) = \rho(\mathbf{x})$, where ρ is the external charge density. That is, it has no effect, regardless of the fields at the particle's location, unless the particle is sitting right on top of some other charge.

This might seem rather mysterious, but it's perfectly comprehensible within classical electromagnetism. Consider a tiny spherical capacitor with places of charge $\pm q$. This charge configuration produces exactly zero electromagnetic field everywhere outside it, which means that it does not interact with external charges until they are right inside the capacitor, in which case they feel the potential inside.

If the particle already has a net charge, then superposing the above configuration spreads the charge out in space. For this reason, this term is said to produce a "charge radius". The charge radius is one way to quantify the size of particles like the proton. (The "proton radius puzzle" is the fact that two independent measurements of this quantity seemed to give different results.) When we say the electron is a pointlike particle, we mean that it has no charge radius, as far as we've detected.

d) What is the physical effect of a term proportional to $\bar{\Psi}\gamma^\mu\gamma^5\Psi\partial^\nu F_{\mu\nu}$?

Solution:

In this case the leading term in the nonrelativistic limit is that of 1(d), but with $\mathbf{B}$ replaced with $\mathbf{J}$, corresponding to an interaction of the form $\mathbf{S}\cdot\mathbf{J}(\mathbf{x})$. In other words, we have an interaction with currents directly on top of the particle.

To physically interpret this, imagine the particle is a tiny toroidal solenoid with its axis of symmetry aligned with $\mathbf{S}$. Such a configuration produces zero magnetic field outside itself, but the magnetic field affects currents that pass right over the toroid. When the current flows parallel to $\mathbf{S}$, it gets pushed directly towards (or away from) the toroid's axis, while a current flowing perpendicular to $\mathbf{S}$ experiences no net force; this therefore reproduces the $\mathbf{S}\cdot\mathbf{J}$ potential.

Depending on the field of physics, such a current configuration is called an anapole moment, or a polar toroidal dipole moment. You have probably not seen such a thing in electromagnetism classes, because it doesn't show up in the multipole expansion - it is a near field effect, not a far field effect. But anapole moments are important observables in nuclear physics.

One can keep going. The anapole moment produces a localized parity-violating magnetic field, while the so-called Schiff moment, also important in nuclear physics, produces a localized parity-violating electric field. It corresponds to a term in H_I^{nr} proportional to $\mathbf{S}\cdot\nabla\rho$.

Stanford.exam-2022.5*/5. Kaluza-Klein theory. (5/40p)

Some theories of physics beyond the Standard Model involve compactified extra spacetime dimensions. In this problem, you will see why this generically gives rise to many new particles, and why the extra dimensions are hard to detect when they are very small.

Consider a massless real scalar field on a five-dimensional spacetime, where the extra dimension is compactified on a circle of radius R. That is, points are labeled by (x, w) where $x = (t, x, y, z)$ is the usual four-dimensional spacetime coordinate, and $w \in [0, 2\pi R]$, with the points $w = 0$ and $w = 2\pi R$ identified. The metric on this spacetime is still mostly negative, so $\eta^{00} = 1$ with the other diagonal elements negative. The action is

$$S = \int d^4x \int_0^{2\pi R} dw \frac{1}{2} \left(\partial_M \phi(x, w)\right) \left(\partial^M \phi(x, w)\right)$$

where the index M ranges from 0 to 4.

a) Using Fourier series, the w-dependence of the field can be written as

$$\phi(x, w) = \sum_n \phi^{(n)}(x) e^{ik_n w}$$

where the sum is over integer n. What are the k_n, and how is $\phi^{(n)}$ related to $\phi^{(-n)}$?

b) Plug this decomposition into the action and perform the w integral to yield an ordinary four-dimensional action, written in terms of the fields $\phi^{(n)}(x)$ for $n \geq 0$. What are the physical masses m_n of these fields?

(Hint: you should rescale the fields to give the kinetic terms the usual normalizations.)

c) Show that we can use a gauge transformation to set $A_4^{(n)} = 0$ when $n \neq 0$, and explain why this is not possible when $n = 0$.

d) Using this gauge, find all the resulting four-dimensional fields and their masses.

Solution a)

Because the function is periodic in w with period $2\pi R$, we must have $k_n = n/R$.

Since the field is real, we must have $\left(\phi^{(n)}\right)^* = \phi^{(-n)}$.

Solution b)

The result is

$$\begin{aligned}
S &= \int d^4x \sum_{n,n'} \int_0^{2\pi R} dw \frac{1}{2} \partial_M \left(\phi^{(n)} e^{inw/R}\right) \partial^M \left(\phi^{(n')} e^{in'w/R}\right) \\
&= \int d^4x \sum_{n,n'} \int_0^{2\pi R} dw \frac{1}{2} \left(\partial_\mu \phi^{(n)} \partial^\mu \phi^{(n')} - \frac{in}{R}\frac{in'}{R} \phi^{(n)} \phi^{(n')}\right) e^{inw/R} e^{in'w/R} \\
&= \int d^4x \sum_{n,n'} \frac{1}{2} \left(\partial_\mu \phi^{(n)} \partial^\mu \phi^{(n')} - \frac{in}{R}\frac{in'}{R} \phi^{(n)} \phi^{(n')}\right) 2\pi R \delta_{n,-n'} \\
&= \int d^4x \sum_{n} \frac{1}{2} \left(\partial_\mu \phi^{(n)} \partial^\mu \phi^{(-n)} - \frac{n^2}{R^2} \phi^{(n)} \phi^{(-n)}\right) 2\pi R.
\end{aligned}$$

Pairing up terms with positive and negative n, we get

$$S = 2\pi R \int d^4x \frac{1}{2} \partial_\mu \phi^{(0)} \partial^\mu \phi^{(0)} + \sum_{n>0} \left(\left|\partial_\mu \phi^{(n)}\right|^2 - \frac{n^2}{R^2} \left|\phi^{(n)}\right|^2\right)$$

This is almost the action for a real scalar field and a bunch of complex scalar fields, but we need to rescale to get the right kinetic terms. Defining $\varphi^{(n)} = \sqrt{2\pi R}\phi^{(n)}$, we conclude

$$S = \int d^4x \frac{1}{2} \partial_\mu \varphi^{(0)} \partial^\mu \varphi^{(0)} + \sum_{n>0} \left(\left|\partial_\mu \varphi^{(n)}\right|^2 - \frac{n^2}{R^2} \left|\varphi^{(n)}\right|^2\right)$$

from which we see $m_0 = 0$ and $m_n = n/R$. We get an infinite tower of increasingly massive particles, whose masses are very high if R is very small.

Now consider the case of a massless vector field, which can be decomposed as

$$A_M(x,w) = \sum_n A_M^{(n)}(x) e^{ik_n w}$$

The action is the five-dimensional analogue of the usual electromagnetic action,

$$S = -\frac{1}{4}\int d^4x \int_0^{2\pi R} dw F_{MN}F^{MN}$$

and we have the five-dimensional analogue of the usual gauge symmetry, $A_M \to A_M + \partial_M \alpha$

Solution c)

We can set $A_4^{(n)} = 0$ for all $n \neq 0$ by taking

$$\alpha = -\sum_{n\neq 0} A_M^{(n)}(x)\frac{e^{ik_n w}}{ik_n}$$

This doesn't work for $n = 0$ because it would result in division by zero. Or, in gratuitously fancy language, $A_4^{(0)}$ corresponds to the constant function on the circle, which has zero derivative, yet itself is not the derivative of anything; such a function exists because of the nontrivial cohomology of the circle.

Solution d)

Since we're going to have to rescale the fields later anyway, let's preemptively define

$$A_\mu(x,w) = \sqrt{2\pi R}\sum_n A_\mu^{(n)}(x)e^{ik_n w}, \quad \phi = \sqrt{2\pi R}A_4^{(0)}.$$

The action becomes

$$S = \int d^4x \int_0^{2\pi R} dw \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}F_{\mu 4}F^{\mu 4}\right)$$

The first term is

$$\begin{aligned} S_1 &= \int d^4x \int_0^{2\pi R}\frac{dw}{2\pi R}\left(-\frac{1}{4}\sum_{n,n'} e^{i(n+n')w/R}F_{\mu\nu}^{(n)}F^{\mu\nu(n')}\right) \\ &= -\frac{1}{4}\int d^4x F_{\mu\nu}^{(0)}F^{\mu\nu(0)} - \frac{1}{2}\sum_{n>0}\int d^4x \left(F_{\mu\nu}^{(n)}\right)^* F^{\mu\nu(n)} \end{aligned}$$

which is just the kinetic term for a real vector field $A_\mu^{(0)}$ and a tower of complex vector fields $A_\mu^{(n)}$ for $n > 0$. Next, the second term is

$$\begin{aligned} S &= -\frac{1}{2}\int d^4x \int_0^{2\pi R}\frac{dw}{2\pi R}\left(\partial_\mu\phi - \sum_n \partial_4\left(A_\mu^{(n)}e^{inw/R}\right)\right)\left(-\partial^\mu\phi - \sum_{n'}\partial^4\left(A^{\mu(n')}e^{in'w/R}\right)\right) \\ &= \int d^4x \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\int d^4x \int_0^{2\pi R}\frac{dw}{2\pi R}\sum_{n,n'} e^{i(n+n')w/R}\frac{nn'}{R^2}A_\mu^{(n)}A^{\mu(n')} \\ &= \int d^4x \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \sum_{n>0}\int d^4x \frac{n^2}{R^2}\left(A_\mu^{(n)}\right)^* A^{\mu(n)}. \end{aligned}$$

From this we conclude we have:
• One massless real scalar field ϕ.
• One massless real vector field $A_\mu^{(0)}$.
• A tower of massive complex vector fields $A_\mu^{(n)}$ with masses n/R.
By the way, if you do the analogous calculation starting with a five-dimensional metric, the massless degrees of freedom you'll get are a real scalar, a real vector, and a four-dimensional metric, suggesting a unification between electromagnetism and gravity; that's why Einstein was a big fan of this theory.

28.3.3 Problems about QFT Formalism in Condensed Media

(тоже много интересных и важных для подготовки)

28.4 Other Problems about Messing with Exotic QFT Formalism

(a lot of exotic questions are here)

28.5 Other, unusual problems

28.5.1 Problems about Deep Understanding of QFT

(?? тут то, что когда-то я отвечу, но это вообще не актуально, так что вряд ли это будет скоро.)

Я. Поворот фотона

Если фотон рассеивается назад, то как так получается, что он при этом не останавливается и всегда движется со скорость света? Мы ведь всегда, чтобы пойти назад должны сперва остановиться.

Solution (?????)

(стану умнее - подумаю.)

Я. Классическая виртуальная частица

Если посчитать сечение рассеяния с передачей частицы, но только в рамках классической механики, то как будет результат отличаться от результата в рамках квантовой теории поля?

Solution (?????)

(стану умнее - подумаю.)

Я. Анизотропность рассеяния

Как так вообще получается, что две частицы если налетают друг на друга, будучи полностью одинаковыми, в одни углы рассеиваются больше, в другие меньше?

Solution (?????)

(как бы понятно, можно говорить слова, типа квантовые флуктуации, но я хотел бы количественно это тут объяснить еще, с формулами.)

Я. Квантование нелинейного поля

The FD pulse was first established as a solution to the homogenous Maxwell's equations by Hellwarth and Nouchi [13]. As space-time non-separable solutions to Maxwells equations, FD pulses can be classified in TE and TM field configurations, with the electric and magnetic fields for the TE case given in cylindrical coordinates (ρ, θ, z) as:

$$E_\theta = -4if_0\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{\rho\left(q_1+q_2-2ict\right)}{\left[\rho^2+\left(q_1+i\tau\right)\left(q_2-i\sigma\right)\right]^3}$$

$$H_\rho = 4if_0\frac{\rho\left(q_2-q_1-2iz\right)}{\left[\rho^2+\left(q_1+i\tau\right)\left(q_2-i\sigma\right)\right]^3}$$

$$H_z = -4f_0\frac{\rho^2-\left(q_1+i\tau\right)\left(q_2-i\sigma\right)}{\left[\rho^2+\left(q_1+i\tau\right)\left(q_2-i\sigma\right)\right]^3}$$

Where $\sigma = z+ct, \tau = z-ct$ and f_0 is an arbitrary normalisation constant. The parameters q_1 and q_2 have the dimensions of length and represent respectively the effective wavelength

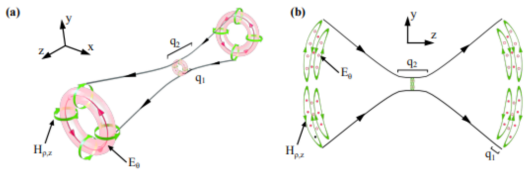


Fig. 1. Field topology and focusing properties of the "focused doughnut" pulse. The electric E and magnetic H fields are represented by green and red arrows respectively. The effective wavelength parameter q_1 and the pulse focal region depth q_2 are indicated on both diagrams, along with the region of maximum energy concentration at $z = 0, t = 0$. The pulse envelope is shown by the black lines and arrows.
Question: how to quantize it?

Solution

(не знаю я)

28.5.2 Problems of Classical Mechanics in QFT Formalism (???)

(очень крутая есть идея их прорешать, там ведь столкновения и распады и тут. пока не тот уровень, но когда-то мб дойду.)

28.5.3 Problems about Computing Correlations

(я хз, что тут, но мб там такое часто считают это?)

Part VIII
Other Parts and Topics of QFT

Посмотрим на некоторые другие темы КТП и следующие теории.
(тут очень много специфических потом будет методов)

29 Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry by Dreiner, Haber

Abstract

Two-component spinors are the basic ingredients for describing fermions in quantum field theory in $3+1$ spacetime dimensions. We develop and review the techniques of the two-component spinor formalism and provide a complete set of Feynman rules for fermions using two-component spinor notation. These rules are suitable for practical calculations of cross-sections, decay rates, and radiative corrections in the Standard Model and its extensions, including supersymmetry, and many explicit examples are provided. The unified treatment presented in this review applies to massless Weyl fermions and massive Dirac and Majorana fermions. We exhibit the relation between the two-component spinor formalism and the more traditional four-component spinor formalism, and indicate their connections to the spinor helicity method and techniques for the computation of helicity amplitudes.

29.1 Introduction

A crucial feature of the Standard Model of particle physics is the chiral nature of fermion quantum numbers and interactions. According to the modern understanding of the electroweak interactions, the fundamental degrees of freedom for quarks and leptons are two-component Weyl-van der Waerden fermions [1], i.e. two-component Lorentz spinors that transform as irreducible representations under the gauge group $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. Furthermore, within the context of supersymmetric field theories, two-component spinors enter naturally, due to the spinorial nature of the symmetry generators themselves, and the holomorphic structure of the superpotential. Despite this, most pedagogical treatments and practical calculations in high-energy physics continue to use the four-component Dirac spinor notation, which combines distinct irreducible representations of the Lorentz symmetry algebra. Parity-conserving theories such as QED and QCD are well-suited to the four-component fermion methods. There is also a certain perceived advantage to familiarity. However, as we progress to phenomena at and above the scale of electroweak symmetry breaking, it seems increasingly natural to employ two-component fermion notation, in harmony with the irreducible transformation properties dictated by the physics.

One occasionally encounters the misconception that two-component fermion notations are somehow inherently ill-suited or unwieldy for practical use. Perhaps this is due in part to a lack of examples of calculations using two-component language in the pedagogical literature. In this review, we seek to dispel this idea by presenting Feynman rules for fermions using two-component spinor notation, intended for practical calculations of cross-sections, decays, and radiative corrections. This formalism employs a unified framework that applies equally well to Dirac fermions [2] such as the Standard Model quarks and charged leptons, and to Majorana fermions [3] such as the light neutrinos of the seesaw extension of the Standard Model [4, 5] or the neutralinos of the minimal supersymmetric extension of the Standard Model (MSSM) [6, 7, 8, 9, 10].

Spinors were introduced by E. Cartan in 1913 as projective representations of the rotation group [11, 12], and entered into physics via the Dirac equation in 1928 [2]. In the same year, H. Weyl

discussed the representations of the Lorentz group [13], including the two-component spinor representations, in terms of stereographic projective coordinates [14]. The extension of the tensor calculus (or tensor analysis) to spinor calculus (or spinor analysis) was given by B.L. van der Waerden [1], upon the instigation of P. Ehrenfest. It is in this paper that van der Waerden (not Weyl as often claimed in the literature) first introduced the notation of dotted and undotted indices for the irreducible $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of the Lorentz group. Both Weyl [15] and van der Waerden independently considered the decomposition of the Dirac equation into two coupled differential equations for two-component spinors. In the 1930s, more pedagogical presentations of two-component spinors were given in refs. [16, 17, 18]. In particular, [16] was the first paper in English to employ the dotted and undotted index notation. Ref. [17] also presents the first two-component spinor analysis for general relativity. In the early 1950s, comprehensive reviews of two-component spinor techniques were published in English by Bade and Jehle [19] and in German by Cap [20]. Shortly thereafter, Bergmann reintroduced two-component spinors into the formalism of general relativity [21], which was followed by significant developments by Penrose [22].[7] Two-component spinor techniques in curved space are reviewed in refs. [23, 24], with an extensive bibliography given in [25]. A recent mathematical treatment of two-component spinors and their geometry can be found in [26]. Two-component spinors also play a central role in the covariant formulation of relativistic wave equations [27].

The formalism of two-component spinors has also been discussed in many textbooks on relativistic quantum mechanics, quantum field theory, elementary particle physics, group theoretical methods in physics, general relativity, and supersymmetry. For a guide to the non-supersymmetric literature, see for example, refs. [14, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68]. Among the early books, we would like to draw attention to [28], which has an extensive discussion of two-component spinor methods. Scheck [42] includes a short discussion of the field theory of two-component spinors, including the propagator. A more extensive field theoretic treatment, including Feynman rules and applications, is given by Ticciati [50]. A modern textbook on quantum field theory by Srednicki [66] includes a comprehensive treatment of two-component fermions and their quantization. Most textbooks and introductory reviews of supersymmetry [6, 7, 8, 9, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90] include a discussion of two-component spinors on some level, with a treatment of dotted and undotted indices and a collection of identities involving two-component spinors and the sigma matrices. Particularly extensive and useful sets of identities can be found in refs. [69, 73, 75, 78, 84, 86]. Finally, some mathematically sophisticated textbook treatments of spinors can be found in refs. [91, 92, 93].

The standard technique for computing scattering cross-sections with initial and final state fermions involves squaring the quantum S-matrix amplitude, summing over the spin states and then computing the traces of products of gamma matrices (in the four-component spinor formalism), or products of sigma matrices (in the two-component spinor formalism). We employ this latter technique throughout this paper (with a translation to the four-component formalism provided in an appendix). However, the computational effort rises rapidly as the number of interfering diagrams increases. The standard techniques typically become impractical with four or more particles in the final state. One approach to make such extensive calculations manageable is the helicity amplitude technique. Here the scattering process is decomposed into the scattering of helicity eigenstates. Then the individual amplitudes are computed analytically in terms of Lorentz scalar invariants, i.e. a complex number that can be readily computed. It is then a simple numerical task to sum all the contributing amplitudes and compute the square of the complex magnitude of the resulting sum. Such methods were first explored in refs. [94, 95, 96, 97], using four-component spinors (see also refs. [98, 99, 100, 101, 102]). Spinor techniques in the helicity formalism were also developed in ref. [103]. In fact, the natural spinor formalism for the helicity amplitude techniques makes use of the two-component Weyl-van der Waerden spinors, which we discuss in detail in this review. They were implemented in the helicity amplitude technique in refs. [104, 105, 106, 107, 108, 109, 110]. Recently, the two-component formalism has been implemented in a computer program for the numerical computation of amplitudes and cross-sections for event generators multi-particle

[7] For typographical reasons, Penrose replaced the dotted indices with primed indices, a notation still employed by most general relativists today.

processes [111].

This review is outlined as follows. In Section 2, we present our conventions and notation (with some additional discussion of our conventions in Appendix A). We also establish numerous identities involving sigma matrices, epsilon symbols and two-component spinors. In Section 3, we derive the basic properties of the quantized two-component fermion fields. For a generic collection of N two-component fermion fields with identical conserved quantum numbers, the corresponding mass matrix is an $N \times N$ complex symmetric matrix. To identify the corresponding mass eigenstates, one must perform a fermion-mass diagonalization that differs from the usual unitary similarity transformation of an hermitian matrix that is employed for a collection of scalar fields. In Section 4, we derive the Feynman rules for two-component spinors and describe how to write down amplitudes in our formalism. We demonstrate how to employ the two-component formalism for both tree-level and loop-level processes. In Section 5, we establish a naming convention for fermion and antifermion particle states and the corresponding fields. This is important as it provides an unambiguous procedure for obtaining the amplitudes for a given physical process, and for comparing these computations in the two-component and four-component spinor formalisms. In Section 6 we provide an extensive number of examples of computations using the two-component spinor formalism. This is the central part of our review.

We have relegated many details to a set of twelve appendices. In Appendix A, we summarize our metric and sigma matrix conventions and indicate how to translate between conventions with opposite metric signature. With our definition of the sigma matrices, one can switch easily between the two conventions by computing one overall sign factor. In Appendix B, we provide a comprehensive list of sigma matrix identities, and indicate which of these identities can be generalized to $d \neq 4$ dimensions required for loop computations that employ dimensional regularization. Explicit forms for the two-component spinor wave functions are given in Appendix C (where we exhibit two of the most common phase conventions employed in the literature). The mathematics of fermion mass diagonalization is discussed in Appendix D. In contrast to the unitary similarity transformation of the scalar squared-mass matrix, fermion mass diagonalization involves the Takagi diagonalization [112] of a complex symmetric matrix (for neutral fermions) or the singular value decomposition of a complex matrix (for charged fermions). In Appendix E, we review some of the basic facts of Lie groups and Lie algebras needed in the treatment of gauge theories. The two-component fermion propagators (derived in Section 4 using canonical field theory techniques) can also be obtained by path integral methods, as exhibited in Appendix F.

As most textbooks on quantum field theory and elementary particle physics employ the four-component spinor formalism for fermions, we provide in Appendix G a dictionary that allows one to translate between the two-component and four-component spinor techniques. We use the two-component spinor methods developed in this review to establish a generalization of the standard four-component spinor Feynman rules that incorporate Majorana fermions in a natural way. In Appendix H, we develop a method for computing helicity amplitudes in terms of Lorentz-invariant scalar quantities. This method, which makes use of the Bouchiat-Michel formulae [113] (originally established in the four-component spinor formalism) is developed in the language of two-component spinors. However, these methods are somewhat limited in scope and must be generalized in the case of multi-particle final states. This was accomplished by Hagiwara and Zeppenfeld (HZ) based on a two-component spinor treatment [106]. In Appendix I, we provide a translation between the HZ formalism and the two-component spinor formalism of this review. We also demonstrate that the spinor helicity method that is now commonly used in obtaining compact expressions for helicity amplitudes of multi-particle processed has a very simple development within the two-component spinor formalism. Finally, the two-component spinor Feynman rules for the Standard Model, the seesaw-extended Standard Model (which incorporates massive neutrinos), the minimal supersymmetric extension of the Standard Model (MSSM), and the R-parity-violating extension of the MSSM are given in Appendices J, K and L.

29.2 Essential conventions, notations and two-component spinor identities

We begin with a discussion of necessary conventions. The metric tensor is taken to be:[8]

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)\,, \tag{29.1}$$

where $\mu, \nu = 0, 1, 2, 3$ are spacetime vector indices. Contravariant four-vectors (e.g. positions and momenta) are defined with raised indices, and covariant four-vectors (e.g. derivatives) with lowered indices:

$$x^\mu = (t\,;\,\vec{\boldsymbol{x}})\,, \tag{29.2}$$
$$p^\mu = (E\,;\,\vec{\boldsymbol{p}})\,, \tag{29.3}$$
$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\partial/\partial t\,;\,\vec{\boldsymbol{\nabla}})\,, \tag{29.4}$$

in units with $c = 1$. The totally antisymmetric pseudo-tensor $\epsilon^{\mu\nu\rho\sigma}$ is defined such that

$$\epsilon^{0123} = -\epsilon_{0123} = +1\,. \tag{29.5}$$

More details on our conventions can be found in Appendix A .

The irreducible building blocks for spin-1/2 fermions are fields that transform either under the left-handed $(\frac{1}{2}, 0)$ or the right-handed $(0, \frac{1}{2})$ representation of the Lorentz group. Hermitian conjugation interchanges these two representations. A Majorana fermion field can be constructed from either representation; this is the spin-1/2 analogue of a real scalar field. A Dirac fermion field combines two equal mass two-component fields into a reducible representation of the form $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$; this is the spin-1/2 analogue of a complex scalar field. It is also possible to use four-component notation to describe a Majorana fermion by imposing a reality condition on the spinor in order to reduce the number of degrees of freedom in half. Details of this construction are given in Appendix G.1. However, in this review, we shall focus primarily on two-component spinor notation for all fermions. In the following, $(\frac{1}{2}, 0)$ spinors carry undotted indices $\alpha, \beta, ... = 1, 2$, and $(0, \frac{1}{2})$ spinors carry dotted indices $\dot{\alpha}, \dot{\beta}, ... = 1, 2$.

We first provide a brief introduction to the Lorentz group and its two-dimensional spinor representations. Under an active Lorentz transformation, a contravariant four-vector x^μ transforms as

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu\,, \tag{29.6}$$

where $\Lambda \in \mathrm{SO}(3,1)$ satisfies $\Lambda^\mu{}_\nu g_{\mu\rho} \Lambda^\rho{}_\lambda = g_{\nu\lambda}$. It then follows that the transformation of the corresponding covariant four-vector $x_\mu \equiv g_{\mu\nu} x^\nu$ satisfies:

$$x_\nu = x'_\mu \Lambda^\mu{}_\nu\,. \tag{29.7}$$

The most general proper orthochronous Lorentz transformation (which is continuously connected to the identity), corresponding to a rotation by an angle θ about an axis $\widehat{\boldsymbol{n}}$ $[\vec{\boldsymbol{\theta}} \equiv \theta\widehat{\boldsymbol{n}}]$ and a boost vector $\vec{\boldsymbol{\zeta}} \equiv \hat{\boldsymbol{v}} \tanh^{-1}\beta$ [where $\hat{\boldsymbol{v}} \equiv \vec{\boldsymbol{v}}/|\vec{\boldsymbol{v}}|$ and $\beta \equiv |\vec{\boldsymbol{v}}|$], is a 4×4 matrix given by:

$$\Lambda = \exp\left(-\tfrac{1}{2} i\theta^{\rho\sigma}\mathcal{S}_{\rho\sigma}\right) = \exp\left(-i\vec{\boldsymbol{\theta}}\cdot\vec{\boldsymbol{\mathcal{S}}} - i\vec{\boldsymbol{\zeta}}\cdot\vec{\boldsymbol{\mathcal{K}}}\right)\,, \tag{29.8}$$

[8] The published version of this paper employs the $(+,-,-,-)$ Minkowski space metric. An otherwise identical version, using the $(-,+,+,+)$ metric favored by one of the authors (SPM), may be found at http://www.niu.edu/spmartin/spinors/. It can also be constructed by changing a single macro at the beginning of the LaTeX source file [114], in an obvious way. You can tell which version you are presently reading from eq. (29.1). See Appendix A pinors for further details and rules for translating between metric conventions.

where $\theta^{\rho\sigma} = -\theta^{\sigma\rho}$ and $\mathcal{S}_{\rho\sigma} = -\mathcal{S}_{\sigma\rho}$. In particular, $\theta^i \equiv \frac{1}{2}\epsilon^{ijk}\theta_{jk}$, $\zeta^i \equiv \theta^{i0} = -\theta^{0i}$, $\mathcal{S}^i \equiv \frac{1}{2}\epsilon^{ijk}\mathcal{S}_{jk}$, $\mathcal{K}^i \equiv \mathcal{S}^{0i} = -\mathcal{S}^{i0}$, and

$$(\mathcal{S}_{\rho\sigma})^{\mu}{}_{\nu} = i(\delta^{\mu}_{\rho}\, g_{\sigma\nu} - \delta^{\mu}_{\sigma}\, g_{\rho\nu})\,. \tag{29.9}$$

Here, the indices $i, j, k = 1, 2, 3$ and $\epsilon^{123} = +1$.

Thus, an infinitesimal orthochronous Lorentz transformation is given by

$$\Lambda^{\mu}{}_{\nu} \simeq \delta^{\mu}_{\nu} - \tfrac{1}{2}i\theta^{\rho\sigma}(\mathcal{S}_{\rho\sigma})^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \tfrac{1}{2}\theta^{\rho\sigma}(\delta^{\mu}_{\rho}\, g_{\sigma\nu} - \delta^{\mu}_{\sigma}\, g_{\rho\nu}) = \delta^{\mu}_{\nu} + \tfrac{1}{2}(\theta^{\mu}{}_{\nu} - \theta_{\nu}{}^{\mu})\,. \tag{29.10}$$

Since $\theta^{\mu}{}_{\nu} = g_{\alpha\nu}\theta^{\mu\alpha} = -g_{\alpha\nu}\theta^{\alpha\mu} = -\theta_{\nu}{}^{\mu}$, it follows that

$$\Lambda^{\mu}{}_{\nu} \simeq \delta^{\mu}_{\nu} + \theta^{\mu}{}_{\nu}\,. \tag{29.11}$$

Moreover, the infinitesimal boost parameter is $\vec{\boldsymbol{\zeta}} \equiv \hat{\boldsymbol{v}}\tanh^{-1}\beta \simeq \beta\hat{\boldsymbol{v}} \equiv \vec{\boldsymbol{\beta}}$, since $\beta \ll 1$ for an infinitesimal boost. Hence, the actions of the infinitesimal boosts and rotations on the spacetime coordinates are

$$\text{Rotations:} \qquad \begin{cases} \vec{\boldsymbol{x}} \to \vec{\boldsymbol{x}}' \simeq \vec{\boldsymbol{x}} + (\vec{\boldsymbol{\theta}} \times \vec{\boldsymbol{x}})\,, \\ t \to\ t' \simeq t\,, \end{cases} \tag{29.12}$$

$$\text{Boosts:} \qquad \begin{cases} \vec{\boldsymbol{x}} \to \vec{\boldsymbol{x}}' \simeq \vec{\boldsymbol{x}} + \vec{\boldsymbol{\beta}}\, t\,, \\ t \to\ t' \simeq t + \vec{\boldsymbol{\beta}}\cdot\vec{\boldsymbol{x}}\,, \end{cases} \tag{29.13}$$

with exactly analogous transformations for any contravariant four-vector.

With respect to the Lorentz transformation Λ, a general n-component field Φ transforms according to a representation R of the Lorentz group as $\Phi(x^\mu) \to \Phi'(x'^\mu) = M_R(\Lambda)\,\Phi(x^\mu)$, where $M_R(\Lambda)$ is the corresponding (finite) d_R-dimensional matrix representation. Equivalently, the functional form of the transformed field Φ obeys

$$\Phi'(x^\mu) = M_R(\Lambda)\Phi([\Lambda^{-1}]^{\mu}{}_{\nu}x^{\nu})\,, \tag{29.14}$$

after using eq. (29.6). For proper orthochronous Lorentz transformations,

$$M_R = \exp\left(-\tfrac{1}{2}i\theta_{\mu\nu}J^{\mu\nu}\right) \simeq \mathbb{1}_{d_R\times d_R} - i\vec{\boldsymbol{\theta}}\cdot\vec{\boldsymbol{J}} - i\vec{\boldsymbol{\zeta}}\cdot\vec{\boldsymbol{K}}\,, \tag{29.15}$$

where $\mathbb{1}_{d_R\times d_R}$ is the $d_R \times d_R$ identity matrix and $\theta_{\mu\nu}$ parameterizes the Lorentz transformation Λ [eq. (29.8)]. The six independent components of the matrix-valued antisymmetric tensor $J^{\mu\nu}$ are the d_R-dimensional generators of the Lorentz group and satisfy the commutation relations:

$$[J^{\mu\nu}\,,\, J^{\lambda\kappa}] = i(g^{\mu\kappa}\, J^{\nu\lambda} + g^{\nu\lambda}\, J^{\mu\kappa} - g^{\mu\lambda}\, J^{\nu\kappa} - g^{\nu\kappa}\, J^{\mu\lambda})\,. \tag{29.16}$$

We identify $\vec{\boldsymbol{J}}$ and $\vec{\boldsymbol{K}}$ as the generators of rotations parameterized by $\vec{\boldsymbol{\theta}}$ and boosts parameterized by $\vec{\boldsymbol{\zeta}}$, respectively, where

$$J^i \equiv \tfrac{1}{2}\epsilon^{ijk}J_{jk}\,, \qquad\qquad K^i \equiv J^{0i}\,. \tag{29.17}$$

Here we focus on the simplest non-trivial irreducible representations of the Lorentz algebra. These are the two-dimensional (inequivalent) representations: $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. In the $(\frac{1}{2}, 0)$ representation, $\vec{\boldsymbol{J}} = \vec{\boldsymbol{\sigma}}/2$ and $\vec{\boldsymbol{K}} = -i\vec{\boldsymbol{\sigma}}/2$ in eq. (29.15), which yields

$$M_{(\frac{1}{2},0)} \equiv M \simeq \mathbb{1}_{2\times 2} - i\vec{\boldsymbol{\theta}}\cdot\vec{\boldsymbol{\sigma}}/2 - \vec{\boldsymbol{\zeta}}\cdot\vec{\boldsymbol{\sigma}}/2\,, \tag{29.18}$$

where $\vec{\boldsymbol{\sigma}} \equiv (\sigma^1\,,\, \sigma^2\,,\, \sigma^3)$ are the Pauli matrices [cf. eq. (29.29)]. By definition M carries undotted spinor indices, as indicated by $M_\alpha{}^\beta$. A two-component $(\frac{1}{2}, 0)$ spinor is denoted by ψ_α and transforms

as $\psi_\alpha \to M_\alpha{}^\beta \psi_\beta$, omitting the coordinate arguments of the fields, which are as in eq. (29.14). In our conventions for the location of the spinor indices, we sum implicitly over a repeated index pair in which one index is lowered and one index is raised.

In the $(0, \frac{1}{2})$ representation, $\vec{\boldsymbol{J}} = -\vec{\boldsymbol{\sigma}}^*/2$ and $\vec{\boldsymbol{K}} = -i\vec{\boldsymbol{\sigma}}^*/2$ in eq. (29.15), so that its representation matrix is M^*, the complex conjugate of eq. (29.18). By definition, the indices carried by M^* are dotted, as indicated by $(M^*)_{\dot\alpha}{}^{\dot\beta}$. A two-component $(0, \frac{1}{2})$ spinor is denoted by $\psi^\dagger_{\dot\alpha}$ and transforms as $\psi^\dagger_{\dot\alpha} \to (M^*)_{\dot\alpha}{}^{\dot\beta}\psi^\dagger_{\dot\beta}$, again suppressing the coordinate arguments of the fields, which are as in eq. (29.14). We distinguish between the undotted and dotted spinor index types because they cannot be directly contracted with each other to form a Lorentz invariant quantity.

It follows that the $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ representations are related by hermitian conjugation. That is, if ψ_α is a $(\frac{1}{2}, 0)$ fermion, then $(\psi_\alpha)^\dagger$ transforms as a $(0, \frac{1}{2})$ fermion. This means that we can, and will, describe all fermion degrees of freedom using only fields defined as left-handed $(\frac{1}{2}, 0)$ fermions ψ_α, and their conjugates. In combining spinors to make Lorentz tensors [as in eq. (29.40)], it is useful to regard $\psi^\dagger_{\dot\alpha}$ as a row vector, and ψ_α as a column vector, with:[9] [10]

$$\psi^\dagger_{\dot\alpha} \equiv (\psi_\alpha)^\dagger. \tag{29.19}$$

The Lorentz transformation property of $\psi^\dagger_{\dot\alpha}$ then follows from $(\psi_\alpha)^\dagger \to (\psi_\beta)^\dagger (M^\dagger)^{\dot\beta}{}_{\dot\alpha}$ [with coordinate arguments of the fields again suppressed], where $(M^\dagger)^{\dot\beta}{}_{\dot\alpha} = (M^*)_{\dot\alpha}{}^{\dot\beta}$ reflects the definition of the hermitian adjoint matrix as the complex conjugate transpose of the matrix. Again the coordinate arguments of the fields have been suppressed, and are as in eq. (29.14).

In this review, we shall employ the dotted-index notation in association with the dagger to denote hermitian conjugation, as specified in eq. (29.19). This is the notation for hermitian conjugation of spinors found in most field theory textbooks (e.g., see refs. [66, 89, 115]). However, it should be noted that many references in the supersymmetry literature (e.g., see refs. [69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 88, 87]) employ the bar notation made popular by Wess and Bagger [69] where $\overline{\psi}_{\dot\alpha} \equiv \psi^\dagger_{\dot\alpha} \equiv (\psi_\alpha)^\dagger$.

Spinors labeled with one undotted or one dotted index are sometimes called spinors of rank one [or more precisely, spinors of rank $(1, 0)$ or $(0, 1)$, respectively]. One can also define spinors of higher rank, which possess more than one spinor index, with Lorentz transformation properties that depend on the number of undotted and dotted spinor indices[16, 19, 20, 23, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 51, 53, 61, 79, 116]. In particular, for a spinor of rank (m, n) denoted by $S_{\alpha_1\alpha_2\cdots\alpha_m\dot\beta_1\dot\beta_2\cdots\dot\beta_n}$, each lowered undotted α-index transforms separately according to $M_{\alpha'_i}{}^{\alpha_i}$ in eq. (29.18) and each lowered dotted $\dot\beta$-index transforms according to $(M^*)_{\dot\beta'_i}{}^{\dot\beta_i}$.

There are two additional spin-1/2 irreducible representations of the Lorentz group, $(M^{-1})^{\mathsf{T}}$ and $(M^{-1})^\dagger$, but these are equivalent representations to the $(\frac{1}{2}, 0)$ and the $(0, \frac{1}{2})$ representations, respectively. The spinors that transform under these representations have raised spinor indices, ψ^α and $\psi^{\dagger\dot\alpha}$, with transformation laws $\psi^\alpha \to [(M^{-1})^{\mathsf{T}}]^\alpha{}_\beta \psi^\beta$ and $\psi^{\dagger\dot\alpha} \to [(M^{-1})^\dagger]^{\dot\alpha}{}_{\dot\beta}\psi^{\dagger\dot\beta}$, respectively (with coordinate arguments of the fields again suppressed). It is convenient to rewrite the transformation law for the undotted spinor as $\psi^\alpha \to \psi^\beta (M^{-1})_\beta{}^\alpha$. In combining spinors to make Lorentz

[9] In the early literature that employed the van der Waerden spinor index notation (surveyed in Section 29.1), no dagger was used in conjunction with the dotted index. The advantage to attaching the dagger to the dotted spinor field is that it permits the development of a spinor-index-free notation for Lorentz-covariant spinor products [see eqs. (29.37)–(29.41) and the accompanying text].

[10] Other conventions for the dotted spinor are possible. For example, in [75], the hermitian conjugate of ψ_α yields a spinor with a *raised* dotted index, i.e. $\psi^{\dagger\,\dot\alpha} \equiv (\psi_\alpha)^\dagger$, which the authors rewrite as $\psi^{\dagger\,\dot\alpha} = \overline{\sigma}^{\dot\alpha\beta}(\psi_\beta)^\dagger$ in an attempt to maintain the same index structure on both sides of the equation. Here, $\overline{\sigma}^0 \equiv \mathbb{1}_{2\times 2}$ [defined in eq. (29.29)] appears as a formal device. This latter convention leads to a number of complications; e.g., $(M_\alpha{}^\beta)^* \neq (M^*)_{\dot\alpha}{}^{\dot\beta}$, etc. (see [75] for further details). Although this alternative convention seems self-consistent, we have adopted the more convenient eqs. (29.19) and (29.20) in this review.

tensors [as in eq. (29.41)], it is useful to regard ψ^α as a row vector, and $\psi^{\dagger\,\dot\alpha}$ as a column vector, with:

$$\psi^{\dagger\,\dot\alpha} \equiv (\psi^\alpha)^\dagger\,. \tag{29.20}$$

The Lorentz transformation property of $\psi^{\dagger\,\dot\alpha}$ then follows from $(\psi^\alpha)^\dagger \to [(M^{-1})^\dagger]^{\dot\alpha}{}_{\dot\beta}(\psi^\beta)^\dagger$.

The spinor indices are raised and lowered with the two-index antisymmetric epsilon symbol with non-zero components,[11]

$$\epsilon^{12} = -\epsilon^{21} = \epsilon_{21} = -\epsilon_{12} = 1\,, \tag{29.21}$$

and the same set of sign conventions for the corresponding dotted spinor indices. In particular, we formally define $\epsilon^{\dot\alpha\dot\beta} \equiv (\epsilon^{\alpha\beta})^*$ and $\epsilon_{\dot\alpha\dot\beta} \equiv (\epsilon_{\alpha\beta})^*$. Viewed as a 2×2 matrix, the epsilon symbol with lowered undotted [dotted] indices is the inverse of the epsilon symbol with raised undotted [dotted] indices. Thus, consistent with eqs. (29.19) and (29.20), one can write:[12] [13]

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta\,, \qquad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta\,, \qquad \psi^\dagger_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\psi^{\dagger\dot\beta}\,, \qquad \psi^{\dagger\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\psi^\dagger_{\dot\beta}\,, \tag{29.22}$$

which respects Lorentz covariance due to the properties of M given in eqs. (29.103) and (29.104). The epsilon symbols $\epsilon^{\alpha\beta}$ ($\epsilon_{\alpha\beta}$) and $\epsilon^{\dot\alpha\dot\beta}$ ($\epsilon_{\dot\alpha\dot\beta}$), first introduced in this context in [1], are also called the *spinor metric tensors*, as they raise (lower) the undotted and dotted spinor indices, respectively. Note that in raising or lowering an index of a spinor quantity, *adjacent* spinor indices are summed over when multiplied on the *left* by the appropriate epsilon symbol.

The epsilon symbols can also be used to raise or lower undotted or dotted indices of spinors of higher rank. For example, for an object with two undotted indices it is natural to define

$$A^{\gamma\delta} = \epsilon^{\gamma\alpha}\epsilon^{\delta\beta}A_{\alpha\beta}\,, \qquad\qquad A_{\gamma\delta} = \epsilon_{\gamma\alpha}\epsilon_{\delta\beta}A^{\alpha\beta}\,. \tag{29.23}$$

In the special case that $A^{\alpha\beta} = \psi^\alpha\chi^\beta$ is a product of rank-one spinors, eq. (29.23) is not just natural but necessary, as it follows directly from eq. (29.22). However, in other cases there can be a different sign associated (by convention) with raising and lowering spinor indices, because of the antisymmetry of the epsilon symbols (in contrast to the symmetry of the spacetime metric used to raise and lower spacetime indices). This sign convention can be defined independently for distinct higher-rank spinors (even in the case where the higher-rank spinors possess the same index

[11] For related earlier work on the epsilon symbol and its properties, see refs. [17, 16, 19, 117]. Various subsets of the subsequent identities in this section involving commuting and anticommuting two-component spinors, as well as the ϵ symbol and the sigma matrices appear in many books and reviews (e.g., see refs. [18, 69, 70, 71, 73, 72, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 86, 88, 87, 89]) and in papers (e.g., see refs. [104, 105, 106, 107, 108, 109, 110]).

[12] In the general relativity literature (see e.g., refs. [22, 23, 38, 43, 44, 47, 53, 61, 63, 65]), the more common convention for the epsilon symbol (also adopted in refs. [19, 28, 30, 37, 46, 48, 93, 110]) is $\epsilon^{\alpha\beta} = \epsilon_{\alpha\beta}$ with $\epsilon^{12} = -\epsilon^{21} = 1$, and similarly for the epsilon symbol with dotted spinor indices. In this convention, one writes $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$ as above, but in contrast to eq. (29.22), $\psi_\alpha = \psi^\beta\epsilon_{\beta\alpha}$, and similarly for the corresponding equations with dotted spinor indices. That is, in raising [lowering] an index of a spinor quantity, *adjacent* spinor indices are summed over when multiplied on the left [right] by the appropriate epsilon symbol. The various identities involving the epsilon symbols given in this review must then be modified by a minus sign for every epsilon symbol with lowered spinor indices. There are some benefits for this alternative convention; e.g., the minus signs appearing in eq. (29.24) are absent. However, one must keep track of other minus signs that arise because $\epsilon_{\alpha\beta}$ is the *negative* of the inverse of $\epsilon^{\alpha\beta}$. In this review, we have adopted the convention of eq. (29.21), which is consistent with most of the supersymmetry literature.

[13] In refs. [51, 70], one finds yet another convention in which the spinor indices are raised and lowered by a two-index antisymmetric quantity, $C_{\alpha\beta} = -C^{\alpha\beta} = C_{\dot\alpha\dot\beta} = -C^{\dot\alpha\dot\beta} = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$, which play the role of the epsilon symbols. As in footnote 12, $C_{\alpha\beta}$ is the negative inverse of $C^{\alpha\beta}$ in which case $\psi^\alpha = C^{\alpha\beta}\psi_\beta$ whereas $\psi_\alpha = \psi^\beta C_{\beta\alpha}$, and similarly for the corresponding equations with dotted spinor indices. However, in this convention where C is pure imaginary, if $\psi^{\dagger\,\dot\alpha} \equiv (\psi^\alpha)^\dagger$ as in eq. (29.20), then $\psi^\dagger_{\dot\alpha} = -(\psi_\alpha)^\dagger$ in contrast to eq. (29.19). We choose not to pursue the alternative epsilon symbol conventions of footnotes 12 or 13 in this review.

structure). Indeed, as a consequence of our epsilon symbol conventions of eq. (29.21), the epsilon symbols themselves satisfy:

$$\epsilon^{\gamma\delta} = -\epsilon^{\gamma\alpha}\epsilon^{\delta\beta}\epsilon_{\alpha\beta}\,, \qquad\qquad \epsilon_{\gamma\delta} = -\epsilon_{\gamma\alpha}\epsilon_{\delta\beta}\epsilon^{\alpha\beta}\,, \tag{29.24}$$

in contrast to eq. (29.23). The above results (and similar ones with dotted indices) show that some care is required [33], since the extra overall minus signs of eq. (29.24) in comparison to eq. (29.23) might otherwise have been unexpected [e.g., see eqs. (29.42) and (29.43) below].[14] This reflects an awkwardness imposed by the epsilon symbol conventions of eq. (29.21), rather than an inconsistency. Practitioners of spinor algebra in the conventions used in this review should be wary of this sign issue when using the epsilon symbols to explicitly raise or lower two or more spinor indices of higher-rank spinors.[15] Fortunately, such manipulations are quite rare in practical calculations.

We also introduce the two-index symmetric Kronecker delta symbol,

$$\delta^1_1 = \delta^2_2 = 1\,, \qquad\qquad \delta^1_2 = \delta^1_2 = 0\,, \tag{29.25}$$

and $\delta^{\dot\beta}_{\dot\alpha} \equiv (\delta^\beta_\alpha)^*$. Eq. (29.25) implies that the numerical values of the undotted and dotted Kronecker delta symbols coincide. The epsilon symbols with undotted and with dotted indices respectively satisfy:

$$\epsilon_{\alpha\beta}\epsilon^{\gamma\delta} = -\delta^\gamma_\alpha\delta^\delta_\beta + \delta^\delta_\alpha\delta^\gamma_\beta, \qquad\qquad \epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\gamma\dot\delta} = -\delta^{\dot\gamma}_{\dot\alpha}\delta^{\dot\delta}_{\dot\beta} + \delta^{\dot\delta}_{\dot\alpha}\delta^{\dot\gamma}_{\dot\beta}\,, \tag{29.26}$$

from which it follows that:

$$\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \epsilon^{\gamma\beta}\epsilon_{\beta\alpha} = \delta^\gamma_\alpha, \qquad\qquad \epsilon_{\dot\alpha\dot\beta}\epsilon^{\dot\beta\dot\gamma} = \epsilon^{\dot\gamma\dot\beta}\epsilon_{\dot\beta\dot\alpha} = \delta^{\dot\gamma}_{\dot\alpha}\,, \tag{29.27}$$

$$\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + \epsilon_{\alpha\gamma}\epsilon_{\delta\beta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma} = 0\,, \qquad\qquad \epsilon_{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\delta} + \epsilon_{\dot\alpha\dot\gamma}\epsilon_{\dot\delta\dot\beta} + \epsilon_{\dot\alpha\dot\delta}\epsilon_{\dot\beta\dot\gamma} = 0\,. \tag{29.28}$$

In the literature, eq. (29.28) is often referred to as the Schouten identities.[16]

To construct Lorentz invariant Lagrangians and observables, one needs to first combine products of spinors to make objects that transform as Lorentz tensors. In particular, Lorentz vectors are obtained by introducing the sigma matrices $\sigma^\mu_{\alpha\dot\beta}$ and $\overline{\sigma}^{\mu\,\dot\alpha\beta}$ defined by [14, 1, 17, 18]

$$\sigma^0 = \overline{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,, \qquad\qquad \sigma^1 = -\overline{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,,$$

$$\sigma^2 = -\overline{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \qquad\qquad \sigma^3 = -\overline{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{29.29}$$

The sigma matrices are hermitian, and have been defined above with an upper (contravariant) index. We denote the 2×2 identity matrix by $\mathbb{1}_{2\times2}$ and the three-vector of Pauli matrices by $\vec{\boldsymbol{\sigma}} \equiv (\sigma^1\,,\,\sigma^2\,,\,\sigma^3)$. Hence, eq. (29.29) is equivalent to:

$$\sigma^\mu = (\mathbb{1}_{2\times2}\,;\,\vec{\boldsymbol{\sigma}})\,, \qquad\qquad \overline{\sigma}^\mu = (\mathbb{1}_{2\times2}\,;\,-\vec{\boldsymbol{\sigma}})\,. \tag{29.30}$$

[14] It would be perhaps more transparent to simply replace the symbol $\epsilon_{\alpha\beta}$ with $\epsilon^{-1}_{\alpha\beta}$, in which case $\epsilon^{\alpha\beta}$ is used to raise spinor indices and $\epsilon^{-1}_{\alpha\beta}$ is used to lower spinor indices (cf. [39]). Although this convention avoids an apparent conflict between eqs. (29.23) and (29.24), it doubles the number of distinct epsilon symbols. We shall not adopt such an approach in this review.

[15] In the alternative convention mentioned in footnote 12, this particular awkwardness is absent; the minus signs in the analogue of eq. (29.24) do not occur, in which case the rules for raising and lowering the spinor indices in eqs. (29.23) and (29.24) are identical. More generally, in the convention of footnote 12, the indices of all higher-rank spinors can be raised [lowered] via multiplication on the left [right] by the appropriate epsilon symbol, including the epsilon symbols themselves, with no extra signs.

[16] The Schouten identities also follow from the observation that a rank-four spinor must vanish if it is antisymmetric with respect to more than two undotted or dotted two-component spinor indices.

We also define the corresponding quantities with lower (covariant) indices:

$$\sigma_\mu = g_{\mu\nu}\sigma^\nu = (\mathbb{1}_{2\times 2}\,;\,-\vec{\boldsymbol{\sigma}})\,, \qquad\qquad \overline{\sigma}_\mu = g_{\mu\nu}\overline{\sigma}^\nu = (\mathbb{1}_{2\times 2}\,;\,\vec{\boldsymbol{\sigma}})\,. \tag{29.31}$$

The relations between σ^μ and $\overline{\sigma}^\mu$ are

$$\sigma^\mu_{\alpha\dot\alpha} = \epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\overline{\sigma}^{\mu\,\dot\beta\beta}\,, \qquad\qquad \overline{\sigma}^{\mu\,\dot\alpha\alpha} = \epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta}\,, \tag{29.32}$$

$$\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\overline{\sigma}^{\mu\dot\beta\alpha}\,, \qquad\qquad \epsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\alpha\dot\beta} = \epsilon_{\alpha\beta}\overline{\sigma}^{\mu\dot\alpha\beta}\,. \tag{29.33}$$

Consider a spinor of rank (n,n) denoted by $S_{\alpha_1\alpha_2\ldots\alpha_n\dot\beta_1\dot\beta_2\ldots\dot\beta_n}$. The object obtained by multiplying S by $\overline{\sigma}^{\mu_1\,\dot\beta_1\alpha_1}\cdots\overline{\sigma}^{\mu_n\,\dot\beta_n\alpha_n}$ has the transformation properties of an nth rank contravariant Lorentz tensor [29, 32, 116]. For example, there is a one-to-one correspondence between each bi-spinor $V_{\alpha\dot\beta}$ and the associated Lorentz four-vector V^μ [1, 17, 19, 20, 28, 29],[17]

$$V^\mu \equiv \tfrac{1}{2}\overline{\sigma}^{\mu\dot\beta\alpha}V_{\alpha\dot\beta}\,, \qquad\qquad V_{\alpha\dot\beta} = V^\mu\sigma_{\mu\alpha\dot\beta}\,. \tag{29.34}$$

In particular, if V^μ is a real four-vector then $V_{\alpha\dot\beta}$ is hermitian (and vice versa). To clarify this last remark, consider the bi-spinor $V_{\alpha\dot\beta}$ regarded as a 2×2 matrix. Then,[18]

$$(V^{\mathsf{T}})_{\alpha\dot\beta} \equiv V_{\beta\dot\alpha}\,, \qquad (V^*)_{\alpha\dot\beta} \equiv (V_{\alpha\dot\beta})^*\,, \qquad (V^\dagger)_{\alpha\dot\beta} \equiv (V_{\beta\dot\alpha})^*\,. \tag{29.35}$$

An hermitian bi-spinor satisfies $V = V^\dagger$, or equivalently $V_{\alpha\dot\beta} = (V_{\beta\dot\alpha})^*$.

Rank-two spinors (with two undotted or with two dotted indices) can also be interpreted as 2×2 matrices. In the case of the rank-two spinor $W_\alpha{}^\beta$, it is convenient to define:

$$(W^{\mathsf{T}})^\beta{}_\alpha \equiv W_\alpha{}^\beta\,, \qquad (W^*)_{\dot\alpha}{}^{\dot\beta} \equiv (W_\alpha{}^\beta)^*\,, \qquad (W^\dagger)^{\dot\beta}{}_{\dot\alpha} \equiv (W_\alpha{}^\beta)^* = (W^*)_{\dot\alpha}{}^{\dot\beta}\,. \tag{29.36}$$

Note that the matrix transposition of $W_\alpha{}^\beta$ interchanges the rows and columns of W without altering the relative heights of the α and β indices. Similar results hold for $W_{\alpha\beta}$ and $W^{\alpha\beta}$ by either lowering or raising the relevant spinor indices with the appropriate epsilon symbol.

When constructing Lorentz tensors from fermion fields, the heights of spinor indices must be consistent in the sense that lowered indices must only be contracted with raised indices. As a convention, *descending* contracted undotted indices and *ascending* contracted dotted indices,

$$\ {}^\alpha{}_\alpha \qquad\qquad \text{and} \qquad\qquad {}_{\dot\alpha}{}^{\dot\alpha}\,, \tag{29.37}$$

can be suppressed. In all spinor products given in this paper, contracted indices always have heights that conform to eq. (29.37). For example, in an index-free notation, we define:

$$\xi\eta \equiv \xi^\alpha\eta_\alpha, \tag{29.38}$$

$$\xi^\dagger\eta^\dagger \equiv \xi^\dagger_{\dot\alpha}\eta^{\dagger\dot\alpha}, \tag{29.39}$$

$$\xi^\dagger\overline{\sigma}^\mu\eta \equiv \xi^\dagger_{\dot\alpha}\overline{\sigma}^{\mu\dot\alpha\beta}\eta_\beta, \tag{29.40}$$

$$\xi\sigma^\mu\eta^\dagger \equiv \xi^\alpha\sigma^\mu_{\alpha\dot\beta}\eta^{\dagger\dot\beta}. \tag{29.41}$$

[17] In the general relativity literature [43, 47, 61, 93], the more common normalization is $V^\mu \equiv \frac{1}{\sqrt{2}}\overline{\sigma}^{\mu\dot\beta\alpha}V_{\alpha\dot\beta}$, which yields $V_{\alpha\dot\beta} = \frac{1}{\sqrt{2}}V^\mu\sigma_{\mu\alpha\dot\beta}$. In this context, the $\frac{1}{\sqrt{2}}\sigma^\mu_{\alpha\dot\beta}$ are often called the Infeld-van der Waerden symbols.

[18] As stressed in [19], taking the transpose of $V_{\alpha\dot\beta}$ interchanges its rows and columns without altering the fact that the first spinor index is undotted and the second spinor index is dotted. Moreover, it is often useful to further simplify the notation by *defining* $V_{\dot\alpha\beta} \equiv (V_{\alpha\dot\beta})^*$. In this notation, an hermitian bi-spinor satisfies $V_{\alpha\dot\beta} = V_{\dot\beta\alpha}$.

All the spinor-index-contracted products above have natural interpretations as products of matrices and vectors by regarding η_α and $\eta^{\dagger\dot\alpha}$ as column vectors and $\xi^\dagger_{\dot\alpha}$ and ξ^α as row vectors of the two-dimensional spinor space. However, the reader is cautioned that in the index-free notation (with undotted and dotted indices suppressed), the undaggered and daggered spinors cannot be uniquely identified as column or row vectors until their locations within the spinor product are specified. Nevertheless, the proper identifications are straightforward, as any spinor on the left end of a spinor product can be identified as a row vector and any spinor on the right end of a spinor product can be identified as a column vector.

For an *anticommuting* two-component spinor ψ, the product $\psi^\alpha\psi^\beta$ is antisymmetric with respect to the interchange of the spinor indices α and β. Hence, this product of spinors must be proportional to $\epsilon^{\alpha\beta}$. Similar conclusions hold for the corresponding spinor products with raised undotted indices and with lowered and raised dotted indices, respectively. Thus,

$$\psi^\alpha\psi^\beta = -\tfrac{1}{2}\epsilon^{\alpha\beta}\psi\psi\,, \qquad \psi_\alpha\psi_\beta = \tfrac{1}{2}\epsilon_{\alpha\beta}\psi\psi\,, \tag{29.42}$$

$$\psi^{\dagger\,\dot\alpha}\psi^{\dagger\,\dot\beta} = \tfrac{1}{2}\epsilon^{\dot\alpha\dot\beta}\psi^\dagger\psi^\dagger\,, \qquad \psi^\dagger_{\dot\alpha}\psi^\dagger_{\dot\beta} = -\tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}\psi^\dagger\psi^\dagger\,, \tag{29.43}$$

where $\psi\psi \equiv \psi^\alpha\psi_\alpha$ and $\psi^\dagger\psi^\dagger \equiv \psi^\dagger_{\dot\alpha}\psi^{\dagger\,\dot\alpha}$ as in eqs. (29.38) and (29.39). Note that the minus signs above can be understood to be a consequence of the extra minus sign that arises when the indices of the epsilon symbol are lowered or raised [cf. eqs. (29.23) and (29.24)].

The behavior of the spinor products under hermitian conjugation (for quantum field operators) or complex conjugation (for classical fields) is as follows:

$$(\xi\eta)^\dagger = \eta^\dagger\xi^\dagger\,, \tag{29.44}$$

$$(\xi\sigma^\mu\eta^\dagger)^\dagger = \eta\sigma^\mu\xi^\dagger, \tag{29.45}$$

$$(\xi^\dagger\overline{\sigma}^\mu\eta)^\dagger = \eta^\dagger\overline{\sigma}^\mu\xi, \tag{29.46}$$

$$(\xi\sigma^\mu\overline{\sigma}^\nu\eta)^\dagger = \eta^\dagger\overline{\sigma}^\nu\sigma^\mu\xi^\dagger\,, \tag{29.47}$$

where we have used the hermiticity properties, $(\sigma^\mu)^\dagger = \sigma^\mu$ and $(\overline{\sigma}^\mu)^\dagger = \overline{\sigma}^\mu$. More generally,

$$(\xi\Sigma\eta)^\dagger = \eta^\dagger\Sigma_r\xi^\dagger\,, \qquad (\xi\Sigma\eta^\dagger)^\dagger = \eta\Sigma_r\xi^\dagger\,, \qquad (\xi^\dagger\Sigma\eta)^\dagger = \eta^\dagger\Sigma_r\xi\,, \tag{29.48}$$

where in each case Σ stands for any sequence of alternating σ and $\overline{\sigma}$ matrices, and Σ_r is obtained from Σ by reversing the order of all of the σ and $\overline{\sigma}$ matrices, since the sigma matrices are hermitian. Eqs. (29.44)–(29.48) are applicable both to anticommuting and to commuting spinors.

The properties of the two-component spinor fields under the discrete C, P and T transformations are elucidated in refs. [40, 118]. The corresponding behaviors of the spinor products under C, P and T are easily obtained (and are left as an exercise for the reader).

The following identities can be used to systematically simplify expressions involving products of σ and $\overline{\sigma}$ matrices:[19]

$$\sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}^{\dot\beta\beta}_\mu = 2\delta_\alpha{}^\beta\delta^{\dot\beta}{}_{\dot\alpha}\,, \tag{29.49}$$

$$\sigma^\mu_{\alpha\dot\alpha}\sigma_{\mu\beta\dot\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,, \tag{29.50}$$

$$\overline{\sigma}^{\mu\dot\alpha\alpha}\overline{\sigma}^{\dot\beta\beta}_\mu = 2\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\,, \tag{29.51}$$

$$[\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu]_\alpha{}^\beta = 2g^{\mu\nu}\delta_\alpha{}^\beta\,, \tag{29.52}$$

$$[\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu]^{\dot\alpha}{}_{\dot\beta} = 2g^{\mu\nu}\delta^{\dot\alpha}{}_{\dot\beta}\,, \tag{29.53}$$

[19] Since the Kronecker delta symbol is symmetric under the interchange of its two indices, naively there is nothing gained in writing $\delta_\alpha{}^\beta$ and $\delta^{\dot\beta}{}_{\dot\alpha}$, with the spinor indices staggered as shown, instead of δ^β_α and $\delta^{\dot\beta}_{\dot\alpha}$, respectively. Nevertheless, we often prefer to employ the former rather than the latter as it provides some insight into the spinor index structure of the equation. For example, in eq. (29.52), α labels the row and β labels the column of the product of sigma matrices. Neither $\sigma^\mu\overline{\sigma}^\nu$ nor $\sigma^\nu\overline{\sigma}^\mu$ is symmetric under the interchange of the (suppressed) spinor indices (although the sum of the two is symmetric). By writing $\delta_\alpha{}^\beta$ on the right-hand side of eq. (29.52), one formally maintains the index structure of each of the separate terms of the equation.

In the literature, one sometimes sees eqs. (29.50) and (29.51) rewritten using the identity $\epsilon_{ab}\epsilon_{cd} = \delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$. However, as this latter result does not formally respect covariance with respect to the dotted and undotted indices, we shall not make use of it here. Products of three or more sigma matrices can be reduced to sums of terms involving at most two sigma matrices by employing the identities,

$$\sigma^\mu\overline{\sigma}^\nu\sigma^\rho = g^{\mu\nu}\sigma^\rho - g^{\mu\rho}\sigma^\nu + g^{\nu\rho}\sigma^\mu + i\epsilon^{\mu\nu\rho\kappa}\sigma_\kappa\,, \tag{29.54}$$
$$\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho = g^{\mu\nu}\overline{\sigma}^\rho - g^{\mu\rho}\overline{\sigma}^\nu + g^{\nu\rho}\overline{\sigma}^\mu - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa\,. \tag{29.55}$$

Computations of cross-sections and decay rates generally require traces of alternating products of σ and $\overline{\sigma}$ matrices (e.g., see [105]):

$$\mathrm{Tr}[\sigma^\mu\overline{\sigma}^\nu] = \mathrm{Tr}[\overline{\sigma}^\mu\sigma^\nu] = 2g^{\mu\nu}\,, \tag{29.56}$$
$$\mathrm{Tr}[\sigma^\mu\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}^\kappa] = 2\left(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho} + i\epsilon^{\mu\nu\rho\kappa}\right)\,, \tag{29.57}$$
$$\mathrm{Tr}[\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho\sigma^\kappa] = 2\left(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho} - i\epsilon^{\mu\nu\rho\kappa}\right)\,. \tag{29.58}$$

Traces involving a larger even number of σ and $\overline{\sigma}$ matrices can be systematically obtained from eqs. (29.56)–(29.58) by repeated use of eqs. (29.52) and (29.53) and the cyclic property of the trace. Traces involving an odd number of σ and $\overline{\sigma}$ matrices cannot arise, since there is no way to connect the spinor indices consistently.

In addition to manipulating expressions containing anticommuting fermion quantum fields, we often must deal with products of *commuting* spinor wave functions that arise when evaluating the Feynman rules. In the following expressions we denote the generic spinor by z_i. In the various identities listed below, an extra minus sign arises when interchanging the order of two anticommuting fermion fields of a given spinor index height. It is convenient to introduce the notation:

$$(-1)^A \equiv \begin{cases} +1\,, & \text{commuting spinors,} \\ -1\,, & \text{anticommuting spinors.} \end{cases} \tag{29.59}$$

The following identities hold for the z_i:

$$z_1 z_2 = -(-1)^A z_2 z_1\,, \tag{29.60}$$
$$z_1^\dagger z_2^\dagger = -(-1)^A z_2^\dagger z_1^\dagger\,, \tag{29.61}$$
$$z_1\sigma^\mu z_2^\dagger = (-1)^A z_2^\dagger\overline{\sigma}^\mu z_1\,, \tag{29.62}$$
$$z_1\sigma^\mu\overline{\sigma}^\nu z_2 = -(-1)^A z_2\sigma^\nu\overline{\sigma}^\mu z_1\,, \tag{29.63}$$
$$z_1^\dagger\overline{\sigma}^\mu\sigma^\nu z_2^\dagger = -(-1)^A z_2^\dagger\overline{\sigma}^\nu\sigma^\mu z_1^\dagger\,, \tag{29.64}$$
$$z_1^\dagger\overline{\sigma}^\mu\sigma^\rho\overline{\sigma}^\nu z_2 = (-1)^A z_2\sigma^\nu\overline{\sigma}^\rho\sigma^\mu z_1^\dagger\,, \tag{29.65}$$

and so on.[20] The hermiticity properties of the spinor products given in eqs. (29.44)–(29.48) hold for both commuting and anticommuting spinors, with no additional sign factor.

Two-component spinor products can often be simplified by using Fierz identities. Due to the antisymmetry of the suppressed two-index epsilon symbol [or equivalently, using the Schouten identities given in eq. (29.28)], the following identities are obtained:

$$(z_1 z_2)(z_3 z_4) = -(z_1 z_3)(z_4 z_2) - (z_1 z_4)(z_2 z_3)\,, \tag{29.66}$$
$$(z_1^\dagger z_2^\dagger)(z_3^\dagger z_4^\dagger) = -(z_1^\dagger z_3^\dagger)(z_4^\dagger z_2^\dagger) - (z_1^\dagger z_4^\dagger)(z_2^\dagger z_3^\dagger)\,, \tag{29.67}$$

where we have used eqs. (29.60) and (29.61) to eliminate any residual factors of $(-1)^A$. Similarly, eqs. (29.49)–(29.51) can be used to derive additional Fierz identities,

$$(z_1\sigma^\mu z_2^\dagger)(z_3^\dagger\overline{\sigma}_\mu z_4) = -2(z_1 z_4)(z_2^\dagger z_3^\dagger)\,, \tag{29.68}$$
$$(z_1^\dagger\overline{\sigma}^\mu z_2)(z_3^\dagger\overline{\sigma}_\mu z_4) = 2(z_1^\dagger z_3^\dagger)(z_4 z_2)\,, \tag{29.69}$$

$$(z_1\sigma^\mu z_2^\dagger)(z_3\sigma_\mu z_4^\dagger) = 2(z_1 z_3)(z_4^\dagger z_2^\dagger)\,. \tag{29.70}$$

[20] In particular, if z is a *commuting* spinor, then $zz = z^\dagger z^\dagger = 0$, as emphasized in refs. [16, 19].

Having eliminated all factors of $(-1)^A$, eqs. (29.66)–(29.70) hold for both commuting and anti-commuting spinors.

From the sigma matrices, one can construct the antisymmetrized products:[21]

$$(\sigma^{\mu\nu})_\alpha{}^\beta \equiv \frac{i}{4}\left(\sigma^\mu_{\alpha\dot\gamma}\overline{\sigma}^{\nu\dot\gamma\beta} - \sigma^\nu_{\alpha\dot\gamma}\overline{\sigma}^{\mu\dot\gamma\beta}\right)\,, \tag{29.71}$$

$$(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \equiv \frac{i}{4}\left(\overline{\sigma}^{\mu\dot\alpha\gamma}\sigma^\nu_{\gamma\dot\beta} - \overline{\sigma}^{\nu\dot\alpha\gamma}\sigma^\mu_{\gamma\dot\beta}\right)\,. \tag{29.72}$$

Equivalently, we can use eqs. (29.52) and (29.53) to write:

$$(\sigma^\mu\overline{\sigma}^\nu)_\alpha{}^\beta = g^{\mu\nu}\delta_\alpha{}^\beta - 2i(\sigma^{\mu\nu})_\alpha{}^\beta\,, \tag{29.73}$$

$$(\overline{\sigma}^\mu\sigma^\nu)^{\dot\alpha}{}_{\dot\beta} = g^{\mu\nu}\delta^{\dot\alpha}{}_{\dot\beta} - 2i(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\,. \tag{29.74}$$

The components of $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ are easily evaluated:

$$\sigma^{ij} = \overline{\sigma}^{ij} = \tfrac{1}{2}\epsilon^{ijk}\sigma^k\,, \qquad \sigma^{i0} = -\sigma^{0i} = -\overline{\sigma}^{i0} = \overline{\sigma}^{0i} = \tfrac{1}{2}i\sigma^i\,. \tag{29.75}$$

The matrices $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ satisfy self-duality relations,

$$\sigma^{\mu\nu} = -\tfrac{1}{2}i\epsilon^{\mu\nu\rho\kappa}\sigma_{\rho\kappa}\,, \qquad \overline{\sigma}^{\mu\nu} = \tfrac{1}{2}i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_{\rho\kappa}\,. \tag{29.76}$$

The self-duality relations can be used to obtain the following two identities:

$$g^{\kappa\rho}\sigma^{\mu\nu} - g^{\nu\rho}\sigma^{\mu\kappa} + g^{\mu\rho}\sigma^{\nu\kappa} - i\epsilon^{\mu\nu\kappa}{}_\lambda\sigma^{\lambda\rho} = 0\,, \tag{29.77}$$

$$g^{\kappa\rho}\overline{\sigma}^{\mu\nu} - g^{\nu\rho}\overline{\sigma}^{\mu\kappa} + g^{\mu\rho}\overline{\sigma}^{\nu\kappa} + i\epsilon^{\mu\nu\kappa}{}_\lambda\overline{\sigma}^{\lambda\rho} = 0\,. \tag{29.78}$$

A number of useful properties and identities involving $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ can be derived. For example, eq. (29.26) implies that:

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \epsilon_{\alpha\tau}\epsilon^{\beta\gamma}(\sigma^{\mu\nu})_\gamma{}^\tau\,, \qquad (\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \epsilon^{\dot\alpha\dot\tau}\epsilon_{\dot\beta\dot\gamma}(\overline{\sigma}^{\mu\nu})^{\dot\gamma}{}_{\dot\tau}\,, \tag{29.79}$$

$$\epsilon^{\tau\alpha}(\sigma^{\mu\nu})_\alpha{}^\beta = \epsilon^{\beta\gamma}(\sigma^{\mu\nu})_\gamma{}^\tau\,, \qquad \epsilon_{\dot\tau\dot\alpha}(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \epsilon_{\dot\beta\dot\gamma}(\overline{\sigma}^{\mu\nu})^{\dot\gamma}{}_{\dot\tau}\,, \tag{29.80}$$

$$\epsilon_{\gamma\beta}(\sigma^{\mu\nu})_\alpha{}^\beta = \epsilon_{\alpha\tau}(\sigma^{\mu\nu})_\gamma{}^\tau\,, \qquad \epsilon^{\dot\gamma\dot\beta}(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \epsilon^{\dot\alpha\dot\tau}(\overline{\sigma}^{\mu\nu})^{\dot\gamma}{}_{\dot\tau}\,. \tag{29.81}$$

Using eqs. (29.49)–(29.55), the following identities can be obtained:

$$(\sigma^{\mu\nu})_\alpha{}^\beta(\sigma_{\mu\nu})_\gamma{}^\tau = \epsilon_{\alpha\gamma}\,\epsilon^{\beta\tau} + \delta_\alpha{}^\tau\delta_\gamma{}^\beta = 2\delta_\alpha{}^\tau\delta_\gamma{}^\beta - \delta_\alpha{}^\beta\delta_\gamma{}^\tau\,, \tag{29.82}$$

$$(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}(\overline{\sigma}_{\mu\nu})^{\dot\gamma}{}_{\dot\tau} = \epsilon^{\dot\alpha\dot\gamma}\,\epsilon_{\dot\beta\dot\tau} + \delta^{\dot\alpha}{}_{\dot\tau}\,\delta^{\dot\gamma}{}_{\dot\beta} = 2\delta^{\dot\alpha}{}_{\dot\tau}\,\delta^{\dot\gamma}{}_{\dot\beta} - \delta^{\dot\alpha}{}_{\dot\beta}\,\delta^{\dot\gamma}{}_{\dot\tau}\,, \tag{29.83}$$

$$(\sigma^{\mu\nu})_\alpha{}^\beta(\overline{\sigma}_{\mu\nu})^{\dot\gamma}{}_{\dot\tau} = 0\,, \tag{29.84}$$

$$\sigma^{\mu\nu}\sigma^\rho = \tfrac{1}{2}i\left(g^{\nu\rho}\sigma^\mu - g^{\mu\rho}\sigma^\nu + i\epsilon^{\mu\nu\rho\kappa}\sigma_\kappa\right)\,, \tag{29.85}$$

$$\overline{\sigma}^{\mu\nu}\overline{\sigma}^\rho = \tfrac{1}{2}i\left(g^{\nu\rho}\overline{\sigma}^\mu - g^{\mu\rho}\overline{\sigma}^\nu - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa\right)\,, \tag{29.86}$$

$$\overline{\sigma}^\mu\sigma^{\nu\rho} = \tfrac{1}{2}i\left(g^{\mu\nu}\overline{\sigma}^\rho - g^{\mu\rho}\overline{\sigma}^\nu - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa\right)\,, \tag{29.87}$$

$$\sigma^\mu\overline{\sigma}^{\nu\rho} = \tfrac{1}{2}i\left(g^{\mu\nu}\sigma^\rho - g^{\mu\rho}\sigma^\nu + i\epsilon^{\mu\nu\rho\kappa}\sigma_\kappa\right)\,, \tag{29.88}$$

$$\sigma^{\mu\nu}\sigma^{\rho\kappa} = -\tfrac{1}{4}\left(g^{\nu\rho}g^{\mu\kappa} - g^{\mu\rho}g^{\nu\kappa} + i\epsilon^{\mu\nu\rho\kappa}\right) + \tfrac{1}{2}i\left(g^{\nu\rho}\sigma^{\mu\kappa} + g^{\mu\kappa}\sigma^{\nu\rho} - g^{\mu\rho}\sigma^{\nu\kappa} - g^{\nu\kappa}\sigma^{\mu\rho}\right)\,, \tag{29.89}$$

$$\overline{\sigma}^{\mu\nu}\overline{\sigma}^{\rho\kappa} = -\tfrac{1}{4}\left(g^{\nu\rho}g^{\mu\kappa} - g^{\mu\rho}g^{\nu\kappa} - i\epsilon^{\mu\nu\rho\kappa}\right) + \tfrac{1}{2}i\left(g^{\nu\rho}\overline{\sigma}^{\mu\kappa} + g^{\mu\kappa}\overline{\sigma}^{\nu\rho} - g^{\mu\rho}\overline{\sigma}^{\nu\kappa} - g^{\nu\kappa}\overline{\sigma}^{\mu\rho}\right)\,. \tag{29.90}$$

[21] The reader is cautioned that $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ are sometimes defined in the literature without the factor of i in eqs. (29.71) and (29.72) (as in [78]), or with an overall factor of $\frac{1}{2}i$ (as in [72]) instead of $\frac{1}{4}i$.

Eqs. (29.89) and (29.90) and the antisymmetry of $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ yield the following trace formulae:

$$\mathrm{Tr}\ \sigma^{\mu\nu} = \mathrm{Tr}\ \overline{\sigma}^{\mu\nu} = 0\,, \tag{29.91}$$

$$\mathrm{Tr}[\sigma^{\mu\nu}\sigma^{\rho\kappa}] = \tfrac{1}{2}\left[g^{\mu\rho}g^{\nu\kappa} - g^{\mu\kappa}g^{\nu\rho} - i\epsilon^{\mu\nu\rho\kappa}\right]\,, \tag{29.92}$$

$$\mathrm{Tr}[\overline{\sigma}^{\mu\nu}\overline{\sigma}^{\rho\kappa}] = \tfrac{1}{2}\left[g^{\mu\rho}g^{\nu\kappa} - g^{\mu\kappa}g^{\nu\rho} + i\epsilon^{\mu\nu\rho\kappa}\right]\,. \tag{29.93}$$

The properties of spinor products involving $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ are easily derived. Under hermitian conjugation (for quantum field operators) or complex conjugation (for classical fields),

$$(\xi\sigma^{\mu\nu}\eta)^\dagger = \eta^\dagger\overline{\sigma}^{\mu\nu}\xi^\dagger\,, \tag{29.94}$$

due to the hermiticity relation, $(\sigma^{\mu\nu})^\dagger = \overline{\sigma}^{\mu\nu}$. Next, we use eqs. (29.63) and (29.64) to obtain:

$$z_1\sigma^{\mu\nu}z_2 = (-1)^A z_2\sigma^{\mu\nu}z_1\,, \tag{29.95}$$

$$z_1^\dagger\overline{\sigma}^{\mu\nu}z_2^\dagger = (-1)^A z_2^\dagger\overline{\sigma}^{\mu\nu}z_1^\dagger\,. \tag{29.96}$$

One can also derive additional Fierz identities, which follow from eqs. (29.82)–(29.84),

$$(z_1\sigma^{\mu\nu}z_2)(z_3\sigma_{\mu\nu}z_4) = -2(z_1z_4)(z_2z_3) - (z_1z_2)(z_3z_4)\,, \tag{29.97}$$

$$(z_1^\dagger\overline{\sigma}^{\mu\nu}z_2^\dagger)(z_3^\dagger\overline{\sigma}_{\mu\nu}z_4^\dagger) = -2(z_1^\dagger z_4^\dagger)(z_2^\dagger z_3^\dagger) - (z_1^\dagger z_2^\dagger)(z_3^\dagger z_4^\dagger)\,, \tag{29.98}$$

$$(z_1\sigma^{\mu\nu}z_2)(z_3^\dagger\overline{\sigma}_{\mu\nu}z_4^\dagger) = 0\,, \tag{29.99}$$

where we have again used eqs. (29.60) and (29.61) to eliminate any residual factors of $(-1)^A$. Thus, eqs. (29.97)–(29.99) hold for both commuting and anticommuting spinors. A more comprehensive list of sigma matrix identities and their associated Fierz identities are given in Appendix B.1 (see also Appendix B of [78]).

The $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ satisfy the commutation relations of the $J^{\mu\nu}$ [cf. eq. (29.16)], and thus can be identified as the generators of the Lorentz group in the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations, respectively. That is, for the $(\frac{1}{2},0)$ representation with a lowered undotted index (e.g. ψ_α), $J^{\mu\nu} = \sigma^{\mu\nu}$, while for the $(0,\frac{1}{2})$ representation with a raised dotted index (e.g. $\psi^{\dagger\dot\alpha}$), $J^{\mu\nu} = \overline{\sigma}^{\mu\nu}$. In particular, the infinitesimal forms for the 4×4 Lorentz transformation matrix Λ [cf. eq. (29.11)] and the corresponding matrices M and $(M^{-1})^\dagger$ that transform the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ spinors, respectively, are given by:

$$\Lambda^\mu{}_\nu \simeq \delta^\mu_\nu + \theta^\mu{}_\nu\,, \tag{29.100}$$

$$M \simeq \mathbb{1}_{2\times 2} - \tfrac{1}{2}i\theta_{\mu\nu}\sigma^{\mu\nu}\,, \tag{29.101}$$

$$(M^{-1})^\dagger \simeq \mathbb{1}_{2\times 2} - \tfrac{1}{2}i\theta_{\mu\nu}\overline{\sigma}^{\mu\nu}\,. \tag{29.102}$$

The inverses of these quantities are obtained (to first order in θ) by replacing $\theta\to-\theta$ in the above formulae. Using eqs. (29.79), (29.101) and (29.102), it follows that:

$$(M^{-1})_\gamma{}^\tau = \epsilon^{\tau\alpha}M_\alpha{}^\beta\epsilon_{\beta\gamma}\,, \tag{29.103}$$

$$(M^{-1\,\dagger})^{\dot\gamma}{}_{\dot\tau} = \epsilon_{\dot\tau\dot\alpha}\,(M^\dagger)^{\dot\alpha}{}_{\dot\beta}\,\epsilon^{\dot\beta\dot\gamma}\,. \tag{29.104}$$

These results can be used to demonstrate the covariance (with respect to Lorentz transformations) of the spinor index raising and lowering properties of the epsilon symbols defined in eq. (29.22). The infinitesimal forms given by eqs. (29.100)–(29.102) can also be used [with the assistance of eqs. (29.85)–(29.87)] to establish the following two results:

$$M^\dagger\overline{\sigma}^\mu M = \Lambda^\mu{}_\nu\,\overline{\sigma}^\nu\,, \tag{29.105}$$

$$M^{-1}\sigma^\mu(M^{-1})^\dagger = \Lambda^\mu{}_\nu\,\sigma^\nu\,. \tag{29.106}$$

Using the Lorentz transformation properties of the undotted and dotted two-component spinor fields, eqs. (29.105) and (29.106) can be used, respectively, to prove that the spinor products $\xi^\dagger\overline{\sigma}^\mu\eta$ and $\xi\sigma^\mu\eta^\dagger$ transform as Lorentz four-vectors.

As an example, consider a pure boost from the rest frame to a frame where $p^\mu = (E_{\boldsymbol{p}}\,, \vec{\boldsymbol{p}})$, which corresponds to $\theta_{ij} = 0$ and $\zeta^i = \theta^{i0} = -\theta^{0i}$. We assume that the mass-shell condition is satisfied, i.e. $p^0 = E_{\vec{\boldsymbol{p}}} \equiv (|\vec{\boldsymbol{p}}|^2 + m^2)^{1/2}$. The matrices $M_\alpha{}^\beta$ and $[(M^{-1})^\dagger]^{\dot\alpha}{}_{\dot\beta}$ that govern the Lorentz transformations of spinor fields with a lowered undotted index and spinor fields with a raised dotted index, respectively, are given by:

$$\exp\left(-\tfrac{1}{2}i\theta_{\mu\nu}J^{\mu\nu}\right) = \begin{cases} M = \exp\left(-\tfrac{1}{2}\vec{\boldsymbol{\zeta}}\cdot\vec{\boldsymbol{\sigma}}\right) = \sqrt{\dfrac{p\cdot\sigma}{m}}\,, & \text{for } (\tfrac{1}{2},0)\,, \\[2ex] (M^{-1})^\dagger = \exp\left(\tfrac{1}{2}\vec{\boldsymbol{\zeta}}\cdot\vec{\boldsymbol{\sigma}}\right) = \sqrt{\dfrac{p\cdot\overline{\sigma}}{m}}\,, & \text{for } (0,\tfrac{1}{2})\,, \end{cases} \tag{29.107}$$

where

$$\sqrt{p\cdot\sigma} \equiv \frac{(E_{\boldsymbol{p}} + m)\,\mathbb{1}_{2\times2} - \vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{p}}}{\sqrt{2(E_{\boldsymbol{p}} + m)}}\,, \tag{29.108}$$

$$\sqrt{p\cdot\overline{\sigma}} \equiv \frac{(E_{\boldsymbol{p}} + m)\,\mathbb{1}_{2\times2} + \vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{p}}}{\sqrt{2(E_{\boldsymbol{p}} + m)}}\,. \tag{29.109}$$

These matrix square roots are defined to be the unique non-negative definite hermitian matrices (i.e., with non-negative eigenvalues) whose squares are equal to the non-negative definite hermitian matrices $p\cdot\sigma$ and $p\cdot\overline{\sigma}$, respectively.[22]

According to eq. (29.107), the spinor index structure of $\sqrt{p\cdot\sigma}$ and $\sqrt{p\cdot\overline{\sigma}}$ corresponds to that of $M_\alpha{}^\beta$ and $[(M^{-1})^\dagger]^{\dot\alpha}{}_{\dot\beta}$, respectively. In this case, we can rewrite eqs. (29.108) and (29.109) as:

$$\left[\sqrt{p\cdot\sigma}\,\right]_\alpha{}^\beta \equiv \left[\sqrt{p\cdot\sigma\,\overline{\sigma}^0}\,\right]_\alpha{}^\beta = \frac{(p\cdot\sigma_{\alpha\dot\alpha})\overline{\sigma}^{0\,\dot\alpha\beta} + m\delta_\alpha^\beta}{\sqrt{2(E_{\boldsymbol{p}} + m)}}\,, \tag{29.110}$$

$$\left[\sqrt{p\cdot\overline{\sigma}}\,\right]^{\dot\alpha}{}_{\dot\beta} \equiv \left[\sqrt{p\cdot\overline{\sigma}\,\sigma^0}\,\right]^{\dot\alpha}{}_{\dot\beta} = \frac{(p\cdot\overline{\sigma}^{\dot\alpha\alpha})\sigma^0_{\alpha\dot\beta} + m\delta^{\dot\alpha}_{\dot\beta}}{\sqrt{2(E_{\boldsymbol{p}} + m)}}\,, \tag{29.111}$$

since $\sigma^0 = \overline{\sigma}^0 = \mathbb{1}_{2\times2}$. Using eqs. (29.54) and (29.55), one can easily verify that:

$$\left[\sqrt{p\cdot\sigma}\,\right]_\alpha{}^\gamma\left[\sqrt{p\cdot\sigma}\,\right]_\gamma{}^\beta = (p\cdot\sigma\,\overline{\sigma}^0)_\alpha{}^\beta\,, \tag{29.112}$$

$$\left[\sqrt{p\cdot\overline{\sigma}}\,\right]^{\dot\alpha}{}_{\dot\gamma}\left[\sqrt{p\cdot\overline{\sigma}}\,\right]^{\dot\gamma}{}_{\dot\beta} = (p\cdot\overline{\sigma}\,\sigma^0)^{\dot\alpha}{}_{\dot\beta}\,, \tag{29.113}$$

where implicit factors of $\overline{\sigma}^0$ and σ^0 inside the square roots of eq. (29.112) have been suppressed.

Due to the fact that $p\cdot\sigma$ and $p\cdot\overline{\sigma}$ are hermitian, we could have defined their hermitian matrix square roots by the hermitian conjugate of eq. (29.107). In this case, the spinor index structure of $\sqrt{p\cdot\sigma}$ and $\sqrt{p\cdot\overline{\sigma}}$ would correspond to that of $[(M^\dagger]^{\dot\alpha}{}_{\dot\beta}$ and $[M^{-1}]_\alpha{}^\beta$, respectively. That is, instead of eqs. (29.110) and (29.111), we would now rewrite eqs. (29.108) and (29.109) in the following form:

$$\left[\sqrt{p\cdot\sigma}\,\right]^{\dot\alpha}{}_{\dot\beta} \equiv \left[\sqrt{\overline{\sigma}^0\,p\cdot\sigma}\,\right]^{\dot\alpha}{}_{\dot\beta} = \frac{\overline{\sigma}^{0\,\dot\alpha\beta}(p\cdot\sigma_{\beta\dot\beta}) + m\delta^{\dot\alpha}_{\dot\beta}}{\sqrt{2(E_{\boldsymbol{p}} + m)}}\,, \tag{29.114}$$

$$\left[\sqrt{p\cdot\overline{\sigma}}\,\right]_\alpha{}^\beta \equiv \left[\sqrt{\sigma^0\,p\cdot\overline{\sigma}}\,\right]_\alpha{}^\beta = \frac{\sigma^0_{\alpha\dot\beta}(p\cdot\overline{\sigma}^{\dot\beta\beta}) + m\delta_\alpha^\beta}{\sqrt{2(E_{\boldsymbol{p}} + m)}}\,. \tag{29.115}$$

[22] Note that $p\cdot\sigma$ and $p\cdot\overline{\sigma}$ are non-negative matrices due to the implicit mass-shell condition satisfied by p^μ.

Using eqs. (29.54) and (29.55), one can again confirm that:

$$\big[\sqrt{p\cdot\sigma}\,\big]^{\dot\alpha}{}_{\dot\gamma}\big[\sqrt{p\cdot\sigma}\,\big]^{\dot\gamma}{}_{\dot\beta} = (\overline\sigma^0\, p\cdot\sigma)^{\dot\alpha}{}_{\dot\beta}\,, \qquad \big[\sqrt{p\cdot\overline\sigma}\,\big]_{\alpha}{}^{\gamma}\big[\sqrt{p\cdot\overline\sigma}\,\big]_{\gamma}{}^{\beta} = (\sigma^0\, p\cdot\overline\sigma)_{\alpha}{}^{\beta}\,, \tag{29.116}$$

where implicit factors of $\overline\sigma^0$ and σ^0 inside the square roots of eq. (29.116) have been suppressed.

The proper choice of the spinor index structure for $\sqrt{p\cdot\sigma}$ and $\sqrt{p\cdot\overline\sigma}$ can always be determined for any covariant expression. That is, if we employ the spinor index-free notation (and suppress the factors of σ^0 and $\overline\sigma^0$), it will always be clear from the context which spinor index structure for $\sqrt{p\cdot\sigma}$ and $\sqrt{p\cdot\overline\sigma}$ is implicit.

As an example that will prove valuable later on, consider an arbitrary four-vector S^μ, defined in a reference frame where $p^\mu = (E\,;\vec{\boldsymbol{p}})$, whose rest frame value is S^μ_R, i.e.

$$S^\mu = \Lambda^\mu{}_\nu S^\nu_R\,, \qquad \text{with} \qquad \Lambda = \begin{pmatrix} E/m & p^j/m \\ p^i/m & \delta_{ij} + \dfrac{p^i p^j}{m(E+m)} \end{pmatrix}\,. \tag{29.117}$$

Then, using eqs. (29.7), (29.106) and (29.107), it follows that:

$$\sqrt{p\cdot\sigma}\, S\cdot\overline\sigma\, \sqrt{p\cdot\sigma} = mS_R\cdot\overline\sigma\,, \qquad \sqrt{p\cdot\overline\sigma}\, S\cdot\sigma\, \sqrt{p\cdot\overline\sigma} = mS_R\cdot\sigma\,. \tag{29.118}$$

The spinor index structures of eq. (29.118) are easily established:

$$\big[\sqrt{p\cdot\sigma}\,\big]^{\dot\beta}{}_{\dot\gamma}\, S\cdot\overline\sigma^{\dot\gamma\alpha}\big[\sqrt{p\cdot\sigma}\,\big]_{\alpha}{}^{\beta} = mS_R\cdot\overline\sigma^{\dot\beta\beta}\,, \tag{29.119}$$

$$\big[\sqrt{p\cdot\overline\sigma}\,\big]_{\beta}{}^{\gamma}\, S\cdot\sigma_{\gamma\dot\alpha}\big[\sqrt{p\cdot\overline\sigma}\,\big]^{\dot\alpha}{}_{\dot\beta} = mS_R\cdot\sigma_{\beta\dot\beta}\,. \tag{29.120}$$

Using eqs. (29.110)–(29.117) and (29.54)–(29.55), one can directly verify the above results.

The two-component spinor formalism established in this section will be applied to the quantum field theory of fermions in Minkowski space of one time and three space dimensions in this review. We also direct the reader's attention to Appendices G.1 and G.2, which provide details of the correspondence between the two-component and four-component spinor notation.

For certain applications, the spinor formalism in four-dimensional Minkowski space is not sufficient. For example, in order to obtain instanton solutions [119, 120, 121], it is necessary to formulate quantum field theory in Euclidean space. One also needs the Euclidean space formalism for a rigorous definition of the path integral [122, 123]. The Green functions derived from the Euclidean path integral can be related to the Green functions of the Minkowski space theory by a Wick rotation [124]. In addition, to evaluate the loop-corrected Green functions of the theory, it is often most convenient to apply a regularization scheme that involves dimensional continuation away from $d = 4$ spacetime dimensions [125]. Thus, we also need to generalize the spinor results of this section to $d \neq 4$.

The treatment of fermions in Euclidean space is subtle [126, 127, 128, 129]. Here, we focus briefly on the mathematics of fermions in $d = 4$ Euclidean dimensions, where the relevant spacetime symmetry group is SO(4) rather than SO(3,1). The two-dimensional representations of SO(3,1)$\cong$SL(2,$\mathbb{C}$), denoted in this section by $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, respectively, are complex representations that are related by hermitian conjugation. In contrast, the two-dimensional representations of SO(4)$\cong$SU(2)$\times$SU(2), also denoted by $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, respectively,[23] are independent pseudo-real representations, i.e. *not* related by hermitian conjugation. A two-component spinor notation can be formulated for fields that transform respectively under the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of SO(4). Details can be found in refs. [120, 130, 131].

[23] These SO(4) representations transform as a doublet under one of the SU(2) groups and as a singlet under the other SU(2) group.

In Feynman diagram calculations, one can adopt the standard procedure for the Wick rotation in order to evaluate the loop integrals in Euclidean space. We shall employ the standard Euclidean metric $\delta^{\mu\nu}$ in computing scalar products of four-vectors. Moreover, one can define Euclidean sigma matrices, $\sigma_E^\mu = (-i\vec{\boldsymbol{\sigma}}\,, \sigma_E^4)$ and $\overline{\sigma}_E^\mu = (i\vec{\boldsymbol{\sigma}}\,, \overline{\sigma}_E^4)$, where $\sigma_E^4 = \overline{\sigma}_E^4 \equiv \mathbb{1}_{2\times 2}$. In this convention, the Wick-rotated versions of eqs. (29.52)–(29.58) are preserved [after making the replacements $g^{\mu\nu} \to \delta^{\mu\nu}$ and $i\epsilon^{ijk0} \to \epsilon^{ijk4}$, with $\epsilon^{1234} = \epsilon_{1234} = +1$].[24] Further details of our Euclidean space conventions are provided at the end of Appendix A.

The generalization of the spinor results of this section to $d \neq 4$, useful for dimensional continuation regularization schemes, is discussed in Appendix B.2 . In particular, the identities of Appendix B.1 used to derive Fierz identities [cf. eqs. (29.66)–(29.70) and (29.97)–(29.99)] and any identities involving the four-dimensional Levi-Civita ϵ-tensor are not valid unless μ is a Lorentz vector index in exactly four dimensions. In our treatment of two-component spinor identities in $d \neq 4$ dimensions given in Appendix B.2 , we take the Lorentz vector indices to formally run over d values, whereas the undotted and dotted spinor indices continue to take on two possible values. This is sufficient when used as a regularization procedure for divergent integrals that arise in loop computations. However in generic d-dimensional field theories where d is an integer greater than 4, the two-component spinor formalism of this review is no longer applicable. Suitable methods for treating spinors in diverse spacetime dimensions and signatures [132, 91, 133, 134, 135, 136, 137, 138, 139, 140, 92, 141, 142, 143, 144] are briefly presented in Appendix G.3 .

29.3 Properties of fermion fields

In this review, we refer to spin-1/2 particles as Majorana or Dirac fermions depending on the nature of the global symmetry[25] that governs the fermion Lagrangian and dictates the form of the fermion mass terms. A *Majorana fermion* is a two-component massive field that is completely neutral (i.e. a singlet with respect to the symmetry group) or transforms as a non-trivial real representation of the symmetry group (cf. footnote 37). A *Dirac fermion* consists of a pair of two-component massive fields that are oppositely charged with respect to a conserved O(2) symmetry. As shown in Section 29.3.2, Dirac fermions arise when a multiplet of two-component fermions transforms as a complex or pseudo-real representation of the symmetry group.[26]

The case of a massless fermion is special, as the absence of mass terms leads to an enhanced global symmetry group. Each physical spin-1/2 zero-mass eigenstate is fundamentally a two-component spinor. Thus, following the standard nomenclature used for massless neutrinos, it is common to employ the term *massless Weyl fermion* to describe any massless spin-1/2 particle.[27]

29.3.1 The two-component fermion field and spinor wave functions

We begin by describing the properties of a free neutral massive anticommuting spin-1/2 field, denoted $\xi_\alpha(x)$, which transforms as $(\frac{1}{2}, 0)$ under the Lorentz group. The field ξ_α therefore describes

[24] In practical computations of one-loop matrix elements, one can carry out all the sigma matrix algebra in Minkowski space *before* Wick-rotating to Euclidean space in order to perform the loop integrals.

[25] A subgroup of the global symmetry group may be gauged (and hence promoted to a local symmetry). Degrees of freedom not associated with the gauged subgroup are typically referred to as flavor degrees of freedom.

[26] Majorana and Dirac fermions can also be described in terms of four-component Majorana and Dirac spinor fields, as in Appendix G . However, keep in mind that the terms *Majorana spinor* and *Dirac spinor* are defined strictly in the context of the four-component spinor formalism as in Appendix G.1 , or in the more general context of a d-dimensional spacetime as in Appendix G.3 .

[27] Two-component fermions are often called Weyl fermions, due to their association with the two-dimensional spinor representations of the Lorentz group introduced by Weyl in refs. [14, 15]. It is now common practice to define a Weyl spinor as the left or right-handed projection of a four-component spinor [as in eq. (G.1.9)]. Of course, there is a one-to-one correspondence between these two definitions.

a Majorana fermion [3]. The free-field Lagrangian density is [16]:

$$\mathscr{L} = i\xi^\dagger\overline{\sigma}^\mu\partial_\mu\xi - \tfrac{1}{2}m(\xi\xi + \xi^\dagger\xi^\dagger)\,. \tag{29.3.1}$$

On-shell, ξ satisfies the free-field Dirac equation [2, 14, 1, 145, 146],

$$i\overline{\sigma}^{\mu\dot\alpha\beta}\partial_\mu\xi_\beta = m\xi^{\dagger\dot\alpha}\,. \tag{29.3.2}$$

Consequently after quantization, ξ_α can be expanded in a Fourier series [145]:

$$\xi_\alpha(x) = \sum_s \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}}\left[x_\alpha(\vec{\boldsymbol{p}},s)a(\vec{\boldsymbol{p}},s)e^{-ip\cdot x} + y_\alpha(\vec{\boldsymbol{p}},s)a^\dagger(\vec{\boldsymbol{p}},s)e^{ip\cdot x}\right]\,, \tag{29.3.3}$$

where $E_{\boldsymbol{p}} \equiv (|\vec{\boldsymbol{p}}|^2 + m^2)^{1/2}$, and the creation and annihilation operators $a^\dagger$ and a satisfy anticommutation relations:

$$\{a(\vec{\boldsymbol{p}},s),\, a^\dagger(\vec{\boldsymbol{p}}',s')\} = \delta^3(\vec{\boldsymbol{p}}-\vec{\boldsymbol{p}}')\delta_{ss'}\,, \tag{29.3.4}$$

and all other anticommutators vanish. It follows that

$$\xi^\dagger_{\dot\alpha}(x) \equiv (\xi_\alpha)^\dagger = \sum_s \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}}\left[x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},s)a^\dagger(\vec{\boldsymbol{p}},s)e^{ip\cdot x} + y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},s)a(\vec{\boldsymbol{p}},s)e^{-ip\cdot x}\right]\,. \tag{29.3.5}$$

We employ covariant normalization of the one-particle states, i.e., we act with one creation operator on the vacuum with the following convention

$$|\vec{\boldsymbol{p}},s\rangle \equiv (2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}a^\dagger(\vec{\boldsymbol{p}},s)\,|0\rangle\,, \tag{29.3.6}$$

so that $\langle\vec{\boldsymbol{p}},s|\vec{\boldsymbol{p}}',s'\rangle = (2\pi)^3(2E_{\boldsymbol{p}})\delta^3(\vec{\boldsymbol{p}}-\vec{\boldsymbol{p}}')\delta_{ss'}$. Therefore,

$$\langle 0|\,\xi_\alpha(x)\,|\vec{\boldsymbol{p}},s\rangle = x_\alpha(\vec{\boldsymbol{p}},s)e^{-ip\cdot x}\,, \qquad \langle 0|\,\xi^\dagger_{\dot\alpha}(x)\,|\vec{\boldsymbol{p}},s\rangle = y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},s)e^{-ip\cdot x}\,, \tag{29.3.7}$$

$$\langle\vec{\boldsymbol{p}},s|\,\xi_\alpha(x)\,|0\rangle = y_\alpha(\vec{\boldsymbol{p}},s)e^{ip\cdot x}\,, \qquad \langle\vec{\boldsymbol{p}},s|\,\xi^\dagger_{\dot\alpha}(x)\,|0\rangle = x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},s)e^{ip\cdot x}\,. \tag{29.3.8}$$

It should be emphasized that $\xi_\alpha(x)$ is an anticommuting spinor field, whereas x_α and y_α are *commuting* two-component spinor wave functions. The anticommuting properties of the fields are carried by the creation and annihilation operators.

Applying eq. (29.3.2) to eq. (29.3.3), we find that the x_α and y_α satisfy momentum space Dirac equations. These conditions can be written down in a number of equivalent ways:

$$(p\cdot\overline{\sigma})^{\dot\alpha\beta}x_\beta = my^{\dagger\dot\alpha}\,, \qquad (p\cdot\sigma)_{\alpha\dot\beta}y^{\dagger\dot\beta} = mx_\alpha\,, \tag{29.3.9}$$

$$(p\cdot\sigma)_{\alpha\dot\beta}x^{\dagger\dot\beta} = -my_\alpha\,, \qquad (p\cdot\overline{\sigma})^{\dot\alpha\beta}y_\beta = -mx^{\dagger\dot\alpha}\,, \tag{29.3.10}$$

$$x^\alpha(p\cdot\sigma)_{\alpha\dot\beta} = -my^\dagger_{\dot\beta}\,, \qquad y^\dagger_{\dot\alpha}(p\cdot\overline{\sigma})^{\dot\alpha\beta} = -mx^\beta\,, \tag{29.3.11}$$

$$x^\dagger_{\dot\alpha}(p\cdot\overline{\sigma})^{\dot\alpha\beta} = my^\beta\,, \qquad y^\alpha(p\cdot\sigma)_{\alpha\dot\beta} = mx^\dagger_{\dot\beta}\,. \tag{29.3.12}$$

Using the identities $[(p\cdot\sigma)(p\cdot\overline{\sigma})]_\alpha{}^\beta = p^2\,\delta_\alpha{}^\beta$ and $[(p\cdot\overline{\sigma})(p\cdot\sigma)]^{\dot\alpha}{}_{\dot\beta} = p^2\,\delta^{\dot\alpha}{}_{\dot\beta}$, one can check that both x_α and y_α must satisfy the mass-shell condition, $p^2 = m^2$ (or equivalently, $p^0 = E_{\boldsymbol{p}}$). We will later see that eqs. (29.3.9)–(29.3.12) are often useful for simplifying matrix elements.

The quantum number s labels the spin or helicity of the spin-1/2 fermion. We shall examine two approaches for constructing the spin-1/2 states. In the first approach, we consider the particle in its rest frame and quantize the spin along a fixed axis specified by the unit vector $\hat{\boldsymbol{s}} \equiv (\sin\theta\cos\phi\,,\, \sin\theta\sin\phi\,,\, \cos\theta)$ with polar angle θ and azimuthal angle ϕ with respect to a fixed

z-axis.[28] The corresponding spin states will be called fixed-axis spin states. The relevant basis of two-component spinors χ_s are eigenstates of $\frac{1}{2}\vec{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{s}}$, i.e.,

$$\tfrac{1}{2}\vec{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{s}}\,\chi_s = s\chi_s\,, \qquad s = \pm\tfrac{1}{2}\,. \tag{29.3.13}$$

Explicit forms for the two-component spinors χ_s and their properties are given in Appendix C .

The fixed-axis spin states described above are not very convenient for particles in relativistic motion. Moreover, these states cannot be employed for massless particles since no rest frame exists. Thus, a second approach is to consider helicity states and the corresponding basis of two-component helicity spinors χ_λ that are eigenstates of $\frac{1}{2}\vec{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{p}}$, i.e.,

$$\tfrac{1}{2}\vec{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{p}}\,\chi_\lambda = \lambda\chi_\lambda, \qquad \lambda = \pm\tfrac{1}{2}\,. \tag{29.3.14}$$

Here $\hat{\boldsymbol{p}}$ is the unit vector in the direction of the three-momentum, with polar angle θ and azimuthal angle ϕ with respect to a fixed z-axis. That is, the two-component helicity spinors can be obtained from the fixed-axis spinors by replacing $\hat{\boldsymbol{s}}$ by $\hat{\boldsymbol{p}}$ and identifying θ and ϕ as the polar and azimuthal angles of $\hat{\boldsymbol{p}}$.

For fermions of mass $m \neq 0$, one can define a spin four-vector S^μ, which reduces to $(0;\hat{\boldsymbol{s}})$ in the rest frame. For fixed-axis spin states, the unit three-vector $\hat{\boldsymbol{s}}$ corresponds to the axis of spin quantization. In an arbitrary reference frame, the spin four-vector satisfies $S\cdot p = 0$ and $S\cdot S = -1$. Boosting from the rest frame to a frame where $p^\mu = (E\,,\,\vec{\boldsymbol{p}})$ [using eq. (29.117)] yields

$$S^\mu = \left(\frac{\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{s}}}{m}\,;\,\hat{\boldsymbol{s}} + \frac{(\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{s}})\,\vec{\boldsymbol{p}}}{m(E+m)}\right)\,. \tag{29.3.15}$$

If necessary, we shall write $S^\mu(\hat{\boldsymbol{s}})$ to emphasize the dependence of S^μ on $\hat{\boldsymbol{s}}$.

The spin four-vector for helicity states is defined by taking $\hat{\boldsymbol{s}} = \hat{\boldsymbol{p}}$. Eq. (29.3.15) then reduces to

$$S^\mu = \frac{1}{m}\,(|\vec{\boldsymbol{p}}|\,;\,E\hat{\boldsymbol{p}})\,. \tag{29.3.16}$$

In the non-relativistic limit, the spin four-vector for helicity states is $S^\mu \simeq (0\,;\,\hat{\boldsymbol{p}})$, as expected.[29] In the high energy limit ($E \gg m$), $S^\mu = p^\mu/m + \mathcal{O}(m/E)$. For a massless fermion, the spin four-vector does not exist (as there is no rest frame). Nevertheless, one can obtain consistent results by working with massive helicity states and taking the $m \to 0$ limit at the end of the computation. In this case, one can simply use $S^\mu = p^\mu/m + \mathcal{O}(m/E)$; in practical computations the final result will be well-defined in the zero mass limit. In contrast, for massive fermions at rest, the helicity state does not exist without reference to some particular boost direction as noted in footnote 29.

Using eq. (29.118) with $S^\mu_R = (0\,;\,\hat{\boldsymbol{s}})$, two important formulae are obtained:

$$\sqrt{p\cdot\sigma}\,S\cdot\overline{\sigma}\,\sqrt{p\cdot\sigma} = m\,\vec{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{s}}\,, \qquad \sqrt{p\cdot\overline{\sigma}}\,S\cdot\sigma\,\sqrt{p\cdot\overline{\sigma}} = -m\,\vec{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{s}}\,. \tag{29.3.17}$$

These results can also be derived directly by employing the explicit form for the spin vector S^μ [eq. (29.3.15)] and the results of eqs. (29.108) and (29.109).

The two-component spinor wave functions x and y can now be given explicitly in terms of the χ_s defined in eq. (C.1.11). First, we note that eq. (29.3.9) when evaluated in the rest frame yields $x_1 = y^{\dagger 1}$ and $x_2 = y^{\dagger 2}$. That is, as column vectors, $x_\alpha(\vec{\boldsymbol{p}} = 0) = y^{\dagger\dot{\alpha}}(\vec{\boldsymbol{p}} = 0)$ can be expressed in general as some linear combination of the χ_s ($s = \pm\frac{1}{2}$). Hence, we may choose $x_\alpha(\vec{\boldsymbol{p}} = 0, s) = y^{\dagger\dot{\alpha}}(\vec{\boldsymbol{p}} = 0, s) = \sqrt{m}\chi_s$, where the factor of $\sqrt{m}$ reflects the standard relativistic

[28] In the literature, it is a common practice to choose $\hat{\boldsymbol{s}} = \hat{\boldsymbol{z}}$. However in order to be somewhat more general, we shall not assume this convention here.

[29] Strictly speaking, $\hat{\boldsymbol{p}}$ is not defined in the rest frame. In practice, helicity states are defined in some moving frame with momentum $\vec{\boldsymbol{p}}$. The rest frame is achieved by boosting in the direction of $-\vec{\boldsymbol{p}}$.

normalization of the rest frame spin states. These wave functions can be boosted to an arbitrary frame using eq. (29.107). The resulting undotted spinor wave functions are given by:[30]

$$x_\alpha(\vec{\boldsymbol{p}},s)=\sqrt{p\cdot\sigma}\,\chi_s\,,\qquad x^\alpha(\vec{\boldsymbol{p}},s)=-2s\chi^\dagger_{-s}\sqrt{p\cdot\overline{\sigma}}\,, \tag{29.3.18}$$

$$y_\alpha(\vec{\boldsymbol{p}},s)=2s\sqrt{p\cdot\sigma}\,\chi_{-s}\,,\qquad y^\alpha(\vec{\boldsymbol{p}},s)=\chi^\dagger_s\sqrt{p\cdot\overline{\sigma}}\,, \tag{29.3.19}$$

and the dotted spinor wave functions are given by

$$x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)=-2s\sqrt{p\cdot\overline{\sigma}}\,\chi_{-s}\,,\qquad x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},s)=\chi^\dagger_s\sqrt{p\cdot\sigma}\,, \tag{29.3.20}$$

$$y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)=\sqrt{p\cdot\overline{\sigma}}\,\chi_s\,,\qquad y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},s)=2s\chi^\dagger_{-s}\sqrt{p\cdot\sigma}\,, \tag{29.3.21}$$

where $\sqrt{p\cdot\sigma}$ and $\sqrt{p\cdot\overline{\sigma}}$ are defined either by eqs. (29.110) and (29.111) or by eqs. (29.114) and (29.115), respectively (as mandated by the spinor index structure). Note that eqs. (29.3.18)–(29.3.21) imply that the x and y spinors are related:

$$y(\vec{\boldsymbol{p}},s)=2sx(\vec{\boldsymbol{p}},-s)\,,\qquad y^\dagger(\vec{\boldsymbol{p}},s)=2sx^\dagger(\vec{\boldsymbol{p}},-s)\,. \tag{29.3.22}$$

The phase choices in eqs. (29.3.18)–(29.3.21) are consistent with those employed for four-component spinor wave functions [see Appendix G]. We again emphasize that in eqs. (29.3.18)–(29.3.21), one may either choose χ_s to be an eigenstate of $\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}$, where the spin is measured in the rest frame along the quantization axis $\hat{\boldsymbol{s}}$, or choose χ_s to be an eigenstate of $\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{p}}$ (in this case we shall write $s=\lambda$), which yields the helicity spinor wave functions.

The following equations can now be derived:

$$(S\cdot\overline{\sigma})^{\dot\alpha\beta}x_\beta(\vec{\boldsymbol{p}},s)=2sy^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)\,,\qquad (S\cdot\sigma)_{\alpha\dot\beta}y^{\dagger\dot\beta}(\vec{\boldsymbol{p}},s)=-2sx_\alpha(\vec{\boldsymbol{p}},s)\,, \tag{29.3.23}$$

$$(S\cdot\sigma)_{\alpha\dot\beta}x^{\dagger\dot\beta}(\vec{\boldsymbol{p}},s)=-2sy_\alpha(\vec{\boldsymbol{p}},s)\,,\qquad (S\cdot\overline{\sigma})^{\dot\alpha\beta}y_\beta(\vec{\boldsymbol{p}},s)=2sx^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)\,, \tag{29.3.24}$$

$$x^\alpha(\vec{\boldsymbol{p}},s)(S\cdot\sigma)_{\alpha\dot\beta}=-2sy^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s)\,,\qquad y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},s)(S\cdot\overline{\sigma})^{\dot\alpha\beta}=2sx^\beta(\vec{\boldsymbol{p}},s)\,, \tag{29.3.25}$$

$$x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},s)(S\cdot\overline{\sigma})^{\dot\alpha\beta}=2sy^\beta(\vec{\boldsymbol{p}},s)\,,\qquad y^\alpha(\vec{\boldsymbol{p}},s)(S\cdot\sigma)_{\alpha\dot\beta}=-2sx^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s)\,. \tag{29.3.26}$$

For example, using eq. (29.3.17) and the definitions above for $x_\alpha(\vec{\boldsymbol{p}},s)$ and $y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)$, we find (suppressing spinor indices),

$$\sqrt{p\cdot\sigma}\,S\cdot\overline{\sigma}\,x(\vec{\boldsymbol{p}},s)=\sqrt{p\cdot\sigma}\,S\cdot\overline{\sigma}\,\sqrt{p\cdot\sigma}\,\chi_s=m\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}\,\chi_s=2sm\,\chi_s\,. \tag{29.3.27}$$

Multiplying both sides of eq. (29.3.27) by $\sqrt{p\cdot\overline{\sigma}}$ and noting that $\sqrt{p\cdot\overline{\sigma}}\sqrt{p\cdot\sigma}=m$, we end up with

$$S\cdot\overline{\sigma}\,x(\vec{\boldsymbol{p}},s)=2s\sqrt{p\cdot\overline{\sigma}}\,\chi_s=2sy^\dagger(\vec{\boldsymbol{p}},s)\,. \tag{29.3.28}$$

All the results of eqs. (29.3.23)–(29.3.26) can be derived in this manner.

The consistency of eqs. (29.3.23)–(29.3.26) can also be checked as follows. First, each of these equations yields

$$(S\cdot\sigma)_{\alpha\dot\alpha}(S\cdot\overline{\sigma})^{\dot\alpha\beta}=-\delta^\beta_\alpha\,,\qquad (S\cdot\overline{\sigma})^{\dot\alpha\alpha}(S\cdot\sigma)_{\alpha\dot\beta}=-\delta^{\dot\alpha}_{\dot\beta}\,, \tag{29.3.29}$$

after noting that $4s^2=1$ (for $s=\pm\frac{1}{2}$). From eqs. (29.52) and (29.53) it follows that $S\cdot S=-1$, as required. Second, if one applies

$$(p\cdot\sigma\,S\cdot\overline{\sigma}+S\cdot\sigma\,p\cdot\overline{\sigma})_\alpha{}^\beta=2p\cdot S\,\delta_\alpha{}^\beta\,, \tag{29.3.30}$$

$$(p\cdot\overline{\sigma}\,S\cdot\sigma+S\cdot\overline{\sigma}\,p\cdot\sigma)^{\dot\alpha}{}_{\dot\beta}=2p\cdot S\,\delta^{\dot\alpha}{}_{\dot\beta}\,, \tag{29.3.31}$$

[30] Explicit forms for two-component spinor wave functions have been exhibited a number of times in the literature. For example, see refs. [105, 106] and Appendix I.1 .

to eqs. (29.3.9)–(29.3.12) and eqs. (29.3.23)–(29.3.26), it follows that $p\cdot S = 0$.

It is useful to combine the results of eqs. (29.3.9)–(29.3.12) and eqs. (29.3.23)–(29.3.26) as follows:

$$(p^\mu - 2smS^\mu)\overline{\sigma}_\mu^{\dot\alpha\beta}x_\beta(\vec{\boldsymbol{p}},s) = 0\,, \qquad (p_\mu - 2smS_\mu)\sigma^\mu_{\alpha\dot\beta}x^{\dagger\dot\beta}(\vec{\boldsymbol{p}},s) = 0\,, \tag{29.3.32}$$

$$(p^\mu + 2smS^\mu)\overline{\sigma}_\mu^{\dot\alpha\beta}y_\beta(\vec{\boldsymbol{p}},s) = 0\,, \qquad (p_\mu + 2smS_\mu)\sigma^\mu_{\alpha\dot\beta}y^{\dagger\dot\beta}(\vec{\boldsymbol{p}},s) = 0\,, \tag{29.3.33}$$

$$x^\alpha(\vec{\boldsymbol{p}},s)\sigma^\mu_{\alpha\dot\beta}(p_\mu - 2smS_\mu) = 0\,, \qquad x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},s)\overline{\sigma}_\mu^{\dot\alpha\beta}(p^\mu - 2smS^\mu) = 0\,, \tag{29.3.34}$$

$$y^\alpha(\vec{\boldsymbol{p}},s)\sigma^\mu_{\alpha\dot\beta}(p_\mu + 2smS_\mu) = 0\,, \qquad y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},s)\overline{\sigma}_\mu^{\dot\alpha\beta}(p^\mu + 2smS^\mu) = 0\,. \tag{29.3.35}$$

Eqs. (29.3.18)–(29.3.35) also apply to the helicity wave functions $x(\vec{\boldsymbol{p}},\lambda)$ and $y(\vec{\boldsymbol{p}},\lambda)$ simply by replacing s with λ and $S^\mu(\hat{\boldsymbol{s}})$ [eq. (29.3.15)] with $S^\mu(\hat{\boldsymbol{p}})$ [eq. (29.3.16)].

The above results are applicable only for massive fermions (where the spin four-vector S^μ exists). We may treat the case of massless fermions directly by employing helicity spinors in eqs. (29.3.18)–(29.3.21). Putting $E = |\vec{\boldsymbol{p}}|$ and $m = 0$, we easily obtain:

$$x_\alpha(\vec{\boldsymbol{p}},\lambda) = \sqrt{2E}\,\left(\tfrac{1}{2} - \lambda\right)\chi_\lambda\,, \qquad x^\alpha(\vec{\boldsymbol{p}},\lambda) = \sqrt{2E}\,\left(\tfrac{1}{2} - \lambda\right)\chi^\dagger_{-\lambda}\,, \tag{29.3.36}$$

$$y_\alpha(\vec{\boldsymbol{p}},\lambda) = \sqrt{2E}\,\left(\tfrac{1}{2} + \lambda\right)\chi_{-\lambda}\,, \qquad y^\alpha(\vec{\boldsymbol{p}},\lambda) = \sqrt{2E}\,\left(\tfrac{1}{2} + \lambda\right)\chi^\dagger_{\lambda}\,, \tag{29.3.37}$$

or equivalently,

$$x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},\lambda) = \sqrt{2E}\,\left(\tfrac{1}{2} - \lambda\right)\chi_{-\lambda}\,, \qquad x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda) = \sqrt{2E}\,\left(\tfrac{1}{2} - \lambda\right)\chi^\dagger_{\lambda}\,, \tag{29.3.38}$$

$$y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},\lambda) = \sqrt{2E}\,\left(\tfrac{1}{2} + \lambda\right)\chi_{\lambda}\,, \qquad y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda) = \sqrt{2E}\,\left(\tfrac{1}{2} + \lambda\right)\chi^\dagger_{-\lambda}\,. \tag{29.3.39}$$

It follows that:

$$\left(\tfrac{1}{2} + \lambda\right)x(\vec{\boldsymbol{p}},\lambda) = 0\,, \qquad \left(\tfrac{1}{2} + \lambda\right)x^\dagger(\vec{\boldsymbol{p}},\lambda) = 0\,, \tag{29.3.40}$$

$$\left(\tfrac{1}{2} - \lambda\right)y(\vec{\boldsymbol{p}},\lambda) = 0\,, \qquad \left(\tfrac{1}{2} - \lambda\right)y^\dagger(\vec{\boldsymbol{p}},\lambda) = 0\,. \tag{29.3.41}$$

The significance of eqs. (29.3.40) and (29.3.41) is clear; for massless fermions, only one helicity component of x and y is non-zero. Applying this result to neutrinos, we find that massless neutrinos are left-handed ($\lambda = -1/2$), while antineutrinos are right-handed ($\lambda = +1/2$).

Eqs. (29.3.40) and (29.3.41) can also be derived by carefully taking the $m \to 0$ limit of eqs. (29.3.32) and (29.3.33) applied to the helicity wave functions $x(\vec{\boldsymbol{p}},\lambda)$ and $y(\vec{\boldsymbol{p}},\lambda)$ [i.e., replacing s with λ]. We then replace mS^μ with p^μ, which is the leading term in the limit of $E \gg m$. Using the results of eqs. (29.3.9) and (29.3.10) and dividing out by an overall factor of m (before finally taking the $m \to 0$ limit) reproduces eqs. (29.3.40) and (29.3.41).

Having defined explicit forms for the two-component spinor wave functions, we can now write down the spin projection matrices. Noting that $\frac{1}{2}(1 + 2s\,\vec{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{s}})\chi_{s'} = \frac{1}{2}(1 + 4ss')\chi_{s'} = \delta_{ss'}\chi_{s'}$ (since s, $s' = \pm\frac{1}{2}$), one can write:

$$\chi_s\chi_s^\dagger = \tfrac{1}{2}\,(1 + 2s\,\vec{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{s}})\sum_{s'}\chi_{s'}\chi_{s'}^\dagger = \tfrac{1}{2}\,(1 + 2s\,\vec{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{s}})\,, \tag{29.3.42}$$

where at the second step, we have employed the completeness relation given in eq. (C.1.21). Making use of eq. (29.3.17) for $\vec{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{s}}$, it follows that

$$\chi_s\chi_s^\dagger = \tfrac{1}{2}\left(1 + \frac{2s}{m}\sqrt{p\cdot\sigma}\,S\cdot\overline{\sigma}\,\sqrt{p\cdot\sigma}\right)\,. \tag{29.3.43}$$

Hence, with both spinor indices in the lowered position,

$$\begin{aligned} x(\vec{\boldsymbol{p}},s)x^\dagger(\vec{\boldsymbol{p}},s) &= \sqrt{p\cdot\sigma}\,\chi_s\chi_s^\dagger\sqrt{p\cdot\sigma} \\ &= \tfrac{1}{2}\sqrt{p\cdot\sigma}\left[1+\frac{2s}{m}\sqrt{p\cdot\sigma}S\cdot\overline{\sigma}\sqrt{p\cdot\sigma}\right]\sqrt{p\cdot\sigma} \\ &= \tfrac{1}{2}\left[p\cdot\sigma+\frac{2s}{m}p\cdot\sigma S\cdot\overline{\sigma}p\cdot\sigma\right] \\ &= \tfrac{1}{2}\left[p\cdot\sigma-2smS\cdot\sigma\right]. \end{aligned} \tag{29.3.44}$$

In the final step above, we simplified the product of three dot products by noting that $p\cdot S=0$ implies that $S\cdot\overline{\sigma}\;p\cdot\sigma=-p\cdot\overline{\sigma}\;S\cdot\sigma$. The other spin projection formulae for massive fermions can be similarly derived. The complete set of such formulae is given below:[31]

$$x_\alpha(\vec{\boldsymbol{p}},s)x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}(p_\mu-2smS_\mu)\sigma^\mu_{\alpha\dot\beta}\,, \tag{29.3.45}$$

$$y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)y^\beta(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}(p^\mu+2smS^\mu)\overline{\sigma}^{\dot\alpha\beta}_\mu\,, \tag{29.3.46}$$

$$x_\alpha(\vec{\boldsymbol{p}},s)y^\beta(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}\left(m\delta_\alpha{}^\beta-2s[S\cdot\sigma\,p\cdot\overline{\sigma}]_\alpha{}^\beta\right), \tag{29.3.47}$$

$$y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}\left(m\delta^{\dot\alpha}{}_{\dot\beta}+2s[S\cdot\overline{\sigma}\,p\cdot\sigma]^{\dot\alpha}{}_{\dot\beta}\right). \tag{29.3.48}$$

By taking the hermitian conjugate of the above results, one obtains an equivalent set of formulae,

$$x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)x^\beta(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}(p^\mu-2smS^\mu)\overline{\sigma}^{\dot\alpha\beta}_\mu\,, \tag{29.3.49}$$

$$y_\alpha(\vec{\boldsymbol{p}},s)y^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}(p_\mu+2smS_\mu)\sigma^\mu_{\alpha\dot\beta}\,, \tag{29.3.50}$$

$$y_\alpha(\vec{\boldsymbol{p}},s)x^\beta(\vec{\boldsymbol{p}},s) = -\tfrac{1}{2}\left(m\delta_\alpha{}^\beta+2s[S\cdot\sigma\,p\cdot\overline{\sigma}]_\alpha{}^\beta\right), \tag{29.3.51}$$

$$x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)y^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s) = -\tfrac{1}{2}\left(m\delta^{\dot\alpha}{}_{\dot\beta}-2s[S\cdot\overline{\sigma}\,p\cdot\sigma]^{\dot\alpha}{}_{\dot\beta}\right). \tag{29.3.52}$$

For the case of massless spin-1/2 fermions, we must use helicity spinor wave functions. The corresponding massless projection operators can be obtained directly from the explicit forms for the two-component spinor wave functions given in eqs. (29.3.36)–(29.3.39):

$$x_\alpha(\vec{\boldsymbol{p}},\lambda)x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = (\tfrac{1}{2}-\lambda)p\cdot\sigma_{\alpha\dot\beta}\,, \qquad x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},\lambda)x^\beta(\vec{\boldsymbol{p}},\lambda) = (\tfrac{1}{2}-\lambda)p\cdot\overline{\sigma}^{\dot\alpha\beta}\,, \tag{29.3.53}$$

$$y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},\lambda)y^\beta(\vec{\boldsymbol{p}},\lambda) = (\tfrac{1}{2}+\lambda)p\cdot\overline{\sigma}^{\dot\alpha\beta}\,, \qquad y_\alpha(\vec{\boldsymbol{p}},\lambda)y^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = (\tfrac{1}{2}+\lambda)p\cdot\sigma_{\alpha\dot\beta}\,, \tag{29.3.54}$$

$$x_\alpha(\vec{\boldsymbol{p}},\lambda)y^\beta(\vec{\boldsymbol{p}},\lambda) = 0\,, \qquad y_\alpha(\vec{\boldsymbol{p}},\lambda)x^\beta(\vec{\boldsymbol{p}},\lambda) = 0\,, \tag{29.3.55}$$

$$y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},\lambda)x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = 0\,, \qquad x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},\lambda)y^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = 0\,. \tag{29.3.56}$$

As a check, one can verify that the above results follow from eqs. (29.3.45)–(29.3.52), by replacing s with λ, setting $mS^\mu=p^\mu$, and taking the $m\to 0$ limit at the end of the computation.

Having listed the projection operators for definite spin projection or helicity, we may now sum over spins to derive the spin sum identities. These arise when computing squared matrix elements for unpolarized scattering and decay. There are only four basic identities, but for convenience we list each of them with the two-index height permutations that can occur in squared amplitudes by following the rules given in this paper. The results can be derived by inspection of the spin projection operators, since summing over $s=\pm\frac{1}{2}$ simply removes all terms linear in the spin

[31] Similar formulae for the products of two-component spinor wave functions are given in ref. [105].

four-vector S^μ.

$$\sum_s x_\alpha(\vec{\boldsymbol{p}},s)x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s) = p\cdot\sigma_{\alpha\dot\beta}\,, \qquad \sum_s x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)x^\beta(\vec{\boldsymbol{p}},s) = p\cdot\overline{\sigma}^{\dot\alpha\beta}\,, \tag{29.3.57}$$

$$\sum_s y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)y^\beta(\vec{\boldsymbol{p}},s) = p\cdot\overline{\sigma}^{\dot\alpha\beta}\,, \qquad \sum_s y_\alpha(\vec{\boldsymbol{p}},s)y^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s) = p\cdot\sigma_{\alpha\dot\beta}\,, \tag{29.3.58}$$

$$\sum_s x_\alpha(\vec{\boldsymbol{p}},s)y^\beta(\vec{\boldsymbol{p}},s) = m\delta_\alpha{}^\beta\,, \qquad \sum_s y_\alpha(\vec{\boldsymbol{p}},s)x^\beta(\vec{\boldsymbol{p}},s) = -m\delta_\alpha{}^\beta\,, \tag{29.3.59}$$

$$\sum_s y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s) = m\delta^{\dot\alpha}{}_{\dot\beta}\,, \qquad \sum_s x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)y^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s) = -m\delta^{\dot\alpha}{}_{\dot\beta}\,. \tag{29.3.60}$$

These results are applicable both to spin sums and helicity sums, and hold for both massive and massless spin-1/2 fermions.

One can generalize the above massive and massless projection operators by considering products of two-component spinor wave functions, where the spin or helicity of each spinor can be different. These are the Bouchiat-Michel formulae [113], which are derived in Appendix H.3 .

29.3.2 Fermion mass diagonalization in a general theory

Consider a collection of free anticommuting two-component spin-1/2 fields, $\hat\xi_{\alpha i}(x)$, which transform as $(\frac12,0)$ fields under the Lorentz group. Here, α is the spinor index, and i labels the distinct fields of the collection. The free-field Lagrangian is given by (e.g., see ref. [5]):

$$\mathscr{L} = i\hat\xi^{\dagger i}\overline{\sigma}^\mu\partial_\mu\hat\xi_i - \tfrac12 M^{ij}\hat\xi_i\hat\xi_j - \tfrac12 M_{ij}\hat\xi^{\dagger i}\hat\xi^{\dagger j}\,, \tag{29.3.1}$$

where

$$M_{ij} \equiv (M^{ij})^*. \tag{29.3.2}$$

Note that M is a complex symmetric matrix, since the product of anticommuting two-component fields satisfies $\hat\xi_i\hat\xi_j = \hat\xi_j\hat\xi_i$ [with the spinor contraction rule according to eq. (29.37)].

In eq. (29.3.1), we have employed the U(N)-covariant tensor calculus [45, 147] for "flavor-tensors" labeled by the flavor indices i and j. Each left-handed $(\frac12,0)$ fermion always has an index with the opposite height of the corresponding right-handed $(0,\frac12)$ fermion. Raised indices can only be contracted with lowered indices and vice versa. Flipping the heights of all flavor indices of an object corresponds to complex conjugation, as in eq. (29.3.2). In particular, we generalize eq. (29.19) as follows:[32]

$$\psi^{\dagger\,i}_\alpha \equiv (\psi_{\alpha i})^\dagger\,. \tag{29.3.3}$$

If $M = 0$, then the free-field Lagrangian is invariant under a global U(N) symmetry. That is, for a unitary matrix U, with matrix elements $U_i{}^j$, and its hermitian conjugate defined by:

$$(U^\dagger)_i{}^j = (U_j{}^i)^* \equiv U^j{}_i\,, \tag{29.3.4}$$

with $U_i{}^k(U^\dagger)_k{}^j = \delta_i^j$, the massless free-field Lagrangian is invariant under the transformations:

$$\hat\xi_i \longrightarrow U_i{}^j\hat\xi_j\,, \qquad \hat\xi^{\dagger i} \longrightarrow U^i{}_j\hat\xi^{\dagger j}\,. \tag{29.3.5}$$

[32] In the case at hand, we have more specifically chosen all of the left-handed fermions to have lowered flavor indices, which implies that all of the right-handed fermions have raised flavor indices. However, in cases where a subset of left-handed fermions transform according to some representation R of a (global) symmetry and a different subset of left-handed fermions transform according to the conjugate representation R^*, it is often more convenient to employ a raised flavor index for the latter subset of left-handed fields.

For $M \neq 0$, eq. (29.3.1) remains formally invariant under the global U(N)-symmetry if M acts as a spurion field [148] with the appropriate tensorial transformation law, $M^{ij} \longrightarrow U^i{}_k U^j{}_\ell M^{k\ell}$.

Expressions consisting of flavor-vectors and second-rank flavor-tensors have natural interpretations as products of vectors and matrices. As a result, the flavor indices can be suppressed, and the resulting expressions can be written in an index-free matrix notation. To accomplish this, one must first assign a particular flavor index structure to the matrices that will appear in the index-free expression. For example, given the second-rank flavor-tensors introduced above, we define the matrix elements of M to be M^{ij} and the matrix elements of U to be $U_i{}^j$. Note that $(U^\dagger)_i{}^j$ has the same flavor-index structure as U.[33]

As a simple example, in an index-free notation eq. (29.3.5) reads: $\hat{\xi} \longrightarrow U\hat{\xi}$ and $\hat{\xi}^\dagger \longrightarrow \hat{\xi}^\dagger U^\dagger$. A slightly more complicated example is exhibited below:

$$U^i{}_k M^{k\ell} = (U^\dagger)_k{}^i M^{k\ell} = (U^* M)^{i\ell}\,, \tag{29.3.6}$$

where we have used $(U^\dagger)^{\mathsf{T}} = U^*$ in obtaining the final result. That is, in matrix notation with suppressed indices, $U^i{}_k M^{k\ell}$ corresponds to the matrix $U^* M$. Thus, in an index-free notation, the tensorial transformation law for the spurion field M is given by $M \longrightarrow U^* M U^\dagger$.

We can diagonalize the mass matrix M and rewrite the Lagrangian in terms of mass eigenstates $\xi_{\alpha i}$ and (real non-negative) masses m_i. To do this, we introduce a unitary matrix Ω,

$$\hat{\xi}_i = \Omega_i{}^k \xi_k\,, \tag{29.3.7}$$

and demand that $M^{ij}\Omega_i{}^k\Omega_j{}^\ell = m_k \delta^{k\ell}$ (no sum over k), where the m_k are real and non-negative. Equivalently, in matrix notation with suppressed indices, $\hat{\xi} = \Omega\xi$ and[34]

$$\Omega^{\mathsf{T}} M\, \Omega = \boldsymbol{m} = \mathrm{diag}(m_1, m_2, \ldots). \tag{29.3.8}$$

This is the Takagi diagonalization [112, 149] of an arbitrary complex symmetric matrix, which is discussed in more detail in Appendix D.2 . To compute the values of the diagonal elements of $\boldsymbol{m}$, note that

$$\Omega^\dagger M^\dagger M \Omega = \boldsymbol{m}^2\,. \tag{29.3.9}$$

Indeed $M^\dagger M$ is hermitian and thus it can be diagonalized by a unitary matrix. Hence, the elements of the diagonal matrix $\boldsymbol{m}$ are the non-negative square roots of the corresponding eigenvalues of $M^\dagger M$. However, in cases where $M^\dagger M$ has degenerate eigenvalues, eq. (29.3.9) *cannot* be employed to determine the unitary matrix Ω that satisfies eq. (29.3.8). A more general technique for determining Ω that works in all cases is given in Appendix D.2 .

In terms of the mass eigenstates,

$$\mathscr{L} = i\xi^{\dagger i}\overline{\sigma}^\mu \partial_\mu \xi_i - \tfrac{1}{2} m_i(\xi_i \xi_i + \xi^{\dagger i}\xi^{\dagger i})\,, \tag{29.3.10}$$

where the sum over i is implicit. If the $m_i \neq 0$ are non-degenerate, then the corresponding field ξ_i describes a neutral Majorana fermion consisting of two on-shell real degrees of freedom. The case of mass degeneracies will be treated explicitly below. If $m_i = 0$, then we shall denote the corresponding field ξ_i as a massless Weyl fermion [15].

Each $\xi_{\alpha i}$ can now be expanded in a Fourier series, exactly as in eq. (29.3.3):

$$\xi_{\alpha i}(x) = \sum_s \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}i})^{1/2}} \left[x_\alpha(\vec{\boldsymbol{p}}, s) a_i(\vec{\boldsymbol{p}}, s) e^{-ip\cdot x} + y_\alpha(\vec{\boldsymbol{p}}, s) a_i^\dagger(\vec{\boldsymbol{p}}, s) e^{ip\cdot x}\right]\,, \tag{29.3.11}$$

[33] The reader should not be tempted to substitute $U^\dagger$ for U in eq. (29.3.4), as the resulting flavor-index structure for U and $U^\dagger$ would then disagree with the original flavor-index assignments.

[34] In general, the m_i are *not* the eigenvalues of M. Rather, they are the *singular values* of the matrix M, which are defined to be the non-negative square roots of the eigenvalues of $M^\dagger M$. See Appendix D for further details.

where $E_{\boldsymbol{p}i} \equiv (|\vec{\boldsymbol{p}}|^2 + m_i^2)^{1/2}$, and the creation and annihilation operators, $a_i^\dagger$ and a_i satisfy anticommutation relations:

$$\{a_i(\vec{\boldsymbol{p}}, s), a_j^\dagger(\vec{\boldsymbol{p}}', s')\} = \delta^3(\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}')\delta_{ss'}\delta_{ij}\,. \tag{29.3.12}$$

We employ covariant normalization of the one-particle states, i.e., we act with one creation operator on the vacuum with the following convention

$$|\vec{\boldsymbol{p}}, i, s\rangle \equiv (2\pi)^{3/2}(2E_{\boldsymbol{p}i})^{1/2} a_i^\dagger(\vec{\boldsymbol{p}}, s)\,|0\rangle\ , \tag{29.3.13}$$

so that $\langle \vec{\boldsymbol{p}}, i, s\,|\,\vec{\boldsymbol{p}}', j, s'\rangle = (2\pi)^3(2E_{\boldsymbol{p}i})\delta^3(\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}')\delta_{ij}\delta_{ss'}$.

In the case of two mass-degenerate massive fermion fields, $m_1 = m_2 \neq 0$, eq. (29.3.10) possesses a global internal O(2) flavor symmetry, $\xi_i \to \mathcal{O}_i{}^j\xi_j$ $(i = 1, 2)$, where $\mathcal{O}^{\mathsf{T}}\mathcal{O} = \mathbb{1}_{2\times 2}$. Corresponding to this symmetry is a conserved hermitian Noether current:

$$J^\mu = i(\xi^{\dagger 1}\overline{\sigma}^\mu\xi_2 - \xi^{\dagger 2}\overline{\sigma}^\mu\xi_1)\,, \tag{29.3.14}$$

with a corresponding conserved charge, $Q = \int J^0 d^3x$. In the ξ_1–ξ_2 basis, the Noether current is off-diagonal. However, it is convenient to define a new basis of fields:

$$\chi \equiv \frac{1}{\sqrt{2}}(\xi_1 + i\xi_2)\,, \qquad \eta \equiv \frac{1}{\sqrt{2}}(\xi_1 - i\xi_2)\,. \tag{29.3.15}$$

With respect to the χ–η basis, the Noether current is diagonal:

$$J^\mu = \chi^\dagger\overline{\sigma}^\mu\chi - \eta^\dagger\overline{\sigma}^\mu\eta\,. \tag{29.3.16}$$

That is, the fermions χ and η are eigenstates of the charge operator Q with corresponding eigenvalues ± 1. In terms of the fermion fields of definite charge, the free-field fermion Lagrangian [eq. (29.3.10) with $i = 1, 2$ and $m_1 = m_2 \equiv m$] is given by [16]:[35]

$$\mathscr{L} = i\chi^\dagger\overline{\sigma}^\mu\partial_\mu\chi + i\eta^\dagger\overline{\sigma}^\mu\partial_\mu\eta - m(\chi\eta + \chi^\dagger\eta^\dagger)\,. \tag{29.3.17}$$

On-shell, χ and η satisfy the free-field Dirac equations:

$$i\,\overline{\sigma}^\mu\partial_\mu\chi - m\eta^\dagger = 0\,, \qquad i\,\overline{\sigma}^\mu\partial_\mu\eta - m\chi^\dagger = 0\,. \tag{29.3.18}$$

In the χ–η basis, the global internal SO(2) symmetry (which is continuously connected to the identity) is realized as the U(1) symmetry $\chi \to e^{i\theta}\chi$ and $\eta \to e^{-i\theta}\eta$, where θ is the rotation angle that defines the SO(2) rotation matrix.

Together, χ and $\eta^\dagger$ constitute a single Dirac fermion. We can then write:

$$\chi_\alpha(x) = \sum_s \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}}\left[x_\alpha(\vec{\boldsymbol{p}}, s)a(\vec{\boldsymbol{p}}, s)e^{-ip\cdot x} + y_\alpha(\vec{\boldsymbol{p}}, s)b^\dagger(\vec{\boldsymbol{p}}, s)e^{ip\cdot x}\right]\ , \tag{29.3.19}$$

$$\eta_\alpha(x) = \sum_s \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}}\left[x_\alpha(\vec{\boldsymbol{p}}, s)b(\vec{\boldsymbol{p}}, s)e^{-ip\cdot x} + y_\alpha(\vec{\boldsymbol{p}}, s)a^\dagger(\vec{\boldsymbol{p}}, s)e^{ip\cdot x}\right]\ , \tag{29.3.20}$$

where $E_{\boldsymbol{p}} \equiv (|\vec{\boldsymbol{p}}|^2 + m^2)^{1/2}$, the creation and annihilation operators, $a^\dagger$, $b^\dagger$, a and b satisfy anticommutation relations:

$$\{a(\vec{\boldsymbol{p}}, s), a^\dagger(\vec{\boldsymbol{p}}', s')\} = \{b(\vec{\boldsymbol{p}}, s), b^\dagger(\vec{\boldsymbol{p}}', s')\} = \delta^3(\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}')\delta_{s,s'}\,, \tag{29.3.21}$$

[35] Although the fermion mass matrix is not diagonal in the χ–η basis, this is not an obstacle to the subsequent analysis, as one only needs a diagonal *squared*-mass matrix, $M^\dagger M$, to ensure that the denominators of propagators are diagonal. Eq. (29.3.15) provides the explicit Takagi diagonalization of the Dirac fermion matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. See Appendix D.3 for the mathematical interpretation of this special case.

and all other anticommutators vanish. We now must distinguish between two types of one-particle states, which we can call fermion (F) and antifermion ($\overline{F}$):

$$|\vec{\boldsymbol{p}}, s; F\rangle \equiv (2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}a^\dagger(\vec{\boldsymbol{p}}, s)\,|0\rangle\ , \qquad \left|\vec{\boldsymbol{p}}, s; \overline{F}\right\rangle \equiv (2\pi)^{3/2}(2E_{\boldsymbol{p}})^{1/2}b^\dagger(\vec{\boldsymbol{p}}, s)\,|0\rangle\ . \tag{29.3.22}$$

Note that both $\eta(x)$ and $\chi^\dagger(x)$ can create $|\vec{\boldsymbol{p}}, s; F\rangle$ from the vacuum, while $\eta^\dagger(x)$ and $\chi(x)$ can create $\left|\vec{\boldsymbol{p}}, s; \overline{F}\right\rangle$. The one-particle wave functions are given by:

$$\langle 0|\,\chi_\alpha(x)\,|\vec{\boldsymbol{p}}, s; F\rangle = x_\alpha(\vec{\boldsymbol{p}}, s)e^{-ip\cdot x}\,, \qquad \langle 0|\,\eta^\dagger_{\dot\alpha}(x)\,|\vec{\boldsymbol{p}}, s; F\rangle = y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}}, s)e^{-ip\cdot x}\,, \tag{29.3.23}$$

$$\langle F; \vec{\boldsymbol{p}}, s|\,\eta_\alpha(x)\,|0\rangle = y_\alpha(\vec{\boldsymbol{p}}, s)e^{ip\cdot x}\,, \qquad \langle F; \vec{\boldsymbol{p}}, s|\,\chi^\dagger_{\dot\alpha}(x)\,|0\rangle = x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}}, s)e^{ip\cdot x}\,, \tag{29.3.24}$$

$$\langle 0|\,\eta_\alpha(x)\,\left|\vec{\boldsymbol{p}}, s; \overline{F}\right\rangle = x_\alpha(\vec{\boldsymbol{p}}, s)e^{-ip\cdot x}\,, \qquad \langle 0|\,\chi^\dagger_{\dot\alpha}(x)\,\left|\vec{\boldsymbol{p}}, s; \overline{F}\right\rangle = y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}}, s)e^{-ip\cdot x}\,, \tag{29.3.25}$$

$$\left\langle \overline{F}; \vec{\boldsymbol{p}}, s\right|\,\chi_\alpha(x)\,|0\rangle = y_\alpha(\vec{\boldsymbol{p}}, s)e^{ip\cdot x}\,, \qquad \left\langle \overline{F}; \vec{\boldsymbol{p}}, s\right|\,\eta^\dagger_{\dot\alpha}(x)\,|0\rangle = x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}}, s)e^{ip\cdot x}\,, \tag{29.3.26}$$

and the eight other single-particle matrix elements vanish.

More generally, consider a collection of free anticommuting charged Dirac fermions, which can be represented by pairs of two-component fields $\hat\chi_{\alpha i}(x)$, $\hat\eta^i_{\dot\alpha}(x)$. These fields transform in (possibly reducible) representations of the unbroken symmetry group that are conjugates of each other. This accounts for the opposite flavor index heights of $\hat\chi_i$ and $\hat\eta^i$ [cf. footnote 32]. The free-field Lagrangian is given by

$$\mathscr{L} = i\hat\chi^{\dagger i}\overline\sigma^\mu\partial_\mu\hat\chi_i + i\hat\eta^\dagger_i\overline\sigma^\mu\partial_\mu\hat\eta^i - M^i{}_j\hat\chi_i\hat\eta^j - M_i{}^j\hat\chi^{\dagger i}\hat\eta^\dagger_j\,, \tag{29.3.27}$$

where M is an arbitrary complex matrix with matrix elements $M^i{}_j$, and

$$M_i{}^j \equiv (M^i{}_j)^*\,. \tag{29.3.28}$$

If $M = 0$, then the free-field Lagrangian is invariant under a global U(N)×U(N) symmetry. That is, for a pair of unitary matrices U_L and U_R, with matrix elements given respectively by $(U_L)_i{}^j$ and $(U_R)^i{}_j$, and the corresponding hermitian conjugates defined by:

$$(U_L^\dagger)_j{}^i = [(U_L)_i{}^j]^* \equiv (U_L)^i{}_j\,, \qquad (U_R^\dagger)^j{}_i = [(U_R)^i{}_j]^* \equiv (U_R)_i{}^j\,, \tag{29.3.29}$$

the massless free-field Lagrangian is invariant under the transformations:

$$\hat\chi_i \longrightarrow (U_L)_i{}^j\hat\chi_j\,, \qquad \hat\chi^{\dagger i} \longrightarrow (U_L)^i{}_j\hat\chi^{\dagger j}\,, \qquad \hat\eta^i \longrightarrow (U_R)^i{}_j\hat\eta^j\,, \qquad \hat\eta^\dagger_i \longrightarrow (U_R)_i{}^j\hat\eta^\dagger_j\,. \tag{29.3.30}$$

For $M \neq 0$, eq. (29.3.27) remains formally invariant under the U(N)×U(N) symmetry if M acts as a spurion field [148] with the appropriate tensorial transformation law, $M^i{}_j \to (U_L)^i{}_k(U_R)_j{}^\ell M^k{}_\ell$ (or equivalently, in an index-free matrix notation with suppressed flavor indices, $M \longrightarrow U_L^* M U_R^\dagger$).

In order to diagonalize the mass matrix, we introduce the mass eigenstates χ_i and η^i and unitary matrices L and R, with matrix elements given respectively by $L_i{}^k$ and $R^i{}_k$, such that

$$\hat\chi_i = L_i{}^k\chi_k\,, \qquad \hat\eta^i = R^i{}_k\eta^k\,, \tag{29.3.31}$$

and demand that $M^i{}_jL_i{}^kR^j{}_\ell = m_k\delta^k_\ell$ (no sum over k), where the m_k are real and non-negative. Equivalently, in matrix notation with suppressed indices, $\hat\chi = L\chi$, $\hat\eta = R\eta$ and

$$L^{\mathsf T}MR = \boldsymbol{m} = \mathrm{diag}(m_1, m_2, \ldots), \tag{29.3.32}$$

with the m_i real and non-negative (cf. footnote 34). The singular value decomposition of linear algebra, discussed more fully in Appendix D.1 , states that for any complex matrix M, unitary matrices L and R exist such that eq. (29.3.32) is satisfied. It then follows that:

$$L^{\mathsf T}(MM^\dagger)L^* \;=\; R^\dagger(M^\dagger M)R \;=\; \boldsymbol{m}^2. \tag{29.3.33}$$

That is, since $MM^\dagger$ and $M^\dagger M$ are both hermitian, they can be diagonalized by unitary matrices. The diagonal elements of $\boldsymbol{m}$ are therefore the non-negative square roots of the corresponding eigenvalues of $MM^\dagger$ (or equivalently, $M^\dagger M$). In terms of the mass eigenstates,

$$\mathscr{L} = i\chi^{\dagger i}\overline{\sigma}^\mu\partial_\mu\chi_i + i\eta_i^\dagger\overline{\sigma}^\mu\partial_\mu\eta^i - m_i(\chi_i\eta^i + \chi^{\dagger i}\eta_i^\dagger)\,. \tag{29.3.34}$$

The mass matrix now consists of 2×2 blocks $\left(\begin{smallmatrix} 0 & m_i \\ m_i & 0\end{smallmatrix}\right)$ along the diagonal. More importantly, the squared-mass matrix is diagonal with doubly degenerate entries m_i^2 that will appear in the denominators of the propagators of the theory. For $m_i \neq 0$, each χ_i–η^i pair describes a charged Dirac fermion consisting of four on-shell real degrees of freedom.[36] In addition, eq. (29.3.34) yields an even number of massless Weyl fermions.

Given an arbitrary collection of two-component left-handed $(\frac{1}{2},0)$ fermions, the distinction between Majorana and Dirac fermions depends on whether the Lagrangian is invariant under a global (or local) continuous symmetry group G, and the corresponding multiplet structure of the fermion fields [150]. If no such continuous symmetry exist, then the fermion mass eigenstates will consist of Majorana fermions. If the Lagrangian is invariant under a symmetry group G, then the collection of two-component fermions will break up into a sum of multiplets that transform irreducibly under G. As described in Appendix E , a representation R can be either a real, pseudo-real, or complex representation of G. If a multiplet transforms under a real representation of G, then the corresponding fermion mass eigenstates are Majorana fermions.[37] If a multiplet transforms under a complex representation of G, then the corresponding fermion mass eigenstates are Dirac fermions. In particular [as noted above eq. (29.3.27)], if the χ_i transform under the representation R, then the η^i transform under the conjugate representation R^*.

The case where a multiplet of two-component left-handed fermions transform under a pseudo-real representation of G has not been explicitly treated above. The simplest example of this kind is a model of $2n$ multiplets (or "flavors") of two-component SU(2)-doublet[38] fermions, $\hat{\psi}_{ia}$ (where $i = 1, 2, ..., 2n$ labels the flavor index and a labels the SU(2) doublet index). The free-field Lagrangian is given by:

$$\mathscr{L} = i\hat{\psi}^{\dagger ia}\overline{\sigma}^\mu\partial_\mu\hat{\psi}_{ia} - \tfrac{1}{2}\left(M^{ij}\epsilon^{ab}\hat{\psi}_{ia}\hat{\psi}_{jb} + \text{h.c.}\right)\,, \tag{29.3.35}$$

where ϵ^{ab} is the antisymmetric SU(2)-invariant tensor, defined such that $\epsilon^{12} = -\epsilon^{21} = +1$. As $\epsilon^{ab}\hat{\psi}_{ia}\hat{\psi}_{jb}$ is antisymmetric under the interchange of flavor indices i and j, it follows that M is a complex antisymmetric matrix. To identify the fermion mass eigenstates ψ_{ja}, we introduce a unitary matrix U (with matrix elements $U_i{}^j$) such that $\hat{\psi}_{ia} = U_i{}^j\psi_{ja}$ and demand that:

$$U^{\mathsf{T}}MU = \boldsymbol{N} \equiv \text{diag}\left\{\begin{pmatrix} 0 & m_1 \\ -m_1 & 0\end{pmatrix}, \begin{pmatrix} 0 & m_2 \\ -m_2 & 0\end{pmatrix}, \cdots, \begin{pmatrix} 0 & m_n \\ -m_n & 0\end{pmatrix}\right\}, \tag{29.3.36}$$

where $\boldsymbol{N}$ is written in block-diagonal form consisting of 2×2 matrix blocks appearing along the diagonal, and the m_j are real and non-negative. Eq. (29.3.36) corresponds to the reduction of a complex antisymmetric matrix to its real normal form [151], which is discussed in more detail in Appendix D.4 . In order to compute the m_k, we first note that

$$U^\dagger M^\dagger MU = \text{diag}(m_1^2\,,\, m_1^2\,,\, m_2^2\,,\, m_2^2\,,\, \ldots\,,\, m_n^2\,,\, m_n^2)\,. \tag{29.3.37}$$

[36] Of course, one could always choose instead to treat the Dirac fermions in a non-charge-eigenstate basis with a fully diagonalized mass matrix, as in eq. (29.3.10). Inverting eq. (29.3.15) for each Dirac fermion yields $\xi_{2i-1} = (\chi_i + \eta^i)/\sqrt{2}$ and $\xi_{2i} = i(\eta_i - \chi^i)/\sqrt{2}$. However, it is rarely, if ever, convenient to do so; practical calculations only require that the squared-mass matrix $M^\dagger M$ is diagonal, and it is of course more convenient to employ fields that carry well-defined charges.

[37] This is a slight generalization of the more restrictive definition that requires Majorana fermions to transform trivially under the group G. Gluinos, which transform under the (real) adjoint representation of the color SU(3) group, are Majorana fermions according to our more general definition.

[38] The doublet representation of SU(2) is pseudo-real.

Hence, the m_j are the non-negative square roots of the corresponding eigenvalues of $M^\dagger M$. Since the dimension of the doublet representation of SU(2) provides an additional degeneracy factor of 2, eq. (29.3.37) implies that the mass spectrum consists of $2n$ pairs of mass-degenerate two-component fermions, which are equivalent to $2n$ Dirac fermions. In particular,

$$\mathscr{L} = \sum_{i=1}^{2n} i\psi^{\dagger ia}\overline{\sigma}^\mu\partial_\mu\psi_{ia} - \sum_{i=1}^{n}\left(m_i\epsilon^{ab}\psi_{2i-1,\,a}\psi_{2i,\,b} + \text{h.c.}\right) . \tag{29.3.38}$$

In the general case of a pseudo-real representation R (of dimension d_R), the SU(2)-invariant ϵ-tensor is replaced by a more general $d_R \times d_R$ unitary antisymmetric matrix, C [defined in eq. (E.1.9)]. Thus, the analysis above can be repeated virtually unchanged. By defining

$$\chi_{ia} \equiv \psi_{2i-1,\,a}\,, \qquad \eta^{ia} \equiv C^{ab}\psi_{2i,\,b}\,, \qquad i = 1, 2, ..., n\,; \quad a = 1, 2, ..., d_R\,, \tag{29.3.39}$$

with an implicit sum over the repeated index b, the resulting Lagrangian given by

$$\mathscr{L} = \sum_{i=1}^{n} i\chi^{\dagger ia}\overline{\sigma}^\mu\partial_\mu\chi_{ia} + i\eta^\dagger_{ia}\overline{\sigma}^\mu\partial_\mu\eta^{ia} - m_i\left(\chi_{ia}\eta^{ia} + \chi^{\dagger ia}\eta^\dagger_{ia}\right) , \tag{29.3.40}$$

describes a free field theory of nd_R Dirac fermions [cf. eq. (29.3.34)]. Therefore, if a multiplet of two-component left-handed fermions transforms under a pseudo-real representation of G, then the corresponding fermion mass eigenstates are Dirac fermions [150]. If eq. (29.3.35) contains an odd number of pseudo-real fermion multiplets, then the (antisymmetric) mass matrix M is odd-dimensional and thus has an odd number of zero eigenvalues [according to eq. (D.4.1)]. But as d_R must be even, it follows that the pseudo-real fermion multiplet contains an even number of massless Weyl fermions.

In conclusion, the mass diagonalization procedure of an arbitrary field theory of fermions yields (in general) a set of massless Weyl fermions, a set of massive neutral Majorana fermions [as in eq. (29.3.10)], and a set of massive charged Dirac fermions [as in eq. (29.3.34)]. The Feynman rules for these mass eigenstate two-component fermion fields are given in Section 29.4.

For completeness, we review the squared-mass matrix diagonalization procedure for scalar fields. First, consider a collection of free commuting real spin-0 fields, $\hat{\varphi}_i(x)$, where the flavor index i labels the distinct scalar fields of the collection. The free-field Lagrangian is given by[39]

$$\mathscr{L} = \tfrac{1}{2}\partial_\mu\hat{\varphi}_i\partial^\mu\hat{\varphi}_i - \tfrac{1}{2}M^2_{ij}\hat{\varphi}_i\hat{\varphi}_j\,, \tag{29.3.41}$$

where M^2 is a real symmetric matrix. We diagonalize the scalar squared-mass matrix by introducing mass eigenstates φ_i and the orthogonal matrix Q such that $\hat{\varphi}_i = Q_{ij}\varphi_j$, with $M^2_{ij}Q_{ik}Q_{j\ell} = m^2_k\delta_{k\ell}$ (no sum over k). In matrix form,

$$Q^{\mathsf{T}}M^2Q = \boldsymbol{m}^2 = \text{diag}(m_1^2, m_2^2, ...)\,, \tag{29.3.42}$$

where the squared-mass eigenvalues m_k^2 are real.[40] This is the standard diagonalization problem for a real symmetric matrix.

Next, consider a collection of free commuting complex spin-0 fields, $\hat{\Phi}_i(x)$. For complex fields, we follow the conventions for flavor indices enunciated below eq. (29.3.2) [e.g. $\hat{\Phi}^i = (\hat{\Phi}_i)^\dagger$]. The free-field Lagrangian is given by

$$\mathscr{L} = \partial_\mu\hat{\Phi}^i\partial^\mu\hat{\Phi}_i - (M^2)^i{}_j\hat{\Phi}_i\hat{\Phi}^j\,, \tag{29.3.43}$$

[39] Since the scalar fields are real, there is no need to distinguish between raised and lowered flavor indices.

[40] If the vacuum corresponds to a local minimum (or flat direction) of the scalar potential, then the squared-mass eigenvalues of M^2 are real *and* non-negative.

where M^2 is an hermitian matrix [i.e., $(M^2)^i{}_j = (M^2)_j{}^i$ in the notation of eq. (29.3.29)].

We diagonalize the scalar squared-mass matrix by introducing mass eigenstates Φ_i and the unitary matrix W such that $\hat{\Phi}_i = W_i{}^k \Phi_k$ (and $\hat{\Phi}^i = W^i{}_k \Phi^k$), with $(M^2)^i{}_j W_i{}^k W^j{}_\ell = m_k^2 \delta_\ell^k$ (no sum over k). In matrix form,

$$W^\dagger M^2 W = \boldsymbol{m}^2 = \text{diag}(m_1^2, m_2^2, \ldots)\,. \tag{29.3.44}$$

where the squared-mass eigenvalues m_k^2 are real (cf. footnote 40). This is the standard diagonalization problem for an hermitian matrix.

29.4 Feynman rules with two-component spinors

In order to systematically perform perturbative calculations using two-component spinors, we present the basic Feynman rules. The Feynman rules for the Standard Model (and its seesaw extension) and the MSSM (including possible R-parity-violating interactions) are given in Appendices J, K and L. Feynman rules for two-component spinors have also been treated in refs. [50, 107, 110].

29.4.1 External fermion and boson rules

Consider a general theory, for which we may assume that the mass matrix for fermions has been diagonalized as discussed in Section 29.3.2. The rules for assigning two-component external state spinors are then as follows:

- For an initial state (incoming) left-handed $(\frac{1}{2}, 0)$ fermion: x
- For an initial state (incoming) right-handed $(0, \frac{1}{2})$ fermion: $y^\dagger$
- For a final state (outgoing) left-handed $(\frac{1}{2}, 0)$ fermion: $x^\dagger$
- For a final state (outgoing) right-handed $(0, \frac{1}{2})$ fermion: y

where we have suppressed the momentum and spin arguments of the spinor wave functions. These rules are summarized in the mnemonic diagram of Fig. 29.4.1.

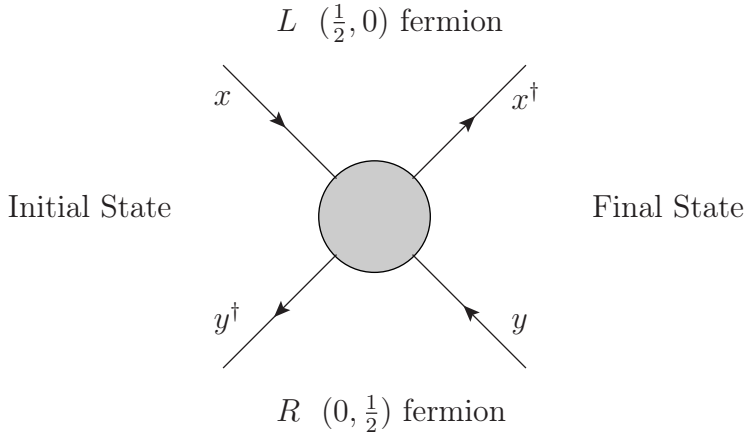


Figure 29.4.1: The external wave function spinors should be assigned as indicated here, for initial state and final state left-handed $(\frac{1}{2}, 0)$ and right-handed $(0, \frac{1}{2})$ fermions.

In general, the two-component external state fermion wave functions are distinguished by their Lorentz group transformation properties, rather than by their particle or antiparticle status as in four-component Feynman rules. This helps to explain why two-component notation is especially convenient for (i) theories with Majorana particles, in which there is no fundamental distinction

between particles and antiparticles, and (ii) theories like the Standard Model and MSSM in which the left and right-handed fermions transform under different representations of the gauge group and (iii) problems with polarized particle beams.

In contrast to four-component Feynman rules (given in Appendix G.5), the direction of the arrow does *not* correspond to the flow of charge or fermion number. The two-component Feynman rules for external fermion lines simply correspond to the formulae for the one-particle wave functions exhibited in eqs. (29.3.7) and (29.3.8) [with the convention that $|\vec{\boldsymbol{p}}, s\rangle$ is an initial state fermion and $\langle\vec{\boldsymbol{p}}, s|$ is a final state fermion]. In particular, the arrows indicate the spinor index structure, with fields of undotted indices flowing *into* any vertex and fields of dotted indices flowing *out* of any vertex.

The rules above apply to any mass eigenstate two-component fermion external wave functions. It is noteworthy that the same rules apply for the two-component fermions governed by the Lagrangians of eq. (29.3.10) [Majorana] and eqs. (29.3.34) or (29.3.40) [Dirac].

The corresponding rules for external boson lines are well-known (see, e.g [115]).

- For an initial state (incoming) or final state (outgoing) spin-0 boson : 1
- For an initial state (incoming) spin-1 boson of momentum $\vec{\boldsymbol{k}}$ and helicity λ : $\varepsilon^\mu(\vec{\boldsymbol{k}}\,,\,\lambda)$
- For a final state (outgoing) spin-1 boson of momentum $\vec{\boldsymbol{k}}$ and helicity λ : $\varepsilon^\mu(\vec{\boldsymbol{k}}\,,\,\lambda)^*$

The explicit form of the helicity ±1 (massless or massive) spin-1 polarization vector ε^μ is given in eq. (I.2.41). The helicity zero massive spin-1 polarization vector is given in eq. (I.2.43).

29.4.2 Propagators

Next we turn to the subject of fermion propagators for two-component fermions. A derivation of the two-component fermion propagators using path integral techniques is given in Appendix F . Here, we will follow the more elementary approach typically given in an initial textbook treatment of quantum field theory.

Fermion propagators are the Fourier transforms of the free-field vacuum expectation values of time-ordered products of two fermion fields. They are obtained by inserting the free-field expansion of the two-component fermion field and evaluating the spin sums using the formulae given in eqs. (29.3.57) and (29.3.60). For the case of a single neutral two-component fermion field $\xi(x)$ of mass m, eqs. (29.3.11) and (29.3.12) yield [145, 50, 152, 107, 109, 110]:

$$\langle 0|\,T\xi_\alpha(x)\xi^\dagger_{\dot\beta}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2+i\epsilon}\sum_s x_\alpha(\vec{\boldsymbol{p}},s)x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s) = \frac{i}{p^2-m^2+i\epsilon}\,p{\cdot}\sigma_{\alpha\dot\beta}\,, \quad (29.4.1)$$

$$\langle 0|\,T\xi^{\dagger\dot\alpha}(x)\xi^\beta(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2+i\epsilon}\sum_s y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)y^\beta(\vec{\boldsymbol{p}},s) = \frac{i}{p^2-m^2+i\epsilon}\,p{\cdot}\overline{\sigma}^{\dot\alpha\beta}\,, \quad (29.4.2)$$

$$\langle 0|\,T\xi^{\dagger\dot\alpha}(x)\xi^\dagger_{\dot\beta}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2+i\epsilon}\sum_s y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s) = \frac{i}{p^2-m^2+i\epsilon}\,m\delta^{\dot\alpha}{}_{\dot\beta}\,, \quad (29.4.3)$$

$$\langle 0|\,T\xi_\alpha(x)\xi^\beta(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2+i\epsilon}\sum_s x_\alpha(\vec{\boldsymbol{p}},s)y^\beta(\vec{\boldsymbol{p}},s) = \frac{i}{p^2-m^2+i\epsilon}\,m\delta_\alpha{}^\beta\,, \quad (29.4.4)$$

where FT indicates the Fourier transform from position to momentum space.[41] These results have a clear diagrammatic representation, as shown in Fig. 29.4.1. Note that the direction of the

[41] The Fourier transform of a translationally invariant function $f(x,y)\equiv f(x-y)$ is given by

$$f(x,y) = \int \frac{d^4p}{(2\pi)^4}\,\widehat{f}(p)\,e^{-ip\cdot(x-y)}\,, \qquad \text{where} \qquad \widehat{f}(p) = \int d^4x\,f(x)e^{ip\cdot x}\,.$$

In the notation of the text above, $f(x,y)_{\rm FT}\equiv\widehat{f}(p)$.

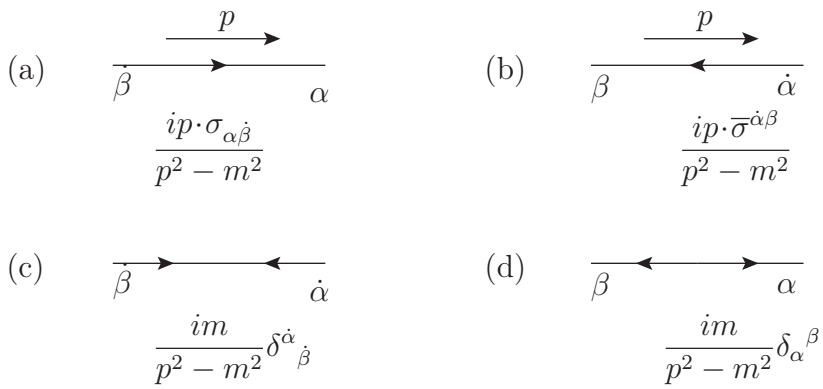


Figure 29.4.1: Feynman rules for propagator lines of a neutral two-component fermion with mass m. (For simplicity, the $+i\epsilon$ terms in the denominators are omitted in all propagator rules.)

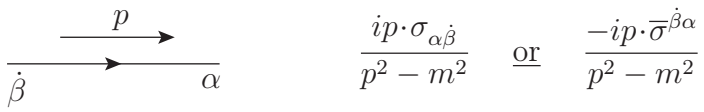


Figure 29.4.2: This rule summarizes the results of both Figs. 29.4.1(a) and (b) for a neutral two-component fermion with mass m.

momentum flow p^μ here is determined by the creation operator that appears in the evaluation of the free-field propagator. Arrows on fermion lines always run away from dotted indices at a vertex and toward undotted indices at a vertex.

There are clearly two types of fermion propagators. The first type preserves the direction of arrows, so it has one dotted and one undotted index. For this type of propagator, it is convenient to establish a convention where p^μ in the diagram is defined to be the momentum flowing in the direction of the arrow on the fermion propagator. With this convention, the two rules above for propagators of the first type can be summarized by one rule, as shown in Fig. 29.4.2. Here the choice of the σ or the $\overline{\sigma}$ version of the rule is uniquely determined by the height of the indices on the vertex to which the propagator is connected.[42] These heights should always be chosen so that they are contracted as in eq. (29.37). It should be noted that in diagrams (a) and (b) of Fig. 29.4.1 as drawn, the indices on the σ and $\overline{\sigma}$ read from right to left. In particular, the Feynman rules for the propagator can be employed with the spinor indices suppressed provided that the arrow-preserving propagator lines are traversed in the direction parallel [antiparallel] to the arrowed line segment for the $\overline{\sigma}$ [σ] version of the rule, respectively.

The second type of propagator shown in diagrams (c) and (d) of Fig. 29.4.1 does not preserve the direction of arrows, and corresponds to an odd number of mass insertions. The indices on $\delta^{\dot\alpha}{}_{\dot\beta}$ and $\delta_\alpha{}^\beta$ are staggered as shown to indicate that $\dot\alpha$ and α are to be contracted with expressions to the left, while $\dot\beta$ and β are to be contracted with expressions to the right, in accord with eq. (29.37).[43]

Starting with massless fermion propagators, one can also derive the massive fermion propagators by employing mass insertions as interaction vertices, as shown in Fig. 29.4.3. By summing up an infinite chain of such mass insertions between massless fermion propagators, one can reproduce the massive fermion propagators of both types.

The above results for the propagator of a Majorana fermion can be generalized to a multiplet of mass eigenstate Majorana fermions, $\xi_{\alpha a}(x)$ [such as a color octet of gluinos], which transforms as

[42] The second form of the rule in Fig. 29.4.2 arises when one flips diagram (b) of Fig. 29.4.1 around by a 180° rotation (about an axis perpendicular to the plane of the diagram), and then relabels $p \to -p$, $\dot\alpha \to \dot\beta$ and $\beta \to \alpha$.

[43] As in Fig. 29.4.2, alternative and equivalent versions of the rules corresponding to diagrams (c) and (d) of Fig. 29.4.1 can be given for which the indices on the Kronecker deltas are staggered as $\delta^{\dot\beta}{}_{\dot\alpha}$ and $\delta_\beta{}^\alpha$. These versions correspond to flipping the two respective diagrams by 180° and relabeling the indices $\dot\alpha \to \dot\beta$ and $\beta \to \alpha$.

Figure 29.4.3: Fermion mass insertions (indicated by the crosses) can be treated as a type of interaction vertex, using the Feynman rules shown here.

a real representation R of a (gauge or flavor) group G (where $a = 1, 2, ..., d_R$ for a representation of dimension d_R). In this case, the Feynman graphs given in Figs. 29.4.1–29.4.3 are modified simply by specifying a group index a and b at either end of the propagator line. The corresponding Feynman rules then include an additional Kronecker delta factor in the group indices. In particular, if we associate the index a with the spinor indices α, $\dot\alpha$ and the index b with the spinor indices β, $\dot\beta$, then the rules exhibited in Fig. 29.4.1(a) and (b) would include the following Kronecker delta factors:

$$(a)\quad \delta^b_a\,, \qquad\qquad (b)\quad \delta^a_b\,, \tag{29.4.5}$$

and the factors of m in the rules exhibited in Fig. 29.4.1(c) and (d) would be replaced by

$$(c)\quad \delta^a_c m^{cd}\delta^b_d = m_a\delta^{ab}\,, \qquad\qquad (d)\quad \delta^c_a m_{cd}\delta^d_b = m_a\delta_{ab}\,, \tag{29.4.6}$$

(with no sum over the repeated index a), where m^{cd} and $m_{cd} \equiv m^{cd}$ are diagonal matrices with real non-negative diagonal elements m_c. Here, we have introduced the separate symbol m_{cd} in order to maintain the convention that two repeated group indices are summed when one index is raised and one index is lowered. Of course, if the Lagrangian is invariant under the symmetry group G, then a multiplet of Majorana fermions corresponding to an irreducible representation R has a common mass $m = m_a$.

It is convenient to treat separately the case of charged massive fermions. Consider a charged Dirac fermion of mass m, which is described by a pair of two-component fields $\chi(x)$ and $\eta(x)$ [cf. eq. (29.3.17)]. Using the free-field expansions [eqs. (29.3.19) and (29.3.20)] and the spin sums [eqs. (29.3.57)–(29.3.60)], the two-component free-field propagators are obtained:

$$\langle 0|\,T\chi_\alpha(x)\chi^\dagger_{\dot\beta}(y)\,|0\rangle_{\rm FT} = \langle 0|\,T\eta_\alpha(x)\eta^\dagger_{\dot\beta}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2}\,p{\cdot}\sigma_{\alpha\dot\beta}\,, \tag{29.4.7}$$

$$\langle 0|\,T\chi^{\dagger\dot\alpha}(x)\chi^\beta(y)\,|0\rangle_{\rm FT} = \langle 0|\,T\eta^{\dagger\dot\alpha}(x)\eta^\beta(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2}\,p{\cdot}\overline\sigma^{\dot\alpha\beta}\,, \tag{29.4.8}$$

$$\langle 0|\,T\chi_\alpha(x)\eta^\beta(y)\,|0\rangle_{\rm FT} = \langle 0|\,T\eta_\alpha(x)\chi^\beta(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2}\,m\,\delta_\alpha{}^\beta\,, \tag{29.4.9}$$

$$\langle 0|\,T\chi^{\dagger\dot\alpha}(x)\eta^\dagger_{\dot\beta}(y)\,|0\rangle_{\rm FT} = \langle 0|\,T\eta^{\dagger\dot\alpha}(x)\chi^\dagger_{\dot\beta}(y)\,|0\rangle_{\rm FT} = \frac{i}{p^2-m^2}\,m\,\delta^{\dot\alpha}{}_{\dot\beta}\,. \tag{29.4.10}$$

For all other combinations of fermion bilinears, the corresponding two-point functions vanish. These results again have a simple diagrammatic representation, as shown in Fig. 29.4.4. Note that for Dirac fermions, the propagators with opposing arrows (proportional to a mass) necessarily change the identity (χ or η) of the two-component fermion, while the single-arrow propagators are diagonal in the fields. In processes involving such a charged fermion, one must of course distinguish between the χ and η fields.

The above results for the propagator of a Dirac fermion can be generalized to a multiplet of mass eigenstate Dirac fermions, $\chi_{\alpha i}$, η^i_α, which transform under a (gauge or flavor) group G. In this case, the Feynman graphs given in Fig. 29.4.4 are modified simply by specifying a group index i and j at either end of the propagator line. The corresponding Feynman rules then include an

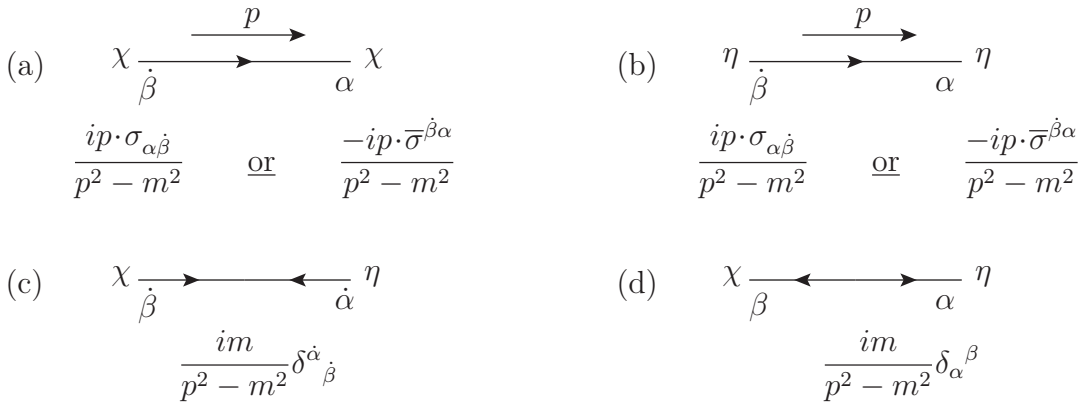


Figure 29.4.4: Feynman rules for propagator lines of a pair of charged two-component fermions with a Dirac mass m. As in Fig. 29.4.2, the direction of the momentum is taken to flow from the dotted to the undotted index in diagrams (a) and (b).

additional Kronecker delta factor in the group indices. In particular, if we associate the group index i with the spinor indices α, $\dot\alpha$ and the index j with the spinor indices β, $\dot\beta$, then the rules exhibited in Fig. 29.4.4(a) and (b) would include the following Kronecker delta factors:

$$(a)\ \ \delta_i^j \qquad\qquad (b)\ \ \delta_j^i\,, \tag{29.4.11}$$

and the factors of m in the rules exhibited in Fig. 29.4.4(c) and (d) would be replaced by

$$(c)\ \ \delta_i^\ell m_\ell{}^n \delta_n^j = m_i \delta_i^j\,, \qquad\qquad (d)\ \ \delta_\ell^i m^\ell{}_n \delta_j^n = m_i \delta_j^i\,, \tag{29.4.12}$$

where $m^\ell{}_n$ and $m_\ell{}^n \equiv m^\ell{}_n$ are diagonal matrices with real non-negative diagonal elements m_ℓ, and there is no sum over the repeated index i. (Here, we have introduced the separate symbol $m_\ell{}^n$ in order to maintain the convention that two repeated group indices are summed when one index is raised and one index is lowered.) As before, if the Lagrangian is invariant under the symmetry group G, then an irreducible multiplet of Dirac fermions has a common mass $m = m_i$.

For completeness, we exhibit in Fig. 29.4.5 the Feynman rules for the propagators of the (neutral or charged) scalar boson and gauge boson in the R_ξ gauge, with gauge parameter ξ [153].

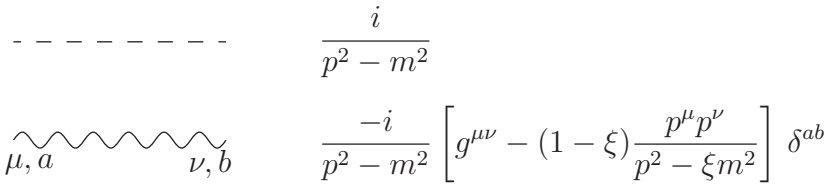


Figure 29.4.5: Feynman rules for the (neutral or charged) scalar and gauge boson propagators, in the R_ξ gauge, where p^μ is the propagating four-momentum. In the gauge boson propagator, $\xi = 1$ defines the 't Hooft-Feynman gauge, $\xi = 0$ defines the Landau gauge, and $\xi \to \infty$ defines the unitary gauge. For the propagation of a non-abelian gauge boson, one must also specify the adjoint representation indices a, b.

29.4.3 Fermion interactions with bosons

We next discuss the interaction vertices for fermions with bosons. Renormalizable Lorentz-invariant interactions involving fermions must consist of bilinears in the fermion fields, which

transform as a Lorentz scalar or vector, coupled to the appropriate bosonic scalar or vector field to make an overall Lorentz scalar quantity.

Let us write all of the two-component left-handed $(\frac{1}{2},0)$ fermions of the theory as $\hat{\psi}_i$, where i runs over all of the gauge group representation and flavor degrees of freedom. In general, the $(\frac{1}{2},0)$-fermion fields $\hat{\psi}_i$ consist of Majorana fermions $\hat{\xi}_i$, and Dirac fermion pairs $\hat{\chi}_i$ and $\hat{\eta}^i$ after mass terms (both explicit and coming from spontaneous symmetry breaking) are taken into account. Likewise, consider a multiplet of scalar fields $\hat{\phi}_I$, where I runs over all of the gauge group representation and flavor degrees of freedom. In general, the scalar fields $\hat{\phi}_I$ consist of real scalar fields $\hat{\varphi}_I$ and pairs of complex scalar fields $\hat{\Phi}_I$ and $\hat{\Phi}^I \equiv (\hat{\Phi}_I)^\dagger$. In matrix form,

$$\hat{\psi} \equiv \begin{pmatrix} \hat{\xi} \\ \hat{\chi} \\ \hat{\eta} \end{pmatrix}, \qquad \hat{\phi} \equiv \begin{pmatrix} \hat{\varphi} \\ \hat{\Phi} \\ \hat{\Phi}^\dagger \end{pmatrix}. \tag{29.4.1}$$

By dividing up the fermions into Majorana and Dirac fermions and the spin-zero fields into real and complex scalars, we are assuming implicitly that some of the indices I and i correspond to states of a definite (global) U(1)-charge (denoted in the following by q_I and q_i, respectively).

The most general set of Yukawa interactions of the scalar fields with a pair of fermion fields is then given by:

$$\mathscr{L}_{\rm int} = -\tfrac{1}{2}\hat{Y}^{Ijk}\hat{\phi}_I\hat{\psi}_j\hat{\psi}_k - \tfrac{1}{2}\hat{Y}_{Ijk}\hat{\phi}^I\hat{\psi}^{\dagger j}\hat{\psi}^{\dagger k}, \tag{29.4.2}$$

where $\hat{Y}_{Ijk} = (\hat{Y}^{Ijk})^*$ is symmetric under the interchange of j and k. We have suppressed the spinor indices; the product of two-component spinors is always performed according to the index convention indicated in eq. (29.37). The Yukawa Lagrangian [eq. (29.4.2)] must be invariant under:

$$\hat{\xi}_i \to \hat{\xi}_i\,, \qquad \hat{\chi}_i \to e^{iq_i\theta}\hat{\chi}_i\,, \qquad \hat{\eta}^i \to e^{-iq_i\theta}\hat{\eta}^i\,, \qquad \hat{\varphi}_i \to \hat{\varphi}_i\,, \qquad \hat{\Phi}_I \to e^{iq_I\theta}\hat{\Phi}_I\,, \qquad \hat{\Phi}^I \to e^{-iq_I\theta}\hat{\Phi}^I\,, \tag{29.4.3}$$

where the q_i are the U(1)-charges of the corresponding Dirac fermions and the q_I are the U(1)-charges of the corresponding complex scalars. Consequently, the form of the $\hat{Y}^{Ijk}$ is constrained:

$$\hat{Y}^{Ijk} = 0\,, \quad \text{unless} \quad q_I + q_j + q_k = 0\,. \tag{29.4.4}$$

Of course, any other conserved symmetries will impose additional selection rules on the Yukawa couplings $\hat{Y}^{Ijk}$.

The hatted fields are the interaction eigenstate fields. However, in general the mass eigenstates can be different, as discussed in Section 29.3.2. The computation of matrix elements for physical processes is more conveniently done in terms of the propagating mass eigenstate fields. The mass eigenstate basis ψ is related to the interaction eigenstate basis $\hat{\psi}$ by a unitary rotation $U_i{}^j$ on the flavor indices. In matrix form:

$$\hat{\psi} \equiv \begin{pmatrix} \hat{\xi}_i \\ \hat{\chi}_i \\ \hat{\eta}^i \end{pmatrix} = U\psi \equiv \begin{pmatrix} \Omega_i{}^j & 0 & 0 \\ 0 & L_i{}^j & 0 \\ 0 & 0 & R^i{}_j \end{pmatrix} \begin{pmatrix} \xi_j \\ \chi_j \\ \eta^j \end{pmatrix}, \tag{29.4.5}$$

where Ω, L, and R are constructed as described previously in Section 29.3.2 [see eqs. (29.3.8) and (29.3.32)]. Likewise, the mass eigenstate basis ϕ is related to the interaction eigenstate basis $\hat{\phi}$ by a unitary rotation $V_I{}^J$ on the flavor indices. In matrix form,

$$\hat{\phi} \equiv \begin{pmatrix} \hat{\varphi}_I \\ \hat{\Phi}_I \\ \hat{\Phi}^I \end{pmatrix} = V\phi \equiv \begin{pmatrix} Q_I{}^J & 0 & 0 \\ 0 & W_I{}^J & 0 \\ 0 & 0 & W^I{}_J \end{pmatrix} \begin{pmatrix} \varphi_J \\ \Phi_J \\ \Phi^J \end{pmatrix}, \tag{29.4.6}$$

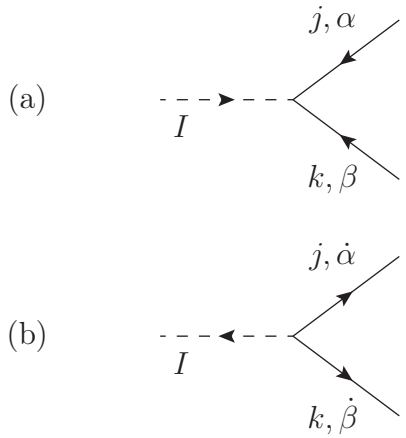


Figure 29.4.1: Feynman rules for Yukawa couplings of scalars to two-component fermions in a general field theory. The choice of which rule to use depends on how the vertex connects to the rest of the amplitude. When indices are suppressed, the spinor index part is always just proportional to the identity matrix.

where $W^I{}_J = (W_I{}^J)^*$, and Q and W are constructed according to eqs. (29.3.42) and (29.3.44).

Thus, we may rewrite eq. (29.4.2) in terms of mass eigenstate fields:

$$\mathscr{L}_{\rm int} = -\tfrac{1}{2}Y^{Ijk}\phi_I\psi_j\psi_k - \tfrac{1}{2}Y_{Ijk}\phi^I\psi^{\dagger j}\psi^{\dagger k}\,, \tag{29.4.7}$$

where

$$Y^{Ijk} = Y^{Ikj} \equiv V_J{}^I U_m{}^j U_n{}^k \hat{Y}^{Jmn}\,. \tag{29.4.8}$$

Note that eq. (29.4.4) implies that $Y^{Ijk} = 0$ unless $q_I + q_j + q_k = 0$. The corresponding Feynman rules that arise from the Yukawa interaction Lagrangian are shown in Fig. 29.4.1. If the scalar ϕ_I is complex, then one can associate an arrow with the flow of analyticity, which would point into the vertex in (a) and would point out of the vertex in (b). That is, the arrow on the scalar line keeps track of the height of the scalar flavor index entering or leaving the vertex.

In Fig. 29.4.1, two versions are given for each Feynman rule. The choice of which rule to use is dictated by the height of the indices on the fermion lines that connect to the vertex. These heights should always be chosen so that they are contracted as in eq. (29.37). However, when all spinor indices are suppressed, the scalar-fermion-fermion rules will have an identical appearance for both cases, since they are just proportional to the identity matrix of the 2×2 spinor space.

To provide a more concrete example of the above results, consider a real neutral scalar field ϕ and a (possibly) complex charged scalar field Φ (with U(1)-charge q_Φ) that interact with a multiplet of Majorana fermions ξ_i and Dirac fermion pairs χ_j and η^j (with U(1)-charges q_j and $-q_j$, respectively). We assume that all fields are given in the mass eigenstate basis. The Yukawa interaction Lagrangian is given by:

$$\begin{aligned}\mathscr{L}_{\rm int} = &-\tfrac{1}{2}(\lambda^{ij}\xi_i\xi_j + \lambda_{ij}\xi^{\dagger i}\xi^{\dagger j})\phi - \kappa^i{}_j\chi_i\eta^j\Phi + \kappa_i{}^j\chi^{\dagger i}\eta^\dagger_j\Phi^\dagger \\ &-[(\kappa_1)^i{}_j\xi_i\eta^j + (\kappa_2)_{ij}\xi^{\dagger i}\chi^{\dagger j}]\Phi - [(\kappa_2)^{ij}\xi_i\chi_j + (\kappa_1)_i{}^j\xi^{\dagger i}\eta^\dagger_j]\Phi^\dagger\,,\end{aligned} \tag{29.4.9}$$

where λ is a complex symmetric matrix, and κ, κ_1 and κ_2 are complex matrices such that $\kappa^i{}_j = 0$ unless $q_\Phi = q_j - q_i$ and $(\kappa_1)^i{}_j = (\kappa_2)_{ij} = 0$ unless $q_\Phi = q_j$ [flavor index conventions are specified in eqs. (29.3.2) and (29.3.28)]. The corresponding Feynman rules of Fig. 29.4.1(a) are obtained by identifying $Y^{Iij} = \lambda^{ij}$, $\kappa^i{}_j$, $(\kappa_1)^i{}_j$ and $(\kappa_2)^{ij}$ for the undotted fermion vertices $\phi\xi_i\xi_j$, $\Phi\chi_i\eta^j$, $\Phi\xi_i\eta^j$ and $\Phi^\dagger\xi_i\chi_j$, respectively.[44] The corresponding Feynman rules of Fig. 29.4.1(b) for the dotted fermion vertices are governed by the complex-conjugated Yukawa couplings, $Y_{Ijk} \equiv (Y^{Ijk})^*$.

[44]For the $\Phi^\dagger\xi_i\chi_j$ vertex, we should reverse the direction of the arrow on the scalar line in Fig. 29.4.1(a) [and

The renormalizable interactions of vector bosons with fermions and scalars arise from gauge interactions. These interaction terms of the Lagrangian derive from the respective kinetic energy terms of the fermions and scalars when the derivative is promoted to the covariant derivative:

$$(D_\mu)_i{}^j \equiv \delta_i{}^j\partial_\mu + ig_a A_\mu^a (\boldsymbol{T^a})_i{}^j\,, \tag{29.4.10}$$

where the index a labels the real (interaction eigenstate) vector bosons A_a^μ and is summed over. The index a runs over the adjoint representation of the gauge group,[45] and the $(\boldsymbol{T^a})_i{}^j$ are hermitian representation matrices of the generators of the Lie algebra of the gauge group acting on the left-handed fermions (for further details, see Appendix E). For a $U(1)$ gauge group, the $\boldsymbol{T^a}$ are replaced by real numbers corresponding to the U(1) charges of the left-handed $(\frac{1}{2},0)$ fermions. There is a separate coupling g_a for each simple group or U(1) factor of the gauge group G.[46]

In the gauge-interaction basis for the left-handed $(\frac{1}{2},0)$ two-component fermions the corresponding interaction Lagrangian is given by

$$\mathscr{L}_{\rm int} = -g_a A_a^\mu \hat{\psi}^{\dagger i}\,\overline{\sigma}_\mu (\boldsymbol{T}^a)_i{}^j \hat{\psi}_j\,. \tag{29.4.11}$$

In the case of spontaneously broken gauge theories, one must diagonalize the vector boson squared-mass matrix. The form of eq. (29.4.11) still applies where A_μ^a are gauge boson fields of definite mass, although in this case for a fixed value of a, $g_a\boldsymbol{T}^a$ [which multiplies A_μ^a in eq. (29.4.11)] is some linear combination of the original $g_a\boldsymbol{T}^a$ of the unbroken theory. That is, the hermitian matrix gauge field $(A_\mu)_i{}^j \equiv A_\mu^a(\boldsymbol{T^a})_i{}^j$ appearing in eq. (29.4.11) can always be re-expressed in terms of the *physical* mass eigenstate gauge boson fields.

If an unbroken U(1) (global or local) symmetry exists, then the physical gauge bosons will be eigenstates of the conserved U(1)-charge.[47] If the U(1) symmetry group is orthogonal to the gauge group under which the A_a^μ transform, then all the gauge bosons are neutral with respect to the U(1)-charge. For example, in the case of the interaction of a gluon with a pair of Majorana fermion gluinos, the gluon is a gauge boson that transforms under the SU(3) color group, which is orthogonal to the conserved $\mathrm{U}(1)_{\rm EM}$. That is, gluinos are color octet, electrically neutral fermions. In contrast, in the case of the interaction of a Z^0 with pair of Majorana neutralinos, $\mathrm{U}(1)_{\rm EM}$ is not orthogonal to the electroweak SU(2)×U(1) gauge group. Nevertheless, the Z^0-gauge boson interactions of the neutralinos are allowed as they conserve electric charge.

To obtain the desired Feynman rule, we rewrite eq. (29.4.11) in terms of mass eigenstate fermion fields. The resulting interaction Lagrangian can be rewritten as

$$\mathscr{L}_{\rm int} = -A_a^\mu \psi^{\dagger i}\,\overline{\sigma}_\mu (G^a)_i{}^j \psi_j\,, \tag{29.4.13}$$

likewise for the corresponding hermitian-conjugated vertex of Fig. 29.4.1(b)], in which case all arrows on the charged scalar and fermion lines would represent the direction of flow of the conserved U(1)-charge.

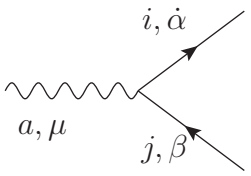
[45] Since the adjoint representation is a real representation, the height of the adjoint index a is not significant. The choice of a subscript or superscript adjoint index is based solely on typographical considerations.

[46] That is, the generators $\boldsymbol{T^a}$ separate out into distinct classes, each of which is associated with a simple group or one of the U(1) factors contained in the direct product that defines G. In particular, $g_a = g_b$ if $\boldsymbol{T^a}$ and $\boldsymbol{T^b}$ are in the same class. If G is simple, then $g_a = g$ for all a.

[47] In terms of the physical gauge boson fields, $A_\mu^a\boldsymbol{T^a}$ consists of a sum over real neutral gauge fields multiplied by hermitian generators, and complex charged gauge fields multiplied by non-hermitian generators. For example, in the electroweak Standard Model, G=SU(2)×U(1) with gauge bosons and generators W_μ^a and $\boldsymbol{T}^a = \frac{1}{2}\tau^a$ for SU(2) and B_μ and $\boldsymbol{Y}$ for U(1), where the τ^a are the usual Pauli matrices. After diagonalizing the gauge boson squared-mass matrix[153]:

$$gW_\mu^a\boldsymbol{T^a} + g'B_\mu\boldsymbol{Y} = \frac{g}{\sqrt{2}}(W_\mu^+\boldsymbol{T^+} + W_\mu^-\boldsymbol{T^-}) + \frac{g}{\cos\theta_W}\left(\boldsymbol{T^3} - \boldsymbol{Q}\sin^2\theta_W\right)Z_\mu + e\boldsymbol{Q}A_\mu\,, \tag{29.4.12}$$

where $\boldsymbol{Q} = \boldsymbol{T^3} + \boldsymbol{Y}$ is the generator of the unbroken $\mathrm{U}(1)_{\rm EM}$, $\boldsymbol{T^\pm} \equiv \boldsymbol{T^1} \pm i\boldsymbol{T^2}$, and $e = g\sin\theta_W = g'\cos\theta_W$. The massive gauge boson charge-eigenstate fields of the broken theory consist of a charged massive gauge boson pair, $W^\pm \equiv (W^1 \mp iW^2)/\sqrt{2}$, a neutral massive gauge boson, $Z \equiv W^3\cos\theta_W - B\sin\theta_W$, and the massless photon, $A \equiv W^3\sin\theta_W + B\cos\theta_W$.



$$-i(G^a)_i{}^j\,\overline{\sigma}_\mu^{\dot\alpha\beta} \quad \text{or} \quad i(G^a)_i{}^j\,\sigma_{\mu\beta\dot\alpha}$$

Figure 29.4.2: The Feynman rules for two-component fermion interactions with gauge bosons. The choice of which rule to use depends on how the vertex connects to the rest of the amplitude. The G^a are defined in eq. (29.4.14). The index a runs over both neutral and charged (mass eigenstate) gauge bosons, consistent with charge conservation at the vertex.

where the A_a^μ are the mass eigenstate gauge fields (of definite U(1)-charge, if relevant), and

$$(G^a)_i{}^j = g_a U^k{}_i (\boldsymbol{T^a})_k{}^m U_m{}^j\,, \tag{29.4.14}$$

or in matrix form, $G^a = g_a U^\dagger \boldsymbol{T^a} U$ (no sum over a). For values of a corresponding to the neutral gauge fields, the G^a are hermitian matrices. The corresponding Feynman rule is shown in Fig. 29.4.2.

The above treatment of the gauge interactions of (two-component) fermions is general. Nevertheless, it is useful to consider separately three cases where the gauge bosons couple to a pair of Majorana fermions, a pair of Dirac fermions, and a fermion pair consisting of one Majorana and one Dirac fermion.

First, consider the gauge interactions of neutral Majorana fermions. The Majorana fermions consist of left-handed $(\frac{1}{2},0)$ interaction eigenstate fermions $\hat\xi_i$ that transform under a real representation of the gauge group. After converting from the interaction eigenstates $\hat\xi_i$ to the mass eigenstates ξ_i using eq. (29.3.7), the Lagrangian for the gauge interactions of Majorana fermions is given by:

$$\mathscr{L}_{\rm int} = -A_\mu^a \xi^{\dagger i}\,\overline{\sigma}^\mu (G^a)_i{}^j \xi_j\,, \tag{29.4.15}$$

where the A_μ^a are *neutral* (real) mass eigenstate gauge fields, and

$$(G^a)_i{}^j = g_a \Omega^k{}_i (\boldsymbol{T^a})_k{}^m \Omega_m{}^j\,, \tag{29.4.16}$$

or in matrix form, $G^a = g_a \Omega^\dagger \boldsymbol{T^a} \Omega$ (no sum over a). Note that the G^a are hermitian matrices. The corresponding Feynman rule takes the same form as the generalized rule shown in Fig. 29.4.2, with a restricted to values corresponding to the neutral mass eigenstate gauge bosons.

Next, consider the gauge interactions of charged Dirac fermions. The Dirac fermions consist of pairs of left-handed $(\frac{1}{2},0)$ interaction eigenstate fermions $\hat\chi_i$ and $\hat\eta^i$ that transform as conjugate representations of the gauge group (hence the opposite flavor index heights). The fermion mass matrix couples χ and η type fields as in eq. (29.3.27). In the coupling to the interaction eigenstate gauge fields, if the $(\boldsymbol{T}^a)_i{}^j$ are matrix elements of the hermitian representation matrices of the generators acting on the $\hat\chi_i$, then the $\hat\eta^i$ transform in the complex conjugate representation with the corresponding generator matrices $-(\boldsymbol{T}^a)^* = -(\boldsymbol{T}^a)^{\sf T}$, i.e. with matrix elements $-(\boldsymbol{T}^a)_j{}^i$. Hence, the Lagrangian for the gauge interactions of Dirac fermions can be written in the form:

$$\mathscr{L}_{\rm int} = -g_a A_a^\mu \hat\chi^{\dagger i}\,\overline{\sigma}_\mu (\boldsymbol{T}^a)_i{}^j \hat\chi_j + g_a A_a^\mu \hat\eta_i^\dagger\,\overline{\sigma}_\mu (\boldsymbol{T}^a)_j{}^i \hat\eta^j\,. \tag{29.4.17}$$

We now rewrite eq. (29.4.17) in terms of mass eigenstate fermion fields using eq. (29.3.31), and express the hermitian matrix gauge field $A^\mu \equiv A_a^\mu \boldsymbol{T^a}$ in terms of mass eigenstate gauge fields (of definite U(1)-charge, if relevant). The resulting interaction Lagrangian is then given by:

$$\mathscr{L}_{\rm int} = -A_a^\mu \left[\chi^{\dagger i}\,\overline{\sigma}_\mu (G_L^a)_i{}^j \chi_j - \eta_i^\dagger\,\overline{\sigma}_\mu (G_R^a)_j{}^i \eta^j\right]\,, \tag{29.4.18}$$

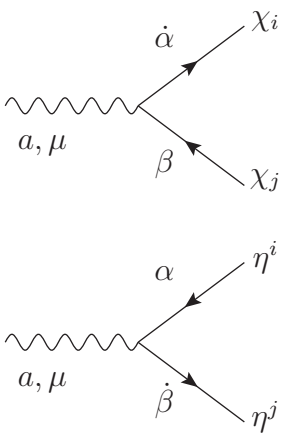


Figure 29.4.3: The Feynman rules for the interaction of a gauge boson and a pair of Dirac fermions (each formed by χ and η of the appropriate flavor index). The fermion lines are labeled by the corresponding two-component left-handed $(\frac{1}{2},0)$ fermion fields. The matrices G_L^a and G_R^a depend on the group generators for the representation carried by the χ_i according to eqs. (29.4.19) and (29.4.20). The index a runs over both neutral and charged (mass eigenstate) gauge bosons, consistent with charge conservation at the vertex.

where $A_a^\mu G_L^a$ and $A_a^\mu G_R^a$ are hermitian matrix-valued gauge fields, with:

$$(G_L^a)_i{}^j = g_a L^k{}_i (\boldsymbol{T^a})_k{}^m L_m{}^j\,, \tag{29.4.19}$$

$$(G_R^a)_j{}^i = g_a R^m{}_j (\boldsymbol{T^a})_m{}^k R_k{}^i\,. \tag{29.4.20}$$

In matrix form, eqs. (29.4.19) and (29.4.20) read: $G_L^a = g_a L^\dagger \boldsymbol{T^a} L$ and $G_R^a = g_a R^\dagger \boldsymbol{T^a} R$ (no sum over a). For values of a corresponding to the neutral gauge fields, G_L^a and G_R^a are hermitian matrices. The corresponding Feynman rules for the gauge interactions of Dirac fermions are shown in Fig. 29.4.3. Note that χ_i with its arrow pointing out of the vertex and η^i with its arrow pointing into the vertex represent the same Dirac fermion.

Finally, consider the interaction of a charged vector boson W (with U(1)-charge q_W) with a fermion pair consisting of one Majorana and one Dirac fermion. As before, we denote the Majorana fermion by ξ_i and the Dirac fermion pair by χ_j and η^j (with U(1)-charges q_j and $-q_j$, respectively). All fields are assumed to be in the mass eigenstate basis. The interaction Lagrangian is given by:[48]

$$\mathscr{L}_{\rm int} = -W_\mu[(G_1)_j{}^i \chi^{\dagger j}\overline\sigma^\mu \xi_i - (G_2)_{ij}\xi^{\dagger i}\overline\sigma^\mu\eta^j] - W_\mu^\dagger[(G_1)^j{}_i\,\xi^{\dagger i}\overline\sigma^\mu\chi_j - (G_2)^{ij}\eta_j^\dagger\overline\sigma^\mu\xi_i]\,, \tag{29.4.21}$$

where G_1 and G_2 are arbitrary complex matrices, with $(G_1)^i{}_j \equiv [(G_1)_i{}^j]^*$ and $(G_2)^{ij} \equiv [(G_2)_{ij}]^*$, such that $(G_1)_j{}^i = (G_2)_{ij} = 0$ unless $q_W = q_j$. The interactions of eq. (29.4.21) yield the Feynman rules exhibited in Fig. 29.4.4. Note that rules (c) and (d) are the complex conjugates of rules (a) and (b), respectively, corresponding to a reversal of the flow of the U(1)-charge through the interaction vertex.

In Figs. 29.4.2–29.4.4, two versions are given for each of the boson-fermion-fermion Feynman rules. The correct version to use depends in a unique way on the heights of indices used to connect each fermion line to the rest of the diagram. For example, the way of writing the vector-fermion-fermion interaction rule depends on whether we used $\psi^{\dagger i}\overline\sigma^\mu\psi_j$, or its equivalent form $-\psi_j\sigma^\mu\psi^{\dagger i}$, in eq. (29.4.11). Note the different heights of the undotted and dotted spinor indices that adorn σ^μ and $\overline\sigma^\mu$. The choice of which rule to use is thus dictated by the height of the indices on the lines that connect to the vertex. These heights should always be chosen so that they are contracted as in eq. (29.37).

The application of the rules of this subsection will be exhibited in Section 29.4.5. Many additional examples involving Standard Model and MSSM processes can be found in Section 29.6.

[48] The sign in front of G_2 is conventionally chosen to match the sign of the term proportional to G_R^a in eq. (29.4.18).

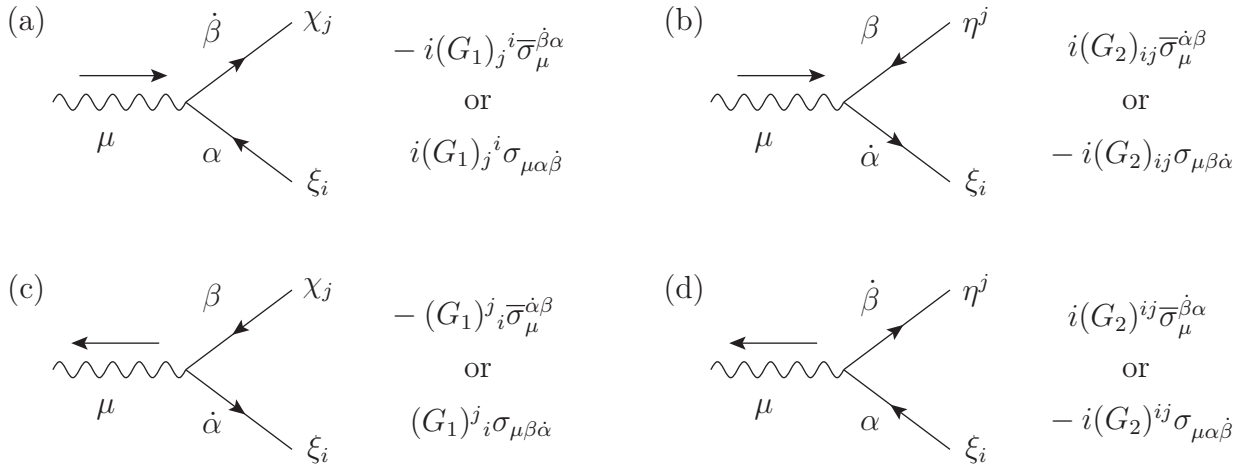


Figure 29.4.4: The Feynman rules for the interactions of a charged vector boson (with U(1)-charge q_W) with a fermion pair consisting of one Majorana fermion ξ_i and one Dirac fermion formed by χ_j and η^j (with corresponding U(1)-charges q_j and $-q_j$). The fermion lines are labeled by the corresponding two-component left-handed $(\frac{1}{2},0)$ fermion fields. The matrix couplings G_1 and G_2 are defined in eq. (29.4.21). Note that $(G_1)_j{}^i = (G_2)_{ij} = 0$ unless $q_W = q_j$. The arrows indicate the direction of flow of the U(1)-charges of the fermion and boson fields.

29.4.4 General structure and rules for Feynman graphs

When computing an amplitude for a given process, all possible diagrams should be drawn that conform with the rules given in Sections 29.4.1–29.4.3 for external wave functions, propagators, and interactions, respectively. Starting from any external wave function spinor (or from any vertex on a fermion loop), factors corresponding to each propagator and vertex should be written down from left to right, following the line until it ends at another external state wave function (or at the original point on the fermion loop). If one starts a fermion line at an x or y external state spinor, it should have a raised undotted index in accord with eq. (29.37). Or, if one starts with an $x^\dagger$ or $y^\dagger$, it should have a lowered dotted spinor index. Then, all spinor indices should always be contracted as in eq. (29.37). If one ends with an x or y external state spinor, it will have a lowered undotted index, while if one ends with an $x^\dagger$ or $y^\dagger$ spinor, it will have a raised dotted index. For arrow-preserving fermion propagators and gauge vertices, the preceding determines whether the σ or $\overline\sigma$ rule should be used.

With only a little practice, one can write down amplitudes immediately with all spinor indices suppressed. In particular, the following must be satisfied:

- For any scattering matrix amplitude, factors of σ and $\overline\sigma$ must alternate. If one or more factors of σ and/or $\overline\sigma$ are present, then x and y must be followed [preceded] by a σ [$\overline\sigma$], and $x^\dagger$ and $y^\dagger$ must be followed [preceded] by a $\overline\sigma$ [σ]. (29.4.1)

These requirements automatically dictate whether the σ or $\overline\sigma$ version of the rule for arrow-preserving fermion propagators and gauge vertices are employed in any tree-level Feynman diagram. In loop diagrams, we must add one further requirement that governs the order of the σ and $\overline\sigma$ factors as one traverses around the loop.

- Arrow-preserving propagator lines must be traversed in a direction parallel [antiparallel] to the arrowed line segment for the $\overline\sigma$ [σ] version of the propagator rule.[49] (29.4.2)

[49] This rule is simply a consequence of the order of the spinor indices in Fig. 29.4.2, as noted in Section 29.4.2.

For fermion lines that are not closed loops, this last requirement is realized automatically provided that the requirements of eq. (29.4.1) are satisfied. However, for closed fermion loops, one must use the correct fermion propagator corresponding to the direction around the loop one has chosen to follow in writing down the spinor trace with suppressed indices. For example, having employed a σ [$\overline{\sigma}$] rule at one vertex attached to the loop, one must then traverse the loop from that vertex point in a direction parallel [antiparallel] to the arrow-preserving propagator lines in the loop. Indeed, this rule is crucial for obtaining the correct sign for the triangle anomaly calculation in Section 29.6.26.

Symmetry factors for identical particles are implemented in the usual way. Fermi-Dirac statistics are implemented by the following rules:

- Each closed fermion loop gets a factor of -1.
- A relative minus sign is imposed between terms contributing to a given amplitude whenever the ordering of external state spinors (written left-to-right in a formula) differs by an odd permutation.

Amplitudes generated according to these rules will contain objects of the form:

$$a = z_1 \Sigma z_2 \tag{29.4.3}$$

where z_1 and z_2 are each commuting external spinor wave functions x, $x^\dagger$, y, or $y^\dagger$, and Σ is a sequence of alternating σ and $\overline{\sigma}$ matrices. The complex conjugate of this quantity is obtained by applying the results of eqs. (29.44)–(29.48), and is given by[50]

$$a^* = z_2^\dagger \Sigma_r z_1^\dagger \tag{29.4.4}$$

where Σ_r is obtained from Σ by reversing the order of all the σ and $\overline{\sigma}$ matrices, and using the same rule for suppressed spinor indices. [Notice that this rule for taking complex conjugates has the same form as for anticommuting spinors; cf. eqs. (29.44)–(29.48).] We emphasize that in principle, it does not matter in what direction a diagram is traversed while applying the rules. However, for each diagram one must include a sign that depends on the ordering of the external fermions. This sign can be fixed by first choosing some canonical ordering of the external fermions. Then for any graph that contributes to the process of interest, the corresponding sign is positive (negative) if the ordering of external fermions is an even (odd) permutation with respect to the canonical ordering. If one chooses a different canonical ordering, then the resulting amplitude changes by an overall phase (is unchanged) if this ordering is an odd (even) permutation of the original canonical ordering.[51] This is consistent with the fact that the S-matrix element is only defined up to an overall sign, which is not physically observable.[52]

Note that different graphs contributing to the same process will often have different external state wave function spinors, with different arrow directions, for the same external fermion. Furthermore, there are no arbitrary choices to be made for arrow directions, as there are in some four-component Feynman rules for Majorana fermions (as discussed in Appendix G .) Instead, one must add together *all* Feynman graphs that obey the rules.

[50] For Lorentz-scalar quantities of the form given by eq. (29.4.3), there is no distinction between complex conjugation and hermitian conjugation.

[51] For a process with exactly two external fermions, it is convenient to apply the Feynman rules by starting from the same fermion external state in all diagrams. That way, all terms in the amplitude have the same canonical ordering of fermions and there are no additional minus signs between diagrams. However, if there are four or more external fermions, it often happens that there is no way to choose the same ordering of external state spinors for all graphs when the amplitude is written down. Then the relative signs between different graphs must be chosen according to the relative sign of the permutation of the corresponding external fermion spinors. This guarantees that the total amplitude is antisymmetric under the interchange of any pair of external fermions.

[52] The S-matrix element is related to the invariant matrix element $\mathcal{M}_{fi}$ by $S_{fi} = \delta_{fi} + (2\pi)^4 \delta^{(4)}(p_f - p_i)\, i\mathcal{M}_{fi}$, where p_f (p_i) is the total four-momentum of the final (initial) state. If $f \neq i$ (i.e. the final and initial states are distinct), then $\delta_{fi} = 0$ in which case the invariant matrix element is only defined up to an overall (unphysical) sign. However, if $f = i$, the most convenient choice for the canonical ordering of external fermions is the one that yields $\langle f|i\rangle = \delta_{fi}$ (with no extra minus sign), which then fixes the absolute sign of the invariant matrix element.

29.4.5 Basic examples of writing down diagrams and amplitudes

Some simple examples will help clarify the rules of Section 29.4.4. In the tree-level Feynman graphs of this subsection, we label all two-component fermion lines by their corresponding left-handed $(\frac{1}{2},0)$ fields. (We shall propose a slightly different labeling convention in Section 29.5.) A larger number of examples, drawn from practical calculations, are given in Section 29.6.

Scalar boson decay to fermion pairs

Let us first consider a theory with a multiplet of uncharged, massive $(\frac{1}{2},0)$ fermions ξ_i, and a real scalar ϕ, with interaction

$$\mathscr{L}_{\rm int} = -\tfrac{1}{2}\left(\lambda^{ij}\xi_i\xi_j + \lambda_{ij}\xi^{\dagger i}\xi^{\dagger j}\right)\phi\,, \tag{29.4.1}$$

where $\lambda_{ij} \equiv (\lambda^{ij})^*$ and $\lambda^{ij} = \lambda^{ji}$. Consider the decay $\phi \to \xi_i(\vec{\boldsymbol{p}}_1, s_1)\xi_j(\vec{\boldsymbol{p}}_2, s_2)$ [for a fixed choice of i and j], where by $\xi_i(\vec{\boldsymbol{p}}, s)$ we mean the one particle state given by eq. (29.3.13).

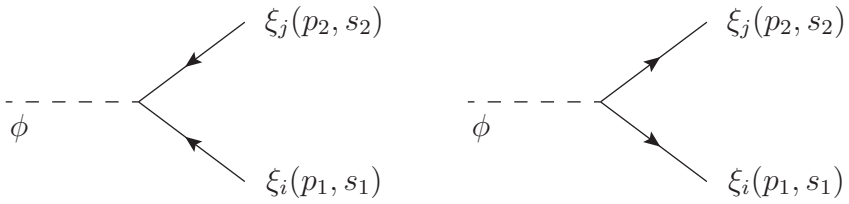


Figure 29.4.1: The two tree-level Feynman diagrams contributing to the decay of a neutral scalar into a pair of Majorana fermions.

Two diagrams contribute to this process, as shown in Fig. 29.4.1. The matrix element is:

$$\begin{aligned} i\mathcal{M} &= y(\vec{\boldsymbol{p}}_1, s_1)^\alpha(-i\lambda^{ij}\delta_\alpha{}^\beta)y(\vec{\boldsymbol{p}}_2, s_2)_\beta + x^\dagger(\vec{\boldsymbol{p}}_1, s_1)_{\dot\alpha}(-i\lambda_{ij}\delta^{\dot\alpha}{}_{\dot\beta})x^\dagger(\vec{\boldsymbol{p}}_2, s_2)^{\dot\beta} \\ &= -i\lambda^{ij}y(\vec{\boldsymbol{p}}_1, s_1)y(\vec{\boldsymbol{p}}_2, s_2) - i\lambda_{ij}x^\dagger(\vec{\boldsymbol{p}}_1, s_1)x^\dagger(\vec{\boldsymbol{p}}_2, s_2)\,. \end{aligned} \tag{29.4.2}$$

The second line could be written down directly by recalling that the sum over suppressed spinor indices is taken according to eq. (29.37). Note that if we reverse the ordering for the external fermions, the overall sign of the amplitude changes sign. This is easily checked, since for the commuting spinor wave functions (x and y), the spinor products in eq. (29.4.2) change sign when the order is reversed [see eqs. (29.60) and (29.61)]. This overall sign is not significant and depends on the order used in constructing the two particle state. One could even make the choice of starting the first diagram from fermion 1, and the second diagram from fermion 2:

$$i\mathcal{M} = -i\lambda^{ij}y(\vec{\boldsymbol{p}}_1, s_1)y(\vec{\boldsymbol{p}}_2, s_2) - (-1)i\lambda_{ij}x^\dagger(\vec{\boldsymbol{p}}_2, s_2)x^\dagger(\vec{\boldsymbol{p}}_1, s_1)\,. \tag{29.4.3}$$

Here, the first term establishes the canonical ordering of fermions (12), and the contribution from the second diagram therefore includes the relative minus sign in parentheses. Indeed, eqs. (29.4.2) and (29.4.3) are equal. In the computation of the total decay rate for the case of $i = j$, one must multiply the integral over the total phase space by 1/2 to account for the identical particles.

Next, we consider a theory of a massive neutral scalar boson that couples to a multiplet of Dirac fermions. We denote the corresponding two-component fields by χ_i and η^i. For simplicity, we take all the U(1)-charges of the χ_i to be equal (and opposite to the charges of the η^i). The corresponding U(1)-invariant interaction is:

$$\mathscr{L}_{\rm int} = -(\kappa^i{}_j\chi_i\eta^j + \kappa_i{}^j\chi^{\dagger i}\eta^\dagger_j)\phi\,, \tag{29.4.4}$$

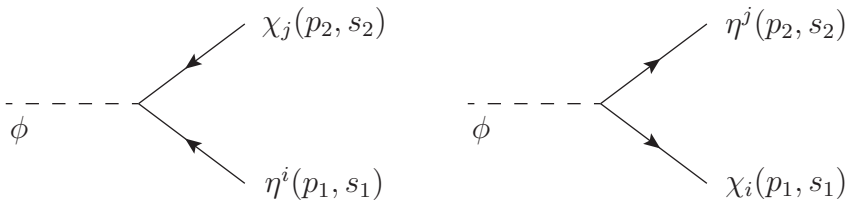


Figure 29.4.2: The two tree-level Feynman diagrams contributing to the decay of a neutral scalar into a pair of Dirac fermions. The χ_i–η^i and χ_j–η^j pairs, each with oppositely directed arrows, comprise Dirac fermion states with flavor indices i and j, respectively.

where $\kappa_i{}^j = (\kappa^i{}_j)^*$. Consider the decay $\phi \to f_i(\vec{\boldsymbol{p}}_1, s_1)\overline{f}{}^j(\vec{\boldsymbol{p}}_2, s_2)$ [for a fixed choice of i and j], where by $f(\vec{\boldsymbol{p}}, s)$ and $\overline{f}(\vec{\boldsymbol{p}}, s)$ we mean the one particle states given by eq. (29.3.22). Two diagrams contribute to this process, as shown in Fig. 29.4.2. Note that the outgoing fermion lines are distinguished by their U(1)-charges. The matrix element is then given by

$$i\mathcal{M} = -i\kappa^j{}_i y(\vec{\boldsymbol{p}}_1, s_1)y(\vec{\boldsymbol{p}}_2, s_2) - i\kappa_i{}^j x^\dagger(\vec{\boldsymbol{p}}_1, s_1)x^\dagger(\vec{\boldsymbol{p}}_2, s_2)\,. \tag{29.4.5}$$

The matrix element for $\phi \to f_i(\vec{\boldsymbol{p}}_1, s_1)\overline{f}{}^j(\vec{\boldsymbol{p}}_2, s_2)$ is identical to that of $\phi \to \xi_i(\vec{\boldsymbol{p}}_1, s_1)\xi_j(\vec{\boldsymbol{p}}_2, s_2)$ after replacing λ^{ij} with $\kappa^i{}_j$. However for fixed $i = j$, the rate for scalar boson decay to $f_i\overline{f}{}^i$ is twice that of $\xi_i\xi_i$ due to the final state identical particles in the latter case, as noted above. One also arrives at the same conclusion if one treats a single Dirac fermion as a pair of mass-degenerate two-component fields ξ_1 and ξ_2 [cf. eq. (29.3.15)]. Due to the U(1)-symmetry, the scalar Yukawa interactions are diagonal in the ξ_1–ξ_2 basis, so the rate for scalar decay into the Dirac fermion pair is equal to the incoherent sum of the rate for decay into $\xi_1\xi_1$ and $\xi_2\xi_2$.

Fermion pair annihilation into a scalar boson

It is also instructive to consider the corresponding $2 \to 1$ scattering (annihilation) processes $\xi(\vec{\boldsymbol{p}}_1, s_1)\xi(\vec{\boldsymbol{p}}_2, s_2) \to \phi$ and $f(\vec{\boldsymbol{p}}_1, s_1)\overline{f}(\vec{\boldsymbol{p}}_2, s_2) \to \phi$, respectively. The corresponding amplitudes are given by eqs. (29.4.2) and (29.4.5) with $y \to x$ and $x^\dagger \to y^\dagger$ (for simplicity, we neglect flavor). In the computation of the cross-sections, there is no extra factor required to account for the case of identical particles in the initial state. That is, the cross-section for $f(\vec{\boldsymbol{p}}_1, s_1)\overline{f}(\vec{\boldsymbol{p}}_2, s_2) \to \phi$ is equal to the cross-section for $\xi(\vec{\boldsymbol{p}}_1, s_1)\xi(\vec{\boldsymbol{p}}_2, s_2) \to \phi$ after replacing λ with κ.

This may at first seem puzzling given that a Dirac fermion can be represented by a pair of mass-degenerate two-component fields χ_1 and χ_2. But, recall the standard procedure for the calculation of decay rates and cross-sections in field theory—*average over unobserved degrees of freedom of the initial state and sum over unobserved degrees of freedom of the final state*. This mantra is well-known for dealing with spin and color degrees of freedom, but it is also applicable to degrees of freedom associated with global internal symmetries. Thus, the cross-section for the annihilation of a Dirac fermion pair into a neutral scalar boson can be obtained by computing the *average* of the cross-sections for $\xi_1(\vec{\boldsymbol{p}}_1, s_1)\xi_1(\vec{\boldsymbol{p}}_2, s_2) \to \phi$ and $\xi_2(\vec{\boldsymbol{p}}_1, s_1)\xi_2(\vec{\boldsymbol{p}}_2, s_2) \to \phi$. Since the annihilation cross-sections for $\xi_1\xi_1$ and $\xi_2\xi_2$ are equal, we confirm the annihilation cross-section for the Dirac fermion pair obtained above in the χ–η basis. Since the latter is conceptually simpler, subsequent computations involving Dirac fermions will be performed in the χ–η basis.

The annihilation rate of fermions enters in the analysis of the event flux due to the annihilation of dark matter in the halo of our galaxy. Let us compare the rates in the case that the dark matter is either a Majorana or a Dirac fermion. Suppose the annihilation involves two fermions whose number densities are n_1 and n_2 respectively. Then the observer on Earth who integrates along the line of sight to the annihilation events that are detected sees a flux of events proportional to [154]

$$\frac{dN_{\rm events}}{dA\,dt} \sim \int n_1 n_2 \,\langle \sigma_{\rm ann} v_{\rm rel}\rangle \, d\ell\,, \tag{29.4.6}$$

where $v_{\rm rel}$ is the relative velocity of the annihilating initial state particles, $\sigma_{\rm ann}$ is the annihilation cross-section and $\langle\cdots\rangle$ refers to a thermal average [155] over the velocity distribution of dark matter particles in the halo. We now compare the case of the annihilation of a single species of Majorana particles and the annihilation of a Dirac fermion-antifermion pair (assumed to have the same mass and couplings). We assume that the number density of Dirac fermions and antifermions and the corresponding number density of Majorana fermions are all the same (and denoted by n). Above, we showed that $\sigma_{\rm ann}$ is the same for the annihilation of a single species of Majorana and Dirac fermions. For the Dirac case, $n_1n_2 = n^2$. For the Majorana case, because the Majorana fermions are identical particles, given N initial state fermions in a volume V, there are $N(N-1)/2$ possible scatterings. In the thermodynamic limit where $N, V \to \infty$ at fixed $n \equiv N/V$, we conclude that $n_1n_2 = \frac{1}{2}n^2$ for a single species of annihilating Majorana fermions.[53] Hence the event flux rate for the annihilation of a Dirac fermion-antifermion pair is double that of a single species of Majorana fermions.[54] The extra factor of $1/2$ can also be understood by noting that in the case of annihilating dark matter particles (in the large N limit), all possible scattering axes occur and are implicitly integrated over. But, integrating over 4π steradians double counts the annihilation of identical particles (in the same way it does in the computation of the decay rate of a scalar into identical fermions discussed above). Hence, one must include a factor of $\frac{1}{2}$ in this case by replacing $n_1n_2 = n^2$ by $\frac{1}{2}n^2$ in eq. (29.4.6).

The relic abundance of primordial dark matter particles in the universe is inversely proportional to $\langle\sigma_{\rm ann}v_{\rm rel}\rangle$ [157]. By similar arguments to the ones just presented, it follows that the relic abundance of a single species of Majorana fermions would be twice that of a single species of Dirac fermions.

Vector boson decay into fermion pairs

Consider next the decay of a massive neutral vector boson A_μ into a pair of Majorana fermions, $A_\mu \to \xi_i(\vec{\boldsymbol{p}}_1, s_1)\xi_j(\vec{\boldsymbol{p}}_2, s_2)$, following from the interaction,

$$\mathscr{L}_{\rm int} = -G_i{}^j A^\mu \xi^{\dagger i}\overline{\sigma}_\mu \xi_j \,, \tag{29.4.7}$$

where G is an hermitian coupling matrix. The two diagrams shown in Fig. 29.4.3 contribute.

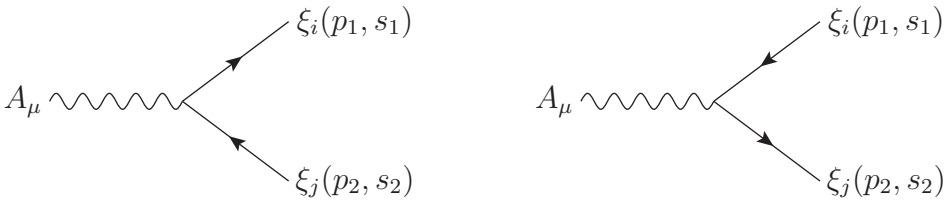


Figure 29.4.3: The two tree-level Feynman diagrams contributing to the decay of a massive neutral vector boson A_μ into a pair of Majorana fermions.

We start from the fermion with momentum p_1 and spin vector s_1 and end at the fermion with momentum p_2 and spin vector s_2, using the rules of Fig. 29.4.2. The resulting amplitude for the decay is

$$i\mathcal{M} = \varepsilon^\mu \left[-iG_i{}^j x^\dagger(\vec{\boldsymbol{p}}_1, s_1)\overline{\sigma}_\mu y(\vec{\boldsymbol{p}}_2, s_2) + iG_j{}^i y(\vec{\boldsymbol{p}}_1, s_1)\sigma_\mu x^\dagger(\vec{\boldsymbol{p}}_2, s_2)\right] \,, \tag{29.4.8}$$

where ε^μ is the vector boson polarization vector. We have used the $\overline{\sigma}$-version of the vector-fermion-fermion rule [see Fig. 29.4.2] for the first diagram of Fig. 29.4.3 and the σ-version for the

[53] The factor of $1/2$, which has been erroneously omitted in many papers in the literature, was correctly employed and explained in footnote 1 of [156].

[54] This is also consistent with the interpretation of a Dirac fermion as a pair of mass-degenerate Majorana fermions.

second diagram of Fig. 29.4.3, as dictated by the implicit spinor indices, which we have suppressed. However, we could have chosen to evaluate the second diagram of Fig. 29.4.3 using the $\overline{\sigma}$-version of the vector-fermion-fermion rule by starting from the fermion with momentum p_2 and spin vector s_2. In that case, the term $iG_j{}^i y(\vec{\boldsymbol{p}}_1, s_1)\sigma_\mu x^\dagger(\vec{\boldsymbol{p}}_2, s_2)$ in eq. (29.4.8) is replaced by

$$(-1)[-iG_j{}^i x^\dagger(\vec{\boldsymbol{p}}_2, s_2)\overline{\sigma}_\mu y(\vec{\boldsymbol{p}}_1, s_1)]\,. \tag{29.4.9}$$

In eq. (29.4.9), the factor of $-iG_j{}^i$ arises from the use of the $\overline{\sigma}$-version of the vector-fermion-fermion rule, and the overall factor of -1 appears because the order of the fermion wave functions has been reversed; i.e. (21) is an odd permutation of (12). This is in accord with the ordering rule stated at the end of Section 29.4.4. Thus, the resulting amplitude for the decay of the vector boson into the pair of Majorana fermions now takes the form:

$$i\mathcal{M} = \varepsilon^\mu \left[-iG_i{}^j x^\dagger(\vec{\boldsymbol{p}}_1, s_1)\overline{\sigma}_\mu y(\vec{\boldsymbol{p}}_2, s_2) + iG_j{}^i x^\dagger(\vec{\boldsymbol{p}}_2, s_2)\overline{\sigma}_\mu y(\vec{\boldsymbol{p}}_1, s_1)\right]\,, \tag{29.4.10}$$

which coincides with eq. (29.4.8) after using $y\sigma^\mu x^\dagger = x^\dagger\overline{\sigma}^\mu y$ [cf. eq. (29.62) with commuting spinors]. Eq. (29.4.10) explicitly exhibits the property that the amplitude is antisymmetric under the interchange of the two external identical fermions. Again, the absolute sign of the total amplitude is not significant and depends on the choice of ordering of the outgoing states.

Next, we consider the decay of a massive neutral vector boson into a pair of Dirac fermions. Each Dirac fermion is described by the two-component fields χ_i and η^i, which possess equal and opposite U(1)-charges, respectively. The corresponding interaction Lagrangian is given by:

$$\mathscr{L}_{\rm int} = -A^\mu[(G_L)_i{}^j\,\chi^{\dagger i}\overline{\sigma}_\mu\chi_j - (G_R)_j{}^i\,\eta_i^\dagger\overline{\sigma}_\mu\eta^j\,]\,, \tag{29.4.11}$$

where G_L and G_R are hermitian. There are two contributing graphs, as shown in Fig. 29.4.4.

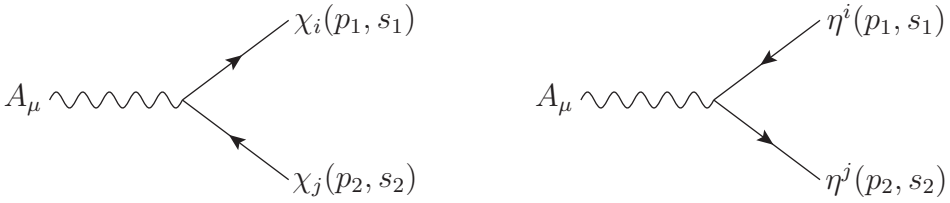


Figure 29.4.4: The two tree-level Feynman diagrams contributing to the decay of a massive neutral vector boson A_μ into a pair of Dirac fermions. The χ_i–η^i and χ_j–η^j pairs, each with oppositely directed arrows, comprise Dirac fermion states with flavor indices i and j, respectively.

To evaluate the amplitude, we start with the fermion of momentum p_1 and spin vector s_1, and end at the fermion with momentum p_2 and spin vector s_2. Note that the outgoing χ_i with the arrow pointing outward from the vertex and the outgoing η^i with the arrow pointing inward to the vertex both correspond to the same outgoing Dirac fermion. The amplitude for the decay is given by:

$$\begin{aligned} i\mathcal{M} &= \varepsilon^\mu \left[-i(G_L)_i{}^j x^\dagger(\vec{\boldsymbol{p}}_1, s_1)\overline{\sigma}_\mu y(\vec{\boldsymbol{p}}_2, s_2) - i(G_R)_i{}^j y(\vec{\boldsymbol{p}}_1, s_1)\sigma_\mu x^\dagger(\vec{\boldsymbol{p}}_2, s_2)\right] \\ &= \varepsilon^\mu \left[-i(G_L)_i{}^j x^\dagger(\vec{\boldsymbol{p}}_1, s_1)\overline{\sigma}_\mu y(\vec{\boldsymbol{p}}_2, s_2) - i(G_R)_i{}^j x^\dagger(\vec{\boldsymbol{p}}_2, s_2)\overline{\sigma}_\mu y(\vec{\boldsymbol{p}}_1, s_1)\right]\,. \end{aligned} \tag{29.4.12}$$

As in the case of the decay to a pair of Majorana fermions, we have exhibited a second form for the amplitude in eq. (29.4.12) in which the $\overline{\sigma}$-version of the vertex Feynman rule has been employed in both diagrams. Of course, the resulting amplitude must be the same in each method (up to a possible overall sign of the total amplitude that is not determined).

The computation of the amplitude for the decay of a charged vector boson to a fermion pair consisting of one Majorana fermion and one Dirac fermion, due to the interactions given in eq. (29.4.21), is straightforward and will not be given explicitly here.

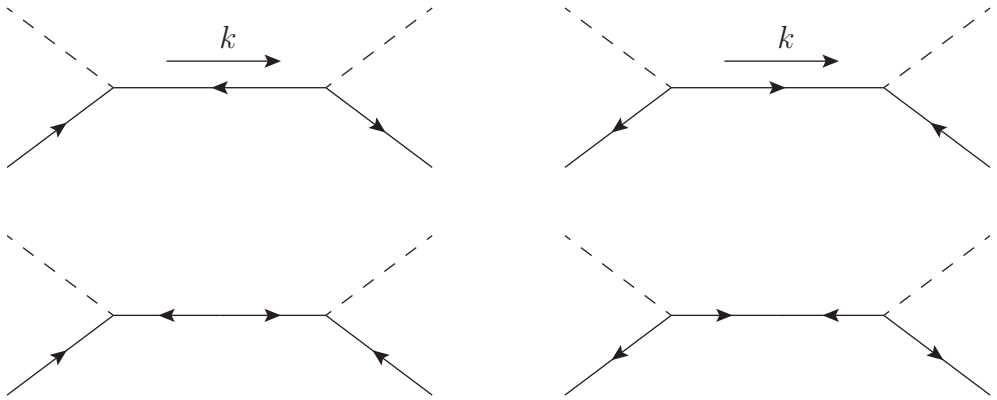


Figure 29.4.5: Tree-level Feynman diagrams contributing to the elastic scattering of a neutral scalar and a Majorana fermion. There are four more diagrams, obtained from these by crossing the initial and final scalar lines.

Two-body scattering of a boson and a neutral fermion

The next level of complexity consists of diagrams that involve fermion propagators. In the examples that follow in this and in the next subsection, we shall ignore the flavor index and consider scattering processes that involve a single flavor of Majorana or Dirac fermion. For our first example of this type, consider the tree-level matrix element for the scattering of a neutral scalar and a two-component neutral massive fermion ($\phi\xi \to \phi\xi$), with the interaction Lagrangian given above in eq. (29.4.1). Using the corresponding Feynman rules, there are eight contributing diagrams. Four are depicted in Fig. 29.4.5; there are another four diagrams (not shown) where the initial and final state scalars are crossed (i.e., the initial state scalar is attached to the same vertex as the final state fermion).

We shall write down the amplitudes for the four diagrams shown in Fig. 29.4.5, starting with the final state fermion line and moving toward the initial state fermion line. Then,

$$i\mathcal{M} = \frac{i}{k^2 - m_\xi^2}\Big\{(-i\lambda)(-i\lambda^*)\left[x^\dagger(\vec{\boldsymbol{p}}_2, s_2)\,\overline{\sigma}\cdot k\, x(\vec{\boldsymbol{p}}_1, s_1) + y(\vec{\boldsymbol{p}}_2, s_2)\,\sigma\cdot k\, y^\dagger(\vec{\boldsymbol{p}}_1, s_1)\right] + m_\xi\left[(-i\lambda)^2 y(\vec{\boldsymbol{p}}_2, s_2)x(\vec{\boldsymbol{p}}_1, s_1) + (-i\lambda^*)^2 x^\dagger(\vec{\boldsymbol{p}}_2, s_2)y^\dagger(\vec{\boldsymbol{p}}_1, s_1)\right]\Big\} + (\text{crossed})\,, \tag{29.4.13}$$

where k^μ is the sum of the two incoming (or outgoing) four-momenta, (p_1, s_1) are the momentum and spin four-vectors of the incoming fermion, and (p_2, s_2) are those of the outgoing fermion. The notation "(crossed)" refers to the contribution to the amplitude from diagrams which have the initial and final scalars interchanged. Note that we could have evaluated the diagrams above by starting with the initial vertex and moving toward the final vertex. It is easy to check that the resulting amplitude is the negative of the one obtained in eq. (29.4.13); the overall sign change simply corresponds to swapping the order of the two fermions and has no physical consequence. The overall minus sign is a consequence of eqs. (29.60)–(29.62) and the minus sign difference between the two ways of evaluating the propagator that preserves the arrow direction.

Next, we compute the tree-level matrix element for the scattering of a neutral vector boson and a neutral massive two-component fermion ξ with the interaction Lagrangian of eq. (29.4.7). Again there are eight diagrams: the four diagrams depicted in Fig. 29.4.6 plus another four (not shown) where the initial and final state vector bosons are crossed.

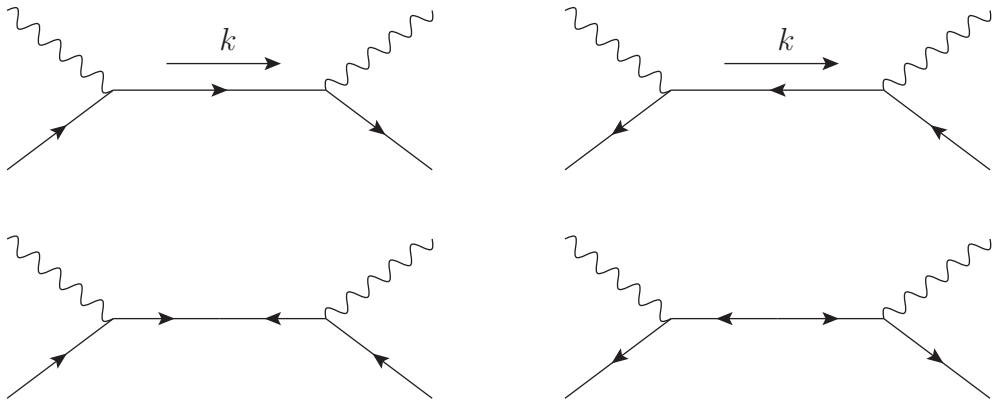


Figure 29.4.6: Tree-level Feynman diagrams contributing to the elastic scattering of a neutral vector boson and a Majorana fermion. There are four more diagrams, obtained from these by crossing the initial and final scalar lines.

Starting with the final state fermion line and moving toward the initial state, we obtain

$$i\mathcal{M} = \frac{i}{k^2 - m_\xi^2}\Big\{(-iG)^2 x^\dagger(\vec{\boldsymbol{p}}_2, s_2)\,\overline{\sigma}\cdot\varepsilon_2^*\,\sigma\cdot k\,\overline{\sigma}\cdot\varepsilon_1\,x(\vec{\boldsymbol{p}}_1, s_1) + (iG)^2 y(\vec{\boldsymbol{p}}_2, s_2)\,\sigma\cdot\varepsilon_2^*\,\overline{\sigma}\cdot k\,\sigma\cdot\varepsilon_1 y^\dagger(\vec{\boldsymbol{p}}_1, s_1)$$
$$+(-iG)(iG)m_\xi\left[y(\vec{\boldsymbol{p}}_2, s_2)\,\sigma\cdot\varepsilon_2^*\,\overline{\sigma}\cdot\varepsilon_1\,x(\vec{\boldsymbol{p}}_1, s_1) + x^\dagger(\vec{\boldsymbol{p}}_2, s_2)\,\overline{\sigma}\cdot\varepsilon_2^*\,\sigma\cdot\varepsilon_1\,y^\dagger(\vec{\boldsymbol{p}}_1, s_1)\right]\Big\}$$
$$+(\text{crossed})\,, \tag{29.4.14}$$

where ε_1 and ε_2 are the initial and final vector boson polarization four-vectors, respectively. As before, k^μ is the sum of the two incoming (or outgoing) four-momenta, (p_1, s_1) and (p_2, s_2) are the momentum and spin four-vectors of the incoming and outgoing fermions, respectively, and "(crossed)" indicates the terms from diagrams in which the initial and final vector bosons are interchanged. Alternatively, if one starts with an initial state fermion and moves toward the final state, the resulting amplitude is the negative of the one obtained in eq. (29.4.14), as expected.

The computation of the amplitude for the scattering of a charged scalar or vector boson and a Majorana fermion is straightforward and will not be given explicitly here.

Two-body scattering of a boson and a charged fermion

We first consider the scattering of a Dirac fermion with a neutral scalar. We denote the Dirac mass of the fermion by m_D. The left-handed fields χ and η have opposite charges (which we take to be $Q = +1$ and -1 respectively), and interact with the scalar ϕ according to

$$\mathscr{L}_{\text{int}} = -\phi[\kappa\chi\eta + \kappa^*\chi^\dagger\eta^\dagger]\,, \tag{29.4.15}$$

where κ is a coupling parameter. Then, for the elastic scattering of the $Q = +1$ fermion and a scalar, the diagrams of Fig. 29.4.7 contribute at tree level plus another four diagrams (not shown) where the initial and final state scalars are crossed. Now, these diagrams match precisely those of Fig. 29.4.5. Thus, applying the Feynman rules yields the same matrix element, eq. (29.4.13), previously obtained for the scattering of a neutral scalar and neutral two-component fermion, with the replacement of λ with κ and m_ξ with m_D.

We next examine the scattering of a Dirac fermion and a charged scalar, where both the scalar and fermion have the same absolute value of the charge. As above, we denote the charged $Q = \pm 1$ fermion by the pair of two-component fermions χ and η and the (intermediate state) neutral two-component fermion by ξ. The charged $Q = \pm 1$ scalar is represented by the complex scalar field Φ

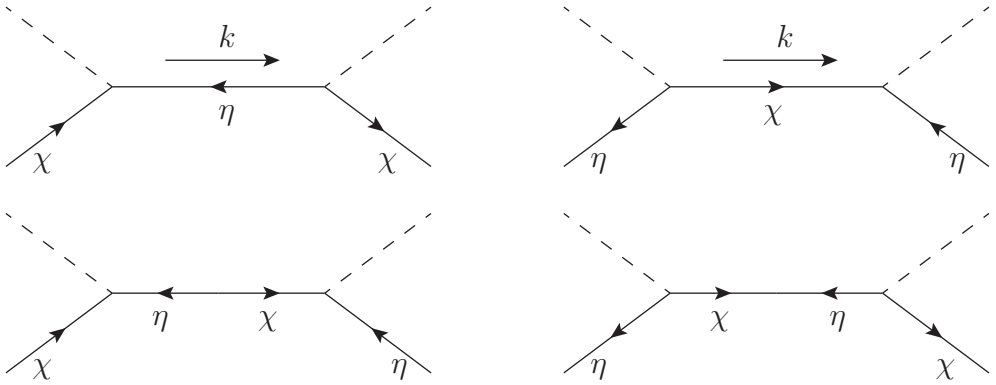


Figure 29.4.7: Tree-level Feynman diagrams contributing to the elastic scattering of a neutral scalar and a charged fermion. There are four more diagrams, obtained from these by crossing the initial and final scalar lines.

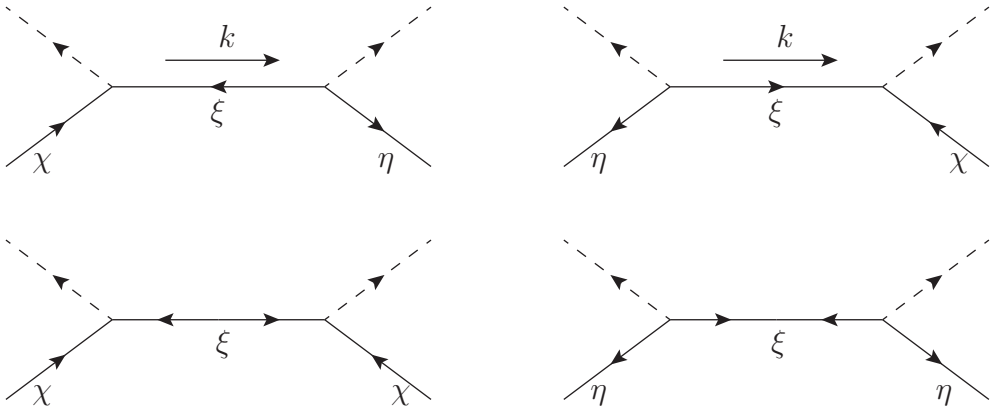


Figure 29.4.8: Tree-level Feynman diagrams contributing to the scattering of an initial charged scalar and a charged fermion into its charge-conjugated final state. The unlabeled intermediate state is a neutral fermion. There are four more diagrams, obtained from these by crossing the initial and final scalar lines.

and its hermitian conjugate. The interaction Lagrangian takes the form:

$$\mathscr{L}_{\rm int} = -\Phi[\kappa_1\eta\xi + \kappa_2^*\chi^\dagger\xi^\dagger] - \Phi^\dagger[\kappa_2\chi\xi + \kappa_1^*\eta^\dagger\xi^\dagger]\,. \tag{29.4.16}$$

Consider the scattering of an initial boson-fermion state into its charge-conjugated final state via the exchange of a neutral fermion. The relevant diagrams are shown in Fig. 29.4.8 plus the corresponding diagrams with the initial and final scalars crossed. We define the four-momentum k to be the sum of the two initial state four-momenta as shown in Fig. 29.4.8. The derivation of the amplitude is similar to the ones given previously, and we end up with

$$\begin{aligned} i\mathcal{M} = \frac{-i}{k^2 - m_\xi^2}\Big\{ &\kappa_1^*\kappa_2[x^\dagger(\vec{\boldsymbol{p}}_2, s_2)\,\overline{\sigma}\cdot k\, x(\vec{\boldsymbol{p}}_1, s_1) + y(\vec{\boldsymbol{p}}_2, s_2)\,\sigma\cdot k\, y^\dagger(\vec{\boldsymbol{p}}_1, s_1)] \\ &+ m_\xi\left[\kappa_2^2 y(\vec{\boldsymbol{p}}_2, s_2)x(\vec{\boldsymbol{p}}_1, s_1) + (\kappa_1^*)^2 x^\dagger(\vec{\boldsymbol{p}}_2, s_2)y^\dagger(\vec{\boldsymbol{p}}_1, s_1)\right]\Big\} + (\text{crossed})\,. \end{aligned} \tag{29.4.17}$$

The scattering of a charged fermion and a neutral spin-1 vector boson can be similarly treated. For example, consider the amplitude for the elastic scattering of a charged fermion of mass m_D

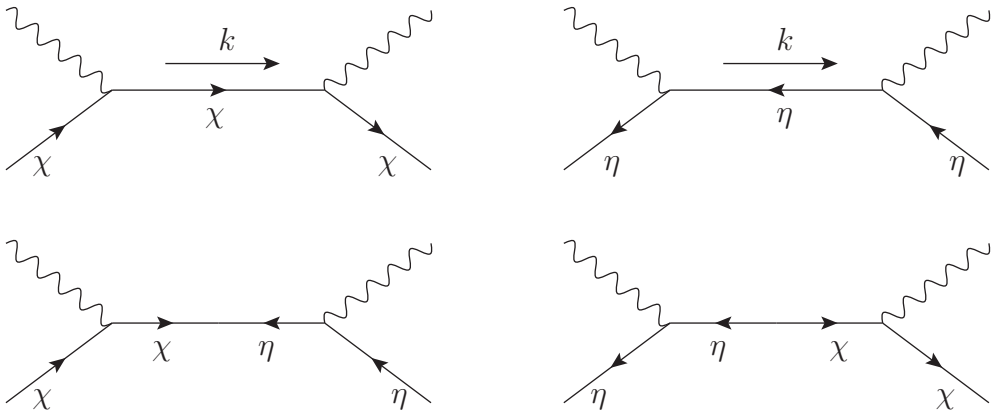


Figure 29.4.9: Tree-level Feynman diagrams contributing to the elastic scattering of a neutral vector boson and a Dirac fermion. There are four more diagrams, obtained from these by crossing the initial and final vector lines.

and a neutral vector boson. Again taking the interactions as given in eq. (29.4.11), the relevant diagrams are those shown in Fig. 29.4.9, plus four diagrams (not shown) obtained from these by crossing the initial and final state vector bosons. Applying the Feynman rules of Fig. 29.4.3, one obtains the following matrix element:

$$i\mathcal{M} = \frac{-i}{k^2 - m_D^2}\Big\{ G_L^2 x^\dagger(\vec{\boldsymbol{p}}_2, s_2)\, \overline{\sigma}\cdot\varepsilon_2^*\, \sigma\cdot k\, \overline{\sigma}\cdot\varepsilon_1\, x(\vec{\boldsymbol{p}}_1, s_1) + G_R^2 y(\vec{\boldsymbol{p}}_2, s_2)\, \sigma\cdot\varepsilon_2^*\, \overline{\sigma}\cdot k\, \sigma\cdot\varepsilon_1 y^\dagger(\vec{\boldsymbol{p}}_1, s_1)$$
$$+ m_D G_L G_R \left[y(\vec{\boldsymbol{p}}_2, s_2)\, \sigma\cdot\varepsilon_2^*\, \overline{\sigma}\cdot\varepsilon_1\, x(\vec{\boldsymbol{p}}_1, s_1) + x^\dagger(\vec{\boldsymbol{p}}_2, s_2)\, \overline{\sigma}\cdot\varepsilon_2^*\, \sigma\cdot\varepsilon_1\, y^\dagger(\vec{\boldsymbol{p}}_1, s_1)\right]\Big\} + (\text{crossed})\,, \tag{29.4.18}$$

and the assignments of momenta and spins are as before.

The computation of the amplitude for the scattering of a charged fermion and a charged vector boson is straightforward and will not be given explicitly here.

Two-body fermion–fermion scattering

Finally, let us work out an example with four external state fermions. Consider the case of elastic scattering of two identical Majorana fermions due to scalar exchange, governed by the interaction of eq. (29.4.1). The diagrams for scattering initial fermions labeled $1, 2$ into final state fermions labeled $3, 4$ are shown in Fig. 29.4.10.

The resulting invariant matrix element is:

$$i\mathcal{M} = (-1)\frac{-i}{s - m_\phi^2}\left\{\lambda^2 (x_1 x_2)(y_3 y_4) + (\lambda^*)^2 (y_1^\dagger y_2^\dagger)(x_3^\dagger x_4^\dagger) + |\lambda|^2 \left[(x_1 x_2)(x_3^\dagger x_4^\dagger) + (y_1^\dagger y_2^\dagger)(y_3 y_4)\right]\right\}$$
$$+ \frac{-i}{t - m_\phi^2}\left\{\lambda^2 (y_3 x_1)(y_4 x_2) + (\lambda^*)^2 (x_3^\dagger y_1^\dagger)(x_4^\dagger y_2^\dagger) + |\lambda|^2 \left[(x_3^\dagger y_1^\dagger)(y_4 x_2) + (y_3 x_1)(x_4^\dagger y_2^\dagger)\right]\right\}$$
$$+ (-1)\frac{-i}{u - m_\phi^2}\left\{\lambda^2 (y_4 x_1)(y_3 x_2) + (\lambda^*)^2 (x_4^\dagger y_1^\dagger)(x_3^\dagger y_2^\dagger)\right.$$
$$\left. + |\lambda|^2 \left[(x_4^\dagger y_1^\dagger)(y_3 x_2) + (y_4 x_1)(x_3^\dagger y_2^\dagger)\right]\right\}\,, \tag{29.4.19}$$

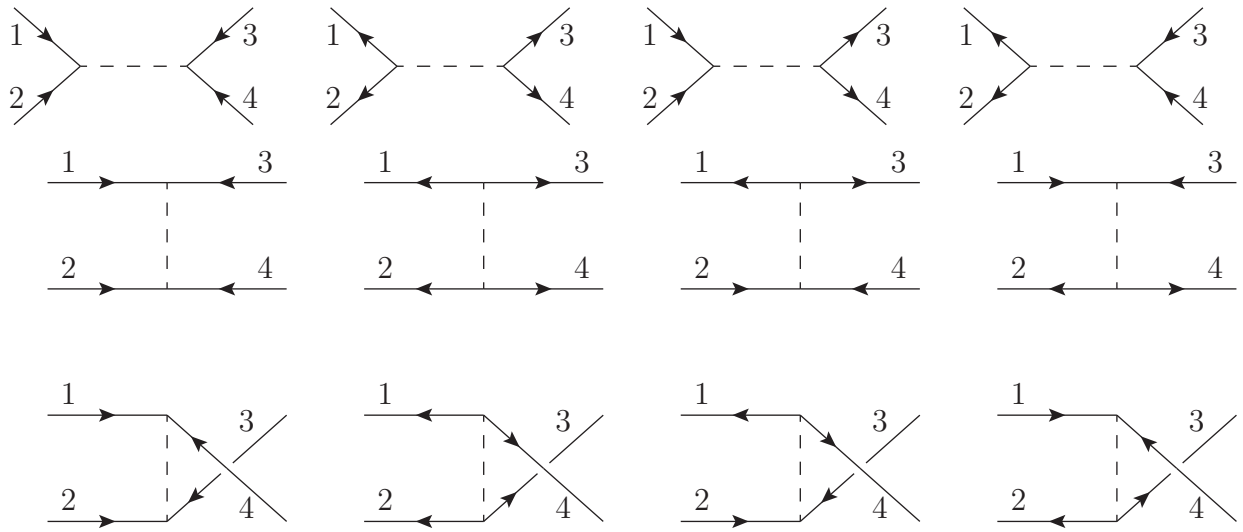


Figure 29.4.10: Tree-level Feynman diagrams contributing to the elastic scattering of identical Majorana fermions via scalar exchange in the s-channel (top row), t-channel (middle row), and u-channel (bottom row).

where $x_i \equiv x(\vec{\boldsymbol{p}}_i, s_i)$, $y_i \equiv y(\vec{\boldsymbol{p}}_i, s_i)$, m_ϕ is the mass of the exchanged scalar, $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$. We have chosen the canonical ordering of external fermions to be 3142 (corresponding to the t-channel contribution). For elastic scattering, this choice of canonical ordering guarantees that if no scattering occurs then the S-matrix is just equal to the unit operator with no extraneous minus sign (cf. footnote 52). The relative minus signs between the t-channel diagram and the s and u-channel diagrams [shown in parentheses in eq. (29.4.19)] are obtained by observing that both 1234 and 4132 are both odd permutations of 3142. Note that we would have obtained the same relative signs for the u-channel diagrams had we crossed the initial state fermion lines instead of the final state fermion lines.

Eq. (29.4.19) can be factorized with respect to the scalar line:

$$i\mathcal{M} = \frac{i}{s - m_\phi^2}(\lambda x_1 x_2 + \lambda^* y_1^\dagger y_2^\dagger)(\lambda y_3 y_4 + \lambda^* x_3^\dagger x_4^\dagger) + \frac{-i}{t - m_\phi^2}(\lambda y_3 x_1 + \lambda^* x_3^\dagger y_1^\dagger)(\lambda y_4 x_2 + \lambda^* x_4^\dagger y_2^\dagger)$$
$$+ \frac{i}{u - m_\phi^2}(\lambda y_4 x_1 + \lambda^* x_4^\dagger y_1^\dagger)(\lambda y_3 x_2 + \lambda^* x_3^\dagger y_2^\dagger)\,. \tag{29.4.20}$$

This is a common feature of Feynman graphs with a virtual boson. This example also illustrates that in contrast to the four-component fermion formalism, the two-component fermion Feynman rules typically yield many more diagrams, but the contribution of each of the diagrams is correspondingly simpler.

Non-relativistic potential due to scalar or pseudoscalar exchange

Consider two distinguishable fermions, and a scalar-fermion-fermion Yukawa interaction given by eq. (29.4.9). We can compute the force law that the fermions experience due to exchange of a spinless boson. That is, we shall derive the Yukawa potential as a function of the separation distance of the two fermions in the static limit.

To carry out this computation, we compute the invariant matrix element for two-body fermion-fermion elastic scattering in the non-relativistic limit. The relevant diagrams are shown in Fig. 29.4.10. As our two fermions are distinguishable, only the t-channel graphs (shown in the

middle row of Fig. 29.4.10) are relevant. As a result, the matrix element for the elastic scattering of two Majorana fermions is given by the t-channel contribution of eq. (29.4.20),

$$i\mathcal{M} = \frac{i}{m_\phi^2 - t}(\lambda y_3 x_1 + \lambda^* x_3^\dagger y_1^\dagger)(\lambda y_4 x_2 + \lambda^* x_4^\dagger y_2^\dagger)\,. \tag{29.4.21}$$

The choice of the overall sign is fixed by the canonical ordering of the external fermions.[55] Although the two fermions are distinguishable, we have assumed for simplicity that their (complex) Yukawa coupling strengths are the same and given by λ. For the scattering of two distinguishable Dirac fermions, the resulting expression for the scattering amplitude is identical to eq. (29.4.21), with λ replaced by the appropriate complex Yukawa coupling κ.

We denote the masses of the distinguishable fermions by m_1 and m_2. In the non-relativistic limit, $p_1 \simeq (m_1\,;\,\vec{\boldsymbol{p}}_{\boldsymbol{1}})$ and $p_3 \simeq (m_1\,;\,\vec{\boldsymbol{p}}_{\boldsymbol{3}})$, so that

$$m_\phi^2 - t \simeq |\vec{\boldsymbol{p}}_{\boldsymbol{1}} - \vec{\boldsymbol{p}}_{\boldsymbol{3}}|^2 + m_\phi^2 \equiv |\vec{\boldsymbol{q}}|^2 + m_\phi^2\,, \tag{29.4.22}$$

where

$$\vec{\boldsymbol{q}} \equiv \vec{\boldsymbol{p}}_{\boldsymbol{3}} - \vec{\boldsymbol{p}}_{\boldsymbol{1}} = \vec{\boldsymbol{p}}_{\boldsymbol{2}} - \vec{\boldsymbol{p}}_{\boldsymbol{4}} \tag{29.4.23}$$

is the momentum-transfer three-vector. Two separate cases will be considered.

In the first case, λ is a real coupling. This corresponds to the exchange of a $J^{PC} = 0^{++}$ scalar. Using the non-relativistic forms of eqs. (C.2.17) and (C.2.23) for the spinor bilinears, it is only necessary to keep the leading term. We then find:

$$i\mathcal{M} = \frac{4i|\lambda|^2 m_1 m_2}{|\vec{\boldsymbol{q}}|^2 + m_\phi^2}\delta_{s_1 s_3}\delta_{s_2 s_4}\,, \tag{29.4.24}$$

in agreement with eq. (4.123) of [115].

In the second case, λ is purely imaginary, and we will write $\lambda = i|\lambda|$ (the overall sign is not significant). This corresponds to the exchange of a $J^{PC} = 0^{-+}$ pseudoscalar. Again, we use the non-relativistic forms of eqs. (C.2.17) and (C.2.23) for the spinor bilinears. However, in this case the leading term cancels and we must retain the $\mathcal{O}(|\vec{\boldsymbol{p}}|/m)$ terms appearing in the non-relativistic limit of the spinor bilinears. In this case, we find

$$i\mathcal{M} = \frac{i|\lambda|^2}{|\vec{\boldsymbol{q}}|^2 + m_\phi^2}(\vec{\boldsymbol{q}}\cdot\hat{\boldsymbol{s}}^{\boldsymbol{a}}\tau^a_{s_3 s_1})\,(\vec{\boldsymbol{q}}\cdot\hat{\boldsymbol{s}}^{\boldsymbol{b}}\tau^b_{s_4 s_2})\,. \tag{29.4.25}$$

We choose the spin quantization axis to lie along the z-direction. That is, according to eq. (C.1.27), we choose

$$(\hat{\boldsymbol{s}}^{\boldsymbol{1}}\,,\,\hat{\boldsymbol{s}}^{\boldsymbol{2}}\,,\,\hat{\boldsymbol{s}}^{\boldsymbol{3}}) = (\hat{\boldsymbol{x}}\,,\,\hat{\boldsymbol{y}}\,,\,\hat{\boldsymbol{z}})\,, \tag{29.4.26}$$

in which case one can rewrite eq. (29.4.25) in the more traditional way,

$$i\mathcal{M} = \frac{i|\lambda|^2}{|\vec{\boldsymbol{q}}|^2 + m_\phi^2}(\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{\sigma}}_{s_3 s_1})\,(\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{\sigma}}_{s_4 s_2})\,, \tag{29.4.27}$$

where $\vec{\boldsymbol{\sigma}} \equiv \hat{\boldsymbol{x}}\tau^1 + \hat{\boldsymbol{y}}\tau^2 + \hat{\boldsymbol{z}}\tau^3$ are the usual spin-1/2 Pauli matrices.[56]. Thus, pseudoscalar exchange yields a spin-dependent force law.

The non-relativistic potential that arises from the t-channel scalar or pseudoscalar exchange is obtained by comparing the relativistic scattering amplitude $\mathcal{M}$ with the Born approximation for scattering off a potential $V(\vec{\boldsymbol{x}})$ in non-relativistic quantum mechanics. Taking into account

[55] As noted in Section 29.4.5, the canonical ordering of the external fermions in two-body elastic scattering is determined by the requirement that $\langle f|i\rangle = +1$ for $f = i$ (cf. footnote 52).

[56] The subscripted spin labels on $\vec{\boldsymbol{\sigma}}$ should be interpreted in the same way as outlined in footnote 102.

the difference between the conventions for the normalization of relativistic and non-relativistic single-particle states, one finds that the static potential is given by [158]

$$V(\vec{\boldsymbol{x}}) = -\frac{1}{4m_1m_2}\int \frac{d^3q}{(2\pi)^3}\,\mathcal{M}(\vec{\boldsymbol{q}})e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{x}}}\,, \tag{29.4.28}$$

in a convention where the invariant amplitude is defined as in footnote 52. Inserting the scattering amplitude for scalar (S) exchange, one obtains the well-known attractive spin-independent Yukawa potential

$$V(\vec{\boldsymbol{x}})_{\rm S} = -\frac{|\lambda|^2}{4\pi r}\,e^{-m_\phi r}\,\delta_{s_1s_3}\delta_{s_2s_4}\,, \tag{29.4.29}$$

where $r \equiv |\vec{\boldsymbol{x}}|$. For the case of pseudoscalar (PS) exchange, one can easily evaluate the integral in eq. (29.4.28) by writing $q_jq_ke^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{x}}} = -\nabla_j\nabla_k e^{i\vec{\boldsymbol{q}}\cdot\vec{\boldsymbol{x}}}$. The end result is [159]:

$$\begin{aligned} V(\vec{\boldsymbol{x}})_{\rm PS} &= \frac{|\lambda|^2}{16\pi m_1m_2}(\vec{\boldsymbol{\sigma}}_{s_3s_1}\cdot\vec{\boldsymbol{\nabla}})(\vec{\boldsymbol{\sigma}}_{s_4s_2}\cdot\vec{\boldsymbol{\nabla}})\,\frac{e^{-m_\phi r}}{r} \\ &= \frac{|\lambda|^2 m_\phi^2}{16\pi m_1m_2}\Bigg\{\left[-\frac{4\pi}{3m_\phi^2}\delta^{(3)}(\vec{\boldsymbol{x}}) + \frac{e^{-m_\phi r}}{r}\right]\vec{\boldsymbol{\sigma}}_{s_3s_1}\cdot\vec{\boldsymbol{\sigma}}_{s_4s_2} \\ &\qquad + \left[\frac{1}{(m_\phi r)^2} + \frac{1}{(m_\phi r)} + \frac{1}{3}\right]\left[\frac{3\,(\vec{\boldsymbol{\sigma}}_{s_3s_1}\cdot\vec{\boldsymbol{x}})(\vec{\boldsymbol{\sigma}}_{s_4s_2}\cdot\vec{\boldsymbol{x}})}{r^2} - \vec{\boldsymbol{\sigma}}_{s_3s_1}\cdot\vec{\boldsymbol{\sigma}}_{s_4s_2}\right]\frac{e^{-m_\phi r}}{r}\Bigg\}\,, \end{aligned} \tag{29.4.30}$$

where we have used [160]:

$$\nabla_i\nabla_j\left(\frac{1}{r}\right) = -\frac{4\pi}{3}\delta_{ij}\,\delta^{(3)}(\vec{\boldsymbol{x}}) + \frac{3x_ix_j - r^2\delta_{ij}}{r^5}\,. \tag{29.4.31}$$

29.4.6 Self-energy functions and pole masses for two-component fermions

In this section, we discuss the self-energy functions for fermions in two-component notation, taking into account the possibilities of loop-induced mixing and absorptive parts corresponding to decays to intermediate states. This formalism is useful in the computation of loop-corrected physical pole masses.

Consider a theory with left-handed fermion degrees of freedom $\hat{\psi}_i$ labeled by an index $i = 1, 2, ..., N$. Associated with each $\hat{\psi}_i$ is a right-handed fermion $\hat{\psi}^{\dagger i}$, where the flavor labels are treated as described below eq. (29.3.2). The theory is assumed to contain arbitrary interactions, which we will not need to refer to explicitly. As discussed in Section 29.3.2, we diagonalize the fermion mass matrix and identify the fermion mass eigenstates ψ_i as indicated in eq. (29.4.5). In general, the mass eigenstates consist of Majorana fermions ξ_k ($k = 1, ...N-2n$) and Dirac fermion pairs χ_ℓ and η_ℓ ($\ell = 1, ..., n$).[57] With respect to this basis, the symmetric $N \times N$ tree-level fermion mass matrix, $\boldsymbol{m}^{ij}$, is made up of diagonal elements m_k and 2×2 blocks $\left(\begin{smallmatrix} 0 & m_\ell \\ m_\ell & 0\end{smallmatrix}\right)$ along the diagonal, where the m_k and m_ℓ are real and non-negative. Since $\boldsymbol{m}^{ij}$ is real, the height of the flavor indices is not significant. Nevertheless, it is useful to define $\overline{\boldsymbol{m}}_{ij} \equiv \boldsymbol{m}^{ij}$ in order to maintain the convention

[57] In order to have a unified description, we shall take the flavor index of all left-handed fields (including η_k) in the lowered position (in contrast to the convention adopted in Sections 29.3.2 and 29.4.3) when considering a collection of two-component fermion fields that contains both Majorana and Dirac fermions.

that two repeated flavor indices are summed when one index is raised and the other is lowered.[58] Note that $\overline{\boldsymbol{m}}_{ik}\boldsymbol{m}^{kj} = \boldsymbol{m}^{ik}\overline{\boldsymbol{m}}_{kj} = m_i^2\delta_i^j$ is a diagonal matrix.

The full, loop-corrected Feynman propagators with four-momentum p^μ are defined by the Fourier transforms of vacuum expectation values of time-ordered products of bilinears of the fully interacting two-component fermion fields [cf. footnote 41]. Following eqs. (29.4.1)–(29.4.4), we define:

$$\langle 0|\, T\psi_{\alpha i}(x)\psi^{\dagger j}_{\dot\beta}(y)\,|0\rangle_{\rm FT} = ip{\cdot}\sigma_{\alpha\dot\beta}\,\boldsymbol{C}_i{}^j(s)\,, \tag{29.4.1}$$

$$\langle 0|\, T\psi^{\dagger\dot\alpha i}(x)\psi^{\beta}_{j}(y)\,|0\rangle_{\rm FT} = ip{\cdot}\overline{\sigma}^{\dot\alpha\beta}\,(\boldsymbol{C}^{\mathsf T})^{i}{}_{j}(s)\,, \tag{29.4.2}$$

$$\langle 0|\, T\psi^{\dagger\dot\alpha i}(x)\psi^{\dagger j}_{\dot\beta}(y)\,|0\rangle_{\rm FT} = i\delta^{\dot\alpha}{}_{\dot\beta}\,\boldsymbol{D}^{ij}(s)\,, \tag{29.4.3}$$

$$\langle 0|\, T\psi_{\alpha i}(x)\psi^{\beta}_{j}(y)\,|0\rangle_{\rm FT} = i\delta_\alpha{}^{\beta}\,\overline{\boldsymbol{D}}_{ij}(s)\,, \tag{29.4.4}$$

where $s \equiv p^2$ and

$$(\boldsymbol{C}^{\mathsf T})^{i}{}_{j} \equiv \boldsymbol{C}_j{}^{i}\,. \tag{29.4.5}$$

One can derive eq. (29.4.2) from eq. (29.4.1) by first writing

$$\psi^{\dagger\dot\alpha i}(x)\psi^{\beta}_{j}(y) = -\epsilon^{\beta\alpha}\,\epsilon^{\dot\alpha\dot\beta}\psi_{\alpha j}(y)\psi^{\dagger i}_{\dot\beta}(x)\,, \tag{29.4.6}$$

where the minus sign arises due to the anticommutativity of the fields, and then using eq. (29.32); the interchange of x and y (after FT) simply changes p^μ to $-p^\mu$.

In general, $\boldsymbol{D}$ and $\overline{\boldsymbol{D}}$ are complex symmetric matrices, and $\overline{\boldsymbol{D}} = \boldsymbol{D}^\star$. The matrix $\boldsymbol{C}$ satisfies the hermiticity condition $[\boldsymbol{C}^{\mathsf T}]^\star = \boldsymbol{C}$. Here, we have introduced the star symbol to mean that a quantity $Q^\star$ is obtained from Q by taking the complex conjugate of all Lagrangian parameters appearing in its calculation, but not taking the complex conjugates of Euclideanized loop integral functions, whose imaginary (absorptive) parts correspond to fermion decay widths to multi-particle intermediate states. That is, the dispersive part of $\boldsymbol{C}$ is hermitian and the absorptive part of $\boldsymbol{C}$ is anti-hermitian.

The diagrammatic representations of the full propagators are displayed in Fig. 29.4.1, where $\boldsymbol{C}_i{}^j$, $\boldsymbol{D}^{ij}$, and $\overline{\boldsymbol{D}}_{ij}$ defined above are each $N\times N$ matrix functions. Note that the second diagram of Fig. 29.4.1, when flipped by 180° about the axis that bisects the diagram, is equivalent to the first diagram of Fig. 29.4.1 (with $p\to -p$, $\alpha\to\beta$, $\dot\beta\to\dot\alpha$ and $i\leftrightarrow j$). In analogy with Fig. 29.4.2, one could replace the first two diagrammatic rules of Fig. 29.4.1 with a single rule shown in Fig. 29.4.2, where we have used eq. (29.4.5) to rewrite the second version of the rule in terms of $\boldsymbol{C}^{\mathsf T}$. Indeed,

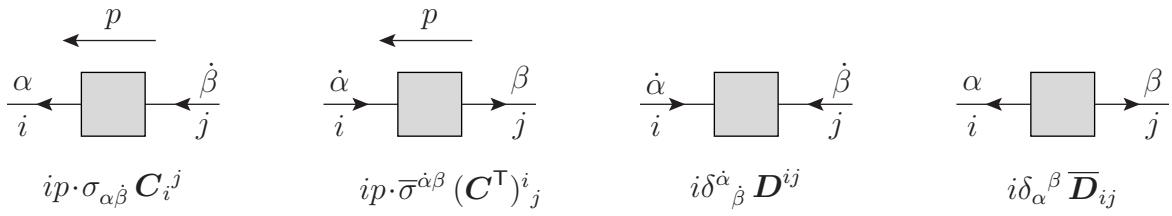


Figure 29.4.1: The full, loop-corrected propagators for two-component fermions are associated with functions $\boldsymbol{C}(p^2)_i{}^j$ [and its matrix transpose], $\boldsymbol{D}(p^2)^{ij}$, and $\overline{\boldsymbol{D}}(p^2)_{ij}$, as shown. The shaded boxes represent the sum of all connected Feynman diagrams, with external legs included. The four-momentum p flows from right to left.

[58] We will soon be suppressing the indices, so it is convenient to employ the bar on $\overline{\boldsymbol{m}}_{ij}$ to indicate its lowered index structure.

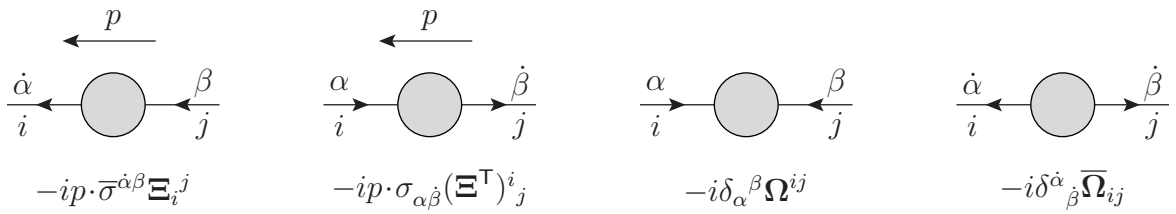
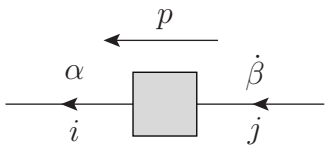


Figure 29.4.3: The self-energy functions for two-component fermions are associated with functions $\Xi(s)_i{}^j$ [and its matrix transpose], $\Omega(s)^{ij}$, and $\overline\Omega(s)_{ij}$, as shown. The shaded circles represent the sum of all one-particle irreducible, connected Feynman diagrams, and the external legs are amputated. The four-momentum p flows from right to left.

by using the $\overline\sigma$-version of the rule shown in Fig. 29.4.2 and flipping the corresponding diagram by 180° as described above, one reproduces the rule of the second diagram of Fig. 29.4.1.



$$ip\cdot\sigma_{\alpha\dot\beta}\,\boldsymbol{C}_i{}^j \quad \underline{\text{or}} \quad -ip\cdot\overline\sigma^{\dot\beta\alpha}\,(\boldsymbol{C}^{\mathsf T})^j{}_i$$

Figure 29.4.2: The first two diagrammatic rules of Fig. 29.4.1 can be summarized by a single diagram. Here, the choice of the σ or $\overline\sigma$ version of the rule is uniquely determined by the height of the spinor indices on the vertex to which the full loop-corrected propagator is connected (cf. Fig. 29.4.2 and the accompanying text).

In what follows, we prefer to keep the first two rules of Fig. 29.4.1 as separate entities. This will permit us to conveniently assemble the four diagrams of Fig. 29.4.1 into a 2×2 block matrix of two-component propagators [cf. eq. (G.5.2)]. In addition, by choosing the momentum flow in the two-component propagators from right to left, the left-to-right orderings of the spinor labels of the diagrams coincide with the ordering of spinor indices that appear in the corresponding algebraic representations. Thus, we can multiply diagrams together and interpret them as the product of the respective algebraic quantities taken from left to right in the normal fashion.

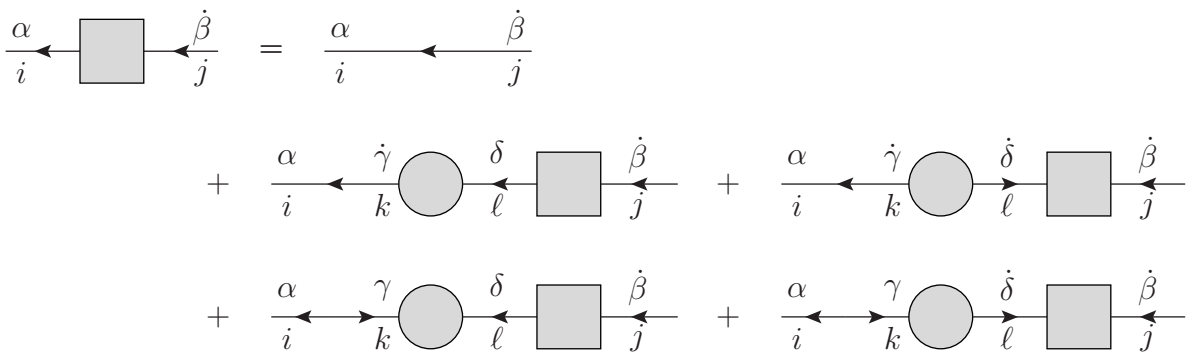
Given the tree-level propagators of Fig. 29.4.1, the full propagator functions are given by:

$$\boldsymbol{C}_i{}^j = \delta_i{}^j/(s-m_i^2)+\ldots \tag{29.4.7}$$
$$\boldsymbol{D}^{ij} = \boldsymbol{m}^{ij}/(s-m_i^2)+\ldots \tag{29.4.8}$$
$$\overline{\boldsymbol{D}}_{ij} = \overline{\boldsymbol{m}}_{ij}/(s-m_i^2)+\ldots, \tag{29.4.9}$$

with no sum on i in each case. They are functions of the external momentum invariant s and the masses and couplings of the theory. Inserting the leading terms [eqs. (29.4.7)–(29.4.9)] into Fig. 29.4.1 and organizing the result in a 2×2 block matrix of two-component propagators reproduces the usual four-component fermion tree-level propagator given in eq. (G.5.2).

The computation of the full propagators can be organized, as usual in quantum field theory, in terms of one-particle irreducible (1PI) self-energy functions. These are formally defined to be the sum of Feynman diagrams to all orders in perturbation theory (with the corresponding tree-level graph *excluded*) that contribute to the 1PI two-point Green function. Diagrammatically, the 1PI self-energy functions are defined in Fig. 29.4.3. As in the case of the full loop-corrected propagators, $\boldsymbol{\Omega}$ and $\overline{\boldsymbol{\Omega}}$ are complex symmetric matrices and the self-energy function satisfy hermiticity conditions, $[\boldsymbol{\Xi}^{\mathsf T}]^\star=\boldsymbol{\Xi}$ and $\overline{\boldsymbol{\Omega}}=\boldsymbol{\Omega}^\star$, where the star symbol was defined in the paragraph following eq. (29.4.6) and $(\boldsymbol{\Xi}^{\mathsf T})^i{}_j\equiv\boldsymbol{\Xi}_j{}^i$.

We illustrate the computation of the full propagator by considering first the following diagrammatic identity (with momentum p flowing from right to left):

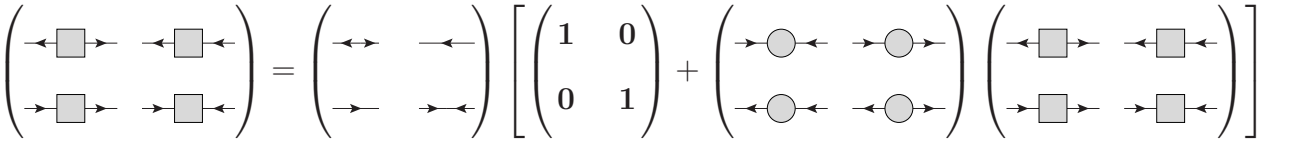


(29.4.10)

Similar diagrammatic identities can be constructed for the three other full loop-corrected propagators of Fig. 29.4.1. The resulting four equations can be neatly summarized by:

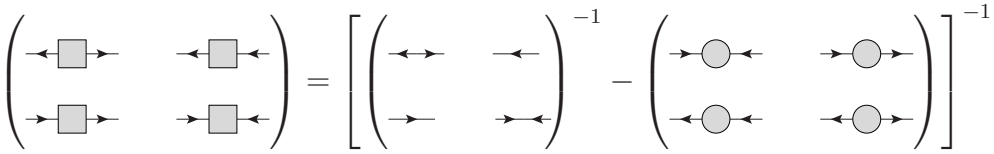
$$F = T + TSF\,, \tag{29.4.11}$$

where F is the matrix of full loop-corrected propagators, T is the matrix of tree-level propagators and S is the matrix of self-energy functions. Expressing eq. (29.4.11) in terms of diagrams,



(29.4.12)

which, when expanded out, yields eq. (29.4.10) and the corresponding identities for the three other full loop-corrected propagators of Fig. 29.4.1. Note that we have chosen the labeling and momentum flow in Figs. 29.4.1 and 29.4.3 such that the spinor and flavor labels of the diagrams appear in the appropriate left-to-right order to permit the interpretation of eq. (29.4.12) as a matrix equation. To solve for F,[59] we multiply eq. (29.4.11) on the left by T^{-1} and on the right by F^{-1} to obtain $T^{-1} = F^{-1} + S$. Thus, $F = [T^{-1} - S]^{-1}$. In pictures:



(29.4.14)

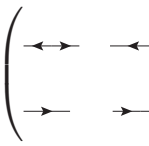
We evaluate the tree-level propagator matrix and its inverse using eqs. (29.4.7)–(29.4.9), keeping

[59] Alternatively, one can solve eq. (29.4.12) by iteration and summing the resulting geometric series. This yields:

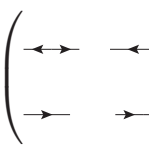
$$\begin{aligned} F &= T + TS(T + TS(T + TS(\cdots))) = T + TST + TSTST + \ldots \\ &= T[1 + ST + (ST)^2 + \ldots] \\ &= T[1 - ST]^{-1} = (T^{-1})^{-1}[1 - ST]^{-1} = [(1 - ST)T^{-1}]^{-1} = [T^{-1} - S]^{-1}\,, \end{aligned} \tag{29.4.13}$$

which is equivalent to eq. (29.4.14).

in mind that the direction of momentum flow is from right to left:



$$= \frac{1}{s - m_i^2} \begin{pmatrix} i\,\overline{\boldsymbol{m}}_{ij}\,\delta_\alpha{}^\beta & ip\cdot\sigma_{\alpha\dot\beta}\,\delta_i{}^j \\ ip\cdot\overline{\sigma}^{\dot\alpha\beta}\,\delta^i{}_j & i\,\boldsymbol{m}^{ij}\,\delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}, \tag{29.4.15}$$



$$= \begin{pmatrix} i\,\boldsymbol{m}^{ij}\,\delta_\alpha{}^\beta & -ip\cdot\sigma_{\alpha\dot\beta}\,\delta^i{}_j \\ -ip\cdot\overline{\sigma}^{\dot\alpha\beta}\,\delta_i{}^j & i\,\overline{\boldsymbol{m}}_{ij}\,\delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}, \tag{29.4.16}$$

where we follow the index structure defined in Figs. 29.4.1 and 29.4.3. Inserting eq. (29.4.16) into eq. (29.4.14), one obtains a $4N \times 4N$ matrix equation for the full propagator functions:

$$\begin{pmatrix} i\overline{\boldsymbol{D}} & ip\cdot\sigma\,\boldsymbol{C} \\ ip\cdot\overline{\sigma}\,\boldsymbol{C}^{\mathsf{T}} & i\boldsymbol{D} \end{pmatrix} = \begin{pmatrix} i(\boldsymbol{m}+\boldsymbol{\Omega}) & -ip\cdot\sigma\,(\boldsymbol{1}-\boldsymbol{\Xi}^{\mathsf{T}}) \\ -ip\cdot\overline{\sigma}\,(\boldsymbol{1}-\boldsymbol{\Xi}) & i(\overline{\boldsymbol{m}}+\overline{\boldsymbol{\Omega}}) \end{pmatrix}^{-1}, \tag{29.4.17}$$

where $\boldsymbol{1}$ is the $N \times N$ identity matrix. The right hand side of eq. (29.4.17) can be evaluated by employing the following identity for the inverse of a block-partitioned matrix [161]:

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}^{-1} = \begin{pmatrix} (P - QS^{-1}R)^{-1} & (R - SQ^{-1}P)^{-1} \\ (Q - PR^{-1}S)^{-1} & (S - RP^{-1}Q)^{-1} \end{pmatrix}, \tag{29.4.18}$$

under the assumption that all inverses appearing in eq. (29.4.18) exist. Applying this result to eq. (29.4.17), we obtain

$$\boldsymbol{C}^{-1} = s(\boldsymbol{1}-\boldsymbol{\Xi}) - (\overline{\boldsymbol{m}}+\overline{\boldsymbol{\Omega}})(\boldsymbol{1}-\boldsymbol{\Xi}^{\mathsf{T}})^{-1}(\boldsymbol{m}+\boldsymbol{\Omega}), \tag{29.4.19}$$

$$\boldsymbol{D}^{-1} = s(\boldsymbol{1}-\boldsymbol{\Xi})(\boldsymbol{m}+\boldsymbol{\Omega})^{-1}(\boldsymbol{1}-\boldsymbol{\Xi}^{\mathsf{T}}) - (\overline{\boldsymbol{m}}+\overline{\boldsymbol{\Omega}}), \tag{29.4.20}$$

$$\overline{\boldsymbol{D}}^{-1} = s(\boldsymbol{1}-\boldsymbol{\Xi}^{\mathsf{T}})(\overline{\boldsymbol{m}}+\overline{\boldsymbol{\Omega}})^{-1}(\boldsymbol{1}-\boldsymbol{\Xi}) - (\boldsymbol{m}+\boldsymbol{\Omega}). \tag{29.4.21}$$

Note that eq. (29.4.21) is consistent with eq. (29.4.20) as $\boldsymbol{\Xi}^\star = \boldsymbol{\Xi}^{\mathsf{T}}$.

The pole mass can be found most easily by considering the rest frame of the (off-shell) fermion, in which the space components of p^μ vanish. This reduces the spinor index dependence to a triviality. Setting $p^\mu = (\sqrt{s}\,;\,\boldsymbol{0})$, we search for values of s where the inverse of the full propagator has a zero eigenvalue. This is equivalent to setting the determinant of the inverse of the full propagator to zero. Here we shall use the well-known formula for the determinant of a block-partitioned matrix [161]:

$$\det \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \det P \, \det\,(S - RP^{-1}Q). \tag{29.4.22}$$

The end result is that the poles of the full propagator (which are in general complex),

$$s_{\text{pole},j} \equiv M_j^2 - i\Gamma_j M_j, \tag{29.4.23}$$

are formally the solutions to the non-linear equation[60]

$$\det\,\left[s\boldsymbol{1} - (\boldsymbol{1}-\boldsymbol{\Xi}^{\mathsf{T}})^{-1}(\boldsymbol{m}+\boldsymbol{\Omega})(\boldsymbol{1}-\boldsymbol{\Xi})^{-1}(\overline{\boldsymbol{m}}+\overline{\boldsymbol{\Omega}})\right] = 0. \tag{29.4.24}$$

[60]The determinant of the inverse of the full propagator [the inverse of eq. (29.4.17)] is equal to eq. (29.4.24) multiplied by $\det\,[-(\boldsymbol{1}-\boldsymbol{\Xi})(\boldsymbol{1}-\boldsymbol{\Xi}^{\mathsf{T}})]$. We assume that the latter does not vanish. This must be true perturbatively since the eigenvalues of $\boldsymbol{\Xi}$ are one-loop (or higher) quantities, which one assumes cannot be as large as 1.

Some care is required in using eq. (29.4.24), since the pole squared mass always has a *non-positive* imaginary part, while the loop integrals used to find the self-energy functions are complex functions of a real variable s that is given an infinitesimal *positive* imaginary part. Therefore, eq. (29.4.24) should be solved iteratively by first expanding the self-energy function matrices $\boldsymbol{\Xi}$, $\boldsymbol{\Omega}$ and $\overline{\boldsymbol{\Omega}}$ in a series in s about either $m_j^2 + i\epsilon$ or $M_j^2 + i\epsilon$. The complex quantities $s_{\text{pole},j}$, which can be identified as the complex pole squared masses, are renormalization group and gauge invariant physical observables. Examples are given in Sections 29.6.24 and 29.6.25.

The results of this section can be applied to an arbitrary collection of fermions (both Majorana or Dirac). However, it is convenient to treat separately the case where all fermions are Dirac fermions (consisting of pairs of two-component fields χ_i and η^i). As discussed in Section 29.3.2, the Dirac fermion mass eigenstates are defined in eq. (29.3.31) and are determined by the singular value decomposition of the Dirac fermion mass matrix. With respect to the mass basis, we denote the diagonal Dirac fermion mass matrix by $\boldsymbol{M}^{ij}$. The elements of this matrix are real and non-negative. Nevertheless, it will be convenient as before to define $\overline{\boldsymbol{M}}_{ij} \equiv \boldsymbol{M}^{ij}$ to maintain covariance when manipulating tensors with flavor indices.

At tree level, there are four propagators for each pair of χ and η fields as shown in Fig. 29.4.4. The corresponding full, loop-corrected propagators are shown in Fig. 29.4.4. The naming and sign

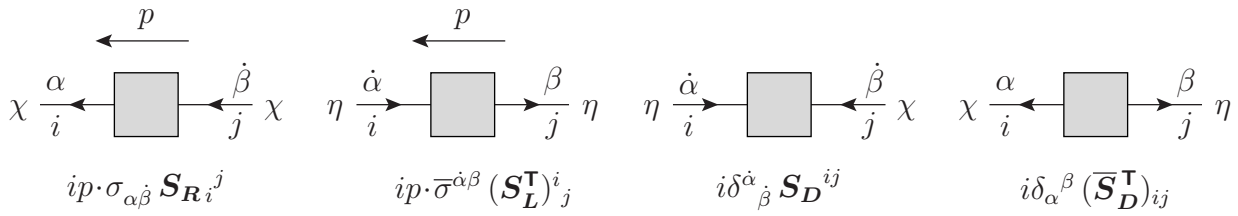


Figure 29.4.4: The full, loop-corrected propagators for Dirac fermions, represented by pairs of two-component (oppositely charged) fermion fields χ_i and η_i, are associated with functions $\boldsymbol{S_R}(s)_i{}^j$, $\boldsymbol{S_L^\mathsf{T}}(s)^i{}_j$, $\boldsymbol{S_D}(s)^{ij}$, and $\overline{\boldsymbol{S}}_{\boldsymbol{D}}^{\mathsf{T}}(s)_{ij}$, as shown. The shaded boxes represent the sum of all connected Feynman diagrams, with external legs included. The four-momentum p and the charge of χ flow from right to left.

conventions employed for the full, loop-corrected Dirac fermion propagator functions in Fig. 29.4.4 derives from the corresponding functions used in the more traditional four-component treatment presented in Appendix G [cf. eq. (G.7.2)].

In general, the complex matrices $\boldsymbol{S_R}$ and $\boldsymbol{S_L}$ satisfy hermiticity conditions $[\boldsymbol{S_R^\mathsf{T}}]^\star = \boldsymbol{S_R}$ and $[\boldsymbol{S_L^\mathsf{T}}]^\star = \boldsymbol{S_L}$, whereas the complex matrices $\boldsymbol{S_D}$ and $\overline{\boldsymbol{S}}_{\boldsymbol{D}}$ are related by $\overline{\boldsymbol{S}}_{\boldsymbol{D}} = \boldsymbol{S_D^\star}$, where the star symbol is defined in the paragraph below eq. (29.4.6). In contrast to the general case of an arbitrary collection of fermions treated earlier, $\boldsymbol{S_R}$ and $\boldsymbol{S_L}$ are unrelated and $\boldsymbol{S}_D$ is a complex matrix (not necessarily symmetric).

Instead of working in a χ–η basis for the two-component Dirac fermion fields, one can Takagi-diagonalize the fermion mass matrix. In the new ψ-basis, the loop-corrected propagators of Fig. 29.4.1 are applicable. It is easy to check that the number of independent functions is the same in both methods for treating Dirac fermions. In particular, the loop-corrected propagator functions in the ψ-basis are given in terms of the corresponding functions in the χ–η basis by:[61]

$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{S}_R & 0 \\ 0 & \boldsymbol{S}_L \end{pmatrix}, \qquad \boldsymbol{D} = \begin{pmatrix} 0 & \boldsymbol{S}_D^\mathsf{T} \\ \boldsymbol{S}_D & 0 \end{pmatrix}, \qquad \overline{\boldsymbol{D}} = \begin{pmatrix} 0 & \overline{\boldsymbol{S}}_{\boldsymbol{D}}^{\mathsf{T}} \\ \overline{\boldsymbol{S}}_{\boldsymbol{D}} & 0 \end{pmatrix}. \tag{29.4.25}$$

We similarly introduce the 1PI self-energy matrix functions for the Dirac fermions in the χ–η basis, where the corresponding self-energy functions are defined in Fig. 29.4.5. As before, the naming and sign conventions employed for the Dirac fermion self-energy functions above derives from

[61] The simple forms of $\boldsymbol{C}$ in eq. (29.4.25) and $\boldsymbol{\Xi}$ in eq. (29.4.26) motivate our definitions of $\boldsymbol{S_L}$ and $\boldsymbol{\Sigma_R}$ with the transpose as indicated in Figs. 29.4.4 and 29.4.5, respectively.

the corresponding functions used in the more traditional four-component treatment of Appendix G [cf. eq. (G.7.3)].

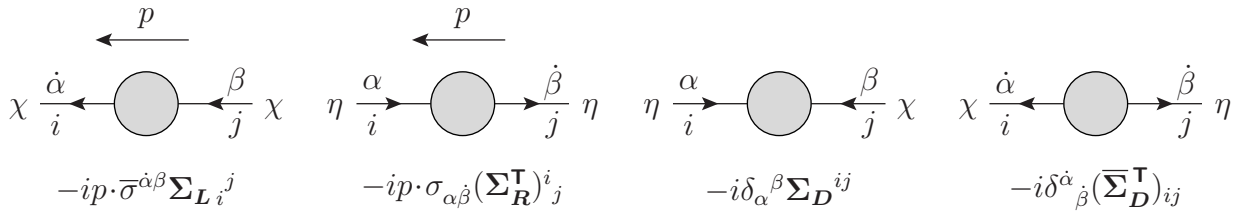


Figure 29.4.5: The self-energy functions for two-component Dirac fermions, represented by pairs of two-component (oppositely charged) fermion fields χ_i and η_i, are associated with functions $\boldsymbol{\Sigma_L}(s)_i{}^j$, $\boldsymbol{\Sigma_R^\mathsf{T}}(s)^i{}_j$, $\boldsymbol{\Sigma_D}(s)^{ij}$, and $\overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}}^{\mathsf{T}}(s)_{ij}$, as shown. The shaded circles represent the sum of all one-particle irreducible, connected Feynman diagrams, and the external legs are amputated. The four-momentum p flows from right to left.

Once again, the complex matrices $\boldsymbol{\Sigma_L}$ and $\boldsymbol{\Sigma_R}$ satisfy hermiticity conditions $[\boldsymbol{\Sigma_L^\mathsf{T}}]^\star = \boldsymbol{\Sigma_L}$ and $[\boldsymbol{\Sigma_R^\mathsf{T}}]^\star = \boldsymbol{\Sigma_R}$, whereas the complex matrices $\boldsymbol{\Sigma_D}$ and $\overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}}$ are related by $\overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}} = \boldsymbol{\Sigma_D^\star}$, where the star symbol is defined in the paragraph below eq. (29.4.6). Likewise, $\boldsymbol{\Sigma_L}$ and $\boldsymbol{\Sigma_R}$ are unrelated and $\boldsymbol{\Sigma_D}$ is a complex matrix (not necessarily symmetric). The self-energy functions in the ψ-basis are given in terms of the corresponding functions in the χ–η basis by:[61]

$$\boldsymbol{\Xi} = \begin{pmatrix} \boldsymbol{\Sigma}_L & 0 \\ 0 & \boldsymbol{\Sigma}_R \end{pmatrix}, \qquad \boldsymbol{\Omega} = \begin{pmatrix} 0 & \boldsymbol{\Sigma}_D^\mathsf{T} \\ \boldsymbol{\Sigma}_D & 0 \end{pmatrix}, \qquad \overline{\boldsymbol{\Omega}} = \begin{pmatrix} 0 & \overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}}^{\mathsf{T}} \\ \overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}} & 0 \end{pmatrix}. \tag{29.4.26}$$

In the case of Dirac fermions fields, eq. (29.4.14) still holds in the χ–η basis, which yields:

$$\begin{pmatrix} i\overline{\boldsymbol{S}}_{\boldsymbol{D}}^{\mathsf{T}} & ip\cdot\sigma\,\boldsymbol{S}_R \\ ip\cdot\overline{\sigma}\,\boldsymbol{S_L^\mathsf{T}} & i\boldsymbol{S}_D \end{pmatrix} = \begin{pmatrix} i(\boldsymbol{M}+\boldsymbol{\Sigma}_D) & -ip\cdot\sigma\,(\mathbf{1}-\boldsymbol{\Sigma}_R^\mathsf{T}) \\ -ip\cdot\overline{\sigma}\,(\mathbf{1}-\boldsymbol{\Sigma}_L) & i(\overline{\boldsymbol{M}}+\overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}}^{\mathsf{T}}) \end{pmatrix}^{-1}. \tag{29.4.27}$$

Using eq. (29.4.18), it follows that:

$$\boldsymbol{S_L}^{-1} = s(\mathbf{1}-\boldsymbol{\Sigma_R}) - (\overline{\boldsymbol{M}}+\overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}})(\mathbf{1}-\boldsymbol{\Sigma_L^\mathsf{T}})^{-1}(\boldsymbol{M}+\boldsymbol{\Sigma_D^\mathsf{T}}), \tag{29.4.28}$$

$$\boldsymbol{S_R}^{-1} = s(\mathbf{1}-\boldsymbol{\Sigma_L}) - (\overline{\boldsymbol{M}}+\overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}}^{\mathsf{T}})(\mathbf{1}-\boldsymbol{\Sigma_R^\mathsf{T}})^{-1}(\boldsymbol{M}+\boldsymbol{\Sigma_D}), \tag{29.4.29}$$

$$\boldsymbol{S_D}^{-1} = s(\mathbf{1}-\boldsymbol{\Sigma_L})(\boldsymbol{M}+\boldsymbol{\Sigma_D})^{-1}(\mathbf{1}-\boldsymbol{\Sigma_R^\mathsf{T}}) - (\overline{\boldsymbol{M}}+\overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}}^{\mathsf{T}}), \tag{29.4.30}$$

$$\overline{\boldsymbol{S}}_{\boldsymbol{D}}{}^{-1} = s(\mathbf{1}-\boldsymbol{\Sigma_L^\mathsf{T}})(\overline{\boldsymbol{M}}+\overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}})^{-1}(\mathbf{1}-\boldsymbol{\Sigma_R}) - (\boldsymbol{M}+\boldsymbol{\Sigma_D^\mathsf{T}}). \tag{29.4.31}$$

Note that eq. (29.4.31) is consistent with eq. (29.4.30) as $\boldsymbol{\Sigma_{L,R}^\star} = \boldsymbol{\Sigma_{L,R}^\mathsf{T}}$.

The pole mass is now easily computed using the technique previously outlined. In particular, eq. (29.4.24) becomes:

$$\det\left[s\mathbf{1} - (\mathbf{1}-\boldsymbol{\Sigma}_R^\mathsf{T})^{-1}(\boldsymbol{M}+\boldsymbol{\Sigma}_D)(\mathbf{1}-\boldsymbol{\Sigma}_L)^{-1}(\overline{\boldsymbol{M}}+\overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}}^{\mathsf{T}})\right] = 0, \tag{29.4.32}$$

which determines the complex pole squared masses, s_{pole}, of the corresponding Dirac fermions. Again, the self-energy functions should be expanded in a series in s about a point with an infinitesimal positive imaginary part.

Finally, we examine the special case of a parity-conserving vectorlike theory of Dirac fermions (such as QED or QCD). In this case, the following relations hold among the loop-corrected propagator functions and self-energy functions, respectively:[62]

$$\boldsymbol{S_{R}}_{i}{}^{j} = (\boldsymbol{S_{L}^{\mathsf{T}}})^{i}{}_{j}\,, \qquad \boldsymbol{S_{D}}^{ij} = (\overline{\boldsymbol{S}}_{\boldsymbol{D}}^{\mathsf{T}})_{ij}\,, \tag{29.4.33}$$

$$\boldsymbol{\Sigma_{L}}_{i}{}^{j} = (\boldsymbol{\Sigma_{R}^{\mathsf{T}}})^{i}{}_{j}\,, \qquad \boldsymbol{\Sigma_{D}}^{ij} = (\overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}}^{\mathsf{T}})_{ij}\,. \tag{29.4.34}$$

By imposing eq. (29.4.34) on eqs. (29.4.28)–(29.4.31) and recalling that $\overline{\boldsymbol{M}}_{ij} = \boldsymbol{M}^{ij}$, it is straightforward to verify that eq. (29.4.33) is satisfied.

29.5 Conventions for fermion and antifermion names and fields

In this section, we establish conventions for labeling Feynman diagrams that contain two-component fermion fields of the Standard Model (SM) and its minimal supersymmetric extension (MSSM). In the case of Majorana fermions, there is a one-to-one correspondence between the particle names and the undaggered $(\frac{1}{2},0)$ [left-handed] fields. In contrast, for Dirac fermions there are always two distinct two-component fields that correspond to each particle name. For a quark or lepton generically denoted by f, we employ the two-component undaggered $(\frac{1}{2},0)$ [left-handed] fields f and $\bar{f}$ (where the bar is part of the field name and does *not* refer to complex conjugation of any kind). This is illustrated in Table 29.1, which lists the SM and MSSM fermion particle names together with the corresponding two-component fields. For each particle, we list the two-component field with the same quantum numbers, i.e., the field that contains the annihilation operator for that one-particle state (which creates the one-particle state when acting to the *left* on the vacuum $\langle 0|$).

There is an option of labeling fermion lines in Feynman diagrams by particle names or by field names; each choice has advantages and disadvantages.[63] In all of the examples that follow, we have chosen to eliminate the possibility of ambiguity as follows. We always label fermion lines with two-component fields (rather than particle names), and adopt the following conventions:

- In the Feynman rules for interaction vertices, the external lines are always labeled by the undaggered $(\frac{1}{2},0)$ [left-handed] field, regardless of whether the corresponding arrow is pointed in or out of the vertex. Two-component fermion lines with arrows pointing away from the vertex correspond to dotted indices, and two-component fermion lines with arrows pointing toward the vertex always correspond to undotted indices. This also applies to Feynman diagrams where the roles of the initial state and the final state are ambiguous (such as self-energy diagrams).
- Internal fermion lines in Feynman diagrams are also always labeled by the undaggered $(\frac{1}{2},0)$ [left-handed] field(s). Internal fermion lines containing a propagator with opposing arrows can carry two labels (e.g., see Fig. 29.4.7).
- Initial state external fermion lines (which always have physical three-momenta pointing into the vertex) in Feynman diagrams for complete processes are labeled by the corresponding undaggered $(\frac{1}{2},0)$ [left-handed] field if the arrow is into the vertex, and by the daggered $(0,\frac{1}{2})$ [right-handed] field if the arrow is away from the vertex.
- Final state external fermion lines (which always have physical three-momenta pointing out of the vertex) in Feynman diagrams for complete processes are labeled by the corresponding daggered $(0,\frac{1}{2})$ [right-handed] field if the arrow is into the vertex, and by the undaggered $(\frac{1}{2},0)$ [left-handed] field if the arrow is away from the vertex.

The rules for labeling external Dirac fermions are summarized in Fig. 29.1. These labeling conventions differ slightly from the ones employed in Section 29.4.5, where *all* internal and external

[62] These relations are derived using four-component spinor methods in Appendix G [cf. eqs. (G.7.10) and (G.7.11)].

[63] Unfortunately, the notation for fermion names can be ambiguous because some of the symbols used also appear as names for one of the two-component fermion fields. In practice, it should be clear from the context which set of names are being employed.

Table 29.1: Fermion and antifermion names and two-component fields in the Standard Model and the MSSM. In the listing of two-component fields, the first is an undaggered $(\frac{1}{2},0)$ [left-handed] field and the second is a daggered $(0,\frac{1}{2})$ [right-handed] field. The bars on the two-component (antifermion) fields are part of their names, and do not denote some form of complex conjugation. (In this table, neutrinos are considered to be exactly massless and the left-handed antineutrino $\bar{\nu}$ is absent from the spectrum.)

Fermion name	Two-component fields
ℓ^- (lepton)	$\ell\,,\ \bar{\ell}^\dagger$
ℓ^+ (antilepton)	$\bar{\ell}\,,\ell^\dagger$
ν (neutrino)	$\nu\,,\ -$
$\bar{\nu}$ (antineutrino)	$-\,,\ \nu^\dagger$
q (quark)	$q\,,\ \bar{q}^\dagger$
$\bar{q}$ (antiquark)	$\bar{q}\,,\ q^\dagger$
f (quark or lepton)	$f\,,\ \bar{f}^\dagger$
$\bar{f}$ (antiquark or antilepton)	$\bar{f}\,,\ f^\dagger$
$\widetilde{N}_i$ (neutralino)	$\chi_i^0\,,\ \chi_i^{0\dagger}$
$\widetilde{C}_i^+$ (chargino)	$\chi_i^+\,,\ \chi_i^{-\dagger}$
$\widetilde{C}_i^-$ (anti-chargino)	$\chi_i^-\,,\ \chi_i^{+\dagger}$
$\widetilde{g}$ (gluino)	$\widetilde{g}\,,\ \widetilde{g}^\dagger$

initial state and final state fermion lines were labeled by the corresponding *undaggered* $(\frac{1}{2},0)$ left-handed fields. In this latter convention, the conserved quantities (charges, lepton numbers, baryon numbers, etc.) of the labeled fields follow the direction of the arrow that adorns the corresponding fermion line in the diagram. In contrast, in the convention of Fig. 29.1, the field labels used for external fermion lines always correspond to the physical particle, and the corresponding conserved quantities of the labeled fields follow the direction of the particle three-momentum. As an example, for either initial or final states, the two-component fields e and $\bar{e}^\dagger$ both represent a negatively charged electron, conventionally denoted by e^-, whereas both $\bar{e}$ and $e^\dagger$ represent a positively charged positron, conventionally denoted by e^+ (cf. Table 29.1).

The application of our labeling conventions to processes involving Majorana fermions is completely straightforward. For example, the conventions for employing the neutralino states as external particles are summarized in Fig. 29.2.

As a simple example, consider Bhabha scattering ($e^-e^+ \to e^-e^+$) [162]. We require the two-component Feynman rules for the QED coupling of electrons and positrons to the photon, which are exhibited in Fig. 29.3. Consider the s-channel tree-level Feynman diagrams that contribute to the invariant amplitude for $e^-e^+ \to e^-e^+$. If we were to label the external fermion lines according to the corresponding particle names (which does *not* conform to the conventions introduced above), the result is shown in Fig. 29.4. One can find the identity of the external two-component fermion fields by carefully observing the direction of the arrow of each fermion line. For contrast, the same

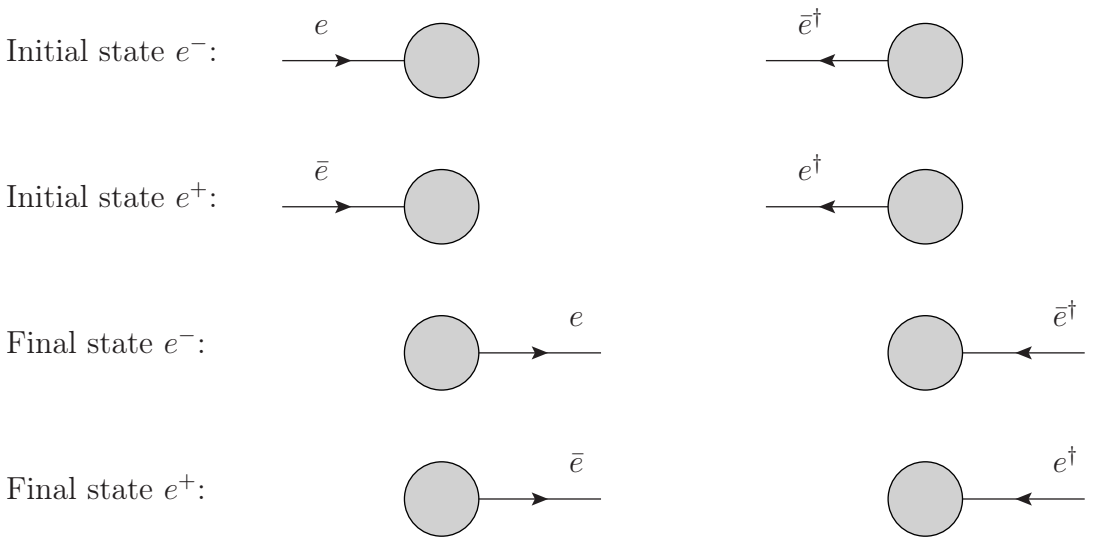


Figure 29.1: The two-component field labeling conventions for external Dirac fermion lines in a Feynman diagram for a physical process. The top row corresponds to an initial state electron, the second row to an initial state positron, the third row to a final state electron, and the fourth row to a final state positron. The labels above each line are the two-component field names. The corresponding conventions for a massless neutrino are obtained by deleting the diagrams with $\bar{e}$ or $\bar{e}^\dagger$, and changing e and $e^\dagger$ to ν and $\nu^\dagger$, respectively.

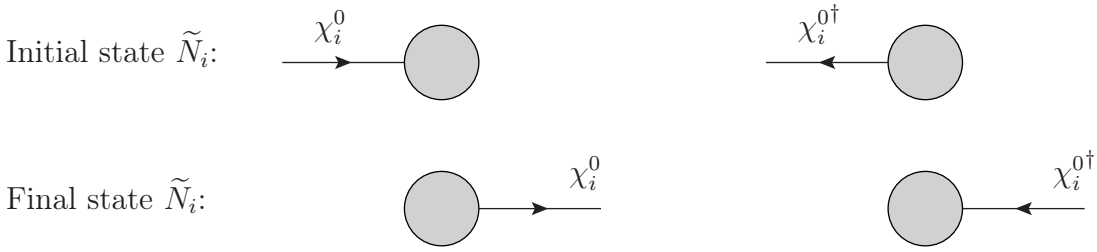


Figure 29.2: The two-component field labeling conventions for external Majorana fermion lines in a Feynman diagram for a physical process. The top row corresponds to an initial state neutralino, and the second row to a final state neutralino. The labels above each line are the two-component field names. (The neutralino is its own antiparticle.)

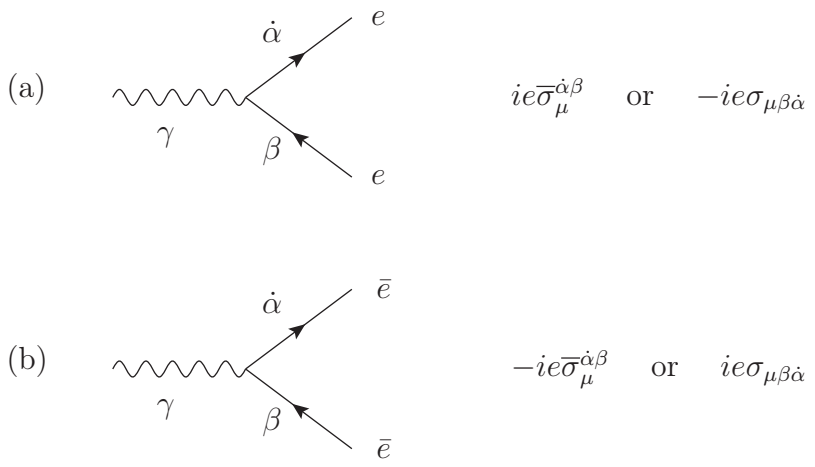


Figure 29.3: The two-component Feynman rules for the QED vertex. Following the conventions outlined in Section 29.5, we label these rules with the $(\frac{1}{2}, 0)$ [left-handed] fields e and $\bar{e}$, which comprise the Dirac electron. Note that $Q_e = -1$, and the electromagnetic coupling constant e (not to be confused with the two-component electron field that is denoted by the same letter) is conventionally defined such that $e > 0$ [cf. Fig. J.1.2].

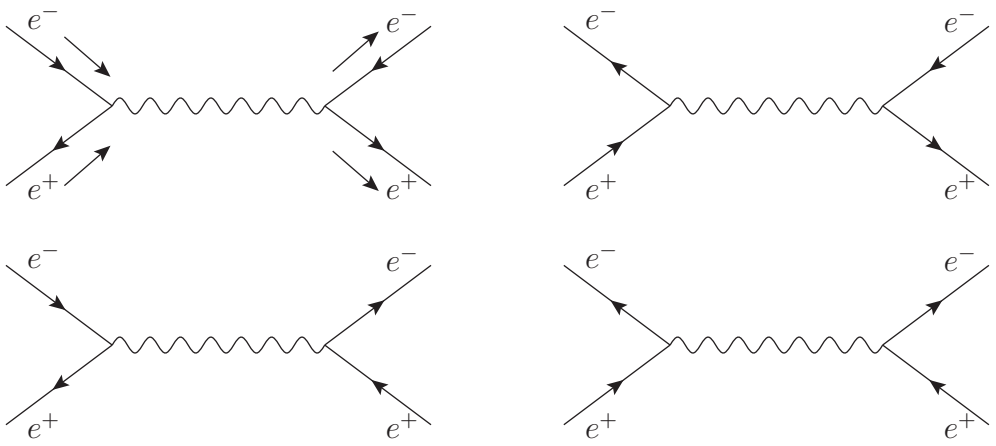


Figure 29.4: Tree-level s-channel Feynman diagrams for $e^-e^+ \to e^-e^+$, with the external lines labeled according to the particle names. The initial state is on the left, and the final state is on the right. Thus, the physical momentum flow of the external particles, as well as the flow of the labeled charges, are indicated by the arrows adjacent to the corresponding fermion lines in the upper left diagram.

diagrams, relabeled with two-component fields following the conventions established in this section (cf. Fig. 29.1), are shown in Fig. 29.5. An explicit computation of the invariant amplitude is given in Section 29.6.3.

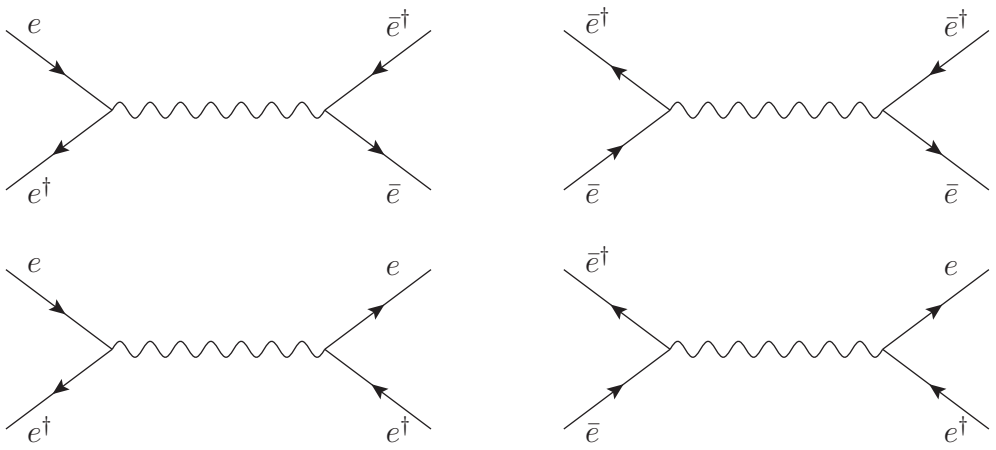


Figure 29.5: Tree-level s-channel Feynman diagrams for $e^+e^- \to e^+e^-$. These diagrams are the same as in Fig. 29.4, but with the external lines relabeled by the two-component fermion fields according to the conventions of Fig. 29.1.

29.6 Practical examples from the Standard Model and its SUSY extension

In this section we will present some examples to illustrate the use of the rules presented in this paper. These examples are chosen from the Standard Model [163] and its supersymmetric (SUSY) extension [6, 7, 8, 9, 10], in order to provide an unambiguous point of reference. In all cases, the fermion lines in Feynman diagrams are labeled by two-component field names, rather than the particle names, as explained in Section 29.5.

29.6.1 Top quark decay: $t \to bW^+$

We begin by calculating the decay width of a top quark into a bottom quark and W^+ vector boson. For simplicity, we treat this as a one-generation problem and ignore Cabibbo-Kobayashi-Maskawa (CKM) [164] mixing among the three quark generations [see eq. (J.1.16) and the surrounding text]. Let the four-momenta and helicities of these particle be (p_t, λ_t), (k_b, λ_b) and (k_W, λ_W), respectively. Then $p_t^2 = m_t^2$, $k_b^2 = m_b^2$ and $k_W^2 = m_W^2$ and

$$2p_t{\cdot}k_W = m_t^2 - m_b^2 + m_W^2\,, \tag{29.6.1}$$
$$2p_t{\cdot}k_b = m_t^2 + m_b^2 - m_W^2\,, \tag{29.6.2}$$
$$2k_W{\cdot}k_b = m_t^2 - m_b^2 - m_W^2\,. \tag{29.6.3}$$

Because only left-handed top quarks couple to the W boson, the only Feynman diagram for $t \to bW^+$ is the one shown in Fig. 29.6.1. The corresponding amplitude can be read off of the Feynman rule of Fig. J.1.2 in Appendix J . Here the initial state top quark is a two-component field t going into the vertex and the final state bottom quark is created by a two-component field $b^\dagger$. Therefore the amplitude is given by:

$$i\mathcal{M} = -i\frac{g}{\sqrt{2}}\varepsilon_\mu^* x_b^\dagger \overline{\sigma}^\mu x_t\,, \tag{29.6.4}$$

where $\varepsilon_\mu^* \equiv \varepsilon_\mu(k_W, \lambda_W)^*$ is the polarization vector of the W^+, and $x_b^\dagger \equiv x^\dagger(\vec{\boldsymbol{k}}_b, \lambda_b)$ and $x_t \equiv x(\vec{\boldsymbol{p}}_t, \lambda_t)$ are the external state wave function factors for the bottom and top quark. Squaring this amplitude using eq. (29.46) yields:

$$|\mathcal{M}|^2 = \frac{g^2}{2}\varepsilon_\mu^*\varepsilon_\nu (x_b^\dagger \overline{\sigma}^\mu x_t)\,(x_t^\dagger \overline{\sigma}^\nu x_b)\,. \tag{29.6.5}$$

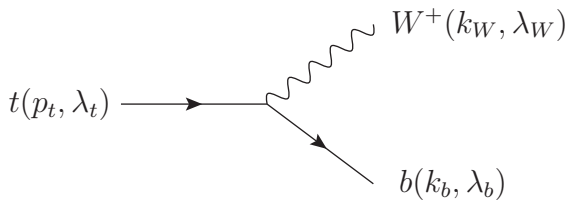


Figure 29.6.1: The Feynman diagram for $t \to bW^+$ at tree level.

Next, we can average over the top quark spin polarizations using eq. (29.3.57):

$$\frac{1}{2}\sum_{\lambda_t} |\mathcal{M}|^2 = \frac{g^2}{4}\varepsilon_\mu^* \varepsilon_\nu x_b^\dagger \overline{\sigma}^\mu \, p_t \cdot \sigma \, \overline{\sigma}^\nu x_b \, . \tag{29.6.6}$$

Summing over the bottom quark spin polarizations in the same way yields a trace over spinor indices:

$$\begin{aligned}\frac{1}{2}\sum_{\lambda_t,\lambda_b} |\mathcal{M}|^2 &= \frac{g^2}{4}\varepsilon_\mu^* \varepsilon_\nu \, \mathrm{Tr}[\overline{\sigma}^\mu p_t \cdot \sigma \, \overline{\sigma}^\nu k_b \cdot \sigma] \\ &= \frac{g^2}{2}\varepsilon_\mu^* \varepsilon_\nu \left(p_t^\mu k_b^\nu + k_b^\mu p_t^\nu - g^{\mu\nu} p_t \cdot k_b - i\epsilon^{\mu\rho\nu\kappa} p_{t\rho} k_{b\kappa}\right) ,\end{aligned} \tag{29.6.7}$$

where we have used eq. (29.58). Finally we can sum over the W^+ polarizations according to:

$$\sum_{\lambda_W} \varepsilon_\mu^* \varepsilon_\nu = -g_{\mu\nu} + (k_W)_\mu (k_W)_\nu / m_W^2 \, . \tag{29.6.8}$$

The end result is:

$$\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{g^2}{2}\left[p_t \cdot k_b + 2(p_t \cdot k_W)(k_b \cdot k_W)/m_W^2\right] . \tag{29.6.9}$$

After performing the phase space integration, one obtains:

$$\begin{aligned}\Gamma(t \to bW^+) &= \frac{1}{16\pi m_t^3}\lambda^{1/2}(m_t^2, m_W^2, m_b^2)\left(\frac{1}{2}\sum_{\text{spins}} |\mathcal{M}|^2\right) \\ &= \frac{g^2}{64\pi m_W^2 m_t^3}\lambda^{1/2}(m_t^2, m_W^2, m_b^2)\left[(m_t^2 + 2m_W^2)(m_t^2 - m_W^2) + m_b^2(m_W^2 - 2m_t^2) + m_b^4\right] ,\end{aligned} \tag{29.6.10}$$

where the kinematical triangle function $\lambda^{1/2}$ is defined by [165]:

$$\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2xy - 2xz - 2yz. \tag{29.6.11}$$

In the approximation $m_b \ll m_W, m_t$, one ends up with the well-known result [166]

$$\Gamma(t \to bW^+) = \frac{g^2 m_t}{64\pi}\left(2 + \frac{m_t^2}{m_W^2}\right)\left(1 - \frac{m_W^2}{m_t^2}\right)^2 , \tag{29.6.12}$$

which exhibits the Nambu-Goldstone enhancement factor (m_t^2/m_W^2) for the longitudinal W contribution compared to the two transverse W contributions [166].

29.6.2 Z^0 vector boson decay: $Z^0 \to f\bar{f}$

Consider the partial decay width of the Z^0 boson into a Standard Model fermion-antifermion pair. As in the generic example of Fig. 29.4.4, there are two contributing Feynman diagrams, shown in Fig. 29.6.1. In diagram (a), the fermion particle f in the final state is created by a two-component field f in the Feynman rule, and the antifermion particle $\bar{f}$ by a two-component field $f^\dagger$. In diagram (b), the fermion particle f in the final state is created by a two-component field $\bar{f}$, and the antifermion particle $\bar{f}$ by a two-component field $\bar{f}^\dagger$. Denote the initial Z^0 four-momentum and helicity $(p,\, \lambda_Z)$ and the final state fermion (f) and antifermion $(\bar{f})$ momentum and helicities (k_f, λ_f) and $(k_{\bar{f}}, \lambda_{\bar{f}})$, respectively. Then, $k_f^2 = k_{\bar{f}}^2 = m_f^2$ and $p^2 = m_Z^2$, and

$$k_f \cdot k_{\bar{f}} = \frac{1}{2} m_Z^2 - m_f^2 \,, \tag{29.6.1}$$

$$p \cdot k_f = p \cdot k_{\bar{f}} = \tfrac{1}{2} m_Z^2 \,. \tag{29.6.2}$$

According to the rules of Fig. J.1.2, the matrix elements for the two Feynman graphs are:

$$i\mathcal{M}_a \;=\; -i \frac{g}{c_W} (T_3^f - s_W^2 Q_f)\, \varepsilon_\mu x_f^\dagger \overline{\sigma}^\mu y_{\bar{f}} \,, \tag{29.6.3}$$

$$i\mathcal{M}_b \;=\; ig \frac{s_W^2}{c_W} Q_f\, \varepsilon_\mu y_f \sigma^\mu x_{\bar{f}}^\dagger \,, \tag{29.6.4}$$

where $x_i \equiv x(\vec{\boldsymbol{k}}_i, \lambda_i)$ and $y_i \equiv y(\vec{\boldsymbol{k}}_i, \lambda_i)$, for $i = f, \bar{f}$, and $\varepsilon_\mu \equiv \varepsilon_\mu(p, \lambda_Z)$.

Using the Bouchiat-Michel formulae developed in Appendix H.3 , one can explicitly evaluate $\mathcal{M}_a$ and $\mathcal{M}_b$ as a function of the final state fermion helicities. The result of this computation is given in eqs. (H.3.40) and (H.3.41). If the final state helicities are not measured, then it is simpler to square the amplitude and sum over the final state spins.

It is convenient to define:

$$a_f \equiv T_3^f - Q_f s_W^2 \,, \qquad\qquad b_f \equiv -Q_f s_W^2 \,. \tag{29.6.5}$$

Then the squared matrix element for the decay is, using eqs. (29.45) and (29.46),

$$|\mathcal{M}|^2 = \frac{g^2}{c_W^2} \varepsilon_\mu \varepsilon_\nu^* \left(a_f x_f^\dagger \overline{\sigma}^\mu y_{\bar{f}} + b_f y_f \sigma^\mu x_{\bar{f}}^\dagger \right) \left(a_f y_{\bar{f}}^\dagger \overline{\sigma}^\nu x_f + b_f x_{\bar{f}} \sigma^\nu y_f^\dagger \right) . \tag{29.6.6}$$

Summing over the antifermion helicity using eqs. (29.3.57)–(29.3.60) gives:

$$\sum_{\lambda_{\bar{f}}} |\mathcal{M}|^2 = \frac{g^2}{c_W^2} \varepsilon_\mu \varepsilon_\nu^* \Big(a_f^2 x_f^\dagger \overline{\sigma}^\mu k_{\bar{f}} \cdot \sigma \overline{\sigma}^\nu x_f + b_f^2 y_f \sigma^\mu k_{\bar{f}} \cdot \overline{\sigma} \sigma^\nu y_f^\dagger$$

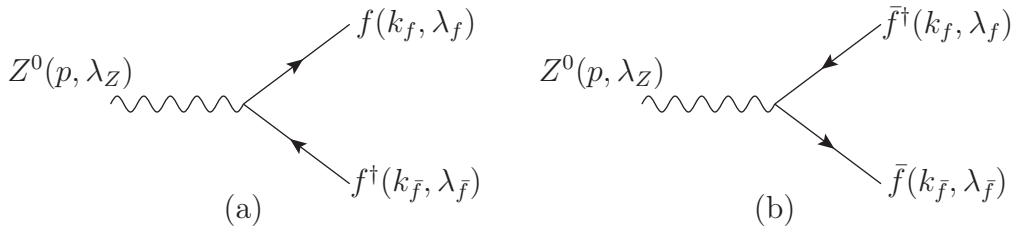


Figure 29.6.1: The Feynman diagrams for Z^0 decay into a fermion-antifermion pair. Fermion lines are labeled according to the two-component fermion field labeling convention established in Section 29.5.

$$-m_f a_f b_f x_f^\dagger \overline{\sigma}^\mu \sigma^\nu y_f^\dagger - m_f a_f b_f y_f \sigma^\mu \overline{\sigma}^\nu x_f \Big) \,. \tag{29.6.7}$$

Next, we sum over the fermion helicity:

$$\sum_{\lambda_f,\lambda_{\bar f}} |\mathcal{M}|^2 = \frac{g^2}{c_W^2} \varepsilon_\mu \varepsilon_\nu^* \Big(a_f^2 \mathrm{Tr}[\overline{\sigma}^\mu k_{\bar f}\cdot\sigma \overline{\sigma}^\nu k_f\cdot\sigma] + b_f^2 \mathrm{Tr}[\sigma^\mu k_{\bar f}\cdot\overline{\sigma} \sigma^\nu k_f\cdot\overline{\sigma}]$$
$$- m_f^2 a_f b_f \mathrm{Tr}[\overline{\sigma}^\mu \sigma^\nu] - m_f^2 a_f b_f \mathrm{Tr}[\sigma^\mu \overline{\sigma}^\nu] \Big) \,. \tag{29.6.8}$$

Averaging over the Z^0 polarization using

$$\frac{1}{3} \sum_{\lambda_Z} \varepsilon_\mu \varepsilon_\nu^* = \frac{1}{3} \left(-g_{\mu\nu} + \frac{p_\mu p_\nu}{m_Z^2} \right) \,, \tag{29.6.9}$$

and applying eqs. (29.56)–(29.58), one gets:

$$\frac{1}{3} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{g^2}{3c_W^2} \left[(a_f^2 + b_f^2) \left(2k_f\cdot k_{\bar f} + 4\, k_f\cdot p\, k_{\bar f}\cdot p/m_Z^2 \right) + 12 a_f b_f m_f^2 \right]$$
$$= \frac{2g^2}{3c_W^2} \left[(a_f^2 + b_f^2)(m_Z^2 - m_f^2) + 6 a_f b_f m_f^2 \right] \,, \tag{29.6.10}$$

where we have used eqs. (29.6.1) and (29.6.2). After the standard phase space integration, we obtain the well-known result for the partial width of the Z^0:

$$\Gamma(Z^0 \to f\bar f) = \frac{N_c^f}{16\pi m_Z} \left(1 - \frac{4m_f^2}{m_Z^2} \right)^{1/2} \left(\frac{1}{3} \sum_{\text{spins}} |\mathcal{M}|^2 \right)$$
$$= \frac{N_c^f g^2 m_Z}{24\pi c_W^2} \left(1 - \frac{4m_f^2}{m_Z^2} \right)^{1/2} \left[(a_f^2 + b_f^2) \left(1 - \frac{m_f^2}{m_Z^2} \right) + 6 a_f b_f \frac{m_f^2}{m_Z^2} \right] \,. \tag{29.6.11}$$

Here we have also included a factor of N_c^f (equal to 1 for leptons and 3 for quarks) for the sum over colors. Since the Z^0 is a color singlet, the color factor is simply equal to the dimension of the color representation of the outgoing fermions.

29.6.3 Bhabha scattering: $e^-e^+ \to e^-e^+$

In our next example, we consider the computation of Bhabha scattering in QED (that is, we consider photon exchange but neglect Z^0-exchange) [162]. Bhabha scattering has also been computed using two-component spinors in [105]. We denote the initial state electron and positron momenta and helicities by (p_1, λ_1) and (p_2, λ_2) and the final state electron and positron momenta and helicities by (p_3, λ_3) and (p_4, λ_4), respectively. Neglecting the electron mass, we have in terms of the usual Mandelstam variables s, t, u:

$$p_1\cdot p_2 = p_3\cdot p_4 \equiv \tfrac{1}{2} s \,, \tag{29.6.1}$$
$$p_1\cdot p_3 = p_2\cdot p_4 \equiv -\tfrac{1}{2} t \,, \tag{29.6.2}$$
$$p_1\cdot p_4 = p_2\cdot p_3 \equiv -\tfrac{1}{2} u \,, \tag{29.6.3}$$

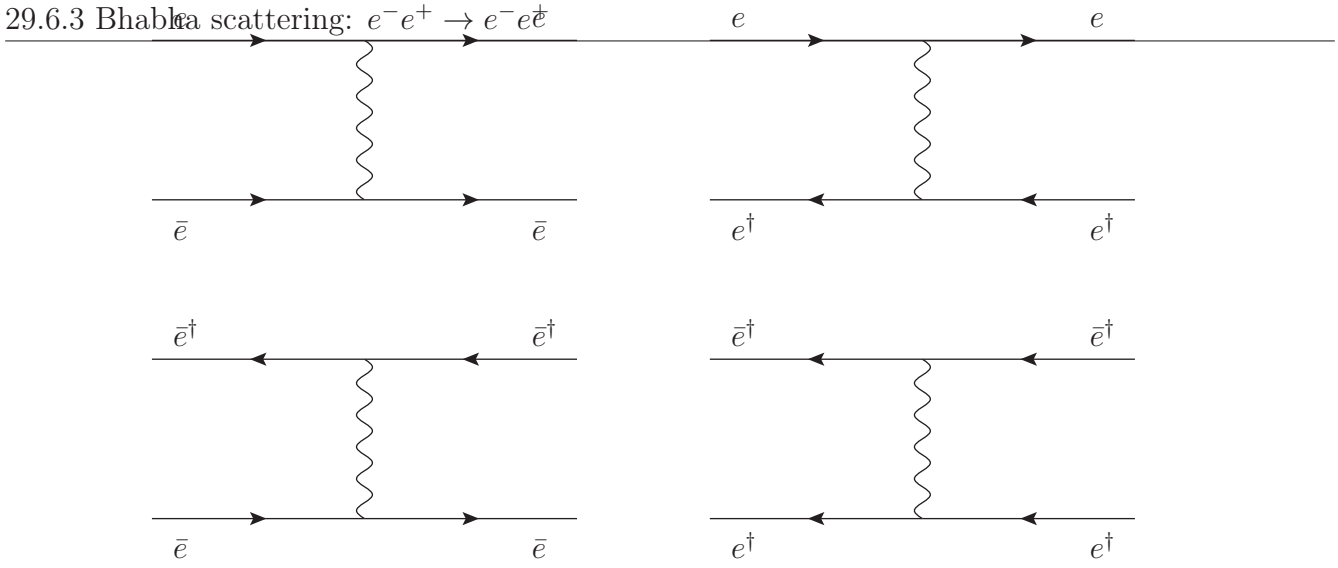


Figure 29.6.1: Tree-level t-channel Feynman diagrams for $e^-e^+ \to e^-e^+$, with the external lines labeled according to the two-component field names. The momentum flow of the external particles is from left to right.

and $p_i^2 = 0$ for $i = 1, ..., 4$. There are eight distinct Feynman diagrams. First, there are four s-channel diagrams, as shown in Fig. 29.5 with amplitudes that follow from the Feynman rules of Fig. 29.3 (more generally, see Fig. J.1.2 in Appendix J):

$$i\mathcal{M}_s = \left(\frac{-ig^{\mu\nu}}{s}\right)\Big[(-ie\,x_1\sigma_\mu y_2^\dagger)(ie\,y_3\sigma_\nu x_4^\dagger) + (-ie\,y_1^\dagger\overline{\sigma}_\mu x_2)(ie\,y_3\sigma_\nu x_4^\dagger) \\ +(-ie\,x_1\sigma_\mu y_2^\dagger)(ie\,x_3^\dagger\overline{\sigma}_\nu y_4) + (-ie\,y_1^\dagger\overline{\sigma}_\mu x_2)(ie\,x_3^\dagger\overline{\sigma}_\nu y_4)\Big]\,, \tag{29.6.4}$$

where $x_i \equiv x(\vec{\boldsymbol{p}}_{\boldsymbol{i}}, \lambda_i)$ and $y_i \equiv y(\vec{\boldsymbol{p}}_{\boldsymbol{i}}, \lambda_i)$, for $i = 1, 4$. The photon propagator in Feynman gauge is $-ig^{\mu\nu}/(p_1 + p_2)^2 = -ig^{\mu\nu}/s$. Here, we have chosen to write the external fermion spinors in the order $1, 2, 3, 4$. This dictates in each term the use of either the $\overline{\sigma}$ or σ forms of the Feynman rules of Fig. 29.3. One can group the terms of eq. (29.6.4) together more compactly:

$$i\mathcal{M}_s = e^2\left(\frac{-ig^{\mu\nu}}{s}\right)\left(x_1\sigma_\mu y_2^\dagger + y_1^\dagger\overline{\sigma}_\mu x_2\right)\left(y_3\sigma_\nu x_4^\dagger + x_3^\dagger\overline{\sigma}_\nu y_4\right). \tag{29.6.5}$$

There are also four t-channel diagrams, as shown in Fig. 29.6.1. The corresponding amplitudes for these four diagrams can be written:

$$i\mathcal{M}_t = (-1)e^2\left(\frac{-ig^{\mu\nu}}{t}\right)\left(x_1\sigma_\mu x_3^\dagger + y_1^\dagger\overline{\sigma}_\mu y_3\right)\left(x_2\sigma_\nu x_4^\dagger + y_2^\dagger\overline{\sigma}_\nu y_4\right). \tag{29.6.6}$$

Here, the overall factor of (-1) comes from Fermi-Dirac statistics, since the external fermion wave functions are written in an odd permutation $(1, 3, 2, 4)$ of the original order $(1, 2, 3, 4)$ established by the first term in eq. (29.6.4).

Fierzing each term using eqs. (29.68)–(29.70), and using eqs. (29.60) and (29.61), the total amplitude can be written as:

$$\mathcal{M} = \mathcal{M}_s + \mathcal{M}_t = 2e^2\Big[\frac{1}{s}(x_1y_3)(y_2^\dagger x_4^\dagger) + \frac{1}{s}(y_1^\dagger x_3^\dagger)(x_2y_4) + \left(\frac{1}{s}+\frac{1}{t}\right)(y_1^\dagger x_4^\dagger)(x_2y_3) \\ + \left(\frac{1}{s}+\frac{1}{t}\right)(x_1y_4)(y_2^\dagger x_3^\dagger) - \frac{1}{t}(x_1x_2)(x_3^\dagger x_4^\dagger) - \frac{1}{t}(y_1^\dagger y_2^\dagger)(y_3y_4)\Big]. \tag{29.6.7}$$

Squaring this amplitude and summing over spins, all of the cross terms will vanish in the $m_e \to 0$ limit. This is because each cross term will have an x or an $x^\dagger$ for some electron or positron combined with a y or a $y^\dagger$ for the same particle, and the corresponding spin sum is proportional to m_e [see eqs. (29.3.59) and (29.3.60)]. Hence, summing over final state spins and averaging over initial state spins, the end result contains only the sum of the squares of the six terms in eq. (29.6.7):

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = e^4 \sum_{\lambda_1,\lambda_2,\lambda_3,\lambda_4} \Bigg\{ &\frac{1}{s^2}\Big[(x_1y_3)(y_3^\dagger x_1^\dagger)(y_2^\dagger x_4^\dagger)(x_4y_2) + (y_1^\dagger x_3^\dagger)(x_3y_1)(x_2y_4)(y_4^\dagger x_2^\dagger)\Big] \\ &+\left(\frac{1}{s}+\frac{1}{t}\right)^2\Big[(y_1^\dagger x_4^\dagger)(x_4y_1)(x_2y_3)(y_3^\dagger x_2^\dagger) + (x_1y_4)(y_4^\dagger x_1^\dagger)(y_2^\dagger x_3^\dagger)(x_3y_2)\Big] \\ &+\frac{1}{t^2}\Big[(x_1x_2)(x_2^\dagger x_1^\dagger)(x_3^\dagger x_4^\dagger)(x_4x_3) + (y_1^\dagger y_2^\dagger)(y_2y_1)(y_3y_4)(y_4^\dagger y_3^\dagger)\Big]\Bigg\}. \end{aligned} \tag{29.6.8}$$

Here we have used eq. (29.44) to get the complex square of the fermion bilinears. Performing these spin sums using eqs. (29.3.57) and (29.3.58) and using the trace identities eq. (B.2.5):

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= 8e^4\left[\frac{p_2\cdot p_4\, p_1\cdot p_3}{s^2} + \frac{p_1\cdot p_2\, p_3\cdot p_4}{t^2} + \left(\frac{1}{s}+\frac{1}{t}\right)^2 p_1\cdot p_4\, p_2\cdot p_3\right] \\ &= 2e^4\left[\frac{t^2}{s^2}+\frac{s^2}{t^2}+\left(\frac{u}{s}+\frac{u}{t}\right)^2\right]. \end{aligned} \tag{29.6.9}$$

Thus, the differential cross-section for Bhabha scattering is given by:

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\left(\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\right) = \frac{2\pi\alpha^2}{s^2}\left[\frac{t^2}{s^2}+\frac{s^2}{t^2}+\left(\frac{u}{s}+\frac{u}{t}\right)^2\right]. \tag{29.6.10}$$

This agrees with the result given in problem 5.2 of ref. [115].

29.6.4 Polarized muon decay

So far we have only treated cases where the initial state fermion spins are averaged and the final state spins are summed. In the case of the polarized decay of a particle or polarized scattering we must project out the appropriate polarization of the particles in the spin sums. This is achieved by replacing the spin sums given in eqs. (29.3.57)–(29.3.60) by the relevant polarized spin projections exhibited in eqs. (29.3.45)–(29.3.56). As an example, we consider the decay of a polarized muon. Polarized muon decay has also been computed using two-component spinors in [105], however with an effective four-fermion interaction.[64] The leading order Feynman diagram for muon decay is shown in Fig. 29.6.1 (and the relevant four-momenta are indicated).

In our computation, the mass of the muon is denoted by m_μ and the electron mass is neglected. The spin of the muon is measured in its rest frame with respect to a fixed z-axis. Assume that the muon at rest is polarized such that its spin component along the $\hat{\boldsymbol{z}}$-direction is $s = +\frac{1}{2}$. Then, the decay amplitude is given by[65]

$$i\mathcal{M} = \left(\frac{-ig}{\sqrt{2}}\right)^2 \left(x_{\nu_\mu}^\dagger \overline{\sigma}_\rho x_\mu\right)\left(x_e^\dagger \overline{\sigma}_\tau y_{\bar{\nu}_e}\right)\left(\frac{-ig^{\rho\tau}}{D_W}\right), \tag{29.6.1}$$

[64] In a related calculation given in [167], two component spinor techniques are applied to the computation of matrix element for $\nu + n \to p + e^-$ using an effective four-fermion $V-A$ interaction.

[65] Throughout this subsection μ and ν are particle labels. Hence, we employ ρ and τ as Lorentz vector indices.

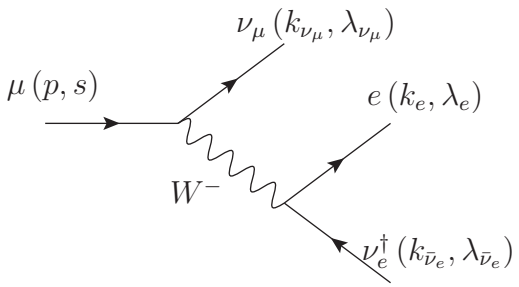


Figure 29.6.1: Feynman diagram for electroweak muon decay.

where $D_W = (p - k_{\nu_\mu})^2 - m_W^2$ is the denominator of the W-boson propagator. In eq. (29.6.1), $x_\mu \equiv x(\vec{\boldsymbol{p}}, s = \frac{1}{2})$ for the spin-polarized initial state muon, and $x^\dagger_{\nu_\mu} \equiv x(\vec{\boldsymbol{k}}_{\nu_\mu}, \lambda_{\nu_\mu})$, $x_e^\dagger \equiv x^\dagger(\vec{\boldsymbol{k}}_e, \lambda_e)$, and $y_{\bar\nu_e} \equiv y(\vec{\boldsymbol{k}}_{\bar\nu_e}, \lambda_{\bar\nu_e})$. Squaring the amplitude using eq. (29.46), we obtain

$$|\mathcal{M}|^2 = \frac{g^4}{4D_W^2}\left(x^\dagger_{\nu_\mu}\overline{\sigma}^\rho x_\mu\right)\left(x^\dagger_\mu\overline{\sigma}^\tau x_{\nu_\mu}\right)\left(x^\dagger_e\overline{\sigma}_\rho y_{\bar\nu_e}\right)\left(y^\dagger_{\bar\nu_e}\overline{\sigma}_\tau x_e\right). \tag{29.6.2}$$

Summing over the neutrino and electron spins using eqs. (29.3.57)–(29.3.58), and using eq. (29.3.45) for the muon spin (with $s = \frac{1}{2}$) yields:

$$\begin{aligned}\sum_{\lambda_{\nu_\mu}\lambda_e\lambda_{\bar\nu_e}} |\mathcal{M}|^2 &= \frac{g^4}{8D_W^2}\mathrm{Tr}[k_{\nu_\mu}\cdot\sigma\,\overline{\sigma}^\rho(p\cdot\sigma - m_\mu S\cdot\sigma)\,\overline{\sigma}^\tau]\,\mathrm{Tr}[k_e\cdot\sigma\,\overline{\sigma}_\rho k_{\bar\nu_e}\cdot\sigma\,\overline{\sigma}_\tau] \\ &= \frac{2g^4}{D_W^2}\,k_e\cdot k_{\nu_\mu}\,k_{\bar\nu_e}\cdot(p - m_\mu S)\,,\end{aligned} \tag{29.6.3}$$

where S^μ in an arbitrary frame is given by eq. (29.3.15) [with $\hat{\boldsymbol{s}} = \hat{\boldsymbol{z}}$]. To obtain the second line we have used the trace identity eq. (29.57) twice; note that the resulting terms linear in the antisymmetric tensor do not contribute, but the term quadratic in the antisymmetric tensor does.

The differential decay amplitude is now given by

$$d\Gamma = \frac{1}{2m_\mu}|\mathcal{M}|^2\frac{d^3\vec{\boldsymbol{k}}_e}{(2\pi)^3 2E_e}\frac{d^3\vec{\boldsymbol{k}}_{\bar\nu_e}}{(2\pi)^3 2E_{\bar\nu_e}}\frac{d^3\vec{\boldsymbol{k}}_{\nu_\mu}}{(2\pi)^3 2E_{\nu_\mu}}(2\pi)^4\delta^4(p - k_e - k_{\bar\nu_e} - k_{\nu_\mu})\,, \tag{29.6.4}$$

where E_i, $i = e, \bar\nu_e, \nu_\mu$ are the energies of the final state particles in the muon rest frame. In the following we shall neglect both the electron mass and the momentum in the W-propagator compared to the W-boson mass, so $D_W^2 \to m_W^4$. We can now use the following identity to integrate over the neutrino momenta [168]

$$\int \frac{d^3\vec{\boldsymbol{k}}_{\bar\nu_e}}{(2\pi)^3 2E_{\bar\nu_e}}\frac{d^3\vec{\boldsymbol{k}}_{\nu_\mu}}{(2\pi)^3 2E_{\nu_\mu}}(2\pi)^4\delta^4(q - k_{\bar\nu_e} - k_{\nu_\mu})k^\rho_{\bar\nu_e}k^\tau_{\nu_\mu} = \frac{1}{96\pi}(q^2 g^{\rho\tau} + 2q^\rho q^\tau)\,, \tag{29.6.5}$$

where $q = p - k_e$. It follows that

$$d\Gamma = \frac{g^4}{1536\pi^4 m_\mu m_W^4}\left[q^2\,k_e\cdot(p - m_\mu S) \;+\; 2q\cdot k_e\,q\cdot(p - m_\mu S)\right]\frac{d^3\vec{\boldsymbol{k}}_e}{E_e}\,. \tag{29.6.6}$$

In the muon rest frame, $k_e = E_e(1; \cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta)$ and $S = (0; 0, 0, 1)$, so that $q^2 = m_\mu^2 - 2E_e m_\mu$ and $k_e\cdot(p - m_\mu S) = m_\mu E_e(1 + \cos\theta)$ and $q\cdot k_e = m_\mu E_e$ and $q\cdot(p - m_\mu S) = m_\mu(m_\mu -$

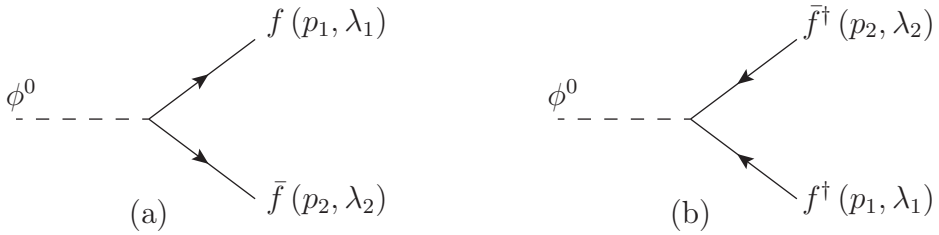


Figure 29.6.1: The Feynman diagrams for the decays $\phi^0 \to f\bar{f}$, where $\phi^0 = h^0, H^0, A^0$ are the neutral Higgs scalar bosons of the MSSM, and f is a Standard Model quark or lepton, and $\bar{f}$ is the corresponding antiparticle. We have labeled the external fermions according to the two-component field names.

$E_e - E_e\cos\theta$). Noting that the maximum energy of the electron is $m_\mu/2$ (when the neutrino and antineutrino both recoil in the opposite direction), we obtain

$$\frac{d\Gamma}{d(\cos\theta)} = \frac{g^4 m_\mu^2}{768\pi^3 m_W^4}\int_0^{m_\mu/2} dE_e E_e^2\left[3 - \frac{4E_e}{m_\mu} + \left(1 - \frac{4E_e}{m_\mu}\right)\cos\theta\right]$$
$$= \frac{g^4 m_\mu^5}{3\cdot 2^{12}\pi^3 m_W^4}\left(1 - \tfrac{1}{3}\cos\theta\right)\,, \tag{29.6.7}$$

in agreement with ref. [168]. Introducing the Fermi constant, $G_F \equiv \sqrt{2}g^2/(8m_W^2)$, we can rewrite eq. (29.6.7) as:

$$\frac{d\Gamma}{d(\cos\theta)} = \frac{G_F^2 m_\mu^5}{384\pi^3}\left(1 - \tfrac{1}{3}\cos\theta\right)\,. \tag{29.6.8}$$

Integrating over $\cos\theta$ reproduces the well-known total muon decay width,

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}\,. \tag{29.6.9}$$

29.6.5 Neutral MSSM Higgs boson decays: $\phi^0 \to f\bar{f}$ for $\phi^0 = h^0, H^0, A^0$

In this subsection, we consider the decays of the neutral Higgs scalar bosons $\phi^0 = h^0$, H^0, and A^0 of the MSSM into Standard Model fermion-antifermion pairs. The relevant tree-level Feynman diagrams are shown in Fig. 29.6.1. The final state fermion is assigned four-momentum p_1 and polarization λ_1, and the antifermion is assigned four-momentum p_2 and polarization λ_2. We will first work out the case that f is a charge $-1/3$ quark or a charged lepton, and later note the simple change needed for charge $+2/3$ quarks. The Feynman rules of Fig. K.1.1 of Appendix K tell us that the amplitudes are:

$$i\mathcal{M}_a = -\frac{i}{\sqrt{2}}\,Y_f\,k^*_{d\phi^0}\,x_1^\dagger x_2^\dagger\,, \tag{29.6.1}$$

$$i\mathcal{M}_b = -\frac{i}{\sqrt{2}}\,Y_f\,k_{d\phi^0}\,y_1 y_2\,. \tag{29.6.2}$$

Here Y_f is the Yukawa coupling of the fermion, $k_{d\phi^0}$ is the Higgs mixing parameter from eq. (K.1.8), and the external wave functions are denoted $x_1 \equiv x(\vec{\boldsymbol{p}}_1, \lambda_1)$, $y_1 \equiv y(\vec{\boldsymbol{p}}_1, \lambda_1)$ for the fermion and

$x_2 \equiv x(\vec{\boldsymbol{p}}_2, \lambda_2)$, $y_2 \equiv y(\vec{\boldsymbol{p}}_2, \lambda_2)$ for the antifermion. Squaring the total amplitude $i\mathcal{M} = i\mathcal{M}_a + i\mathcal{M}_b$ using eq. (29.44) results in:

$$|\mathcal{M}|^2 = \frac{1}{2}|Y_f|^2 \left[|k_{d\phi^0}|^2 (y_1 y_2\, y_2^\dagger y_1^\dagger + x_1^\dagger x_2^\dagger\, x_2 x_1) + (k^*_{d\phi^0})^2 x_1^\dagger x_2^\dagger\, y_2^\dagger y_1^\dagger + (k_{d\phi^0})^2 y_1 y_2\, x_2 x_1 \right]. \tag{29.6.3}$$

Summing over the final state antifermion spin using eqs. (29.3.57)–(29.3.60) gives:

$$\sum_{\lambda_2} |\mathcal{M}|^2 = \frac{1}{2}|Y_f|^2 \left[|k_{d\phi^0}|^2 (y_1 p_2\cdot\sigma y_1^\dagger + x_1^\dagger p_2\cdot\overline{\sigma} x_1) - (k^*_{d\phi^0})^2 m_f x_1^\dagger y_1^\dagger - (k_{d\phi^0})^2 m_f y_1 x_1 \right]. \tag{29.6.4}$$

Summing over the fermion spins in the same way yields:

$$\begin{aligned} \sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 &= \frac{1}{2}|Y_f|^2 \left\{ |k_{d\phi^0}|^2 (\mathrm{Tr}[p_2\cdot\sigma p_1\cdot\overline{\sigma}] + \mathrm{Tr}[p_2\cdot\overline{\sigma} p_1\cdot\sigma]) - 2(k^*_{d\phi^0})^2 m_f^2 - 2(k_{d\phi^0})^2 m_f^2 \right\} \\ &= |Y_f|^2 \left\{ 2|k_{d\phi^0}|^2 p_1\cdot p_2 - 2\mathrm{Re}[(k_{d\phi^0})^2] m_f^2 \right\} \\ &= |Y_f|^2 \left\{ |k_{d\phi^0}|^2 (m_{\phi^0}^2 - 2m_f^2) - 2\mathrm{Re}[(k_{d\phi^0})^2] m_f^2 \right\}, \end{aligned} \tag{29.6.5}$$

where we have used the trace identity eq. (29.56) to obtain the second equality. The corresponding expression for charge +2/3 quarks can be obtained by simply replacing $k_{d\phi^0}$ with $k_{u\phi^0}$. The total decay rates now follow from integration over phase space [169]

$$\Gamma(\phi^0 \to f\bar{f}) = \frac{N_c^f}{16\pi m_{\phi^0}} \left(1 - 4m_f^2/m_{\phi^0}^2\right)^{1/2} \sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2. \tag{29.6.6}$$

The factor of $N_c^f = 3$ for quarks and 1 for leptons comes from the sum over colors.

Results for special cases are obtained by putting in the relevant values for the couplings and the mixing parameters from eqs. (K.1.7) and (K.1.8). In particular, for the CP-even Higgs bosons h^0 and H^0, $k_{d\phi^0}$ and $k_{u\phi^0}$ are real, so one obtains:

$$\Gamma(h^0 \to b\bar{b}) = \frac{3}{16\pi} Y_b^2 \sin^2\!\alpha\, m_{h^0} \left(1 - 4m_b^2/m_{h^0}^2\right)^{3/2}, \tag{29.6.7}$$

$$\Gamma(h^0 \to c\bar{c}) = \frac{3}{16\pi} Y_c^2 \cos^2\!\alpha\, m_{h^0} \left(1 - 4m_c^2/m_{h^0}^2\right)^{3/2}, \tag{29.6.8}$$

$$\Gamma(h^0 \to \tau^+\tau^-) = \frac{1}{16\pi} Y_\tau^2 \sin^2\!\alpha\, m_{h^0} \left(1 - 4m_\tau^2/m_{h^0}^2\right)^{3/2}, \tag{29.6.9}$$

$$\Gamma(H^0 \to t\bar{t}) = \frac{3}{16\pi} Y_t^2 \sin^2\!\alpha\, m_{H^0} \left(1 - 4m_t^2/m_{H^0}^2\right)^{3/2}, \tag{29.6.10}$$

$$\Gamma(H^0 \to b\bar{b}) = \frac{3}{16\pi} Y_b^2 \cos^2\alpha\, m_{H^0} \left(1 - 4m_b^2/m_{H^0}^2\right)^{3/2}, \tag{29.6.11}$$

etc., which check with the expressions in Appendix C of ref. [170]. For the CP-odd Higgs boson A^0, the mixing parameters $k_{uA^0} = i\cos\beta_0$ and $k_{dA^0} = i\sin\beta_0$ are purely imaginary, so

$$\Gamma(A^0 \to t\bar{t}) = \frac{3}{16\pi} Y_t^2 \cos^2\!\beta_0\, m_{A^0} \left(1 - 4m_t^2/m_{A^0}^2\right)^{1/2}, \tag{29.6.12}$$

$$\Gamma(A^0 \to b\bar{b}) = \frac{3}{16\pi} Y_b^2 \sin^2\!\beta_0\, m_{A^0} \left(1 - 4m_b^2/m_{A^0}^2\right)^{1/2}, \tag{29.6.13}$$

$$\Gamma(A^0 \to \tau^+\tau^-) = \frac{1}{16\pi} Y_\tau^2 \sin^2\!\beta_0\, m_{A^0} \left(1 - 4m_\tau^2/m_{A^0}^2\right)^{1/2}. \tag{29.6.14}$$

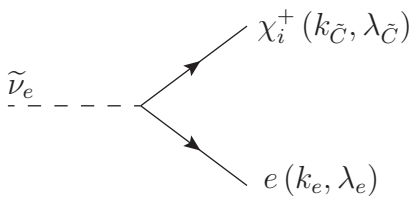


Figure 29.6.1: The Feynman diagram for $\widetilde{\nu}_e \to \widetilde{C}_i^+ e^-$ in the MSSM.

Note that the differing kinematic factors for the CP-odd Higgs decays came about because of the different relative sign between the two Feynman diagrams. For example, in the case of $h^0 \to b\bar{b}$, the matrix element is

$$i\mathcal{M} = \frac{i}{\sqrt{2}} Y_b \sin\alpha\, (y_1 y_2 + x_1^\dagger x_2^\dagger), \tag{29.6.15}$$

while for $A^0 \to b\bar{b}$, it is

$$i\mathcal{M} = \frac{1}{\sqrt{2}} Y_b \sin\beta_0\, (y_1 y_2 - x_1^\dagger x_2^\dagger). \tag{29.6.16}$$

The differing relative sign between $y_1 y_2$ and $x_1^\dagger x_2^\dagger$ follows from the imaginary pseudoscalar Lagrangian coupling, which is complex conjugated in the second diagram.

29.6.6 Sneutrino decay: $\widetilde{\nu}_e \to \widetilde{C}_i^+ e^-$

Next we consider the process of sneutrino decay $\widetilde{\nu}_e \to \widetilde{C}_i^+ e^-$ in the MSSM. Because only the left-handed electron can couple to the chargino and sneutrino (with the excellent approximation that the electron Yukawa coupling vanishes), there is just one Feynman diagram, shown in Fig. 29.6.1. The external wave functions of the electron and chargino are denoted as $x_e \equiv x(\vec{\boldsymbol{k}}_e, \lambda_e)$, and $x_{\tilde{C}} \equiv x(\vec{\boldsymbol{k}}_{\tilde{C}}, \lambda_{\tilde{C}})$, respectively. From the corresponding Feynman rule given in Fig. K.4.1 of Appendix K , the amplitude is:

$$i\mathcal{M} = -igV_{i1}\, x_{\tilde{C}}^\dagger x_e^\dagger, \tag{29.6.1}$$

where V_{ij} is one of the two matrices used to diagonalize the chargino masses [cf. eq. (K.2.6)]. Squaring this using eq. (29.44) yields:

$$|\mathcal{M}|^2 = g^2 |V_{i1}|^2\, (x_{\tilde{C}}^\dagger x_e^\dagger)(x_e x_{\tilde{C}})\,. \tag{29.6.2}$$

Summing over the electron and chargino spin polarizations using eq. (29.3.57) yields

$$\sum_{\lambda_e, \lambda_{\tilde{C}}} |\mathcal{M}|^2 = g^2 |V_{i1}|^2 \mathrm{Tr}[k_e \cdot \overline{\sigma}\, k_{\tilde{C}} \cdot \sigma] = 2g^2 |V_{i1}|^2\, k_e \cdot k_{\tilde{C}} = g^2 |V_{i1}|^2 (m_{\widetilde{\nu}_e}^2 - m_{\widetilde{C}_i}^2)\,, \tag{29.6.3}$$

where we have used $2k_e \cdot k_{\tilde{C}} = m_{\widetilde{\nu}_e}^2 - m_{\widetilde{C}_i}^2$, neglecting the electron mass. Therefore, after integrating over phase space in the standard way, the decay width is:

$$\Gamma(\widetilde{\nu}_e \to \widetilde{C}_i^+ e^-) = \frac{1}{16\pi m_{\widetilde{\nu}_e}} \left(1 - \frac{m_{\widetilde{C}_i}^2}{m_{\widetilde{\nu}_e}^2}\right) \left[\sum_{\lambda_e, \lambda_{\tilde{C}}} |\mathcal{M}|^2\right] = \frac{g^2}{16\pi} |V_{i1}|^2 m_{\widetilde{\nu}_e} \left(1 - \frac{m_{\widetilde{C}_i}^2}{m_{\widetilde{\nu}_e}^2}\right)^2, \tag{29.6.4}$$

which agrees with ref. [171] and eq. (3.8) in ref. [7].

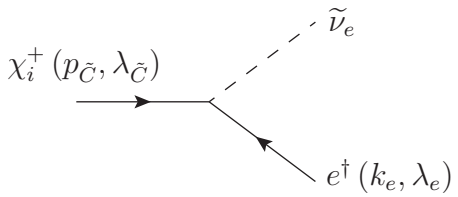


Figure 29.6.1: The Feynman diagram for $\widetilde{C}_i^+ \to \widetilde{\nu}_e e^+$ in the MSSM.

29.6.7 Chargino decay: $\widetilde{C}_i^+ \to \widetilde{\nu}_e e^+$

Here again, there is just one Feynman diagram (neglecting the electron mass in the Yukawa coupling) shown in Fig. 29.6.1. The external wave functions for the chargino and the positron are denoted by $x_{\tilde{C}} \equiv x(\vec{\boldsymbol{p}}_{\tilde{C}}, \lambda_{\tilde{C}})$ and $y_e \equiv y(\vec{\boldsymbol{k}}_e, \lambda_e)$, respectively. The fermion momenta and helicities are denoted as in Fig. 29.6.1. As in the previous example, the amplitude can be directly determined using the Feynman rule given in Fig. K.4.1 in Appendix K :

$$\mathcal{M} = -igV_{i1}^* \, x_{\tilde{C}} \, y_e \, . \tag{29.6.1}$$

Squaring this using eq. (29.44) yields:

$$|\mathcal{M}|^2 = g^2 |V_{i1}|^2 \, (x_{\tilde{C}} y_e) \, (y_e^\dagger x_{\tilde{C}}^\dagger) \, . \tag{29.6.2}$$

Summing over the electron helicity and averaging over the chargino helicity using eqs. (29.3.57) and (29.3.58) we obtain:

$$\tfrac{1}{2} \sum_{\lambda_e, \lambda_{\tilde{C}}} |\mathcal{M}|^2 = \tfrac{1}{2} g^2 |V_{i1}|^2 \mathrm{Tr}[k_e \cdot \sigma \, p_{\tilde{C}} \cdot \overline{\sigma}] = g^2 |V_{i1}|^2 k_e \cdot p_{\tilde{C}} = \frac{g^2}{2} |V_{i1}|^2 (m_{\tilde{C}_i}^2 - m_{\tilde{\nu}_e}^2) \, . \tag{29.6.3}$$

So the decay width is, neglecting the electron mass:

$$\Gamma(\widetilde{C}_i^+ \to \widetilde{\nu} e^+) = \frac{1}{16\pi m_{\tilde{C}_i}} \left(1 - \frac{m_{\tilde{\nu}_e}^2}{m_{\tilde{C}_i}^2}\right) \left(\frac{1}{2} \sum_{\lambda_e, \lambda_{\tilde{C}}} |\mathcal{M}|^2\right) = \frac{g^2}{32\pi} |V_{i1}|^2 m_{\tilde{C}_i} \left(1 - \frac{m_{\tilde{\nu}_e}^2}{m_{\tilde{C}_i}^2}\right)^2 , \tag{29.6.4}$$

which agrees with ref. [171].

29.6.8 Neutralino decays: $\widetilde{N}_i \to \phi^0 \widetilde{N}_j$ for $\phi^0 = h^0, H^0, A^0$

Next we consider the decay of a neutralino to a lighter neutralino and neutral Higgs boson $\phi^0 = h^0$, H^0, or A^0. The two tree-level Feynman graphs are shown in Fig. 29.6.1, where we have also labeled the momenta and helicities. We denote the masses for the neutralinos and the Higgs boson as $m_{\widetilde{N}_i}$, $m_{\widetilde{N}_j}$, and m_{ϕ^0}. Using the Feynman rules of Fig. K.3.1, the amplitudes are respectively given by

$$i\mathcal{M}_1 = -iY \, x_i y_j \, , \tag{29.6.1}$$

$$i\mathcal{M}_2 = -iY^* \, y_i^\dagger x_j^\dagger \, , \tag{29.6.2}$$

where the coupling $Y \equiv Y^{\phi^0 \chi_i^0 \chi_j^0}$ is defined in eq. (K.3.1), and the external wave functions are $x_i \equiv x(\vec{\boldsymbol{p}}_i, \lambda_i)$, $y_i^\dagger \equiv y^\dagger(\vec{\boldsymbol{p}}_i, \lambda_i)$, $y_j \equiv y(\vec{\boldsymbol{k}}_j, \lambda_j)$, and $x_j^\dagger \equiv x^\dagger(\vec{\boldsymbol{k}}_j, \lambda_j)$.

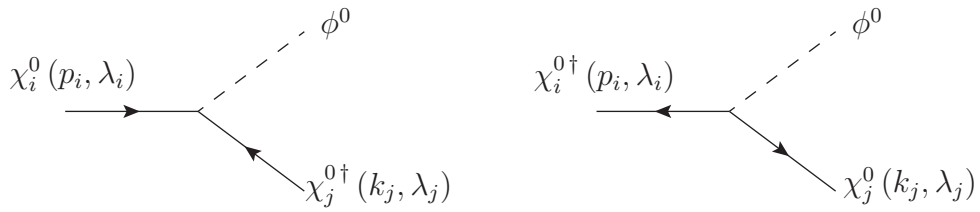


Figure 29.6.1: The Feynman diagrams for $\widetilde{N}_i \to \widetilde{N}_j \phi^0$ in the MSSM.

Taking the square of the total matrix element using eq. (29.44) gives:

$$|\mathcal{M}|^2 = |Y|^2(x_i y_j y_j^\dagger x_i^\dagger + y_i^\dagger x_j^\dagger x_j y_i) + Y^2 x_i y_j x_j y_i + Y^{*2} y_i^\dagger x_j^\dagger y_j^\dagger x_i^\dagger. \tag{29.6.3}$$

Summing over the final state neutralino spins using eqs. (29.3.57)–(29.3.60) yields

$$\sum_{\lambda_j} |\mathcal{M}|^2 = |Y|^2(x_i k_j\cdot\sigma x_i^\dagger + y_i^\dagger k_j\cdot\overline{\sigma} y_i) - Y^2 m_{\widetilde{N}_j} x_i y_i - Y^{*2} m_{\widetilde{N}_j} y_i^\dagger x_i^\dagger. \tag{29.6.4}$$

Averaging over the initial state neutralino spins in the same way gives

$$\begin{aligned}\frac{1}{2}\sum_{\lambda_i,\lambda_j} |\mathcal{M}|^2 &= \frac{1}{2}|Y|^2(\mathrm{Tr}[k_j\cdot\sigma p_i\cdot\overline{\sigma}] + \mathrm{Tr}[k_j\cdot\overline{\sigma} p_i\cdot\sigma]) + \mathrm{Re}[Y^2] m_{\widetilde{N}_i} m_{\widetilde{N}_j} \mathrm{Tr}[\mathbb{1}_{2\times 2}] \\ &= 2|Y|^2 p_i\cdot k_j + 2\mathrm{Re}[Y^2] m_{\widetilde{N}_i} m_{\widetilde{N}_j} \\ &= |Y|^2(m_{\widetilde{N}_i}^2 + m_{\widetilde{N}_j}^2 - m_{\phi^0}^2) + 2\mathrm{Re}[Y^2] m_{\widetilde{N}_i} m_{\widetilde{N}_j},\end{aligned} \tag{29.6.5}$$

where we have used eq. (29.56) to obtain the second equality. The total decay rate is therefore

$$\begin{aligned}\Gamma(\widetilde{N}_i \to \phi^0 \widetilde{N}_j) &= \frac{1}{16\pi m_{\widetilde{N}_i}^3} \lambda^{1/2}(m_{\widetilde{N}_i}^2, m_{\phi^0}^2, m_{\widetilde{N}_j}^2) \left(\frac{1}{2}\sum_{\lambda_i,\lambda_j} |\mathcal{M}|^2\right) \\ &= \frac{m_{\widetilde{N}_i}}{16\pi} \lambda^{1/2}(1, r_\phi, r_j) \left[|Y^{\phi^0 \chi_i^0 \chi_j^0}|^2 (1 + r_j - r_\phi) + 2\mathrm{Re}\left[\left(Y^{\phi^0 \chi_i^0 \chi_j^0}\right)^2\right]\sqrt{r_j}\right],\end{aligned} \tag{29.6.6}$$

where the triangle function $\lambda^{1/2}$ is defined in eq. (29.6.11), $r_j \equiv m_{\widetilde{N}_j}^2/m_{\widetilde{N}_i}^2$ and $r_\phi \equiv m_{\phi^0}^2/m_{\widetilde{N}_i}^2$. The results for $\phi^0 = h^0, H^0, A^0$ can now be obtained by using eqs. (K.1.7) and (K.1.8) in eq. (K.3.1). In comparing eq. (29.6.6) with the original calculation in [172], it is helpful to employ eqs. (4.51) and (4.53) of [173]. The results agree.

29.6.9 $\widetilde{N}_i \to Z^0 \widetilde{N}_j$

For this two-body decay there are two tree-level Feynman diagrams, shown in Fig. 29.6.1 with the definitions of the helicities and the momenta. Using the Feynman rules of Fig. K.2.1, the two amplitudes are given by[66]

$$i\mathcal{M}_1 = -i\frac{g}{c_W} \mathcal{O}''^L_{ji} x_i \sigma^\mu x_j^\dagger \varepsilon_\mu^*\,, \tag{29.6.1}$$

$$i\mathcal{M}_2 = i\frac{g}{c_W} \mathcal{O}''^L_{ij} y_i^\dagger \overline{\sigma}^\mu y_j \varepsilon_\mu^*\,, \tag{29.6.2}$$

[66] When comparing with the four-component Feynman rule in ref. [7] note that $\mathcal{O}''^L_{ij} = -\mathcal{O}''^{R*}_{ij}$ [cf. eq. (K.2.5)].

where the external wave functions are $x_i = x(\vec{\boldsymbol{p}}_i, \lambda_i)$, $y_i^\dagger = y^\dagger(\vec{\boldsymbol{p}}_i, \lambda_i)$, $x_j^\dagger = x^\dagger(\vec{\boldsymbol{k}}_j, \lambda_j)$, $y_j = y(\vec{\boldsymbol{k}}_j, \lambda_j)$, and $\varepsilon_\mu^* = \varepsilon_\mu(\vec{\boldsymbol{k}}_Z, \lambda_Z)^*$. Noting that $\mathcal{O}''^L_{ji} = \mathcal{O}''^{L*}_{ij}$ [see eq. (K.2.5)], and applying eqs. (29.45) and (29.46), we find that the squared matrix element is:

$$|\mathcal{M}|^2 = \frac{g^2}{c_W^2}\varepsilon_\mu^*\varepsilon_\nu\Big[|\mathcal{O}''^L_{ij}|^2(x_i\sigma^\mu x_j^\dagger x_j\sigma^\nu x_i^\dagger + y_i^\dagger\overline{\sigma}^\mu y_j y_j^\dagger\overline{\sigma}^\nu y_i) - \left(\mathcal{O}''^L_{ij}\right)^2 y_i^\dagger\overline{\sigma}^\mu y_j x_j\sigma^\nu x_i^\dagger - \left(\mathcal{O}''^{L*}_{ij}\right)^2 x_i\sigma^\mu x_j^\dagger y_j^\dagger\overline{\sigma}^\nu y_i\Big]. \tag{29.6.3}$$

Summing over the final state neutralino spin using eqs. (29.3.57)–(29.3.60) yields:

$$\sum_{\lambda_j}|\mathcal{M}|^2 = \frac{g^2}{c_W^2}\varepsilon_\mu^*\varepsilon_\nu\Big[|\mathcal{O}''^L_{ij}|^2(x_i\sigma^\mu k_j\cdot\overline{\sigma}\sigma^\nu x_i^\dagger + y_i^\dagger\overline{\sigma}^\mu k_j\cdot\sigma\overline{\sigma}^\nu y_i) + \left(\mathcal{O}''^L_{ij}\right)^2 m_{\widetilde{N}_j} y_i^\dagger\overline{\sigma}^\mu\sigma^\nu x_i^\dagger + \left(\mathcal{O}''^{L*}_{ij}\right)^2 m_{\widetilde{N}_j} x_i\sigma^\mu\overline{\sigma}^\nu y_i\Big]. \tag{29.6.4}$$

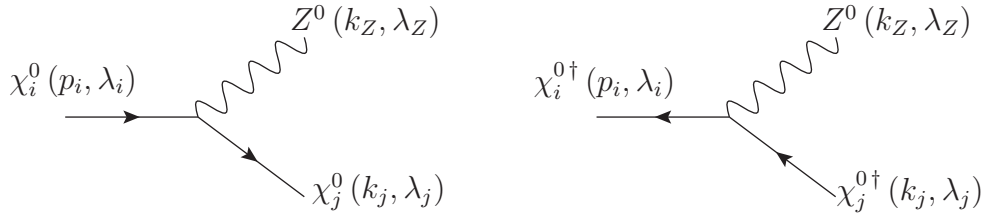


Figure 29.6.1: The Feynman diagrams for $\widetilde{N}_i \to \widetilde{N}_j Z^0$ in the MSSM.

Averaging over the initial state neutralino spins in the same way gives

$$\begin{aligned}\frac{1}{2}\sum_{\lambda_i,\lambda_j}|\mathcal{M}|^2 &= \frac{g^2}{2c_W^2}\varepsilon_\mu^*\varepsilon_\nu\Big[|\mathcal{O}''^L_{ij}|^2\Big(\mathrm{Tr}[\sigma^\mu k_j\cdot\overline{\sigma}\sigma^\nu p_i\cdot\overline{\sigma}] + \mathrm{Tr}[\overline{\sigma}^\mu k_j\cdot\sigma\overline{\sigma}^\nu p_i\cdot\sigma]\Big) \\ &\quad - \left(\mathcal{O}''^L_{ij}\right)^2 m_{\widetilde{N}_i}m_{\widetilde{N}_j}\mathrm{Tr}[\overline{\sigma}^\mu\sigma^\nu] - \left(\mathcal{O}''^{L*}_{ij}\right)^2 m_{\widetilde{N}_i}m_{\widetilde{N}_j}\mathrm{Tr}[\sigma^\mu\overline{\sigma}^\nu]\Big] \\ &= \frac{2g^2}{c_W^2}\varepsilon_\mu^*\varepsilon_\nu\Big\{|\mathcal{O}''^L_{ij}|^2\left(k_j^\mu p_i^\nu + p_i^\mu k_j^\nu - p_i\cdot k_j g^{\mu\nu}\right) - \mathrm{Re}\Big[\left(\mathcal{O}''^L_{ij}\right)^2\Big]m_{\widetilde{N}_i}m_{\widetilde{N}_j}g^{\mu\nu}\Big\},\end{aligned} \tag{29.6.5}$$

where in the last equality we have applied eqs. (29.56)–(29.58). Using

$$\sum_{\lambda_Z}\varepsilon^{\mu*}\varepsilon^\nu = -g^{\mu\nu} + k_Z^\mu k_Z^\nu/m_Z^2\,, \tag{29.6.6}$$

we obtain

$$\frac{1}{2}\sum_{\lambda_i,\lambda_j,\lambda_Z}|\mathcal{M}|^2 = \frac{2g^2}{c_W^2}\Big\{|\mathcal{O}''^L_{ij}|^2\left(p_i\cdot k_j + 2p_i\cdot k_Z k_j\cdot k_Z/m_Z^2\right) + 3m_{\widetilde{N}_i}m_{\widetilde{N}_j}\mathrm{Re}\big[\left(\mathcal{O}''^L_{ij}\right)^2\big]\Big\}. \tag{29.6.7}$$

Using $2k_j\cdot k_Z = m^2_{\widetilde{N}_i} - m^2_{\widetilde{N}_j} - m^2_Z$, $2p_i\cdot k_j = m^2_{\widetilde{N}_i} + m^2_{\widetilde{N}_j} - m^2_Z$, and $2p_i\cdot k_Z = m^2_{\widetilde{N}_i} - m^2_{\widetilde{N}_j} + m^2_Z$, we obtain the total decay width:

$$\begin{aligned}\Gamma(\widetilde{N}_i \to Z^0 \widetilde{N}_j) &= \frac{1}{16\pi m^3_{\widetilde{N}_i}} \lambda^{1/2}\big(m^2_{\widetilde{N}_i}, m^2_Z, m^2_{\widetilde{N}_j}\big) \left(\frac{1}{2}\sum_{\lambda_i,\lambda_j,\lambda_Z} |\mathcal{M}|^2\right) \\ &= \frac{g^2 m_{\widetilde{N}_i}}{16\pi c^2_W} \lambda^{1/2}(1, r_Z, r_j)\left[|\mathcal{O}''^L_{ij}|^2\left(1 + r_j - 2r_Z + (1-r_j)^2/r_Z\right) + 6\mathrm{Re}\big[\big(\mathcal{O}''^L_{ij}\big)^2\big]\sqrt{r_j}\right], \quad (29.6.8)\end{aligned}$$

where

$$r_j \equiv m^2_{\widetilde{N}_j}/m^2_{\widetilde{N}_i}\,, \qquad r_Z \equiv m^2_Z/m^2_{\widetilde{N}_i}\,, \quad (29.6.9)$$

and the triangle function $\lambda^{1/2}$ is defined in eq. (29.6.11). The result obtained in eq. (29.6.8) agrees with the original calculation in [172].

29.6.10 Selectron pair production in electron-electron collisions

$e^-e^- \to \widetilde{e}^-_L \widetilde{e}^-_R$

Here there are two Feynman graphs (neglecting the electron mass and Yukawa couplings), shown in Fig. 29.6.1. Note that these two graphs are related by interchange of the identical initial state electrons. Let the electrons have momenta p_1 and p_2 and the selectrons have momenta $k_{\widetilde{e}_L}$ and $k_{\widetilde{e}_R}$, so that $p_1^2 = p_2^2 = 0$; $k_1^2 = m^2_{\widetilde{e}_L}$; $k_2^2 = m^2_{\widetilde{e}_R}$; $s = (p_1+p_2)^2 = (k_1+k_2)^2$; $t = (k_1-p_1)^2 = (k_2-p_2)^2$; $u = (k_1-p_2)^2 = (k_2-p_1)^2$.

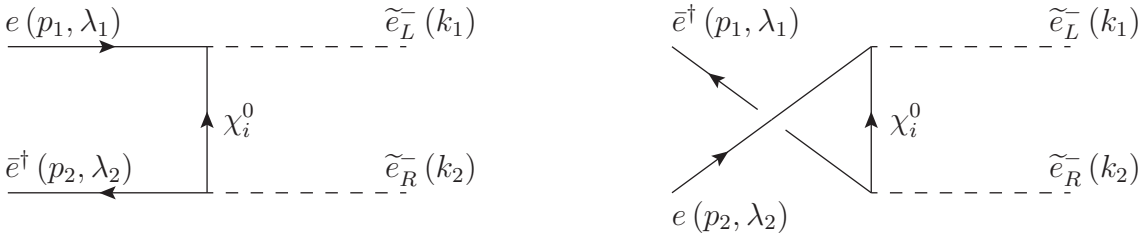


Figure 29.6.1: Feynman diagrams for $e^-e^- \to \widetilde{e}^-_L\widetilde{e}^-_R$.

Using the Feynman rules of Fig. K.4.2, the matrix element for the first graph, for each neutralino $\widetilde{N}_i$ exchanged in the t channel, is:

$$i\mathcal{M}_t = \left[i\frac{g}{\sqrt{2}}\left(N^*_{i2} + \frac{s_W}{c_W}N^*_{i1}\right)\right]\left[-i\sqrt{2}g\frac{s_W}{c_W}N_{i1}\right]\, x_1\left[\frac{i(k_1-p_1)\cdot\sigma}{(k_1-p_1)^2 - m^2_{\widetilde{N}_i}}\right] y_2^\dagger\,. \quad (29.6.1)$$

We employ the notation for the external wave functions $x_i = (\vec{\boldsymbol{p}}_i, \lambda_i)$, $i = 1, 2$ and analogously for $y_i, x_i^\dagger, y_i^\dagger$. The matrix element for the second (u-channel) graph is the same with the two incoming electrons exchanged, $e_1 \leftrightarrow e_2$:

$$i\mathcal{M}_u = (-1)\left[i\frac{g}{\sqrt{2}}\left(N^*_{i2} + \frac{s_W}{c_W}N^*_{i1}\right)\right]\left[-i\sqrt{2}g\frac{s_W}{c_W}N_{i1}\right]\, x_2\left[\frac{i(k_1-p_2)\cdot\sigma}{(k_1-p_2)^2 - m^2_{\widetilde{N}_i}}\right] y_1^\dagger\,. \quad (29.6.2)$$

Note that since we have written the fermion wave function spinors in the opposite order in $\mathcal{M}_2$ compared to $\mathcal{M}_1$, there is a factor (-1) for Fermi-Dirac statistics. Alternatively, starting at the electron with momentum p_1 and using the Feynman rules as above, we can directly write:

$$i\mathcal{M}_u = \left[i\frac{g}{\sqrt{2}}\left(N_{i2}^* + \frac{s_W}{c_W}N_{i1}^*\right)\right]\left[-i\sqrt{2}g\frac{s_W}{c_W}N_{i1}\right]\, y_1^\dagger\left[\frac{-i(k_1-p_2)\cdot\overline{\sigma}}{(k_1-p_2)^2 - m_{\tilde{N}_i}^2}\right] x_2\,. \tag{29.6.3}$$

This has no Fermi-Dirac factor (-1) because the wave function spinors are written in the same order as in $\mathcal{M}_t$. However, now the Feynman rule for the propagator has an extra minus sign, as can be seen in Fig. 29.4.2. We can also obtain eq. (29.6.3) from eq. (29.6.2) by using eq. (29.62). So we can write for the total amplitude:

$$\mathcal{M} = \mathcal{M}_t + \mathcal{M}_u = x_1 a\cdot\sigma y_2^\dagger + y_1^\dagger b\cdot\overline{\sigma} x_2\,, \tag{29.6.4}$$

where

$$a^\mu \equiv \frac{g^2 s_W}{c_W}(k_1^\mu - p_1^\mu)\sum_{i=1}^4 N_{i1}(N_{i2}^* + \frac{s_W}{c_W}N_{i1}^*)\frac{1}{t - m_{\tilde{N}_i}^2}\,, \tag{29.6.5}$$

$$b^\mu \equiv -\frac{g^2 s_W}{c_W}(k_1^\mu - p_2^\mu)\sum_{i=1}^4 N_{i1}(N_{i2}^* + \frac{s_W}{c_W}N_{i1}^*)\frac{1}{u - m_{\tilde{N}_i}^2}\,. \tag{29.6.6}$$

Hence, using eqs. (29.45) and (29.46):

$$\begin{aligned}|\mathcal{M}|^2 = &\left(x_1 a\cdot\sigma y_2^\dagger\right)\left(y_2 a^*\cdot\sigma x_1^\dagger\right) + \left(y_1^\dagger b\cdot\overline{\sigma} x_2\right)\left(x_2^\dagger b^*\cdot\overline{\sigma} y_1\right)\\ &+ \left(x_1 a\cdot\sigma y_2^\dagger\right)\left(x_2^\dagger b^*\cdot\overline{\sigma} y_1\right) + \left(y_1^\dagger b\cdot\overline{\sigma} x_2\right)\left(y_2 a^*\cdot\sigma x_1^\dagger\right)\,.\end{aligned} \tag{29.6.7}$$

Averaging over the initial state electron spins using eqs. (29.3.57)–(29.3.60), the a, b^* and a^*, b cross terms are proportional to m_e and can thus be neglected in our approximation. We get:

$$\frac{1}{4}\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2 = \frac{1}{4}\mathrm{Tr}[a\cdot\sigma\; p_2\cdot\overline{\sigma}\; a^*\cdot\sigma\; p_1\cdot\overline{\sigma}] + \frac{1}{4}\mathrm{Tr}[b\cdot\overline{\sigma}\; p_2\cdot\sigma\; b^*\cdot\overline{\sigma}\; p_1\cdot\sigma]\,. \tag{29.6.8}$$

These terms can be simplified using the identities:

$$\begin{aligned}\mathrm{Tr}[(k_1-p_1)\cdot\sigma\; p_2\cdot\overline{\sigma}\; (k_1-p_1)\cdot\sigma\; p_1\cdot\overline{\sigma}] &= \mathrm{Tr}[(k_1-p_2)\cdot\overline{\sigma}\; p_2\cdot\sigma\; (k_1-p_2)\cdot\overline{\sigma}\; p_1\cdot\sigma]\\ &= tu - m_{\tilde{e}_L}^2 m_{\tilde{e}_R}^2,\end{aligned} \tag{29.6.9}$$

which follow from eq. (29.57) and (29.58), resulting in:

$$\begin{aligned}\frac{1}{4}\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2 = \frac{g^4 s_W^2}{4c_W^2}(tu - m_{\tilde{e}_L}^2 m_{\tilde{e}_R}^2)\sum_{i,j=1}^4 N_{j1}N_{i1}^*(N_{j2}^* + \frac{s_W}{c_W}N_{j1}^*)(N_{i2} + \frac{s_W}{c_W}N_{i1})\\ \left[\frac{1}{(t - m_{\tilde{N}_i}^2)(t - m_{\tilde{N}_j}^2)} + \frac{1}{(u - m_{\tilde{N}_i}^2)(u - m_{\tilde{N}_j}^2)}\right]\,.\end{aligned} \tag{29.6.10}$$

To get the differential cross-section $d\sigma/dt$, multiply this by $1/(16\pi s^2)$:

$$\frac{d\sigma}{dt} = \frac{\pi\alpha^2}{4s_W^2 c_W^2}\left(\frac{tu - m_{\tilde{e}_L}^2 m_{\tilde{e}_R}^2}{s^2}\right)\sum_{i,j=1}^4 N_{j1}N_{i1}^*(N_{j2}^* + \frac{s_W}{c_W}N_{j1}^*)(N_{i2} + \frac{s_W}{c_W}N_{i1})$$

$$\left[\frac{1}{(t-m^2_{\widetilde{N}_i})(t-m^2_{\widetilde{N}_j})}+\frac{1}{(u-m^2_{\widetilde{N}_i})(u-m^2_{\widetilde{N}_j})}\right]. \qquad (29.6.11)$$

To compare with the original calculation in [174] and with eq. E26, p. 244 in ref. [7], note that for a pure photino exchange, $N_{i1}\to c_W\delta_{i1}$ and $N_{i2}\to s_W\delta_{i1}$, so that

$$\frac{1}{4s_W^2c_W^2}|N_{i1}|^2|N_{i2}+\frac{s_W}{c_W}N_{i1}|^2 \;\to\; 1\,. \qquad (29.6.12)$$

Also note that in [174] polarized electron beams are assumed. The result checks.

$e^-e^-\to\widetilde{e}^-_R\widetilde{e}^-_R$

For this process, there are again two Feynman graphs, which are related by the exchange of identical electrons in the initial state or equivalently by exchange of the identical selectrons in the final state, as shown in Fig. 29.6.2. (We again neglect the electron mass and thus the higgsino coupling to the electron.) Let the electrons have momenta p_1 and p_2 and the selectrons have momenta k_1 and k_2, so that $p_1^2=p_2^2=0$; $k_1^2=k_2^2=m^2_{\widetilde{e}_R}$; $s=(p_1+p_2)^2$; $t=(k_1-p_1)^2$; $u=(k_1-p_2)^2$.

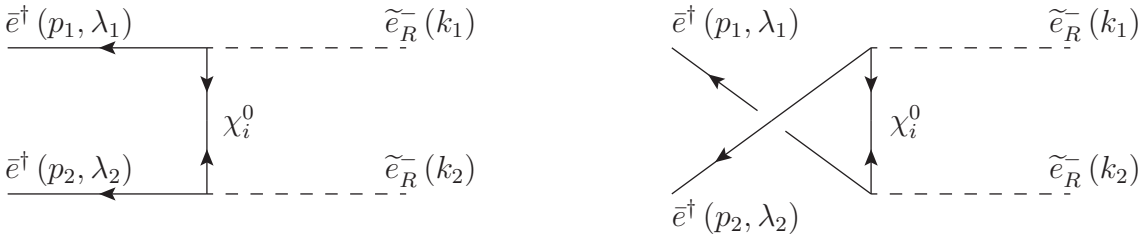


Figure 29.6.2: The two Feynman diagrams for $e^-e^-\to\widetilde{e}^-_R\widetilde{e}^-_R$ in the limit where $m_e\to 0$.

Using the Feynman rules of Fig. K.4.2, the amplitude for the first graph is:

$$i\mathcal{M}_t=\left(-i\sqrt{2}g\frac{s_W}{c_W}N_{i1}\right)^2\left[\frac{i\,m_{\widetilde{N}_i}}{(k_1-p_1)^2-m^2_{\widetilde{N}_i}}\right]y_1^\dagger y_2^\dagger\,, \qquad (29.6.13)$$

for each exchanged neutralino. The amplitude for the second graph is the same, but with the electrons interchanged:

$$i\mathcal{M}_u=\left(-i\sqrt{2}g\frac{s_W}{c_W}N_{i1}\right)^2\left[\frac{i\,m_{\widetilde{N}_i}}{(k_1-p_2)^2-m^2_{\widetilde{N}_i}}\right]y_1^\dagger y_2^\dagger\,. \qquad (29.6.14)$$

Since we have chosen to write the external state wave function spinors in the same order in $\mathcal{M}_t$ and $\mathcal{M}_u$, there is no factor of (-1) for Fermi-Dirac statistics. So, applying eq. (29.44), the total amplitude squared is:

$$|\mathcal{M}|^2=\frac{4g^4s_W^4}{c_W^4}(y_1^\dagger y_2^\dagger)(y_2y_1)\left|\sum_{i=1}^4(N_{i1})^2m_{\widetilde{N}_i}\left(\frac{1}{t-m^2_{\widetilde{N}_i}}+\frac{1}{u-m^2_{\widetilde{N}_i}}\right)\right|^2. \qquad (29.6.15)$$

The sum over the electron spins is obtained from

$$\sum_{\lambda_1,\lambda_2}(y_1^\dagger y_2^\dagger)(y_2y_1)=\mathrm{Tr}[p_2{\cdot}\overline{\sigma}p_1{\cdot}\sigma]=2p_2{\cdot}p_1=s\,. \qquad (29.6.16)$$

So, using eq. (29.3.58), the spin-averaged differential cross-section is:

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\left(\frac{1}{4}\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2\right) = \frac{\pi\alpha^2}{2c_W^4 s}\left|\sum_{i=1}^{4}(N_{i1})^2 m_{\widetilde{N}_i}\left(\frac{1}{t-m^2_{\widetilde{N}_i}}+\frac{1}{u-m^2_{\widetilde{N}_i}}\right)\right|^2 . \tag{29.6.17}$$

After integrating over t to obtain the total cross-section, the resulting expression must be multiplied by a factor of $1/2$ to account for the identical sleptons in the final state (to avoid the double counting of indistinguishable particles).

To compare with [174] and also with eq. E27 of ref. [7], note that for a pure photino exchange, $N_{i1} \to c_W \delta_{i1}$, so it checks.

$e^-e^- \to \widetilde{e}_L^-\widetilde{e}_L^-$

Again, in the limit of vanishing electron mass, there are two Feynman graphs, which are related by the exchange of identical electrons in the initial state or equivalently by exchange of the identical selectrons in the final state. As shown in Fig. 29.6.3, they are exactly like the previous example, but with all arrows reversed.

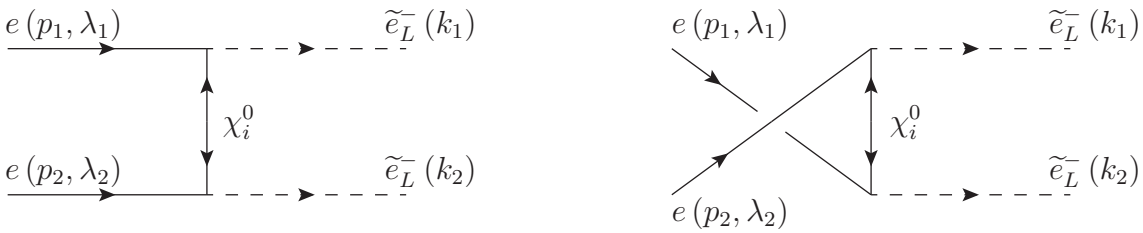


Figure 29.6.3: The two Feynman diagrams for $e^-e^- \to \widetilde{e}_L^-\widetilde{e}_L^-$ in the limit of vanishing electron mass.

Using the Feynman rules of Fig. K.4.2, the amplitude for the first graph is:

$$\mathcal{M}_t = \left(i\frac{g}{\sqrt{2}}[N^*_{i2}+\frac{s_W}{c_W}N^*_{i1}]\right)^2\left[\frac{i\,m_{\widetilde{N}_i}}{(p_1-k_1)^2-m^2_{\widetilde{N}_i}}\right]x_1x_2\,, \tag{29.6.18}$$

for each exchanged neutralino. The amplitude for the second graph is the same, but with $p_1 \leftrightarrow p_2$:

$$\mathcal{M}_u = \left(i\frac{g}{\sqrt{2}}[N^*_{i2}+\frac{s_W}{c_W}N^*_{i1}]\right)^2\left[\frac{i\,m_{\widetilde{N}_i}}{(p_2-k_1)^2-m^2_{\widetilde{N}_i}}\right]x_1x_2\,. \tag{29.6.19}$$

Since we have chosen to write the external state wave function spinors in the same order in $\mathcal{M}_1$ and $\mathcal{M}_2$, there is no factor of (-1) for Fermi-Dirac statistics. The total amplitude squared is:

$$|\mathcal{M}|^2 = \frac{g^4}{4}(x_1x_2)(x_2^\dagger x_1^\dagger)\left|\sum_{i=1}^{4}(N^*_{i2}+\frac{s_W}{c_W}N^*_{i1})^2 m_{\widetilde{N}_i}\left(\frac{1}{t-m^2_{\widetilde{N}_i}}+\frac{1}{u-m^2_{\widetilde{N}_i}}\right)\right|^2 . \tag{29.6.20}$$

The average over the electron spins follows from eq. (29.3.57):

$$\sum_{\lambda_1,\lambda_2}(x_1x_2)(x_2^\dagger x_1^\dagger) = \mathrm{Tr}[p_2{\cdot}\sigma p_1{\cdot}\overline{\sigma}] = 2p_2{\cdot}p_1 = s\,. \tag{29.6.21}$$

So the spin-averaged differential cross-section is:

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\left(\frac{1}{4}\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2\right) = \frac{\pi\alpha^2}{16 s_W^4 s}\left|\sum_{i=1}^{4}(N_{i2}^* + \frac{s_W}{c_W}N_{i1}^*)^2 m_{\widetilde{N}_i}\left(\frac{1}{t-m_{\widetilde{N}_i}^2}+\frac{1}{u-m_{\widetilde{N}_i}^2}\right)\right|^2 . \tag{29.6.22}$$

After integrating over t to obtain the total cross-section, one must multiply the resulting expression by a factor of 1/2 to account for the identical sleptons in the final state [as noted below eq. (29.6.17)]. To compare with [174] and also with eq. (E27) of ref. [7], note that for a pure photino exchange, $N_{i1} \to c_W\delta_{i1}$ and $N_{i2} \to s_W\delta_{i1}$, so it checks.

29.6.11 $e^-e^+ \to \widetilde{\nu}\widetilde{\nu}^*$

Consider now the pair production of sneutrinos in electron-positron collisions. There are two graphs featuring the s-channel exchange of the Z^0. We will neglect the electron mass and Yukawa coupling, so there is only one graph involving the t-channel exchange of the charginos. These three Feynman diagrams are shown in Fig. 29.6.1, where we have also defined the helicities and momenta of the particles. The Mandelstam variables can be expressed in terms of the external momenta and the sneutrino mass:

$$2p_1\cdot p_2 = s\,, \qquad 2k_1\cdot k_2 = s - 2m_{\widetilde{\nu}}^2\,, \tag{29.6.1}$$

$$2p_1\cdot k_1 = 2p_2\cdot k_2 = m_{\widetilde{\nu}}^2 - t\,, \qquad 2p_1\cdot k_2 = 2p_2\cdot k_1 = m_{\widetilde{\nu}}^2 - u\,. \tag{29.6.2}$$

Using the Feynman rules of Fig. J.1.2, the amplitudes for the two s-channel Z boson exchange diagrams are:[67]

$$i\mathcal{M}_1 = \left[-i\frac{g}{2c_W}(k_1-k_2)_\mu\right]\left[\frac{-ig^{\mu\nu}}{D_Z}\right]\left[i\frac{g}{c_W}(s_W^2-\tfrac{1}{2})\right]x_1\sigma_\nu y_2^\dagger\,, \tag{29.6.3}$$

$$i\mathcal{M}_2 = \left[-i\frac{g}{2c_W}(k_1-k_2)_\mu\right]\left[\frac{-ig^{\mu\nu}}{D_Z}\right]\left[i\frac{g s_W^2}{c_W}\right]y_1^\dagger\overline{\sigma}_\nu x_2\,, \tag{29.6.4}$$

where the first factor in each case is the Feynman rule from the Z boson coupling to the sneutrinos (see Fig. 72c, ref. [7]), and $D_Z \equiv s - m_Z^2 + i\Gamma_Z m_Z$ is the denominator of the Z boson propagator.[68] The t-channel diagram due to each chargino gives a contribution

$$i\mathcal{M}_3 = (-igV_{i1}^*)\,(-igV_{i1})\,x_1\left[\frac{i(k_1-p_1)\cdot\sigma}{(k_1-p_1)^2 - m_{\widetilde{C}_i}^2}\right]y_2^\dagger, \tag{29.6.5}$$

using the rules of Fig. K.4.1. Therefore, the total amplitude can be rewritten as:

$$\mathcal{M} = c_1 x_1(k_1-k_2)\cdot\sigma y_2^\dagger + c_2 y_1^\dagger(k_1-k_2)\cdot\overline{\sigma}x_2 + c_3 x_1(k_1-p_1)\cdot\sigma y_2^\dagger\,, \tag{29.6.6}$$

where

$$c_1 \equiv \frac{g^2(1-2s_W^2)}{4c_W^2 D_Z}, \qquad c_2 \equiv -\frac{g^2 s_W^2}{2c_W^2 D_Z}, \qquad c_3 \equiv g^2\sum_{i=1}^{2}\frac{|V_{i1}|^2}{m_{\widetilde{C}_j}^2 - t}\,. \tag{29.6.7}$$

[67] Because we neglect the electron mass, we may drop the $Q^\mu Q^\nu$ term of the Z propagator, where $Q \equiv p_1 + p_2$ is the propagating four-momentum in the s-channel [cf. Fig. 29.4.5].

[68] The explicit inclusion of the finite decay width in the propagator of an unstable particle involves subtle issues of gauge invariance and unitarity, particularly in higher loop computations. The authors of [175] recommend the complex-mass scheme for perturbative calculations with unstable particles, first introduced in [176].

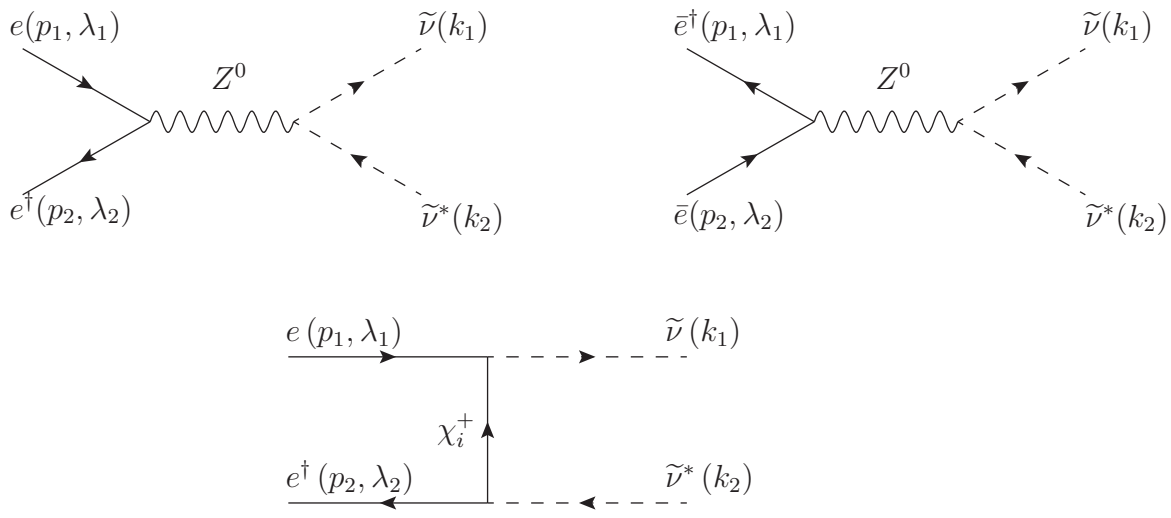


Figure 29.6.1: The Feynman diagrams for $e^-e^+ \to \widetilde{\nu}\widetilde{\nu}^*$.

Squaring the amplitude and summing over the electron and positron spins, the interference terms involving c_2 will vanish in the massless electron limit due to eqs. (29.3.59) and (29.3.60). Therefore, we obtain

$$\begin{aligned}\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2 &= \sum_{\lambda_1,\lambda_2}\Big\{|c_1|^2\,x_1(k_1-k_2)\cdot\sigma y_2^\dagger\,y_2(k_1-k_2)\cdot\sigma x_1^\dagger + |c_2|^2\,y_1^\dagger(k_1-k_2)\cdot\overline{\sigma}x_2\,x_2^\dagger(k_1-k_2)\cdot\overline{\sigma}y_1 \\ &\qquad + c_3^2\,x_1(k_1-p_1)\cdot\sigma y_2^\dagger\,y_2(k_1-p_1)\cdot\sigma x_1^\dagger + 2\mathrm{Re}[c_1c_3\,x_1(k_1-k_2)\cdot\sigma y_2^\dagger\,y_2(k_1-p_1)\cdot\sigma x_1^\dagger]\Big\} \\ &= |c_1|^2\,\mathrm{Tr}[(k_1-k_2)\cdot\sigma p_2\cdot\overline{\sigma}(k_1-k_2)\cdot\sigma p_1\cdot\overline{\sigma}] + |c_2|^2\,\mathrm{Tr}[(k_1-k_2)\cdot\overline{\sigma}p_2\cdot\sigma(k_1-k_2)\cdot\overline{\sigma}p_1\cdot\sigma] \\ &\quad + c_3^2\,\mathrm{Tr}[(k_1-p_1)\cdot\sigma p_2\cdot\overline{\sigma}(k_1-p_1)\cdot\sigma p_1\cdot\overline{\sigma}] + 2\mathrm{Re}[c_1]c_3\,\mathrm{Tr}[(k_1-k_2)\cdot\sigma p_2\cdot\overline{\sigma}(k_1-p_1)\cdot\sigma p_1\cdot\overline{\sigma}],\end{aligned} \tag{29.6.8}$$

where we have used eqs. (29.3.57) and (29.3.58) to perform the spin sums. Applying the trace identities eqs. (29.57) and (29.58) and simplifying the results using eqs. (29.6.1)–(29.6.2) and $u = 2m_{\tilde{\nu}}^2 - s - t$, we get

$$\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2 = -[st + (t-m_{\tilde{\nu}}^2)^2]\left(4|c_1|^2 + 4|c_2|^2 + c_3^2 + 4\mathrm{Re}[c_1]c_3\right). \tag{29.6.9}$$

When $m_{\tilde{C}_1} = m_{\tilde{C}_2}$, this agrees with eqs. (E46)–(E48) of [7] [69] and with [177]. The differential cross-section follows in the standard way by averaging over the initial state spins:

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\left(\frac{1}{4}\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2\right). \tag{29.6.10}$$

Note that

$$t = m_{\tilde{\nu}}^2 - \tfrac{1}{2}(1-\beta\cos\theta)s\,, \qquad \beta \equiv \left(1 - \frac{4m_{\tilde{\nu}}^2}{s}\right)^{1/2}, \tag{29.6.11}$$

[69] There is a typographical error in eq. (E48) of [7]; the right-hand side should be multiplied by $1/\cos^2\theta_W$.

where θ is the angle between the initial state electron and the final state sneutrino in the center-of-momentum frame. The upper and lower limits t_+ and t_- are obtained by inserting $\cos\theta = \pm 1$ above, respectively.

Performing the integration over t to obtain the total cross-section, one obtains

$$\sigma = \int_{t_-}^{t_+} \frac{d\sigma}{dt} dt = \frac{g^4}{64\pi s}\left(S_Z + \sum_{i,j=1}^{2} S_{ij} + \sum_{i=1}^{2} S_{Zi}\right), \tag{29.6.12}$$

where

$$S_Z = \frac{\beta^3}{24 c_W^4}(8s_W^4 - 4s_W^2 + 1)\frac{s^2}{|D_Z|^2}, \tag{29.6.13}$$

$$S_{ii} = |V_{i1}|^4 \left[(1-2\gamma_i)L_i - 2\beta\right], \tag{29.6.14}$$

$$S_{12} = S_{21} = |V_{11}V_{12}|^2 \left\{ \frac{(m^2_{\tilde{C}_2} + s\gamma_2^2)L_2 - (m^2_{\tilde{C}_1} + s\gamma_1^2)L_1}{m^2_{\tilde{C}_2} - m^2_{\tilde{C}_1}} - \beta \right\}, \tag{29.6.15}$$

$$S_{Zi} = \frac{(2s_W^2 - 1)}{c_W^2}|V_{i1}|^2 \left[(m^2_{\tilde{C}_i} + s\gamma_i^2)L_i + s\beta(\gamma_i - 1/2)\right] \frac{(s - m_Z^2)}{|D_Z|^2}, \tag{29.6.16}$$

with

$$\gamma_i \equiv \frac{m^2_{\tilde{\nu}} - m^2_{\tilde{C}_i}}{s}, \qquad\qquad L_i \equiv \ln\left(\frac{m^2_{\tilde{C}_i} - t_-}{m^2_{\tilde{C}_i} - t_+}\right). \tag{29.6.17}$$

This agrees with eqs. (E49)-(E52) of ref. [7] in the limit of degenerate charginos, or of a single wino chargino with $|V_{11}| = 1$ and $V_{12} = 0$. It also agrees with [177].

29.6.12 $e^-e^+ \to \widetilde{N}_i\widetilde{N}_j$

Next we consider the pair production of neutralinos via e^-e^+ annihilation. There are four Feynman graphs for s-channel Z^0 exchange, shown in Fig. 29.6.1, and four for t/u-channel selectron exchange, shown in Fig. 29.6.2. The momenta and polarizations are as labeled in the graphs. We denote the neutralino masses as $m_{\tilde{N}_i}, m_{\tilde{N}_j}$ and the selectron masses as $m_{\tilde{e}_L}$ and $m_{\tilde{e}_R}$. The electron mass will again be neglected. The kinematic variables are then given by

$$s = 2p_1\cdot p_2 = m^2_{\tilde{N}_i} + m^2_{\tilde{N}_j} + 2k_i\cdot k_j, \tag{29.6.1}$$

$$t = m^2_{\tilde{N}_i} - 2p_1\cdot k_i = m^2_{\tilde{N}_j} - 2p_2\cdot k_j, \tag{29.6.2}$$

$$u = m^2_{\tilde{N}_i} - 2p_2\cdot k_i = m^2_{\tilde{N}_j} - 2p_1\cdot k_j. \tag{29.6.3}$$

By applying the Feynman rules of Figs. J.1.2 and K.2.1, we obtain for the sum of the s-channel diagrams in Fig. 29.6.1 [cf. footnote 67],

$$i\mathcal{M}_Z = \frac{-ig^{\mu\nu}}{D_Z}\left[\frac{ig(s_W^2 - \frac{1}{2})}{c_W} x_1\sigma_\mu y_2^\dagger + \frac{igs_W^2}{c_W} y_1^\dagger\overline{\sigma}_\mu x_2\right]\left[\frac{ig}{c_W}O''^L_{ij} x_i^\dagger\overline{\sigma}_\nu y_j - \frac{ig}{c_W}O''^L_{ji} y_i\sigma_\nu x_j^\dagger\right], \tag{29.6.4}$$

where O''_{ij} is given in eq. (K.2.5), and $D_Z \equiv s - m_Z^2 + i\Gamma_Z m_Z$. The fermion spinors are denoted by $x_1 \equiv x(\vec{\boldsymbol{p}}_1, \lambda_1)$, $y_2^\dagger \equiv y^\dagger(\vec{\boldsymbol{p}}_2, \lambda_2)$, $x_i^\dagger \equiv x^\dagger(\vec{\boldsymbol{k}}_i, \lambda_i)$, $y_j \equiv y(\vec{\boldsymbol{k}}_j, \lambda_j)$, etc. Note that we have combined the

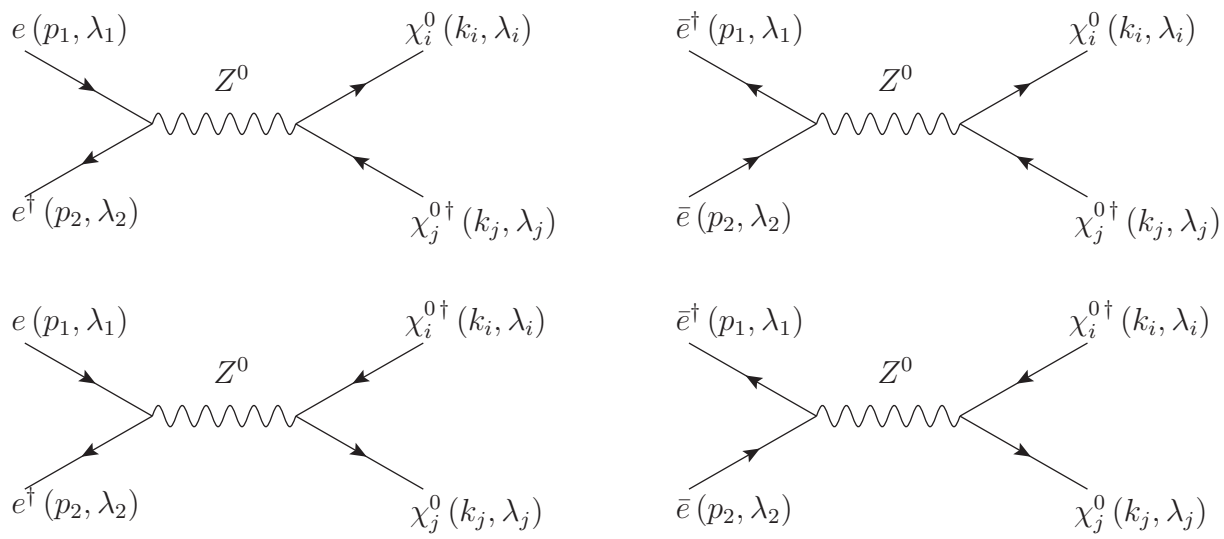


Figure 29.6.1: The four Feynman diagrams for $e^-e^+ \to \widetilde{N}_i\widetilde{N}_j$ via s-channel Z^0 exchange.

matrix elements of the four diagrams by factorizing with respect to the common boson propagator. For the four t/u-channel diagrams, we obtain, by applying the rules of Fig. K.4.2:

$$i\mathcal{M}^{(t)}_{\tilde{e}_L} = (-1)\left[\frac{i}{t-m^2_{\tilde{e}_L}}\right]\left[\frac{ig}{\sqrt{2}}\Big(N^*_{i2}+\frac{s_W}{c_W}N^*_{i1}\Big)\right]\left[\frac{ig}{\sqrt{2}}\Big(N_{j2}+\frac{s_W}{c_W}N_{j1}\Big)\right]x_1y_iy_2^\dagger x_j^\dagger, \tag{29.6.5}$$

$$i\mathcal{M}^{(u)}_{\tilde{e}_L} = \left[\frac{i}{u-m^2_{\tilde{e}_L}}\right]\left[\frac{ig}{\sqrt{2}}\Big(N^*_{j2}+\frac{s_W}{c_W}N^*_{j1}\Big)\right]\left[\frac{ig}{\sqrt{2}}\Big(N_{i2}+\frac{s_W}{c_W}N_{i1}\Big)\right]x_1y_jy_2^\dagger x_i^\dagger, \tag{29.6.6}$$

$$i\mathcal{M}^{(t)}_{\tilde{e}_R} = (-1)\left[\frac{i}{t-m^2_{\tilde{e}_R}}\right]\Big(-i\sqrt{2}g\frac{s_W}{c_W}N_{i1}\Big)\Big(-i\sqrt{2}g\frac{s_W}{c_W}N^*_{j1}\Big)y_1^\dagger x_i^\dagger x_2y_j, \tag{29.6.7}$$

$$i\mathcal{M}^{(u)}_{\tilde{e}_R} = \left[\frac{i}{u-m^2_{\tilde{e}_R}}\right]\Big(-i\sqrt{2}g\frac{s_W}{c_W}N_{j1}\Big)\Big(-i\sqrt{2}g\frac{s_W}{c_W}N^*_{i1}\Big)y_1^\dagger x_j^\dagger x_2y_i. \tag{29.6.8}$$

The first factors of (-1) in each of eqs. (29.6.5) and (29.6.7) are present because the order of the spinors in each case is an odd permutation of the ordering $(1,2,i,j)$ established by the s-channel contribution. The other contributions have spinors in an even permutation of that ordering.

The s-channel diagram contribution of eq. (29.6.4) can be profitably rearranged using the Fierz identities of eqs. (29.68) and (29.69). Then, combining the result with the t/u-channel and s-channel contributions, we have for the total:

$$\mathcal{M} = c_1x_1y_jy_2^\dagger x_i^\dagger + c_2x_1y_iy_2^\dagger x_j^\dagger + c_3y_1^\dagger x_i^\dagger x_2y_j + c_4y_1^\dagger x_j^\dagger x_2y_i, \tag{29.6.9}$$

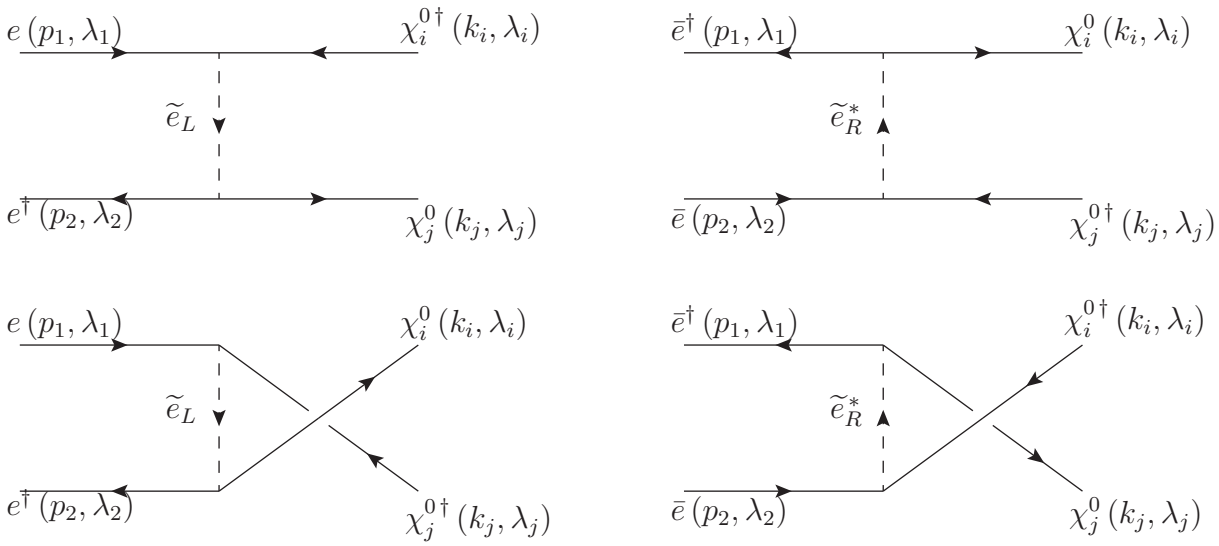


Figure 29.6.2: The four Feynman diagrams for $e^-e^+ \to \widetilde{N}_i\widetilde{N}_j$ via t/u-channel selectron exchange.

where

$$c_1 = \frac{g^2}{c_W^2}\left[(1-2s_W^2)O''^L_{ij}/D_Z - \tfrac{1}{2}(c_W N_{i2} + s_W N_{i1})(c_W N^*_{j2} + s_W N^*_{j1})/(u - m^2_{\tilde{e}_L})\right], \tag{29.6.10}$$

$$c_2 = \frac{g^2}{c_W^2}\left[(2s_W^2-1)O''^L_{ji}/D_Z + \tfrac{1}{2}(c_W N^*_{i2} + s_W N^*_{i1})(c_W N_{j2} + s_W N_{j1})/(t - m^2_{\tilde{e}_L})\right], \tag{29.6.11}$$

$$c_3 = \frac{2g^2 s_W^2}{c_W^2}\left[-O''^L_{ij}/D_Z + N_{i1}N^*_{j1}/(t - m^2_{\tilde{e}_R})\right], \tag{29.6.12}$$

$$c_4 = \frac{2g^2 s_W^2}{c_W^2}\left[O''^L_{ji}/D_Z - N^*_{i1}N_{j1}/(u - m^2_{\tilde{e}_R})\right]. \tag{29.6.13}$$

Squaring the amplitude and averaging over electron and positron spins, only terms involving $x_1x_1^\dagger$ or $y_1y_1^\dagger$, and $x_2x_2^\dagger$ or $y_2y_2^\dagger$ survive in the massless electron limit. Thus,

$$\begin{aligned}\sum_{\lambda_1,\lambda_2}|\mathcal{M}|^2 = \sum_{\lambda_1,\lambda_2}\Big(&|c_1|^2 y_j^\dagger x_1^\dagger x_1 y_j x_i y_2 y_2^\dagger x_i^\dagger + |c_2|^2 y_i^\dagger x_1^\dagger x_1 y_i x_j y_2 y_2^\dagger x_j^\dagger \\ &+|c_3|^2 x_i y_1 y_1^\dagger x_i^\dagger y_j^\dagger x_2^\dagger x_2 y_j + |c_4|^2 x_j y_1 y_1^\dagger x_j^\dagger y_i^\dagger x_2^\dagger x_2 y_i \\ &+2\mathrm{Re}\big[c_1 c_2^* y_i^\dagger x_1^\dagger x_1 y_j x_j y_2 y_2^\dagger x_i^\dagger\big] + 2\mathrm{Re}\big[c_3 c_4^* x_j y_1 y_1^\dagger x_i^\dagger y_i^\dagger x_2^\dagger x_2 y_j\big]\Big) \\ = |c_1|^2 y_j^\dagger p_1\cdot\overline{\sigma} y_j\, x_i p_2\cdot\sigma x_i^\dagger &+ |c_2|^2 y_i^\dagger p_1\cdot\overline{\sigma} y_i\, x_j p_2\cdot\sigma x_j^\dagger \\ &+|c_3|^2 x_i p_1\cdot\sigma x_i^\dagger\, y_j^\dagger p_2\cdot\overline{\sigma} y_j + |c_4|^2 x_j p_1\cdot\sigma x_j^\dagger\, y_i^\dagger p_2\cdot\overline{\sigma} y_i \\ &+2\mathrm{Re}\big[c_1 c_2^* y_i^\dagger p_1\cdot\overline{\sigma} y_j\, x_j p_2\cdot\sigma x_i^\dagger\big] + 2\mathrm{Re}\big[c_3 c_4^* x_j p_1\cdot\sigma x_i^\dagger\, y_i^\dagger p_2\cdot\overline{\sigma} y_j\big],\end{aligned} \tag{29.6.14}$$

after employing the results of eqs. (29.3.57)–(29.3.60).

We now perform the remaining spin sums using eqs. (29.3.57)–(29.3.60) again, obtaining:

$$\begin{aligned}\sum_{\lambda_1,\lambda_2,\lambda_i,\lambda_j} |\mathcal{M}|^2 = {}& |c_1|^2 \mathrm{Tr}[p_1\cdot\overline{\sigma}k_j\cdot\sigma]\mathrm{Tr}[p_2\cdot\sigma k_i\cdot\overline{\sigma}] + |c_2|^2\mathrm{Tr}[p_1\cdot\overline{\sigma}k_i\cdot\sigma]\mathrm{Tr}[p_2\cdot\sigma k_j\cdot\overline{\sigma}] \\ & + |c_3|^2\mathrm{Tr}[p_1\cdot\sigma k_i\cdot\overline{\sigma}]\mathrm{Tr}[p_2\cdot\overline{\sigma}k_j\cdot\sigma] + |c_4|^2\mathrm{Tr}[p_1\cdot\sigma k_j\cdot\overline{\sigma}]\mathrm{Tr}[p_2\cdot\overline{\sigma}k_i\cdot\sigma] \\ & + 2\mathrm{Re}[c_1c_2^*]m_{\widetilde{N}_i}m_{\widetilde{N}_j}\mathrm{Tr}[p_2\cdot\sigma p_1\cdot\overline{\sigma}] + 2\mathrm{Re}[c_3c_4^*]m_{\widetilde{N}_i}m_{\widetilde{N}_j}\mathrm{Tr}[p_1\cdot\sigma p_2\cdot\overline{\sigma}]. \end{aligned} \quad (29.6.15)$$

Applying the trace identity of eq. (29.56) to this yields

$$\begin{aligned}\sum_{\text{spins}} |\mathcal{M}|^2 = {}& (|c_1|^2 + |c_4|^2)4p_1\cdot k_j\, p_2\cdot k_i + (|c_2|^2 + |c_3|^2)4p_1\cdot k_i\, p_2\cdot k_j \\ & + 4\mathrm{Re}[c_1c_2^* + c_3c_4^*]m_{\widetilde{N}_i}m_{\widetilde{N}_j}p_1\cdot p_2 \\ = {}& (|c_1|^2 + |c_4|^2)(u - m^2_{\widetilde{N}_i})(u - m^2_{\widetilde{N}_j}) + (|c_2|^2 + |c_3|^2)(t - m^2_{\widetilde{N}_i})(t - m^2_{\widetilde{N}_j}) \\ & + 2\mathrm{Re}[c_1c_2^* + c_3c_4^*]m_{\widetilde{N}_i}m_{\widetilde{N}_j}s. \end{aligned} \quad (29.6.16)$$

The differential cross-section then follows:

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\left(\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\right). \quad (29.6.17)$$

This agrees with the first complete calculation presented in [178]. For the case of pure photino pair production, i.e. $N_{i1} \to c_W\delta_{i1}$ and $N_{i2} \to s_W\delta_{i1}$ and for degenerate selectron masses this also agrees with eq. (E9) of the erratum of [7]. Other earlier calculations with some simplifications are given in refs. [179, 180].

Defining $\cos\theta = \hat{\boldsymbol{p}}_{\boldsymbol{1}}\cdot\hat{\boldsymbol{k}}_{\boldsymbol{i}}$ (the cosine of the angle between the initial state electron and one of the neutralinos in the center-of-momentum frame), the Mandelstam variables t, u are given by

$$t = \frac{1}{2}\left[m^2_{\widetilde{N}_i} + m^2_{\widetilde{N}_j} - s + \lambda^{1/2}(s, m^2_{\widetilde{N}_i}, m^2_{\widetilde{N}_j})\cos\theta\right], \quad (29.6.18)$$

$$u = \frac{1}{2}\left[m^2_{\widetilde{N}_i} + m^2_{\widetilde{N}_j} - s - \lambda^{1/2}(s, m^2_{\widetilde{N}_i}, m^2_{\widetilde{N}_j})\cos\theta\right], \quad (29.6.19)$$

where the triangle function $\lambda^{1/2}$ is defined in eq. (29.6.11). Taking into account the identical fermions in the final state when $i = j$, the total cross-section is

$$\sigma = \frac{1}{1+\delta_{ij}}\int_{t_-}^{t_+}\frac{d\sigma}{dt}dt\,, \quad (29.6.20)$$

where t_- and t_+ are obtained by inserting $\cos\theta = \mp 1$ in eq. (29.6.18), respectively.

29.6.13 $\widetilde{N}_1\widetilde{N}_1 \to f\bar{f}$

In this section, we compute the annihilation rate for $\widetilde{N}_1\widetilde{N}_1 \to f\bar{f}$, where f is any kinematically allowed quark, charged lepton or neutrino. The case of $f = e^-$ is the reversed reaction of the process examined in Section 29.6.12 (with $i = j = 1$). In R-parity-conserving supersymmetric models in which $\widetilde{N}_1$ is the lightest supersymmetric particle (and hence is stable), the $\widetilde{N}_1\widetilde{N}_1$ annihilation process is relevant for the computation of the neutralino relic density [181]. In particular, $\widetilde{N}_1\widetilde{N}_1 \to f\bar{f}$ can be an important contribution to cold dark matter annihilation [181, 182, 183, 184]. Neutralino dark matter is typically heavier than about 6 GeV [185]; for lighter neutralinos see [186].

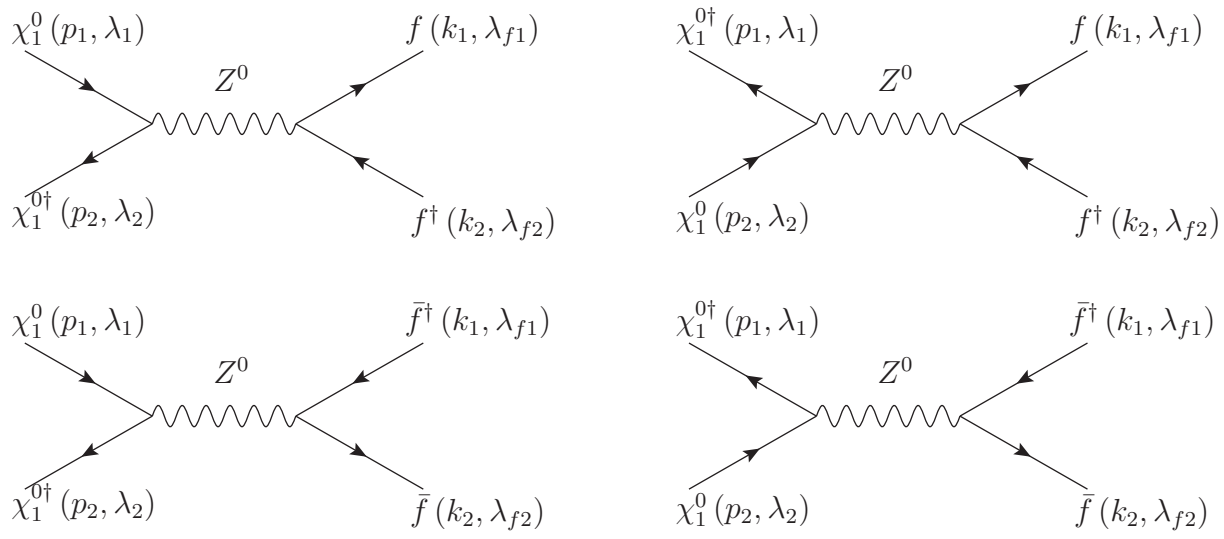


Figure 29.6.1: The four Feynman diagrams for $\widetilde{N}_1\widetilde{N}_1 \to f\bar{f}$ via s–channel Z^0 exchange, where f is a quark or lepton.

In the computation of the relic density, one computes $v_{\rm rel}\sigma_{\rm ann}$, where $\sigma_{\rm ann}$ is the $\widetilde{N}_1\widetilde{N}_1$ annihilation cross-section and $v_{\rm rel}$ is the relative velocity of the two neutralinos in the center-of-momentum frame. The square of the relative velocity is taken to be its thermal average, $v_{\rm rel}^2 \simeq 6k_BT/m_{\widetilde{N}_1}$ [181], which is typically non-relativistic when the temperature is of order the freeze-out temperature [183] (where the neutralino falls out of thermal equilibrium). Hence, it is sufficient to compute the annihilation cross-section for $\widetilde{N}_1\widetilde{N}_1 \to f\bar{f}$ in the non-relativistic limit.

As in Section 29.6.12, there are four Feynman graphs for s-channel Z^0 exchange, shown in Fig. 29.6.1. In addition, there are s-channel neutral Higgs exchange graphs, shown in Fig. 29.6.2, that yield contributions to the annihilation amplitude proportional to the fermion mass, m_f.[70] Likewise, as in Section 29.6.12, there are four Feynman graphs for t/u-channel $\widetilde{f}_L$ and $\widetilde{f}_R$ exchange, shown in Fig. 29.6.3. However, because we do not set m_f to zero, four additional t/u-channel graphs contribute, shown in Fig. 29.6.4, that are sensitive to the higgsino components of the neutralino.

The neutralino and the final state fermion four-momenta and polarizations are as labeled in the Feynman graphs. In the center-of-momentum (CM) frame, the four-momenta are

$$p_1^\mu = (E\,;\,\vec{\boldsymbol{p}})\,, \qquad p_2^\mu = (E\,;\,-\vec{\boldsymbol{p}})\,, \qquad k_1^\mu = E(1\,;\,\beta\hat{\boldsymbol{k}})\,, \qquad k_2^\mu = E(1\,;\,-\beta\hat{\boldsymbol{k}})\,, \tag{29.6.1}$$

where

$$\beta \equiv \sqrt{1 - \frac{m_f^2}{E^2}}\,. \tag{29.6.2}$$

In the non-relativistic limit where $|\vec{\boldsymbol{p}}| \ll m_{\widetilde{N}_1}$,

$$E \simeq m_{\widetilde{N}_1} + \frac{|\vec{\boldsymbol{p}}|^2}{2m_{\widetilde{N}_1}}\,, \tag{29.6.3}$$

[70] In regions of parameter space where $m_{\widetilde{N}_1} \simeq \frac{1}{2}m_Z$ or $m_{\widetilde{N}_1} \simeq \frac{1}{2}m_{\phi^0}$ (where $\phi^0 = h^0$, H^0 or A^0), the resonant $2 \to 1$ annihilation $\widetilde{N}_1\widetilde{N}_1 \to Z^0$ or $\widetilde{N}_1\widetilde{N}_1 \to \phi^0$ dominates the $2 \to 2$ annihilation processes considered here.

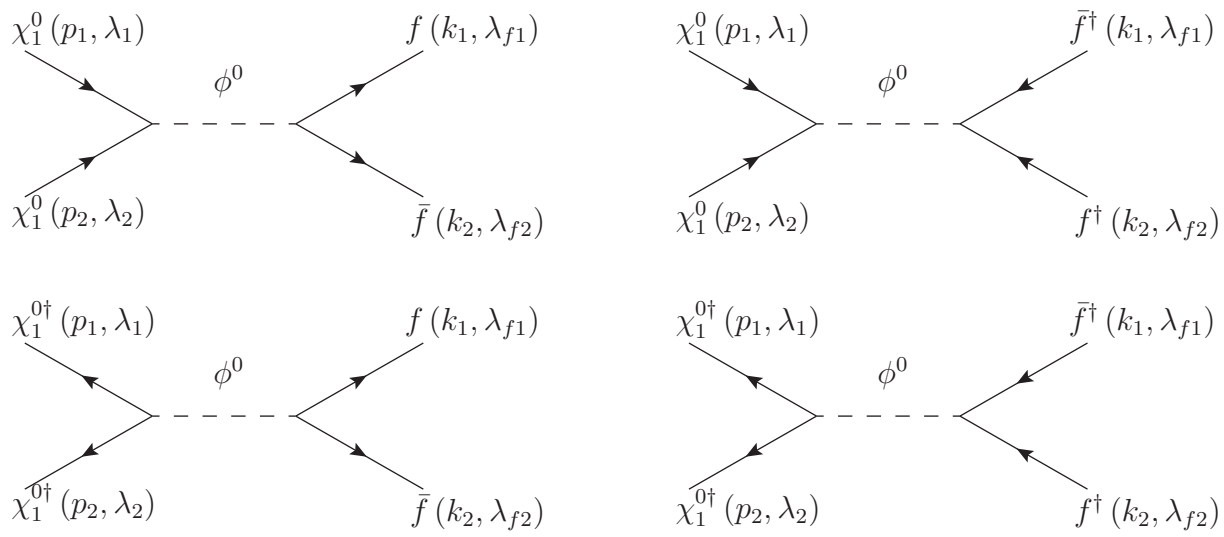


Figure 29.6.2: Feynman diagrams for $\widetilde{N}_1\widetilde{N}_1 \to f\bar{f}$ via s–channel Higgs exchange. There are four diagrams for each possible neutral Higgs state $\phi^0 = h^0$, H^0 and A^0.

and the kinematic invariants are given by

$$s = (p_1+p_2)^2 = 4E^2 = 4m_{\widetilde{N}_1}^2 + 4|\vec{\boldsymbol{p}}|^2\,, \tag{29.6.4}$$

$$t = (p_1-k_1)^2 = m_{\widetilde{N}_1}^2 + m_f^2 - 2p_1\cdot k_1 \simeq -m_{\widetilde{N}_1}^2 + m_f^2 + 2\beta m_{\widetilde{N}_1}|\vec{\boldsymbol{p}}|\cos\theta - 2|\vec{\boldsymbol{p}}|^2\,, \tag{29.6.5}$$

$$u = (p_1-k_2)^2 = m_{\widetilde{N}_1}^2 + m_f^2 - 2p_1\cdot k_2 \simeq -m_{\widetilde{N}_1}^2 + m_f^2 - 2\beta m_{\widetilde{N}_1}|\vec{\boldsymbol{p}}|\cos\theta - 2|\vec{\boldsymbol{p}}|^2\,, \tag{29.6.6}$$

where θ is the CM scattering angle. Subsequently, we shall neglect the subdominant $\mathcal{O}(|\vec{\boldsymbol{p}}|)$ terms in the t and u-channel propagator denominators by setting $t \simeq u \simeq -m_{\widetilde{N}_1}^2 + m_f^2$.

By applying the Feynman rules of Figs. J.1.2 and K.2.1, and using the unitary gauge for the Z-boson propagator, we obtain for the sum of the s-channel Z-exchange diagrams of Fig. 29.6.1,

$$i\mathcal{M}_Z = \frac{i\,(-g^{\mu\nu} + Q^\mu Q^\nu/m_Z^2)}{D_Z}\left(\frac{-ig}{c_W}\right)^2 O_{11}''^L\Big[x_1\sigma_\mu y_2^\dagger - y_1^\dagger\overline{\sigma}_\mu x_2\Big]\Big[(T_3^f - s_W^2 Q_f)x_{f1}^\dagger\overline{\sigma}_\nu y_{f2} - s_W^2 Q_f y_{f1}\sigma_\nu x_{f2}^\dagger\Big]\,, \tag{29.6.7}$$

where $O_{11}''^L$ is given in eq. (K.2.5), $D_Z \equiv s - m_Z^2 + i\Gamma_Z m_Z$, and $Q \equiv p_1 + p_2 = k_1 + k_2$. The spinor wave functions are denoted by $x_1 \equiv x(\vec{\boldsymbol{p}}_1,\lambda_1)$, $y_2^\dagger \equiv y^\dagger(\vec{\boldsymbol{p}}_2,\lambda_2)$, $x_{f1}^\dagger \equiv x^\dagger(\vec{\boldsymbol{k}}_1,\lambda_{f1})$, $y_{f2} \equiv y(\vec{\boldsymbol{k}}_2,\lambda_{f2})$, etc. In obtaining eq. (29.6.7), we have combined the matrix elements of the four diagrams by factorizing with respect to the common Z-boson propagator. Note that all four terms in eq. (29.6.7) have the same order of spinor wave functions (1,2,f1,f2). Thus, no additional relative signs arise (beyond the sign associated with the choice of the σ or $\overline{\sigma}$ version of the vertex Feynman rules). One can simplify the terms that originate from the $Q^\mu Q^\nu$ part of the Z-boson propagator by writing $Q^\mu = (p_1+p_2)^\mu$ and $Q^\nu = (k_1+k_2)^\nu$. Contracting the μ and ν indices with the help of eqs. (29.3.9)–(29.3.12) yields:

$$(p_1+p_2)^\mu\Big[x_1\sigma_\mu y_2^\dagger - y_1^\dagger\overline{\sigma}_\mu x_2\Big] = 2m_{\widetilde{N}_1}\Big(x_1x_2 - y_1^\dagger y_2^\dagger\Big)\,, \tag{29.6.8}$$

$$(k_1+k_2)^\nu\Big[(T_3^f - s_W^2Q_f)x_{f1}^\dagger\overline{\sigma}_\nu y_{f2} - s_W^2Q_f y_{f1}\sigma_\nu x_{f2}^\dagger\Big] = T_3^f m_f\Big(y_{f1}y_{f2} - x_{f1}^\dagger x_{f2}^\dagger\Big)\,. \tag{29.6.9}$$

Hence, we shall write

$$\mathcal{M}_Z \equiv \mathcal{M}_Z^{(1)} + \mathcal{M}_Z^{(2)}\,, \tag{29.6.10}$$

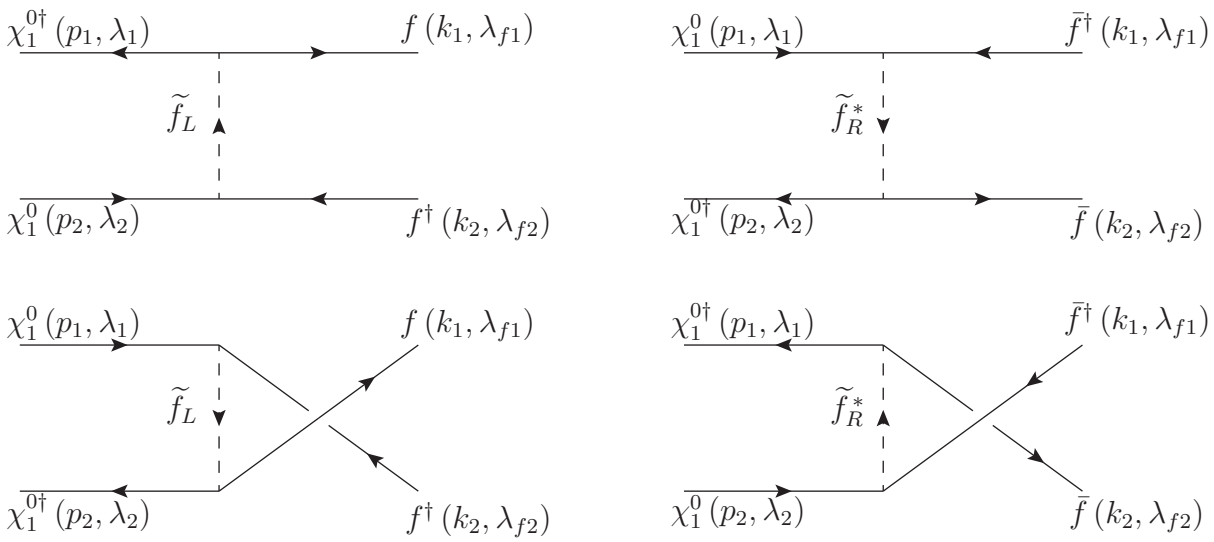


Figure 29.6.3: The four Feynman diagrams for $\widetilde{N}_i\widetilde{N}_j \to f\bar{f}$ via t/u–channel $\widetilde{f}_L$ and $\widetilde{f}_R$ exchange, where $\widetilde{f}_L$ and $\widetilde{f}_R$ couple to the gaugino components of the neutralino.

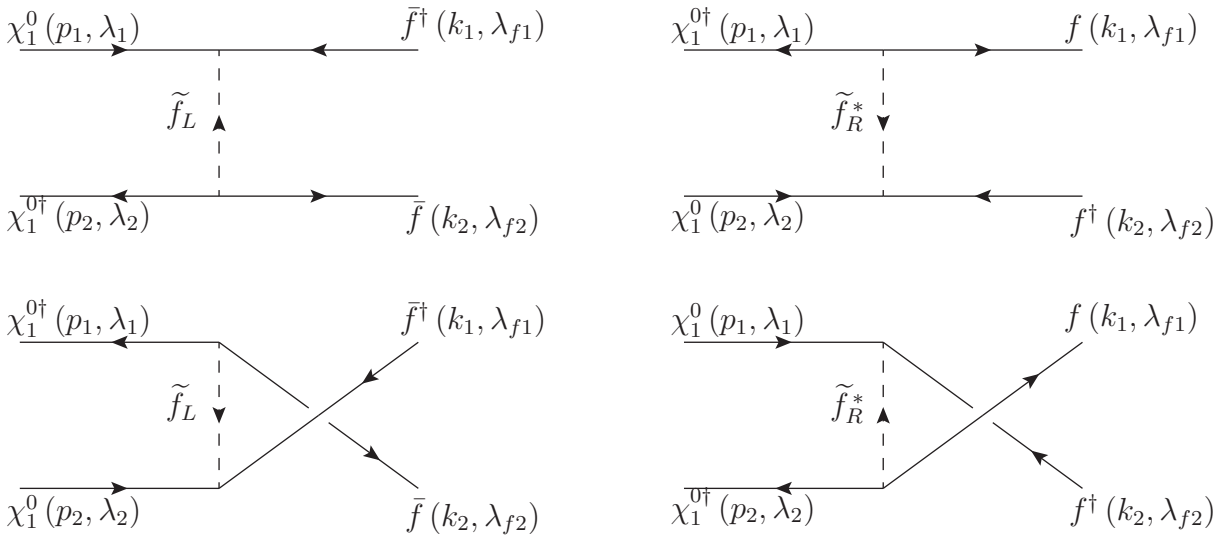


Figure 29.6.4: The four Feynman diagrams for $\widetilde{N}_i\widetilde{N}_j \to f\bar{f}$ via t/u–channel $\widetilde{f}_L$ and $\widetilde{f}_R$ exchange, where $\widetilde{f}_L$ and $\widetilde{f}_R$ couple to the higgsino components of the neutralino.

where

$$i\mathcal{M}_Z^{(1)} = \frac{-ig^{\mu\nu}}{D_Z}\left(\frac{-ig}{c_W}\right)^2 O''^L_{11}\Big[x_1\sigma_\mu y_2^\dagger - y_1^\dagger\overline{\sigma}_\mu x_2\Big]\Big[(T_3^f - s_W^2 Q_f)x_{f1}^\dagger\overline{\sigma}_\nu y_{f2} - s_W^2 Q_f y_{f1}\sigma_\nu x_{f2}^\dagger\Big], \tag{29.6.11}$$

$$i\mathcal{M}_Z^{(2)} = \frac{im_f m_{\widetilde{N}_1}}{m_Z^2 D_Z}\left(\frac{-ig}{c_W}\right)^2 O''^L_{11}(2T_3^f)\left(x_1x_2 - y_1^\dagger y_2^\dagger\right)\left(y_{f1}y_{f2} - x_{f1}^\dagger x_{f2}^\dagger\right). \tag{29.6.12}$$

Next, we apply the Feynman rules of Figs. K.1.1 and K.3.1 to obtain the sum of the four s-channel Higgs exchange diagrams (for $\phi^0 = h^0$, H^0 and A^0) of Fig. 29.6.2,

$$i\mathcal{M}_H = \sum_{\phi^0=h^0,H^0,A^0} \frac{i}{D_{\phi^0}} \left(\frac{-m_f}{\sqrt{2}\,v_f}\right) \left[(Y^{\phi^0\chi_1^0\chi_1^0})x_1x_2 + (Y^{\phi^0\chi_1^0\chi_1^0})^* y_1^\dagger y_2^\dagger\right] \left[k_{f\phi^0}y_{f1}y_{f2} + k^*_{f\phi^0}x^\dagger_{f1}x^\dagger_{f2}\right], \tag{29.6.13}$$

where $Y^{\phi^0\chi_1^0\chi_1^0}$ is given by eq. (K.3.1), and $D_{\phi^0} \equiv s - m^2_{\phi^0} + i\Gamma_{\phi^0}m_{\phi^0}$. In addition, we have introduced the following notation

$$k_{f\phi^0} \equiv \begin{cases} k_{d\phi^0}\,, & \text{for } f = d\,, e^-\,, \\ k_{u\phi^0}\,, & \text{for } f = u\,, \\ 0\,, & \text{for } f = \nu\,, \end{cases} \qquad v_f \equiv \begin{cases} v_d\,, & \text{for } f = d\,, e^-\,, \\ v_u\,, & \text{for } f = u\,, \nu\,, \end{cases} \tag{29.6.14}$$

where v_u, v_d are the neutral Higgs vacuum expectation values [cf. eq. (K.1.9)] and $k_{u\phi^0}$ and $k_{d\phi^0}$ are defined in eqs. (K.1.7) and (K.1.8). As the order of the spinor wave functions is $(1, 2, f1, f2)$ for all four terms of $\mathcal{M}_H$, no extra minus signs appear.

A good check of the above calculations is to repeat the analysis in the 't Hooft–Feynman gauge (where the gauge parameter $\xi = 1$). In this gauge, $\mathcal{M}_Z = \mathcal{M}_Z^{(1)}$, since the term proportional to $Q^\mu Q^\nu$ is absent from the gauge boson propagator. However, we must now include the diagrams of Fig. 29.6.2 with $\phi^0 = G^0$. In the 't Hooft–Feynman gauge, $m_{G^0} = m_Z$ and $D_{G^0} = D_Z$. Moreover, using eqs. (K.1.7) and (K.1.8),

$$\frac{k_{fG^0}}{v_f} = \frac{2iT_f}{v}\,. \tag{29.6.15}$$

Hence, using eq. (29.6.13) with $\phi^0 = G^0$,

$$i\mathcal{M}_G = \frac{m_f}{\sqrt{2}\,vD_Z}(2T_f)\,Y^{G^0\chi_1^0\chi_1^0}\left(x_1x_2 - y_1^\dagger y_2^\dagger\right)\left(y_{f1}y_{f2} - x^\dagger_{f1}x^\dagger_{f2}\right), \tag{29.6.16}$$

where we have noted that $iY^{G^0\chi_1^0\chi_1^0}$ is real. In particular, using eq. (K.3.14) and recalling that $m_W^2 = m_Z^2c_W^2 = \frac{1}{2}g^2v^2$, we confirm that $\mathcal{M}_G = \mathcal{M}_Z^{(2)}$ as expected from gauge invariance.

We next evaluate the t/u-channel exchange diagrams shown in Figs. 29.6.3 and 29.6.4. We neglect $\widetilde{f}_L$–$\widetilde{f}_R$ mixing. Eight Feynman graphs contribute, and we denote the total invariant amplitude by:

$$\mathcal{M}_{\tilde{f}} = \sum_{j=1}^{2}(\mathcal{M}^{(tj)}_{\tilde{f}_L} + \mathcal{M}^{(uj)}_{\tilde{f}_L} + \mathcal{M}^{(tj)}_{\tilde{f}_R} + \mathcal{M}^{(uj)}_{\tilde{f}_R})\,, \tag{29.6.17}$$

where $j = 1, 2$ labels the contributions of Figs. 29.6.3 and 29.6.4, respectively, and the other superscripts (t or u) and subscripts ($\widetilde{f}_L$ or $\widetilde{f}_R$) indicate the exchange channel and the exchanged particle, respectively. These matrix elements are evaluated by applying the rules of Fig. K.4.2.

The graphs of Fig. 29.6.3 are sensitive to the gaugino components of $\widetilde{N}_1$, and yield

$$i\mathcal{M}^{(t1)}_{\tilde{f}_L} = (-1)\left(-ig\sqrt{2}\right)^2\left(\frac{i}{t - m^2_{\tilde{f}_L}}\right)\left|T_3^f N_{12} + \frac{s_W}{c_W}(Q_f - T_3^f)N_{11}\right|^2 (y_1^\dagger x^\dagger_{f1})(x_2y_{f2})\,, \tag{29.6.18}$$

$$i\mathcal{M}^{(u1)}_{\tilde{f}_L} = \left(-ig\sqrt{2}\right)^2\left|T_3^f N_{12} + \frac{s_W}{c_W}(Q_f - T_3^f)N_{11}\right|^2\left(\frac{i}{u - m^2_{\tilde{f}_L}}\right)(x_1y_{f2})(y_2^\dagger x^\dagger_{f1})\,, \tag{29.6.19}$$

$$i\mathcal{M}^{(t1)}_{\tilde{f}_R} = (-1)\left(i\sqrt{2}g\frac{s_W}{c_W}Q_f\right)^2\left(\frac{i}{t - m^2_{\tilde{f}_R}}\right)|N_{11}|^2\,(x_1y_{f1})(y_2^\dagger x^\dagger_{f2})\,, \tag{29.6.20}$$

$$i\mathcal{M}^{(u1)}_{\tilde{f}_R} = \left(i\sqrt{2}g\frac{s_W}{c_W}Q_f\right)^2\left(\frac{i}{u - m^2_{\tilde{f}_R}}\right)|N_{11}|^2\,(y_1^\dagger x^\dagger_{f2})(x_2y_{f1})\,. \tag{29.6.21}$$

The explicit factors of (-1) in eqs. (29.6.18) and (29.6.20) are present because the order of the spinor wave functions in these cases is an odd permutation of the ordering $(1, 2, f1, f2)$ established in the computation of the s-channel amplitudes.

The graphs of Fig. 29.6.4 are sensitive to the higgsino components of $\widetilde{N}_1$, and yield

$$i\mathcal{M}^{(t2)}_{\tilde{f}_L} = (-1)\left(\frac{-im_f}{v_f}\right)^2\left(\frac{i}{t-m^2_{\tilde{f}_L}}\right)|N_{1f}|^2\,(x_1y_{f1})(y_2^\dagger x_{f2}^\dagger)\,, \tag{29.6.22}$$

$$i\mathcal{M}^{(u2)}_{\tilde{f}_L} = \left(\frac{-im_f}{v_f}\right)^2\left(\frac{i}{u-m^2_{\tilde{f}_L}}\right)|N_{1f}|^2\,(y_1^\dagger x_{f2}^\dagger)(x_2y_{f1})\,, \tag{29.6.23}$$

$$i\mathcal{M}^{(t2)}_{\tilde{f}_R} = (-1)\left(\frac{-im_f}{v_f}\right)^2\left(\frac{i}{t-m^2_{\tilde{f}_R}}\right)|N_{1f}|^2\,(y_1^\dagger x_{f1}^\dagger)(x_2y_{f2})\,, \tag{29.6.24}$$

$$i\mathcal{M}^{(u2)}_{\tilde{f}_R} = \left(\frac{-im_f}{v_f}\right)^2\left(\frac{i}{u-m^2_{\tilde{f}_R}}\right)|N_{1f}|^2\,(x_1y_{f2})(y_2^\dagger x_{f1}^\dagger)\,, \tag{29.6.25}$$

where v_f is defined in eq. (29.6.14), and

$$N_{1f} \equiv \begin{cases} N_{13}\,, & \text{for } f = d\,,\, e^-\,, \\ N_{14}\,, & \text{for } f = u\,, \\ 0\,, & \text{for } f = \nu\,. \end{cases} \tag{29.6.26}$$

As before, the explicit factors of (-1) are due to the ordering of the spinor wave functions.

It is convenient to write the total matrix element for $\widetilde{N}_1\widetilde{N}_1 \to f\bar{f}$ as the sum of products of separate neutralino and final state fermionic currents. The contributions of the s-channel diagrams are already in this form. The contributions of the t– and u–channel diagrams given in eqs. (29.6.18)–(29.6.25) can be rearranged using the Fierz identities of eqs. (29.68)–(29.70),

$$y_1^\dagger x_{f1}^\dagger x_2 y_{f2} = -\tfrac{1}{2}(y_1^\dagger\overline{\sigma}^\mu x_2)(x_{f1}^\dagger\overline{\sigma}_\mu y_{f2})\,, \tag{29.6.27}$$

$$x_1 y_{f2} y_2^\dagger x_{f1}^\dagger = -\tfrac{1}{2}(x_1\sigma^\mu y_2^\dagger)(x_{f1}^\dagger\overline{\sigma}_\mu y_{f2})\,, \tag{29.6.28}$$

$$x_1 y_{f1} y_2^\dagger x_{f2}^\dagger = -\tfrac{1}{2}(x_1\sigma^\mu y_2^\dagger)(y_{f1}\sigma_\mu x_{f2}^\dagger)\,, \tag{29.6.29}$$

$$y_1^\dagger x_{f2}^\dagger x_2 y_{f1} = -\tfrac{1}{2}(y_1^\dagger\overline{\sigma}^\mu x_2)(y_{f1}\sigma_\mu x_{f2}^\dagger)\,. \tag{29.6.30}$$

Combining the result of the s, t, and u–channel contributions, we have for the total amplitude:

$$\begin{aligned}\mathcal{M} = {}& \frac{m_f m_{\widetilde{N}_1}}{m_Z^2}c_0\left(x_1x_2 - y_1^\dagger y_2^\dagger\right)\left(y_{f1}y_{f2} - x_{f1}^\dagger x_{f2}^\dagger\right) \\ & + c_1(y_1^\dagger\overline{\sigma}^\mu x_2)(x_{f1}^\dagger\overline{\sigma}_\mu y_{f2}) + c_2(x_1\sigma^\mu y_2^\dagger)(x_{f1}^\dagger\overline{\sigma}_\mu y_{f2}) + c_3(x_1\sigma^\mu y_2^\dagger)(y_{f1}\sigma_\mu x_{f2}^\dagger) + c_4(y_1^\dagger\overline{\sigma}^\mu x_2)(y_{f1}\sigma_\mu x_{f2}^\dagger) \\ & + m_f\left[c_5(x_1x_2)(y_{f1}y_{f2}) + c_6(x_1x_2)(x_{f1}^\dagger x_{f2}^\dagger) + c_7(y_1^\dagger y_2^\dagger)(y_{f1}y_{f2}) + c_8(y_1^\dagger y_2^\dagger)(x_{f1}^\dagger x_{f2}^\dagger)\right]\,,\end{aligned} \tag{29.6.31}$$

where the coefficients c_0 , c_1 , ... , c_4 are given by

$$c_0 = -g^2 \frac{2T_3^f O''^L_{11}}{c_W^2 D_Z} \,, \tag{29.6.32}$$

$$c_1 = -g^2 \left[\frac{(T_3^f - s_W^2 Q_f) O''^L_{11}}{c_W^2 D_Z} + \frac{|T_3^f N_{12} + \frac{s_W}{c_W}(Q_f - T_3^f) N_{11}|^2}{t - m^2_{\tilde{f}_L}} \right] - \frac{m_f^2}{2v_f^2} \left(\frac{|N_{1f}|^2}{t - m^2_{\tilde{f}_R}} \right) \,, \tag{29.6.33}$$

$$c_2 = g^2 \left[\frac{(T_3^f - s_W^2 Q_f) O''^L_{11}}{c_W^2 D_Z} + \frac{|T_3^f N_{12} + \frac{s_W}{c_W}(Q_f - T_3^f) N_{11}|^2}{u - m^2_{\tilde{f}_L}} \right] + \frac{m_f^2}{2v_f^2} \left(\frac{|N_{1f}|^2}{u - m^2_{\tilde{f}_R}} \right) \,, \tag{29.6.34}$$

$$c_3 = -g^2 \frac{s_W^2}{c_W^2} Q_f \left[\frac{O''^L_{11}}{D_Z} + \frac{Q_f |N_{11}|^2}{t - m^2_{\tilde{f}_R}} \right] - \frac{m_f^2}{2v_f^2} \left(\frac{|N_{1f}|^2}{t - m^2_{\tilde{f}_L}} \right) \,, \tag{29.6.35}$$

$$c_4 = g^2 \frac{s_W^2}{c_W^2} Q_f \left[\frac{O''^L_{11}}{D_Z} + \frac{Q_f |N_{11}|^2}{u - m^2_{\tilde{f}_R}} \right] + \frac{m_f^2}{2v_f^2} \left(\frac{|N_{1f}|^2}{u - m^2_{\tilde{f}_L}} \right) \,. \tag{29.6.36}$$

The coefficients c_5, ..., c_8 are obtained from eq. (29.6.13) and represent the s-channel Higgs exchange contributions to the annihilation matrix element.

In the non–relativistic limit, $|\vec{\boldsymbol{p}}| \ll m_{\widetilde{N}_1}$. Then $t \simeq u \simeq -m^2_{\widetilde{N}_1} + m_f^2$, and we can approximate[71] $c_1 = -c_2$ and $c_3 = -c_4$. Hence, the total amplitude, eq. (29.6.31), can be written as

$$\begin{aligned} \mathcal{M} = {} & \frac{m_f m_{\widetilde{N}_1}}{m_Z^2} c_0 \left(x_1 x_2 - y_1^\dagger y_2^\dagger \right) \left(y_{f1} y_{f2} - x_{f1}^\dagger x_{f2}^\dagger \right) \\ & + \left[y_1^\dagger \overline{\sigma}^\mu x_2 - x_1 \sigma^\mu y_2^\dagger \right] \left[c_1 (x_{f1}^\dagger \overline{\sigma}_\mu y_{f2}) - c_3 (y_{f1} \sigma_\mu x_{f2}^\dagger) \right] + \mathcal{M}_H \,, \end{aligned} \tag{29.6.37}$$

where the s-channel Higgs exchange contributions, $\mathcal{M}_H$, will be neglected for simplicity in the subsequent analysis. The spin-averaged squared matrix element for $\widetilde{N}_1\widetilde{N}_1 \to f\bar{f}$ then takes the following form:

$$\begin{aligned} \tfrac{1}{4} \sum_{s_1,s_2,s_{f1},s_{f2}} |\mathcal{M}_Z + \mathcal{M}_{\tilde{f}}|^2 = {} & N_{\mu\nu} \left[|c_1|^2 F_1^{\mu\nu} + |c_3|^2 F_2^{\mu\nu} - 2\mathrm{Re}(c_1 c_3^*) F_{12}^{\mu\nu} \right] + \frac{m_f^2 m^2_{\widetilde{N}_1}}{m_Z^4} |c_0|^2 N F \\ & + \frac{2 m_f m_{\widetilde{N}_1}}{m_Z^2} \mathrm{Re}[c_0^* (c_1 + c_3)] N_\mu F^\mu \,, \end{aligned} \tag{29.6.38}$$

where $N_{\mu\nu}$, N_μ and N are spin-averaged tensor, vector and scalar quantities that depend on the initial state neutralino kinematics and $F^{\mu\nu}_{1,2,12}$, F^μ and F are spin-summed tensor, vector and scalar quantities that depend on the final state fermion kinematics. These quantities are easily computed using the projection operators of eqs. (29.3.57)–(29.3.60) and the standard trace techniques to perform the spin averages and sums. Explicitly, the spin-averaged neutralino quantities are

$$N \equiv \tfrac{1}{4} \sum_{s_1,s_2} (x_1 x_2 - y_1^\dagger y_2^\dagger)(x_2^\dagger x_1^\dagger - y_2 y_1) = p_1 \cdot p_2 + m^2_{\widetilde{N}_1} = 2E^2 \,, \tag{29.6.39}$$

$$N^\mu \equiv \tfrac{1}{4} \sum_{s_1,s_2} (y_1^\dagger \overline{\sigma}^\mu x_2 - x_1 \sigma^\mu y_2^\dagger)(x_2^\dagger x_1^\dagger - y_2 y_1) = -m_{\widetilde{N}_1} (p_1 + p_2)^\mu = \begin{cases} -2 m_{\widetilde{N}_1} E \,, & \mu = 0 \,, \\ 0 \,, & \mu = i \,, \end{cases} \tag{29.6.40}$$

[71] In particular, we assume that $\widetilde{f}_L$ and $\widetilde{f}_R$ are significantly heavier than all other particles in the annihilation process. Consequently, we can ignore all $\mathcal{O}(|\vec{\boldsymbol{p}}|/m_{\tilde{f}_{L,R}})$ terms in $c_1 + c_2$ and $c_3 + c_4$.

and a symmetric second-rank tensor,

$$N^{\mu\nu} \equiv \tfrac{1}{4}\sum_{s_1,s_2}(y_1^\dagger\overline{\sigma}^\mu x_2 - x_1\sigma^\mu y_2^\dagger)(x_2^\dagger\overline{\sigma}^\nu y_1 - y_2\sigma^\nu x_1^\dagger) = p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - g^{\mu\nu}(p_1\cdot p_2 - m_{\widetilde{N}_1}^2)$$

$$= \begin{cases} 2m_{\widetilde{N}_1}^2\,, & \mu=\nu=0\,, \\ 0\,, & \mu=0\,,\ \nu=j \ \text{ or } \ \mu=i\,,\ \nu=0\,, \\ 2\left[|\vec{\boldsymbol{p}}|^2\,\delta^{ij} - p^i p^j\right]\,, & \mu=i,\ \nu=j\,, \end{cases} \tag{29.6.41}$$

where the final results given in eqs. (29.6.39)–(29.6.41) have been evaluated in the CM frame. Similarly, the spin-summed final state fermion quantities are

$$F \equiv \sum_{s_{f1},s_{f2}}(y_{f1}y_{f2} - x_{f1}^\dagger x_{f2}^\dagger)(y_{f2}^\dagger y_{f1}^\dagger - x_{f2}x_{f1}) = 4(k_1\cdot k_2 + m_f^2) = 8E^2\,, \tag{29.6.42}$$

$$F^\mu \equiv \sum_{s_{f1},s_{f2}}(x_{f1}^\dagger\overline{\sigma}^\mu y_{f2})(y_{f2}^\dagger y_{f1}^\dagger - x_{f2}x_{f1}) = -\sum_{s_{f1},s_{f2}}(y_{f1}\sigma^\mu x_{f2}^\dagger)(y_{f2}^\dagger y_{f1}^\dagger - x_{f2}x_{f1})$$

$$= 2m_f(k_1+k_2)^\mu = \begin{cases} 4m_f E\,, & \mu=0\,, \\ 0\,, & \mu=i\,, \end{cases} \tag{29.6.43}$$

after evaluating the above quantities in the CM frame, and

$$F_1^{\mu\nu} \equiv \sum_{s_{f1},s_{f2}}(x_{f1}^\dagger\overline{\sigma}^\mu y_{f2})(y_{f2}^\dagger\overline{\sigma}^\nu x_{f1}) = k_{1\rho}k_{2\lambda}\,\mathrm{Tr}(\sigma^\rho\overline{\sigma}^\mu\sigma^\lambda\overline{\sigma}^\nu)\,, \tag{29.6.44}$$

$$F_2^{\mu\nu} \equiv \sum_{s_{f1},s_{f2}}(y_{f1}\sigma^\mu x_{f2}^\dagger)(x_{f2}\sigma^\nu y_{f1}^\dagger) = k_{1\rho}k_{2\lambda}\,\mathrm{Tr}(\overline{\sigma}^\rho\sigma^\mu\overline{\sigma}^\lambda\sigma^\nu)\,, \tag{29.6.45}$$

$$F_{12}^{\mu\nu} \equiv \sum_{s_{f1},s_{f2}}(y_{f1}\sigma^\mu x_{f2}^\dagger)(y_{f2}^\dagger\overline{\sigma}^\nu x_{f1}) = \sum_{s_{f1},s_{f2}}(x_{f1}^\dagger\overline{\sigma}^\mu y_{f2})(x_{f2}\sigma^\nu y_{f1}^\dagger) = -m_f^2\,\mathrm{Tr}(\sigma^\mu\overline{\sigma}^\nu). \tag{29.6.46}$$

Since $N^{\mu\nu}$ is symmetric, the antisymmetric parts of $F_1^{\mu\nu}$ and $F_2^{\mu\nu}$ do not contribute in eq. (29.6.38). The symmetric parts of $F_1^{\mu\nu}$ and $F_2^{\mu\nu}$ are equal and given by:

$$[F_1^{\mu\nu}]_{\rm symm} = [F_2^{\mu\nu}]_{\rm symm} = 2(k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - k_1\cdot k_2 g^{\mu\nu})$$

$$= \begin{cases} 2m_f^2\,, & \mu=\nu=0\,, \\ 0\,, & \mu=0\,,\ \nu=j \ \text{ or } \ \mu=i\,,\ \nu=0\,, \\ 2m_f^2(2\hat{\boldsymbol{k}}^i\hat{\boldsymbol{k}}^j - \delta^{ij}) - 4E^2(\hat{\boldsymbol{k}}^i\hat{\boldsymbol{k}}^j - \delta^{ij})\,, & \mu=i,\ \nu=j\,, \end{cases} \tag{29.6.47}$$

and $F_{12}^{\mu\nu} = -2m_f^2 g^{\mu\nu}$. The spin-averaged squared matrix element for $\widetilde{N}_1\widetilde{N}_1 \to f\bar{f}$ given by eq. (29.6.38) can now be fully evaluated, resulting in

$$\tfrac{1}{4}\sum_{s_1,s_2,s_{f1},s_{f2}}|\mathcal{M}_Z + \mathcal{M}_{\tilde{f}}|^2 = 4(|c_1|^2+|c_3|^2)\left[m_{\widetilde{N}_1}^2 m_f^2 + 2|\vec{\boldsymbol{p}}|^2(E^2(1+\cos^2\theta) - m_f^2\cos^2\theta)\right]$$

$$+8m_f^2\,\mathrm{Re}(c_1c_3^*)\left[m_{\widetilde{N}_1}^2 - 2|\vec{\boldsymbol{p}}|^2\right]$$

$$+\frac{16m_f^2 m_{\widetilde{N}_1}^2}{m_Z^4}\,E^2\left[E^2|c_0|^2 - m_Z^2\mathrm{Re}[c_0^*(c_1+c_3)]\right], \tag{29.6.48}$$

where $\cos\theta = \vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{k}}/|\vec{\boldsymbol{p}}|$. In the non-relativistic limit, we use eq. (29.6.3) and drop terms of $\mathcal{O}(|\vec{\boldsymbol{p}}|^4)$.

To compute $v_{\rm rel}\sigma_{\rm ann}$, we make use of the following result for the differential annihilation cross-section in the CM frame:

$$v_{\rm rel}\left(\frac{d\sigma}{d\Omega}\right)_{\rm CM} = \frac{1}{32\pi^2 s}\left(1-\frac{4m_f^2}{s}\right)^{1/2}|\mathcal{M}|^2_{\rm ave}\,, \tag{29.6.49}$$

where $|\mathcal{M}|^2_{\rm ave}$ is the squared matrix element for the annihilation process, averaged over initial spins and summed over final spins, and the relative velocity of the initial state neutralinos in the CM frame is given by $v_{\rm rel} = 4|\vec{\boldsymbol{p}}|/\sqrt{s} \simeq 2|\vec{\boldsymbol{p}}|/m_{\widetilde{N}_1}$, after noting that $\sqrt{s} \simeq 2m_{\widetilde{N}_1}$ in the non-relativistic limit. Inserting the squared matrix element obtained above into eq. (29.6.49) and integrating over solid angles, we end up with:

$$\begin{aligned} v_{\rm rel}\sigma_{\rm ann} = \frac{1}{8\pi E^2}\left(1-\frac{m_f^2}{E^2}\right)^{1/2}\Bigg\{&(|c_1|^2+|c_3|^2)\left[m_{\widetilde{N}_1}^2 m_f^2 + \frac{2|\vec{\boldsymbol{p}}|^2}{3}\left(4m_{\widetilde{N}_1}^2 - m_f^2\right)\right] \\ &+\frac{4m_f^2 m_{\widetilde{N}_1}^2}{m_Z^4}\left[m_{\widetilde{N}_1}^2(m_{\widetilde{N}_1}^2+2|\vec{\boldsymbol{p}}|^2)|c_0|^2 - m_Z^2(m_{\widetilde{N}_1}^2+|\vec{\boldsymbol{p}}|^2)\,{\rm Re}[c_0^*(c_1+c_3)]\right] \\ &+2m_f^2\,{\rm Re}(c_1c_3^*)\left[m_{\widetilde{N}_1}^2 - 2|\vec{\boldsymbol{p}}|^2\right] + \mathcal{O}(|\vec{\boldsymbol{p}}|^4)\Bigg\}\,, \end{aligned} \tag{29.6.50}$$

where the effects of the s-channel Higgs boson exchanges have been omitted.

The momentum dependence of eq. (29.6.50) reflects the famous p-wave suppression of the annihilation cross-section in the $m_f = 0$ limit noted in [181].[72] In general, the annihilation cross-section in the non-relativistic limit behaves as $v_{\rm rel}\,\sigma_{\rm ann} \propto |\vec{\boldsymbol{p}}|^{2\ell}$. Applying this result to eq. (29.6.50) in the $m_f = 0$ limit implies that $\ell = 1$. This is a consequence of the Majorana nature of the neutralino. In particular, in the limit of $m_f = 0$, the $f\bar{f}$ pair is in a $J = 1$ angular momentum state. However, Fermi statistics dictates that at threshold, a pair of identical Majorana fermions in a $J = 1$ state must have relative orbital angular momentum $\ell = 1$ (corresponding to p-wave annihilation). The s-wave annihilation (corresponding to the Majorana fermion pair in a $J = 0$ state) is suppressed by a factor of m_f^2, as is evident from eq. (29.6.50).

We have checked that eq. (29.6.50) corresponds to a result first obtained in [182] (although the latter reference omits the terms in eq. (29.6.50) proportional to c_0). However, we emphasize that this formula neglects the effects of s-channel Higgs boson exchanges. We invite the reader to complete the computation of the annihilation cross-section by including these terms (along with the effects of interference between the neglected contributions and the ones computed above).

The annihilation of $\widetilde{N}_1\widetilde{N}_1$ into heavy quarks (c, b and t), followed by the decay of the heavy quarks, can yield observable signatures suitable for indirect dark matter detection. For example, the annihilation of neutralinos in the galaxy provides a possible source of indirect dark matter detection via the observation of positrons in cosmic rays [187]. Neutralino dark matter can also be captured in the sun [188]. The neutrinos that arise (either directly or indirectly) from the neutralino annihilation in the sun can be detected on Earth (see, e.g., [189]).

29.6.14 $e^-e^+ \to \widetilde{C}_i^-\widetilde{C}_j^+$

Next we consider the pair production of charginos in electron-positron collisions. The s-channel Feynman diagrams are shown in Fig. 29.6.1, where we have also introduced the notation for the

[72] In [181], the annihilation rate for photinos is computed, corresponding to $N_{11} = c_W$, $N_{12} = s_W$ and $N_{13} = N_{14} = 0$. In this case, the Z boson and Higgs boson s-channel exchange diagrams are absent. The result presented in [181] should be multiplied by a factor of two (H. Goldberg, private communication)—the corrected expression then agrees with eq. (29.6.50).

fermion momenta and polarizations. The Mandelstam variables are given by

$$s = 2p_1{\cdot}p_2 = m^2_{\widetilde{C}_i} + m^2_{\widetilde{C}_j} + 2k_i{\cdot}k_j, \tag{29.6.1}$$

$$t = m^2_{\widetilde{C}_i} - 2p_1{\cdot}k_i = m^2_{\widetilde{C}_j} - 2p_2{\cdot}k_j, \tag{29.6.2}$$

$$u = m^2_{\widetilde{C}_i} - 2p_2{\cdot}k_i = m^2_{\widetilde{C}_j} - 2p_1{\cdot}k_j. \tag{29.6.3}$$

Note that the negatively charged chargino carries momentum and polarization (k_i, λ_i), while the positively charged one carries (k_j, λ_j).

Using the Feynman rules of Figs. J.1.2 and K.2.1, the sum of the photon-exchange diagrams is given by:

$$i\mathcal{M}_\gamma = \frac{-ig^{\mu\nu}}{s}\left(-ie\, x_1\sigma_\mu y_2^\dagger - ie\, y_1^\dagger\overline{\sigma}_\mu x_2\right)\left(ie\,\delta_{ij}y_i\sigma_\nu x_j^\dagger + ie\,\delta_{ij}x_i^\dagger\overline{\sigma}_\nu y_j\right). \tag{29.6.4}$$

The Z-exchange diagrams yields [cf. footnote 67]:

$$i\mathcal{M}_Z = \frac{-ig^{\mu\nu}}{D_Z}\left[\frac{ig}{c_W}(s_W^2 - \tfrac{1}{2})\, x_1\sigma_\mu y_2^\dagger + \frac{igs_W^2}{c_W}\, y_1^\dagger\overline{\sigma}_\mu x_2\right]\left[-\frac{ig}{c_W}O'^L_{ji}\, y_i\sigma_\nu x_j^\dagger - \frac{ig}{c_W}O'^R_{ji}\, x_i^\dagger\overline{\sigma}_\nu y_j\right], \tag{29.6.5}$$

where $D_Z \equiv s - m_Z^2 + i\Gamma_Z m_Z$. The t-channel Feynman diagram via sneutrino exchange is shown in Fig. 29.6.2. Applying the rules of Fig. K.4.1, we find:

$$i\mathcal{M}_{\tilde{\nu}_e} = (-1)\frac{i}{t - m^2_{\tilde{\nu}_e}}\left(-igV_{i1}^* x_1 y_i\right)\left(-igV_{j1}y_2^\dagger x_j^\dagger\right). \tag{29.6.6}$$

The Fermi-Dirac factor (-1) in this equation arises because the spinors appear an order which is an odd permutation of the order used in all of the s-channel diagram results.

One can now apply the Fierz transformation identities eqs. (29.68)–(29.70) to eqs. (29.6.4) and (29.6.5) to remove the σ and $\overline{\sigma}$ matrices. The result can be combined with the t-channel

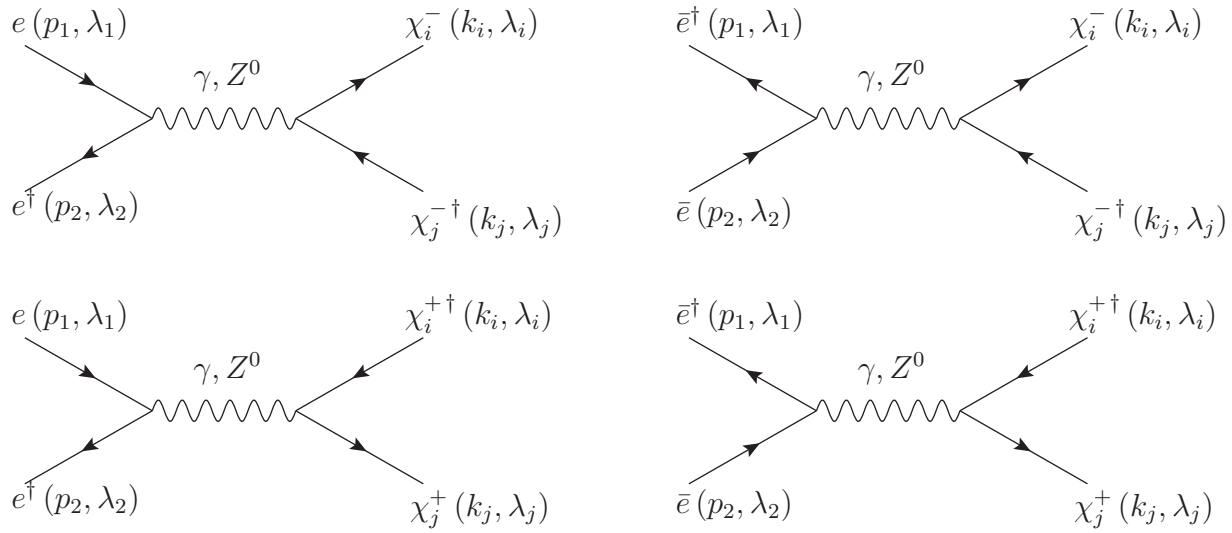


Figure 29.6.1: Feynman diagrams for $e^-e^+ \to \widetilde{C}_i^-\widetilde{C}_j^+$ via s-channel γ and Z^0 exchange.

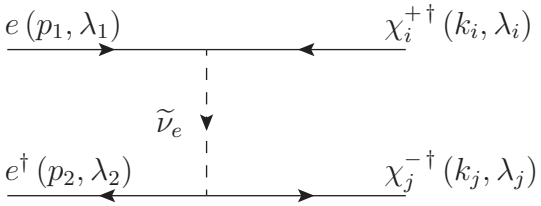


Figure 29.6.2: The Feynman diagram for $e^-e^+ \to \widetilde{C}_i^- \widetilde{C}_j^+$ via the t-channel exchange of a sneutrino.

contribution to obtain a total matrix element $\mathcal{M}$ with exactly the same form as eq. (29.6.9), but now with:

$$c_1 = 2\frac{e^2\delta_{ij}}{s} - \frac{g^2}{c_W^2 D_Z}(1-2s_W^2)O'^R_{ji}, \tag{29.6.7}$$

$$c_2 = \frac{2e^2\delta_{ij}}{s} - \frac{g^2}{c_W^2 D_Z}(1-2s_W^2)O'^L_{ji} + \frac{g^2 V^*_{i1}V_{j1}}{t-m^2_{\widetilde{\nu}_e}}, \tag{29.6.8}$$

$$c_3 = \frac{2e^2\delta_{ij}}{s} + \frac{2g^2 s_W^2}{c_W^2 D_Z}O'^R_{ji}\,, \tag{29.6.9}$$

$$c_4 = \frac{2e^2\delta_{ij}}{s} + \frac{2g^2 s_W^2}{c_W^2 D_Z}O'^L_{ji}. \tag{29.6.10}$$

The rest of this calculation is identical in form to eqs. (29.6.9)–(29.6.16), so that the result is:

$$\sum_{\text{spins}} |\mathcal{M}|^2 = (|c_1|^2+|c_4|^2)(u-m^2_{\widetilde{C}_i})(u-m^2_{\widetilde{C}_j}) + (|c_2|^2+|c_3|^2)(t-m^2_{\widetilde{C}_i})(t-m^2_{\widetilde{C}_j})$$
$$+2\text{Re}[c_1c_2^* + c_3c_4^*]m_{\widetilde{C}_i}m_{\widetilde{C}_j}s\,. \tag{29.6.11}$$

The differential cross-section then follows:

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\left(\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\right). \tag{29.6.12}$$

As in the previous subsection, we define $\cos\theta = \hat{\boldsymbol{p}}_{\boldsymbol{1}}\cdot\hat{\boldsymbol{k}}_{\boldsymbol{i}}$ (where θ is the angle between the initial state electron and $\widetilde{C}_i^-$ in the center-of-momentum frame). The Mandelstam variables t, u are given by

$$t = \frac{1}{2}\left[m^2_{\widetilde{C}_i} + m^2_{\widetilde{C}_j} - s + \lambda^{1/2}(s, m^2_{\widetilde{C}_i}, m^2_{\widetilde{C}_j})\cos\theta\right], \tag{29.6.13}$$

$$u = \frac{1}{2}\left[m^2_{\widetilde{C}_i} + m^2_{\widetilde{C}_j} - s - \lambda^{1/2}(s, m^2_{\widetilde{C}_i}, m^2_{\widetilde{C}_j})\cos\theta\right]. \tag{29.6.14}$$

The total cross-section can now be computed as

$$\sigma = \int_{t_-}^{t_+}\frac{d\sigma}{dt}dt\,, \tag{29.6.15}$$

where t_- and t_+ are obtained with $\cos\theta = -1$ and $+1$ in eq. (29.6.13), respectively. Our results agree with the original first complete calculation in [190]. Earlier work with simplifying assumptions is given in [191]. An extended calculation for the production of polarized charginos is given in [192].

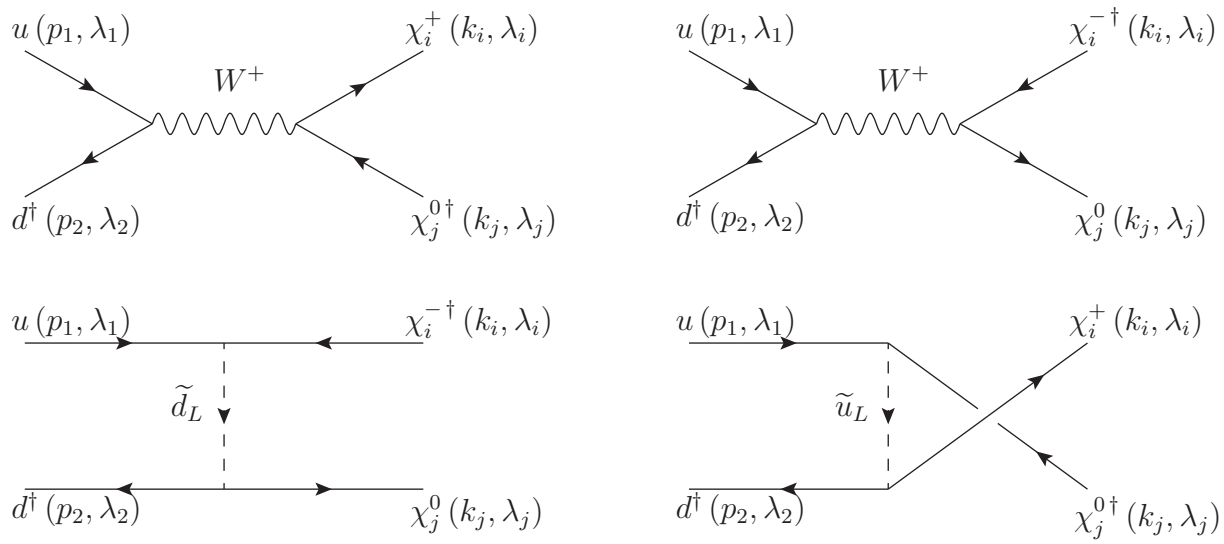


Figure 29.6.1: The four tree-level Feynman diagrams for $u\bar{d} \to \widetilde{C}_i^+ \widetilde{N}_j$.

29.6.15 $u\bar{d} \to \widetilde{C}_i^+ \widetilde{N}_j$

Next we consider the associated production of a chargino and a neutralino in quark, antiquark collisions. The leading order Feynman diagrams are shown in Fig. 29.6.1, where we have also defined the momenta and the helicities. The corresponding Mandelstam variables are

$$s = 2p_1{\cdot}p_2 = m^2_{\widetilde{C}_i} + m^2_{\widetilde{N}_j} + 2k_i{\cdot}k_j, \tag{29.6.1}$$

$$t = m^2_{\widetilde{C}_i} - 2p_1{\cdot}k_i = m^2_{\widetilde{N}_j} - 2p_2{\cdot}k_j, \tag{29.6.2}$$

$$u = m^2_{\widetilde{C}_i} - 2p_2{\cdot}k_i = m^2_{\widetilde{N}_j} - 2p_1{\cdot}k_j. \tag{29.6.3}$$

The matrix elements for the s-channel diagrams are obtained by applying the Feynman rules of Figs. J.1.2 and K.2.2:

$$i\mathcal{M}_s = \frac{-ig^{\mu\nu}}{s - m_W^2}\left(\frac{ig}{\sqrt{2}}x_1\sigma_\mu y_2^\dagger\right)\left(igO_{ji}^{L*}x_i^\dagger\overline{\sigma}_\nu y_j + igO_{ji}^{R*}y_i\sigma_\nu x_j^\dagger\right). \tag{29.6.4}$$

The external spinors are denoted by $x_1 \equiv x(\vec{\boldsymbol{p}}_1,\lambda_1)$, $y_2^\dagger \equiv y^\dagger(\vec{\boldsymbol{p}}_2,\lambda_2)$, $x_i^\dagger \equiv x^\dagger(\vec{\boldsymbol{k}}_i,\lambda_i)$, $y_j \equiv y(\vec{\boldsymbol{k}}_j,\lambda_j)$, etc. The matrix elements for the t and u channel graphs follow from the rules of Figs. K.4.1 and K.4.2:

$$i\mathcal{M}_t = (-1)\frac{i}{t - m^2_{\tilde{d}_L}}(-igU_{i1}^*)\left(\frac{ig}{\sqrt{2}}\Big[N_{j2} - \frac{s_W}{3c_W}N_{j1}\Big]\right)x_1y_iy_2^\dagger x_j^\dagger\,, \tag{29.6.5}$$

$$i\mathcal{M}_u = \frac{i}{u - m^2_{\tilde{u}_L}}(-igV_{i1})\left(\frac{ig}{\sqrt{2}}\Big[-N_{j2}^* - \frac{s_W}{3c_W}N_{j1}^*\Big]\right)x_1y_jy_2^\dagger x_i^\dagger\,. \tag{29.6.6}$$

The first factor of (-1) in eq. (29.6.5) is required because the order of the spinors $(1,i,2,j)$ is in an odd permutation of the order $(1,2,i,j)$ used in the s-channel and u-channel results.

Now we can use the Fierz relations eqs. (29.68) and (29.70) to rewrite the s-channel amplitude in a form without σ or $\overline{\sigma}$ matrices. Combining the result with the t-channel and u-channel

contributions yields a total $\mathcal{M}$ with exactly the same form as eq. (29.6.9), but now with

$$c_1 = -\sqrt{2} g^2 \left[\frac{O^{L*}_{ji}}{s - m_W^2} + \left(\frac{1}{2} N^*_{j2} + \frac{s_W}{6 c_W} N^*_{j1} \right) \frac{V_{i1}}{u - m_{\tilde{u}_L}} \right], \tag{29.6.7}$$

$$c_2 = -\sqrt{2} g^2 \left[\frac{O^{R*}_{ji}}{s - m_W^2} + \left(\frac{1}{2} N^*_{j2} - \frac{s_W}{6 c_W} N^*_{j1} \right) \frac{U^*_{i1}}{t - m_{\tilde{d}_L}} \right], \tag{29.6.8}$$

$$c_3 = c_4 = 0. \tag{29.6.9}$$

The rest of this calculation is identical in form to that of eqs. (29.6.9)–(29.6.16), leading to:

$$\sum_{\text{spins}} |\mathcal{M}|^2 = |c_1|^2 (u - m^2_{\widetilde{C}_i})(u - m^2_{\widetilde{N}_j}) + |c_2|^2 (t - m^2_{\widetilde{C}_i})(t - m^2_{\widetilde{N}_j}) + 2\text{Re}[c_1 c_2^*] m_{\widetilde{C}_i} m_{\widetilde{N}_j} s. \tag{29.6.10}$$

From this, one can obtain:

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2} \left(\frac{1}{3 \cdot 4} \sum_{\text{spins}} |\mathcal{M}|^2 \right), \tag{29.6.11}$$

where we have included a factor of 1/3 from the color average for the incoming quarks. As in the previous two subsections, eq. (29.6.11) can be expressed in terms of the angle between the u quark and the chargino in the center-of-momentum frame, using

$$t = \frac{1}{2} \left[m^2_{\widetilde{C}_i} + m^2_{\widetilde{N}_j} - s + \lambda^{1/2}(s, m^2_{\widetilde{C}_i}, m^2_{\widetilde{N}_j}) \cos\theta \right], \tag{29.6.12}$$

$$u = \frac{1}{2} \left[m^2_{\widetilde{C}_i} + m^2_{\widetilde{N}_j} - s - \lambda^{1/2}(s, m^2_{\widetilde{C}_i}, m^2_{\widetilde{N}_j}) \cos\theta \right]. \tag{29.6.13}$$

This process occurs in proton-antiproton and proton-proton collisions, where $\sqrt{s}$ is not fixed, and the angle θ is different than the lab frame angle. The observable cross-section depends crucially on experimental cuts. Our result in eq. (29.6.11) agrees with the complete computation in [193]. Earlier calculations in special supersymmetric scenarios, e.g. with photino mass eigenstates, are given in refs. [180, 194].

29.6.16 $\widetilde{N}_i \to \widetilde{N}_j \widetilde{N}_k \widetilde{N}_\ell$

Next we consider the decay of a neutralino $\widetilde{N}_i$ to three lighter neutralinos: $\widetilde{N}_j, \widetilde{N}_k, \widetilde{N}_\ell$. To the best of our knowledge, this process has not been computed in the literature. This decay is not likely to be phenomenologically relevant, because a variety of two-body decay modes will always be available. Furthermore, the calculation itself is quite complicated because of the large number of Feynman diagrams involved. Therefore, we consider this only as a matter-of-principle example of a process with four external state Majorana fermions, and will restrict ourselves to writing down the contributing matrix element amplitudes.

At tree level, the decay can proceed via a virtual Z^0 boson; the Feynman graphs are shown in Fig. 29.6.1. In addition, it can proceed via the exchange of any of the neutral scalar Higgs bosons of the MSSM, $\phi^0 = h^0, H^0, A^0$, as shown in Fig. 29.6.2. Since any of the final state neutralinos can directly couple to the initial state neutralino there are two more diagrams for each one shown in Figs. 29.6.1 and 29.6.2, for a total of 48 tree-level diagrams (counting each intermediate Higgs boson state as distinct). In all cases, the four-momenta of the neutralinos $\widetilde{N}_i$, $\widetilde{N}_j$, $\widetilde{N}_k$, $\widetilde{N}_\ell$ are denoted p_i, k_j, k_k, k_ℓ respectively.

We obtain the sum of the four diagrams in Fig. 29.6.1 by implementing the rules of Fig. K.2.1, and using the 't Hooft-Feynman gauge:

$$i\mathcal{M}_Z^{(1)} = \frac{-ig^2/c_W^2}{(p_i - k_j)^2 - m_Z^2}\Big(O''^L_{ji} x_i \sigma_\mu x_j^\dagger - O''^L_{ij} y_i^\dagger \overline{\sigma}_\mu y_j\Big)\Big(O''^L_{k\ell} x_k^\dagger \overline{\sigma}^\mu y_\ell - O''^L_{\ell k} y_k \sigma^\mu x_\ell^\dagger\Big)\,. \tag{29.6.1}$$

The external wave functions are $x_i \equiv x(\vec{\boldsymbol{p}}_i, \lambda_i)$, $x_{j,k,\ell} \equiv x(\vec{\boldsymbol{k}}_{j,k,\ell}, \lambda_{j,k,\ell})$, and analogously for $x^\dagger_{i,j,k,\ell}$, and $y_{i,j,k,\ell}$ and $y^\dagger_{i,j,k,\ell}$. Note that we have factorized the sum of the four diagrams, taking advantage of the common virtual boson line propagator. By a judicious use of the σ or $\overline{\sigma}$ version of the vertex rule, we have ensured that the order of the four spinor wave functions is the same for each of the four diagrams. Hence, no additional relative minus signs are required.

The contributions from the diagrams related to these by permutations can now be obtained from the appropriate substitutions $(j \leftrightarrow k)$ and $(j \leftrightarrow \ell)$:

$$i\mathcal{M}_Z^{(2)} = (-1)\frac{-ig^2/c_W^2}{(p_i - k_k)^2 - m_Z^2}\Big(O''^L_{ki} x_i \sigma_\mu x_k^\dagger - O''^L_{ik} y_i^\dagger \overline{\sigma}_\mu y_k\Big)\Big(O''^L_{j\ell} x_j^\dagger \overline{\sigma}^\mu y_\ell - O''^L_{\ell j} y_j \sigma^\mu x_\ell^\dagger\Big), \tag{29.6.2}$$

$$i\mathcal{M}_Z^{(3)} = (-1)\frac{-ig^2/c_W^2}{(p_i - k_\ell)^2 - m_Z^2}\Big(O''^L_{\ell i} x_i \sigma_\mu x_\ell^\dagger - O''^L_{i\ell} y_i^\dagger \overline{\sigma}_\mu y_\ell\Big)\Big(O''^L_{kj} x_k^\dagger \overline{\sigma}^\mu y_j - O''^L_{jk} y_k \sigma^\mu x_j^\dagger\Big). \tag{29.6.3}$$

The first factors of (-1) in $i\mathcal{M}_Z^{(2)}$ and $i\mathcal{M}_Z^{(3)}$ are present because the order of the spinors in each case appear in an odd permutation of the canonical order set by $i\mathcal{M}_Z^{(1)}$. Note that if we were to proceed to a computation of the decay rate, the very first step would be to apply the Fierz relations of eqs. (29.68)–(29.70) to eliminate all of the σ and $\overline{\sigma}$ matrices in the above amplitudes.

The diagrams in Fig. 29.6.2 combine to give a contribution:

$$i\mathcal{M}_{\phi^0}^{(1)} = \frac{-i}{(p_i - k_j)^2 - m_{\phi^0}^2}(Y^{ij} x_i y_j + Y_{ij} y_i^\dagger x_j^\dagger)(Y^{k\ell} y_k y_\ell + Y_{k\ell} x_k^\dagger x_\ell^\dagger)\,, \tag{29.6.4}$$

where we have used the Feynman rules of Fig. K.3.1, and adopted the shorthand notation $Y^{ij} = (Y_{ij})^* = Y^{\phi^0 \chi_i^0 \chi_j^0}$. Again we have factored the amplitude using the common virtual boson propagator. As in the Z-exchange diagrams, the other contributions can be obtained by the appropriate

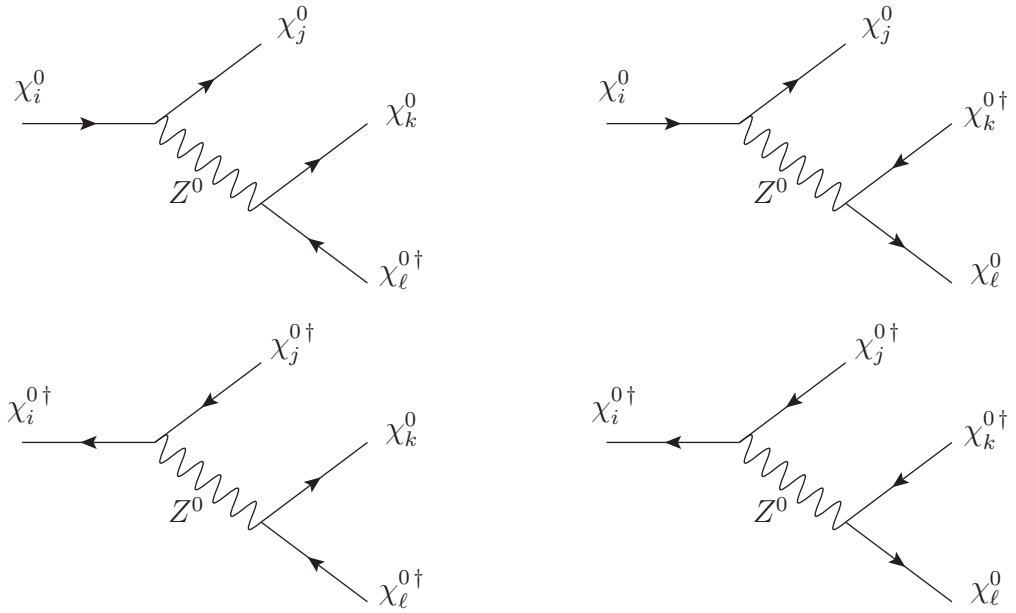


Figure 29.6.1: Four Feynman diagrams for $\widetilde{N}_i \to \widetilde{N}_j \widetilde{N}_k \widetilde{N}_\ell$ in the MSSM via Z^0 exchange. There are four more where $\widetilde{N}_j \leftrightarrow \widetilde{N}_k$ and another four where $\widetilde{N}_j \leftrightarrow \widetilde{N}_\ell$.

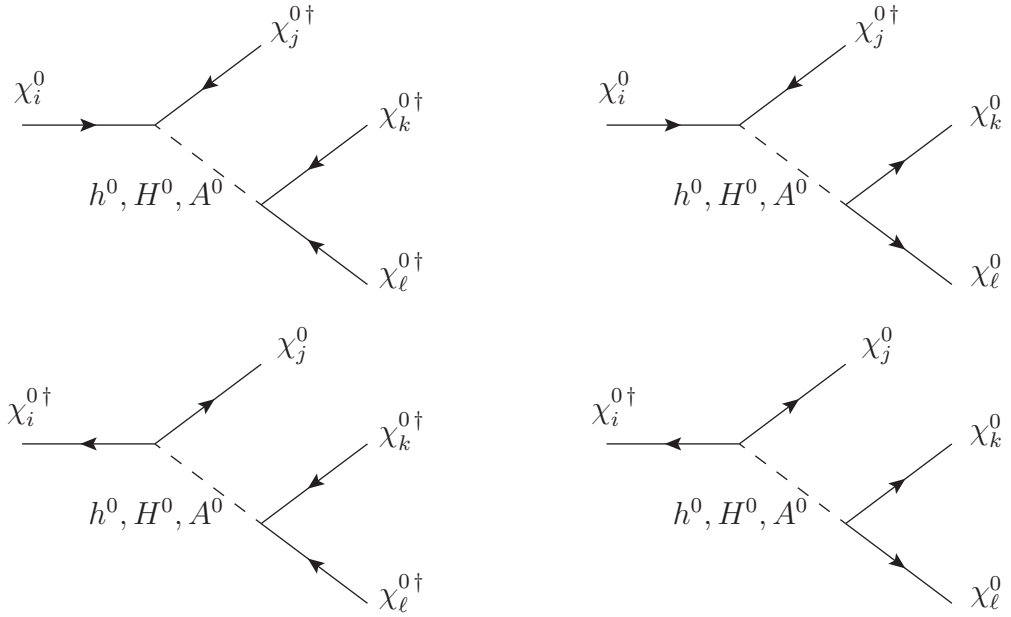


Figure 29.6.2: Four Feynman diagrams for $\widetilde{N}_i \to \widetilde{N}_j \widetilde{N}_k \widetilde{N}_\ell$ in the MSSM via $\phi^0 = h^0$, H^0, A^0 exchange. There are four more where $\widetilde{N}_j \leftrightarrow \widetilde{N}_k$ and another four where $\widetilde{N}_j \leftrightarrow \widetilde{N}_\ell$.

substitutions:

$$i\mathcal{M}^{(2)}_{\phi^0} = (-1)\frac{-i}{(p_i - k_k)^2 - m^2_{\phi^0}}(Y^{ik}x_i y_k + Y_{ik}y_i^\dagger x_k^\dagger)(Y^{j\ell}y_j y_\ell + Y_{j\ell}x_j^\dagger x_\ell^\dagger)\,, \tag{29.6.5}$$

$$i\mathcal{M}^{(3)}_{\phi^0} = (-1)\frac{-i}{(p_i - k_\ell)^2 - m^2_{\phi^0}}(Y^{i\ell}x_i y_\ell + Y_{i\ell}y_i^\dagger x_\ell^\dagger)(Y^{kj}y_k y_j + Y_{kj}x_k^\dagger x_j^\dagger)\,. \tag{29.6.6}$$

The first factors of (-1) in $i\mathcal{M}^{(2)}_{\phi^0}$ and $i\mathcal{M}^{(3)}_{\phi^0}$ are needed because the spinors in each case are in an odd permutation of the canonical order established earlier.

The total matrix element is obtained by adding all the contributing diagrams:

$$\mathcal{M} = \sum_{n=1}^{3}\mathcal{M}^{(n)}_Z + \sum_{\phi^0}\sum_{n=1}^{3}\mathcal{M}^{(n)}_{\phi^0}\,. \tag{29.6.7}$$

Squaring the matrix element, dividing by $2M_{\widetilde{N}_i}$, and integrating over phase space yields the total decay rate. Note that final states differing by the interchange of identical particles must be considered as a single state, counted once [39]. Given an N-body final state made up of ν_r particles of type r (where $r \le N$), we define a statistical factor S,

$$S = \prod_r \nu_r!\,, \qquad \text{where} \qquad \sum_r \nu_r = N\,. \tag{29.6.8}$$

Then, in computing the total decay rate, the integration over the total phase space must be divided by S to avoid over-counting. In the present example, $N = 3$ with $S = 2$ [or $S = 6$] in the case of two [or three] identical neutralinos in the final state, respectively

29.6.17 Three-body slepton decays: $\widetilde{\ell}_R^- \to \ell^- \tau^\pm \widetilde{\tau}_1^\mp$ for $\ell = e, \mu$

We next consider the three-body decays of sleptons through a virtual neutralino. The usual assumption in supersymmetric phenomenology is that these decays will have a very small branching

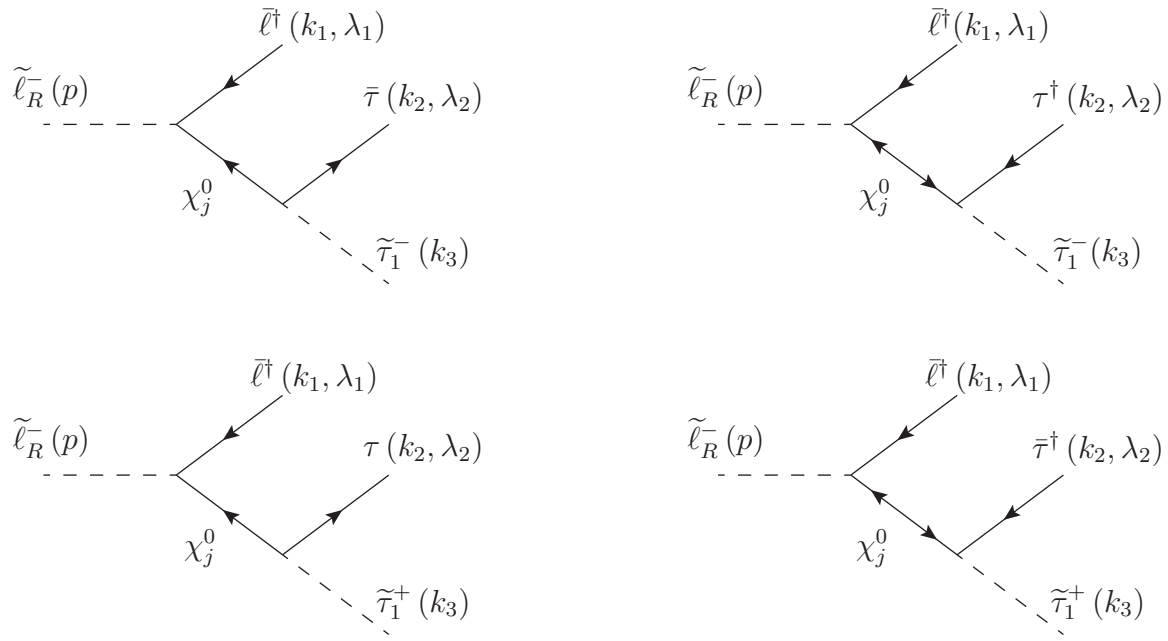


Figure 29.6.1: Feynman diagrams for the three-body slepton decays $\widetilde{\ell}_R^- \to \ell^- \tau^+ \widetilde{\tau}_1^-$ (top row) and $\widetilde{\ell}_R^- \to \ell^- \tau^- \widetilde{\tau}_1^+$ (bottom row) in the MSSM.

fraction, because a two-body decay to a lighter neutralino and lepton is always open. However, in Gauge Mediated Supersymmetry Breaking models with a non-minimal messenger sector, the sleptons can be lighter than the lightest neutralino [195, 196]. In that case, the mostly R-type smuon and selectron, $\widetilde{\mu}_R$ and $\tilde{e}_R$, will decay by $\widetilde{\ell}_R^- \to \ell^- \tau^\pm \widetilde{\tau}_1^\mp$. The lightest stau mass eigenstate, $\widetilde{\tau}_1^\pm$, is a mixture of the weak eigenstates $\widetilde{\tau}_L^\pm$ and $\widetilde{\tau}_R^\pm$, as described in Appendix K.4 :

$$\widetilde{\tau}_1^- = R^*_{\tilde{\tau}_1} \widetilde{\tau}_R^- + L^*_{\tilde{\tau}_1} \widetilde{\tau}_L^-, \tag{29.6.1}$$

and $\widetilde{\tau}_1^+ = (\widetilde{\tau}_1^-)^*$, while the $\widetilde{\mu}_R$ and $\tilde{e}_R$ are taken to be unmixed.

First consider the decay $\widetilde{\ell}_R^- \to \ell^- \tau^+ \widetilde{\tau}_1^-$, which proceed by the diagrams in the top row of Fig. 29.6.1. The momenta and polarizations of the particles are also indicated on the diagram. Using the Feynman rules of Fig. K.4.4, we find that the amplitudes of these two diagrams, for each neutralino $\tilde{N}_j$ exchanged, are:

$$i\mathcal{M}_1 = (-ia_j^{\tilde{\ell}*})(-ia_j^{\tilde{\tau}})\, y_1 \left[\frac{-i(p-k_1)\cdot\sigma}{(p-k_1)^2 - m^2_{\tilde{N}_j}}\right] x_2^\dagger, \tag{29.6.2}$$

$$i\mathcal{M}_2 = (-ia_j^{\tilde{\ell}*})(-ib_j^{\tilde{\tau}})\, y_1 \left[\frac{i m_{\tilde{N}_j}}{(p-k_1)^2 - m^2_{\tilde{N}_j}}\right] y_2. \tag{29.6.3}$$

where

$$a_j^{\tilde{\ell}} = \sqrt{2} g' N_{j1}, \tag{29.6.4}$$

$$a_j^{\tilde{\tau}} = Y_\tau N_{j3} L^*_{\tilde{\tau}_1} + \sqrt{2} g' N_{j1} R^*_{\tilde{\tau}_1}, \tag{29.6.5}$$

$$b_j^{\tilde{\tau}} = Y_\tau N^*_{j3} R^*_{\tilde{\tau}_1} - \frac{1}{\sqrt{2}} (g N^*_{j2} + g' N^*_{j1}) L^*_{\tilde{\tau}_1}. \tag{29.6.6}$$

The spinor wave function factors are $y_1 = y(\vec{\boldsymbol{k}}_1, \lambda_1)$, $y_2 = y(\vec{\boldsymbol{k}}_2, \lambda_2)$, and $x_2^\dagger = x^\dagger(\vec{\boldsymbol{k}}_2, \lambda_2)$.

In the following, we will use the kinematic variables

$$z_\ell \equiv 2p\cdot k_1/m^2_{\tilde{\ell}_R} = 2E_\ell/m_{\tilde{\ell}_R}, \qquad z_\tau \equiv 2p\cdot k_2/m^2_{\tilde{\ell}_R} = 2E_\tau/m_{\tilde{\ell}_R}, \tag{29.6.7}$$

$$r_{\tilde{N}_j} \equiv m_{\tilde{N}_j}/m_{\tilde{\ell}_R}, \qquad r_{\tilde{\tau}} \equiv m_{\tilde{\tau}_1}/m_{\tilde{\ell}_R}, \tag{29.6.8}$$

$$r_\tau \equiv m_\tau/m_{\tilde{\ell}_R}, \qquad r_\ell \equiv m_\ell/m_{\tilde{\ell}_R}. \tag{29.6.9}$$

The total amplitude then can be written as

$$\mathcal{M} = \sum_{j=1}^{4} \left[c_j y_1 (p-k_1)\cdot\sigma x_2^\dagger + d_j y_1 y_2 \right] , \tag{29.6.10}$$

where

$$c_j = -a_j^{\tilde{\ell}*} a_j^{\tilde{\tau}}/[m^2_{\ell_R}(r^2_{\tilde{N}_j} - 1 + z_\ell)], \tag{29.6.11}$$

$$d_j = a_j^{\tilde{\ell}*} b_j^{\tilde{\tau}} m_{\tilde{N}_j}/[m^2_{\ell_R}(r^2_{\tilde{N}_j} - 1 + z_\ell)]. \tag{29.6.12}$$

We consistently neglect the electron and muon masses and Yukawa couplings (so $r_\ell = 0$) in the matrix elements, but not below in the kinematic integration over phase space, where the muon mass can be important.

Using eqs. (29.44) and (29.45), we find

$$\begin{aligned} |\mathcal{M}|^2 = \sum_{j,k} \Big[& c_j c_k^*\, y_1 (p-k_1)\cdot\sigma x_2^\dagger\, x_2 (p-k_1)\cdot\sigma y_1^\dagger + d_j d_k^* y_1 y_2\, y_2^\dagger y_1^\dagger \\ & + c_j d_k^* y_1 (p-k_1)\cdot\sigma x_2^\dagger\, y_2^\dagger y_1^\dagger + c_j^* d_k x_2 (p-k_1)\cdot\sigma y_1^\dagger\, y_1 y_2 \Big]. \end{aligned} \tag{29.6.13}$$

Summing over the lepton spins using eqs. (29.3.57)–(29.3.60) gives

$$\begin{aligned} \sum_{\lambda_1,\lambda_2} |\mathcal{M}|^2 = \sum_{j,k} \Big[& c_j c_k^* \mathrm{Tr}[(p-k_1)\cdot\sigma k_2\cdot\overline{\sigma}(p-k_1)\cdot\sigma k_1\cdot\overline{\sigma}] + d_j d_k^* \mathrm{Tr}[k_2\cdot\sigma k_1\cdot\overline{\sigma}] \\ & - c_j d_k^* m_\tau \mathrm{Tr}[(p-k_1)\cdot\sigma k_1\cdot\overline{\sigma}] - c_j^* d_k m_\tau \mathrm{Tr}[(p-k_1)\cdot\sigma k_1\cdot\overline{\sigma}] \Big]. \end{aligned} \tag{29.6.14}$$

Taking the traces using eqs. (29.56) and (29.57) yields

$$\begin{aligned} \sum_{\text{spins}} |\mathcal{M}|^2 &= \sum_{j,k} \Big\{ c_j c_k^* [4k_1\cdot(p-k_1) k_2\cdot(p-k_1) - 2k_1\cdot k_2 (p-k_1)^2] + 2 d_j d_k^* k_1\cdot k_2 \\ &\qquad - 4\mathrm{Re}[c_j d_k^*] m_\tau k_1\cdot(p-k_1) \Big\} \\ &= \sum_{j,k} \Big\{ c_j c_k^* m^4_{\tilde{\ell}_R} [(1-z_\ell)(1-z_\tau) - r_{\tilde{\tau}}^2 + r_\tau^2] \\ &\qquad + d_j d_k^* m^2_{\tilde{\ell}_R} (z_\ell + z_\tau - 1 + r_{\tilde{\tau}}^2 - r_\tau^2) - 2\mathrm{Re}[c_j d_k^*] m_\tau m^2_{\tilde{\ell}_R} z_\ell \Big\}. \end{aligned} \tag{29.6.15}$$

The differential decay rate for $\widetilde{\ell}_R^- \to \ell^- \tau^+ \widetilde{\tau}_1^-$ then follows:

$$\frac{d^2\Gamma}{dz_\ell dz_\tau} = \frac{m_{\tilde{\ell}_R}}{256\pi^3} \left(\sum_{\text{spins}} |\mathcal{M}|^2 \right). \tag{29.6.16}$$

The total decay rate in that channel can be found by integrating over z_ℓ, z_τ, with the limits (see for example ref. [166]):

$$2r_\ell < z_\ell < 1 + r_\ell^2 - (r_\tau + r_{\tilde{\tau}})^2, \tag{29.6.17}$$

$$(z_\tau)_{\rm min} < z_\tau < (z_\tau)_{\rm max}\,, \tag{29.6.18}$$

where

$$(z_\tau)_{\rm min\,,max} = \frac{1}{2(1 - z_\ell + r_\ell^2)}\Big[(2 - z_\ell)(1 + r_\ell^2 + r_\tau^2 - r_{\tilde{\tau}}^2 - z_\ell) \mp (z_\ell^2 - 4r_\ell^2)^{1/2}\lambda^{1/2}(1 + r_\ell^2 - z_\ell, r_\tau^2, r_{\tilde{\tau}}^2)\Big]\,, \tag{29.6.19}$$

and the triangle function $\lambda^{1/2}$ is defined in eq. (29.6.11).

Now we turn to the competing decay $\widetilde{\ell}_R^- \to \ell^-\tau^-\widetilde{\tau}_1^+$, with diagrams appearing in the bottom row of Fig. 29.6.1. By appealing again to the Feynman rules of Fig. K.4.3, we find that the amplitude has exactly the same form as in eqs. (29.6.2) and (29.6.3), except now with $a_j^{\tilde{\tau}} \leftrightarrow b_j^{\tilde{\tau}*}$. Therefore, the entire previous calculation goes through precisely as before, but now with

$$c_j = \frac{-a_j^{\tilde{\ell}*} b_j^{\tilde{\tau}*}}{m_{\tilde{\ell}_R}^2 (r_{\tilde{N}_j}^2 - 1 + z_\ell)}, \tag{29.6.20}$$

$$d_j = \frac{a_j^{\tilde{\ell}*} a_j^{\tilde{\tau}*} m_{\tilde{N}_j}}{m_{\tilde{\ell}_R}^2 (r_{\tilde{N}_j}^2 - 1 + z_\ell)}. \tag{29.6.21}$$

The differential decay widths found above can be integrated to find the total decay widths. The results agree with ref. [197], except that the signs of the coefficient $c_{ij}^{(3)}$ and $c_{ij}^{(4)}$ in the published version of that paper are incorrect; the arXiv eprint version has been corrected. (Also, the notations for the sfermion mixing angle are different in that paper.) If $m_{\tilde{\ell}_R} - m_{\tilde{\tau}_1} - m_\tau$ is not too large, the resulting decays can have a macroscopic length in a detector, and the ratio of the two decay modes can provide an interesting probe of the supersymmetric Lagrangian.

29.6.18 Neutralino decay to photon and Goldstino: $\widetilde{N}_i \to \gamma\widetilde{G}$

The Goldstino $\widetilde{G}$ is a massless Weyl fermion that couples to the neutralino and photon fields according to the non-renormalizable Lagrangian term [198]:

$$\mathscr{L} = -\frac{a_i}{2}(\chi_i^0 \sigma^\mu \overline{\sigma}^\rho \sigma^\nu \partial_\mu \widetilde{G}^\dagger)\,(\partial_\nu A_\rho - \partial_\rho A_\nu) + \text{h.c.} \tag{29.6.1}$$

Here χ_i^0 is the left-handed two-component fermion field that corresponds to the neutralino $\widetilde{N}_i$ particle, $\widetilde{G}$ is the two-component fermion field corresponding to the (nearly) massless Goldstino, and the effective coupling is

$$a_i \equiv \frac{1}{\sqrt{2}\langle F\rangle}(N_{i1}^* \cos\theta_W + N_{i2}^* \sin\theta_W), \tag{29.6.2}$$

where N_{ij} the mixing matrix for the neutralinos [see eq. (K.2.8)], and $\langle F\rangle$ is the F-term expectation value associated with supersymmetry breaking. Therefore $\widetilde{N}_i$ can decay to γ plus $\widetilde{G}$ through the diagrams shown in Fig. 29.6.1, with amplitudes:

$$i\mathcal{M}_1 = i\frac{a_i}{2}\, x_{\tilde{N}} k_{\tilde{G}}\cdot\sigma\,(\varepsilon^*\cdot\overline{\sigma}\, k_\gamma\cdot\sigma - k_\gamma\cdot\overline{\sigma}\,\varepsilon^*\cdot\sigma)\, x_{\tilde{G}}^\dagger\,, \tag{29.6.3}$$

$$i\mathcal{M}_2 = -i\frac{a_i^*}{2}\, y_{\tilde{N}}^\dagger k_{\tilde{G}}\cdot\overline{\sigma}\,(\varepsilon^*\cdot\sigma\, k_\gamma\cdot\overline{\sigma} - k_\gamma\cdot\sigma\,\varepsilon^*\cdot\overline{\sigma})\, y_{\tilde{G}}\,. \tag{29.6.4}$$

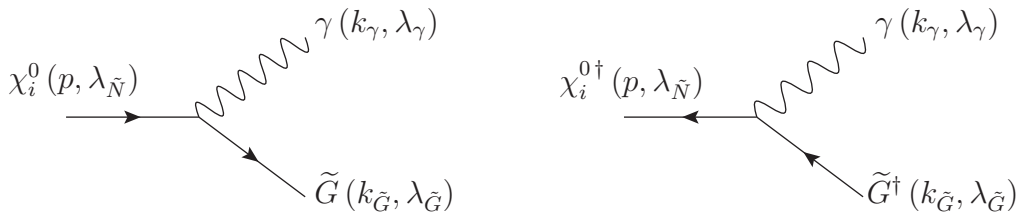


Figure 29.6.1: The two Feynman diagrams for $\widetilde{N}_i \to \gamma\widetilde{G}$ in supersymmetric models with a light Goldstino.

Here $x_{\tilde{N}} \equiv x(\vec{\boldsymbol{p}}, \lambda_{\tilde{N}})$, $y^\dagger_{\tilde{N}} \equiv y^\dagger(\vec{\boldsymbol{p}}, \lambda_{\tilde{N}})$, and $x^\dagger_{\tilde{G}} \equiv x^\dagger(\vec{\boldsymbol{k}}_{\tilde{G}}, \lambda_{\tilde{G}})$, $y_{\tilde{G}} \equiv y(\vec{\boldsymbol{k}}_{\tilde{G}}, \lambda_{\tilde{G}})$, and $\varepsilon^* = \varepsilon^*(\vec{\boldsymbol{k}}_\gamma, \lambda_\gamma)$ are the external wave function factors for the neutralino, Goldstino, and photon, respectively. Using the on-shell condition $k_\gamma\cdot\varepsilon^* = 0$, we have $k_\gamma\cdot\sigma\varepsilon^*\cdot\overline{\sigma} = -\varepsilon^*\cdot\sigma k_\gamma\cdot\overline{\sigma}$ and $k_\gamma\cdot\overline{\sigma}\varepsilon^*\cdot\sigma = -\varepsilon^*\cdot\overline{\sigma}k_\gamma\cdot\sigma$ from eqs. (29.52) and (29.53). So we can rewrite the total amplitude as

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 = x_{\tilde{N}} A x^\dagger_{\tilde{G}} + y^\dagger_{\tilde{N}} B y_{\tilde{G}}\,, \tag{29.6.5}$$

where

$$A = a_i\, k_{\tilde{G}}\cdot\sigma\, \varepsilon^*\cdot\overline{\sigma}\, k_\gamma\cdot\sigma, \tag{29.6.6}$$
$$B = -a_i^*\, k_{\tilde{G}}\cdot\overline{\sigma}\, \varepsilon^*\cdot\sigma\, k_\gamma\cdot\overline{\sigma}. \tag{29.6.7}$$

The complex square of the matrix element is therefore

$$|\mathcal{M}|^2 = x_{\tilde{N}} A x^\dagger_{\tilde{G}} x_{\tilde{G}} \hat{A} x^\dagger_{\tilde{N}} + y^\dagger_{\tilde{N}} B y_{\tilde{G}} y^\dagger_{\tilde{G}} \hat{B} y_{\tilde{N}} + x_{\tilde{N}} A x^\dagger_{\tilde{G}} y^\dagger_{\tilde{G}} \hat{B} y_{\tilde{N}} + y^\dagger_{\tilde{N}} B y_{\tilde{G}} x_{\tilde{G}} \hat{A} x^\dagger_{\tilde{N}}, \tag{29.6.8}$$

where $\hat{A}$ and $\hat{B}$ are obtained from A and B by reversing the order of the σ and $\overline{\sigma}$ matrices and taking the complex conjugates of a_i and ε [cf. eq. (29.4.4) and the associated text].

Summing over the Goldstino spins using eqs. (29.3.57)–(29.3.60) now yields:

$$\sum_{\lambda_{\tilde{G}}} |\mathcal{M}|^2 = x_{\tilde{N}} A k_{\tilde{G}}\cdot\overline{\sigma}\hat{A} x^\dagger_{\tilde{N}} + y^\dagger_{\tilde{N}} B k_{\tilde{G}}\cdot\sigma\hat{B} y_{\tilde{N}}. \tag{29.6.9}$$

(The $A, \hat{B}$ and $\hat{A}, B$ cross terms vanish because of $m_{\tilde{G}} = 0$.) Averaging over the neutralino spins using eqs. (29.3.57) and (29.3.58), we find

$$\begin{aligned}\frac{1}{2}\sum_{\lambda_{\tilde{N}},\lambda_{\tilde{G}}} |\mathcal{M}|^2 &= \frac{1}{2}\mathrm{Tr}[A k_{\tilde{G}}\cdot\overline{\sigma}\hat{A} p\cdot\overline{\sigma}] + \frac{1}{2}\mathrm{Tr}[B k_{\tilde{G}}\cdot\sigma\hat{B} p\cdot\sigma] \\ &= \frac{1}{2}|a_i|^2\mathrm{Tr}[\varepsilon^*\cdot\overline{\sigma}\, k_\gamma\cdot\sigma\, k_{\tilde{G}}\cdot\overline{\sigma}\, k_\gamma\cdot\sigma\, \varepsilon\cdot\overline{\sigma}\, k_{\tilde{G}}\cdot\sigma\, p\cdot\overline{\sigma}\, k_{\tilde{G}}\cdot\sigma] + (\sigma \leftrightarrow \overline{\sigma}).\end{aligned} \tag{29.6.10}$$

We now use

$$k_\gamma\cdot\sigma\, k_{\tilde{G}}\cdot\overline{\sigma}\, k_\gamma\cdot\sigma = 2k_{\tilde{G}}\cdot k_\gamma\, k_\gamma\cdot\sigma, \tag{29.6.11}$$
$$k_{\tilde{G}}\cdot\sigma\, p\cdot\overline{\sigma}\, k_{\tilde{G}}\cdot\sigma = 2k_{\tilde{G}}\cdot p\, k_{\tilde{G}}\cdot\sigma, \tag{29.6.12}$$

which follow from eq. (29.54), and the corresponding identities with $\sigma \leftrightarrow \overline{\sigma}$, to obtain:

$$\frac{1}{2}\sum_{\lambda_{\tilde{N}},\lambda_{\tilde{G}}} |\mathcal{M}|^2 = 2|a_i|^2(k_{\tilde{G}}\cdot k_\gamma)(k_{\tilde{G}}\cdot p)\mathrm{Tr}[\varepsilon^*\cdot\overline{\sigma}\, k_\gamma\cdot\sigma\, \varepsilon\cdot\overline{\sigma}\, k_{\tilde{G}}\cdot\sigma] + (\sigma \leftrightarrow \overline{\sigma}). \tag{29.6.13}$$

Applying the photon spin-sum identity

$$\sum_{\lambda_\gamma} \varepsilon^\mu \varepsilon^{\nu *} = -g^{\mu\nu}\,, \tag{29.6.14}$$

and the trace identities eq. (29.57) and (29.58), we get

$$\frac{1}{2}\sum_{\lambda_\gamma,\lambda_{\tilde{N}},\lambda_{\tilde{G}}} |\mathcal{M}|^2 = 16|a_i|^2 (k_{\tilde{G}}\cdot k_\gamma)^2 (k_{\tilde{G}}\cdot p) = 2|a_i|^2 m_{\widetilde{N}_i}^6\,. \tag{29.6.15}$$

So, the decay rate is [199, 195]:

$$\Gamma(\widetilde{N}_i \to \gamma\widetilde{G}) = \frac{1}{16\pi m_{\widetilde{N}_i}}\left(\frac{1}{2}\sum_{\lambda_\gamma,\lambda_{\tilde{N}},\lambda_{\tilde{G}}} |\mathcal{M}|^2\right) = |N_{i1}\cos\theta_W + N_{i2}\sin\theta_W|^2 \frac{m_{\widetilde{N}_i}^5}{16\pi|\langle F\rangle|^2}. \tag{29.6.16}$$

29.6.19 Gluino pair production from gluon fusion: $gg \to \widetilde{g}\widetilde{g}$

In this subsection we will compute the cross-section for the process $gg \to \widetilde{g}\widetilde{g}$. The relevant Feynman diagrams are shown in Fig. 29.6.1. The initial state gluons have $SU(3)_c$ adjoint representation indices a and b, with momenta p_1 and p_2 and polarization vectors $\varepsilon_1^\mu = \varepsilon^\mu(\vec{\boldsymbol{p}}_1, \lambda_1)$ and $\varepsilon_2^\mu = \varepsilon^\mu(\vec{\boldsymbol{p}}_2, \lambda_2)$, respectively. The final state gluinos carry adjoint representation indices c and d, with momenta k_1 and k_2 and wave function spinors $x_1^\dagger = x^\dagger(\vec{\boldsymbol{k}}_1, \lambda_1')$ or $y_1 = y(\vec{\boldsymbol{k}}_1, \lambda_1')$ and $x_2^\dagger = x^\dagger(\vec{\boldsymbol{k}}_2, \lambda_2')$ or $y_2 = y(\vec{\boldsymbol{k}}_2, \lambda_2')$, respectively.

The Feynman rules for the gluino couplings in the supersymmetric extension of QCD are given in Fig. K.5.1. For the two s-channel amplitudes, we obtain:

$$\begin{aligned} i\mathcal{M}_s = {} & \big(-g_s f^{abe}[g_{\mu\nu}(p_1-p_2)_\rho + g_{\nu\rho}(p_1+2p_2)_\mu - g_{\mu\rho}(2p_1+p_2)_\nu]\big)\left(\frac{-ig^{\rho\kappa}}{s}\right)\varepsilon_1^\mu\varepsilon_2^\nu \\ & \times \Big[\big(-g_s f^{cde}\big)\, x_1^\dagger \overline{\sigma}_\kappa y_2 \;+\; \big(g_s f^{dce}\big)\, y_1 \sigma_\kappa x_2^\dagger\Big]\,. \end{aligned} \tag{29.6.1}$$

The first factor is the Feynman rule for the three-gluon interaction of standard QCD, and the second factor is the gluon propagator. The next four (t-channel) diagrams have a total amplitude:

$$\begin{aligned} i\mathcal{M}_t = {} & \big(-g_s f^{cea}\varepsilon_1^\mu\big)\big(-g_s f^{edb}\varepsilon_2^\nu\big)\, x_1^\dagger \overline{\sigma}_\mu \left[\frac{i(k_1-p_1)\cdot\sigma}{(k_1-p_1)^2 - m_{\tilde{g}}^2}\right]\overline{\sigma}_\nu y_2 \\ & + \big(g_s f^{eca}\varepsilon_1^\mu\big)\big(g_s f^{deb}\varepsilon_2^\nu\big)\, y_1 \sigma_\mu \left[\frac{i(k_1-p_1)\cdot\overline{\sigma}}{(k_1-p_1)^2 - m_{\tilde{g}}^2}\right]\sigma_\nu x_2^\dagger \\ & + \big(-g_s f^{cea}\varepsilon_1^\mu\big)\big(g_s f^{deb}\varepsilon_2^\nu\big)\, x_1^\dagger \overline{\sigma}_\mu \left[\frac{im_{\tilde{g}}}{(k_1-p_1)^2 - m_{\tilde{g}}^2}\right]\sigma_\nu x_2^\dagger \\ & + \big(g_s f^{eca}\varepsilon_1^\mu\big)\big(-g_s f^{edb}\varepsilon_2^\nu\big)\, y_1 \sigma_\mu \left[\frac{im_{\tilde{g}}}{(k_1-p_1)^2 - m_{\tilde{g}}^2}\right]\overline{\sigma}_\nu y_2. \end{aligned} \tag{29.6.2}$$

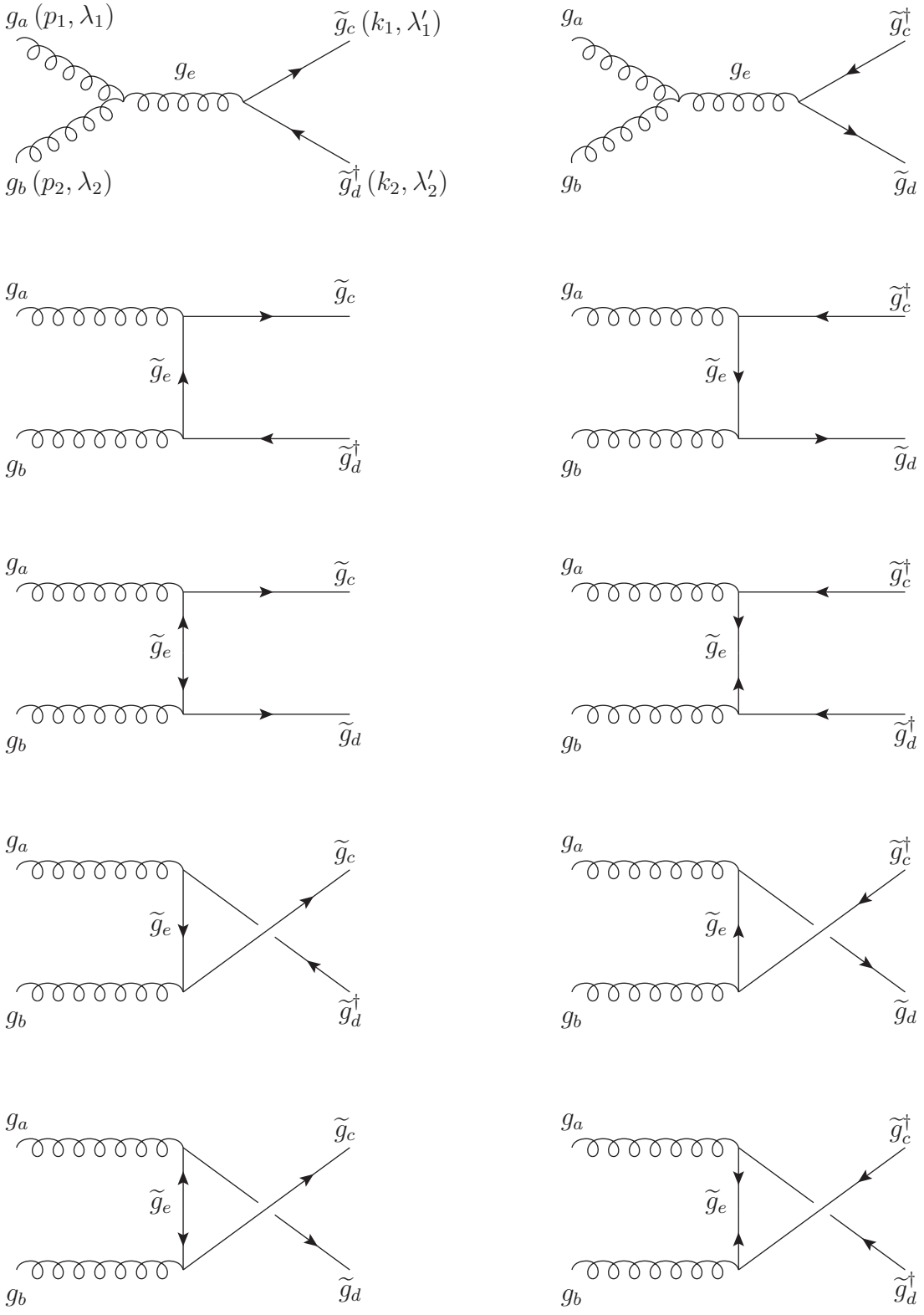


Figure 29.6.1: The ten Feynman diagrams for $gg \to \widetilde{g}\widetilde{g}$. The momentum and spin polarization assignments are indicated on the first diagram.

Finally, the u-channel Feynman diagrams result in:

$$\begin{aligned} i\mathcal{M}_u = {} & \big(-g_s f^{eda}\varepsilon_1^\mu\big)\big(-g_s f^{ceb}\varepsilon_2^\nu\big)\, x_1^\dagger \overline{\sigma}_\nu \left[\frac{i(k_1-p_2)\cdot\sigma}{(k_1-p_2)^2-m_{\tilde{g}}^2}\right]\overline{\sigma}_\mu y_2 \\ & + \big(g_s f^{dea}\varepsilon_1^\mu\big)\big(g_s f^{ecb}\varepsilon_2^\nu\big)\, y_1 \sigma_\nu \left[\frac{i(k_1-p_2)\cdot\overline{\sigma}}{(k_1-p_2)^2-m_{\tilde{g}}^2}\right]\sigma_\mu x_2^\dagger \\ & + \big(g_s f^{dea}\varepsilon_1^\mu\big)\big(-g_s f^{ceb}\varepsilon_2^\nu\big)\, x_1^\dagger \overline{\sigma}_\nu \left[\frac{im_{\tilde{g}}}{(k_1-p_2)^2-m_{\tilde{g}}^2}\right]\sigma_\mu x_2^\dagger \\ & + \big(-g_s f^{eda}\varepsilon_1^\mu\big)\big(g_s f^{ecb}\varepsilon_2^\nu\big)\, y_1 \sigma_\nu \left[\frac{im_{\tilde{g}}}{(k_1-p_2)^2-m_{\tilde{g}}^2}\right]\overline{\sigma}_\mu y_2. \end{aligned} \tag{29.6.3}$$

We choose to work with *real* transverse polarization vectors ε_1, ε_2. These vectors must both be orthogonal to the initial state collision axis in the center-of-momentum frame. Hence,

$$\varepsilon_1\cdot\varepsilon_1 = \varepsilon_2\cdot\varepsilon_2 = -1\,, \tag{29.6.4}$$
$$\varepsilon_1\cdot p_1 = \varepsilon_2\cdot p_1 = \varepsilon_1\cdot p_2 = \varepsilon_2\cdot p_2 = 0, \tag{29.6.5}$$
$$\varepsilon_1\cdot k_2 = -\varepsilon_1\cdot k_1, \tag{29.6.6}$$
$$\varepsilon_2\cdot k_2 = -\varepsilon_2\cdot k_1, \tag{29.6.7}$$

for each choice of λ_1, λ_2. The sums over gluon polarizations will be performed using [cf. eq. (I.2.61)]:

$$\sum_{\lambda_1}\varepsilon_1^\mu\varepsilon_1^\nu = \sum_{\lambda_2}\varepsilon_2^\mu\varepsilon_2^\nu = -g^{\mu\nu} + \frac{2\,(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu)}{s}. \tag{29.6.8}$$

Note that in QCD processes with two or more external gluons, the term $2\,(p_1^\mu p_2^\nu + p_2^\mu p_1^\nu)\,/s$ in eq. (29.6.8) cannot in general be dropped [200]. This is to be contrasted to the photon polarization sum [cf. eq. (29.6.14)], where this latter term can always be neglected (due to a Ward identity of quantum electrodynamics).

Before taking the complex square of the amplitude, it is convenient to rewrite the last two terms in each of eqs. (29.6.2) and (29.6.3) by using the identities [see eq. (29.3.12)]:

$$m_{\tilde{g}} x_1^\dagger = y_1(k_1\cdot\sigma)\,, \qquad\qquad m_{\tilde{g}} y_1 = x_1^\dagger(k_1\cdot\overline{\sigma})\,. \tag{29.6.9}$$

Using eqs. (29.54) and (29.55), the resulting total matrix element is then reduced to a sum of terms that each contain exactly one σ or $\overline{\sigma}$ matrix. We define convenient factors:

$$G_s \equiv g_s^2 f^{abe} f^{cde}/s, \tag{29.6.10}$$
$$G_t \equiv g_s^2 f^{ace} f^{bde}/(t-m_{\tilde{g}}^2), \tag{29.6.11}$$
$$G_u \equiv g_s^2 f^{ade} f^{bce}/(u-m_{\tilde{g}}^2). \tag{29.6.12}$$

where the usual Mandelstam variables are:

$$s = (p_1+p_2)^2 = (k_1+k_2)^2, \tag{29.6.13}$$
$$t = (k_1-p_1)^2 = (k_2-p_2)^2, \tag{29.6.14}$$
$$u = (k_1-p_2)^2 = (k_2-p_1)^2. \tag{29.6.15}$$

Then the total amplitude is (noting that the gluon polarizations $\varepsilon_1, \varepsilon_2$ were chosen real):

$$\mathcal{M} = \mathcal{M}_s + \mathcal{M}_t + \mathcal{M}_u = x_1^\dagger a\cdot\overline{\sigma} y_2 + y_1 a^*\cdot\sigma x_2^\dagger, \tag{29.6.16}$$

where

$$a^\mu \equiv -(G_t+G_s)\varepsilon_1\cdot\varepsilon_2\, p_1^\mu - (G_u-G_s)\varepsilon_1\cdot\varepsilon_2\, p_2^\mu - 2G_t k_1\cdot\varepsilon_1\,\varepsilon_2^\mu - 2G_u k_1\cdot\varepsilon_2\,\varepsilon_1^\mu \\ -i\epsilon^{\mu\nu\rho\kappa}\varepsilon_{1\nu}\varepsilon_{2\rho}(G_t p_1 - G_u p_2)_\kappa. \tag{29.6.17}$$

Squaring the amplitude using eqs. (29.45) and (29.46), we get:

$$|\mathcal{M}|^2 = x_1^\dagger a\cdot\overline{\sigma} y_2 y_2^\dagger a^*\cdot\overline{\sigma} x_1 + y_1 a^*\cdot\sigma x_2^\dagger x_2 a\cdot\sigma y_1^\dagger + x_1^\dagger a\cdot\overline{\sigma} y_2 x_2 a\cdot\sigma y_1^\dagger + y_1 a^*\cdot\sigma x_2^\dagger y_2^\dagger a^*\cdot\overline{\sigma} x_1. \tag{29.6.18}$$

Summing over the gluino spins using eqs. (29.3.57)–(29.3.60), we find:

$$\sum_{\lambda_1',\lambda_2'} |\mathcal{M}|^2 = \mathrm{Tr}[a\cdot\overline{\sigma} k_2\cdot\sigma a^*\cdot\overline{\sigma} k_1\cdot\sigma] + \mathrm{Tr}[a^*\cdot\sigma k_2\cdot\overline{\sigma} a\cdot\sigma k_1\cdot\overline{\sigma}] \\ -m_{\tilde{g}}^2\mathrm{Tr}[a\cdot\overline{\sigma} a\cdot\sigma] - m_{\tilde{g}}^2\mathrm{Tr}[a^*\cdot\sigma a^*\cdot\overline{\sigma}]. \tag{29.6.19}$$

Performing the traces with eqs. (29.56)–(29.58) then yields:

$$\sum_{\lambda_1',\lambda_2'} |\mathcal{M}|^2 = 8\,\mathrm{Re}[a\cdot k_1 a^*\cdot k_2] - 4a\cdot a^*\, k_1\cdot k_2 - 4i\epsilon^{\mu\nu\rho\kappa} k_{1\mu} k_{2\nu} a_\rho a_\kappa^* - 4m_{\tilde{g}}^2\mathrm{Re}[a^2]. \tag{29.6.20}$$

Inserting the explicit form for a^μ [eq. (29.6.17)] into the above result, we obtain:

$$\sum_{\lambda_1',\lambda_2'} |\mathcal{M}|^2 = 2(t-m_{\tilde{g}}^2)(u-m_{\tilde{g}}^2)[(G_t+G_u)^2 + 4(G_s+G_t)(G_s-G_u)(\varepsilon_1\cdot\varepsilon_2)^2] \\ +16(G_t+G_u)[G_s(t-u)+G_t(t-m_{\tilde{g}}^2)+G_u(u-m_{\tilde{g}}^2)](\varepsilon_1\cdot\varepsilon_2)(k_1\cdot\varepsilon_1)(k_1\cdot\varepsilon_2) \\ -32(G_t+G_u)^2(k_1\cdot\varepsilon_1)^2(k_1\cdot\varepsilon_2)^2. \tag{29.6.21}$$

The sums over gluon polarizations can be done using eq. (29.6.8), which implies:

$$\sum_{\lambda_1,\lambda_2} 1 = 4, \qquad \sum_{\lambda_1,\lambda_2} (\varepsilon_1\cdot\varepsilon_2)^2 = 2, \tag{29.6.22}$$

$$\sum_{\lambda_1,\lambda_2} (\varepsilon_1\cdot\varepsilon_2)(k_1\cdot\varepsilon_1)(k_1\cdot\varepsilon_2) = m_{\tilde{g}}^2 - (t-m_{\tilde{g}}^2)(u-m_{\tilde{g}}^2)/s, \tag{29.6.23}$$

$$\sum_{\lambda_1,\lambda_2} (k_1\cdot\varepsilon_1)^2(k_1\cdot\varepsilon_2)^2 = \left(m_{\tilde{g}}^2 - (t-m_{\tilde{g}}^2)(u-m_{\tilde{g}}^2)/s\right)^2. \tag{29.6.24}$$

Summing over colors using $f^{abe}f^{cde}f^{abe'}f^{cde'} = 2f^{abe}f^{cde}f^{ace'}f^{bde'} = N_c^2(N_c^2-1) = 72$,

$$\sum_{\text{colors}} G_s^2 = \frac{72g_s^4}{s^2}, \qquad \sum_{\text{colors}} G_t^2 = \frac{72g_s^4}{(t-m_{\tilde{g}}^2)^2}, \tag{29.6.25}$$

$$\sum_{\text{colors}} G_u^2 = \frac{72g_s^4}{(u-m_{\tilde{g}}^2)^2}, \qquad \sum_{\text{colors}} G_sG_t = \frac{36g_s^4}{s(t-m_{\tilde{g}}^2)}, \tag{29.6.26}$$

$$\sum_{\text{colors}} G_sG_u = -\frac{36g_s^4}{s(u-m_{\tilde{g}}^2)}, \qquad \sum_{\text{colors}} G_tG_u = \frac{36g_s^4}{(t-m_{\tilde{g}}^2)(u-m_{\tilde{g}}^2)}. \tag{29.6.27}$$

Putting all the factors together, and averaging over the initial state colors and spins, we have:

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2}\left(\frac{1}{64}\sum_{\text{colors}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2\right) \\ = \frac{9\pi\alpha_s^2}{4s^4}\left[2(t-m_{\tilde{g}}^2)(u-m_{\tilde{g}}^2) - 3s^2 - 4m_{\tilde{g}}^2 s + \frac{s^2(s+2m_{\tilde{g}}^2)^2}{(t-m_{\tilde{g}}^2)(u-m_{\tilde{g}}^2)} - \frac{4m_{\tilde{g}}^4 s^4}{(t-m_{\tilde{g}}^2)^2(u-m_{\tilde{g}}^2)^2}\right], \tag{29.6.28}$$

which agrees with the result of [201, 180] (after some rearrangement). Note that in the center-of-momentum frame, the Mandelstam variable t is related to the scattering angle θ between an initial state gluon and a final state gluino by:

$$t = m_{\tilde{g}}^2 + \frac{s}{2}\left(\cos\theta\sqrt{1-4m_{\tilde{g}}^2/s} - 1\right). \tag{29.6.29}$$

Since the final state has identical particles, the total cross-section can now be obtained by:

$$\sigma = \frac{1}{2}\int_{t_-}^{t_+} \frac{d\sigma}{dt} dt\,, \tag{29.6.30}$$

where $t_\pm$ are obtained by inserting $\cos\theta = \pm 1$ into eq. (29.6.29).

29.6.20 R-parity violating stau decay: $\widetilde{\tau}_R^+ \to e^+\bar{\nu}_\mu$

In an R-parity-violating extension of the MSSM (denoted henceforth by RPV-MSSM), new Yukawa couplings can arise [see eqs. (L.1)–(L.3)] that violate either a global U(1) lepton number L or baryon number B. The corresponding Feynman rules are derived in Appendix L . Consider the decay of a right-handed scalar tau via an L-violating $LL\bar{e}$ coupling governed by eq. (L.1). This is particularly relevant when the scalar tau is the lightest supersymmetric particle (LSP) [202, 203] and in the case of resonant slepton production [204, 205]. Note that in R-parity violation the LSP need not be the lightest neutralino and in a minimal supergravity embedding often it is not [206, 207]. The Feynman diagram is shown in Fig. **??**, where we have also defined the momenta and the helicities of the fermions.

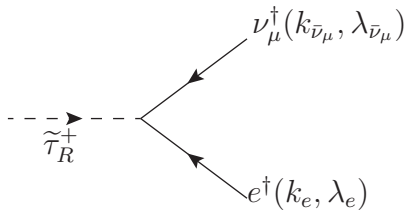


Figure 29.6.1: Feynman diagram for the R-parity-violating decay $\widetilde{\tau}_R^+ \to e^+\bar{\nu}_\mu$

The amplitude for the R-parity-violating $\tilde{\tau}_R^+$ decay is given by:

$$i\mathcal{M} = -i\lambda y_e y_{\bar{\nu}_\mu}\,. \tag{29.6.1}$$

Here we have defined $\lambda \equiv \lambda_{123}$, and the external wave functions are denoted by $y_e \equiv y(\vec{\boldsymbol{k}}_e, \lambda_e)$, and $y_{\bar{\nu}_\mu} \equiv y(\vec{\boldsymbol{k}}_{\bar{\nu}_\mu}, \lambda_{\bar{\nu}_\mu})$, respectively. Using eq. (29.44), the amplitude squared is

$$|\mathcal{M}|^2 = |\lambda|^2 y_e y_{\bar{\nu}_\mu} y_{\bar{\nu}_\mu}^\dagger y_e^\dagger. \tag{29.6.2}$$

Summing over the fermion spins using eq. (29.3.58) gives:

$$\sum_{\lambda_e, \lambda_{\bar{\nu}_\mu}} |\mathcal{M}|^2 = |\lambda|^2 \mathrm{Tr}[k_e\cdot\sigma\, k_{\bar{\nu}_\mu}\cdot\overline{\sigma}] = |\lambda|^2 m_{\tilde{\tau}_R}^2\,, \tag{29.6.3}$$

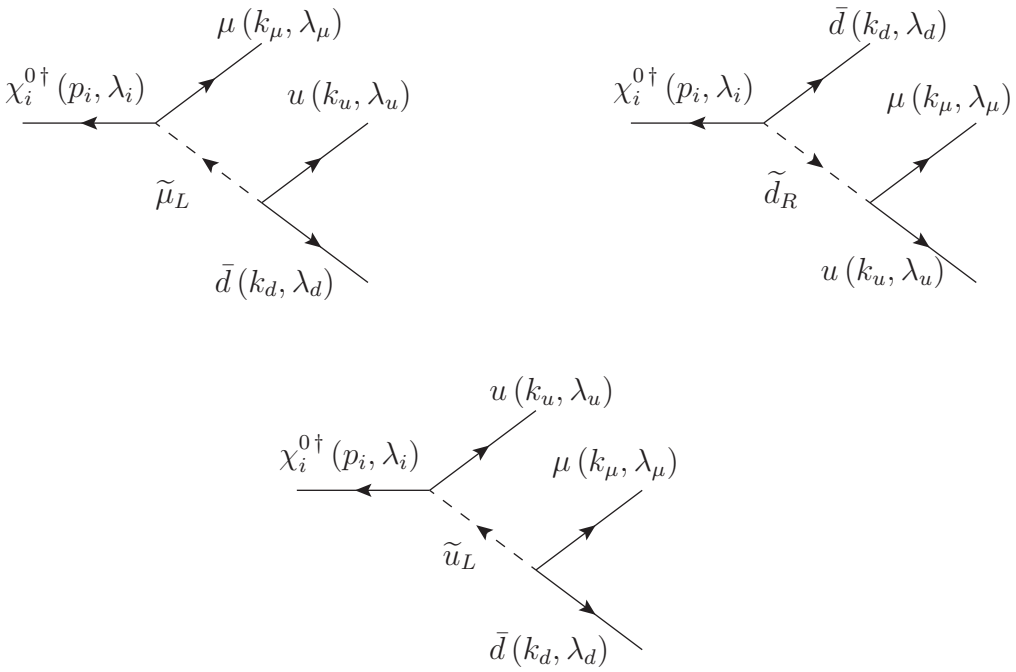


Figure 29.6.1: Feynman diagrams for the R-parity violating decay $\widetilde{N}_i \to \mu^- u \bar{d}$.

where in the last step we have used the trace formula eq. (29.56), and neglected the mass of the electron and the neutrino. The total decay rate is then given by

$$\Gamma = \frac{1}{16\pi m_{\tilde{\tau}_R}} \left(\sum_{\lambda_e, \lambda_{\bar{\nu}_\mu}} |\mathcal{M}|^2 \right) = \frac{|\lambda|^2}{16\pi} m_{\tilde{\tau}_R} \,, \tag{29.6.4}$$

which agrees with the computation in refs. [208, 209, 210]. Completely analogously we can obtain the total rate for the decays $\widetilde{\nu}_\mu \to \tau^- e^+$ and $\widetilde{e}_L^- \to \tau^- \bar{\nu}_\mu$, which proceed via the same operator, by replacing $m_{\tilde{\tau}_R} \to (m_{\tilde{e}_L}, m_{\tilde{\nu}_\mu})$, respectively.

In general the two-body decay rate of a sfermion $\widetilde{f}$ via the L-violating $LQ\bar{d}$ coupling governed by eq. (L.2) or the B-violating $\bar{u}\bar{d}\bar{d}$ coupling governed by eq. (L.3) is given by:

$$\Gamma(\tilde{f} \to f_1 f_2) = \frac{C|\lambda|^2}{16\pi} m_{\tilde{f}} \,, \tag{29.6.5}$$

where we have neglected the masses $m_{1,2}$ of the final state fermions. The factor C denotes the color factor. For the slepton decays via the $LQ\bar{d}$ coupling which are summed over the final state quark colors, $C = \delta^{ij}\delta_{ij} = 3$, where $i, j = 1, 2, 3$ and δ_{ij} is the symmetric invariant tensor of color SU(3). For the squark decays via the $LQ\bar{d}$ where the initial state color is averaged over and the final state color is summed, $C = 1$. For the squark decays via the $\bar{u}\bar{d}\bar{d}$ coupling, $C = \frac{1}{3}\epsilon^{ijk}\epsilon_{ijk} = 2$, where the Levi-Civita tensor, $\epsilon^{ijk} = \epsilon_{ijk}$, is the antisymmetric invariant tensor of color SU(3). In realistic cases, one must also include the effects of mixing for the third-family sfermions, which we have omitted here for simplicity.

29.6.21 R-parity-violating neutralino decay: $\widetilde{N}_i \to \mu^- u \bar{d}$

Next we consider the R-parity-violating three-body decay of a neutralino $\widetilde{N}_i \to \mu^- u \bar{d}$, which arises due to the L-violating $LQ\bar{d}$ coupling governed by eq. (L.2). This is of particular interest

when the neutralino is the LSP, since it determines the final state signatures [211, 212, 213]. The three Feynman diagrams are shown in Fig. 29.6.1, including the definitions of the momenta and helicities. We have neglected sfermion mixing, i.e. we assume $\widetilde{\mu}_L$, $\widetilde{u}_L$, and $\widetilde{d}_R$ are mass eigenstates. Using the Feynman rules given in Figs. L.2 and K.4.2 (or K.4.4), we obtain the corresponding contributions to the decay amplitude,

$$i\mathcal{M}_1 = (i\lambda'^*)\left[\frac{i}{\sqrt{2}}(gN_{i2} + g'N_{i1})\right]\left[\frac{i}{(p_i - k_\mu)^2 - m^2_{\tilde{\mu}_L}}\right] y_i^\dagger x_\mu^\dagger x_u^\dagger x_d^\dagger\,, \tag{29.6.1}$$

$$i\mathcal{M}_2 = (i\lambda'^*)\left[-\frac{i\sqrt{2}}{3}g'N_{i1}\right]\left[\frac{i}{(p_i - k_d)^2 - m^2_{\tilde{d}_R}}\right] y_i^\dagger x_d^\dagger x_\mu^\dagger x_u^\dagger\,, \tag{29.6.2}$$

$$i\mathcal{M}_3 = (i\lambda'^*)\left[-\frac{i}{\sqrt{2}}(gN_{i2} + g'N_{i1}/3)\right]\left[\frac{i}{(p_i - k_u)^2 - m^2_{\tilde{u}_L}}\right] y_i^\dagger x_u^\dagger x_d^\dagger x_\mu^\dagger\,. \tag{29.6.3}$$

Here we have defined $\lambda' \equiv \lambda'_{211}$, and the external wave functions are denoted by $y_i^\dagger \equiv y^\dagger(\vec{\boldsymbol{p}}_i, \lambda_i)$, $x_\mu^\dagger \equiv x^\dagger(\vec{\boldsymbol{k}}_\mu, \lambda_\mu)$, $x_u^\dagger \equiv x^\dagger(\vec{\boldsymbol{k}}_u, \lambda_u)$, and $x_d^\dagger \equiv x^\dagger(\vec{\boldsymbol{k}}_d, \lambda_d)$, respectively. In the following, we will neglect all of the final state fermion masses. The results will be expressed in terms of the kinematic variables

$$z_\mu \equiv 2p_i{\cdot}k_\mu/m^2_{\tilde{N}_i} = 2E_\mu/m_{\tilde{N}_i}, \tag{29.6.4}$$

$$z_d \equiv 2p_i{\cdot}k_d/m^2_{\tilde{N}_i} = 2E_d/m_{\tilde{N}_i}, \tag{29.6.5}$$

$$z_u \equiv 2p_i{\cdot}k_u/m^2_{\tilde{N}_i} = 2E_u/m_{\tilde{N}_i}, \tag{29.6.6}$$

which satisfy $z_\mu + z_d + z_u = 2$. Then we can rewrite the total matrix element as:

$$\mathcal{M} = c_1 y_i^\dagger x_\mu^\dagger x_u^\dagger x_d^\dagger + c_2 y_i^\dagger x_d^\dagger x_\mu^\dagger x_u^\dagger + c_3 y_i^\dagger x_u^\dagger x_d^\dagger x_\mu^\dagger\,, \tag{29.6.7}$$

where

$$c_1 \equiv \frac{1}{\sqrt{2}}\lambda'^*(gN_{i2} + g'N_{i1})/[m^2_{\tilde{\mu}_L} - m^2_{\tilde{N}_i}(1 - z_\mu)], \tag{29.6.8}$$

$$c_2 \equiv -\frac{\sqrt{2}}{3}\lambda'^* g'N_{i1}/[m^2_{\tilde{d}_R} - m^2_{\tilde{N}_i}(1 - z_d)], \tag{29.6.9}$$

$$c_3 \equiv -\frac{1}{\sqrt{2}}\lambda'^*(gN_{i2} + g'N_{i1}/3)/[m^2_{\tilde{u}_L} - m^2_{\tilde{N}_i}(1 - z_u)]. \tag{29.6.10}$$

Before squaring the amplitude, it is convenient to use the Fierz identity [eq. (29.67)] to reduce the number of terms:

$$\mathcal{M} = (c_1 - c_3) y_i^\dagger x_\mu^\dagger x_u^\dagger x_d^\dagger + (c_2 - c_3) y_i^\dagger x_d^\dagger x_\mu^\dagger x_u^\dagger. \tag{29.6.11}$$

Using eq. (29.44), we obtain

$$\begin{aligned}|\mathcal{M}|^2 = {}& |c_1 - c_3|^2 y_i^\dagger x_\mu^\dagger x_\mu y_i x_u^\dagger x_d^\dagger x_d x_u + |c_2 - c_3|^2 y_i^\dagger x_d^\dagger x_d y_i x_\mu^\dagger x_u^\dagger x_u x_\mu \\ & -2\mathrm{Re}[(c_1 - c_3)(c_2^* - c_3^*) y_i^\dagger x_\mu^\dagger x_\mu x_u x_u^\dagger x_d^\dagger x_d y_i]\,,\end{aligned} \tag{29.6.12}$$

where eq. (29.60) was used on the last term. Summing over the fermion spins using eqs. (29.3.57)–(29.3.60), we obtain:

$$\begin{aligned}\sum_{\text{spins}} |\mathcal{M}|^2 = {}& |c_1 - c_3|^2 \mathrm{Tr}[k_\mu{\cdot}\overline{\sigma} p_i{\cdot}\sigma]\mathrm{Tr}[k_d{\cdot}\overline{\sigma} k_u{\cdot}\sigma] + |c_2 - c_3|^2 \mathrm{Tr}[k_d{\cdot}\overline{\sigma} p_i{\cdot}\sigma]\mathrm{Tr}[k_u{\cdot}\overline{\sigma} k_\mu{\cdot}\sigma] \\ & -2\mathrm{Re}\big[(c_1 - c_3)(c_2^* - c_3^*)\mathrm{Tr}[k_\mu{\cdot}\overline{\sigma} k_u{\cdot}\sigma k_d{\cdot}\overline{\sigma} p_i{\cdot}\sigma]\big]\,.\end{aligned} \tag{29.6.13}$$

Applying the trace formulae, eqs. (29.56) and (29.58), we obtain

$$\begin{aligned}\sum_{\text{spins}}|\mathcal{M}|^2 &= 4|c_1-c_3|^2 p_i{\cdot}k_\mu\, k_d{\cdot}k_u + 4|c_2-c_3|^2 p_i{\cdot}k_d\, k_\mu{\cdot}k_u \\ &\quad -4\text{Re}\big[(c_1-c_3)(c_2^*-c_3^*)\big](k_\mu{\cdot}k_u\, p_i{\cdot}k_d + p_i{\cdot}k_\mu\, k_d{\cdot}k_u - k_\mu{\cdot}k_d\, p_i{\cdot}k_u) \\ &= m_{\tilde{N}_i}^4\Big[|c_1|^2 z_\mu(1-z_\mu) + |c_2|^2 z_d(1-z_d) + |c_3|^2 z_u(1-z_u) \\ &\quad -2\text{Re}[c_1c_2^*](1-z_\mu)(1-z_d) - 2\text{Re}[c_1c_3^*](1-z_\mu)(1-z_u) \\ &\quad -2\text{Re}[c_2c_3^*](1-z_d)(1-z_u)\Big]\,, \end{aligned} \tag{29.6.14}$$

where in the last equality we have used eqs. (29.6.4)–(29.6.6) and

$$2k_\mu{\cdot}k_d = (1-z_u)m_{\tilde{N}_i}^2, \qquad 2k_\mu{\cdot}k_u = (1-z_d)m_{\tilde{N}_i}^2, \qquad 2k_d{\cdot}k_u = (1-z_\mu)m_{\tilde{N}_i}^2. \tag{29.6.15}$$

The differential decay rate follows:

$$\frac{d^2\Gamma}{dz_\mu dz_d} = \frac{N_c m_{\tilde{N}_i}}{2^8\pi^3}\left(\frac{1}{2}\sum_{\text{spins}}|\mathcal{M}|^2\right), \tag{29.6.16}$$

where a factor of $N_c = 3$ has been included for the sum over colors, a factor of $1/2$ to average over the neutralino spin, and the kinematic limits are

$$0 < z_\mu < 1, \tag{29.6.17}$$
$$1 - z_\mu < z_d < 1. \tag{29.6.18}$$

In the limit of heavy sfermions, the integrations over z_d and then z_μ are simple, with the result for the total decay width:

$$\Gamma = \frac{N_c m_{\tilde{N}_i}^5}{2^{11}\cdot 3\pi^3}\left(|c_1'|^2 + |c_2'|^2 + |c_3'|^2 - \text{Re}[c_1'c_2'^* + c_1'c_3'^* + c_2'c_3'^*]\right), \tag{29.6.19}$$

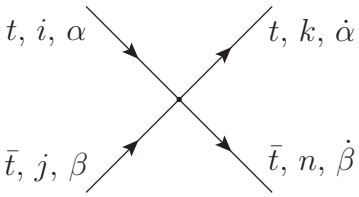
where the c_i' are obtained from c_i of eqs. (29.6.8)–(29.6.10) by neglecting $m_{\tilde{N}_i}^2$ in the denominators. Our results agree with the complete computation (which includes mixing) given in refs. [214, 209, 210]. Earlier calculations with some simplifications are given in refs. [212, 215].

29.6.22 Top-quark condensation from a Nambu-Jona-Lasinio model gap equation

The previous examples have involved renormalizable field theories. However, there are cases in which it is preferable to use effective four-fermion interactions. The obvious historical example is the four-fermion Fermi theory of weak decays. This has been superseded by a more complete and accurate theory of the weak interactions but is still useful for leading order calculations of low-energy processes. Another case of some interest is the use of strong coupling four-fermion interactions to drive symmetry breaking via a Nambu-Jona-Lasinio model [216], as in the top quark condensate approach [217, 218, 219, 220, 221] to electroweak symmetry breaking.

Consider an effective four-fermion Lagrangian involving the top quark [219], written in two-component fermion form as:

$$\mathscr{L} = it^\dagger\overline{\sigma}^\mu\partial_\mu t + i\bar{t}^\dagger\overline{\sigma}^\mu\partial_\mu\bar{t} + \frac{G}{\Lambda^2}(t\bar{t})(t^\dagger\bar{t}^\dagger). \tag{29.6.1}$$



$$i\frac{G}{\Lambda^2}\delta_i^j\delta_n^k\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}$$

Figure 29.6.1: Feynman rule for the four-fermion interaction in the top quark condensate model. The indices $i, j, k, n = 1, 2, 3$ are for color in the fundamental representation of $SU(3)$, and the indices $\alpha, \beta, \dot\alpha, \dot\beta$ are two-component spinor indices.

Here the Standard Model gauge interactions have been suppressed; the quantities within parentheses are color singlets. Note also that there is no top quark Yukawa coupling to a Higgs scalar boson, nor a top quark mass term, which would normally appear in the form $-m_t(t\bar t + t^\dagger \bar t^\dagger)$. Instead, the effective top quark mass is supposed to be driven by a non-perturbatively large and positive dimensionless coupling G, with Λ the cutoff scale at which G arises from some more fundamental physics such as topcolor [221].

The Feynman rule for the four-fermion interaction can be derived from the mode expansion results of Section 29.3, and is given in Fig. 29.6.1. The resulting gap equation for the dynamically generated top quark mass is shown in Fig. 29.6.2.

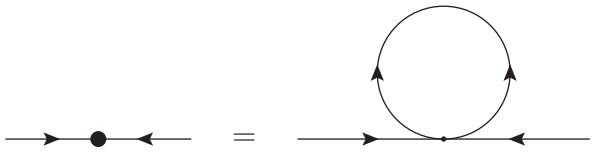


Figure 29.6.2: The Nambu-Jona-Lasinio gap equation for a possible dynamically generated top quark mass m_t.

Evaluating this using the Feynman rules of Figs. 29.4.3 and 29.4.4, one finds:

$$-im_t\delta_i^j\delta_\alpha^\beta = (-1)\int^\Lambda \frac{d^4k}{(2\pi)^4}\left(i\frac{G}{\Lambda^2}\delta_i^j\delta_n^k\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}\right)\left(\delta_k^n\delta_{\dot\beta}^{\dot\alpha}\frac{im_t}{k^2-m_t^2+i\epsilon}\right). \tag{29.6.2}$$

Here i, j, k, n are color indices of the fundamental representation of $SU(3)$, and $\alpha, \beta, \dot\alpha, \dot\beta$ are two-component spinor indices. The factor of (-1) on the right-hand side is due to the presence of a fermion loop.

Euclideanizing the loop integration over k^μ by $k^2 \to -k_E^2$ and $\int d^4k \to i\int d^4k_E$, and then rewriting the integration in terms of $x = k_E^2$, this amounts to [219]:

$$m_t = \frac{2N_cGm_t}{16\pi^2\Lambda^2}\int_0^{\Lambda^2} dx/(1+m_t^2/x) = \frac{3Gm_t}{8\pi^2}[1-(m_t^2/\Lambda^2)\ln(\Lambda^2/m_t^2)+\ldots], \tag{29.6.3}$$

where $N_c = 3$ is the number of colors, and a factor of two arises from the sum over dotted spinor indices of $\delta_{\dot\alpha}^{\dot\beta}\delta_{\dot\beta}^{\dot\alpha}$.

For small or negative G, only the trivial solution $m_t = 0$ is possible. However, for $G \geq G_{\rm critical} = 8\pi^2/3 \approx 26$, there is a positive solution for m_t^2/Λ^2 [219]. It is now known that this minimal version of the model cannot explain the top quark mass and the observed features of electroweak symmetry breaking, but extensions of it may be viable [222].

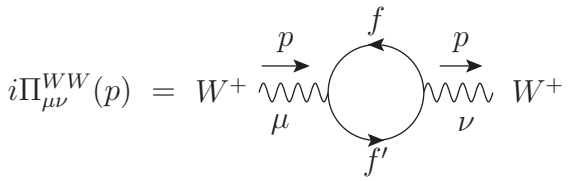


Figure 29.6.1: Contributions to the self-energy function for the W boson in the Standard Model, from loops involving the left-handed quark and lepton pairs $(f, f') = (e, \nu_e)$, (μ, ν_μ), (τ, ν_τ), (d, u), (s, c), and (b, t). The momentum of the positively charged W^+ flows from left to right.

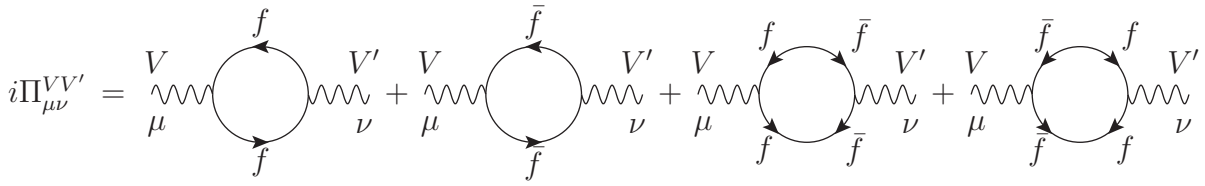


Figure 29.6.2: Contributions to the diagonal and off-diagonal self-energy functions for the neutral vector bosons $V, V' = \gamma, Z$ in the Standard Model, from loops involving the three generations of leptons and quarks: $f = e, \nu_e, \mu, \nu_\mu, \tau, \nu_\tau, d, u, s, c, b, t$.

29.6.23 Electroweak vector boson self-energies from fermion loops

In this subsection, we consider the contributions to the self-energy functions of the Standard Model electroweak vector bosons coming from quark and lepton loops. (For a derivation of equivalent results in the four-component fermion formalism, see for example Section 21.3 of [115].) The independent self-energies are given by $\Pi^{WW}_{\mu\nu}$, $\Pi^{ZZ}_{\mu\nu}$, $\Pi^{\gamma Z}_{\mu\nu} = \Pi^{Z\gamma}_{\mu\nu}$, and $\Pi^{\gamma\gamma}_{\mu\nu}$, as shown in Figs. 29.6.1 and 29.6.2. In each case, $i\Pi_{\mu\nu}$ is equal to the sum of Feynman diagrams for two-point functions with amputated external legs, and is implicitly a function of the external momentum p^μ.

First consider the self-energy function for the W boson, shown in Fig. 29.6.1. The W boson only couples to left-handed fermions, so there is only one Feynman diagram for each Standard model weak isodoublet. Taking the external momentum flowing from left to right to be p, and the loop momentum flowing counterclockwise in the upper fermion line (f) to be k, we have from the Feynman rules of Fig. J.1.2:

$$i\Pi^{WW}_{\mu\nu} = (-1)\,\mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \sum_{(f,f')} N_c^f \mathrm{Tr}\left[\left(-i\frac{g}{\sqrt{2}}\overline{\sigma}_\mu\right)\left(\frac{ik\cdot\sigma}{k^2 - m_f^2}\right)\left(-i\frac{g}{\sqrt{2}}\overline{\sigma}_\nu\right)\left(\frac{i(k+p)\cdot\sigma}{(k+p)^2 - m_{f'}^2}\right)\right]. \tag{29.6.1}$$

Here μ is a regularization scale for dimensional regularization in $d \equiv 4 - 2\epsilon$ dimensions. The sum in eq. (29.6.1) is over the six isodoublet pairs $(f, f') = (e, \nu_e)$, (μ, ν_μ), (τ, ν_τ), (d, u), (s, c), and (b, t) with CKM mixing neglected, and

$$N_c^f = \begin{cases} 3\,, & f = \text{quarks}\,, \\ 1\,, & f = \text{leptons}\,. \end{cases} \tag{29.6.2}$$

The first factor of (-1) in eq. (29.6.1) is due to the presence of a closed fermion loop. The trace is taken over the two-component dotted spinor indices. Using eq. (B.2.27), it follows that

$$\Pi^{WW}_{\mu\nu} = \frac{g^2}{32\pi^2} \sum_f N_c^f I_{\mu\nu}(m_f^2, m_{f'}^2)\,, \tag{29.6.3}$$

where we have defined

$$I_{\mu\nu}(x,y) = i(16\pi^2)\,\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\,\frac{4k_\mu k_\nu + 2k_\mu p_\nu + 2k_\nu p_\mu - 2k\cdot(k+p)\,g_{\mu\nu}}{(k^2-x)[(k+p)^2-y]}\,. \tag{29.6.4}$$

We do not explicitly exhibit above the term proportional to $\epsilon_{\mu\nu\alpha\beta}$, as it integrates to zero. The integral $I_{\mu\nu}$ can be evaluated by the standard dimensional regularization methods [115, 223],

$$I_{\mu\nu}(x,y) = (p^2 g_{\mu\nu} - p_\mu p_\nu) I_1(p^2;x,y) + g_{\mu\nu} I_2(p^2;x,y)\,, \tag{29.6.5}$$

where

$$I_1(s;x,y) = -\frac{2}{3\epsilon} + \frac{2}{3s^2}\Big\{(2x-2y-s)A(x) + (2y-2x-s)A(y) + \Big[2(x-y)^2 - s(x+y) - s^2\Big]B(s;x,y) - s(x+y) + s^2/3\Big\}, \tag{29.6.6}$$

$$I_2(s;x,y) = \frac{x+y}{\epsilon} - \frac{1}{s}\Big\{(x-y)\big[A(x)-A(y)\big] + \Big[(x-y)^2 - s(x+y)\Big]B(s;x,y)\Big\}\,, \tag{29.6.7}$$

after neglecting terms that vanish as $\epsilon \to 0$. The functions

$$A(x) \equiv x\ln(x/Q^2) - x, \tag{29.6.8}$$

$$B(s;x,y) \equiv -\int_0^1 dt\,\ln\left(\frac{tx + (1-t)y - t(1-t)s - i\varepsilon}{Q^2}\right), \tag{29.6.9}$$

are the finite parts of one-loop Passarino-Veltman functions [224], with the renormalization scale Q related to the regularization scale μ by the modified minimal subtraction relation

$$\mu^2 = Q^2 e^\gamma/4\pi, \tag{29.6.10}$$

where $\gamma = 0.577216\ldots$ is Euler's constant.

The photon and Z boson have mixed self-energy functions, defined in Fig. 29.6.2. Applying the pertinent Feynman rules from Fig. J.1.2, we obtain:

$$\begin{aligned} i\Pi^{VV'}_{\mu\nu} \;=\; (-1)\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\sum_f N_c^f \mathrm{Tr}\Big\{ &\big(-iG_V^f\overline{\sigma}_\mu\big)\left(\frac{ik\cdot\sigma}{k^2-m_f^2}\right)\big(-iG_{V'}^f\overline{\sigma}_\nu\big)\left(\frac{i(k+p)\cdot\sigma}{(k+p)^2-m_f^2}\right) \\ &+\big(-iG_V^{\bar f}\overline{\sigma}_\mu\big)\left(\frac{ik\cdot\sigma}{k^2-m_f^2}\right)\big(-iG_{V'}^{\bar f}\overline{\sigma}_\nu\big)\left(\frac{i(k+p)\cdot\sigma}{(k+p)^2-m_f^2}\right) \\ &+\big(-iG_V^f\overline{\sigma}_\mu\big)\left(\frac{im_f}{k^2-m_f^2}\right)\big(iG_{V'}^{\bar f}\sigma_\nu\big)\left(\frac{im_f}{(k+p)^2-m_f^2}\right) \\ &+\big(-iG_V^{\bar f}\overline{\sigma}_\mu\big)\left(\frac{im_f}{k^2-m_f^2}\right)\big(iG_{V'}^{f}\sigma_\nu\big)\left(\frac{im_f}{(k+p)^2-m_f^2}\right)\Big\}, \end{aligned} \tag{29.6.11}$$

where V and V' can each be either γ or Z, and $\sum_f$ is taken over the 12 Standard Model fermions. The corresponding Vff and $V\bar f\bar f$ couplings are:[73]

$$G_\gamma^f = -G_\gamma^{\bar f} = eQ_f, \tag{29.6.12}$$

$$G_Z^f = \frac{g}{c_W}(T_3^f - s_W^2 Q_f), \qquad G_Z^{\bar f} = \frac{g}{c_W}s_W^2 Q_f. \tag{29.6.13}$$

[73] Note that there is no contribution from the left-handed two-component antineutrino fields, $\bar\nu_e$, $\bar\nu_\mu$, $\bar\nu_\tau$, which do not exist in the Standard Model.

The four terms in eq. (29.6.11) correspond to the four diagrams in Fig. 29.6.2, in the same order.

The first two terms in eq. (29.6.11) are computed exactly as for $\Pi^{WW}_{\mu\nu}$, while in the last two terms we use eq. (B.2.5) to compute the trace. It follows that the neutral electroweak vector boson self-energy function matrix, after dropping terms that vanish as $\epsilon \to 0$, is given by

$$\Pi^{VV'}_{\mu\nu} = \frac{1}{16\pi^2}\sum_f N_c^f \Big[(G_V^f G_{V'}^f + G_V^{\bar f} G_{V'}^{\bar f}) I_{\mu\nu}(m_f^2, m_f^2) + g_{\mu\nu}(G_V^f G_{V'}^{\bar f} + G_V^{\bar f} G_{V'}^f) m_f^2 I_3(m_f^2, m_f^2)\Big], \tag{29.6.14}$$

where $I_{\mu\nu}(x,y)$ was defined in eqs. (29.6.5)–(29.6.7), and we have defined the function

$$I_3(x,y) = -i(16\pi^2)\,\mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\,\frac{2}{(k^2-x)[(k+p)^2-y]} = \frac{2}{\epsilon} + 2B(p^2;x,y). \tag{29.6.15}$$

The photon self-energy function is a simple special case of eq. (29.6.14):

$$\Pi^{\gamma\gamma}_{\mu\nu} = \frac{1}{16\pi^2}\sum_f 2N_c^f (eQ_f)^2 \big[I_{\mu\nu}(m_f^2, m_f^2) - g_{\mu\nu} m_f^2 I_3(m_f^2, m_f^2)\big]\,. \tag{29.6.16}$$

Evaluating the integrals $I_{\mu\nu}$ and I_3 yields

$$\Pi^{\gamma\gamma}_{\mu\nu} = \frac{\alpha}{3\pi}\sum_f N_c^f Q_f^2 \left(p^2 g_{\mu\nu} - p_\mu p_\nu\right)\left\{-\frac{1}{\epsilon} + \frac{1}{3} - \frac{2}{p^2}\left[A(m_f^2) + m_f^2\right] - \left(1 + \frac{2m_f^2}{p^2}\right) B(p^2; m_f^2, m_f^2)\right\}, \tag{29.6.17}$$

in agreement with the result given in, for example, eq. (7.90) of [115]. This formula satisfies $p^\mu \Pi^{\gamma\gamma}_{\mu\nu} = p^\nu \Pi^{\gamma\gamma}_{\mu\nu} = 0$ as required by the Ward identity of QED, and is regular in the limit $p^2 \to 0$.

In each of eqs. (29.6.3), (29.6.14), and (29.6.17), there are $1/\epsilon$ poles, contained in the loop integral functions. In the $\overline{\rm MS}$ renormalization scheme, these poles are simply removed by counterterms, which have no other effect.

In eqs. (29.6.1) and (29.6.11), we chose to write a $\overline{\sigma}_\mu$ for the left vertex in the Feynman diagram in each case. This is an arbitrary choice; we could also have chosen to use instead $-\sigma_\mu$ for the left vertex in any given diagram, as mentioned in the caption for Fig. J.1.2. This would have dictated the replacements $\overline{\sigma} \leftrightarrow -\sigma$ throughout the expression for the diagram, including for the fermion propagators, as was indicated in Fig. 29.4.4. It is not hard to check that the result after computing the spinor index traces is unaffected. Note that the contribution proportional to $\epsilon_{\mu\nu\rho\kappa}$ from eq. (B.2.26) or eq. (B.2.27) vanishes; this is clear because the self-energy function is symmetric under interchange of vector indices, and there is only one independent momentum in the problem.

29.6.24 Self-energy and pole mass of the top quark

We next consider the one-loop calculation of the self-energy and the pole mass of the top quark in the Standard Model, including the effects of the gauge interactions and the top and bottom quark Yukawa couplings. As in Section 29.6.1, we treat this as a one-generation problem, neglecting CKM mixing. Consequently, the corresponding Yukawa couplings Y_t and Y_b are real and positive (by a suitable phase redefinition of the Higgs field[74]). Using the formalism of Section 29.4.6 for Dirac fermions, the independent 1PI self-energy functions are given by[75] Σ_{Lt}, Σ_{Rt} and Σ_{Dt} (defined in Fig. 29.4.5) as shown in Fig. 29.6.1.

[74] As shown in Section 29.3.2, after the fermion mass matrix diagonalization procedure, the tree-level fermion masses are real and non-negative. If CKM mixing is neglected, it follows from eq. (J.1.9) that the corresponding diagonal Yukawa couplings are real and positive if the phase of the Higgs field is chosen such that the neutral Higgs vacuum expectation value $v > 0$.

[75] Since the Yukawa couplings can be chosen real (in the one-generation model), $\overline{\Sigma}_{Lt} = \Sigma_{Lt}$. Note that after suppressing the color degrees of freedom, Σ_{Lt}, Σ_{Rt} and Σ_{Dt} are one-dimensional matrices, so we do not employ boldface letters in this case.

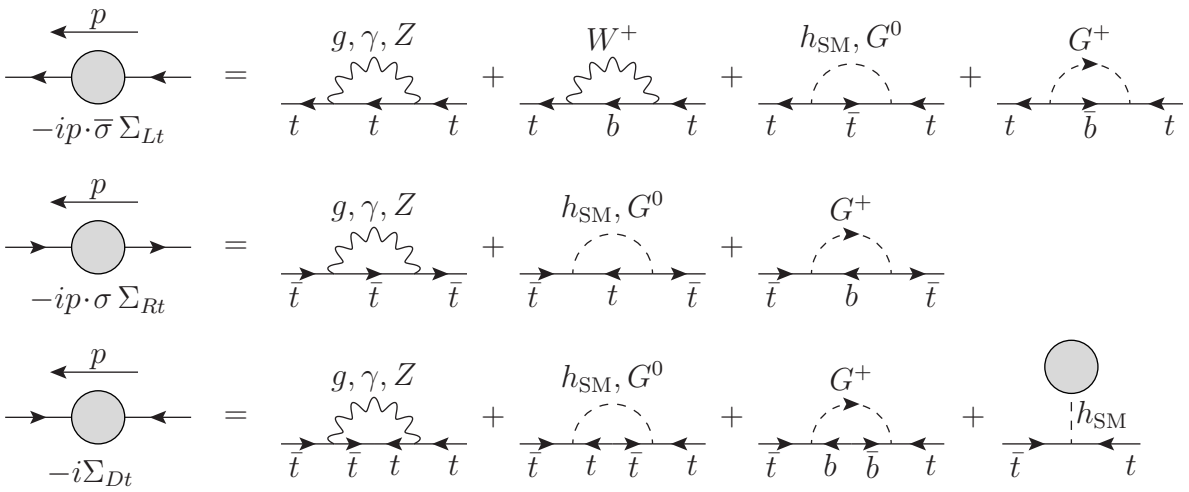


Figure 29.6.1: One-loop contributions to the 1PI self-energy functions for the top quark in the Standard Model. The external momentum of the physical top quark, p^μ, flows from the right to the left. The loop momentum k^μ in the text is taken to flow clockwise. Spinor and color indices are suppressed. The external legs are amputated. The last diagram contains one-loop tadpole contributions.

Note that in these diagrams, the physical top quark moves from right to left, carrying momentum p^μ. Then according to the general formula obtained in eq. (29.4.32), the complex pole squared mass of the top quark is given by:

$$M_t^2 - i\Gamma_t M_t = \frac{(m_t + \Sigma_{Dt})^2}{(1 - \Sigma_{Lt})(1 - \Sigma_{Rt})}\,, \tag{29.6.1}$$

where m_t is the tree-level mass. Working consistently to one-loop order, this yields[76]

$$M_t^2 - i\Gamma_t M_t = \left[m_t^2(1 + \Sigma_{Lt} + \Sigma_{Rt}) + 2m_t\Sigma_{Dt}\right]\Big|_{s=m_t^2+i\varepsilon}. \tag{29.6.2}$$

It remains to calculate the self-energy functions Σ_{Lt}, Σ_{Rt} and Σ_{Dt}. Two regularization procedures will be used simultaneously—the $\overline{\rm MS}$ scheme [225] based on dimensional regularization [125] and the $\overline{\rm DR}$ scheme based on dimensional reduction [226]. This is accomplished by integrating over the loop momentum in

$$d \equiv 4 - 2\epsilon \tag{29.6.3}$$

dimensions, but with the vector bosons possessing

$$D \equiv 4 - 2\epsilon\delta_{\overline{\rm MS}} \tag{29.6.4}$$

components, where

$$\delta_{\overline{\rm MS}} \equiv \begin{cases} 1 & \text{for } \overline{\rm MS}\,, \\ 0 & \text{for } \overline{\rm DR}\,. \end{cases} \tag{29.6.5}$$

In other words, the metric $g^{\mu\nu}$ appearing explicitly in the vector propagator is treated as four dimensional in $\overline{\rm DR}$, but as d-dimensional in $\overline{\rm MS}$. The renormalization scale Q is related to the regularization scale μ in both cases by the modified minimal subtraction relation of eq. (29.6.10).

[76] It would be just as valid to substitute in $s = M_t^2 + i\varepsilon$ here, as two-loop order effects are being neglected.

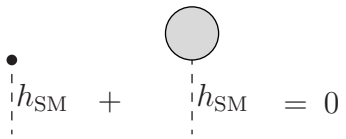


Figure 29.6.2: The tree-level Higgs tadpole cancels against the one-loop Higgs tadpole, provided that one expands around a Higgs vacuum expectation value that minimizes the one-loop effective potential (rather than the tree-level Higgs potential, which would yield no tree-level tadpole).

The calculation of the non-tadpole contributions to the self-energy functions will be performed below in a general R_ξ gauge, with a vector boson propagator as in Fig. 29.4.5. There are different ways to treat the tadpole contributions, corresponding to different choices for the Higgs vacuum expectation value around which the tree-level Lagrangian is expanded. If one chooses to expand around the minimum of the tree-level Higgs potential, then there are no tree-level tadpoles, but there will be non-zero contributions from the last diagram shown in Fig. 29.6.1. (This corresponds to the treatment given, for example, in ref. [227].) Alternatively, one can choose to expand around the Higgs vacuum expectation value v that minimizes the one-loop Landau gauge[77] effective potential. In that case, the one-loop tadpole contribution is precisely canceled by the tree-level Higgs tadpole, as shown in Fig. 29.6.2. Here, we have in mind the latter prescription; the calculation for the pole mass is therefore complete without tadpole contributions provided that the tree-level top quark mass is taken to be

$$m_t = Y_t v, \tag{29.6.6}$$

where Y_t is the $\overline{\rm MS}$ or $\overline{\rm DR}$ Yukawa coupling, and v is the Higgs vacuum expectation value at the minimum of the one-loop effective potential in Landau gauge. To be consistent with this choice, $\xi = 0$ should be taken in all formulae below that involve electroweak gauge bosons or Goldstone bosons. (The gluon contribution is naturally independent of ξ because the gauge symmetry is unbroken, providing a check of gauge-fixing invariance.) Nevertheless, for the sake of generality we will keep the dependence on ξ in the computation of the individual non-tadpole self-energy diagrams below.

Consider the one-loop calculation of the self-energy Σ_{Lt}, which is the sum of individual diagram contributions $\Sigma_{Lt} = [\Sigma_{Lt}]_g + [\Sigma_{Lt}]_\gamma + [\Sigma_{Lt}]_Z + [\Sigma_{Lt}]_W + [\Sigma_{Lt}]_{h_{\rm SM}} + [\Sigma_{Lt}]_{G^0} + [\Sigma_{Lt}]_{G^+}$. First, consider the diagrams involving exchanges of the scalars $\phi = h_{\rm SM}, G^0, G^\pm$. These contributions all have the same form

$$-ip\cdot\overline{\sigma}\,[\Sigma_{Lt}]_\phi \;=\; \mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\,(-iY^*)\left(\frac{i(k+p)\cdot\overline{\sigma}}{(k+p)^2-m_f^2}\right)(-iY)\left(\frac{i}{k^2-m_\phi^2}\right), \tag{29.6.7}$$

where the loop momentum k^μ flows clockwise, and the couplings and propagator masses are, using the Feynman rules of Figs. J.1.3 and J.1.4,

$$\text{for } \phi = h_{\rm SM}: \qquad Y = Y_t/\sqrt{2}; \qquad m_f = m_t; \qquad m_\phi^2 = m_{h_{\rm SM}}^2, \tag{29.6.8}$$

$$\text{for } \phi = G^0: \qquad Y = iY_t/\sqrt{2}; \qquad m_f = m_t; \qquad m_\phi^2 = \xi m_Z^2, \tag{29.6.9}$$

$$\text{for } \phi = G^\pm: \qquad Y = Y_b; \qquad m_f = m_b; \qquad m_\phi^2 = \xi m_W^2. \tag{29.6.10}$$

Multiplying both sides by $p\cdot\sigma$ and taking the trace over spinor indices using eq. (B.2.5), one finds

$$[\Sigma_{Lt}]_\phi \;=\; i|Y|^2\frac{\mu^{2\epsilon}}{p^2}\int\frac{d^dk}{(2\pi)^d}\frac{p\cdot(k+p)}{[(k+p)^2-m_f^2][k^2-m_\phi^2]}\,. \tag{29.6.11}$$

[77] This procedure is considerably more involved outside of Landau gauge, because the propagators mix the longitudinal components of the vector boson with the Nambu-Goldstone bosons for $\xi \neq 0$ if one expands around a Higgs vacuum expectation value that does not minimize the tree-level potential. This is the same reason the effective potential is traditionally calculated specifically in Landau gauge.

Performing the loop momentum integration in the standard way [115, 223], and expanding in ϵ up to constant terms, one finds that in each case

$$[\Sigma_{Lt}]_\phi = -\frac{1}{16\pi^2}|Y|^2\, I_{FS}(s; m_f^2, m_\phi^2). \tag{29.6.12}$$

Here we have introduced some notation for the loop integral:

$$I_{FS}(s;x,y) \equiv \frac{1}{2\epsilon} + \frac{(s+x-y)B(s;x,y) + A(x) - A(y)}{2s}\,, \tag{29.6.13}$$

where the Passarino-Veltman functions $A(x)$ and $B(s;x,y)$ were defined in eqs. (29.6.8) and (29.6.9). These functions depend on the renormalization scale Q, which is related to μ via eq. (29.6.10). It can be checked that $I_{FS}(s;x,y)$ has a smooth limit as $s \to 0$.

Next, let us consider the contributions to Σ_{Lt} involving the vector bosons $V = g, \gamma, Z, W$. These have the common form:

$$\begin{aligned} -ip\cdot\overline{\sigma}[\Sigma_{Lt}]_V \;=\; \mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\; (-iG\,\overline{\sigma}_\mu)\left(\frac{i(k+p)\cdot\sigma}{(k+p)^2-m_f^2}\right)(-iG\,\overline{\sigma}_\nu) \\ \left(\frac{-i}{k^2-m_V^2}\right)\left(g^{\mu\nu} + \frac{(\xi-1)k^\mu k^\nu}{k^2-\xi m_V^2}\right), \end{aligned} \tag{29.6.14}$$

where again the loop momentum k flows clockwise, and, using the rules of Figs. J.1.2 and K.5.1:

$$\text{for } V = g: \qquad G = g_s T^a\,, \qquad m_f = m_t, \tag{29.6.15}$$

$$\text{for } V = \gamma: \qquad G = eQ_t\,, \qquad m_f = m_t, \tag{29.6.16}$$

$$\text{for } V = Z: \qquad G = g(T_3^t - s_W^2 Q_t)/c_W\,, \qquad m_f = m_t, \tag{29.6.17}$$

$$\text{for } V = W: \qquad G = g/\sqrt{2}\,, \qquad m_f = m_b. \tag{29.6.18}$$

In the case of gluon exchange ($V = g$), the T^a are the $SU(3)_C$ generators (with color indices suppressed). The adjoint representation index a is summed over, producing a factor of the Casimir invariant $(T^aT^a)_{ij} = C_F\delta_{ij} = \frac{4}{3}\delta_{ij}$. We now use $\overline{\sigma}_\mu\,\sigma_\rho\,\overline{\sigma}_\nu\,g^{\mu\nu} = -(D-2)\,\overline{\sigma}_\rho$ [see eq. (B.2.11)]; note that this introduces a difference between the $\overline{\rm MS}$ and $\overline{\rm DR}$ schemes. Also, we use $k\cdot\overline{\sigma}(k+p)\cdot\sigma k\cdot\overline{\sigma} = (k^2+2k\cdot p)k\cdot\overline{\sigma} - k^2p\cdot\overline{\sigma}$, which follows from eq. (29.55). One therefore obtains, after multiplying by $p\cdot\sigma$ and taking the trace over spinor indices:

$$\begin{aligned} [\Sigma_{Lt}]_V = -i\,G^2\frac{\mu^{2\epsilon}}{p^2}\int\frac{d^dk}{(2\pi)^d}\,\frac{1}{[(k+p)^2-m_f^2][k^2-m_V^2]}\Big[(2-D)p\cdot(k+p) \\ + \left(k^2k\cdot p + 2(k\cdot p)^2 - k^2p^2\right)\frac{(\xi-1)}{k^2-\xi m_V^2}\Big]. \end{aligned} \tag{29.6.19}$$

Performing the loop momentum integration, one finds that

$$[\Sigma_{Lt}]_V = -\frac{1}{16\pi^2}G^2 I_{FV}(s; m_f^2, m_V^2), \tag{29.6.20}$$

where we have introduced the notation

$$\begin{aligned} I_{FV}(s;x,y) = \frac{\xi}{\epsilon} + [(s+x-y)B(s;x,y) + A(x) - A(y)]/s - \delta_{\overline{\rm MS}} + \big\{(s-x)[A(y) - A(\xi y)] \\ +[(s-x)^2 - y(s+x)]B(s;x,y) - [(s-x)^2 - \xi y(s+x)]B(s;x,\xi y)\big\}/2ys, \end{aligned} \tag{29.6.21}$$

after dropping terms that vanish as $\epsilon \to 0$. Combining the results of eqs. (29.6.12) and (29.6.20):

$$\begin{aligned}\Sigma_{Lt} = -\frac{1}{16\pi^2}\Big[&\Big(g_s^2 C_F + e^2 Q_t^2\Big) I_{FV}(m_t^2; m_t^2, 0) + [g(T_3^t - s_W^2 Q_t)/c_W]^2 I_{FV}(m_t^2; m_t^2, m_Z^2) \\ &+\frac{1}{2} g^2 I_{FV}(m_t^2; m_b^2, m_W^2) + \frac{1}{2} Y_t^2 I_{FS}(m_t^2; m_t^2, m_{h_{\rm SM}}^2) \\ &+\frac{1}{2} Y_t^2 I_{FS}(m_t^2; m_t^2, \xi m_Z^2) + Y_b^2 I_{FS}(m_t^2; m_b^2, \xi m_W^2)\Big],\end{aligned} \tag{29.6.22}$$

where we have now substituted $s = m_t^2$. It is useful to note that for massless gauge bosons,

$$I_{FV}(x; x, 0) = \xi\left[\frac{1}{\epsilon} - \ln(x/Q^2) + 2\right] + 1 - \delta_{\overline{\rm MS}}. \tag{29.6.23}$$

The contributions to $\Sigma_{Rt} = [\Sigma_{Rt}]_g + [\Sigma_{Rt}]_\gamma + [\Sigma_{Rt}]_Z + [\Sigma_{Rt}]_{h_{\rm SM}} + [\Sigma_{Rt}]_{G^0} + [\Sigma_{Rt}]_{G^\pm}$ are obtained similarly. [Note that there is no W boson contribution, since the right-handed top quark is an $SU(2)_L$ singlet.] For the scalar exchange diagrams with $\phi = h_{\rm SM}, G^0, G^\pm$, the general form is:

$$-ip\cdot\sigma[\Sigma_{Rt}]_\phi \;=\; \mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\,(-iY)\left(\frac{i(k+p)\cdot\sigma}{(k+p)^2 - m_f^2}\right)(-iY^*)\left(\frac{i}{k^2 - m_\phi^2}\right), \tag{29.6.24}$$

which yields

$$[\Sigma_{Rt}]_\phi = -\frac{1}{16\pi^2}|Y|^2\, I_{FS}(s; m_f^2, m_\phi^2). \tag{29.6.25}$$

Here the couplings and propagator masses for $h_{\rm SM}$ and G^0 are the same as in eqs. (29.6.8), (29.6.9), but now instead of eq. (29.6.10),

$$\text{for } \phi = G^\pm: \qquad Y = -Y_t\,, \qquad m_f = m_b\,, \qquad m_\phi^2 = \xi m_W^2\,, \tag{29.6.26}$$

from Fig. J.1.4. For the contributions due to exchanges of vectors $v = g, \gamma, Z$, the general form is given by

$$\begin{aligned}-ip\cdot\sigma[\Sigma_{Rt}]_V \;=\; \mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\;&(iG\,\sigma_\mu)\left(\frac{i(k+p)\cdot\overline{\sigma}}{(k+p)^2 - m_f^2}\right)(iG\,\sigma_\nu) \\ &\left(\frac{-i}{k^2 - m_V^2}\right)\left(g^{\mu\nu} + \frac{(\xi-1)k^\mu k^\nu}{k^2 - \xi m_V^2}\right),\end{aligned} \tag{29.6.27}$$

where

$$\text{for } V = g: \qquad G = -g_s \boldsymbol{T^a}\,, \tag{29.6.28}$$
$$\text{for } V = \gamma: \qquad G = -eQ_t\,, \tag{29.6.29}$$
$$\text{for } V = Z: \qquad G = g s_W^2 Q_t/c_W\,, \tag{29.6.30}$$

after using the rules of Figs. J.1.2 and K.5.1 with $m_f = m_t$ in each case. We then make use of $\sigma_\mu\,\overline{\sigma}_\rho\,\sigma_\nu\,g^{\mu\nu} = -(D-2)\,\sigma_\rho$ [cf. eq. (B.2.10)] and $k\cdot\sigma(k+p)\cdot\overline{\sigma}k\cdot\sigma = (k^2 + 2k\cdot p)k\cdot\sigma - k^2 p\cdot\sigma$ [cf. eq. (29.54)]. After multiplying by $p\cdot\overline{\sigma}$ and taking the trace over spinor indices [using eq. (B.2.5)], we obtain

$$[\Sigma_{Rt}]_V = -\frac{1}{16\pi^2} G^2 I_{FV}(s; m_t^2, m_V^2)\,, \tag{29.6.31}$$

in terms of the same function appearing in eqs. (29.6.21) and (29.6.23). Adding up these contributions and taking $s = m_t^2$ yields

$$\Sigma_{Rt} = -\frac{1}{16\pi^2}\Big[\Big(g_s^2 C_F + e^2 Q_t^2\Big) I_{FV}(m_t^2; m_t^2, 0) + (g^2 Q_t^2 s_W^4/c_W^2) I_{FV}(m_t^2; m_t^2, m_Z^2) + \frac{1}{2} Y_t^2 I_{FS}(m_t^2; m_t^2, m_{h_{\rm SM}}^2) + \frac{1}{2} Y_t^2 I_{FS}(m_t^2; m_t^2, \xi m_Z^2) + Y_t^2 I_{FS}(m_t^2; m_b^2, \xi m_W^2)\Big]. \quad (29.6.32)$$

Next, consider the contributions to $\Sigma_{Dt} = [\Sigma_{Dt}]_g + [\Sigma_{Dt}]_\gamma + [\Sigma_{Dt}]_Z + [\Sigma_{Dt}]_{h_{\rm SM}} + [\Sigma_{Dt}]_{G^0} + [\Sigma_{Dt}]_{G^\pm}$, ignoring the tadpole contribution for now. The diagrams involving the exchange of scalars $\phi = h_{\rm SM}, G^0, G^\pm$ have the form:

$$-i[\Sigma_{Dt}]_\phi \;=\; \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d}\; (-iY_1) \left(\frac{i m_f}{(k+p)^2 - m_f^2}\right) (-iY_2) \left(\frac{i}{k^2 - m_\phi^2}\right), \quad (29.6.33)$$

so that

$$[\Sigma_{Dt}]_\phi = i m_f Y_1 Y_2 \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{[(k+p)^2 - m_f^2][k^2 - m_\phi^2]} = \frac{1}{16\pi^2} m_f Y_1 Y_2 I_{\overline{F}S}(s; m_f^2, m_\phi^2)\,, \quad (29.6.34)$$

where we have introduced the notation:

$$I_{\overline{F}S}(s; x, y) \equiv -\frac{1}{\epsilon} - B(s; x, y), \quad (29.6.35)$$

after dropping terms that vanish as $\epsilon \to 0$. The relevant couplings and masses are, from Figs. J.1.3 and J.1.4:

$$\text{for } \phi = h_{\rm SM}: \quad Y_1 = Y_2 = Y_t/\sqrt{2}\,, \quad m_f = m_t\,, \quad m_\phi^2 = m_{h_{\rm SM}}^2, \quad (29.6.36)$$

$$\text{for } \phi = G^0: \quad Y_1 = Y_2 = iY_t/\sqrt{2}, \quad m_f = m_t\,, \quad m_\phi^2 = \xi m_Z^2, \quad (29.6.37)$$

$$\text{for } \phi = G^\pm: \quad Y_1 = Y_b, \quad Y_2 = -Y_t, \quad m_f = m_b\,, \quad m_\phi^2 = \xi m_W^2. \quad (29.6.38)$$

The contributions from vector boson exchanges are of the form

$$-i[\Sigma_{Dt}]_V \;=\; \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d}\; (iG_1\sigma_\mu) \left(\frac{i m_f}{(k+p)^2 - m_f^2}\right) (-iG_2\overline{\sigma}_\nu) \left(\frac{-i}{k^2 - m_V^2}\right) \left(g^{\mu\nu} + \frac{(\xi-1)k^\mu k^\nu}{k^2 - \xi m_V^2}\right), \quad (29.6.39)$$

Using $\sigma_\mu \overline{\sigma}_\nu g^{\mu\nu} = D$ [see eq. (B.2.8)] and $k\cdot\sigma k\cdot\overline{\sigma} = k^2$ [from eq. (29.52)] yields

$$[\Sigma_{Dt}]_V = i m_f G_1 G_2 \mu^{2\epsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{[(k+p)^2 - m_f^2][k^2 - m_V^2]} \left[D + \frac{(\xi-1)k^2}{k^2 - \xi m_V^2}\right] = \frac{1}{16\pi^2} m_f G_1 G_2 I_{\overline{F}V}(s; m_f^2, m_V^2)\,, \quad (29.6.40)$$

where

$$I_{\overline{F}V}(s; x, y) \equiv -\frac{3+\xi}{\epsilon} - 3B(s; x, y) - \xi B(s; x, \xi y) + 2\delta_{\overline{\rm MS}}, \quad (29.6.41)$$

after dropping terms that vanish as $\epsilon \to 0$. It is useful to note that for massless gauge bosons

$$I_{\overline{F}V}(x;x,0) \equiv -\frac{3+\xi}{\epsilon} + (3+\xi)[\ln(x/Q^2) - 2] + 2\delta_{\overline{\rm MS}}. \tag{29.6.42}$$

The relevant couplings are obtained from the rules of Figs. J.1.2 and K.5.1:

$$\text{for } V = g: \qquad G_1 = -G_2 = g_s T^a, \tag{29.6.43}$$
$$\text{for } V = \gamma: \qquad G_1 = -G_2 = eQ_t, \tag{29.6.44}$$
$$\text{for } V = Z: \qquad G_1 = g(T_3^t - s_W^2 Q_t)/c_W, \qquad G_2 = g s_W^2 Q_t/c_W, \tag{29.6.45}$$

and $m_f = m_t$ in each case. Adding up these contributions and taking $s = m_t^2$, we have:

$$\Sigma_{Dt} = \frac{m_t}{16\pi^2}\Big\{ g^2\left[(T_3^t - s_W^2 Q_t)s_W^2 Q_t/c_W^2\right] I_{\overline{F}V}(m_t^2; m_t^2, m_Z^2) - (g_s^2 C_F + e^2 Q_t^2) I_{\overline{F}V}(m_t^2; m_t^2, 0)$$
$$+\tfrac{1}{2}Y_t^2 I_{\overline{F}S}(m_t^2; m_t^2, m_{h_{\rm SM}}^2) - \tfrac{1}{2}Y_t^2 I_{\overline{F}S}(m_t^2; m_t^2, \xi m_Z^2) - Y_b^2 I_{\overline{F}S}(m_t^2; m_b^2, \xi m_W^2)\Big\}, \tag{29.6.46}$$

where $Y_t = m_t Y_b/m_b$ was used on the last term.

In each of the self-energy functions above, there are poles in $1/\epsilon$, contained within the functions I_{FV}, I_{FS}, $I_{\overline{F}V}$ and $I_{\overline{F}S}$. In the $\overline{\rm MS}$ or $\overline{\rm DR}$ schemes, these poles are simply canceled by counterterms, which have no other effect at one-loop order. The one-loop top quark pole mass can now be obtained by plugging eqs. (29.6.22), (29.6.32), and (29.6.46) into eq. (29.6.2) with $\xi = 0$, as discussed earlier. It is not hard to check that the terms from massless Nambu-Goldstone boson exchange just cancel against the terms from the vector exchange diagrams that came from ξm_W^2 and ξm_Z^2.

As a simple example, consider the one-loop pole mass with only QCD effects included. Then the result of eq. (29.6.2) has no imaginary part. Taking the square root (and dropping a two-loop order part) yields the well-known result [228]:

$$\begin{aligned} M_{t,\rm pole} &= m_t(1 + \tfrac{1}{2}\Sigma_{Lt} + \tfrac{1}{2}\Sigma_{Rt}) + \Sigma_{Dt} \\ &= m_t\Big(1 - \frac{C_F g_s^2}{16\pi^2}\Big[I_{FV}(m_t^2; m_t^2, 0) + I_{\overline{F}V}(m_t^2; m_t^2, 0)\Big]\Big) \\ &= m_t\Big(1 + \frac{\alpha_s}{4\pi} C_F\Big[5 - \delta_{\overline{\rm MS}} - 3\ln(m_t^2/Q^2)\Big]\Big). \end{aligned} \tag{29.6.47}$$

As another check, consider the imaginary part of the pole squared mass of the top quark. At leading order, eq. (29.6.2) implies:

$$\begin{aligned} \Gamma_t &= -{\rm Im}[m_t(\Sigma_{Lt} + \Sigma_{Rt}) + 2\Sigma_{Dt}] \\ &= \frac{m_t}{16\pi^2}{\rm Im}\Big[\frac{g^2}{2} I_{FV}(m_t^2; m_b^2, m_W^2) + (Y_t^2 + Y_b^2) I_{FS}(m_t^2; m_b^2, \xi m_W^2) + 2Y_b^2 I_{\overline{F}S}(m_t^2; m_b^2, \xi m_W^2)\Big] \\ &= \frac{1}{32\pi^2 m_t}\left\{(g^2 + Y_t^2 + Y_b^2)(m_t^2 + m_b^2 - m_W^2) - 4Y_b^2 m_t^2\right\} {\rm Im}[B(m_t^2; m_b^2, m_W^2)]. \end{aligned} \tag{29.6.48}$$

The fact that the ξ dependence canceled here is a successful check of gauge-fixing invariance, since the tadpole diagram in Fig. 29.6.1 does not contribute to the absorptive part of the self-energy. One can express ${\rm Im}[B(s;x,y)]$ in terms of the triangle function [cf. eq. (29.6.11)],

$${\rm Im}[B(s;x,y)] = \begin{cases} 0 & \text{for} \quad s \le (\sqrt{x} + \sqrt{y})^2, \\ \pi\lambda^{1/2}(s,x,y)/s & \text{for} \quad s > (\sqrt{x} + \sqrt{y})^2. \end{cases} \tag{29.6.49}$$

Eq. (29.6.48) then reproduces the result of eq. (29.6.10) for the top quark width at leading order.

29.6.25 Self-energy and pole mass of the gluino

The Feynman diagrams for the gluino self-energy are shown in Fig. 29.6.1. Since the gluino is a Majorana fermion, we can use the general formalism of Section 29.4.6. We will compute the self-energy functions $\Xi_{\tilde{g}} \equiv \Xi_{\tilde{g}}^{\tilde{g}}$ and $\Omega_{\tilde{g}} \equiv \Omega^{\tilde{g}\tilde{g}}$ defined in Fig. 29.4.3, and infer $\overline{\Omega}_{\tilde{g}} \equiv \overline{\Omega}_{\tilde{g}\tilde{g}}$ from the latter by replacing all Lagrangian parameters by their complex conjugates.[78] At one-loop order, it follows from the general result of eq. (29.4.24) that the complex pole squared mass of the gluino is related to the tree-level mass $m_{\tilde{g}}$ by

$$M_{\tilde{g}}^2 - iM_{\tilde{g}}\Gamma_{\tilde{g}} = \left[m_{\tilde{g}}^2(1+2\Xi_{\tilde{g}}) + m_{\tilde{g}}(\Omega_{\tilde{g}} + \overline{\Omega}_{\tilde{g}})\right]\Big|_{s=m_{\tilde{g}}^2+i\varepsilon}. \tag{29.6.1}$$

It is convenient to split the self-energy functions into gluon/gluino loop and squark/quark loop contributions, as

$$\Xi_{\tilde{g}} = [\Xi_{\tilde{g}}]_g + \sum_q \sum_{x=1,2} [\Xi_{\tilde{g}}]_{\tilde{q}_x}, \qquad \text{and} \qquad \Omega_{\tilde{g}} = [\Omega_{\tilde{g}}]_g + \sum_q \sum_{x=1,2} [\Omega_{\tilde{g}}]_{\tilde{q}_x}, \tag{29.6.2}$$

where the sum over q runs over the six squark flavors u, d, s, c, b, t, and $x = 1, 2$ corresponds to the two squark mass eigenstates [i.e., the two appropriate linear combinations (for fixed squark flavor) of $\widetilde{q}_L$ and $\widetilde{q}_R$]. The gluon exchange contributions, following from the Feynman rules of Fig. K.5.1, are:

$$\begin{aligned} -ip\cdot\overline{\sigma}\,[\Xi_{\tilde{g}}]_g\,\delta^{ab} = \mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\,(-g_s f^{aec}\overline{\sigma}_\mu)\left(\frac{i(k+p)\cdot\sigma}{(k+p)^2 - m_{\tilde{g}}^2}\right)(-g_s f^{ebc}\overline{\sigma}_\nu) \\ \left(\frac{-i}{k^2}\right)\left(g^{\mu\nu} + (\xi-1)\frac{k^\mu k^\nu}{k^2}\right), \end{aligned} \tag{29.6.3}$$

$$\begin{aligned} -i\,[\Omega_{\tilde{g}}]_g\,\delta^{ab} = \mu^{2\epsilon}\int \frac{d^dk}{(2\pi)^d}\,(g_s f^{eac}\sigma_\mu)\left(\frac{im_{\tilde{g}}}{(k+p)^2 - m_{\tilde{g}}^2}\right)(-g_s f^{ebc}\overline{\sigma}_\nu) \\ \left(\frac{-i}{k^2}\right)\left(g^{\mu\nu} + (\xi-1)\frac{k^\mu k^\nu}{k^2}\right). \end{aligned} \tag{29.6.4}$$

The internal gluon and gluino lines carry $SU(3)_c$ adjoint representation indices c and e respectively, while the external gluinos on the left and right carry indices a and b respectively. The gluino external momentum p^μ flows from right to left, and the loop momentum k^μ flows clockwise. Comparing with the derivations of eqs. (29.6.20) and (29.6.40) in the previous subsection, and using $-f^{aec}f^{ebc} = f^{eac}f^{ebc} = \delta^{ab}C_A$ [with $C_A = 3$ for $SU(3)_c$], we can immediately conclude that

$$[\Xi_{\tilde{g}}]_g = -\frac{\alpha_s}{4\pi}C_A I_{FV}(s; m_{\tilde{g}}^2, 0), \tag{29.6.5}$$

$$[\Omega_{\tilde{g}}]_g = -\frac{\alpha_s}{4\pi}C_A m_{\tilde{g}} I_{\overline{F}V}(s; m_{\tilde{g}}^2, 0), \tag{29.6.6}$$

where the loop integral functions I_{FV} and $I_{\overline{F}V}$ were defined in eqs. (29.6.21) and (29.6.41).

[78] Suppressing the color degrees of freedom, Ξ, Ω and $\overline{\Omega}$ are one-dimensional matrices, so we do not employ boldface letters in this case.

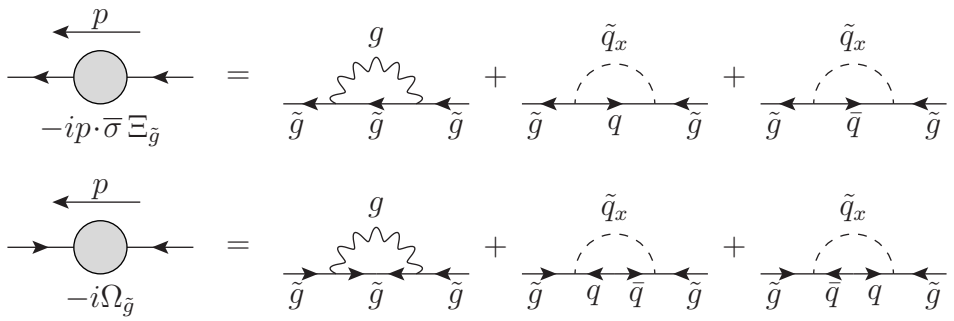


Figure 29.6.1: Self-energy functions for the gluino in supersymmetry. The external momentum p^μ flows from the right to the left. The loop momentum k^μ in the text is taken to flow clockwise. Spinor and color indices are suppressed. The index $x = 1, 2$ labels the two squark mass eigenstates of a given flavor $q = u, d, s, c, b, t$. Both x and q must be summed over. The external legs are amputated.

Next consider the virtual squark exchange diagrams contributing to $\Xi_{\tilde{g}}$. Labeling the quark and squark with color indices j, k respectively, we have for each squark mass eigenstate:

$$-ip\cdot\overline{\sigma}\,[\Xi_{\tilde{g}}]_{\tilde{q}_x}\,\delta^{ab} = \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}(-i\sqrt{2}g_sT_j^{ak}L_{\tilde{q}_x})\left(\frac{i(k+p)\cdot\overline{\sigma}}{(k+p)^2-m_q^2}\right)(-i\sqrt{2}g_sT_k^{bj}L_{\tilde{q}_x}^*)\left(\frac{i}{k^2-m_{\tilde{q}_x}^2}\right)$$
$$+\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\left(i\sqrt{2}g_sT_k^{aj}R_{\tilde{q}_x}^*\right)\left(\frac{i(k+p)\cdot\overline{\sigma}}{(k+p)^2-m_q^2}\right)\left(i\sqrt{2}g_sT_j^{bk}R_{\tilde{q}_x}\right)\left(\frac{i}{k^2-m_{\tilde{q}_x}^2}\right). \quad (29.6.7)$$

This uses the Feynman rules shown in Fig. K.5.3, given in terms of the squark mixing parameters $L_{\tilde{q}_x}$ and $R_{\tilde{q}_x}$ defined in eq. (K.4.1). Using $\mathrm{Tr}[T^aT^b] = \frac{1}{2}\delta^{ab}$ and $|L_{\tilde{q}_x}|^2 + |R_{\tilde{q}_x}|^2 = 1$, and comparing to the derivation of eq. (29.6.12) of the previous subsection, we obtain:

$$[\Xi_{\tilde{g}}]_{\tilde{q}_x} = -\frac{\alpha_s}{4\pi}I_{FS}(s; m_q^2, m_{\tilde{q}_x}^2). \quad (29.6.8)$$

Similarly, for the last two diagrams of Fig. 29.6.1, we obtain:

$$-i[\Omega_{\tilde{g}}]_{\tilde{q}_x}\,\delta^{ab} = \mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\left(-i\sqrt{2}g_sT_k^{aj}L_{\tilde{q}_x}^*\right)\left(\frac{im_q}{(k+p)^2-m_q^2}\right)\left(i\sqrt{2}g_sT_j^{bk}R_{\tilde{q}_x}\right)\left(\frac{i}{k^2-m_{\tilde{q}_x}^2}\right)$$
$$+\mu^{2\epsilon}\int\frac{d^dk}{(2\pi)^d}\left(i\sqrt{2}g_sT_j^{ak}R_{\tilde{q}_x}\right)\left(\frac{im_q}{(k+p)^2-m_q^2}\right)\left(-i\sqrt{2}g_sT_k^{bj}L_{\tilde{q}_x}^*\right)\left(\frac{i}{k^2-m_{\tilde{q}_x}^2}\right), \quad (29.6.9)$$

again using the Feynman rules shown in Fig. K.5.3. As before, j and k are the color indices for the quark and the squark, respectively. Comparing to the derivation of eq. (29.6.34) of the previous subsection, we obtain:

$$[\Omega_{\tilde{g}}]_{\tilde{q}_x} = -\frac{\alpha_s}{2\pi}L_{\tilde{q}_x}^*R_{\tilde{q}_x}m_qI_{\overline{F}S}(s; m_q^2, m_{\tilde{q}_x}^2)\,. \quad (29.6.10)$$

Summing up the results obtained above, and taking $s = m_{\tilde{g}}^2$, we have:

$$\Xi_{\tilde{g}} = -\frac{\alpha_s}{4\pi}\left[C_AI_{FV}(m_{\tilde{g}}^2; m_{\tilde{g}}^2, 0) + \sum_q\sum_{x=1,2}I_{FS}(m_{\tilde{g}}^2; m_q^2, m_{\tilde{q}_x}^2)\right], \quad (29.6.11)$$

$$\Omega_{\tilde{g}} = -\frac{\alpha_s}{4\pi}\left[C_Am_{\tilde{g}}I_{\overline{F}V}(m_{\tilde{g}}^2; m_{\tilde{g}}^2, 0) + 2\sum_q\sum_{x=1,2}L_{\tilde{q}_x}^*R_{\tilde{q}_x}m_qI_{\overline{F}S}(m_{\tilde{g}}^2; m_q^2, m_{\tilde{q}_x}^2)\right]. \quad (29.6.12)$$

As previously noted, we can now write down $\overline{\Omega}_{\tilde{g}}$ by replacing the Lagrangian parameters of eq. (29.6.12) by their complex conjugates:

$$\overline{\Omega}_{\tilde{g}} = -\frac{\alpha_s}{4\pi}\left[C_A m_{\tilde{g}} I_{\overline{F}V}(m_{\tilde{g}}^2; m_{\tilde{g}}^2, 0) + 2\sum_q \sum_{x=1,2} L_{\tilde{q}_x} R_{\tilde{q}_x}^* m_q I_{\overline{F}S}(m_{\tilde{g}}^2; m_q^2, m_{\tilde{q}_x}^2)\right]. \tag{29.6.13}$$

Inserting the results of eqs. (29.6.11)–(29.6.13) into eq. (29.6.1), one obtains the result [229, 230]:

$$M_{\tilde{g}}^2 - iM_{\tilde{g}}\Gamma_{\tilde{g}} = m_{\tilde{g}}^2\Bigg[1 + \frac{\alpha_s}{2\pi}\Big\{C_A\left[5 - \delta_{\overline{\rm MS}} - 3\ln\big(m_{\tilde{g}}^2/Q^2\big)\right] - \sum_q \sum_{x=1,2}\left[I_{FS}(m_{\tilde{g}}^2; m_q^2, m_{\tilde{q}_x}^2) + 2{\rm Re}[L_{\tilde{q}_x}^* R_{\tilde{q}_x}]\frac{m_q}{m_{\tilde{g}}} I_{\overline{F}S}(m_{\tilde{g}}^2; m_q^2, m_{\tilde{q}_x}^2)\right]\Big\}\Bigg], \tag{29.6.14}$$

with $\delta_{\overline{\rm MS}}$ defined in eq. (29.6.5).

29.6.26 Triangle anomaly from chiral fermion loops

As our final example, we consider the anomaly in chiral symmetries for fermions, arising from the triangle diagram involving three currents carrying vector indices.[79] Since the anomaly is independent of the fermion masses, we simplify the computation by setting all fermion masses to zero. In four-component notation,[80] the treatment of the anomaly requires care because of the difficulty in defining a consistent and unambiguous γ_5 and the epsilon tensor in dimensional regularization [233, 234]. The same subtleties arise in two-component language, of course, but in a slightly different form since γ_5 does not appear explicitly.

We shall assemble all the $(\frac{1}{2}, 0)$ [left-handed] two-component fermion fields of the theory into a multiplet ψ_j. For example, the fermions of the Standard Model are: $\psi_j = (\ell_k\,,\ \bar{\ell}_k\,,\ \nu_k\,,\ q_{i\ell}\,,\ \bar{q}_{i\ell})$, where $k = 1, 2, 3$ and $i = 1, 2, ..., 6$ are flavor labels and $\ell = 1, 2, 3$ are color labels [see Table 29.1]. The two-component spinor indices are suppressed here. Let the symmetry generators be given by hermitian matrices $\boldsymbol{T^a}$, so that the ψ_j transform as:

$$\delta\psi_j = -i\theta^a(\boldsymbol{T^a})_j{}^k\psi_k, \tag{29.6.1}$$

for infinitesimal parameters θ^a. The matrices $\boldsymbol{T^a}$ form a representation R of the generators of the Lie algebra of the symmetry group. In general R will be reducible, in which case the $\boldsymbol{T^a}$ have a block diagonal structure, where each block separately transforms (irreducibly) the corresponding field of ψ_j according to its symmetry transformation properties. Some or all of these symmetries may be gauged. The Feynman rule for the corresponding currents is the same as for external gauge bosons, as in Fig. 29.4.2 (but without the gauge couplings), and is shown in Fig. 29.6.1.

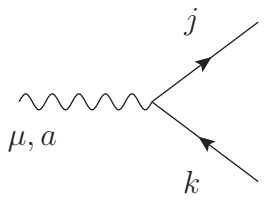
Fig. 29.6.2 exhibits the two Feynman diagrams that contribute at one-loop to the three-point function of the symmetry currents. Applying the $\overline{\sigma}$-version of the Feynman rule for the currents given in Fig. 29.6.1, and employing the Feynman rules of Fig. 29.4.1 (with $m = 0$) for the propagators [traversing the loop in the direction dictated by eq. (29.4.2)], the sum of the two triangle diagrams shown in Fig. 29.6.2 can be evaluated.

The resulting sum of loop integrals is

$$i\Gamma^{abc}_{\mu\nu\rho} = (-1)\int \frac{d^4k}{(2\pi)^4}{\rm Tr}\Bigg\{(-i\overline{\sigma}_\mu \boldsymbol{T}^a)\frac{i(k-p+A)\cdot\sigma}{(k-p+A)^2}(-i\overline{\sigma}_\nu \boldsymbol{T}^b)\frac{i(k+A)\cdot\sigma}{(k+A)^2}(-i\overline{\sigma}_\rho \boldsymbol{T}^c)\frac{i(k+q+A)\cdot\sigma}{(k+q+A)^2}$$
$$+(-i\overline{\sigma}_\mu \boldsymbol{T}^a)\frac{i(k-q+B)\cdot\sigma}{(k-q+B)^2}(-i\overline{\sigma}_\rho \boldsymbol{T}^c)\frac{i(k+B)\cdot\sigma}{(k+B)^2}(-i\overline{\sigma}_\nu \boldsymbol{T}^b)\frac{i(k+p+B)\cdot\sigma}{(k+p+B)^2}\Bigg\}, \tag{29.6.2}$$

[79] The discussion here parallels that given in ref. [231], Section 22.3.

[80] For an excellent review of the computation of the chiral anomaly via four-component massless and massive spinor triangle loops, see [232].



$$-i(\boldsymbol{T^a})_j{}^k\,\overline{\sigma}_\mu \quad \text{or} \quad i(\boldsymbol{T^a})_j{}^k\,\sigma_\mu$$

Figure 29.6.1: Feynman rule for the coupling of a current carrying vector index μ and corresponding to the symmetry generator $\boldsymbol{T^a}$ acting on $(\frac{1}{2},0)$ [left-handed] fermions. Spinor indices are suppressed.

where the overall factor of (-1) is due to the presence of a closed fermion loop. The trace is taken over fermion flavor/group and spinor indices, both of which are suppressed. Because the individual integrals are linearly divergent, we must allow for arbitrary constant four-vectors A^μ and B^μ as offsets for the loop momentum when defining the loop integrations for the two diagrams [235, 236].

The persistence of the symmetry in the quantum theory for the currents labeled by μ, a and ν, b and ρ, c implies the naive Ward identities:[81]

$$(p+q)^\mu\, i\Gamma^{abc}_{\mu\nu\rho}(-p-q,p,q) = f^{abd}\Pi^{dc}_{\nu\rho}(q) + f^{acd}\Pi^{bd}_{\nu\rho}(p)\,, \tag{29.6.3}$$

$$-p^\nu\, i\Gamma^{abc}_{\mu\nu\rho}(-p-q,p,q) = f^{bcd}\Pi^{da}_{\rho\mu}(p+q) + f^{bad}\Pi^{cd}_{\rho\mu}(q)\,, \tag{29.6.4}$$

$$-q^\rho\, i\Gamma^{abc}_{\mu\nu\rho}(-p-q,p,q) = f^{cad}\Pi^{db}_{\mu\nu}(p) + f^{cbd}\Pi^{ad}_{\mu\nu}(p+q)\,, \tag{29.6.5}$$

where $i\Pi^{ab}_{\mu\nu}(p)$ is the one-loop current-current two-point function shown in Fig. 29.6.3.

By Lorentz covariance, $\Pi^{ab}_{\mu\nu}(p)$ is a rank-two symmetric tensor that is an even function of the four-momentum p [cf. eq. (29.6.41)]. In eqs. (29.6.3)–(29.6.5), we have employed a convention in which the arguments of $i\Gamma$ correspond to the *outgoing* momentum of the external legs of the corresponding one-loop Feynman diagrams, and the order of the momentum arguments matches the order of the indices.

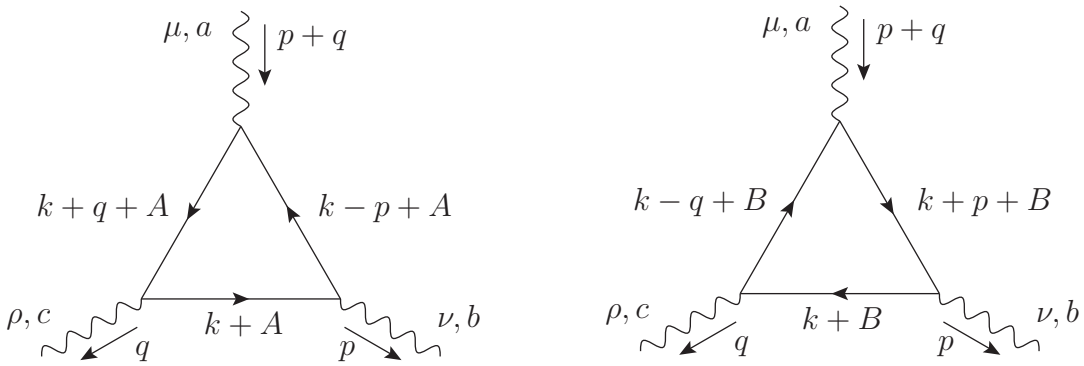


Figure 29.6.2: Triangle Feynman diagrams leading to the chiral fermion anomaly. Fermion spinor and flavor indices are suppressed. The fermion momenta, as labeled, flow in the arrow directions.

[81] The derivation of the Ward identities is most easily achieved by writing the three-point function in position space as a vacuum expectation value of the time-ordered product of three currents. After taking the divergence (with respect to the position of any one of the three currents) of the time-ordered product and using the fact that the currents are conserved ($\partial_\mu j^{a\mu} = 0$), the surviving terms can be evaluated using the equal-time commutation relations, $\delta(x^0-y^0)[j^{a0}(x), j^{b\nu}(y)] = if^{abc}j^{c\nu}(x)\delta^4(x-y)$. Fourier-transforming the result yields the terms on the right-hand side of eqs. (29.6.3)–(29.6.5). See refs. [237, 238] for further details.

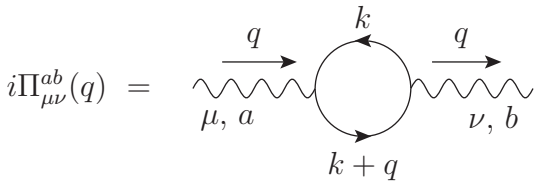


Figure 29.6.3: The one-loop contribution to the current-current two-point function. The fermion momenta, as labeled, flow along the corresponding arrow directions.

It is convenient to define the *symmetrized* three-point function by symmetrizing over the indices a, b and c:

$$\mathcal{A}^{abc}_{\mu\nu\rho} = \tfrac{1}{6} i\Gamma^{abc}_{\mu\nu\rho} + [\text{five permutations of } a, b, c]. \tag{29.6.6}$$

In terms of the symmetrized three-point function, the naive Ward identities imply

$$(p+q)^\mu \mathcal{A}^{abc}_{\mu\nu\rho} = 0\,, \qquad -p^\nu \mathcal{A}^{abc}_{\mu\nu\rho} = 0\,, \quad \text{and} \quad -q^\rho \mathcal{A}^{abc}_{\mu\nu\rho} = 0\,. \tag{29.6.7}$$

We now perform the explicit diagrammatic computation to show that the naive Ward identities exhibited in eq. (29.6.7) are violated due to a quantum anomaly. Although the symmetrized three-point function is ultraviolet finite, the individual loop momentum integrals are divergent, and must be defined with care. We do not regularize them by the usual procedure of continuing to $d = 4 - 2\epsilon$ dimensions, because the trace over sigma matrices crucially involves the antisymmetric tensor with four indices, brought in by eqs. (B.2.26) and (B.2.27), for which there is no consistent and unambiguous generalization outside of four dimensions. (This is related to the difficulty of defining γ_5 in the four-component spinor formalism.) The existence of the vectors A and B corresponds to an ambiguity in the regulation procedure, which can be fixed to preserve some of the symmetries, as we will see below.

Starting from eq. (29.6.2), it follows from eq. (E.2.7) that the symmetrized three-point function is proportional to the group theory factor (often called the anomaly coefficient),

$$D^{abc} = \tfrac{1}{2}\mathrm{Tr}[\{\boldsymbol{T^a}, \boldsymbol{T^b}\}\boldsymbol{T^c}]\,, \tag{29.6.8}$$

where the numerical values of the D^{abc} depend on the representation R. As discussed in Appendix E , D^{abc} vanishes for all simple Lie groups, with the exception of SU(N) for $N \geq 3$. The D^{abc} are also non-vanishing in general for any non-semisimple compact Lie group, which contains at least one U(1) factor.

First, consider the result for $(p+q)^\mu \mathcal{A}^{abc}_{\mu\nu\rho}$. This can be simplified by rewriting

$$(p+q)^\mu = (k+q+A)^\mu - (k-p+A)^\mu\,, \tag{29.6.9}$$
$$(p+q)^\mu = (k+p+B)^\mu - (k-q+B)^\mu\,, \tag{29.6.10}$$

in the first and second diagram terms, respectively, and then applying the formulae

$$v{\cdot}\sigma\, v{\cdot}\overline{\sigma} \;=\; v^2\,, \qquad v{\cdot}\overline{\sigma}\, v{\cdot}\sigma \;=\; v^2\,, \tag{29.6.11}$$

which follow from eqs. (B.2.1) and (B.2.2). After rearranging the terms using the cyclic property of the trace, we obtain:

$$\begin{aligned}(p+q)^\mu \mathcal{A}^{abc}_{\mu\nu\rho} \;&=\; -D^{abc}\,\mathrm{Tr}[\sigma_\kappa \overline{\sigma}_\nu \sigma_\lambda \overline{\sigma}_\rho]\, X^{\kappa\lambda}, \\ &= -2D^{abc}\left[X_{\nu\rho} + X_{\rho\nu} - g_{\nu\rho} X_\lambda{}^\lambda + i\epsilon_{\kappa\nu\lambda\rho} X^{\kappa\lambda}\right]\,,\end{aligned} \tag{29.6.12}$$

after applying eq. (B.2.26). (In our conventions, $\epsilon_{0123} = -1$.) The integral $X^{\kappa\lambda}$ is given by:

$$X^{\kappa\lambda} = \int \frac{d^4k}{(2\pi)^4} \left[\frac{(k-p+A)^\kappa}{(k-p+A)^2} \frac{(k+A)^\lambda}{(k+A)^2} - \frac{(k+q+A)^\kappa}{(k+q+A)^2} \frac{(k+A)^\lambda}{(k+A)^2} \right.$$
$$\left. + \frac{(k+B)^\kappa}{(k+B)^2} \frac{(k-q+B)^\lambda}{(k-q+B)^2} - \frac{(k+B)^\kappa}{(k+B)^2} \frac{(k+p+B)^\lambda}{(k+p+B)^2} \right]. \tag{29.6.13}$$

Naively, this integral appears to vanish, because the first term is equal to the negative of the fourth term after a momentum shift $k \to k-p+A-B$, and the second term is equal to the negative of the third term after $k \to k+q+A-B$. However, these momentum shifts are not valid for the individually divergent integrals. Instead, $X^{\kappa\lambda}$ can be evaluated by a Wick rotation to Euclidean space, followed by isolating the terms that contribute for large k^2 and are responsible for the integral not vanishing, and then employing the divergence (Gauss') theorem in four dimensions to rewrite $X^{\kappa\lambda}$ as an angular integral over a three-sphere with radius tending to infinity. This integral is initially evaluated at large but finite Euclidean k, with the limit $k \to \infty$ taken at the end of the computation. For example, consider a smooth function $f(k)$ of the four-momentum k with the property that the integral

$$\int d^4k f(k) \tag{29.6.14}$$

is at worst quadratically divergent. We define the even and odd parts of $f(k)$, respectively, by:

$$f_e(k) \equiv \tfrac{1}{2}\left[f(k) + f(-k)\right], \qquad f_o(k) \equiv \tfrac{1}{2}\left[f(k) - f(-k)\right]. \tag{29.6.15}$$

It then follows that [237, 239, 240]

$$\int \frac{d^4k}{(2\pi)^4} \left[f(k+a) - f(k)\right] = \frac{i}{(2\pi)^4} \left[2\pi^2 a_\mu \lim_{k\to\infty} k^\mu k^2 f_o(k) + \pi^2 a_\mu a_\nu \lim_{k\to\infty} k^\mu k^2 \frac{\partial}{\partial k_\nu} f_e(k) \right] \tag{29.6.16}$$

has a finite limit.[82] In deriving this result, we have expanded $f(k+a)$ in a Taylor expansion and follow the procedure outlined above eq. (29.6.14). Note that the angular integration removes the even parts of $f(k)$ and $\partial f/\partial k^\nu \equiv 2k_\nu\, \partial f/\partial k^2$ from the right-hand side of eq. (29.6.16). The "limits" in eq. (29.6.16) actually correspond to an average over the three-sphere at large Euclidean k, and thus should be interpreted by the use of:

$$\lim_{k\to\infty} \frac{k^\mu k^\nu}{k^2} = \tfrac{1}{4} g^{\mu\nu}, \tag{29.6.17}$$

$$\lim_{k\to\infty} \frac{k^\mu k^\nu k^\rho k^\lambda}{(k^2)^2} = \frac{1}{24}\left(g^{\mu\nu}g^{\rho\lambda} + g^{\mu\rho}g^{\nu\lambda} + g^{\mu\lambda}g^{\nu\rho}\right). \tag{29.6.18}$$

For example, if

$$f(k) = \frac{(k-p+A)^\kappa (k+A)^\lambda}{(k-p+A)^2 (k+A)^2}, \tag{29.6.19}$$

then in evaluating eq. (29.6.16), it is sufficient to write:

$$f_o(k) \simeq \tfrac{1}{2}(k-p+A)^\kappa (k+A)^\lambda \left[\frac{1}{(k^2)^2} + \frac{2k\cdot(p-2A)}{(k^2)^3} \right] - (k \to -k)$$
$$\simeq \frac{k^\kappa A^\lambda - k^\lambda (p-A)^\kappa}{(k^2)^2} + \frac{2k^\kappa k^\lambda\, k\cdot(p-2A)}{(k^2)^3}, \tag{29.6.20}$$

[82] If eq. (29.6.14) is linearly divergent, then the second term on the right-hand side of eq. (29.6.16) is zero. If eq. (29.6.14) is logarithmically divergent or finite, then the right-hand side of eq. (29.6.16) vanishes.

where we have dropped terms that do not contribute to eq. (29.6.16) in the limit of $k \to \infty$. Similarly,

$$\frac{\partial f_e}{\partial k_\nu} \simeq \frac{g^{\kappa\nu}k^\lambda + g^{\lambda\nu}k^\kappa}{(k^2)^2} - \frac{4k^\kappa k^\lambda k^\nu}{(k^2)^3}\,. \tag{29.6.21}$$

The evaluation of $X^{\kappa\lambda}$ is now straightforward [after using eqs. (29.6.17) and (29.6.18)]:

$$X^{\kappa\lambda} = \frac{i}{96\pi^2}\Big[g^{\kappa\lambda}(p+q)\cdot(A+B) + (A-2B)^\kappa(p+q)^\lambda + (p+q)^\kappa(B-2A)^\lambda\Big]. \tag{29.6.22}$$

Hence, eq. (29.6.12) yields the result for the anomaly in the current labeled by μ, a:

$$(p+q)^\mu \mathcal{A}^{abc}_{\mu\nu\rho} = \frac{i}{48\pi^2}D^{abc}\Big[(p+q)_\nu(A+B)_\rho + (A+B)_\nu(p+q)_\rho + g_{\nu\rho}(p+q)\cdot(A+B) \\ -3i\epsilon_{\nu\rho\kappa\lambda}(p+q)^\kappa(A-B)^\lambda\Big]. \tag{29.6.23}$$

Repeating all of the steps starting with eq. (29.6.9), we similarly obtain:[83]

$$-p^\nu \mathcal{A}^{abc}_{\mu\nu\rho} = -\frac{i}{48\pi^2}D^{abc}\Big[p_\rho(A+B)_\mu + p_\mu(A+B)_\rho + g_{\mu\rho}p\cdot(A+B) - 3i\epsilon_{\rho\mu\kappa\lambda}p^\kappa(A-B+2q)^\lambda\Big], \tag{29.6.24}$$

$$-q^\rho \mathcal{A}^{abc}_{\mu\nu\rho} = -\frac{i}{48\pi^2}D^{abc}\Big[q_\mu(A+B)_\nu + q_\nu(A+B)_\mu + g_{\mu\nu}q\cdot(A+B) - 3i\epsilon_{\mu\nu\kappa\lambda}q^\kappa(A-B-2p)^\lambda\Big]. \tag{29.6.25}$$

Non-chiral anomalies will arise for all three of the currents (assuming D^{abc} is non-vanishing), unless we choose the arbitrary constant vectors A and B such that

$$A + B = 0\,, \tag{29.6.26}$$

with the result:

$$(p+q)^\mu \mathcal{A}^{abc}_{\mu\nu\rho} = \frac{1}{8\pi^2}D^{abc}\epsilon_{\nu\rho\kappa\lambda}(p+q)^\kappa A^\lambda, \tag{29.6.27}$$

$$-p^\nu \mathcal{A}^{abc}_{\mu\nu\rho} = -\frac{1}{8\pi^2}D^{abc}\epsilon_{\rho\mu\kappa\lambda}p^\kappa(A+q)^\lambda, \tag{29.6.28}$$

$$-q^\rho \mathcal{A}^{abc}_{\mu\nu\rho} = -\frac{1}{8\pi^2}D^{abc}\epsilon_{\mu\nu\kappa\lambda}q^\kappa(A-p)^\lambda. \tag{29.6.29}$$

If D^{abc} is non-vanishing, it is not possible to avoid an anomaly simultaneously in all three symmetries, but one can still arrange for two of the symmetries to be non-anomalous. If one wants an anomaly to arise only in the current labeled by μ, a (for example, if the symmetries labeled by b, c are gauged), one must now choose $A = p - q$. The standard result follows:

$$(p+q)^\mu \mathcal{A}^{abc}_{\mu\nu\rho} = -\frac{1}{4\pi^2}D^{abc}\epsilon_{\nu\rho\kappa\lambda}p^\kappa q^\lambda, \tag{29.6.30}$$

$$-p^\nu \mathcal{A}^{abc}_{\mu\nu\rho} = 0, \tag{29.6.31}$$

$$-q^\rho \mathcal{A}^{abc}_{\mu\nu\rho} = 0. \tag{29.6.32}$$

[83] Alternatively, one can simply note that eq. (29.6.24) follows from eq. (29.6.23) by making the replacements $\mu \to \nu$, $\nu \to \rho$, $\rho \to \mu$, $A \to A+q$, $B \to B-q$, $p \to q$, and $q \to -p-q$, while eq. (29.6.25) follows from eq. (29.6.23) by making the replacements $\mu \to \rho$, $\nu \to \mu$, $\rho \to \nu$, $A \to A-p$, $B \to B+p$, $p \to -p-q$, and $q \to p$.

In particular, one cannot gauge all three symmetries labeled by a, b, c unless $D^{abc} = 0$.

If all three currents are identical, then by Bose symmetry the anomalies of the three currents must coincide. This can be achieved by choosing $A = \frac{1}{3}(p - q)$, in which case,

$$(p+q)^\mu \mathcal{A}^{abc}_{\mu\nu\rho} = -\frac{1}{12\pi^2} D^{abc} \epsilon_{\nu\rho\kappa\lambda} p^\kappa q^\lambda, \tag{29.6.33}$$

$$-p^\nu \mathcal{A}^{abc}_{\mu\nu\rho} = -\frac{1}{12\pi^2} D^{abc} \epsilon_{\rho\mu\kappa\lambda} p^\kappa q^\lambda, \tag{29.6.34}$$

$$-q^\rho \mathcal{A}^{abc}_{\mu\nu\rho} = -\frac{1}{12\pi^2} D^{abc} \epsilon_{\mu\nu\kappa\lambda} p^\kappa q^\lambda. \tag{29.6.35}$$

Returning briefly to the original naive Ward identities given in eqs. (29.6.3)–(29.6.5), the analysis above shows that these identities must be modified by an additional additive contribution given by the right-hand side of eqs. (29.6.27)–(29.6.29). In particular, there is no anomalous contribution proportional to f^{abc}. This can be checked explicitly by a diagrammatic computation of the two-point and three-point functions that appear in eqs. (29.6.3) and (29.6.5). We use eqs. (E.2.11) and (E.2.14) to write

$$\mathrm{Tr}(\boldsymbol{T^a T^b T^c}) = D^{abc}(R) + \frac{i}{2} I_2(R) f^{abc}\,, \tag{29.6.36}$$

where $I_2(R)$ is the index defined in eq. (E.2.1) and R is the representation of the generators $\boldsymbol{T^a}$. For example, inserting this result in eq. (29.6.2), it follows that:

$$(p+q)^\mu \, i\Gamma^{abc}_{\mu\nu\rho} = -\left[D^{abc}\, X^{\kappa\lambda} + \frac{i}{2} I_2(R) f^{abc}\, Y^{\kappa\lambda} \right] \mathrm{Tr}[\sigma_\kappa \overline{\sigma}_\nu \sigma_\lambda \overline{\sigma}_\rho]\,, \tag{29.6.37}$$

where the integral $Y^{\kappa\lambda}$ is given by:[84]

$$Y^{\kappa\lambda} = \int \frac{d^4k}{(2\pi)^4} \left[\frac{(k-p)^\kappa}{(k-p)^2} \frac{k^\lambda}{k^2} - \frac{(k+q)^\kappa}{(k+q)^2} \frac{k^\lambda}{k^2} - \frac{k^\kappa}{k^2} \frac{(k-q)^\lambda}{(k-q)^2} + \frac{k^\kappa}{k^2} \frac{(k+p)^\lambda}{(k+p)^2} \right]. \tag{29.6.38}$$

By letting $k \to -k$ in the third and fourth term in the integrand of eq. (29.6.38), we see that $Y^{\kappa\lambda} = Y^{\lambda\kappa}$, and hence by eq. (B.2.26),

$$-\frac{i}{2} I_2(R) f^{abc}\, Y^{\kappa\lambda} \mathrm{Tr}[\sigma_\kappa \overline{\sigma}_\nu \sigma_\lambda \overline{\sigma}_\rho] = -i I_2(R) f^{abc} \left[2Y_{\nu\rho} - g_{\nu\rho} Y_\lambda{}^\lambda \right]\,. \tag{29.6.39}$$

Since no ϵ-tensor appears, we can evaluate this integral in $d \neq 4$ dimensions using the standard techniques of dimensional regularization.

One can check that this result matches the diagrammatic calculation of the right-hand side of eq. (29.6.27). In particular, Fig. 29.6.3 yields

$$\begin{aligned} i\Pi^{ab}_{\mu\nu}(q) &= (-1) \int \frac{d^4k}{(2\pi)^4} \,\mathrm{Tr}\left[(-i\overline{\sigma}_\mu \boldsymbol{T^a}) \frac{ik\cdot\sigma}{k^2} (-i\overline{\sigma}_\nu \boldsymbol{T^b}) \frac{i(k+q)\cdot\sigma}{(k+q)^2} \right] \\ &= -I_2(R) \delta^{ab}\, \mathrm{Tr}(\overline{\sigma}_\mu \sigma_\rho \overline{\sigma}_\nu \sigma_\lambda) \int \frac{d^4k}{(2\pi)^4} \frac{k^\rho (k+q)^\lambda}{k^2 (k+q)^2}\,, \end{aligned} \tag{29.6.40}$$

where we have used eq. (E.2.1). Lorentz covariance implies that

$$i\Pi^{ab}_{\mu\nu}(q) = \delta^{ab} \left[C_1(q^2) g_{\mu\nu} + C_2(q^2) q_\mu q_\nu \right]\,, \tag{29.6.41}$$

[84] Here $Y^{\kappa\lambda}$ is obtained from $X^{\kappa\lambda}$ by setting $A = B = 0$, since we can use dimensional regularization for this part of the computation as explained below eq. (29.6.38).

for some scalar functions C_1 and C_2. It follows that $\Pi^{ab}_{\mu\nu}(q) = \Pi^{ab}_{\mu\nu}(-q)$ and $\Pi^{ab}_{\mu\nu}(q) = \Pi^{ab}_{\nu\mu}(q)$. Consequently, we can write:

$$\Pi^{ab}_{\mu\nu}(q) = \frac{i}{2} I_2(R)\delta^{ab}\,\mathrm{Tr}(\overline{\sigma}_\mu\sigma_\rho\overline{\sigma}_\nu\sigma_\lambda + \overline{\sigma}_\nu\sigma_\rho\overline{\sigma}_\mu\sigma_\lambda)\int \frac{d^4k}{(2\pi)^4}\frac{k^\rho(k+q)^\lambda}{k^2(k+q)^2}\,, \tag{29.6.42}$$

and so no ϵ-tensor appears in the evaluation of the trace. As above, we are now free to evaluate the integral in $d \neq 4$ dimensions. Comparing eqs. (29.6.37) and (29.6.38) to eq. (29.6.42), and using eq. (29.6.27), the end result is

$$(p+q)^\mu\, i\Gamma^{abc}_{\mu\nu\rho}(-p-q,p,q) = I_2(R) f^{abc}\,[\Pi_{\nu\rho}(q) - \Pi_{\nu\rho}(p)] + \frac{1}{8\pi^2} D^{abc}(R)\epsilon_{\nu\rho\kappa\lambda}(p+q)^\kappa A^\lambda\,, \tag{29.6.43}$$

where we have written $\Pi^{ab}_{\nu\rho} \equiv I_2(R)\delta^{ab}\Pi_{\nu\rho}$. Indeed the terms on the right-hand side proportional to f^{abc} match those of the naive Ward identity given in eq. (29.6.3). As previously asserted, the anomaly only resides in the contributions to the Ward identity proportional to D^{abc}.

In writing down eq. (29.6.2), we chose to use the rules with $\overline{\sigma}$ matrices for the current vertices and σ matrices for the massless fermion propagators. If we had chosen the opposite prescription (i.e., σ matrices for the current vertices and $\overline{\sigma}$ matrices for the massless fermion propagators), then the order of the factors inside the trace of eq. (29.6.2) would have been reversed.[85] Instead of eq. (29.6.12), we would have obtained

$$(p+q)^\mu \mathcal{A}^{abc}_{\mu\nu\rho} \;=\; -D^{abc}\,\mathrm{Tr}[\overline{\sigma}_\kappa\sigma_\nu\overline{\sigma}_\lambda\sigma_\rho]\,\bar{X}^{\kappa\lambda} = -2D^{abc}\left[\bar{X}_{\nu\rho} + \bar{X}_{\rho\nu} - g_{\nu\rho}\bar{X}_\lambda{}^\lambda - i\epsilon_{\kappa\nu\lambda\rho}\bar{X}^{\kappa\lambda}\right], \tag{29.6.44}$$

after applying eq. (B.2.27). The integral $\bar{X}^{\kappa\lambda}$ is simply related to $X^{\kappa\lambda}$ by:

$$\bar{X}^{\kappa\lambda} = X^{\lambda\kappa}\,. \tag{29.6.45}$$

Inserting eq. (29.6.45) into eq. (29.6.44), we immediately reproduce the result of eq. (29.6.12), as expected.

It is instructive to examine the case of massless QED. The terms of the Lagrangian involving the electron fields is given by

$$\mathscr{L} = i\chi^\dagger\overline{\sigma}^\mu D_\mu\chi + i\eta^\dagger\overline{\sigma}^\mu D_\mu\eta\,, \tag{29.6.46}$$

where $D_\mu \equiv \partial_\mu + iQA_\mu$ is the covariant derivative, and Q is the charge operator. Here, we identify χ as the two-component (left-handed) electron field and η as the two-component (left-handed) positron field. The corresponding eigenvalues of the charge operator are: $Q\chi = -e\chi$ and $Q\eta = +e\eta$ (where $e > 0$ is the electromagnetic gauge coupling constant, or equivalently the electric charge of the positron).

At the classical level, the massless QED Lagrangian [eq. (29.6.46)] is invariant under a $\mathrm{U}(1)_V\times\mathrm{U}(1)_A$ global symmetry. Under a $\mathrm{U}(1)_V\times\mathrm{U}(1)_A$ transformation specified by the infinitesimal parameters θ_V and θ_A,

$$\mathrm{U}(1)_V: \qquad \delta\chi = ie\theta_V\chi\,, \qquad\qquad \delta\eta = -ie\theta_V\eta\,, \tag{29.6.47}$$

$$\mathrm{U}(1)_A: \qquad \delta\chi = i\theta_A\chi\,, \qquad\qquad \delta\eta = i\theta_A\eta\,. \tag{29.6.48}$$

We can combine these equations into a two-dimensional matrix equation,

$$\delta\psi_j = -i\theta_a(\boldsymbol{T_a})_j{}^k\psi_k\,, \qquad \text{where} \qquad \psi = \begin{pmatrix}\chi\\ \eta\end{pmatrix}\,, \tag{29.6.49}$$

[85] The arrowed fermion lines in the loop must be traversed in the direction parallel [antiparallel] to the arrow directions when the $\overline{\sigma}$ [σ] versions of the propagator rule are employed, as indicated in eq. (29.4.2) [and in the discussion that follows]. This rule determines the order of the factors inside the spinor trace.

and the index a takes on two values, $a = V$, A. It follows that the $\mathrm{U}(1)_V \times \mathrm{U}(1)_A$ generators are given by

$$T_V = e \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad \text{for} \quad \mathrm{U}(1)_V \,, \tag{29.6.50}$$

$$T_A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} , \qquad \text{for} \quad \mathrm{U}(1)_A \,. \tag{29.6.51}$$

The classically conserved Noether currents corresponding to the $\mathrm{U}(1)_V \times \mathrm{U}(1)_A$ global symmetry are the vector and axial currents:[86]

$$J_V^\mu = -e(\chi^\dagger \overline{\sigma}^\mu \chi - \eta^\dagger \overline{\sigma}^\mu \eta) \,, \tag{29.6.52}$$

$$J_A^\mu = -\chi^\dagger \overline{\sigma}^\mu \chi - \eta^\dagger \overline{\sigma}^\mu \eta \,. \tag{29.6.53}$$

Since the $\mathrm{U}(1)_V$ symmetry is gauged, we demand that this symmetry should be anomaly free. Thus, we make use of eqs. (29.6.30)–(29.6.32), where we identify the index pair μ, a with the axial vector current and the index pairs ν, b and ρ, c with the vector current. Thus, we compute:

$$D^{AVV} = \mathrm{Tr}\,(T_A T_V T_V) = -2e^2 \,. \tag{29.6.54}$$

Moreover, for an abelian symmetry group, $f^{abc} = 0$. Hence, using eq. (29.6.30) [which also applies in this case to the unsymmetrized three-point function], the U(1) axial vector anomaly equation reads:

$$(p+q)^\mu \, i\Gamma^{AVV}_{\mu\nu\rho} \;=\; \frac{e^2}{2\pi^2} \epsilon_{\nu\rho\kappa\lambda} p^\kappa q^\lambda \,, \tag{29.6.55}$$

in agreement with the well-known result.[87]

We now convert eq. (29.6.55) into an operator equation. Consider the process of two photon production by an axial vector current source [242]. First, we note that $\partial_\mu J_A^\mu(x) = i[P^\mu, J_{A\mu}(x)]$, where P^μ is the momentum operator. It follows that:

$$\langle p\,,\, q\,|\,\partial_\mu J_A^\mu(0)\,|\,0\rangle = i\,\langle p\,,\, q\,|\,[P^\mu\,,\, J_{A\mu}(0)]\,|\,0\rangle = i(p+q)^\mu\,\langle p\,,\, q\,|\,J_{A\mu}(0)|\,0\rangle \;. \tag{29.6.56}$$

We identify the S-matrix amplitude for the two photon production as:

$$i\Gamma^{AVV}_{\mu\nu\rho}\;\varepsilon^{\nu\,*}(p)\varepsilon^{\rho\,*}(q) = \langle p\,,\, q\,|\,-iJ_{A\mu}(0)|\,0\rangle \;, \tag{29.6.57}$$

where $\varepsilon(p)$ and $\varepsilon(q)$ are the polarization vectors for the final state photons. Note that the factor of $-i$ on the right-hand side of eq. (29.6.57) has been inserted to be consistent with the Feynman rule for the axial vector current insertion given in Fig. 29.6.1. Thus, using eqs. (29.6.55)–(29.6.57), we end up with [115]:

$$\begin{aligned} \langle p\,,\, q\,|\,\partial_\mu J_A^\mu(0)\,|\,0\rangle &= -\,\frac{e^2}{2\pi^2}\epsilon_{\nu\rho\kappa\lambda}\varepsilon^{\nu\,*}(p)\varepsilon^{\rho\,*}(q)p^\kappa q^\lambda \\ &= -\,\frac{e^2}{16\pi^2}\langle p\,,\, q\,|\,\epsilon_{\kappa\nu\lambda\rho}F^{\kappa\nu}F^{\lambda\rho}(0)\,|0\rangle \,, \end{aligned} \tag{29.6.58}$$

[86] Note that the interaction Lagrangian for massless QED is $\mathscr{L}_{\rm int} = -J_V^\mu A_\mu$, as expected. This accounts for the factor of e in the definition of the vector current. The axial vector current does not couple to the photon field; hence no coupling constant is included in its definition.

[87] This result was first obtained by Adler [241]. In comparing eq. (29.6.55) with Adler's result, note that the normalization of the triangle amplitude in [241] differs by a factor of $(2\pi)^4$ and the opposite sign convention for ϵ_{0123} is employed.

where $\epsilon_{\kappa\nu\lambda\rho}F^{\kappa\nu}F^{\lambda\rho} = 4\epsilon_{\kappa\nu\lambda\rho}(\partial^\kappa A^\nu)(\partial^\lambda A^\rho)$ has been used to eliminate the photon fields in favor of a product of electromagnetic field strength tensors. In deriving eq. (29.6.58), an additional factor of two arises due to two possible contractions of the photon fields with the external states. We thus obtain the operator form for the axial vector anomaly:[88]

$$\partial_\mu J^\mu_A = -\frac{e^2}{8\pi^2}F^{\lambda\rho}\widetilde{F}_{\lambda\rho}\,, \tag{29.6.59}$$

where the dual electromagnetic field strength tensor is defined by $\widetilde{F}_{\lambda\rho} \equiv \frac{1}{2}\epsilon_{\kappa\nu\lambda\rho}F^{\kappa\nu}$.

As a final example, we examine the anomalous baryon number and lepton number currents in the theory of electroweak interactions [243, 244, 245]. For simplicity of notation, we consider a one-generation model. The baryon number current is a vector current given by:

$$J^\mu_B = \tfrac{1}{3}\left[u^\dagger\overline{\sigma}^\mu u + d^\dagger\overline{\sigma}^\mu d - \bar{u}^\dagger\overline{\sigma}^\mu\bar{u} - \bar{d}^\dagger\overline{\sigma}^\mu\bar{d}\right]\,, \tag{29.6.60}$$

following the particle naming conventions of Table 29.1. Consider the process of gauge boson pair production by a baryon number current source. It is convenient to work in the interaction basis of gauge fields, $\{W^{\mu\,a}\,,\,B^\mu\}$, where $W^{\mu\,a}$ is an SU(2)-triplet of gauge fields and B^μ is a $\mathrm{U(1)_Y}$ hypercharge gauge field. We consider triangle diagrams where one generation of quarks runs in the loop. The external vertices consist of the baryon number current source and the two gauge bosons.

The generators corresponding to the SU(2) gauge boson vertices are given in block diagonal form by:

$$\boldsymbol{T^b} = g\,\mathrm{diag}\left(\frac{\tau^b}{2}\otimes\mathbb{1}_{3\times3}\,,\;0\,,\;0\right)\,, \tag{29.6.61}$$

where the τ^b are the Pauli matrices, $\mathbb{1}_{3\times3}$ is the identity matrix in color space, and $\otimes$ is the Kronecker product.[89] We have included a factor of the weak SU(2) coupling g in the definition of $\boldsymbol{T^b}$, since the Feynman rule given by Fig. 29.6.1 does not explicitly include the gauge coupling. Likewise, the generators corresponding to the $\mathrm{U(1)_Y}$ gauge boson vertices are given in block diagonal form by (cf. Table J.1):

$$\boldsymbol{Y} = g'\,\mathrm{diag}\left(\tfrac{1}{6}\mathbb{1}_{2\times2}\otimes\mathbb{1}_{3\times3}\,,\;-\tfrac{2}{3}\mathbb{1}_{3\times3}\,,\;\tfrac{1}{3}\mathbb{1}_{3\times3}\right)\,, \tag{29.6.62}$$

where $\mathbb{1}_{2\times2}$ is the identity matrix in weak isospin space, and g' is the $\mathrm{U(1)_Y}$ hypercharge gauge coupling. Finally, the generator corresponding to the baryon number current source is given in block diagonal form by:

$$\boldsymbol{B} = \tfrac{1}{3}\mathrm{diag}\left(\mathbb{1}_{2\times2}\otimes\mathbb{1}_{3\times3}\,,\;-\mathbb{1}_{3\times3}\,,\;-\mathbb{1}_{3\times3}\right)\,. \tag{29.6.63}$$

Consider first the production of two SU(2)-triplet gauge fields. We put $\boldsymbol{T^a} = \boldsymbol{B}$ and associate the indices b and c with the SU(2)-triplet gauge bosons. A simple calculation yields

$$D^{Bbc} = g^2\,\mathrm{Tr}(\boldsymbol{B}\boldsymbol{T^b}\boldsymbol{T^c}) = \tfrac{1}{2}g^2\delta^{bc}\,, \tag{29.6.64}$$

where the superscript index B refers to the baryon number current. Since the gauged weak SU(2) and hypercharge $\mathrm{U(1)_Y}$ currents must be anomaly free for the mathematical consistency of

[88] In the literature, eq. (29.6.59) often occurs with the opposite sign due to a sign convention for the Levi-Civita ϵ-tensor that is opposite to the one employed in this review. Here, we have reproduced the form given in [115].

[89] The Kronecker product of an $n\times n$ matrix and an $m\times m$ matrix is an $nm\times nm$ matrix. In addition, the following two properties of the Kronecker product are noteworthy [161, 246]: (i) $(A\otimes B)(C\otimes D) = AC\otimes BD$, and (ii) $\mathrm{Tr}(A\otimes B) = \mathrm{Tr}\,A\;\mathrm{Tr}\,B$.

the electroweak theory, it follows that eqs. (29.6.30)–(29.6.32) apply. That is, the symmetrized amplitude for the production of SU(2) gauge boson pairs by a baryon number source is anomalous:

$$(p+q)^\mu \mathcal{A}^{Bbc}_{\mu\nu\rho} = -\frac{g^2}{8\pi^2}\delta^{bc}\epsilon_{\nu\rho\kappa\lambda}p^\kappa q^\lambda\,. \tag{29.6.65}$$

Next, consider the production of two $\mathrm{U}(1)_\mathrm{Y}$ hypercharge gauge fields. A simple calculation yields

$$D^{BYY} = g'^2\,\mathrm{Tr}(\boldsymbol{BY^2}) = -\tfrac{1}{2}g'^2\,. \tag{29.6.66}$$

That is, the symmetrized amplitude for the production of $\mathrm{U}(1)_\mathrm{Y}$ gauge boson pairs by a baryon number source is anomalous:

$$(p+q)^\mu \mathcal{A}^{BYY}_{\mu\nu\rho} = \frac{g'^2}{8\pi^2}\epsilon_{\nu\rho\kappa\lambda}p^\kappa q^\lambda\,. \tag{29.6.67}$$

Finally, the symmetrized amplitude for the associated production of an SU(2)-triplet and $\mathrm{U}(1)_\mathrm{Y}$ hypercharge gauge field exhibits no anomaly as the corresponding $D^{BYc} = gg'\,\mathrm{Tr}(\boldsymbol{BYT^c}) = 0$.

The symmetrized amplitudes of the triangle diagrams involving a baryon number current source and a pair of SU(2) or $\mathrm{U}(1)_\mathrm{Y}$ gauge bosons are anomalous. Since the baryon number current is a vector current, we conclude that the source of the anomaly is a VVA triangle diagram in which one of the gauge boson currents is vector (V) and the other gauge boson current is axial vector (A). Nevertheless, the gauge boson axial vector current must be conserved, as noted above. Hence, the baryon number vector current must be anomalous [243]. In eqs. (29.6.55)–(29.6.58), we showed how to derive the operator form of the anomaly equation from the anomalous non-conservation of the symmetrized triangle amplitude. Following the same set of steps starting with eqs. (29.6.65) and (29.6.67), one obtains the anomalous non-conservation of the baryon number vector current, in a model with N_g quark generations [244, 56, 245]:

$$\partial_\mu J^\mu_B = \frac{g^2 N_g}{32\pi^2}W^{\lambda\rho b}\widetilde{W}^b_{\lambda\rho} - \frac{g'^2 N_g}{32\pi^2}B^{\lambda\rho}\widetilde{B}_{\lambda\rho}\,, \tag{29.6.68}$$

where $B_{\lambda\rho}$ and

$$W^b_{\lambda\rho} = \partial_\lambda W^b_\rho - \partial_\rho W^b_\lambda - g\epsilon^{bca}W^c_\lambda W^a_\rho\,, \tag{29.6.69}$$

are the field strength tensors for the hypercharge $\mathrm{U}(1)_\mathrm{Y}$ gauge boson and SU(2) gauge boson fields, respectively.[90] Note that for the non-abelian SU(2) gauge fields W^a_μ,

$$\begin{aligned} W^{\lambda\rho b}\,\widetilde{W}^b_{\lambda\rho} &= 2\epsilon^{\kappa\nu\lambda\rho}\left[(\partial_\kappa W^b_\nu)(\partial_\lambda W^b_\rho) - g\epsilon^{abc}(\partial_\kappa W^a_\nu)W^b_\lambda W^c_\rho\right] \\ &= 2\epsilon^{\kappa\nu\lambda\rho}\partial_\kappa\left[W^b_\nu(\partial_\lambda W^b_\rho) - \tfrac{1}{3}g\epsilon^{abc}W^a_\nu W^b_\lambda W^c_\rho\right]\,. \end{aligned} \tag{29.6.70}$$

Strictly speaking, the triangle graphs yield only the terms on the right-hand side of eq. (32.5) that are quadratic in the gauge fields. To obtain the corresponding terms that are cubic in the gauge terms, one must compute the anomalies that arise from VVVA and VAAA box diagrams [247, 237].

For completeness, we re-express the anomalous non-conservation of the baryon number current in terms of the mass eigenstate $\mathrm{SU}(2)\times\mathrm{U}(1)_\mathrm{Y}$ gauge fields:

$$\partial_\mu J^\mu_B = \frac{g^2 N_g}{16\pi^2}W^{\lambda\rho+}\widetilde{W}^-_{\lambda\rho} - \frac{g^2 N_g}{32\pi^2 c_W^2}Z^{\lambda\rho}\widetilde{Z}_{\lambda\rho} - \frac{egN_g}{32\pi^2 c_W}\left[Z^{\lambda\rho}\widetilde{F}_{\lambda\rho} + F^{\lambda\rho}\widetilde{Z}_{\lambda\rho}\right]\,, \tag{29.6.71}$$

[90] We again caution the reader that a different overall sign in eq. (32.5) often appears in the literature due to a sign convention for the Levi-Civita ϵ-tensor that is opposite to the one employed in this review. Here, we have chosen $\epsilon^{0123} = +1$.

where $c_W \equiv \cos\theta_W$, and $W^\pm_{\lambda\rho}$, $Z_{\lambda\rho}$ and $F_{\lambda\rho}$ are the $W^\pm$, Z and the electromagnetic field strength tensors, respectively.

By a similar analysis, one can also compute the anomalous non-conservation of the lepton number current,

$$J^\mu_L = \ell^\dagger\overline{\sigma}^\mu\ell + \nu^\dagger\overline{\sigma}^\mu\nu - \bar{\ell}^\dagger\overline{\sigma}^\mu\bar{\ell}\,, \tag{29.6.72}$$

due to triangle diagrams with N_g generations of leptons running in the loop. In the one-generation calculation, the relevant generators are:

$$\boldsymbol{T^b} = g\,\mathrm{diag}\left(\frac{\tau^b}{2}\,,\ 0\right)\,, \qquad \boldsymbol{Y} = g'\,\mathrm{diag}\left(-\tfrac{1}{2}\mathbb{1}_{2\times2}\,,\ 1\right)\,, \qquad \boldsymbol{L} = \mathrm{diag}\left(\mathbb{1}_{2\times2}\,,\ -1\right)\,. \tag{29.6.73}$$

Thus, we end up with:

$$D^{Lbc} = \tfrac{1}{2}g^2\delta^{bc}\,, \qquad D^{LYY} = -\tfrac{1}{2}g'^2\,, \qquad D^{LYc} = 0\,. \tag{29.6.74}$$

Thus, in the Standard Model with N_g generations of quarks and leptons,

$$\partial_\mu J^\mu_L = \partial_\mu J^\mu_B\,. \tag{29.6.75}$$

Hence, the $B-L$ current is conserved. However, $B-L$ is *not* anomaly free, due to the fact that the lepton number current exhibited in eq. (29.6.72) has both vector and axial vector pieces. In particular, the symmetrized amplitude of the triangle diagrams with three lepton number current sources is anomalous. To avoid this anomaly, one can add a right-handed neutrino to the Standard Model, in which case the leptonic current, $J^\mu_L = \ell^\dagger\overline{\sigma}^\mu\ell + \nu^\dagger\overline{\sigma}^\mu\nu - \bar{\ell}^\dagger\overline{\sigma}^\mu\bar{\ell} - \bar{\nu}^\dagger\overline{\sigma}^\mu\bar{\nu}$, is a vector current. In this case, eq. (29.6.75) still holds, and the $B-L$ current is conserved and anomaly free.

29.7 Appendixes

29.7.1 Metric and sigma matrix conventions

In this review, the metric tensor of four-dimensional Minkowski space is taken to be:[91]

$$g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)\,, \tag{A.1}$$

where $\mu,\nu = 0,1,2,3$ are spacetime vector indices. Contravariant four-vectors (e.g. positions, momenta, gauge fields and currents) are defined with raised indices, and covariant four-vectors (e.g. derivatives) with lowered indices:

$$x^\mu = (t\,;\,\vec{\boldsymbol{x}})\,, \tag{A.2}$$
$$p^\mu = (E\,;\,\vec{\boldsymbol{p}})\,, \tag{A.3}$$
$$A^\mu(x) = (\Phi(\vec{\boldsymbol{x}},t)\,;\,\vec{\boldsymbol{A}}(\vec{\boldsymbol{x}},t))\,, \tag{A.4}$$
$$J^\mu(x) = (\rho(\vec{\boldsymbol{x}},t)\,;\,\vec{\boldsymbol{J}}(\vec{\boldsymbol{x}},t))\,, \tag{A.5}$$
$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\partial/\partial t\,;\,\vec{\boldsymbol{\nabla}})\,, \tag{A.6}$$

in units with $c=1$. The totally antisymmetric pseudo-tensor $\epsilon^{\mu\nu\rho\sigma}$ is defined such that

$$\epsilon^{0123} = -\epsilon_{0123} = +1\,. \tag{A.7}$$

Eqs. (A.2)–(A.7) are taken to be independent of the metric signature convention.

[91] An otherwise identical version of this paper with the opposite metric signature is also available; see footnote 8.

The sigma matrices are defined with a raised (contravariant) index to be independent of the metric signature convention,

$$\sigma^\mu = (\mathbb{1}_{2\times2}\,;\,\vec{\boldsymbol{\sigma}})\,, \qquad\qquad \overline{\sigma}^\mu = (\mathbb{1}_{2\times2}\,;\,-\vec{\boldsymbol{\sigma}})\,, \tag{A.8}$$

where the three-vector of Pauli matrices is given by $\vec{\boldsymbol{\sigma}} \equiv (\sigma^1\,,\,\sigma^2\,,\,\sigma^3)$ [cf. eq. (29.29)] and $\mathbb{1}_{2\times2}$ is the 2×2 identity matrix. The corresponding quantities with lower (covariant) index are:

$$\sigma_\mu = g_{\mu\nu}\sigma^\nu = (\mathbb{1}_{2\times2}\,;\,-\vec{\boldsymbol{\sigma}})\,, \qquad\qquad \overline{\sigma}_\mu = g_{\mu\nu}\overline{\sigma}^\nu = (\mathbb{1}_{2\times2}\,;\,\vec{\boldsymbol{\sigma}})\,. \tag{A.9}$$

Various identities involving products of sigma matrices are given in Appendix B . The generators of the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of the Lorentz group are, respectively, given by:

$$\sigma^{\mu\nu} \equiv \frac{i}{4}(\sigma^\mu\overline{\sigma}^\nu - \sigma^\nu\overline{\sigma}^\mu)\,, \qquad\qquad \overline{\sigma}^{\mu\nu} \equiv \frac{i}{4}(\overline{\sigma}^\mu\sigma^\nu - \overline{\sigma}^\nu\sigma^\mu)\,. \tag{A.10}$$

In adopting the above definition of the sigma matrices, we differ from the corresponding conventions of Wess and Bagger [69] and Bilal [83]. The Wess/Bagger and Bilal (WBB) definition of the sigma matrices can be written (with lowered index μ) as:[92]

$$(\sigma^{\rm WBB})_{\mu\,\alpha\dot\beta} = \sigma_{0\alpha\dot\gamma}\overline{\sigma}_\mu^{\dot\gamma\delta}\sigma_{0\delta\dot\beta} = (\mathbb{1}_{2\times2}\,;\,\vec{\boldsymbol{\sigma}})\,, \tag{A.11}$$

$$(\overline{\sigma}^{\rm WBB})_\mu^{\dot\alpha\beta} = \overline{\sigma}_0^{\dot\alpha\gamma}\sigma_{\mu\gamma\dot\delta}\overline{\sigma}_0^{\dot\delta\beta} = (\mathbb{1}_{2\times2}\,;\,-\vec{\boldsymbol{\sigma}})\,. \tag{A.12}$$

One consequence of the WBB definition of σ and $\overline{\sigma}$ is that $\gamma_5 = \mathrm{diag}(\mathbb{1}_{2\times2}\,,\,-\mathbb{1}_{2\times2})$ in the chiral representation [cf. eq. (G.1.2)]. This associates a lowered undotted [raised dotted] two-component spinor with a right-handed [left-handed] four-component spinor [cf. eqs. (G.1.7) and (G.1.9)]. Indeed, this was the common convention in the older literature (e.g., see refs. [36, 37, 39, 40, 42, 59]).[93] However, in the modern formulation of electroweak theory in terms of left-handed fermions, it is now more common to associate a lower undotted [raised dotted] two-component spinor with a left-handed [right-handed] four-component spinor. This is the motivation for our conventions for the sigma matrices given in eqs. (A.8) and (A.9).

In order to facilitate the comparison with the metric signature with $g_{00} = -1$, we provide the key ingredients needed for translating between Minkowski metrics of opposite signature. In our conventions [cf. eqs. (A.2)–(A.9)], each of the following objects (with the Lorentz index heights as shown) is defined *independently* of the metric signature:

$$x^\mu\,,\,p^\mu\,,\,J^{\mu\nu}\,,\,J_{\mu\nu}\,,\,\partial_\mu\,,\,\sigma^\mu\,,\,\overline{\sigma}^\mu\,,\,S^\mu\,,\,J^\mu\,,\,A^\mu\,,\,D_\mu\,,\,G^\mu{}_\nu\,,\,\gamma^\mu\,,\,\gamma_5\,,\,\delta^\mu_\nu\,,\,\epsilon^{\mu\nu\rho\sigma}\,,\,\epsilon_{\mu\nu\rho\sigma}\,,\,\theta^\mu{}_\nu \qquad \text{[no sign change]}, \tag{A.13}$$

whereas the following objects change sign when the Minkowski metric signature is reversed:

$$g_{\mu\nu}\,,\,g^{\mu\nu}\,,\,x_\mu\,,\,p_\mu\,,\,\partial^\mu\,,\,\sigma_\mu\,,\,\overline{\sigma}_\mu\,,\,S_\mu\,,\,J_\mu\,,\,A_\mu\,,\,D^\mu\,,\,G^{\mu\nu}\,,\,G_{\mu\nu}\,,\,\gamma_\mu\,,\,\theta^{\mu\nu}\,,\,\theta_{\mu\nu} \qquad \text{[sign change]}. \tag{A.14}$$

Here, $J^{\mu\nu}$ is the angular momentum tensor, the spin four-vector S^μ is defined in eq. (29.3.15), J^μ is any conserved current, A^μ is any gauge vector potential, and D_μ and $G_{\mu\nu}$ are the corresponding covariant derivative and antisymmetric tensor field strength, respectively. The Dirac gamma matrices are defined in eq. (G.1.2), and the tensor $\theta^{\mu\nu}$ parameterizes Lorentz transformations [cf. eqs. (29.8), (29.15), and (29.100)–(29.102)]. The list of eq. (A.14) can be deduced from eq. (A.13) by using the metric tensor and its inverse to lower and raise Lorentz indices, simply

[92] Although Wess/Bagger and Bilal employ opposite metric signatures of $g_{00} = -1$ and $g_{00} = +1$, respectively, their definitions of σ_μ and $\overline{\sigma}_\mu$ (with covariant index μ) coincide. Note that the spinor structure of the σ and $\overline{\sigma}$ matrices and the definitions of the various (two-index and four-index) epsilon tensors [cf. eqs. (29.21) and (A.7)] are identical in both the WBB conventions and in our conventions.

[93] This convention persists in the literature of the spinor helicity method (cf. footnote 163 in Appendix I.2).

because each metric or inverse metric changes sign when the metric signature is reversed. Given any other object not included in eqs. (A.13) and (A.14), it is straightforward to make the appropriate assignment by considering how the object is defined. For example, we must assign $\sigma_{\mu\nu}$, $\overline{\sigma}_{\mu\nu}$, $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ to the list of eq. (A.13), based on the definitions given in eqs. (29.71) and (29.72). In general, objects that do not carry Lorentz vector indices (including all fermion spinor fields and spinor wave functions) are defined to be the same in the two metric signatures, with the obvious exception of scalar quantities formed from an odd number of objects from the list of eq. (A.14). For example, the dot product of two four-vectors may or may not change sign when the Minkowski metric signature is reversed. By writing out the dot product explicitly using the metric tensor to contract the indices, one can use eqs. (A.13) and (A.14) to determine the behavior of a dot product under the reversal of the metric signature. In particular, $p\cdot A$ changes sign whereas $\sigma\cdot\partial$ and $\overline{\sigma}\cdot\partial$ do not change sign, when the Minkowski metric signature is reversed.

The translation between Minkowski metrics of opposite signatures is now straightforward. Given any relativistic covariant quantity or equation in the convention where $g_{00} = +1$, one need only employ eqs. (A.13) and (A.14) to obtain the same quantity or equation in the convention where $g_{00} = -1$, and vice versa.[94]

As an example, let us verify that under the reversal of the Minkowski metric signature the gauge covariant derivative D_μ does not change sign and the gauge field strength tensor $G^{\mu\nu}$ changes sign. In the metric signature with $g_{00} = +1$, we define

$$D_\mu \equiv \boldsymbol{I}_{d_R}\partial_\mu + igA_\mu\,, \qquad\qquad (g_{00} = +1)\,, \tag{A.15}$$

where $A_\mu \equiv A^a_\mu \boldsymbol{T^a}$ is the matrix gauge field for a representation R of dimension d_R, and $\boldsymbol{I}_{d_R}$ is the $d_R \times d_R$ identity matrix. Since under the reversal of the metric signature, ∂_μ does not change sign [according to eq. (A.13)] whereas A_μ changes sign [according to eq. (A.14)], it follows that the quantity defined in the metric signature where $g_{00} = -1$,

$$D_\mu \equiv \boldsymbol{I}_{d_R}\partial_\mu - igA_\mu\,, \qquad\qquad (g_{00} = -1) \tag{A.16}$$

has the same overall sign as eq. (A.15). It follows that when the metric signature is reversed, D_μ does not change sign whereas $D^\mu \equiv g^{\mu\nu}D_\nu$ does change sign, as indicated in eqs. (A.13) and (A.14). Next, consider the matrix gauge field strength tensor $G_{\mu\nu} \equiv G^a_{\mu\nu}\boldsymbol{T^a}$, defined by

$$G^{\mu\nu} \equiv \frac{-i}{g}\,[D^\mu\,,\,D^\nu] = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu\,,\,A^\nu]\,, \qquad (g_{00} = +1)\,, \tag{A.17}$$

where the commutator $[D^\mu\,,\,D^\nu]$ is an operator that acts on fields that transform with respect to an arbitrary representation R. In the metric signature with $g_{00} = -1$, we define the gauge field strength tensor as a commutator of covariant derivatives with the *opposite* overall sign:

$$G^{\mu\nu} \equiv \frac{i}{g}\,[D^\mu\,,\,D^\nu] = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu\,,\,A^\nu]\,, \qquad (g_{00} = -1)\,, \tag{A.18}$$

where D^μ is now defined as in eq. (A.16). Since under a reversal of the metric signature, A^μ does not change sign [according to eq. (A.13)] whereas ∂^μ changes sign [according to eq. (A.14)], it follows that $G^{\mu\nu}$ and $G_{\mu\nu} \equiv g_{\mu\rho}g_{\nu\sigma}G^{\rho\sigma}$ do indeed change sign when the metric signature is reversed, as stated in eq. (A.14).

As another simple illustration, consider the σ-matrix identity,

$$\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho = g^{\mu\nu}\overline{\sigma}^\rho - g^{\mu\rho}\overline{\sigma}^\nu + g^{\nu\rho}\overline{\sigma}^\mu - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa\,, \qquad (g_{00} = +1)\,, \tag{A.19}$$

[94] Note that for any relativistic covariant term appearing additively in a valid equation, the *relative* sign that results from changing between Minkowski metrics of opposite signature is simply given by $\mathcal{S} = (-1)^{\mathcal{N}}$, where $\mathcal{N} \equiv N_m + N_d + N_G + ...$ Here N_m is the number of metric tensors appearing either explicitly or implicitly through contracted upper and lower indices, N_d is the number of spacetime and/or covariant derivatives, N_G is the number of gauge field strength tensors, and the ellipsis (...) accounts for any additional quantities whose contravariant forms (with all Lorentz indices raised) appear in the list of eq. (A.14).

In the opposite metric signature with $g_{00} = -1$, we apply the results of eqs. (A.13) and (A.14) and then multiply both sides of the equation by -1 to obtain:

$$\overline{\sigma}^\mu \sigma^\nu \overline{\sigma}^\rho = -g^{\mu\nu}\overline{\sigma}^\rho + g^{\mu\rho}\overline{\sigma}^\nu - g^{\nu\rho}\overline{\sigma}^\mu + i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa\,, \qquad (g_{00} = -1)\,. \tag{A.20}$$

Finally, in the sigma matrix conventions of Wess/Bagger [69] and Bilal [83], both eqs. (A.19) and (A.20) are modified by changing the overall sign of $i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa$. In general, to convert the identities of Appendix B to the conventions of WBB, one must first convert (if necessary) to the appropriate metric signature, and then interchange $\sigma \leftrightarrow \overline{\sigma}$ [cf. eqs. (A.11) and (A.12)].

We end this Appendix with a brief summary of our conventions for four-dimensional Euclidean space. The Euclidean components of the coordinates [represented in Minkowski space by the contravariant four-vector, $x^\mu = (x^0\,;\,\vec{\boldsymbol{x}})$, for $\mu = 0, 1, 2, 3$], are defined as

$$x^\mu_E = x_{E\,\mu} = (\vec{\boldsymbol{x}}\,,\, x^4_E)\,, \qquad x^4_E = x_{E4} \equiv ix^0\,, \qquad (\mu = 1, 2, 3, 4)\,. \tag{A.21}$$

The four-momentum operator in Minkowski space is $p^\mu = i\partial^\mu = i(\partial/\partial t\,,\, -\vec{\boldsymbol{\nabla}})$. Following the conventions of [248], the Euclidean counterpart of the momentum operator is

$$p^\mu_E = p_{E\mu} = (\vec{\boldsymbol{p}}\,,\, p^4_E) = -i\partial^\mu_E = -i(\vec{\boldsymbol{\nabla}}\,,\, \partial/\partial x^4_E)\,, \qquad p^4_E = p_{E4} = ip^0\,, \tag{A.22}$$

The Minkowski space Green functions are obtained from Euclidean space Green functions by means of a Wick rotation [124, 249, 248] of $x^4_E \equiv ix^0$ in a counterclockwise sense.[95] Scalar products of Euclidean four-vectors are carried out by employing the Euclidean metric tensor $\delta_{\mu\nu} = \delta^{\mu\nu} = \mathrm{diag}(1\,,\,1\,,\,1\,,\,1)$. For example, the Euclidean counterpart of $-p\cdot x = -p^0x^0 + \vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}}$ is $p^\mu_E x^\mu_E = \vec{\boldsymbol{p}}\cdot\vec{\boldsymbol{x}} + p^4_E x^4_E$, etc. Given any tensorial equation in Euclidean space, the heights of the indices is irrelevant. Consequently, one can simply place all indices at the same height (either all raised or all lowered), with an implied sum over a pair of repeated indices.

One can also introduce Euclidean sigma matrices [250]:

$$\sigma^\mu_E \equiv (-i\vec{\boldsymbol{\sigma}}\,,\, \sigma^4_E)\,, \qquad \overline{\sigma}^\mu_E \equiv (i\vec{\boldsymbol{\sigma}}\,,\, \overline{\sigma}^4_E)\,, \qquad \text{where } \sigma^4_E = \overline{\sigma}^4_E \equiv \mathbb{1}_{2\times2}\,, \tag{A.23}$$

which satisfy:[96]

$$\sigma^\mu_E\overline{\sigma}^\nu_E + \sigma^\nu_E\overline{\sigma}^\mu_E = 2\delta^{\mu\nu}\,, \qquad \overline{\sigma}^\mu_E\sigma^\nu_E + \overline{\sigma}^\nu_E\sigma^\mu_E = 2\delta^{\mu\nu}\,. \tag{A.24}$$

The four-dimensional rotation group in Euclidean space is SO(4), which is locally equivalent to SU(2)×SU(2). It possesses two independent pseudo-real two-dimensional spinor representations $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ [not related by hermitian conjugation in contrast to the Lorentz group], with corresponding hermitian generators $\sigma^{\mu\nu}_E$ and $\overline{\sigma}^{\mu\nu}_E$, respectively:

$$\sigma^{\mu\nu}_E = \frac{i}{4}\,(\sigma^\mu_E\overline{\sigma}^\nu_E - \sigma^\nu_E\overline{\sigma}^\mu_E)\,, \qquad \overline{\sigma}^{\mu\nu}_E = \frac{i}{4}\,(\overline{\sigma}^\mu_E\sigma^\nu_E - \overline{\sigma}^\nu_E\sigma^\mu_E)\,. \tag{A.25}$$

These tensors are anti-self-dual and self-dual, respectively [121],

$$\sigma^{\mu\nu}_E = -\tfrac{1}{2}\epsilon^{\mu\nu\rho\tau}\sigma^{\rho\tau}_E\,, \qquad \overline{\sigma}^{\mu\nu}_E = \tfrac{1}{2}\epsilon^{\mu\nu\rho\tau}\overline{\sigma}^{\rho\tau}_E\,, \tag{A.26}$$

where the totally antisymmetric Levi-Civita tensor is defined in Euclidean space such that $\epsilon^{1234} = \epsilon_{1234} = +1$. One can express $\sigma^{\mu\nu}_E$ and $\overline{\sigma}^{\mu\nu}_E$ in terms of the 't Hooft eta symbols [251],

$$\sigma^{\mu\nu}_E = -\tfrac{1}{2}\overline{\eta}^{k\mu\nu}\sigma^k\,, \qquad \overline{\sigma}^{\mu\nu}_E = -\tfrac{1}{2}\eta^{k\mu\nu}\sigma^k\,, \tag{A.27}$$

[95] Expressing the corresponding Green functions as Fourier transforms of momentum-space Green functions, one must simultaneously Wick-rotate $p^4_E \equiv ip^0$ in a clockwise sense to avoid singularities in the complex p^0-plane.

[96] It is seemingly more natural to define $\sigma^\mu_E \equiv (\vec{\boldsymbol{\sigma}}\,,\, \sigma^4_E)$ and $\overline{\sigma}^\mu_E \equiv (-\vec{\boldsymbol{\sigma}}\,,\, \overline{\sigma}^4_E)$ where $\sigma^4_E = \overline{\sigma}^4_E \equiv i\mathbb{1}_{2\times2}$, in which case one must replace $\delta^{\mu\nu}$ with $-\delta^{\mu\nu}$ in eq. (A.24). Nevertheless we prefer eq. (A.23), which avoids an overall minus sign in the respective anticommutation relations of the Euclidean sigma and gamma matrices [cf. footnote 140].

where $\mu,\nu = 1,2,3,4$ and there is an implicit sum over $k = 1,2,3$. Equivalently,

$$\sigma^\mu_E \overline{\sigma}^\nu_E = \delta^{\mu\nu} + i\overline{\eta}^{k\mu\nu}\sigma^k\,, \qquad\qquad \overline{\sigma}^\mu_E \sigma^\nu_E = \delta^{\mu\nu} + i\eta^{k\mu\nu}\sigma^k\,. \tag{A.28}$$

The 't Hooft symbols η and $\overline{\eta}$ satisfy self-duality and anti-self-duality properties, respectively:

$$\eta^{k\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\rho\lambda}\eta^{k\rho\lambda}\,, \qquad\qquad \overline{\eta}^{k\mu\nu} = -\tfrac{1}{2}\epsilon^{\mu\nu\rho\lambda}\overline{\eta}^{k\rho\lambda}\,, \tag{A.29}$$

and are explicitly given by:

$$\eta^{kij} = \overline{\eta}^{kij} = \epsilon^{ijk}\,, \qquad \eta^{kj4} = -\eta^{k4j} = \overline{\eta}^{k4j} = -\overline{\eta}^{kj4} = \delta^{kj}\,, \qquad \eta^{k44} = \overline{\eta}^{k44} = 0\,. \tag{A.30}$$

For a more comprehensive treatment of two-component spinors in Euclidean space, see [130].

29.7.2 Sigma matrix identities and Fierz identities

In Section 29.2, we derived a number of identities involving σ^μ, $\overline{\sigma}^\mu$, $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$. When considering a theory regularized by dimensional continuation [125], one must give meaning to the sigma matrices and their respective identities in $d \neq 4$ dimensions. In many cases, it is possible to reinterpret the sigma matrix identities for $d \neq 4$. However, the Fierz identities, which depend on the completeness of $\{\mathbb{1}_{2\times 2}\,,\,\sigma^i\}$ in the vector space of 2×2 matrices, do not have a consistent, unambiguous meaning outside of four dimensions (e.g., see refs. [252, 253, 254, 255] and references therein). In Appendix B.1 , we exhibit a comprehensive list of identities from which many generalized Fierz identities can be derived. In Appendix B.2 , we examine the class of sigma matrix identities that can unambiguously be extended to $d \neq 4$ dimension and thus can be employed in the context of dimensional regularization.

Two-component spinor Fierz identities

We begin with the basic identity for 2×2 matrices [78],

$$\delta_{ab}\delta_{cd} = \tfrac{1}{2}\left[\delta_{ad}\delta_{cb} + \sigma^i_{ad}\sigma^i_{cb}\right]\,, \tag{B.1.1}$$

where there is an implicit sum over the repeated superscript $i = 1,2,3$. Eq. (B.1.1) is a consequence of the completeness of $\{\mathbb{1}_{2\times 2}\,,\,\sigma^i\}$ in the four-dimensional vector space of 2×2 matrices. Applying these considerations to matrices that possess two indices, either undotted and/or dotted, one can establish four isomorphic four-dimensional vector spaces, each of which is spanned by four linearly independent hermitian matrices,

$$\mathcal{V} = \{\delta_\alpha{}^\beta\,,\, i(\sigma^{0i})_\alpha{}^\beta\}\,, \qquad \overline{\mathcal{V}} = \{\delta^{\dot\alpha}{}_{\dot\beta}\,,\, i(\overline{\sigma}^{0i})^{\dot\alpha}{}_{\dot\beta}\}\,, \qquad \mathcal{V}' = \{\sigma^\mu_{\alpha\dot\beta}\}\,, \qquad \overline{\mathcal{V}}' = \{\overline{\sigma}^{\mu\,\dot\alpha\beta}\}\,. \tag{B.1.2}$$

Note that the σ^{jk} and $\overline{\sigma}^{jk}$ are completely determined by the six matrices σ^{0i} and $\overline{\sigma}^{0i}$ (where $i,j,k = 1,2,3$) due to the self-duality relations given by eq. (29.76).

It is therefore convenient to consider the set of matrices,

$$\Gamma \equiv \left\{\delta_\alpha{}^\beta\,,\,\sigma^\mu_{\alpha\dot\beta}\,,\,\sigma^{\mu\nu}{}_\alpha{}^\beta\,,\,\delta^{\dot\alpha}{}_{\dot\beta}\,,\,\sigma^{\mu\,\dot\alpha\beta}\,,\,\overline{\sigma}^{\mu\nu\,\dot\alpha}{}_{\dot\beta}\right\}\,. \tag{B.1.3}$$

Elements of Γ will be denoted by $\Gamma^{(n)}$ $(n = 1,2,...,6)$. Starting from eq. (B.1.1), one can establish a set of 21 identities of the following form:

$$(\Gamma^{(k)})^I_{AB}(\Gamma^{(n)})^J_{CD} = \sum_{p,q,K,L} (C^{kn}_{pq})^{IJ}_{KL}\,(\Gamma^{(p)})^K_{AD}(\Gamma^{(q)})^L_{CB}\,, \tag{B.1.4}$$

where each label I, J, K and L can represent zero, one or two Lorentz spacetime indices, and A, B, C and D represent two-component spinor indices, each of which may be undotted or dotted and in the lowered or raised position as appropriate. The sum in eq. (B.1.4) is taken over the matrices specified in eq. (B.1.3), and the C_{pq}^{kn} are numerical coefficients [cf. eqs. (B.1.6)–(B.1.26)].

Let us multiply eq. (B.1.4) by four (commuting or anticommuting) two-component spinors $Z_{1A}Z_{2B}Z_{3C}Z_{4D}$, where Z_i stands for either the undaggered or daggered spinor z_i or $z_i^\dagger$, depending on whether the corresponding spinor index is undotted or dotted. This procedure yields generalized Fierz identities of the form [78, 89, 75, 86]:

$$(Z_1\Gamma^{(k)I}Z_2)(Z_3\Gamma^{(n)J}Z_4) = (-1)^A \sum_{p,q,K,L} (C_{pq}^{kn})_{KL}^{IJ}(Z_1\Gamma^{(p)K}Z_4)(Z_3\Gamma^{(q)L}Z_2)\,, \tag{B.1.5}$$

where $(-1)^A = +1$ [-1] for commuting [anticommuting] spinors.[97]

[97] It is often convenient to reverse the order of the spinors Z_2 and Z_3 on the right-hand side of eq. (B.1.5) by using eqs. (29.60)–(29.62) and (29.95)–(29.96) to eliminate the factor of $(-1)^A$ [cf. eq. (29.68)].

The explicit expressions for the 21 identities represented by eq. (B.1.4) are as follows.

$$\delta_\alpha{}^\beta \delta^{\dot\beta}{}_{\dot\alpha} = \tfrac{1}{2}\sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}_\mu^{\dot\beta\beta}\,, \tag{B.1.6}$$

$$\delta_\alpha{}^\beta \delta_\gamma{}^\tau = \tfrac{1}{2}\left[\delta_\alpha{}^\tau \delta_\gamma{}^\beta + (\sigma^{\mu\nu})_\alpha{}^\tau (\sigma_{\mu\nu})_\gamma{}^\beta\right]\,, \tag{B.1.7}$$

$$\delta^{\dot\alpha}{}_{\dot\beta}\delta^{\dot\gamma}{}_{\dot\tau} = \tfrac{1}{2}\left[\delta^{\dot\alpha}{}_{\dot\tau}\delta^{\dot\gamma}{}_{\dot\beta} + (\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\tau}(\overline{\sigma}_{\mu\nu})^{\dot\gamma}{}_{\dot\beta}\right]\,, \tag{B.1.8}$$

$$\delta_\alpha{}^\beta \sigma^\mu_{\gamma\dot\alpha} = \tfrac{1}{2}\sigma^\mu_{\alpha\dot\alpha}\delta_\gamma{}^\beta - i\sigma_{\nu\,\alpha\dot\alpha}(\sigma^{\mu\nu})_\gamma{}^\beta\,, \tag{B.1.9}$$

$$\delta_\alpha{}^\beta \overline{\sigma}^{\mu\,\dot\beta\gamma} = \tfrac{1}{2}\delta_\alpha{}^\gamma \overline{\sigma}^{\mu\,\dot\beta\beta} + i(\sigma^{\mu\nu})_\alpha{}^\gamma \overline{\sigma}_\nu^{\dot\beta\beta}\,, \tag{B.1.10}$$

$$\delta^{\dot\alpha}{}_{\dot\beta}\sigma^\mu_{\beta\dot\gamma} = \tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\gamma}\sigma^\mu_{\beta\dot\beta} + i(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\gamma}\sigma_{\nu\,\beta\dot\beta}\,, \tag{B.1.11}$$

$$\delta^{\dot\alpha}{}_{\dot\beta}\overline{\sigma}^{\mu\,\dot\gamma\alpha} = \tfrac{1}{2}\overline{\sigma}^{\mu\,\dot\alpha\alpha}\delta^{\dot\gamma}{}_{\dot\beta} - i\overline{\sigma}_\nu^{\dot\alpha\alpha}(\overline{\sigma}^{\mu\nu})^{\dot\gamma}{}_{\dot\beta}\,, \tag{B.1.12}$$

$$\delta_\alpha{}^\beta (\sigma^{\mu\nu})_\gamma{}^\tau = \tfrac{1}{2}\Big\{(\sigma^{\mu\nu})_\alpha{}^\tau \delta_\gamma{}^\beta + \delta_\alpha{}^\tau (\sigma^{\mu\nu})_\gamma{}^\beta - ig_{\rho\kappa}\left[(\sigma^{\mu\kappa})_\alpha{}^\tau (\sigma^{\nu\rho})_\gamma{}^\beta - (\sigma^{\nu\kappa})_\alpha{}^\tau (\sigma^{\mu\rho})_\gamma{}^\beta\right]\Big\}, \tag{B.1.13}$$

$$\delta_\alpha{}^\beta (\overline{\sigma}^{\mu\nu})^{\dot\beta}{}_{\dot\alpha} = -\tfrac{1}{4}i\left[\sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}^{\nu\,\dot\beta\beta} - \sigma^\nu_{\alpha\dot\alpha}\overline{\sigma}^{\mu\,\dot\beta\beta} + i\epsilon^{\mu\nu\rho\kappa}\sigma_{\rho\,\alpha\dot\alpha}\overline{\sigma}_\kappa^{\dot\beta\beta}\right]\,, \tag{B.1.14}$$

$$\delta^{\dot\alpha}{}_{\dot\beta}(\sigma^{\mu\nu})_\beta{}^\alpha = -\tfrac{1}{4}i\left[\overline{\sigma}^{\mu\,\dot\alpha\alpha}\sigma^\nu_{\beta\dot\beta} - \overline{\sigma}^{\nu\,\dot\alpha\alpha}\sigma^\mu_{\beta\dot\beta} - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\rho^{\dot\alpha\alpha}\sigma_{\kappa\,\beta\dot\beta}\right]\,, \tag{B.1.15}$$

$$\delta^{\dot\alpha}{}_{\dot\beta}(\overline{\sigma}^{\mu\nu})^{\dot\gamma}{}_{\dot\tau} = \tfrac{1}{2}\Big\{(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\tau}\delta^{\dot\gamma}{}_{\dot\beta} + \delta^{\dot\alpha}{}_{\dot\tau}(\overline{\sigma}^{\mu\nu})^{\dot\gamma}{}_{\dot\beta} - ig_{\rho\kappa}\left[(\overline{\sigma}^{\mu\kappa})^{\dot\alpha}{}_{\dot\tau}(\overline{\sigma}^{\nu\rho})^{\dot\gamma}{}_{\dot\beta} - (\overline{\sigma}^{\nu\kappa})^{\dot\alpha}{}_{\dot\tau}(\overline{\sigma}^{\mu\rho})^{\dot\gamma}{}_{\dot\beta}\right]\Big\}, \tag{B.1.16}$$

$$\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} = \tfrac{1}{2}\left[\sigma^\mu_{\alpha\dot\beta}\sigma^\nu_{\beta\dot\alpha} + \sigma^\nu_{\alpha\dot\beta}\sigma^\mu_{\beta\dot\alpha} - g^{\mu\nu}\sigma^\lambda_{\alpha\dot\beta}\sigma_{\lambda\,\beta\dot\alpha} + i\epsilon^{\mu\nu\rho\kappa}\sigma_{\rho\,\alpha\dot\beta}\,\sigma_{\kappa\,\beta\dot\alpha}\right]\,, \tag{B.1.17}$$

$$\overline{\sigma}^{\mu\dot\alpha\alpha}\overline{\sigma}^{\nu\dot\beta\beta} = \tfrac{1}{2}\left[\overline{\sigma}^{\mu\dot\alpha\beta}\overline{\sigma}^{\nu\dot\beta\alpha} + \overline{\sigma}^{\nu\dot\alpha\beta}\overline{\sigma}^{\mu\dot\beta\alpha} - g^{\mu\nu}\overline{\sigma}^{\lambda\,\dot\alpha\beta}\overline{\sigma}_\lambda^{\dot\beta\alpha} - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\rho^{\dot\alpha\beta}\,\overline{\sigma}_\kappa^{\dot\beta\alpha}\right]\,, \tag{B.1.18}$$

$$\sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}^{\nu\dot\beta\beta} = \tfrac{1}{2}\left[g^{\mu\nu}\delta_\alpha{}^\beta\delta^{\dot\beta}{}_{\dot\alpha} - 2i(\sigma^{\mu\nu})_\alpha{}^\beta\delta^{\dot\beta}{}_{\dot\alpha} + 2i\delta_\alpha{}^\beta(\overline{\sigma}^{\mu\nu})^{\dot\beta}{}_{\dot\alpha} - 4g_{\rho\kappa}(\sigma^{\mu\kappa})_\alpha{}^\beta(\overline{\sigma}^{\nu\rho})^{\dot\beta}{}_{\dot\alpha}\right]\,, \tag{B.1.19}$$

$$\begin{aligned}(\sigma^{\mu\nu})_\alpha{}^\beta\sigma^\rho_{\gamma\dot\alpha} = \tfrac{1}{2}\big[&\sigma^\nu_{\alpha\dot\alpha}(\sigma^{\mu\rho})_\gamma{}^\beta - \sigma^\mu_{\alpha\dot\alpha}(\sigma^{\nu\rho})_\gamma{}^\beta + i\epsilon^{\mu\nu\kappa}{}_\lambda\sigma_{\kappa\,\alpha\dot\alpha}(\sigma^{\lambda\rho})_\gamma{}^\beta\\ &-\tfrac{1}{2}i\,\left(g^{\mu\rho}\sigma^\nu_{\alpha\dot\alpha} - g^{\nu\rho}\sigma^\mu_{\alpha\dot\alpha} - i\epsilon^{\mu\nu\rho\kappa}\sigma_{\kappa\,\alpha\dot\alpha}\right)\delta_\gamma{}^\beta\big]\,,\end{aligned} \tag{B.1.20}$$

$$\begin{aligned}\overline{\sigma}^{\rho\,\dot\alpha\beta}(\sigma^{\mu\nu})_\gamma{}^\alpha = \tfrac{1}{2}\big[&\overline{\sigma}^{\nu\,\dot\alpha\alpha}(\sigma^{\mu\rho})_\gamma{}^\beta - \overline{\sigma}^{\mu\,\dot\alpha\alpha}(\sigma^{\nu\rho})_\gamma{}^\beta + i\epsilon^{\mu\nu\kappa}{}_\lambda\overline{\sigma}_\kappa^{\dot\alpha\alpha}(\sigma^{\lambda\rho})_\gamma{}^\beta\\ &+\tfrac{1}{2}i\,\left(g^{\mu\rho}\overline{\sigma}^{\nu\,\dot\alpha\alpha} - g^{\nu\rho}\overline{\sigma}^{\mu\,\dot\alpha\alpha} - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa^{\dot\alpha\alpha}\right)\delta_\gamma{}^\beta\big]\,,\end{aligned} \tag{B.1.21}$$

$$\begin{aligned}\sigma^\rho_{\alpha\dot\beta}(\overline{\sigma}^{\mu\nu})^{\dot\gamma}{}_{\dot\alpha} = \tfrac{1}{2}\big[&\sigma^\nu_{\alpha\dot\alpha}(\overline{\sigma}^{\mu\rho})^{\dot\gamma}{}_{\dot\beta} - \sigma^\mu_{\alpha\dot\alpha}(\overline{\sigma}^{\nu\rho})^{\dot\gamma}{}_{\dot\beta} - i\epsilon^{\mu\nu\kappa}{}_\lambda\sigma_{\kappa\,\alpha\dot\alpha}(\overline{\sigma}^{\lambda\rho})^{\dot\gamma}{}_{\dot\beta}\\ &+\tfrac{1}{2}i\,\left(g^{\mu\rho}\sigma^\nu_{\alpha\dot\alpha} - g^{\nu\rho}\sigma^\mu_{\alpha\dot\alpha} + i\epsilon^{\mu\nu\rho\kappa}\sigma_{\kappa\,\alpha\dot\alpha}\right)\delta^{\dot\gamma}{}_{\dot\beta}\big]\,,\end{aligned} \tag{B.1.22}$$

$$\begin{aligned}(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\overline{\sigma}^{\rho\,\dot\gamma\alpha} = \tfrac{1}{2}\big[&\overline{\sigma}^{\nu\,\dot\alpha\alpha}(\overline{\sigma}^{\mu\rho})^{\dot\gamma}{}_{\dot\beta} - \overline{\sigma}^{\mu\,\dot\alpha\alpha}(\overline{\sigma}^{\nu\rho})^{\dot\gamma}{}_{\dot\beta} - i\epsilon^{\mu\nu\kappa}{}_\lambda\overline{\sigma}_\kappa^{\dot\alpha\alpha}(\overline{\sigma}^{\lambda\rho})^{\dot\gamma}{}_{\dot\beta}\\ &-\tfrac{1}{2}i\,\left(g^{\mu\rho}\overline{\sigma}^{\nu\,\dot\alpha\alpha} - g^{\nu\rho}\overline{\sigma}^{\mu\,\dot\alpha\alpha} + i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa^{\dot\alpha\alpha}\right)\delta^{\dot\gamma}{}_{\dot\beta}\big]\,,\end{aligned} \tag{B.1.23}$$

$$
\begin{aligned}
(\sigma^{\mu\nu})_\alpha{}^\beta(\sigma^{\rho\kappa})_\gamma{}^\tau &= \tfrac{1}{2}(\sigma^{\mu\nu})_\alpha{}^\tau(\sigma^{\rho\kappa})_\gamma{}^\beta + \tfrac{1}{8}\delta_\alpha{}^\tau\delta_\gamma{}^\beta\,(g^{\mu\rho}g^{\nu\kappa} - g^{\mu\kappa}g^{\nu\rho} - i\epsilon^{\mu\nu\rho\kappa}) \\
&\quad +\tfrac{1}{4}i\delta_\alpha{}^\tau\,(g^{\mu\rho}\sigma^{\nu\kappa} + g^{\nu\kappa}\sigma^{\mu\rho} - g^{\nu\rho}\sigma^{\mu\kappa} - g^{\mu\kappa}\sigma^{\nu\rho})_\gamma{}^\beta \\
&\quad -\tfrac{1}{4}i\delta_\gamma{}^\beta\,(g^{\mu\rho}\sigma^{\nu\kappa} + g^{\nu\kappa}\sigma^{\mu\rho} - g^{\nu\rho}\sigma^{\mu\kappa} - g^{\mu\kappa}\sigma^{\nu\rho})_\alpha{}^\tau \\
&\quad +\tfrac{1}{4}\left[(\sigma^{\mu\rho})_\alpha{}^\tau(\sigma^{\nu\kappa})_\gamma{}^\beta + (\sigma^{\nu\kappa})_\alpha{}^\tau(\sigma^{\mu\rho})_\gamma{}^\beta - (\sigma^{\nu\rho})_\alpha{}^\tau(\sigma^{\mu\kappa})_\gamma{}^\beta - (\sigma^{\mu\kappa})_\alpha{}^\tau(\sigma^{\nu\rho})_\gamma{}^\beta\right] \\
&\quad +\tfrac{1}{4}g_{\lambda\sigma}\left[g^{\mu\kappa}(\sigma^{\rho\sigma})_\alpha{}^\tau(\sigma^{\nu\lambda})_\gamma{}^\beta + g^{\nu\rho}(\sigma^{\kappa\sigma})_\alpha{}^\tau(\sigma^{\mu\lambda})_\gamma{}^\beta\right. \\
&\qquad \left. -g^{\nu\kappa}(\sigma^{\rho\sigma})_\alpha{}^\tau(\sigma^{\mu\lambda})_\gamma{}^\beta - g^{\mu\rho}(\sigma^{\kappa\sigma})_\alpha{}^\tau(\sigma^{\nu\lambda})_\gamma{}^\beta\right]\,, \qquad \text{(B.1.24)}
\end{aligned}
$$

$$
\begin{aligned}
(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}(\overline{\sigma}^{\rho\kappa})^{\dot\gamma}{}_{\dot\tau} &= \tfrac{1}{2}(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\tau}(\overline{\sigma}^{\rho\kappa})^{\dot\gamma}{}_{\dot\beta} + \tfrac{1}{8}\delta^{\dot\alpha}{}_{\dot\tau}\delta^{\dot\gamma}{}_{\dot\beta}\,(g^{\mu\rho}g^{\nu\kappa} - g^{\mu\kappa}g^{\nu\rho} + i\epsilon^{\mu\nu\rho\kappa}) \\
&\quad +\tfrac{1}{4}i\delta^{\dot\alpha}{}_{\dot\tau}\,(g^{\mu\rho}\overline{\sigma}^{\nu\kappa} + g^{\nu\kappa}\overline{\sigma}^{\mu\rho} - g^{\nu\rho}\overline{\sigma}^{\mu\kappa} - g^{\mu\kappa}\overline{\sigma}^{\nu\rho})^{\dot\gamma}{}_{\dot\beta} \\
&\quad -\tfrac{1}{4}i\delta^{\dot\gamma}{}_{\dot\beta}\,(g^{\mu\rho}\overline{\sigma}^{\nu\kappa} + g^{\nu\kappa}\overline{\sigma}^{\mu\rho} - g^{\nu\rho}\overline{\sigma}^{\mu\kappa} - g^{\mu\kappa}\overline{\sigma}^{\nu\rho})^{\dot\alpha}{}_{\dot\tau} \\
&\quad +\tfrac{1}{4}\left[(\overline{\sigma}^{\mu\rho})^{\dot\alpha}{}_{\dot\tau}(\overline{\sigma}^{\nu\kappa})^{\dot\gamma}{}_{\dot\beta} + (\overline{\sigma}^{\nu\kappa})^{\dot\alpha}{}_{\dot\tau}(\overline{\sigma}^{\mu\rho})^{\dot\gamma}{}_{\dot\beta} - (\overline{\sigma}^{\nu\rho})^{\dot\alpha}{}_{\dot\tau}(\overline{\sigma}^{\mu\kappa})^{\dot\gamma}{}_{\dot\beta} - (\overline{\sigma}^{\mu\kappa})^{\dot\alpha}{}_{\dot\tau}(\overline{\sigma}^{\nu\rho})^{\dot\gamma}{}_{\dot\beta}\right] \\
&\quad +\tfrac{1}{4}g_{\lambda\sigma}\left[g^{\mu\kappa}(\overline{\sigma}^{\rho\sigma})^{\dot\alpha}{}_{\dot\tau}(\overline{\sigma}^{\nu\lambda})^{\dot\gamma}{}_{\dot\beta} + g^{\nu\rho}(\overline{\sigma}^{\kappa\sigma})^{\dot\alpha}{}_{\dot\tau}(\overline{\sigma}^{\mu\lambda})^{\dot\gamma}{}_{\dot\beta}\right. \\
&\qquad \left. -g^{\nu\kappa}(\overline{\sigma}^{\rho\sigma})^{\dot\alpha}{}_{\dot\tau}(\overline{\sigma}^{\mu\lambda})^{\dot\gamma}{}_{\dot\beta} - g^{\mu\rho}(\overline{\sigma}^{\kappa\sigma})^{\dot\alpha}{}_{\dot\tau}(\overline{\sigma}^{\nu\lambda})^{\dot\gamma}{}_{\dot\beta}\right]\,, \qquad \text{(B.1.25)}
\end{aligned}
$$

$$
\begin{aligned}
(\sigma^{\mu\nu})_\alpha{}^\beta(\overline{\sigma}^{\rho\kappa})^{\dot\beta}{}_{\dot\alpha} &= \tfrac{1}{8}\left[(g^{\mu\rho}g^{\nu\kappa} - g^{\mu\kappa}g^{\nu\rho})\sigma^\lambda_{\alpha\dot\alpha}\overline{\sigma}_\lambda^{\dot\beta\beta}\right. \\
&\quad +i\epsilon^{\mu\nu\rho\lambda}\sigma_{\lambda\,\alpha\dot\alpha}\overline{\sigma}^{\kappa\,\dot\beta\beta} - i\epsilon^{\mu\nu\kappa\lambda}\sigma_{\lambda\,\alpha\dot\alpha}\overline{\sigma}^{\rho\,\dot\beta\beta} - i\epsilon^{\mu\rho\kappa\lambda}\sigma^\nu_{\alpha\dot\alpha}\overline{\sigma}_\lambda^{\dot\beta\beta} + i\epsilon^{\nu\rho\kappa\lambda}\sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}_\lambda^{\dot\beta\beta} \\
&\quad -g^{\mu\rho}(\sigma^\kappa_{\alpha\dot\alpha}\overline{\sigma}^{\nu\,\dot\beta\beta} + \sigma^\nu_{\alpha\dot\alpha}\overline{\sigma}^{\kappa\,\dot\beta\beta}) + g^{\nu\rho}(\sigma^\kappa_{\alpha\dot\alpha}\overline{\sigma}^{\mu\,\dot\beta\beta} + \sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}^{\kappa\,\dot\beta\beta}) \\
&\quad \left. +g^{\mu\kappa}(\sigma^\rho_{\alpha\dot\alpha}\overline{\sigma}^{\nu\,\dot\beta\beta} + \sigma^\nu_{\alpha\dot\alpha}\overline{\sigma}^{\rho\,\dot\beta\beta}) - g^{\nu\kappa}(\sigma^\rho_{\alpha\dot\alpha}\overline{\sigma}^{\mu\,\dot\beta\beta} + \sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}^{\rho\,\dot\beta\beta})\right]\,. \qquad \text{(B.1.26)}
\end{aligned}
$$

From eqs. (B.1.6)–(B.1.26), one immediately obtains the corresponding 21 Fierz identities represented by eq. (B.1.5). Eleven of these identities also appear in Appendix A of [78].[98]

The derivation of the 21 identities listed above is straightforward. Eqs. (B.1.6)–(B.1.8) are equivalent to the completeness relation of eq. (B.1.1). The next eight identities [eqs. (B.1.9)–(B.1.16)] are easily derived starting from eqs. (B.1.6)–(B.1.8). As a simple example, using the results of eqs. (B.1.7) and (B.2.20), it follows that

$$
\begin{aligned}
\delta_\alpha{}^\beta\sigma^\mu_{\gamma\dot\alpha} = \delta_\alpha{}^\beta\delta_\gamma{}^\tau\sigma^\mu_{\tau\dot\alpha} &= \tfrac{1}{2}\left[\delta_\alpha{}^\tau\delta_\gamma{}^\beta + (\sigma^{\rho\kappa})_\alpha{}^\tau(\sigma_{\rho\kappa})_\gamma{}^\beta\right]\sigma^\mu_{\tau\dot\alpha} \\
&= \tfrac{1}{2}\left[\sigma^\mu_{\alpha\dot\alpha}\delta_\gamma{}^\beta + (\sigma^{\rho\kappa}\sigma^\mu)_{\alpha\dot\alpha}(\sigma_{\rho\kappa})_\gamma{}^\beta\right] \\
&= \tfrac{1}{2}\left[\sigma^\mu_{\alpha\dot\alpha}\delta_\gamma{}^\beta + \tfrac{1}{2}i(g^{\kappa\mu}\sigma^\rho - g^{\rho\mu}\sigma^\kappa + i\epsilon^{\rho\kappa\mu\nu}\sigma_\nu)_{\alpha\dot\alpha}(\sigma_{\rho\kappa})_\gamma{}^\beta\right] \\
&= \tfrac{1}{2}\delta_\alpha{}^\gamma\overline{\sigma}^{\mu\,\dot\beta\beta} + i(\sigma^{\mu\nu})_\alpha{}^\gamma\overline{\sigma}_\nu^{\dot\beta\beta}\,, \qquad \text{(B.1.27)}
\end{aligned}
$$

where eq. (29.76) was employed in the final step. We can now use eqs. (B.1.9)–(B.1.12) to derive eqs. (B.1.17)–(B.1.23) by a similar technique. Finally, starting from eqs. (B.1.13)–(B.1.16) we may employ the same technique once more to derive eqs. (B.1.24)–(B.1.26).[99] A useful check of the last three identities can be carried out by multiplying these results by $g_{\mu\rho}g_{\nu\kappa}$ and summing over the

[98] Note that in [78], $\epsilon^{\mu\nu\rho\kappa}$ has the opposite sign with respect to our conventions, and $\sigma^{\mu\nu}$ is defined without an overall factor of i. Taking these differences into account, we have confirmed that the results of Appendix A of [78] match the corresponding results obtained here.

[99] In particular, the identities given in eqs. (B.2.18) and (B.2.19) are especially useful in the evaluation of eqs. (B.1.20)–(B.1.25).

two repeated Lorentz index pairs. We then find:

$$(\sigma^{\mu\nu})_\alpha{}^\beta(\sigma_{\mu\nu})_\gamma{}^\tau = -\tfrac{1}{2}(\sigma^{\mu\nu})_\alpha{}^\tau(\sigma_{\mu\nu})_\gamma{}^\beta + \tfrac{3}{2}\delta_\alpha{}^\tau\delta_\gamma{}^\beta\,, \tag{B.1.28}$$

$$(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}(\overline{\sigma}_{\mu\nu})^{\dot\gamma}{}_{\dot\tau} = -\tfrac{1}{2}(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\tau}(\overline{\sigma}_{\mu\nu})^{\dot\gamma}{}_{\dot\beta} + \tfrac{3}{2}\delta^{\dot\alpha}{}_{\dot\tau}\delta^{\dot\gamma}{}_{\dot\beta}\,, \tag{B.1.29}$$

$$(\sigma^{\mu\nu})_\alpha{}^\beta(\overline{\sigma}_{\mu\nu})^{\dot\gamma}{}_{\dot\tau} = 0\,. \tag{B.1.30}$$

Eq. (B.1.30) has already been recorded in eq. (29.84). To verify eqs. (B.1.28) and (B.1.29), we first rewrite these equations with the interchange of $\beta \leftrightarrow \tau$ and $\dot\beta \leftrightarrow \dot\tau$. Inserting the resulting equations back into eqs. (B.1.28) and (B.1.29) then yields the previously obtained eqs. (29.82) and (29.83) [or equivalently, eqs. (B.1.7) and (B.1.8)].

A similar check can be performed on eqs. (B.1.17)–(B.1.19) by multiplying these results by $g_{\mu\nu}$ and summing over the repeated Lorentz index pair [with assistance from eq. (B.1.30)]:

$$\sigma^\mu_{\alpha\dot\alpha}\sigma_{\mu\,\beta\dot\beta} = -\sigma^\mu_{\alpha\dot\beta}\sigma_{\mu\,\beta\dot\alpha}\,, \tag{B.1.31}$$

$$\overline{\sigma}^{\mu\,\dot\alpha\alpha}\overline{\sigma}^{\dot\beta\beta}_\mu = -\overline{\sigma}^{\mu\,\dot\alpha\beta}\overline{\sigma}^{\dot\beta\alpha}_\mu\,, \tag{B.1.32}$$

$$\sigma^\mu_{\alpha\dot\alpha}\overline{\sigma}^{\dot\beta\beta}_\mu = 2\,\delta_\alpha{}^\beta\delta^{\dot\beta}{}_{\dot\alpha}\,. \tag{B.1.33}$$

It follows that:

$$\sigma^\mu_{\alpha\dot\alpha}\sigma_{\mu\,\beta\dot\beta} = 2\,\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}\,, \tag{B.1.34}$$

$$\overline{\sigma}^{\mu\,\dot\alpha\alpha}\overline{\sigma}^{\dot\beta\beta}_\mu = 2\,\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\,, \tag{B.1.35}$$

since eqs. (B.1.31) and (B.1.32) are antisymmetric under the separate interchanges of $\alpha \leftrightarrow \beta$ and $\dot\alpha \leftrightarrow \dot\beta$. The coefficients in eqs. (B.1.34) and (B.1.35) are determined by substituting $\alpha = \dot\alpha = 1$ and $\beta = \dot\beta = 2$. Thus, we have confirmed the results previously obtained in eqs. (29.49)–(29.51).

Eqs. (B.1.6)–(B.1.8) can also be used to derive four additional identities, which yield Fierz identities of a different form. Simply multiply each of these equations by two ϵ symbols (with appropriately chosen undotted and/or dotted spinor indices), and use eqs. (29.32) and (29.79). Two of the resulting identities coincide with eqs. (B.1.34) and (B.1.35), while the other two are:

$$\epsilon_{\alpha\beta}\epsilon^{\gamma\tau} = -\tfrac{1}{2}\left[\delta_\alpha{}^\gamma\delta_\beta{}^\tau - (\sigma^{\mu\nu})_\alpha{}^\gamma(\sigma_{\mu\nu})_\beta{}^\tau\right], \tag{B.1.36}$$

$$\epsilon^{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\tau} = -\tfrac{1}{2}\left[\delta^{\dot\alpha}{}_{\dot\gamma}\delta^{\dot\beta}{}_{\dot\tau} - (\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\gamma}(\overline{\sigma}_{\mu\nu})^{\dot\beta}{}_{\dot\tau}\right]. \tag{B.1.37}$$

One can check that eqs. (B.1.36) and (B.1.37) are equivalent to the previously obtained eqs. (29.82) and (29.83). Multiplying eqs. (B.1.34)–(B.1.37) by four (commuting or anticommuting) two-component spinors $Z_{1A}Z_{2B}Z_{3C}Z_{4D}$ yields the corresponding Fierz identities of the form:

$$(Z_1\Gamma^{(k)I}Z_2)(Z_3\Gamma^{(n)J}Z_4) = (-1)^A\sum_{p,q,K,L}(C^{kn}_{pq})^{IJ}_{KL}(Z_1\Gamma^{(p)K}Z_3)(Z_2\Gamma^{(q)L}Z_4)\,, \tag{B.1.38}$$

which differs from eq. (B.1.5) in the ordering of the spinors on the right-hand side.

Finally, we note that the Schouten identities,

$$\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + \epsilon_{\alpha\gamma}\epsilon_{\delta\beta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma} = 0\,, \qquad \epsilon_{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\delta} + \epsilon_{\dot\alpha\dot\gamma}\epsilon_{\dot\delta\dot\beta} + \epsilon_{\dot\alpha\dot\delta}\epsilon_{\dot\beta\dot\gamma} = 0\,, \tag{B.1.39}$$

are the basis for Fierz identities given by eqs. (29.66) and (29.67), which do not assume the simple forms of either eqs. (B.1.5) or (B.1.38).

Sigma matrix identities in $d \neq 4$ dimensions

When considering a theory regularized by dimensional continuation [125], one must be careful in treating cases with contracted spacetime vector indices $\mu, \nu, \kappa, \rho, \ldots$. Instead of taking on four possible values, these vector indices formally run over d values, where d is infinitesimally different from 4. This means that some identities that would hold in unregularized four-dimensional theories are inconsistent and must not be used; other identities remain valid if d replaces 4 in the appropriate spots; and still other identities hold without modification.

Two important identities that do hold in $d \neq 4$ dimensions are:

$$[\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu]_\alpha{}^\beta = 2g^{\mu\nu}\delta_\alpha{}^\beta\,, \tag{B.2.1}$$
$$[\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu]^{\dot\alpha}{}_{\dot\beta} = 2g^{\mu\nu}\delta^{\dot\alpha}{}_{\dot\beta}\,. \tag{B.2.2}$$

Equivalently,

$$(\sigma^\mu\overline{\sigma}^\nu)_\alpha{}^\beta = g^{\mu\nu}\delta_\alpha{}^\beta - 2i(\sigma^{\mu\nu})_\alpha{}^\beta\,, \tag{B.2.3}$$
$$(\overline{\sigma}^\mu\sigma^\nu)^{\dot\alpha}{}_{\dot\beta} = g^{\mu\nu}\delta^{\dot\alpha}{}_{\dot\beta} - 2i(\overline{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\,, \tag{B.2.4}$$

where $\sigma^{\mu\nu}$ and $\overline{\sigma}^{\mu\nu}$ are defined in eq. (A.10). The trace identities,

$$\mathrm{Tr}[\sigma^\mu\overline{\sigma}^\nu] = \mathrm{Tr}[\overline{\sigma}^\mu\sigma^\nu] = 2g^{\mu\nu}\,, \tag{B.2.5}$$
$$\mathrm{Tr}\,\sigma^{\mu\nu} = \mathrm{Tr}\,\overline{\sigma}^{\mu\nu} = 0\,, \tag{B.2.6}$$

then follow. We also note that the spinor index trace identity,

$$\mathrm{Tr}[\mathbb{1}] = \delta^\alpha_\alpha = \delta^{\dot\alpha}_{\dot\alpha} = 2\,, \tag{B.2.7}$$

continues to hold in dimensional continuation regularization methods. In contrast, the Fierz identities of Appendix B.1 do not have a consistent, unambiguous meaning outside of four dimensions [252, 253, 254, 255]. However, the following identities that are implied by eq. (B.1.6) do consistently generalize to $d \neq 4$ spacetime dimensions:

$$[\sigma^\mu\overline{\sigma}_\mu]_\alpha{}^\beta = d\delta^\beta_\alpha\,, \tag{B.2.8}$$
$$[\overline{\sigma}^\mu\sigma_\mu]^{\dot\alpha}{}_{\dot\beta} = d\delta^{\dot\alpha}_{\dot\beta}\,. \tag{B.2.9}$$

Using eqs. (B.2.8) and (B.2.9) along with the repeated use of eqs. (B.2.1) and (B.2.2) then yields:

$$[\sigma^\mu\overline{\sigma}^\nu\sigma_\mu]_{\alpha\dot\beta} = -(d-2)\sigma^\nu_{\alpha\dot\beta}\,, \tag{B.2.10}$$
$$[\overline{\sigma}^\mu\sigma_\nu\overline{\sigma}_\mu]^{\dot\alpha\beta} = -(d-2)\overline{\sigma}_\nu^{\dot\alpha\beta}\,, \tag{B.2.11}$$
$$[\sigma^\mu\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}_\mu]_\alpha{}^\beta = 4g^{\nu\rho}\delta^\beta_\alpha - (4-d)[\sigma^\nu\overline{\sigma}^\rho]_\alpha{}^\beta\,, \tag{B.2.12}$$
$$[\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho\sigma_\mu]^{\dot\alpha}{}_{\dot\beta} = 4g^{\nu\rho}\delta^{\dot\alpha}_{\dot\beta} - (4-d)[\overline{\sigma}^\nu\sigma^\rho]^{\dot\alpha}{}_{\dot\beta}\,, \tag{B.2.13}$$
$$[\sigma^\mu\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}^\kappa\sigma_\mu]_{\alpha\dot\beta} = -2[\sigma^\kappa\overline{\sigma}^\rho\sigma^\nu]_{\alpha\dot\beta} + (4-d)[\sigma^\nu\overline{\sigma}^\rho\sigma^\kappa]_{\alpha\dot\beta}\,, \tag{B.2.14}$$
$$[\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho\sigma^\kappa\overline{\sigma}_\mu]^{\dot\alpha\beta} = -2[\overline{\sigma}^\kappa\sigma^\rho\overline{\sigma}^\nu]^{\dot\alpha\beta} + (4-d)[\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}^\kappa]^{\dot\alpha\beta}\,. \tag{B.2.15}$$

Identities that involve the (explicitly and inextricably four-dimensional) $\epsilon^{\mu\nu\rho\kappa}$ symbol

$$\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho = g^{\mu\nu}\overline{\sigma}^\rho - g^{\mu\rho}\overline{\sigma}^\nu + g^{\nu\rho}\overline{\sigma}^\mu - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa\,, \tag{B.2.16}$$
$$\sigma^\mu\overline{\sigma}^\nu\sigma^\rho = g^{\mu\nu}\sigma^\rho - g^{\mu\rho}\sigma^\nu + g^{\nu\rho}\sigma^\mu + i\epsilon^{\mu\nu\rho\kappa}\sigma_\kappa\,, \tag{B.2.17}$$
$$\epsilon^{\mu\nu\kappa}{}_\lambda\sigma^{\lambda\rho} = -i\left(g^{\kappa\rho}\sigma^{\mu\nu} - g^{\nu\rho}\sigma^{\mu\kappa} + g^{\mu\rho}\sigma^{\nu\kappa}\right)\,, \tag{B.2.18}$$
$$\epsilon^{\mu\nu\kappa}{}_\lambda\overline{\sigma}^{\lambda\rho} = i\left(g^{\kappa\rho}\overline{\sigma}^{\mu\nu} - g^{\nu\rho}\overline{\sigma}^{\mu\kappa} + g^{\mu\rho}\overline{\sigma}^{\nu\kappa}\right)\,, \tag{B.2.19}$$
$$\sigma^{\mu\nu}\sigma^\rho = \tfrac{1}{2}i\left(g^{\nu\rho}\sigma^\mu - g^{\mu\rho}\sigma^\nu + i\epsilon^{\mu\nu\rho\kappa}\sigma_\kappa\right)\,, \tag{B.2.20}$$
$$\overline{\sigma}^{\mu\nu}\overline{\sigma}^\rho = \tfrac{1}{2}i\left(g^{\nu\rho}\overline{\sigma}^\mu - g^{\mu\rho}\overline{\sigma}^\nu - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa\right)\,, \tag{B.2.21}$$
$$\overline{\sigma}^\mu\sigma^{\nu\rho} = \tfrac{1}{2}i\left(g^{\mu\nu}\overline{\sigma}^\rho - g^{\mu\rho}\overline{\sigma}^\nu - i\epsilon^{\mu\nu\rho\kappa}\overline{\sigma}_\kappa\right)\,, \tag{B.2.22}$$
$$\sigma^\mu\overline{\sigma}^{\nu\rho} = \tfrac{1}{2}i\left(g^{\mu\nu}\sigma^\rho - g^{\mu\rho}\sigma^\nu + i\epsilon^{\mu\nu\rho\kappa}\sigma_\kappa\right)\,, \tag{B.2.23}$$
$$\sigma^{\mu\nu}\sigma^{\rho\kappa} = -\tfrac{1}{4}(g^{\nu\rho}g^{\mu\kappa} - g^{\mu\rho}g^{\nu\kappa} + i\epsilon^{\mu\nu\rho\kappa}) + \tfrac{1}{2}i(g^{\nu\rho}\sigma^{\mu\kappa} + g^{\mu\kappa}\sigma^{\nu\rho} - g^{\mu\rho}\sigma^{\nu\kappa} - g^{\nu\kappa}\sigma^{\mu\rho})\,, \tag{B.2.24}$$
$$\overline{\sigma}^{\mu\nu}\overline{\sigma}^{\rho\kappa} = -\tfrac{1}{4}(g^{\nu\rho}g^{\mu\kappa} - g^{\mu\rho}g^{\nu\kappa} - i\epsilon^{\mu\nu\rho\kappa}) + \tfrac{1}{2}i(g^{\nu\rho}\overline{\sigma}^{\mu\kappa} + g^{\mu\kappa}\overline{\sigma}^{\nu\rho} - g^{\mu\rho}\overline{\sigma}^{\nu\kappa} - g^{\nu\kappa}\overline{\sigma}^{\mu\rho})\,, \tag{B.2.25}$$

are also only meaningful in exactly four dimensions. This applies as well to the trace identities which follow from them.[100] For example,

$$\mathrm{Tr}[\sigma^\mu\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}^\kappa] = 2\left(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho} + i\epsilon^{\mu\nu\rho\kappa}\right)\,, \tag{B.2.26}$$
$$\mathrm{Tr}[\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho\sigma^\kappa] = 2\left(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho} - i\epsilon^{\mu\nu\rho\kappa}\right)\,. \tag{B.2.27}$$

This could lead to ambiguities in loop computations where it is necessary to perform the computation in $d \neq 4$ dimensions (until the end of the calculation where the limit $d \to 4$ is taken). However, in practice one typically finds that the above expressions appear multiplied by the metric and/or other external tensors (such as four-momenta appropriate to the problem at hand). In almost all such cases, two of the indices appearing in eqs. (B.2.26) and (B.2.27) are symmetrized which eliminates the $\epsilon^{\mu\nu\rho\kappa}$ term, rendering the resulting expressions unambiguous. Similarly, the sum of the above trace identities can be assigned an unambiguous meaning in $d \neq 4$ dimensions:

$$\mathrm{Tr}[\sigma^\mu\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}^\kappa] + \mathrm{Tr}[\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho\sigma^\kappa] = 4\left(g^{\mu\nu}g^{\rho\kappa} - g^{\mu\rho}g^{\nu\kappa} + g^{\mu\kappa}g^{\nu\rho}\right)\,. \tag{B.2.28}$$

One can recursively derive trace formulae for products of six or more $\sigma/\overline{\sigma}$ matrices by using the results of eqs. (B.2.16) and (B.2.17) to reduce the number of $\sigma/\overline{\sigma}$ matrices by two. For example,

$$\begin{aligned}\mathrm{Tr}[\sigma^\mu\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}^\kappa\sigma^\lambda\overline{\sigma}^\delta] = {}& g^{\mu\nu}\mathrm{Tr}[\sigma^\rho\overline{\sigma}^\kappa\sigma^\lambda\overline{\sigma}^\delta] - g^{\mu\rho}\mathrm{Tr}[\sigma^\nu\overline{\sigma}^\kappa\sigma^\lambda\overline{\sigma}^\delta] + g^{\nu\rho}\mathrm{Tr}[\sigma^\mu\overline{\sigma}^\kappa\sigma^\lambda\overline{\sigma}^\delta] \\ & + i\epsilon^{\mu\nu\rho\epsilon}\mathrm{Tr}[\sigma_\epsilon\overline{\sigma}^\kappa\sigma^\lambda\overline{\sigma}^\delta]\,,\end{aligned} \tag{B.2.29}$$

$$\begin{aligned}\mathrm{Tr}[\overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho\sigma^\kappa\overline{\sigma}^\lambda\sigma^\delta] = {}& g^{\mu\nu}\mathrm{Tr}[\overline{\sigma}^\rho\sigma^\kappa\overline{\sigma}^\lambda\sigma^\delta] - g^{\mu\rho}\mathrm{Tr}[\overline{\sigma}^\nu\sigma^\kappa\overline{\sigma}^\lambda\sigma^\delta] + g^{\nu\rho}\mathrm{Tr}[\overline{\sigma}^\mu\sigma^\kappa\overline{\sigma}^\lambda\sigma^\delta] \\ & - i\epsilon^{\mu\nu\rho\epsilon}\mathrm{Tr}[\overline{\sigma}_\epsilon\sigma^\kappa\overline{\sigma}^\lambda\sigma^\delta]\,.\end{aligned} \tag{B.2.30}$$

We then use eqs. (B.2.26) and (B.2.27) to evaluate the remaining traces over four $\sigma/\overline{\sigma}$ matrices.

29.7.3 Explicit forms for the two-component spinor wave functions

In this Appendix, we construct the explicit forms for the eigenstates of the spin operator $\frac{1}{2}\vec{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{s}}$, and examine their properties. For massive fermions, it is possible to transform to the rest frame,

[100] This is analogous to the statement that Tr $(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\kappa) = -4i\epsilon^{\mu\nu\rho\kappa}$ [in our convention where $\epsilon^{0123} = +1$, and γ_5 is defined by eq. (G.1.2)] is only meaningful in $d = 4$ dimensions. In two-component notation, the equivalent result is $\mathrm{Tr}[\sigma^\mu\overline{\sigma}^\nu\sigma^\rho\overline{\sigma}^\kappa - \overline{\sigma}^\mu\sigma^\nu\overline{\sigma}^\rho\sigma^\kappa] = 4i\epsilon^{\mu\nu\rho\kappa}$. In the literature various schemes have been proposed for defining the properties of γ_5 in $d \neq 4$ dimensions [234, 255]. In two-component notation, this would translate into a procedure for dealing with general traces involving four or more $\sigma/\overline{\sigma}$ matrices.

and quantize the spin along a fixed axis in space. The corresponding spinor wave functions will be called fixed-axis spinors. For either massive or massive fermions, one can quantize the spin along the direction of momentum. The corresponding spinor wave functions are helicity spinors. Helicity spinor wave functions are most conveniently applied to massless fermions or fermions in the relativistic limit of high energy $E \gg m$. Fixed-axis spinors are most conveniently applied to massive fermions in the non-relativistic limit.

Fixed-axis spinor wave functions

Consider a spin-1/2 fermion in its rest frame and quantize the spin along a fixed axis specified by the unit vector

$$\hat{\boldsymbol{s}} \equiv (\sin\theta\cos\phi\,,\ \sin\theta\sin\phi\,,\ \cos\theta)\,, \tag{C.1.1}$$

with polar angle θ and azimuthal angle ϕ with respect to a fixed z-axis. The corresponding spin states will be called fixed-axis spin states. The relevant basis of two-component fixed-axis spinors χ_s are eigenstates of $\frac{1}{2}\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}$, i.e.,

$$\tfrac{1}{2}\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}\,\chi_s = s\chi_s\,, \qquad s = \pm\tfrac{1}{2}\,. \tag{C.1.2}$$

In order to construct the eigenstates of $\frac{1}{2}\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}$, we first consider the case where $\hat{\boldsymbol{s}} = \hat{\boldsymbol{z}}$. In this case, we define the eigenstates of $\frac{1}{2}\sigma^3$ to be:

$$\chi_{1/2}(\hat{\boldsymbol{z}}) = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \chi_{-1/2}(\hat{\boldsymbol{z}}) = \begin{pmatrix}0\\1\end{pmatrix}. \tag{C.1.3}$$

By convention, we have set an arbitrary overall multiplicative phase factor for each spinor of eq. (C.1.3) to unity. We then determine $\chi_s(\hat{\boldsymbol{s}})$ from $\chi_s(\hat{\boldsymbol{z}})$ by employing the spin-1/2 rotation operator that corresponds to a rotation from $\hat{\boldsymbol{z}}$ to $\hat{\boldsymbol{s}}$. This rotation is represented by a 3×3 matrix $\mathcal{R}$ such that $\hat{\boldsymbol{s}} = \mathcal{R}\hat{\boldsymbol{z}}$. However, this rotation operator is not unique. In its most general form, the rotation operator can be parameterized in terms of three Euler angles (e.g., see refs. [45, 46]):

$$\mathcal{R}(\phi\,,\,\theta\,,\,\gamma) \equiv R(\hat{\boldsymbol{z}}\,,\,\phi)\,R(\hat{\boldsymbol{y}}\,,\,\theta)\,R(\hat{\boldsymbol{z}}\,,\,\gamma)\,, \tag{C.1.4}$$

The Euler angles can be chosen to lie in the range $0 \leq \theta \leq \pi$ and $0 \leq \phi\,,\,\gamma < 2\pi$. In general, $R(\hat{\boldsymbol{n}}\,,\,\theta)$ is a 3×3 orthogonal matrix of unit determinant that represents a rotation by an angle θ about a fixed axis $\hat{\boldsymbol{n}}$,

$$R^{ij}(\hat{\boldsymbol{n}}\,,\,\theta) = \exp(-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\mathcal{S}}}) = n^i n^j + (\delta^{ij} - n^i n^j)\cos\theta - \epsilon^{ijk}n^k\sin\theta\,, \tag{C.1.5}$$

where the $\vec{\boldsymbol{\mathcal{S}}} = (\mathcal{S}^1\,,\,\mathcal{S}^2\,,\,\mathcal{S}^3)$ are three 3×3 matrices whose matrix elements are given by $(\mathcal{S}^i)^{jk} = -i\epsilon^{ijk}$ [cf. eq. (29.9)].

However, the angle γ is arbitrary, since $R(\hat{\boldsymbol{z}}\,,\,\gamma)\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$. Thus,

$$\hat{\boldsymbol{s}} = \mathcal{R}\hat{\boldsymbol{z}} = (\sin\theta\cos\phi\,,\ \sin\theta\sin\phi\,,\ \cos\theta)\,, \tag{C.1.6}$$

independently of the choice of γ. For $\theta = 0$ or $\theta = \pi$, where $\hat{\boldsymbol{s}}$ is parallel to the z-axis, the azimuthal angle ϕ is undefined. Since $\hat{\boldsymbol{s}} \to -\hat{\boldsymbol{s}}$ corresponds in general to $\theta \to \pi - \theta$ and $\phi \to \phi + \pi \pmod{2\pi}$, we shall adopt a convention whereby:

$$\phi = \begin{cases} 0\,, & \text{for } \hat{\boldsymbol{s}} = \hat{\boldsymbol{z}}\,, \quad (\theta = 0)\,, \\ \pi\,, & \text{for } \hat{\boldsymbol{s}} = -\hat{\boldsymbol{z}}\,, \quad (\theta = \pi)\,. \end{cases} \tag{C.1.7}$$

Using the spin-1/2 rotation operator corresponding to $\mathcal{R}(\phi\,,\,\theta\,,\,\gamma)$, one can compute $\chi_s(\hat{\boldsymbol{s}})$,

$$\chi_s(\hat{\boldsymbol{s}}) = \mathcal{D}(\phi\,,\,\theta\,,\,\gamma)\,\chi_s(\hat{\boldsymbol{z}})\,, \tag{C.1.8}$$

where $\mathcal{D}$ is the spin-1/2 unitary representation matrix [256]

$$\mathcal{D}(\phi\,,\,\theta\,,\,\gamma) \equiv D(\hat{\boldsymbol{z}},\phi)\,D(\hat{\boldsymbol{y}},\theta)\,D(\hat{\boldsymbol{z}},\gamma)\,, \tag{C.1.9}$$

and D is the 2×2 unitary matrix

$$D(\hat{\boldsymbol{n}},\theta) \equiv \exp\left(-i\theta\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\sigma}}/2\right) = \cos\frac{\theta}{2} - i\hat{\boldsymbol{n}}\cdot\vec{\boldsymbol{\sigma}}\sin\frac{\theta}{2}\,. \tag{C.1.10}$$

Eq. (C.1.8) yields explicit forms for the eigenstates of $\frac{1}{2}\vec{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{s}}$:

$$\chi_{1/2}(\hat{\boldsymbol{s}}) = \begin{pmatrix} e^{-i(\phi+\gamma)/2}\cos\dfrac{\theta}{2} \\[2ex] e^{i(\phi-\gamma)/2}\sin\dfrac{\theta}{2} \end{pmatrix}, \qquad \chi_{-1/2}(\hat{\boldsymbol{s}}) = \begin{pmatrix} -e^{-i(\phi-\gamma)/2}\sin\dfrac{\theta}{2} \\[2ex] e^{i(\phi+\gamma)/2}\cos\dfrac{\theta}{2} \end{pmatrix}. \tag{C.1.11}$$

The well-known two-to-one mapping between SU(2) and SO(3) implies that for a given rotation matrix $\mathcal{R}$ there are two corresponding spin-1/2 rotation matrices $\mathcal{D}$. In particular,

$$\mathcal{D}(\phi+2\pi\,,\,\theta\,,\,\gamma) = -\mathcal{D}(\phi\,,\,\theta\,,\,\gamma)\,, \tag{C.1.12}$$

which implies that a rotation of a spinor by 2π yields an overall change of sign in the spinor wave function (an effect that can be observed in quantum interference experiments!). Strictly speaking, we should take the range of the Euler angles to be $0\leq\phi<4\pi$, $0\leq\theta\leq\pi$ and $0\leq\gamma<2\pi$. However, when constructing the spinor wave function of a spin-1/2 particle whose spin quantization axis is given by eq. (C.1.6), we will fix the overall sign of the spinor wave function by convention.

More generally, the overall phase of the spinor wave function is unphysical. Noting that $D(\hat{\boldsymbol{z}},\gamma)\,\chi_s(\hat{\boldsymbol{z}}) = e^{-is\gamma}\chi_s(\hat{\boldsymbol{z}})$, the choice of γ is also a matter of convention. First, we will require that when $\hat{\boldsymbol{s}}=\hat{\boldsymbol{z}}$, eq. (C.1.8) should reproduce the spinor wave functions given in eq. (C.1.3). This implies that:

$$\gamma = 0\,, \qquad \text{for } \hat{\boldsymbol{s}}=\hat{\boldsymbol{z}}\,, \quad (\theta=\phi=0)\,. \tag{C.1.13}$$

For $\hat{\boldsymbol{s}}=-\hat{\boldsymbol{z}}$, we use eq. (C.1.7) to obtain:

$$\chi_s(-\hat{\boldsymbol{z}}) = ie^{-is\gamma(-\hat{\boldsymbol{z}})}\,\chi_{-s}(\hat{\boldsymbol{z}})\,, \qquad\qquad s=\pm\tfrac{1}{2}\,, \tag{C.1.14}$$

where the notation $\gamma(-\hat{\boldsymbol{z}})$ has been employed to allow the possibility that the convention for γ depends on the direction indicated by its argument.

Two different conventions are commonly employed in the literature. In the first convention, one chooses $\gamma=-\phi$. This choice has the good feature that $\mathcal{R}(\phi\,,\,0\,,\,-\phi)=\mathbb{1}_{3\times3}$, independently of the angle ϕ, which is undefined when $\theta=0$.[101] Moreover, the rotation matrix $\mathcal{R}(\phi\,,\,\theta\,,\,-\phi)$ and the corresponding spin-1/2 rotation matrix $\mathcal{D}(\phi\,,\,\theta\,,\,-\phi)$ can be expressed simply as a single rotation by an angle θ about a fixed axis that points along a unit vector in the azimuthal direction:

$$\hat{\boldsymbol{\varphi}} \equiv (-\sin\phi\,,\,\cos\phi\,,\,0)\,, \tag{C.1.15}$$

In particular,

$$R(\hat{\boldsymbol{\varphi}}\,,\,\theta) = R(\hat{\boldsymbol{z}}\,,\,\phi)\,R(\hat{\boldsymbol{y}}\,,\,\theta)\,R(\hat{\boldsymbol{z}}\,,\,-\phi)\,, \tag{C.1.16}$$

$$D(\hat{\boldsymbol{\varphi}}\,,\,\theta) = \mathcal{D}(\phi\,,\,\theta\,,\,-\phi)\,. \tag{C.1.17}$$

Hence, in this convention $\chi_s(\hat{\boldsymbol{s}}) = D(\hat{\boldsymbol{\varphi}}\,,\,\theta)\chi_s(\hat{\boldsymbol{z}})$, which is the most common choice for the spinor wave function [37, 257, 258].

[101] However, $\mathcal{R}(\phi\,,\,\pi\,,\,-\phi)\neq\mathbb{1}_{3\times3}$ even though ϕ is also undefined when $\theta=\pi$.

In the second convention, one chooses $\gamma = 0$. One motivation for this choice is that the corresponding rotation matrix is somewhat simpler:

$$\mathcal{R}(\phi\,,\,\theta\,,\,0) = R(\hat{\boldsymbol{z}}\,,\,\phi)\,R(\hat{\boldsymbol{y}}\,,\,\theta) = \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{C.1.18}$$

Employing the corresponding spin-1/2 rotation operator $\mathcal{D}(\phi\,,\,\theta\,,\,0)$ in eq. (C.1.8) yields a slightly more symmetrical form for the spinor wave function [259].

Explicit forms for the spinor wave functions in the two conventions are obtained from eq. (C.1.11) by taking $\gamma(\hat{\boldsymbol{s}}) = -\phi$ and $\gamma(\hat{\boldsymbol{s}}) = 0$, respectively. For example, eq. (C.1.14) reduces to:

$$\chi_s(-\hat{\boldsymbol{z}}) = \begin{cases} -2s\chi_{-s}(\hat{\boldsymbol{z}}) & \text{for } \gamma(-\hat{\boldsymbol{z}}) = -\phi = -\pi\,, \\ i\chi_{-s}(\hat{\boldsymbol{z}}) & \text{for } \gamma(-\hat{\boldsymbol{z}}) = 0\,, \end{cases} \qquad s = \pm\tfrac{1}{2}\,, \tag{C.1.19}$$

in the convention specified by eq. (C.1.7).

Many of the properties of the spinor wave functions are independent of the choice of the Euler angle γ. The spinor wave functions χ_s defined by eq. (C.1.8) are normalized such that

$$\chi_s^\dagger(\hat{\boldsymbol{s}})\chi_{s'}(\hat{\boldsymbol{s}}) = \delta_{ss'}\,, \tag{C.1.20}$$

and satisfy the following completeness relation:

$$\sum_s \chi_s(\hat{\boldsymbol{s}})\chi_s^\dagger(\hat{\boldsymbol{s}}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{C.1.21}$$

The spinor wave functions $\chi_s(\hat{\boldsymbol{s}})$ and $\chi_{-s}(\hat{\boldsymbol{s}})$ are connected by the following relation:

$$\chi_{-s}(\hat{\boldsymbol{s}}) = -2si\sigma^2\,\chi_s^*(\hat{\boldsymbol{s}})\,. \tag{C.1.22}$$

Consider a spin-1/2 fermion with four-momentum $p^\mu = (E\,,\,\vec{\boldsymbol{p}})$, with $E = (|\vec{\boldsymbol{p}}|^2 + m^2)^{1/2}$, and the direction of $\vec{\boldsymbol{p}}$ given by

$$\hat{\boldsymbol{p}} = (\sin\theta_p\cos\phi_p\,,\,\sin\theta_p\sin\phi_p\,,\,\cos\theta_p)\,. \tag{C.1.23}$$

Using eqs. (29.108) and (29.109), one can employ eqs. (29.3.18)–(29.3.21) to obtain explicit expressions for the two-component spinor wave functions $x(\vec{\boldsymbol{p}}, s)$, $y(\vec{\boldsymbol{p}}, s)$, $x^\dagger(\vec{\boldsymbol{p}}, s)$ and $y^\dagger(\vec{\boldsymbol{p}}, s)$.

Additional properties of the χ_s can be derived by introducing an orthonormal set of unit three-vectors $\hat{\boldsymbol{s}}^a$ that provide a basis for a right-handed coordinate system. Explicitly,

$$\hat{\boldsymbol{s}}^{\boldsymbol{a}}\cdot\hat{\boldsymbol{s}}^{\boldsymbol{b}} = \delta^{ab}\,, \tag{C.1.24}$$

$$\hat{\boldsymbol{s}}^{\boldsymbol{a}}\times\hat{\boldsymbol{s}}^{\boldsymbol{b}} = \epsilon^{abc}\hat{\boldsymbol{s}}^{\boldsymbol{c}}\,. \tag{C.1.25}$$

We shall identify

$$\hat{\boldsymbol{s}}^{\boldsymbol{3}} \equiv \hat{\boldsymbol{s}} \tag{C.1.26}$$

as the quantization axis used in defining the third component of the spin of the fermion in its rest frame. The unit vectors $\hat{\boldsymbol{s}}^{\boldsymbol{1}}$ and $\hat{\boldsymbol{s}}^{\boldsymbol{2}}$ are then chosen such that eqs. (C.1.24) and (C.1.25) are satisfied. To explicitly construct the $\hat{\boldsymbol{s}}^{\boldsymbol{a}}$, we begin with the orthonormal set $\{\hat{\boldsymbol{x}}\,,\,\hat{\boldsymbol{y}}\,,\,\hat{\boldsymbol{z}}\}$, and employ the *same* rotation operator $\mathcal{R}$ used to define $\chi_s(\hat{\boldsymbol{s}})$. That is,

$$(\hat{\boldsymbol{s}}^{\boldsymbol{1}}\,,\,\hat{\boldsymbol{s}}^{\boldsymbol{2}}\,,\,\hat{\boldsymbol{s}}^{\boldsymbol{3}}) = (\mathcal{R}\hat{\boldsymbol{x}}\,,\,\mathcal{R}\hat{\boldsymbol{y}}\,,\,\mathcal{R}\hat{\boldsymbol{z}})\,, \qquad \text{where} \quad \mathcal{R} \equiv \mathcal{R}(\phi\,,\,\theta\,,\,\gamma)\,, \tag{C.1.27}$$

and ϕ, θ and γ are the Euler angles used to define the spinor wave function in eq. (C.1.8). From eq. (C.1.27), one can immediately derive the completeness relation (as a consequence of $\mathcal{R}\mathcal{R}^{\mathsf{T}} = \mathbb{1}$),

$$\hat{\boldsymbol{s}}^{\boldsymbol{a}i}\hat{\boldsymbol{s}}^{\boldsymbol{a}j} = \delta^{ij}\,, \tag{C.1.28}$$

where i and j label the space components of the three-vector $\hat{\boldsymbol{s}}^{\boldsymbol{a}}$.

We can use the $\hat{\boldsymbol{s}}^{\boldsymbol{a}}$ to extend the defining equation of χ_s [eq. (C.1.2)]:

$$\tfrac{1}{2}\,\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}^{\boldsymbol{a}}\chi_{s'}(\hat{\boldsymbol{s}}) = \tfrac{1}{2}\tau^a_{ss'}\chi_s(\hat{\boldsymbol{s}})\,, \tag{C.1.29}$$

where the $\tau^a_{ss'}$ are the matrix elements of the Pauli matrices.[102] That is, $\frac{1}{2}\vec{\boldsymbol{\sigma}}\cdot(\boldsymbol{s^1} \pm i\boldsymbol{s^2})$ serve as ladder operators that connect the spinor wave functions $\chi_{1/2}$ and $\chi_{-1/2}$. Using eq. (C.1.20), it follows that eq. (C.1.29) is equivalent to:

$$\chi^\dagger_s(\hat{\boldsymbol{s}})\,\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}^{\boldsymbol{a}}\chi_{s'}(\hat{\boldsymbol{s}}) = \tau^a_{ss'}\,. \tag{C.1.30}$$

It is instructive to prove eq. (C.1.30) directly. Employing eq. (C.1.8) and using the fact that $\mathcal{D}$ is a unitary matrix,

$$\chi^\dagger_s(\hat{\boldsymbol{s}})\,\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}^{\boldsymbol{a}}\chi_{s'}(\hat{\boldsymbol{s}}) = \chi^\dagger_s(\hat{\boldsymbol{z}})\,[\mathcal{D}(\phi\,,\theta\,,\gamma)]^{-1}\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}^{\boldsymbol{a}}\,\mathcal{D}(\phi\,,\theta\,,\gamma)\chi_{s'}(\hat{\boldsymbol{z}})\,. \tag{C.1.31}$$

The above result can be simplified by a repeated use of the following identity,

$$e^{i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2}\,\sigma^j\,e^{-i\theta\hat{\boldsymbol{n}}\boldsymbol{\cdot}\vec{\boldsymbol{\sigma}}/2} = R^{jk}(\hat{\boldsymbol{n}}\,,\theta)\sigma^k\,, \tag{C.1.32}$$

which is valid for any fixed axis $\hat{\boldsymbol{n}}$, where $R(\hat{\boldsymbol{n}}\,,\theta)$ is the rotation matrix defined in eq. (C.1.5). It follows that

$$[\mathcal{D}(\phi\,,\theta\,,\gamma)]^{-1}\,\sigma^j\,\mathcal{D}(\phi\,,\theta\,,\gamma) = \mathcal{R}^{jk}(\phi\,,\theta\,,\gamma)\,\sigma^k\,, \tag{C.1.33}$$

where $\mathcal{R}(\phi\,,\theta\,,\gamma)$ is defined in eq. (C.1.4). Since $\mathcal{R}^{\mathsf{T}} = \mathcal{R}^{-1}$,

$$\chi^\dagger_s(\hat{\boldsymbol{s}})\,\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}^{\boldsymbol{a}}\chi_{s'}(\hat{\boldsymbol{s}}) = \chi^\dagger_s(\hat{\boldsymbol{z}})\,\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\left[\mathcal{R}^{-1}\hat{\boldsymbol{s}}^{\boldsymbol{a}}\right]\,\chi_{s'}(\hat{\boldsymbol{z}})\,. \tag{C.1.34}$$

Eq. (C.1.27) implies that $(\mathcal{R}^{-1}\hat{\boldsymbol{s}}^{\boldsymbol{1}}\,,\,\mathcal{R}^{-1}\hat{\boldsymbol{s}}^{\boldsymbol{2}}\,,\,\mathcal{R}^{-1}\hat{\boldsymbol{s}}^{\boldsymbol{3}}) = (\hat{\boldsymbol{x}}\,,\,\hat{\boldsymbol{y}}\,,\,\hat{\boldsymbol{z}})$, and it follows that

$$\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\left[\mathcal{R}^{-1}\hat{\boldsymbol{s}}^{\boldsymbol{a}}\right] = \sigma^a\,. \tag{C.1.35}$$

Consequently, we end up with

$$\chi^\dagger_s(\hat{\boldsymbol{s}})\,\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}^{\boldsymbol{a}}\chi_{s'}(\hat{\boldsymbol{s}}) = \chi^\dagger_s(\hat{\boldsymbol{z}})\sigma^a\chi_{s'}(\hat{\boldsymbol{z}}) \equiv \tau^a_{ss'}\,, \tag{C.1.36}$$

which defines the matrix elements of the Pauli matrices, and our proof of eq. (C.1.30) is complete.

Using the completeness relation given by eq. (C.1.28), we can rewrite eq. (C.1.30) as

$$\chi^\dagger_s(\hat{\boldsymbol{s}})\,\sigma^i\chi_{s'}(\hat{\boldsymbol{s}}) = \tau^a_{ss'}\hat{\boldsymbol{s}}^{\boldsymbol{a}i}\,. \tag{C.1.37}$$

Taking the hermitian conjugate of eq. (C.1.37) is equivalent to interchanging $s \leftrightarrow s'$, since the σ^i are hermitian matrices and $(\tau^a_{ss'})^* = \tau^a_{s's}$. To evaluate expressions similar to eq. (C.1.37) that contain products of σ-matrices, it is sufficient to use the relation $\sigma^i\sigma^j = \delta^{ij}\mathbb{1} + i\epsilon^{ijk}\sigma^k$ as many

[102] We use the symbol τ rather than σ to emphasize that the indices of the Pauli matrices τ^a are spin labels s, s' and *not* spinor indices α, $\dot{\alpha}$. The first (second) row and column of the τ-matrices correspond to $s = 1/2$ $(-1/2)$. For example, $\tau^3_{ss'} = 2s\delta_{ss'}$ (no sum over s).

times as needed to reduce the final expression to terms containing at most one σ-matrix. For example, using eqs. (C.1.20) and (C.1.37), it follows that

$$\chi_s^\dagger(\hat{\boldsymbol{s}})\,\sigma^i\sigma^j\chi_{s'}(\hat{\boldsymbol{s}}) = \delta_{ss'}\delta^{ij} + i\epsilon^{ijk}\tau^a_{ss'}\hat{\boldsymbol{s}}^{\boldsymbol{a}k}\,. \tag{C.1.38}$$

It is sometimes useful to have a more explicit representation of the $\hat{\boldsymbol{s}}^{\boldsymbol{a}}$. In the convention where $\gamma = -\phi$, eq. (C.1.27) yields:

$$\begin{aligned}
\hat{\boldsymbol{s}}^{\boldsymbol{1}} &= (1 - 2\cos^2\phi\,\sin^2\frac{\theta}{2}\,,\, -\sin 2\phi\,\sin^2\frac{\theta}{2}\,,\, -\sin\theta\cos\phi)\,,\\
\hat{\boldsymbol{s}}^{\boldsymbol{2}} &= (-\sin 2\phi\,\sin^2\frac{\theta}{2}\,,\, 1 - 2\sin^2\phi\,\sin^2\frac{\theta}{2}\,,\, -\sin\theta\sin\phi)\,,\\
\hat{\boldsymbol{s}}^{\boldsymbol{3}} &= (\sin\theta\cos\phi\,,\,\sin\theta\sin\phi\,,\,\cos\theta)\,.
\end{aligned} \tag{C.1.39}$$

The explicit forms for the $\hat{\boldsymbol{s}}^{\boldsymbol{a}}$ are somewhat simpler in the convention where $\gamma = 0$. In this case, eqs. (C.1.18) and (C.1.27) yield:

$$\begin{aligned}
\hat{\boldsymbol{s}}^{\boldsymbol{1}} &= (\cos\theta\cos\phi,\,\cos\theta\sin\phi,\,-\sin\theta)\,,\\
\hat{\boldsymbol{s}}^{\boldsymbol{2}} &= (-\sin\phi,\,\cos\phi,\,0)\,,\\
\hat{\boldsymbol{s}}^{\boldsymbol{3}} &= (\sin\theta\cos\phi,\,\sin\theta\sin\phi,\,\cos\theta)\,.
\end{aligned} \tag{C.1.40}$$

Fixed-axis spinors in the non-relativistic limit

Consider an on-shell massive fermion of three-momentum $\vec{\boldsymbol{p}}$, mass m and spin quantum number s, where $s = \pm\frac{1}{2}$ are the possible projections of the spin vector (in units of $\hbar$) along the fixed $\hat{\boldsymbol{s}}$ direction [cf. eq. (C.1.2)]. The spinor wave functions, x, y, and their hermitian conjugates are given by eqs. (29.3.18)–(29.3.21). In the non-relativistic limit,

$$\sqrt{p\cdot\sigma} \simeq \sqrt{m}\left(\mathbb{1} - \frac{\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{p}}}{2m}\right)\,, \tag{C.2.1}$$

$$\sqrt{p\cdot\overline{\sigma}} \simeq \sqrt{m}\left(\mathbb{1} + \frac{\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{p}}}{2m}\right)\,, \tag{C.2.2}$$

where we keep terms only up to $\mathcal{O}(|\vec{\boldsymbol{p}}|/m)$. Inserting the above results into eqs. (29.3.18)–(29.3.21) yields:

$$x_\alpha(\vec{p},s) \simeq \sqrt{m}\left(\mathbb{1} - \frac{\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{p}}}{2m}\right)\chi_s(\hat{\boldsymbol{s}})\,, \tag{C.2.3}$$

$$x^\alpha(\vec{p},s) \simeq -2s\sqrt{m}\,\chi^\dagger_{-s}(\hat{\boldsymbol{s}})\left(\mathbb{1} + \frac{\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{p}}}{2m}\right)\,, \tag{C.2.4}$$

$$y_\alpha(\vec{p},s) \simeq 2s\sqrt{m}\left(\mathbb{1} - \frac{\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{p}}}{2m}\right)\chi_{-s}(\hat{\boldsymbol{s}})\,, \tag{C.2.5}$$

$$y^\alpha(\vec{p},s) \simeq \sqrt{m}\,\chi^\dagger_s(\hat{\boldsymbol{s}})\left(\mathbb{1} + \frac{\vec{\boldsymbol{\sigma}}\cdot\vec{\boldsymbol{p}}}{2m}\right)\,, \tag{C.2.6}$$

for the undotted spinor wave functions and

$$x^{\dagger\dot\alpha}(\vec p,s)\simeq -2s\sqrt{m}\,\left(\mathbb{1}+\frac{\vec{\boldsymbol\sigma}\cdot\vec{\boldsymbol p}}{2m}\right)\chi_{-s}(\hat{\boldsymbol s})\,, \tag{C.2.7}$$

$$x^{\dagger}_{\dot\alpha}(\vec p,s)\simeq \sqrt{m}\,\chi^{\dagger}_{s}(\hat{\boldsymbol s})\left(\mathbb{1}-\frac{\vec{\boldsymbol\sigma}\cdot\vec{\boldsymbol p}}{2m}\right)\,, \tag{C.2.8}$$

$$y^{\dagger\dot\alpha}(\vec p,s)\simeq \sqrt{m}\,\left(\mathbb{1}+\frac{\vec{\boldsymbol\sigma}\cdot\vec{\boldsymbol p}}{2m}\right)\chi_{s}(\hat{\boldsymbol s})\,, \tag{C.2.9}$$

$$y^{\dagger}_{\dot\alpha}(\vec p,s)\simeq 2s\sqrt{m}\,\chi^{\dagger}_{-s}(\hat{\boldsymbol s})\left(\mathbb{1}-\frac{\vec{\boldsymbol\sigma}\cdot\vec{\boldsymbol p}}{2m}\right)\,, \tag{C.2.10}$$

for the dotted spinor wave functions.

In the computation of the S-matrix amplitudes for scattering and decay processes, one typically must evaluate a bilinear product of spinors, i.e. quantities of the form

$$z_1(\vec{\boldsymbol p}_1,s_1)\,\Gamma\,z_2(\vec{\boldsymbol p}_2,s_2)\,, \tag{C.2.11}$$

where z_1 and z_2 represent one of the two-component spinor wave functions x, y, $x^\dagger$ or $y^\dagger$, and Γ is a 2×2 matrix (in spinor space) that is either the identity matrix, or is made up of alternating products of σ and $\overline{\sigma}$. In the non-relativistic limit, these bilinears take on rather simple forms. In what follows, we work to first order in $|\vec{\boldsymbol p}_i|/m_i$. For example,

$$\begin{aligned} y^{\alpha}(\vec{\boldsymbol p}_1,s_1)x_\alpha(\vec{\boldsymbol p}_2,s_2)&\simeq\sqrt{m_1m_2}\,\chi^{\dagger}_{s_1}(\hat{\boldsymbol s})\left(\mathbb{1}+\frac{\vec{\boldsymbol\sigma}\cdot\vec{\boldsymbol p}}{2m_1}-\frac{\vec{\boldsymbol\sigma}\cdot\vec{\boldsymbol p}}{2m_2}\right)\chi_{s_2}(\hat{\boldsymbol s})\\ &\simeq\sqrt{m_1m_2}\left[\delta_{s_1,s_2}+\left(\frac{\vec{\boldsymbol p}_1}{2m_1}-\frac{\vec{\boldsymbol p}_2}{2m_2}\right)\cdot\hat{\boldsymbol s}^{\boldsymbol a}\tau^a_{s_1,s_2}\right]\,, \end{aligned} \tag{C.2.12}$$

where we have used the results of eqs. (C.1.20) and (C.1.37). Similarly,

$$\begin{aligned} y^{\alpha}(\boldsymbol p_{\boldsymbol 1},s_1)\sigma^\mu_{\alpha\dot\beta}y^{\dagger\dot\beta}(\boldsymbol p_{\boldsymbol 2},s_2)&\simeq\sqrt{m_1m_2}\,\chi^{\dagger}_{s_1}(\hat{\boldsymbol s})\left[\sigma^\mu+\frac{\vec{\boldsymbol\sigma}\cdot\vec{\boldsymbol p}_{\boldsymbol 1}}{2m_1}\sigma^\mu+\sigma^\mu\frac{\vec{\boldsymbol\sigma}\cdot\vec{\boldsymbol p}_{\boldsymbol 2}}{2m_2}\right]\chi_{s_2}(\hat{\boldsymbol s})\\ &\simeq\sqrt{m_1m_2}\,Z^\mu_{s_1,s_2}(\vec{\boldsymbol p}_{\boldsymbol 1},\vec{\boldsymbol p}_{\boldsymbol 2})\,, \end{aligned} \tag{C.2.13}$$

where[103]

$$Z^\mu_{ss'}(\vec{\boldsymbol p}_{\boldsymbol 1},\vec{\boldsymbol p}_{\boldsymbol 2})\equiv\begin{cases}\delta_{ss'}+\left(\dfrac{\vec{\boldsymbol p}_1}{2m_1}+\dfrac{\vec{\boldsymbol p}_2}{2m_2}\right)\cdot\hat{\boldsymbol s}^{\boldsymbol a}\tau^a_{ss'}\,, & \text{for }\mu=0\,,\\ \hat{\boldsymbol s}^{\boldsymbol a i}\tau^a_{ss'}+\left(\dfrac{p_1^i}{2m_1}+\dfrac{p_2^i}{2m_2}\right)\delta_{ss'}+\left(\dfrac{p_2^j}{2m_2}-\dfrac{p_1^j}{2m_1}\right)i\epsilon^{ijk}\hat{\boldsymbol s}^{\boldsymbol a k}\tau^a_{ss'}\,, & \text{for }\mu=i=1,2,3\,,\end{cases} \tag{C.2.14}$$

is obtained after using the results of eqs. (C.1.37) and (C.1.38).

In summary, we list the non-relativistic forms of the spinor bilinears. Referring to eq. (C.2.11),

[103] We also define $Z^\mu_{s's}(\vec{\boldsymbol p}_{\boldsymbol 2},\vec{\boldsymbol p}_{\boldsymbol 1})$ as the expression given by eq. (C.2.14) with the interchange of $\{s\,,\,\vec{\boldsymbol p}_{\boldsymbol 1}\,,\,m_1\}$ and $\{s'\,,\,\vec{\boldsymbol p}_{\boldsymbol 2}\,,\,m_2\}$.

if $\Gamma = \mathbb{1}$, then

$$x^\alpha(\vec{\boldsymbol{p}}_1, s_1)x_\alpha(\vec{\boldsymbol{p}}_2, s_2) \simeq 2s_2\sqrt{m_1m_2}\left[\delta_{-s_2,s_1} + \left(\frac{\vec{\boldsymbol{p}}_1}{2m_1} - \frac{\vec{\boldsymbol{p}}_2}{2m_2}\right)\cdot\hat{\boldsymbol{s}}^{\boldsymbol{a}}\tau^a_{-s_2,s_1}\right], \tag{C.2.15}$$

$$y^\alpha(\vec{\boldsymbol{p}}_1, s_1)y_\alpha(\vec{\boldsymbol{p}}_2, s_2) \simeq 2s_2\sqrt{m_1m_2}\left[\delta_{s_1,-s_2} + \left(\frac{\vec{\boldsymbol{p}}_1}{2m_1} - \frac{\vec{\boldsymbol{p}}_2}{2m_2}\right)\cdot\hat{\boldsymbol{s}}^{\boldsymbol{a}}\tau^a_{s_1,-s_2}\right], \tag{C.2.16}$$

$$x^\alpha(\vec{\boldsymbol{p}}_1, s_1)y_\alpha(\vec{\boldsymbol{p}}_2, s_2) \simeq \sqrt{m_1m_2}\left[-\delta_{s_2,s_1} + \left(\frac{\vec{\boldsymbol{p}}_1}{2m_1} - \frac{\vec{\boldsymbol{p}}_2}{2m_2}\right)\cdot\hat{\boldsymbol{s}}^{\boldsymbol{a}}\tau^a_{s_2,s_1}\right], \tag{C.2.17}$$

$$y^\alpha(\vec{\boldsymbol{p}}_1, s_1)x_\alpha(\vec{\boldsymbol{p}}_2, s_2) \simeq \sqrt{m_1m_2}\left[\delta_{s_1,s_2} + \left(\frac{\vec{\boldsymbol{p}}_1}{2m_1} - \frac{\vec{\boldsymbol{p}}_2}{2m_2}\right)\cdot\hat{\boldsymbol{s}}^{\boldsymbol{a}}\tau^a_{s_1,s_2}\right], \tag{C.2.18}$$

where we have used

$$\tau^a_{s's} = -4ss'\tau^a_{-s,-s'}\,, \qquad\qquad s, s' = \pm\tfrac{1}{2}\,, \tag{C.2.19}$$

to arrive at the final forms given in eqs. (C.2.15) and (C.2.17). However, in using the above results, one must now pay close attention to the ordering of the subscript indices of the τ^a. The corresponding formulae for dotted spinor wave function bilinears are obtained by taking the hermitian conjugates of eqs. (C.2.15)–(C.2.18), which complex-conjugates the τ^a that appear on the right-hand side of these equations. Since $(\tau^a_{ss'})^* = \tau^a_{s's}$, we obtain

$$x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}}_1, s_1)x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}}_2, s_2) \simeq 2s_1\sqrt{m_1m_2}\left[\delta_{s_2,-s_1} - \left(\frac{\vec{\boldsymbol{p}}_1}{2m_1} - \frac{\vec{\boldsymbol{p}}_2}{2m_2}\right)\cdot\hat{\boldsymbol{s}}^{\boldsymbol{a}}\tau^a_{s_2,-s_1}\right], \tag{C.2.20}$$

$$y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}}_1, s_1)y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}}_2, s_2) \simeq 2s_1\sqrt{m_1m_2}\left[\delta_{-s_1,s_2} - \left(\frac{\vec{\boldsymbol{p}}_1}{2m_1} - \frac{\vec{\boldsymbol{p}}_2}{2m_2}\right)\cdot\hat{\boldsymbol{s}}^{\boldsymbol{a}}\tau^a_{-s_1,s_2}\right], \tag{C.2.21}$$

$$y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}}_1, s_1)x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}}_2, s_2) \simeq -\sqrt{m_1m_2}\left[\delta_{s_2,s_1} + \left(\frac{\vec{\boldsymbol{p}}_1}{2m_1} - \frac{\vec{\boldsymbol{p}}_2}{2m_2}\right)\cdot\hat{\boldsymbol{s}}^{\boldsymbol{a}}\tau^a_{s_2,s_1}\right], \tag{C.2.22}$$

$$x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}}_1, s_1)y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}}_2, s_2) \simeq \sqrt{m_1m_2}\left[\delta_{s_1,s_2} - \left(\frac{\vec{\boldsymbol{p}}_1}{2m_1} - \frac{\vec{\boldsymbol{p}}_2}{2m_2}\right)\cdot\hat{\boldsymbol{s}}^{\boldsymbol{a}}\tau^a_{s_1,s_2}\right]. \tag{C.2.23}$$

Likewise, if $\Gamma = \sigma^\mu$, then

$$x^\alpha(\boldsymbol{p_1}, s_1)\sigma^\mu_{\alpha\dot\beta}x^{\dagger\dot\beta}(\boldsymbol{p_2}, s_2) \simeq 4s_1s_2\sqrt{m_1m_2}\,Z^\mu_{-s_1,-s_2}(\vec{\boldsymbol{p}}_{\boldsymbol{1}}, \vec{\boldsymbol{p}}_{\boldsymbol{2}})\,, \tag{C.2.24}$$

$$y^\alpha(\boldsymbol{p_1}, s_1)\sigma^\mu_{\alpha\dot\beta}y^{\dagger\dot\beta}(\boldsymbol{p_2}, s_2) \simeq \sqrt{m_1m_2}\,Z^\mu_{s_1,s_2}(\vec{\boldsymbol{p}}_{\boldsymbol{1}}, \vec{\boldsymbol{p}}_{\boldsymbol{2}})\,, \tag{C.2.25}$$

$$x^\alpha(\boldsymbol{p_1}, s_1)\sigma^\mu_{\alpha\dot\beta}y^{\dagger\dot\beta}(\boldsymbol{p_2}, s_2) \simeq -2s_1\sqrt{m_1m_2}\,Z^\mu_{-s_1,s_2}(\vec{\boldsymbol{p}}_{\boldsymbol{1}}, \vec{\boldsymbol{p}}_{\boldsymbol{2}})\,, \tag{C.2.26}$$

$$y^\alpha(\boldsymbol{p_1}, s_1)\sigma^\mu_{\alpha\dot\beta}x^{\dagger\dot\beta}(\boldsymbol{p_2}, s_2) \simeq -2s_2\sqrt{m_1m_2}\,Z^\mu_{s_1,-s_2}(\vec{\boldsymbol{p}}_{\boldsymbol{1}}, \vec{\boldsymbol{p}}_{\boldsymbol{2}})\,, \tag{C.2.27}$$

where $Z^\mu_{ss'}(\vec{\boldsymbol{p}}_{\boldsymbol{1}}, \vec{\boldsymbol{p}}_{\boldsymbol{2}})$ is defined in eq. (C.2.14). If $\Gamma = \overline{\sigma}^\mu$, one can use $z_1\sigma^\mu z_2^\dagger = z_2^\dagger\overline{\sigma}^\mu z_1$ [i.e. eq. (29.62) for commuting spinors] to obtain the corresponding formulae for the spinor wave function bilinears (cf. footnote 103):

$$x^\dagger_{\dot\alpha}(\boldsymbol{p_1}, s_1)\overline{\sigma}^{\mu\dot\alpha\beta}x_\beta(\boldsymbol{p_2}, s_2) \simeq 4s_1s_2\sqrt{m_1m_2}\,Z^\mu_{-s_2,-s_1}(\vec{\boldsymbol{p}}_{\boldsymbol{2}}, \vec{\boldsymbol{p}}_{\boldsymbol{1}})\,, \tag{C.2.28}$$

$$y^\dagger_{\dot\alpha}(\boldsymbol{p_1}, s_1)\overline{\sigma}^{\mu\dot\alpha\beta}y_\beta(\boldsymbol{p_2}, s_2) \simeq \sqrt{m_1m_2}\,Z^\mu_{s_2,s_1}(\vec{\boldsymbol{p}}_{\boldsymbol{2}}, \vec{\boldsymbol{p}}_{\boldsymbol{1}})\,, \tag{C.2.29}$$

$$y^\dagger_{\dot\alpha}(\boldsymbol{p_1}, s_1)\overline{\sigma}^{\mu\dot\alpha\beta}x_\beta(\boldsymbol{p_2}, s_2) \simeq -2s_2\sqrt{m_1m_2}\,Z^\mu_{-s_2,s_1}(\vec{\boldsymbol{p}}_{\boldsymbol{2}}, \vec{\boldsymbol{p}}_{\boldsymbol{1}})\,, \tag{C.2.30}$$

$$x^\dagger_{\dot\alpha}(\boldsymbol{p_1}, s_1)\overline{\sigma}^{\mu\dot\alpha\beta}y_\beta(\boldsymbol{p_2}, s_2) \simeq -2s_1\sqrt{m_1m_2}\,Z^\mu_{s_2,-s_1}(\vec{\boldsymbol{p}}_{\boldsymbol{2}}, \vec{\boldsymbol{p}}_{\boldsymbol{1}})\,. \tag{C.2.31}$$

These results can also be derived directly from eqs. (C.2.3)–(C.2.10), after employing eq. (C.2.19).

It is straightforward to evaluate the spinor wave function bilinears when Γ is a product of two or more $\sigma/\overline{\sigma}$ matrices. As the corresponding expressions are considerably more complicated, we shall not write them out explicitly here.

Helicity spinor wave functions

All the results of Appendix C.1 apply to the helicity spinors χ_λ, which are defined to be eigenstates of $\frac{1}{2}\vec{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{p}}$, i.e.,

$$\tfrac{1}{2}\vec{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{p}}\,\chi_\lambda(\hat{\boldsymbol{p}}) = \lambda\chi_\lambda(\hat{\boldsymbol{p}})\,, \qquad \lambda = \pm\tfrac{1}{2}\,, \tag{C.3.1}$$

where $\hat{\boldsymbol{p}} = (\sin\theta_p\cos\phi_p\,,\,\sin\theta_p\sin\phi_p\,,\,\cos\theta_p)$. It follows that:

$$\sqrt{p\cdot\sigma}\,\chi_\lambda(\hat{\boldsymbol{p}}) = \omega_{-\lambda}(\vec{\boldsymbol{p}})\,\chi_\lambda(\hat{\boldsymbol{p}})\,, \qquad \sqrt{p\cdot\overline{\sigma}}\,\chi_\lambda(\hat{\boldsymbol{p}}) = \omega_\lambda(\vec{\boldsymbol{p}})\,\chi_\lambda(\hat{\boldsymbol{p}})\,, \tag{C.3.2}$$

where $\omega_\lambda(\vec{\boldsymbol{p}}) \equiv (E + 2\lambda|\vec{\boldsymbol{p}}|\,)^{1/2}$ and $E = \sqrt{|\vec{\boldsymbol{p}}|^2 + m^2}$. As a result, the explicit forms for the two-component helicity spinor wave functions [cf. eqs. (29.3.18)–(29.3.21)] simplify:

$$x_\alpha(\vec{\boldsymbol{p}},\lambda) = \omega_{-\lambda}\,\chi_\lambda(\hat{\boldsymbol{p}})\,, \qquad x^\alpha(\vec{\boldsymbol{p}},\lambda) = -2\lambda\,\omega_{-\lambda}\,\chi^\dagger_{-\lambda}(\hat{\boldsymbol{p}})\,, \tag{C.3.3}$$

$$y_\alpha(\vec{\boldsymbol{p}},\lambda) = 2\lambda\,\omega_\lambda\,\chi_{-\lambda}(\hat{\boldsymbol{p}})\,, \qquad y^\alpha(\vec{\boldsymbol{p}},\lambda) = \omega_\lambda\,\chi^\dagger_\lambda(\hat{\boldsymbol{p}})\,, \tag{C.3.4}$$

$$x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},\lambda) = -2\lambda\,\omega_{-\lambda}\,\chi_{-\lambda}(\hat{\boldsymbol{p}})\,, \qquad x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda) = \omega_{-\lambda}\,\chi^\dagger_\lambda(\hat{\boldsymbol{p}})\,, \tag{C.3.5}$$

$$y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},\lambda) = \omega_\lambda\,\chi_\lambda(\hat{\boldsymbol{p}})\,, \qquad y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda) = 2\lambda\,\omega_\lambda\,\chi^\dagger_{-\lambda}(\hat{\boldsymbol{p}})\,, \tag{C.3.6}$$

where $\omega_{\pm\lambda} = \omega_{\pm\lambda}(\vec{\boldsymbol{p}})$.

In analogy with the $\hat{\boldsymbol{s}}^{\boldsymbol{a}}$, it is convenient to introduce an orthonormal set of unit three-vectors $\hat{\boldsymbol{p}}^{\boldsymbol{a}}$ such that $\hat{\boldsymbol{p}}^{\boldsymbol{3}} = \hat{\boldsymbol{p}}$. Then, eqs. (C.1.24)–(C.1.30) apply as well to the two-component helicity spinors after taking $\hat{\boldsymbol{s}}^{\boldsymbol{a}} = \hat{\boldsymbol{p}}^{\boldsymbol{a}}$.

In scattering processes, it is often convenient to work in the rest frame of the incoming particles, in which the corresponding incoming fermion three-momenta are denoted by $\vec{\boldsymbol{p}}$ and $-\vec{\boldsymbol{p}}$, respectively. The helicity spinor wave function of the second fermion depends on the definition of $\chi_\lambda(-\hat{\boldsymbol{p}})$. In this review, we follow a convention[104] in which $\chi_\lambda(-\hat{\boldsymbol{p}})$ is defined to be the spinor wave function obtained from $\chi_\lambda(\hat{\boldsymbol{z}})$ via a rotation by a polar angle $\pi - \theta_p$ and an azimuthal angle $\phi_p + \pi$ with respect to the $\hat{\boldsymbol{z}}$-direction. Then,

$$\chi_\lambda(-\hat{\boldsymbol{p}}) = \mathcal{D}(\phi_p + \pi\,,\,\pi - \theta_p\,,\,\gamma(-\hat{\boldsymbol{p}}))\,\chi_\lambda(\hat{\boldsymbol{z}})\,, \tag{C.3.7}$$

where we have exhibited the possible dependence of γ on the direction $-\hat{\boldsymbol{p}}$. Using the properties of the spin-1/2 rotation matrices, one can derive

$$\mathcal{D}(\phi_p + \pi\,,\,\pi - \theta_p\,,\,\gamma(-\hat{\boldsymbol{p}})) = -\mathcal{D}(\phi_p\,,\,\theta_p\,,\,\gamma(\hat{\boldsymbol{p}}))\,D(\hat{\boldsymbol{z}}, -\gamma(\hat{\boldsymbol{p}}) - \gamma(-\hat{\boldsymbol{p}}))\,D(\hat{\boldsymbol{x}},\pi)\,. \tag{C.3.8}$$

Inserting this result in eq. (C.3.7) and using the relation

$$D(\hat{\boldsymbol{x}},\pi)\chi_\lambda(\hat{\boldsymbol{z}}) = -i\sigma^1\chi_\lambda(\hat{\boldsymbol{z}}) = -i\chi_{-\lambda}(\hat{\boldsymbol{z}})\,, \tag{C.3.9}$$

we obtain

$$\chi_\lambda(-\hat{\boldsymbol{p}}) = \xi_{-\lambda}(\hat{\boldsymbol{p}})\chi_{-\lambda}(\hat{\boldsymbol{p}})\,, \tag{C.3.10}$$

[104] An alternative convention (called the *second-particle convention*) advocated by Jacob and Wick [260] is to define $\chi_\lambda(-\hat{\boldsymbol{p}})$ by starting with $\chi_{-\lambda}(\hat{\boldsymbol{z}})$ and then rotating the spinor by polar and azimuthal angles θ_p and ϕ_p. In this case, $\chi_\lambda(-\hat{\boldsymbol{p}}) = \chi_{-\lambda}(\hat{\boldsymbol{p}})$, and the extra phase factors of eq. (C.3.10) is absent, i.e. $\xi_\lambda(\hat{\boldsymbol{p}}) = 1$ in eq. (C.3.10). However, this convention is less suited to scattering processes involving final states with more than two fermions. Hence, we do not adopt the second-particle convention in this review.

where the phase factor $\xi_\lambda(\hat{\boldsymbol{p}})$ is given by

$$\xi_\lambda(\hat{\boldsymbol{p}}) = ie^{i\lambda[\gamma(\hat{\boldsymbol{p}})+\gamma(-\hat{\boldsymbol{p}})]}\,, \qquad \lambda = \pm\tfrac{1}{2}\,. \tag{C.3.11}$$

Since γ is a real angle, it follows that:

$$\xi^*_\lambda(\hat{\boldsymbol{p}}) = \frac{1}{\xi_\lambda(\hat{\boldsymbol{p}})} = -\xi_{-\lambda}(\hat{\boldsymbol{p}})\,. \tag{C.3.12}$$

Using eq. (C.3.11), we note that $\chi_\lambda(\hat{\boldsymbol{p}})$ possesses the peculiar property that:

$$\chi_\lambda(-(-\hat{\boldsymbol{p}})) = -\chi_\lambda(\hat{\boldsymbol{p}})\,. \tag{C.3.13}$$

This is a consequence of the fact that the result of two successive inversions is equivalent to $\phi_p \to \phi_p + 2\pi$, which yields an overall change of sign of a spinor wave function [cf. eq. (C.1.12)].[105]

For example, corresponding to the two conventional choices for γ,

$$\xi_\lambda(\hat{\boldsymbol{p}}) = \begin{cases} (-1)^{\frac{1}{2}-\lambda}\, e^{-2i\lambda\phi_p} & \text{for } \ \gamma(\hat{\boldsymbol{p}}) = -\phi_p\,, \quad \gamma(-\hat{\boldsymbol{p}}) = -\phi_p - \pi\,, \\ i & \text{for } \ \gamma(\hat{\boldsymbol{p}}) = \gamma(-\hat{\boldsymbol{p}}) = 0\,, \end{cases} \tag{C.3.14}$$

with the proviso that for $\hat{\boldsymbol{p}} = \pm\hat{\boldsymbol{z}}$, we define ϕ_p according to eq. (C.1.7).

Suppose that the two fermions considered above have equal mass. In the center-of-mass frame, if the four-momentum of one of the fermions is $p^\mu = (E\,;\,\vec{\boldsymbol{p}})$, then the four-momentum of the other fermion is $\widetilde{p}^\mu \equiv (E\,;\,-\vec{\boldsymbol{p}})$. The following *numerical* identities are then satisfied: $\sigma\cdot\widetilde{p} = \overline{\sigma}\cdot p$ and $\overline{\sigma}\cdot\widetilde{p} = \sigma\cdot p$. However, in order to maintain covariance with respect to the undotted and dotted spinor indices, we shall write these identities as:

$$\widetilde{p}\cdot\sigma_{\alpha\dot\beta} = \sigma^0_{\alpha\dot\alpha}(p\cdot\overline{\sigma}^{\dot\alpha\beta})\,\sigma^0_{\beta\dot\beta}\,, \qquad \widetilde{p}\cdot\overline{\sigma}^{\dot\alpha\beta} = \overline{\sigma}^{0\dot\alpha\alpha}(p\cdot\sigma_{\alpha\dot\beta})\,\overline{\sigma}^{0\dot\beta\beta}\,. \tag{C.3.15}$$

Taking the matrix square root of both sides of the equations above removes one of the factors of σ^0 and $\overline{\sigma}^0$, respectively [cf. eqs. (29.110)–(29.116)]. Thus, using eqs. (29.3.18) and (C.3.10),

$$x_\alpha(-\vec{\boldsymbol{p}},-\lambda) = \sqrt{\widetilde{p}\cdot\sigma}\chi_{-\lambda}(-\hat{\boldsymbol{p}}) = \sigma^0\sqrt{p\cdot\overline{\sigma}}\,\xi_\lambda(\hat{\boldsymbol{p}})\,\chi_\lambda(\vec{\boldsymbol{p}}) = \sigma^0_{\alpha\dot\beta}\,\xi_\lambda(\hat{\boldsymbol{p}})\,y^{\dagger\dot\beta}(\vec{\boldsymbol{p}},\lambda)\,. \tag{C.3.16}$$

In this way, we can derive all relations of this kind for the helicity spinor wave functions:

$$x_\alpha(-\vec{\boldsymbol{p}},-\lambda) = \xi_\lambda\sigma^0_{\alpha\dot\beta}\,y^{\dagger\dot\beta}(\vec{\boldsymbol{p}},\lambda) = \omega_\lambda\xi_\lambda\,\chi_\lambda(\hat{\boldsymbol{p}})\,, \tag{C.3.17}$$

$$y_\alpha(-\vec{\boldsymbol{p}},-\lambda) = \xi_{-\lambda}\sigma^0_{\alpha\dot\beta}\,x^{\dagger\dot\beta}(\vec{\boldsymbol{p}},\lambda) = -2\lambda\,\omega_{-\lambda}\xi_{-\lambda}\,\chi_{-\lambda}(\hat{\boldsymbol{p}})\,, \tag{C.3.18}$$

$$x^{\dagger\dot\alpha}(-\vec{\boldsymbol{p}},-\lambda) = \xi_{-\lambda}\overline{\sigma}^{0\dot\alpha\beta}\,y_\beta(\vec{\boldsymbol{p}},\lambda) = 2\lambda\,\omega_\lambda\xi_{-\lambda}\,\chi_{-\lambda}(\hat{\boldsymbol{p}})\,, \tag{C.3.19}$$

$$y^{\dagger\dot\alpha}(-\vec{\boldsymbol{p}},-\lambda) = \xi_\lambda\overline{\sigma}^{0\dot\alpha\beta}\,x_\beta(\vec{\boldsymbol{p}},\lambda) = \omega_{-\lambda}\xi_\lambda\,\chi_\lambda(\hat{\boldsymbol{p}})\,, \tag{C.3.20}$$

where $\omega_{\pm\lambda} \equiv \omega_{\pm\lambda}(\vec{\boldsymbol{p}})$ and $\xi_\lambda \equiv \xi_\lambda(\hat{\boldsymbol{p}})$. Raising [lowering] the undotted [dotted] index yields:

$$x^\alpha(-\vec{\boldsymbol{p}},-\lambda) = -y^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},\lambda)\,\xi_\lambda\overline{\sigma}^{0\dot\beta\alpha} = -2\lambda\,\omega_\lambda\xi_\lambda\chi^\dagger_{-\lambda}(\hat{\boldsymbol{p}})\,, \tag{C.3.21}$$

$$y^\alpha(-\vec{\boldsymbol{p}},-\lambda) = -x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},\lambda)\,\xi_{-\lambda}\overline{\sigma}^{0\dot\beta\alpha} = -\omega_{-\lambda}\xi_{-\lambda}\,\chi^\dagger_\lambda(\hat{\boldsymbol{p}})\,, \tag{C.3.22}$$

$$x^\dagger_{\dot\alpha}(-\vec{\boldsymbol{p}},-\lambda) = -y^\beta(\vec{\boldsymbol{p}},\lambda)\,\xi_{-\lambda}\sigma^0_{\beta\dot\alpha} = -\omega_\lambda\xi_{-\lambda}\,\chi^\dagger_\lambda(\hat{\boldsymbol{p}})\,, \tag{C.3.23}$$

$$y^\dagger_{\dot\alpha}(-\vec{\boldsymbol{p}},-\lambda) = -x^\beta(\vec{\boldsymbol{p}},\lambda)\,\xi_\lambda\sigma^0_{\beta\dot\alpha} = 2\lambda\,\omega_{-\lambda}\xi_\lambda\,\chi^\dagger_{-\lambda}(\hat{\boldsymbol{p}})\,. \tag{C.3.24}$$

Eqs. (C.3.17)–(C.3.24) can also be obtained directly from eqs. (C.3.3)–(C.3.6).

[105] A slightly modified procedure (not adopted in this review) is to take the azimuthal angle of $-\hat{\boldsymbol{p}}$ to be $\phi_p \pm \pi$, where the $\pm$ sign is chosen according to which of the two conditions $0 \leq \phi_p \pm \pi < 2\pi$ is true. This procedure would yield an extra minus sign in the definition of $\xi_\lambda(\hat{\boldsymbol{p}})$ when $\pi \leq \phi_p < 2\pi$. In this convention, two successive inversions are equivalent to the identity rotation so that $\chi_\lambda(-(-\hat{\boldsymbol{p}})) = \chi_\lambda(\hat{\boldsymbol{p}})$.

29.7.4 Matrix decompositions for mass matrix diagonalization

In scalar field theory, the diagonalization of the scalar squared-mass matrix M^2 is straightforward. For a theory of n complex scalar fields, M^2 is an hermitian $n \times n$ matrix, which can be diagonalized by a unitary matrix W:

$$W^\dagger M^2 W = m^2 = \text{diag}(m_1^2, m_2^2, ..., m_n^2)\,. \tag{D.1}$$

For a theory of n real scalar fields, M^2 is a real symmetric $n \times n$ matrix, which can be diagonalized by an orthogonal matrix Q:

$$Q^\mathsf{T} M^2 Q = m^2 = \text{diag}(m_1^2, m_2^2, ..., m_n^2)\,. \tag{D.2}$$

In both cases, the eigenvalues m_k^2 of M^2 are real. These are the standard matrix diagonalization problems that are treated in all elementary linear algebra textbooks.

In spin-1/2 fermion field theory, the most general fermion mass matrix, obtained from the Lagrangian, written in terms of two-component spinors, is complex and symmetric [cf. Section 29.3.2]. If the Lagrangian exhibits a U(1) symmetry, then a basis can be found such that fields that are charged under the U(1) pair up into Dirac fermions. The fermion mass matrix then decomposes into the direct sum of a complex Dirac fermion mass matrix and a complex symmetric neutral fermion mass matrix. In this Appendix, we review the linear algebra theory relevant for the matrix decompositions associated with the general charged and neutral spin-1/2 fermion mass matrix diagonalizations. The diagonalization of the Dirac fermion mass matrix is governed by the singular value decomposition of a complex matrix, as shown in Appendix D.1 . In contrast, the diagonalization of a neutral fermion mass matrix is governed by the Takagi diagonalization of a complex symmetric matrix, which is treated in Appendix D.2 .[106] These two techniques are compared and contrasted in Appendix D.3 . Dirac fermions can also arise in the case of a pseudo-real representation of fermion fields. As shown in Section 29.3.2, this latter case requires the reduction of a complex antisymmetric fermion mass matrix to real normal form. The relevant theorem and its proof are given in Appendix D.4 .

Singular value decomposition

The diagonalization of the charged (Dirac) fermion mass matrix requires the singular value decomposition of an arbitrary complex matrix M.

Theorem: For any complex [or real] $n \times n$ matrix M, unitary [or real orthogonal] matrices L and R exist such that

$$L^\mathsf{T} M R = M_D = \text{diag}(m_1, m_2, ..., m_n), \tag{D.1.1}$$

where the m_k are real and non-negative. This is called the singular value decomposition of the matrix M (e.g., see refs. [149, 261]).

In general, the m_k are *not* the eigenvalues of M. Rather, the m_k are the *singular values* of the general complex matrix M, which are defined to be the non-negative square roots of the eigenvalues of $M^\dagger M$ (or equivalently of $MM^\dagger$). An equivalent definition of the singular values can be established as follows. Since $M^\dagger M$ is an hermitian non-negative matrix, its eigenvalues are real and non-negative and its eigenvectors, v_k, defined by $M^\dagger M v_k = m_k^2 v_k$, can be chosen to be orthonormal.[107] Consider first the eigenvectors corresponding to the non-zero eigenvalues of $M^\dagger M$. Then, we define the vectors w_k such that

$$M v_k = m_k w_k^*\,. \tag{D.1.2}$$

[106] One may choose not to work in a basis where the fermion fields are eigenstates of the U(1) charge operator. In this case, all fermions are governed by a complex symmetric mass matrix, which can be Takagi-diagonalized according to the procedure described in Appendix D.2 .

[107] We define the inner product of two vectors to be $\langle v|w\rangle \equiv v^\dagger w$. Then, v and w are orthonormal if $\langle v|w\rangle = 0$. The norm of a vector is defined by $\|v\| = \langle v|v\rangle^{1/2}$.

It follows that $m_k^2 v_k = M^\dagger M v_k = m_k M^\dagger w_k^*$, which yields: $M^\dagger w_k^* = m_k v_k$. Note that these equations also imply that $MM^\dagger w_k^* = m_k^2 w_k^*$. The orthonormality of the v_k implies the orthonormality of the w_k, and vice versa. For example,

$$\delta_{jk} = \langle v_j | v_k \rangle = \frac{1}{m_j m_k} \langle M^\dagger w_j^* | M^\dagger w_k^* \rangle = \frac{1}{m_j m_k} \langle w_j | M M^\dagger w_k^* \rangle = \frac{m_k}{m_j} \langle w_j^* | w_k^* \rangle \,, \tag{D.1.3}$$

which yields $\langle w_k | w_j \rangle = \delta_{jk}$. If M is a real matrix, then the eigenvectors v_k can be chosen to be real, in which case the corresponding w_k are also real.

If v_i is an eigenvector of $M^\dagger M$ with zero eigenvalue, then $0 = v_i^\dagger M^\dagger M v_i = \langle M v_i | M v_i \rangle$, which implies that $M v_i = 0$. Likewise, if w_i^* is an eigenvector of $MM^\dagger$ with zero eigenvalue, then $0 = w_i^{\mathsf{T}} M M^\dagger w_i^* = \langle M^{\mathsf{T}} w_i | M^{\mathsf{T}} w_i \rangle^*$, which implies that $M^{\mathsf{T}} w_i = 0$. Because the eigenvectors of $M^\dagger M$ [$MM^\dagger$] can be chosen orthonormal, the eigenvectors corresponding to the zero eigenvalues of M [$M^\dagger$] can be taken to be orthonormal.[108] Finally, these eigenvectors are also orthogonal to the eigenvectors corresponding to the non-zero eigenvalues of $M^\dagger M$ [$MM^\dagger$]. That is, if the indices i and j run over the eigenvectors corresponding to the zero and non-zero eigenvalues of $M^\dagger M$ [$MM^\dagger$], respectively, then

$$\langle v_j | v_i \rangle = \frac{1}{m_j} \langle M^\dagger w_j^* | v_i \rangle = \frac{1}{m_j} \langle w_j^* | M v_i \rangle = 0 \,, \tag{D.1.4}$$

and similarly $\langle w_j | w_i \rangle = 0$.

Thus, we can define the singular values of a general complex $n \times n$ matrix M to be the simultaneous solutions (with real non-negative m_k) of:[109]

$$M v_k = m_k w_k^* \,, \qquad\qquad w_k^{\mathsf{T}} M = m_k v_k^\dagger \,. \tag{D.1.5}$$

The corresponding v_k (w_k), normalized to have unit norm, are called the right (left) singular vectors of M. In particular, the number of linearly independent v_k coincides with the number of linearly independent w_k and is equal to n.

Proof of the singular value decomposition theorem: Eqs. (D.1.3) and (D.1.4) imply that the right [left] singular vectors can be chosen to be orthonormal. Consequently, the unitary matrix R [L] can be constructed such that its kth column is given by the right [left] singular vector v_k [w_k]. It then follows from eq. (D.1.5) that:

$$w_k^{\mathsf{T}} M v_\ell = m_k \delta_{k\ell} \,, \qquad\qquad \text{(no sum over } k\text{)}. \tag{D.1.6}$$

In matrix form, eq. (D.1.6) coincides with eq. (D.1.1), and the singular value decomposition is established. If M is real, then the right and left singular vectors, v_k and w_k, can be chosen to be real, in which case eq. (D.1.1) holds for real orthogonal matrices L and R.

The singular values of a complex matrix M are unique (up to ordering), as they correspond to the eigenvalues of $M^\dagger M$ (or equivalently the eigenvalues of $MM^\dagger$). The unitary matrices L and R are not unique. The matrix R must satisfy

$$R^\dagger M^\dagger M R = M_D^2 \,, \tag{D.1.7}$$

which follows directly from eq. (D.1.1) by computing $M_D^\dagger M_D = M_D^2$. That is, R is a unitary matrix that diagonalizes the non-negative definite matrix $M^\dagger M$. Since the eigenvectors of $M^\dagger M$ are orthonormal, each v_k corresponding to a non-degenerate eigenvalue of $M^\dagger M$ can be multiplied by

[108] This analysis shows that the number of linearly independent eigenvectors of $M^\dagger M$ [$MM^\dagger$] with zero eigenvalue coincides with the number of linearly independent eigenvectors of M [$M^\dagger$] with zero eigenvalue.

[109] One can always find a solution to eq. (D.1.5) such that the m_k are real and non-negative. Given a solution where m_k is complex, we simply write $m_k = |m_k| e^{i\theta}$ and redefine $w_k \to w_k e^{i\theta}$ to remove the phase θ.

an arbitrary phase $e^{i\theta_k}$. For the case of degenerate eigenvalues, any orthonormal linear combination of the corresponding eigenvectors is also an eigenvector of $M^\dagger M$. It follows that within the subspace spanned by the eigenvectors corresponding to non-degenerate eigenvalues, R is uniquely determined up to multiplication on the right by an arbitrary diagonal unitary matrix. Within the subspace spanned by the eigenvectors of $M^\dagger M$ corresponding to a degenerate eigenvalue, R is determined up to multiplication on the right by an arbitrary unitary matrix.

Once R is fixed, L is obtained from eq. (D.1.1):

$$L = (M^{\mathsf{T}})^{-1} R^* M_D \,. \tag{D.1.8}$$

However, if some of the diagonal elements of M_D are zero, then L is not uniquely defined. Writing M_D in 2×2 block form such that the upper left block is a diagonal matrix with positive diagonal elements and the other three blocks are equal to the zero matrix of the appropriate dimensions, it follows that, $M_D = M_D W$, where

$$W = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & W_0 \end{pmatrix} , \tag{D.1.9}$$

W_0 is an arbitrary unitary matrix whose dimension is equal to the number of zeros that appear in the diagonal elements of M_D, and $\mathbb{1}$ and $\mathbb{0}$ are respectively the identity matrix and zero matrix of the appropriate size. Hence, we can multiply both sides of eq. (D.1.8) on the right by W, which means that L is only determined up to multiplication on the right by an arbitrary unitary matrix whose form is given by eq. (D.1.9).[110]

If M is a real matrix, then the singular value decomposition of M is given by Eq. (D.1.1), where L and R are real orthogonal matrices. This result is easily established by replacing "phase" with "sign" and replacing "unitary" by "real orthogonal" in the above proof.

Takagi diagonalization

The mass matrix of neutral fermions (or a system of two-component fermions in a generic basis) is complex and symmetric. This mass matrix must be diagonalized in order to identify the physical fermion mass eigenstates and to compute their masses. However, the fermion mass matrix is *not* diagonalized by the standard unitary similarity transformation. Instead a different diagonalization equation is employed that was discovered by Takagi [112], and rediscovered many times since [149].[111]

Theorem: For any complex symmetric $n \times n$ matrix M, there exists a unitary matrix Ω such that:

$$\Omega^{\mathsf{T}} M \, \Omega = M_D = \mathrm{diag}(m_1, m_2, ..., m_n) \,, \tag{D.2.1}$$

where the m_k are real and non–negative. This is the Takagi diagonalization[112] of the complex symmetric matrix M.

[110] Of course, one can reverse the above procedure by first determining the unitary matrix L. Eq. (D.1.1) implies that $L^{\mathsf{T}} M M^\dagger L^* = M_D^2$, in which case L is determined up to multiplication on the right by an arbitrary [diagonal] unitary matrix within the subspace spanned by the eigenvectors corresponding to the degenerate [non-degenerate] eigenvalues of $MM^\dagger$. Having fixed L, one can obtain $R = M^{-1} L^* M_D$ from eq. (D.1.1). As above, R is only determined up to multiplication on the right by a unitary matrix whose form is given by eq. (D.1.9).

[111] Subsequently, it was recognized in Ref. [261] that the Takagi diagonalization was first established for nonsingular complex symmetric matrices by Autonne [262]. In the physics literature, the first proof of eq. (D.2.1) was given in [151]. Applications of Takagi diagonalization to the study of neutrino mass matrices can be found in refs. [5, 263].

[112] In Ref. [149], eq. (D.2.1) is called the Takagi factorization of a complex symmetric matrix. We choose to refer to this as Takagi *diagonalization* to emphasize and contrast this with the more standard diagonalization of normal matrices by a unitary similarity transformation. In particular, not all *complex* symmetric matrices are diagonalizable by a similarity transformation, whereas complex symmetric matrices are *always* Takagi-diagonalizable.

In general, the m_k are *not* the eigenvalues of M. Rather, the m_k are the singular values of the symmetric matrix M. From eq. (D.2.1) it follows that:

$$\Omega^\dagger M^\dagger M \Omega = M_D^2 = \mathrm{diag}(m_1^2, m_2^2, ..., m_n^2)\,. \tag{D.2.2}$$

If all of the singular values m_k are non-degenerate, then one can find a solution to eq. (D.2.1) for Ω from eq. (D.2.2). This is no longer true if some of the singular values are degenerate. For example, if $M = \left(\begin{smallmatrix} 0 & m \\ m & 0 \end{smallmatrix}\right)$, then the singular value $|m|$ is doubly–degenerate, but eq. (D.2.2) yields $\Omega^\dagger\Omega = \mathbb{1}_{2\times 2}$, which does not specify Ω. That is, in the degenerate case, the physical fermion states *cannot* be determined by the diagonalization of $M^\dagger M$. Instead, one must make direct use of eq. (D.2.1). Below, we shall present a constructive method for determining Ω that is applicable in both the non-degenerate and the degenerate cases.

Eq. (D.2.1) can be rewritten as $M\Omega = \Omega^* M_D$, where the columns of Ω are orthonormal. If we denote the kth column of Ω by v_k, then,

$$Mv_k = m_k v_k^*\,, \tag{D.2.3}$$

where the m_k are the singular values and the vectors v_k are normalized to have unit norm. Following Ref. [264], the v_k are called the *Takagi vectors* of the complex symmetric $n \times n$ matrix M. The Takagi vectors corresponding to non–degenerate non–zero [zero] singular values are unique up to an overall sign [phase]. Any orthogonal [unitary] linear combination of Takagi vectors corresponding to a set of degenerate non–zero [zero] singular values is also a Takagi vector corresponding to the same singular value. Using these results, one can determine the degree of non–uniqueness of the matrix Ω. For definiteness, we fix an ordering of the diagonal elements of M_D.[113] If the singular values of M are distinct, then the matrix Ω is uniquely determined up to multiplication by a diagonal matrix whose entries are either ± 1 (i.e., a diagonal orthogonal matrix). If there are degeneracies corresponding to non–zero singular values, then within the degenerate subspace, Ω is unique up to multiplication on the right by an arbitrary orthogonal matrix. Finally, in the subspace corresponding to zero singular values, Ω is unique up to multiplication on the right by an arbitrary unitary matrix.

For a real symmetric matrix M, the Takagi diagonalization [eq. (D.2.1)] still holds for a unitary matrix Ω, which is easily determined as follows. Any real symmetric matrix M can be diagonalized by a real orthogonal matrix Z,

$$Z^{\mathsf{T}} M Z = \mathrm{diag}(\varepsilon_1 m_1\,,\, \varepsilon_2 m_2\,,\, \ldots\,,\, \varepsilon_n m_n)\,, \tag{D.2.4}$$

where the m_k are real and non-negative and the $\varepsilon_k m_k$ are the real eigenvalues of M with corresponding signs $\varepsilon_k = \pm 1$.[114] Then, the Takagi diagonalization of M is achieved by taking:

$$\Omega_{ij} = \varepsilon_j^{1/2} Z_{ij}\,, \qquad \text{no sum over } j\,. \tag{D.2.5}$$

Proof of the Takagi diagonalization. To prove the existence of the Takagi diagonalization of a complex symmetric matrix, it is sufficient to provide an algorithm for constructing the orthonormal Takagi vectors v_k that make up the columns of Ω. This is achieved by rewriting the $n\times n$ complex matrix equation $Mv = mv^*$ [with m real and non–negative] as a $2n\times 2n$ real matrix equation [265, 266]:

$$M_R \begin{pmatrix} \mathrm{Re}\, v \\ \mathrm{Im}\, v \end{pmatrix} \equiv \begin{pmatrix} \mathrm{Re}\, M & -\mathrm{Im}\, M \\ -\mathrm{Im}\, M & -\mathrm{Re}\, M \end{pmatrix} \begin{pmatrix} \mathrm{Re}\, v \\ \mathrm{Im}\, v \end{pmatrix} = m \begin{pmatrix} \mathrm{Re}\, v \\ \mathrm{Im}\, v \end{pmatrix}\,, \quad \text{where } m \geq 0\,. \tag{D.2.6}$$

Since $M = M^{\mathsf{T}}$, the $2n \times 2n$ matrix $M_R \equiv \left(\begin{smallmatrix} \mathrm{Re}\, M & -\mathrm{Im}\, M \\ -\mathrm{Im}\, M & -\mathrm{Re}\, M \end{smallmatrix}\right)$ is a real symmetric matrix.[115] In particular, M_R is diagonalizable by a real orthogonal similarity transformation, and its eigenvalues

[113] Permuting the order of the singular values is equivalent to permuting the order of the columns of Ω.

[114] In the case of $m_k = 0$, we conventionally choose the corresponding $\varepsilon_k = +1$.

[115] The $2n \times 2n$ matrix M_R is a real representation of the $n \times n$ complex matrix M.

are real. Moreover, if m is an eigenvalue of M_R with eigenvector $(\,\mathrm{Re}\,v\,,\ \mathrm{Im}\,v)$, then $-m$ is an eigenvalue of M_R with (orthogonal) eigenvector $(-\,\mathrm{Im}\,v\,,\ \mathrm{Re}\,v)$. This observation implies that M_R has an equal number of positive and negative eigenvalues and an even number of zero eigenvalues.[116] Thus, eq. (D.2.3) has been converted into an ordinary eigenvalue problem for a real symmetric matrix. Since $m \geq 0$, we solve the eigenvalue problem $M_R u = mu$ for the real eigenvectors $u \equiv (\,\mathrm{Re}\,v\,,\ \mathrm{Im}\,v)$ corresponding to the non–negative eigenvalues of M_R,[117] which then immediately yields the complex Takagi vectors, v. It is straightforward to prove that the total number of linearly independent Takagi vectors is equal to n. Simply note that the orthogonality of $(\,\mathrm{Re}\,v_1\,,\ \mathrm{Im}\,v_1)$ and $(-\,\mathrm{Im}\,v_1\,,\ \mathrm{Re}\,v_1)$ with $(\,\mathrm{Re}\,v_2\,,\ \mathrm{Im}\,v_2)$ implies that $v_1^\dagger v_2 = 0$.

Thus, we have derived a constructive method for obtaining the Takagi vectors v_k. If there are degeneracies, one can always choose the v_k in the degenerate subspace to be orthonormal. The Takagi vectors then make up the columns of the matrix Ω in eq. (D.2.1). A numerical package for performing the Takagi diagonalization of a complex symmetric matrix has recently been presented in [267] (see also refs. [264, 268] for previous numerical approaches to Takagi diagonalization).

Relation between Takagi diagonalization and singular value decomposition

The Takagi diagonalization is a special case of the singular value decomposition. If the complex matrix M in eq. (D.1.1) is symmetric, $M = M^{\mathsf{T}}$, then the Takagi diagonalization corresponds to $\Omega = L = R$. In this case, the right and left singular vectors coincide ($v_k = w_k$) and are identified with the Takagi vectors defined in eq. (D.2.3). However as previously noted, the matrix Ω cannot be determined from eq. (D.2.2) in cases where there is a degeneracy among the singular values.[118] For example, one possible singular value decomposition of the matrix $M = \left(\begin{smallmatrix} 0 & m \\ m & 0 \end{smallmatrix}\right)$ [with m assumed real and positive] can be obtained by choosing $R = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $L = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, in which case $L^{\mathsf{T}} M R = \left(\begin{smallmatrix} m & 0 \\ 0 & m \end{smallmatrix}\right) = M_D$. Of course, this is not a Takagi diagonalization because $L \neq R$. Since R is only defined modulo the multiplication on the right by an arbitrary 2×2 unitary matrix $\mathcal{O}$, then at least one singular value decomposition exists that is also a Takagi diagonalization. For the example under consideration, it is not difficult to deduce the Takagi diagonalization: $\Omega^{\mathsf{T}} M \Omega = M_D$, where

$$\Omega = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \mathcal{O}\,, \tag{D.3.1}$$

and $\mathcal{O}$ is any 2×2 orthogonal matrix.

Since the Takagi diagonalization is a special case of the singular value decomposition, it seems plausible that one can prove the former from the latter. This turns out to be correct; for completeness, we provide the proof below. Our second proof depends on the following lemma:

Lemma: For any symmetric unitary matrix V, there exists a unitary matrix U such that $V = U^{\mathsf{T}} U$.

Proof of the Lemma: For any $n \times n$ unitary matrix V, there exists an hermitian matrix H such that $V = \exp(iH)$ (this is the polar decomposition of V). If $V = V^{\mathsf{T}}$ then $H = H^{\mathsf{T}} = H^*$ (since H is hermitian); therefore H is real symmetric. But, any real symmetric matrix can be diagonalized by an orthogonal transformation. It follows that V can also be diagonalized by an orthogonal transformation. Since the eigenvalues of any unitary matrix are pure phases, there exists a real orthogonal matrix Q such that $Q^{\mathsf{T}} V Q = \mathrm{diag}\,(\mathrm{e}^{\mathrm{i}\theta_1}\,,\ \mathrm{e}^{\mathrm{i}\theta_2}\,,\ \ldots\,,\ \mathrm{e}^{\mathrm{i}\theta_n})$. Thus, the unitary

[116] Note that $(-\,\mathrm{Im}\,v\,,\ \mathrm{Re}\,v)$ corresponds to replacing v_k in eq. (D.2.3) by iv_k. However, for $m < 0$ these solutions are not relevant for Takagi diagonalization (where the m_k are by definition non–negative). The case of $m = 0$ is considered in footnote 117.

[117] For $m = 0$, the corresponding vectors $(\,\mathrm{Re}\,v\,,\ \mathrm{Im}\,v)$ and $(-\,\mathrm{Im}\,v\,,\ \mathrm{Re}\,v)$ are two linearly independent eigenvectors of M_R; but these yield only one independent Takagi vector v (since v and iv are linearly dependent).

[118] This is in contrast to the singular value decomposition, where R can be determined from eq. (D.1.7) modulo right multiplication by a [diagonal] unitary matrix in the [non-]degenerate subspace and L is then determined by eq. (D.1.8) modulo multiplication on the right by eq. (D.1.9).

matrix,

$$U = \text{diag}\,(e^{i\theta_1/2}\,,\, e^{i\theta_2/2}\,,\, \ldots\,,\, e^{i\theta_n/2})\,Q^{\mathsf{T}}\,, \tag{D.3.2}$$

satisfies $V = U^{\mathsf{T}}U$ and the lemma is proved. Note that U is unique modulo multiplication on the left by an arbitrary real orthogonal matrix.

Second Proof of the Takagi diagonalization. Starting from the singular value decomposition of M, there exist unitary matrices L and R such that $M = L^*M_DR^\dagger$, where M_D is the diagonal matrix of singular values. Since $M = M^{\mathsf{T}} = R^*M_DL^\dagger$, we have two different singular value decompositions for M. However, as noted below eq. (D.1.7), R is unique modulo multiplication on the right by an arbitrary [diagonal] unitary matrix, V, within the [non-]degenerate subspace. Thus, it follows that a [diagonal] unitary matrix V exists such that $L = RV$. Moreover, $V = V^{\mathsf{T}}$. This is manifestly true within the non-degenerate subspace where V is diagonal. Within the degenerate subspace, M_D is proportional to the identity matrix so that $L^*R^\dagger = R^*L^\dagger$. Inserting $L = RV$ then yields $V^{\mathsf{T}} = V$. Using the Lemma proved above, there exists a unitary matrix U such that $V = U^{\mathsf{T}}U$. That is,

$$L = RU^{\mathsf{T}}U\,, \tag{D.3.3}$$

for some unitary matrix U. Moreover, it is now straightforward to show that

$$M_DU^* = U^*M_D\,. \tag{D.3.4}$$

To see this, note that within the degenerate subspace, eq. (D.3.4) is trivially true since M_D is proportional to the identity matrix. Within the non-degenerate subspace V is diagonal; hence we may choose $U = U^{\mathsf{T}} = V^{1/2}$, so that eq. (D.3.4) is true since diagonal matrices commute. Using eqs. (D.3.3) and (D.3.4), we can write the singular value decomposition of M as follows

$$M = L^*M_DR^\dagger = R^*U^\dagger U^*M_DR^\dagger = (RU^{\mathsf{T}})^*M_DU^*R^\dagger = \Omega^*M_D\Omega^\dagger\,, \tag{D.3.5}$$

where $\Omega \equiv RU^{\mathsf{T}}$ is a unitary matrix. Thus the existence of the Takagi diagonalization of an arbitrary complex symmetric matrix [eq. (D.2.1)] is once again proved.

In the diagonalization of the two-component fermion mass matrix, M, the eigenvalues of $M^\dagger M$ typically fall into two classes—non-degenerate eigenvalues corresponding to neutral fermion mass eigenstates and degenerate pairs corresponding to charged (Dirac) mass eigenstates. In this case, the sector of the neutral fermions corresponds to a non-degenerate subspace of the space of fermion fields. Hence, in order to identify the neutral fermion mass eigenstates, it is sufficient to diagonalize $M^\dagger M$ with a unitary matrix R [as in eq. (D.1.7)], and then adjust the overall phase of each column of R so that the resulting matrix Ω satisfies $\Omega^{\mathsf{T}}M\Omega = M_D$, where M_D is a diagonal matrix of the non-negative fermion masses. This last result is a consequence of eqs. (D.3.3)–(D.3.5), where $\Omega = RV^{1/2}$ and V is a diagonal matrix of phases.

Reduction of a complex antisymmetric matrix to real normal form

In the case of two-component fermions that transform under a pseudo-real representation of a compact Lie group [cf. eq. (29.3.35)], the corresponding mass matrix is in general complex and antisymmetric. In this case, one needs the antisymmetric analogue of the Takagi diagonalization of a complex symmetric matrix [149].

Theorem: For any complex [or real] antisymmetric $n \times n$ matrix M, there exists a unitary [or real orthogonal] matrix U such that:

$$U^{\mathsf{T}}MU = N \equiv \text{diag}\left\{\begin{pmatrix} 0 & m_1 \\ -m_1 & 0 \end{pmatrix}\,,\, \begin{pmatrix} 0 & m_2 \\ -m_2 & 0 \end{pmatrix}\,,\, \cdots\,,\, \begin{pmatrix} 0 & m_p \\ -m_p & 0 \end{pmatrix}\,,\, \mathbb{O}_{n-2p}\right\}\,, \tag{D.4.1}$$

where N is written in block diagonal form with 2×2 matrices appearing along the diagonal, followed by an $(n-2p) \times (n-2p)$ block of zeros (denoted by $\mathbb{O}_{n-2p}$), and the m_j are real and positive. N is called the *real normal form* of an antisymmetric matrix [151, 269, 270].

Proof: A number of proofs can be found in the literature [151, 269, 150, 270, 271]. Here we provide a proof inspired by [269]. Following Appendix D.1 , we first consider the eigenvalue equation for $M^\dagger M$:

$$M^\dagger M v_k = m_k^2 v_k\,, \qquad m_k > 0\,, \qquad \text{and} \qquad M^\dagger M u_k = 0\,, \tag{D.4.2}$$

where we have distinguished the eigenvectors corresponding to positive eigenvalues and zero eigenvalues, respectively. The quantities m_k are the positive singular values of M. Noting that $u_k^\dagger M^\dagger M u_k = \langle M u_k \,|\, M u_k\rangle = 0$, it follows that

$$M u_k = 0\,, \tag{D.4.3}$$

so that the u_k are the eigenvectors corresponding to the zero eigenvalues of M. For each eigenvector of $M^\dagger M$ with $m_k \neq 0$, we define a new vector

$$w_k \equiv \frac{1}{m_k} M^* v_k^*\,. \tag{D.4.4}$$

It follows that $m_k^2 v_k = M^\dagger M v_k = m_k M^\dagger w_k^*$, which yields $M^\dagger w_k^* = m_k v_k$. Comparing with eq. (D.1.5), we identify v_k and w_k as the right and left singular vectors, respectively, corresponding to the non-zero singular values of M. For any antisymmetric matrix, $M^\dagger = -M^*$. Hence,

$$M v_k = m_k w_k^*\,, \qquad\qquad M w_k = -m_k v_k^*\,, \tag{D.4.5}$$

and

$$M^\dagger M w_k = -m_k M^\dagger v_k^* = m_k M^* v_k^* = m_k^2 w_k\,, \qquad m_k > 0\,. \tag{D.4.6}$$

That is, the w_k are also eigenvectors of $M^\dagger M$.

The key observation is that for fixed k the vectors v_k and w_k are orthogonal, since eq. (D.4.5) implies that:

$$\langle w_k | v_k\rangle = \langle v_k | w_k\rangle^* = -\frac{1}{m_k^2}\langle M w_k | M v_k\rangle = -\frac{1}{m_k^2}\langle w_k | M^\dagger M v_k\rangle = -\langle w_k | v_k\rangle\,, \tag{D.4.7}$$

which yields $\langle w_k | v_k\rangle = 0$. Thus, if all the m_k are distinct, it follows that m_k^2 is a doubly degenerate eigenvalue of $M^\dagger M$, with corresponding linearly independent eigenvectors v_k and w_k, where $k = 1, 2, ..., p$ (and $p \leq \frac{1}{2}n$). The remaining zero eigenvalues are $(n-2p)$-fold degenerate, with corresponding eigenvectors u_k (for $k = 1, 2, ..., n-2p$). If some of the m_k are degenerate, these conclusions still apply. For example, suppose that $m_j = m_k$ for $j \neq k$, which means that m_k^2 is at least a three-fold degenerate eigenvalue of $M^\dagger M$. Then, there must exist an eigenvector v_j that is orthogonal to v_k and w_k such that $M^\dagger M v_j = m_k^2 v_j$. We now construct $w_j \equiv M^* v_j^*/m_k$ according to eq. (D.4.4). According to eq. (D.4.7), w_j is orthogonal to v_j. However, we still must show that w_j is also orthogonal to v_k and w_k. But this is straightforward:

$$\langle w_j | w_k\rangle = \langle w_k | w_j\rangle^* = \frac{1}{m_k^2}\langle M v_k | M v_j\rangle = \frac{1}{m_k^2}\langle v_k | M^\dagger M v_j\rangle = \langle v_k | v_j\rangle = 0\,, \tag{D.4.8}$$

$$\langle w_j | v_k\rangle = \langle v_k | w_j\rangle^* = -\frac{1}{m_k^2}\langle M w_k | M v_j\rangle = -\frac{1}{m_k^2}\langle w_k | M^\dagger M v_j\rangle = -\langle w_k | v_j\rangle = 0\,, \tag{D.4.9}$$

where we have used the assumed orthogonality of v_j with v_k and w_k, respectively. It follows that v_j, w_j, v_k and w_k are linearly independent eigenvectors corresponding to a four-fold degenerate eigenvalue m_k^2 of $M^\dagger M$. Additional degeneracies are treated in the same way.

Thus, the number of non-zero eigenvalues of $M^\dagger M$ must be an even number, denoted by $2p$ above. Moreover, one can always choose the complete set of eigenvectors $\{u_k\,,\, v_k\,,\, w_k\}$ of $M^\dagger M$

to be orthonormal. These orthonormal vectors can be used to construct a unitary matrix U with matrix elements:

$$\begin{aligned} &U_{\ell\,,2k-1} = (w_k)_\ell\,, \qquad U_{\ell\,,2k} = (v_k)_\ell\,, \qquad\qquad k = 1\,,2\,,\ldots\,,p\,,\\ &U_{\ell\,,k+2p} = (u_k)_\ell\,, \qquad\qquad\qquad\qquad\quad k = 1\,,2\,,\ldots\,,n-2p\,, \end{aligned} \tag{D.4.10}$$

for $\ell = 1\,,2\,,\ldots\,,n$, where e.g., $(v_k)_\ell$ is the ℓth component of the vector v_k with respect to the standard orthonormal basis. The orthonormality of $\{u_k\,,\,v_k\,,\,w_k\}$ implies that $(U^\dagger U)_{\ell k} = \delta_{\ell k}$ as required. Eqs. (D.4.3) and (D.4.5) are thus equivalent to the matrix equation $MU = U^* N$, which immediately yields eq. (D.4.1), and the theorem is proven. If M is a real antisymmetric matrix, then all the eigenvectors of $M^\dagger M$ can be chosen to be real, in which case U is a real orthogonal matrix.

Finally, we address the non-uniqueness of the matrix U. For definiteness, we fix an ordering of the 2×2 blocks containing the m_k in the matrix N. In the subspace corresponding to a non-zero singular value of degeneracy d, U is unique up to multiplication on the right by a $2d\times 2d$ unitary matrix S that satisfies:

$$S^{\mathsf{T}} J S = J\,, \tag{D.4.11}$$

where the $2d\times 2d$ matrix J, defined by

$$J = \mathrm{diag}\left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}, \tag{D.4.12}$$

is a block diagonal matrix with d blocks of 2×2 matrices. A unitary matrix S that satisfies eq. (D.4.11) is an element of the unitary symplectic group, $\mathrm{Sp}(d)$. If there are no degeneracies among the m_k, then $d = 1$. Identifying $\mathrm{Sp}(1)\cong \mathrm{SU}(2)$, it follows that within the subspace corresponding to a non-degenerate singular value, U is unique up to multiplication on the right by an arbitrary SU(2) matrix. Finally, in the subspace corresponding to the zero eigenvalues of M, U is unique up to multiplication on the right by an arbitrary unitary matrix.

29.7.5 Lie group theoretical techniques for gauge theories

Basic facts about Lie groups, Lie algebras and their representations

Consider a compact connected Lie Group G [272]. The most general form for G is a direct product of compact simple groups and U(1) groups. If no U(1) factors are present, then G is semisimple. For any $U\in G$,

$$U = \exp(-i\theta^a \boldsymbol{T^a})\,, \tag{E.1.1}$$

where the $\boldsymbol{T^a}$ are called the generators of G, and the θ^a are real numbers that parameterize the elements of G. The corresponding real Lie algebra $\mathfrak{g}$ consists of arbitrary real linear combinations of the generators, $\theta^a \boldsymbol{T^a}$. The Lie group generators $\boldsymbol{T^a}$ satisfy the commutation relations:

$$[\boldsymbol{T^a}, \boldsymbol{T^b}] = i f_c^{ab} \boldsymbol{T^c}\,, \tag{E.1.2}$$

where the real structure constants f_c^{ab} define the compact Lie algebra. The generator indices run over a, b, $c = 1, 2, ..., d_G$, where d_G is the dimension of the Lie algebra. For compact Lie algebras, the Killing form $g^{ab} = \mathrm{Tr}(\boldsymbol{T^a T^b})$ is positive definite, so one can always choose a basis for the Lie algebra in which $g^{ab} \propto \delta^{ab}$ (where the proportionality constant is a positive real number). With respect to this new basis, the structure constants $f^{abc} \equiv g^{ad} f_d^{bc}$ are totally antisymmetric with respect to the interchange of the indices a, b and c. Henceforth, we shall always assume that such a *preferred* basis of generators has been chosen.

The elements of the compact Lie group G act on a multiplet of fields that transform under some d_R-dimensional representation R of G. The group elements $U\in G$ are represented by $d_R\times d_R$

unitary matrices, $D_R(U) = \exp(-i\theta^a \boldsymbol{T}_{\boldsymbol{R}}^{\boldsymbol{a}})$, where the $\boldsymbol{T}_{\boldsymbol{R}}^{\boldsymbol{a}}$ are $d_R \times d_R$ hermitian matrices that satisfy eq. (E.1.2) and thus provide a representation of the Lie group generators. For any representation R of a semisimple group, $\mathrm{Tr}\, \boldsymbol{T}_{\boldsymbol{R}}^{\boldsymbol{a}} = 0$ for all a. A representation R' is unitarily equivalent to R if there exists a fixed unitary matrix S such that $D_{R'}(U) = S^{-1} D_R(U) S$ for all $U \in G$. Similarly, the corresponding generators satisfy $\boldsymbol{T}_{\boldsymbol{R'}}^{\boldsymbol{a}} = S^{-1} \boldsymbol{T}_{\boldsymbol{R}}^{\boldsymbol{a}} S$ for all $a = 1, 2, \ldots, d_G$.

For compact semisimple Lie groups, two representations are noteworthy. If G is one of the classical groups, SU(N) [for $N \geq 2$], SO(N) [for $N \geq 3$] or Sp($N/2$) [the latter is defined by eqs. (D.4.11) and (D.4.12) for even $N \geq 2$], then the $N \times N$ matrices that define these groups comprise the *fundamental (or defining) representation* F, with $d_F = N$. For example, the fundamental representation of SU(N) consists of $N \times N$ unitary matrices with determinant equal to one, and the corresponding generators comprise a suitably chosen basis for the $N \times N$ traceless hermitian matrices. Every Lie group G also possesses an *adjoint representation* A, with $d_A = d_G$. The matrix elements of the generators in the adjoint representation are given by[119]

$$(\boldsymbol{T}_{\boldsymbol{A}}^{\boldsymbol{a}})^{bc} = -i f^{abc}\,. \tag{E.1.3}$$

Given the unitary representation matrices $D_R(U)$ of the representation R of G, the matrices $[D_R(U)]^*$ constitute the *conjugate* representation R^*. Equivalently, if the $\boldsymbol{T}_{\boldsymbol{R}}^{\boldsymbol{a}}$ comprise a representation of the Lie algebra $\mathfrak{g}$, then the $-(\boldsymbol{T}_{\boldsymbol{R}}^{\boldsymbol{a}})^* = -(\boldsymbol{T}_{\boldsymbol{R}}^{\boldsymbol{a}})^{\mathsf{T}}$ comprise a representation R^* of $\mathfrak{g}$ of the same dimension d_R. If R and R^* are unitarily equivalent representations, then we say that the representation R is *self-conjugate*. Otherwise, we say that the representation R is *complex*, or "strictly complex" in the language of [273]. However, the representation matrices $D_R(U)$ of a self-conjugate representation can also be complex. We can then define two classes of self-conjugate representations. If R and R^* are unitarily equivalent to a representation R' that satisfies the reality property $[D_{R'}(U)]^* = [D_{R'}(U)]$ for all $U \in G$ (equivalently, the matrices $i\boldsymbol{T}_{\boldsymbol{R'}}^{\boldsymbol{a}}$ are real for all a), then R is said to be *real*, or "strictly real" in the language of [273]. If R and R^* are unitarily equivalent representations, but neither is unitarily equivalent to a representation that satisfies the reality property above, then R is said to be *pseudo-real*.

Henceforth, we drop the adjective "strictly" and simply refer to real, pseudo-real and complex representations. Self-conjugate representations are either real or pseudo-real. An important theorem states that for self-conjugate representations, there exists a constant unitary matrix W such that [273]

$$[D_R(U)]^* = W D_R(U) W^{-1}\,, \quad \text{or equivalently,} \quad (i\boldsymbol{T}_{\boldsymbol{R}}^{\boldsymbol{a}})^* = W (i\boldsymbol{T}_{\boldsymbol{R}}^{\boldsymbol{a}}) W^{-1}\,, \tag{E.1.4}$$

where

$$WW^* = \mathbb{1}\,, \qquad W^{\mathsf{T}} = W\,, \qquad \text{for real representations}\,, \tag{E.1.5}$$

$$WW^* = -\mathbb{1}\,, \qquad W^{\mathsf{T}} = -W\,, \qquad \text{for pseudo-real representations}\,, \tag{E.1.6}$$

and $\mathbb{1}$ is the $d_R \times d_R$ identity matrix. Taking the determinant of eq. (E.1.6), and using the fact that W is unitary (and hence invertible), it follows that $1 = (-1)^{d_R}$. Therefore, a pseudo-real representation must be even-dimensional.

If we redefine the basis for the Lie group generators by $\boldsymbol{T}_{\boldsymbol{R}}^{\boldsymbol{a}} \to V^{-1} \boldsymbol{T}_{\boldsymbol{R}}^{\boldsymbol{a}} V$, where V is unitary, then $W \to V^{\mathsf{T}} W V$. We can make use of this change of basis to transform W to a canonical form. Since W is unitary, its singular values (i.e. the positive square roots of the eigenvalues of $W^\dagger W$) are all equal to 1. Hence, in the two cases corresponding to $W^{\mathsf{T}} = \pm W$, respectively, eqs. (D.2.1) and (D.4.1) yield the following canonical forms (for an appropriately chosen V),

$$W = \mathbb{1}\,, \qquad \text{for a real representation } R\,, \tag{E.1.7}$$

$$W = J\,, \qquad \text{for a pseudo-real representation } R\,, \tag{E.1.8}$$

[119] Since the f^{abc} are real, the $i\boldsymbol{T}_{\boldsymbol{A}}^{\boldsymbol{a}}$ are real antisymmetric matrices. The heights of the adjoint labels a, b and c are not significant, as they can be lowered by the inverse Killing form given by $g_{ab} \propto \delta_{ab}$ in the preferred basis.

where $J \equiv \mathrm{diag}\left\{\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right), \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right), \cdots, \left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\right\}$ is a $d_R \times d_R$ matrix (and d_R is even).

There are many examples of complex, real and pseudo-real representations in mathematical physics. For example, the fundamental representation of SU(N) is complex for $N \geq 3$. The adjoint representation of any compact Lie group is real [cf. footnote 119]. The simplest example of a pseudo-real representation is the two-dimensional representation of SU(2),[120] where $\boldsymbol{T^a} = \frac{1}{2}\tau^a$ (and the τ^a are the usual Pauli matrices). More generally, the generators of a pseudo-real representation must satisfy

$$(i\boldsymbol{T_R^a})^* = C^{-1}(i\boldsymbol{T_R^a})C\,, \tag{E.1.9}$$

for some fixed unitary antisymmetric matrix C [previously denoted by W^{-1} in eqs. (E.1.4) and (E.1.6)]. For the doublet representation of SU(2) just given, $C^{ab} = (i\tau^2)^{ab} \equiv \epsilon^{ab}$ is the familiar SU(2)-invariant tensor.

Finally, we note that for U(1), all irreducible representations are one-dimensional. The structure constants vanish and any d-dimensional representation of the U(1)-generator is given by the $d \times d$ identity matrix multiplied by the corresponding U(1)-charge. For a Lie group that is a direct product of a semisimple group and U(1) groups, $\mathrm{Tr}\,\boldsymbol{T_R^a}$ is non-zero when a corresponds to one of the U(1)-generators, unless the sum of the corresponding U(1)-charges of the states of the representation R vanishes.

The quadratic and cubic index and Casimir operator

In this section, we define the index and Casimir operator of a representation of a compact semisimple Lie algebra $\mathfrak{g}$. The *index* $I_2(R)$ of the representation R is defined by [272, 274, 275, 276]

$$\mathrm{Tr}(\boldsymbol{T_R^a T_R^b}) = I_2(R)\delta^{ab}\,, \tag{E.2.1}$$

where $I_2(R)$ is a positive real number that depends on R. Once $I_2(R)$ is defined for one representation, its value is uniquely fixed for any other representation. In the case of a simple compact Lie algebra $\mathfrak{g}$, it is traditional to normalize the generators of the fundamental (or defining) representation F according to[121]

$$\mathrm{Tr}(\boldsymbol{T_F^a T_F^b}) = \tfrac{1}{2}\delta^{ab}\,. \tag{E.2.2}$$

If the representation R is reducible, it can be decomposed into the direct sum of irreducible representations, $R = \sum\limits_k R_k$. In this case, the index of R is given by

$$I_2(R) = \sum_k I_2(R_k)\,. \tag{E.2.3}$$

The index of a tensor product of two representations R_1 and R_2 is given by [274]

$$I_2(R_1 \otimes R_2) = d_{R_1} I_2(R_2) + d_{R_2} I_2(R_1)\,. \tag{E.2.4}$$

Finally, we note that if R^* is the complex conjugate of the representation R, then $I_2(R^*) = I_2(R)$.

A Casimir operator of a Lie algebra $\mathfrak{g}$ is an operator that commutes with all the generators $\boldsymbol{T^a}$. If the representation of the $\boldsymbol{T^a}$ is *irreducible*, then Schur's lemma implies that the Casimir operator is a multiple of the identity. The proportionality constant depends on the representation R. The quadratic Casimir operator of an *irreducible* representation R is given by

$$(\boldsymbol{T_R^2})_i{}^j \equiv (\boldsymbol{T_R^a})_i{}^k (\boldsymbol{T_R^a})_k{}^j = C_R \delta_i{}^j\,, \tag{E.2.5}$$

[120] No unitary matrix W exists such that the $Wi\tau^a W^{-1}$ are real for all $a = 1, 2, 3$. Thus, the two-dimensional representation of SU(2) is not real. However, $(i\tau^a)^* = (i\tau^2)(i\tau^a)(i\tau^2)^{-1}$ for $a = 1, 2, 3$, which proves that the two-dimensional representation of SU(2) is pseudo-real.

[121] In the literature, the index is often defined as the ratio $I_2(R)/I_2(F)$, where $I_2(F)$ is fixed by some convention. This has the advantage that the index of R is independent of the normalization convention of the generators. In this Appendix, we will simply refer to $I_2(R)$ as the index.

where the sum over the repeated indices are implicit and i, j, $k = 1, 2...d_R$. A simple computation then yields the eigenvalue of the quadratic Casimir operator, C_R,

$$C_R = \frac{I_2(R) d_G}{d_R} \,. \tag{E.2.6}$$

For a simple Lie algebra (where the adjoint representation is irreducible), it immediately follows that $C_A = I_2(A)$. For a reducible representation, $\boldsymbol{T^2}$ is a block diagonal matrix consisting of $d_{R_k} \times d_{R_k}$ blocks given by $C_{R_k} \mathbb{1}$ for each irreducible component R_k of R.

The example of the simple Lie algebra $\mathfrak{su}(N)$ is well known. The dimension of this Lie algebra (equal to the number of generators) is given by N^2-1. As previously noted, $d_F = N$ and $I_2(F) = \frac{1}{2}$. It then follows that $C_F = (N^2-1)/(2N)$. One can also check that $C_A = I_2(A) = N$.

The Lie algebras $\mathfrak{su}(N)$ [$N \geq 3$] are the only simple Lie algebra that possesses a cubic Casimir operator. First, we define the symmetrized trace of three generators [276, 277]:

$$D^{abc} \equiv \mathrm{Str}\ (\boldsymbol{T^a T^b T^c}) = \tfrac{1}{6}\,\mathrm{Tr}(\boldsymbol{T^a T^b T^c} + \mathrm{perm.})\,, \tag{E.2.7}$$

where "perm." indicates five other terms obtained by permuting the indices a, b and c in all possible ways. Due to the properties of the trace, it follows that for a given representation R,

$$D^{abc}(R) = \tfrac{1}{2}\,\mathrm{Tr}\left[\{\boldsymbol{T^a_R}, \boldsymbol{T^b_R}\}\boldsymbol{T^c_R}\right] \,. \tag{E.2.8}$$

For the N-dimensional defining representation of $\mathfrak{su}(N)$, it is conventional to define

$$d^{abc} \equiv 2\,\mathrm{Tr}\left[\{\boldsymbol{T^a_F}, \boldsymbol{T^b_F}\}\boldsymbol{T^c_F}\right] \,. \tag{E.2.9}$$

One important property of the d^{abc} is [278, 279]:

$$d^{abc} d^{abc} = \frac{(N^2-1)(N^2-4)}{N} \,. \tag{E.2.10}$$

In general, $D^{abc}(R)$ is proportional to d^{abc}. In particular, the *cubic index* $I_3(R)$ of a representation R is defined such that [280, 278, 276],

$$D^{abc}(R) = I_3(R) d^{abc} \,. \tag{E.2.11}$$

Having fixed $I_3(F) = \frac{1}{4}$, the cubic index is uniquely determined for all representations of $\mathfrak{su}(N)$ [280, 278, 281, 282]. As in the case of the quadratic index $I_2(R)$, we have:

$$I_3(R) = \sum_k I_3(R_k)\,, \tag{E.2.12}$$

for a reducible representation $R = \sum\limits_k R_k$. The cubic index of a tensor product of two representations R_1 and R_2 is given by [280]

$$I_3(R_1 \otimes R_2) = d_{R_1} I_3(R_2) + d_{R_2} I_3(R_1)\,. \tag{E.2.13}$$

If the generators of the representation R are $\boldsymbol{T^a_R}$, then the generators of the complex conjugate representation R^* are $-\boldsymbol{T^{a\,T}_R}$. It then follows that $I_3(R^*) = -I_3(R)$. In particular, the cubic index of a self-conjugate representation vanishes. Note that the converse is not true. That is, it is possible for the cubic index of a complex representation of $\mathfrak{su}(N)$ to vanish in special circumstances [282].

One can show that among the simple Lie groups, $D^{abc} = 0$ except for the case of SU(N), when $N \geq 3$ [278]. For any non-semisimple Lie group (i.e., a Lie group that is a direct product of simple

Lie groups and at least one U(1) factor), D^{abc} is generally non-vanishing. For example, suppose that the $\boldsymbol{T_R^a}$ constitute an irreducible representation of the generators of $G\times$U(1), where G is a semisimple Lie group. Then the U(1) generator (which we denote by setting $\boldsymbol{a} = \boldsymbol{Q}$) is $\boldsymbol{T_R^Q} \equiv q\mathbb{1}$, where q is the corresponding U(1)-charge. It then follows that $D^{Qab} = qI_2(R)\delta^{ab}$. More generally, for a compact non-semisimple Lie group, D^{abc} can be non-zero when either one or three of its indices corresponds to a U(1) generator.

In the computation of the anomaly [cf. Section 29.6.26], the quantity $\mathrm{Tr}(\boldsymbol{T_R^a T_R^b T_R^c})$ appears. We can evaluate this trace using eqs. (E.1.2) and (E.2.11):

$$\mathrm{Tr}(\boldsymbol{T_R^a T_R^b T_R^c}) = I_3(R)d^{abc} + \frac{i}{2}I_2(R)f^{abc}\,. \tag{E.2.14}$$

The cubic Casimir operator of an *irreducible* representation R is given by

$$(\boldsymbol{T_R^3})_i{}^j \equiv d^{abc}(\boldsymbol{T_R^a T_R^b T_R^c})_i{}^j = C_{3R}\delta_i{}^j\,. \tag{E.2.15}$$

Using eqs. (E.2.10) and (E.2.11), we obtain a relation between the eigenvalue of the cubic Casimir operator, C_{3R} and the cubic index [278]:

$$C_{3R} = \frac{(N^2-1)(N^2-4)I_3(R)}{Nd_R}\,. \tag{E.2.16}$$

Again, we provide two examples. For the fundamental representation of $\mathfrak{su}(N)$, $I_3(F) = \frac{1}{4}$ and $C_{3F} = (N^2-1)(N^2-4)/(4N^2)$. For the adjoint representation, $I_3(A) = C_{3A} = 0$, since the adjoint representation is self-conjugate. A general formula for the eigenvalue of the cubic Casimir operator in an arbitrary $\mathfrak{su}(N)$ representation [or equivalently the cubic index $I_3(R)$, which is related to C_{3R} by eq. (E.2.16)] can be found in refs. [280, 278, 281, 282].

29.7.6 Path integral treatment of two-component fermion propagators

In Section 29.4.2 we derived the two-component fermion propagators in momentum space, which are the Fourier transforms of the free-field expectation values of time-ordered products of two two-component fermion fields, for example,

$$\langle 0|\,T\xi_\alpha(x)\xi^\dagger_{\dot\beta}(y)\,|0\rangle_{\rm FT} \equiv \int d^4w\,\langle 0|\,T\xi_\alpha(x)\xi^\dagger_{\dot\beta}(y)\,|0\rangle\,e^{ip\cdot w}\,, \qquad w\equiv x-y\,, \tag{F.1}$$

where the (translationally invariant) expectation values such as $\langle 0|\,T\xi_\alpha(x)\xi^\dagger_{\dot\beta}(y)\,|0\rangle$ are functions of the coordinate difference $w \equiv x-y$. In Section 29.4.2, the Fourier transforms of these quantities were computed by using the free-field expansion obtained from the canonical quantization procedure, and then evaluating the resulting spin sums. In this Appendix, we provide a derivation of the same result by employing path integral techniques. We follow the analysis given in Appendix C of ref. [283] (with a few minor changes in notation). For a similar textbook treatment of two-component fermion propagators see for example [223]. For the analogous treatment of the four-component fermion propagator, see for example [115].

We first consider the action for a single massive neutral two-component fermion $\xi_\alpha(x)$, coupled to an anticommuting two-component fermionic source term $J_\alpha(x)$ [cf. eq. (29.3.1)]:

$$S = \int d^4x\,(\mathscr{L}+J\xi+\xi^\dagger J^\dagger) = \int d^4x\,\Big\{\tfrac{1}{2}\left[i\xi^\dagger\overline{\sigma}^\mu\partial_\mu\xi + i\xi\sigma^\mu\partial_\mu\xi^\dagger - m(\xi\xi+\xi^\dagger\xi^\dagger)\right]+J\xi+\xi^\dagger J^\dagger\Big\}\,, \tag{F.2}$$

where we have split the kinetic energy term symmetrically into two terms. The generating functional is given by

$$W[J,\,J^\dagger\,] = N\int \mathcal{D}\xi\,\mathcal{D}\xi^\dagger\,e^{iS[\xi,\,\xi^\dagger,\,J,\,J^\dagger]}\,, \tag{F.3}$$

where N is a normalization factor chosen such that $W[0,0] = 1$ and $\mathcal{D}\xi\,\mathcal{D}\xi^\dagger$ is the integration measure. It is convenient to Fourier transform the fields $\xi(x)$, $\xi^\dagger(x)$ and sources $J(x)$, $J^\dagger(x)$ in eq. (F.3), and rewrite the action in terms of the corresponding Fourier coefficients $\widehat{\xi}(p)$, $\widehat{\xi}^\dagger(p)$, $\widehat{J}(p)$ and $\widehat{J}^\dagger(p)$:

$$\xi_\alpha(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\widehat{\xi}_\alpha(p)\,, \qquad \xi^\dagger_{\dot\alpha}(x) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x}\widehat{\xi}^\dagger_{\dot\alpha}(p)\,, \tag{F.4}$$

$$J_\alpha(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\widehat{J}_\alpha(p)\,, \qquad J^\dagger_{\dot\alpha}(x) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x}\widehat{J}^\dagger_{\dot\alpha}(p)\,. \tag{F.5}$$

Furthermore, we introduce the integral representation of the delta function:

$$\delta^{(4)}(x-x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')}\,. \tag{F.6}$$

In order to rewrite eq. (F.3) in a more convenient matrix form, we introduce the following definitions:

$$\Omega(p) \equiv \begin{pmatrix} \widehat{\xi}^{\dagger\dot\alpha}(-p) \\ \widehat{\xi}_\alpha(p) \end{pmatrix}, \qquad X(p) \equiv \begin{pmatrix} \widehat{J}_\alpha(p) \\ \widehat{J}^{\dagger\dot\alpha}(-p) \end{pmatrix}, \qquad \mathcal{M}(p) \equiv \begin{pmatrix} p\cdot\sigma_{\alpha\dot\beta} & -m\,\delta_\alpha{}^\beta \\ -m\,\delta^{\dot\alpha}{}_{\dot\beta} & p\cdot\overline{\sigma}^{\dot\alpha\beta} \end{pmatrix}. \tag{F.7}$$

Note that $\mathcal{M}$ is an hermitian matrix. We can then rewrite the action [eq. (F.2)] in the following matrix form [after using eqs. (29.60) and (29.61) to write the product of the spinor field and the source in a symmetrical fashion]:

$$S = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\,\left(\Omega^\dagger\mathcal{M}\Omega + \Omega^\dagger X + X^\dagger\Omega\right)\,. \tag{F.8}$$

The linear term in the field Ω can be removed by a field redefinition

$$\Omega' = \Omega + \mathcal{M}^{-1}X\,. \tag{F.9}$$

In terms of Ω', the action now takes the convenient form:

$$S = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\,\left(\Omega'^\dagger\mathcal{M}\Omega' - X^\dagger\mathcal{M}^{-1}X\right)\,, \tag{F.10}$$

where the inverse of the matrix $\mathcal{M}$ is given by

$$\mathcal{M}^{-1} = \frac{1}{p^2-m^2}\begin{pmatrix} p\cdot\overline{\sigma}^{\dot\alpha\beta} & m\,\delta^{\dot\alpha}{}_{\dot\beta} \\ m\,\delta_\alpha{}^\beta & p\cdot\sigma_{\alpha\dot\beta} \end{pmatrix}. \tag{F.11}$$

The Jacobian of the field transformation given in eq. (F.9) is unity. Hence, one can insert the new action, eq. (F.10), in the generating functional, eq. (F.3) to obtain (after dropping the primes on the two-component fermion fields):

$$\begin{aligned} W[\widehat{J},\,\widehat{J}^\dagger\,] &= N\int \mathcal{D}\xi\,\mathcal{D}\xi^\dagger \exp\left\{\frac{i}{2}\int\frac{d^4p}{(2\pi)^4}\,\left(\Omega^\dagger\mathcal{M}\Omega - X^\dagger\mathcal{M}^{-1}X\right)\right\} \\ &= N\left[\int \mathcal{D}\xi\,\mathcal{D}\xi^\dagger\,\exp\left\{\frac{i}{2}\Omega^\dagger\mathcal{M}\Omega\right\}\right]\,\exp\left\{-\frac{i}{2}\int\frac{d^4p}{(2\pi)^4}\,X^\dagger\mathcal{M}^{-1}X\right\} \\ &= \exp\left\{-\frac{i}{2}\int\frac{d^4p}{(2\pi)^4}\,X^\dagger\mathcal{M}^{-1}X\right\}\,, \end{aligned} \tag{F.12}$$

where we have defined the normalization constant N such that $W[0,0]=1$. Inserting the explicit forms for X and $\mathcal{M}$ into eq. (F.12), we obtain

$$W[\widehat{J},\,\widehat{J}^\dagger\,]=\exp\biggl\{-\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\biggl(\widehat{J}^\alpha(-p)\frac{ip\cdot\sigma_{\alpha\dot\beta}}{p^2-m^2}\widehat{J}^{\dagger\dot\beta}(-p)+\widehat{J}^\dagger_{\dot\alpha}(p)\frac{ip\cdot\overline{\sigma}^{\dot\alpha\beta}}{p^2-m^2}\widehat{J}_\beta(p)$$
$$+\widehat{J}^\alpha(-p)\frac{im\delta_\alpha{}^\beta}{p^2-m^2}\widehat{J}_\beta(p)+\widehat{J}^\dagger_{\dot\alpha}(p)\frac{im\delta^{\dot\alpha}{}_{\dot\beta}}{p^2-m^2}\widehat{J}^{\dagger\dot\beta}(-p)\biggr)\biggr\}\,. \tag{F.13}$$

Using eq. (29.62), it is convenient to rewrite the first two terms of the integrand on the right-hand side of eq. (F.13) in two different ways:

$$\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\left[\widehat{J}^\alpha(-p)\frac{ip\cdot\sigma_{\alpha\dot\beta}}{p^2-m^2}\widehat{J}^{\dagger\dot\beta}(-p)+\widehat{J}^\dagger_{\dot\alpha}(p)\frac{ip\cdot\overline{\sigma}^{\dot\alpha\beta}}{p^2-m^2}\widehat{J}_\beta(p)\right]$$
$$=\int\frac{d^4p}{(2\pi)^4}\,\widehat{J}^\alpha(-p)\frac{ip\cdot\sigma_{\alpha\dot\beta}}{p^2-m^2}\widehat{J}^{\dagger\dot\beta}(-p)=\int\frac{d^4p}{(2\pi)^4}\,\widehat{J}^\dagger_{\dot\alpha}(p)\frac{ip\cdot\overline{\sigma}^{\dot\alpha\beta}}{p^2-m^2}\widehat{J}_\beta(p)\,, \tag{F.14}$$

where we have changed integration variables from $p\to-p$ in relating the two terms above. The vacuum expectation value of the time-ordered product of two spinor fields in configuration space is obtained by taking two functional derivatives of the generating functional with respect to the sources J and $J^\dagger$ and then setting $J=J^\dagger=0$ at the end of the computation (e.g., see [115]). For example,

$$\left(-i\frac{\overrightarrow{\delta}}{\delta J^\alpha(x_1)}\right)W[J,J^\dagger]\left(-i\frac{\overleftarrow{\delta}}{\delta J^{\dagger\dot\beta}(x_2)}\right)\biggr|_{J=J^\dagger=0}=N\int\mathcal{D}\xi\,\mathcal{D}\xi^\dagger\;\xi_\alpha(x_1)\xi^\dagger_{\dot\beta}(x_2)\exp\left(i\int d^4x\,\mathscr{L}\right)$$
$$=\langle0|T\xi_\alpha(x_1)\xi^\dagger_{\dot\beta}(x_2)|0\rangle\,, \tag{F.15}$$

where the functional derivatives act in the indicated direction (which ensures that no extra minus signs are generated due to the anticommutativity properties of the sources and their functional derivatives). To obtain the two-point functions involving the product of two spinor fields with different combinations of dotted and undotted spinors, it may be more convenient to write $J\xi=\xi J$ and/or $\xi^\dagger J^\dagger=J^\dagger\xi^\dagger$ in eq. (F.3). One can then easily verify the following expressions for the four possible two-point functions:

$$\langle0|T\xi_\alpha(x_1)\xi^\dagger_{\dot\beta}(x_2)|0\rangle=\left(-i\frac{\overrightarrow{\delta}}{\delta J^\alpha(x_1)}\right)W[J,J^\dagger]\left(-i\frac{\overleftarrow{\delta}}{\delta J^{\dagger\dot\beta}(x_2)}\right)\biggr|_{J=J^\dagger=0}\,, \tag{F.16}$$

$$\langle0|T\xi^{\dagger\dot\alpha}(x_1)\xi^\beta(x_2)|0\rangle=\left(-i\frac{\overrightarrow{\delta}}{\delta J^\dagger_{\dot\alpha}(x_1)}\right)W[J,J^\dagger]\left(-i\frac{\overleftarrow{\delta}}{\delta J_\beta(x_2)}\right)\biggr|_{J=J^\dagger=0}\,, \tag{F.17}$$

$$\langle0|T\xi^{\dagger\dot\alpha}(x_1)\xi^\dagger_{\dot\beta}(x_2)|0\rangle=\left(-i\frac{\overrightarrow{\delta}}{\delta J^\dagger_{\dot\alpha}(x_1)}\right)W[J,J^\dagger]\left(-i\frac{\overleftarrow{\delta}}{\delta J^{\dagger\dot\beta}(x_2)}\right)\biggr|_{J=J^\dagger=0}\,, \tag{F.18}$$

$$\langle0|T\xi_\alpha(x_1)\xi^\beta(x_2)|0\rangle=\left(-i\frac{\overrightarrow{\delta}}{\delta J^\alpha(x_1)}\right)W[J,J^\dagger]\left(-i\frac{\overleftarrow{\delta}}{\delta J_\beta(x_2)}\right)\biggr|_{J=J^\dagger=0}\,. \tag{F.19}$$

As an example, we provide details for the evaluation of eq. (F.16). Using eqs. (F.13) and (F.14), we obtain:

$$\langle0|T\xi_\alpha(x_1)\xi^\dagger_{\dot\beta}(x_2)|0\rangle=\frac{\overrightarrow{\delta}}{\delta J^\alpha(x_1)}\left(\int\frac{d^4p}{(2\pi)^4}\,\widehat{J}^\alpha(-p)\frac{ip\cdot\sigma_{\alpha\dot\beta}}{p^2-m^2}\widehat{J}^{\dagger\dot\beta}(-p)\right)\frac{\overleftarrow{\delta}}{\delta J^{\dagger\dot\beta}(x_2)}\,. \tag{F.20}$$

The chain rule for functional differentiation and the inverse Fourier transforms of eq. (F.5) yield:

$$\frac{\delta}{\delta J^\alpha(x_1)} = \int d^4p_1 \, \frac{\delta \widehat{J}^\beta(-p_1)}{\delta J^\alpha(x_1)} \frac{\delta}{\delta \widehat{J}^\beta(-p_1)} = \int d^4p_1 \, e^{-ip_1\cdot x_1} \frac{\delta}{\delta \widehat{J}^\alpha(-p_1)} \,, \tag{F.21}$$

$$\frac{\delta}{\delta J^{\dagger\dot\beta}(x_2)} = \int d^4p_2 \, \frac{\delta \widehat{J}^{\dagger\dot\alpha}(-p_2)}{\delta J^{\dagger\dot\beta}(x_2)} \frac{\delta}{\delta \widehat{J}^{\dagger\dot\alpha}(-p_2)} = \int d^4p_2 \, e^{ip_2\cdot x_2} \frac{\delta}{\delta \widehat{J}^{\dagger\dot\beta}(-p_2)} \,. \tag{F.22}$$

Applying eqs. (F.21) and (F.22) to eq. (F.20), we obtain:

$$\langle 0|T\xi_\alpha(x_1)\xi^\dagger_{\dot\beta}(x_2)|0\rangle = \int \frac{d^4p}{(2\pi)^4} \, e^{-ip\cdot(x_1-x_2)} \frac{ip\cdot\sigma_{\alpha\dot\beta}}{p^2-m^2} \,, \tag{F.23}$$

which is equivalent to eq. (29.4.1) of Section 29.4.2. With the same methods applied to eqs. (F.17)–(F.19), one can easily reproduce the results of eqs. (29.4.2)–(29.4.4).

We next consider the action for a single massive Dirac two-component fermion. We shall work in a basis of fields where the action, including external anticommuting sources, is given by

$$\begin{aligned} S[\chi,\chi^\dagger,\eta,\eta^\dagger,J_\chi,J_\chi^\dagger,J_\eta,J_\eta^\dagger] = \int d^4x \Big[& i\chi^\dagger\overline\sigma^\mu\partial_\mu\chi + i\eta^\dagger\overline\sigma^\mu\partial_\mu\eta - m(\chi\eta+\chi^\dagger\eta^\dagger) \\ & + J_\chi\chi + \chi^\dagger J_\chi^\dagger + J_\eta\eta + \eta^\dagger J_\eta^\dagger \Big]. \end{aligned} \tag{F.24}$$

Following the techniques employed above, we introduce Fourier coefficients for all the fields and sources and define

$$\Omega_c(p) \equiv \begin{pmatrix} \widehat\eta^{\dagger\dot\alpha}(-p) \\ \widehat\chi_\alpha(p) \end{pmatrix} , \qquad X_c(p) \equiv \begin{pmatrix} \widehat J_{\eta\alpha}(p) \\ \widehat J_\chi^{\dagger\dot\alpha}(-p) \end{pmatrix} . \tag{F.25}$$

The action functional, eq. (F.24), can then rewritten in matrix form as before (but with no overall factor of 1/2):

$$S = \int \frac{d^4p}{(2\pi)^4} \, \left(\Omega_c^\dagger \mathcal{M} \Omega_c + \Omega_c^\dagger X_c + X_c^\dagger \Omega_c \right) \,, \tag{F.26}$$

where $\mathcal{M}$ is again given by eq. (F.7). The remaining calculation proceeds as before with few modifications, and yields the Dirac two-component fermion free-field propagators given in eqs. (29.4.7)–(29.4.10).

29.7.7 Correspondence to four-component spinor notation 1

Dirac gamma matrices and four-component spinors

In four-dimensional Minkowski space, four-component spinor notation employs four-component Dirac spinor fields and the 4×4 Dirac gamma matrices, whose defining property is:

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\mathbb{1} \,, \tag{G.1.1}$$

where $\mathbb{1}$ is the 4×4 identity matrix.

The correspondence between the two-component spinor notation and the four-component Dirac spinor notation is most easily exhibited in the basis in which γ_5 is diagonal (this is called the *chiral*

representation[122]). In 2×2 blocks, the gamma matrices are given by:[123]

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu{}_{\alpha\dot\beta} \\ \overline{\sigma}^{\mu\,\dot\alpha\beta} & 0 \end{pmatrix}, \qquad \gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\delta_\alpha{}^\beta & 0 \\ 0 & \delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}, \tag{G.1.2}$$

and the 4×4 identity matrix that appears in eq. (G.1.1) can be written as:

$$\mathbb{1} = \begin{pmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & \delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}. \tag{G.1.3}$$

In addition, we identify the generators of the Lorentz group in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation:[124]

$$\tfrac{1}{2}\Sigma^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu,\gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu}{}_\alpha{}^\beta & 0 \\ 0 & \overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta} \end{pmatrix}, \tag{G.1.4}$$

where $\Sigma^{\mu\nu}$ satisfies the duality relation,

$$\gamma_5\Sigma^{\mu\nu} = \tfrac{1}{2}i\epsilon^{\mu\nu\rho\tau}\Sigma_{\rho\tau}\,. \tag{G.1.5}$$

Eq. (G.1.4) yields the spin-$\frac{1}{2}$ angular momentum matrix representation $\{\frac{1}{2}\Sigma^i\}$, $i=1,2,3$, where

$$\Sigma^i \equiv \tfrac{1}{2}\epsilon^{ijk}\Sigma_{jk} = \gamma^0\gamma^i\gamma_5 = \begin{pmatrix} -(\sigma^0\overline{\sigma}^i)_\alpha{}^\beta & 0 \\ 0 & (\overline{\sigma}^0\sigma^i)^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}. \tag{G.1.6}$$

A four-component *Dirac spinor* field, $\Psi(x)$, is made up of two mass-degenerate two-component spinor fields, $\chi_\alpha(x)$ and $\eta_\alpha(x)$, of opposite U(1)-charge as follows:

$$\Psi(x) \equiv \begin{pmatrix} \chi_\alpha(x) \\ \eta^{\dagger\dot\alpha}(x) \end{pmatrix}. \tag{G.1.7}$$

We next introduce the chiral projections operators,[125]

$$P_L \equiv \tfrac{1}{2}(\mathbb{1}-\gamma_5) = \begin{pmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad P_R \equiv \tfrac{1}{2}(\mathbb{1}+\gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}, \tag{G.1.8}$$

[122] For a review of other representations of the Dirac gamma matrices and their properties, see e.g. refs. [284, 285].

[123] Employing the conventions for the sigma matrices described in Appendix A , it follows that the definition of γ^μ is independent of the choice of metric signature, whereas $\gamma_\mu \equiv g_{\mu\nu}\gamma^\nu$ changes sign under a reversal of the metric signature. In the metric signature convention with $g_{00}=+1$, our gamma matrix conventions follow those of [115], whereas in the convention with $g_{00}=-1$, our gamma matrix conventions follow those of [66].

[124] In most textbooks, $\Sigma^{\mu\nu}$ is called $\sigma^{\mu\nu}$. Here, we use the former symbol so that there is no confusion with $\sigma^{\mu\nu}{}_\alpha{}^\beta$ given in eq. (29.71).

[125] In the earlier literature, a different set of conventions for the sigma matrices in which the roles of σ and $\overline{\sigma}$ were reversed [e.g, as in eqs. (A.11) and (A.12)] resulted in $\gamma_5 = \mathrm{diag}(\mathbb{1}_{2\times2}\,,\,-\mathbb{1}_{2\times2})$ in the chiral representation, which differs from our convention by an overall sign [cf. eq. (G.1.2)]. As a result, in this latter convention, P_L [P_R] projects out the raised dotted [lowered undotted] two-component spinor field. This latter convention is still prevalent in the literature of the spinor helicity method (see footnote 163 in Appendix I.2).

and the (left and right-handed) *Weyl spinor* fields, $\Psi_L(x)$ and $\Psi_R(x)$, which are defined by:

$$\Psi_L(x) \equiv P_L\Psi(x) = \begin{pmatrix} \chi_\alpha(x) \\ 0 \end{pmatrix}, \qquad \Psi_R(x) \equiv P_R\Psi(x) = \begin{pmatrix} 0 \\ \eta^{\dagger\dot\alpha}(x) \end{pmatrix}. \tag{G.1.9}$$

Equivalently, one can define the Weyl spinors Ψ_L and Ψ_R as the four-component spinor eigenstates of γ_5 with corresponding eigenvalues -1 and $+1$, respectively (i.e., $\gamma_5\Psi_{L,R} = \mp\Psi_{L,R}$).

The Dirac conjugate field $\overline{\Psi}(x)$ and the charge conjugate field $\Psi^C(x)$ are defined by:

$$\overline{\Psi}(x) \equiv \Psi^\dagger A = \left(\eta^\alpha(x),\, \chi^\dagger_{\dot\alpha}(x)\right), \tag{G.1.10}$$

$$\Psi^C(x) \equiv C\overline{\Psi}^{\mathsf{T}}(x) = \begin{pmatrix} \eta_\alpha(x) \\ \chi^{\dagger\dot\alpha}(x) \end{pmatrix}, \tag{G.1.11}$$

where the Dirac conjugation matrix A and the charge conjugation matrix C satisfy [286, 287, 288]:

$$A\gamma_\mu A^{-1} = \gamma_\mu^\dagger, \qquad C^{-1}\gamma_\mu C = -\gamma_\mu^{\mathsf{T}}. \tag{G.1.12}$$

It is convenient to introduce a notation for left and right-handed charge-conjugated fields (which are also Weyl spinor fields) following the conventions of refs. [68, 289],[126]

$$\Psi^C_L(x) \equiv P_L\Psi^C(x) = C\overline{\Psi}^{\mathsf{T}}_R(x) = [\Psi_R(x)]^C, \tag{G.1.13}$$

$$\Psi^C_R(x) \equiv P_R\Psi^C(x) = C\overline{\Psi}^{\mathsf{T}}_L(x) = [\Psi_L(x)]^C. \tag{G.1.14}$$

To fix the properties of A and C, it is conventional to impose two additional conditions:

$$\Psi = A^{-1}\overline{\Psi}^\dagger, \qquad (\Psi^C)^C = \Psi. \tag{G.1.15}$$

The first of these conditions together with eq. (G.1.10) is equivalent to the statement that $\overline{\Psi}\Psi$ is hermitian. The second condition corresponds to the statement that the (discrete) charge conjugation transformation applied twice is equal to the identity operator. Using eqs. (G.1.12) and (G.1.15) and the defining property of the gamma matrices [eq. (G.1.1)], one can show (independently of the gamma matrix representation) that the matrices A and C must satisfy:

$$A^\dagger = A, \qquad C^{\mathsf{T}} = -C, \qquad (AC)^{-1} = (AC)^*. \tag{G.1.16}$$

Following [137], it is convenient to introduce a matrix D such that

$$D \equiv CA^{\mathsf{T}}, \qquad D^{-1}\gamma_\mu D = -\gamma_\mu^*, \tag{G.1.17}$$

and $D^*D = DD^* = \mathbb{1}$. The charge-conjugated four-component spinor is then given by:

$$\Psi^C(x) \equiv D\Psi^*(x). \tag{G.1.18}$$

A four-component *Majorana spinor* field, $\Psi_M(x)$, is defined by imposing the constraint $\Psi^C(x) = \Psi(x)$ on a four-component Dirac spinor, which sets $\eta = \chi$. That is, the *Majorana condition* is

$$\Psi_M(x) = D\Psi^*_M(x) = \begin{pmatrix} \chi_\alpha(x) \\ \chi^{\dagger\dot\alpha}(x) \end{pmatrix}. \tag{G.1.19}$$

[126] The reader is warned that the opposite convention is often employed in the literature (e.g., see [290]) in which Ψ^C_L is a right-handed field and Ψ^C_R is a left-handed field.

For a review of the Majorana field and its properties, see e.g. refs. [145, 146].

For completeness, we also introduce a matrix B that satisfies [291, 286, 287, 288]:

$$B \equiv -C^{-1}\gamma_5\,, \qquad B\gamma_\mu B^{-1} = \gamma_\mu^{\mathsf{T}}\,. \tag{G.1.20}$$

The matrix B arises in the study of time reversal invariance of the Dirac equation. In the chiral representation, A, B, C and D are explicitly given by

$$A = \begin{pmatrix} 0 & \delta^{\dot\alpha}{}_{\dot\beta} \\ \delta_\alpha{}^\beta & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix}, \tag{G.1.21}$$

$$B = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & -\epsilon_{\dot\alpha\dot\beta} \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & \epsilon_{\alpha\beta} \\ \epsilon^{\dot\alpha\dot\beta} & 0 \end{pmatrix}. \tag{G.1.22}$$

Note the *numerical* equalities, $A = \gamma^0$, $B = \gamma^1\gamma^3$, $C = i\gamma^0\gamma^2$ and $D = -i\gamma^2$. However these identifications do not respect either the structure of the undotted and dotted spinor indices specified in eqs. (G.1.21) and (G.1.22), or the four-component spinor index structure introduced below [cf. eqs. (G.1.47) and (G.1.48)].[127] In translating between two-component and four-component spinor notation, eqs. (G.1.21) and (G.1.22) should always be used. In practical four-component spinor calculations, there is often no harm in employing the numerical values for A, B, C and D.

Using eqs. (G.1.12)–(G.1.20), the following results are easily derived:

$$A\Gamma A^{-1} = \eta^A_\Gamma \Gamma^\dagger\,, \qquad \eta^A_\Gamma = \begin{cases} +1\,, & \text{for } \Gamma = \mathbb{1}\,,\ \gamma^\mu\,,\ \gamma^\mu\gamma_5\,,\ \Sigma^{\mu\nu}\,, \\ -1\,, & \text{for } \Gamma = \gamma_5\,,\ \Sigma^{\mu\nu}\gamma_5\,, \end{cases} \tag{G.1.23}$$

$$B\Gamma B^{-1} = \eta^B_\Gamma \Gamma^{\mathsf{T}}\,, \qquad \eta^B_\Gamma = \begin{cases} +1\,, & \text{for } \Gamma = \mathbb{1}\,,\ \gamma_5\,,\ \gamma^\mu\,, \\ -1\,, & \text{for } \Gamma = \gamma^\mu\gamma_5\,,\ \Sigma^{\mu\nu}\,,\ \Sigma^{\mu\nu}\gamma_5\,, \end{cases} \tag{G.1.24}$$

$$C^{-1}\Gamma C = \eta^C_\Gamma \Gamma^{\mathsf{T}}\,, \qquad \eta^C_\Gamma = \begin{cases} +1\,, & \text{for } \Gamma = \mathbb{1}\,,\ \gamma_5\,,\ \gamma^\mu\gamma_5\,, \\ -1\,, & \text{for } \Gamma = \gamma^\mu\,,\ \Sigma^{\mu\nu}\,,\ \Sigma^{\mu\nu}\gamma_5\,, \end{cases} \tag{G.1.25}$$

$$D^{-1}\Gamma D = \eta^D_\Gamma \Gamma^*\,, \qquad \eta^D_\Gamma = \begin{cases} +1\,, & \text{for } \Gamma = \mathbb{1}\,,\ \gamma^\mu\gamma_5\,,\ \Sigma^{\mu\nu}\gamma_5\,, \\ -1\,, & \text{for } \Gamma = \gamma^\mu\,,\ \gamma_5\,,\ \Sigma^{\mu\nu}\,. \end{cases} \tag{G.1.26}$$

The Lorentz transformation properties of the four-component spinor field can be determined from those of the two-component spinor fields given in Section 29.2. The 4×4 representation matrices of the Lorentz group in the $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ representation are given by

$$\mathbb{M} = \begin{pmatrix} M & 0 \\ 0 & (M^{-1})^\dagger \end{pmatrix} = \exp\left(-\frac{i}{4}\theta_{\mu\nu}\Sigma^{\mu\nu}\right) \simeq \mathbb{1}_{4\times4} - \tfrac{1}{4}i\theta_{\mu\nu}\Sigma^{\mu\nu}\,, \tag{G.1.27}$$

where the infinitesimal forms of M and $(M^{-1})^\dagger$ are given in eqs. (29.101) and (29.102). Two useful identities that follow from eqs. (G.1.23), (G.1.25) and (G.1.27) are:[128]

$$A\,\mathbb{M}A^{-1} = (\mathbb{M}^{-1})^\dagger\,, \tag{G.1.28}$$

$$C^{-1}\mathbb{M}C = (\mathbb{M}^{-1})^{\mathsf{T}}\,. \tag{G.1.29}$$

[127] When treated as ordinary 4×4 matrices A, B, C and D are unitary. But when written in 2×2 block form [noting that $\delta^{\dot\alpha}{}_{\dot\beta} = (\delta^\alpha{}_\beta)^*$ and $\epsilon^{\dot\alpha\dot\beta} \equiv (\epsilon^{\alpha\beta})^*$, as indicated below eqs. (29.21) and (29.25)], the products $AA^\dagger$, $BB^\dagger$, $CC^\dagger$ and $DD^\dagger$ are not covariant with respect to the dotted and undotted two-component spinor indices. Similarly, these matrix products are not covariant with respect to the four-component spinor indices. In practice, only covariant combinations of A, B, C, D and the four-component spinor fields arise in typical calculations.

[128] Note that eq. (G.1.29) is a direct consequence of the identities in two-component spinor notation given in eqs. (29.103) and (29.104).

The four-component Dirac or Majorana spinor, Ψ_a, is assigned a lowered spinor index a, and is defined in terms of two-component spinors by eqs. (G.1.7) or (G.1.19), respectively. Four-component spinor indices, which will be chosen in general from the beginning of the lower case Roman alphabet, $a, b, c, ...$, can assume integer values $1, 2, 3, 4$. Under a Lorentz transformation, Ψ_a transforms as

$$\Psi_a \to \mathbb{M}_a{}^b \, \Psi_b \,. \tag{G.1.30}$$

In analogy with the conventions for two-component spinor indices, we sum implicitly over a pair of repeated indices consisting of a raised and a lowered spinor index. The transformation law for the Dirac conjugate spinor (often called the Dirac adjoint spinor), $\overline{\Psi} = \Psi^\dagger A$, is obtained from eq. (G.1.30) after employing $A^\dagger = A$ and eq. (G.1.28),

$$\overline{\Psi}^a \to \overline{\Psi}^b \, (\mathbb{M}^{-1})_b{}^a \,. \tag{G.1.31}$$

In particular, $\overline{\Psi}\Psi \equiv \overline{\Psi}^a \Psi_a$ is a Lorentz scalar, which justifies the assignment of a raised spinor index for the Dirac conjugate spinor $\overline{\Psi}^a$.

It is convenient to introduce *barred* four-component spinor indices [292] in the transformation laws of the hermitian-conjugated four-component spinors,[129]

$$\Psi^\dagger_{\bar{a}} \to \Psi^\dagger_{\bar{b}} \, (\mathbb{M}^\dagger)^{\bar{b}}{}_{\bar{a}} \,, \tag{G.1.32}$$

$$\overline{\Psi}^{\dagger\,\bar{a}} \to [(\mathbb{M}^{-1})^\dagger]^{\bar{a}}{}_{\bar{b}} \, \overline{\Psi}^{\dagger\,\bar{b}} \,, \tag{G.1.33}$$

where there is an implicit sum over the repeated lowered and raised barred spinor indices, and

$$\Psi^\dagger_{\bar{a}} \equiv (\Psi_a)^\dagger \,, \qquad\qquad \overline{\Psi}^{\dagger\,\bar{a}} \equiv (\overline{\Psi}^a)^\dagger \,. \tag{G.1.34}$$

The spinor index structure of the Dirac conjugation matrix A is then fixed by noting that the Dirac conjugate spinor, $\overline{\Psi}^b \equiv \Psi^\dagger_{\bar{a}} A^{\bar{a}b}$, has a raised unbarred spinor index, whereas the hermitian-conjugated spinor has a lowered barred spinor index.

The charge conjugation matrix can be used to raise and lower four-component spinor indices [292], which we shall employ in defining the spinors Ψ^a, $\Psi^{\dagger\,\bar{a}}$, $\overline{\Psi}_a$ and $\overline{\Psi}^\dagger_{\bar{a}}$,[130]

$$\Psi_a = C_{ab} \Psi^b \,, \qquad\qquad \Psi^a = (C^{-1})^{ab} \Psi_b \,, \tag{G.1.35}$$

$$\Psi^\dagger_{\bar{a}} = C_{\bar{a}\bar{b}} \Psi^{\dagger\,\bar{b}} \,, \qquad\qquad \Psi^{\dagger\,\bar{a}} = (C^{-1})^{\bar{a}\bar{b}} \Psi^\dagger_{\bar{b}} \,, \tag{G.1.36}$$

where

$$C_{\bar{a}\bar{b}} \equiv (C_{ab})^* \,, \qquad\qquad (C^{-1})^{\bar{a}\bar{b}} \equiv [(C^{-1})^{ab}]^* \,. \tag{G.1.37}$$

Eqs. (G.1.35) and (G.1.36) also apply to $\overline{\Psi}^a$, $\overline{\Psi}_a$ and their hermitian conjugates. In particular, one can identify the Dirac conjugate spinor with a lowered spinor index ($\overline{\Psi}_a$) as the charge-conjugated spinor, $\Psi^C \equiv C \overline{\Psi}^{\mathsf{T}}$, and the Dirac spinor with a raised spinor index (Ψ^a) as the Dirac conjugate of the charge-conjugated spinor, $\overline{\Psi^C} = -\Psi^{\mathsf{T}} C^{-1}$. That is,[131]

$$\Psi^C_a \equiv \overline{\Psi}_a = C_{ab} \overline{\Psi}^b \,, \qquad\qquad \overline{\Psi^C}^{\,a} = \Psi^a = (C^{-1})^{ab} \Psi_b \,. \tag{G.1.38}$$

The rules for raising and lowering spinor indices are consistent with the Lorentz transformation properties of eqs. (G.1.30)–(G.1.33), as a consequence of eq. (G.1.29). In particular, the condition for a self-conjugate four-component (Majorana) spinor, $\overline{\Psi}_a \equiv \Psi^C_a = \Psi_a$, is Lorentz covariant.

[129] Of course, eqs. (G.1.30)–(G.1.33) can also be derived directly from the corresponding two-component spinor transformation laws of Section 29.2.

[130] In contrast to the epsilon symbols of the two-component spinor formalism, here we prefer to explicitly exhibit the inverse symbols in $(C^{-1})^{ab}$ and $(C^{-1})^{\bar{a}\bar{b}}$ [cf. footnote 14].

[131] For a Dirac spinor field defined in eq. (G.1.7), $\overline{\Psi}_a(x) = \Psi^C_a(x)$ is given in terms of two-component spinors by eq. (G.1.11), and $\Psi^a(x) = \overline{\Psi^C}^{\,a}(x) = \Big(\chi^\alpha(x)\,,\, \eta^\dagger_{\dot{\alpha}}(x)\Big)$.

Using eqs. (G.1.16), (G.1.35), (G.1.36), and the definition of the Dirac conjugate spinor, it then follows that:

$$\overline{\Psi}_a = (A^{-1})_{a\bar{b}}\Psi^{\dagger\,\bar{b}}\,, \qquad \overline{\Psi}^{\,a} = \Psi^{\dagger}_{\bar{b}}\,A^{\bar{b}a}\,, \tag{G.1.39}$$

$$\Psi^{\dagger}_{\bar{a}} = \overline{\Psi}^{\,b}(A^{-1})_{b\bar{a}}\,, \qquad \Psi^{\dagger\,\bar{a}} = A^{\bar{a}b}\,\overline{\Psi}_b\,. \tag{G.1.40}$$

One can check that eqs. (G.1.39) and (G.1.40) are consistent with the Lorentz transformation properties of eqs. (G.1.30)–(G.1.33), as a consequence of eq. (G.1.28).

In addition to the Lorentz scalar $\overline{\Psi}\Psi \equiv \overline{\Psi}^{\,a}\Psi_a$, one can construct two additional independent Lorentz scalar quantities,[132]

$$-\Psi^{\mathsf{T}}C^{-1}\,\Psi \equiv -\Psi_a(C^{-1})^{ab}\Psi_b = \Psi^a\Psi_a\,, \tag{G.1.41}$$

and its hermitian conjugate,

$$\overline{\Psi}\,C\,\overline{\Psi}^{\mathsf{T}} \equiv \overline{\Psi}^{\,a}C_{ab}\overline{\Psi}^{\,b} = \overline{\Psi}^{\,a}\overline{\Psi}_a = \Psi^{\dagger}_{\bar{a}}\Psi^{\dagger\,\bar{a}} = (\Psi^a\Psi_a)^{\dagger}\,, \tag{G.1.42}$$

after using C^{-1} and C to raise and lower the appropriate spinor indices, respectively. The penultimate equality in eq. (G.1.42) is a consequence of eq. (G.1.40). The Lorentz invariance of $\overline{\Psi}^{\,a}\Psi_a$, $\Psi^a\Psi_a$ and $\Psi^{\dagger}_{\bar{a}}\Psi^{\dagger\,\bar{a}} = \overline{\Psi}^{\,a}\overline{\Psi}_a$ is manifest and demonstrates the power of the four-component spinor index notation developed above. After invoking eq. (G.1.38), we note that [analogous to eq. (29.37)] *descending* contracted unbarred spinor indices and *ascending* contracted barred spinor indices can be suppressed in spinor-index-contracted products. For example,

$$\overline{\Psi}^{\,a}\Psi_a \equiv \overline{\Psi}\Psi \qquad \Psi^a\Psi_a = \overline{\Psi^C}^{\,a}\Psi_a \equiv \overline{\Psi^C}\Psi \qquad \overline{\Psi}^{\,a}\overline{\Psi}_a = \overline{\Psi}^{\,a}\Psi^C_a \equiv \overline{\Psi}\Psi^C\,, \tag{G.1.43}$$

where the suppression of barred spinor indices is implicit in the definition of $\overline{\Psi}^{\,b} \equiv \Psi^{\dagger}_{\bar{a}}A^{\bar{a}b}$.

The charge-conjugated spinor can also be written as $\Psi^C_a \equiv D_a{}^{\bar{c}}\Psi^{\dagger}_{\bar{c}}$ [cf. eq. (G.1.18)]. The spinor index structure of D (and its inverse) derives from:

$$D_a{}^{\bar{c}} \equiv C_{ab}(A^{\mathsf{T}})^{b\bar{c}} = C_{ab}A^{\bar{c}b}\,, \qquad (D^{-1})_{\bar{a}}{}^{c} \equiv (C^*A)_{\bar{a}}{}^{c} = C_{\bar{a}\bar{b}}A^{\bar{b}c}\,, \tag{G.1.44}$$

where we have used $D^{-1} = D^*$. Combining the results of eqs. (G.1.35), (G.1.36), (G.1.39) and (G.1.40) then yields:

$$\overline{\Psi}_a = D_a{}^{\bar{c}}\,\Psi^{\dagger}_{\bar{c}}\,, \qquad \Psi^{\dagger}_{\bar{a}} = (D^{-1})_{\bar{a}}{}^{c}\,\overline{\Psi}_c\,, \tag{G.1.45}$$

$$\overline{\Psi}^{\,a} = -\Psi^{\dagger\,\bar{c}}(D^{-1})_{\bar{c}}{}^{a}\,, \qquad \Psi^{\dagger\,\bar{a}} = -\overline{\Psi}^{\,c}\,D_c{}^{\bar{a}}\,. \tag{G.1.46}$$

In summary, a four-component spinor Ψ_a and its charge-conjugated spinor Ψ^C_a possess a lowered unbarred spinor index, whereas the corresponding Dirac conjugates, $\overline{\Psi}^{\,a}$ and $\overline{\Psi^C}^{\,a}$, possess a raised unbarred spinor index. The corresponding hermitian-conjugated spinors exhibit barred spinor indices (with the height of each spinor index unchanged). Following eqs. (G.1.35) and (G.1.36), one can also lower or raise a four-component unbarred or barred spinor index by multiplying by the appropriate matrix C, C^{-1}, C^* or $(C^{-1})^*$, respectively.

The identity matrix, the gamma matrices and their products are denoted collectively by Γ. The spinor index structure of these matrices and their inverses is given by:

$$\delta^b_a\,,\ \Gamma_a{}^b\,,\ (\Gamma^{-1})_a{}^b\,, \tag{G.1.47}$$

where the δ^b_a are the matrix elements of the identity matrix $\mathbb{1}$. In this case, the rows are labeled by the lowered index and the columns are labeled by the raised index. Note that the quantities

[132] A fourth possible Lorentz scalar, $\Psi^a\overline{\Psi}_a = (C^{-1})^{ab}C_{ac}\Psi_b\overline{\Psi}^{\,c} = -\Psi_c\overline{\Psi}^{\,c} = \overline{\Psi}^{\,c}\Psi_c$, is not independent. Here, we have used $C^{\mathsf{T}} = -C$ and the anticommutativity of the spinors. Equivalently, $\overline{\Psi^C}\Psi^C = \overline{\Psi}\Psi$.

$\overline{\Psi}^a\,\Gamma_a{}^b\,\Psi_b$, $\Psi^a\,\Gamma_a{}^b\,\Psi_b$, and $\overline{\Psi}^a\,\Gamma_a{}^b\,\overline{\Psi}^b$ transform as Lorentz tensors, whose rank is equal to the number of (suppressed) spacetime indices of Γ.

For the matrices A, B, C, D and their inverses, the spinor index structure is given by:

$$A^{\bar{a}b}\,,\ (A^{-1})_{a\bar{b}}\,,\ B^{ab}\,,\ (B^{-1})_{ab}\,,C_{ab}\,,\ (C^{-1})^{ab}\,,D_a{}^{\bar{b}}\,,\ (D^{-1})_{\bar{a}}{}^b\,. \tag{G.1.48}$$

The corresponding complex-conjugated matrices exhibit the analogous spinor index structure with unbarred spinor indices changed to barred spinor indices and vice versa. Matrix transposition interchanges rows and columns. For example,

$$(\Gamma^{\mathsf{T}})^a{}_b \equiv \Gamma_b{}^a\,,\qquad (A^{\mathsf{T}})^{a\bar{b}} \equiv A^{\bar{b}a}\,,\qquad (C^{\mathsf{T}})_{ab} = C_{ba}\,,\qquad (D^{\mathsf{T}})^{\bar{a}}{}_b \equiv D_b{}^{\bar{a}}\,. \tag{G.1.49}$$

Hermitian conjugation is complex conjugation followed by matrix transposition. For example,

$$(\Gamma^\dagger)^{\bar{a}}{}_{\bar{b}} \equiv (\Gamma_b{}^a)^*\,,\qquad (A^\dagger)^{\bar{a}b} \equiv (A^{\bar{b}a})^*\,,\qquad (C^\dagger)_{\bar{a}\bar{b}} = (C_{ba})^*\,,\qquad (D^\dagger)^a{}_{\bar{b}} \equiv (D_b{}^{\bar{a}})^*\,. \tag{G.1.50}$$

Using the above results, it is straightforward to identify the four-component spinor index structure of eqs. (G.1.1)–(G.1.29). For example, specifying the four-component spinor indices of eq. (G.1.29) yields:

$$(C^{-1})^{ab}\mathbb{M}_b{}^cC_{cd} = [(\mathbb{M}^{-1})^{\mathsf{T}}]^a{}_d \equiv (\mathbb{M}^{-1})_d{}^a\,. \tag{G.1.51}$$

To complete the spinor index formalism, we introduce hybrid quantities L, $\overline{L}$, R and $\overline{R}$ that contain an unbarred four-component spinor index and a two-component undotted or dotted spinor index [293]:

$$L_\beta{}^b = (\mathbb{1}_{2\times2}\quad \mathbb{O}_{2\times2})\,,\qquad R^{\dot\beta\,b} = (\mathbb{O}_{2\times2}\quad \mathbb{1}_{2\times2})\,, \tag{G.1.52}$$

$$\overline{L}_b{}^\beta = \begin{pmatrix}\mathbb{1}_{2\times2}\\ \mathbb{O}_{2\times2}\end{pmatrix}\,,\qquad \overline{R}_{b\dot\beta} = \begin{pmatrix}\mathbb{O}_{2\times2}\\ \mathbb{1}_{2\times2}\end{pmatrix}\,. \tag{G.1.53}$$

These quantities satisfy:

$$\overline{L}_a{}^\alpha L_\alpha{}^b = (P_L)_a{}^b\,,\qquad L_\alpha{}^a\overline{L}_a{}^\beta = \delta_\alpha{}^\beta\,, \tag{G.1.54}$$

$$\overline{R}_{a\dot\alpha}R^{\dot\alpha\,b} = (P_R)_a{}^b\,,\qquad R^{\dot\alpha\,a}\overline{R}_{a\dot\beta} = \delta^{\dot\alpha}{}_{\dot\beta}\,, \tag{G.1.55}$$

where P_L and P_R are the chiral projection operators defined in eq. (G.1.8). It then follows that:

$$L_\alpha{}^a(P_L)_a{}^b = L_\alpha{}^b\,,\qquad (P_L)_a{}^b\overline{L}_b{}^\beta = \overline{L}_a{}^\beta\,, \tag{G.1.56}$$

$$R^{\dot\alpha\,a}(P_R)_a{}^b = R^{\dot\alpha\,b}\,,\qquad (P_R)_a{}^b\overline{R}_{b\dot\beta} = \overline{R}_{a\dot\beta}\,. \tag{G.1.57}$$

The hybrid quantities L, $\overline{L}$, R and $\overline{R}$ connect objects with four-component and two-component spinor indices. For the Dirac spinor defined in eq. (G.1.7), it follows that:

$$\chi_\alpha = L_\alpha{}^b\Psi_b\,,\qquad \eta^\alpha = \overline{\Psi}^b\,\overline{L}_b{}^\alpha\,, \tag{G.1.58}$$

$$\eta^{\dagger\dot\alpha} = R^{\dot\alpha\,b}\Psi_b\,,\qquad \chi^\dagger_{\dot\alpha} = \overline{\Psi}^b\,\overline{R}_{b\dot\alpha}\,. \tag{G.1.59}$$

The corresponding inverse relations are:

$$(P_L)_a{}^b\,\Psi_b = \overline{L}_a{}^\beta\chi_\beta\,,\qquad \overline{\Psi}^a(P_L)_a{}^b = \eta^\beta L_\beta{}^b\,, \tag{G.1.60}$$

$$(P_R)_a{}^b\,\Psi_b = \overline{R}_{a\dot\beta}\eta^{\dagger\dot\beta}\,,\qquad \overline{\Psi}^a(P_R)_a{}^b = \chi^\dagger_{\dot\beta}R^{\dot\beta\,b}\,. \tag{G.1.61}$$

One can use eqs. (G.1.2), (G.1.4) and (G.1.21) to identify:

$$\sigma^{\mu}_{\alpha\dot\beta} = L_\alpha{}^a(\gamma^\mu)_a{}^b\overline{R}_{b\dot\beta}\,, \qquad \overline{\sigma}^{\mu\dot\alpha\beta} = R^{\dot\alpha\, a}(\gamma^\mu)_a{}^b\overline{L}_b{}^\beta\,, \tag{G.1.62}$$

$$\sigma^{\mu\nu}{}_\alpha{}^\beta = L_\alpha{}^a(\tfrac{1}{2}\Sigma^{\mu\nu})_a{}^b\overline{L}_b{}^\beta\,, \qquad \overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta} = R^{\dot\alpha\, a}(\tfrac{1}{2}\Sigma^{\mu\nu})_a{}^b\overline{R}_{b\dot\beta}\,, \tag{G.1.63}$$

$$\delta_\alpha{}^\beta = -L_\alpha{}^a(\gamma_5)_a{}^b\overline{L}_b{}^\beta\,, \qquad \delta^{\dot\alpha}{}_{\dot\beta} = R^{\dot\alpha\, a}(\gamma_5)_a{}^b\overline{R}_{b\dot\beta}\,, \tag{G.1.64}$$

$$\epsilon_{\alpha\beta} = L_\alpha{}^a C_{ab} L_\beta{}^b\,, \qquad \epsilon^{\dot\alpha\dot\beta} = R^{\dot\alpha\, a} C_{ab} R^{\dot\beta\, b}\,, \tag{G.1.65}$$

$$\epsilon^{\alpha\beta} = \overline{L}_a{}^\alpha (C^{-1})^{ab}\overline{L}_b{}^\beta\,, \qquad \epsilon_{\dot\alpha\dot\beta} = \overline{R}_{a\dot\alpha}(C^{-1})^{ab}\overline{R}_{b\dot\beta}\,. \tag{G.1.66}$$

Inverting these results yields:

$$(\gamma^\mu P_L)_c{}^d = \overline{R}_{c\dot\alpha}\overline{\sigma}^{\mu\dot\alpha\beta}L_\beta{}^d\,, \qquad (\gamma^\mu P_R)_c{}^d = \overline{L}_c{}^\alpha\sigma^\mu_{\alpha\dot\beta}R^{\dot\beta d}\,, \tag{G.1.67}$$

$$\tfrac{1}{2}(\Sigma^{\mu\nu}P_L)_c{}^d = \overline{L}_c{}^\alpha\sigma^{\mu\nu}{}_\alpha{}^\beta L_\beta{}^d \qquad \tfrac{1}{2}(\Sigma^{\mu\nu}P_R)_c{}^d = \overline{R}_{c\dot\alpha}\overline{\sigma}^{\mu\nu\dot\alpha}{}_{\dot\beta}R^{\dot\beta d}\,, \tag{G.1.68}$$

$$(AP_L)_c{}^d = \overline{R}_c{}^\beta L_\beta{}^d\,, \qquad (AP_R)_c{}^d = \overline{L}_{c\dot\beta}R^{\dot\beta d}\,, \tag{G.1.69}$$

$$(P_LC)_{cd} = \epsilon_{\alpha\beta}\overline{L}_c{}^\alpha\overline{L}_d{}^\beta\,, \qquad (P_RC)_{cd} = \epsilon^{\dot\alpha\dot\beta}\overline{R}_{c\dot\alpha}\overline{R}_{d\dot\beta}\,, \tag{G.1.70}$$

$$(C^{-1}P_L)^{cd} = \epsilon^{\alpha\beta}L_\alpha{}^c L_\beta{}^d\,, \qquad (C^{-1}P_R)^{cd} = \epsilon_{\dot\alpha\dot\beta}R^{\dot\alpha c}R^{\dot\beta d}\,. \tag{G.1.71}$$

Likewise, one can introduce $L^\dagger$, $\overline{L}^\dagger$, $R^\dagger$ and $\overline{R}^\dagger$, which are hybrid quantities that contain a barred four-component spinor index and a two-component undotted or dotted spinor index:

$$(L^\dagger)^{\bar a}{}_{\dot\beta} \equiv (L_\beta{}^a)^*\,, \qquad (R^\dagger)^{\bar a\beta} \equiv (R^{\dot\beta a})^*\,, \tag{G.1.72}$$

$$(\overline{L}^\dagger)^{\dot\beta}{}_{\bar a} \equiv (\overline{L}_a{}^\beta)^*\,, \qquad (\overline{R}^\dagger)_{\beta\bar a} \equiv (\overline{R}_{a\dot\beta})^*\,. \tag{G.1.73}$$

In particular, using eqs. (G.1.60) and (G.1.61), one can relate the quantities L, $\overline{L}$, R and $\overline{R}$ and their hermitian conjugates:

$$(L^\dagger)^{\bar a}{}_{\dot\beta} = A^{\bar a b}\overline{R}_{b\dot\beta}\,, \qquad (R^\dagger)^{\bar a\beta} = A^{\bar a b}\overline{L}_b{}^\beta\,, \tag{G.1.74}$$

$$(\overline{L}^\dagger)^{\dot\beta}{}_{\bar a} = R^{\dot\beta b}(A^{-1})_{b\bar a}\,, \qquad (\overline{R}^\dagger)_{\beta\bar a} = L_\beta{}^b(A^{-1})_{b\bar a}\,, \tag{G.1.75}$$

after employing $AP_L = P_R^\dagger A$ [cf. eq. (G.1.23)] and $A^\dagger = A$. The set of equations analogous to eqs. (G.1.54)–(G.1.71) involving the corresponding hermitian-conjugated quantities can also be obtained. However, such formulae will rarely be needed in practice.

Eqs. (G.1.54)–(G.1.71) [and their hermitian conjugates] can be employed to translate any expression involving two-component spinors into the corresponding expression involving four-component spinors, and vice versa. With a little practice, both two-component and four-component spinor indices can be suppressed, which greatly simplifies the manipulation of the spinor quantities. In particular, by treating the four-component spinors Ψ_a and Ψ^C_a as column vectors and their hermitian (Dirac) conjugates $\Psi^\dagger_{\bar a}$ and $\Psi^{C\,\dagger}_{\bar a}$ ($\overline{\Psi}^a$ and $\overline{\Psi^C}^a$) as row vectors, all equations in the four-component spinor formalism have a natural interpretation as products of matrices and vectors. Henceforth, we shall suppress all four-component spinor indices.

Multiple species of fermions are indicated with a flavor index such as i and j. Dirac fermions are constructed from two-component fields of opposite charge, χ_i and η^i (hence the opposite flavor index heights). Thus, we establish the following conventions for the flavor indices of four-component Dirac fermions:

$$\Psi_i(x) \equiv \begin{pmatrix}\chi_{\alpha i}(x)\\ \eta_i^{\dagger\dot\alpha}(x)\end{pmatrix}\,, \qquad \overline{\Psi}^i(x) = \begin{pmatrix}\eta^{\alpha i}(x)\,,\ \chi^{\dagger\, i}_{\dot\alpha}(x)\end{pmatrix}\,, \qquad \Psi^{C\,i}(x) \equiv \begin{pmatrix}\eta^i_\alpha(x)\\ \chi^{\dagger\dot\alpha i}(x)\end{pmatrix}\,. \tag{G.1.76}$$

Note that $\chi^{\dagger\,i} = (\chi_i)^\dagger$ and $\eta_i^\dagger \equiv (\eta^i)^\dagger$ following the conventions established in Section 29.3.2. Raised flavor indices can only be contracted with lowered flavor indices and vice versa. In contrast, Majorana fermions are neutral so that there is no a priori distinction between raised and lowered flavor indices. That is,

$$\Psi_{Mi}(x) = \Psi_M^i(x) = \Psi_{Mi}^C(x) = \Psi_M^{C\,i}(x) \equiv \begin{pmatrix} \xi_{\alpha i}(x) \\ \xi^{\dagger\dot\alpha i}(x) \end{pmatrix}, \qquad \overline{\Psi}_{Mi}(x) = \overline{\Psi}_M^i(x) \equiv \Big(\xi_i^\alpha(x)\,,\ \xi_{\dot\alpha}^{\dagger\,i}(x)\Big)\,. \tag{G.1.77}$$

In this case, the contraction of two repeated flavor indices is allowed in all cases, irrespective of the heights of the two indices. In the convention adopted in Section 29.3.2, in which all neutral left-handed $(\frac{1}{2},0)$ [right-handed $(0,\frac{1}{2})$] fermions have lowered [raised] flavor indices, the height of the flavor index of a four-component Majorana fermion field is meaningful when multiplied by a left-handed or right-handed projection operator. Thus, the height of the flavor index for Majorana fermions can be consistently chosen according to one of the following four cases:

$$P_L\Psi_{Mi}\,, \qquad \overline{\Psi}_{Mi}P_L\,, \qquad P_R\Psi_M^i\,, \qquad \overline{\Psi}_M^i P_R\,. \tag{G.1.78}$$

Bilinear covariants are quantities that are quadratic in the spinor fields and transform irreducibly as Lorentz tensors. We first construct a translation table between the two-component form and the four-component form for the bilinear covariants made up of a pair of Dirac fields. Using eqs. (G.1.60) and (G.1.61) to convert the four-component spinor fields into the corresponding two-component spinor fields, and employing the appropriate identities involving products of the hybrid quantities L, $\overline{L}$, R and $\overline{R}$, the following results are then obtained:

$$\overline{\Psi}^i P_L\Psi_j = \eta^{\,i}\chi_j\,, \tag{G.1.79}$$

$$\overline{\Psi}^i P_R\Psi_j = \chi^{\dagger\,i}\eta_j^\dagger\,, \tag{G.1.80}$$

$$\overline{\Psi}^i\gamma^\mu P_L\Psi_j = \chi^{\dagger\,i}\overline{\sigma}^\mu\chi_j\,, \tag{G.1.81}$$

$$\overline{\Psi}^i\gamma^\mu P_R\Psi_j = \eta^{\,i}\sigma^\mu\eta_j^\dagger\,, \tag{G.1.82}$$

$$\overline{\Psi}^i\Sigma^{\mu\nu} P_L\Psi_j = 2\,\eta^{\,i}\sigma^{\mu\nu}\chi_j\,, \tag{G.1.83}$$

$$\overline{\Psi}^i\Sigma^{\mu\nu} P_R\Psi_j = 2\,\chi^{\dagger\,i}\overline{\sigma}^{\mu\nu}\eta_j^\dagger\,. \tag{G.1.84}$$

The first two results above follow immediately after using eqs. (G.1.54) and (G.1.55), respectively, and the last four results are a consequence of eqs. (G.1.62) and (G.1.63).

Eqs. (G.1.79)–(G.1.84) apply to both commuting and anticommuting fermion fields.[133] These results can then be used to express the standard four-component spinor bilinear covariants in terms of two-component spinor bilinears:

$$\overline{\Psi}^i\Psi_j = \eta^{\,i}\chi_j + \chi^{\dagger\,i}\eta_j^\dagger \tag{G.1.85}$$

$$\overline{\Psi}^i\gamma_5\Psi_j = -\eta^{\,i}\chi_j + \chi^{\dagger\,i}\eta_j^\dagger \tag{G.1.86}$$

$$\overline{\Psi}^i\gamma^\mu\Psi_j = \chi^{\dagger\,i}\overline{\sigma}^\mu\chi_j + \eta^{\,i}\sigma^\mu\eta_j^\dagger \tag{G.1.87}$$

$$\overline{\Psi}^i\gamma^\mu\gamma_5\Psi_j = -\chi^{\dagger\,i}\overline{\sigma}^\mu\chi_j + \eta^{\,i}\sigma^\mu\eta_j^\dagger \tag{G.1.88}$$

$$\overline{\Psi}^i\Sigma^{\mu\nu}\Psi_j = 2(\eta^{\,i}\sigma^{\mu\nu}\chi_j + \chi^{\dagger\,i}\overline{\sigma}^{\mu\nu}\eta_j^\dagger) \tag{G.1.89}$$

$$\overline{\Psi}^i\Sigma^{\mu\nu}\gamma_5\Psi_j = 2(-\eta^{\,i}\sigma^{\mu\nu}\chi_j + \chi^{\dagger\,i}\overline{\sigma}^{\mu\nu}\eta_j^\dagger)\,. \tag{G.1.90}$$

[133] In the case of anticommuting spinors, it is often useful to apply eq. (29.62) to eqs. (G.1.82), (G.1.87) and (G.1.88) and rewrite $\eta^{\,i}\sigma^\mu\eta_j^\dagger = -\eta_j^\dagger\overline{\sigma}^\mu\eta^{\,i}$.

Additional identities can be derived that involve the charge-conjugated four-component Dirac fermion fields. As an example, we may use $C^{\mathsf{T}} = -C$ and $\overline{\Psi^C} = -\Psi^{\mathsf{T}} C^{-1}$ to prove that

$$\overline{\Psi^C_i}\,\Gamma\,\Psi^{C\,j} = -(-1)^A \overline{\Psi}^{\,j} C\,\Gamma^{\mathsf{T}}\,C^{-1}\Psi_i = -(-1)^A \eta^C_\Gamma\,\overline{\Psi}^{\,j}\,\Gamma\,\Psi_i\,, \tag{G.1.91}$$

where the sign η^C_Γ is given in eq. (G.1.25). The factor of $(-1)^A = \pm 1$ [for commuting/anticommuting fermion fields, respectively] arises at the second step above after reversing the order of the terms by matrix transposition. Identities involving just one charge-conjugated four-component field can also be easily obtained. For example, using eqs. (G.1.58) and (G.1.71),

$$\overline{\Psi^C_i} P_L \Psi_j = -\Psi_i^{\mathsf{T}} C^{-1} P_L \Psi_j = -\Psi_{ai}\epsilon^{\alpha\beta} L_\alpha{}^a L_\beta{}^b \Psi_{bj} = -\epsilon^{\alpha\beta}\chi_{\alpha i}\chi_{\beta j} = \chi_i\chi_j\,. \tag{G.1.92}$$

In general, if one replaces Ψ_k with $\Psi^{C\,k}$ in eqs. (G.1.79)–(G.1.90), then in the corresponding two-component expression one simply interchanges $\chi_k \leftrightarrow \eta^k$ and $\chi^{\dagger\,k} \leftrightarrow \eta^\dagger_k$.

Eqs. (G.1.79)–(G.1.90) also apply to four-component Majorana spinors, Ψ_{Mi}, by setting $\chi_i = \eta^i \equiv \xi_i$, and $\chi^{\dagger\,i} = \eta^\dagger_i \equiv \xi^{\dagger\,i}$. This implements the Majorana condition, $\Psi_{Mi} = D\Psi^*_{Mi}$, and imposes additional restrictions on the Majorana bilinear covariants. For example, eqs. (G.1.25) and (G.1.91) imply that *anticommuting* Majorana four-component fermions satisfy:[134]

$$\overline{\Psi}_{Mi}\Psi_{Mj} = \overline{\Psi}_{Mj}\Psi_{Mi}\,, \tag{G.1.93}$$
$$\overline{\Psi}_{Mi}\gamma_5\Psi_{Mj} = \overline{\Psi}_{Mj}\gamma_5\Psi_{Mi}\,, \tag{G.1.94}$$
$$\overline{\Psi}_{Mi}\gamma^\mu\Psi_{Mj} = -\overline{\Psi}_{Mj}\gamma^\mu\Psi_{Mi}\,, \tag{G.1.95}$$
$$\overline{\Psi}_{Mi}\gamma^\mu\gamma_5\Psi_{Mj} = \overline{\Psi}_{Mj}\gamma^\mu\gamma_5\Psi_{Mi}\,, \tag{G.1.96}$$
$$\overline{\Psi}_{Mi}\Sigma^{\mu\nu}\Psi_{Mj} = -\overline{\Psi}_{Mj}\Sigma^{\mu\nu}\Psi_{Mi}\,, \tag{G.1.97}$$
$$\overline{\Psi}_{Mi}\Sigma^{\mu\nu}\gamma_5\Psi_{Mj} = -\overline{\Psi}_{Mj}\Sigma^{\mu\nu}\gamma_5\Psi_{Mi}\,. \tag{G.1.98}$$

By setting $i = j$, it follows that $\overline{\Psi}_M\gamma^\mu\Psi_M = \overline{\Psi}_M\Sigma^{\mu\nu}\Psi_M = \overline{\Psi}_M\Sigma^{\mu\nu}\gamma_5\Psi_M = 0$. One additional useful result is:

$$\overline{\Psi}^i_M\gamma^\mu P_L\Psi_{Mj} = -\overline{\Psi}_{Mj}\gamma^\mu P_R\Psi^i_M\,, \tag{G.1.99}$$

which follows immediately from eqs. (G.1.95) and (G.1.96). Note that in eq. (G.1.99), the heights of the flavor indices follow the convention established in eq. (G.1.78).

In the four-component spinor formalism, Fierz identities (first introduced in [294]) consist of relations among products of two bilinear covariants, in which the fermion fields appear in two different orders. The corresponding two-component spinor Fierz identities are treated in detail in Appendix B.1 . In principle, the latter can be converted into four-component spinor Fierz identities using the techniques developed in this Appendix. However, it is easier to derive the four-component spinor Fierz identities directly using the properties of the gamma matrix algebra [291, 284].

Instead of eqs. (B.1.6)–(B.1.8), the equivalent identity relevant for four-component spinors is:

$$\delta^b_a\delta^d_c = \tfrac{1}{4}\left[\delta^d_a\delta^b_c + (\gamma_5)_a{}^d(\gamma_5)_c{}^b + (\gamma^\mu)_a{}^d(\gamma_\mu)_c{}^b - (\gamma^\mu\gamma_5)_a{}^d(\gamma_\mu\gamma_5)_c{}^b + \tfrac{1}{2}(\Sigma^{\mu\nu})_a{}^d(\Sigma_{\mu\nu})_c{}^b\right]. \tag{G.1.100}$$

This is the fundamental identity from which many other such identities can be derived (cf. the Appendix of [288]). One of many possible Fierz identities can be obtained by multiplying eq. (G.1.100) by $\overline{\Psi}^a_1\Psi_{2b}\overline{\Psi}^c_3\Psi_{4d} = (-1)^A\,\overline{\Psi}^a_1\Psi_{4d}\overline{\Psi}^c_3\Psi_{2b}$, where $(-1)^A = +1$ [-1] for commuting [anticommuting] Dirac, Majorana or Weyl spinors. More generally [284, 295, 296],

$$(\overline{\Psi}_1\Gamma^{(k)I}\Psi_2)(\overline{\Psi}_3\Gamma^{(k)}_I\Psi_4) = (-1)^A\sum_{n=1}^{5} F^k{}_n\,(\overline{\Psi}_1\Gamma^{(n)J}\Psi_4)(\overline{\Psi}_3\Gamma^{(n)}_J\Psi_2)\,, \tag{G.1.101}$$

[134] Here, one is free to choose all flavor indices to be in the lowered position [cf. eq. (G.1.77)].

where the sum is taken over the 4×4 matrices, $\Gamma^{(n)}\in\Gamma$, which have been ordered as follows,[135]

$$\Gamma = \{\mathbb{1}\,,\ \gamma^\mu\,,\ \Sigma^{\mu\nu}\,(\mu<\nu)\,,\ \gamma^\mu\gamma_5\,,\ \gamma_5\}\,, \tag{G.1.102}$$

I, J represent zero, one or two spacetime indices (sums over repeated I and J are implied), and

$$F = \frac{1}{4}\begin{pmatrix} 1 & 1 & \frac{1}{2} & -1 & 1\\ 4 & -2 & 0 & -2 & -4\\ 12 & 0 & -2 & 0 & 12\\ -4 & -2 & 0 & -2 & 4\\ 1 & -1 & \frac{1}{2} & 1 & 1\end{pmatrix}. \tag{G.1.103}$$

For example, taking $k=1$ in eq. (G.1.101) yields a result equivalent to eq. (G.1.100):

$$\begin{aligned}(\overline{\Psi}_1\Psi_2)(\overline{\Psi}_3\Psi_4) = \tfrac{1}{4}(-1)^A\big[&(\overline{\Psi}_1\Psi_4)(\overline{\Psi}_3\Psi_2)+(\overline{\Psi}_1\gamma_5\Psi_4)(\overline{\Psi}_3\gamma_5\Psi_2)+(\overline{\Psi}_1\gamma^\mu\Psi_4)(\overline{\Psi}_3\gamma_\mu\Psi_2)\\ &-(\overline{\Psi}_1\gamma^\mu\gamma_5\Psi_4)(\overline{\Psi}_3\gamma_\mu\gamma_5\Psi_2)+\tfrac{1}{2}(\overline{\Psi}_1\Sigma^{\mu\nu}\Psi_4)(\overline{\Psi}_3\Sigma_{\mu\nu}\Psi_2)\big]\,.\end{aligned} \tag{G.1.104}$$

For a comprehensive treatment of all possible four-component spinor Fierz identities, see [297]. Simple derivations of generalized Fierz identities have also been given in refs. [296, 298]. A Mathematica package for performing Fierz transformations is available in [299].

Free-field four-component fermion Lagrangians

The free-field Lagrangian density in four-component spinor notation can be obtained from the corresponding two-component fermion Lagrangian by employing the relevant identities for the bilinear covariants given in eqs. (G.1.79)–(G.1.90). First, consider a collection of free anticommuting four-component Majorana fields, $\Psi_{Mi}=\Psi_{Mi}^C$. The free-field Lagrangian (in terms of mass eigenstate fields) may be obtained from eq. (29.3.10) by converting to four-component spinor notation using eqs. (G.1.85) and (G.1.87) with $\chi=\eta\equiv\xi$, which yields [3]:

$$\mathscr{L} = \tfrac{1}{2}i\overline{\Psi}_{Mi}\gamma^\mu\partial_\mu\Psi_{Mi} - \tfrac{1}{2}m_i\overline{\Psi}_{Mi}\Psi_{Mi}\,, \tag{G.2.1}$$

where the sum over i is implicit. The corresponding free-field equation for Ψ_{Mi} is the Dirac equation:

$$(i\gamma^\mu\partial_\mu - m)\Psi_{Mi} = 0\,. \tag{G.2.2}$$

For simplicity, we focus on a theory of a single four-component Majorana fermion field, $\Psi_M(x)=\Psi_M^C(x)$. One can rewrite the free-field Majorana fermion Lagrangian in terms of a single Weyl fermion, $\Psi_L(x)\equiv P_L\Psi(x)$, where $\Psi(x)$ is a four-component fermion field whose lower two components (in the chiral representation) are irrelevant for the present discussion. The Majorana and Weyl fields are related by:

$$\Psi_{\!M}(x) = \Psi_{\!L}(x) + \Psi_{\!R}^C(x)\,, \tag{G.2.3}$$

where $\Psi_{\!R}^C(x)$ is defined in eq. (G.1.14). The corresponding Dirac conjugate field is given by $\overline{\Psi}_{\!M}(x)=\overline{\Psi}_{\!L}(x)+\overline{\Psi_{\!R}^C}(x)$, where

$$\overline{\Psi}_{\!L}(x) \equiv [P_L\Psi(x)]^\dagger A = \overline{\Psi}(x)P_R\,, \tag{G.2.4}$$

$$\overline{\Psi_{\!R}^C}(x) \equiv \overline{\Psi^C}(x)P_L = -\Psi^{\mathsf{T}}(x)C^{-1}P_L = -\Psi_L^{\mathsf{T}}(x)C^{-1}\,. \tag{G.2.5}$$

Using the identity:[136]

$$\overline{\Psi_{\!R}^C}\gamma^\mu\partial_\mu\Psi_{\!R}^C = -\Psi^{\mathsf{T}}C^{-1}P_L\gamma^\mu\partial_\mu P_RC\overline{\Psi}^{\mathsf{T}} = \overline{\Psi}_{\!L}\gamma^\mu\partial_\mu\Psi_{\!L} + \text{total divergence}\,, \tag{G.2.6}$$

[135] The 16 matrices of Γ constitute a complete set that spans the sixteen-dimensional vector space of 4×4 matrices.

[136] In deriving eq. (G.2.6), we have used eq. (G.1.25) and the anticommutativity of the spinor fields. The total divergence can be dropped from the Lagrangian, as it does not contribute to the field equations.

the Lagrangian for a single Majorana field can be written in terms of a single Weyl field:[137]

$$\mathscr{L} = i\overline{\Psi}_L\gamma^\mu\partial_\mu\Psi_L + \tfrac{1}{2}m\left(\Psi_L^{\mathsf{T}}C^{-1}\Psi_L - \overline{\Psi}_L C\overline{\Psi}_L^{\mathsf{T}}\right) . \tag{G.2.7}$$

The corresponding free-field equation is

$$i\gamma^\mu\partial_\mu\Psi_L = mC\overline{\Psi}_L^{\mathsf{T}} , \tag{G.2.8}$$

where we have used $(\overline{\Psi}_L C)^{\mathsf{T}} = -C\overline{\Psi}_L^{\mathsf{T}}$ and the anticommutativity of Ψ_L, $\overline{\Psi}_L$. The generalization of eqs. (G.2.3)–(G.2.7) to the case of a multiplet of four-component Majorana fields is straightforward and is left as an exercise for the reader.

Of course, one could have chosen instead to rewrite the four-component Majorana fermion Lagrangian in terms of a single Weyl fermion, $\Psi_R(x) \equiv P_R\Psi(x)$, in which case the upper two components (in the chiral representation) of $\Psi(x)$ are not relevant. In this case, the Majorana and Weyl fields are related by:[138]

$$\Psi_M(x) = \Psi_R(x) + \Psi_L^C(x) , \tag{G.2.9}$$

where $\Psi_L^C(x)$ is defined in eq. (G.1.13). The corresponding Dirac conjugate field is given by $\overline{\Psi}_M(x) = \overline{\Psi}_R(x) + \overline{\Psi_L^C}(x)$, where

$$\overline{\Psi}_R(x) \equiv [P_R\Psi(x)]^\dagger A = \overline{\Psi}(x)P_L , \tag{G.2.10}$$

$$\overline{\Psi_L^C}(x) \equiv \overline{\Psi^C}(x)P_R = -\Psi^{\mathsf{T}}(x)C^{-1}P_R = -\Psi_R^{\mathsf{T}}(x)C^{-1} . \tag{G.2.11}$$

The corresponding Weyl fermion Lagrangian is given by eq. (G.2.7) with L replaced by R.

Thus, a Majorana fermion can be represented either by a four-component self-conjugate field $\Psi_M(x)$ or by a single Weyl field [either $\Psi_L(x)$ or $\Psi_R(x)$]. Both descriptions are unitarily equivalent [290, 300]; i.e., one can construct a unitary similarity transformation that connects a Majorana field operator and a Weyl field operator (and vice versa). Of course, this is hardly a surprise in the two-component spinor formalism, where both the Majorana and Weyl forms of the Lagrangian correspond to the same field theory of a single two-component spinor field $\xi_\alpha(x)$.

For $m \neq 0$, the Weyl Lagrangian given by eq. (G.2.7) possesses no global symmetry, and hence no conserved charge. In contrast, for $m = 0$ the Weyl Lagrangian exhibits a U(1) chiral symmetry. In a theory of massless neutrinos, the U(1) chiral charge of the neutrino is correlated with its lepton number L, and one is free to use either a Majorana or Weyl description. In the former, the neutrino is a neutral self-conjugate fermion, which is not an eigenstate of L. In the latter, $\Psi_L(x)$ corresponds to the left-handed neutrino and $\Psi_R^C(x)$ corresponds to the right-handed antineutrino, which are eigenstates of L with opposite sign lepton numbers. No experimental observable can distinguish between these two descriptions.

We now consider a collection of free anticommuting four-component Dirac fields, Ψ_i. The free-field Lagrangian (in terms of mass eigenstate fields) may be obtained from eq. (29.3.34) by converting to four-component spinor notation. We then obtain the standard textbook result:

$$\mathscr{L} = i\overline{\Psi}^i\gamma^\mu\partial_\mu\Psi_i - m_i\overline{\Psi}^i\Psi_i . \tag{G.2.12}$$

By writing $\Psi = \Psi_L + \Psi_R$, we see that the Lagrangian for a single Dirac field can be written in terms of two Weyl fields:

$$\mathscr{L} = i\overline{\Psi}_L\gamma^\mu\partial_\mu\Psi_L + i\overline{\Psi}_R\gamma^\mu\partial_\mu\Psi_R - m\left(\overline{\Psi}_L\Psi_R + \overline{\Psi}_R\Psi_L\right) . \tag{G.2.13}$$

[137] Using eq. (G.1.16), it follows that $(\Psi^{\mathsf{T}}C^{-1}\Psi)^\dagger = -\overline{\Psi}A^{-1}C^{-1\,*}A^{-1\,*}\overline{\Psi}^{\mathsf{T}} = -\overline{\Psi}\,C\,\overline{\Psi}^{\mathsf{T}}$.

[138] If Ψ is an unconstrained four-component spinor, then Ψ_L and Ψ_R are independent Weyl fields, in which case $\Psi_L + \Psi_R^C$ and $\Psi_R + \Psi_L^C$ are independent self-conjugate fields.

The corresponding free-field equations are:

$$i\gamma^\mu\partial_\mu\Psi_L = m\Psi_R\,, \qquad\qquad i\gamma^\mu\partial_\mu\Psi_R = m\Psi_L\,. \tag{G.2.14}$$

Summing these two equations yields the Dirac equation, $(i\gamma^\mu\partial_\mu - m)\Psi = 0$.

As a pedagogical example in which both Dirac and Majorana mass terms are present, we perform the diagonalization of the neutrino mass matrix in a one-generation seesaw model[139] using the four-component spinor formalism. Following Appendix A of [301], we first introduce a four-component anticommuting neutrino field ν_D, and the corresponding Weyl fields,

$$\nu_L \equiv P_L\nu_D\,, \qquad \nu_L^C \equiv P_L\nu_D^C\,, \qquad \nu_R \equiv P_R\nu_D\,, \quad \text{and} \quad \nu_R^C \equiv P_R\nu_D^C\,. \tag{G.2.15}$$

Note that eqs. (G.1.13) and (G.2.5) imply that the anticommuting Weyl fermion fields satisfy:

$$\overline{\nu_R^C}\nu_L^C = \overline{\nu_R}\nu_L\,, \qquad\qquad \overline{\nu_L^C}\nu_R^C = \overline{\nu_L}\nu_R\,. \tag{G.2.16}$$

A Dirac mass term for the neutrinos in the one-generation seesaw model couples ν_L and ν_L^C (and by hermiticity of the Lagrangian, ν_R^C and ν_R), and can be written equivalently as:

$$m_D(\nu_L^{\mathsf{T}}C^{-1}\nu_L^C + \nu_R^{\mathsf{T}}C^{-1}\nu_R^C) = -m_D(\overline{\nu_R^C}\nu_L^C + \overline{\nu_L^C}\nu_R^C) = -m_D(\overline{\nu_R}\nu_L + \overline{\nu_L}\nu_R) = -m_D\overline{\nu}_D\nu_D\,, \tag{G.2.17}$$

after making use of eq. (G.2.16). The Majorana mass term for the neutrinos in the one-generation seesaw model couples ν_L^C to itself (and by hermiticity of the Lagrangian, ν_R to itself), and can be written equivalently as:

$$\tfrac{1}{2}M(\nu_L^{C\,\mathsf{T}}C^{-1}\nu_L^C + \nu_R^{\mathsf{T}}C^{-1}\nu_R) = -\tfrac{1}{2}M(\overline{\nu_R}\nu_L^C + \overline{\nu_L^C}\nu_R)\,. \tag{G.2.18}$$

We shall define the phases of the neutrino fields such that the parameters m_D and M are real and non-negative.

Thus, the mass terms of the one-generation neutrino seesaw Lagrangian, given in eq. (J.2.18) in terms of two-component fermion fields, translates in four-component spinor notation to

$$\begin{aligned}
\mathscr{L}_{\text{mass}} &= -\tfrac{1}{2}m_D(\overline{\nu_L}\nu_R + \overline{\nu_R}\nu_L + \overline{\nu_L^C}\nu_R^C + \overline{\nu_R^C}\nu_L^C) - \tfrac{1}{2}M(\overline{\nu_R}\nu_L^C + \overline{\nu_L^C}\nu_R) \\
&= -\tfrac{1}{2}\begin{pmatrix}\overline{\nu_R^C} & \overline{\nu_R}\end{pmatrix}\begin{pmatrix}0 & m_D\\ m_D & M\end{pmatrix}\begin{pmatrix}\nu_L\\ \nu_L^C\end{pmatrix} - \tfrac{1}{2}\begin{pmatrix}\overline{\nu_L} & \overline{\nu_L^C}\end{pmatrix}\begin{pmatrix}0 & m_D\\ m_D & M\end{pmatrix}\begin{pmatrix}\nu_R^C\\ \nu_R\end{pmatrix} \\
&= \tfrac{1}{2}\begin{pmatrix}\nu_L^{\mathsf{T}} & \nu_L^{C\,\mathsf{T}}\end{pmatrix}C^{-1}\begin{pmatrix}0 & m_D\\ m_D & M\end{pmatrix}\begin{pmatrix}\nu_L\\ \nu_L^C\end{pmatrix} + \text{h.c.}\,,
\end{aligned} \tag{G.2.19}$$

where we have used eq. (G.2.16) to write the first line of eq. (G.2.19) in a symmetrical fashion and eqs. (G.1.13) and (G.2.5) to obtain the final form above. Note that if $M = 0$, then one can write $\mathscr{L}_{\text{mass}} = -m_D\overline{\nu}_D\nu_D$ and identify ν_D as a four-component massive Dirac neutrino.

The Takagi diagonalization of the neutrino mass matrix yields two mass eigenstates, which we designate by ν_ℓ and ν_h, where ℓ and h stand for *light* and *heavy*, respectively. The mass eigenstate Weyl neutrino fields are related to the interaction eigenstate Weyl neutrino fields via

$$\begin{pmatrix}\nu_L\\ \nu_L^C\end{pmatrix} = \mathcal{U}\begin{pmatrix}P_L\nu_\ell\\ P_L\nu_h^C\end{pmatrix}\,, \tag{G.2.20}$$

[139] In Appendix J.2 , the seesaw model of neutrino masses is introduced using the two-component spinor formalism.

where $\mathcal{U}$ is a 2×2 unitary matrix that is chosen such that

$$\mathcal{U}^{\mathsf{T}}\begin{pmatrix} 0 & m_D \\ m_D & M \end{pmatrix}\mathcal{U} = \begin{pmatrix} m_{\nu_\ell} & 0 \\ 0 & m_{\nu_h} \end{pmatrix}. \tag{G.2.21}$$

For $M \neq 0$, the neutrino mass eigenstates are *not* Dirac fermions. In the seesaw limit of $M \gg m_D$, the corresponding neutrino masses are $m_{\nu_\ell} \simeq m_D^2/M$ and $m_{\nu_h} \simeq M + m_D^2/M$, with $m_{\nu_\ell} \ll m_{\nu_h}$. In terms of the mass eigenstates, the neutrino mass Lagrangian is:

$$\mathscr{L}_{\rm mass} = \tfrac{1}{2}\left[m_{\nu_\ell}\nu_\ell^{\mathsf{T}}C^{-1}P_L\nu_\ell + m_{\nu_h}\nu_h^{C\,\mathsf{T}}C^{-1}P_L\nu_h^C\right] + \text{h.c.}\,, \tag{G.2.22}$$

after using eq. (G.1.25). We now define four-component self-conjugate Majorana neutrino fields, denoted by Ψ_ℓ and Ψ_h respectively, according to eqs. (G.2.3) and (G.2.9),

$$\Psi_\ell \equiv P_L\nu_\ell + P_R\,C\overline{\nu}_\ell^{\mathsf{T}}\,, \qquad \overline{\Psi}_\ell \equiv \overline{\nu}_\ell P_R - \nu_\ell^{\mathsf{T}}C^{-1}P_L\,, \tag{G.2.23}$$

$$\Psi_h \equiv P_R\nu_h + P_L C\overline{\nu}_h^{\mathsf{T}}\,, \qquad \overline{\Psi}_h \equiv \overline{\nu}_h P_L - \nu_h^{\mathsf{T}}C^{-1}P_R\,. \tag{G.2.24}$$

Then, eq. (G.2.22) reduces to the expected form:

$$\mathscr{L}_{\rm mass} = -\tfrac{1}{2}\left[m_{\nu_\ell}\overline{\Psi}_\ell\Psi_\ell + m_{\nu_h}\overline{\Psi}_h\Psi_h\right]. \tag{G.2.25}$$

A comparison with the analysis of the neutrino mass matrix given in Appendix J.2 exhibits the power and the simplicity of the two-component spinor formalism, as compared to the rather awkward four-component spinor analysis presented above.

29.7.8 Correspondence to four-component spinor notation 2

Gamma matrices and spinors in spacetimes of diverse dimensions and signatures

The translation from two-component to four-component spinor notation given in Appendix G.1 is specific to $3+1$ spacetime dimensions. In $d=4$ Euclidean space dimensions (independently of the choice of convention for the Minkowski metric), the Dirac gamma matrix algebra is defined by $\{\gamma_E^\mu\,,\,\gamma_E^\nu\} = 2\delta^{\mu\nu}\mathbb{1}$, where $\delta^{\mu\nu} \equiv \text{diag}(1\,,\,1\,,\,1\,,\,1)$. Using eqs. (A.23) and (G.1.2), the Euclidean gamma matrices (defined for $\mu,\,\nu = 1\,,\,\ldots\,,\,4$) are hermitian and given by $\gamma_E^k \equiv -i\gamma^k$ $(k=1,2,3)$, $\gamma_E^4 \equiv \gamma^0$ and $\gamma_{5E} \equiv -\gamma_E^1\gamma_E^2\gamma_E^3\gamma_E^4 = \gamma_5$ (e.g, see Appendix A.1.2 of [302]).[140] The four-dimensional reducible (Dirac) spinor representation corresponds to the $(\frac{1}{2},0)\otimes(0,\frac{1}{2})$ representation of SO(4), although the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations are independent pseudo-real representations of SO(4) not related by hermitian conjugation, as noted at the end of Section 29.2. A complete treatment of Euclidean two-component spinors can be found in [130].

The Euclidean space formalism for fermions is necessary for a rigorous definition of the path integral in quantum field theory [122, 123]. Using the Euclidean Dirac gamma matrices introduced above, one can express the four-component Dirac Lagrangian directly in Euclidean space [223]. Carrying out the same procedure for the four-component Majorana Lagrangian is problematical. Because the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations of SO(4) are not hermitian conjugates of each other, a self-conjugate Euclidean Majorana fermion does not exist. Nevertheless, it is possible to devise a continuous Wick rotation from Minkowski spacetime to Euclidean space for Dirac, Majorana

[140] One can also choose to define the Euclidean Dirac algebra by $\{\gamma_E^\mu\,,\,\gamma_E^\nu\} = -2\delta^{\mu\nu}\mathbb{1}$ (simply by multiplying all gamma matrices by a factor of i), in which case the Euclidean gamma matrices, $\gamma_E^k \equiv \gamma^k$ and $\gamma_E^4 \equiv i\gamma^0$ are anti-hermitian, and $\gamma_{5E} \equiv -\gamma_E^1\gamma_E^2\gamma_E^3\gamma_E^4 = \gamma_5$ is hermitian (e.g., see [303]). These conventions arise more naturally in the general treatment of gamma matrices in d spacetime dimensions as defined in eq. (G.3.1). The corresponding Euclidean sigma matrices would then be defined as in footnote 96.

and Weyl spinor fields and the gamma matrices. In particular, one can construct a non-hermitian Euclidean action for a single Majorana or Weyl field whose Green functions are related to the usual Minkowski space Green functions by analytic continuation and a Wick rotation of the spinor fields. Further details can be found in refs. [127, 128, 129].[141]

The two-component spinor technology of this review is specifically designed to treat spinors in three space and one time dimension. In theories of d spacetime dimensions (where d is any positive integer), more general techniques are required. By considering spinors in this more general setting, one gains insight into the concepts of Majorana, Weyl and Dirac spinors and their distinguishing features.

The mathematics of spinors [132] in spacetimes of dimension $d = t + s$ (where t is the number of time dimensions and s is the number of space dimensions) is most easily treated by introducing higher-dimensional analogues of the gamma matrices, Γ^μ, which satisfy the Clifford algebra [91, 92, 129, 133, 134, 135, 136, 137, 138, 142, 143, 144, 292, 304],[142]

$$\{\Gamma^\mu\,,\,\Gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}\,, \qquad \eta^{\mu\nu} = \mathrm{diag}(\underbrace{+\,+\cdots+}_{t}\,,\underbrace{-\,-\cdots-}_{s})\,, \tag{G.3.1}$$

where the identity matrix $\mathbb{1}$ and the Γ^μ are $2^{[d/2]}\times 2^{[d/2]}$ matrices, and $[d/2]$ is the integer part of $d/2$,

$$[d/2] \equiv \begin{cases} d/2\,, & \text{for } d \text{ even},\\ (d-1)/2\,, & \text{for } d \text{ odd}. \end{cases} \tag{G.3.2}$$

The choice of (s,t) denotes the signature of the spacetime. One can choose $\Gamma^{\mu\,\dagger} = \Gamma^\mu$ for $\mu = 1,2,...,t$ and $\Gamma^{\mu\,\dagger} = -\Gamma^\mu$ for $\mu = t{+}1\,,\ t{+}2\,,\ ...\,,\ d$. We identify $\frac{1}{2}\Sigma^{\mu\nu} \equiv \frac{1}{4}\,i\,[\Gamma^\mu\,,\,\Gamma^\nu]$ as the generators of $\mathrm{SO}(s,t)$ in the spinor representation. Next, we introduce the $[d/2]$-component (complex) Dirac spinor Ψ and its Dirac conjugate $\overline{\Psi} \equiv \Psi^\dagger A$, where $A = \Gamma^1\Gamma^2\cdots\Gamma^t$ is a unitary matrix that satisfies:[143]

$$A\Gamma^\mu A^{-1} = (-1)^{t+1}\Gamma^{\mu\,\dagger}\,, \qquad A^\dagger = (-1)^{t(t-1)/2}A\,. \tag{G.3.3}$$

One can now build $\mathrm{SO}(s,t)$-covariant bilinears, $\overline{\Psi}\Gamma\Psi$, where Γ is a product of gamma matrices. Biquadratic spinor Fierz identities involving quantities such as $(\overline{\Psi}_1\Gamma^I\Psi_2)(\overline{\Psi}_3\Gamma_I\Psi_4)$ can also be derived [305], where the $\Gamma^I = \big\{\mathbb{1}\,,\,\Gamma^\mu, \Gamma^{\mu\nu}\ (\mu<\nu)\,,\,\Gamma^\mu\Gamma^\nu\Gamma^\lambda\ (\mu<\nu<\lambda)\,,\,...,\Gamma^1\Gamma^2\cdots\Gamma^{2[d/2]}\big\}$ are a complete set of $2^{2[d/2]}$ linearly independent matrices [which generalizes eq. (G.1.102)].

If d is even, one can also introduce the d-dimensional analogue of γ_5 by defining[144]

$$\Gamma_{d+1} \equiv i^{(s-t)/2}\,\Gamma^1\Gamma^2\cdots\Gamma^d\,, \tag{G.3.4}$$

which is hermitian and satisfies $(\Gamma_{d+1})^2 = \mathbb{1}$ and $\{\Gamma^\mu\,,\,\Gamma_{d+1}\} = 0$. In the case of even-dimensional spacetimes, there are two possible choices for the charge-conjugated spinor Ψ^C,[145]

$$\Psi^C = B_\eta^{-1}\Psi^*\,, \qquad \text{where} \quad \eta = \pm 1\,, \tag{G.3.5}$$

and the B_η are unitary matrices that satisfy:

$$B_\eta\Gamma^\mu B_\eta^{-1} = \eta\Gamma^{\mu\,*}\,, \qquad \eta = \pm 1\,. \tag{G.3.6}$$

[141] Previous attempts in the literature to define Euclidean Majorana field theories can be found in [126].

[142] This includes the Euclidean case [141] corresponding to $t = 0$ and $s = d$ [cf. footnote 140], and the Minkowski case corresponding to $t = 1$ and $s = d - 1$.

[143] In d-dimensional Euclidean space (where $t = 0$), $\Gamma^{\mu\,\dagger} = -\Gamma^\mu$ for all $\mu = 1,2,...,d$. As a result, we may choose $A = \mathbb{1}$, in which case $\overline{\Psi} = \Psi^\dagger$.

[144] For $t = 1$ and d even, one traditionally takes $\mu = 0,1,2,...,d-1$ (where 0 is the time index), in which case, $\Gamma_{d+1} \equiv i^{(d-2)/2}\,\Gamma^0\Gamma^1\cdots\Gamma^{d-1}$.

[145] In four-dimensional Minkowski spacetime, we identify $D = B_-^{-1}$ [cf. eq. (G.1.18)] and $\gamma_5 D = B_+^{-1}$.

For even d, a convenient choice is $B_+ = B_-\Gamma_{d+1}$ [137].

If d is odd with signature (s,t), then the $2^{(d-1)/2}\times 2^{(d-1)/2}$ gamma matrices Γ^μ $(\mu = 1,2,...,d)$ consist of $\{\Gamma^1,\Gamma^2,...,\Gamma^{d-1},\pm i\Gamma_{d+1}\}$ of the $(d-1)$-dimensional theory of signature $(s-1,t)$. By assumption, $\mu = d$ is a space index, so that $\Gamma^d \equiv \pm i\Gamma_{d+1}$ is anti-hermitian. In the case of odd d, only one sign choice for η, namely $\eta = (-1)^{(s-t+1)/2}$, is consistent with eq. (G.3.6) as applied to Γ^d.[146] Consequently, only one definition of the charge-conjugated spinor is viable, namely $\Psi^C = B_-^{-1}\Psi^*$ for $s-t = 1,\,5 \pmod 8$ and $\Psi^C = B_+^{-1}\Psi^*$ for $s-t = 3,\,7 \pmod 8$.

One important property of the B_η is [129, 133, 136, 142, 143]:

$$B_\eta^* B_\eta = \varepsilon_\eta\,, \qquad \varepsilon_\eta = \pm 1\,, \tag{G.3.7}$$

for $\eta = \pm 1$ in even-dimensional spacetimes and $\eta = (-1)^{(s-t+1)/2}$ in odd-dimensional spacetimes. In particular [136],[147]

$$\varepsilon_- = \begin{cases} +1\,, & \text{for } s-t = 0,1,2 \pmod 8\,,\\ -1\,, & \text{for } s-t = 4,5,6 \pmod 8\,,\end{cases} \qquad \varepsilon_+ = \begin{cases} +1\,, & \text{for } s-t = 0,6,7 \pmod 8\,,\\ -1\,, & \text{for } s-t = 2,3,4 \pmod 8\,.\end{cases} \tag{G.3.8}$$

Using the charge-conjugated spinor defined in eq. (G.3.5), one can define a self-conjugate spinor, $\Psi^C = \Psi$. Two cases arise depending on the sign of η [136, 142, 143, 144],

$$\text{Majorana spinor:} \qquad \Psi = B_-^{-1}\Psi^*\,, \tag{G.3.9}$$

$$\text{pseudo-Majorana spinor:} \qquad \Psi = B_+^{-1}\Psi^*\,. \tag{G.3.10}$$

Due to the reality conditions [eqs. (G.3.9) and (G.3.10)], the (pseudo-)Majorana spinor possesses $2^{[d/2]}$ real degrees of freedom. Using eq. (G.3.7), one immediately sees that eqs. (G.3.9) and (G.3.10) are respectively consistent if and only if $\varepsilon_\eta = +1$.[148] The possible existence of Majorana [pseudo-Majorana] spinors in d-dimensional spacetime depends on the choice of $s-t$ such that $\varepsilon_- = +1$ [$\varepsilon_+ = +1$]. Using eq. (G.3.8), it follows that Majorana spinors can only exist in spacetimes where $s-t = 0,\,1,\,2 \pmod 8$, and pseudo-Majorana can only exist in spacetimes where $s-t = 0,\,6,\,7 \pmod 8$.[149] In particular, a Majorana spinor cannot exist in four-dimensional Euclidean space.

Given a choice of sign for $\eta = \pm 1$, one can define a corresponding charge conjugation matrix C_η, which is unitary and is defined by[150]

$$C_\eta \equiv B_\eta^{\mathsf T} A\,, \qquad \text{where} \quad C_\eta \Gamma^\mu C_\eta^{-1} = \eta(-1)^{t+1}\Gamma^{\mu\,\mathsf T}\,. \tag{G.3.11}$$

Eq. (G.3.5) then yields $\Psi^C = C_\eta^*\,\overline{\Psi}^{\mathsf T}$. The unitary matrices A, B_η and C_η satisfy the following useful identities[136, 142]:

$$B_\eta^{\mathsf T} = \varepsilon_\eta B_\eta\,, \qquad C_\eta^{\mathsf T} = \varepsilon_\eta \eta^t (-1)^{t(t-1)/2} C_\eta\,, \qquad A^* B_\eta = \eta^t B_\eta A\,, \qquad A^{\mathsf T} C_\eta = \eta^t C_\eta A^{-1}\,. \tag{G.3.12}$$

[146] The two sign choices for Γ^d correspond to two inequivalent representations of the Clifford algebra [eq. (G.3.1)] for d odd. Nevertheless, the corresponding $\Sigma^{\mu\nu}$ yield equivalent spinor representations of $\mathrm{SO}(s,t)$.

[147] For d even, one can use $B_+ = B_-\Gamma_{d+1}$ and $B_\eta\Gamma_{d+1}B_\eta^{-1} = (-1)^{(s-t)/2}\Gamma_{d+1}^*$ to derive $\varepsilon_+ = (-1)^{(s-t)/2}\,\varepsilon_-$.

[148] If $\varepsilon_\eta = -1$ then one can introduce a generalized reality condition [cf. eq. (G.3.16)], which constrains the structure of a multiplet of Dirac fermions that transforms under a pseudo-real representation of the flavor group. In this case, the corresponding (generalized) self-conjugate spinors are called symplectic (pseudo-)Majorana spinors, as discussed below eq. (G.3.20).

[149] As shown in [136, 142], no $\mathrm{SO}(s,t)$-invariant mass term is allowed for a pseudo-Majorana spinor.

[150] In four-dimensional Minkowski spacetime, we identify $C = (C_-^{\mathsf T})^{-1} = C_-^*$ [cf. eq. (G.1.11)] and $B = C_+$ [cf. eq. (G.1.20)]. In this case, one cannot use C_+ to consistently define a self-conjugate spinor, as the corresponding $\varepsilon_+ = -1$.

In the case of even d, one can define left and right-handed chiral projection operators:

$$P_L \equiv \tfrac{1}{2}(1-\Gamma_{d+1})\,, \qquad P_R \equiv \tfrac{1}{2}(1+\Gamma_{d+1})\,, \tag{G.3.13}$$

and introduce Weyl fermions, Ψ_L and Ψ_R, which satisfy $\Gamma_{d+1}\Psi_{R,L} = \pm\Psi_{R,L}$. Equivalently,

$$\Psi_L \equiv P_L\Psi\,, \qquad \Psi_R \equiv P_R\Psi\,, \tag{G.3.14}$$

so that Ψ_L (and likewise Ψ_R) possesses $2^{(d-2)/2}$ complex degrees of freedom. It is possible for a spinor to be simultaneously a (pseudo) Majorana and a Weyl spinor if the spinor and its charge conjugate have the same chirality, in which case $B_\eta\Gamma_{d+1}B_\eta^{-1} = \Gamma_{d+1}^*$ (for even d). The latter condition holds when $i^{s-t} = 1$ or equivalently $s-t = 0$ (mod 4). Combining this requirement with the condition for the existence of a (pseudo) Majorana spinor, it follows that a (pseudo) Majorana-Weyl spinor, which possesses $2^{(d-2)/2}$ real degrees of freedom, can only exist in spacetimes where $s-t = 0$ (mod 8). For further details, see refs. [92, 133, 134, 135, 136, 142, 143, 144, 292].

As in Section 29.3.2, one can also consider a multiplet of fermions Ψ_i that transforms under a complex, real or pseudo-real representation R of the flavor group G as

$$\Psi_i \to (D_R)_i{}^j\Psi_j\,, \qquad D_R = \exp(-i\theta^a\boldsymbol{T_R^a})\,, \qquad i,j = 1,2,...,d_R\,, \tag{G.3.15}$$

where D_R is unitary and the corresponding generators $\boldsymbol{T_R^a}$ are hermitian. The dimension of R is denoted by d_R, which must be even in the pseudo-real case. In both the real and pseudo-real cases, one can also impose a reality condition that generalizes the Majorana conditions of eqs. (G.3.9) and (G.3.10),

$$(\Psi_i)^* \equiv \Psi^{*\,i} = W^{ij}B_\eta\Psi_j\,, \tag{G.3.16}$$

where W is a unitary matrix and B_η acts on the (suppressed) spinor indices of Ψ_j. Additional constraints on the form of W are obtained as follows. First, taking the complex conjugate of eq. (G.3.16) and inserting the result back into the same equation, it follows that

$$W^*W = \varepsilon_\eta\mathbb{1}\,, \tag{G.3.17}$$

after making use of eq. (G.3.7). Second, eq. (G.3.16) must hold true if Ψ is replaced by $D_R\Psi$ on both sides of the equation, in order to be compatible with the flavor symmetry group transformation law [eq. (G.3.15)]. This latter requirement combined with eq. (G.3.17) yields:

$$D_R = \varepsilon_\eta W^* D_R^* W = W^{-1}D_R^* W\,. \tag{G.3.18}$$

Eq. (G.3.18) can be expressed in terms of the flavor group generators,

$$i\boldsymbol{T_R^a} = W^{-1}(i\boldsymbol{T_R^a})^*\,W\,. \tag{G.3.19}$$

Comparing with eqs. (E.1.4)–(E.1.6), we conclude that the unitary matrix W satisfies:

$$W = \varepsilon_\eta W^{\mathsf{T}}\,, \qquad \varepsilon_\eta = \begin{cases} +1\,, & R \text{ is a real representation}\,, \\ -1\,, & R \text{ is a pseudo-real representation}\,. \end{cases} \tag{G.3.20}$$

When R is a real representation, $W = W^{\mathsf{T}}$, and a basis for the flavor group generators can be chosen such that $W = \mathbb{1}$ [cf. eq. (E.1.7)], in which case D_R is a real orthogonal matrix. Since $\varepsilon_\eta = +1$, eq. (G.3.16) yields (pseudo-)Majorana spinors (depending on the sign of η) as defined previously in eqs. (G.3.9) and (G.3.10).

When R is a pseudo-real representation, $W = -W^{\mathsf{T}}$, and a basis for the flavor group generators can be chosen such that $W = J \equiv \mathrm{diag}\left\{\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right), \cdots, \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\right\}$ is a $d_R \times d_R$ matrix, where d_R is even [cf. eq. (E.1.8)]. In this case, $D_R^{\mathsf{T}}JD_R = J$, which implies that D_R is a unitary

symplectic matrix [138]. Moreover, $\varepsilon_\eta = -1$, which was incompatible with the reality conditions of eqs. (G.3.9) and (G.3.10), but is compatible with the generalized reality condition of eq. (G.3.16).

Therefore, we define *symplectic* (pseudo-)Majorana spinors [139, 136, 138, 144, 292] to be spinors that transform as a pseudo-real representation under some flavor group and satisfy the generalized reality condition of eq. (G.3.16), where W is a unitary antisymmetric matrix, depending on the choice of $\eta = \pm 1$ (with $\eta = -1$ yielding the "pseudo" designation). As suggested by eqs. (29.3.35)–(29.3.40), $2d_R$ symplectic (pseudo-)Majorana spinors are equivalent to d_R Dirac fermions. The possible existence of symplectic (pseudo-)Majorana spinors in a d-dimensional spacetime is governed by eq. (G.3.8). Requiring that $\varepsilon_\eta = -1$ implies that symplectic Majorana spinors exist in spacetimes where $s-t = 4, 5, 6 \pmod 8$, and symplectic pseudo-Majorana spinors exist in spacetimes where $s-t = 2, 3, 4 \pmod 8$. Using this nomenclature, the fermions described by the four-dimensional Minkowski space Lagrangian given in eq. (29.3.35) are symplectic pseudo-Majorana spinors.

Four-component spinor wave functions

In four-dimensional Minkowski space, the free four-component Majorana field can be expanded in a Fourier series; each positive [negative] frequency mode is multiplied by a *commuting* spinor wave function $u(\vec{\boldsymbol{p}}, s)$ [$v(\vec{\boldsymbol{p}}, s)$] as in eq. (29.3.11),[151]

$$\Psi_{Mi}(x) = \sum_s \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}i})^{1/2}} \left[u(\vec{\boldsymbol{p}}, s)a_i(\vec{\boldsymbol{p}}, s)e^{-ip\cdot x} + v(\vec{\boldsymbol{p}}, s)a_i^\dagger(\vec{\boldsymbol{p}}, s)e^{ip\cdot x}\right] , \tag{G.4.1}$$

where $E_{\boldsymbol{p}i} \equiv (|\vec{\boldsymbol{p}}|^2 + m_i^2)^{1/2}$, and the creation operators $a_i^\dagger$ and the annihilation operators a_i satisfy anticommutation relations:

$$\{a_i(\vec{\boldsymbol{p}}, s), a_j^\dagger(\vec{\boldsymbol{p}}\,', s')\} = \delta^3(\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}')\delta_{ss'}\delta_{ij} , \tag{G.4.2}$$

with all other anticommutation relations vanishing. We employ covariant normalization of the one-particle states given by eq. (29.3.13). It then follows that

$$\langle 0|\, \Psi_M(x)\, |\vec{\boldsymbol{p}}, s\rangle = u(\vec{\boldsymbol{p}}, s)e^{-ip\cdot x} , \qquad \langle 0|\, \overline{\Psi}_M(x)\, |\vec{\boldsymbol{p}}, s\rangle = \bar{v}(\vec{\boldsymbol{p}}, s)e^{-ip\cdot x} , \tag{G.4.3}$$

$$\langle \vec{\boldsymbol{p}}, s|\, \overline{\Psi}_M(x)\, |0\rangle = \bar{u}(\vec{\boldsymbol{p}}, s)e^{ip\cdot x} , \qquad \langle \vec{\boldsymbol{p}}, s|\, \Psi_M(x)\, |0\rangle = v(\vec{\boldsymbol{p}}, s)e^{ip\cdot x} . \tag{G.4.4}$$

These results are the four-component spinor versions of eqs. (29.3.7) and (29.3.8).

Likewise, the free Dirac field can be expanded in a Fourier series,

$$\Psi_i(x) = \sum_s \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}i})^{1/2}} \left[u(\vec{\boldsymbol{p}}, s)a_i(\vec{\boldsymbol{p}}, s)e^{-ip\cdot x} + v(\vec{\boldsymbol{p}}, s)b_i^\dagger(\vec{\boldsymbol{p}}, s)e^{ip\cdot x}\right] , \tag{G.4.5}$$

where the creation operators $a_i^\dagger$ and $b_i^\dagger$ and the annihilation operators a_i and b_i satisfy anticommutation relations:

$$\{a_i(\vec{\boldsymbol{p}}, s), a_j^\dagger(\vec{\boldsymbol{p}}\,', s')\} = \delta^3(\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}')\delta_{ss'}\delta_{ij} , \tag{G.4.6}$$

$$\{b_i(\vec{\boldsymbol{p}}, s), b_j^\dagger(\vec{\boldsymbol{p}}\,', s')\} = \delta^3(\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}')\delta_{ss'}\delta_{ij} , \tag{G.4.7}$$

with all other anticommutation relations vanishing. We employ covariant normalization of the fermion (F) and antifermion ($\overline{F}$) one-particle states given by eq. (29.3.22). It then follows that

$$\langle 0|\, \Psi(x)\, |\vec{\boldsymbol{p}}, s; F\rangle = u(\vec{\boldsymbol{p}}, s)e^{-ip\cdot x} , \qquad \langle 0|\, \overline{\Psi}(x)\, \big|\vec{\boldsymbol{p}}, s; \overline{F}\big\rangle = \bar{v}(\vec{\boldsymbol{p}}, s)e^{-ip\cdot x} , \tag{G.4.8}$$

$$\langle \vec{\boldsymbol{p}}, s; F|\, \overline{\Psi}(x)\, |0\rangle = \bar{u}(\vec{\boldsymbol{p}}, s)e^{ip\cdot x} , \qquad \big\langle \vec{\boldsymbol{p}}, s; \overline{F}\big|\, \Psi(x)\, |0\rangle = v(\vec{\boldsymbol{p}}, s)e^{ip\cdot x} , \tag{G.4.9}$$

[151] Some subtleties arise in the choice of relative phases of the creation and annihilation operators, which are related to the C, CP and CPT transformation properties of the Majorana field. For further details, see [306].

and the four other single-particle matrix elements vanish. These results are the four-component spinor versions of eqs. (29.3.23)–(29.3.26). The Fourier expansion of the charge-conjugated free Dirac field $\Psi_i^C(x) = C\overline{\Psi}_i^{\mathsf{T}}(x)$ is given by:

$$\Psi_i^C(x) = \sum_s \int \frac{d^3\vec{\boldsymbol{p}}}{(2\pi)^{3/2}(2E_{\boldsymbol{p}i})^{1/2}} \left[u(\vec{\boldsymbol{p}},s)b_i(\vec{\boldsymbol{p}},s)e^{-ip\cdot x} + v(\vec{\boldsymbol{p}},s)a_i^\dagger(\vec{\boldsymbol{p}},s)e^{ip\cdot x}\right], \tag{G.4.10}$$

where we have used eq. (G.4.13). That is, the charge conjugation transformation interchanges the annihilation and creation operators, $a_i \leftrightarrow b_i$ and $a_i^\dagger \leftrightarrow b_i^\dagger$. Thus, if $\Psi^C(x) = \Psi(x)$, then we must identify $a = b$ and $a^\dagger = b^\dagger$, corresponding to the free Majorana field given in eq. (G.4.1).

The two-component spinor momentum space wave functions are related to the traditional four-component spinor wave functions according to:

$$u(\vec{\boldsymbol{p}},s) = \begin{pmatrix} x_\alpha(\vec{\boldsymbol{p}},s) \\ y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)\end{pmatrix}, \qquad \bar u(\vec{\boldsymbol{p}},s) = (y^\alpha(\vec{\boldsymbol{p}},s),\ x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},s)), \tag{G.4.11}$$

$$v(\vec{\boldsymbol{p}},s) = \begin{pmatrix} y_\alpha(\vec{\boldsymbol{p}},s) \\ x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s)\end{pmatrix}, \qquad \bar v(\vec{\boldsymbol{p}},s) = (x^\alpha(\vec{\boldsymbol{p}},s),\ y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},s)), \tag{G.4.12}$$

where the u and v-spinors are related by

$$v(\vec{\boldsymbol{p}},s) = C\bar u(\vec{\boldsymbol{p}},s)^{\mathsf{T}}, \qquad u(\vec{\boldsymbol{p}},s) = C\bar v(\vec{\boldsymbol{p}},s)^{\mathsf{T}}, \tag{G.4.13}$$

$$\bar v(\vec{\boldsymbol{p}},s) = -u(\vec{\boldsymbol{p}},s)^{\mathsf{T}}C^{-1}, \qquad \bar u(\vec{\boldsymbol{p}},s) = -v(\vec{\boldsymbol{p}},s)^{\mathsf{T}}C^{-1}. \tag{G.4.14}$$

The spin quantum number takes on values $s = \pm\frac{1}{2}$, and refers either to the component of the spin as measured in the rest frame with respect to a fixed axis or to the helicity (as discussed in Section 29.3.1 and Appendix C). Note that the u and v-spinors also satisfy:

$$v(\vec{\boldsymbol{p}},s) = -2s\gamma_5 u(\vec{\boldsymbol{p}},-s), \qquad u(\vec{\boldsymbol{p}},s) = 2s\gamma_5 v(\vec{\boldsymbol{p}},-s), \tag{G.4.15}$$

which follows from eq. (29.3.22). Explicit forms for the four-component spinor wave functions in the chiral representation can be obtained using eqs. (29.3.18)–(29.3.21), where $\chi_s(\hat{\boldsymbol{s}})$ is given in eq. (C.1.11). For helicity spinors, further simplifications result by employing eqs. (C.3.3)–(C.3.6).

One can check that u and v satisfy the Dirac equations

$$(\not p - m)\,u(\vec{\boldsymbol{p}},s) = (\not p + m)\,v(\vec{\boldsymbol{p}},s) = 0, \tag{G.4.16}$$

$$\bar u(\vec{\boldsymbol{p}},s)\,(\not p - m) = \bar v(\vec{\boldsymbol{p}},s)\,(\not p + m) = 0, \tag{G.4.17}$$

corresponding to eqs. (29.3.9)–(29.3.12), and

$$(2s\gamma_5 \not S - 1)\,u(\vec{\boldsymbol{p}},s) = (2s\gamma_5 \not S - 1)\,v(\vec{\boldsymbol{p}},s) = 0, \tag{G.4.18}$$

$$\bar u(\vec{\boldsymbol{p}},s)\,(2s\gamma_5 \not S - 1) = \bar v(\vec{\boldsymbol{p}},s)\,(2s\gamma_5 \not S - 1) = 0, \tag{G.4.19}$$

corresponding to eqs. (29.3.23)–(29.3.26), where the spin vector S^μ is defined in eq. (29.3.15).[152] For massive fermions, eqs. (29.3.45)–(29.3.48) correspond to

$$u(\vec{\boldsymbol{p}},s)\bar u(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}(1 + 2s\gamma_5 \not S)\,(\not p + m), \tag{G.4.20}$$

$$v(\vec{\boldsymbol{p}},s)\bar v(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}(1 + 2s\gamma_5 \not S)\,(\not p - m). \tag{G.4.21}$$

[152] We use the standard Feynman slash notation: $\not p \equiv \gamma_\mu p^\mu$ and $\not S \equiv \gamma_\mu S^\mu$.

To apply the above formulae to the massless case we must employ helicity states, where s is replaced by the helicity quantum number λ, and S^μ is defined by eq. (29.3.16). In particular, in the $m \to 0$ limit, $S^\mu = p^\mu/m + \mathcal{O}(m/E)$. Inserting this result in eqs. (G.4.18) and (G.4.19) and using the Dirac equations, it follows that the massless helicity spinors are eigenstates of γ_5,

$$\gamma_5 u(\vec{\boldsymbol{p}},\lambda) = 2\lambda u(\vec{\boldsymbol{p}},\lambda)\,, \qquad\qquad \gamma_5 v(\vec{\boldsymbol{p}},\lambda) = -2\lambda v(\vec{\boldsymbol{p}},\lambda)\,. \tag{G.4.22}$$

Combining these results with eq. (G.4.15) [with s replaced by λ] yields:

$$v(p,\lambda) = -2\lambda\gamma_5 u(p,-\lambda) = u(p,-\lambda)\,, \qquad\qquad \lambda = \pm\tfrac{1}{2}\,, \tag{G.4.23}$$

and we see that the massless u and v spinors of opposite helicity are the same.

Applying the above $m \to 0$ limiting procedure to eqs. (G.4.20) and (G.4.21) and using the mass-shell condition ($\not{p}\not{p} = p^2 = m^2$), one obtains the massless helicity projection operators corresponding to eqs. (29.3.53)–(29.3.56):

$$u(\vec{\boldsymbol{p}},\lambda)\bar{u}(\vec{\boldsymbol{p}},\lambda) = \tfrac{1}{2}(1+2\lambda\gamma_5)\,\not{p}\,, \tag{G.4.24}$$

$$v(\vec{\boldsymbol{p}},\lambda)\bar{v}(\vec{\boldsymbol{p}},\lambda) = \tfrac{1}{2}(1-2\lambda\gamma_5)\,\not{p}\,. \tag{G.4.25}$$

Summing over the spin degree of freedom, we obtain the spin-sum identities corresponding to eqs. (29.3.57)–(29.3.60),

$$\sum_s u(\vec{\boldsymbol{p}},s)\bar{u}(\vec{\boldsymbol{p}},s) = \not{p} + m\,, \tag{G.4.26}$$

$$\sum_s v(\vec{\boldsymbol{p}},s)\bar{v}(\vec{\boldsymbol{p}},s) = \not{p} - m\,, \tag{G.4.27}$$

$$\sum_s u(\vec{\boldsymbol{p}},s)v^{\mathsf{T}}(\vec{\boldsymbol{p}},s) = (\not{p}+m)C^{\mathsf{T}}\,, \tag{G.4.28}$$

$$\sum_s \bar{u}^{\mathsf{T}}(\vec{\boldsymbol{p}},s)\bar{v}(\vec{\boldsymbol{p}},s) = C^{-1}(\not{p}-m)\,, \tag{G.4.29}$$

$$\sum_s \bar{v}^{\mathsf{T}}(\vec{\boldsymbol{p}},s)\bar{u}(\vec{\boldsymbol{p}},s) = C^{-1}(\not{p}+m)\,, \tag{G.4.30}$$

$$\sum_s v(\vec{\boldsymbol{p}},s)u^{\mathsf{T}}(\vec{\boldsymbol{p}},s) = (\not{p}-m)C^{\mathsf{T}}\,, \tag{G.4.31}$$

which are valid for both the massive case and the massless $m \to 0$ limit.

As previously noted, the results for the bilinear covariants obtained in eqs. (G.1.79)–(G.1.90) can also be applied to expressions involving the commuting spinor wave functions. Various relations among the possible bilinear covariants can be established by using eqs. (G.4.13) and (G.4.14). As an example, for $\Gamma = \mathbb{1}\,,\ \gamma_5\,,\ \gamma^\mu\,,\ \gamma^\mu\gamma_5\,,\ \Sigma^{\mu\nu}\,,\ \Sigma^{\mu\nu}\gamma_5$,

$$\bar{u}(\vec{\boldsymbol{p}}_1,s_1)\Gamma v(\vec{\boldsymbol{p}}_2,s_2) = -v(\vec{\boldsymbol{p}}_1,s_1)^{\mathsf{T}}C^{-1}\Gamma C\bar{u}(\vec{\boldsymbol{p}}_2,s_2)^{\mathsf{T}} = -\eta_\Gamma^C\,\bar{u}(\vec{\boldsymbol{p}}_2,s_2)\Gamma v(\vec{\boldsymbol{p}}_1,s_1)\,, \tag{G.4.32}$$

$$\bar{u}(\vec{\boldsymbol{p}}_1,s_1)\Gamma u(\vec{\boldsymbol{p}}_2,s_2) = -v(\vec{\boldsymbol{p}}_1,s_1)^{\mathsf{T}}C^{-1}\Gamma C\bar{v}(\vec{\boldsymbol{p}}_2,s_2)^{\mathsf{T}} = -\eta_\Gamma^C\,\bar{v}(\vec{\boldsymbol{p}}_2,s_2)\Gamma v(\vec{\boldsymbol{p}}_1,s_1)\,, \tag{G.4.33}$$

where the sign η_Γ^C [defined in eq. (G.1.25)] arises after taking the transpose and applying eq. (G.1.25). In particular, the (commuting) u and v spinors satisfy the following relations:

$$\bar{u}(\vec{\boldsymbol{p}}_1,s_1)P_L v(\vec{\boldsymbol{p}}_2,s_2) = -\bar{u}(\vec{\boldsymbol{p}}_2,s_2)P_L v(\vec{\boldsymbol{p}}_1,s_1)\,, \tag{G.4.34}$$

$$\bar{u}(\vec{\boldsymbol{p}}_1,s_1)P_R v(\vec{\boldsymbol{p}}_2,s_2) = -\bar{u}(\vec{\boldsymbol{p}}_2,s_2)P_R v(\vec{\boldsymbol{p}}_1,s_1)\,, \tag{G.4.35}$$

$$\bar{u}(\vec{\boldsymbol{p}}_1,s_1)\gamma^\mu P_L v(\vec{\boldsymbol{p}}_2,s_2) = \bar{u}(\vec{\boldsymbol{p}}_2,s_2)\gamma^\mu P_R v(\vec{\boldsymbol{p}}_1,s_1)\,, \tag{G.4.36}$$

$$\bar{u}(\vec{\boldsymbol{p}}_1,s_1)\gamma^\mu P_R v(\vec{\boldsymbol{p}}_2,s_2) = \bar{u}(\vec{\boldsymbol{p}}_2,s_2)\gamma^\mu P_L v(\vec{\boldsymbol{p}}_1,s_1)\,, \tag{G.4.37}$$

and four similar relations obtained by interchanging $v(\vec{\boldsymbol{p}}_2,s_2) \leftrightarrow u(\vec{\boldsymbol{p}}_2,s_2)$.

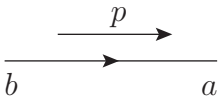
Feynman rules for four-component fermions

We now illustrate some basic applications of the above formalism. In particular, we shall establish a set of Feynman rules for four-component fermions that treat both Dirac and Majorana fermions on the same footing. These rules generalize the standard Feynman rules for four-component Dirac fermions found in most quantum field theory textbooks. Two advantages of the rules presented here are: (i) no factors of the charge conjugation matrix C are required for fermion interaction vertices and propagators, and (ii) the *relative* sign between different diagrams corresponding to the same physical process is simply determined. Our rules have been obtained by translating our two-component fermion Feynman rules into the four-component spinor language. The resulting Feynman rules for four-component Majorana fermions are equivalent to the set of rules independently obtained in [307] (see also refs. [308, 309]).

Consider first the Feynman rule for the four-component fermion propagator. Virtual Dirac fermion lines can either correspond to Ψ or Ψ^C. Here, there is no ambiguity in the propagator Feynman rule, since for free Dirac fermion fields,[153]

$$\langle 0|\, T[\Psi_a(x)\overline{\Psi}^b(y)]\,|0\rangle = \langle 0|\, T[\Psi^C_a(x)\overline{\Psi^C}^{\,b}(y)]\,|0\rangle \ , \tag{G.5.1}$$

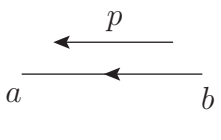
so that the Feynman rules for the propagator of a Ψ and Ψ^C line, exhibited in Fig. G.5.1, are identical. The same rule also applies to a four-component Majorana fermion.



$$\frac{i(\not{p}+m)_a{}^b}{p^2-m^2+i\epsilon}$$

Figure G.5.1: Feynman rule for the propagator of a four-component fermion with mass m. The same rule applies to a Majorana, Dirac and charge-conjugated Dirac fermion. The four-component spinor labels a and b are specified.

Using eq. (G.1.2), the four-component fermion propagator Feynman rule can be expressed as a partitioned matrix of 2×2 blocks,



$$= \left(\begin{array}{cc} \leftarrow\!\!\rightarrow & \leftarrow \\ \rightarrow & \rightarrow\!\!\leftarrow \end{array}\right) = \frac{i}{p^2-m^2+i\epsilon}\begin{pmatrix} m\,\delta_\alpha{}^\beta & p\cdot\sigma_{\alpha\dot\beta} \\ p\cdot\overline{\sigma}^{\dot\alpha\beta} & m\,\delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix} , \tag{G.5.2}$$

where a and b are four-component spinor indices. That is, eq. (G.5.2) is a partitioned matrix whose blocks consist of two-component fermion propagators defined in Fig. 29.4.1, with the undotted and dotted α [β] indices on the left [right] and with the momentum flowing from right to left.

The derivation of the four-component Dirac fermion propagator is treated in most modern textbooks of quantum field theory (see, e.g., [115]). Here, we briefly sketch the path integral derivation of the four-component fermion propagator by exploiting the path integral treatment of the two-component fermion propagators outlined in Appendix F . Consider a single massive Dirac fermion $\Psi(x)$ coupled to an anticommuting four-component Dirac fermionic source term

$$J_\psi(x) \equiv \begin{pmatrix} J_{\eta\alpha}(x) \\ J^{\dagger\dot\alpha}_\chi(x) \end{pmatrix} . \tag{G.5.3}$$

[153] In deriving eq. (G.5.1), we have used $\mathcal{C}\,\Psi_a\,\mathcal{C}^{-1} = \eta_c\Psi^C_a$ and $\mathcal{C}\,\overline{\Psi}^{\,a}\,\mathcal{C}^{-1} = \eta_c^*\,\overline{\Psi^C}^{\,a}$, where $\mathcal{C}$ is the charge conjugation *operator* that acts on the quantum Hilbert space and η_c is a convention-dependent phase factor [37, 39]. Note that $\mathcal{C}$ is a unitary operator and $\mathcal{C}\,|0\rangle = |0\rangle$ in the free-field vacuum.

The corresponding action [eq. (F.2)] in four-component notation is given by

$$S = \int d^4x\,(\mathscr{L} + \overline{J}_\psi\Psi + \overline{\Psi}\,J_\psi) = \int d^4x\,\left[\overline{\Psi}(i\partial\!\!\!/ - m)\Psi + \overline{J}_\psi\Psi + \overline{\Psi}\,J_\psi\right]\,. \tag{G.5.4}$$

Introducing the momentum space Fourier coefficients:

$$\Psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\widehat{\Psi}(p)\,, \qquad J_\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\widehat{J}_\psi(p)\,, \tag{G.5.5}$$

we can identify the following four-component quantities with matrices of two-component quantities given in eqs. (F.7) and (F.25):

$$\widehat{\Psi}(p) = A^{-1}\Omega_c(p)\,, \qquad \widehat{J}_\psi(p) = X_c(p)\,, \qquad p\!\!\!/ - m = \mathcal{M}(p)A\,, \tag{G.5.6}$$

where A is the Dirac conjugation matrix defined in eqs. (G.1.10) and (G.1.12). Using the results of Appendix F , one easily derives:

$$\langle 0|T(\Psi(x_1)\overline{\Psi}(x_2))|0\rangle = \left(-i\frac{\overrightarrow{\delta}}{\delta\overline{J}_\psi(x_1)}\right) W[J,\overline{J}] \left(-i\frac{\overleftarrow{\delta}}{\delta J_\psi(x_2)}\right)\Bigg|_{J_\psi=\overline{J}_\psi=0}\,, \tag{G.5.7}$$

where

$$W[J_\psi\,,\,\overline{J}_\psi] = \exp\left\{-i\int \frac{d^4p}{(2\pi)^4}\widehat{\overline{J}}_\psi(p)\frac{p\!\!\!/ + m}{p^2 - m^2}\widehat{J}_\psi(p)\right\}\,. \tag{G.5.8}$$

Using the analogues of eqs. (F.21) and (F.22), we end up with the expected result

$$\langle 0|T(\Psi(x_1)\overline{\Psi}(x_2))|0\rangle = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x_1-x_2)}\,\frac{i(p\!\!\!/ + m)}{p^2 - m^2}\,. \tag{G.5.9}$$

In principle, the analogous computation can be carried out for a single four-component Majorana fermion field $\Psi_M(x)$ coupled to a Majorana fermionic source, $J_\xi(x)$. The corresponding action is similar to that of eq. (G.5.4), with an extra overall factor of 1/2. However, in evaluating the functional derivative in eq. (G.5.8), one must take into account that the Majorana fermionic source $J_\xi(x)$ satisfies $J_\xi^C \equiv C\overline{J}_\xi^{\mathsf{T}} = J_\xi$. Consequently, the functional derivative with respect to $\overline{J}_\xi$ is related to the corresponding functional derivative with respect to J_ξ. Hence, the calculation of eq. (G.5.8) will yield two equal terms that will cancel the overall factor of 1/2, resulting again in eq. (G.5.9). Nevertheless, this computation is somewhat awkward using four-component spinor notation, in contrast to the straightforward calculation of Appendix F .

We next examine the various interactions involving four-component fermions. First, we consider the interactions of a neutral scalar ϕ or a gauge boson A^a_μ with a pair of Majorana fermions To obtain the interactions of the four-component fermion fields, we first identify the neutral two-component fermion mass eigenstate neutral fields ξ_i. Using eqs. (29.4.9) and (29.4.15), the interaction Lagrangian in two-component form is given by:

$$\mathscr{L}_{\rm int} = -\tfrac{1}{2}(\lambda^{ij}\xi_i\xi_j + \lambda_{ij}\xi^{\dagger i}\xi^{\dagger j})\phi - (G^a)_i{}^j\,\xi^{\dagger i}\overline{\sigma}^\mu\xi_j A^a_\mu\,, \tag{G.5.10}$$

where λ is a complex symmetric matrix with $\lambda^{ij} \equiv \lambda^*_{ij}$ [cf. eq. (29.3.2)], the A^a_μ are the mass eigenstate gauge fields, and the corresponding hermitian matrices G^a are defined in eq. (29.4.16). It is now simple to convert this result into four-component notation:

$$\mathscr{L}_{\rm int} = -\tfrac{1}{2}(\lambda^{ij}\overline{\Psi}_{Mi}P_L\Psi_{Mj} + \lambda_{ij}\overline{\Psi}^i_M P_R\Psi^j_M)\phi - (G^a)_i{}^j\overline{\Psi}^i_M\gamma^\mu P_L\Psi_{Mj}A^a_\mu\,, \tag{G.5.11}$$

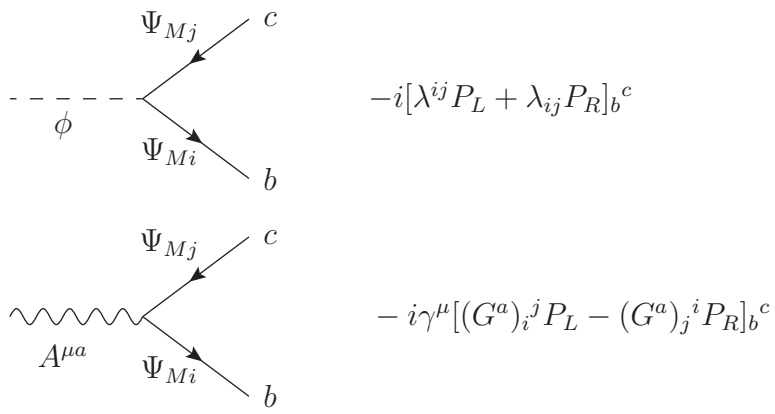


Figure G.5.2: Feynman rules for neutral scalar and gauge boson interactions with a pair of four-component Majorana fermions (labeled by four-component spinor indices b and c). The G^a are defined in eq. (29.4.16). The index a runs over the neutral (mass eigenstate) gauge bosons.

where the Ψ_{Mj} are a set of (neutral) Majorana four-component fermions. It is convenient to use eq. (G.1.99) to rewrite the term proportional to $(G^a)_i{}^j$ in eq. (G.5.11) as follows

$$\begin{aligned}(G^a)_i{}^j\overline{\Psi}{}^i_M\gamma^\mu P_L\Psi_{Mj} &= \tfrac{1}{2}(G^a)_i{}^j\left[\overline{\Psi}{}^i_M\gamma^\mu P_L\Psi_{Mj} - \overline{\Psi}_{Mj}\gamma^\mu P_R\Psi^i_M\right]\\ &= \tfrac{1}{2}\overline{\Psi}_{Mi}\gamma^\mu\left[(G^a)_i{}^j P_L - (G^a)_j{}^i P_R\right]\Psi_{Mj}\,.\end{aligned}\tag{G.5.12}$$

In the last step above, we have lowered the flavor indices of the four-component Majorana fermion fields, as the heights of these indices can be arbitrarily chosen [cf. eq. (G.1.77)].

Using standard four-component spinor methods, the corresponding four-component spinor Feynman rules are displayed in Fig. G.5.2. A Majorana fermion is neutral under all conserved charges (and thus equal to its own antiparticle). Hence, an arrow on a Majorana fermion line simply reflects the structure of the interaction Lagrangian; i.e., $\overline{\Psi}_M$ [Ψ_M] is represented by an arrow pointing out of [into] the vertex. These arrows are then used for determining the placement of the u and v spinors in an invariant amplitude, according to the rules of Appendix G.6 . In particular, the four-component spinor labels of Fig. G.5.2 indicate that one should traverse any continuous fermion line by moving antiparallel to the direction of the fermion arrows.

Next, we consider the interactions of a (possibly complex) scalar Φ or a gauge boson A^a_μ with a pair of Dirac fermions. The Dirac fermions are charged with respect to some global or local U(1) symmetry, which is assumed to be a symmetry of the Lagrangian. To obtain the interactions of the four-component fermion fields, we first identify the mass-degenerate oppositely charged pairs χ_j and η_j (with U(1)-charges q_j and $-q_j$, respectively) that combine to form the mass eigenstate Dirac fermions. The scalar field Φ carries a U(1)-charge q_Φ. We also identify the gauge boson mass eigenstates of definite U(1)-charge by A^a_μ as described in Section 29.4.3 (cf. footnote 47). Using eqs. (29.4.9) and (29.4.18), the interaction Lagrangian in two-component form is given by:

$$\mathscr{L}_{\rm int} = -\kappa^i{}_j\chi_i\eta^j\Phi - \kappa_i{}^j\chi^{\dagger i}\eta^\dagger_j\Phi^\dagger - \left[(G^a_L)_i{}^j\chi^{\dagger i}\overline{\sigma}^\mu\chi_j - (G^a_R)_j{}^i\eta^\dagger_i\overline{\sigma}^\mu\eta^j\right]A^a_\mu\,,\tag{G.5.13}$$

where $\kappa_i{}^j \equiv (\kappa^i{}_j)^*$ [cf. eq. (29.3.28)] and κ is an arbitrary complex matrix coupling, subject to the conditions that $\kappa^i{}_j = 0$ unless $q_\Phi = q_j - q_i$. For the gauge boson couplings, we follow the notation of eqs. (29.4.19) and (29.4.20). In particular, $A^a_\mu G^a_L$ and $A^a_\mu G^a_R$ are hermitian matrix-valued gauge fields, which when summed over a can contain both neutral and charged [with respect to U(1)] mass eigenstate gauge boson fields. Converting to four-component notation yields:

$$\mathscr{L}_{\rm int} = -\kappa^i{}_j\overline{\Psi}{}^j P_L\Psi_i\Phi - \kappa_i{}^j\overline{\Psi}{}^i P_R\Psi_j\Phi^\dagger - \left[(G^a_L)_i{}^j\overline{\Psi}{}^i\gamma^\mu P_L\Psi_j + (G^a_R)_i{}^j\overline{\Psi}{}^i\gamma^\mu P_R\Psi_j\right]A^a_\mu\,,\tag{G.5.14}$$

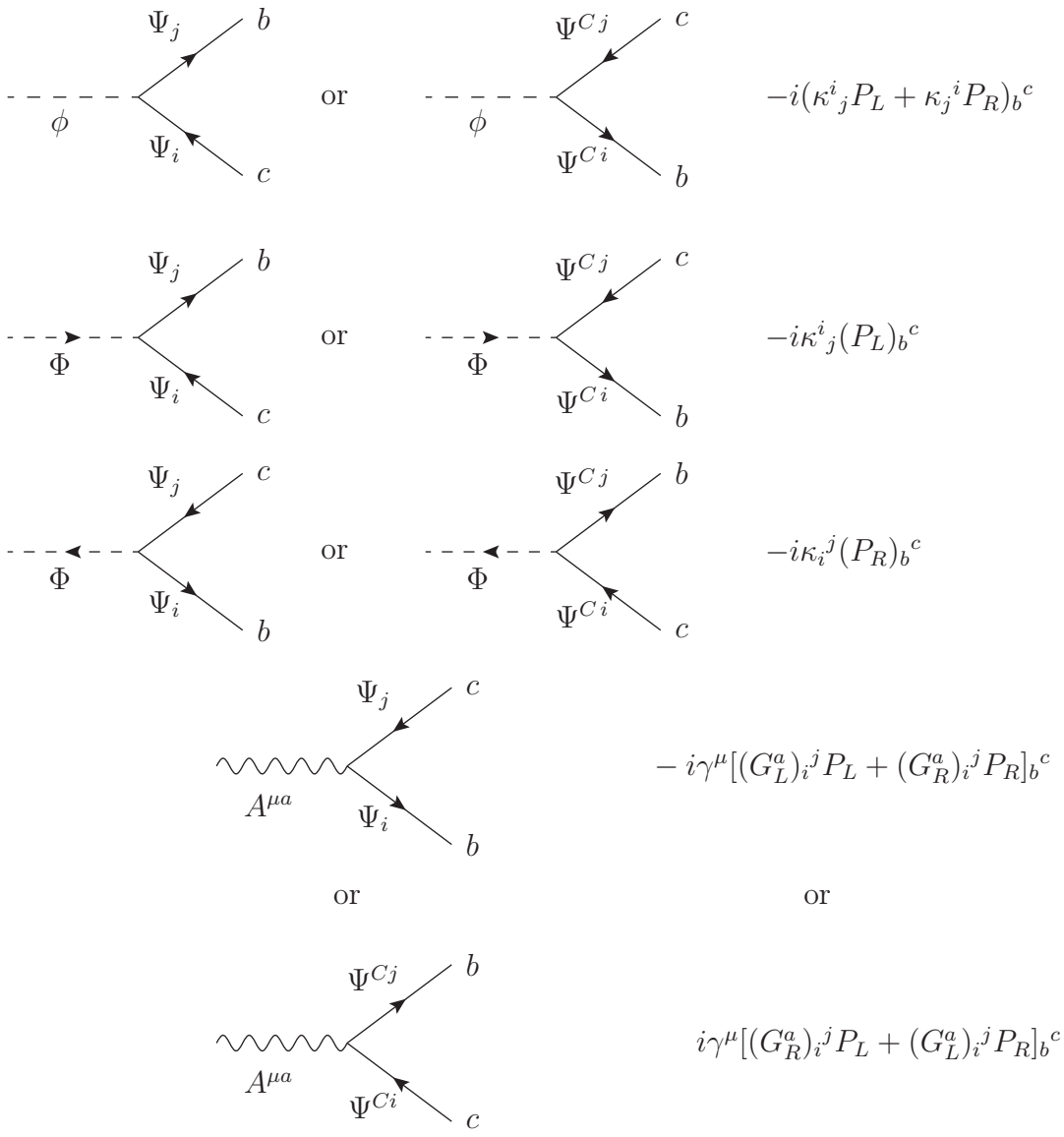


Figure G.5.3: Feynman rules for neutral scalar (ϕ), charged scalar (Φ) and gauge boson ($A^{\mu a}$) interactions with a pair of four-component Dirac fermions (labeled by four-component spinor indices b and c). In each case, one has two choices for the corresponding Feynman rule: one involving Ψ and one involving the oppositely charged Ψ^C (with the arrows of the corresponding Ψ and Ψ^C lines pointing in opposite directions). The arrows indicate the direction of flow of the U(1)-charges of the Dirac fermion and charged scalar fields. The index a runs over both neutral and charged (mass eigenstate) gauge bosons, consistent with charge conservation at the vertex.

where the Ψ_j are a set of Dirac four-component fermions. If Φ is a real (neutral) scalar field, then we shall write $\phi \equiv \Phi = \Phi^\dagger$. The corresponding four-component spinor Feynman rules are exhibited in Fig. G.5.3. The rules involving the charge-conjugated Dirac fields have been obtained by using eq. (G.1.91). Note that the arrows on the charged scalar and Dirac fermion lines depict the flow of the conserved U(1)-charge.

Finally, we treat the interaction of a charged scalar boson Φ (with U(1)-charge q_Φ) or a charged vector boson W (with U(1)-charge q_W) with a fermion pair consisting of one Majorana and one

Dirac fermion. We denote the neutral fermion mass eigenstate fields by ξ_i and pairs of oppositely charged fermion mass eigenstate fields by χ_j and η^j (with U(1)-charges q_j and $-q_j$, respectively). Using eqs. (29.4.9) and (29.4.21), the interaction Lagrangian is given by:

$$\begin{aligned}\mathscr{L}_{\rm int} = &-(\kappa_1)^i{}_j\xi_i\eta^j + (\kappa_2)_{ij}\xi^{\dagger i}\chi^{\dagger j}]\Phi - [(\kappa_2)^{ij}\xi_i\chi_j + (\kappa_1)_i{}^j\xi_i^{\dagger i}\eta_j^\dagger]\Phi^\dagger \\ &-[(G_1)_j{}^i\chi^{\dagger j}\overline{\sigma}^\mu\xi_i - (G_2)_{ij}\xi^{\dagger i}\overline{\sigma}^\mu\eta^j]W_\mu - [(G_1)^j{}_i\xi^{\dagger i}\overline{\sigma}^\mu\chi_j - (G_2)^{ij}\eta_j^\dagger\overline{\sigma}^\mu\xi_i]W_\mu^\dagger\,, \qquad \text{(G.5.15)}\end{aligned}$$

where κ_1, κ_2, G_1, and G_2 are arbitrary complex coupling matrices, subject to the conditions that $(\kappa_1)^i{}_j = (\kappa_2)_{ij} = 0$ unless $q_\Phi = q_j$, and $(G_1)_j{}^i = (G_2)_{ij} = 0$ unless $q_W = q_j$. Converting to four-component spinor notation yields:

$$\begin{aligned}\mathscr{L}_{\rm int} = &-\left[(\kappa_1)^i{}_j\overline{\Psi}^j P_L\Psi_{Mi} + (\kappa_2)_{ij}\overline{\Psi}^j P_R\Psi_M^i\right]\Phi \\ &-\left[(G_1)_j{}^i\overline{\Psi}^j\gamma^\mu P_L\Psi_{Mi} + (G_2)_{ij}\overline{\Psi}^j\gamma^\mu P_R\Psi_M^i\right]W_\mu + \text{h.c.} \qquad \text{(G.5.16)}\end{aligned}$$

The corresponding four-component spinor Feynman rules are exhibited in Fig. G.5.4.

There is an equivalent form for the interactions given by eqs. (G.5.13) and (G.5.16) where $\mathscr{L}_{\rm int}$ is written in terms of charge-conjugated Dirac fields [after using eq. (G.1.91)]. The Feynman rules involving Dirac fermions can take two possible forms, as shown in Figs. G.5.3 and G.5.4. As previously noted, the direction of an arrow on a Dirac fermion line indicates the direction of the fermion charge flow (whereas the arrow on the Majorana fermion line is unconnected to charge flow). However, we are free to choose either a Ψ or Ψ^C line to represent a Dirac fermion at any place in a given Feynman graph.[154] For any decay or scattering process, a suitable choice of either the Ψ-rule or the Ψ^C-rule at each vertex (the choice can be different at different vertices), will guarantee that the arrow directions on fermion lines flow continuously through the Feynman diagram. Then, to evaluate an invariant amplitude, one should traverse *any* continuous fermion line (either Ψ or Ψ^C) by moving antiparallel to the direction of the fermion arrows, as indicated by the order of the four-component spinor labels in the Feynman rules of Figs. G.5.3 and G.5.4. Examples will be provided in Appendix G.6 .

29.7.9 Correspondence to four-component spinor notation 3

Applications of four-component spinor Feynman rules

For a given process, there may be a number of distinct choices for the arrow directions on the Majorana fermion lines, which may depend on whether one represents a given Dirac fermion by Ψ or Ψ^C. However, different choices do *not* lead to independent Feynman diagrams.[155] When computing an invariant amplitude, one first writes down the relevant Feynman diagrams with no arrows on any Majorana fermion line. The number of distinct graphs contributing to the process is then determined. Finally, one makes some choice for how to distribute the arrows on the Majorana fermion lines and how to label Dirac fermion lines (either as the field Ψ or its charge conjugate Ψ^C) in a manner consistent with the rules of Figs. G.5.2 and G.5.4. The end result for the invariant amplitude (apart from an overall unobservable phase) does not depend on the choices made for the direction of the fermion arrows.

Using the above procedure, the Feynman rules for the external fermion wave functions are the same for Dirac and Majorana fermions:

- $u(\vec{\boldsymbol{p}}, s)$: incoming Ψ [or Ψ^C] with momentum $\vec{\boldsymbol{p}}$ parallel to the arrow direction,

[154] Since the charge of Ψ^C is opposite in sign to the charge of Ψ, the corresponding arrow directions of the Ψ and Ψ^C lines must point in opposite directions.

[155] In contrast, the two-component Feynman rules developed in Section 29.4 require that two vertices differing by the direction of the arrows on the two-component fermion lines must both be included in the calculation of the matrix element.

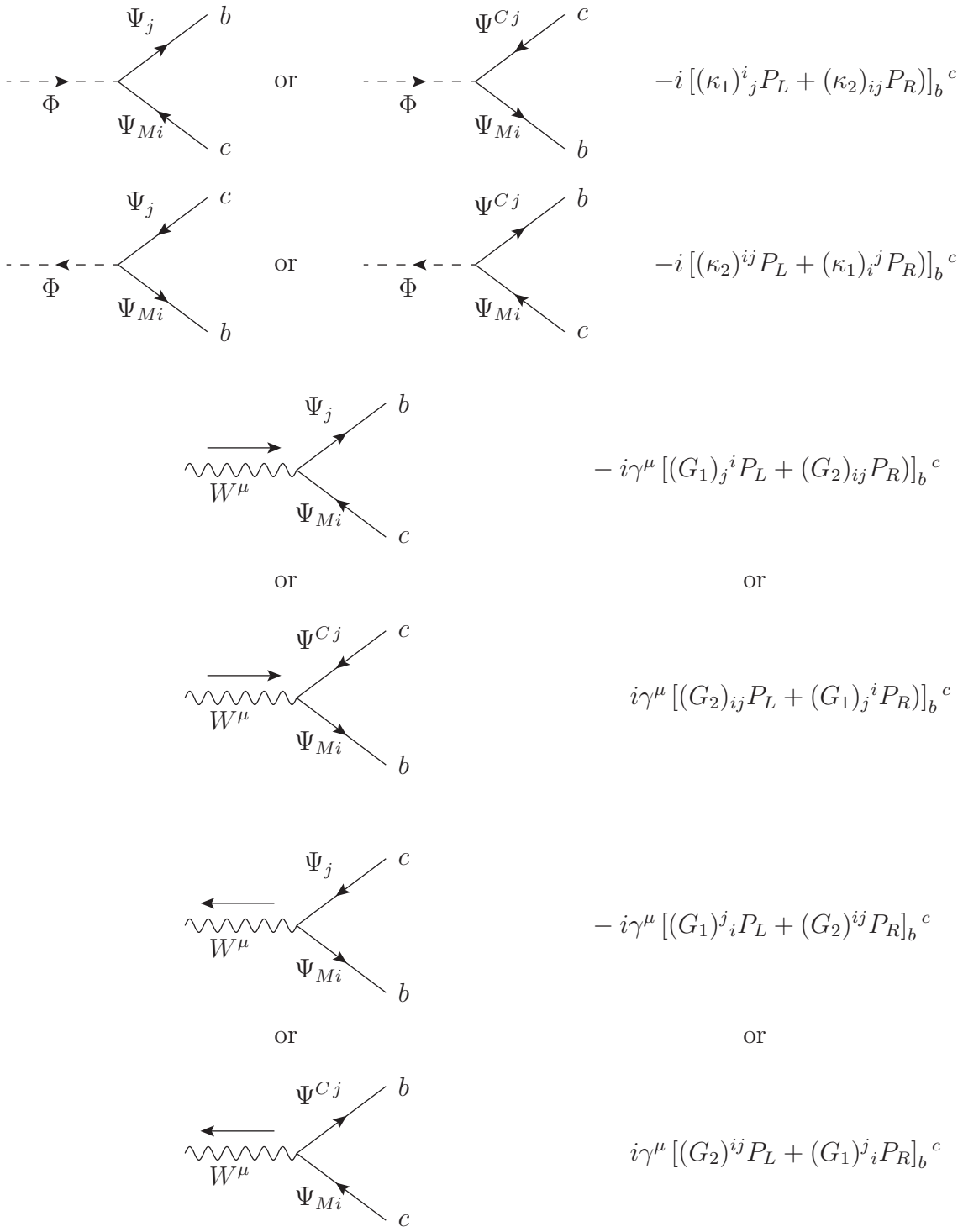


Figure G.5.4: Feynman rules for charged scalar and vector boson interactions with a fermion pair consisting of one Majorana and one Dirac four-component fermion (labeled by four-component spinor indices b and c). In each case, one has two choices for the corresponding Feynman rule: one involving Ψ and one involving the oppositely charged Ψ^C (with the arrows of the Ψ and Ψ^C lines pointing in opposite directions). The arrows of the Dirac fermion and charged bosons indicate the direction of flow of the corresponding U(1)-charges. That is, the charge of the boson (either Φ or W above) must coincide with the charge of Ψ_j. The arrows of the Majorana fermions satisfy the requirement that the fermion line arrow directions flow continuously through the vertex.

- $\bar{u}(\vec{\boldsymbol{p}}, s)$: outgoing Ψ [or Ψ^C] with momentum $\vec{\boldsymbol{p}}$ parallel to the arrow direction,
- $v(\vec{\boldsymbol{p}}, s)$: outgoing Ψ [or Ψ^C] with momentum $\vec{\boldsymbol{p}}$ antiparallel to the arrow direction,
- $\bar{v}(\vec{\boldsymbol{p}}, s)$: incoming Ψ [or Ψ^C] with momentum $\vec{\boldsymbol{p}}$ antiparallel to the arrow direction.

The proof that the above rules for external wave functions apply unambiguously to Majorana fermions is straightforward. Simply insert the plane wave expansion of the Majorana field given by eq. (G.4.1) into eq. (G.5.11), and evaluate matrix elements for, e.g., the decay of a scalar or vector particle into a pair of Majorana fermions.

We now reconsider the matrix elements for scalar and vector particle decays into fermion pairs and $2 \to 2$ elastic scattering of a fermion off a scalar and vector boson, respectively. We shall compute the matrix elements using the Feynman rules of Fig. G.5.2, and check that the results agree with the ones obtained by two-component methods in Section 29.4.5.

Consider first the decay of a neutral scalar boson ϕ into a pair of Majorana fermions, $\phi \to \Psi_{Mi}(\vec{\boldsymbol{p}}_1, s_1)\Psi_{Mj}(\vec{\boldsymbol{p}}_2, s_2)$, of flavor i and j, respectively. Here, $\Psi_{Mi}(\vec{\boldsymbol{p}}, s)$ denotes the one-particle state given by eq. (29.3.13). The matrix element for the decay is given by

$$i\mathcal{M} = -i\bar{u}(\vec{\boldsymbol{p}}_1, s_1)(\lambda^{ij} P_L + \lambda_{ij} P_R) v(\vec{\boldsymbol{p}}_2, s_2)\,. \tag{G.6.1}$$

One can easily check that this result matches with eq. (29.4.2), which was derived using two-component spinor techniques. Note that if one had chosen to switch the two final states (equivalent to switching the directions of the Majorana fermion arrows), then the resulting matrix element would simply exhibit an overall sign change [due to the results of eqs. (G.4.34) and (G.4.35)]. This overall sign change is a consequence of the Fermi-Dirac statistics, and corresponds to changing which order one uses to construct the two-particle final state.

Consider next the decay of a (neutral or charged) scalar boson Φ into a pair of Dirac fermions, $\Phi \to F_i(\vec{\boldsymbol{p}}_1, s_1)\overline{F}^j(\vec{\boldsymbol{p}}_2, s_2)$, where by $F(\vec{\boldsymbol{p}}, s)$ and $\overline{F}(\vec{\boldsymbol{p}}, s)$ we mean the one-particle states given by eq. (29.3.22). The matrix element for the decay is given by

$$i\mathcal{M} = -i\bar{u}(\vec{\boldsymbol{p}}_1, s_1)(\kappa^j{}_i P_L + \kappa_i{}^j P_R) v(\vec{\boldsymbol{p}}_2, s_2)\,, \tag{G.6.2}$$

which is equivalent to eq. (29.4.5), which was derived using two-component spinor techniques.

For the decay of a neutral vector boson (denoted by A_μ) into a pair of Majorana fermions, $A_\mu \to \Psi_{Mi}(\vec{\boldsymbol{p}}_1, s_1)\Psi_{Mj}(\vec{\boldsymbol{p}}_2, s_2)$, we use the Feynman rules of Fig. G.5.2 to obtain:

$$i\mathcal{M} = -i\bar{u}(\vec{\boldsymbol{p}}_1, s_1)\gamma^\mu \left[G_i{}^j P_L - G_j{}^i P_R\right] v(\vec{\boldsymbol{p}}_2, s_2)\varepsilon_\mu\,, \tag{G.6.3}$$

The above result is equivalent to eq. (29.4.8), which was derived using two-component spinor techniques. Again, we note that if one had chosen to switch the two final states (equivalent to switching the directions of the Majorana fermion arrows), then the resulting matrix element would simply exhibit an overall sign change [due to the results of eqs. (G.4.36) and (G.4.37)].

For $i = j$, eq. (G.6.3) simplifies to

$$i\mathcal{M} = iG\bar{u}(\vec{\boldsymbol{p}}_1, s_1)\gamma^\mu\gamma_5 v(\vec{\boldsymbol{p}}_2, s_2)\varepsilon_\mu\,, \tag{G.6.4}$$

where $G \equiv G_i{}^i$. The absence of a vector coupling of the vector boson to a pair of identical Majorana fermions is a consequence of the identity $\overline{\Psi}_M\gamma^\mu\Psi_M = 0$ noted below eq. (G.1.98).

For the decay of a (neutral or charged) vector particle A_μ into a fermion pair consisting of a Dirac fermion and antifermion, $A_\mu \to F_i(\vec{\boldsymbol{p}}_1, s_1)\overline{F}^j(\vec{\boldsymbol{p}}_2, s_2)$, the matrix element is given by:

$$i\mathcal{M} = -i\bar{u}(\vec{\boldsymbol{p}}_1, s_1)\gamma^\mu \left[(G_L)_i{}^j P_L + (G_R)_i{}^j P_R\right] v(\vec{\boldsymbol{p}}_2, s_2)\varepsilon_\mu\,, \tag{G.6.5}$$

which matches the result of eq. (29.4.12).

Finally, we consider the decay of a charged boson to a fermion pair consisting of one Dirac fermion and one Majorana fermion. Using the Feynman rules of Fig. G.5.4, we see that we have a choice of two rules for each decay process. As an example, consider the decay $W \to \Psi_{Mi}(\vec{\boldsymbol{p}}_1, s_1)F_j(\vec{\boldsymbol{p}}_2, s_2)$. If we apply the $W\Psi_M\Psi$ Feynman rule of Fig. G.5.4, we obtain:

$$i\mathcal{M} = -i\bar{u}(\vec{\boldsymbol{p}}_2, s_2)\left[(G_1)_j{}^i P_L + (G_2)_{ij}P_R\right] v(\vec{\boldsymbol{p}}_1, s_1)\,. \tag{G.6.6}$$

If we apply the corresponding $W\Psi_M\Psi^C$ Feynman rule, we obtain the negative of eq. (G.6.6) with $P_L \leftrightarrow P_R$ and $(\vec{\boldsymbol{p}}_1, s_1) \leftrightarrow (\vec{\boldsymbol{p}}_2, s_2)$. Using eqs. (G.4.36) and (G.4.37), the resulting amplitude is the negative of eq. (G.6.6), as expected since the order of the spinor wave functions in the two computations is reversed. A similar conclusion is obtained for the decay $\Phi \to \Psi_{Mi}F_j$.

Turning to the elastic scattering of a Majorana fermion and a neutral scalar, we shall examine two equivalent ways for computing the amplitude. Following the rules previously stated, there are two possible choices for the direction of arrows on the Majorana fermion lines. Thus, we may evaluate either one of the following two diagrams:

plus a second set of diagrams (not shown) where the initial and final state scalars are crossed.

Evaluating the first diagram above, the matrix element for $\phi\Psi_M \to \phi\Psi_M$ is given by:

$$\begin{aligned} i\mathcal{M} &= \frac{-i}{s-m^2}\,\bar{u}(\vec{\boldsymbol{p}}_2, s_2)(\lambda P_L + \lambda^* P_R)(\not{p}+m)(\lambda P_L + \lambda^* P_R)u(\vec{\boldsymbol{p}}_1, s_1) + (\text{crossed}) \\ &= \frac{-i}{s-m^2}\,\bar{u}(\vec{\boldsymbol{p}}_2, s_2)\left[|\lambda|^2\not{p} + \left(\lambda^2 P_L + (\lambda^*)^2 P_R\right)m\right]u(\vec{\boldsymbol{p}}_1, s_1) + (\text{crossed})\,, \end{aligned} \tag{G.6.7}$$

where m is the Majorana fermion mass and $\sqrt{s}$ is the center-of-mass energy. Using eqs. (G.1.2) and (G.4.11), one recovers the results of eq. (29.4.13). Had we chosen to evaluate the second diagram instead, the resulting amplitude would have been given by:

$$i\mathcal{M} = \frac{-i}{s-m^2}\,\bar{v}(\vec{\boldsymbol{p}}_1, s_1)\left[-|\lambda|^2\not{p} + \left(\lambda^2 P_L + (\lambda^*)^2 P_R\right)m\right]v(\vec{\boldsymbol{p}}_2, s_2) + (\text{crossed})\,. \tag{G.6.8}$$

Using eq. (G.4.33), it follows that:

$$\bar{v}(\vec{\boldsymbol{p}}_1, s_1)v(\vec{\boldsymbol{p}}_2, s_2) = -\bar{u}(\vec{\boldsymbol{p}}_2, s_2)u(\vec{\boldsymbol{p}}_1, s_1)\,, \tag{G.6.9}$$

$$\bar{v}(\vec{\boldsymbol{p}}_1, s_1)\gamma^\mu v(\vec{\boldsymbol{p}}_2, s_2) = \bar{u}(\vec{\boldsymbol{p}}_2, s_2)\gamma^\mu u(\vec{\boldsymbol{p}}_1, s_1)\,. \tag{G.6.10}$$

Consequently, the amplitude computed in eq. (G.6.8) is just the negative of eq. (G.6.7). This is expected, since the order of spinor wave functions in eq. (G.6.8) is an odd permutation of the order of spinor wave functions in eq. (G.6.7) [(12) and (21), respectively]. As in the two-component Feynman rules, the overall sign of the amplitude is arbitrary, but the relative signs of any pair of diagrams is unambiguous. This relative sign is positive [negative] if the permutation of the order of spinor wave functions of one diagram relative to the other diagram is even [odd].

Next, we consider the elastic scattering of a charged fermion and a neutral scalar. Again, we examine two equivalent ways for computing the amplitude. Following our rules, there are two possible choices for the directions of the fermion line arrows, depending on whether we represent the fermion by Ψ or Ψ^C. Thus, we may evaluate either one of the following two diagrams:

plus a second set of diagrams (not shown) where the initial and final state scalars are crossed. Evaluating the first diagram above, the matrix element for $\phi F \to \phi F$ is given by eq. (G.6.7), with λ replaced by κ. Had we chosen to evaluate the second diagram instead, the resulting amplitude would have been given by eq. (G.6.8), with λ replaced by κ. Thus, the discussion above in the case of neutral fermion scattering processes also applies to charged fermion scattering processes.

In processes that only involve vertices with two Dirac fields, it is never necessary to use charge-conjugated Dirac fermion lines. In contrast, consider the following process that involves a vertex with one Dirac and one Majorana fermion. Specifically, we examine the scattering of a Dirac fermion and a charged scalar into its charge-conjugated final state, via the exchange of a Majorana fermion: $\Phi^\dagger F \to \Phi \overline{F}$. If one attempts to draw the relevant Feynman diagram employing Dirac fermion lines but with no charge-conjugated Dirac fermion lines, one finds that there is no possible choice of arrow direction for the Majorana fermion that is consistent with the vertex rules of Fig. G.5.4. The resolution is simple: one can choose the incoming line to be Ψ and the outgoing line to be Ψ^C or vice versa. Thus, the two possible choices are given by:

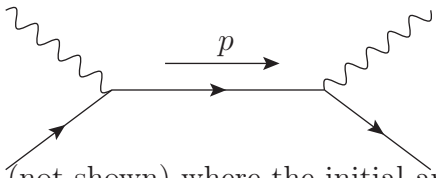
plus a second diagram in each case (not shown) in which the initial and final scalars are crossed. If we evaluate the first diagram, the resulting amplitude is given by:

$$\begin{aligned} i\mathcal{M} &= \frac{-i}{s-m^2}\,\bar{u}(\vec{\boldsymbol{p}}_2,s_2)(\kappa_2 P_L+\kappa_1^* P_R)(\not{p}+m)(\kappa_2 P_L+\kappa_1^* P_R)u(\vec{\boldsymbol{p}}_1,s_1)+(\text{crossed}) \\ &= \frac{-i}{s-m^2}\,\bar{u}(\vec{\boldsymbol{p}}_2,s_2)\left[\kappa_1^*\kappa_2\not{p}+\left(\kappa_2^2 P_L+(\kappa_1^*)^2 P_R\right)m\right]u(\vec{\boldsymbol{p}}_1,s_1)+(\text{crossed})\,, \end{aligned} \tag{G.6.11}$$

where m is the Majorana fermion mass. This result is equivalent to eq. (29.4.17) obtained via the two-component spinor methods. Had we chosen to evaluate the second diagram instead, the resulting amplitude [after using eqs. (G.6.9) and (G.6.10)] would have been found to be the negative of eq. (G.6.11), as expected. As before, the relative sign between diagrams for the same process is unambiguous.

In the literature, there are a number of alternative methods for dealing with scattering processes involving Majorana particles. For example, one can define a fermion number violating propagator for four-component fermions (see, *e.g.*, [7]). Using the methods of [7], factors of the charge conjugation matrix C appear both in fermion-number-violating propagators and vertices. However, all such factors of C eventually cancel out by the end of the computation of any S-matrix amplitude. Moreover, such methods often involve subtle choices of signs that require first-principles computations to verify. As previously noted, our four-component fermion diagrammatic techniques do not suffer from either of these complications.

In the case of elastic scattering of a Majorana fermion and a neutral vector boson, the two contributing diagrams include the following diagram:



plus a second diagram (not shown) where the initial and final state vector bosons are crossed. Consider first the scattering of a neutral Majorana fermion of mass m. Using the Feynman rules of Fig. G.5.2, the Feynman rule for the $A_\mu \overline{\Psi}_M \Psi_M$ vertex is given by $iG\gamma^\mu\gamma_5$. Hence, the corresponding matrix element is given by

$$i\mathcal{M} = \frac{-iG^2}{s-m^2}\bar{u}(\vec{\boldsymbol{p}}_2, s_2)\,\gamma\cdot\varepsilon_2^*\,(\not{p}-m)\,\gamma\cdot\varepsilon_1 u(\vec{\boldsymbol{p}}_1, s_1) + (\text{crossed})\,, \tag{G.6.12}$$

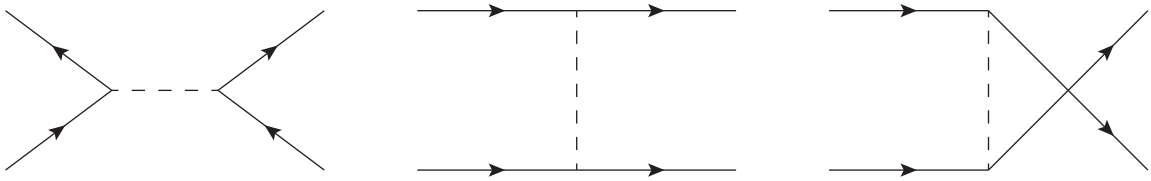
where we have used $\gamma^\nu\gamma_5(\not{p}+m)\gamma^\mu\gamma_5 = \gamma^\nu(\not{p}-m)\gamma^\mu$. Using eqs. (G.1.2) and (G.4.11), one easily recovers the results of eq. (29.4.14).

The scattering of a Dirac fermion of mass m and a neutral vector boson can be similarly treated. The relevant Feynman graphs are the same as in the previous case, and the matrix element is given by

$$\begin{aligned} i\mathcal{M} &= \frac{-i}{s-m^2}\bar{u}(\vec{\boldsymbol{p}}_2, s_2)\,\gamma\cdot\varepsilon_2^*\,(G_L P_L + G_R P_R)(\not{p}+m)\,\gamma\cdot\varepsilon_1\,(G_L P_L + G_R P_R)u(\vec{\boldsymbol{p}}_1, s_1) + (\text{crossed}) \\ &= \frac{-i}{s-m^2}\bar{u}(\vec{\boldsymbol{p}}_2, s_2)\,\gamma\cdot\varepsilon_2^*\left[(G_L^2 P_L + G_R^2 P_R)\not{p} + G_L G_R m\right]\,\gamma\cdot\varepsilon_1 u(\vec{\boldsymbol{p}}_1, s_1) + (\text{crossed})\,. \end{aligned} \tag{G.6.13}$$

One can easily check that this result coincides with that of eq. (29.4.18).

Finally, we examine the elastic scattering of two identical Majorana fermions via scalar exchange. The three contributing diagrams are:



and the corresponding matrix element is given by

$$\begin{aligned} i\mathcal{M} = {} & \frac{-i}{s-m_\phi^2}\left[\bar{v}_1(\lambda P_L + \lambda^* P_R)u_2\,\bar{u}_3(\lambda P_L + \lambda^* P_R)v_4\right] \\ & + (-1)\frac{-i}{t-m_\phi^2}\left[\bar{u}_3(\lambda P_L + \lambda^* P_R)u_1\,\bar{u}_4(\lambda P_L + \lambda^* P_R)u_2\right] \\ & + \frac{-i}{u-m_\phi^2}\left[\bar{u}_4(\lambda P_L + \lambda^* P_R)u_1\,\bar{u}_3(\lambda P_L + \lambda^* P_R)u_2\right]\,, \end{aligned} \tag{G.6.14}$$

where $u_i \equiv u(\vec{\boldsymbol{p}}_i, s_i)$, $v_j \equiv v(\vec{\boldsymbol{p}}_j, s_j)$ and m_ϕ is the exchanged scalar mass. The relative minus sign of the t-channel graph relative to the s and u-channel graphs is obtained by noting that 3142 [4132] is an odd [even] permutation of 1234. Using eqs. (G.1.8), (G.4.11) and (G.4.12), one easily recovers the results of eq. (29.4.19).

Self-energy functions and pole masses for four-component fermions

In this section, we examine the self-energy functions and the pole masses for a set of four-component fermions. We first consider four-component Dirac fermion fields Ψ_{ai}, where a is the

four-component spinor index and i is the flavor index. The full, loop-corrected Feynman propagators with four-momentum p^μ are defined by the Fourier transforms [cf. footnote 41] of vacuum expectation values of time-ordered products of bilinears of the fully interacting four-component fermion fields:

$$\langle 0|\, T\Psi_{ai}(x)\overline{\Psi}^{bj}(y)\,|0\rangle_{\rm FT} = i(\boldsymbol{S}_a{}^b)_i{}^j(p)\,, \tag{G.7.1}$$

with [310, 311, 312, 313, 314, 315, 316, 317]

$$\boldsymbol{S}(p) \equiv \not{p}\left[P_L\boldsymbol{S}_{\boldsymbol{L}}^{\mathsf{T}}(p^2) + P_R\boldsymbol{S}_{\boldsymbol{R}}(p^2)\right] + P_L\overline{\boldsymbol{S}}_{\boldsymbol{D}}^{\mathsf{T}}(p^2) + P_R\boldsymbol{S}_{\boldsymbol{D}}(p^2)\,, \tag{G.7.2}$$

where the four-component spinor indices a and b and the flavor indices i and j have been suppressed. As in Section 29.4.6, we shall organize the computation of the full propagator in terms of the 1PI self-energy function[313]:[156]

$$\boldsymbol{\Sigma}(p) \equiv \not{p}\left[P_L\boldsymbol{\Sigma}_{\boldsymbol{L}}(p^2) + P_R\boldsymbol{\Sigma}_{\boldsymbol{R}}^{\mathsf{T}}(p^2)\right] + P_L\boldsymbol{\Sigma}_{\boldsymbol{D}}(p^2) + P_R\overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}}^{\mathsf{T}}(p^2)\,. \tag{G.7.3}$$

Diagrammatically, $i\boldsymbol{S}$ and $-i\boldsymbol{\Sigma}$ are shown in Fig. G.7.1.

Figure G.7.1: The full, loop-corrected propagator for four-component Dirac fermions, $i(\boldsymbol{S}_a{}^b)_i{}^j(p)$, is denoted by the shaded box, which represents the sum of all connected Feynman diagrams, with external legs included. The self-energy function for four-component Dirac fermions, $-i(\boldsymbol{\Sigma}_a{}^b)_i{}^j(p)$, is denoted by the shaded circle, which represents the sum of all one-particle irreducible, connected Feynman diagrams with the external legs amputated. In both cases, The four-momentum p flows from right to left.

The hermiticity of the effective action implies that $\boldsymbol{S}$ and $\boldsymbol{\Sigma}$ satisfy hermiticity conditions [318, 302]

$$[\boldsymbol{S}^{\mathsf{T}}]^\star = A\boldsymbol{S}A^{-1}\,, \qquad [\boldsymbol{\Sigma}^{\mathsf{T}}]^\star = A\boldsymbol{\Sigma}A^{-1}\,, \tag{G.7.4}$$

where A is the Dirac conjugation matrix [$A = \gamma^0$ in the standard representations; see eq. (G.1.21) and the text that follows] and the star symbol was defined in the paragraph below eq. (29.4.6). Applying eq. (G.7.4) to eqs. (G.7.2) and (G.7.3) then yields the following conditions for the complex matrix functions:

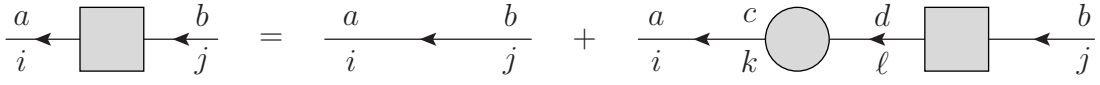
$$[\boldsymbol{S}_{\boldsymbol{L}}^{\mathsf{T}}]^\star = \boldsymbol{S}_{\boldsymbol{L}}\,, \qquad [\boldsymbol{S}_{\boldsymbol{R}}^{\mathsf{T}}]^\star = \boldsymbol{S}_{\boldsymbol{R}}\,, \qquad \overline{\boldsymbol{S}}_{\boldsymbol{D}} = \boldsymbol{S}_{\boldsymbol{D}}^\star\,, \tag{G.7.5}$$

$$[\boldsymbol{\Sigma}_{\boldsymbol{L}}^{\mathsf{T}}]^\star = \boldsymbol{\Sigma}_{\boldsymbol{L}}\,, \qquad [\boldsymbol{\Sigma}_{\boldsymbol{R}}^{\mathsf{T}}]^\star = \boldsymbol{\Sigma}_{\boldsymbol{R}}\,, \qquad \overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}} = \boldsymbol{\Sigma}_{\boldsymbol{D}}^\star\,. \tag{G.7.6}$$

Starting at tree level and comparing with Fig. G.5.1, the full propagator function is given by:

$$\boldsymbol{S}_i{}^j(p) = (\not{p} + m)\delta_i^j/(p^2 - m_i^2) + \ldots\,, \tag{G.7.7}$$

with no sum over i implied. The full loop-corrected propagator can be expressed diagrammatically in terms of the 1PI self-energy function:



(G.7.8)

[156] Our notation in eq. (G.7.3) differs from that of [313], as we employ $\boldsymbol{\Sigma}_{\boldsymbol{R}}^{\mathsf{T}}$ instead of $\boldsymbol{\Sigma}_{\boldsymbol{R}}$. Our motivation for this choice is that in the case of Majorana fermions [cf. eq. (G.7.15)], we simply have $\boldsymbol{\Sigma}_{\boldsymbol{L}} = \boldsymbol{\Sigma}_{\boldsymbol{R}}$, without an extra transpose (or conjugation). We have also chosen to employ $\boldsymbol{S}_{\boldsymbol{L}}^{\mathsf{T}}$ in eq. (G.7.2) for similar reasons.

As in Section 29.4.6, the algebraic representation of eq. (G.7.8) can be written as [cf. footnote 59]:

$$\boldsymbol{S} = \boldsymbol{T} + \boldsymbol{T}\boldsymbol{\Sigma}\boldsymbol{S} = (\boldsymbol{T}^{-1} - \boldsymbol{\Sigma})^{-1}\,, \tag{G.7.9}$$

where $\boldsymbol{T}_i{}^j \equiv (\not{p} + m)\delta_i^j/(p^2 - m_i^2)$ is the tree-level contribution to $\boldsymbol{S}$ given in eq. (G.7.7). By writing the expressions for $\boldsymbol{S}$ and $\boldsymbol{\Sigma}$ given in eqs. (G.7.2) and (G.7.3) and $\boldsymbol{T}$ in block matrix form using eq. (G.1.2), one can verify that eq. (G.7.9) is equivalent to eq. (29.4.27). Consequently, the complex pole masses of the corresponding Dirac fermions are again determined from eq. (29.4.32).

In the special case of a parity-conserving vectorlike theory of Dirac fermions (such as QED or QCD), the pseudoscalar and pseudovector parts of $\boldsymbol{S}(p)$ and $\boldsymbol{\Sigma}(p)$ must be absent. Thus, the following relations must hold among the loop-corrected propagator functions and self-energy functions, respectively:

$$\boldsymbol{S_R} = \boldsymbol{S_L^{\mathsf{T}}}\,, \qquad \boldsymbol{S_D} = [\boldsymbol{S_D^{\mathsf{T}}}]^\star\,, \tag{G.7.10}$$

$$\boldsymbol{\Sigma_L} = \boldsymbol{\Sigma_R^{\mathsf{T}}}\,, \qquad \boldsymbol{\Sigma_D} = [\boldsymbol{\Sigma_D^{\mathsf{T}}}]^\star\,, \tag{G.7.11}$$

in agreement with eqs. (29.4.33) and (29.4.34).

In the case of a set of four-component Majorana fermion fields, we can still use the results of eqs. (G.7.2)–(G.7.9). However, one obtains additional constraints on the full propagator and self-energy matrix functions due to the Majorana condition $\Psi_{Mi} = C\overline{\Psi}_{Mi}^{\mathsf{T}}$. Inserting this result into eq. (G.7.1), and making use of the anticommutativity of the fermion fields, one easily derives:

$$\langle 0|\, T\Psi_{Mai}(x)\overline{\Psi}_{Mj}^{\,b}(y)\,|0\rangle_{\rm FT} = C_{ae}\,\langle 0|\, T\Psi_{Mdi}(x)\overline{\Psi}_{Mj}^{\,e}(y)\,|0\rangle_{\rm FT}\,(C^{-1})^{db}\,. \tag{G.7.12}$$

Consequently,

$$C\boldsymbol{S}^{\mathsf{T}}C^{-1} = \boldsymbol{S}\,, \qquad C\boldsymbol{\Sigma}^{\mathsf{T}}C^{-1} = \boldsymbol{\Sigma}\,. \tag{G.7.13}$$

Inserting the expressions for $\boldsymbol{S}$ and $\boldsymbol{\Sigma}$ [eqs. (G.7.2) and (G.7.3)] and using the result of eq. (G.1.91), it follows that:

$$\boldsymbol{S_L} = \boldsymbol{S_R}\,, \qquad \boldsymbol{S_D} = \boldsymbol{S_D^{\mathsf{T}}}\,, \qquad \overline{\boldsymbol{S}}_{\boldsymbol{D}} = \overline{\boldsymbol{S}}_{\boldsymbol{D}}^{\mathsf{T}}\,, \tag{G.7.14}$$

$$\boldsymbol{\Sigma_L} = \boldsymbol{\Sigma_R}\,, \qquad \boldsymbol{\Sigma_D} = \boldsymbol{\Sigma_D^{\mathsf{T}}}\,, \qquad \overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}} = \overline{\boldsymbol{\Sigma}}_{\boldsymbol{D}}^{\mathsf{T}}\,. \tag{G.7.15}$$

As expected, with these constraints the form of eq. (29.4.27) matches precisely with the form of eq. (29.4.17), corresponding to the equation for the full propagator functions for a theory of generic two-component fermion fields. In the notation of Section 29.4.6, we can therefore identify: $\boldsymbol{C} \equiv \boldsymbol{S_L} = \boldsymbol{S_R}$, $\boldsymbol{D} \equiv \boldsymbol{S_D}$, $\boldsymbol{\Xi} \equiv \boldsymbol{\Sigma_L} = \boldsymbol{\Sigma_R}$, and $\boldsymbol{\Omega} \equiv \boldsymbol{\Sigma_D}$.

29.7.10 Covariant spin operators and the Bouchiat-Michel formulae

Bouchiat and Michel derived a useful set of formulae [113] that generalize the spin projection operators used in four-component spinor computations. In this Appendix, we establish the two-component analogues of the Bouchiat-Michel formulae, and demonstrate their equivalence to the corresponding four-component spinor formulae.

The covariant spin operators for a spin-1/2 fermion

Consider a massive spin-1/2 fermion of mass m and four-momentum p. We define a set of three four-vectors $S^{a\mu}$ $(a = 1, 2, 3)$ such that the $S^{a\mu}$ and p^μ/m form an orthonormal set of four-vectors. In the rest frame of the fermion, where $p^\mu = (m\,;\,\vec{\boldsymbol{0}})$, we can define

$$S^{a\mu} \equiv (0\,;\,\hat{\boldsymbol{s}}^{\boldsymbol{a}})\,, \qquad a = 1, 2, 3\,, \tag{H.1.1}$$

where the unit vectors $\hat{\boldsymbol{s}}^{\boldsymbol{a}}$ are a mutually orthonormal set of unit three-vectors that form a basis for a right-handed coordinate system. Explicit forms for the $\hat{\boldsymbol{s}}^{\boldsymbol{a}}$ depend on the Euler angle γ used to define the spinor wave function $\chi_s(\hat{\boldsymbol{s}})$. Two common choices corresponding to $\gamma = -\phi$ and $\gamma = 0$ are given in eqs. (C.1.39) and (C.1.40), respectively. Using eq. (29.117), the three four-vectors $S^{a\mu}$, in a reference frame in which the four-momentum of the fermion is $p^\mu = (E\,;\,\vec{\boldsymbol{p}})$, are given by:

$$S^{a\mu} = \left(\frac{\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{s}}^{\boldsymbol{a}}}{m}\,;\,\hat{\boldsymbol{s}}^{\boldsymbol{a}} + \frac{(\vec{\boldsymbol{p}}\cdot\hat{\boldsymbol{s}}^{\boldsymbol{a}})\,\vec{\boldsymbol{p}}}{m(E+m)}\right), \qquad a = 1,2,3\,. \tag{H.1.2}$$

As discussed in Appendix C , we identify $\hat{\boldsymbol{s}} = \hat{\boldsymbol{s}}^{\boldsymbol{3}}$ as the quantization axis used in defining the third component of the spin of the fermion in its rest frame. It then follows that the spin four-vector, previously introduced in eq. (29.3.15) is given by $S^\mu = S^{3\mu}$.

The orthonormal set of four four-vectors p^μ/m and the $S^{a\mu}$ satisfy the following Lorentz-covariant relations:

$$p\cdot S^a = 0\,, \tag{H.1.3}$$

$$S^a\cdot S^b = -\delta^{ab}\,, \tag{H.1.4}$$

$$\epsilon^{\mu\nu\lambda\sigma}p_\mu S^1_\nu S^2_\lambda S^3_\sigma = -m\,, \tag{H.1.5}$$

$$S^a_\mu S^b_\nu - S^a_\nu S^b_\mu = \epsilon^{abc}\epsilon_{\mu\nu\rho\sigma}S^{c\rho}\,\frac{p^\sigma}{m}\,, \tag{H.1.6}$$

$$S^a_\mu\, S^a_\nu = -g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\,, \tag{H.1.7}$$

where the sum over the repeated indices is implicit. It is convenient to define a matrix-valued spin four-vector $\mathscr{S}^\mu$, whose matrix elements are given by:

$$\mathscr{S}^\mu_{ss'} \equiv S^{a\mu}\tau^a_{ss'}\,, \qquad s, s' = \pm\tfrac{1}{2}\,, \tag{H.1.8}$$

where $\tau^a_{ss'}$ are the matrix elements of the Pauli matrices (see footnote 102). Then, we can rewrite eqs. (H.1.4) and (H.1.6) as:

$$\tfrac{1}{3}\,g_{\mu\nu}\mathscr{S}^\mu\mathscr{S}^\nu = -\mathbb{1}_{2\times2}\,, \tag{H.1.9}$$

$$\mathscr{S}^\mu\mathscr{S}^\nu - \mathscr{S}^\nu\mathscr{S}^\mu = \frac{2i}{m}\epsilon^{\mu\nu\rho\sigma}\mathscr{S}_\rho p_\sigma\,, \tag{H.1.10}$$

where the product $\mathscr{S}^\mu\mathscr{S}^\nu$ corresponds to ordinary 2×2 matrix multiplication. The $\mathscr{S}^\mu$ serve as covariant spin operators for a spin-1/2 fermion. In particular, in the rest frame, the $\frac{1}{2}\mathscr{S}^i$ satisfy the usual SU(2) commutation relations, with $(\frac{1}{2}\vec{\boldsymbol{\mathscr{S}}})^2 = \frac{3}{4}$ as expected for a spin-1/2 particle.

It is often desirable to work with helicity states. In this case, we choose:

$$\hat{\boldsymbol{s}}^{\boldsymbol{a}} = \hat{\boldsymbol{p}}^{\boldsymbol{a}}\,, \tag{H.1.11}$$

where the $\hat{\boldsymbol{p}}^{\boldsymbol{a}}$ are an orthonormal triad of unit three-vectors with $\hat{\boldsymbol{p}}^{\boldsymbol{3}} \equiv \hat{\boldsymbol{p}}$. Moreover, since $\hat{\boldsymbol{p}}^{\boldsymbol{a}}\cdot\hat{\boldsymbol{p}} = 0$ for $a \neq 3$, it follows that $S^{a\mu} = (0\,;\,\hat{\boldsymbol{p}}^{\boldsymbol{a}})$ for $a = 1,2$ in all reference frames obtained from the rest frame by a boost in the $\hat{\boldsymbol{p}}$ direction. Hence, in a reference frame where $p^\mu = (E\,;\,\vec{\boldsymbol{p}})$, eq. (H.1.2) yields,

$$S^{1\mu} = (0\,;\,\hat{\boldsymbol{p}}^{\boldsymbol{1}})\,, \tag{H.1.12}$$

$$S^{2\mu} = (0\,;\,\hat{\boldsymbol{p}}^{\boldsymbol{2}})\,, \tag{H.1.13}$$

$$S^{3\mu} = \left(\frac{|\vec{\boldsymbol{p}}|}{m}\,;\,\frac{E}{m}\,\hat{\boldsymbol{p}}\right), \tag{H.1.14}$$

in a coordinate system where $\hat{\boldsymbol{p}} = (\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta)$. One can check that eqs. (H.1.1)–(H.1.7) are also satisfied by the $S^{a\mu}$ defined in eqs. (H.1.12)–(H.1.14).

As expected, $S^{3\mu}$ is the spin four-vector for helicity states obtained in eq. (29.3.16). In the high energy limit ($E \gg m$),

$$mS^{a\mu} = p^\mu\,\delta^{a3} + \mathcal{O}(m)\,. \tag{H.1.15}$$

Explicit forms for $\hat{\boldsymbol{p}}^{\boldsymbol{1}}$ and $\hat{\boldsymbol{p}}^{\boldsymbol{2}}$ are convention dependent and depend on the conventional choice of the Euler angle γ. For example, consider the quantities:

$$S^\mu_- \equiv \tfrac{1}{2}S^{a\mu}\tau^a_{\frac{1}{2},-\frac{1}{2}} = \tfrac{1}{2}(S^{1\mu} - iS^{2\mu})\,, \qquad S^\mu_+ \equiv \tfrac{1}{2}S^{a\mu}\tau^a_{-\frac{1}{2},\frac{1}{2}} = \tfrac{1}{2}(S^{1\mu} + iS^{2\mu})\,. \tag{H.1.16}$$

Using eqs. (H.1.11)–(H.1.14) and employing eq. (C.1.27) with $\mathcal{R}$ given by eq. (C.1.4),

$$\sigma\cdot S_- = e^{i\gamma}\begin{pmatrix} \frac{1}{2}\sin\theta & e^{-i\phi}\sin^2\frac{\theta}{2} \\ -e^{i\phi}\cos^2\frac{\theta}{2} & -\frac{1}{2}\sin\theta \end{pmatrix}, \qquad \overline{\sigma}\cdot S_+ = e^{-i\gamma}\begin{pmatrix} \frac{1}{2}\sin\theta & -e^{-i\phi}\cos^2\frac{\theta}{2} \\ e^{i\phi}\sin^2\frac{\theta}{2} & -\frac{1}{2}\sin\theta \end{pmatrix}. \tag{H.1.17}$$

In the convention of eq. (C.1.39) [eq. (C.1.40)], we take $\gamma = -\phi$ [$\gamma = 0$], respectively.

Two-component spinor wave function relations

In Section 29.3.1, we wrote down explicit forms for the undotted spinor wave functions

$$x_\alpha(\vec{\boldsymbol{p}}, s) = \sqrt{p\cdot\sigma}\,\chi_s\,, \qquad x^\alpha(\vec{\boldsymbol{p}}, s) = -2s\chi^\dagger_{-s}\sqrt{p\cdot\overline{\sigma}}\,, \tag{H.2.1}$$

$$y_\alpha(\vec{\boldsymbol{p}}, s) = 2s\sqrt{p\cdot\sigma}\,\chi_{-s}\,, \qquad y^\alpha(\vec{\boldsymbol{p}}, s) = \chi^\dagger_s\sqrt{p\cdot\overline{\sigma}}\,, \tag{H.2.2}$$

and the dotted spinor wave functions

$$x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}}, s) = -2s\sqrt{p\cdot\overline{\sigma}}\,\chi_{-s}\,, \qquad x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}}, s) = \chi^\dagger_s\sqrt{p\cdot\sigma}\,, \tag{H.2.3}$$

$$y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}}, s) = \sqrt{p\cdot\overline{\sigma}}\,\chi_s\,, \qquad y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}}, s) = 2s\chi^\dagger_{-s}\sqrt{p\cdot\sigma}\,, \tag{H.2.4}$$

where $\sqrt{p\cdot\sigma}$ and $\sqrt{p\cdot\overline{\sigma}}$ are defined by eqs. (29.108) and (29.109), respectively.

As shown in Appendix C , the two-component spinors χ_s satisfy:

$$\tfrac{1}{2}\,\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}^{\boldsymbol{a}}\chi_{s'} = \tfrac{1}{2}\tau^a_{ss'}\chi_s\,, \qquad \chi^\dagger_s(\hat{\boldsymbol{s}})\chi_{s'}(\hat{\boldsymbol{s}}) = \delta_{ss'}\,, \qquad s\,, s' = \pm\tfrac{1}{2}\,. \tag{H.2.5}$$

Next, we use eq. (29.118) to obtain:

$$\sqrt{p\cdot\sigma}\,S^a\cdot\overline{\sigma}\,\sqrt{p\cdot\sigma} = m\,\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}^{\boldsymbol{a}}\,, \tag{H.2.6}$$

$$\sqrt{p\cdot\overline{\sigma}}\,S^a\cdot\sigma\,\sqrt{p\cdot\overline{\sigma}} = -m\,\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}^{\boldsymbol{a}}\,, \tag{H.2.7}$$

which extends the results of eq. (29.3.17). As a result, we obtain a generalization of eqs. (29.3.23)–(29.3.26):

$$(S^a\cdot\overline{\sigma})^{\dot\alpha\beta}x_\beta(\vec{\boldsymbol{p}}, s') = \tau^a_{ss'}y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}}, s)\,, \qquad (S^a\cdot\sigma)_{\alpha\dot\beta}y^{\dagger\dot\beta}(\vec{\boldsymbol{p}}, s') = -\tau^a_{ss'}x_\alpha(\vec{\boldsymbol{p}}, s)\,, \tag{H.2.8}$$

$$(S^a\cdot\sigma)_{\alpha\dot\beta}x^{\dagger\dot\beta}(\vec{\boldsymbol{p}}, s') = -\tau^a_{s's}y_\alpha(\vec{\boldsymbol{p}}, s)\,, \qquad (S^a\cdot\overline{\sigma})^{\dot\alpha\beta}y_\beta(\vec{\boldsymbol{p}}, s') = \tau^a_{s's}x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}}, s)\,, \tag{H.2.9}$$

$$x^\alpha(\vec{\boldsymbol{p}}, s')(S^a\cdot\sigma)_{\alpha\dot\beta} = -\tau^a_{s's}y^\dagger_{\dot\beta}(\vec{\boldsymbol{p}}, s)\,, \qquad y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}}, s')(S^a\cdot\overline{\sigma})^{\dot\alpha\beta} = \tau^a_{s's}x^\beta(\vec{\boldsymbol{p}}, s)\,, \tag{H.2.10}$$

$$x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}}, s')(S^a\cdot\overline{\sigma})^{\dot\alpha\beta} = \tau^a_{ss'}y^\beta(\vec{\boldsymbol{p}}, s)\,, \qquad y^\alpha(\vec{\boldsymbol{p}}, s')(S^a\cdot\sigma)_{\alpha\dot\beta} = -\tau^a_{ss'}x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}}, s)\,, \tag{H.2.11}$$

where there are implicit sums over the repeated labels $s = \pm\frac{1}{2}$. As expected, the case of $a = 3$ simply reproduces the results of eqs. (29.3.23)–(29.3.26) obtained previously. The above equations also apply to helicity wave functions $x(\vec{\boldsymbol{p}}, \lambda)$ and $y(\vec{\boldsymbol{p}}, \lambda)$ by replacing s, s' with λ, λ' and defining the $S^{a\mu}$ by eqs. (H.1.12)–(H.1.14).

The derivation of eqs. (H.2.8)–(H.2.11) for arbitrary a closely follows the corresponding derivation for $a = 3$ previously given. For example, using eqs. (H.2.6) and (H.2.7) and the definitions for $x_\alpha(\vec{\boldsymbol{p}}, s)$ and $y^{\dagger\dot{\alpha}}(\vec{\boldsymbol{p}}, s)$, we find (suppressing spinor indices),

$$\sqrt{p\cdot\sigma}\, S^a\cdot\overline{\sigma}\, x(\vec{\boldsymbol{p}}, s') = \sqrt{p\cdot\sigma}\, S^a\cdot\overline{\sigma}\, \sqrt{p\cdot\sigma}\, \chi_{s'} = m\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}^{\boldsymbol{a}}\, \chi_{s'} = m\tau^a_{ss'}\, \chi_s\,, \tag{H.2.12}$$

after using eq. (H.2.5). Multiplying both sides of eq. (H.2.12) by $\sqrt{p\cdot\overline{\sigma}}$, we end up with

$$S^a\cdot\overline{\sigma}\, x(\vec{\boldsymbol{p}}, s') = \tau^a_{ss'}\sqrt{p\cdot\overline{\sigma}}\, \chi_s = \tau^a_{ss'}\, y^\dagger(\vec{\boldsymbol{p}}, s)\,. \tag{H.2.13}$$

Similarly,

$$S^a\cdot\sigma x^\dagger(\vec{\boldsymbol{p}}, s') = 2s'\tau^a_{-s,-s'}\sqrt{p\cdot\sigma}\, \chi_{-s} = -\tau^a_{s's}\, y(\vec{\boldsymbol{p}}, s)\,, \tag{H.2.14}$$

where we have used:

$$4ss'\tau^a_{-s,-s'} = -\tau^a_{s's}\,, \qquad \text{for} \quad s, s' = \pm 1/2\,. \tag{H.2.15}$$

All the results of eqs. (H.2.8)–(H.2.11) can be derived in this manner.

Two-component Bouchiat-Michel formulae

To establish the Bouchiat-Michel formulae, we begin with the following identity:

$$\tfrac{1}{2}(\delta_{ss'} + \vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}^a\, \tau^a_{ss'}) \sum_{t=\pm 1/2} \chi_t\chi_t^\dagger = \chi_{s'}\chi_s^\dagger\,. \tag{H.3.1}$$

To verify eq. (H.3.1), we first make use of eq. (H.2.5) to write $\vec{\boldsymbol{\sigma}}\boldsymbol{\cdot}\hat{\boldsymbol{s}}^a\chi_t = \tau^a_{t't}\chi_{t'}$ and evaluate the product of two Pauli matrices:

$$\tau^a_{ss'}\tau^a_{t't} = 2\,\delta_{st}\delta_{s't'} - \delta_{ss'}\delta_{tt'}\,. \tag{H.3.2}$$

We then use eq. (H.2.6) and the completeness relation given in eq. (C.1.21) to rewrite eq. (H.3.1) in terms of $\mathscr{S}^\mu_{ss'}$ defined in eq. (H.1.8),

$$\chi_{s'}\chi_s^\dagger = \tfrac{1}{2}\left(\delta_{ss'} + \frac{1}{m}\sqrt{p\cdot\sigma}\,\mathscr{S}_{ss'}\cdot\overline{\sigma}\,\sqrt{p\cdot\sigma}\right)\,. \tag{H.3.3}$$

Hence, with both spinor indices in the lowered position,

$$\begin{aligned} x(\vec{\boldsymbol{p}}, s')x^\dagger(\vec{\boldsymbol{p}}, s) &= \sqrt{p\cdot\sigma}\,\chi_{s'}\chi_s^\dagger\,\sqrt{p\cdot\sigma} \\ &= \tfrac{1}{2}\sqrt{p\cdot\sigma}\left[\delta_{ss'} + \frac{1}{m}\sqrt{p\cdot\sigma}\,\mathscr{S}_{ss'}\cdot\overline{\sigma}\,\sqrt{p\cdot\sigma}\right]\sqrt{p\cdot\sigma} \\ &= \tfrac{1}{2}\left[p\cdot\sigma\delta_{ss'} + \frac{1}{m}\,p\cdot\sigma\,\mathscr{S}_{ss'}\cdot\overline{\sigma}\,p\cdot\sigma\right] \\ &= \tfrac{1}{2}\left(p\cdot\sigma\delta_{ss'} - m\mathscr{S}_{ss'}\cdot\sigma\right)\,. \end{aligned} \tag{H.3.4}$$

In the final step of eq. (H.3.4), we simplified the product of three dot products by noting that $p\cdot S^a = 0$ implies that $\mathscr{S}_{ss'}\cdot\overline{\sigma}\; p\cdot\sigma = -p\cdot\overline{\sigma}\;\mathscr{S}_{ss'}\cdot\sigma$. Eq. (H.3.4) is the two-component version of one

of the Bouchiat-Michel formulae. We list below a complete set of Bouchiat-Michel formulae, which can be derived by similar techniques:

$$x_\alpha(\vec{\boldsymbol{p}},s')x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}(p\,\delta_{ss'} - m\mathscr{S}_{ss'})\cdot\sigma_{\alpha\dot\beta}\,, \tag{H.3.5}$$

$$y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s')y^\beta(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}(p\,\delta_{ss'} + m\mathscr{S}_{ss'})\cdot\overline{\sigma}^{\dot\alpha\beta}\,, \tag{H.3.6}$$

$$x_\alpha(\vec{\boldsymbol{p}},s')y^\beta(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}\left[m\delta_{ss'}\delta_\alpha{}^\beta - [(\sigma\cdot\mathscr{S}_{ss'})\,(\overline{\sigma}\cdot p)]_\alpha{}^\beta\right]\,, \tag{H.3.7}$$

$$y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s')x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}\left[m\delta_{ss'}\delta^{\dot\alpha}{}_{\dot\beta} + [(\overline{\sigma}\cdot\mathscr{S}_{ss'})\,(\sigma\cdot p)]^{\dot\alpha}{}_{\dot\beta}\right]\,. \tag{H.3.8}$$

If we set $s = s'$, we recover eqs. (29.3.45)–(29.3.48) as expected. The Bouchiat-Michel formulae can also be verified directly by using the explicit forms for the two-component spinor wave functions [eq. (C.1.11)] and the $\mathscr{S}^\mu_{ss'}$ [defined in eq. (H.1.8)]. The latter depends on the explicit form of the $\hat{\boldsymbol{s}}^{\boldsymbol{a}}$ via eq. (H.1.2).

An equivalent set of Bouchiat-Michel formulae can be obtained by raising and/or lowering the appropriate free spinor indices using eqs. (29.32) and (29.79):

$$x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s')x^\beta(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}(p\,\delta_{s's} - m\mathscr{S}_{s's})\cdot\overline{\sigma}^{\dot\alpha\beta}\,, \tag{H.3.9}$$

$$y_\alpha(\vec{\boldsymbol{p}},s')y^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}(p\,\delta_{s's} + m\mathscr{S}_{s's})\cdot\sigma_{\alpha\dot\beta}\,, \tag{H.3.10}$$

$$y_\alpha(\vec{\boldsymbol{p}},s')x^\beta(\vec{\boldsymbol{p}},s) = -\tfrac{1}{2}\left[m\delta_{s's}\delta_\alpha{}^\beta + [(\sigma\cdot\mathscr{S}_{s's})\,(\overline{\sigma}\cdot p)]_\alpha{}^\beta\right]\,, \tag{H.3.11}$$

$$x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},s')y^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},s) = -\tfrac{1}{2}\left[m\delta_{s's}\delta^{\dot\alpha}{}_{\dot\beta} - [(\overline{\sigma}\cdot\mathscr{S}_{s's})\,(\sigma\cdot p)]^{\dot\alpha}{}_{\dot\beta}\right]\,. \tag{H.3.12}$$

In this derivation, the spin labels in eqs. (H.3.9)–(H.3.12) are reversed relative to those in eqs. (H.3.5)–(H.3.8) due to eq. (H.2.15). Eight additional relations of the Bouchiat-Michel type can be obtained by replacing one x-spinor with a y-spinor (or vice versa). Recalling that the x and y spinors are related by [cf. eq. (29.3.22)],

$$y(\vec{\boldsymbol{p}},s) = 2sx(\vec{\boldsymbol{p}},-s)\,, \qquad\qquad y^\dagger(\vec{\boldsymbol{p}},s) = 2sx^\dagger(\vec{\boldsymbol{p}},-s)\,, \tag{H.3.13}$$

all possible spinor bilinears can be obtained from eqs. (H.3.5)–(H.3.12).

Note that eqs. (H.3.5)–(H.3.12) also apply to helicity spinor wave functions $x(\vec{\boldsymbol{p}},\lambda)$ and $y(\vec{\boldsymbol{p}},\lambda)$ after replacing s, s' with λ, λ' and using the $S^{a\mu}$ as defined in eqs. (H.1.12)–(H.1.14). Strictly speaking, all results involving the spinor wave functions obtained up to this point apply in the case of a massive spin-1/2 fermion. If we take the massless limit, then the four-vector $S^{3\mu}$ does not exist, as its definition depends on the existence of a rest frame. (In contrast, the four-vectors $S^{1\mu}$ and $S^{2\mu}$ do exist in the massless limit.) Nevertheless, massless helicity spinor wave functions are well defined; explicit forms can be found in eqs. (29.3.36)–(29.3.39). Using these forms, one can derive the Bouchiat-Michel formulae for a massless spin-1/2 fermion:

$$x_\alpha(\vec{\boldsymbol{p}},\lambda')x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = (\tfrac{1}{2}-\lambda)\,\delta_{\lambda\lambda'}\,p\cdot\sigma_{\alpha\dot\beta}\,, \tag{H.3.14}$$

$$y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},\lambda')y^\beta(\vec{\boldsymbol{p}},\lambda) = (\tfrac{1}{2}+\lambda)\,\delta_{\lambda\lambda'}\,p\cdot\overline{\sigma}^{\dot\alpha\beta}\,, \tag{H.3.15}$$

$$x_\alpha(\vec{\boldsymbol{p}},\lambda')y^\beta(\vec{\boldsymbol{p}},\lambda) = -(\tfrac{1}{2}-\lambda')(\tfrac{1}{2}+\lambda)\,[(\sigma\cdot S_-)(\overline{\sigma}\cdot p)]_\alpha{}^\beta\,, \tag{H.3.16}$$

$$y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},\lambda')x^\dagger_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = (\tfrac{1}{2}+\lambda')(\tfrac{1}{2}-\lambda)\,[(\overline{\sigma}\cdot S_+)(\sigma\cdot p)]^{\dot\alpha}{}_{\dot\beta}\,, \tag{H.3.17}$$

where S^μ_- and S^μ_+ are defined in eq. (H.1.16). The equivalent set of Bouchiat-Michel formulae, obtained by raising and/or lowering the appropriate free spinor indices, is given by:

$$x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},\lambda')x^{\beta}(\vec{\boldsymbol{p}},\lambda) = (\tfrac{1}{2}-\lambda)\,\delta_{\lambda\lambda'}\,p{\cdot}\overline{\sigma}^{\dot\alpha\beta}\,, \tag{H.3.18}$$

$$y_{\alpha}(\vec{\boldsymbol{p}},\lambda')y^{\dagger}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = (\tfrac{1}{2}+\lambda)\,\delta_{\lambda\lambda'}\,p{\cdot}\sigma_{\alpha\dot\beta}\,, \tag{H.3.19}$$

$$y_{\alpha}(\vec{\boldsymbol{p}},\lambda')x^{\beta}(\vec{\boldsymbol{p}},\lambda) = -(\tfrac{1}{2}+\lambda')(\tfrac{1}{2}-\lambda)\,[(\sigma{\cdot}S_-)(\overline{\sigma}{\cdot}p)]_{\alpha}{}^{\beta}\,, \tag{H.3.20}$$

$$x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},\lambda')y^{\dagger}_{\dot\beta}(\vec{\boldsymbol{p}},\lambda) = (\tfrac{1}{2}-\lambda')(\tfrac{1}{2}+\lambda)\,[(\overline{\sigma}{\cdot}S_+)(\sigma{\cdot}p)]^{\dot\alpha}{}_{\dot\beta}\,. \tag{H.3.21}$$

Eight additional relations of the Bouchiat-Michel type can be obtained by replacing one x-spinor with a y-spinor (or vice versa), using the results of eq. (H.3.13). As a check, one can verify that the above results follow from eqs. (H.3.5)–(H.3.12) by replacing s with λ, setting $mS^{a\mu} = p^{\mu}\,\delta^{a3}$, applying the mass-shell condition ($p^2 = m^2$), and taking the $m \to 0$ limit at the end of the computation.

We now demonstrate how to use the Bouchiat-Michel formulae to evaluate helicity amplitudes involving two equal mass spin-1/2 fermions. A typical amplitude involving a fermion-antifermion pair, evaluated in the center-of-mass frame of the pair has the generic structure:

$$z(\vec{\boldsymbol{p}},\lambda)\,\Gamma\,z'(-\vec{\boldsymbol{p}},\lambda')\,, \tag{H.3.22}$$

where z is one of the two-component spinor wave functions x, $x^\dagger$, y, or $y^\dagger$, and Γ is a 2×2 matrix (in spinor space) that is either the identity matrix, or is made up of alternating products of σ and $\overline{\sigma}$. As an illustration, we evaluate:

$$x^{\dagger}_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\,\Gamma^{\dot\alpha\beta}\,y_{\beta}(-\vec{\boldsymbol{p}},\lambda') = 2\lambda'\,\Gamma^{\dot\alpha\beta}\,x_{\beta}(-\vec{\boldsymbol{p}},-\lambda')x^{\dagger}_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda) = 2\lambda'\,\xi_{\lambda'}(\hat{\boldsymbol{p}})\;\Gamma^{\dot\alpha\beta}\sigma^{0}_{\beta\dot\beta}\,y^{\dagger\dot\beta}(\vec{\boldsymbol{p}},\lambda')x^{\dagger}_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\,, \tag{H.3.23}$$

where $\xi_{\lambda'}(\hat{\boldsymbol{p}})$ is defined in eq. (C.3.14), and we have used eqs. (C.3.17) and (29.3.22). We can now employ the Bouchiat-Michel formula to convert the above result into a trace. By a similar computation, all expressions of the form of eq. (H.3.22) can be expressed as a trace:

$$x^{\dagger}_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\,\Gamma^{\dot\alpha\beta}\,y_{\beta}(-\vec{\boldsymbol{p}},\lambda') = \lambda'\,\xi_{\lambda'}(\hat{\boldsymbol{p}})\;\mathrm{Tr}\left[\Gamma\,\sigma^{0}(m\delta_{\lambda\lambda'}+\overline{\sigma}{\cdot}\mathscr{S}_{\lambda\lambda'}\,\sigma{\cdot}p)\right]\,, \tag{H.3.24}$$

$$y^{\alpha}(\vec{\boldsymbol{p}},\lambda)\,\Gamma_{\alpha\dot\beta}\,x^{\dagger\dot\beta}(-\vec{\boldsymbol{p}},\lambda') = -\lambda'\,\xi_{\lambda'}(\hat{\boldsymbol{p}})\;\mathrm{Tr}\left[\Gamma\,\overline{\sigma}^{0}(m\delta_{\lambda\lambda'}-\sigma{\cdot}\mathscr{S}_{\lambda\lambda'}\,\overline{\sigma}{\cdot}p)\right]\,, \tag{H.3.25}$$

$$y^{\alpha}(\vec{\boldsymbol{p}},\lambda)\,\Gamma_{\alpha}{}^{\beta}\,y_{\beta}(-\vec{\boldsymbol{p}},\lambda') = \lambda'\,\xi_{\lambda'}(\hat{\boldsymbol{p}})\;\mathrm{Tr}\left[\Gamma\,\sigma^{0}(\overline{\sigma}{\cdot}p\,\delta_{\lambda\lambda'}+m\overline{\sigma}{\cdot}\mathscr{S}_{\lambda\lambda'})\right]\,, \tag{H.3.26}$$

$$x^{\dagger}_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\,\Gamma^{\dot\alpha}{}_{\dot\beta}\,x^{\dagger\dot\beta}(-\vec{\boldsymbol{p}},\lambda') = -\lambda'\,\xi_{\lambda'}(\hat{\boldsymbol{p}})\;\mathrm{Tr}\left[\Gamma\,\overline{\sigma}^{0}(\sigma{\cdot}p\,\delta_{\lambda\lambda'}-m\sigma{\cdot}\mathscr{S}_{\lambda\lambda'})\right]\,, \tag{H.3.27}$$

after making use of eqs. (H.3.5) and (H.3.8). Similarly, there are four additional results that make use of eqs. (H.3.9) and (H.3.12):

$$y^{\dagger}_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\,\Gamma^{\dot\alpha\beta}\,x_{\beta}(-\vec{\boldsymbol{p}},\lambda') = \lambda'\,\xi_{-\lambda'}(\hat{\boldsymbol{p}})\;\mathrm{Tr}\left[\Gamma\,\sigma^{0}(m\delta_{\lambda'\lambda}-\overline{\sigma}{\cdot}\mathscr{S}_{\lambda'\lambda}\,\sigma{\cdot}p)\right]\,, \tag{H.3.28}$$

$$x^{\alpha}(\vec{\boldsymbol{p}},\lambda)\,\Gamma_{\alpha\dot\beta}\,y^{\dagger\dot\beta}(-\vec{\boldsymbol{p}},\lambda') = -\lambda'\,\xi_{-\lambda'}(\hat{\boldsymbol{p}})\;\mathrm{Tr}\left[\Gamma\,\overline{\sigma}^{0}(m\delta_{\lambda'\lambda}+\sigma{\cdot}\mathscr{S}_{\lambda'\lambda}\,\overline{\sigma}{\cdot}p)\right]\,, \tag{H.3.29}$$

$$x^{\alpha}(\vec{\boldsymbol{p}},\lambda)\,\Gamma_{\alpha}{}^{\beta}\,x_{\beta}(-\vec{\boldsymbol{p}},\lambda') = -\lambda'\,\xi_{-\lambda'}(\hat{\boldsymbol{p}})\;\mathrm{Tr}\left[\Gamma\,\sigma^{0}(\overline{\sigma}{\cdot}p\,\delta_{\lambda'\lambda}-m\overline{\sigma}{\cdot}\mathscr{S}_{\lambda'\lambda})\right]\,, \tag{H.3.30}$$

$$y^{\dagger}_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\,\Gamma^{\dot\alpha}{}_{\dot\beta}\,y^{\dagger\dot\beta}(-\vec{\boldsymbol{p}},\lambda') = \lambda'\,\xi_{-\lambda'}(\hat{\boldsymbol{p}})\;\mathrm{Tr}\left[\Gamma\,\overline{\sigma}^{0}(\sigma{\cdot}p\,\delta_{\lambda'\lambda}+m\sigma{\cdot}\mathscr{S}_{\lambda'\lambda})\right]\,. \tag{H.3.31}$$

For amplitudes involving equal mass fermions (or equal mass antifermions), other combinations of spinor bilinears appear in which one x-spinor above is replaced by a y-spinor or vice versa. These amplitudes can be reduced to one of the eight listed above by using eq. (29.3.22).

In the massless limit, one can again put $mS^{a\mu} = p^{\mu}\delta^{a3}$, set $p^2 = m^2$ and take $m \to 0$ at the end of the computation. Alternatively, one can repeat the derivation of eqs. (H.3.24)–(H.3.31) using

the results of eqs. (H.3.14) and (H.3.21). For completeness, we record the end result here.

$$x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\,\Gamma^{\dot\alpha\beta}\,y_\beta(-\vec{\boldsymbol{p}},\lambda') = (\tfrac{1}{2}+\lambda')(\tfrac{1}{2}-\lambda)\,\xi_{\lambda'}(\hat{\boldsymbol{p}})\ \mathrm{Tr}(\Gamma\,\sigma^0\overline{\sigma}\cdot S_-\,\sigma\cdot p)\,, \tag{H.3.32}$$

$$y^\alpha(\vec{\boldsymbol{p}},\lambda)\,\Gamma_{\alpha\dot\beta}\,x^{\dagger\dot\beta}(-\vec{\boldsymbol{p}},\lambda') = -(\tfrac{1}{2}-\lambda')(\tfrac{1}{2}+\lambda)\,\xi_{\lambda'}(\hat{\boldsymbol{p}})\ \mathrm{Tr}(\Gamma\,\overline{\sigma}^0\sigma\cdot S_-\,\overline{\sigma}\cdot p)\,, \tag{H.3.33}$$

$$y^\alpha(\vec{\boldsymbol{p}},\lambda)\,\Gamma_\alpha{}^\beta\,y_\beta(-\vec{\boldsymbol{p}},\lambda') = (\tfrac{1}{2}+\lambda)\,\delta_{\lambda\lambda'}\,\xi_{\lambda'}(\hat{\boldsymbol{p}})\ \mathrm{Tr}(\Gamma\,\sigma^0\,\overline{\sigma}\cdot p)\,, \tag{H.3.34}$$

$$x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\,\Gamma^{\dot\alpha}{}_{\dot\beta}\,x^{\dagger\dot\beta}(-\vec{\boldsymbol{p}},\lambda') = (\tfrac{1}{2}-\lambda)\,\delta_{\lambda\lambda'}\,\xi_{\lambda'}(\hat{\boldsymbol{p}})\ \mathrm{Tr}(\Gamma\,\overline{\sigma}^0\,\sigma\cdot p)\,. \tag{H.3.35}$$

The equivalent set of formulae, obtained by raising and/or lowering the appropriate free spinor indices as before, is given by:

$$y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\,\Gamma^{\dot\alpha\beta}\,x_\beta(-\vec{\boldsymbol{p}},\lambda') = (\tfrac{1}{2}-\lambda')(\tfrac{1}{2}+\lambda)\,\xi_{-\lambda'}(\hat{\boldsymbol{p}})\ \mathrm{Tr}(\Gamma\,\sigma^0\overline{\sigma}\cdot S_+\,\sigma\cdot p)\,, \tag{H.3.36}$$

$$x^\alpha(\vec{\boldsymbol{p}},\lambda)\,\Gamma_{\alpha\dot\beta}\,y^{\dagger\dot\beta}(-\vec{\boldsymbol{p}},\lambda') = -(\tfrac{1}{2}+\lambda')(\tfrac{1}{2}-\lambda)\,\xi_{-\lambda'}(\hat{\boldsymbol{p}})\ \mathrm{Tr}(\Gamma\,\overline{\sigma}^0\sigma\cdot S_+\,\overline{\sigma}\cdot p)\,, \tag{H.3.37}$$

$$x^\alpha(\vec{\boldsymbol{p}},\lambda)\,\Gamma_\alpha{}^\beta\,x_\beta(-\vec{\boldsymbol{p}},\lambda') = (\tfrac{1}{2}-\lambda)\,\delta_{\lambda\lambda'}\,\xi_{-\lambda'}(\hat{\boldsymbol{p}})\ \mathrm{Tr}(\Gamma\,\sigma^0\,\overline{\sigma}\cdot p)\,, \tag{H.3.38}$$

$$y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\,\Gamma^{\dot\alpha}{}_{\dot\beta}\,y^{\dagger\dot\beta}(-\vec{\boldsymbol{p}},\lambda') = (\tfrac{1}{2}+\lambda)\,\delta_{\lambda\lambda'}\,\xi_{-\lambda'}(\hat{\boldsymbol{p}})\ \mathrm{Tr}(\Gamma\,\overline{\sigma}^0\,\sigma\cdot p)\,. \tag{H.3.39}$$

The traces are easily evaluated using the results of Appendix B . Here, we apply the above results to the amplitude for the decay $Z^0 \to f\bar f$ [see Section 29.6.2]. The corresponding center-of-mass frame helicity amplitude is a linear combination of eqs. (H.3.24) and (H.3.25) with $\Gamma = \overline{\sigma}$ and $\Gamma = \sigma$, respectively. Evaluating the corresponding terms, we find for $\Gamma = \overline{\sigma}$,

$$x^\dagger(\vec{\boldsymbol{p}},\lambda)\overline{\sigma}^\mu y(-\vec{\boldsymbol{p}},\lambda') = 2\lambda'\,\xi_{\lambda'}(\hat{\boldsymbol{p}})\,\left[mg^{\mu 0}\delta_{\lambda\lambda'} + p^\mu \mathscr{S}^0_{\lambda\lambda'} - p^0\mathscr{S}^\mu_{\lambda\lambda'} - 2m(\mathscr{S}^\mu\mathscr{S}^0 - \mathscr{S}^0\mathscr{S}^\mu)_{\lambda\lambda'}\right]\,, \tag{H.3.40}$$

where we have used eq. (H.1.10) to replace the term with the Levi-Civita tensor. Similarly, we calculate for $\Gamma = \sigma$,

$$y(\vec{\boldsymbol{p}},\lambda)\sigma^\mu x^\dagger(-\vec{\boldsymbol{p}},\lambda') = 2\lambda'\,\xi_{\lambda'}(\hat{\boldsymbol{p}})\,\left[-mg^{\mu 0}\delta_{\lambda\lambda'} + p^\mu \mathscr{S}^0_{\lambda\lambda'} - p^0\mathscr{S}^\mu_{\lambda\lambda'} + 2m(\mathscr{S}^\mu\mathscr{S}^0 - \mathscr{S}^0\mathscr{S}^\mu)_{\lambda\lambda'}\right]\,. \tag{H.3.41}$$

Eqs. (H.3.40) and (H.3.41) provide explicit forms for the $Z^0 \to f\bar f$ decay helicity amplitudes defined in eqs. (29.6.3) and (29.6.4).

The above method is not applicable if the two fermions have unequal mass. In order to compute the helicity amplitudes of the form given by eq. (H.3.22) for unequal masses, a generalization of the above techniques is required. Some methods for four-component spinor wave functions have been proposed in [319]. We leave it as an exercise for the reader to translate these techniques so that they are applicable to helicity amplitudes expressed in terms of two-component spinor wave functions. An alternative approach, which is applicable to the computation of helicity amplitudes for processes involving multi-fermion final states of arbitrary mass, is reviewed in Appendix I.1 .

Four-component Bouchiat-Michel formulae

Using the results of Appendix G , the translation of the results of Appendix H.3 into four-component spinor notation is straightforward. First, we consider a massive spin-1/2 fermion. Eqs. (H.2.8)–(H.2.11) yield [288]:

$$\gamma_5 \not{S}^a\, u(\vec{\boldsymbol{p}},s') = \tau^a_{ss'}\,u(\vec{\boldsymbol{p}},s)\,, \qquad \gamma_5\not{S}^a\, v(\vec{\boldsymbol{p}},s') = \tau^a_{s's}\,v(\vec{\boldsymbol{p}},s)\,, \tag{H.4.1}$$

$$\bar u(\vec{\boldsymbol{p}},s')\,\gamma_5\not{S}^a = \tau^a_{ss'}\,\bar u(\vec{\boldsymbol{p}},s)\,, \qquad \bar v(\vec{\boldsymbol{p}},s')\,\gamma_5\not{S}^a = \tau^a_{s's}\,\bar v(\vec{\boldsymbol{p}},s)\,. \tag{H.4.2}$$

In the case of $a = 3$, eqs. (H.4.1) and (H.4.2) reduce to those of eqs. (G.4.18) and (G.4.19).

The four-component Bouchiat-Michel formulae [113, 320, 319] can be obtained from eqs. (H.3.5)–(H.3.12):

$$u(\vec{\boldsymbol{p}},s')\bar u(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}\left[\delta_{ss'} + \gamma_5\gamma_\mu\mathscr{S}^\mu_{ss'}\right](\not p + m)\,, \tag{H.4.3}$$

$$v(\vec{\boldsymbol{p}},s')\bar v(\vec{\boldsymbol{p}},s) = \tfrac{1}{2}\left[\delta_{s's} + \gamma_5\gamma_\mu\mathscr{S}^\mu_{s's}\right](\not p - m)\,, \tag{H.4.4}$$

where $\mathscr{S}^{\mu}_{ss'} \equiv S^{a\mu}\tau^{a}_{ss'}$. As expected, the above results for $s = s'$ correspond to the spin projection operators given in eqs. (G.4.20) and (G.4.21). Related formulae involving products of u and v-spinors can be obtained by using [cf. eq. (G.4.15)]:

$$v(\vec{\boldsymbol{p}}, s) = -2s\gamma_5 u(\vec{\boldsymbol{p}}, -s)\,, \qquad\qquad u(\vec{\boldsymbol{p}}, s) = 2s\gamma_5 v(\vec{\boldsymbol{p}}, -s)\,. \tag{H.4.5}$$

Eqs. (H.4.1)–(H.4.4) also apply to helicity u and v-spinors, after replacing s, s' with λ, λ' and using the S^a as defined in eq. (H.1.14). The four-component versions of eqs. (C.3.17)–(C.3.20) yield:

$$u(-\boldsymbol{p}\,,\,-\lambda) = \xi_{\lambda}(\hat{\boldsymbol{p}})\,\gamma^0\,u(\boldsymbol{p}\,,\,\lambda)\,, \qquad\qquad v(-\boldsymbol{p}\,,\,-\lambda) = \xi_{-\lambda}(\hat{\boldsymbol{p}})\,\gamma^0\,v(\boldsymbol{p}\,,\,\lambda)\,, \tag{H.4.6}$$

$$\bar{u}(-\boldsymbol{p}\,,\,-\lambda) = -\bar{u}(\boldsymbol{p}\,,\,\lambda)\,\gamma^0\,\xi_{-\lambda}(\hat{\boldsymbol{p}})\,, \qquad\qquad \bar{v}(-\boldsymbol{p}\,,\,-\lambda) = -\bar{v}(\boldsymbol{p}\,,\,\lambda)\,\gamma^0\,\xi_{\lambda}(\hat{\boldsymbol{p}})\,, \tag{H.4.7}$$

where the phase $\xi_\lambda(\hat{\boldsymbol{p}})$ was defined in eq. (C.3.11). In order to consider the massless limit, one must employ helicity spinors, as discussed in Appendix H.3 . For $a = 1, 2$, eqs. (H.4.1) and (H.4.2) apply in the $m \to 0$ limit as written. The corresponding massless limit for the case of $a = 3$ is smooth and results in eq. (G.4.22). Similarly, the massless limit of the Bouchiat-Michel formulae for helicity spinors can be obtained by setting $mS^{a\mu} = p^\mu\,\delta^{a3}$, applying the mass-shell condition ($p^2 = m^2$), and taking the $m \to 0$ limit at the end of the computation. The end result is given by

$$u(p,\lambda')\bar{u}(p,\lambda) = \tfrac{1}{2}(1 + 2\lambda\gamma_5)\,\not{p}\,\delta_{\lambda\lambda'} + \tfrac{1}{2}\gamma_5[\not{S}^1\tau^1_{\lambda\lambda'} + \not{S}^2\tau^2_{\lambda\lambda'}]\,\not{p}\,, \tag{H.4.8}$$

$$v(p,\lambda')\bar{v}(p,\lambda) = \tfrac{1}{2}(1 - 2\lambda\gamma_5)\,\not{p}\,\delta_{\lambda'\lambda} + \tfrac{1}{2}\gamma_5[\not{S}^1\tau^1_{\lambda'\lambda} + \not{S}^2\tau^2_{\lambda'\lambda}]\,\not{p}\,. \tag{H.4.9}$$

As expected, when $\lambda = \lambda'$, we recover the helicity projection operators for massless spin-1/2 particles given in eqs. (G.4.24) and (G.4.25).

As before, we can use the Bouchiat-Michel formulae to evaluate helicity amplitudes involving two equal mass spin-1/2 fermions. A typical amplitude involving a fermion-antifermion pair, evaluated in the center-of-mass frame of the pair, has the generic structure:

$$\overline{w}(\vec{\boldsymbol{p}}, \lambda)\,\Gamma\,w'(-\vec{\boldsymbol{p}}, \lambda')\,, \tag{H.4.10}$$

where w is either a u or v spinor, w' is respectively either a v or u spinor, and Γ is a product of Dirac gamma matrices. For example,

$$\bar{u}(\vec{\boldsymbol{p}}, \lambda)\,\Gamma\,v(-\vec{\boldsymbol{p}}, \lambda') = -2\lambda'\bar{u}(\vec{\boldsymbol{p}}, \lambda)\,\Gamma\,\gamma_5\,u(-\vec{\boldsymbol{p}}, -\lambda') = -2\lambda'\,\xi_{\lambda'}(\hat{\boldsymbol{p}})\,\bar{u}(\vec{\boldsymbol{p}}, \lambda)\,\Gamma\,\gamma_5\,\gamma^0\,u(\vec{\boldsymbol{p}}, \lambda')\,, \tag{H.4.11}$$

where we have used the results of eqs. (H.4.5) and (H.4.6). We can now employ the Bouchiat-Michel formula to convert the above result into a trace. By a similar computation, all expressions of the form of eq. (H.4.10) can be expressed as a trace:

$$\bar{u}(\vec{\boldsymbol{p}}, \lambda)\,\Gamma\,v(-\vec{\boldsymbol{p}}, \lambda') = -\lambda'\,\xi_{\lambda'}(\hat{\boldsymbol{p}})\,\mathrm{Tr}\left[\Gamma\gamma_5\gamma^0(\delta_{\lambda\lambda'} + \gamma_5\gamma_\mu\mathscr{S}^{\mu}_{\lambda\lambda'})(\not{p} + m)\right], \tag{H.4.12}$$

$$\bar{v}(\vec{\boldsymbol{p}}, \lambda)\,\Gamma\,u(-\vec{\boldsymbol{p}}, \lambda') = \lambda'\,\xi_{-\lambda'}(\hat{\boldsymbol{p}})\,\mathrm{Tr}\left[\Gamma\gamma_5\gamma^0(\delta_{\lambda'\lambda} + \gamma_5\gamma_\mu\mathscr{S}^{\mu}_{\lambda'\lambda})(\not{p} - m)\right]. \tag{H.4.13}$$

These results are the four-component analogues of eqs. (H.3.24)–(H.3.27) and eqs. (H.3.28)–(H.3.31), respectively. For amplitudes that involve a pair of equal mass fermions [or equal mass antifermions], w and w' in eq. (H.4.10) are both u-spinors [or v-spinors]. Using eq. (G.4.15), these amplitudes can then be evaluated using the results of eqs. (H.4.12) and (H.4.13) above.

In the massless limit, one can again put $mS^{a\mu} = p^\mu\delta^{a3}$, set $p^2 = m^2$ and take $m \to 0$ at the end of the computation. Alternatively, one can repeat the derivation of eqs. (H.4.12)–(H.4.13) using the results of eqs. (H.4.8) and (H.4.9). For completeness, we record the end result here.

$$\bar{u}(\vec{\boldsymbol{p}}, \lambda)\,\Gamma\,v(-\vec{\boldsymbol{p}}, \lambda') = \xi_{\lambda'}(\hat{\boldsymbol{p}})\Big\{\tfrac{1}{2}\delta_{\lambda\lambda'}\ \mathrm{Tr}[\Gamma\gamma^0(1 + 2\lambda\gamma_5)\not{p}] + \lambda'\ \mathrm{Tr}[\Gamma\gamma^0(\not{S}^1\tau^1_{\lambda\lambda'} + \not{S}^2\tau^2_{\lambda\lambda'})\,\not{p}]\Big\}, \tag{H.4.14}$$

$$\bar{v}(\vec{\boldsymbol{p}}, \lambda)\,\Gamma\,u(-\vec{\boldsymbol{p}}, \lambda') = \xi_{-\lambda'}(\hat{\boldsymbol{p}})\Big\{\tfrac{1}{2}\delta_{\lambda'\lambda}\ \mathrm{Tr}[\Gamma\gamma^0(1 - 2\lambda\gamma_5)\not{p}] - \lambda'\ \mathrm{Tr}[\Gamma\gamma^0(\not{S}^1\tau^1_{\lambda'\lambda} + \not{S}^2\tau^2_{\lambda'\lambda})\,\not{p}]\Big\}. \tag{H.4.15}$$

As an example, we consider once again the decay $Z^0 \to f\bar{f}$. The decay amplitude is equal to eq. (H.4.12), where Γ is a linear combination of $\frac{1}{2}\gamma^\mu(1-\gamma_5)$ and $\frac{1}{2}\gamma^\mu(1+\gamma_5)$. Evaluating the corresponding traces yields:

$$\bar{u}(\vec{\boldsymbol{p}},\lambda)\,\tfrac{1}{2}\gamma^\mu(1-\gamma_5)\,v(-\vec{\boldsymbol{p}},\lambda') = 2\lambda'\,\xi_{\lambda'}(\hat{\boldsymbol{p}})\,\left[mg^{\mu 0}\delta_{\lambda\lambda'} + p^\mu \mathscr{S}^0_{\lambda\lambda'} - p^0\mathscr{S}^\mu_{\lambda\lambda'} + i\epsilon^{0\mu\nu\rho}(\mathscr{S}_{\lambda\lambda'})_\nu p_\rho\right]\,, \tag{H.4.16}$$

$$\bar{u}(\vec{\boldsymbol{p}},\lambda)\,\tfrac{1}{2}\gamma^\mu(1+\gamma_5)\,v(-\vec{\boldsymbol{p}},\lambda') = 2\lambda'\,\xi_{\lambda'}(\hat{\boldsymbol{p}})\,\left[-mg^{\mu 0}\delta_{\lambda\lambda'} + p^\mu \mathscr{S}^0_{\lambda\lambda'} - p^0\mathscr{S}^\mu_{\lambda\lambda'} - i\epsilon^{0\mu\nu\rho}(\mathscr{S}_{\lambda\lambda'})_\nu p_\rho\right]\,. \tag{H.4.17}$$

Using eq. (H.1.10), we see that eqs. (H.4.16) and (H.4.17) reproduce exactly the results of eqs. (H.3.40) and (H.3.41), respectively.

Finally, we note that if the two fermions do not have the same mass, then the method presented above is not applicable. However, generalizations of the above method exist in the literature that can be employed to evaluate helicity amplitudes of the form of eq. (H.4.10) for unequal mass fermions; see, e.g., [319]. An alternative approach due to Hagiwara and Zeppenfeld [106] is reviewed in Appendix I.1 .

29.7.11 Helicity amplitudes and the spinor helicity method

In Appendix H , we showed how to use the Bouchiat-Michel formulae (with versions applicable to both two-component and four-component spinor wave functions) to construct helicity amplitudes for processes with two initial state and two final state equal mass fermions (or a fermion-antifermion pair) in the center-of-mass frame of the two fermions. For practical applications, it is important to extend these techniques to allow for final states with an arbitrary number of particles. The techniques should be powerful enough to allow for pairs of fermions of unequal mass, and both massless and massive spin-1 particles. Ideally, these techniques should produce simple analytic results (when possible) and yield efficient numerical algorithms for the evaluation of the helicity amplitudes.

The helicity amplitude technique of Hagiwara and Zeppenfeld

One method for computing helicity amplitudes for multi-particle final states that is easily amenable to numerical analysis was developed by Hagiwara and Zeppenfeld (HZ) [106]. The HZ formalism was subsequently employed in refs. [321, 322] in developing a fast numerical algorithm for the computation of multi-parton processes. In this section, we demonstrate how our two-component formalism (denoted by DHM) can be connected to theirs. In particular, we present a translation between the two formalisms in Table I.1.1.

After removing the propagator factors, an arbitrary tree amplitude with external fermions can be expressed in terms of a four-component fermion string

$$\overline{\Psi}_i P_\tau \not{\phi}_1 \not{\phi}_2 \ldots \not{\phi}_n \Psi_j\,, \qquad\qquad \tau = \pm 1\,, \tag{I.1.1}$$

where Ψ_j is a four-component spinor wave function $u(p_j,\lambda_j)$ or $v(p_j,-\lambda_j)$,[157] and $P_\pm = \frac{1}{2}(1\pm\gamma_5)$ are the chiral projection operators. Furthermore, $\not{a}_k \equiv \gamma_\mu a_k^\mu$ where a_k represents an arbitrary Lorentz four-vector, which can be a four-momentum p_k^μ, a vector boson wave function $\epsilon^\mu(p_k,\lambda_k)$, an axial vector (e.g., $\epsilon^{\mu\nu\rho\sigma}p_{k\nu}p_{m\rho}p_{n\sigma}$) or another fermion string with uncontracted Lorentz indices (e.g., $\overline{\Psi}_m\gamma^\mu\Psi_n$).

[157] HZ defines $v(p,\lambda) = C\bar{u}^{\mathsf{T}}(p,\lambda)$, where $C = i\gamma^2\gamma^0$, which differs by an overall minus sign from the conventions employed in this review [cf. eq. (G.1.21)]. In this section, we will modify the HZ results in order to be consistent with our sign convention.

DHM Formalism	HZ Formalism
$x_\alpha(p,\lambda)$	$u(p,\lambda)_-$
$x^\alpha(p,\lambda)$	$v^\dagger(p,\lambda)_+$
$x^{\dagger\dot\alpha}(p,\lambda)$	$v(p,\lambda)_+$
$x^\dagger_{\dot\alpha}(p,\lambda)$	$u^\dagger(p,\lambda)_-$
$y_\alpha(p,\lambda)$	$v(p,\lambda)_-$
$y^\alpha(p,\lambda)$	$u^\dagger(p,\lambda)_+$
$y^{\dagger\dot\alpha}(p,\lambda)$	$u(p,\lambda)_+$
$y^\dagger_{\dot\alpha}(p,\lambda)$	$v^\dagger(p,\lambda)_-$
$p\cdot\sigma$	$(\not p)_+$
$p\cdot\overline{\sigma}$	$(\not p)_-$
σ^μ	σ^μ_+
$\overline{\sigma}^\mu$	σ^μ_-
P_R	P_+
P_L	P_-
$\lambda=\pm\frac{1}{2}$	$\lambda=\pm1$
$\chi_\lambda(-\hat{\boldsymbol{z}})$	$-\chi_\lambda(-\hat{\boldsymbol{z}})$

Table I.1.1: Translation between our notation (denoted by DHM) and the notation of Hagiwara and Zeppenfeld (HZ) [106]. The sign convention governing the definition of $v(p,\lambda)_\pm$ is opposite to that of HZ (cf. footnote 157).

In order to rewrite the fermion string, eq. (I.1.1), in terms of two-component spinors, HZ decomposes the four-component spinors as follows:

$$\Psi_j \equiv \begin{pmatrix} (\psi_j)_- \\ (\psi_j)_+ \end{pmatrix}, \qquad u(p_j,\lambda_j) \equiv \begin{pmatrix} u(p_j,\lambda_j)_- \\ u(p_j,\lambda_j)_+ \end{pmatrix}, \qquad v(p_j,\lambda_j) \equiv \begin{pmatrix} v(p_j,\lambda_j)_- \\ v(p_j,\lambda_j)_+ \end{pmatrix}. \tag{I.1.2}$$

Comparing with eqs. (G.4.11) and (G.4.12), the corresponding expressions in our notation are given in Table I.1.1. Note that $\lambda=\pm1$ in the notation of HZ, whereas in our notation (which we follow below) $\lambda=\pm\frac{1}{2}$.

The four-component fermion string is then replaced by the two-component fermion string:

$$\overline{\Psi}_i P_\tau \not a_1 \not a_2 \ldots \not a_n \Psi_j = (\psi_i)^\dagger_\tau (\not a_1)_\tau (\not a_2)_{-\tau} \ldots (\not a_n)_{-\delta_n\tau} (\psi_j)_{-\delta_n\tau}\,, \qquad \tau=\pm1\,, \quad \text{where } \delta_n \equiv (-1)^n\,. \tag{I.1.3}$$

In the notation of HZ,

$$(\not a)_\pm = a_\mu \sigma^\mu_\pm\,, \tag{I.1.4}$$

where $\sigma_+^\mu \equiv \sigma^\mu$ and $\sigma_-^\mu \equiv \overline{\sigma}^\mu$. In eq. (I.1.3), the helicity labels are suppressed; more explicitly,

$$(\psi_k)_\tau \equiv \psi_k(p_k\,,\,\lambda_k)_\tau = u(p_k\,,\,\lambda_k)_\tau \quad \text{or} \quad v(p_k\,,\,-\lambda_k)_\tau\,. \tag{I.1.5}$$

This convention of HZ (note the $-\lambda_k$ argument of v) allows one to write simple generic formulae in terms of $(\psi)_\pm$ that are applicable to both $u_\pm$ and $v_\pm$.

Using the results of Table I.1.1, one can verify that eq. (I.1.3) is covariant with respect to dotted and undotted indices. That is, the sign τ of $\psi_i^\dagger$ must match the sign of the first σ-matrix in the string $(\not{\phi}_1)_\tau(\not{\phi}_2)_{-\tau}...(\not{\phi}_n)_{-\delta_n\tau}$. The signs of the sigma matrices within this string alternate (either $+-+-...$ or $-+-+...$ in the case of $\tau = +1$ or -1, respectively). Finally, the sign of the last σ-matrix in the string [which must be equal to $-\delta_n\tau$ in light of the previous statement] must match the sign of ψ_j as indicated.

As noted above, it is possible that one of the $(\not{\phi}_i)_\tau$ could be of the form σ_τ^μ multiplied by another fermion string with a free μ-index. One can uncouple the two fermion strings by employing the Fierz identities given by eqs. (29.68)–(29.70). For example,

$$\begin{aligned}&[(u_1)_-^\dagger(\not{\phi}_1)_-\sigma_+^\mu(\not{\phi}_2)_-(u_2)_-][(u_3)_-^\dagger(\not{\phi}_3)_-\sigma_{+\mu}(\not{\phi}_4)_-(u_4)_-]\\ &\qquad = [(u_1)_-^\dagger(\not{\phi}_1)_-(\not{\phi}_3)_+(v_3)_+][(v_4)_+^\dagger(\not{\phi}_4)_+(\not{\phi}_2)_-(u_2)_-]\,,\end{aligned} \tag{I.1.6}$$

which is easily derived after translating to the DHM notation.[158] As a result, the helicity tree amplitude for any process can be expressed as the product of *uncoupled* strings of two-component fermion spinors

$$\mathrm{FS} = (\psi_i)_\tau^\dagger(\not{\phi}_1)_\tau(\not{\phi}_2)_{-\tau}...(\not{\phi}_n)_{-\delta_n\tau}(\psi_j)_{-\delta_n\tau}\,. \tag{I.1.7}$$

To evaluate the fermion string FS, we employ the explicit forms for the two-component helicity spinor wave functions given in eqs. (C.3.3)–(C.3.6), which can be rewritten as

$$\psi(p_k,\lambda_k)_\tau = C_k\,\omega_{\tau\lambda_k}(\vec{\boldsymbol{p}})\,\chi_{\lambda_k}(\hat{\boldsymbol{p}}_{\boldsymbol{k}})\,, \tag{I.1.8}$$

where, following the convention of eq. (I.1.5),

$$C_k = \begin{cases} 1 & \text{for} \quad (\psi_k)_\tau = u(p_k,\lambda_k)_\tau\,,\\ 2\lambda_k\tau & \text{for} \quad (\psi_k)_\tau = v(p_k,-\lambda_k)_\tau\,,\end{cases} \qquad \tau = \pm 1, \qquad \lambda_k = \pm 1/2\,, \tag{I.1.9}$$

and $\omega_\lambda(\vec{\boldsymbol{p}}) \equiv (E + 2\lambda|\vec{\boldsymbol{p}}|)^{1/2}$ for $\lambda = \pm 1/2$. Hence, the fermion string [eq. (I.1.7)] is given by[106]

$$\mathrm{FS} = C_i\,C_j\,\omega_{\tau\lambda_i}(\vec{\boldsymbol{p}}_{\boldsymbol{i}})\,\omega_{-\delta_n\tau\lambda_j}(\vec{\boldsymbol{p}}_{\boldsymbol{j}})\,S(p_i,a_1,a_2,...,a_n,p_j)^\tau_{\lambda_i\lambda_j}\,, \tag{I.1.10}$$

where the function S is defined as

$$S(p_i,a_1,a_2,...,a_n,p_j)^\tau_{\lambda_i\lambda_j} \equiv \chi^\dagger_{\lambda_i}(\hat{\boldsymbol{p}}_{\boldsymbol{i}})\left[\prod_{k=1}^n(\not{\phi}_k)_{-\delta_k\tau}\right]\chi_{\lambda_j}(\hat{\boldsymbol{p}}_{\boldsymbol{j}})\,, \tag{I.1.11}$$

where $\delta_k \equiv (-1)^k$. In the absence of the $\not{\phi}_{\pm\tau}$ factors, we define

$$S(p_i\,,\,p_j)_{\lambda_i\lambda_j} \equiv T(\hat{\boldsymbol{p}}_{\boldsymbol{i}}\,,\,\hat{\boldsymbol{p}}_{\boldsymbol{j}})_{2\lambda_i,2\lambda_j} = \chi^\dagger_{\lambda_i}(\hat{\boldsymbol{p}}_{\boldsymbol{i}})\chi_{\lambda_j}(\hat{\boldsymbol{p}}_{\boldsymbol{j}})\,, \tag{I.1.12}$$

where the $T(\hat{\boldsymbol{p}}_{\boldsymbol{i}}\,,\,\hat{\boldsymbol{p}}_{\boldsymbol{j}})_{2\lambda_i,2\lambda_j}$ are proportional to the (massless) spinorial products introduced by Kleiss [99] [cf. eqs. (I.2.22) and (I.2.23)].

[158] Here, we differ from HZ, who employ a Fierz identity that is not covariant with respect to the dotted and undotted indices [note the comment below eq. (29.55)]. Thus, eq. (I.1.6) differs from the result obtained in eq. (3.17b) of HZ.

To evaluate S, we assume that the four-vectors a_k^μ are real.[159] Then, we may employ the following identity:[160]

$$(\not{a})_\tau = \sum_{\tau'=\pm} \left[a^0 - \tau'\tau|\vec{\boldsymbol{a}}|\right] \chi_{\tau'/2}(\hat{\boldsymbol{a}})\chi^\dagger_{\tau'/2}(\hat{\boldsymbol{a}})\,, \tag{I.1.13}$$

where $\chi_{\tau'/2}(\hat{\boldsymbol{a}})$ is a two-component helicity spinor with three-momentum $\vec{\boldsymbol{a}}$. Using eq. (I.1.13) in eq. (I.1.11), we end up with the desired expression:

$$\begin{aligned} S(p_i, a_1, a_2, ..., a_n, p_j)^\tau_{\lambda_i\lambda_j} \;=\; & \left[\prod_{k=1}^{n}\sum_{\tau_k=\pm}\left[a_k^0 + \tau_k\delta_k\tau|\vec{\boldsymbol{a}}_{\boldsymbol{k}}|\right]\right] T(\hat{\boldsymbol{p}}_i, \hat{\boldsymbol{a}}_1)_{2\lambda_i,\tau_1} T(\hat{\boldsymbol{a}}_1, \hat{\boldsymbol{a}}_2)_{\tau_1\tau_2} \\ & \times \cdots T(\hat{\boldsymbol{a}}_{n-1}, \hat{\boldsymbol{a}}_n)_{\tau_{n-1}\tau_n} T(\hat{\boldsymbol{a}}_n, \hat{\boldsymbol{p}}_j)_{\tau_n,2\lambda_j}\,. \end{aligned} \tag{I.1.14}$$

All that remains is to evaluate the spinorial product $T(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})_{\tau_a\,\tau_b}$ (τ_a, $\tau_b = \pm 1$) for arbitrary unit three-vectors $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$. Two properties of the spinorial product $T(\hat{\boldsymbol{a}}\,, \hat{\boldsymbol{b}})_{\tau_a,\tau_b}$ are noteworthy. First, as this product is a scalar, it follows that $T(\hat{\boldsymbol{a}}\,, \hat{\boldsymbol{b}})^*_{\tau_a,\tau_b} = T(\hat{\boldsymbol{a}}\,, \hat{\boldsymbol{b}})^\dagger_{\tau_a,\tau_b}$. Hence, eq. (I.1.12) implies that

$$T(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})_{\tau_a\,\tau_b} = T(\hat{\boldsymbol{b}}, \hat{\boldsymbol{a}})^*_{\tau_b\,\tau_a}\,. \tag{I.1.15}$$

Second, we use eq. (C.1.8) to write:

$$\begin{aligned} T(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})_{\tau_a\,\tau_b} = {} & \chi^\dagger_{\tau_a/2}(\hat{\boldsymbol{z}})\, \exp(i\gamma_a\sigma^3/2)\, \exp(i\theta_a\sigma^2/2)\, \exp(i\phi_a\sigma^3/2) \\ & \times \exp(-i\phi_b\sigma^3/2)\, \exp(-i\theta_b\sigma^2/2)\, \exp(-i\gamma_b\sigma^3/2)\, \chi_{\tau_b/2}(\hat{\boldsymbol{z}})\,. \end{aligned} \tag{I.1.16}$$

Complex conjugating this result, and using the fact that $\chi(\hat{\boldsymbol{z}})$, σ^1 and σ^3 are real and σ^2 is pure imaginary,

$$\begin{aligned} T(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})^*_{\tau_a\,\tau_b} = {} & \tau_a\tau_b\, \chi^\dagger_{-\tau_a/2}(\hat{\boldsymbol{z}})\, \sigma^2\, \exp(-i\gamma_a\sigma^3/2)\, \exp(i\theta_a\sigma^2/2)\, \exp(-i\phi_a\sigma^3/2) \\ & \times \exp(i\phi_b\sigma^3/2)\, \exp(-i\theta_b\sigma^2/2)\, \exp(i\gamma_b\sigma^3/2)\, \sigma^2\, \chi_{-\tau_b/2}(\hat{\boldsymbol{z}})\,, \end{aligned} \tag{I.1.17}$$

after using eq. (C.1.22). Since σ^2 anticommutes with σ^3, we end up with:

$$\begin{aligned} T(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})^*_{\tau_a\,\tau_b} &= \tau_a\tau_b\, \chi^\dagger_{-\tau_a/2}(\hat{\boldsymbol{z}})\, \exp(i\gamma_a\sigma^3/2)\, \exp(i\theta_a\sigma^2/2)\, \exp(i\phi_a\sigma^3/2) \\ &\quad \times \exp(-i\phi_b\sigma^3/2)\, \exp(-i\theta_b\sigma^2/2)\, \exp(-i\gamma_b\sigma^3/2)\, \chi_{-\tau_b/2}(\hat{\boldsymbol{z}}) \\ &= \tau_a\tau_b\, T(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})_{-\tau_a\,,\,-\tau_b}\,. \end{aligned} \tag{I.1.18}$$

Since τ_a, $\tau_b = \pm 1$, it follows that

$$T(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})_{-\tau_a,-\tau_b} = \tau_a\, \tau_b\, T(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})^*_{\tau_a\,\tau_b}\,. \tag{I.1.19}$$

Using eqs. (I.1.15) and (I.1.19), it is sufficient to give explicit forms for only two of the spinorial products [100, 106]. Eq. (I.1.16) yields:

$$T(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})_{++} = e^{i(\phi_a-\phi_b+\gamma_a-\gamma_b)/2}\, \cos\frac{\theta_a}{2}\cos\frac{\theta_b}{2} + e^{-i(\phi_a-\phi_b-\gamma_a+\gamma_b)/2}\, \sin\frac{\theta_a}{2}\sin\frac{\theta_b}{2}\,, \tag{I.1.20}$$

$$T(\hat{\boldsymbol{a}}, \hat{\boldsymbol{b}})_{-+} = e^{-i(\phi_a-\phi_b+\gamma_a+\gamma_b)/2}\, \cos\frac{\theta_a}{2}\sin\frac{\theta_b}{2} - e^{i(\phi_a-\phi_b-\gamma_a-\gamma_b)/2}\, \sin\frac{\theta_a}{2}\cos\frac{\theta_b}{2}\,, \tag{I.1.21}$$

[159] In the case of complex a^μ, one should decompose a^μ into its real and imaginary parts and evaluate separately the real and imaginary parts of S.

[160] To obtain eq. (I.1.13), we make use of eq. (29.3.42) applied to helicity spinors: $\chi_\lambda\chi^\dagger_\lambda = \frac{1}{2}(1 + 2\lambda\vec{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{p}})$.

where (θ_p, ϕ_p) are the polar and azimuthal angles of $\hat{\boldsymbol{p}}$ (for $\hat{\boldsymbol{p}} = \hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$, respectively). In the case where $\hat{\boldsymbol{a}}$ and/or $\hat{\boldsymbol{b}}$ are parallel to the negative z-axis, we employ the convention of eq. (C.1.7) and choose the corresponding azimuthal angle equal to π.[161] Note that HZ employ a convention for their spinor wave functions [cf. eq. (C.1.8)] in which $\gamma = -\phi$, although the convention in which $\gamma = 0$ yields a slightly more symmetrical form for the spinorial products.

Eqs. (I.1.7), (I.1.10) and (I.1.11) can be written in a form that is reminiscent of the results obtained in Appendix H.3 . For example, using eqs. (C.3.3)–(C.3.6),

$$x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\,\Gamma^{\dot\alpha\beta}\,y_\beta(\vec{\boldsymbol{p}}',-\lambda') = -2\lambda'\omega_{-\lambda}(\vec{\boldsymbol{p}})\omega_{-\lambda'}(\vec{\boldsymbol{p}}')\chi^\dagger_\lambda(\hat{\boldsymbol{p}})\Gamma\chi_{\lambda'}(\hat{\boldsymbol{p}}')\,, \tag{I.1.22}$$

$$y^\alpha(\vec{\boldsymbol{p}},\lambda)\,\Gamma_{\alpha\dot\beta}\,x^{\dagger\dot\beta}(\vec{\boldsymbol{p}}',-\lambda') = 2\lambda'\omega_{\lambda}(\vec{\boldsymbol{p}})\omega_{\lambda'}(\vec{\boldsymbol{p}}')\chi^\dagger_\lambda(\hat{\boldsymbol{p}})\Gamma\chi_{\lambda'}(\hat{\boldsymbol{p}}')\,, \tag{I.1.23}$$

$$y^\alpha(\vec{\boldsymbol{p}},\lambda)\,\Gamma_\alpha{}^\beta\,y_\beta(\vec{\boldsymbol{p}}',-\lambda') = -2\lambda'\omega_{\lambda}(\vec{\boldsymbol{p}})\omega_{-\lambda'}(\vec{\boldsymbol{p}}')\chi^\dagger_\lambda(\hat{\boldsymbol{p}})\Gamma\chi_{\lambda'}(\hat{\boldsymbol{p}}')\,, \tag{I.1.24}$$

$$x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\lambda)\,\Gamma^{\dot\alpha}{}_{\dot\beta}\,x^{\dagger\dot\beta}(\vec{\boldsymbol{p}}',-\lambda') = 2\lambda'\omega_{-\lambda}(\vec{\boldsymbol{p}})\omega_{\lambda'}(\vec{\boldsymbol{p}}')\chi^\dagger_\lambda(\hat{\boldsymbol{p}})\Gamma\chi_{\lambda'}(\hat{\boldsymbol{p}}')\,, \tag{I.1.25}$$

where Γ is a product of alternating σ and $\overline{\sigma}$ matrices. The spinor index structure determines the identity of the first and last matrix (e.g., $\Gamma^{\dot\alpha}{}_{\dot\beta}$ indicates a string of matrices that begins with a $\overline{\sigma}$ and ends with a σ, etc.). By suitable interchanges of x and y, twelve additional equations of similar type may be written. Note that $\chi^\dagger_\lambda\Gamma\chi_{\lambda'}$ [appearing on the right-hand side of eqs. (I.1.22)–(I.1.25)] corresponds precisely to the $S(p,a_1,a_2,...,a_n,p')^\tau_{\lambda\lambda'}$ of eq. (I.1.11), where the four possible $(\tau\,,\,\delta_n)$ combinations are in one-to-one correspondence with the four possible spinor index structures of Γ. If $\vec{\boldsymbol{p}}' = -\vec{\boldsymbol{p}}$, then one should recover eqs. (H.3.24)–(H.3.27). Thus, the HZ method provides a powerful generalization of the helicity amplitude methods derived in Appendix H.3 .

The spinor helicity method

In many practical calculations, the masses of the fermions can be neglected. In this case the computation of multi-particle helicity amplitudes simplifies considerably. In this section, we give a brief introduction to the spinor helicity method; for a review, see refs. [323, 324]. The spinor helicity method is a powerful technique for computing helicity amplitudes for multi-particle processes involving massless spin-1/2 and spin-1 particles. Although initially applied to tree-level processes, more general techniques have also been developed that are applicable to one-loop (and multiloop) diagrams [325]. Rules for computing dimensionally regularized amplitudes within the framework of the spinor helicity method have been given by [326]. The spinor helicity techniques are ideal for QCD where light quark masses can almost always be neglected. Generalizations of these methods that incorporate massive spin-1/2 and spin-1 particles exist, although they tend to be quite cumbersome [327, 328]. A Mathematica implementation of the spinor helicity formalism can be found in [329]. In this section, we restrict the discussion to the massless case.

The spinor helicity technique described below is based on a formalism developed by Xu, Zhang and Chang [102] (denoted henceforth by XZC), which is a modification of techniques established by the CALKUL collaboration [330]. The XZC formalism (which was also independently developed in refs. [100, 331]) is based on the four-component spinor formalism. Using eq. (G.4.23), XZC introduce a very useful notation for massless spinors

$$|p\pm\rangle \equiv u(p,\pm\tfrac{1}{2}) = v(p,\mp\tfrac{1}{2})\,, \tag{I.2.1}$$

$$\langle p\pm| \equiv \bar u(p,\pm\tfrac{1}{2}) = \bar v(p,\mp\tfrac{1}{2})\,. \tag{I.2.2}$$

Using these spinor wave functions, they define two non-trivial (massless) spinor products (which are equivalent to the spinorial products introduced by Kleiss [99]):[162]

$$\langle p\,q\rangle \equiv \langle p-|q+\rangle = \bar u(p,-\tfrac{1}{2})\,u(q,+\tfrac{1}{2})\,, \tag{I.2.3}$$

$$[p\,q] \equiv \langle p+|q-\rangle = \bar u(p,+\tfrac{1}{2})\,u(q,-\tfrac{1}{2})\,. \tag{I.2.4}$$

[161] This convention yields a value of $\chi_\lambda(-\hat{\boldsymbol{z}})$ that is opposite in sign to the convention adopted by HZ.

[162] Note that $\langle p-|q-\rangle = \langle p+|q+\rangle = 0$ due to $P_LP_R = P_RP_L = 0$.

The $\pm$ notation specified by the bra and ket indicates the chirality (i.e. the eigenvalue of γ_5) of the corresponding four-component spinor [cf. eq. (G.4.22)].

However, the two-component spinor formalism is especially economical in the case of massless spin-1/2 fermions. Hence, we shall reformulate the XZC approach using two-component spinor notation. First, we consider the explicit forms for the two-component helicity spinor wave functions [given by eqs. (29.3.36)–(29.3.39)] in the massless limit:

$$x_\alpha(\vec{\boldsymbol{p}},-\tfrac{1}{2}) = y_\alpha(\vec{\boldsymbol{p}},\tfrac{1}{2}) = (2E)^{1/2}\,\chi_{-1/2}(\hat{\boldsymbol{p}})\,, \qquad x^\alpha(\vec{\boldsymbol{p}},-\tfrac{1}{2}) = y^\alpha(\vec{\boldsymbol{p}},\tfrac{1}{2}) = (2E)^{1/2}\,\chi^\dagger_{1/2}(\hat{\boldsymbol{p}})\,, \tag{I.2.5}$$

$$x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},-\tfrac{1}{2}) = y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},\tfrac{1}{2}) = (2E)^{1/2}\,\chi_{1/2}(\hat{\boldsymbol{p}})\,, \qquad x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},-\tfrac{1}{2}) = y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\tfrac{1}{2}) = (2E)^{1/2}\,\chi^\dagger_{-1/2}(\hat{\boldsymbol{p}})\,, \tag{I.2.6}$$

where $E = |\vec{\boldsymbol{p}}|$. For all other choices of helicities, the corresponding helicity spinor wave functions vanish. Hence, we define:[163]

$$|p+\rangle = y^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},\tfrac{1}{2}) = x^{\dagger\dot\alpha}(\vec{\boldsymbol{p}},-\tfrac{1}{2})\,, \qquad \langle p+| = y^\alpha(\vec{\boldsymbol{p}},\tfrac{1}{2}) = x^\alpha(\vec{\boldsymbol{p}},-\tfrac{1}{2})\,, \tag{I.2.7}$$

$$|p-\rangle = x_\alpha(\vec{\boldsymbol{p}},-\tfrac{1}{2}) = y_\alpha(\vec{\boldsymbol{p}},\tfrac{1}{2})\,, \qquad \langle p-| = x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},-\tfrac{1}{2}) = y^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}},\tfrac{1}{2})\,. \tag{I.2.8}$$

The $|p\pm\rangle$ and $\langle p\pm|$ satisfy the massless Dirac equation [cf. eqs. (29.3.9)–(29.3.12)]:

$$p\cdot\sigma_\pm\,|p\pm\rangle = 0, \qquad \langle p\pm|\,p\cdot\sigma_\pm = 0\,, \tag{I.2.9}$$

where $\sigma_+ \equiv \sigma$ and $\sigma_- \equiv \overline{\sigma}$ as indicated in Table I.1.1. The above and the following equations should each be read as two separate equations corresponding to the upper and lower set of signs, respectively. The following properties are also noteworthy:

$$|p\pm\rangle\,\langle p\pm| = p\cdot\sigma_\mp\,, \tag{I.2.10}$$

$$\langle p\pm|\,\sigma^\mu_\pm\,|p\pm\rangle = 2p^\mu\,, \tag{I.2.11}$$

$$\langle p\pm|q\mp\rangle = -\,\langle q\pm|p\mp\rangle\ , \tag{I.2.12}$$

$$\langle p+|\,\sigma^\mu_+\,|q+\rangle = \langle q-|\,\sigma^\mu_-\,|p-\rangle\ , \tag{I.2.13}$$

$$\langle p\pm|\,\sigma^\mu_\pm\sigma^\nu_\mp\,|q\mp\rangle = -\,\langle q\pm|\,\sigma^\nu_\pm\sigma^\mu_\mp\,|p\mp\rangle\ . \tag{I.2.14}$$

Note that eqs. (I.2.9)–(I.2.14) are covariant with respect to the undotted and dotted spinor indices. Eqs. (I.2.10) and (I.2.11) follow from eqs. (29.3.53) and (29.3.54). For example,

$$\langle p+|\,\sigma^\mu_+\,|p+\rangle = y^\alpha(\vec{\boldsymbol{p}},\tfrac{1}{2})\sigma^\mu_{\alpha\dot\beta}\,y^{\dagger\dot\beta}(\vec{\boldsymbol{p}},\tfrac{1}{2}) = \sigma^\mu_{\alpha\dot\beta}\,y^{\dagger\dot\beta}(\vec{\boldsymbol{p}},\tfrac{1}{2})y^\alpha(\vec{\boldsymbol{p}},\tfrac{1}{2}) = \,\mathrm{Tr}\,(\sigma^\mu p\cdot\overline{\sigma}) = 2p^\mu\,, \tag{I.2.15}$$

and similarly for $\langle p-|\,\sigma^\mu_-\,|p-\rangle$. Eqs. (I.2.12)–(I.2.14) follow immediately from eqs. (29.60)–(29.64). Eqs. (I.2.13) and (I.2.14) generalize easily to the case of a product of an even and odd number of $\sigma/\overline{\sigma}$ matrices. For any positive integer n,

$$\langle p+|\,\sigma^{\mu_1}_+\sigma^{\mu_2}_-\cdots\sigma^{\mu_{2n-1}}_+\,|q+\rangle = \langle q-|\,\sigma^{\mu_{2n-1}}_-\cdots\sigma^{\mu_2}_+\sigma^{\mu_1}_-\,|p-\rangle\ , \tag{I.2.16}$$

$$\langle p\pm|\,\sigma^{\mu_1}_\pm\sigma^{\mu_2}_\mp\cdots\sigma^{\mu_{2n}}_\mp\,|q\mp\rangle = -\,\langle q\pm|\,\sigma^{\mu_{2n}}_\pm\cdots\sigma^{\mu_2}_\pm\sigma^{\mu_1}_\mp\,|p\mp\rangle\ . \tag{I.2.17}$$

[163] The association of undotted and dotted indices in eqs. (I.2.7) and (I.2.8) is a consequence of our convention for the Dirac gamma matrices given in Appendix G [cf. eq. (G.1.2)]. Note that in this convention, the left-handed [right-handed] projection operator P_L [P_R] projects out the lowered undotted [raised dotted] index components of the four-component spinor [cf. eq. (G.1.7)]. However, the reader is warned that in the literature on the spinor helicity method, one often finds $|p+\rangle$ associated with a lowered undotted index and $|p-\rangle$ associated with an upper dotted index. This is due to a different convention for the sigma matrices, such as the Wess and Bagger definition given in eqs. (A.11) and (A.12). Numerically, this is equivalent to a convention for the Dirac gamma matrices in which σ^μ and $\overline{\sigma}^\mu$ are interchanged in eq. (G.1.2), resulting in an overall change of sign in the matrix representation of γ_5. As a result, in this latter convention the lowered undotted [raised dotted] index components are associated with positive [negative] chirality. For an historical perspective, see the discussion following eq. (A.12).

Spinor products can be formed from the bras and kets in the usual way and satisfy:

$$\langle p \pm |q\mp\rangle^* = \langle q \mp |p\pm\rangle \,, \tag{I.2.18}$$
$$\langle p \pm |\sigma^\mu_\pm| q\pm\rangle^* = \langle q \pm |\sigma^\mu_\pm| p\pm\rangle \,, \tag{I.2.19}$$

where we have used the fact that the $\sigma^\mu_\pm$ are hermitian. Covariance with respect to the undotted and dotted spinors allows only two possible spinor products:[164]

$$\langle p\, q\rangle \equiv \langle p - |q+\rangle = x^\dagger(\vec{\boldsymbol{p}}, -\tfrac{1}{2})\, y^\dagger(\vec{\boldsymbol{q}}, \tfrac{1}{2})\,, \tag{I.2.20}$$
$$[p\, q] \equiv \langle p + |q-\rangle = y(\vec{\boldsymbol{p}}, \tfrac{1}{2})\, x(\vec{\boldsymbol{q}}, -\tfrac{1}{2})\,. \tag{I.2.21}$$

In particular, the products $\langle p+|q+\rangle$ and $\langle p-|q-\rangle$ never arise in a computation using two-component spinor notation. In terms of the spinorial products defined in eq. (I.1.12),

$$\langle p\, q\rangle \equiv \langle p - |q+\rangle = (2E_p)^{1/2}(2E_q)^{1/2} T(\hat{\boldsymbol{p}}\,,\, \hat{\boldsymbol{q}})_{-+}\,, \tag{I.2.22}$$
$$[p\, q] \equiv \langle p + |q-\rangle = (2E_p)^{1/2}(2E_q)^{1/2} T(\hat{\boldsymbol{p}}\,,\, \hat{\boldsymbol{q}})_{+-}\,, \tag{I.2.23}$$

where $E_p = |\vec{\boldsymbol{p}}|$ and $E_q = |\vec{\boldsymbol{q}}|$. Explicit forms for $T(\hat{\boldsymbol{p}}\,,\, \hat{\boldsymbol{q}})_{-+}$ and $T(\hat{\boldsymbol{p}}\,,\, \hat{\boldsymbol{q}})_{+-} = -T(\hat{\boldsymbol{p}}\,,\, \hat{\boldsymbol{q}})^*_{-+}$ can be obtained from eq. (I.1.21). Using eqs. (I.1.15) and (I.1.19) [or equivalently, using eqs. (I.2.12) and (I.2.18)], the spinor products satisfy the following relations:

$$\langle p\, q\rangle = -\langle q\, p\rangle \,, \tag{I.2.24}$$
$$[p\, q] = -[q\, p]\,, \tag{I.2.25}$$
$$\langle p\, q\rangle^* = -[p\, q]\,. \tag{I.2.26}$$

One immediate consequence of the above results is:

$$\langle p\, p\rangle = \langle p - |p+\rangle = 0\,, \tag{I.2.27}$$
$$[p\, p] = \langle p + |p-\rangle = 0\,. \tag{I.2.28}$$

We next compute the absolute square of the spinor product:

$$\begin{aligned} |\,\langle p\, q\rangle\,|^2 &= x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}}, -\tfrac{1}{2})\; y^{\dagger\dot\alpha}(\vec{\boldsymbol{q}}, \tfrac{1}{2})\, x_\alpha(\vec{\boldsymbol{p}}, -\tfrac{1}{2})\, y^\alpha(\vec{\boldsymbol{q}}, \tfrac{1}{2}) = x_\alpha(\vec{\boldsymbol{p}}, -\tfrac{1}{2})\, x^\dagger_{\dot\alpha}(\vec{\boldsymbol{p}}, -\tfrac{1}{2})\, y^{\dagger\dot\alpha}(\vec{\boldsymbol{q}}, \tfrac{1}{2})\, y^\alpha(\vec{\boldsymbol{q}}, \tfrac{1}{2}) \\ &= p{\cdot}\sigma_{\alpha\dot\alpha}\, q{\cdot}\overline{\sigma}^{\dot\alpha\alpha} = p_\mu q_\nu\; \mathrm{Tr}(\sigma^\mu\overline{\sigma}^\nu) = 2p{\cdot}q\,. \end{aligned} \tag{I.2.29}$$

Using this result and eq. (I.2.26) yields

$$|\,\langle p\, q\rangle\,|^2 = |[p\, q]|^2 = 2p{\cdot}q\,, \tag{I.2.30}$$

which indicates that the spinor products are roughly the square roots of the corresponding dot products. One other noteworthy relation is:

$$\begin{aligned} \langle p_1\, p_2\rangle\, [p_2\, p_3]\, \langle p_3\, p_4\rangle\, [p_4\, p_1] &= \mathrm{Tr}\,(\sigma{\cdot}p_1\, \overline{\sigma}{\cdot}p_2\, \sigma{\cdot}p_3\, \overline{\sigma}{\cdot}p_4) \\ &= 2(g_{\mu\nu}g_{\rho\kappa} - g_{\mu\rho}g_{\nu\kappa} + g_{\mu\kappa}g_{\nu\rho} + i\epsilon_{\mu\nu\rho\kappa})p_1^\mu p_2^\nu p_3^\rho p_4^\kappa\,, \end{aligned} \tag{I.2.31}$$

where the trace has been evaluated using eq. (B.2.26). Note that the first line of eq. (I.2.31) immediately follows from eqs. (29.3.53) and (29.3.54) after plugging in the definition of the spinor products.

[164] Since we wish to preserve the definition of the spinor products given in eq. (I.2.3), $\langle p\, q\rangle$ is a sum over dotted indices and $[p\, q]$ is a sum over undotted indices in our two-component spinor conventions. This is to be contrasted with most of the literature on the spinor helicity method, in which $\langle p\, q\rangle$ is written as a sum over undotted indices and $[p\, q]$ as a sum over dotted indices. The origin of this difference is explained in footnote 163.

In Appendix I.1 , we showed that a fermion string can be expressed in terms of products of the spinorial products T [cf. eq. (I.1.14)]. When applied to massless spinors, eq. (I.2.30) indicates that the square of the helicity amplitude of a multi-fermion scattering process can be expressed in terms of products of dot products of pairs of fermion momenta. If more than one diagram contributes to a helicity amplitude, then it is often possible to combine the contributions after a rearrangement of momenta via the Fierz identities. Using eqs. (29.66)–(29.70), it follows that:

$$\langle p_1\, p_2\rangle\, \langle p_3\, p_4\rangle = \langle p_1\, p_3\rangle\, \langle p_2\, p_4\rangle + \langle p_1\, p_4\rangle\, \langle p_3\, p_2\rangle\ , \tag{I.2.32}$$

$$[p_1\, p_2]\, [p_3\, p_4] \ = \ [p_1\, p_3]\, [p_2\, p_4] \ + \ [p_1\, p_4]\, [p_3\, p_2]\,, \tag{I.2.33}$$

$$\langle p_1 + |\sigma_+^\mu| p_2 +\rangle\, \langle p_3 + |\sigma_{+\mu}| p_4 +\rangle = 2\, [p_1\, p_3]\ \langle p_4\, p_2\rangle\ , \tag{I.2.34}$$

$$\langle p_1 - |\sigma_-^\mu|\, p_2 -\rangle\, \langle p_3 - |\sigma_{-\mu}| p_4 -\rangle = 2\, \langle p_1\, p_3\rangle\ [p_4\, p_2]\,, \tag{I.2.35}$$

$$\langle p_1 + |\sigma_+^\mu| p_2 +\rangle\, \langle p_3 - |\sigma_{-\mu}| p_4 -\rangle = 2\, [p_1\, p_4]\ \langle p_3\, p_2\rangle\ . \tag{I.2.36}$$

Eqs. (I.2.32) and (I.2.33) are often called the Schouten identities, as they follow from eq. (29.28).

It is desirable to extend the spinor helicity formalism to multi-particle processes involving massless fermions and massless spin-1 bosons. In particular, XZC developed a simple technique for expressing the squares of the corresponding helicity amplitudes in terms of ratios of products of dot products. Their trick was to introduce a convenient expression for the massless spin-1 polarization vector in terms of products of massless spin-1/2 spinor wave functions. Before exhibiting their result, we provide a brief review of spin-1 polarization vectors in the helicity basis.

We first consider a massless spin-1 particle moving in the z-direction with four-momentum $k^\mu = E(1\,;\,0\,,\,0\,,\,1)$. The textbook expression for the helicity ±1 polarization vectors of a massless spin-1 boson is given by [257, 258, 259, 332]:

$$\varepsilon^\mu(\hat{\boldsymbol{z}}, \pm1) = \frac{1}{\sqrt{2}}\,(0\,;\,\mp1\,,\,-i\,,\,0)\ . \tag{I.2.37}$$

Note that the $\varepsilon^\mu(\hat{\boldsymbol{z}}, \lambda)$ are normalized eigenvectors of the spin-1 operator $\vec{\boldsymbol{\mathcal{S}}}\cdot\hat{\boldsymbol{z}}$,

$$(\vec{\boldsymbol{\mathcal{S}}}\cdot\hat{\boldsymbol{z}})^\mu{}_\nu\, \varepsilon^\nu(\hat{\boldsymbol{z}}, \lambda) = \lambda\, \varepsilon^\mu(\hat{\boldsymbol{z}}, \lambda)\,, \tag{I.2.38}$$

where $\mathcal{S}^i \equiv \frac{1}{2}\epsilon^{ijk}\mathcal{S}_{jk}$, and the matrix elements of the 4×4 matrices $\mathcal{S}_{jk}$ are given by eq. (29.9).

If we transform $\varepsilon^\mu(\hat{\boldsymbol{z}}, \lambda)$ by employing a three-dimensional rotation $\mathcal{R}$ such that $\hat{\boldsymbol{k}} = \mathcal{R}\,\hat{\boldsymbol{z}}$, then we can obtain the polarization vector for a massless spin-1 boson of energy E moving in the direction $\hat{\boldsymbol{k}} = (\sin\theta\cos\phi\,,\ \sin\theta\sin\phi\,,\ \cos\theta)$. That is,

$$\varepsilon^\mu(\hat{\boldsymbol{k}}, \lambda) = \Lambda^\mu{}_\nu(\phi\,,\theta\,,\,\gamma)\, \varepsilon^\nu(\hat{\boldsymbol{z}}, \lambda)\,, \tag{I.2.39}$$

where

$$\Lambda^0{}_0 = 1\,, \qquad \Lambda^i{}_0 = \Lambda^0{}_i = 0\,, \quad \text{and} \quad \Lambda^i{}_j = \mathcal{R}^{ij}(\phi\,,\theta\,,\,\gamma)\,, \tag{I.2.40}$$

and $\mathcal{R}(\phi\,,\theta\,,\,\gamma)$ is the rotation matrix introduced in eq. (C.1.4). A simple computation yields:

$$\varepsilon^\mu(\hat{\boldsymbol{k}}, \pm1) = \frac{1}{\sqrt{2}}\, e^{\mp i\gamma}\,(0\,;\,\mp\cos\theta\cos\phi + i\sin\phi\,,\,\mp\cos\theta\sin\phi - i\cos\phi\,,\,\pm\sin\theta)\ . \tag{I.2.41}$$

Note that $\varepsilon^\mu(\hat{\boldsymbol{k}}, \pm1)$ depends only on the direction of $\vec{\boldsymbol{k}}$ and not on its magnitude $E = |\vec{\boldsymbol{k}}|$. One can easily check that the $\varepsilon^\mu(\hat{\boldsymbol{k}}, \pm1)$ are normalized eigenstates of $\vec{\boldsymbol{\mathcal{S}}}\cdot\hat{\boldsymbol{k}}$ with corresponding eigenvalues ±1.

Similar to the corresponding discussion in Appendix C for the spin-1/2 spinor wave functions, the Euler angle γ is arbitrary. In the literature, one typically finds conventions where $\gamma = -\phi$ [37, 257, 258] or $\gamma = 0$ [259], and we will consider both possibilities below.

Although we will not need it here, the expressions given by eqs. (I.2.37) and (I.2.41) also apply in the case of a massive spin-1 particle. In addition, there is a helicity $\lambda = 0$ polarization vector which depends on the magnitude of the momentum as well as its direction:

$$\varepsilon^\mu(|\vec{\boldsymbol{k}}|\hat{\boldsymbol{z}}\,,\,0) = (|\vec{\boldsymbol{k}}|/m\,;\,0\,,\,0\,,\,E/m)\,, \tag{I.2.42}$$

where $E = (|\vec{\boldsymbol{k}}|^2 + m^2)^{1/2}$. One can use eq. (I.2.39) to obtain the helicity zero polarization vector for a massive spin-1 particle moving in an arbitrary direction

$$\varepsilon^\mu(\vec{\boldsymbol{k}}\,,\,0) = \frac{1}{m}\left(|\vec{\boldsymbol{k}}|\,;\,E\sin\theta\cos\phi\,,\,E\sin\theta\sin\phi\,,\,E\cos\theta\right)\,. \tag{I.2.43}$$

Note that both the massless and massive spin-1 polarization vectors satisfy:[165]

$$\varepsilon^\mu(\vec{\boldsymbol{k}}\,,\,\lambda)^* = (-1)^\lambda\varepsilon^\mu(\vec{\boldsymbol{k}}\,,\,-\lambda)\,. \tag{I.2.44}$$

One can check that the $\varepsilon^\mu(\vec{\boldsymbol{k}},\lambda)$ also satisfies the standard conditions for a valid polarization four-vector:

$$k\cdot\varepsilon(\vec{\boldsymbol{k}}\,,\,\lambda) = 0\,, \qquad\qquad \varepsilon(\vec{\boldsymbol{k}}\,,\,\lambda)\cdot\varepsilon(\vec{\boldsymbol{k}}\,,\,\lambda')^* = -\delta_{\lambda\lambda'}\,. \tag{I.2.45}$$

If the spin-1 boson three-momentum is $-\vec{\boldsymbol{k}}$, then its polarization vector can be obtained from eqs. (I.2.41) and (I.2.43) by taking $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$. It can also be derived from eqs. (I.2.39) and (I.2.40) by making use of the spin-1 analogue of eq. (C.3.8),

$$\mathcal{R}(\phi+\pi\,,\,\pi-\theta\,,\,\gamma(-\hat{\boldsymbol{k}})) = \mathcal{R}(\phi\,,\,\theta\,,\,\gamma(\hat{\boldsymbol{k}}))\,R(\hat{\boldsymbol{z}},-\gamma(\hat{\boldsymbol{k}})-\gamma(-\hat{\boldsymbol{k}}))\,R(\hat{\boldsymbol{x}},\pi)\,, \tag{I.2.46}$$

where we have exhibited the possible dependence of γ on the direction of $\hat{\boldsymbol{k}}$, and R is the rotation matrix given by eq. (C.1.5). Introducing the notation $\varepsilon^\mu \equiv (\varepsilon^0\,;\,\hat{\boldsymbol{\varepsilon}})$, and noting the relations:

$$R(\hat{\boldsymbol{x}},\pi)\hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{z}},\lambda) = -\hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{z}},-\lambda)\,, \tag{I.2.47}$$

$$R(\hat{\boldsymbol{z}},\beta)\hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{z}},\lambda) = e^{-i\lambda\beta}\,\hat{\boldsymbol{\varepsilon}}(\hat{\boldsymbol{z}},\lambda)\,, \tag{I.2.48}$$

it follows that:

$$\varepsilon^\mu(-\vec{\boldsymbol{k}}\,,\,\lambda) = -g_{\mu\mu}\,\xi_{-\lambda}(\hat{\boldsymbol{k}})\,\varepsilon^\mu(\vec{\boldsymbol{k}}\,,\,-\lambda)\,, \qquad \lambda = 0\,,\pm1\,, \tag{I.2.49}$$

where there is no sum over the repeated index μ, and

$$\xi_\lambda(\hat{\boldsymbol{k}}) = -e^{i\lambda[\gamma(\hat{\boldsymbol{k}})+\gamma(-\hat{\boldsymbol{k}})]}\,, \qquad \lambda = 0,\pm1\,. \tag{I.2.50}$$

Note that for $\lambda = \pm1$, the phase factor $\xi_\lambda(\hat{\boldsymbol{k}})$ depends on the convention for the definition of the Euler angle γ used to define the spin-1 polarization vector. As an example, corresponding to the two conventional choices for γ,

$$\xi_\lambda(\hat{\boldsymbol{k}}) = \begin{cases} (-1)^{1-\lambda}e^{-2i\lambda\phi} & \text{for}\;\; \gamma(\hat{\boldsymbol{k}}) = -\phi\,, \quad \gamma(-\hat{\boldsymbol{k}}) = -\phi-\pi\,, \\ -1 & \text{for}\;\; \gamma(\hat{\boldsymbol{k}}) = \gamma(-\hat{\boldsymbol{k}}) = 0\,. \end{cases} \tag{I.2.51}$$

To motivate the XZC form for the massless spin-1 polarization vectors, we first introduce a four-vector

$$\widetilde{k}^\mu \equiv E(1\,;\,-\hat{\boldsymbol{k}})\,, \tag{I.2.52}$$

[165] Some authors introduce polarization vectors where the sign factor $(-1)^\lambda$ in eq. (I.2.44) is omitted. One motivation for eq. (I.2.44) is to maintain consistency with the Condon-Shortley phase conventions [333] for the eigenfunctions of the spin angular momentum operators $\vec{\boldsymbol{S}}^{\boldsymbol{2}}$ and S_z (for spin-1 particles). In particular, note the relation $\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\varepsilon}}^\mu(\hat{\boldsymbol{z}},\pm1) = (4\pi/3)^{1/2}Y_{1,\pm1}(\theta,\phi)$ between the polarization three-vector and the $\ell = 1$ spherical harmonics without any additional sign factors.

corresponding to the four-momentum $k^\mu = E(1\,;\,\hat{\boldsymbol{k}})$ of the massless spin-1 boson. A straightforward calculation then shows that

$$\varepsilon^\mu(k\,,\pm 1) = \frac{1}{\sqrt{2}}\,\frac{\langle k\mp|\sigma^\mu_\mp|\,\widetilde{k}\mp\rangle}{\langle k\pm|\widetilde{k}\mp\rangle} \tag{I.2.53}$$

precisely reproduces the result of eq. (I.2.41), where the massless spinor wave functions are defined according to eq. (C.1.8). Eq. (I.2.53) is somewhat inconvenient because the four-vector $\widetilde{k}$ cannot be covariantly defined in terms of k. XZC finessed this problem by introducing a "reference" four-vector p (in practical computations, p is taken to be another four-momentum vector in the scattering process of interest), with the properties that $p^2 = 0$ and $p\cdot k \neq 0$. The XZC spin-1 polarization vectors are given by [cf. eq. (29.34)] [166]

$$\varepsilon^\mu(k\,,\pm 1) = \frac{1}{\sqrt{2}}\,\frac{\langle k\mp|\,\sigma^\mu_\mp\,|p\mp\rangle}{\langle k\pm|p\mp\rangle}\,. \tag{I.2.54}$$

One can immediately check that $\varepsilon^\mu(k,\lambda)$ so defined satisfy the standard conditions for a valid polarization four-vector given in eq. (I.2.45) and the phase convention of eq. (I.2.44). The representation of the massless spin-1 polarization vector in terms of spinor products [eqs. (I.2.53) and (I.2.54)] is an application of the spinor calculus that was first developed by van der Waerden [1].

The significance of the reference four-vector p can be discerned from the property that if a different reference momentum is chosen then ε^μ is shifted by a factor proportional to k^μ. Explicitly, if $\varepsilon^\mu(k\,,\,p\,,\,\lambda)$ is a polarization vector with reference momentum p, then[167]

$$\varepsilon^\mu(k\,,\,q\,,\,\pm 1) = \varepsilon^\mu(k\,,\,p\,,\,\pm 1) + \frac{\sqrt{2}\,\langle q\pm|p\mp\rangle}{\langle k\pm|q\mp\rangle\,\langle k\pm|p\mp\rangle}\,k^\mu\,. \tag{I.2.55}$$

In particular, if we choose $q = \widetilde{k}$, we see that the difference of the XZC spin-1 polarization vector and the polarization vector given by eq. (I.2.41) is proportional to k^μ. This shift of the reference momentum from p to q in the XZC definition of the polarization vector does not affect eq. (I.2.45) since $k^2 = 0$ for massless spin-1 particles. Moreover, this shift does not affect the final result for any observable (in particular the sum of amplitudes of any gauge invariant set of Feynman diagrams remains unchanged). Thus, the presence of the arbitrary four-vector p just reflects the gauge invariance of the theory of massless spin-1 particles.

We can also verify that $\varepsilon^\mu(k\,,\,p\,,\,\lambda)$ defined in eq. (I.2.54) behaves as expected under rotations. Using eq. (C.1.8), massless spinors transform as:

$$|k\pm\rangle \longrightarrow \mathcal{D}(\phi,\theta,\gamma)\,|k\pm\rangle\,, \qquad \langle k\pm| \longrightarrow \langle k\pm|\,[\mathcal{D}(\phi,\theta,\gamma)]^{-1}\,, \tag{I.2.56}$$

under a rotation specified by the Euler angles ϕ, θ and γ. We shall rotate the spin-1 polarization vectors by rotating both $\vec{\boldsymbol{k}}$ and the reference momentum $\vec{\boldsymbol{p}}$ simultaneously (since one is always free to shift the reference vector with no physical consequence). Using eq. (C.1.33), it follows that:

$$[\mathcal{D}(\phi,\theta,\gamma)]^{-1}\,\sigma^\mu_\pm\,\mathcal{D}(\phi,\theta,\gamma) = \Lambda^\mu{}_\nu\sigma^\nu_\pm\,, \tag{I.2.57}$$

[166] In the literature on the spinor helicity method, the spin-1 polarization vector ε is employed in Feynman diagram computations for an *outgoing* final state boson, in contrast to the standard conventions of most quantum field theory textbooks. In this review, we subscribe to the latter [as indicated at the end of Section 29.4.1]. Hence, to be consistent with our conventions above for the spin-1 polarization vector, we have taken the complex conjugate of the original definition of the XZC spin-1 polarization vectors. In addition, we have removed an overall $\pm$ sign in order to conform to eq. (I.2.44) [cf. footnote 165].

[167] To derive eq. (I.2.55), evaluate $\varepsilon^\mu(k\,,\,q\,,\,\lambda) - \varepsilon^\mu(k\,,\,p\,,\,\lambda)$, and simplify the resulting expression using eqs. (I.2.10), (I.2.13) and (I.2.14).

where $\Lambda^\mu{}_\nu$ is specified by eq. (I.2.40). Indeed, if we simultaneously rotate both k and p via $k^\mu \to \Lambda^\mu{}_\nu k^\nu$ and $p^\mu \to \Lambda^\mu{}_\nu p^\nu$, then

$$\varepsilon^\mu(k\,,\,p\,,\,\lambda) \longrightarrow \Lambda^\mu{}_\nu\,\varepsilon^\nu(k\,,\,p\,,\,\lambda)\,, \tag{I.2.58}$$

as expected. By a similar computation, one can check that under $\vec{\boldsymbol{k}} \to -\vec{\boldsymbol{k}}$ and $\vec{\boldsymbol{p}} \to -\vec{\boldsymbol{p}}$, eq. (I.2.49) is satisfied.[168] In terms of the $\xi_{\pm 1/2}$ defined in eq. (C.3.11), we find

$$\xi_\lambda(\vec{\boldsymbol{k}}) = -\left(\frac{\xi_{-\lambda/2}(\vec{\boldsymbol{k}})}{\xi_{\lambda/2}(\vec{\boldsymbol{k}})}\right)^* = \Big[\xi_{\lambda/2}(\vec{\boldsymbol{k}})\Big]^2\,, \qquad \lambda = \pm 1\,, \tag{I.2.59}$$

which agrees with eq. (I.2.50).

The following additional properties of $\varepsilon^\mu(k\,,\,p\,,\,\lambda)$ defined in eq. (I.2.54) are noteworthy:

$$p{\cdot}\varepsilon(k\,,\,p\,,\,\lambda) = 0\,, \tag{I.2.60}$$

$$\sum_{\lambda=\pm 1} \varepsilon_\mu(k\,,\,p\,,\,\lambda)\varepsilon_\nu(k\,,\,p\,,\,\lambda)^* = -g_{\mu\nu} + \frac{p_\mu k_\nu + p_\nu k_\mu}{p{\cdot}k}\,. \tag{I.2.61}$$

For example, to prove eq. (I.2.61), we use eqs. (I.2.18) and (I.2.19), and simplify the resulting expression with the help of eqs. (I.2.10) and (I.2.16), which yields:

$$\sum_{\lambda=\pm 1} \varepsilon_\mu(k\,,\,p\,,\,\lambda)\varepsilon_\nu(k\,,\,p\,,\,\lambda)^* = \frac{\langle k+|\,(\sigma_\mu p{\cdot}\overline{\sigma}\sigma_\nu + \sigma_\nu p{\cdot}\overline{\sigma}\sigma_\mu)\,|k+\rangle}{2\,\langle k+|\,p{\cdot}\sigma\,|k+\rangle}\,. \tag{I.2.62}$$

Using eq. (B.2.17) to simplify the product of three $\sigma/\overline{\sigma}$ matrices, and employing eq. (I.2.11) then yields eq. (I.2.61).

Finally, using the Fierz identities given in eqs. (B.1.6)–(B.1.8), one derives from eq. (I.2.54) that

$$\sigma_\pm{\cdot}\varepsilon(k\,,\,\pm 1) = \frac{\sqrt{2}\,|p\mp\rangle\,\langle k\,\mp|}{\langle k\,\pm\,|p\mp\rangle}\,, \qquad \sigma_\pm{\cdot}\varepsilon(k\,,\,\pm 1)^* = \frac{\sqrt{2}\,|k\mp\rangle\,\langle p\mp|}{\langle p\,\mp\,|k\pm\rangle}\,, \tag{I.2.63}$$

$$\sigma_\mp{\cdot}\varepsilon(k\,,\,\pm 1) = \frac{\sqrt{2}\,|k\pm\rangle\,\langle p\pm|}{\langle k\,\pm\,|p\mp\rangle}\,, \qquad \sigma_\mp{\cdot}\varepsilon(k\,,\,\pm 1)^* = \frac{\sqrt{2}\,|p\pm\rangle\,\langle k\pm|}{\langle p\,\mp\,|k\pm\rangle}\,. \tag{I.2.64}$$

Note that each equation in eqs. (I.2.63) and (I.2.64) represents two separate equations, corresponding to the upper and lower signs in each equation, respectively.

It should now be clear how to convert the square of a helicity amplitude for a multi-particle process involving massless spin-1/2 and massless spin-1 particles into a ratio of products of dot products of momenta. By writing all massless spin-1 polarization vectors in the form of eq. (I.2.54) and using the properties given above, the helicity amplitudes can easily be expressed as a ratio of two quantities, each of which is a product of spinor products. Squaring the corresponding amplitude then yields a ratio of products of dot products of four-momenta. A following simple example will demonstrate the technique.

Consider Compton scattering in QED, $e^-(\vec{\boldsymbol{p}}_1,\lambda_1)\gamma(\vec{\boldsymbol{k}}_1,\lambda_1') \to e^-(\vec{\boldsymbol{p}}_2,\lambda_2)\gamma(\vec{\boldsymbol{k}}_2,\lambda_2')$, in the limit of massless electrons. The amplitude for this process is given by eq. (29.4.18) with $m = 0$ and $G_L = G_R = -e$. Writing out the "crossed" term explicitly, and noting that for massless particles, $s \equiv (p_1+k_1)^2 = 2p_1{\cdot}k_1$ and $u \equiv (p_1-k_2)^2 = -2p_1{\cdot}k_2$,

$$\begin{aligned} i\mathcal{M} = {}& \frac{-ie^2}{2p_1{\cdot}k_1}\Big\{x^\dagger(\vec{\boldsymbol{p}}_2,\lambda_2)\,\overline{\sigma}{\cdot}\varepsilon_2^*\,\sigma{\cdot}(p_1+k_1)\,\overline{\sigma}{\cdot}\varepsilon_1\,x(\vec{\boldsymbol{p}}_1,\lambda_1) + y(\vec{\boldsymbol{p}}_2,\lambda_2)\,\sigma{\cdot}\varepsilon_2^*\,\overline{\sigma}{\cdot}(p_1+k_1)\,\sigma{\cdot}\varepsilon_1 y^\dagger(\vec{\boldsymbol{p}}_1,\lambda_1)\Big\} \\ & + \frac{ie^2}{2p_1{\cdot}k_2}\Big\{x^\dagger(\vec{\boldsymbol{p}}_2,\lambda_2)\,\overline{\sigma}{\cdot}\varepsilon_1\,\sigma{\cdot}(p_1-k_2)\,\overline{\sigma}{\cdot}\varepsilon_2^*\,x(\vec{\boldsymbol{p}}_1,\lambda_1) + y(\vec{\boldsymbol{p}}_2,\lambda_2)\,\sigma{\cdot}\varepsilon_1\,\overline{\sigma}{\cdot}(p_1-k_2)\,\sigma{\cdot}\varepsilon_2^* y^\dagger(\vec{\boldsymbol{p}}_1,\lambda_1)\Big\}\,. \end{aligned} \tag{I.2.65}$$

[168] Here, we have used eqs. (B.2.16) and (B.2.17) to write $\sigma_\pm^0\sigma_\mp^\mu\sigma_\pm^0 = -\sigma_\pm^\mu + 2g^{\mu 0}\sigma_\pm^0 = g^{\mu\mu}\sigma_\pm^\mu$ (no sum over μ).

The results of eqs. (29.3.36)–(29.3.39) imply that the helicity amplitudes with $\lambda_1 \neq \lambda_2$ vanish. Using eqs. (I.2.7) and (I.2.8), we identify:

$$i\mathcal{M}(\lambda_1=\lambda_2=\tfrac{1}{2}) = \frac{-ie^2}{2p_1\cdot k_1}\,\langle p_2+|\sigma_+\cdot\varepsilon_2^*\,\sigma_-\cdot(p_1+k_1)\,\sigma_+\cdot\varepsilon_1|p_1+\rangle \\ +\frac{ie^2}{2p_1\cdot k_2}\,\langle p_2+|\sigma_+\cdot\varepsilon_1\,\sigma_-\cdot(p_1-k_2)\,\sigma_+\cdot\varepsilon_2^*|p_1+\rangle\ , \tag{I.2.66}$$

$$i\mathcal{M}(\lambda_1=\lambda_2=-\tfrac{1}{2}) = \frac{-ie^2}{2p_1\cdot k_1}\,\langle p_2-|\sigma_-\cdot\varepsilon_2^*\,\sigma_+\cdot(p_1+k_1)\,\sigma_-\cdot\varepsilon_1|p_1-\rangle \\ +\frac{ie^2}{2p_1\cdot k_2}\,\langle p_2-|\sigma_-\cdot\varepsilon_1\,\sigma_+\cdot(p_1-k_2)\,\sigma_-\cdot\varepsilon_2^*|p_1-\rangle\ . \tag{I.2.67}$$

Further simplification ensues when we apply the results of eqs. (I.2.63) and (I.2.64). To use these results, we must select a reference momentum p, which can be any lightlike four-vector that is not parallel to the corresponding photon polarization vector. One is free to choose a different reference momentum for each photon polarization vector. Moreover, when computing two different helicity amplitudes (each of which are gauge invariant quantities), one may select a different reference momentum for the *same* photon polarization vector in the two computations. The decision on which reference momenta to choose is somewhat of an art; experience will teach you which choices lead to the most simplification in a given calculation.

We shall consider two possible choices for the reference momenta for ε_1 and ε_2, which we denote as $p^{(1)}$ and $p^{(2)}$, respectively:

1. $p^{(1)}=p_1$ and $p^{(2)}=p_2$,
2. $p^{(1)}=p_2$ and $p^{(2)}=p_1$.

With either choice, it is straightforward to show that $\mathcal{M}(\lambda_1=\lambda_2=\pm\frac{1}{2})$ vanish unless the photon helicities are equal, i.e. $\lambda_1'=\lambda_2'$. This leaves only four possible non-vanishing helicity amplitudes.

For the case of $\lambda_1=\lambda_2=\pm\frac{1}{2}$ and $\lambda_1'=\lambda_2'=\pm1$ (i.e., $\lambda_1\lambda_1'>0$), we choose reference momenta $p^{(1)}=p_2$ and $p^{(2)}=p_1$. Then, the second term vanishes on the right-hand side of eqs. (I.2.66) and (I.2.67), respectively. Making use of eqs. (I.2.10), (I.2.63) and (I.2.64), we find

$$i\mathcal{M}(\lambda_1=\lambda_2=\tfrac{1}{2}\,,\,\lambda_1'=\lambda_2'=1) = \frac{-ie^2}{p_1\cdot k_1}\,\frac{\langle p_2+|k_2-\rangle\,\langle p_1-|k_1+\rangle\,\langle k_1+|p_2-\rangle\,\langle k_1-|p_1+\rangle}{\langle p_1-|k_2+\rangle\,\langle k_1+|p_2-\rangle} \\ = \frac{-ie^2}{p_1\cdot k_1}\,\frac{\langle p_1\,k_1\rangle\,\langle k_1\,p_1\rangle\,[p_2\,k_2]}{\langle p_1\,k_2\rangle}\,. \tag{I.2.68}$$

Using eqs. (I.2.24) and (I.2.30) to write the dot product in terms of spinor products, we obtain:

$$i\mathcal{M}(\lambda_1=\lambda_2=\tfrac{1}{2}\,,\,\lambda_1'=\lambda_2'=1) = 2ie^2\,\frac{\langle p_1\,k_1\rangle}{\langle p_1\,k_1\rangle^*}\,\frac{[p_2\,k_2]}{\langle p_1\,k_2\rangle}\,. \tag{I.2.69}$$

A similar computation yields

$$i\mathcal{M}(\lambda_1=\lambda_2=-\tfrac{1}{2}\,,\,\lambda_1'=\lambda_2'=-1) = 2ie^2\,\frac{[p_1\,k_1]}{[p_1\,k_1]^*}\,\frac{\langle p_2\,k_2\rangle}{[p_1\,k_2]}\,. \tag{I.2.70}$$

For the case of $\lambda_1=\lambda_2=\pm\frac{1}{2}$ and $\lambda_1'=\lambda_2'=\mp1$ (i.e., $\lambda_1\lambda_1'<0$), we choose reference momenta $p^{(1)}=p_1$ and $p^{(2)}=p_2$. Then, the first term vanishes on the right-hand side of eqs. (I.2.66) and

(I.2.67), respectively. A similar calculation to the one given above yields:

$$i\mathcal{M}(\lambda_1=\lambda_2=-\tfrac{1}{2}\,,\ \lambda_1'=\lambda_2'=1)=2ie^2\,\frac{[p_1\,k_2]}{[p_1\,k_2]^*}\,\frac{\langle p_2\,k_1\rangle}{[p_1\,k_1]}\,, \tag{I.2.71}$$

$$i\mathcal{M}(\lambda_1=\lambda_2=\tfrac{1}{2}\,,\ \lambda_1'=\lambda_2'=-1)=2ie^2\,\frac{\langle p_1\,k_2\rangle}{\langle p_1\,k_2\rangle^*}\,\frac{[p_2\,k_1]}{\langle p_1\,k_1\rangle}\,. \tag{I.2.72}$$

Note that each pair of helicity amplitudes above is simply related:

$$\mathcal{M}_{\lambda_1\,,\,\lambda_1';\lambda_2\,,\,\lambda_2'}(s,\theta,\phi)^*=\mathcal{M}_{-\lambda_1\,,\,-\lambda_1';-\lambda_2\,,\,-\lambda_2'}(s,\theta,\phi)\,, \tag{I.2.73}$$

which is a consequence of rotational and parity invariance (as shown below). Thus in this example, we only need to evaluate two non-zero helicity amplitudes. It is clear that we have simplified the computation enormously by our choice of reference momenta. With a less judicious choice, the calculation is significantly more tedious, although gauge invariance guarantees that one must arrive at the same result for the helicity amplitudes quoted above.

One can easily evaluate the spinor products above in the center-of-mass system. Writing $p_1^\mu=E(1\,;\,\hat{\boldsymbol{z}})$, $k_1^\mu=E(1\,;\,-\hat{\boldsymbol{z}})$, $p_2^\mu=E(1\,;\,\hat{\boldsymbol{p}}_{\rm CM})$ and $k_2^\mu=E(1\,;\,-\hat{\boldsymbol{p}}_{\rm CM})$, and using the results of eqs. (I.1.21) and (I.2.22), we obtain:

$$\langle p_1\,k_1\rangle=2E\xi_{-1/2}(\hat{\boldsymbol{z}})\,,\qquad \langle p_1\,k_2\rangle=2Ee^{i[\phi+\gamma(\hat{\boldsymbol{p}}_{\rm CM})]/2}\xi_{-1/2}(\hat{\boldsymbol{p}}_{\rm CM})\cos(\theta/2)\,, \tag{I.2.74}$$

$$\langle p_2\,k_2\rangle=2E\xi_{-1/2}(\hat{\boldsymbol{p}}_{\rm CM})\,,\qquad \langle p_2\,k_1\rangle=2Ee^{-i[\phi+\gamma(\hat{\boldsymbol{p}}_{\rm CM})]/2}\xi_{-1/2}(\hat{\boldsymbol{z}})\cos(\theta/2)\,, \tag{I.2.75}$$

where θ and ϕ are the polar and azimuthal angles of $\hat{\boldsymbol{p}}_{\rm CM}$. Phase factors involving $\xi_{-1/2}$ arise from the use of eqs. (C.3.10) and (C.3.11). For example, corresponding to the two conventional choices for γ, we use eq. (C.3.14) to obtain

$$\xi_{-1/2}(\hat{\boldsymbol{z}})=\begin{cases}-1 & \text{for } \gamma(\hat{\boldsymbol{z}})=0\,,\quad \gamma(-\hat{\boldsymbol{z}})=-\pi\,,\\ i & \text{for } \gamma(\hat{\boldsymbol{z}})=\gamma(-\hat{\boldsymbol{z}})=0\,,\end{cases} \tag{I.2.76}$$

$$\xi_{-1/2}(\hat{\boldsymbol{p}}_{\rm CM})=\begin{cases}-e^{i\phi} & \text{for } \gamma(\hat{\boldsymbol{p}}_{\rm CM})=-\phi\,,\quad \gamma(-\hat{\boldsymbol{p}}_{\rm CM})=-\phi-\pi\\ i & \text{for } \gamma(\hat{\boldsymbol{p}}_{\rm CM})=\gamma(-\hat{\boldsymbol{p}}_{\rm CM})=0\,.\end{cases} \tag{I.2.77}$$

All other relevant spinor products can be found using eqs. (I.2.24)–(I.2.26).

It is always possible to define the plane of the scattering process to be the x–z plane, in which case $\phi=0$ and all the spinor products in eqs. (I.2.74) and (I.2.75) are manifestly real. Nevertheless, by keeping the explicit ϕ-dependence, one maintains a useful check of the calculation. Inserting the explicit forms for the spinor products into eqs. (I.2.69)–(I.2.72), we confirm that the ϕ-dependence of the helicity amplitudes is given by [334, 259]:

$$\mathcal{M}_{\lambda_1\,,\,\lambda_1';\lambda_2\,,\,\lambda_2'}(s,\theta,\phi)=\begin{cases}e^{i(\lambda_1-\lambda_1'-\lambda_2-\lambda_2')\phi}\mathcal{M}_{\lambda_1\,,\,\lambda_1';\lambda_2\,,\,\lambda_2'}(s,\theta)\,, & \text{for } \gamma(\hat{\boldsymbol{p}}_{\rm CM})=-\phi\,,\text{ and}\\ & \quad\gamma(-\hat{\boldsymbol{p}}_{\rm CM})=-\phi-\pi\,,\\ e^{i(\lambda_1-\lambda_1')\phi}\mathcal{M}_{\lambda_1\,,\,\lambda_1';\lambda_2\,,\,\lambda_2'}(s,\theta)\,, & \text{for } \gamma(\hat{\boldsymbol{p}}_{\rm CM})=\gamma(-\hat{\boldsymbol{p}}_{\rm CM})=0\,,\end{cases} \tag{I.2.78}$$

as a consequence of rotational invariance [260].[169] The remaining θ-dependent amplitudes are easily evaluated and are in agreement with the results of refs. [259, 335]. Note that parity invariance implies that eqs. (I.2.69)–(I.2.72) must satisfy [260, 259, 335]

$$\mathcal{M}_{\lambda_1\,,\,\lambda_1';\lambda_2\,,\,\lambda_2'}(s,\theta)=\mathcal{M}_{-\lambda_1\,,\,-\lambda_1';-\lambda_2\,,\,-\lambda_2'}(s,\theta)\,. \tag{I.2.79}$$

[169] In the first case, where $\gamma(\hat{\boldsymbol{p}}_{\rm CM})=-\phi$ and $\gamma(-\hat{\boldsymbol{p}}_{\rm CM})=-\phi-\pi$, the sign of λ_2' in the ϕ-dependent phase factor of eq. (I.2.78) is opposite to the one given in [260], due to the Jacob-Wick second-particle convention, which we do not employ here. Since $\lambda_1=\lambda_2$ and $\lambda_1'=\lambda_2'$, the latter would imply that the ϕ-dependent phase cancels exactly if the Jacob-Wick second-particle convention is used. This is easily checked by putting $\gamma(\hat{\boldsymbol{p}}_{\rm CM})=-\phi$ and $\xi_\lambda=1$ in eqs. (I.2.74) and (I.2.75), in which case all the spinor products are real.

Indeed, in our computation above, eq. (I.2.73) is satisfied, which is consistent with eq. (I.2.79) in light of eq. (I.2.78).

To compute the unpolarized cross-section for Compton scattering, one must sum the absolute squares of the helicity amplitudes and divide by 4 to average over the initial helicities. Since quantities such as $\langle p_1\, k_1\rangle \,/\, \langle p_1\, k_1\rangle^*$ are pure phases, one immediately obtains:

$$|\mathcal{M}(\lambda_1 = \lambda_2 = \tfrac{1}{2}\,,\, \lambda_1' = \lambda_2' = 1)|^2 = |\mathcal{M}(\lambda_1 = \lambda_2 = -\tfrac{1}{2}\,,\, \lambda_1' = \lambda_2' = -1)|^2 = 4e^4 \frac{p_1\cdot k_1}{p_1\cdot k_2}\,, \quad \text{(I.2.80)}$$

$$|\mathcal{M}(\lambda_1 = \lambda_2 = -\tfrac{1}{2}\,,\, \lambda_1' = \lambda_2' = 1)|^2 = |\mathcal{M}(\lambda_1 = \lambda_2 = \tfrac{1}{2}\,,\, \lambda_1' = \lambda_2' = -1)|^2 = 4e^4 \frac{p_1\cdot k_2}{p_1\cdot k_1}\,, \quad \text{(I.2.81)}$$

after employing eq. (I.2.30) and noting that $p_1\cdot k_1 = p_2\cdot k_2$ and $p_1\cdot k_2 = p_2\cdot k_1$ (which follow from four-momentum conservation, $p_1 + k_1 = p_2 + k_2$, for the scattering of massless particles). Thus,

$$\tfrac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4 \left(\frac{p_1\cdot k_1}{p_1\cdot k_2} + \frac{p_1\cdot k_2}{p_1\cdot k_1}\right)\,, \quad \text{(I.2.82)}$$

which coincides with the well-known result quoted in [115].

29.7.12 The Standard Model and its seesaw extension 1

In the Standard Model, three generations of quarks and leptons are described by the two-component fermion fields listed in Table J.1, where Y is the weak hypercharge, T_3 is the third component of the weak isospin, and $Q = T_3 + Y$ is the electric charge. After $\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$ breaking, the quark and lepton fields gain mass in such a way that the above two-component fields combine to make up four-component Dirac fermions:

$$U_i = \begin{pmatrix} u_i \\ \bar{u}^{\dagger i} \end{pmatrix}\,, \qquad D_i = \begin{pmatrix} d_i \\ \bar{d}^{\dagger i} \end{pmatrix}\,, \qquad L_i = \begin{pmatrix} \ell_i \\ \bar{\ell}^{\dagger i} \end{pmatrix}\,, \quad \text{(J.1)}$$

while the neutrinos ν_i remain massless. The extension of the Standard Model to include neutrino masses will be treated in Appendix J.2 .

Here, we follow the convention for particle symbols established in Table 29.1. Note that u, $\bar{u}$, d, $\bar{d}$, ℓ and $\bar{\ell}$ are two-component fields, whereas the usual four-component quark and charged lepton fields are denoted by capital letters U, D and E. Consider a generic four-component field expressed in terms of the corresponding two-component fields:

$$F = \begin{pmatrix} f \\ \bar{f}^{\dagger} \end{pmatrix}\,. \quad \text{(J.2)}$$

The electroweak quantum numbers of f are denoted by T_3^f, Y_f and Q_f, whereas the corresponding quantum numbers for $\bar{f}$ are $T_3^{\bar{f}} = 0$ and $Q_{\bar{f}} = Y_{\bar{f}} = -Q_f$. Thus we have the correspondence to our general notation [eq. (G.1.7)]

$$f \longleftrightarrow \chi, \qquad \bar{f} \longleftrightarrow \eta\,. \quad \text{(J.3)}$$

We can then immediately translate the couplings given in the general case in Fig. 29.4.3 to the Standard Model.

Two-component fermion fields	SU(3)	$SU(2)_L$	Y	T_3	$Q = T_3 + Y$
$\boldsymbol{Q}_i \equiv \begin{pmatrix} u_i \\ d_i \end{pmatrix}$	triplet	doublet	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{2}{3}$
	triplet		$\frac{1}{6}$	$-\frac{1}{2}$	$-\frac{1}{3}$
$\bar{u}^i$	anti-triplet	singlet	$-\frac{2}{3}$	0	$-\frac{2}{3}$
$\bar{d}^i$	anti-triplet	singlet	$\frac{1}{3}$	0	$\frac{1}{3}$
$\boldsymbol{L}_i \equiv \begin{pmatrix} \nu_i \\ \ell_i \end{pmatrix}$	singlet	doublet	$-\frac{1}{2}$	$\frac{1}{2}$	0
	singlet		$-\frac{1}{2}$	$-\frac{1}{2}$	-1
$\bar{\ell}^i$	singlet	singlet	1	0	1

Table J.1: Fermions of the Standard Model (following the naming conventions of Table 29.1) and their $SU(3)\times SU(2)_L\times U(1)_Y$ quantum numbers. The generation indices run over $i = 1, 2, 3$. Color indices for the quarks are suppressed. The bars on the two-component antifermion fields are part of their names, and do not denote some form of complex conjugation.

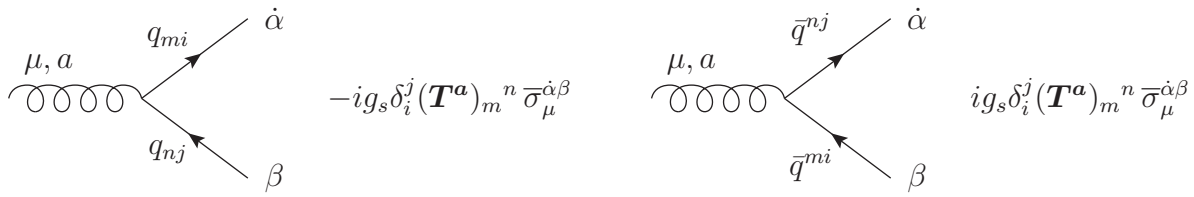


Figure J.1.1: Fermionic Feynman rules for QCD that involve the gluon, with $q = u, d, c, s, t, b$. Lowered (raised) indices m, n correspond to the fundamental (anti-fundamental) representation of $SU(3)_c$. The gluon interactions are flavor-diagonal (where i, j are flavor indices). For each rule, a corresponding one with lowered spinor indices is obtained by $\overline{\sigma}_\mu^{\dot{\alpha}\beta} \to -\sigma_{\mu\beta\dot{\alpha}}$.

Standard Model fermion interaction vertices

The QCD color interactions of the quarks are governed by the following interaction Lagrangian:

$$\mathscr{L}_{\rm int} = -g_s A_a^\mu q^{\dagger mi}\,\overline{\sigma}_\mu(\boldsymbol{T}^a)_m{}^n q_{ni} + g_s A_a^\mu \bar{q}_{ni}^\dagger\,\overline{\sigma}_\mu(\boldsymbol{T}^a)_m{}^n \bar{q}^{mi}\,, \tag{J.1.1}$$

summed over the generations i, where q is a (mass eigenstate) quark field, m and n are SU(3) color triplet indices, A_a^μ is the gluon field (with the corresponding gluons denoted by g_a), and $\boldsymbol{T}^a$ are the color generators in the triplet representation of SU(3). The corresponding Feynman rules are given in Fig. J.1.1.

Next, we write out the Feynman rules for the electroweak interactions of quarks and leptons.

Using eqs. (29.4.11) and (29.4.12), the interactions of the gauge bosons and quarks are given by:

$$\begin{aligned}\mathscr{L}_{\rm int} = &-\frac{g}{\sqrt{2}}\left[(\hat{u}^{\dagger i}\overline{\sigma}^\mu \hat{d}_i + \hat{\nu}^{\dagger i}\overline{\sigma}^\mu \hat{\ell}_i)W^+_\mu + (\hat{d}^{\dagger i}\overline{\sigma}^\mu \hat{u}_i + \hat{\ell}^{\dagger i}\overline{\sigma}^\mu \hat{\nu}_i)W^-_\mu\right] \\ &-\frac{g}{c_W}\sum_{f=u,d,\nu,\ell}\left\{(T_3^f - s_W^2 Q_f)\hat{f}^{\dagger i}\overline{\sigma}^\mu \hat{f}_i + s_W^2 Q_f \hat{\hat{f}}^{\dagger i}\overline{\sigma}^\mu \hat{\hat{f}}_i\right\} Z_\mu \\ &-e\sum_{f=u,d,\ell} Q^f(\hat{f}^{\dagger i}\overline{\sigma}^\mu \hat{f}_i - \hat{\hat{f}}^{\dagger i}\overline{\sigma}^\mu \hat{\hat{f}}_i)A_\mu\,, \end{aligned} \tag{J.1.2}$$

where $s_W \equiv \sin\theta_W$, $c_W \equiv \cos\theta_W$, the hatted symbols indicate fermion interaction eigenstates and i labels the generations. Following the discussion of Section 29.3.2, we must convert from fermion interaction eigenstates to mass eigenstates. In order to accomplish this step, we must first identify the quark and lepton mass matrices. In the electroweak theory, the fermion mass matrices originate from the fermion-Higgs Yukawa interactions.

The Higgs field of the Standard Model is a complex $\mathrm{SU}(2)_L$ doublet of hypercharge $Y = \frac{1}{2}$,

$$\boldsymbol{\Phi}_a \equiv \begin{pmatrix}\Phi^+ \\ \Phi^0\end{pmatrix}\,, \tag{J.1.3}$$

where the $\mathrm{SU}(2)_L$ index $a = 1, 2$ is defined such that $\boldsymbol{\Phi}_1 \equiv \Phi^+$ and $\boldsymbol{\Phi}_2 \equiv \Phi^0$. Here, the superscripts $+$ and 0 refer to the electric charge of the Higgs field, $Q = T_3 + Y$, with $Y = \frac{1}{2}$ and $T_3 = \pm\frac{1}{2}$. Since $\boldsymbol{\Phi_a}$ is complex, we can also introduce the complex conjugate Higgs doublet field with hypercharge $Y = -\frac{1}{2}$,

$$\boldsymbol{\Phi}^{\dagger\, a} \equiv \left(\Phi^-\,,\ \ (\Phi^0)^\dagger\right)\,, \tag{J.1.4}$$

where $\Phi^- \equiv (\Phi^+)^\dagger$. The $\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$ gauge invariant Yukawa interactions of the quarks and leptons with the Higgs field are then given by:

$$\mathscr{L}_{\rm Y} = \epsilon^{ab}(\boldsymbol{Y}_u)^i{}_j \boldsymbol{\Phi}_a \hat{\boldsymbol{Q}}_{bi}\bar{u}^j - (\boldsymbol{Y}_d)^i{}_j \boldsymbol{\Phi}^{\dagger\, a}\hat{\boldsymbol{Q}}_{ai}\hat{\bar{d}}^j - (\boldsymbol{Y}_\ell)^i{}_j \boldsymbol{\Phi}^{\dagger\, a}\hat{\boldsymbol{L}}_{ai}\hat{\bar{\ell}}^j + \text{h.c.} \tag{J.1.5}$$

where ϵ^{ab} is the antisymmetric invariant tensor of $\mathrm{SU}(2)_L$, defined such that $\epsilon^{12} = -\epsilon^{21} = +1$. Using the definitions of the $\mathrm{SU}(2)_L$ doublet quark and lepton fields given in Table J.1, one can rewrite eq. (J.1.5) more explicitly as:

$$-\mathscr{L}_{\rm Y} = (\boldsymbol{Y}_u)^i{}_j\left[\Phi^0\hat{u}_i\hat{\bar{u}}^j - \Phi^+\hat{d}_i\hat{\bar{u}}^j\right] + (\boldsymbol{Y}_d)^i{}_j\left[\Phi^-\hat{u}_i\hat{\bar{d}}^j + \Phi^{0*}\hat{d}_i\hat{\bar{d}}^j\right] + (\boldsymbol{Y}_\ell)^i{}_j\left[\Phi^-\hat{\nu}_i\hat{\bar{\ell}}^j + \Phi^{0*}\hat{\ell}_i\hat{\bar{\ell}}^j\right] + \text{h.c.} \tag{J.1.6}$$

The Higgs fields can be written in terms of the physical Higgs scalar $h_{\rm SM}$ and Nambu-Goldstone bosons $G^0, G^\pm$ as

$$\Phi^0 = v + \frac{1}{\sqrt{2}}(h_{\rm SM} + iG^0)\,, \tag{J.1.7}$$

$$\Phi^+ = G^+ = (\Phi^-)^\dagger = (G^-)^\dagger. \tag{J.1.8}$$

where $v = \sqrt{2}m_W/g \simeq 174$ GeV. In the unitary gauge appropriate for tree-level calculations, the Nambu-Goldstone bosons become infinitely heavy and decouple. We identify the quark and lepton mass matrices by setting $\Phi^0 = v$ and $\Phi^+ = \Phi^- = 0$ in eq. (J.1.6):

$$(\boldsymbol{M}_u)^i{}_j = v(\boldsymbol{Y}_u)^i{}_j\,, \qquad (\boldsymbol{M}_d)^i{}_j = v(\boldsymbol{Y}_d)^i{}_j\,, \qquad (\boldsymbol{M}_\ell)^i{}_j = v(\boldsymbol{Y}_\ell)^i{}_j\,. \tag{J.1.9}$$

The neutrinos remain massless. An extension of the Standard Model that incorporates massive neutrinos is treated in Appendix J.2 .

To diagonalize the quark and lepton mass matrices, we introduce four unitary matrices for the quark mass diagonalization, L_u, L_d, R_u and R_d, and two unitary matrices for the lepton mass diagonalization, L_ℓ and R_ℓ [cf. eq. (29.3.31)] such that

$$\hat{u}_i = (L_u)_i{}^j u_j\,, \qquad \hat{d}_i = (L_d)_i{}^j d_j\,, \qquad \hat{\bar{u}}^i = (R_u)^i{}_j \bar{u}^j\,, \qquad \hat{\bar{d}}^i = (R_d)^i{}_j \bar{d}^j\,, \tag{J.1.10}$$

$$\hat{\ell}_i = (L_\ell)_i{}^j \ell_j\,, \qquad \hat{\bar{\ell}}^i = (R_\ell)^i{}_j \bar{\ell}^j\,, \tag{J.1.11}$$

where the unhatted fields u, d, $\bar{u}$ and $\bar{d}$ are the corresponding quark mass eigenstates and ν, ℓ and $\bar{\ell}$ are the corresponding lepton mass eigenstates. The fermion mass diagonalization procedure consists of the singular value decomposition of the quark and lepton mass matrices:

$$L_u^\mathsf{T} \boldsymbol{M}_u R_u = \mathrm{diag}(m_u\,,\, m_c\,,\, m_t)\,, \tag{J.1.12}$$

$$L_d^\mathsf{T} \boldsymbol{M}_d R_d = \mathrm{diag}(m_d\,,\, m_s\,,\, m_b)\,, \tag{J.1.13}$$

$$L_\ell^\mathsf{T} \boldsymbol{M}_\ell R_\ell = \mathrm{diag}(m_e\,,\, m_\mu\,,\, m_\tau)\,, \tag{J.1.14}$$

where the diagonalized masses are real and non-negative (cf. Appendix D.1). Since the neutrinos are massless, we are free to define the physical neutrino fields, ν_i, as the weak SU(2) partners of the corresponding charged lepton mass eigenstate fields. That is,

$$\hat{\nu}_i = (L_\ell)_i{}^j \nu_j\,. \tag{J.1.15}$$

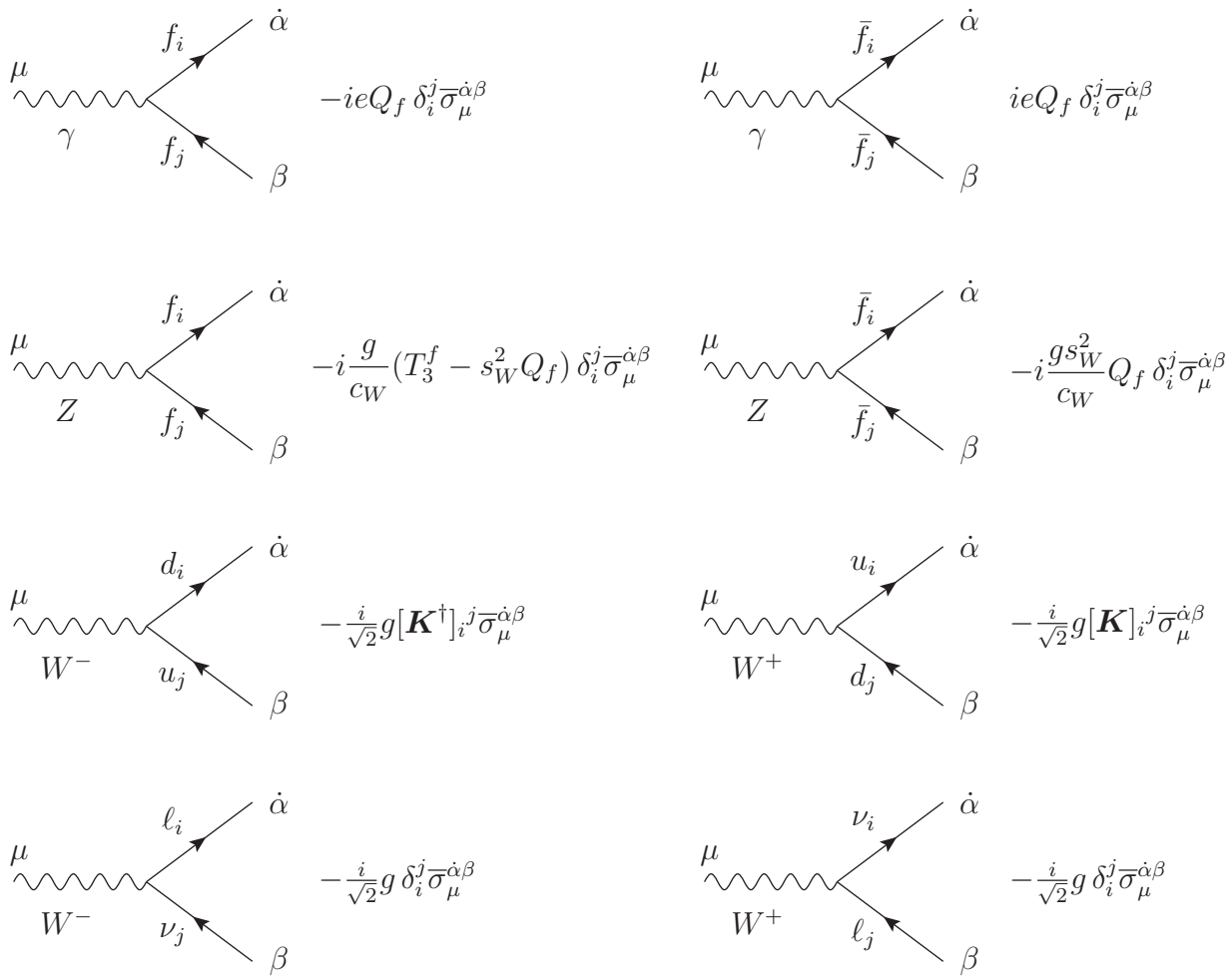


Figure J.1.2: Feynman rules for the two-component fermion interactions with electroweak gauge bosons in the Standard Model. The couplings of the fermions to γ and Z are flavor-diagonal. In all couplings, i and j label the fermion generations; an upper [lowered] flavor index in the corresponding Feynman rule is associated with a fermion line that points into [out from] the vertex. For the $W^\pm$ bosons, the charge indicated is flowing into the vertex. The electric charge is denoted by Q_f (in units of $e>0$), with $Q_e=-1$ for the electron. $T_3^f=1/2$ for $f=u$, ν, and $T_3^f=-1/2$ for $f=d$, ℓ. The CKM mixing matrix is denoted by $\boldsymbol{K}$, and $s_W\equiv\sin\theta_W$, $c_W\equiv\cos\theta_W$ and $e\equiv g\sin\theta_W$. For each rule, a corresponding one with lowered spinor indices is obtained by $\overline\sigma_\mu^{\dot\alpha\beta}\to-\sigma_{\mu\beta\dot\alpha}$.

We can now write out the couplings of the mass eigenstate quarks and leptons to the gauge bosons and Higgs bosons. Consider first the charged current interactions of the quarks and leptons. Using eq. (J.1.10), it follows that $\hat u^{\dagger i}\overline\sigma^\mu\hat d_i=\boldsymbol{K}_i{}^j u^{\dagger i}\overline\sigma^\mu d_j$, where

$$\boldsymbol{K}=L_u^\dagger L_d \tag{J.1.16}$$

is the unitary Cabibbo-Kobayashi-Maskawa (CKM) matrix [336].[170] Due to eq. (J.1.15), the corresponding leptonic CKM matrix is the unit matrix. Hence, the charged current interactions take the form

$$\mathscr{L}_{\rm int}=-\frac{g}{\sqrt2}\left[\boldsymbol{K}_i{}^j u^{\dagger i}\overline\sigma^\mu d_j W_\mu^+ + (\boldsymbol{K}^\dagger)_i{}^j d^{\dagger i}\overline\sigma^\mu u_j W_\mu^- + \nu^{\dagger i}\overline\sigma^\mu\ell_i W_\mu^+ + \ell^{\dagger i}\overline\sigma^\mu\nu_i W_\mu^-\right], \tag{J.1.17}$$

[170] The CKM matrix elements V_{ij} as defined in ref. [337] are related by, for example, $V_{tb}=\boldsymbol{K}_3{}^3$ and $V_{us}=\boldsymbol{K}_1{}^2$.

where $[\boldsymbol{K}^\dagger]_i{}^j \equiv [\boldsymbol{K}_j{}^i]^*$. Note that in the Standard Model, $\bar{u}$, $\bar{d}$ and $\bar{\ell}$ do not couple to the $W^\pm$.

To obtain the neutral current interactions, we insert eqs. (J.1.10)–(J.1.15) into eq. (J.1.2). All factors of the *unitary* matrices L_f and R_f ($f = u, d, \ell$) cancel out, and the resulting interactions are flavor-diagonal. That is, we may simply remove the hats from the quark and lepton fields that couple to the Z and photon fields in eq. (J.1.2). This is the well-known Glashow-Iliopoulos-Maiani (GIM) mechanism for the flavor-conserving neutral currents [338].[171]

The Feynman rules for the interactions of the quarks and leptons with the charged and neutral gauge bosons are exhibited in Fig. J.1.2. For each of the rules of Fig. J.1.2, we have chosen to employ $\overline{\sigma}_\mu^{\dot\alpha\beta}$. If the indices are lowered one should take $\overline{\sigma}_\mu^{\dot\alpha\beta} \to -\sigma_{\mu\beta\dot\alpha}$.

Finally, we exhibit the interactions of the quark and lepton mass eigenstates with the Higgs fields. The diagonalization of the fermion mass matrices is equivalent to the diagonalization of the Yukawa couplings [cf. eqs. (J.1.9) and (J.1.12)–(J.1.14)]. Thus, we define[172]

$$Y_{fi} = m_{fi}/v\,, \qquad f = u, d, \ell\,, \tag{J.1.18}$$

where i labels the fermion generation. It is convenient to rewrite eqs. (J.1.12)–(J.1.14) as follows:

$$(L_f)_k{}^j(\boldsymbol{Y}_f)^k{}_m(R_f)^m{}_i = Y_{fi}\delta_i^j\,, \qquad f = u, d, \ell\,, \tag{J.1.19}$$

with no sum over the repeated index i. Using the unitarity of L_f ($f = u$, d), eq. (J.1.19) is equivalent to the following convenient form:

$$(\boldsymbol{Y}_f R_f)^k{}_i = Y_{fi}(L_f^\dagger)_i{}^k\,. \tag{J.1.20}$$

Inserting eqs. (J.1.10), (J.1.15) and (J.1.19) into eq. (J.1.6), the resulting Higgs-fermion Lagrangian is flavor-diagonal:

$$\mathscr{L}_{\rm int} = -\frac{1}{\sqrt{2}} h_{\rm SM}\left[Y_{ui}u_i\bar{u}^i + Y_{di}d_i\bar{d}^i + Y_{\ell i}\ell_i\bar{\ell}^i\right] + \text{h.c.} \tag{J.1.21}$$

The corresponding Feynman rules for the Higgs-fermion interaction are shown in Fig. J.1.3.

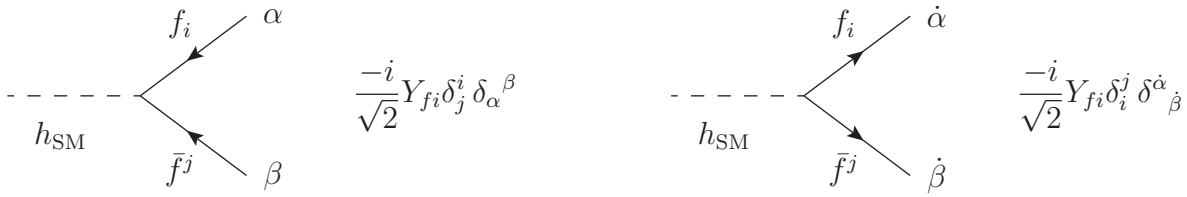


Figure J.1.3: Feynman rules for the Standard Model Higgs boson interactions with fermions, where $Y_{fi} \equiv m_{fi}/v$, and i, j label the generations.

In the case of more general covariant gauge-fixing (e.g., the 't Hooft-Feynman gauge or Landau gauge), the Goldstone bosons appear explicitly in internal lines of Feynman diagrams. The Feynman rules for G^0-fermion interactions are flavor-diagonal, whereas the corresponding rules for $G^\pm$ exhibit flavor-changing interactions that depend on the CKM matrix elements, as shown in Fig. J.1.4. In the derivation of the couplings of the Nambu-Goldstone bosons to the fermion mass eigenstates [cf. eqs. (J.1.6)–(J.1.8)], the following quantities appear:

$$(L_d)_k{}^j(\boldsymbol{Y}_u)^k{}_m(R_u)^m{}_i = Y_{ui}(L_d)_k{}^j(L_u^\dagger)_i{}^k = Y_{ui}(L_u^\dagger L_d)_i{}^j = [\boldsymbol{K}]_i{}^j Y_{ui}\,, \tag{J.1.22}$$

$$(L_u)_k{}^j(\boldsymbol{Y}_d)^k{}_m(R_d)^m{}_i = Y_{di}(L_u)_k{}^j(L_d^\dagger)_i{}^k = Y_{di}(L_d^\dagger L_u)_i{}^j = [\boldsymbol{K}^\dagger]_i{}^j Y_{di}\,, \tag{J.1.23}$$

[171] This also provides the justification for employing mass eigenstate quark fields in the QCD interaction Lagrangian in eq. (J.1.1).

[172] Boldfaced symbols are used for the non-diagonal Yukawa matrices, while non-boldfaced symbols are used for the diagonalized Yukawa couplings.

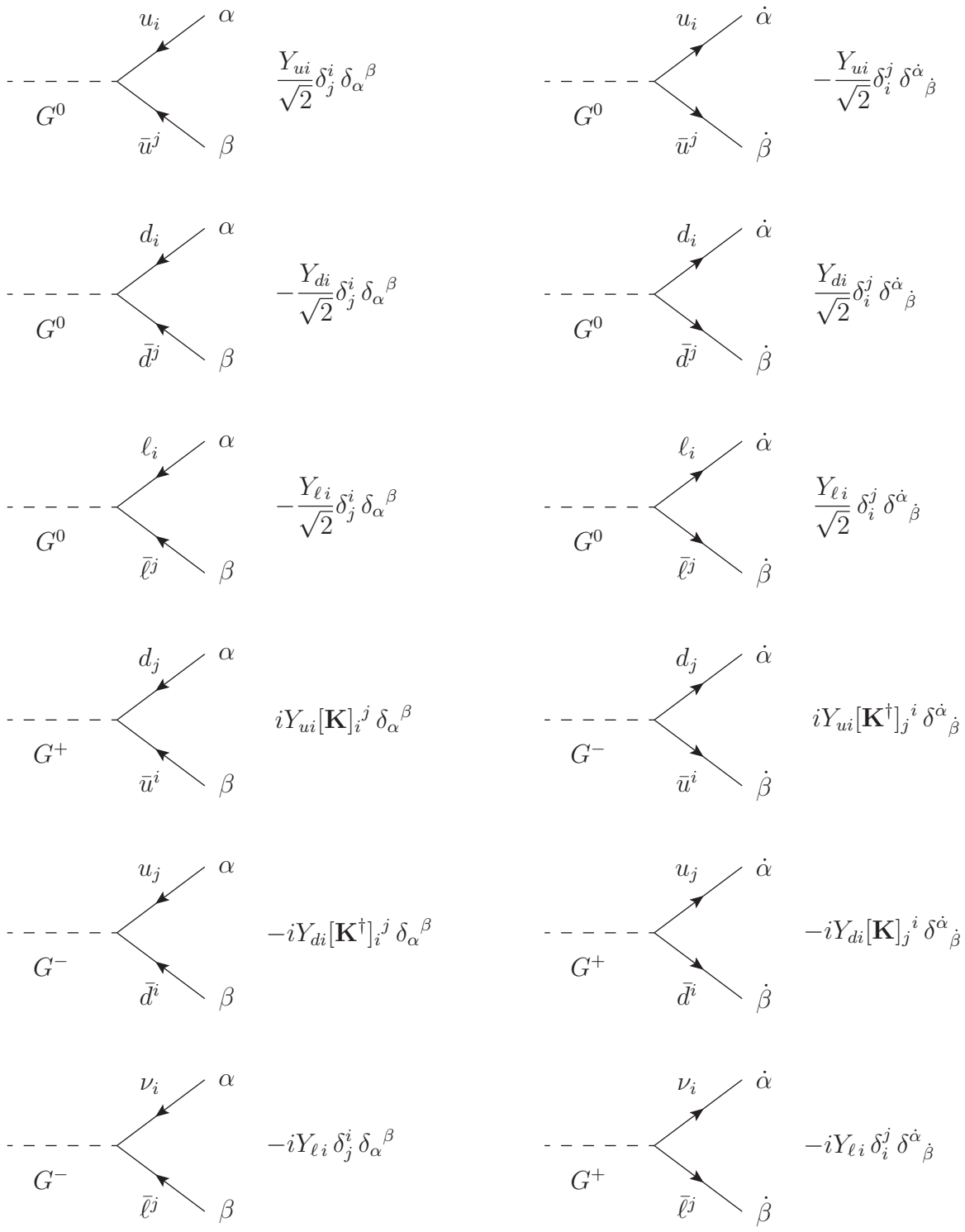


Figure J.1.4: Feynman rules for the Standard Model Nambu-Goldstone boson interactions with quarks and leptons, where $Y_{fi} \equiv m_{fi}/v$, and i, j label the generations.

with no sum over the repeated index i. The CKM matrix, $\boldsymbol{K}$, appears by virtue of eqs. (J.1.16) and (J.1.20). Hence, the interaction Lagrangian for the coupling of the Nambu-Goldstone bosons

to the fermion mass eigenstates is given by:

$$\mathscr{L}_{\rm int} = Y_{ui}[\boldsymbol{K}]_i{}^j d_j \bar{u}^i G^+ - Y_{di}[\boldsymbol{K}^\dagger]_i{}^j u_j \bar{d}^i G^- - Y_{\ell i}\nu_i \bar{\ell}^i G^- + \frac{i}{\sqrt{2}}\left[Y_{di} d_i \bar{d}^i - Y_{ui} u_i \bar{u}^i + Y_{\ell\, i}\ell_i \bar{\ell}^i\right] G^0 + \text{h.c.}, \tag{J.1.24}$$

which yields the diagrammatic Feynman rules shown in Fig. J.1.4.

29.7.13 The Standard Model and its seesaw extension 2

Incorporating massive neutrinos into the Standard Model

To accommodate massive neutrinos, we must slightly extend the Standard Model [339]. The simplest approach is to introduce an SU(2)×U(1) gauge invariant dimension-five operator [340],

$$\begin{aligned}\mathscr{L}_5 &= -\frac{\hat{\boldsymbol{F}}^{ij}}{2\Lambda}(\epsilon^{ab}\boldsymbol{\Phi}_a \hat{\boldsymbol{L}}_{bi})(\epsilon^{cd}\boldsymbol{\Phi}_c \hat{\boldsymbol{L}}_{dj}) + \text{h.c.} \\ &= -\frac{\hat{\boldsymbol{F}}^{ij}}{2\Lambda}(\Phi^0\hat{\nu}_i - \Phi^+\hat{\ell}_i)(\Phi^0\hat{\nu}_j - \Phi^+\hat{\ell}_j) + \text{h.c.}\,,\end{aligned} \tag{J.2.1}$$

where $\hat{\boldsymbol{F}}^{ij}$ are generalized Yukawa couplings, the hatted fields indicate two-component fermion interaction eigenstates (with spinor indices suppressed), and i, j label the three generations. After electroweak symmetry breaking, the neutral component of the doublet Higgs field acquires a vacuum expectation value, and a Majorana mass matrix for the neutrinos is generated.

The diagonalization of the charged lepton mass matrix is unmodified from the treatment given in Appendix J.1 , where the unhatted mass eigenstate charged lepton fields are given by eq. (J.1.11), and L_ℓ and R_ℓ satisfy eq. (J.1.14). However, the unhatted neutrino field introduced in eq. (J.1.15) is *not* a neutrino mass eigenstate field when the effect of the dimension-five Lagrangian, eq. (J.2.1), is taken into account. To avoid confusion, we replace the unhatted neutrino fields of eq. (J.1.15) with new neutrino fields $\breve{\nu}_j$. That is, we define

$$\hat{\nu}_i = (L_\ell)_i{}^j \breve{\nu}_j\,. \tag{J.2.2}$$

We then rewrite eq. (J.2.1) in terms of the charged lepton mass eigenstate field and the new neutrino field defined by eq. (J.2.2):

$$\mathscr{L}_5 = -\frac{\boldsymbol{F}^{ij}}{2\Lambda}(\Phi^0\breve{\nu}_i - \Phi^+\ell_i)(\Phi^0\breve{\nu}_j - \Phi^+\ell_j) + \text{h.c.}\,, \tag{J.2.3}$$

where $\boldsymbol{F} \equiv L_\ell^{\mathsf{T}}\hat{\boldsymbol{F}}L_\ell$. Setting $\Phi^0 = v$ and $\Phi^+ = \Phi^- = 0$, we identify the 3×3 complex symmetric effective light neutrino mass matrix, $\boldsymbol{M}_{\boldsymbol{\nu\ell}}$, by

$$-\mathscr{L}_{m_\nu} = \tfrac{1}{2}(\boldsymbol{M}_{\boldsymbol{\nu\ell}})^{ij}\breve{\nu}_i\breve{\nu}_j + \text{h.c.}\,, \tag{J.2.4}$$

where

$$\boldsymbol{M}_{\boldsymbol{\nu\ell}} = \frac{v^2}{\Lambda}\boldsymbol{F}\,. \tag{J.2.5}$$

Current bounds on light neutrino masses suggest that $v^2/\Lambda \lesssim 1$ eV, or $\Lambda \gtrsim 10^{13}$ GeV [337, 341].

The physical neutrino mass eigenstate fields can be identified by introducing the unitary Maki-Nakagawa-Sakata (MNS) matrix, $\boldsymbol{U}_{\rm MNS}$, such that [342],[173]

$$(\breve{\nu}_\ell)_i = (\boldsymbol{U}_{\rm MNS})_i{}^j(\nu_\ell)_j\,, \tag{J.2.6}$$

[173] In the literature, the MNS matrix is often defined such that $\boldsymbol{U}^*_{\rm MNS}$ (and *not* $\boldsymbol{U}_{\rm MNS}$) appears in eq. (J.2.6).

where the unhatted $(\nu_\ell)_j$ fields [$j = 1, 2, 3$] denote the physical (mass eigenstate) Majorana neutrino fields. $\boldsymbol{U}_{\rm MNS}$ is determined by the Takagi diagonalization of $\boldsymbol{M}_{\boldsymbol{\nu\ell}}$ [cf. Appendix D.2]:

$$\boldsymbol{U}_{\rm MNS}^{\mathsf{T}}\boldsymbol{M}_{\boldsymbol{\nu\ell}}\boldsymbol{U}_{\rm MNS} = {\rm diag}(m_{\nu_\ell 1}\,,\, m_{\nu_\ell 2}\,,\, m_{\nu_\ell 3})\,, \tag{J.2.7}$$

where the $m_{\nu_\ell j}$ are the (real non-negative) masses of the physical neutrinos.

The interaction Lagrangian of the neutrino mass eigenstates can now be determined. The charged current neutrino interactions are given by [cf. eq. (J.1.17)]:

$$\begin{aligned}\mathscr{L}_{\rm int} &= -\frac{g}{\sqrt{2}}\left[\breve{\nu}^{\dagger i}\overline{\sigma}^\mu\ell_i W_\mu^+ + \ell^{\dagger i}\overline{\sigma}^\mu\breve{\nu}_i W_\mu^-\right] \\ &= -\frac{g}{\sqrt{2}}\left[(\boldsymbol{U}_{\rm MNS}^\dagger)_j{}^i\nu_\ell^{\dagger j}\overline{\sigma}^\mu\ell_i W_\mu^+ + (\boldsymbol{U}_{\rm MNS})_i{}^j\ell^{\dagger i}\overline{\sigma}^\mu\nu_{\ell j}W_\mu^-\right]\,,\end{aligned} \tag{J.2.8}$$

where we have used eq. (J.2.6) to express the interaction Lagrangian in terms of the neutrino mass eigenstate fields. The neutral current neutrino interactions are flavor-diagonal (which follows from the unitarity of $U_{\rm MNS}$), and are thus equivalent to those of the Standard Model. Finally, the couplings of the neutrinos to the Higgs and Nambu-Goldstone fields arise from eq. (J.2.3) and from the term in eq. (J.1.6) proportional to $\boldsymbol{Y}_\ell$. Neglecting terms of $\mathcal{O}(m_\nu^2/v^2)$, one obtains:

$$\begin{aligned}\mathscr{L}_{\rm int} = \frac{1}{v}\sum_{i,j}\Big[&(m_{\nu_\ell})_j(\boldsymbol{U}_{\rm MNS}^\dagger)_j{}^i(\nu_\ell)_j\,\ell_i\,G^+ - (m_\ell)_i(\boldsymbol{U}_{\rm MNS})_i{}^j(\nu_\ell)_j\,\bar{\ell}^i\,G^- + {\rm h.c.}\Big] \\ -\frac{1}{\sqrt{2}v}\sum_j&(m_{\nu_\ell})_j\left[(\nu_\ell)_j(\nu_\ell)_j(h_{\rm SM}+iG^0) + {\rm h.c.}\right]\,.\end{aligned} \tag{J.2.9}$$

The Feynman rules for the interactions of the neutrino with the electroweak gauge bosons, the Higgs boson and the Nambu-Goldstone bosons are exhibited in Fig. J.2.1.

The dimension-five Lagrangian, eq. (J.2.1), is generated by new physics beyond the Standard Model at the scale Λ. A possible realization of eq. (J.2.1) is the seesaw mechanism, which was independently discovered on a number of occasions [4]. In the seesaw extension of the Standard Model [5], one introduces the SU(3)×SU(2)×U(1) gauge singlet two-component neutrino fields $\bar{\nu}^I$ ($I = 1, 2, ..., n$) and writes down the most general renormalizable couplings of the $\bar{\nu}^I$ to the Standard Model fields:

$$\mathscr{L}_{\rm seesaw} = \epsilon^{ab}(\hat{\boldsymbol{Y}}_\nu)^i{}_J\boldsymbol{\Phi}_a\hat{\boldsymbol{L}}_{bi}\hat{\bar{\nu}}^J - \tfrac{1}{2}\hat{\boldsymbol{M}}_{IJ}\hat{\bar{\nu}}^I\hat{\bar{\nu}}^J + {\rm h.c.}\,, \tag{J.2.10}$$

where the Yukawa coupling proportional to $\hat{\boldsymbol{Y}}_\nu$ is the leptonic analogue of the Higgs-quark Yukawa coupling proportional to $\hat{\boldsymbol{Y}}_u$ [cf. eq. (J.1.5)]. In eq. (J.2.10), we have distinguished the flavor labels of three generations of Standard Model neutrino and charged lepton fields (denoted by lower case Roman letters i, j, ...) and the flavor labels of singlet neutrino fields (denoted by upper case Roman letters I, J, ...). Note that $\hat{\boldsymbol{Y}}_\nu$ is a $3\times n$ matrix and $\hat{\boldsymbol{M}}$ is an $n\times n$ matrix, where n is the number of singlet neutrino flavors. In general, we shall not specify the value of n, which may differ from the number of Standard Model lepton generations.

If $\Lambda \equiv \|\hat{\boldsymbol{M}}\| \gg v$,[174] then a dimension-five operator of the form given by eq. (J.2.1) is generated in the effective theory at energy scales below Λ. In this limit, we may neglect the kinetic energy term of the gauge singlet neutrino fields. Using the Lagrange field equations, we may solve for $\hat{\bar{\nu}}^I$. Inserting the solution back into eq. (J.2.10) then yields eq. (J.2.1), with $\hat{\boldsymbol{F}}/\Lambda$ given by

$$\hat{\boldsymbol{F}}^{ij}/\Lambda = -(\hat{\boldsymbol{Y}}_\nu)^i{}_K(\hat{\boldsymbol{Y}}_\nu)^j{}_N(\hat{\boldsymbol{M}}^{-1})^{KN}\,. \tag{J.2.11}$$

[174]The Euclidean matrix norm is defined by $\|A\| \equiv \left[{\rm Tr}(A^\dagger A)\right]^{1/2} = \left[\sum_{i,j}|a_{ij}|^2\right]^{1/2}$, for a matrix A whose matrix elements are given by a_{ij}.

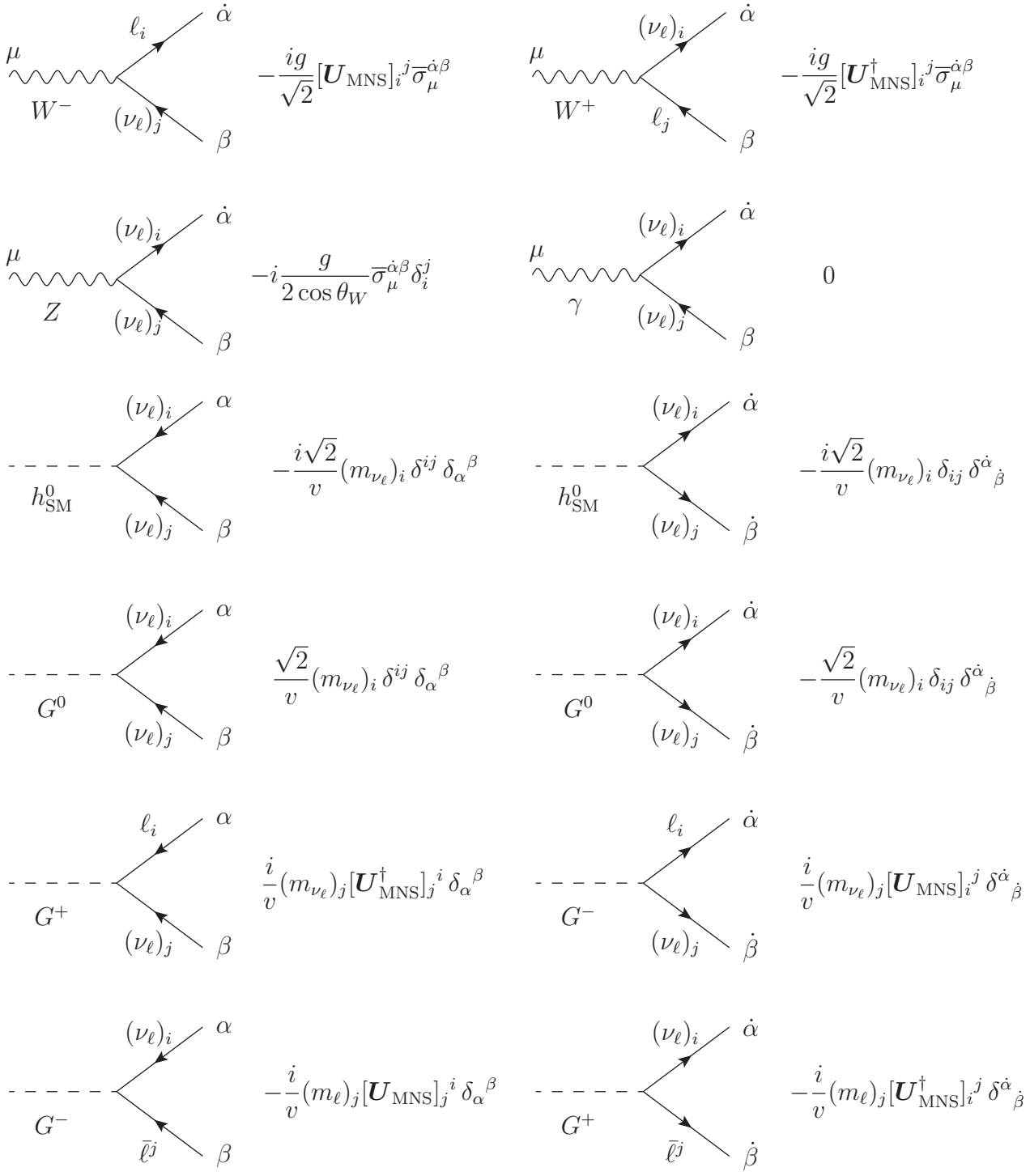


Figure J.2.1: Feynman rules for the interactions of the two-component light neutrino (ν_ℓ) with electroweak gauge bosons, the Standard Model Higgs boson and the Nambu-Goldstone bosons, where i, j label the generation. For the $W^\pm$ bosons and $G^\pm$ scalars, the charge indicated is flowing into the vertex. The MNS mixing matrix is denoted by $\boldsymbol{U}_{\rm MNS}$. For the rules involving $W^\pm$ and Z bosons, a corresponding one with lowered spinor indices is obtained by $\overline{\sigma}_\mu^{\dot\alpha\beta} \to -\sigma_{\mu\beta\dot\alpha}$. In the $h^0_{\rm SM}$ and G^0 interactions, a factor of 2 is included to account for the identical neutrinos.

Using the definition of the $\mathrm{SU}(2)_L$ doublet lepton field given in Table J.1, one can rewrite

eq. (J.2.10) more explicitly as:

$$\mathscr{L}_{\text{seesaw}} = -(\hat{\boldsymbol{Y}}_\nu)^i{}_J \left[\Phi^0 \hat{\nu}_i \hat{\bar{\nu}}^J - \Phi^+ \hat{\ell}_i \hat{\bar{\nu}}^J\right] - \tfrac{1}{2}\hat{\boldsymbol{M}}_{IJ}\hat{\bar{\nu}}^I \hat{\bar{\nu}}^J + \text{h.c.} \tag{J.2.12}$$

To analyze the physical consequences of the seesaw Lagrangian, we first express eq. (J.2.12) in terms of the unhatted mass eigenstate charged lepton fields [cf. eq. (J.1.11)], and the light neutrino fields $\breve{\nu}_i$ introduced in eq. (J.2.2). It is also convenient to introduce new gauge singlet neutrino fields $\breve{\bar{\nu}}^J$ by defining

$$\hat{\bar{\nu}}^I = N^I{}_J \breve{\bar{\nu}}^J\,, \tag{J.2.13}$$

where N is the unitary matrix that Takagi-diagonalizes the complex symmetric matrix $\hat{\boldsymbol{M}}$. That is,

$$\boldsymbol{M} \equiv N^{\mathsf{T}} \hat{\boldsymbol{M}} N = \text{diag}(M_1\,,\, M_2\,, \ldots\,, M_n)\,, \tag{J.2.14}$$

where the M_I are the singular values of $\hat{\boldsymbol{M}}$ (i.e., the non-negative square roots of the eigenvalues of $\hat{\boldsymbol{M}}^\dagger \hat{\boldsymbol{M}}$). In terms of the mass eigenstate charged lepton fields ℓ_i and the neutrino fields $\breve{\nu}_i$ and $\breve{\bar{\nu}}^I$, the seesaw Lagrangian [eq. (J.2.12)] is then given by:

$$\mathscr{L}_{\text{seesaw}} = -(\boldsymbol{Y}_\nu)^i{}_J \left[\Phi^0 \breve{\nu}_i \breve{\bar{\nu}}^J - \Phi^+ \ell_i \breve{\bar{\nu}}^J\right] - \tfrac{1}{2}\boldsymbol{M}_{IJ}\breve{\bar{\nu}}^I \breve{\bar{\nu}}^J + \text{h.c.}\,, \tag{J.2.15}$$

where

$$\boldsymbol{Y}_\nu \equiv L_\ell^{\mathsf{T}} \hat{\boldsymbol{Y}}_\nu N\,. \tag{J.2.16}$$

As above, in the limit of $\Lambda \equiv \|\hat{\boldsymbol{M}}\| = \|\boldsymbol{M}\| \gg v$, it is also possible to directly generate the effective dimension-five operator [eq. (J.2.3)] in terms of the mass eigenstate charged lepton fields and the new neutrino fields $\breve{\nu}_j$. We then identify the corresponding coefficient, $\boldsymbol{F}/\Lambda$, as

$$\boldsymbol{F}^{ij}/\Lambda = -(\boldsymbol{Y}_\nu)^i{}_K (\boldsymbol{Y}_\nu)^j{}_N (\boldsymbol{M}^{-1})^{KN}\,. \tag{J.2.17}$$

Recalling that $\boldsymbol{F} = L_\ell^{\mathsf{T}} \hat{\boldsymbol{F}} L_\ell$, one can check that eq. (J.2.17) indeed follows from eqs. (J.2.11), (J.2.14) and (J.2.16).

To identify the neutrino mass matrix, we set $\Phi^0 = v$ and $\Phi^+ = \Phi^- = 0$ in eq. (J.2.15):

$$-\mathscr{L}_{m_\nu} = \tfrac{1}{2}\begin{pmatrix}\breve{\nu}_i & \breve{\bar{\nu}}^J\end{pmatrix} \mathcal{M}_\nu \begin{pmatrix}\breve{\nu}_k \\ \breve{\bar{\nu}}^M\end{pmatrix} + \text{h.c.} \tag{J.2.18}$$

The neutrino mass matrix $\mathcal{M}_\nu$ is a $(3+n)\times(3+n)$ complex symmetric matrix given in block form by:

$$\mathcal{M}_\nu \equiv \begin{pmatrix} \mathbb{O} & \boldsymbol{M_D} \\ \boldsymbol{M_D}^{\mathsf{T}} & \boldsymbol{M}\end{pmatrix}\,, \tag{J.2.19}$$

where $\mathbb{O}$ is the 3×3 zero matrix, $\boldsymbol{M}$ is the diagonal matrix defined in eq. (J.2.14) and $\boldsymbol{M_D}$ is a $3\times n$ complex matrix (called the Dirac neutrino mass matrix),

$$(\boldsymbol{M_D})^i{}_J \equiv v(\boldsymbol{Y}_\nu)^i{}_J\,. \tag{J.2.20}$$

Note that if $n=3$ and $\boldsymbol{M}=\mathbb{O}$, then $\boldsymbol{M_D}$ is a 3×3 matrix that is simply the leptonic analogue of the up-type quark mass matrix $\boldsymbol{M}_u$. In this case, we would perform a singular value decomposition of $\boldsymbol{M_D}$ and identify the unhatted neutrino mass eigenstate fields, which can be assembled into three generations of four-component Dirac neutrinos,

$$N_i = \begin{pmatrix}\nu_i \\ \bar{\nu}^{\dagger i}\end{pmatrix}\,, \qquad i=1,2,3\,. \tag{J.2.21}$$

In the seesaw model (with n not specified), we assume that $\|\boldsymbol{M}\| \gg \|\boldsymbol{M_D}\|$. In this case, the neutrino mass matrix can be perturbatively Takagi-block-diagonalized as follows [283, 343, 301]. Introduce the $(3+n)\times(3+n)$ (approximate) unitary matrix:

$$\mathcal{U} = \begin{pmatrix} \mathbb{1}_{3\times 3} - \frac{1}{2}\boldsymbol{M_D^*}\boldsymbol{M}^{-2}\boldsymbol{M_D^{\mathsf{T}}} & \boldsymbol{M_D^*}\boldsymbol{M}^{-1} \\ -\boldsymbol{M}^{-1}\boldsymbol{M_D^{\mathsf{T}}} & \mathbb{1}_{n\times n} - \frac{1}{2}\boldsymbol{M}^{-1}\boldsymbol{M_D^{\mathsf{T}}}\boldsymbol{M_D^*}\boldsymbol{M}^{-1} \end{pmatrix}, \tag{J.2.22}$$

where $\mathbb{1}$ is the identity matrix (whose dimension is explicitly specified above). We define transformed [light (ℓ) and heavy (h)] neutrino states $(\breve{\nu}_\ell)_i$ and $(\breve{\breve{\nu}}_h)^j$ by:

$$\begin{pmatrix} \breve{\nu}_i \\ \breve{\breve{\nu}}^J \end{pmatrix} = \mathcal{U}\begin{pmatrix} (\breve{\nu}_\ell)_k \\ (\breve{\breve{\nu}}_h)^M \end{pmatrix}. \tag{J.2.23}$$

By straightforward matrix multiplication, one can verify that to second order accuracy in perturbation theory,

$$\mathcal{U}^{\mathsf{T}}\mathcal{M}_\nu\mathcal{U} \simeq \begin{pmatrix} -\boldsymbol{M_D}\boldsymbol{M}^{-1}\boldsymbol{M_D^{\mathsf{T}}} & \mathbb{O} \\ \mathbb{O}^{\mathsf{T}} & \boldsymbol{M} + \frac{1}{2}(\boldsymbol{M}^{-1}\boldsymbol{M_D^\dagger}\boldsymbol{M_D} + \boldsymbol{M_D^{\mathsf{T}}}\boldsymbol{M_D^*}\boldsymbol{M}^{-1}) \end{pmatrix}, \tag{J.2.24}$$

where $\mathbb{O}$ is the $3\times n$ zero matrix.

We now can identify an effective 3×3 complex symmetric mass matrix $\boldsymbol{M_{\nu_\ell}}$ for the three light neutrinos as the upper left-hand block of eq. (J.2.24),

$$\boldsymbol{M_{\nu_\ell}} \simeq -\boldsymbol{M_D}\boldsymbol{M}^{-1}\boldsymbol{M_D^{\mathsf{T}}}, \tag{J.2.25}$$

where corrections of $\mathcal{O}(v^4/\Lambda^3)$ have been neglected. Using eqs. (J.2.17) and (J.2.20), we see that the light neutrino mass matrix obtained in eq. (J.2.5) has been correctly reproduced to leading order in v^2/Λ^2.

The physical light neutrino mass eigenstate fields and their masses are identified by eqs. (J.2.6) and (J.2.7). At energy scales below the heavy neutrino mass scale, $\Lambda \equiv \|\boldsymbol{M}\|$, and we can set $\breve{\breve{\nu}}_h = 0$. Neglecting corrections of $\mathcal{O}(v^2/\Lambda^2)$, eqs. (J.2.20)–(J.2.25) imply that[175]

$$\breve{\nu}_i \simeq (\boldsymbol{U}_{\rm MNS})_i{}^j(\nu_\ell)_j\,, \tag{J.2.26}$$

$$(\boldsymbol{Y}_\nu)^i{}_J\breve{\breve{\nu}}^J \simeq \frac{1}{v}(\boldsymbol{M_{\nu_\ell}}\boldsymbol{U}_{\rm MNS})^{ik}(\nu_\ell)_k = \frac{1}{v}\sum_k (\boldsymbol{U}^\dagger_{\rm MNS})_k{}^i\,(m_{\nu_\ell})_k(\nu_\ell)_k\,, \tag{J.2.27}$$

where in the last step above we have used eq. (J.2.7) and $(\boldsymbol{U}^\dagger_{\rm MNS})_j{}^i \equiv [(\boldsymbol{U}_{\rm MNS})_i{}^j]^*$. Using eqs. (J.2.26) and (J.2.27) to express the seesaw Lagrangian in terms of the light neutrino mass eigenstate fields, one can verify that the resulting interactions of the light neutrinos (and charged leptons) to gauge bosons, the Higgs boson and the Nambu-Goldstone bosons reproduce the results of eqs. (J.2.8) and (J.2.9) at leading order in v^2/Λ^2.

For completeness, we examine the effective $n\times n$ complex symmetric mass matrix of the heavy neutrino states, $\boldsymbol{M_{\nu_h}}$, which is identified as the lower right-hand block in eq. (J.2.24),

$$\boldsymbol{M_{\nu_h}} \simeq \boldsymbol{M} + \frac{1}{2}(\boldsymbol{M}^{-1}\boldsymbol{M_D^\dagger}\boldsymbol{M_D} + \boldsymbol{M_D^{\mathsf{T}}}\boldsymbol{M_D^*}\boldsymbol{M}^{-1})\,. \tag{J.2.28}$$

[175] Strictly speaking, eq. (J.2.27) should be written as:

$$(\boldsymbol{Y}_\nu)^i{}_J\breve{\breve{\nu}}^J \simeq \frac{1}{v}\sum_{k,n}(\boldsymbol{U}^\dagger_{\rm MNS})_n{}^i\,\delta^{nk}(m_{\nu_\ell})_k(\nu_\ell)_k\,,$$

to maintain covariance in the flavor indices.

Although $\boldsymbol{M}$ is diagonal by definition [cf. eq. (J.2.14)], the right-hand side of eq. (J.2.28) is no longer diagonal due to the second order perturbative correction. However, we do not have to perform another Takagi diagonalization, since the off-diagonal elements of the lower right-hand block only affect the physical (diagonal) masses at higher order in perturbation theory. Thus, we identify the physical heavy neutrino mass eigenstates to leading order by the unhatted fields,

$$\bar{\nu}_h^J \simeq \breve{\bar{\nu}}_h^J \,, \tag{J.2.29}$$

with masses

$$m_{\nu_{hJ}} \simeq M_J \left(1 + \frac{1}{M_J^2} \sum_i |(\boldsymbol{M_D})^i{}_J|^2 \right) , \tag{J.2.30}$$

where the M_J are the diagonal elements of $\boldsymbol{M}$ (and no sum over the repeated index J is implied). That is, the masses of the heavy neutrinos are simply given by $m_{\nu_{hJ}} \simeq M_J$, up to corrections that are of the same order as the light neutrino masses.

The interactions of the heavy neutrinos can be likewise obtained. The only unsuppressed interactions are heavy neutrino couplings to the Higgs boson and Nambu-Goldstone bosons that are proportional to the Dirac neutrino mass matrix,

$$\mathscr{L}_{\rm int} = -\frac{1}{\sqrt{2}v}(\boldsymbol{U}_{\rm MNS}^{\mathsf{T}}\boldsymbol{M_D})^k{}_J\bar{\nu}_h^J(\nu_\ell)_k(h^0_{\rm SM} + iG^0) + \frac{1}{v}(\boldsymbol{M_D})^i{}_J\ell_i\bar{\nu}_h^J G^+ + \text{h.c.} \tag{J.2.31}$$

All other couplings of the heavy neutrinos to the $W^\pm$ and Z bosons (and additional contributions to the couplings of the heavy neutrinos to the Higgs boson and Nambu-Goldstone bosons) are suppressed by (at least) a factor of $\mathcal{O}(v/\Lambda)$.

29.7.14 MSSM fermion interaction vertices

In this section, we provide the Feynman rules for the MSSM interaction vertices. To complete the tabulation of all MSSM Feynman rules, one requires the rules for the purely bosonic interactions of the MSSM. These can be found in refs. [344, 345].

Higgs-fermion interaction vertices in the MSSM

The MSSM Higgs sector is a two Higgs doublet model containing eight real scalar degrees of freedom: one complex $Y = -\frac{1}{2}$ doublet, $\boldsymbol{H_d} = (H_d^0\,,\, H_d^-)$ and one complex $Y = +\frac{1}{2}$ doublet, $\boldsymbol{H_u} = (H_u^+\,,\, H_u^0)$. The notation reflects the form of the MSSM Higgs sector coupling to fermions:

$$\mathscr{L}_{\rm Y} = \epsilon^{ab}\left[(\boldsymbol{Y}_u)^i{}_j(\boldsymbol{H_u})_a\hat{\boldsymbol{Q}}_{bi}\hat{\bar{u}}^j - (\boldsymbol{Y}_d)^i{}_j(\boldsymbol{H_d})_a\hat{\boldsymbol{Q}}_{bi}\hat{\bar{d}}^j - (\boldsymbol{Y}_\ell)^i{}_j(\boldsymbol{H_d})_a\hat{\boldsymbol{L}}_{bi}\hat{\bar{\ell}}^j\right] + \text{h.c.}\,, \tag{K.1.1}$$

where the hatted fields are interaction eigenstate quark and lepton fields (with generation labels i and j), a and b are $\mathrm{SU}(2)_L$ indices and the invariant $\mathrm{SU}(2)_L$ tensor ϵ^{ab} is defined below eq. (J.1.5). That is, the neutral Higgs fields H_d^0 [H_u^0] couple exclusively to down-type [up-type] fermion pairs, respectively. In the supersymmetric model, both hypercharge $Y = -\frac{1}{2}$ and $Y = +\frac{1}{2}$ complex Higgs doublets are required in order that the theory (which now contains the corresponding higgsino superpartners) remain anomaly free. The supersymmetric structure of the theory forbids the coupling of $\boldsymbol{H_u^\dagger}$ to $\hat{\bar{d}}^j$ and $\hat{\bar{\ell}}^j$ or the coupling of $\boldsymbol{H_d^\dagger}$ to $\hat{\bar{u}}^j$, as such couplings would not be holomorphic. Consequently, (at least) two Higgs doublets are required in the MSSM to generate mass for both "up"-type and "down"-type quarks and charged leptons [346, 347, 173].

To find the couplings of the Higgs fields, we expand them around the neutral Higgs field vacuum expectation values $v_d \equiv \langle H_d^0\rangle$ and $v_u \equiv \langle H_u^0\rangle$. Depending on the application, these may be chosen to be the minimum of the tree-level scalar potential, or of the full loop-corrected effective potential,

or just left arbitrary. It is always possible to choose the phases of the Higgs fields such that v_u and v_d are real and positive. We then define

$$\beta \equiv \tan^{-1}\left(\frac{v_u}{v_d}\right), \qquad 0 \leq \beta \leq \frac{\pi}{2}. \tag{K.1.2}$$

The one potentially complex squared-mass parameter that appears in the tree-level MSSM Higgs scalar potential is necessarily real in the convention where the vacuum expectation values of the neutral Higgs fields are real and positive.[176] Consequently, the tree-level MSSM Higgs sector conserves CP, which implies that the neutral Higgs mass eigenstates possess definite CP quantum numbers.[177] Spontaneous electroweak symmetry breaking results in three Goldstone bosons $G^\pm$, G^0 (the neutral Goldstone boson is a CP-odd scalar field), which are absorbed and become the longitudinal components of the $W^\pm$ and Z. The remaining five physical Higgs particles consist of a charged Higgs pair $H^\pm$, one CP-odd scalar A^0, and two CP-even scalars h^0 and H^0.

It is convenient to define $H_u^- \equiv (H_u^+)^\dagger$ and $H_d^+ \equiv (H_d^-)^\dagger$. One can then parameterize the mixing angles between Higgs gauge eigenstates and mass eigenstates by writing:

$$H_u^0 = v_u + \frac{1}{\sqrt{2}}\sum_{\phi^0} k_{u\phi^0}\phi^0, \qquad H_u^\pm = \sum_{\phi^\pm} k_{u\phi^\pm}\phi^\pm, \tag{K.1.3}$$

$$H_d^0 = v_d + \frac{1}{\sqrt{2}}\sum_{\phi^0} k_{d\phi^0}\phi^0, \qquad H_d^\pm = \sum_{\phi^\pm} k_{d\phi^\pm}\phi^\pm. \tag{K.1.4}$$

For $\phi^\pm = (H^\pm,\, G^\pm)$,[178]

$$k_{u\phi^\pm} = (\cos\beta_\pm,\ \sin\beta_\pm), \tag{K.1.5}$$

$$k_{d\phi^\pm} = (\sin\beta_\pm,\ -\cos\beta_\pm), \tag{K.1.6}$$

and for $\phi^0 = (h^0,\, H^0,\, A^0,\, G^0)$,

$$k_{u\phi^0} = (\cos\alpha,\ \sin\alpha,\ i\cos\beta_0,\ i\sin\beta_0), \tag{K.1.7}$$

$$k_{d\phi^0} = (-\sin\alpha,\ \cos\alpha,\ i\sin\beta_0,\ -i\cos\beta_0), \tag{K.1.8}$$

where the mixing angle α parameterizes the orthogonal matrix that diagonalizes the 2×2 CP-even Higgs squared-mass matrix $\mathcal{M}_0^2$ [defined in eq. (K.1.11) below].

In eqs. (K.1.3) and (K.1.4), the normalization of the vacuum expectation values is

$$v_d^2 + v_u^2 = 2m_W^2/g^2 \simeq (174\ \text{GeV})^2, \tag{K.1.9}$$

if one chooses v_u, v_d to be near the true minimum of the Higgs effective potential. Note that in the special case that v_u and v_d are at the minimum of the *tree-level* potential, the mixing angles $\beta_\pm$ in the charged Higgs sector and β_0 in the CP-odd neutral Higgs sectors coincide such that $\beta_\pm = \beta_0 = \beta$, where β is defined in eq. (K.1.2). However, if one expands around a more general choice of v_u, v_d, including for example the minimum of the full effective potential, then the tree-level mixing angles β_0 and $\beta_\pm$ are distinct from each other and from β. (Depending on the choice of renormalization scale for a particular calculation, the tree-level potential in the MSSM may have a very different minimum from the true minimum of the full effective potential, or may not have

[176] The coefficients of the quartic terms of the tree-level MSSM Higgs potential are related to the electroweak gauge couplings and are manifestly real, independently of the convention for the phases of the Higgs fields.

[177] When one-loop corrections are taken into account, new MSSM phases can enter in the loops that cannot be removed. In this case, the physical neutral Higgs states can be mixtures of CP-even and CP-odd scalar states [348].

[178] Note that $\phi^- \equiv (\phi^+)^\dagger$. Since the $k_{f\phi^\pm}$ (for $f = u, d$) are real quantities, we adopt the notation in which $k_{f\phi^+} = k_{f\phi^-} \equiv k_{\phi^\pm}$ and $\beta_+ = \beta_- \equiv \beta_\pm$.

a proper minimum at all.) Therefore, we do not assume anything specific about v_u and v_d except that they are real and positive by convention.

All MSSM Higgs boson masses and the mixing angle α are determined at tree level by two Higgs sector parameters, usually taken to be the ratio of the tree-level vacuum expectation values, $\tan\beta = v_u/v_d$, and the mass of the CP-odd Higgs scalar, m_A [347, 173]. The tree-level value of the squared mass of the charged Higgs boson is given by

$$m^2_{H^\pm} = m^2_A + m^2_W \,. \tag{K.1.10}$$

The CP-even Higgs bosons h^0 and H^0 are eigenstates of the tree-level squared-mass matrix,

$$\mathcal{M}^2_0 = \begin{pmatrix} m^2_A \sin^2\beta + m^2_Z \cos^2\beta & -(m^2_A + m^2_Z)\sin\beta\cos\beta \\ -(m^2_A + m^2_Z)\sin\beta\cos\beta & m^2_A\cos^2\beta + m^2_Z\sin^2\beta \end{pmatrix} . \tag{K.1.11}$$

The eigenvalues of $\mathcal{M}^2_0$ are the tree-level squared masses of the two CP-even Higgs scalars,

$$m^2_{H,h} = \tfrac{1}{2}\left(m^2_A + m^2_Z \pm \sqrt{(m^2_A + m^2_Z)^2 - 4m^2_Z m^2_A \cos^2 2\beta}\,\right) , \tag{K.1.12}$$

with $m_h \leq m_H$. The angle α of the orthogonal matrix that diagonalizes $\mathcal{M}^2_0$ is given by [349]:

$$\sin 2\alpha = -\sin 2\beta \left(\frac{m^2_A + m^2_Z}{m^2_H - m^2_h}\right) , \qquad \cos 2\alpha = -\cos 2\beta \left(\frac{m^2_A - m^2_Z}{m^2_H - m^2_h}\right) . \tag{K.1.13}$$

Since $\sin 2\alpha \leq 0$, the tree-level value of α is restricted to lie in the range $-\pi/2 \leq \alpha \leq 0$.

Radiative corrections can have a significant impact on the tree-level Higgs masses and mixing angle α [348, 350]. For example, the tree-level bound $m_h \leq m_Z|\cos 2\beta| \leq m_Z$ [which follows from eq. (K.1.12)] is significantly modified by an incomplete cancellation of top quark and top squark loop corrections. Including the latter implies that $m_h \lesssim 135$ GeV [351], which (in contrast to the tree-level prediction) is not experimentally excluded.

The Higgs-fermion Yukawa couplings in the gauge-interaction basis are given by eq. (K.1.1). Explicitly,

$$\begin{aligned} -\mathscr{L}_{\rm Y} = (\boldsymbol{Y}_u)^i{}_j \left[\hat{u}_i \hat{\bar{u}}^j H^0_u - \hat{d}_i \hat{\bar{u}}^j H^+_u\right] + (\boldsymbol{Y}_d)^i{}_j \left[\hat{d}_i \hat{\bar{d}}^j H^0_d - \hat{u}_i \hat{\bar{d}}^j H^-_d\right] \\ + (\boldsymbol{Y}_\ell)^i{}_j \left[\hat{\ell}_i \hat{\bar{\ell}}^j H^0_d - \hat{\nu}_i \hat{\bar{\ell}}^j H^-_d\right] + \text{h.c.} \end{aligned} \tag{K.1.14}$$

We use eqs. (K.1.3) and (K.1.4) to express the interaction-eigenstate Higgs fields in terms of the physical Higgs fields and Goldstone fields. We can identify the quark and lepton mass matrices simply by setting $H^0_u = v_u$, $H^0_d = v_d$ and $H^+_u = H^-_d = 0$ in eq. (K.1.14) ,

$$(\boldsymbol{M}_u)^i{}_j = v_u(\boldsymbol{Y}_u)^i{}_j \,, \qquad (\boldsymbol{M}_d)^i{}_j = v_d(\boldsymbol{Y}_d)^i{}_j \,, \qquad (\boldsymbol{M}_\ell)^i{}_j = v_d(\boldsymbol{Y}_\ell)^i{}_j \,. \tag{K.1.15}$$

We then use eqs. (J.1.10) and (J.1.11) to express the interaction-eigenstate quark and lepton fields in terms of the corresponding mass eigenstate fields. Eqs. (J.1.12) and (J.1.14) ensure that the fermion mass matrices are diagonal (with real non-negative elements) in the fermion mass eigenstate basis. In this basis, the resulting neutral Higgs-fermion interactions are diagonal. Here, the diagonalized Higgs-fermion Yukawa coupling matrices appear:

$$\text{diag}(Y_{u1}, Y_{u2}, Y_{u3}) \equiv \text{diag}(Y_u, Y_c, Y_t) = L^{\mathsf{T}}_u \boldsymbol{Y}_u R_u \,, \tag{K.1.16}$$

$$\text{diag}(Y_{d1}, Y_{d2}, Y_{d3}) \equiv \text{diag}(Y_d, Y_s, Y_b) = L^{\mathsf{T}}_d \boldsymbol{Y}_d R_d \,, \tag{K.1.17}$$

$$\text{diag}(Y_{\ell 1}, Y_{\ell 2}, Y_{\ell 3}) \equiv \text{diag}(Y_e, Y_\mu, Y_\tau) = L^{\mathsf{T}}_\ell \boldsymbol{Y}_\ell R_\ell \,. \tag{K.1.18}$$

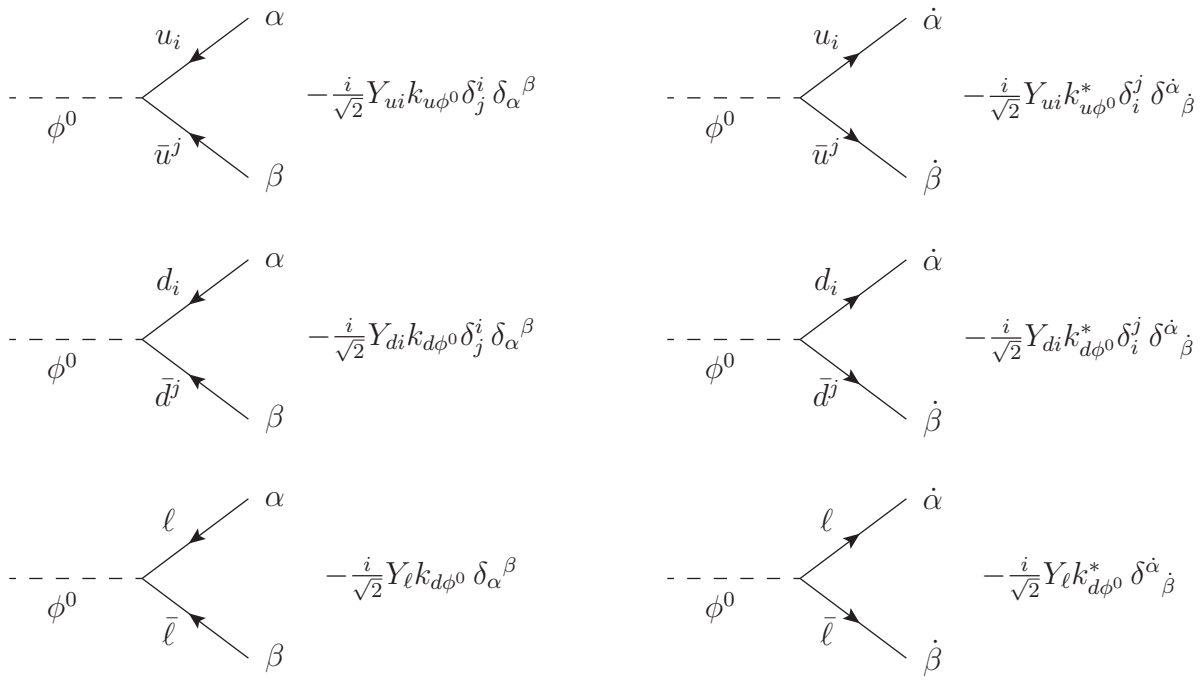


Figure K.1.1: Feynman rules for the interactions of neutral Higgs bosons $\phi^0 = (h^0, H^0, A^0, G^0)$ with fermion-antifermion pairs in the MSSM. The repeated index i is not summed.

The diagonalized Yukawa couplings are related to the corresponding fermion masses by

$$Y_{ui} = m_{u_i}/v_u\,, \qquad Y_{di} = m_{d_i}/v_d\,, \qquad Y_{\ell i} = m_{\ell_i}/v_d\,. \tag{K.1.19}$$

We have used the same symbol for the Yukawa couplings in the MSSM as we did for the Standard Model Yukawa couplings in Appendix J.1 . However, it is important to note that the MSSM Yukawa couplings are normalized differently because of the presence of two neutral Higgs field vacuum expectation values. Using a superscript SM to denote the Standard Model Yukawa couplings of Appendix J.1 , the MSSM Yukawa couplings defined here are related by:

$$Y_{ui} = Y^{\rm SM}_{ui}/\sin\beta\,, \qquad Y_{di} = Y^{\rm SM}_{di}/\cos\beta\,, \qquad Y_{\ell i} = Y^{\rm SM}_{\ell i}/\cos\beta\,. \tag{K.1.20}$$

The interactions of the neutral Higgs and Goldstone scalars $\phi^0 = (h^0, H^0, A^0, G^0)$ with Standard Model fermions are given in Fig. K.1.1. Note that the rules involving undotted spinor indices are proportional to either couplings $k_{d\phi^0}$ and $k_{u\phi^0}$, whereas the rules involving dotted spinor indices are proportional to the corresponding complex conjugated couplings. For the CP-even scalars, h^0 and H^0, the corresponding couplings are real. Hence, starting with the rule for the coupling of the CP-even neutral scalars to fermions with undotted indices, one obtains the corresponding rule for the coupling to fermions with dotted indices (with the direction of the arrows reversed) by taking $\delta_\alpha{}^\beta \to \delta^{\dot\alpha}{}_{\dot\beta}$. In contrast, for the CP-odd scalars, A^0 and G^0, the corresponding couplings $k_{d\phi^0}$ and $k_{u\phi^0}$ are purely imaginary. Therefore, starting with the rule for the coupling of the CP-odd neutral scalars to fermions with undotted indices, one obtains the corresponding rule for the coupling to fermions with dotted indices (with the direction of the arrows reversed) by taking $\delta_\alpha{}^\beta \to -\delta^{\dot\alpha}{}_{\dot\beta}$. The latter minus sign is a signal that A^0 and G^0 are CP-odd scalars. In particular, due to the fact that the Feynman rules for A^0 and G^0 arise from a term in $\mathscr{L}_{\rm int}$ proportional to i Im H^0, the latter i flips sign when the rule is conjugated resulting in the extra minus sign noted above. As an additional consequence, since the Feynman rules are obtained from $i\mathscr{L}_{\rm int}$, the overall A^0 and G^0 rules are real.

The couplings of the charged Higgs and Goldstone bosons to quark-antiquark pairs are not flavor-diagonal and involve the CKM matrix $\boldsymbol{K}$. Starting with eq. (K.1.14), and changing to the

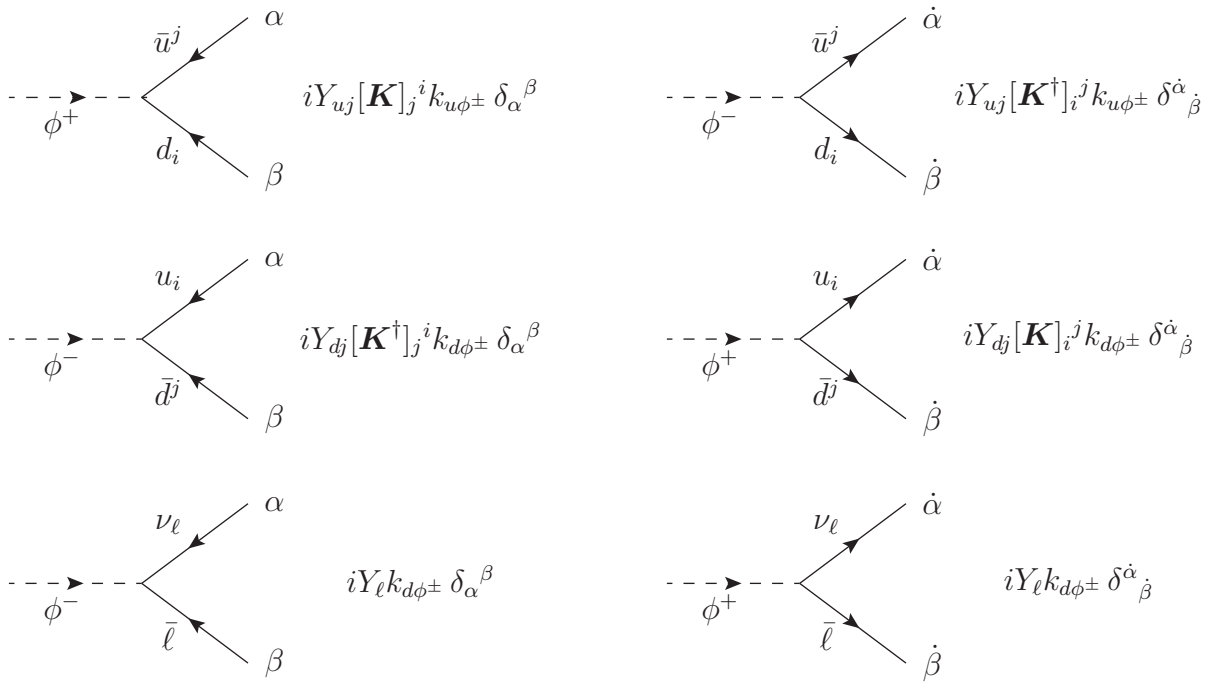


Figure K.1.2: Feynman rules for the interactions of charged Higgs bosons $\phi^\pm = (H^\pm, G^\pm)$ with fermion-antifermion pairs in the MSSM. The repeated index j is not summed.

mass eigenstate basis as before, we make use of eqs. (J.1.22) and (J.1.23) to obtain

$$\begin{aligned}\mathscr{L}_{\rm int} = {} & Y_{ui}[\boldsymbol{K}]_i{}^j d_j \bar{u}^i H^+ \cos\beta_\pm + Y_{di}[\boldsymbol{K}^\dagger]_i{}^j u_j \bar{d}^i H^- \sin\beta_\pm + Y_{\ell i}\nu_i \bar{\ell}^i H^- \sin\beta_\pm \\ & + Y_{ui}[\boldsymbol{K}]_i{}^j d_j \bar{u}^i G^+ \sin\beta_\pm - Y_{di}[\boldsymbol{K}^\dagger]_i{}^j u_j \bar{d}^i G^- \cos\beta_\pm - Y_{\ell i}\nu_i \bar{\ell}^i G^- \cos\beta_\pm + \text{h.c.}\end{aligned} \tag{K.1.21}$$

The resulting charged scalar Feynman rules of the MSSM are given in Fig. K.1.2. Note that when eq. (K.1.20) is taken into account, the fermion couplings to the neutral and charged Goldstone bosons are equivalent to those of the Standard Model [cf. eq. (J.1.24)] if we choose $\beta_0 = \beta_\pm = \beta$.

Gauge interaction vertices for neutralinos and charginos

Following eqs. (C83) and (C88) of [7], we define:

$$O^L_{ij} = -\tfrac{1}{\sqrt{2}} N_{i4} V^*_{j2} + N_{i2} V^*_{j1}\,, \tag{K.2.1}$$

$$O^R_{ij} = \tfrac{1}{\sqrt{2}} N^*_{i3} U_{j2} + N^*_{i2} U_{j1}\,, \tag{K.2.2}$$

$$O'^L_{ij} = -V_{i1} V^*_{j1} - \tfrac{1}{2} V_{i2} V^*_{j2} + \delta_{ij} s^2_W\,, \tag{K.2.3}$$

$$O'^R_{ij} = -U^*_{i1} U_{j1} - \tfrac{1}{2} U^*_{i2} U_{j2} + \delta_{ij} s^2_W\,, \tag{K.2.4}$$

$$O''^L_{ij} = -O''^R_{ji} = \tfrac{1}{2}(N_{i4} N^*_{j4} - N_{i3} N^*_{j3})\,, \tag{K.2.5}$$

where $s_W \equiv \sin\theta_W$. Here U and V are the unitary matrices that diagonalize the chargino mass matrix via the singular value decomposition:

$$U^* M_{\psi^\pm} V^{-1} = \text{diag}(m_{\widetilde{C}_1}, m_{\widetilde{C}_2})\,, \tag{K.2.6}$$

with

$$M_{\psi^\pm} = \begin{pmatrix} M_2 & g v_u \\ g v_d & \mu \end{pmatrix}\,. \tag{K.2.7}$$

Similarly, N is a unitary matrix that Takagi-diagonalizes the neutralino mass matrix,

$$N^* M_{\psi^0} N^{-1} = \mathrm{diag}(m_{\widetilde{N}_1}, m_{\widetilde{N}_2}, m_{\widetilde{N}_3}, m_{\widetilde{N}_4})\,, \tag{K.2.8}$$

with

$$M_{\psi^0} = \begin{pmatrix} M_1 & 0 & -g'v_d/\sqrt{2} & g'v_u/\sqrt{2} \\ 0 & M_2 & gv_d/\sqrt{2} & -gv_u/\sqrt{2} \\ -g'v_d/\sqrt{2} & gv_d/\sqrt{2} & 0 & -\mu \\ g'v_u/\sqrt{2} & -gv_u/\sqrt{2} & -\mu & 0 \end{pmatrix}. \tag{K.2.9}$$

As noted above eq. (K.1.2), we work in a convention in which v_u and v_d are real and positive. The gaugino mass parameters M_1, M_2 and the higgsino mass parameter μ are potentially complex.

We now list the gauge boson interactions with the neutralinos and charginos in the form of Feynman rules. Here, we make use of the results presented in Figs. 29.4.2–29.4.4. The Feynman rules for Z and γ interactions with charginos and neutralinos are given in Fig. K.2.1 and the corresponding rules for $W^\pm$ interactions are given in Fig. K.2.2. For each of these rules, one has a version with lowered spinor indices by replacing $\overline{\sigma}_\mu^{\dot\alpha\beta} \to -\sigma_{\mu\beta\dot\alpha}$. We label fermion lines with the symbols of the two-component fermion fields as given in Table 29.1. The $Z\widetilde{N}_i\widetilde{N}_j$ interaction vertex also subsumes the O''^R_{ij} interaction found in four-component Majorana Feynman rules as in ref. [7], due to the result of eq. (G.1.99) and the relation $O''^R_{ij} = -O''^L_{ji}$ of eq. (K.2.5).

The chargino sector is CP-conserving if $\mathrm{Im}\,(M_2\mu^*) = 0$. In this case, the chargino fields can be rephased such that M_2 and μ are real, and the chargino mixing matrices U and V can be chosen to be real orthogonal. In particular, the couplings O'^L and O'^R are manifestly real. Likewise, the

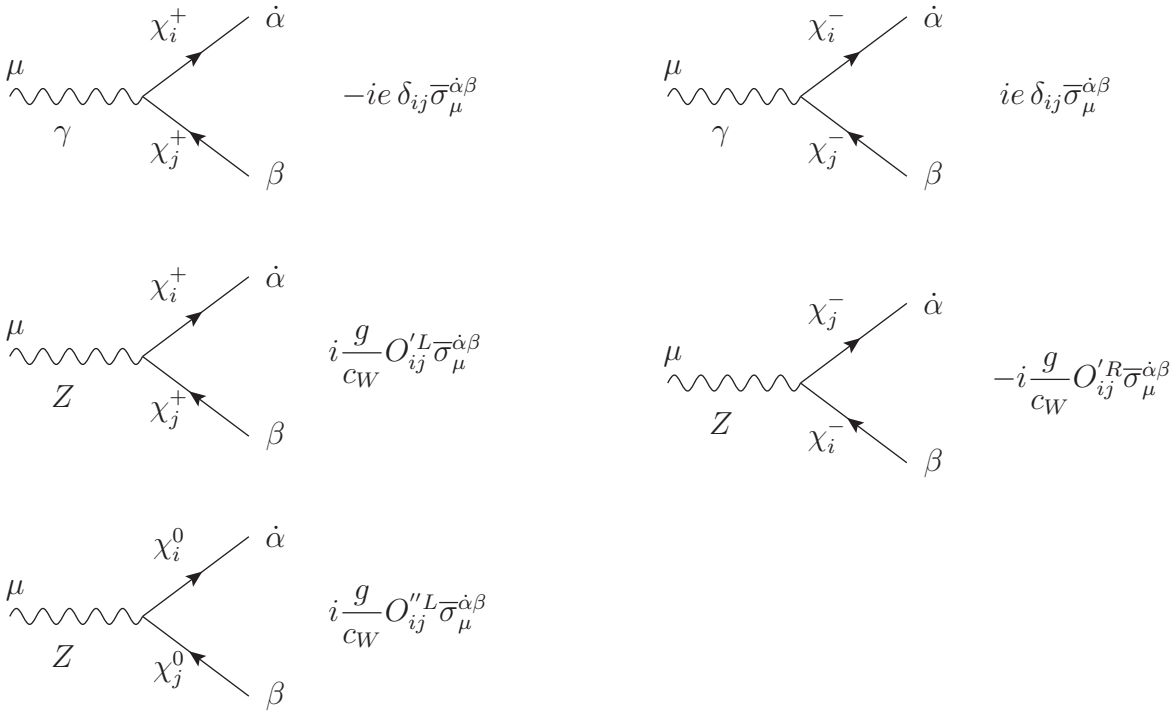


Figure K.2.1: Feynman rules for the chargino and neutralino interactions with neutral gauge bosons. The coupling matrices are defined in eqs. (K.2.3)–(K.2.5) and $c_W \equiv \cos\theta_W$. For each rule, a corresponding one with lowered spinor indices is obtained by $\overline{\sigma}_\mu^{\dot\alpha\beta} \to -\sigma_{\mu\beta\dot\alpha}$.

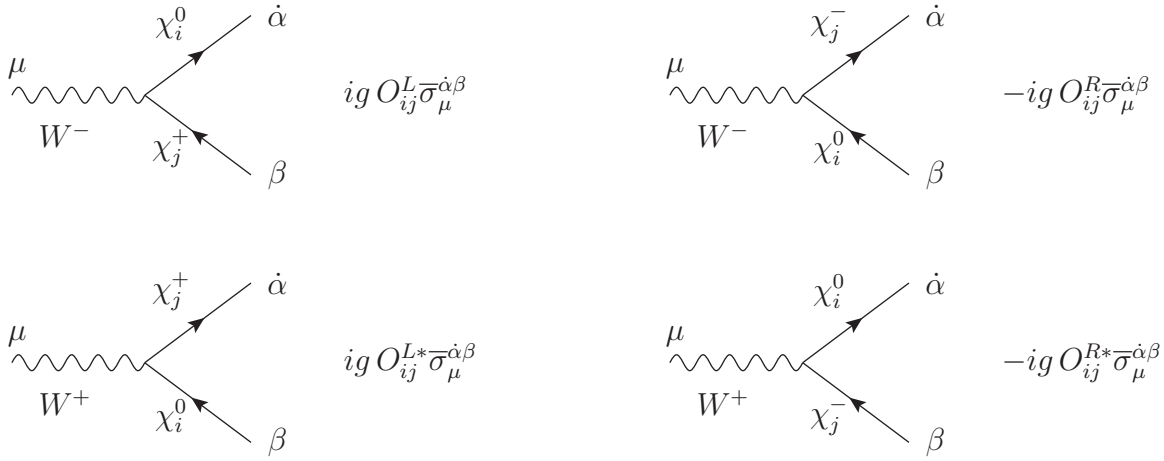


Figure K.2.2: Feynman rules for the chargino and neutralino interactions with $W^\pm$ gauge bosons. The charge indicated on the W boson is flowing into the vertex in each case. The coupling matrices are defined in eqs. (K.2.1) and (K.2.2). For each rule, a corresponding one with lowered spinor indices is obtained by $\overline\sigma_\mu^{\dot\alpha\beta} \to -\sigma_{\mu\beta\dot\alpha}$.

neutralino sector is CP-conserving if $\mathrm{Im}\,(M_1\mu^*) = \mathrm{Im}\,(M_2\mu^*) = \mathrm{Im}\,(M_1 M_2^*) = 0$.[179] In this case, the neutralino fields can be rephased such that M_1, M_2 and μ are all real, and the neutralino mixing matrix can be chosen [cf. eqs. (D.2.4) and (D.2.5)] such that [173]:

$$N_{ij} = \varepsilon_i^{1/2} Z_{ij}\,, \qquad \text{no sum over } i\,, \tag{K.2.10}$$

where Z is a real orthogonal matrix, and ε_i is the sign (either ± 1) of the ith eigenvalue of the real symmetric neutralino mass matrix, M_{ψ^0}. That is, the ith row of N is purely real [imaginary] if $\varepsilon_i = +1$ [-1]. In particular, the matrix element O''^L_{ij} is purely real [imaginary] if $\varepsilon_i\varepsilon_j = +1$ [-1]. More generally, the neutralino and chargino interactions with the electroweak gauge bosons are CP-conserving if the corresponding Feynman rules for the interaction vertices are either purely real or purely imaginary.

In the CP-violating case, the matrices U and V cannot be chosen to be real orthogonal, and N cannot be written in the form of eq. (K.2.10).[180] Nevertheless, the diagonal couplings O'^L_{ii}, O'^R_{ii} and O''^L_{ii} are manifestly real. This indicates that the diagonal $Z^0\widetilde C_i^+\widetilde C_i^-$ and $Z^0\widetilde N_i\widetilde N_i$ couplings are CP-conserving at tree level, even in the presence of a CP-violating chargino and neutralino sector. Similarly, the diagonal $\gamma\widetilde C_i^+\widetilde C_i^-$ couplings are CP-conserving, whereas the off-diagonal $\gamma\widetilde C_i^\pm\widetilde C_j^\mp$ couplings ($i \neq j$) vanish at tree level, as expected from gauge invariance.

Higgs interactions with charginos and neutralinos

The couplings of chargino and neutralino mass eigenstates to the Higgs mass eigenstates can be written, in terms of the Higgs mixing parameters of eqs. (K.1.7) and (K.1.8) and the neutralino

[179] If all three of the potentially complex parameters M_1, M_2 and μ are non-zero, then only two of the three conditions for a CP-conserving neutralino sector are independent, since the third condition follows automatically from the first two conditions.

[180] Since M_{ψ^0} is in general a *complex* symmetric matrix, its eigenvalues are not necessarily all real. In particular, if the ith eigenvalue is not real, then there is no longer any meaning to the sign ε_i.

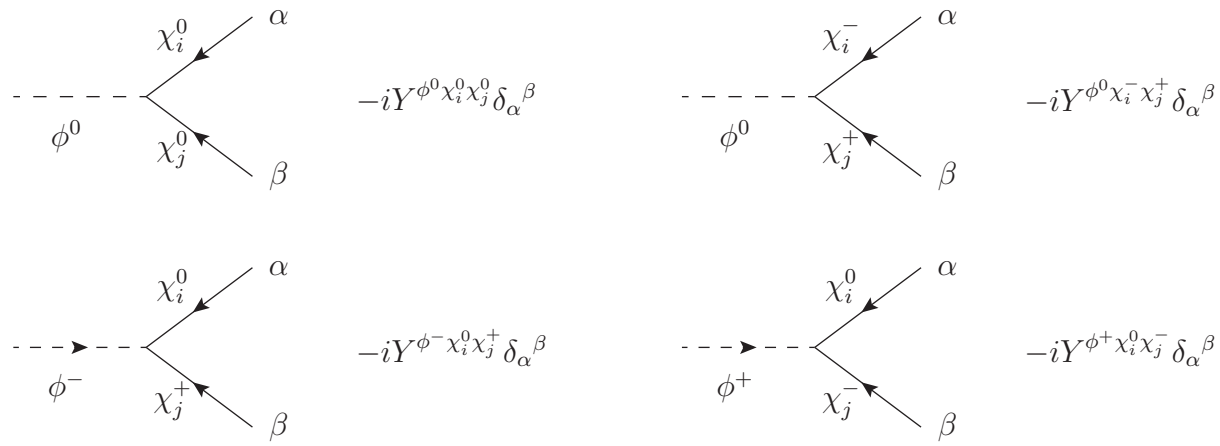


Figure K.3.1: Feynman rules for the interactions of neutral Higgs bosons $\phi^0 = (h^0, H^0, A^0, G^0)$ with neutralino pairs and chargino pairs, respectively, and the interaction of charged Higgs bosons $\phi^\pm = (H^\pm, G^\pm)$ with chargino-neutralino pairs. For each rule, there is a corresponding one with all arrows reversed, undotted indices changed to dotted indices with the opposite height, and the Y coupling (without the explicit $-i$) replaced by its complex conjugate.

and chargino mixing matrices of Appendix K.2 , as [173]:

$$Y^{\phi^0\chi_i^0\chi_j^0} = \frac{1}{2}(k^*_{d\phi^0}N^*_{i3} - k^*_{u\phi^0}N^*_{i4})(gN^*_{j2} - g'N^*_{j1}) + (i \leftrightarrow j)\,, \tag{K.3.1}$$

$$Y^{\phi^0\chi_i^-\chi_j^+} = \frac{g}{\sqrt{2}}(k^*_{u\phi^0}U^*_{i1}V^*_{j2} + k^*_{d\phi^0}U^*_{i2}V^*_{j1})\,, \tag{K.3.2}$$

$$Y^{\phi^+\chi_i^0\chi_j^-} = k_{d\phi^\pm}\big[g(N^*_{i3}U^*_{j1} - \frac{1}{\sqrt{2}}N^*_{i2}U^*_{j2}) - \frac{g'}{\sqrt{2}}N^*_{i1}U^*_{j2}\big]\,, \tag{K.3.3}$$

$$Y^{\phi^-\chi_i^0\chi_j^+} = k_{u\phi^\pm}\big[g(N^*_{i4}V^*_{j1} + \frac{1}{\sqrt{2}}N^*_{i2}V^*_{j2}) + \frac{g'}{\sqrt{2}}N^*_{i1}V^*_{j2}\big]\,, \tag{K.3.4}$$

for $\phi^0 = h^0, H^0, A^0, G^0$ and $\phi^\pm = H^\pm, G^\pm$. We exhibit the Higgs boson and Goldstone boson interactions with the neutralinos and charginos in Fig. K.3.1. For each of the Feynman rules in Fig. K.3.1, one can reverse all arrows by taking $\delta_\alpha{}^\beta \to \delta^{\dot\alpha}{}_{\dot\beta}$ and complex conjugating the corresponding coupling (but not the overall factor of $-i$).

Goldstone bosons may appear as internal lines in of Feynman graphs that are evaluated in the 't Hooft-Feynman gauge. The propagation of a Goldstone boson yields a result that is identical to the propagation of the corresponding longitudinal gauge boson in the unitary gauge. It is thus convenient to express the Goldstone boson couplings to the neutralinos and charginos in terms of the corresponding gauge boson couplings. To accomplish this, we first record a number of identities among the neutralino and chargino mixing matrices. First, we use eqs. (K.2.6) and (K.2.7) to derive:

$$M_2U^*_{i1} + gv_dU^*_{i2} = m_{\widetilde{C}_i}V_{i1}\,, \qquad gv_uU^*_{i1} + \mu U^*_{i2} = m_{\widetilde{C}_i}V_{i2}\,, \tag{K.3.5}$$

$$M_2V^*_{i1} + gv_uV^*_{i2} = m_{\widetilde{C}_i}U_{i1}\,, \qquad gv_dV^*_{i1} + \mu V^*_{i2} = m_{\widetilde{C}_i}U_{i2}\,. \tag{K.3.6}$$

Next, we use eqs. (K.2.8) and (K.2.9) to derive:

$$m_{\widetilde{N}_i} N_{i4} = \sum_{j=1}^{4} N_{ij}^* (M_{\psi^0})_{j4} = \frac{v_u}{\sqrt{2}} \left(g' N_{i1}^* - g N_{i2}^* \right) - \mu N_{i3}^* \,, \tag{K.3.7}$$

$$m_{\widetilde{N}_i} N_{i3} = \sum_{j=1}^{4} N_{ij}^* (M_{\psi^0})_{j3} = -\frac{v_d}{\sqrt{2}} \left(g' N_{i1}^* - g N_{i2}^* \right) - \mu N_{i4}^* \,, \tag{K.3.8}$$

$$m_{\widetilde{N}_i} N_{i2} = \sum_{j=1}^{4} N_{ij}^* (M_{\psi^0})_{j2} = N_{i2}^* M_2 + \frac{g}{\sqrt{2}} \left(v_d N_{i3}^* - v_u N_{i4}^* \right) \,. \tag{K.3.9}$$

By a judicious combination of the above identities, μ and M_2 can be eliminated. One can then rewrite the Goldstone boson couplings of eqs. (K.3.1)–(K.3.4) in terms of the gauge boson couplings $O^{L,R}$, $O'^{L,R}$ and $O''^{L,R}$ defined in eqs. (K.2.1)–(K.2.5). It then follows that:

$$iY^{G^0\chi_i^0\chi_j^0} = \frac{\sqrt{2}}{v} \left(m_{\widetilde{N}_i} O''^{L}_{ij} - m_{\widetilde{N}_j} O''^{R}_{ij} \right) \,, \tag{K.3.10}$$

$$iY^{G^0\chi_i^-\chi_j^+} = \frac{\sqrt{2}}{v} \left(m_{\widetilde{C}_i} O'^{L}_{ij} - m_{\widetilde{C}_j} O'^{R}_{ij} \right) \,, \tag{K.3.11}$$

$$Y^{G^+\chi_i^0\chi_j^-} = \frac{\sqrt{2}}{v} \left(m_{\widetilde{C}_j} O^{L\,*}_{ij} - m_{\widetilde{N}_i} O^{R\,*}_{ij} \right) \,, \tag{K.3.12}$$

$$Y^{G^-\chi_i^0\chi_j^+} = -\frac{\sqrt{2}}{v} \left(m_{\widetilde{N}_i} O^{L}_{ij} - m_{\widetilde{C}_j} O^{R}_{ij} \right) \,. \tag{K.3.13}$$

Note that by using $O''^{R}_{ij} = -O''^{L}_{ji}$, it follows from eq. (K.3.10) that $iY^{G^0\chi_i^0\chi_j^0}$ is symmetric under the interchange of i and j, as expected.

In general, for a CP-violating chargino and neutralino sector, the couplings $Y^{\phi^0\chi_i^0\chi_i^0}$ and $Y^{\phi^0\chi_i^+\chi_i^-}$ for $\phi^0 = h^0, H^0, A^0$ are neither purely real nor purely imaginary. That is, the diagonal neutralino and chargino couplings to the physical neutral Higgs bosons are generically CP-violating. However for $\phi^0 = G^0$, the diagonal neutralino and chargino couplings to the neutral Goldstone boson (when multiplied by i) are manifestly real. In particular, eqs. (K.3.10) and (K.3.11) yield:

$$iY^{G^0\chi_i^0\chi_i^0} = \frac{2\sqrt{2} m_{\widetilde{N}_i}}{v} O''^{L}_{ii} = \frac{\sqrt{2}\, m_{\widetilde{N}_i}}{v} \left[|N_{i4}|^2 - |N_{i3}|^2 \right] \,, \tag{K.3.14}$$

$$iY^{G^0\chi_i^-\chi_i^+} = \frac{\sqrt{2}\, m_{\widetilde{C}_i}}{v} (O'^{L}_{ii} - O'^{R}_{ii}) = \frac{m_{\widetilde{C}_i}}{\sqrt{2}\, v} \left[|V_{i2}|^2 - |U_{i2}|^2 \right] \,, \tag{K.3.15}$$

where the unitarity of U and V has been used to obtain the final expression in eq. (K.3.15). It follows that the diagonal neutralino and chargino couplings to the neutral Goldstone boson are CP-conserving. This result is not surprising, as the corresponding diagonal tree-level couplings of the (longitudinal) Z^0 boson are always CP-conserving as noted at the end of Appendix K.2 .

Chargino and neutralino interactions with fermions and sfermions

In the MSSM, the scalar partners of the two-component fields q and $\bar{q}^\dagger$ are the squarks, denoted by $\widetilde{q}_L$ and $\widetilde{q}_R$, respectively. In our notation, $\widetilde{q}_L^*$ and $\widetilde{q}_R^*$ denote both the complex conjugate fields and the names of the corresponding anti-squarks. Thus u, $\widetilde{u}_L$ and $\widetilde{u}_R$ all have electric charges $+2/3$, whereas $\bar{u}$, $\widetilde{u}_L^*$ and $\widetilde{u}_R^*$ all have electric charges $-2/3$. Likewise, the scalar partners of the two-component fields ℓ and $\bar{\ell}^\dagger$ are the charged sleptons, denoted by $\widetilde{\ell}_L$ and $\widetilde{\ell}_R$, respectively, with

$\ell = e, \mu, \tau$. The sneutrino, $\widetilde{\nu}$ is the superpartner of the neutrino. There is no $\widetilde{\nu}_R$, since there is no $\bar{\nu}$ in the theory.[181]

The Feynman rules for the chargino-quark-squark interactions are given in Fig. K.4.1, and the rules for the neutralino-quark-squark interactions are given in Fig. K.4.2. Here we have taken the quark and lepton two-component fields to be in a mass eigenstate basis, and the squark and slepton field basis consists of the superpartners of these fields, as described above. Therefore, in practical applications, one must include unitary rotation matrix elements relating the squarks and sleptons as given to the mass eigenstates, which can be different.

In principle, all sfermions with a given electric charge can mix with each other. However, there is a popular, and perhaps phenomenologically and theoretically favored, approximation in which only the sfermions of the third family have significant mixing. For $f = t, b, \tau$, one can then write the relationship between the gauge eigenstates $\tilde{f}_L$, $\tilde{f}_R$ and the mass eigenstates $\tilde{f}_1$, $\tilde{f}_2$ as [352]

$$\begin{pmatrix} \tilde{f}_R \\ \tilde{f}_L \end{pmatrix} = X_{\tilde{f}} \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix}, \qquad X_{\tilde{f}} \equiv \begin{pmatrix} R_{\tilde{f}_1} & R_{\tilde{f}_2} \\ L_{\tilde{f}_1} & L_{\tilde{f}_2} \end{pmatrix}, \tag{K.4.1}$$

where X is a 2×2 unitary matrix. Then one can choose $R_{\tilde{f}_1} = L^*_{\tilde{f}_2} = c_{\tilde{f}}$, and $L_{\tilde{f}_1} = -R^*_{\tilde{f}_2} = s_{\tilde{f}}$ with

$$|c_{\tilde{f}}|^2 + |s_{\tilde{f}}|^2 = 1. \tag{K.4.2}$$

If there is no CP violation, then $c_{\tilde{f}}$ and $s_{\tilde{f}}$ can be taken real, and they are the cosine and sine of a sfermion mixing angle.[182] For the other charged sfermions ($\widetilde{f} = \widetilde{u}, \widetilde{d}, \widetilde{c}, \widetilde{s}, \widetilde{e}, \widetilde{\mu}$), one can use the same notation, and approximate $L_{\tilde{f}_2} = R_{\tilde{f}_1} = 1$ and $L_{\tilde{f}_1} = R_{\tilde{f}_2} = 0$. The resulting Feynman rules for squarks and sleptons that mix within each generation are shown in Figs. K.4.3 and K.4.4.

For each Feynman rule in Figs. K.4.1–K.4.4, one can reverse all arrows by taking $\delta_\alpha{}^\beta \to \delta^{\dot{\alpha}}{}_{\dot{\beta}}$ and complex conjugating the corresponding rule (but leaving the explicit factor of i intact).

SUSY-QCD Feynman rules

In supersymmetric (SUSY) QCD, the Lagrangian governing the gluon interactions with colored fermions (gluinos and quarks) in two-component spinor notation, which derives from the covariant derivatives in the kinetic terms, is given by

$$\mathscr{L}_{\rm int} \;=\; i g_s f^{abd} (\tilde{g}^\dagger_a \overline{\sigma}_\mu \tilde{g}_b) A^\mu_d - g_s T^{ak}_j \sum_q \left[q^{\dagger j} \overline{\sigma}_\mu q_k - \bar{q}^\dagger_k \overline{\sigma}_\mu \bar{q}^j \right] A^\mu_a \,. \tag{K.5.1}$$

Here g_s is the strong coupling constant, $a, b, d = 1, 2, ..., 8$ are $SU(3)_c$ adjoint representation indices, and f^{abd} are the $SU(3)$ structure constants. Raised (lowered) indices $j, k = 1, 2, 3$ are color indices in the fundamental (anti-fundamental) representation. We have denoted the two-component gluino field by $\widetilde{g}_a$ as in Table 29.1 and the gluon field by A^μ_a. The sum $\sum_q$ is over the six flavors $q = u, d, s, c, b, t$ (in either the mass eigenstate or electroweak gauge-eigenstate basis). The corresponding Feynman rules are shown in Fig. K.5.1. The gluino-squark-quark Lagrangian is given by:

$$\mathscr{L}_{\rm int} = -\sqrt{2} g_s T^{ak}_j \sum_q \left[\tilde{g}_a q_k \, \tilde{q}^{*j}_L + \tilde{g}^\dagger_a q^{\dagger j} \, \tilde{q}_{Lk} - \tilde{g}_a \bar{q}^j \, \tilde{q}_{Rk} - \tilde{g}^\dagger_a \bar{q}^\dagger_k \, \tilde{q}^{*j}_R \right], \tag{K.5.2}$$

[181] It is possible to construct a seesaw-extended MSSM that would be the minimal supersymmetric extension of the seesaw-extended Standard Model described in Appendix J.2 . In the seesaw-extended MSSM, both $\bar{\nu}$ and its supersymmetric partner $\widetilde{\nu}_R$ exist. For further details on the sneutrino sector of the seesaw-extended MSSM, see [301].

[182] Our convention for $c_{\tilde{f}}, s_{\tilde{f}}$ has the property that for zero mixing angle, $\widetilde{f}_1 = \widetilde{f}_R$ and $\widetilde{f}_2 = \widetilde{f}_L$. The conventions most commonly found in the literature unfortunately do not have this nice property.

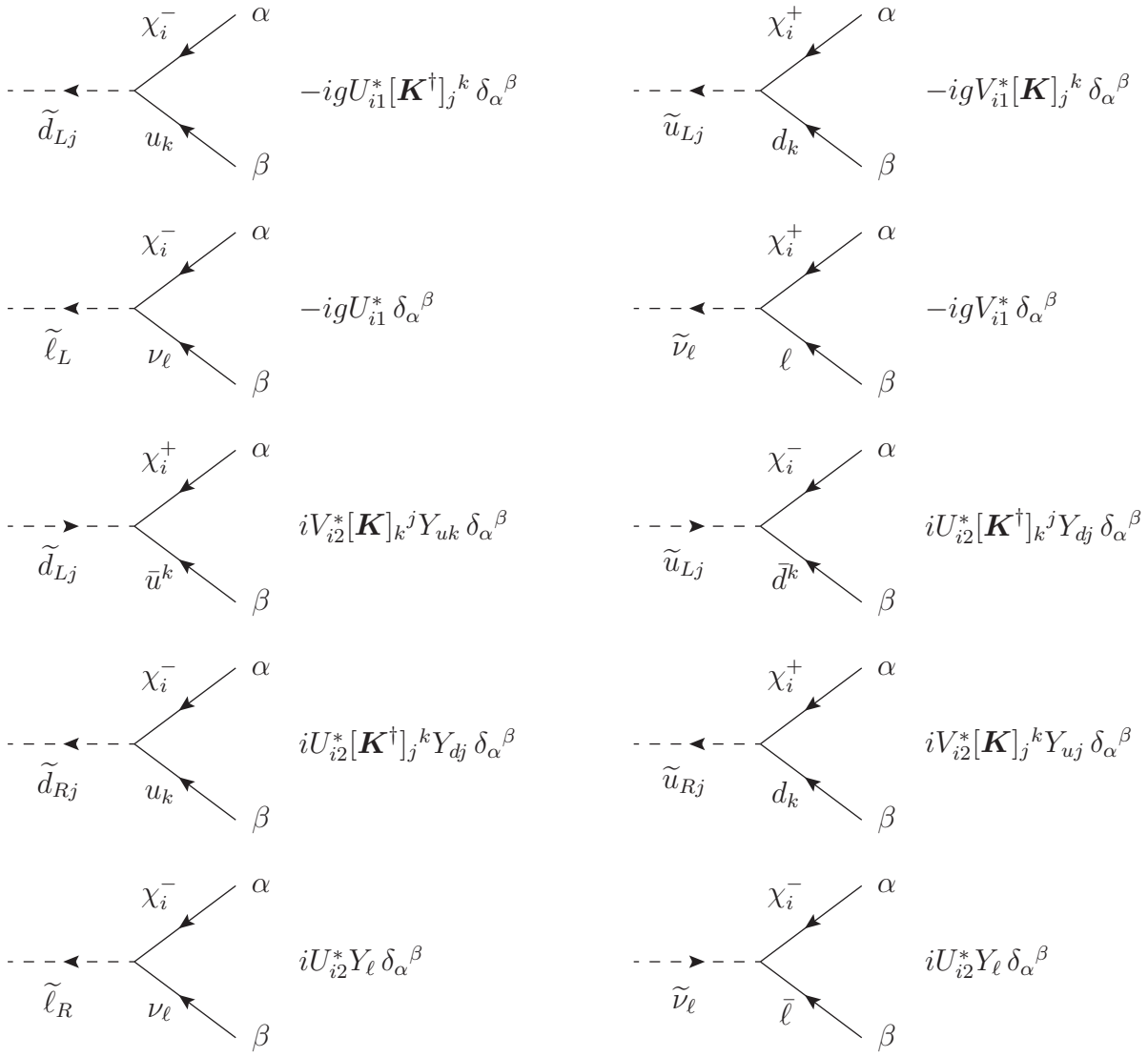


Figure K.4.1: Feynman rules for the interactions of charginos with fermion/sfermion pairs in the MSSM. The fermions are taken to be in a mass eigenstate basis, and the sfermions are in a basis whose elements are the supersymmetric partners of them. For each rule, there is a corresponding one with all arrows reversed, undotted indices changed to dotted indices with the opposite height, and the coupling (without the explicit i) replaced by its complex conjugate. Note that chargino interaction vertices involving $\bar{u}\widetilde{d}_R$ and $\bar{d}\widetilde{u}_R$ do not occur in the MSSM. An alternative version of these rules, for the case that mixing is allowed only among third-family sfermions, is given in Fig. K.4.3.

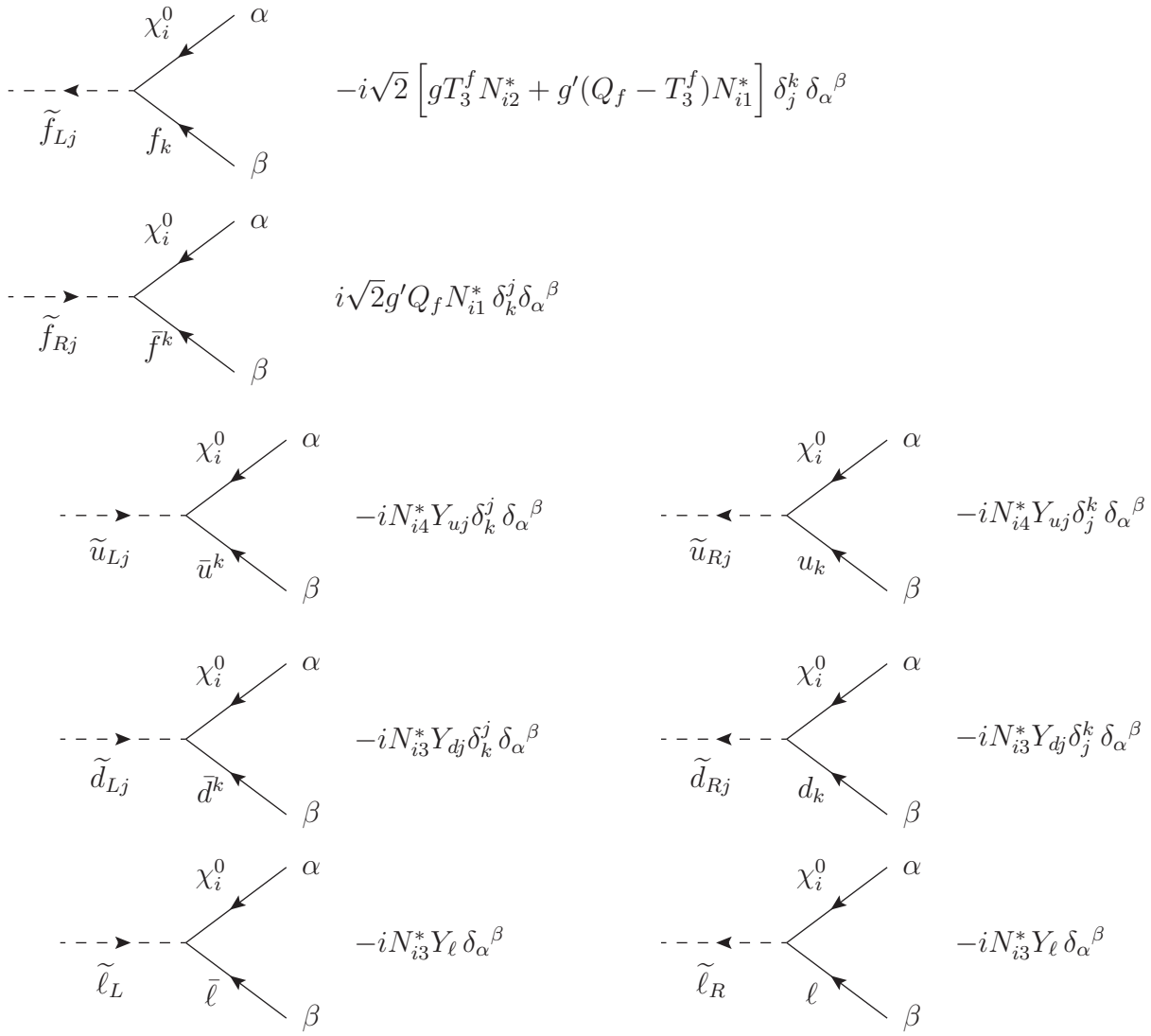


Figure K.4.2: Feynman rules for the interactions of neutralinos with fermion/sfermion pairs in the MSSM. The fermions are taken to be in a mass eigenstate basis, and the sfermions are in a basis whose elements are the supersymmetric partners of them. For each rule, there is a corresponding one with all arrows reversed, undotted indices changed to dotted indices with the opposite height, and the coupling (without the explicit i) replaced by its complex conjugate. An alternative version of these rules, for the case that mixing is allowed only among third-family sfermions, is given in Fig. K.4.4.

where the squark fields are taken to be in the same basis as the quarks. The Feynman rules resulting from these Lagrangian terms are shown in Fig. K.5.2.

For practical applications, one typically takes the quark fields as the familiar mass eigenstates, and then performs a unitary rotation on the squarks in the corresponding basis to obtain their mass eigenstate basis. In the approximation described at the end of Appendix K.4 [cf eqs. (K.4.1) and (K.4.2) and the accompanying text], one obtains the Feynman rules of Fig. K.5.3, as an alternative to those of Fig. K.5.2.

29.7.15 Trilinear R-parity-violating Yukawa interactions

In the MSSM, a multiplicative R-parity invariance is imposed, where $R = (-1)^{3(B-L)+2S}$ for a particle of baryon number B, lepton number L and spin S[353]. Equivalently, R-parity can be

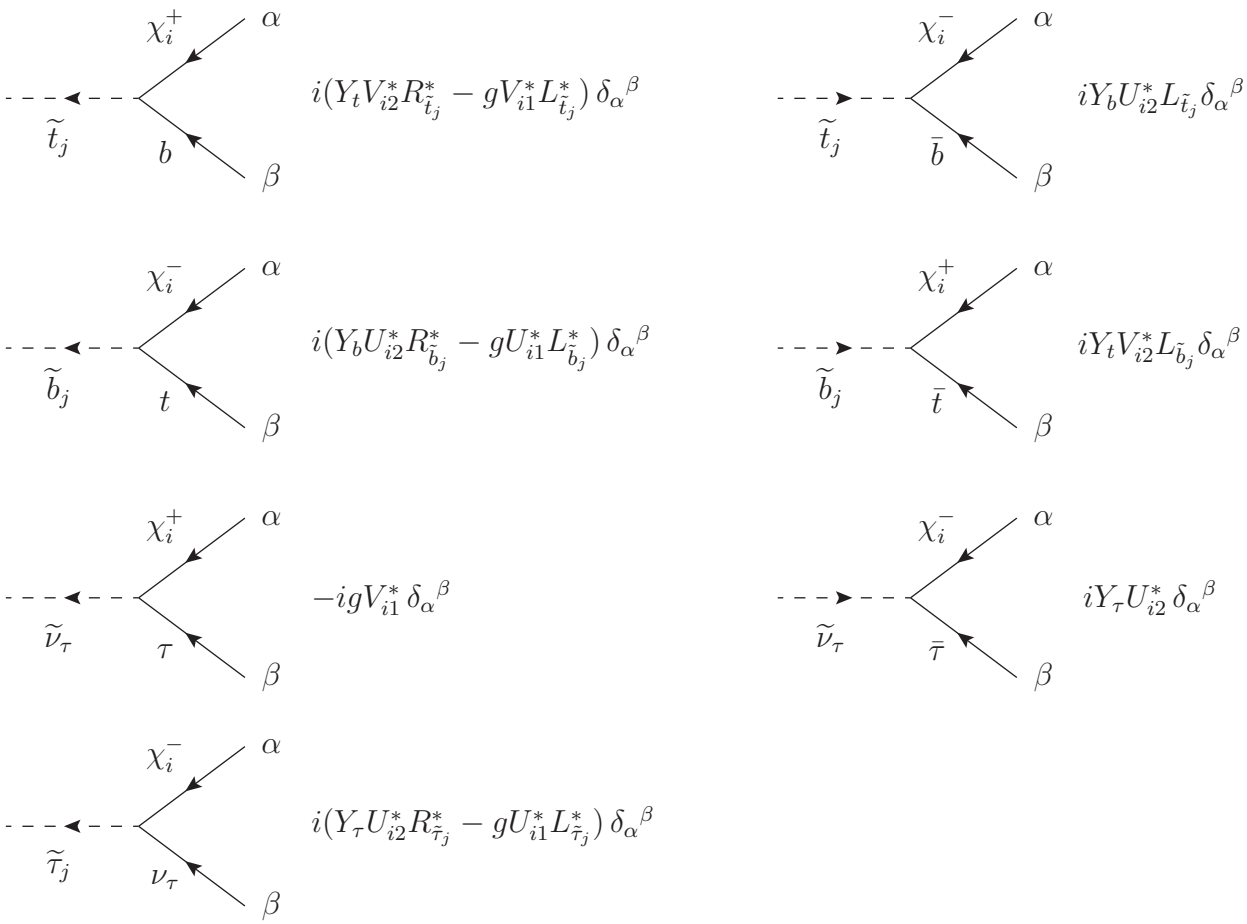
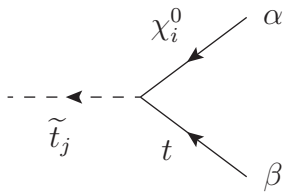
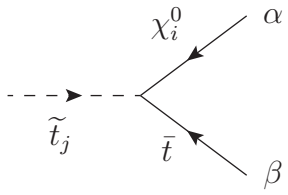


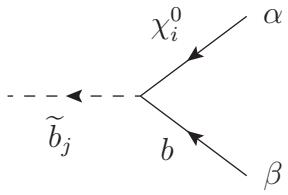
Figure K.4.3: Feynman rules for the interactions of charginos with third-family fermion/sfermion pairs in the MSSM. The fermions are taken to be in a mass eigenstate basis. CKM mixing is neglected, and the sfermions are assumed to only mix within the third family. The corresponding rules for the first and second families with the approximation of no mixing and vanishing fermion masses can be obtained from these by setting $Y_f = 0$ and $L_{\tilde{f}_2} = R_{\tilde{f}_1} = 1$ and $L_{\tilde{f}_1} = R_{\tilde{f}_2} = 0$ (so that $\widetilde{f}_1 = \widetilde{f}_R$ and $\widetilde{f}_2 = \widetilde{f}_L$). For each rule, there is a corresponding one with all arrows reversed, undotted indices changed to dotted indices with the opposite height, and the coupling (without the explicit i) replaced by its complex conjugate.



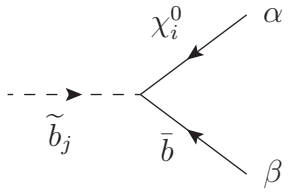
$$-i\left[Y_t N^*_{i4} R^*_{\tilde{t}_j} + \tfrac{1}{\sqrt{2}}(gN^*_{i2} + \tfrac{1}{3}g'N^*_{i1})L^*_{\tilde{t}_j}\right]\delta_\alpha{}^\beta$$



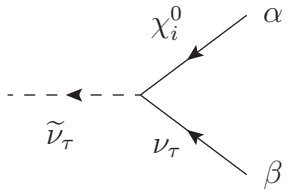
$$-i\left[Y_t N^*_{i4} L_{\tilde{t}_j} - \tfrac{2\sqrt{2}}{3}g'N^*_{i1}R_{\tilde{t}_j}\right]\delta_\alpha{}^\beta$$



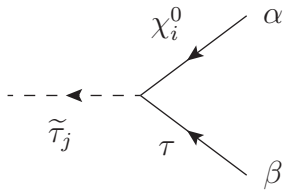
$$-i\left[Y_b N^*_{i3} R^*_{\tilde{b}_j} + \tfrac{1}{\sqrt{2}}(-gN^*_{i2} + \tfrac{1}{3}g'N^*_{i1})L^*_{\tilde{b}_j}\right]\delta_\alpha{}^\beta$$



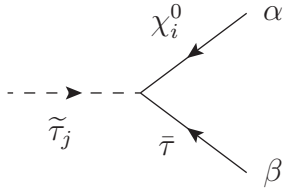
$$-i\left[Y_b N^*_{i3} L_{\tilde{b}_j} + \tfrac{\sqrt{2}}{3}g'N^*_{i1}R_{\tilde{b}_j}\right]\delta_\alpha{}^\beta$$



$$-\tfrac{i}{\sqrt{2}}(gN^*_{i2} - g'N^*_{i1})\,\delta_\alpha{}^\beta$$



$$-i\left[Y_\tau N^*_{i3} R^*_{\tilde{\tau}_j} - \tfrac{1}{\sqrt{2}}(gN^*_{i2} + g'N^*_{i1})L^*_{\tilde{\tau}_j}\right]\delta_\alpha{}^\beta$$



$$-i\left[Y_\tau N^*_{i3} L_{\tilde{\tau}_j} + \sqrt{2}g'N^*_{i1}R_{\tilde{\tau}_j}\right]\delta_\alpha{}^\beta$$

Figure K.4.4: Feynman rules for the interactions of neutralinos with third-family fermion/s-fermion pairs in the MSSM. The comments of the caption of Fig. K.4.3 also apply here.

defined to be an additive quantum number modulo 2, where $R = +1$ corresponds to an even R-parity and $R = -1$ corresponds to an odd R-parity. In particular, all the ordinary Standard Model particles are R parity even, whereas the corresponding supersymmetric partners are R parity odd. In the R-parity-violating extension of the MSSM (denoted below as RPV-MSSM), new interactions are allowed that violate R-parity. Such interactions necessarily violate the $B - L$ global symmetry. R-parity-violating interactions can significantly alter the phenomenology at colliders (see for example [208, 211]), especially as the lightest supersymmetric particle (LSP) is no longer stable [354, 212]. Moreover, the LSP need not be restricted to the lightest neutralino (or perhaps the sneutrino) as in the MSSM, but can be any supersymmetric particle [207].

In this appendix, we focus on new trilinear supersymmetric Yukawa interactions that can appear in an RPV-MSSM [355, 354, 356, 357]:

$$\mathscr{L}_{LL\bar{e}} = -\tfrac{1}{2}\lambda_{ijk}\left(\widetilde{\ell}^*_{Rk}\nu_i\ell_j + \widetilde{\nu}_i\ell_j\bar{\ell}_k + \widetilde{\ell}_{Lj}\bar{\ell}_k\nu_i - \widetilde{\ell}^*_{Rk}\ell_i\nu_j - \widetilde{\nu}_j\bar{\ell}_k\ell_i - \widetilde{\ell}_{Li}\nu_j\bar{\ell}_k\right) + \text{h.c.}\,, \tag{L.1}$$

$$\mathscr{L}_{LQ\bar{d}} = -\lambda'_{ijk}\left(\widetilde{d}^*_{Rk}\nu_i d_j + \widetilde{\nu}_i d_j\bar{d}_k + \widetilde{d}_{Lj}\bar{d}_k\nu_i - \widetilde{d}^*_{Rk}\ell_i u_j - \widetilde{u}_{Lj}\bar{d}_k\ell_i - \widetilde{\ell}_{Li}u_j\bar{d}_k\right) + \text{h.c.}\,, \tag{L.2}$$

$$\mathscr{L}_{\bar{u}\bar{d}\bar{d}} = -\tfrac{1}{2}\lambda''_{ijk}\epsilon_{pqr}\left[\widetilde{u}^{p*}_{Ri}\bar{d}^q_j\bar{d}^r_k + \widetilde{d}^{q*}_{Rj}\bar{u}^p_i\bar{d}^r_k + \widetilde{d}^{r*}_{Rk}\bar{u}^p_i\bar{d}^q_j\right] + \text{h.c.}\,, \tag{L.3}$$

where repeated indices are summed over.[183] In eqs. (L.1)–(L.3), λ_{ijk}, λ'_{ijk}, λ''_{ijk} are dimensionless coupling constants, i, j, k are generation indices, and $p, q, r = 1, 2, 3$ are color $SU(3)$ indices, respectively. The couplings proportional to λ and λ' violate L and conserve B, whereas the couplings proportional to λ'' violate B and conserve L. Various phenomenological constraints on these couplings are summarized in refs. [357].

In addition to λ_{ijk}, λ'_{ijk}, λ''_{ijk}, the Lagrangian of the RPV-MSSM contains one additional supersymmetric L-violating mass parameter, κ_i, which leads to slepton–Higgs mixing and lepton–higgsino mixing. Finally, supersymmetry-breaking R-parity-violating parameters would also contribute to slepton–Higgs mixing and yields new trilinear scalar interactions. These effects modify the Feynman rules of Appendix K through additional mixing matrices, which we do not include here (for further details, see e.g. ref. [202]).

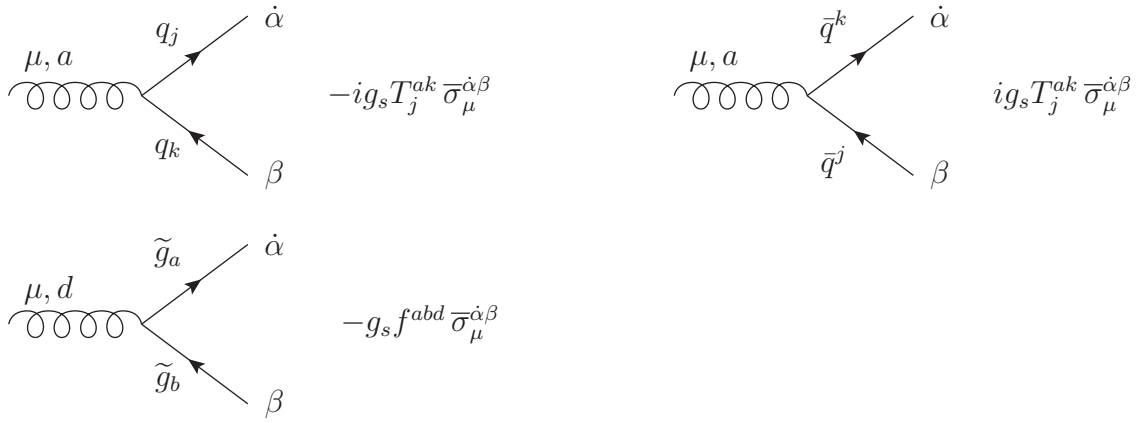


Figure K.5.1: Fermionic Feynman rules for SUSY-QCD that involve the gluon, with $q = u, d, c, s, t, b$. Lowered (raised) indices j, k correspond to the fundamental (anti-fundamental) representation of $SU(3)_c$. For each rule, a corresponding one with lowered spinor indices is obtained by $\overline{\sigma}^{\dot{\alpha}\beta}_\mu \to -\sigma_{\mu\beta\dot{\alpha}}$.

[183] The extra factors of $\frac{1}{2}$ in eqs. (L.1) and (L.3) have been chosen for convenience, due to the antisymmetry properties of the corresponding couplings: $\lambda_{ijk} = -\lambda_{jik}$, $\lambda''_{ijk} = -\lambda''_{ikj}$.

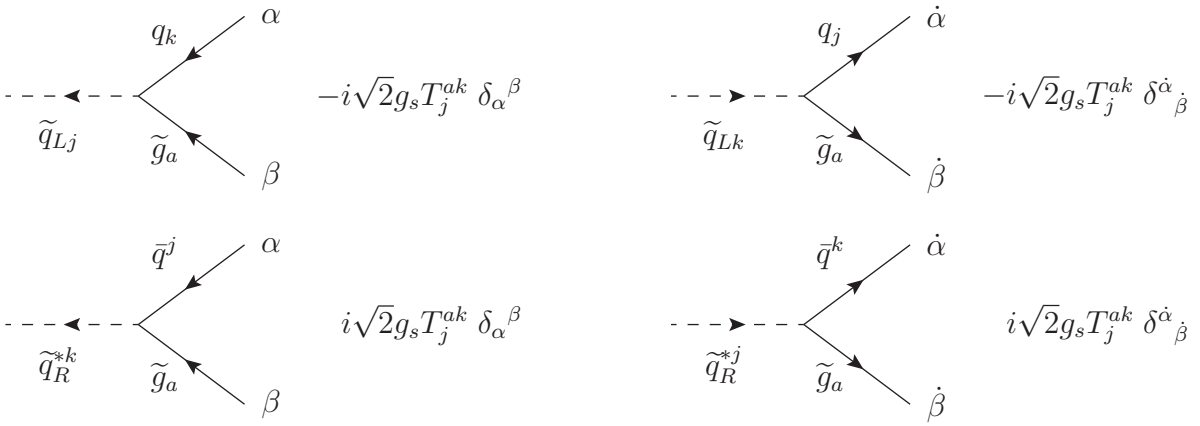


Figure K.5.2: Fermionic Feynman rules for SUSY-QCD that involve the squarks, in a basis corresponding to the quark mass eigenstates $q = u, d, c, s, t, b$. Lowered (raised) indices j, k correspond to the fundamental (anti-fundamental) representation of $SU(3)_c$, and the index a labels the adjoint representation carried by the gluino. The spinor index heights can be exchanged in each case, by replacing $\delta_\alpha{}^\beta \to \delta_\beta{}^\alpha$ or $\delta^{\dot\alpha}{}_{\dot\beta} \to \delta^{\dot\beta}{}_{\dot\alpha}$. For an alternative set of rules, incorporating $\widetilde{q}_L$–$\widetilde{q}_R$ mixing, see Fig. K.5.3.

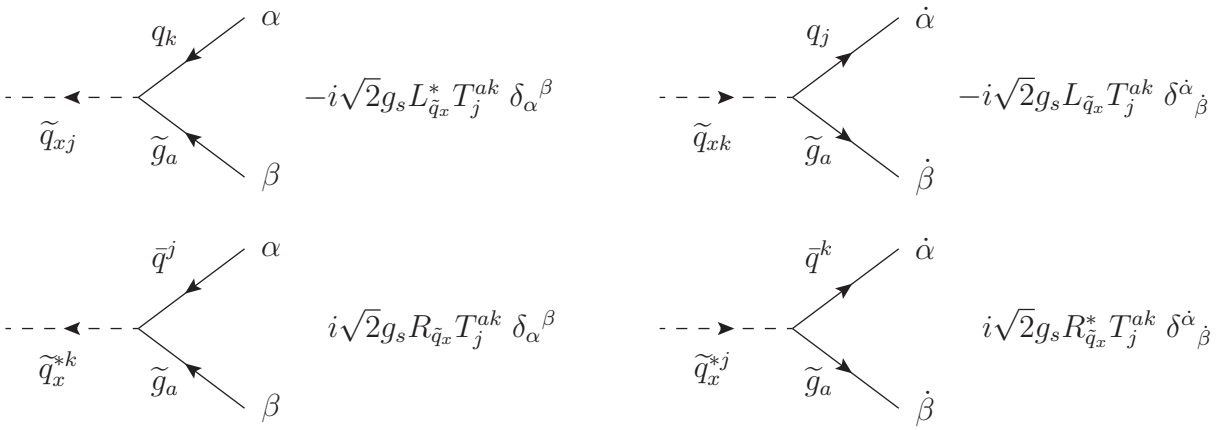


Figure K.5.3: Fermionic Feynman rules for SUSY-QCD that involve the squarks in the mass eigenstate basis labeled by $x = 1, 2$ and $q = u, d, c, s, t, b$, in the approximation where mixing is allowed only within a given flavor (typically, for the third family only), as in eq. (K.4.1). Lowered (raised) indices j, k correspond to the fundamental (anti-fundamental) representation of $SU(3)_c$, and the index a labels the adjoint representation carried by the gluino. The spinor index heights can be exchanged in each case, by replacing $\delta_\alpha{}^\beta \to \delta_\beta{}^\alpha$ or $\delta^{\dot\alpha}{}_{\dot\beta} \to \delta^{\dot\beta}{}_{\dot\alpha}$.

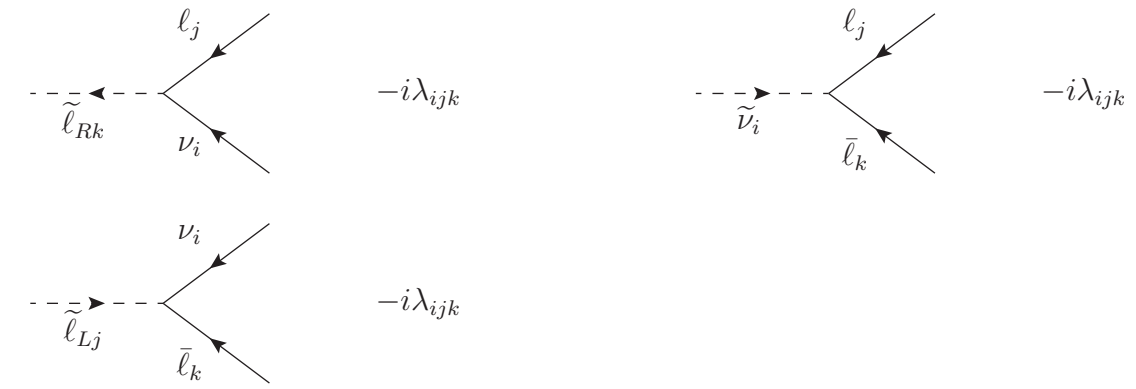


Figure L.1: Feynman rules for the Yukawa couplings of two-component fermions due to the supersymmetric, R-parity-violating Yukawa Lagrangian $\mathscr{L}_{LL\bar{e}}$ [cf. eq. (L.1)]. For each diagram, there is another with all arrows reversed and $\lambda_{ijk} \to \lambda^*_{ijk}$.

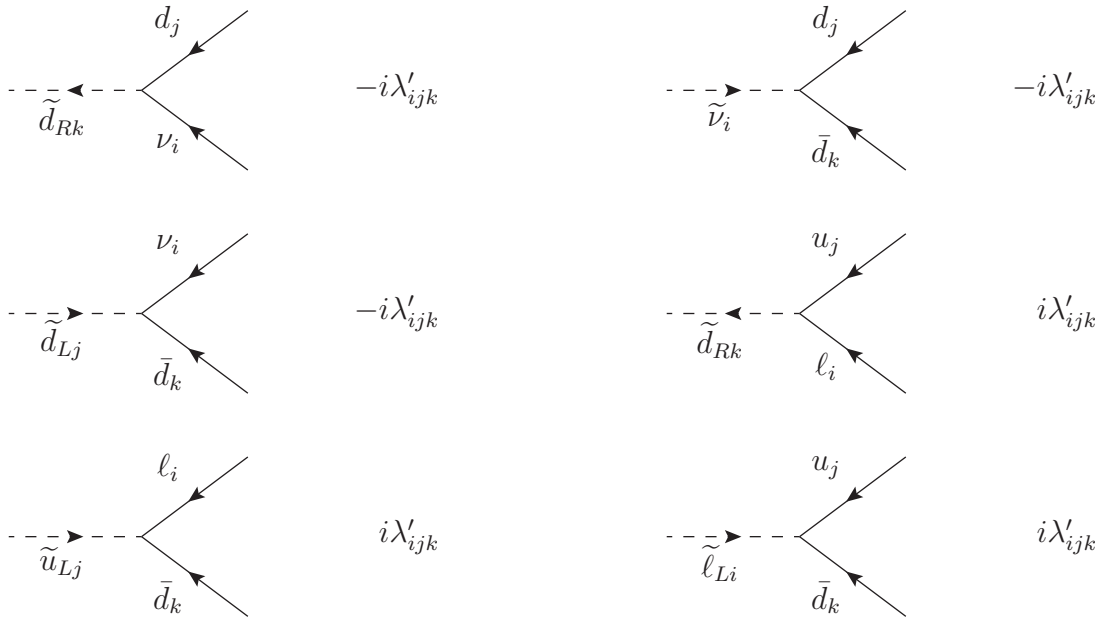


Figure L.2: Feynman rules for the Yukawa couplings of two-component fermions for the supersymmetric, R-parity-violating Yukawa Lagrangian $\mathscr{L}_{LQ\bar{d}}$ [cf. eq. (L.2)]. For each diagram, there is another with all arrows reversed and $\lambda'_{ijk} \to \lambda'^*_{ijk}$.

Recently, the two-component fermion Feynman rules for the neutral fermions have been given in refs. [358, 283]. Using eq. (29.4.2) and Fig. 29.4.1 we can now directly determine the corresponding Feynman rules. These are given in Figs. L.1, L.2, and L.3. The same Lagrangian for the Yukawa interactions is given in terms of four-component fermions in refs. [209, 210]. Two sample computations that make use of these rules are presented in Sections 29.6.20 and 29.6.21.

Figure L.3: Feynman rules for the Yukawa couplings of two-component fermions due to the supersymmetric, R-parity-violating Yukawa Lagrangian $\mathscr{L}_{\bar{u}\bar{d}\bar{d}}$ [cf. eq. (L.3)]. For each diagram, there is another with all arrows reversed and $\lambda''_{ijk} \to \lambda''^*_{ijk}$.

30 Special Topics by Zi

(тут очень много всего годного, потом обязательно разберу!)

30.1 Zi.IV Symmetry and Symmetry Breaking

30.1.1 Zi.IV.1 Symmetry Breaking

Как скучен был бы симметричный мир

Нам бы хотелось, чтобы фундаментальные законы природы были симметричными, хотя сами не представляем, насколько скучным был бы симметричный мир. На самом деле реальный мир не совсем симметричен. Точнее, хорошо было бы, если бы симметричным был лагранжиан, а не мир, описываемый лагранжианом. Главной целью современной физики действительно является изучение того, как могут нарушиться свойства симметрии лагранжиана. В следующих главах мы увидим, что наши знания фундаментальных законов физики основаны на понимании процессов нарушения симметрии.

Рассмотрим лагранжиан, который мы изучали в главе I.9:

$$\mathcal{L} = \frac{1}{2}\left[(\partial\vec{\varphi})^2 - \mu^2\vec{\varphi}^2\right] - \frac{\lambda}{4}\left(\vec{\varphi}^2\right)^2,$$

где $\vec{\varphi} = (\varphi_1, \varphi_2, ..., \varphi_N)$. Этот лагранжиан обладает $O(N)$-симметрией, при которой $\vec{\varphi}$ преобразуется как N-компонентный вектор.

Мы можем легко добавить в лагранжиан члены, не обладающие свойством симметрии. Например, добавим члены φ_1^2, φ_1^4 и $\varphi_1^2\vec{\varphi}^2$ и нарушим $O(N)$-симметрию до $O(N-1)$, при которой $\varphi_2, ..., \varphi_N$ преобразуются как $N-1$-компонентный вектор. Такой способ нарушения симметрии («руками») называется явным нарушением.

Можно нарушать симметрию поэтапно. Очевидно, что по желанию мы можем нарушить ее до $O(N-M)$ для любого $M < N$.

Обратите внимание, что в этом примере добавление новых членов не нарушает симметрию относительно отражений $\varphi_a \to -\varphi_a$ (при любом a). Этот вид симметрии также легко нарушить, если добавить, например, член φ_a^3.

Явное нарушение симметрии не представляет никакого интереса. С тем же успехом мы могли бы начать наши рассуждения со случая несимметричного лагранжиана.

Спонтанное нарушение симметрии

Гораздо интереснее позволить системе «самой нарушить симметрию», т. е. дождаться так называемого спонтанного нарушения симметрии. Поясню на примере. Изменим знак члена $\vec{\varphi}^2$ в лагранжиане (1) на противоположный и запишем:

$$\mathcal{L} = \frac{1}{2}\left[(\partial\vec{\varphi})^2 + \mu^2\vec{\varphi}^2\right] - \frac{\lambda}{4}\left(\vec{\varphi}^2\right)^2$$

Наивно для малых λ поле φ создает частицу массой $\sqrt{-\mu^2} = i\mu$. Что-то, очевидно, не так.

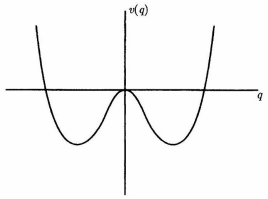


Рис. IV.1.1

Физический смысл этого наблюдения аналогичен тому, что произошло бы, если бы мы записали коэффициент пружины для ангармонического осциллятора с неверным знаком, т. е. $L = \frac{1}{2}\left(\dot{q}^2 + kq^2\right) - (\lambda/4)q^4$. Все знают, что делать в этом случае в классической механике. Потенциальная энергия $V(q) = -\frac{1}{2}kq^2 + (\lambda/4)q^4$ [известная как двухъямный потенциал (рис. IV.1.1)] имеет два минимума при $q = \pm v$, где $v \equiv (k/\lambda)^{\frac{1}{2}}$. При низких энергиях мы выбираем любой из двух минимумов и анализируем малые колебания относительно этого минимума. Привязка к одному или другому минимуму нарушает симметрию относительно отражений $q \to -q$ системы.

В квантовой механике, однако, частица может проникать сквозь потенциальный барьер между двумя минимумами; высота барьера равна $V(0)$ - $-V(\pm v)$. Вероятность нахождения в каждом из двух минимумов должна быть равной, т. е. симметрия гамильтониана $q \to -q$ гамильтониана сохраняется. В частности, волновая функция основного состояния $\psi(q) = \psi(-q)$ является четной.

Попробуем применить подобные рассуждения для квантовой теории поля. Найдем для лагранжиана собственного скалярного поля $\mathcal{L} = \frac{1}{2}\left(\partial_0\varphi\right)^2 - \frac{1}{2}\left(\partial_i\varphi\right)^2 - V(\varphi)$ минимальное значение потенциальной энергии $\int d^D x \left[\frac{1}{2}\left(\partial_i\varphi\right)^2 + V(\varphi)\right]$, где D - размерность пространства. Ясно, что любое пространственное изменение φ лишь увеличит энергию, поэтому считаем $\varphi(x)$ независимой от пространства-времени величиной φ и ищем минимум $V(\varphi)$. В частности, для лагранжиана (2) имеем:

$$V(\varphi) = -\frac{1}{2}\mu^2\vec{\varphi}^2 + \frac{\lambda}{4}\left(\vec{\varphi}^2\right)^2$$

Далее мы увидим, что случаи $N = 1$ и $N \geqslant 2$ существенно отличаются друг от друга.

Отличия между квантовой механикой и квантовой теорией поля

Рассмотрим сначала случай $N = 1$. Потенциал $V(\varphi)$ аналогичен потенциалу на рис. IV.1.1, за исключением того, что горизонтальная ось теперь обозначена параметром φ. Существуют два минимума при $\varphi = \pm v = \pm\left(\mu^2/\lambda\right)^{\frac{1}{2}}$

Некоторые моменты, однако, существенно отличаются между собой для квантовой теории поля и квантовой механики. Туннельный барьер теперь равен $[V(0) - V(\pm v)] \int d^D x$($D$ - размерность пространства) и, следовательно, бесконечен (или, точнее говоря, пропорционален размеру системы)! Туннельный эффект отсутствует, волновая функция основного состояния локализуется в окрестности $+v$ или $-v$. Необходимо выбрать одну из двух возможностей для основного состояния и построить теорию возмущений относительно него. Не важно, какой вариант мы выберем: будет одна и та же физика. Однако, осуществляя выбор, мы нарушаем симметрию $\varphi \to -\varphi$ лагранжиана.

Симметрия относительно отражений нарушается спонтанно! Мы не вводили в лагранжиан нарушающие симметрию члены, а она все равно нарушилась 1.

[1] Сделаю не очень важное техническое замечание для придирчивых читателей: строго говоря. в теории поля волновая функция основного состояния должна по идее именоваться волновым функционалом, поскольку $\Psi[\varphi(\vec{x})]$ есть функционал функции $\varphi(\vec{x})$. Выберем основное состояние в $+v$ и запишем $\varphi = v + \varphi'$. Разлагая в ряд по φ', получаем, что:

$$\mathcal{L} = \frac{\mu^4}{4\lambda} + \frac{1}{2}\left(\partial\varphi'\right)^2 - \mu^2\varphi^2 - O\left(\varphi^3\right)$$

Физическая частица, отвечающая сдвинутому полю φ', имеет массу $\sqrt{2}\mu$. Квадрат физической массы должен быть положительным, поскольку это, как легко сообразить, всего-навсего $V''(\varphi)|_{\varphi=v}$.

Аналогично вы увидите, что первый член лагранжиана (4) есть $-V(\varphi)|_{\varphi=v}$. Если нас интересует лишь рассеяние мезонов, связанное с φ', этот член не должен входить в лагранжиан. Действительно, мы всегда можем добавить в $\mathcal{L}$ произвольную константу. Мы довольно произвольно выбирали $V(\varphi = 0)$ равным нулю. Такая же ситуация возникает в квантовой механике: у гармонического осциллятора, энергия нулевых колебаний $\frac{1}{2}\hbar\omega$ ненаблюдаема; физическими являются лишь переходы между энергетическими уровнями энергии. Мы вернемся к этому вопросу в главе VIII.2.

Еще одна точка зрения на (2) состоит в том, что в квантовой теории поля вычисляют евклидов функциональный интеграл

$$Z = \int D\varphi e^{-\int d^dx\left\{\frac{1}{2}\left[(\partial\varphi)^2-\mu^2\varphi^2\right]+\frac{\lambda}{4}\left(\varphi^2\right)^2\right\}}$$

а теория возмущений сводится к изучению малых колебаний относительно минимума евклидова действия. Обычно мы раскладываем лагранжиан относительно минимума $\varphi = 0$, если перед φ^2 стоит знак минус. Если φ^2 входит с положительным знаком, $\varphi = 0$ является не минимумом, а локальным максимумом.

В квантовой теории поля основное состояние называют вакуумом, так как это буквально такое состояние, в котором поле «покоится» и нет ни одной частицы. В нашем случае мы имеем два физически эквивалентных вакуума, из которых мы должны выбрать один. Значение φ в основном состоянии (либо v или $-v$ в рассматриваемом примере) называют вакуумным средним для φ. Говорят, что поле φ получило вакуумное среднее.

Непрерывная симметрия

Вернемся к формуле (2) для случая $N \geqslant 2$. На рис. IV.1.2 изображен потенциал (3) при $N = 2$. Форму потенциала сравнивают с дном бутылки или сомбреро. Потенциал имеет минимум при $\vec{\varphi}^2 = \mu^2/\lambda$. Интересно, что у нас есть бесконечное число вакуумов, характеризуемых направлением $\vec{\varphi}$ в данном вакууме. Благодаря $O(2)$-симметрии лагранжиана все они физически эквивалентны. Желательно, чтобы результат не зависел от нашего выбора. Пусть $\vec{\varphi}$ направлен вдоль первой координаты, т. е. $\varphi_1 = v \equiv +\sqrt{\mu^2/\lambda}$ и $\varphi_2 = 0$.

Теперь рассмотрим флуктуации относительно этой конфигурации поля, другими словами, запишем $\varphi_1 = v+\varphi_1'$ и $\varphi_2 = \varphi_2'$. Подставим эти выражения в (2) для случая $N = 2$ и разложим $\mathcal{L}$ в ряд. Предлагаю вам самим проделать простые выкладки. У вас должно получиться (опустив штрихи у полей, чтобы не перегружать обозначения):

$$\mathcal{L} = \frac{\mu^4}{4\lambda} + \frac{1}{2}\left[(\partial\varphi_1)^2 + (\partial\varphi_2)^2\right] - \mu^2\varphi_1^2 + O\left(\varphi^3\right)$$

Постоянный член в точности как и в (4), как и поле φ' в (4), поле φ_1 имеет массу $\sqrt{2}\mu$. Теперь обратите внимание на отличительное свойство лагранжиана (5): отсутствует член φ_2^2. Поле φ_2 безмассово!

Появление безмассового бозона

Тот факт, что поле φ_2 оказалось безмассовым, не случаен. Я сейчас объясню, что безмассовость - это общее и точное явление.

Из рис. IV.1.2 мы можем легко выяснить спектр частиц. Возбуждения поля φ_1 соответствуют флуктуациям в радиальном направлении (так сказать, «забираемся на стену»), а возбуждения поля φ_2 соответствуют флуктуациям в угловом направлении (так сказать, «вращаемся вдоль желоба»). Чтобы шарик катился вдоль минимумов потенциальной энергии, переходя от одного минимума к другому, не требуется затрат энергии. Чтобы пояснить данное утверждение, изобразим длинноволновое возбуждение вида $\varphi_2 = a\sin(\omega t - \vec{k}\vec{x})$, где a мало. В области, где масштаб длины меньше $|\vec{k}|^{-1}$, поле φ_2 постоянно и, следовательно, поле $\vec{\varphi}$ просто слегка отклоняется от направления 1, эквивалентного вакууму вследствие $O(2)$-симметрии. Только в тех областях, где масштаб длины больше $|\vec{k}|^{-1}$, для возбуждения требуются затраты энергии. Таким образом, при $|\vec{k}| \to 0$ можно ожидать, что энергия возбуждения обратится в нуль.

Теперь мы понимаем ключевое различие между случаями $N = 1$ и $N = 2$: для первого из них характерна дискретная симметрия относительно отражений, а для второго - непрерывная $O(2)$-симметрия.

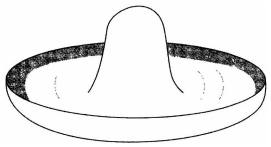


Рис. IV.1.2

После того как мы детально проанализировали случай $N = 2$, вы сможете обобщить наши рассуждения до произвольного $N \geqslant 2$ (см. упражнение (IV.1.1)).

Между тем взглянем на случай $N = 2$ с другой стороны. Многие теории поля могут быть записаны более чем одним способом, и важно узнавать их под разными видами. Зададим комплексное поле $\varphi = (1/\sqrt{2})\,(\varphi_1 + i\varphi_2)$; имеем $\varphi^\dagger\varphi = \frac{1}{2}\,(\varphi_1^2 + \varphi_2^2)$, поэтому перепишем (2) в виде:

$$\mathcal{L} = \partial\varphi^\dagger\partial\varphi + \mu^2\varphi^\dagger\varphi - \lambda\left(\varphi^\dagger\varphi\right)^2$$

явно инвариантном относительно $U(1)$-преобразования $\varphi \to e^{i\alpha}\varphi$ (вспомните главу I.9). Это соответствует условию локальной изоморфности групп $O(2)$ и $U(1)$. Вектор можно записать не только в декартовых или полярных координатах, но и параметризовать поле $\varphi(x) = \rho(x)e^{i\theta(x)}$ (как в главе III.5), чтобы получить $\partial_\mu\varphi = (\partial_\mu\rho + i\rho\partial_\mu\theta)\,e^{i\theta}$. Приходим к лагранжиану $\mathcal{L} = \rho^2(\partial\theta)^2 + (\partial\rho)^2 + \mu^2\rho^2 - \lambda\rho^4$. Спонтанное нарушение симметрии предполагает задание $\rho = v + \chi$, где $v = +\sqrt{\mu^2/2\lambda}$. Тогда [2]

$$\mathcal{L} = v^2(\partial\theta)^2 + \left[(\partial\chi)^2 - 2\mu^2\chi^2 - 4\sqrt{\frac{\mu^2\lambda}{2}}\chi^3 - \lambda\chi^4\right] + \left(\sqrt{\frac{2\mu^2}{\lambda}}\chi + \chi^2\right)(\partial\theta)^2.$$

Видим, что фаза $\theta(x)$ есть безмассовое поле. Мы объединили члены лагранжиана в три группы: кинетическая энергия безмассового поля θ, кинетическая и потенциальная энергия массивного поля χ и члены взаимодействия θ и χ.

[2] Автор опустил постоянный член в лагранжиане. - Прим. ред.

Теорема Голдстоуна

Теперь докажем теорему Голдстоуна, согласно которой при спонтанном нарушении непрерывной симметрии появляются безмассовые поля, известные как намбу-голдстоуновские бозоны.

Вспомним, что с любой непрерывной симметрией связан сохраняющийся заряд Q. Тот факт, что Q генерирует симметрию, выражается условием:

$$[H, Q] = 0.$$

Пусть вакуум (или основное состояние в квантовой механике) обозначается как $|0\rangle$. Добавив в гамильтониан соответствующую константу $H \to H+ +c$, мы всегда можем записать $H|0\rangle = 0$. Обычно вакуум инвариантен относительно преобразования симметрии $e^{i\theta Q}|0\rangle = |0\rangle$, другими словами, $Q|0\rangle = 0$.

Однако предположим, что симметрия спонтанно нарушается и вакуум перестает быть инвариантным относительно преобразования симметрии, т. е. $Q|0\rangle \neq 0$. Рассмотрим состояние $Q|0\rangle$. Какова его энергия? Итак,

$$HQ|0\rangle = [H, Q]|0\rangle = 0.$$

Первое равенство следует из условия $H|0\rangle = 0$, а второе - из условия (8). Следовательно, мы нашли другое состояние $Q|0\rangle$ с той же энергией, как и $|0\rangle$.

Обратите внимание, что доказательство не опиралось на релятивистскую инвариантность или поля. Видим также, что оно просто формализует картину с вращением шарика вдоль желоба.

В квантовой теории поля есть локальные токи, поэтому

$$Q = \int d^D x J^0(\vec{x}, t)$$

где D - размерность пространства. Сохранение заряда Q означает, что вычисление интеграла может проводиться в произвольный момент времени. Рассмотрим состояние

$$|s\rangle = \int d^D x e^{i\vec{k}\vec{x}} J^0(\vec{x}, t)|0\rangle$$

с пространственным импульсом $\vec{k}$[3]. Если $\vec{k}$ стремится к нулю, состояние

[3] Действуя на него с помощью P^i (упражнение I.10.3) и учитывая условие $P^i|0\rangle = 0$, после интегрирования по частям получаем

$$P^i|s\rangle = \int d^D x e^{i\vec{k}\vec{x}} \left[P^i, J^0(\vec{x}, t)\right] |0\rangle = k^i|s\rangle$$

переходит в состояние $Q|0\rangle$ с нулевой энергией (что следует из (9)). Значит, если импульс состояния $|s\rangle$ стремится к нулю, его энергия также стремится к нулю. В релятивистской теории это в точности означает, что состояние $|s\rangle$ описывает безмассовую частицу.

Из доказательства следует, что теорема практически универсальна: она справедлива для любой спонтанно нарушенной непрерывной симметрии.

Подсчет намбу-голдстоуновских бозонов

Из нашего доказательства следует, что число намбу-голдстоуновских бозонов, очевидно, равно числу сохраняющихся зарядов, которые не оставляют вакуум инвариантным, т. е. не уничтожают $|0\rangle$. Для каждого такого заряда Q^α можно построить состояние с нулевой энергией $Q^\alpha|0\rangle$.

В нашем примере присутствует лишь один ток $J_\mu = i(\varphi_1\partial_\mu\varphi_2 - -\varphi_2\partial_\mu\varphi_1)$ и, следовательно, единственный намбу-голдстоуновский бозон. В общем случае, если лагранжиан остается инвариантен относительно группы симметрии G с $n(G)$-генераторами, а вакуум остается инвариантен относительно подгруппы H группы G с $n(H)$ генераторами, то число намбу-голдстоуновских бозонов равно $n(G) - n(H)$. Если вы хотите продемонстрировать свое владение математической лексикой, то можете сказать, что: намбу-голдстоуновские бозоны живут в фактор-пространстве G/H.

Ферромагнетики и спиновая волна

Из общности доказательства следует, что теорема Голдстоуна применима не только к физике элементарных частиц. В действительности она берет начало из физики конденсированных сред, классическим примером является ферромагнетик. Гамильтониан, в который входит только взаимодействие нерелятивистских электронов с ионами в твердом теле, является инвариантным относительно группы вращения $SO(3)$, но вектор намагничивания $\vec{M}$ выбирает одно направление, так что ферромагнетик остается инвариантным только относительно подгруппы $SO(2)$, которую составляют вращения вокруг оси $\vec{M}$. Теорему Намбу-Голдстоуна легко визуализировать с физической точки зрения. Рассмотрим «спиновую волну», в которой локальные намагниченности $\vec{M}(\vec{x})$ медленно меняются от точки к точке. Физик, живущий в мире, малом по сравнению с длиной волны, даже не представляет себе, что он больше не находится в «вакууме». Следовательно, частота волны должна стремиться к нулю по мере того, как длина волны стремится к бесконечности. Это, конечно, такой же эвристический аргумент, к которым мы прибегали ранее. Обратите внимание, что квантовая механика нужна только для того, чтобы сопоставить волновому вектору $\vec{k}$ импульс, а частоте ω - энергию. К вопросу о магнетиках и спиновых волнах вернемся в главах V.3 и VI.5.

Квантовые флуктуации и размерность пространства-времени

Наше обсуждение спонтанного нарушения симметрии является, по существу, классическим. Что происходит при учете квантовых флуктуаций? Более детально рассмотрим этот вопрос в главе IV.3, а пока обратимся снова к лагранжиану (5). В основном состоянии $\varphi_1 = v$ и $\varphi_2 = 0$. Вспомним, что в матрацной модели теории скалярного поля масса обусловлена действием пружин, приводящим матрац в равновесное положение. Член $-\mu^2\varphi_1'^2$ (обратите внимание на штрих) в выражении (5) означает, что для ухода поля φ_1' из основного состояния, в котором $\varphi_1' = 0$, необходимо затратить действие. Но теперь нас беспокоит следующее: поле φ_2 безмассово. Сможет ли оно уйти из своего основного состояния? Чтобы ответить на этот вопрос, вычислим среднеквадратичную флуктуацию

$$\begin{aligned}\left\langle(\varphi_2(0))^2\right\rangle &= \frac{1}{Z}\int D\varphi e^{iS(\varphi)}\left[\varphi_2(0)\right]^2 = \\ &= \lim_{x\to 0}\frac{1}{Z}\int D\varphi e^{iS(\varphi)}\varphi_2(x)\varphi_2(0) = \\ &= \lim_{x\to 0}\int\frac{d^dk}{(2\pi)^d}\frac{e^{ikx}}{k^2}.\end{aligned}$$

(Мы узнаем функциональный интеграл, определяющий пропагатор; см. главу I.7.) Верхний предел интеграла в выражении (10) обрезается на некотором значении Λ (соответствующем обратному значению шага решетки, если говорить о ферромагнетике). Как говорилось в главе III.1 (и будет говориться в главе VIII.3), нас не должна волновать ультрафиолетовая расходимость для больших значений k. Мы должны беспокоиться по поводу возможной инфракрасной расходимости для малых значений k. (Замечу, что в случае массивного поля член $1/k^2$ в выражении (10) следовало бы заменить членом $1/\left(k^2+\mu^2\right)$ и тогда бы не стало инфракрасной расходимости.) Видим, что при $d > 2$ инфракрасной расходимости нет. Рассмотренная нами схема спонтанного нарушения непрерывной симметрии справедлива в нашем (3+1)-мерном мире. Однако при $d \leqslant 2$ среднеквадратичная флуктуация поля φ_2 оказывается бесконечной, так что наша схема уже не действует. Приходим к теореме Коулмена-Мермина-Вагнера (независимо доказанной теоретиком в области физики элементарных частиц и двумя теоретиками в области физики конденсированных сред), которая гласит, что для $d = 2$ спонтанное нарушение непрерывной симметрии невозможно. Обратите внимание, что данная теорема справедлива не только

для рассмотренной нами $O(2)$-симметрии, но и для любой непрерывной симметрии, поскольку она зависит исключительно от присутствия намбу-голдстоуновских полей.

В наших примерах спонтанного нарушения симметрии спонтанно нарушалась скалярным полем φ, но ничего не говорится о том, что поле φ должно быть элементарным. Во многих конденсированных средах, например в сверхпроводниках, симметрии нарушаются спонтанно, хотя система состоит из электронов и атомных ядер. Поле φ генерируется динамически, например, как связанное состояние двух электронов в сверхпроводниках. Подробнее об этом в главе V.4. Спонтанное нарушение симметрии динамически генерируемым полем иногда называют динамическим нарушением симметрии [4].

Упражнения

IV.1.1. Явно покажите существование $N-1$ намбу-голдстоуновских бозонов в $G=O(N)$ в примере (2).

IV.1.2. Постройте аналог выражения (2) с N комплексными скалярными полями инвариантный относительно $SU(N)$. Найдите число намбу-голдстоуновских бозонов для случая, когда одно из скалярных полей получает вакуумное среднее.

[4] Эта глава посвящена памяти покойного Йорге Свиека.

30.1.2 Zi.IV.2 Pion as a Nambu–Goldstone Boson

Кризис теории поля

В 1950-1960-х гг. после нескольких впечатляющих успехов квантовой теории поля в случае электромагнитного взаимодействия ученые, естественно, были решительно настроены на то, чтобы применить ее к случаям сильного и слабого взаимодействий. Как мы уже знаем, теория поля оказывается неперенормируемой в случае слабого взаимодействия. Что касается сильных взаимодействий, то теория поля оказывается несостоятельной по другим причинам. С одной стороны, по мере того как экспериментальной оценке стало поддаваться все больше и больше адронов (т. е. сильно взаимодействующих частиц), ученые пришли к выводу, что, если каждому адрону поставить в соответствие поле, результирующая теория поля окажется запутанной, с огромным количеством произвольных констант связи. Даже если мы ограничимся рассмотрением нуклонов и пионов, известная константа связи во взаимодействии между пионами и нуклонами будет большой. (Отсюда термин «сильное взаимодействие».) Пертурбативный подход, так впечатляюще эффективный в квантовой электродинамике, обречен на неудачу.

Многие выдающиеся физики того времени выступали за отказ от квантовой теории поля, а в некоторых высших учебных заведениях квантовую теорию поля даже исключили из учебной программы. И лишь в начале 1970-х гг. произошло триумфальное возвращение квантовой теории поля. Была сформулирована теория поля для сильных взаимодействий в терминах кварков и глюонов, а не адронов. Мы рассмотрим ее в главе VII.3.

Слабый распад пионов

Чтобы понять кризис, который наступил для теории поля, вернемся назад во времени и представим себе, что мог бы делать физик-теоретик в конце 1950-х гг. Поскольку эта книга вовсе не посвящена физике элементарных частиц, изложу лишь основные факты. Подробности найдете в соответствующих монография [1]. К тому времени были измерены многие полулептонные распады, такие как $n\to p+e^-+\bar\nu, \pi^-\to e^-+\bar\nu$ и $\pi^-\to\pi^0+e^-+\bar\nu$.$\beta$-распад нейтрона $n\to p+e^-+\bar\nu$; тот самый процесс, для которого Ферми предложил свою теорию, в то время описывался лагранжианом вида $\mathcal{L}=G\left[\bar e\gamma^\mu\left(1-\gamma_5\right)\nu\right]\left[\bar p\gamma_\mu\left(1-\gamma_5\right)n\right]$, где n - поле нейтрона, поглощающее нейтрон, p - поле протона, поглощающее протон, ν - поле нейтрино, поглощающее нейтрино (или рождающее антинейтрино, как в β-распаде), а e - поле электрона, поглощающее электрон.

Становится понятно, почему теоретики не могли написать поле для каждого адрона и лагранжиан для каждого процесса распада, хотя фактически занимались этим все время. Вместо этого следует записать

$$\mathcal{L} = G\left[\bar{e}\gamma^{\mu}\left(1-\gamma_5\right)\nu\right]\left(J_{\mu}-J_{5\mu}\right)$$

где J_{μ} и $J_{5\mu}$ - два тока, преобразующихся как лоренцовский и аксиальный векторы соответственно. В канонической формулировке теории поля токи J_{μ} и $J_{5\mu}$ рассматриваются как квантовые операторы. Наша задача тогда сводилась бы к вычислению матричных элементов между адронными состояниями, $\langle p\left|\left(J_{\mu}-J_{5\mu}\right)\right|n\rangle, \langle 0\left|\left(J_{\mu}-J_{5\mu}\right)\right|\pi^{-}\rangle, \langle\pi^{0}\left|\left(J_{\mu}-J_{5\mu}\right)\right|\pi^{-}\rangle$ и т. д., которые соответствуют трем распадам, упоминаемым выше. (Следует пояснить, что, хотя речь идет о слабых распадах, вычисление этих матричных элементов является задачей о сильном взаимодействии. Другими словами, чтобы понять природу распада, сильное взаимодействие необходимо учитывать во всех порядках константы сильного взаимодействия, а слабое взаимодействие достаточно учесть в низшем порядке по константе слабого взаимодействия G.) На самом деле существует прецедент, из-за которого мы придерживаемся этой точки зрения. Чтобы вычислить ядерный β-распад $(Z,A)\rightarrow(Z+1,A)+e^{-}+\bar{\nu}$, Ферми вовсе не записывал отдельный лагранжиан для каждого ядра. Вычислением матричного элемента $\langle Z+1,A\left|\left[\bar{p}\gamma_{\mu}\left(1-\gamma_5\right)n\right]\right|Z,A\rangle$ должен был заниматься скорее теоретикспециалист по атомному ядру. Аналогично, вычисление матричных элементов $\langle p\left|\left(J_{\mu}-J_{5\mu}\right)\right|n\rangle$ есть задача теоретика-специалиста по сильным взаимодействиям.

Для нашего сюжета я сосредоточусь на вычислении матричного элемента аксиального тока J_5^{μ} между нейтроном и протоном. Слегка изменим обозначения: отныне мы не будем говорить, что нейтрон находится в начальном состоянии, а протон - в конечном состоянии, вместо этого будем указывать на то, что импульс нейтрона равен p, а импульс протона - p'.

[1] См., например. E. Commins and P. H. Bucksbaum, Weak Interactions of Leptons and Quarks (Комминс Ю., Буксбаум Б. Слабые взаимодействия лептонов и кварков. - М.: Энергоатомиздат, 1987.). Не забывайте, к тому же, что все поля и токи в выражении (1) являются функциями от координаты пространства-времени x. Следовательно, необходимо вычислить $\langle p'\left|J_5^{\mu}(x)\right|p\rangle$, что, однако, вследствие трансляционной инвариантности равно $\langle p'\left|J_5^{\mu}(0)\right|p\rangle e^{+i(p'-p)\cdot x}$. Поэтому просто вычисляем $\langle p'\left|J_5^{\mu}(0)\right|p\rangle$ и опускаем 0. Заметим, что спиновые значки уже опущены.

Лоренцевская инвариантность и четность позволяют нам несколько продвинуться. Из них следует, что [2]:

$$\langle p'\left|J_5^{\mu}\right|p\rangle = \bar{u}\left(p'\right)\left[\gamma^{\mu}\gamma^{5}F\left(q^2\right)+q^{\mu}\gamma^{5}G\left(q^2\right)\right]u(p)$$

где $q\equiv p'-p$ [сравните с выражением (III.6.7)]. Но лоренцевская инвариантность и четность больше ничего не могут для нас сделать, поскольку мы ничего не знаем о «форм-факторах» $F\left(q^2\right)$ и $G\left(q^2\right)$.

Аналогично для матричного элемента $\langle 0\left|J_5^{\mu}\right|\pi^{-}\rangle$лоренцевская инвариантность означает следующее:

$$\langle 0\left|J_5^{\mu}\right|k\rangle = fk^{\mu}$$

Начальное состояние я обозначил импульсом k для пиона. Правая часть уравнения (3) должна быть вектором и, так как k - единственный имеющийся вектор, пропорциональной k. Подобно параметрам $F\left(q^2\right)$ и $G\left(q^2\right)$, константа f - это величина, определяемая сильным взаимодействием, вычислять которую мы не умеем. С другой стороны, значения $F\left(q^2\right), G\left(q^2\right)$ и f можно установить экспериментальным путем. Например, ясно, что скорость распада $\pi^{-}\rightarrow e^{-}+\bar{\nu}$ зависит от f^2.

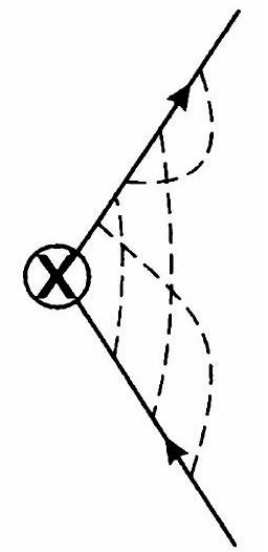
Слишком много диаграмм

Давайте заглянем чрез плечо теоретика конца 1950-х гг., пытающегося вычислить $\langle p'\left|J_6^{\mu}\right|p\rangle$ и $\langle 0\left|J_5^{\mu}\right|k\rangle$ в выражении (2). Он бы начал строить фейнмановские диаграммы,

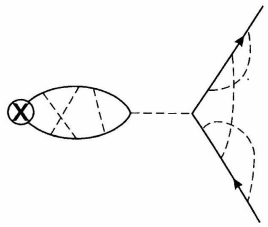
подобные изображенным на рис. IV.2.1 и IV.2.2, и вскоре бы осознал всю безнадежность своей попытки. Из-за сильной связи ему пришлось бы вычислить бесконечное число диаграмм, даже если бы сильное взаимодействие описывалось теорией поля, понятием уже отвергнутым многими светилами того времени.

Я не собираюсь вдаваться во все подробности, связанные с захватывающей историей развития данной области, полной недоразумений и тупиковых моментов. Вместо этого я, с оглядкой на прошлое, выбираю педагогически наилучший подход для изложения сюжета.

[2] Другой возможный член вида $(p' + p)^\mu \gamma^5$ зануляется вследствие симметрии относительно зарядового сопряжения и изоспиновой симметрии.



(a)



(b)

Рис. IV.2.1

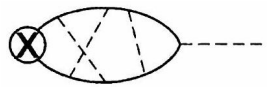


Рис. IV.2.2

Пион очень легкий

Переломным моментом послужило наблюдение, которое показало, что масса π^-, равная 139 МэВ, значительно меньше массы протона, равной 938 МэВ. Долгое время это воспринимали просто как факт и не пытались никак объяснять. В конце концов некоторые теоретики задумались, почему это один адрон должен быть гораздо легче другого.

Наконец они предприняли смелый шаг, вообразив «идеальный мир», в котором π^-безмассов. Идея была в том, что этот идеальный мир является хорошим приближением реального мира с точностью около 15% ($\sim 139/938$).

Помните ли вы одно обстоятельство, при котором происходит естественное появление безмассовой и бесспиновой частицы? Конечно, это спонтанное нарушение симметрии! В одном из тех ярких озарений, которые определяли историю физики элеметарных частиц, теоретики предположили, что π-мезоны есть намбу-голдстоуновские бозоны некоторой спонтанно нарушенной симметрии. Действительно, умножим (3) на k_μ:

$$k_\mu \langle 0 | J_5^\mu | k \rangle = f k^2 = f m_\pi^2,$$

что равно нулю в идеальном мире. Из обсуждения трансляционной инвариантности вспомним, что

$$\langle 0 | J_5^\mu(x) | k \rangle = \langle 0 | J_5^\mu(0) | k \rangle e^{-ik\cdot x}$$

и, следовательно,

$$\langle 0 | \partial_\mu J_5^\mu(x) | k \rangle = -ik_\mu \langle 0 | J_5^\mu(0) | k \rangle e^{-ik\cdot x}.$$

Таким образом, если в идеальном мире сохраняется аксиальный ток, $\partial_\mu J_5^\mu(x) = 0$, из условия $k_\mu \langle 0 | J_5^\mu | k \rangle = 0$ и уравнения (5) следует $m_\pi^2 = 0$.

Для рассматриваемого нами идеального мира характерна симметрия, известная под названием киральной симметрии сильного взаимодействия. В основном состоянии, в котором мы живем, симметрия нарушается спонтанно и образуется π-мезон как намбу-голдстоуновский бозон. Связанный с этой симметрией нетеровский ток есть сохраняющийся ток J_5^μ.

Вы, наверное, уже догадались, что совершенные нами выкладки тесно связаны с доказательством теоремы Намбу-Голдстоуна, приведенным в главе (IV.1).

Соотношение Голдбергера-Треймана

Подходим к кульминации. Умножим (2) на $(p'-p)_\mu$. Согласно той же трансляционной инвариантности имеем

$$(p'-p)_\mu \langle p' | J_5^\mu(0) | p \rangle = -i \langle p' | \partial_\mu J_5^\mu(x) | p \rangle e^{i(p'-p)\cdot x}.$$

Если $\partial_\mu J_5^\mu = 0$, полученное выражение зануляется. С другой стороны, умножая правую часть уравнения (2) на $(p'-p)_\mu$, получим $\bar{u}(p')\left[(\not{p}' - \not{p})\gamma^5 F(q^2) + q^2\gamma^5 G(q^2)\right] u(p)$. С учетом уравнения Дирака приходим к выводу, что

$$0 = 2m_N F\left(q^2\right) + q^2 G\left(q^2\right)$$

где m_N - масса нуклона.

Каждый из форм-факторов $F(q^2)$ и $G(q^2)$ определяется бесконечным числом фейнмановских диаграмм. Мы не можем их вычислить, но все же нам удалось связать их соотношением! Такая распространенная стратегия используется во многих областях физики: если сталкиваешься с различными величинами, которые не знаешь как вычислить, можно попытаться связать их.

Мы можем еще продвинуться, положив (5) $q \to 0$. Возвращаясь к (2), видим, что $F(0)$ измеряется экспериментально в распаде $n \to p + e^- + +\bar{\nu}$ (в масштабе сильных взаимодействий передача импульса пренебрежимо мала). Однако сталкиваемся с другой проблемой: масса нуклона m_N получается равной нулю!

Решить проблему помогает IV.2.1b: существует бесконечное количество диаграмм, полюс которых обусловлен не чем иным, как безмассовым π-мезоном, который дает

$$f q^\mu \frac{1}{q^2} g_{\pi NN} \bar{u}(p')\gamma^5 u(p).$$

Когда π-пропагатор соединяется с нуклонной линией, сумма бесконечного количества диаграмм дает экспериментально наблюдаемую константу связи $g_{\pi NN}$ пион-нуклонного

взаимодействия. Следовательно, возвращаясь к уравнению (2), видим, что при $q \sim 0$ форм-фактор равен $G(q^2) \sim f(1/q^2)\, g_{\pi NN}$. Подставляя это выражение в (5), приходим к известному соотношению Голдбергера-Треймана:

$$2m_N F(0) + f g_{\pi NN} = 0,$$

которое связывает четыре экспериментально измеримые величины. Как и ожидалось, его точность составляет 15%; погрешность обусловлена тем, что на самом деле мы не живем в мире с безмассовым π-мезоном.

К теории сильных взаимодействий

Искусство установления связи между бесконечными наборами фейнмановских диаграмм без их вычисления, и это вид искусства, требующий большого мастерства, было развито в теории дисперсионных соотношений и теории S-матрицы. Эти теории лежат в основе нашего современного понимания сильных взаимодействий. Стоит заметить, что важной составляющей дисперсионных соотношений являлось изучение аналитических свойств фейнмановских диаграмм. Например, в главе III.7 мы показали, что диаграмма поляризации вакуума имеет разрез. Суть аргументов Голдбергера-Треймана состоит в разделении бесконечного количества диаграмм на диаграммы с полюсом в комплексной q^2-плоскости и диаграммы без полюса (но с разрезом). Открытие того факта, что у сильного взаимодействия есть спонтанно нарушенная симметрия, явилось ключом к построению теории, лежащей в основе сильных взаимодействий, и неизбежно привело к появлению понятий кварков и глюонов.

Замечание для историка науки: отношение физиков-теоретиков к некоторой величине как к большой или как к малой зависит (очевидно) от культурной и интеллектуальной среды, в которой они выросли. Как-то Трейман сказал мне, что для поколения, выросшего вместе с атомной бомбой (сам Трейман был в составе вооруженных сил в Тихом океане), казалось верхом абсурдности считать 138 МэВ равными нулю, когда энергия, выделяемая при распаде одного ядра, оказывается порядка 10МВ. А сегодня новое поколение молодых теоретиков, работающих в области теории струн, совершенно спокойно считает равным нулю все, что меньше энергии Планка в 10^{19} ГэВ.

30.1.3 Zi.IV.3 Effective Potential

Квантовые флуктуации и нарушение симметрии

Такое важное явление, как спонтанное нарушение симметрии, основывается на нахождении минимумов классической потенциальной энергии $V(\varphi)$ квантовой теории поля. Естественно задаться вопросом, как на описанную картину повлияют квантовые флуктуации?

Рассмотрим еще раз лагранжиан (III.3.4)

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}\mu^2\varphi^2 - \frac{1}{4!}\lambda\varphi^4 + A(\partial\varphi)^2 + B\varphi^2 + C\varphi^4.$$

(Когда речь идет о квантовых флуктуациях, необходимо включать контрчлены, что мы и сделали.) Что вы знаете об этой теории? Для $\mu^2 > 0$ экстремум действия находится при $\varphi = 0$, и, квантуя малые флуктуации относительно $\varphi = 0$, получаем скалярные частицы, рассеивающиеся друг на друге. Для $\mu^2 < 0$ экстремум действия находится при некотором значении $\varphi_{\min}$, а дискретная симметрия $\varphi \to -\varphi$ спонтанно нарушается (см. главу IV.1). Что происходит при $\mu = 0$? Нарушать или не нарушать - вот в чем вопрос.

Первое предположение таково, что квантовые флуктуации нарушат симметрию. Теория $\mu = 0$ находится на грани нарушения симметрии и квантовые флуктуации должны бы вытолкнуть ее за эту грань. Представьте себе карандаш, балансирующий на своем острие, а затем «включите» квантовую механику.

Житейская мудрость

Следуя Швингеру и Йона-Лозиньо, построим формализм, который позволит ответить на поставленный вопрос. Рассмотрим теорию скалярного поля, задаваемую как

$$Z = e^{iW(J)} = \int D\varphi e^{i[S(\varphi)+J\varphi]}$$

[с удобным сокращением $J\varphi = \int d^4x J(x)\varphi(x)$]. Если вычислить функциональный интеграл, получим производящий функционал $W(J)$. Как говорилось в главе I.7, путем повторного дифференцирования W по источнику $J(x)$ можно получить любую функцию Грина и, следовательно, любую желаемую амплитуду рассеяния. В частности,

$$\varphi_c(x) \equiv \frac{\delta W}{\delta J(x)} = \frac{1}{Z}\int D\varphi e^{i[S(\varphi)+J\varphi]}\varphi(x).$$

Подстрочный индекс c традиционно используется для того, чтобы напомнить (см. приложение к главе I.8), что в каноническом формализме $\varphi_c(x)$ обозначает среднее $\langle 0|\hat{\varphi}|0\rangle$ квантового оператора $\hat{\varphi}$. Не путайте с переменной интегрирования φ из уравнения (3). Чтобы не загромождать запись, далее будем опускать индекс c без какого бы то ни было риска создать путаницу.

Если задан функционал W для J, можно осуществить преобразование Лежандра и получить функционал Γ для φ (т.е. φ_c). Под преобразованием Лежандра понимают простое соотношение

$$\Gamma(\varphi) = W(J) - \int d^4x J(x)\varphi(x).$$

Хоть соотношение и простое, будьте осторожны, когда раскрываете его смысл: оно определяет функционал для $\varphi(x)$ через неявную зависимость J от φ. В правой части выражения (4) необходимо выразить J через φ путем решения уравнения (3). Раскладываем функционал $\Gamma(\varphi)$ следующим образом:

$$\Gamma(\varphi) = \int d^4x \left[-V_{3\phi.}.(\varphi) + Z(\varphi)(\partial\varphi)^2 + ...\right]$$

где (..) обозначает члены с более высоким порядком ∂. Вскоре мы убедимся в справедливости обозначения $V_{.}(\varphi)$.

У преобразования Лежандра простая и удобная форма функциональной производной от Γ:

$$\frac{\delta\Gamma(\varphi)}{\delta\varphi(y)} = \int d^4x \frac{\delta J(x)}{\delta\varphi(y)}\frac{\delta W(J)}{\delta J(x)} - \int d^4x \frac{\delta J(x)}{\delta\varphi(y)}\varphi(x) - J(y) = -J(y).$$

Это соотношение можно считать «дуальным» соотношению $\delta W(J)/\delta J(x) = \varphi(x)$.

Если вам начинает казаться, что подобные преобразования вам уже встречались раньше в вашем физическом образовании, то вы правы! В курсе термодинамики вы прибегали к преобразованию Лежандра, связавшему свободную энергию с энергией $F = E - TS$, где F - функция от температуры T, а E - функция от энтропии S. Следовательно, J и φ образуют «сопряженные» пары, подобные паре T и S (или, что еще более очевидно, паре напряженность магнитного поля H и намагниченность M). Можете удостовериться, что это не просто совпадение.

Для J и φ, независимых от x, из уравнения (5) следует, что условие (6) сводится к виду:

$$V'_{\text{эфф.}}(\varphi) = J.$$

Это соотношение проясняет, для чего служит эффективный потенциал $V_{\text{эфф.}}(\varphi)$. Посмотрим, что произойдет в отсутствие внешнего источника J. Ответ очевиден: из соотношения (7) следует, что

$$V'(\varphi) = 0.$$

Другими словами, вакуумное среднее значение φ в отсутствие внешнего источника определяется минимумами $V(\varphi)$.

Первый порядок по квантовым флуктуациям

Все эти формальные манипуляции ничего не стоят, если мы не можем вычислить $W(J)$. В большинстве случаев мы можем найти $e^{iW(J)} = \int D\varphi e^{i[S(\varphi)+J\varphi]}$ лишь в приближении наискорейшего спуска (см. главу I.2). Найдем «точку» наискорейшего спуска $\varphi_s(x)$, т. е. решение уравнения

$$\left.\frac{\delta\left[S(\varphi) + \int d^4y J(y)\varphi(y)\right]}{\delta\varphi(x)}\right|_{\varphi_s} = 0$$

или, в более явной форме,

$$\partial^2\varphi_s(x) + V'\left[\varphi_s(x)\right] = J(x).$$

Запишем переменную интегрирования в (2) как $\varphi = \varphi_s + \widetilde{\varphi}$ и разложим до членов второго порядка $\widetilde{\varphi}$:

$$\begin{aligned} Z = e^{(i/\hbar)W(J)} &= \int D\varphi e^{(i/\hbar)[S(\varphi)+J\varphi]} \simeq \\ &\simeq e^{(i/\hbar)[S(\varphi)+J\varphi]} \int D\widetilde{\varphi} e^{(i/\hbar)\int d^4x \frac{1}{2}\left[(\partial\widetilde{\varphi})^2 - V''(\varphi_*)\widetilde{\varphi}^2\right]} = \\ &= e^{(i/\hbar)[S(\varphi)+J\varphi] - \frac{1}{2}\operatorname{tr}\log\left[\partial^2 + V''(\varphi_s)\right]}. \end{aligned}$$

Для представления детерминанта, возникшего в результате интегрирования по $\widetilde{\varphi}$, мы воспользовались тождеством (II.5.2). Обратите внимание, что я вернул постоянную Планка $\hbar$. Здесь φ_s, решение (10), следует рассматривать как функцию от J.

Теперь, когда мы определили

$$W(J) = \left[S\left(\varphi_s\right) + J\varphi_s\right] + \frac{i\hbar}{2}\operatorname{tr}\log\left[\partial^2 + V''\left(\varphi_s\right)\right] + O\left(\hbar^2\right),$$

переходим к преобразованию Лежандра. Опишу следующие действия более подробно:

$$\varphi = \frac{\delta W}{\delta J} = \frac{\delta\left[S\left(\varphi_s\right) + J\varphi_s\right]}{\delta\varphi_s}\frac{\delta\varphi_s}{\delta J} + \varphi_s + O(\hbar) = \varphi_s + O(\hbar).$$

В главном порядке $\hbar$ значение φ (известное раньше как φ_c) равно φ_s. Таким образом, из соотношения (4) получаем:

$$\Gamma(\varphi) = S(\varphi) + \frac{i\hbar}{2}\operatorname{tr}\log\left[\partial^2 + V''(\varphi)\right] + O\left(\hbar^2\right).$$

Несмотря на то что эта формула выглядит достаточно простой, на практике оказывается невозможным вычислить след для произвольного $\varphi(x)$: необходимо найти все собственные значения оператора $\partial^2 + V''(\varphi)$, найти их логарифм и просуммировать. Задача существенно упрощается, если мы анализируем функционал $\Gamma(\varphi)$ для φ, который не зависит от x, потому что в этом случае значение $V''(\varphi)$ постоянно, а оператор $\partial^2 + V''(\varphi)$ трансляционно-инвариантен и легко вычисляется в импульсном пространстве:

$$\begin{aligned}\operatorname{tr}\log\left[\partial^2+V''(\varphi)\right] &= \int d^4x\,\langle x\,|\log\left[\partial^2+V''(\varphi)\right]|\,x\rangle = \\ &= \int d^4x\int\frac{d^4k}{(2\pi)^4}\langle x|k\rangle\,\langle k\,|\log\left[\partial^2+V''(\varphi)\right]|\,k\rangle\,\langle k|x\rangle = \\ &= \int d^4x\int\frac{d^4k}{(2\pi)^4}\log\left[-k^2+V''(\varphi)\right].\end{aligned}$$

С учетом (5) получаем:

$$V_{3\phi\phi.}(\varphi)=V(\varphi)-\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\log\left[\frac{k^2-V''(\varphi)}{k^2}\right]+O\left(\hbar^2\right),$$

известный как эффективный потенциал Коулмена-Вайнберга. Итак, мы вычислили поправку к классическому потенциалу $V(\varphi)$ до порядка $\hbar$. Напоминаю, что мы добавили независимую от φ константу, чтобы сделать аргумент логарифма безразмерным.

Можно дать красивую физическую интерпретацию формулы (14). Определим так называемое фоновое поле, заполнив Вселенную скалярным полем $\varphi(x)$, принимающим значение φ. Для $V(\varphi)=\frac{1}{2}\mu^2\varphi^2+(1/4!)\lambda\varphi^4$ имеем $V''(\varphi)=\mu^2+\frac{1}{2}\lambda\varphi^2\equiv\mu(\varphi)^2$, что соответствует квадрату зависящей от φ эффективной массы скалярной частицы, распространяющейся в фоновом поле φ. Квадрат массы μ^2 в лагранжиане получает поправку $\frac{1}{2}\lambda\varphi^2$, обусловленным взаимодействием частицы с фоновым полем φ. Теперь становится очевидным смысл выражения (14): первый член $V(\varphi)$ представляет собой плотность классической энергии, содержащейся в фоновом поле φ, а второй член есть плотность энергии вакуума для скалярного поля, квадрат массы которого равен $V''(\varphi)$ (см. выражение (II.5.3) и упражнение IV.3.4).

Ваша теория перенормировок в деле

Интеграл в (14) квадратично расходится, а если точнее, квадратично зависит от обрезания. Ничего страшного, мы ввели три контрчлена, лишь два из которых нам потребуются, поскольку φ не зависит от x. Поэтому на самом деле получаем:

$$\begin{aligned}V_{\text{эф.}}(\varphi)=V(\varphi)+\frac{\hbar}{2}\int\frac{d^4k_E}{(2\pi)^4}\log\left[\frac{k_E^2+V''(\varphi)}{k_E^2}\right]+ \\ +B\varphi^2+C\varphi^4+O\left(\hbar^2\right),\end{aligned}$$

где сделали виковский поворот к евклидову интегралу (см. приложение D). Используя соотношение (D.9) и интегрируя до $k_E^2=\Lambda^2$, получаем (опуская $\hbar$)

$$V_{.}(\varphi)=V(\varphi)+\frac{\Lambda^2}{32\pi^2}V''(\varphi)-\frac{\left[V''(\varphi)\right]^2}{64\pi^2}\log\frac{e^{\frac{1}{2}}\Lambda^2}{V''(\varphi)}+B\varphi^2+C\varphi^4$$

Как и ожидалось, интеграл квадратично и логарифмически зависит от обрезания Λ^2, поскольку подынтегральное выражение в (15) для больших k_E^2 имеет вид $1/k_E^2$.

Такова теория перенормировки в действии! Так как V есть многочлен четвертой степени от $\varphi, V''(\varphi)$ - это квадратичный многочлен, а $\left[V''(\varphi)\right]^2$ - многочлен четвертой степени. Следовательно, для устранения зависимости от обрезания достаточно ввести два контрчлена $B\varphi^2+C\varphi^4$. Рассмотренный пример, в частности, ясно демонстрирует, как работает метод добавления контрчленов.

Чтобы понять, что плохого может произойти в неперенормируемой теории, представим себе, что V есть многочлен шестой степени от φ. Мы могли мы добавить три контрчлена $B\varphi^2+C\varphi^4+D\varphi^6$, но этого будет недостаточно, поскольку $\left[V''(\varphi)\right]^2$ будет являться многочленом восьмой степени. Это значит, что нам следовало начать с V как многочлена

восьмой степени, но тогда выражение $[V''(\varphi)]^2$ было бы многочленом двенадцатой степени. Очевидно, что в итоге придем к многочлену бесконечной степени. В этом состоит отличительное свойство неперенормируемой теории: в ее неутолимой потребности в контрчленах.

Наложение условий перенормируемости

Вернемся из кошмара бесконечного количества окружающих нас контрчленов и обратимся к простой перенормируемой теории φ^4. В главе III.3 мы фиксировали контрчлены заданием условий для различных амплитуд рассеяния. Теперь нам надо зафиксировать коэффициенты B и C заданием двух условий для $V.(\varphi)$ при определенных значениях φ. Мы, так сказать, работаем с пространством полей, а не пространством импульсов, но с прежней концептуальной основой.

Можно было бы начать с общего многочлена четвертой степени $V(\varphi)$, но лучше сначала ответим на главный вопрос, поставленный в главе: что произойдет при $\mu = 0$, т. е. когда $V(\varphi) = (1/4!)\lambda\varphi^4$? Выкладки тоже упрощаются.

Вычисляя (16), получаем:

$$V_{\phi..}(\varphi) = \left(\frac{\Lambda^2}{64\pi^2}\lambda + B\right)\varphi^2 + \left(\frac{1}{4!}\lambda + \frac{\lambda^2}{(16\pi)^2}\log\frac{\varphi^2}{\Lambda^2} + C\right)\varphi^4 + O\left(\lambda^3\right)$$

(после включения нескольких независимых от φ коэффициентов в C). Видим, что зависимость от Λ можно включить в B и C.

Мы начали с многочлена четвертой степени $V(\varphi)$. Квантовые флуктуации породили квадратично расходящийся член φ^2, который можно сократить с контрчленом B. Что означает условие $\mu = 0$? Оно означает, что член $(d^2V/d\varphi^2)|_{\varphi=0}$ исчезает. Теория $\mu = 0$ предполагает обязательное сокращение квадрата ренормированной массы, обозначаемой в нашем случае коэффициентом при φ^2. Таким образом, приходим к первому условию:

$$\left.\frac{d^2V_{\text{Эфф.}}}{d\varphi^2}\right|_{\varphi=0} = 0$$

Иначе говоря, мы хотим, чтобы $B = -\left(\Lambda^2/64\pi^2\right)\lambda$ в данном порядке.

Аналогично можно было бы сформулировать второе условие как требование равенства $(d^4V_{\text{эфф.}}/d\varphi^4)|_{\varphi=0}$ некоторой константе связи. Однако,

дифференцируя член $\varphi^4\log\varphi$ в выражении для $V_{\text{эфф.}}$ четыре раза, получим

член $\log\varphi$, значение которого не определено при $\varphi = 0$. Мы должны наложить условие на $d^4V_{\text{эфф.}}/d\varphi^4$ не для $\varphi = 0$, а для φ, равного некоторой произвольно выбранной массе M. (Вспомним, что размерность φ совпадает с размерностью массы.) Следовательно, второе условие будет иметь вид:

$$\left.\frac{d^4V_{\text{эфф.}}}{d\varphi^4}\right|_{\varphi=M} = \lambda(M)$$

где $\lambda(M)$ - явно зависимая от M константа связи. Подставляя в (18) выражение

$$V_{\text{эфф.}}(\varphi) = \left(\frac{1}{4!}\lambda + \frac{\lambda^2}{(16\pi)^2}\log\frac{\varphi^2}{\Lambda^2} + C\right)\varphi^4 + O\left(\lambda^3\right)$$

видим, что $\lambda(M)$ равно λ плюс поправки $O\left(\lambda^2\right)$, среди которых фигурирует член $\lambda^2\log M$. Дифференцируя $\lambda(M)$, получаем явное соотношение:

$$\begin{aligned} M\frac{d\lambda(M)}{dM} &= \frac{3}{16\pi^2}\lambda^2 + O\left(\lambda^3\right) = \\ &= \frac{3}{16\pi^2}\lambda(M)^2 + O\left[\lambda(M)^3\right] \end{aligned}$$

где второе равенство справедливо в данном порядке. Анализ соотношения показывает, как константа связи $\lambda(M)$ зависит от масштаба массы M, при котором она задается. Вспомните упражнение III.1.3. К данному соотношению мы вернемся в главе VI.7, когда речь пойдет о ренорм-группе.

А пока продолжим. Выразим из условия (18) константу C и, подставив

ее в выражение для $V_{\text{эфф.}}$, получим:

$$V_{\text{эфф.}}\,(\varphi) = \frac{1}{4!}\lambda(M)\varphi^4 + \frac{\lambda(M)^2}{(16\pi)^2}\varphi^4\left(\log\frac{\varphi^2}{M^2} - \frac{25}{6}\right) + O\left[\lambda(M)^3\right].$$

Надеюсь, вас не удивляет, почему больше нет константы C и обрезания Λ. Вот вам и перенормируемая теория!

Тот факт, что $V_{\text{эфф.}}$ не зависит от произвольно выбранной массы M, т. е.

$M\,(dV_{\text{эфф.}}\,/dM) = 0$, в указанном порядке выражается соотношением (19).

Нарушение симметрии квантовыми флуктуациями

Теперь ответим на главный вопрос: нарушать или не нарушать?

Квантовые флуктуации генерируют поправку к потенциалу вида $+\varphi^4\log\varphi^2$, но значение $\log\varphi^2$ чрезвычайно велико и отрицательно для малых φ. Поправка $O(\hbar)$ превосходит классический $O\,(\hbar^0)$-потенциал $+\varphi^4$ вблизи $\varphi = 0$. Квантовые флуктуации нарушают дискретную симметрию $\varphi \to -\varphi$.

Достаточно легко определить минимумы $\pm\varphi_{\min}$ эффективного потенциала $F_{3\phi\phi}(\varphi)$ (чтобы иметь представлсние, нарисуйте потенциал от φ). Более детальный анализ показывает, что нельзя всерьез рассматривать точ-

ное значение $\varphi_{\min}$; $V_{\text{эфф.}}$ имеет вид $\lambda\varphi^4(1+\lambda\log\varphi+\ldots)$, т. е. параметром

разложения в действительности будет $\lambda\log\varphi$. [Постарайтесь убедиться, что

$(\ldots)$ начинаются с $[\lambda\log\varphi]^2$.] Минимум $\varphi_{\min}$ функции $V_{\text{эфф.}}$ лежит там, где

параметр разложения порядка единицы. В упражнении к главе IV.6 вы познакомитесь с остроумным способом обойти эту проблему.

Фермионы

В выражении (11) φ_s выступает в качестве внешнего поля, а $\widetilde{\varphi}$ соответствует квантовому полю, по которому проводится интегрирование. Роль $\widetilde{\varphi}$ может играть поле фермиона ψ. Добавим в лагранжиан член $\bar{\psi}(i\not{\partial} - m - -f\varphi)\psi$. В интеграле по траекториям

$$Z = \int D\varphi D\bar{\psi} D\psi e^{i\int d^4x\left[\frac{1}{2}(\partial\varphi)^2 - V(\varphi) + \bar{\psi}(i\not{\partial} - m - f\varphi)\psi\right]}$$

можно сначала проинтегрировать по ψ и получить

$$Z = \int D\varphi e^{i\int d^4x\left[\frac{1}{2}(\partial\varphi)^2 - V(\varphi)\right] + \operatorname{tr}\log(i\not{\partial} - m - f\varphi)}.$$

Совершая те же шаги, что и в (13), приходим к выводу, что вклад фермионного поля в $V_{\text{эфф.}}\,(\varphi)$ равен:

$$V_F(\varphi) = +i\int\frac{d^4p}{(2\pi)^4}\operatorname{tr}\log\frac{\not{p} - m - f\varphi}{\not{p}}.$$

(След в (23) берется по гамма-матрицам.) Из главы II.5 следует, что физически параметр $V_F(\varphi)$ дает вакуумную энергию фермиона с эффективной массой $m(\varphi) \equiv m + f\varphi$.

Получить след логарифма можно с помощью $\operatorname{tr}\log M = \log\det M$ (II.5.2), циклически переставляя множители в детерминанте:

$$\begin{aligned}\operatorname{tr}\log(\not p - a) &= \operatorname{th}\log\gamma^5(\not p - a)\gamma^5 = \operatorname{tr}\log(-\not p - a) = \\ &= \frac{1}{2}\operatorname{tr}(\log(\not p - a) + \log(\not p + a)) + \frac{1}{2}\operatorname{tr}\log(-1) = \\ &= \frac{1}{2}\operatorname{tr}\log(-1)\left(p^2 - a^2\right).\end{aligned}$$

Следовательно,

$$\operatorname{tr}\log\frac{(\not p - a)}{\not p} = \frac{1}{2}\operatorname{tr}\log\frac{p^2 - a^2}{p^2} = 2\log\frac{p^2 - a^2}{p^2}$$

и поэтому

$$V_F(\varphi) = 2i\int\frac{d^4p}{(2\pi)^4}\log\frac{p^2 - m(\varphi)^2}{p^2}.$$

Обратите внимание на то, что общий знак противоположен знаку в (14): противоположность знаков между фермионной и бозонной петлями объяснялась в главе II.5.

Итак, эффективный потенциал, порождаемый квантовыми флуктуациями, интерпретируется следующим образом: он соответствует плотности энергии, обусловленной флуктуирующей энергией, которая полностью аналогична энергии нулевых колебаний гармонического осциллятора, для квантовых полей в фоновом поле φ (см. упражнение IV.3.5).

Упражнения

IV.3.1. Рассмотрите эффективный потенциал в (0+1)-мерном пространстве-времени:

$$V_{\text{s}\phi\phi.}(\varphi) = V(\varphi) + \frac{\hbar}{2}\int\frac{dk_E}{(2\pi)}\log\frac{k_E^2 + V''(\varphi)}{k_E^2} + O\left(\hbar^2\right).$$

Поскольку интеграл сходится, нет необходимости добавлять контрчлены. Однако $(0+1)$-мерная теория поля - это всего лишь квантовая механика. Вычислите интеграл и покажите, что $V_{\text{жф.}}$ не противоречит вашим знаниям о квантовой механике.

IV.3.2. Изучите $V_{\text{эфф. в}}$ $(1+1)$-мерном пространстве-времени.

IV.3.3. Рассмотрите безмассовое фермионное поле ψ, взаимодействующее со скалярным полем φ посредством $f\varphi\bar\psi\psi$ в (1+1)-мерном пространстве-времени. Покажите, что

$$V_F = \frac{1}{2\pi}(f\varphi)^2\log\frac{\varphi^2}{M^2}$$

после добавления подходящего контрчлена. В главе V.5, посвященной нестабильности Пайерлса, мы увидим, что полученный результат имеет большое значение для физики конденсированных сред.

IV.3.4. Попробуйте понять смысл выражения (14) с помощью фейнмановских диаграмм. Покажите, что $V_{\text{эф}}$ порождается бесконечным количеством диаграмм. [Указание: разложите логарифм в (14) в ряд по $V''(\varphi)/k^2$ и постарайтесь сопоставить для каждого члена ряда фейнмановскую диаграмму.]

IV.3.5. Рассмотрите электродинамику комплексного скалярного поля

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left[\left(\partial^\mu + ieA^\mu\right)\varphi^\dagger\right]\left[\left(\partial_\mu - ieA_\mu\right)\varphi\right] + \mu^2\varphi^\dagger\varphi - \lambda\left(\varphi^\dagger\varphi\right)^2.$$

Во Вселенной, заполненной скалярным полем $\varphi(x)$, принимающим независимое от x значение φ, лагранжиан будет содержать член $\left(e^2\varphi^\dagger\varphi\right)A_\mu A^\mu$, так что квадрат эффективной массы фотонного поля становится равным $M(\varphi)^2 \equiv e^2\varphi^\dagger\varphi$. Покажите, что его вклад в $V_.(\varphi)$ равен:

$$\int \frac{d^4k}{(2\pi)^4}\log\frac{k^2 - M(\varphi)^2}{k^2}$$

Сравните с выражениями (14) и (26). [Указание: для упрощения вычислений воспользуйтесь калибровкой Ландау.] Если вам нужна помощь, настоятельно рекомендую прочесть работу Коулмена и Вайнберга (S. Coleman and E. Weinberg, Phys. Rev. D7: 1888, 1973), которая очень понятно и четко написана.

30.1.4 Zi.IV.4 Magnetic Monopole

Квантовая механика и магнитные монополи

Довольно любопытно, хотя электрические заряды стали вполне привычным делом, никто никогда не видел магнитный заряд или монополь. В классической физике мы можем совершенно спокойно модифицировать одно из уравнений Максвелла, записав его в виде $\vec{\nabla}\cdot\vec{B} = \rho_M$, где ρ_M обозначает плотность магнитных монополей. При этом мы больше не можем представлять поле $\vec{B}$ как $\vec{B} = \vec{\nabla}\times\vec{A}$, поскольку иначе $\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot\vec{\nabla}\times\vec{A} = \varepsilon_{ijk}\partial_i\partial_j A_k = 0$. Ньютон и Лейбниц сообщили нам, что производные коммутируют друг с другом.

И что из этого? Действительно, кого волнует, что $\vec{B}$ больше нельзя записать в виде $\vec{\nabla}\times\vec{A}$? Векторный потенциал $\vec{A}$ ввели в физику исключительно из математических соображений, причем до сих пор студентам так и говорят в курсе классического электромагнетизма. Как сказал выдающийся физик девятнадцатого века Хэвисайд: «Физику надо очистить от такой чепухи, как скалярные и векторные потенциалы; физический смысл имеют лишь поля $\vec{E}$ и $\vec{B}$ ».

С приходом квантовой механики, однако, оказалось, что Хэвисайд был не прав. Вспомните, например, нерелятивистское уравнение Шредингера для заряженной частицы в электромагнитном поле:

$$\left[-\frac{1}{2m}(\vec{\nabla} - ie\vec{A})^2 + e\phi\right]\psi = E\psi$$

Заряженные частицы непосредственно взаимодействуют с векторным и скалярным потенциалами $\vec{A}$ и ϕ, которые теперь рассматриваются иногда как более фундаментальные понятия, чем электромагнитные поля $\vec{E}$ и $\vec{B}$ (см. главу III.4). Для квантовой физики просто необходим векторный потенциал. Дирак гениально заметил, что эти утверждения ведут к внутреннему конфликту между квантовой механикой и понятием магнитных монополей. При более тщательном анализе он пришел к выводу, что квантовая механика вовсе не запрещает существование магнитных монополей. Просто она разрешает существование только таких монополей, которые несут определенное количество магнитного заряда.

Дифференциальные формы

Для дальнейшего обсуждения вопроса магнитных монополей и теории Янга-Миллса в следующей главе очень удобно пользоваться языком дифференциальных форм. Не бойтесь, нам понадобится лишь несколько элементарных понятий. Пусть x^μ обозначает D действительных переменных (т. е. индекс μ может принимать D разных значений), а A_μ (не обязательно электромагнитный калибровочный потенциал в нашем чисто математическом разделе) обозначает D функций от x. В наших приложениях x^μ - это координаты и, как мы увидим, дифференциальные формы имеют естественную геометрическую интерпретацию.

Назовем объект $A \equiv A_\mu dx^\mu$ 1-формой. Дифференциалы dx^μ рассматриваются следуя Ньютону и Лейбницу. При замене координат $x \to x'$, как обычно, $dx^\mu = (\partial x^\mu/\partial x'^\nu)\, dx'^\nu$, так что $A \equiv A_\mu dx^\mu = A_\mu (\partial x^\mu/\partial x'^\nu)\, dx'^\nu \equiv A'_\nu dx'^\nu$. Мы записали стандартный закон преобразования векторов при координатном преобразовании $A'_\nu = A_\mu (\partial x^\mu/\partial x'^\nu)$. В качестве примера рассмотрим $A = \cos\theta d\varphi$. Считая θ и φ угловыми координатами на 2-сфере (т. е. поверхности 3-шара), получаем $A_\theta = 0$ и $A_\varphi = \cos\theta$. Аналогично определяем p-форму как $H = (1/p!) H_{\mu_1\mu_2\ldots\mu_p} dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p}$. (Повторяющиеся индексы, как обычно, суммируются.) «Вырожденным» примером является 0-форма, назовем ее Λ, которая представляет собой скалярную функцию от координат x^μ. Примером 2-формы служит $F = (1/2!) F_{\mu\nu} dx^\mu dx^\nu$.

Приходим к вопросу, как рассматривать произведения дифференциалов. Из элементарного курса дифференциального исчисления мы знаем, 'то $dxdy$ представляет собой область бесконечно малого прямоугольника с длиной dx и шириной dy. Тогда мы чисто автоматически считали, что $dydx$ равняется $dxdy$. Порядок написания дифференциалов не был важен. ()днако если осуществить координатное преобразование, то $x = x(x', y')$ $y = y(x', y')$ будут функциями от новых координат x' и y'. Теперь посмотрим на

$$dxdy = \left(\frac{\partial x}{\partial x'}dx' + \frac{\partial x}{\partial y'}dy'\right)\left(\frac{\partial y}{\partial x'}dx' + \frac{\partial y}{\partial y'}dy'\right).$$

() братите внимание, что коэффициент при $dx'dy'$ равен $(\partial x/\partial x')(\partial y/\partial y')$, а коэффициент при $dy'dx'$ равен $(\partial x/\partial y')(\partial y/\partial x')$. Видим, что дифференциалы dx^μ лучше рассматривать как антикоммутирующие объекты [кото- рые математики называют грассмановыми переменными (вспомните главу II.5)], поэтому $dy'dx' = -dx'dy'$ и $dx'dx' = 0 = dy'dy'$. Тогда (2) сводится к виду:

$$dxdy = \left(\frac{\partial x}{\partial x'}\frac{\partial y}{\partial y'} - \frac{\partial x}{\partial y'}\frac{\partial y}{\partial x'}\right) dx'dy' \equiv J(x, y; x', y')\, dx'dy'.$$

Мы получили правильный якобиан $J(x, y; x', y')$, преобразующий элемент площади $dxdy$ в элемент площади $dx'dy'$.

Во многих текстах $dxdy$ записывают как $dx^\wedge dy$. Мы будем опускать знак $\wedge$, чтобы не загромождать записи.

Это небольшое упражнение показывает, что нам следует определить $dx^\mu dx^\nu = -dx^\nu dx^\mu$ и считать элемент площади $dx^\mu dx^\nu$ направленным. Элементы площади $dx^\mu dx^\nu$ и $dx^\nu dx^\mu$ одинаковы по величине и противоположны по направлению.

Теперь зададим дифференциальный оператор d, действующий на любую форму. Действуя им на p-форму H, получаем, по определению,

$$dH = \frac{1}{p!}\partial_\nu H_{\mu_1\mu_2\ldots\mu_p} dx^\nu dx^{\mu_1} dx^{\mu_2} \ldots dx^{\mu_p}.$$

Следовательно, $d\Lambda = \partial_\nu \Lambda dx^\nu$ и

$$dA = \partial_\nu A_\mu dx^\nu dx^\mu = \frac{1}{2}(\partial_\nu A_\mu - \partial_\mu A_\nu)\, dx^\nu dx^\mu.$$

Видим, что этот математический формализм сделан почти как по заказу для описания электромагнетизма. Если назвать 1-форму $A \equiv A_\mu dx^\mu$ потенциалом, а A_μ считать электромагнитным потенциалом, тогда $F = dA$ конечно будет 2-формой поля. Если

записать F через его компоненты $F = (1/2!)F_{\mu\nu}dx^\mu dx^\nu$, то $F_{\mu\nu}$ действительно будет электромагнитным полем.

Замечу, что x^μ не является формой, а dx^μ не обозначает действие d на форму.

Если хотите, можете считать дифференциальные формы всего лишь элегантным и компактным обозначением. Главное - рассматривать объекты A и F как нечто целое, не привязанное к определенной координатной системе. Такой подход удобен в том случае, когда приходится иметь дело с объектами, более сложными, чем A и F, например, в теории струн. Дифференциальные формы освобождают нас от использования целого моря индексов. Важным тождеством является следующее:

$$dd = 0$$

Оно означает, что двукратное действие d на любую форму дает нуль. В качестве упражнения можете проверить это утверждение. В частности, $dF = ddA = 0$. Если записать его в компонентах, придем к стандартному тождеству электромагнетизма - «тождеству Бьянки».

Замкнутая форма не обязательно точна глобально

Удобно сейчас ввести несколько новых терминов. Говорят, что p-форма α замкнута, если $d\alpha = 0$. Говорят, что она точна, если существует такая $(p-1)$-форма β, что $\alpha = d\beta$.

В этих терминах (4) означает, что точные формы замкнуты.

Верно ли обратное? В некотором смысле. Согласно лемме Пуанкаре, замкнутая форма локально точна. Другими словами, если $dH = 0$, где H - некоторая p-форма, то локально справедливо

$$H = dK$$

для некоторой $(p-1)$-формы K. Однако не обязательно, что $H = dK$ истинно в глобальном смысле, т. е. везде. На самом деле мы уже сталкивались с леммой Пуанкаре. Например, я уверен, вы знаете, что, если равен нулю ротор векторного поля, векторное поле локально становится градиентом некоторого скалярного поля.

Для интегрирования форм не нужна никакая дополнительная информация. Например, задана 2-форма $F = (1/2!)F_{\mu\nu}dx^\mu dx^\nu$. Для любого 2-многообразия M можно записать $\int\limits_M F$. Обратите внимание, что в интеграл уже включена мера, так что нет необходимости делать координатный выбор. Помните вы это или нет, но вас уже знакомили с одной важной теоремой:

$$\int\limits_M dH = \int\limits_{\partial M} H$$

где H – p-форма, а ∂M - граница $(p+1)$-мерного многообразия M.

Дираковское квантование магнитного заряда

После такого экскурса в математику мы готовы заняться физикой. Рассмотрим сферу, окружающую магнитный монополь с магнитным зарядом g. Тогда 2-форма электромагнитного поля задается как $F = (g/4\pi)d\cos\theta d\varphi$, что практически совпадает с определением магнитного монополя (см. упражнение IV.4.3). В частности, вычислим магнитный поток путем интегрирования F по сфере S^2:

$$\int\limits_{S^2} F = g$$

Я уже говорил, что элемент площади включается автоматически. Вы, должно быть, уже узнали в $d\cos\theta d\varphi = -\sin\theta d\theta d\varphi$ элемент площади на единичной сфере. Замечу, что

при «обычном обозначении» в выражении (7) будет использоваться магнитное поле $\vec{B} = (g/4\pi r^2)\,\widehat{r}$, где $\widehat{r}$ - единичный вектор в радиальном направлении.

Теперь приведу довольно математический, но строгий вывод дираковского квантования магнитного заряда g, впервые полученный Ву и Янгом.

Сначала вспомним, как работает калибровочная инвариантность на примере (II.7.3). При преобразовании поля электрона $\psi(x) \to e^{i\Lambda(x)}\psi(x)$ электромагнитное калибровочное поле изменяется на

$$A_\mu(x) \to A_\mu(x) + \frac{1}{ie} e^{-i\Lambda(x)} \partial_\mu e^{i\Lambda(x)}$$

или, в терминах дифференциальных форм,

$$A \to A + \frac{1}{ie} e^{-i\Lambda} d e^{i\Lambda}.$$

Дифференцируя, получаем:

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \Lambda(x).$$

Использованная здесь форма напоминает нам, что калибровочное преобразование определяется умножением на фазовый множитель $e^{i\Lambda(x)}$, так что $\Lambda(x)$ и $\Lambda(x) + 2\pi$ описывают одно и то же преобразование.

В квантовой механике A физическое (при всем нашем уважении к Хэвисайду), поэтому нам лучше задаться вопросом, какое значение A привело бы к $F = (g/4\pi) d\cos\theta d\varphi$. Ответ кажется очевидным: $A = (g/4\pi)\cos\theta d\varphi$. (Если решите проверить это, вычислив dA, не забывайте, что $dd = 0$.)

Однако не все так быстро; ваш друг-математик утверждает, что $d\varphi$ не определено у Северного и Южного полюсов. На бытовом языке это бы соответствовало ситуации, при которой вы стоите на Северном полюсе и пытаетесь определить свою долготу. То есть, строго говоря, нельзя писать $A = (g/4\pi)\cos\theta d\varphi$. Но вы ведь достаточно умны, чтобы возразить: как насчет $A_N = (g/4\pi)(\cos\theta - 1)d\varphi$? Если подействовать на A_N оператором d, получим желаемую F; дополнительный член $(g/4\pi)(-1)d\varphi$ зануляется d вследствие тождества (4).

Хорошо, но ваш друг-математик указывает на то, что наше A_N не определено на Южном полюсе.

- Все верно, - отвечаете вы, - я это учел, добавив индекс N[1]. Теперь зададим $A_S = (g/4\pi)(\cos\theta + 1)d\varphi$. Обратите внимание, что, действуя на A_S оператором d, мы снова получаем желаемое F. Однако теперь A_S определено везде, за исключением Северного полюса.

На математическом языке мы скажем, что калибровочное поле A определено локально, но не глобально. Калибровочное поле A_N определен на карте, которая покрывает северную полусферу и простирается на юг так далеко, как мы того захотим, но не включая Северный полюс. Аналогично, A_S определено на карте, которая покрывает южную полусферу и простирается так далеко на север, как мы того захотим, за исключением Северного полюса.

Что произойдет при наложении друг на друга двух участков координатной сетки, к примеру, вдоль экватора? Калибровочные потенциалы A_N и A_S не одинаковы:

$$A_S - A_N = 2\frac{g}{4\pi} d\varphi$$

Что теперь? Ах да, это калибровочная теория! Если A_S и A_N связаны калибровочным преобразованием, то все хорошо. Итак, с учетом (8) задаем условие $2(g/4\pi)d\varphi = (1/ie)e^{-i\Lambda}de^{i\Lambda}$ для некоторой фазовой функции $e^{i\Lambda}$. Из прямой проверки следует, что $e^{i\Lambda} = e^{i2(eg/4\pi)\varphi}$.

Однако значения $\varphi = 0$ и $\varphi = 2\pi$ описывают в точности одну и ту же точку. Чтобы $e^{i\Lambda}$ имело смысл, необходимо выполнение условия $e^{i2(eg/4\pi)(2\pi)} = e^{i2(eg/4\pi)(0)=1}$; иначе говоря, $e^{ieg} = 1$ или

$$g = \frac{2\pi}{e}n$$

где n - целое число. В этом состоит знаменитое открытие Дирака: магнитный заряд на магнитном монополе квантуется в единицах $2\pi/e$. Другими словами, если существует монополь, то электрический заряд квантуется в единицах $2\pi/g$.

Не забывайте, что форма F является точной локально, а не глобально; в противном случае, согласно (6), магнитный заряд $g = \int\limits_{S^2} F$ был бы равен нулю.

[1] От англ. «north» - «север». - Прим. пер. Я привел здесь строгий математический вывод частично для того, чтобы, во-первых, избавить вас от путаницы, с которой связывают данный вывод в более простых текстах. Во-вторых, подобного рода доказательства встречаются в более передовых областях физики, например, теории струн.

Электромагнитная дуальность

Тот факт, что между электрическим и магнитным полями может существовать дуальность, приводило физиков-теоретиков в затруднительное положение на протяжении полутора веков. Кстати, если вы почитаете Максвелла, то обнаружите, что он часто говорил о магнитных зарядах. Можете проверить, что уравнения Максвелла инвариантны относительно следующего изящного преобразования: $(\vec{E} + i\vec{B}) \to e^{i\theta}(\vec{E} + i\vec{B})$ в случае существования магнитных зарядов.

Одним из интригующих свойств уравнения (10) является следующее: если e мало, то g велико, и наоборот. Как бы выглядели магнитные заряды, если бы существовали? Они ничем бы не отличались от электрических зарядов: они тоже взаимодействуют с помощью потенциала $1/r$, отталкиваясь при одноименных и притягиваясь при разноименных зарядах. В принципе мы могли бы эффективно сформулировать теорию электромагнетизма в терминах магнитных зарядов, которые выступали бы в качестве электрических зарядов. В этом случае теория была бы сильно взаимодействующей, но роль константы связи играл бы параметр g, а не e.

Физиков-теоретиков интересует дуальность по той причине, что она позволяет получить представление о теориях поля в режиме сильной связи. При преобразовании дуальности теория поля в режиме слабой связи отображается в сильно взаимодействущую теорию поля. Именно поэтому, когда несколько лет назад ученые узнали о дуальности теорий струн другим теориям, это произвело огромное впечатление на специалистов в области теории струн: теперь мы знаем, как ведут себя теории струн в режиме сильной связи. Более подробно о дуальности читайте в главе VI.3.

Формы и геометрия

Проясним геометрический характер дифференциальных форм путем анализа электромагнитного тока заряженной частицы, движущейся вдоль мировой линии $X^\mu(\tau)$ в D-мерном пространстве-времени (см. рис. IV.4.1a):

$$J^\mu(x) = \int d\tau \frac{dX^\mu}{d\tau}\delta^{(D)}[x - X(\tau)]$$

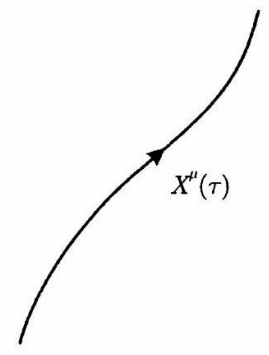
Физический смысл этой простой формулы из электродинамики очевиден: $dX^\mu/d\tau$ есть 4-скорость для заданного значения параметра τ («собственное время»), а дельта-функция гарантирует, что у x ток исчезает, если частица не проходит через x. Обратите внимание, что $J^\mu(x)$ инвариантно относительно репараметризации $\tau \to \tau'(\tau)$.

Более или менее очевидно обобщение на протяженный объект. Рассмотрим струну. Она движется вдоль мирового листа $X^\mu(\tau, \sigma)$ в пространстве-времени (см. рис. 1b), где σ - это

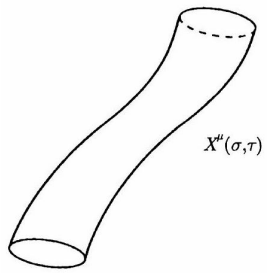
параметр, характеризующий положение вдоль струны. [Например, для замкнутой струны значение σ традиционно считается изменяющимся в диапазоне от 0 до 2π с $X^\mu(\tau,0) = X^\mu(\tau,2\pi)$.] Соответствующий струне ток равен

$$J^{\mu\nu}(x) = \int d\tau d\sigma \det \begin{pmatrix} \partial_\tau X^\mu & \partial_\tau X^\nu \\ \partial_\sigma X^\mu & \partial_\sigma X^\nu \end{pmatrix} \delta^{(D)}[x - X(\tau,\sigma)]$$

где $\partial_\tau \equiv \partial/\partial\tau$ и т. д.



(a)



(b)

Рис. IV.4.1

Детерминант обусловлен условием инвариантности относительно репараметризации $\tau \to \tau'(\tau,\sigma), \sigma \to \sigma'(\tau,\sigma)$. Отсюда следует, что $J^{\mu\nu}$ - это антисимметричный тензор. Значит, аналогом электромагнитного потенциала A_μ, взаимодействующего с током J^μ, является поле антисимметричного тензора $B_{\mu\nu}$, взаимодействующего с током $J^{\mu\nu}$. Таким образом, теория струн содержит 2-форму потенциала $B = \frac{1}{2}B_{\mu\nu}dx^\mu dx^\nu$ и соответствующую 3-форму поля $H = dB$. На самом деле в теории струн есть много p-форм.

Эффект Ааронова-Бома

Реальность калибровочного потенциала A продемонстрировали в 1959 году Ааронов и Бом. Рассмотрим магнитное поле B, ограниченное областью Ω, как изображено на рис. IV.4.2. Квантовая физика электрона описывается решением уравнения Шредингера (1). В формализме фейнмановского интеграла по траекториям амплитуда, соответствующая пути P, модифицируется за счет множитля $e^{ie\int_1 \vec{A}\cdot d\vec{x}}$, в котором одномсрный интсграл вычисляется вдоль траектории P. Таким образом, при вычислении интеграла по траекториям, определяющим вероятность перехода электрона из точки a в точку b (рис. IV.4.2), будет существовать интерференция между вкладами от траекторий 1 и 2 вида

$$\left(e^{ie\int_{P_1}\vec{A}\cdot d\vec{x}}\right)\left(e^{ie\int_{P_2}\vec{A}\cdot d\vec{x}}\right)^* = \left(e^{ie\oint \vec{A}\cdot d\vec{x}}\right)$$

Однако $\oint \vec{A} \cdot d\vec{x} = \int \vec{B} \cdot d\vec{S}$ соответствует потоку, ограниченному замкнутой кривой $(P_1 - P_2)$. Удивительно, но электрон чувствует влияние магнитного поля даже в том случае, когда он не попадает в область присутствия магнитного поля.

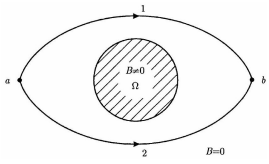


Рис. IV.4.2

Первая публикация результатов работы Аронова-Бома привела в смущение самого Нильса Бора. Впоследствии их выводы подтвердились опытами Тономуры и его сотрудников.

Однажды Коулмен рассказал о мысленном розыгрыше, позволяющем связать эффект Аронова-Бома с дираковским квантованием магнитного за- ряда. Допустим, мы помещаем очень тонкий невидимый соленоид в лабораторию ничего не подозревающего экспериментатора, например, нашего друга из главы III.1. Включаем ток и генерируем магнитное поле в соленоиде. Когда экспериментатор вдруг видит, как из ниоткуда начинает течь магнитный поток, то приходит в такой неописуемый восторг, что начинает планировать свою поездку в Стокгольм (за Нобелевской премией).

При каком условии экспериментатор не сможет обнаружить розыгрыш? Дотошный экспериментатор может начать рассеивать электроны, чтобы попытаться найти соленоид. Условием, гарантирующим, что он не увидит эффект Аронова-Бома и, следовательно, не обнаружит розыгрыш, служит тот факт, что поток, текущий через соленоид, должен быть кратен $2\pi/e$. То есть заряд магнитного монополя точно соответствует магнитному заряду, предсказанному Дираком!

Упражнения

IV.4.1. Докажите, что $dd = 0$.

IV.4.2. Запишите в компонентах уравнение $dF = 0$ и убедитесь, что раньше вы уже встречались с ним, но не узнали в краткой записи.

IV.4.3. Рассмотрим $F = (g/4\pi)d\cos\theta d\varphi$. Перейдя в декартовы координаты, покажите, что оно описывает магнитное поле, направленное наружу, вдоль радиуса.

IV.4.4. Восстановите коэффициенты $\hbar$ и c в условии дираковского квантования.

IV.4.5. Запишите инвариантный относительно репараметризации ток $J^{\mu\nu\lambda}$ в мембране.

IV.4.6. Пусть $g(x)$ - это элемент группы G. 1-форма $v = gdg^\dagger$ известна как форма Маурера-Картана. Тогда след $\operatorname{tr} v^N$ замкнут на N-мерном многообразии, поскольку сам является N-формой. Рассмотрите $Q = \int\limits_{S^N} \operatorname{tr} v^N$, где S^N N-мерная сфера. Выясните и топологический смысл Q. Это окажется полезным для дальнейшего обсуждения топологии в теории поля в главе V.7. [Указание: проанализируйте случай $N = 3$ и $G = SU(2)$.]

30.1.5 Zi.IV.5 Non-Abelian Gauge Theory

От большинства подобных идей в конечном итоге отказываются или откладывают на неопределенное время. Но некоторые продолжают существовать и могут стать наваждением. Изредка наваждение в конце концов может обернуться чем-то хорошим.

Ч. Н. Янг об идее, которая впервые пришла к нему в голову в студенческие времена и к которой он возвращался год от года [1].

Локальное преобразование

Речь идет о довольно простой идее.

Чтобы объяснить, о какой идее говорит Янг, напомним наше обсуждение симметрии в главе I.9. Для определенности зададим N-компонентное комплексное скалярное поле $\varphi(x) = \{\varphi_1(x), \varphi_2(x), ..., \varphi_N(x)\}$, преобразующееся как $\varphi(x) \to U\varphi(x)$, где U - элемент группы $SU(N)$. Так как $\varphi^\dagger \to \varphi^\dagger U^\dagger$ и $U^\dagger U = 1$, имеем $\varphi^\dagger\varphi \to \varphi^\dagger\varphi$ и $\partial\varphi^\dagger\partial\varphi \to \partial\varphi^\dagger\partial\varphi$. Для любого многочлена V очевидна инвариантность лагранжиана $\mathcal{L} = \partial\varphi^\dagger\partial\varphi - V\left(\varphi^\dagger\varphi\right)$ относительно группы преобразований $SU(N)$.

Среди физиков-теоретиков больше тех, кто умеет отвечать на хорошо поставленные вопросы, чем тех, кто умеет задавать действительно важные вопросы. Физики второго типа, к тому же, умеют многое из того, что делают физики первого типа, тогда как обратное утверждение заведомо невсрно.

В 1954 году Ч.Н. Янг и Р. Миллс задались вопросом, что произойдет, если преобразование будет меняться от точки к точке в пространстве-времени, другими словами, если $U = U(x)$ будет зависеть от x.

'C. N. Yang, Selected Papers 1945-1980 with Commentary, p. 19. Ясно, что $\varphi^\dagger\varphi$ остается инвариантным. Но $\partial\varphi^\dagger\partial\varphi$, наоборот, больше не будет инвариантным. Действительно,

$$\partial_\mu\varphi \to \partial_\mu(U\varphi) = U\partial_\mu\varphi + (\partial_\mu U)\,\varphi = U\left[\partial_\mu\varphi + \left(U^\dagger\partial_\mu U\right)\varphi\right].$$

Чтобы сократить нежелательный для нас член $\left(U^\dagger\partial_\mu U\right)\varphi$, мы обобщаем обычную производную ∂_μ до ковариантной производной D_μ, которая при действии на φ дает

$$D_\mu\varphi(x) = \partial_\mu\varphi(x) - iA_\mu(x)\varphi(x).$$

Поле A_μ называется калибровочным потенциалом по аналогии с электромагнетизмом.

Как необходимо преобразовать A_μ, чтобы оказалось справедливым $D_\mu\varphi(x) \to U(x)D_\mu\varphi(x)$? Иначе говоря, нам необходимо преобразовать $D_\mu\varphi(x)$ так же, как преобразовано $\partial_\mu\varphi(x)$, при условии, что U не зависит от x. Если так, то выражение $[D_\mu\varphi(x)]^\dagger D_\mu\varphi(x) \to [D_\mu\varphi(x)]^\dagger D_\mu\varphi(x)$ можно использовать в качестве инвариантного кинетического члена для поля φ.

Двигаясь в обратном направлении, видим, что преобразование $D_\mu\varphi(x) \to U(x)D_\mu\varphi(x)$ оказывается справедливым (не стоит и напоминать, но вам лучше это проверить) в случае, когда

$$A_\mu \to UA_\mu U^\dagger - i\,(\partial_\mu U)\,U^\dagger = UA_\mu U^\dagger + iU\partial_\mu U^\dagger.$$

(Равенство следует из условия $UU^\dagger = 1$.) A_μ мы называем неабелевым калибровочным потенциалом, а выражение (2) неабелевым калибровочным преобразованием.

Сделаем ряд простых наблюдений.

1. Очевидно, что A_μ должны быть матрицами $N \times N$. Из (2) получите закон преобразования для $A_\mu^\dagger$ и покажите, что условие $A_\mu - A_\mu^\dagger = 0$ сохраняется при калибровочном преобразовании. Поэтому можно считать A_μ эрмитовым. В частности, установите смысл сказанного для группы $SU(2)$, такой что $U = e^{i\theta\cdot\tau/2}$, где $\theta\cdot\tau = \theta^a\tau^a$ и τ^a - известные вам матрицы Паули.

2. Записав $U = e^{i\theta\cdot T}$, где T^a - генераторы группы $SU(N)$, получим

$$A_\mu \to A_\mu + i\theta^a\,[T^a, A_\mu] + \partial_\mu\theta^a T^a$$

относительно инфинитезимального преобразования $U \simeq 1 + i\theta\cdot T$. Как правило, инфинитезимальной формы (3) бываст достаточно. 3. Вычислив след (3), видим, что след A_μ не преобразуется, поэтому можно считать, мы можем выбрать A_μ эрмитовым и

бесследовым. Это означает, что мы можем всегда написать $A_\mu = A^a_\mu T^a$ и таким образом разложить матричное поле A_μ по компонентам A^a_μ. Существует столько компонент A^a_μ, сколько генераторов у группы [3 для $SU(2), 8$ для $SU(3)$ и т. д.].

4. В приложении B мы напомним, что алгебра Ли для группы задается выражением $\left[T^a, T^b\right] = if^{abc}T^c$, где числа f^{abc} называются структурными константами. Например, для $SU(2)$ имеем $f^{abc} = \varepsilon^{abc}$. Значит, выражение (3) можно переписать в виде:

$$A^a_\mu \to A^a_\mu - f^{abc}\theta^b A^c_\mu + \partial_\mu \theta^a$$

Если θ не зависит от x, компоненты поля A^a_μ преобразуются в присоединенном представлении группы.

5. Если $U(x) = e^{i\theta(x)}$ - всего лишь элемент абелевой группы $U(1)$, то все эти выражения упрощаются, A_μ есть просто абелев калибровочный потенциал, известный из электродинамики, а выражение (2) описывает обычное абелево калибровочное преобразование. Поэтому A_μ известен как неабелев калибровочный потенциал.

Преобразование U, зависящее от пространственно-временных координат x называют калибровочным преобразованием или локальным преобразованием. Говорят, что инвариантный относительно калибровочного преобразования лагранжиан $\mathcal{L}$ является калибровочно инвариантным.

Построение напряженности поля

Теперь мы можем записать калибровочно-инвариантный лагранжиан

$$\mathcal{L} = \left(D_\mu \varphi\right)^\dagger \left(D_\mu \varphi\right) - V\left(\varphi^\dagger \varphi\right)$$

но калибровочный потенциал A_μ не будет иметь собственной динамики. В знакомом примере $U(1)$ калибровочной инвариантности мы уже записали взаимодействие электромагнитного потенциала A_μ с полем материи φ, но еще мы должны были записать в лагранжиане максвеллов член $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. Наша первая задача - построить напряженность поля $F_{\mu\nu}$ из A_μ. Как это сделать? По-видимому, Янг и Миллс воспользовались методом проб и ошибок. В качестве упражнения попробуйте сделать это до прочтения текста. Для этого будет полезен язык дифференциальных форм, с которым мы познакомились в главе IV.4. Удобно избавиться от множителя $-i$, переопределив $A^M_\mu \equiv -iA^P_\mu$, где A^P_μ обозначает знакомый нам калибровочный потенциал. Впредь, до особого указания, под обозначением A_μ будем понимать A^M_μ. Из (1) видим, что ковариантная производная равна $D_\mu = \partial_\mu + A_\mu$. (Кстати, индексы M и P означают потенциал, появляющийся в математической и физической литературе соответственно.) Как и раньше, введем матричную 1-форму $A = A_\mu dx^\mu$, т. е. форму, являющуюся, к тому же, матрицей в фундаментальном представлении алгебры Ли [например, для $SU(N)$ эрмитовой матрицей $N \times N$ с нулевым следом]. Обратите внимание, что

$$A^2 = A_\mu A_\nu dx^\mu dx^\nu = \frac{1}{2}\left[A_\mu, A_\nu\right] dx^\mu dx^\nu$$

не равно нулю для неабелева калибровочного потенциала. (Очевидно, что в электродинамике такого объекта нет.)

Наша задача - получить 2-форму $F = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$ из 1-формы A. Воспользуемся прямым методом. Из A можно получить лишь две возможные 2-формы: dA и A^2. Таким образом, F должна быть линейной комбинацией этих двух форм.

В наших обозначениях закон преобразования (2) сводится к виду:

$$A \to UAU^\dagger + UdU^\dagger$$

где U – 0-форма (и, следовательно, $dU^\dagger = \partial_\mu U^\dagger dx^\mu$). Действуя на (6) оператором d, получаем:

$$dA \to UdAU^\dagger + dUAU^\dagger - UAdU^\dagger + dUdU^\dagger$$

Обратите внимание на знак минус перед третьим членом. С другой стороны, возводя (6) в квадрат, получаем:

$$A^2 \to UA^2U^\dagger + UAdU^\dagger + UdU^\dagger UAU^\dagger + UdU^\dagger UdU^\dagger$$

Действуя на $UU^\dagger = 1$ оператором d, получаем $UdU^\dagger = -dUU^\dagger$. Поэтому можно переписать (8) в виде:

$$A^2 \to UA^2U^\dagger + UAdU^\dagger - dUAU^\dagger - dUdU^\dagger$$

Внимание! Если сложить (7) и (9), шесть членов взаимно сократятся, и у нас останется нечто очень простое:

$$dA + A^2 \to U\left(dA + A^2\right)U^\dagger$$

Математическая структура, таким образом, привела Янга и Миллса к следующему определению напряженности поля:

$$F = dA + A^2$$

В отличие от A, 2-форма напряженности поля F преобразуется однородно:

$$F \to UFU^\dagger$$

В абелевом случае A^2 исчезает, и выражение для F сводится к обычному электромагнитному виду. В неабелевом случае F не калибровочно инвариантна, но калибровочно ковариантна. Конечно, вы можете построить $F^a_{\mu\nu}$, без использования дифференциальных форм. В качестве упражнения проделайте это, начав с выражения (4). После этого вы оцените по достоинству дифференциальные формы! Мы должны ценить дифференциальные формы, по крайней мере как изящное компактное обозначение, позволяющее опустить индексы a и μ в выражении (4). В то же время тот факт, что (11) возникает так естественно, очевидно означает, что в основе лежит глубокая математическая структура. Действительно, существует взаимно-однозначное соответствие между языком физика, рассуждающего в терминах калибровочной теории, и языком математика, рассуждающего в терминах расслоений.

К формуле (11) можно прийти другим путем. По аналогии с оператором d определим оператор $D = d + A$, действующий на форму слева. Вычислим

$$D^2 = (d + A)(d + A) = d^2 + dA + Ad + A^2.$$

Первый член равен нулю, второй можно записать в виде $dA = (dA) - Ad$; круглые скобки обозначают, что d действует только на A. Таким образом,

$$D^2 = (dA) + A^2 = F$$

Ловко, не правда ли? Я оставляю вам в качестве упражнения показать, что D^2 преобразуется однородно и, следовательно, также преобразуется F.

Несмотря на то что дифференциальные формы очень изящны, в физике часто бывает желательно использовать более явные формулы. Выражение (11) можно переписать в виде:

$$F = \left(\partial_\mu A_\nu + A_\mu A_\nu\right) dx^\mu dx^\nu = \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\right) dx^\mu dx^\nu.$$

Если определить $F \equiv \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu$, получим:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

Здесь можно вернуться к физической системе обозначений. Вспомним, что A_μ в выражении (15) фактически представляет собой $A_\mu^M \equiv -iA_\mu^P$, поэтому по аналогии определим $F_{\mu\nu} = -iF_{\mu\nu}^P$. Получим:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\,[A_\mu, A_\nu]$$

где A_μ теперь и впредь обозначает A_μ^P. (В выражении (16) необходимо использование i по той причине, что физики любят выражать A_μ эрмитовой матрицей, а коммутатор двух эрмитовых матриц является антиэрмитовым.)

Мы можем сделать формулы еще более явными, выписав групповые и лоренцевы индексы. По аналогии с $A_\mu = A_\mu^a T^a$ запишем $F_{\mu\nu} = F_{\mu\nu}^a T^a$. Тогда (16) сводится к виду:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c.$$

Я уже говорил, что для $SU(2) A$ и F преобразуются как векторы, а структурная константа f^{abc} равна ε^{abc}, поэтому в этом случае часто используется векторное обозначение $\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + \vec{A}_\mu \times \vec{A}_\nu$.

Лагранжиан Янга-Миллса

С учетом того что F преобразуется однородно (12), мы можем сразу записать аналог лагранжиана Максвелла, называемый лагранжианом ЯнгаМиллса

$$\mathcal{L} = -\frac{1}{2g^2} \operatorname{tr} F_{\mu\nu} F^{\mu\nu}$$

Мы нормируем $T^a \operatorname{tr} T^a T^b = \frac{1}{2}\delta^{ab}$, так что $\mathcal{L} = -(1/4g^2)\, F_{\mu\nu}^a F^{a\mu\nu}$. Теория, описываемая этим лагранжианом известна, как чистая теория ЯнгаМиллса, или неабелева калибровочная теория.

Помимо квадратичного члена $\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)^2$, лагранжиан $\mathcal{L} = -(1/4g^2)\, F_{\mu\nu}^a F^{a\mu\nu}$ также содержит кубический член $f^{abc} A^{b\mu} A^{c\nu} \left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right)$ и член четвертой степени $\left(f^{abc} A_\mu^a A_\nu^c\right)$. Как в электродинамке квадратичный член описывает распространение безмассового векторного бозона, несущего внутренний индекс a и называемого неабелевым калибровочным бозоном, или бозоном Янга-Миллса. Члены третьей и четвертой степеней, отсутствующие в электродинамике, описывают самодействие неабелева калибровочного бозона. На рис. IV.5.1a,b,c приведены соответствующие диаграммы Фейнмана.

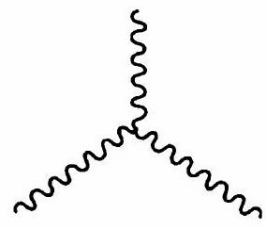
Несложно понять физику, стоящую за этим самодействием бозонов Янта-Миллса. Фотон взаимодействует с заряженными полями, хотя сам не является заряженным. Подобно тому, как по заряду поля можно говорить о характере преобразования поля относительно калибровочной группы $U(1)$, аналогом заряда поля в неабелевой калибровочной теории является то, к какому представлению поле принадлежит. Бозоны Янга-Миллса взаимодействуют со всеми полями, нетривиально преобразующимися относительно калибровочной группы. Однако бозоны Янга-Миллса сами по себе преобразуются нетривиально: мы уже говорили, что они преобразуются по присоединенному представлению. Получается, что они должны взаимодействовать сами с собой.

Чистая теория Максвелла является свободной и, по существу, элементарной. Она описывает невзаимодействующий фотон. Теория ЯнгаМиллса, напротив, содержит самодействие, поэтому ее никак нельзя назвать элементарной. Обратите внимание, что структурные коэффициенты f^{abc} полностью фиксируются теорией групп, поэтому, в отличие от теории скалярного поля, самодействия калибровочных бозонов третьей и четвертой степени, включая их относительные величины, полностью фиксируются

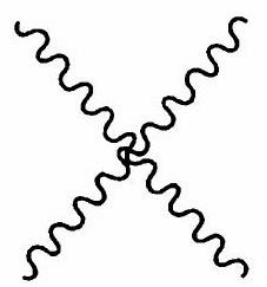
симметрией. Если можно точно решить какую-нибудь 4-мерную теорию поля, она может оказаться чистой теорией Янга-Миллса. Но, несмотря на все колоссальные усилия теоретиков, она остается нерешенной (см. главы VII.3 и VII.4).

arourara

(a)



(b)



(c)

Рис. IV.5.1

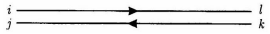
Формализм двойных линий 't Хоофта

Хотя для многих целей удобно пользоваться компонентными полями A^a_μ, матричное поле $A_\mu = A^a_\mu T^a$ более изящным образом реализует математическую структуру неабелевой калибровочной теории. Пропагато- ры компонент матричного поля в калибровочной теории $U(N)$ имеют вид:

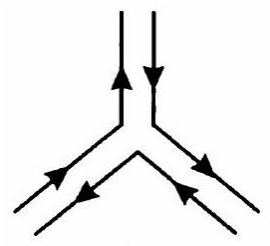
$$\begin{aligned}&\left\langle 0\left|TA_\mu(x)^i_j A_\nu(0)^k_l\right|0\right\rangle = \\ &= \left\langle 0\left|TA^a_\mu(x)A^b_\nu(0)\right|0\right\rangle (T^a)^i_j \left(T^b\right)^k_l \propto \\ &\propto \delta^{ab}\,(T^a)^i_j \left(T^b\right)^k_l \propto \delta^i_l \delta^k_j.\end{aligned}$$

Матричная структура $A^i_{\mu j}$ естественным образом побуждает нас, вслед за 't Хоофтом, ввести формализм двойных линий, в котором калибровочный потенциал описывается двумя линиями, каждая из которых соответствует одному из двух индексов i или j. Пусть верхний индекс входит в диаграмму, а нижний индекс - выходит из диаграммы. На рис. IV.5.2a изображен пропагатор из (19). Формализм двойных линий позволяет нам естественным образом представить структуру индексов $\delta^i_l \delta^k_j$. На рис. IV.5.2b,c изображены взаимодействия третьей и четвертой степеней.

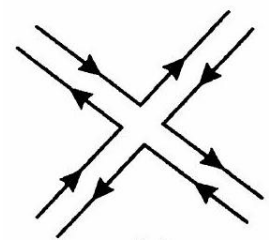
Константа g, введенная в (18), известна как константа связи ЯнгаМиллса. Мы всегда можем написать квадратичный член в (18) в нормировке, обычно используемой в электродинамике, тривиально переопределяя $A \to gA$. После этого взаимодействия третьей и четвертой степеней для бозона Янга-Миллса будут пропорциональны g и g^2 соответственно. Ковариантная производная в (1) принимает вид $D_\mu\varphi = \partial_\mu\varphi - igA_\mu\varphi$, из которого видно, что g характеризует также взаимодействие бозона ЯнгаМиллса с полем материи.



(a)



(b)



(c)
Рис. IV.5.2

Однако используемая нами нормировка, болсс явно раскрывает математическую структуру неабелевой калибровочной тсории. (18), g^2 характеризует легкость, с которой распространяется бозон Янга-Миллса. Из главы III.7 вспомним, что в электродинамике определение константы связи, как меры распространения, также считалось удобным. В главе VIII.1 мы увидим, что ньютоновская константа появляется таким же образом в действии Эйнштейна-Гильберта для гравитации.

θ-член

Помимо $\operatorname{tr} F_{\mu\nu}F^{\mu\nu}$, можно написать член, имеющий размерность 4 $\varepsilon^{\mu\nu\lambda\rho}\operatorname{tr} F_{\mu\nu}F_{\lambda\rho}$. Очевидно, что он нарушает инвариантность относительно обращения времени T и четности P, поскольку содержит единственный индекс для времени и три пространственных индекса. Позже мы увидим, что сильное взаимодействие описывается неабелевой калибровочной теорией, лагранжиан в которой содержит так называемый θ-член $(\theta/32\pi^2)\,\varepsilon^{\mu\nu\lambda\rho}\operatorname{tr} F_{\mu\nu}F_{\lambda\rho}$. В упражнении IV.5.3 вам предстоит показать, что этот член является полной производной и не влияет на уравнения движения. Тем не менее он индуцирует электрический дипольный момент для нейтрона. Максимальное экспериментальное ограничение сверху для электрического дипольного момента для нейтрона приводит к ограничению сверху на θ, которое порядка 10^{-9}. Не буду вдаваться в подробности, как специалисты в области физики элементарных частиц разрешают проблему малости значения θ или его полного зануления.

Взаимодействие с полями материи

Возьмем скалярное поле φ в фундаментальном представлении группы. В общем случае φ может преобразовываться в произвольном представлении $\mathcal{R}$ калибровочной группы G. Запишем ковариантную производную в виде:

$$D_\mu\varphi = \left(\partial_\mu - iA^a_\mu T^a_{(\mathcal{R})}\right)\varphi$$

где $T^a_{(\mathcal{R})}$ представляет собой a-ый генератор в представлении $\mathcal{R}$ (см. упражнение IV.5.1).

Очевидно, что для перевода глобально симметричной теории в локально симметричную теорию необходимо заменить обычную производную ∂_μ, действующую на любое поле (бозон или фермион), которое принадлежит представлению $\mathcal{R}$, ковариантной производной $D_\mu = \left(\partial_\mu - iA^a_\mu T^a_{(\mathcal{R})}\right)$. Таким образом, взаимодействие неабелева калибровочного потенциала с фермионным полем описывается лагранжианом

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi = \bar{\psi}\left(i\gamma^\mu\partial_\mu + \gamma^\mu A^a_\mu T^a_{(\mathcal{R})} - m\right)\psi$$

Поля, принадлежащие представлению $\mathcal{R}$, «слышат» калибровочные бозоны Янга-Миллса, а поля, принадлежащие тривиальному единичному представлению, не слышат зов калибровочных бозонов. В специальном случае калибровочной теории $U(1)$, известном как теория электромагнетизма, $\mathcal{R}$ соответствует электрическому заряду поля. Поля, преобразующиеся относительно $U(1)$ тривиально, являются электрически нейтральными.

Приложение

Рассмотрим появление структуры Янга-Миллса [2] в другом контексте, несколько необычном на первый взгляд. Рассмотрим уравнение Шредингера

$$i\frac{\partial}{\partial t}\Psi(t) = H(t)\Psi(t)$$

с зависящим от времени гамильтонианом $H(t)$. Будем придерживаться общего подхода: можно, например, говорить о спиновых состояниях в магнитном поле или об одночастичном нерелятивистском гамильтониане с волновой функцией $\Psi(\vec{x},t)$. Будем опускать в обозначении зависимость H и Ψ от всех переменных, кроме времени t.

Сначала решим задачу на собственные значения $H(t)$. Предположим, что вследствие симметрии или какой-либо другой причины, спектр $H(t)$ содержит n-кратное вырождение, другими словами, существует n различных решений уравнения $H(t)\psi_a(t) = E(t)\psi_a(t)$, где $a = 1, ..., n$. Обратите внимание, что $E(t)$ может изменяться с течением времени, но мы предполагаем, что вырождение сохраняется с течением времени, т. е. вырождение не может возникнуть «случайным образом» в какой-то момент времени. Всегда можно заменить $H(t)$ на $H(t) - E(t)$, поэтому с этого момента $H(t) = 0$. К тому же, состояния могут быть выбраны ортогональными, так что $\langle\psi_b(t)|\psi_a(t)\rangle = \delta_{ba}$. (Для ясности обозначений удобно переключаться между шредингеровскими и дираковскими обозначениями. Чтобы внести окончательную ясность, запишем

$$\langle\psi_b(t)|\psi_a(t)\rangle = \int d\vec{x}\psi^*_b(\vec{x},t)\psi_a(\vec{x},t)$$

для случая одночастичной квантовой механики.)

Теперь изучим уравнение (22) в адиабатическом пределе, т. е. примем, что временной масштаб, в котором меняется $H(t)$, гораздо больше, чем $1/\Delta E$, где ΔE - энергетическая щель между состояниями $\psi_a(t)$ и соселними с ними состояниями. Если в этом случае функция $\Psi(t)$ начинает эволюционировать в подпространстве, образованном состояниями $\{\psi_a(t)\}$, то она останстся в нем, поэтому можно записать $\Psi(t) = \sum_a c_a(t)\psi_a(t)$. Подставив это выраженис в уравнение (22), получим

"Wilczek and A. Zee. "Appearance of Gauge Structure in Simple Dynamical Systems." Phys. Rev. Lett. 52:2111. 1984. $\sum_a \left[\left(dc_a/dt\right)\psi_a(t) + c_a(t)\left(\partial\psi_a/\partial t\right)\right] = 0$. Если скалярно умножить на $\psi_b(t)$, придем к уравнению:

$$\frac{dc_b}{dt} = -\sum_a A_{ba}c_a$$

с матрицей $n \times n$

$$A_{ba}(t) \equiv \langle \psi_b(t)| \, \frac{\partial \psi_a}{\partial t}$$

Допустим, кто-то еще решает использовать другой базис, $\psi'_a(t) = U^*_{ac}(t) \, (\psi_c(t)$, связанный с нашим унитарным преобразованием. (Матрица, комплексно-сопряженная унитарной матрице U, нужна нам для того, чтобы наше окончательное уравнение совпадало по виду с уравнением, полученным ранее; см. далее.) Также я просуммировал члены с повторяющимися индексами. Дифференцируя, получаем $(\partial \psi'_a/\partial t) = U^*_{ac}(t) \, (\partial \psi_c/\partial t) + (dU^*_{ac}/dt) \, \psi_c(t)$. Сворачивая с $\psi'_b(t) = U_{bd}(t)\psi^*_d(t)$ и умножая иа i. ириходим к:

$$A' = UAU^\dagger + iU\frac{\partial U^\dagger}{\partial t}$$

Пусть гамильтониан $H(t)$ зависит от d параметров $\lambda^1, ..., \lambda^d$. Изменяем значения параметров, отмечая путь, задаваемый как $\{\lambda^\mu(t), \mu = 1, ..., d\}$, в d-мерном пространстве параметров. Например, для спинового гамильтониана λ^μ может соответствовать внешнему магнитному полю. Итак, выражение (23) сводится к виду:

$$\frac{dc_b}{dt} = -\sum_a (A_\mu)_{ba} \, c_a \frac{d\lambda^\mu}{dt}$$

если задать $(A_\mu)_{ba} \equiv i \, \langle \psi_b | \partial_\mu \psi_a \rangle$, где $\partial_\mu \equiv \partial/\partial\lambda^\mu$. В общем виде уравнение (25) будет следующим:

$$A'_\mu = UA_\mu U^\dagger + iU\partial_\mu U^\dagger$$

Оно совпадает с (IV.5.2). Вот так, на наших собственных глазах появляется калибровочный потенциал Янга-Миллса A_μ !

Уравнение «переноса» (26) можно формально решить, если записать $c(\lambda) = Pe^{-\int A_\mu d\lambda''}$, где одномерный интеграл берется по траектории, соединяющей начальную точку в пространстве параметров с некоторой конечной точкой λ, а P обозначает упорядочение вдоль пути. Разбиваем траекторию на бесконечно малые сегменты и перемножаем между собой некоммутирующие составляющие $e^{-A_{1\nu}\Delta\lambda^\mu}$ из каждого сегмента, упорядоченные вдоль траектории. В частности, если траектория явзяется замкнутой кривой, через некоторое время мы вернемся к начальным значениям параметров; волновая функция будет содержать матричный фазовый множитель, называемый неабелевой фазой Берри. Обратите внимание на тесную связь с фазой Ааронова-Бома из предыдущей главы.

Чтобы увидеть неабелеву фазу, все, что мы должны сделать, - это взять некоторую квантовую систему с вырожденностью в спектре и изменить какой-нибудь внешний параметр, например, магнитное поле [3]. В своей работе Янг и Миллс говорят о вырожденности протона и нейтрона с изоспином в идеализированном мире и рассматривают процесс переноса протона из одной точки Вселенной в другую. Чтобы была возможность интерпретировать протон в одной точке как нейтрон в другой, необходимо введение неабелева калибровочного потенциала. Разве не удивительно, что сейчас аналогичным образом можно в лабораторных условиях реализовать такой воображаемый перенос?

Теперь проведем параллель между текущими пояснениями и вышеизложенным в тексте материалом, когда было получено уравнение (IV.5.2). Зависящее от пространства-времени преобразование симметрии соответствует зависящему от параметра изменению базиса. Когда в главе VIII.1 речь пойдет о гравитации, вы увидите, что движение базиса $\{\psi_a\}$ в пространстве параметров аналогично параллельному переносу локальной системы координат в дифференциальной геометрии и общей теории относительности. В главе VII.1 мы снова столкнемся с величиной $Pe^{-\int A_\mu d\lambda'}$, под названием петли Вильсона.

Упражнения

IV.5.1. Запишите лагранжиан для $SU(2)$ калибровочной теории со скалярным полем в представлении $I = 2$.

IV.5.2. Докажите тождество Бьянки $DF \equiv dF + [A, F] = 0$. Запишите его в явном виде с использованием индексов и докажите, что в абелевом случае оно сводится к половине уравнений Максвелла.

IV.5.3. В четырех измерениях $\varepsilon^{\mu\nu\lambda\rho} \operatorname{tr} F_{\mu\nu} F_{\lambda\rho}$ можно записать как $\operatorname{tr} F^2$. Покажите, что в любом числе измерений $d \operatorname{tr} F^2 = 0$.

IV.5.4. Используя лемму Пуанкаре (IV.4.5) и результат упражнения IV.5.3, покажите, что $F^2 = d \operatorname{tr}\left(AdA + \frac{2}{3}A^3\right)$. Запишите его в явном виде с индексами. Определите эти величины в случае электродинамики.

IV.5.5. Попытаетесь показать, что все $\operatorname{tr} F^n$, которые появляются в теориях более высокой размерности (например, теории струн), являются полными производными. Другими словами, существует $(2\pi - 1)$-форма $\omega_{2n-1}(A)$, такая что $\operatorname{tr} F^n = d\omega_{2n-1}(A)$. [Указание: существует краткая запись формы $\omega_{2n-1}(A) = \int\limits_0^1 dt f_{2n-1}(t, A)$.] Запишите в явном виде выражение для $\omega_5(A)$ и попробуйте обобщить его, зная ω_3 и ω_5. Найдите $(2n-1)$-форму $f_{2n-1}(t, A)$. За помощью обращайтесь к работе B. Zumino et al., Nucl. Phys. B239:477, 1984.

IV.5.6. Запишите лагранжиан для калибровочной теории $SU(3)$ с фермионным полем в фундаментальном триплетном представлении.

[3] A. Zee. "On the Non-Abelian Gauge Structure in Nuclear Quadrupole Resonanse." Phvs. Rev. A38:1, 1988. Позже предложенный эксперимент провел Л. Пайнс.

30.1.6 Zi.IV.6 Anderson-Higgs Mechanism

Калибровочный потенциал поглощает намбу-голдстоуновский бозон

Как я уже говорил, в физике принципиально важной является способность задавать хорошие вопросы. Вот отличный вопрос: как проявляется спонтанное нарушение симметрии в калибровочных теориях?

Возвращаясь к главе IV.1, прокалибруем теорию $U(1)$ в (IV.1.6), заменяя $\partial_\mu\varphi$ на $D_\mu\varphi = (\partial_\mu - ieA_\mu)\,\varphi$, так что

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D\varphi)^\dagger D\varphi + \mu^2\varphi^\dagger\varphi - \lambda\left(\varphi^\dagger\varphi\right)^2$$

При переходе к полярным координатам $\varphi = \rho e^{i\theta}$ получим $D_\mu\varphi = [\partial_\mu\rho + \ + i\rho\,(\partial_\mu\theta - eA_\mu)]\,e^{i\theta}$ и, таким образом

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \rho^2\,(\partial_\mu\theta - eA_\mu)^2 + (\partial\rho)^2 + \mu^2\rho^2 - \lambda\rho^4.$$

(Сравните с лагранжианом $\mathcal{L} = \rho^2\,(\partial_\mu\theta)^2 + (\partial\rho)^2 + \mu^2\rho^2 - \lambda\rho^4$ для случая, когда отсутствует калибровочное поле.) При калибровочном преобразовании $\varphi \to e^{i\alpha}\varphi$ (так что $\theta \to \theta + \alpha$) и $eA_\mu \to eA_\mu + \partial_\mu\alpha$, поэтому комбинация $B_\mu \equiv A_\mu - (1/e)\partial_\mu\theta$ является калибровочно-инвариантной. Значит, первые два члена в $\mathcal{L}$ сводятся к виду $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2\rho^2B_\mu^2$. Обратите внимание, что в терминах поля B_μ выражение для $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu B_\nu - \partial_\nu B_\mu$ имеет аналогичную форму.

При спонтанном нарушении симметрии запишем $\rho = (1/\sqrt{2})(v + \chi)$, где $v = \sqrt{\mu^2/\lambda}$. Следовательно,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2B_\mu^2 + e^2v\chi B_\mu^2 + \frac{1}{2}e^2\chi^2B_\mu^2 + \\ + \frac{1}{2}(\partial\chi)^2 - \mu^2\chi^2 - \sqrt{\lambda}\mu\chi^3 - \frac{\lambda}{4}\chi^4 + \frac{\mu^4}{4\lambda}.$$

Теперь теория состоит из векторного поля B_μ массой

$$M = ev$$

взаимодействующего со скалярным полем χ массой $\sqrt{2}\mu$. Фаза θ, которая стала бы намбу-голдстоуновским бозоном в некалиброванной теории, исчезла. Мы говорим, что калибровочное поле A_μ поглотило намбуголдстоуновский бозон; оно приобрело массу и изменило обозначение на B_μ.

Вспомним, что безмассовое калибровочное поле имеет всего 2 степени свободы, тогда как массивное калибровочное поле - 3 степени свободы. Безмассовое калибровочное поле должно поглотить намбуголдстоуновский бозон, чтобы приобрести необходимое количество степеней свободы. Намбу-голдстоуновский бозон становится продольной степенью свободы массивного калибровочного поля. Мы не теряем ни одну степень свободы, что радует.

Явление превращения безмассового калибровочного поля в массивное за счет поглощения намбу-голдстоуновского бозона носит название механизма Хиггса. Его открыли специалисты в области физики элементарных частиц'. Поле Хиггса обозначают как φ или реже χ. Аналогичное явление было открыто в физике конденсированных сред Ландау, Гинзбургом и Андерсоном - и стало известно как механизм Андерсона.

Приведем более сложный пример. Рассмотрим калибровочную теорию $O(3)$ с полем Хигтса $\varphi^a (a = 1, 2, 3)$, преобразующимся в векторном представлении. Лагранжиан содержит член кинетической энергии $\frac{1}{2}\left(D_\mu\varphi^a\right)^2$, где $D_\mu\varphi^a = \partial_\mu\varphi^a + g\varepsilon^{abc}A_\mu^b\varphi^c$ согласно (IV.5.20). При спонтанном нарушении симметрии $\vec{\varphi}$ получает среднее вакуумное значение, которое мы можем выбрать совпадающим с третьим направлением, так что $\langle\varphi^a\rangle = v\delta^{a3}$. Задаем $\varphi^3 = v$ и видим, что

$$\frac{1}{2}\left(D_\mu\varphi^a\right)^2 \to \frac{1}{2}(gv)^2\left(A_\mu^1A^{\mu 1} + A_\mu^2A^{\mu 2}\right)$$

Калибровочные поля A_μ^1 и A_μ^2 приобретают массу gv [сравните с (4)], тогда как поле A_μ^3 остается безмассовым.

Еще более сложный пример - калибровочная теория $SU(5)$ с полем φ, в 24-мерном присоединенном представлении. (См. в приложении В необходимые сведения из теории групп.) Поле φ представляет собой эрмитову

' Включая Хиггса. Энглерта. Броута, Гуралника. Хагена и Киббла. бесследовую матрицу 5×5. Поскольку присоединенное представление преобразуется как $\varphi \to \varphi + i\theta^a\left[T^a, \varphi\right]$, получаем $D_\mu\varphi = \partial_\mu\varphi - igA_\mu^a\left[T^a, \varphi\right] (a = 1, ..., 24$ пробегает по 24 генераторам группы $SU(5))$. Вследствие преобразования симметрии, вакуумное значение φ можно выбрать диагональным $\left\langle\varphi_j^i\right\rangle = v_j\delta_j^i (i, j = 1, ..., 5)$, где $\sum\limits_j v_j = 0$. (Это аналогично нашему выбору $\langle\vec{\varphi}\rangle$ вдоль третьей оси в предыдущем примере.) В лагранжиане имеем

$$\operatorname{tr}\left(D_\mu\varphi\right)\left(D^\mu\varphi\right) \to g^2\operatorname{tr}\left[T^a, \langle\varphi\rangle\right]\left[\langle\varphi\rangle, T^b\right]A_\mu^aA^{\mu b}.$$

Квадраты массы калибровочных бозонов совпадают с собственными значениями 24×24-матрицы, $g^2\operatorname{tr}\left[T^a, \langle\varphi\rangle\right]\left[\langle\varphi\rangle, T^b\right]$, которые можно, потрудившись, вычислить для любого заданного $\langle\varphi\rangle$.

Легко увидеть, однако, какие калибровочные бозоны остаются безмассовыми. В качестве характерного примера (который будет представлять для нас интерес в главе VII.6) рассмотрим поле

$$\langle\varphi\rangle = v\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}.$$

Какие из генераторов T^a коммутируют с $\langle\varphi\rangle$? Ясно, что это делают генераторы вида $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ и $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$. Здесь A обозначает эрмитовы матрицы 3×3 с нулевым следом (из которых $3^2-1=8$ так называемых матриц Гелл-Манна), а B обозначает эрмитовы матрицы 2×2 с нулевым следом (из которых $2^2-1=3$ так называемых матриц Паули). Более того, генератор

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

пропорционален $\langle\varphi\rangle$ и, следовательно, коммутирует с $\langle\varphi\rangle$. Очевидно, что эти генераторы образуют группы $SU(3), SU(2)$ и $U(1)$ соответственно. Таким образом, в матрице 24×24 из квадратов массы $g^2\,\mathrm{tr}\,[T^a,\langle\varphi\rangle]\left[\langle\varphi\rangle,T^b\right]$ выделяются зануляющиеся блоки, а именно блок 8×8, блок 3×3 и блок 1×1. Получаем $8+3+1=12$ безмассовых калибровочных бозонов. Оставшиеся $24-12=12$ калибровочных бозонов получают массу.

Подсчет безмассовых калибровочных бозонов

Рассмотрим теорию с глобальной группой симметрии G, спонтанно нарушенной в подгруппу H. Из главы IV.1 мы знаем, что появляются $n(G)-n(H)$ намбу-голдстоуновских бозонов. Теперь допустим, что группа симметрии G калибруется. Начнем с $n(G)$ безмассовых калибровочных бозонов, по одному на каждый генератор. При спонтанном нарушении симметрии $n(G)-n(H)$ намбу-голдстоуновских бозонов поглощается $n(G)$ - $-n(H)$ калибровочными бозонами, в результате чего $n(H)$ калибровочных бозонов остаются безмассовыми. Похоже, полученное число является верным, поскольку связанные с выжившей калибровочной группой H калибровочные бозоны должны остаться безмассовыми.

В первом нашем примере $G=U(1), H=$ отсутствует: $n(G)=1$ и $n(H)=0$. Во втором примере $G=O(3), H=O(2)\simeq U(1): n(G)=3$ и $n(H)=1$, так что остается один безмассовый калибровочный бозон. В третьем примере $G=SU(5), H=SU(3)\otimes SU(2)\otimes U(1)$, так что $n(G)=24$ и $n(H)=12$. Другие примеры и обобщения предложены в качестве упражнений.

Спектр масс калибровочного бозона

Довольно легко проанализировать в явном виде спектр масс. Ковариантная производная поля Хиггса равна $D_\mu\varphi=\partial_\mu\varphi+gA^a_\mu T^a\varphi$, где g - калибровочная константа связи, T^a - генераторы группы G при действии на φ, а A^a_μ - калибровочный потенциал, соответствующий a-ому генератору. При спонтанном нарушении симметрии мы заменяем φ его вакуумным средним значением $\langle\varphi\rangle=v$. В результате член $D_\mu\varphi$ заменяется выражением $gA^a_\mu T^a v$. Кинетический член $\frac{1}{2}\left(D^\mu\varphi\cdot D_\mu\varphi\right)$ [здесь $(\cdot)$ обозначает скалярное произведение в группе G] в лагранжиане сводится к виду:

$$\frac{1}{2}g^2\left(T^a v\cdot T^b v\right)A^{\mu a}A^b_\mu\equiv\frac{1}{2}A^{\mu a}\left(\mu^2\right)^{ab}A^b_\mu,$$

где мы ввели матрицу квадратов массы

$$\left(\mu^2\right)^{ab} = g^2 \left(T^a v \cdot T^b v\right)$$

для калибровочных бозонов. [Выражение (9) является обобщением (4); сравните также (5) и (6).] Чтобы найти массы калибровочных бозонов, необходимо диагонализовать $\left(\mu^2\right)^{ab}$. По собственным векторам можно сказать, какие линейные комбинации A^a_μ соответствуют массивным собственным состояниям. Обратите внимание, что μ^2 является матрицей $n(G) \times n(G)$ с $n(H)$ нулевыми собственными значениями, существование которых можно установить явным образом. Пусть T^c - генератор H. Утверждение, согласно которому H не нарушается вакуумным средним значением v, соответствует случаю, при котором v остается инвариантным относительно преобразования симметрии, с генератором T^c. Другими словами, $T^c v = 0$ и, следовательно, калибровочный бозон, связанный с T^c, остается безмассовым. Все эти выводы очевидны в рассмотренном нами $SU(5)$ примере.

Правила Фейнмана в спонтанно нарушенных калибровочных теориях

Несложно сформулировать правила Фейнмана для спонтанно нарушснных калибровочных теорий. Возьмем, например, лагранжиан (3). Как обычно, найдем в нем члены второго порядка, преобразуем их по Фурье и обратим. Видим, что пропагатор калибровочного бозона задается выражением

$$\frac{-i}{k^2 - M^2 + i\varepsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{M^2}\right)$$

а пропагатор χ - выражением

$$\frac{i}{k^2 - 2\mu^2 + i\varepsilon}$$

Попробуйте самостоятельно вывести правила для вершин взаимодействия. Как я уже говорил по другому поводу, теории поля часто существуют в нескольких эквивалентных формах. Возьмем теорию $U(1)$ в (1) и вместо полярных координат перейдем к декартовым $\varphi = (1/\sqrt{2})\,(\varphi_1 + i\varphi_2)$. Получим:

$$D_\mu\varphi = \partial_\mu\varphi - ieA_\mu\varphi = \frac{1}{\sqrt{2}}\left[\left(\partial_\mu\varphi_1 + eA_\mu\varphi_2\right) + i\left(\partial_\mu\varphi_2 - eA_\mu\varphi_1\right)\right]$$

Тогда (1) сводится к виду:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left[\left(\partial_\mu\varphi_1 + eA_\mu\varphi_2\right)^2 + \left(\partial_\mu\varphi_2 - eA_\mu\varphi_1\right)^2\right] + \\ &+ \frac{1}{2}\mu^2\left(\varphi_1^2 + \varphi_2^2\right) - \frac{1}{4}\lambda\left(\varphi_1^2 + \varphi_2^2\right).\end{aligned}$$

Спонтанное нарушение симметрии означает, что $\varphi_1 \to v + \varphi_1'$, где $v = \sqrt{\mu^2/\lambda}$

Физический смысл (12) и (3) должен быть одинаковым. Действительно, разложим лагранжиан (12) до второго порядка по полям:

$$\begin{aligned}\mathcal{L} = \frac{\mu^4}{4\lambda} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2A_\mu^2 - MA_\mu\partial^\mu\varphi_2 + \frac{1}{2}\left[\left(\partial_\mu\varphi_1'\right)^2 - 2\mu^2\varphi_1'\right] + \\ + \frac{1}{2}\left(\partial_\mu\varphi_2\right)^2 + \ldots\end{aligned}$$

Спектр - калибровочный бозон A массой $M = ev$ и скалярный бозон φ_1' массой $\sqrt{2}\mu$ совпадает со спектром в (3). (Там частицы обозначались как B и χ.

Однако вы могли заметить нечто очень странное: появление члена $-MA_\mu\partial^\mu\varphi_2$, который смешивает поля A_μ и φ_2. К тому же, возникает вопрос, почему вообще продолжает существовать поле φ_2? Разве его не должны были поглотить? Что делать?

Конечно, можно прибегнуть к диагонализации, но гораздо удобнее совсем избавиться от смешанного члена. Применяя квантование калибровочных теорий Фаддеева-Попова, рассмотренное нами в главе III.4, приходим к выводу, что член, фиксирующий калибровку, обуславливает введение в лагранжиан $\mathcal{L}$ дополнительного члена. Если калибровочную функцию принять равной $f(A) = \partial A + \xi ev\varphi_2 - \sigma$, то нежелательный для нас смешанный член сократится. В конечном итоге получим лагранжиан $\mathcal{L}_{\text{эфф.}} = \mathcal{L} - (1/2\xi)\left(\partial A + \xi M\varphi_2\right)^2$ [сравните с (III.4.7)]. Теперь нежелательный член $-MA_\mu\partial^\mu\varphi_2$ в лагранжиане $\mathcal{L}$ сокращается после интегрирования по частям. В $\mathcal{L}_{\text{эфф.}}$ члены второго порядка по A принимают вид $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2A_\mu^2 - (1/2\xi)(\partial A)^2$, тогда как члены второго порядка по φ_2 равны $\frac{1}{2}\left[\left(\partial_\mu\varphi_2\right)^2 - \xi M^2\varphi_2^2\right]$. С учетом этого пропагатор калибровочного бозона будет иметь вид:

$$\frac{-i}{k^2 - M^2 + i\varepsilon}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M^2 + i\varepsilon}\right]$$

а пропагатор φ_2 - вид:

$$\frac{i}{k^2 - \xi M^2 + i\varepsilon}$$

Это однопараметрическое семейство калибровок называют R_ξ-калибровкой. Обратите внимание, что хотя голдстоуновское поле φ_2 остается в лагранжиане, но сам факт, что его масса зависит от параметра калибровки ξ, делает его нефизическим. В любом физическом процессе зависимость пропагаторов φ_2 и A от параметра ξ должна сокращаться, чтобы физическая амплитуда оставалась независимой от ξ. В упражнении IV.6.9 на простом примере мы проверим, что это действительно так.

Разные преимущества у разных калибровок

Вы, должно быть, удивляетесь, зачем прибегать к R_ξ-калибровке, когда можно воспользоваться эквивалентной формулировкой теории (3), известной под названием унитарной калибровки? В унитарной калибровке пропагатор калибровочного бозона (10) имеет гораздо более простой вид, по сравнению с выражением (14), и мы не встречаемся с нефизическим полсм φ_2. Причина в том, что R_ξ-калибровка и унитарная калибровка дополняют друг друга. В R_ξ-калбировке пропагатор калибровочного бозона (14) при больших значениях k ведет себя как $1/k^2$, так что легко доказать перенормируемость теории. С другой стороны, для унитарной калибровки все поля являются физическими (потому ее и называют «унитарной»), но пропагатор калибровочного бозона (10) при больших значениях k пропорционален $k_\mu k_\nu/k^2$; для доказательства перенормируемости мы должны показать, что $k_\mu k_\nu$ в пропагаторе не вносит вклада. Используя обе калибровки, легко доказать как перенормируемость, так и унитарность теории. Кстати, в пределе $\xi \to \infty$ выражение (14) стремится к (10), а поле φ_2 исчезает, по крайней мере формально.

В практических вычислениях приходится иметь дело с большим количеством диаграмм. В R_ξ-калибровке параметр ξ должен изчезнуть в выражении для физической амплитуды на массовой поверхности. R_ξ-калибровка столь популярна именно потому, что это требование обеспечивает мощную проверку практических вычислений.

Ранее отмечалось, что, строго говоря, калибровочная инвариантность не является симметрией, а отражает избыточность используемых степеней свободы. (У фотона всего две степени свободы, но мы используем поле A_μ с четырьмя компонентами.) В том же духе можно утверждать, что не существует такой вещи, как спонтанное нарушение калибровочной симметрии. Напомню, что спонтанное нарушение симметрии означает, что $\rho \equiv |\varphi|$ равно v, а $\theta - 0$ в уравнении (2). Условие $|\varphi| = v$ является инвариантным относительно $U(1)$, поскольку определяет окружность в пространстве φ. Выбирая на окружности точку $\theta = 0$ в теории с глобальной симметрией, мы нарушаем эту симметрию.

В калибровочной теории, напротив, мы можем использовать калибровочную свободу, чтобы фиксировать $\theta = 0$ в любой точке пространства-времени. Отсюда и невозможность спонтанного нарушения калибровочной симметрии. Я не буду столь педантичен в этой книге и продолжу использовать удобный язык нарушения симметрии даже в калибровочной теории.

Упражнения

IV.6.1. Рассмотрим $SU(5)$ калибровочную теориюо с полем Хиггса φ, преобразующимся по 5-мерному представлению: $\varphi^i, i = 1, 2, ..., 5$. Покажите, что вакуумное среднее φ нарушает $SU(5)$ до $SU(4)$. Затем введите дополнительное поле Хиггса φ', также преобразующееся как 5-мерное представление. Покажите, что $SU(4)$ симметрия либо сохраняется, либо нарушается до $SU(3)$.

IV.6.2. В общем случае может существовать несколько полей Хиггса, принадлежащих разным представлениям α. Покажите, что матрица квадратов массы для калибровочного бозона в общем виде равна $(\mu^2)^{ab} = \sum_\alpha g^2 \left(T^a_\alpha v_\alpha \cdot T^b_\alpha v_\alpha\right)$, где v_α - вакуумное среднее поля φ_α, а $T^a_\alpha - a$-ый генератор, в представлении φ_α. Объедините случаи из упражнений IV.6.1 и IV.6.2 и получите спектр масс калибровочных бозонов.

IV.6.3. Калибровочная группа G не обязана быть простой; она может иметь вид $G_1 \otimes G_2 \otimes ... \otimes G_k$ с константами связи $g_1, g_2, ..., g_k$. Рассмотрите случай $G = SU(2) \otimes U(1)$, когда поле Хиггса φ преобразуется как дублет относительно $SU(2)$ и как поле с зарядом $\frac{1}{2}$ относительно $U(1)$, так что $D_\mu\varphi = \partial_\mu\varphi - i\left[gA^a_\mu(\tau^a/2) + g'B_\mu\frac{1}{2}\right]\varphi$. Пусть $\langle\varphi\rangle = \begin{pmatrix} o \\ v \end{pmatrix}$. Найдите линейные комбинации калибровочных бозонов A^a_μ и B_μ, которые приобретают массу.

IV.6.4. В главе IV.5 вы работали с калибровочной теорией $SU(2)$, со скалярным полем φ в представлении $I = 2$. Запишите наиболее общее выражение для потенциала $V(\varphi)$ четвертой степени и проанализируйте возможную схему нарушения симметрии.

IV.6.5. Завершите вывод правил Фейнмана для теории (3) и вычислите амплитуду для физического процесса $\chi + \chi \to B + B$.

IV.6.6. Выведите (14). [Указание: процедура аналогична выводу уравнения (III.4.9).] Запишите $\mathcal{L} = \frac{1}{2}A_\mu Q^{\mu\nu}A_\nu$, где $Q^{\mu\nu} = (\partial^2 + M^2)\,g^{\mu\nu} - [1 - (1/\xi)]\partial^\mu\partial^\nu$ или, в импульсном пространстве, $Q^{\mu\nu} = -\left(k^2 - M^2\right)g^{\mu\nu} + [1 - (1/\xi)]k^\mu k^\nu$. Пропагатор есть обратный к $Q^{\mu\nu}$.

IV.6.7. Получите (...) в (13) и правила Фейнмана для разных вершин взаимодействия.

IV.6.8. Используя правила Фейнмана из упражнения IV.6.8, вычислите амплитуду для физического процесса $\varphi'_1 + \varphi'_1 \to A + A$ и покажите, что зависимость от ξ сокращается. Сравните с результатом упражнения IV.6.6. [Указание: есть две диаграммы, одна с обменом A, а другая с обменом φ_2.] IV.6.9. Рассмотрите теорию, задаваемую в (12), при $\mu = 0$. Основываясь на результатах упражнения IV.3.5, покажите, что

$$V_{3\phi.}(\varphi) = \frac{1}{4}\lambda\varphi^4 + \frac{1}{64\pi^2}\left(10\lambda^2 + 3e^4\right)\varphi^4\left(\log\frac{\varphi^2}{M^2} - \frac{25}{6}\right) + ...$$

где $\varphi^2 = \varphi_1^2 + \varphi_2^2$. Минимальное значение данного потенциала находится вне точки $\varphi = 0$, поэтому калибровочная симметрия спонтанно нарушается квантовыми флуктуациями. В главе IV.3, где у нас не было члена e^4, мы показали, что полученному значению минимума нельзя доверять. Теперь мы можем компенсировать $\lambda\varphi^4$ членом $e^4\varphi^4\log(\varphi^2/M^2)$ для λ того же порядка, что и e^4. Найденному минимуму можно доверять. Покажите, что спектр этой теории содержит массивный скалярный бозон и массивный векторный бозон, такие что

$$\frac{m^2\ (\text{скалярн.})}{m^2\ (\text{векторн.})} = \frac{3}{2\pi}\frac{e^2}{4\pi}.$$

За помощью обращайтесь к работе Коулмена и Вайнберга (S. Coleman and E. Weinberg, Phys. Rev. D7: 1888, 1973.)

30.1.7 Zi.IV.7 Chiral Anomaly (????)

Классическая симметрия против квантовой симметрии

Я уже подчеркивал, насколько важно задавать хорошие вопросы. Вот очередной: всегда ли симметрия в классической физике является симметрией в квантовой физике?

О симметрии в классической физике мы говорим тогда, когда действие $S(\varphi)$ остается инвариантным относительно преобразования $\varphi \to \varphi + +\delta\varphi$. О симметрии в квантовой физике мы говорим тогда, когда преобразование оставляет интеграл по траекториям $\int D\varphi e^{iS(\varphi)}$ инвариантным.

Если сформулировать наш вопрос в терминах интеграла по траекториям, то ответ кажется очевидным: не обязательно. Действительно, мера $D\varphi$ может быть, а может и не быть инвариантной.

Исторически теоретики-полевики считали почти самоочевидным, что любая симметрия в классической физике с необходимостью является симметрией в квантовой физике. Почти все симметрии, о которых было известно физикам на раннем этапе развития теории поля, оставались симметриями как классической, так и квантовой физики. Например, квантовую механику мы, естественно, считаем инвариантной относительно вращения. Было бы очень странно, если бы квантовые флуктуации выбирали себе определенное направление.

Теперь представьте себе, каким шоком в конце 1960-х гг. послужило для теоретиков-полевиков открытие, согласно которому квантовые флуктуации на самом деле могут нарушать классические симметрии. Они были настолько шокированы, что ошибочно назвали данное явление «аномалией». Оглядываясь назад, мы понимаем, что аномалия не представляет собой никакого концептуального вреда и что, изменяя в интеграле переменные интегрирования, мы просто не должны забывать о якобиане.

Со временем теоретики развили много разных подходов к описанию аномалии, несколько способов борьбы с аномалией. Все они полезны, поскольку по-разному раскрывают природу появления аномалии. В этой книге я решил для примера продемонстрировать существование аномалии явным вычислением фейнмановских диаграмм. Диаграммный метод более трудоемок, в сравнении с другими методами, но у него есть одно преимущество вы своими собственными глазами увидите, как исчезает классическая симметрия! Без всяких формальных аргументов.

Меньшее из двух зол

Рассмотрим теорию с одним безмассовым фермионом $\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi$. Более простой теории и придумать сложно! Из главы II.1 известно, что лагранжиан $\mathcal{L}$ инвариантен относительно преобразований $\psi \to e^{i\theta}\psi$ и $\psi \to e^{i\theta\gamma^5}\psi$, соответствующих сохраняющемуся векторному току $J^\mu = \bar{\psi}\gamma^\mu\psi$ и сохраняющемуся аксиальному току $J_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ соответственно. Вы можетс проверить, что $\partial_\mu J^\mu = 0$ и $\partial_\mu J_5^\mu = 0$ сразу следуют из классических уравнений движения $i\gamma^\mu\partial_\mu\psi = 0$.

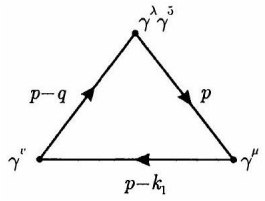
Вычислим амплитуду пространственно-временного процесса, в котором в x_1 рождается пара фермион-антифермион, а в x_2 векторным током создается другая такая пара. Фермион из одной пары аннигилирует с антифермионом из другой пары, а оставшаяся пара фермион-антифермион уничтожается аксиальным током. Для описания этой амплитуды $\left\langle 0 \left| T J_5^\lambda(0) J^\mu(x_1) J^\nu(x_2) \right| 0 \right\rangle$ требуется так много слов, но я хочу быть уверенным, что вы понимаете, о чем я говорю. Согласно Фейнману, фурье-преобразование амплитуды определяется двумя «треугольными» диаграммами на рис. IV.7.1a и b.

$$\Delta^{\lambda\mu\nu}(k_1,k_2) = (-1)i^3 \int \frac{d^4p}{(2\pi)^4} \times$$
$$\times \operatorname{tr}\left(\gamma^\lambda\gamma^5 \frac{1}{\not p - \not q}\gamma^\nu \frac{1}{\not p - \not k_1}\gamma^\mu \frac{1}{\not p} + \gamma^\lambda\gamma^5 \frac{1}{\not p - \not q}\gamma^\mu \frac{1}{\not p - \not k_2}\gamma^\nu \frac{1}{\not p} \right)$$

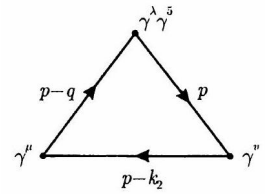
где $q = k_1 + k_2$. Обратите внимание, что появление двух членов обусловлено статистикой Бозе. Общий множитель (-1) обусловлен замкнутой фермионной петлей.

В классическом случае у нас есть две симметрии $\partial_\mu J^\mu = 0$ и $\partial_\mu J_5^\mu = 0$. В квантовой теории, если условие $\partial_\mu J^\mu = 0$ продолжает выполняться, $k_{1\mu}\Delta^{\lambda\mu\nu} = 0$ и $k_{2\nu}\Delta^{\lambda\mu\nu} = 0$, а выполнение $\partial_\mu J_5^\mu = 0$ означает, что $q_\lambda \Delta^{\lambda\mu\nu} = 0$. Теперь можно приступать к вычислению $\Delta^{\lambda\mu\nu}$, результаты которого нам покажут, сохраняют ли квантовые флуктуации эти две симметрии. Ничего сложного. Прежде чем погрузиться в вычисления, зададимся вопросом, огорчит ли нас тот факт, что один из двух токов J^μ и J_5^μ окажется несохраняющимся? Мы сильно огорчимся, если векторный ток не будет сохраняться. Соответствующий заряд $Q = \int d^3x J^0$ считает число фермионов. Мы бы не хотели, чтобы наши фермионы бесследно исчезали и появлялись из ничего. Более того, было бы хорошо, если бы фотон взаимодействовал с фермионным полем ψ. В этом случае нам понадобится условие $\partial_\mu J^\mu = 0$ для сохранения калибровочной инвариантности (см. главу II.6) и, как следствие, всего две поляризации для фотона. Более явно, представьте себе фотонную линию, входящую в вершину μ на рис. IV.7.1a и b, имеющую пропагатор $(i/k_1^2)\left[(1-\xi)\left(k_{1\mu}k_{1\rho}/k_1^2\right) - g_{\mu\rho}\right]$. Зависящий от калибровки член $\xi\left(k_{1\mu}k_{1\rho}/k_1^2\right)$ никуда не исчезнет, если векторный ток будет несохраняющимся, т. е. если не занулится член $k_{1\mu}\Delta^{\lambda\mu\nu}$.

С другой стороны, если говорить откровенно, только между нами, друзья, мы не слишком-то и огорчимся, если вследствие квантовых флуктуаций не будет сохраняться аксиальный ток. Кого волнует, что аксиальный заряд $Q^5 = \int d^3x J_5^0$ не будет постоянным во времени?



(a)



(b)
Рис. IV.7.1.

Сдвиг переменной интегрирования

Итак, зануляются ли члены $k_{1\mu}\Delta^{\lambda\mu\nu}$ и $k_{2\nu}\Delta^{\lambda\mu\nu}$? Посмотрим за плечо профессора Конфузио, когда он вычисляет $k_{1\mu}\Delta^{\lambda\mu\nu}$. (Мы в 1960-х годах, спустя много времени после появления теории перенормируемости, описанной в главе III.1; и Конфузио получил пятилетнюю позицию ассистента [1].) Он сворачивает $\Delta^{\lambda\mu\nu}$ в (7) с $k_{1\mu}$ и, основываясь на знаниях из главы II.7, записывает k_1 в первом члене как $\not{p} - (\not{p} - \not{k}_1)$, а во втором

[1] tenure track assistant professorship. - Прим. ped.

члене как $(\not{p} - \not{k}_2) - (\not{p} - \not{q})$. В итоге получает выражение:

$$k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2) = \\ = i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^\lambda\gamma^5\frac{1}{\not{p}-\not{q}}\gamma^\nu\frac{1}{\not{p}-k_1} - \gamma^\lambda\gamma^5\frac{1}{\not{p}-\not{k}_2}\gamma^\nu\frac{1}{\not{p}}\right).$$

Как и в главе II.7, профессор Конфузио узнает в первом члене подынтегрального выражения второй член, но со сдвинутой переменной интегрирования $p \to p - k_1$. Два члена сокращаются друг с другом, и профессор Конфузио публикует статью, утверждая, что $k_{1\mu}\Delta^{\lambda\mu\nu} = 0$, как мы все и ожидали.

В главе II. 7 я говорил, что позже нам предстоит озаботиться тем, имеем ли право сдвигать переменные интегрирования. Вот и настало это время!

Вполне возможно, что вы уже задавали такой вопрос своим преподавателям математического анализа. В каких случаях $\int\limits_{-\infty}^{+\infty} dpf(p+a)$ равняется $\int\limits_{-\infty}^{+\infty} dpf(p)$? Разница двух интегралов равна

$$\int\limits_{-\infty}^{+\infty} dp\left(a\frac{d}{dp}f(p)+...\right) = a(f(+\infty)-f(-\infty))+...$$

Видим, что сдвиг недопустим, когда $(+\infty)$ и $f(-\infty)$ есть две разные константы. Но если интеграл $\int\limits_{-\infty}^{+\infty} dpf(p)$ сходится или даже логарифмически расходится, то сдвиг возможен. В главе II.6 он был допустим, но сейчас, в выражении (2), определенно нет.

Как обычно, повернем фейнмановскую подынтегральную функцию в евклидово пространство. Обобщая наши наблюдения до d-мерного евклидова пространства, получаем

$$\int d_E^d p[f(p+a)-f(p)] = \int d_E^d p\left[a^\mu \partial_\mu f(p)+...\right]$$

которое по теореме Гаусса равно

$$\lim_{P\to\infty} a^\mu \left(\frac{P_\mu}{P}\right) f(P) S_{d-1}(P)$$

где $S_{d-1}(P)$ - площадь $(d-1)$-мерной сферы (см. приложение D), в которой находится среднее по поверхности сферы значение. (Из расчета фейнмановских диаграмм мы знаем, что среднее для $P^\mu P^\nu/P^2$ значение равно $\frac{1}{4}\eta^{\mu\nu}$ вследствие симметрии; нормировка $\frac{1}{4}$ фиксируется сворачиванием с $\eta_{\mu\nu}$). Вращая обратно, получаем 4-мерный интеграл Минковского

$$\int d^d p[f(p+a)-f(p)] = \lim_{P\to\infty} ia^\mu \left(\frac{P_\mu}{P}\right) f(P)\left(2\pi^2 P^3\right).$$

Обратите внимание, что множитель i обусловлен виковским обратным поворотом.

Объединяя (3) с

$$f(p) = \mathrm{tr}\left(\gamma^\lambda\gamma^5 \frac{1}{\not p - \not k_2}\gamma^\nu \frac{1}{\not p}\right) = \frac{\mathrm{tr}\left[\gamma^5(\not p - \not k_2)\gamma^\nu \not p \gamma^\lambda\right]}{(p-k_2)^2 p^2} = \frac{4i\varepsilon^{\tau\nu\sigma\lambda}k_{2\tau}p_\sigma}{(p-k_2)^2 p^2}$$

получаем

$$k_{1\mu}\Delta^{\lambda\mu\nu} = \frac{i}{(2\pi)^4}\lim_{P\to\infty} i(-k_1)^\mu \frac{P_\mu}{P}\frac{4i\varepsilon^{\tau\nu\sigma\lambda}k_{2\tau}P_\sigma}{P^4}2\pi^2 P^3 = \frac{i}{8\pi^2}\varepsilon^{\lambda\nu\tau\sigma}k_{1\tau}k_{2\sigma}.$$

Вопреки словам профессора Конфузио, $k_{1\mu}\Delta^{\lambda\mu\nu} \neq 0$.

Как я уже говорил, это для нас катастрофа. Количество фермионов не сохраняется, и материя вокруг нас разрушается! Как выйти из данного положения?

В действительности мы лишь слегка сообразительнее профессора Конфузио. Мы не заметили того, что интеграл, определяющий $\Delta^{\lambda\mu\nu}$ в (1), линейно расходится и потому не является хорошо определенным.

Гм, прежде чем волноваться по поводу вычисления $k_{1\mu}\Delta^{\lambda\mu\nu}$ и $k_{2\mu}\Delta^{\lambda\mu\nu}$, нам следовало бы задаться вопросом, зависит ли $\Delta^{\lambda\mu\nu}$ от физика, производящего вычисление? Предположим, что другой физик решает [2] сдвинуть переменную интегрирования p в линейно расходящемся интеграле в (1) на произвольный 4-вектор a, определяя как

$$\begin{aligned}&\Delta^{\lambda\mu\nu}(a,k_1,k_2)=\\&=(-1)i^3\int\frac{d^4p}{(2\pi)^4}\operatorname{tr}\left(\gamma^\lambda\gamma^5\frac{1}{\not p+\not a-\not q}\gamma^\nu\frac{1}{\not p+\not a-\not k_1}\gamma^\mu\frac{1}{\not p+\not a}\right)+\\&+\{\mu,k_1\leftrightarrow\nu,k_2\}.\end{aligned}$$

Сколько на свете физиков, столько и разных ответов можно получить для фейнмановских диаграмм на рис. IV.7.1a и b, что, конечно, было бы концом физики или по крайней мере квантовой теории поля.

2 Это свобода выбора обозначения внутреннего импульса. о котором упоминалось в глаВе I.7. Чей результат мы должны объявить правильным?

Единственным разумным ответом на данный вопрос является следующий: надо доверять человеку, который выбирает такое a, что сокращаются $k_{1\mu}\Delta^{\lambda\mu\nu}(a,k_1,k_2)$ и $k_{2\mu}\Delta^{\lambda\mu\nu}(a,k_1,k_2)$, а фотон имеет должное количество степеней свободы.

Вычислим разность $\Delta^{\lambda\mu\nu}(a,k_1,k_2)-\Delta^{\lambda\mu\nu}(k_1,k_2)$, объединяя (3) и $f(p)=\operatorname{tr}\left(\gamma^\lambda\gamma^5\frac{1}{\not p-\not q}\gamma^\nu\frac{1}{\not p-\not k}\gamma^\mu\frac{1}{\not p}\right)$. С учетом того что

$$\begin{aligned}f(P)&=\lim_{P\to\infty}\frac{\operatorname{tr}\left(\gamma^\lambda\gamma^5\not\gamma^\nu\not\gamma^\mu\not P\right)}{P^6}=\\&=\frac{2P^\mu\operatorname{tr}\left(\gamma^\lambda\gamma^5\not\gamma^\nu\not P\right)-P^2\operatorname{tr}\left(\gamma^\lambda\gamma^5\not P\not\gamma^\nu\gamma^\mu\right)}{P^6}=\frac{-4iP^2P_\sigma\varepsilon^{\sigma\nu\mu\lambda}}{P^6}\end{aligned}$$

приходим к выражению:

$$\begin{aligned}\Delta^{\lambda\mu\nu}(a,k_1,k_2)-\Delta^{\lambda\mu\nu}(k_1,k_2)&=\frac{-4i}{8\pi^2}\lim_{P\to\infty}a^\omega\frac{P_\omega P_\sigma}{P^2}\varepsilon^{\sigma\nu\mu\lambda}+\{\mu,k_1\leftrightarrow\nu,k_2\}=\\&=\frac{-i}{8\pi^2}\varepsilon^{\sigma\nu\mu\lambda}a_\sigma+\{\mu,k_1\leftrightarrow\nu,k_2\}\end{aligned}$$

В задаче есть два независимых импульса k_1 и k_2, поэтому можно выбрать $a=\alpha(k_1+k_2)+\beta(k_1-k_2)$. Подставляя это выражение в (5), получаем:

$$\Delta^{\lambda\mu\nu}(a,k_1,k_2)=\Delta^{\lambda\mu\nu}(k_1,k_2)-\frac{i\beta}{4\pi^2}\varepsilon^{\lambda\mu\nu\sigma}(k_1-k_2)_\sigma$$

Обратите внимание, что параметр α выпадает.

Как и ожидалось, $\Delta^{\lambda\mu\nu}(a,k_1,k_2)$ зависит от β и, следовательно, от a. Наше непоколебимое желание иметь сохраняющийся векторный ток, при котором $k_{1\mu}\Delta^{\lambda\mu\nu}(a,k_1,k_2)=0$, фиксирует параметр β выполнением условия

$$k_{1\mu}\Delta^{\lambda\mu\nu}(k_1,k_2)=\frac{i}{8\pi^2}\varepsilon^{\lambda\nu\tau\sigma}k_{1\tau}k_{2\sigma}$$

Таким образом, наш выбор должен падать на $\Delta^{\lambda\mu\nu}(a,k_1,k_2)$ с $\beta=-\frac{1}{2}$. Рассмотрим все вышесказанное с точки зрения правил Фейнмана. При формулировании правил Фейнмана недостаточно определить $\left\langle 0\left|TJ_5^\lambda(0)J^\mu(x_1)J^\nu(x_2)\right|0\right\rangle$. Их необходимо дополнить условием сохранения векторного тока. Амплитуда $\left\langle 0\left|TJ_5^\lambda(0)J^\mu(x_1)J^\nu(x_2)\right|0\right\rangle$ определяется с помощью $\Delta^{\lambda\mu\nu}(a,k_1,k_2)$ при $\beta=-\frac{1}{2}$.

Квантовые флуктуации нарушают сохранение аксиального тока

Подходим к кульминации истории. Мы потребовали сохранение векторного тока. Сохраняется ли аксиальный ток?

Чтобы ответить на этот вопрос, нам нужно просто вычислить:

$$q_\lambda \Delta^{\lambda\mu\nu}(a,k_1,k_2) = q_\lambda \Delta^{\lambda\mu\nu}(k_1,k_2) + \frac{i}{4\pi^2}\varepsilon^{\mu\nu\lambda\sigma}k_{1\lambda}k_{2\sigma}$$

Вы уже знаете, как это сделать:

$$\begin{aligned} q_\lambda \Delta^{\lambda\mu\nu}(k_1,k_2) = i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^5 \frac{1}{\not p - \not q}\gamma^\nu \frac{1}{\not p - \not k_1}\gamma^\mu - \right. \\ \left. -\gamma^5 \frac{1}{\not p - \not k_2}\gamma^\nu \frac{1}{p\gamma^\mu}\right) + \{\mu, k_1 \leftrightarrow \nu\cdot k_2\} = \\ = \frac{i}{4\pi^2}\varepsilon^{\mu\nu\lambda\sigma}k_{1\lambda}k_{2\sigma} \end{aligned}$$

В самом деле, это интегрирование было выполнено (2). В конечном итоге получаем:

$$q_\lambda \Delta^{\lambda\mu\nu}(a,k_1,k_2) = \frac{i}{2\pi^2}\varepsilon^{\mu\nu\lambda\sigma}k_{1\lambda}k_{2\sigma}$$

Аксиальный ток не сохраняется!

Итак, для простой теории $\mathcal{L} = \bar\psi i\gamma^\mu\partial_\mu\psi$ в классическом случае сохраняются и векторный, и аксиальный токи, а в квантовом случае - только векторный. Это явление называют аномалией, аксиальной аномалией или киральной аномалией.

Следетвия аномалии

Как я уже говорил, аномалия представляет собой чрезвычайно богатое явление. Сделаю ряд замечаний, детали которых вы можете проработать в качестве упражнений.

1. Допустим, мы калибруем нашу простую теорию $\mathcal{L} = \bar\psi i\gamma^\mu(\partial_\mu - ieA_\mu)\psi$ и рассматриваем A_μ в качестве фотонного поля. Тогда рис. IV.7.1 будет соответствовать случаю, когда две фотонные линии выходят из вершин μ и ν. Полученный ранее рсзультат (9) можно переписать теперь в виде двух операторных уравнений:

КЛАССИЧЕСКАЯ ФИЗИКА: $\partial_\mu J_5^\mu = 0$,
КВАНТОВАЯ ФИЗИКА:

$$\partial_\mu J_5^\mu = \frac{e^2}{(4\pi)^2}\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}$$

Производная аксиального тока $\partial_\mu J_5^\mu$ равна не нулю, а оператору, способному создать два фотона.

2. Следуя логике главы IV.2, можно вычислить скорость распада $\pi^0 \to \gamma + \gamma$. Раньше физики, основываясь на ошибочном результате (10), полагали, что этот распад, наблюдаемый экспериментальным путем, невозможен! См. упражнение IV.7.2. Разрешение этого несомненного парадокса привело к верному результату (11).

3. Записав лагранжиан в терминах левого ψ_L и правого ψ_R полей и вводя левый и правый токи $J_R^\mu \equiv \bar\psi_R\gamma^\mu\psi_R$ и $J_L^\mu \equiv \bar\psi_L\gamma^\mu\psi_L$, можно сформулировать аномалию в виде:

$$\partial_\mu J^\mu_R = \frac{1}{2}\frac{e^2}{(4\pi)^2}\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}$$

И

$$\partial_\mu J^\mu_L = -\frac{1}{2}\frac{e^2}{(4\pi)^2}\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}$$

(Отсюда название «киральная»!) Мы можем считать, что левый и правый фермионы бегут по петле на рис. IV.7.1, внося противоположный вклад в аномалию.

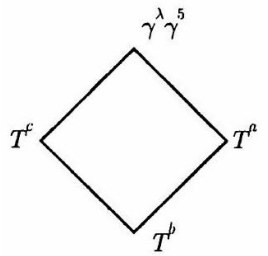
4. Рассмотрим теорию $\mathcal{L} = \bar\psi\left(i\gamma^\mu\partial_\mu - m\right)\psi$. Инвариантность относительно преобразования $\psi \to e^{i\theta\gamma^5}\psi$ нарушается массовым членом. В классическом случае имеем $\partial_\mu J^\mu_5 = 2m\bar\psi i\gamma^5\psi$, т. е. аксиальный ток явно не сохраняется. Аномалия выражается в том, что квантовые флуктуации теперь обуславливают появление дополнительного члена. В теории $\mathcal{L} = \bar\psi\left(i\gamma^\mu\left(\partial_\mu - ieA_\mu\right) - m\right)\psi$ имеем:

$$\partial_\mu J^\mu_5 = 2m\bar\psi i\gamma^5\psi + \frac{e^2}{(4\pi)^2}\varepsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}$$

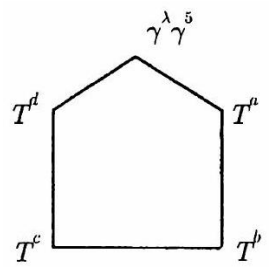
5. Вспомним, что в главе III.7, при вычислении поляризации вакуума, мы вводили регуляторы Паули-Вилларса. Мы вычитали из подынтегрального выражения такое подынтегральное выражение, в котором масса электрона заменяется некоторой массой регулятора. Масса электрона в (1) равна нулю, поэтому вычтем из подынтегрального выражения такое подынтегральное выражение, в котором 0 заменяется на массу регулятора M. Запишем:

$$\begin{aligned}\Delta^{\lambda\mu\nu}(k_1,k_2) =& (-1)i^3\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^\lambda\gamma^5\frac{1}{\not p - \not q}\gamma^\nu\frac{1}{\not p - \not k_1}\gamma^\mu\frac{1}{\not p} - \right.\\ &\left. -\gamma^\lambda\gamma^5\frac{1}{\not p - \not q - M}\gamma^\nu\frac{1}{\not p - \not k_1 - M}\gamma^\mu\frac{1}{\not p - M}\right) + \\ &+ \left\{\mu, k_1 \leftrightarrow \nu, k_2\right\}.\end{aligned}$$

Заметим, что при $p \to \infty$ подынтегральное выражение стремится к нулю быстрее, чем $1/p^3$, что находится в согласии с философией регуляризации, описанной в общих чертах в главах III.1 и III.7: для $p \ll M$, границы неведения, подынтегральное выражение остается неизменным, а для $p \gg M$ подынтегральное выражение обрезается. В нашем случае интеграл в (14) имеет кажущуюся логарифмическую расходимость, поэтому по своему усмотрению можно сдвинуть переменную интегрирования p.



(a)



(b)
Рис. IV.7.2

Как возникает киральная аномалия? Включив массу регулятора M, мы явным образом нарушили условие сохранения аксиального тока. Аномалия заключается в том, что данное нарушение имеет место даже в случае $M \to \infty$. Будет чрезвычайно полезно, если вы проверите это утверждение (упражнение IV.7.4). 6. Рассмотрим неабелеву теорию $\mathcal{L} = \bar{\psi} i\gamma^\mu \left(\partial_\mu - igA_\mu^a T^a\right)\psi$. Достаточно ввести в выражение для фейнмановской амплитуды множитель T^a для вершины μ и множитель T^b для вершины ν. Повторяем все изложенные выше шаги за исключением того, что при суммировании по всем различным фермионам, бегущим по петле, получаем множитель $\operatorname{tr} T^a T^b$. Следовательно, в неабелевой калибровочной теории имеем:

$$\partial_\mu J_5^\mu = \frac{g^2}{(4\pi)^2}\varepsilon^{\mu\nu\lambda\sigma} \operatorname{tr} F_{\mu\nu} F_{\lambda\sigma}$$

где $F_{\mu\nu} = F_{\mu\nu}^a T^a$ - напряженность матричного поля, определенная в главе IV.5. Неабелева симметрия обладает одним замечательным свойством: объект $\varepsilon^{\mu\nu\lambda\sigma} \operatorname{tr} F_{\mu\nu} F_{\lambda\sigma}$ содержит не только член второго порядка по A, но и члены третьей и четвертой степени по A, поэтому наблюдается киральная аномалия с тремя и четырьмя входящими калибровочными бозонами, как на рис. IV.7.2a и b. Некоторые называют аномалии на рис. IV.7.1 и IV.7.2 треугольной, квадратичной и пятиугольной аномалией. Когда была обнаружена треугольная аномалия, шли споры по поводу того, существуют ли вообще квадратичная и пятиугольная аномалии. Видим, что неабелева симметрия разрешает спор очевидным образом, однако в те времена люди вычисляли фейнмановские диаграммы явно и любой неосторожный шаг мог привести к ошибке.

7. В главе V.7 мы увидим, что аномалия тесно связана с топологией.

8. Мы вычислили киральную аномалию в свободной теории $\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi$. Предположим, что фермион взаимодействует со скалярным полем посредством добавления члена $f\varphi\bar{\psi}\psi$ или с электромагнитным полем. Необходимо вычислить диаграммы более высокого порядка, например, трехпетлевую диаграмму на рис. IV.7.3. Можно предположить, что правую часть уравнения (9) следует умножить на $1 + h(f, e, ...)$, где h - некоторая неизвестная функция от всех констант связи, фигурирующих в теории.

А теперь сюрприз! Адлер и Бардин доказали, что $h = 0$. Этот удивительный факт, называемый неперенормируемостью аномалии, можно объяснить эвристически. Перед интегрированием по импульсам скалярных пропагаторов на рис. IV.7.3 (обозначенных как W_1 и W_2) фейнмановское подынтегральное выражение содержит семь фермионных пропагаторов и, следовательно, хорошо сходится. Значит, можно безболезненно сдвинуть переменные интегрирования. Итак, перед интегрированием по W_1 и W_2 выполняются все соответствующие тождества

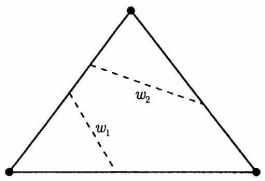


Рис. IV.7.3

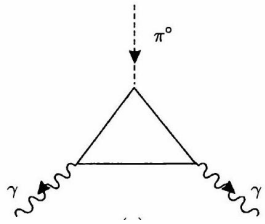
Уорда, например, $q_\lambda \Delta_3^{\lambda\mu\nu}(k_1, k_2; W_1, W_2) = 0$. Попробуйте продолжить доказательство самостоятельно. В упражнении VI.7.13 вы должны будете дать доказательство, основанное на топологии [3].

9. Полученный выше результат имел большое значение для физики элементарных частиц, поскольку непосредственно связан с представлениями о цвете, и речь об этом пойдет в главе VII.3. Условие неперенормируемости аномалии позволило точно вычислить амплитуду распада $\pi^0 \to \gamma + \gamma$ в конце 1960−-х гг. Хотя в кварковой модели того времени амплитуда задается бесконечным числом фейнмановских диаграмм (рис. IV.7.4), условие неперенормируемости аномалии говорит о том, что только диаграмма на рис. IV.7.4a дает вклад. Иначе говоря, амплитуда не зависит от деталей сильного взаимодействия. Тот факт, что амплитуда оказывается втрое меньше требуемой, свидетельствует о существовании трех экземпляров кварков (глава VII.3).

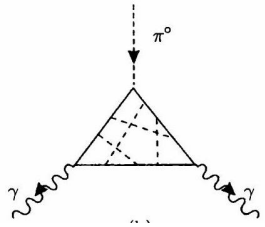
10. Естественно задаться вопросом, являются ли кварки и лептоны составными объектами из более элементарных фермионов, называемых преонами? Условие неперенормируемости киральной аномалии предоставляет нам мощное средство для решения данного вопроса. Не важно, насколько сложными могут быть взаимодействия, но, до тех пор пока они описываются теорией поля, аномалия, вычисленная на преонном уровне, должна совпадать с аномалией, вычисленной на кварк-лептонном уровне. Получаем так называемое условие соответствия аномалии [4], которое существенно ограничивает возможные преонные теории.

11. Исторически теоретики-полевики с большим недоверием относились к интегралу по траекториям, отдавая предпочтение каноническому

[3] Простое доказательство, без привлечения топологии, смотрите в книге Коллинза (. Col-



(a)



(b)
Рис. IV.7.4

подходу. Когда была открыта киральная аномалия, некоторые физики продолжали утверждать, что существование киральной аномалии доказывает некорректность интеграла по траекториям. Они говорили, что интеграл по траекториям

$$\int D\bar{\psi}D\psi e^{i\int d^4x\bar{\psi}i\gamma^\mu(\partial_\mu - iA_\mu)\psi}$$

кажется настолько глупым, что никак не может служить доказательством своей неинвариантности относительно кирального преобразования $\psi \to e^{i\theta\gamma^5}\psi$. Фудзикава [5] разрешил спор, когда доказал, что в интеграле по траекториям учитывается аномалия: при киральном преобразовании мера $D\bar{\psi}D\psi$ изменяется на якобиан. Именно этими словами я начинал главу: действие может быть инвариантно, тогда как интеграл по траекториям - нет.

Зи-у

IV.7.1. Из (9) получите (11). Импульсы $k_{1\lambda}$ и $k_{2\sigma}$, фигурирующие в (9), становятся в (11) двумя производными в $F_{\mu\nu}F_{\lambda\sigma}$.

Зи-у

IV.7.2. Опираясь на рассуждения в главе IV.2 и применяя ошибочное уравнение (10), покажите, что амплитуда распада $\pi^0 \to \gamma + \gamma$ зануляется в идеальном мире, в котором поле π^0 безмассово. Поскольку π^0 действительно распадается, а наш мир приближен к идеальному, получаем первое доказательство некорректности уравнения (10).

Зи-у

IV.7.3. Повторите все вычисления в тексте для теории $\mathcal{L} = \bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi$.

Зи-у

IV.7.4. Возьмите регуляризованную методом Паули-Вилларса амплитуду $\Delta^{\lambda\mu\nu}(k_1, k_2)$ и сверните ее с q_λ. Аналогичным образом в главе II.7 член $\not{q}\gamma^5$ записывался в виде $[2M + (\not{p} - M) - (\not{p} - \not{q} + M)]\gamma^5$. Теперь можно свободно сдвинуть переменные интегрирования. Покажите, что

$$q_\lambda\Delta^{\lambda\mu\nu}(k_1, k_2) = -2M\Delta^{\mu\nu}(k_1, k_2)$$

где

$$\Delta^{\mu\nu}(k_1, k_2) \equiv (-1)i^3\int\frac{d^4p}{(2\pi)^4}$$
$$\mathrm{tr}\left(\gamma^5\frac{1}{\not{p} - \not{q} - M}\gamma^\nu\frac{1}{\not{p} - \not{K}_1 - M}\gamma^\mu\frac{1}{\not{p} - M}\right) + \{\mu, k_1 \leftrightarrow \nu, k_2\}.$$

Вычислите $\Delta^{\mu\nu}$ и покажите, что в пределе $M \to \infty\Delta_{\mu\nu}$ ведет себя как $1/M$, а правая часть уравнения (17) стремится к конечному пределу. Аномалия заключается в том, что после своего исчезновения из низкоэнергетического спектра регулятор оставляет некий след, подобно улыбке чеширского кота. [Даже не приступая к вычислениям, мы фактически можем утверждать, что $\Delta^{\mu\nu}$ будет равна $1/M$. Вследствие лоренцевской инвариантности и наличия γ^5, значение $\Delta^{\mu\nu}$ должно быть пропорционально $\varepsilon^{\mu\nu\lambda\rho}k_{1\lambda}k_{2\rho}$, но анализ размерностей показывает, что $\Delta^{\mu\nu}$ равняется $\varepsilon^{\mu\nu\lambda\rho}k_{1\lambda}k_{2\rho}/M$, умноженному на некоторую константу. Вы,

должно быть, удивляетесь, почему нельзя вместо $1/M$ использовать $1/\left(k_1^2\right)^{\frac{1}{2}}$, чтобы в итоге размерность оказалась верной. Все дело в том, что опыт расчетов фейнмановских диаграмм в (3+1)-мерном пространстве-времени говорит о невозможности получения фактора вида $1/\left(k_1^2\right)^{\frac{1}{2}}$.]

Зи-у

IV.7.5. Существует буквально N способов вывода уравнения аномалии. Вот еще один. Вычислите

$$\Delta^{\lambda\mu\nu}\left(k_1,2\right)=(-1)i^3\int\frac{d^4p}{(2\pi)^4}\,\mathrm{tr}\left(\gamma^\lambda\gamma^5\frac{1}{\not p-\not q-m}\gamma^\nu\frac{1}{\not p-\not K_1-m}\gamma^\mu\frac{1}{\not p-m}\right)+\{\mu,k_1\leftrightarrow\nu,k_2\}$$

для случая массивного фермиона, но не прямым вычислением, а используя сначала лоренцевскую инвариантность, чтобы записать

$$\Delta^{\lambda\mu\nu}\left(k_1,k_2\right)=\varepsilon^{\lambda\mu\nu\sigma}k_{1\sigma}A_1+...+\varepsilon^{\mu\nu\sigma\tau}k_{1\sigma}k_{2\tau}k_2^\lambda A_8$$

где $A_i\equiv A_i\left(k_1^2,k_2^2,q^2\right)$ - восемь функций от трех лоренцевских скаляров. Подсчетом степеней в духе глав III.3 и III.7, покажите, что в двух из восьми функциях появляются интегралы с кажущейся логарифмической расходимостью, тогда как в остальных шести - сходящиеся интегралы. Затем, применяя статистику Бозе и условие сохранения векторного тока $k_{1\mu}\Delta^{\lambda\mu\nu}=0=k_{2\nu}\Delta^{\lambda\mu\nu}$, покажите, что мы можем избежать вычисление интегралов с кажущейся логарифмической расходимостью. Вычислите сходящиеся интегралы и найдите $q_\lambda\Delta^{\lambda\mu\nu}\left(k_1,k_2\right)$.

Зи-у

IV.7.6. Рассмотрите аномалию с точки зрения анализа амплитуды

$$\left\langle 0\left|TJ_5^\lambda(0)J_5^\mu\left(x_1\right)J_5^\mu\left(x_2\right)\right|0\right\rangle$$

в низшем порядке заданной треугольными диаграммами с аксиальными токами в каждой вершине. [Указание: используйте амплитуду в импульсном пространстве $\Delta_5^{\lambda\mu\nu}\left(k_1,k_2\right)$.] Для условия $\left(\gamma^5\right)^2=1$ и симметрии Бозе докажите, что

$$\Delta_5^{\lambda\mu\nu}\left(k_1,k_2\right)=\frac{1}{3}\left[\Delta^{\lambda\mu\nu}\left(a,k_1,k_2\right)+\Delta^{\mu\nu\lambda}\left(a,k_2,-q\right)+\Delta^{\nu\lambda\mu}\left(a,-q,k_1\right)\right].$$

Теперь используйте (9), чтобы вычислить $q_\lambda\Delta_5^{\lambda\mu\nu}\left(k_1,k_2\right)$.

Зи-у

IV.7.7. Аккуратно определите фермионную меру $D\psi$ в (16), перейдя в евклидово пространство. Вычислите якобиан при киральном преобразовании и выведите аномалию.
[Указание: (K. Fujikawa, Phys. Rev. Lett. 42:1195, 1979).]

Зи-у

IV.7.8. Вычислите пятиугольную аномалию с помощью фейнмановских диаграмм для проверки замечания 6 в тексте. Иначе говоря, найдите коэффициент c в уравнении $\partial_\mu J_5^\mu=...+c\varepsilon^{\mu\nu\lambda\sigma}\,\mathrm{tr}\,A_\mu A_\nu A_\lambda A_\sigma$.

30.2 Zi.V Field Theory and Collective Phenomena

Во введении я упоминал, что одним из наиболее интеллектуально ярких достижений за последние 2-3 десятилетия является то, что методы квантовой теории поля играют все более важную роль в физике конденсированных сред. Они составляют богатый и многоплановый предмет исследований; в этой и последующей главах я едва коснусь вершины айсберга и рассмотрю лишь отдельные вопросы.

Исторически теория поля появилась в физике конденсированных сред прямым и непосредственным образом. В системе конденсированных сред нерелятивистские электроны в конденстрованной среде можно описать полем ψ в духе главы III.5. В этом случае появляется возможность написать лагранжианы, построить фейнмановские диаграммы и сформулировать соответствующие правила. Результаты можно найти в некоторых специализированных текстах. Здесь я хочу изложить в значительной степени более современный подход к описанию конденсированных систем, в терминах эффективной теории поля. Система конденсированных сред обладает одним замечательным свойством: вследствие глубоко нетривиальных многочастичных эффектов низкоэнергетические степени свободы могут полностью отличаться от электронов, с которых мы начинали. Особенно ярким примером такой системы (о ней мы поговорим в главе VI.2) является квантовая система Холла, в которой низкоэнергетическая эффективная степень свободы несет дробный заряд и подчиняется дробной статистике.

Есть и другая причина, по которой я существенную часть книги посвятил физике конденсированных сред, - с исторической и педагогической точек зрения, ренормализационную группу гораздо легче понять в рамках физики конденсированных сред, а не физики элементарных частиц.

Я нарочито не буду проводить строгую грань между физикой конденсированных сред и физикой элементарных частиц. Некоторые вопросы, обсуждаемые в пятой и шестой частях, по существу, относятся к физике элементарных частиц. И конечно, я не могу брать на себя большую ответственность за объяснение физики конденсированных сред, чем за свои объяснения физики элементарных частиц в главе IV.2.

30.2.1 Zi.V.1 Superfluid Liquids

Отталкивающиеся бозоны

Рассмотрим конечную плотность $\bar{\rho}$ нерелятивистских бозонов, с короткодействующим отталкиванием. Вернемся к лагранжиану (III.5.11):

$$\mathcal{L} = i\varphi^\dagger \partial_0 \varphi - \frac{1}{2m}\partial_i\varphi^\dagger\partial_i\varphi - g^2\left(\varphi^\dagger\varphi - \bar{\rho}\right)^2$$

Последний член есть в точности потенциал типа сомбреро из главы IV.1. Он вынуждает величину φ быть близкой к $\sqrt{\bar{\rho}}$ и тем самым предлагает нам использовать полярные переменные $\varphi \equiv \sqrt{\rho}e^{i\theta}$, как в уравнении (III.5.7). Делая подстановку и опуская полную производную $(i/2)\partial_0\rho$, получаем лагранжиан в виде

$$\mathcal{L} = -\rho\partial_0\theta - \frac{1}{2m}\left[\frac{1}{4\rho}\left(\partial_i\rho\right)^2 + \rho\left(\partial_i\theta\right)^2\right] - g^2(\rho-\bar{\rho})^2$$

Спонтанное нарушение симметрии

Запишем так, как в главе IV.1, $\sqrt{\rho} = \sqrt{\bar{\rho}} + h$ (вакуумным средним φ является $\sqrt{\bar{\rho}}$). Предположим, что $h \ll \sqrt{\bar{\rho}}$, и разложим [1]:

$$\mathcal{L} = -2\sqrt{\bar{\rho}}h\partial_0\theta - \frac{\bar{\rho}}{2m}\left(\partial_i\theta\right)^2 - \frac{1}{2m}\left(\partial_i h\right)^2 - 4g^2\bar{\rho}h^2 + ...$$

Выбрав в (3) члены второго порядка по h, используем «центральное тождество квантовой теории поля» (см. приложение А), чтобы отынтегриро-

[1] Заметим. что мы отбросили (потенциально интересный) член $-\bar{\rho}\partial_0\theta$, потому что он является полной производной. вать h. В результате имеем

$$\mathcal{L} = \bar{\rho}\partial_0\theta \frac{1}{4g^2\bar{\rho} - (1/2m)\partial_i^2}\partial_0\theta - \frac{\bar{\rho}}{2m}\left(\partial_i\theta\right)^2 + ... =$$
$$= \frac{1}{4g^2}\left(\partial_0\theta\right)^2 - \frac{\bar{\rho}}{2m}\left(\partial_i\theta\right)^2 + ...$$

Допустим, что во втором равенстве нас интересуют процессы с волновым числом k, малым по сравнению с $\sqrt{8g^2\bar{\rho}m}$, так что $(1/2m)\partial_i^2$ пренебрежимо мал, по сравнению с $4g^2\bar{\rho}$. Таким образом, легко видеть, что в этой бозе-жидкости существует бесщелевая мода (часто называемая фононом) с дисперсией

$$\omega^2 = \frac{2g^2\bar{\rho}}{m}\vec{k}^2$$

Самые продвинутые из вас осознают, что мы получили классический результат Боголюбова, даже не выполняя преобразование Боголюбова (см. приложение к этой главе).

Кратко напомню идеализированные рассуждения Ландау [2], согласно которым мода с линейной дисперсией (когда ω линейна относительно k) означает сверхтекучесть. Рассмотрим массу M жидкости, текущей по трубе со скоростью v. Эта жидкость может терять импульс и замедляться до скорости v', создавая возбуждение импульса k : $Mv = Mv' + \hbar k$. Такая ситуация возможна лишь при условии достаточного запаса энергии: $\frac{1}{2}Mv^2 \geqslant \frac{1}{2}Mv'^2 + \hbar\omega(k)$. Исключая v', получаем для M макроскопическую скорость $v \geqslant \omega/k$. Для моды с линейной дисперсией она соответствует критической скорости $v_c \equiv \omega/k$, ниже которой жидкость не может терять импульс и поэтому является сверхтекучей. [Таким образом, из (5) следует, что идеализированная скорость имеет вид $v_c = g\sqrt{2\bar{\rho}/m}$.]

Масштабируя надлежащим образом координату, можно свести низкоэнергетическую физику сверхтекучести к компактному лагранжиану

$$\mathcal{L} = \frac{1}{4g^2}\left(\partial_\mu\theta\right)^2$$

который, как легко видеть, является безмассовым вариантом теории скалярного поля (которая рассматривалась в первой части книги), но при важном условии, что поле θ является фазой, то есть $\theta(x)$ и $\theta(x) + 2\pi$ действи-

2 Л. Д. Ландау. Е. М. Лифшиц. Статическая физика. Часть 2. С. 117. тельно одинаковы. Эта бесщелевая мода, очевидно, является намбу-голдстоуновским бозоном, связанным со спонтанным нарушением глобальной $U(1)$-симметрии $\varphi \to e^{i\alpha}\varphi$.

Бесщелевая мода с линейной дисперсией

Физика становится особенно ясной, если рассуждать в терминах газа свободных бозонов. Передать импульс $\hbar\vec{k}$ любому данному бозону можно за счет проигрыша в энергии $(\hbar\vec{k})^2/2m$. В системе свободных бозонов существует много низкоэнергетических возбуждений. Но как только между бозонами возникает короткодействующее отталкивание, бозон, движущийся с импульсом $\vec{k}$, будет влиять на все другие бозоны. В результате, как было показано в (5), образуется волна плотности с энергией, пропорциональной k. Бесщелевая мода превращается из моды с квадратичной дисперсией в моду с линейной дисперсией. Число низкоэнергетических возбуждений становится гораздо меньше. Вспомним, в частности, что плотность состояний определяется как $N(E) \propto k^{D-1}(dk/dE)$. Например, для $D = 2$ плотность состояний переходит при малых

энергиях от $N(E) \propto \text{const}$ (в случае мод с квадратичной дисперсией) к $N(E) \propto E$ (в случае мод с линейной дисперсией).

Как, наряду с другими, отмечал Фейнман [3], физика сверхтекучести заключается не в присутствии бесщелевых возбуждений, а в их малом числе. (В конце концов, ферми-жидкость имеет непрерывный спектр бесщелевых мод.) Число мод, на которых сверхтекучая жидкость может терять энергию и импульс, слишком мало.

Сравнение нарушения симметрии в релятивистской и нерелятивистской теории

Рассмотрим релятивистскую теорию, $\mathcal{L} = \left(\partial\Phi^\dagger\right)(\partial\Phi) - \lambda\left(\Phi^\dagger\Phi - v^2\right)^2$. Часто удобнее оказывается взять предел $\lambda \to \infty$ при фиксированном v. На языке, использованном в главе IV.1, это будет звучать так: чтобы «подняться по стене», необходимо неограниченно больше энергии, чем для того, чтобы «катиться по желобу». Полученная в результате теория определяется лагранжианом

$$\mathcal{L} = \left(\partial\Phi^\dagger\right)(\partial\Phi)$$

со связью $\Phi^\dagger\Phi = v^2$. Это так называемая нелинейная сигма-модель, (!?!??!?) которая подробно рассматривается в главе VI.4.

Существование намбу-голдстоуновского бозона особенно легко видеть в нелинейной сигма-модели. Связь разрешается с помощью соотношения $\Phi = ve^{i\theta}$, которое при подстановке в $\mathcal{L}$ дает $\mathcal{L} = v^2(\partial\theta)^2$. Это и есть намбу-голдстоуновский бозон θ.

Повторим то же самое для нерелятивистского случая. Возьмем предел $g^2 \to \infty$ при фиксированном $\bar{\rho}$, чтобы лагранжиан (1) имел вид

$$\mathcal{L} = i\varphi^\dagger\partial_0\varphi - \frac{1}{2m}\partial_i\varphi^\dagger\partial_i\varphi$$

со связью $\varphi^\dagger\varphi = \bar{\rho}$. Но теперь, если подставить в $\mathcal{L}$ решение связи $\varphi = \sqrt{\bar{\rho}}e^{i\theta}$ (и опустить полную производную $-\bar{\rho}\partial_0\theta$)), получим $\mathcal{L} = (\bar{\rho}/2m)\left(\partial_i\theta\right)^2$ с уравнением движения $\partial_i^2\theta = 0$. Что же это такое? Это даже не распространяющаяся степень свободы. Где же намбу-голдстоуновский бозон?

Исходя из того, что я уже рассказывал, этот явный парадокс [4] не должен сильно вас смущать. Но поверьте, я ставил с его помощью в тупик немало блестящих умов, занимающихся релятивистской теорией. Намбу-голдстоуновский бозон все еще присутствует, но, как видно из (5), скорость его распространения ω/k стремится к бесконечности как g, и поэтому он исчезает из спектра для любого неравного нулю $\vec{k}$.

Почему же нам позволено переходить к «нелинейному» пределу в релятивистском случае? Потому что есть лоренц-инвариантность! Скорость моды с линейной дисперсией, если такая мода существует, гарантированно равна 1.

Простое обоснование бесщелевой моды (???????)

(на минут 20 подумать место!!!)

Применим вторично квантованный формализм с гамильтонианом

$$H = \sum_k \frac{k^2}{2m}a_k^\dagger a_k + \sum_{k,p,q} G\left(a_k^\dagger a_p^\dagger a_q a_{k+p-q}\right),$$

где $a_k^\dagger$ и a_k обозначают оператор рождения и уничтожения для бозона с 3-импульсом $\vec{k}$ (стрелка в подстрочных индексах опущена для простоты обозначений). Понимая, что макроскопическая плотность бозонов образует конденсат бозонов в основном состоянии с импульсом $\vec{k} = 0$, Боголюбов заменяет a_0 и $a_0^\dagger$ на c-числа $\langle a_0\rangle = \left\langle a_0^\dagger\right\rangle = u$, что приводит к квадратичному гамильтониану

$$H \sim \sum_k \frac{k^2}{2m} a_k^\dagger a_k + \sum_{k\neq 0} G\left(u^2 + \sum_{k\neq 0} a_k^\dagger a_k\right)\left(a_k^\dagger a_k + a_{-k}^\dagger a_{-k} + a_k^\dagger a_{-k}^\dagger + a_{-k} a_k\right).$$

Ясно, что мы получаем бесщелевую моду, поскольку с точностью до аддитивной постоянной второй член можно записать как $\sum_{k\neq 0} Gu^2\left(a_k^\dagger + a_{-k}\right)\left(a_k + a_{-k}^\dagger\right)$, и поэтому ортогональная комбинация $\left(a_k^\dagger - a_{-k}\right)$ создает бесщелевое возбуждение импульса $\vec{k}$. Решающим является то, что $\langle a_0\rangle = \left\langle a_0^\dagger\right\rangle \neq 0$, а не просто $\left\langle a_0^\dagger a_0\right\rangle \neq 0$. В последнем случае мы бы получили только щелевую моду с энергией $(2Gu^2 + k^2/2m)$. Это доказывает, что нарушение $U(1)$-симметрии бозонного числа является ключевым.

Зи-у.V.1.1. О преобразовании в сверхтекучести (????)

Проверьте, что приближение, использованное для получения лагранжиана (3), непротиворечиво.

Зи-у.V.1.2. Об уравнении Гросса-Питаевского (!!!)

Чтобы удержать сверхтекучую жидкость во внешнем потенциале $W(\vec{x})$, добавим в (1) член $-W(\vec{x})\varphi^\dagger(\vec{x},t)\varphi(\vec{\varphi},t)$. Выведите соответствующее уравнение движения для φ. Это уравнение, известное как уравнение Гросса-Питаевского, последние годы много изучалось в связи с конденсатом Бозе-Эйнштейна.

30.2.2 Zi.V.2 Euclid, Boltzmann, Hawking, and Field Theory at Finite Temperature

Статистическая механика и евклидова теория поля

В главе I.2 упоминалось, что для более строгого определения интеграла по траекториям нужно выполнить виковский поворот $t = -it_E$. Тогда теория скалярного поля вместо того, чтобы определяться интегралом по траекториям Минковского

$$\mathcal{Z} = \int D\varphi e^{(i/\hbar)\int d^d x\left[\frac{1}{2}(\partial\varphi)^2 - V(\varphi)\right]}$$

будет определяться евклидовым функциональным интегралом

$$\mathcal{Z} = \int D\varphi e^{-(1/\hbar)\int d_F^d x\left[\frac{1}{2}(\partial\varphi)^2 + V(\varphi)\right]} = \int D\varphi e^{-(1/\hbar)\mathcal{E}(\varphi)},$$

где $d^d x = -id_E^d x$, а $d_E^d x \equiv dt_E d^{(d-1)}x$. В интеграле (1) $(\partial\varphi)^2 = (\partial\varphi/\partial t)^2$ - $-(\vec{\nabla}\varphi)^2$, тогда как в интеграле (2) $(\partial\varphi)^2 = (\partial\varphi/\partial t_E)^2 + (\vec{\nabla}\varphi)^2$. Эта запись несколько сбивает с толку, но я стараюсь не вводить слишком много посторонних символов. Вам может показаться полезным представить $(\vec{\nabla}\varphi)^2 + V(\varphi)$ как одно целое, нетронутое вращением Вика. Я ввел $\mathcal{E}(\varphi) \equiv \int d_E^d x\left[\frac{1}{2}(\partial\varphi)^2 + V(\varphi)\right]$, что может естественно рассматриваться как функционал статистической энергии поля $\varphi(x)$. Таким образом, когда в d-мерном пространстве задана конфигурация $\varphi(x)$, то чем больше она меняется, тем менее вероятно, что она вносит вклад в евклидов функциональный интеграл $\mathcal{Z}$.

Не напоминает ли вам евклидов функциональный интеграл (2) что-нибудь из статистической механики. Действитсльно, герр Больцман учил нас, что в тепловом равновесии при температуре $T = 1/\beta$ вероятность возникновения конфигурации в

классической системе или вероятность возникновения состояния в квантовой системе является нормированным фактором Больцмана $e^{-\beta E}$, где E следует рассматривать как энергию конфигурации в классической системе или как собственное значение энергии состояния в квантовой системе. В частности, вспомним классическую статистическую механику системы из N частиц, для которой

$$E(p,q) = \sum_i \frac{1}{2m} p_i^2 + V\left(q_1, q_2, ..., q_N\right)$$

Статистическая сумма определяется (с точностью до некоторого общего множителя) как

$$Z = \prod_i \int dp_i dq_i e^{-\beta E(p,q)}$$

После интегрирования по p получаем (приведенную) статистическую сумму

$$Z = \prod_i \int dq_i e^{-\beta V(q_1, q_2, ..., q_N)}$$

Преобразуя ее к теории поля так, как в главе 1.3, и допуская, как и раньше, что $i \to x$ и $q_i \to \varphi(x)$, легко видеть, что статистическая сумма классической теории поля с функционалом статической энергии $\mathcal{E}(\varphi)$ имеет такой же вид, как в (2), если отождествить символ h с температурой $T = 1/\beta$. Итак,

евклидова квантовая теория поля в d-мерном пространстве-времени классическая статистическая механика в d-мерном пространстве

Представление квантовой статистической суммы в виде функционального интеграла

Еще более интересная ситуация возникает, когда мы переходим к квантовой статистической механике. Интегрирование по фазовому пространству $\{p, q\}$ заменяется на след, то есть на сумму по состояниям. Таким образом, статистическая сумма квантово-механической системы (для определенности, скажем, одной частицы) с гамильтонианом H определяется как

$$Z = \operatorname{tr} e^{-\beta H} = \sum_n \left\langle n \left| e^{-\beta H} \right| n \right\rangle$$

В главе 1.2 получено интегральное представление $\left\langle F \left| e^{-iHT} \right| I \right\rangle$. (Не следует, конечно, путать время T с температурой T.) Предположим, мы хотим получить интегральное представление статистической суммы. Для этого больше ничего не надо делать! Просто заменяем время T на $-i\beta$, задаем $|I\rangle = |F\rangle = |n\rangle$ и суммируем по $|n\rangle$, в результате получаем

$$Z = \operatorname{tr} e^{-\beta H} = \int\limits_{\text{ПГ}} Dq e^{-\int\limits_0^\beta d\tau L(q)}.$$

Прослеживая шаги от (I.2.3) до (I.2.5), можно убедиться в том, что здесь $L(q) = \frac{1}{2}(dq/d\tau)^2 + V(q)$ в точности является лагранжианом, соответствующим гамильтониану H с евклидовым временем τ. Интеграл по τ берется от 0 до β. Операция взятия следа задает равными начальное и конечное состояния, и поэтому функциональный интеграл необходимо брать по всем траекториям $q(\tau)$ с граничным условием $q(0) = q(\beta)$. Подстрочный индекс ПГУ напоминает нам об этом важном периодическом граничном условии.

Обобщение до теории поля следует немедленно. Если H является гамильтонианом квантовой теории поля в D-мерном пространстве [и, следовательно, в $d = (D+1)$-мерном пространстве-времени], то статистическая сумма (4) имеет вид

$$Z = \operatorname{tr} e^{-\beta H} = \int\limits_{\text{пГу}} D\varphi e^{-\int\limits_0^\beta d\tau \int d^D x \mathcal{L}(\varphi)}$$

с интегралом, вычисленным по всем траекториям $\varphi(\vec{x}, \tau)$, так что

$$\varphi(\vec{x}, 0) = \varphi(\vec{x}, \beta)$$

(Здесь φ представляет собой все бозе-поля в теории.)

Действительно, замечательный результат! Чтобы изучить теорию поля при конечной температуре, все, что надо сделать, - это повернуть ее в евклидово пространство и наложить граничное условие (6). Таким образом,

евклидова квантовая теория поля в $(D+1)$-мерном пространстве-времени, $0 \leqslant \tau < \beta \sim$ квантовая статистическая механика в D-мерном пространстве

В пределе нулевой температуры $\beta \to \infty$ из (5) получаем стандартную квантовую теорию поля с виковским поворотом в бесконечном пространстве-времени, как и должно быть.

Вы, несомненно, имели бы успех у мистиков, если бы сказали им, что температура является эквивалентом циклического мнимого времени. На уровне арифметики такая ассоциация следует из того факта, что основные объекты квантовой физики e^{-iHT} и статфизики $e^{-\beta H}$ формально связаны аналитическим продолжением. Некоторые физики, включая меня самого, чувствуют, что в этом может скрываться нечто очень глубокое, что мы еще не вполне понимаем.

Фейнмановские диаграммы при конечных температурах

При желании из (5) можно развить теорию возмущений при конечных температурах, формулируя правила Фейнмана и т. д. Это делается так же, как и раньше, за исключением одного существенного отличия, связанного с условием (6) $\varphi(\vec{x}, \tau = 0) = \varphi(\vec{x}, \tau = \beta)$. Ясно, что при преобразовании Фурье с множителем $e^{i\omega\tau}$, евклидова частота ω может принимать только дискретные значения $\omega_n \equiv (2\pi/\beta)n$, где n - целое число. Пропагатор скалярного поля приобретает вид $1/\left(k_4^2 + \vec{k}^2\right) \to 1/\left(\omega_n^2 + \vec{k}^2\right)$. Таким образом, чтобы вычислить статистическую сумму, просто запишем соответствующие евклидовы фейнмановские диаграммы и вместо интегрирования по частоте просуммируем по дискретному набору частот $\omega_n (2\pi T) n, n = -\infty, ..., +\infty$. Другими словами, после приведения интеграла Фейнмана к виду $\int d_E^d k F\left(k_E^2\right)$ достаточно заменить его на $2\pi T \sum\limits_n \int d^D k F\left[(2\pi T)^2 n^2 + \vec{k}^2\right]$.

Полезно посмотреть, что происходит в пределе высоких температур $T \to \infty$. При суммировании по ω_n доминирует член $n = 0$, поскольку в знаменателе появляется комбинация $(2\pi T)^2 n^2 + \vec{k}^2$. В результате эти диаграммы эффективно вычисляются в D-мерном пространстве. Мы потеряли размерность! Таким образом,

> евклидова квантовая теория поля в D-мерном пространстве-времени $\sim$ высокотемпературная квантовая статистическая механика в D-мерном пространстве

То, что при высокой температуре квантовая статистическая механика становится классической, - всего лишь утверждение [сравните с (3)].

Важным применением квантовой теории поля при конечной температуре является космология: раннюю Вселенную можно описать как варево из элементарных частиц при некоторой высокой температуре.

Излучение Хокинга

За последние несколько десятилетий хокинговское излучение черными дырами, несомненно, является самым поразительным предсказанием гравитационной физики. Понятие черных дыр существует еще со времен Лапласа, который заметил, что скорость убегания от достаточно массивного объекта может превышать скорость света. С классической точки зрения объекты падают в черные дыры и дело с концом. Но с точки зрения квантовой физики черная дыра на самом деле может излучать как абсолютно черное тело при температуре T.

С учетом того малого, что мы узнали из этой главы и главы I.10, можно фактичсски определить температуру черной дыры. Спешу добавить, что систематический вывод был бы довольно сложным и включал бы в себя много тонкостей; и действительно этому предмету посвящены целые книги. Однако, более или менее ясно, что нужно сделать. Начав с главы I.10, необходимо развить квантовую теорию поля (например, для скалярного поля φ) в искривленном пространстве-времени, в частности, в присутствии черной дыры, и рассмотреть, какое вакуумное состояние (то есть состояние, лишенное квантов φ) в далеком прошлом эволюционирует в далекое будущее. Мы бы нашли состояние, заполненное тепловым распределением квантов φ. Здесь мы это делать не будем.

В прошлом были предложены многочисленные эвристические обоснования излучения Хокинга. Приведу одно из них. Рассмотрим решснис Шварцшильда (см. главу I.10)

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2.$$

На горизонте $r = 2GM$ коэффициенты при dt^2 и dr^2 меняют знак, указывая на то, что время и пространство, а следовательно, энергия и импульс, взаимозаменяемы. Ясно, что должно произойти нечто странное. При квантовых флуктуациях в вакууме всегда появляются и исчезают пары частица-античастица, но обычно, как мы обсуждали ранее, принцип неопределенности ограничивает время существования таких пар Δt до $\sim 1/\Delta E$. Вблизи горизонта черной дыры ситуация совершенно иная. Пара может рождаться из вакуума непосредственно у горизонта, при этом частица находится сразу за пределами горизонта, а античастица - тут же перед горизонтом. Эвристически ограничение Гейзенберга, накладываемое на Δt, можно избежать, поскольку то, что подразумевается под энергией, изменяется, как только мы пересекаем горизонт. Античастица падает в черную дыру, а частица вылетает в пространственную бесконечность. Конечно, такого рода объяснение «на пальцах» должно быть обосновано детальными расчетами.

Если черные дыры действительно излучают при определенной температуре T, а это априори вовсе не очевидно, можно легко определить T из анализа размерностей. Мы видим из (9), что можно получить только комбинацию GM, которая явно имеет размерность длины. Поскольку T имеет размерность массы, то есть обратную длине, то единственной возможностью является $T \propto 1/GM$.

Чтобы точно определить T, используем некоторую хитрость. Предупреждаю сразу, что ее следует воспринимать критически. Она необходима для того, чтобы пробудить у вас желание к более точному рассмотрению.

Представим себе процесс квантования теории скалярного поля в метрике Шварцшильда аналогично тому, как это делалось в главе I.10. Если при виковском повороте это поле «чувствует», что время периодично с периодом β, то в соответствии с тем, что известно из этой главы, кванты скалярного поля будут считать, что живут в тепловой бане при температуре $T = 1/\beta$.

Задавая $t \to -i\tau$, поворачиваем метрику и получаем

$$ds^2 = -\left[\left(1 - \frac{2GM}{r}\right) d\tau^2 + \right.$$
$$\left. + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right].$$

В области сразу же за горизонтом, где $r \gtrsim 2GM$, выполняем общее преобразование координат $(\tau, r) \to (\alpha, R)$, определяемое равенствами $Rd\alpha = (1-2GM/r)^{1/2}d\tau$ и $dR = (1-2GM/r)^{-1/2}dr$, такое что первые два члена в ds^2 будут равны $R^2d\alpha^2 + dR^2$, то есть возведенному в квадрат элементу длины плоского двухмерного евклидова пространства в полярных координатах. Таким образом, евклидово время τ пропорционально полярному углу α. Обозначим период времени τ как β. Поскольку нас интересует, что происходит непосредственно за горизонтом, необходимо определить (α, R) только до главного порядка в $(r-2GM)$. Интегрируя два дифференциальных уравнения, получаем

$$R(2\pi) = \left(1 - \frac{2GM}{r}\right)^{\frac{1}{2}} \beta \simeq (2GM)^{-\frac{1}{2}}(r-2GM)^{\frac{1}{2}}\beta$$

и

$$R = \int\limits_{2GM}^{r} \left(1 - \frac{2GM}{r'}\right)^{-\frac{1}{2}} dr' \simeq (2GM)^{\frac{1}{2}} \int\limits_{2GM}^{r} (r'-2GM)^{-\frac{1}{2}}\, dr' = \\ = (2GM)^{\frac{1}{2}} 2(r-2GM)^{\frac{1}{2}}.$$

Разделив, находим $\beta = (2\pi)^2 2(2GM) = 8\pi GM$ и соответственно температуру Хокинга

$$T = \frac{1}{8\pi GM} = \frac{\hbar c^3}{8\pi GM}$$

Восстанавливая $\hbar$, легко видеть, что излучение Хокинга действительно является квантовым эффектом.

Упражнения

V.2.1. Изучите теорию свободного поля $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2$ при конечной температуре и выведите распределение Бозе-Эйнштейна.

V.2.2. Вас, вероятно, не удивит, что для фермионных полей периодическое граничное условие (6) заменяется антипериодическим граничным условием $\psi(\vec{x}, 0) = -\psi(\vec{x}, \beta)$ с той целью, чтобы можно было воспроизвести результаты, полученные в главе II.5. Докажите это, рассматривая простейший фермионный функциональный интеграл. [Указание: наиболее ясное проявление этого убедительного факта можно найти в приложении A к статье R. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D12: 2443, 1975.]

V.2.3. Интересно рассмотреть квантовую теорию поля для конечной плотности, которая может иметь место в плотных астрофизических объектах или при столкновениях тяжелых ионов. (В предыдущей главе мы изучили систему бозонов при конечной плотности и нулевой температуре.) Изучая статистическую механику, мы научились переходить от статистической суммы к большой статистической сумме $Z = \operatorname{tr} e^{-\beta(H-\mu N)}$, где химический потенциал μ вводится для числа N каждой сохраняющейся частицы. Например, для невзаимодействующих релятивистских фермионов лагранжиан преобразуется к виду $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \mu\bar{\psi}\gamma^0\psi$. Заметим, что конечная плотность, как и конечная температура, нарушает лоренц-инвариантность. Разработайте квантовую теорию поля при конечной плотности в такой степени, какой сможете.

30.2.3 Zi.V.3 Ginzburg–Landau Theory of Critical Phenomena

Появление неаналитичности

Исторически понятие спонтанного нарушения симметрии пришло в физику элементарных частиц из физики конденсированных сред и было впервые использовано в работе Гинзбурга и Ландау о фазовых переходах второго рода.

Рассмотрим ферромагнетик в тепловом равновесии при температуре T. Намагничивание $M(\vec{x})$ определяется как среднее атомных магнитных моментов, взятое по области намного большего размера, чем масштаб длин, характерный для рассматриваемой микроскопической физики. (В этой главе мы обсуждаем нерелятивистскую теорию и x обозначает только пространственные координаты.) Известно, что при низких температурах вращательная инвариантность спонтанно нарушается и вещество проявляет объемную намагниченность, ориентированную в некотором направлении. Когда температура растет выше некоторой критической T_c, объемная намагниченность внезапно исчезает. Понятно, что при более активном тепловом перемешивании атомные магнитные моменты в большей мере ориентированы в случайных направлениях, взаимно уничтожая друг друга. Если точнее, то экспериментально было обнаружено, что при температуре непосредственно ниже T_c намагничивание $|\vec{M}|$ исчезает как $\sim (T_c - T)^{\beta}$, где так называемая критическая экспонента равна $\beta \approx 0,37$.

Это внезапное изменение известно как фазовый переход второго рода и является примером критического явления. Исторически изучение критических явлений представляло собой трудную задачу для физиков-теоретиков. В принципе необходимо вычислить статистическую сумму $Z = \operatorname{tr} e^{-\mathcal{H}/T}$ с микроскопическим гамильтонианом $\mathcal{H}$, но Z очевидно является гладкой от T, за исключением, возможно, случая $T = 0$. Некоторые физики зашли настолько далеко, что стали утверждать невозможность такого неаналитического поведения, как $(T_c - T)^{\beta}$, и что в пределах ошибки эксперимента $|\vec{M}|$ фактически исчезает как гладкая функция от T. Тот факт, что Онсагер нашел в 1944 году всем известное точное решение двухмерной модели Изинга, имеет немаловажное значение, в том числе и в том, что оно определенно ответило на поставленный вопрос. Секрет в том, что бесконечная сумма членов, каждый из которых может быть аналитическим по некоторой переменной, не обязана быть аналитической по этой переменной. След в $\operatorname{tr} e^{-\mathcal{H}/T}$ берется по бесконечному числу членов.

Симметрийные аргументы

В большинстве случаев, как правило, невозможно вычислить Z, если начать с микроскопического гамильтониана. Ландау и Гинзбург проявили блестящую проницательность, полагая, что вид свободной энергии G как функцию $\vec{M}$ для системы с объемом V можно рассматривать исходя из общих принципов. Во-первых, для $\vec{M}$, постоянной по x, получаем в силу вращательной инвариантности выражение

$$G = V\left[a\vec{M}^2 + b\left(\vec{M}^2\right)^2 + \ldots\right]$$

где $a, b, \ldots$ являются неизвестными (но ожидаются гладкими) функциями от T. Ландау и Гинзбург предположили, что при некоторой температуре T_c функция a обращается в нуль. Если нет какой-то особой причины, можно ожидать, что $a = a_1(T - T_c) + \ldots$ [а не, скажем, $a = a_2(T - T_c)^2 + \ldots$]. Но из главы IV.1 уже известно, что произойдет. Для $T > T_c$ свободная энергия G имеет минимум при $\vec{M} = 0$, но, когда температура T падает ниже T_c, внезапно образуется новый минимум при $|\vec{M}| = \sqrt{(-a/2b)} \sim (T_c - T)^{1/2}$. Вращательная симметрия спонтанно нарушается, и таинственное неаналитическое поведение внезапно исчезает.

Чтобы учесть в модели возможность $\vec{M}$ меняться в пространстве, Ландау и Гинзбург предложили использовать G в следующем виде:

$$G = \int d^3x \left\{\partial_i\vec{M}\partial_i\vec{M} + a\vec{M}^2 + b\left(\vec{M}^2\right)^2 + \ldots\right\}$$

где коэффициент члена $\left(\partial_i\vec{M}\right)^2$ положен равным 1 за счет изменения масштаба $\vec{M}$. Легко понять, что выражение (2) является евклидовым вариантом теории скалярного поля, которую мы уже изучали. Из размерного анализа следует, что $1/\sqrt{a}$ определяет масштаб длин.

Скажем точнее: включим для $T > T_c$ возмущающее внешнее магнитное поле $\vec{H}(x)$, добавляя член $-\vec{H} \cdot \vec{M}$. Предполагая, что $\vec{M}$ мало, и минимизируя G, получаем $(-\partial^2 + a)\vec{M} \simeq \vec{H}$ и решение

$$\vec{M}(x) = \int d^3y \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{y})}}{\vec{k}^2 + a} \vec{H}(y) = \int d^3y \frac{1}{4\pi|\vec{x}-\vec{y}|} e^{-\sqrt{a}|\vec{x}-\vec{y}|} \vec{H}(y)$$

[Вспомним, что мы уже вычисляли интеграл (I.4.7), и восхитимся единством физики!]

Чтобы определить корреляционную функцию $\langle \vec{M}(x)\vec{M}(0)\rangle$, обычно задаются вопросом, каким будет намагничивание $\vec{M}(x)$, если использовать магнитное поле, сильно локализованное в начале координат, для создания там намагничивания $\vec{M}(0)$. Ожидается, что корреляционная функция спадает как $e^{-|\vec{x}|/\xi}$ на некоторой корреляционной длине ξ, которая стремится к бесконечности, когда T приближается к T_c сверху. Критическая экспонента ν обычно определяется выражением $\xi \sim 1/(T - T_c)^{\nu}$.

В теории Гинзбурга - Ландау, которую также называют теорией среднего поля, получаем $\xi = 1/\sqrt{a}$ и, следовательно, $\nu = 1/2$.

Важным фактором является не то, насколько хорошо согласуются с экспериментом предсказанные критические экспоненты, такие как β и ν, а то, насколько легко они возникают из теории Гинзбурга-Ландау. Эта теория является отправной для полной теории критических явлений, которая в итоге была разработана с использованием ренорм-группы Кадановым, Фишером, Вильсоном и другими (она будет рассмотрена в главе VI.8).

Говорят, что Ландау оценивал физиков-теоретиков по логарифмической шкале, в которой первое место занимал Эйнштейн; и что после создания теории Гинзбурга-Ландау он переместил себя на полкласса выше [1].

Упражнения

V.3.1. Другая важная критическая экспонента γ определяется тем, что восприимчивость $\chi \equiv (\partial M/\partial H)|_{H=0}$ расходится $\sim 1/|T - T_c|^{\gamma}$, когда T приближается к T_c. Определите γ в теории Гинзбурга-Ландау. [Указание: чтобы это сделать, есть два пути. (a) Добавьте $-\vec{H} \cdot \vec{M}$ в выражение (1) для постоянных в пространстве $\vec{M}$ и $\vec{H}$ и найдите $\vec{M}(\vec{H})$. (b) Вычислите функцию восприимчивости $\chi_{ij}(x-y) \equiv [\partial M_i(x)/\partial H_j(y)]|_{H=0}$ и проинтегрируйте по пространству.]

[1] Из второго класса. - Прим. ped.

30.2.4 Zi.V.4 Superconductivity

(see my special note about it)

Спаривание и конденсация

Когда некоторые материалы охлаждаются до температуры ниже критической T_c, они внезапно становятся сверхпроводящими. Исторически физики давно предполагали, что переход в сверхпроводящее состояние, как и переход в сверхтекучее состояние, как-то связан с конденсацией БозеЭйнштейна. Но электроны являются фермионами, а не бозонами, и поэтому они вначале должны спариться с образованием бозонов, которые затем конденсируются. Теперь мы знаем, что подобное общее представление является, по существу, правильным. Электроны образуют куперовские пары, конденсация которых отвечает за сверхпроводимость.

С блестящей проницательностью Ландау и Гинзбург поняли, что, изучая поле $\varphi(x)$, связанное с конденсирующимися бозонами, явление сверхпроводимости можно достаточно хорошо понять и без знания детального механизма, определяющего спаривание электронов с образованием бозонов. По аналогии с ферромагнитным переходом, при котором

намагничивание $\vec{M}(x)$ в ферромагнетике внезапно меняется от нуля до ненулевого значения при падении температуры ниже некоторого критического значения, они предположили, что поле $\varphi(x)$ становится отличным от нуля при температурах ниже T_c. (В этой главе x обозначает только пространственные координаты.) В статистической физике такие величины, как $\vec{M}(x)$ и $\varphi(x)$, которые изменяются при фазовом переходе, называют параметрами порядка.

Поле $\varphi(x)$ имеет две единицы электрического заряда и поэтому является комплексным. Теперь рассмотрение будет во многом таким же, как в главе V.3, за исключением того, что $\partial_i\varphi$ надо заменить на $D_i\varphi = (\partial_i - \ -i2eA_i)\,\varphi$, поскольку поле φ заряжено. Следуя Ландау и Гинзбургу, а также включая энергию внешнего магнитного поля, запишем свободную энергию в виде уравнения

$$\mathcal{F} = \frac{1}{4}F_{ij}^2 + |D_i\varphi|^2 + a|\varphi|^2 + \frac{b}{2}|\varphi|^4 + ...$$

которое явно инвариантно относительно $U(1)$-калибровочного преобразования $\varphi \to e^{i2e\Lambda}\varphi$ и $A_i \to A_i + \partial_i\Lambda$. Как и прежде, требование равенства коэффициента при $|D_i\varphi|^2$ единице эквивалентно выбору нормировки для φ. Сходство между (1) и (IV.6.1) очевидно.

Эффект Мейсснера

Отличительной чертой сверхпроводимости является эффект Мейсснера, когда проникающее сквозь вещество внешнее магнитное поле $\vec{B}$ вытесняется из него при падении температуры ниже T_c. Это означает, что постоянное магнитное поле внутри вещества энергетически невыгодно. Действующие законы электромагнетизма в таком веществе при T_c должны некоторым образом модифицироваться. Как правило, постоянное магнитное поле требует затрат энергии порядка $\sim \vec{B}^2V$, где V - объем вещества. Прсдположим, что плотность энергии меняется от стандартной $\vec{B}^2$ до $\vec{A}^2$ (где $\vec{\nabla}\times\vec{A} = \vec{B}$). Значение $\vec{A}$ увеличивается с расстоянием в постоянном магнитном поле $\vec{B}$, и поэтому полная энергия растет быстрее, чем V. Когда вещество становится сверхпроводящим, для поддержания постоянного магнитного поля необходимо так много дополнительной энергии, что магнитное поле выгоднее вытеснить.

Заметим, что такой член, как $\vec{A}^2$ в эффективной плотности энергии, сохраняет вращательную и трансляционную инвариантность, но нарушает электромагнитную калибровочную инвариантность. Однако из главы IV.6 мы знаем, как нарушить калибровочную инвариантность. Действительно, описанная там калибровочная теория U(1) и описанная здесь теория сверхпроводимости, по существу, одинаковы и связаны виковским поворотом.

Как и в главе V.3, предположим, что $a \simeq a_1\,(T - T_c)$, а b остается положительным при температуре $T \simeq T_c$. Свободная энергия $\mathcal{F}$ минимизируется выше T_c за счет $\varphi = 0$, а ниже T_c за счет $|\varphi| = \sqrt{-a/b} \equiv v$. Все вышесказанное хорошо знакомо тем, кто знает, что калибровочное поле приобретает массу при нарушении симметрии в калибровочной теории. Из выражения (1) просто получаем

$$\mathcal{F} = \frac{1}{4}F_{ij}^2 + (2ev)^2A_i^2 + ...$$

т. е. именно то, что необходимо для объяснения эффекта Мейсснера.

Лондоновская длина проникновения и длина когерентности

Физически магнитное поле не резко уменьшается от некоторого ненулевого значения вне сверхпроводника до нуля внутри него, а в пределах некоторого характеристического масштаба длин, называемого лондоновской длиной проникновения. Магнитное поле проникает в сверхпроводник на глубину, немного большую масштаба длины l, что

определяется конкуренцией между энергией в магнитном поле $F_{ij}^2 \sim (\partial A)^2 A^2/l^2$ и мейсснеровским членом $(2ev)^2A^2$ в (2). Так, Ландау и Гинзбург получили лондоновскую длину проникновения $l_L \sim (1/ev) = (1/e)\sqrt{b/-a}$.

Аналогично характеристический масштаб длин, в пределе которого изменяется параметр порядка φ, известен как длина когерентности l_φ. Ее можно оценить приравнивая второй и третий члены в (1), например так: $(\partial\varphi)^2 \sim \varphi^2/l_\varphi^2$ и $a\varphi^2$, что дает длину когерентности порядка $l_\varphi \sim 1/\sqrt{-a}$.

Объединяя все вместе, получаем

$$\frac{l_L}{l_\varphi} \sim \frac{\sqrt{b}}{e}$$

Заметим, что согласно главе IV.6 это есть просто отношение массы скалярного поля к массе векторного поля.

Как говорилось выше, концепция спонтанного нарушения симметрии пришла в физику элементарных частиц из физики конденсированных сред. Намбу сыграл важную роль тем, что привнес понятие спонтанного нарушения симметрии в сообщество специалистов по физике элементарных частиц после того, как услышал в университете Чикаго доклад молодого Шриффера по теории сверхпроводимости Бардина-Купера-Шриффера.

Упражнения

V.4.1. Проварьируйте (1) чтобы получить уравнение для A, и более аккуратно получите лондоновскую глубину проникновения.

V.4.2. Выведите более аккуратно выражение для длины когерентности.

30.2.5 Zi.V.5 Peierls Instability

Невзаимодействующие прыгающие электроны

Появление уравнения Дирака и релятивистской теории поля для твердого тела кажется поистине удивительным, но это на самом деле это возможно.

Рассмотрим гамильтониан

$$H = -t\sum_j \left(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1}\right)$$

описывающий невзаимодействующие прыгающие электроны на 1-мерной решетке (рис. V.5.1). Здесь c_j уничтожает электрон в узле j. Таким образом, первый член описывает электрон, скачущий из узла j в узел $j+1$ с амплитудой t. Спиновые значки мы опустили. Это лишь простейшая модель твердого тела; она хорошо изложена в фейнмановских лекциях по физике.

Выполняя Фурье-преобразование $c_j = \sum\limits_k e^{ikaj}c(k)$ (где a - расстояние между узлами), сразу же находим энергетический спектр $\varepsilon(k) = -2t\cos ka$ (рис. V.5.2). Налагая периодическое граничное условие на решетку с числом узлов N, получаем $k = (2\pi/Na)n$, где n - целое число от $-1/2N$ до $1/2N$. При $N \to \infty k$ становится непрерывной, а не дискретной переменной. Как обычно, зона Бриллюэна определяется как $-\pi/a < k \leqslant \pi/a$.

Во всем вышеизложенном нет ничего релятивистского. В самом деле, в нижней части спектра энергия (с точностью до несущественной аддитивной постоянной) определяется как $\varepsilon(k) \simeq 2t\frac{1}{2}(ka)^2 \equiv k^2/2m_{\text{eff.}}$ Электрон распространяется нерелятивистски с эффективной массой $m_{\text{eff.}}$.

Теперь заполним систему электронами до некоторого ферми-уровня ε_F (см. рис. V.5.2). Сосредоточим внимание на электроне вблизи ферми-поверхности и измерим его энергию от

ε_F и импульс от $+k_F$. Предположим, что нас интересуют электроны, энергия которых мала по сравнению с энергией ε_F, то есть $E \equiv \varepsilon - \varepsilon_F \ll \varepsilon_F$, а импульс которых мал по сравнению

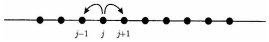


Рис. V.5.1

с k_F, то есть $p \equiv k - k_F \ll k_F$. Эти электроны подчиняются линейному закону дисперсии энергии $E = v_F p$ с ферми-скоростью $v_F = (\partial\varepsilon/\partial k)|_{k=k_F}$. Обозначим поле, связанное с этими электронами, как ψ_R, где нижний индекс означает, что электроны «движутся направо», что удовлетворяет уравнению движения $(\partial/\partial t + v_F \partial/\partial x)\,\psi_R = 0$.

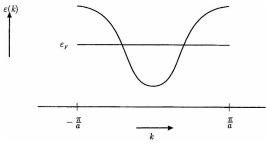


Рис. V.5.2

Аналогично электроны с импульсом порядка $-k_F$ подчиняются закону дисперсии $E = -v_F p$. Обозначим поле, связанное с этими электронами, как ψ_L, где L означает «движение налево», что удовлетворяет уравнению движения $(\partial/\partial t - v_F \partial/\partial x)\,\psi_L = 0$.

Появление уравнения Дирака

Лагранжиан, объединяющий все это, имеет вид

$$\mathcal{L} = i\psi_R^\dagger \left(\frac{\partial}{\partial t} + v_F \frac{\partial}{\partial x}\right)\psi_R + i\psi_L^\dagger \left(\frac{\partial}{\partial t} - v_F \frac{\partial}{\partial x}\right)\psi_L$$

Вводя двухкомпонентное поле $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \bar{\psi} \equiv \psi^\dagger\gamma^0 \equiv \psi^\dagger\sigma_2$ и выбирая единицы так, чтобы $v_F = 1$, лагранжиан $\mathcal{L}$ можно записать в более ком- пактном виде

$$\mathcal{L} = i\psi^\dagger \left(\frac{\partial}{\partial t} - \sigma_3 \frac{\partial}{\partial x}\right)\psi = \bar{\psi} i\gamma^\mu \partial_\mu \psi$$

где $\gamma^0 = \sigma_2$ и $\gamma^1 = i\sigma_1$ удовлетворяют алгебре Клиффорда $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

Удивительно, что (1 + 1)-мерный лагранжиан Дирака появляется в совершенно нерелятивистской ситуации!

Рис. V.5.3

Неустойчивость

Теперь перейдем к рассмотрению важного явления, известного как пайерлсовская неустойчивость. Я буду вынужден использовать несколько схематичный подход. Нет необходимости повторять, что это не учебник по физике твердого тела, но в любом случае вам нетрудно будет восполнить пробелы.

Пайерлс рассмотрел искажение решетки для случая, когда ион в узле j смещен относительно своего равновесного положения на $\cos[q(ja)]$. (Напоминает наш матрац из главы I.1!) Искажение решетки с волновым вектором $q = 2k_F$ соединяет электроны, имеющие импульс k_F, с электронами, имеющими импульс $-k_F$. Другими словами, это искажение соединяет электроны, движущиеся направо, с электронами, движущимися налево, или ψ_R с ψ_L в соответствии с используемой здесь терминологией теории поля. Поскольку электроны, движущиеся направо и налево, имеют одинаковую энергию на

ферми поверхности (а именно ε_F, что не ново), имеем интересный случай теории возмущений при наличии вырождения $\begin{pmatrix} \varepsilon_F & 0 \\ 0 & \varepsilon_F \end{pmatrix} + \begin{pmatrix} 0 & \delta \\ \delta & 0 \end{pmatrix}$ с собственными значениями $\varepsilon_F \pm \delta$. На поверхности Ферми образуется щель. Здесь δ является возмущением. Таким образом, Пайерлс пришел к выводу, что спектр изменяется радикально и система является нестабильной при возмущении с волновым вектором $2k_F$.

Особенно интересная ситуация возникает, когда система наполовину заполнена электронами (так что плотность равна одному электрону на каждый узел - напомним, что электроны имеют спин, направленный вверх или вниз). Другими словами, $k_F = \pi/2a$, и поэтому $2k_F = \pi/a$. Искажение решетки, которое показано на рис. V.5.3, имеет точно такой же волно- вой вектор. Пайерлс показал, что наполовину заполненная система будет стрсмиться исказить решетку, удваивая элементарную ячейку. Поучительно обратить внимание на то, как это физическое явление появляется в теоретико-полевой формулировке.

Обозначим смещение иона в узле j как d_j. В непрерывном пределе мы должны заменить d_j на скалярное поле. Покажем, что возмущение, связывающее ψ_R и ψ_L, взаимодействует с $\bar{\psi}\psi$ и $\bar{\psi}\gamma^5\psi$ и что линейную комбинацию $\bar{\psi}\psi$ и $\bar{\psi}\gamma^5\psi$ киральным преобразованием можно всегда повернуть к $\bar{\psi}\psi$ (см. упражнение V.5.1.). Таким образом, мы расширяем (3) до

$$\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi + \frac{1}{2}\left[(\partial_t\varphi)^2 - v^2\,(\partial_x\varphi)^2\right] - \frac{1}{2}\mu^2\varphi^2 + g\varphi\bar{\psi}\psi + \ldots$$

Вспомним, что в упражнении (IV.3.2) вы уже вычисляли эффективный потенциал $V_{\text{eff}}(\varphi)$ такой $(1+1)$-мерной теории поля: $V_{\text{eff}}(\varphi)$ для малого φ имеет вид $\varphi^2 \log \varphi^2$, который превосходит член $\frac{1}{2}\mu^2\varphi^2$. Таким образом, симметрия $\varphi \to -\varphi$ динамически нарушается. Поле φ приобретает вакуумное среднее, и ψ становится массивным. Другими словами, в электронном спектре образуется щель.

Упражнения

V.5.1. С учетом рассуждений, приведенных в главе II.1, вы легко можете увидеть, что пространство 2×2-матриц образовано четырьмя матрицами I, γ^μ и $\gamma^5 \equiv \gamma^0\gamma^1 = \sigma_3$. (Обратите внимание на своеобразное, но стандартное обозначение γ^5.) Убедитесь в том, что $\frac{1}{2}(I \pm \gamma^5)$ проецирует правые и левые поля точно так, как в $(3+1)$-мерном пространстве-времени. Покажите, что в билинейном члене $\bar{\psi}\gamma^\mu\psi$ левые поля связаны с левыми полями, а правые с правыми и что в скаляре $\bar{\psi}\psi$ и псевдоскаляре $\bar{\psi}\gamma^5\psi$ правые поля связаны с левыми и наоборот. Наконец, обратите внимание, что при преобразовании $\psi \to e^{i\theta\gamma^5}\psi$ скаляр и псевдоскаляр обращаются друг в друга. Проверьте, что это преобразование оставляет безмассовый лагранжиан Дирака (3) инвариантным.

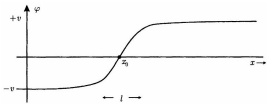
30.2.6 Zi.V.6 Solitons

Я люблю говорить студентам, которым преподаю квантовую теорию поля, что к середине 1970-х годов специалисты по теории поля начали избавляться от оков фейнмановских диаграмм. Звучит это несколько мелодраматично, но к тому времени фейнмановские диаграммы, благодаря их впечатляющему успеху в квантовой электродинамике, доминировали (возможно, даже слишком) в мышлении многих специалистов по теории поля. В студенчестве мне даже говорили, что фейнмановские диаграммы определяют квантовую теорию поля, что квантовые поля являются просто «ломтиками оленины» [1], которые использовали для выведения правил Фейнмана и от которых надо отказаться, раз уже эти правила получены. Превалировало мнение, что вряд ли имеет смысл записывать $\varphi(x)$. Открытие топологических солитонов навсегда опровергло это мнение, как мы сейчас увидим. (!?!??!?)

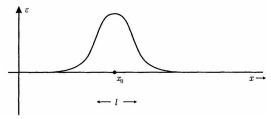
Малые колебания в сравнении со сгустками

Рассмотрим еще раз нашу любимую игрушечную модель $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - V(\varphi)$ с пресловутым двухъямным потенциалом $V(\varphi) = (\lambda/4)\times\times\left(\varphi^2 - v^2\right)^2$ в (1+1)-мерном пространстве-времени. Из главы IV.1 известно, что из двух вакуумов $\varphi = \pm v$ нужно выбрать один и изучить относительно него малые колебания. Итак, выберем один вакуум и запишем $\varphi = v + \chi$, разложим $\mathcal{L}$ по χ и изучим динамику χ-мезона с массой $\mu = (2\lambda v^2)^{1/2}$. Тогда физика будет заключаться в квантованных соответствующим образом волнах, колеблющихся вблизи вакуума v.

' Гелл-Манн имел обыкновение говорить. что во Франции мясо фазана готовят между двумя ломтиками оленины. которые затем выбрасывают. Он убедительно рекомендовал использовать такой подход для извлечения и изучения алгебраической структуры квантовых теорий поля. а затем от них избавляться. Но это еще не все. Можно также иметь не зависящую от времени конфигурацию поля с $\varphi(x)$ (в этой и следующей главах x обозначает только пространство, если из контекста не следует что-то другое), принимающую значение $-v$, когда $x \to -\infty$, и $+v$, когда $x \to +\infty$, и изменяющуюся от $-v$ до $+v$ относительно некоторой точки x_0 на некотором масштабе длин l, как показано на рис. V.6.1a. [Заметим, что если рассматривать евклидов вариант теории поля, определять временную координату как координату y и считать $\varphi(x, y)$ намагничиванием (как в главе V.4), то такая конфигурация будет описывать «доменную стенку» в двумерной магнитной системе.]



(a)



(b)
Рис. V.6.1

Рассмотрим энергию, приходящуюся на единицу длины,

$$\varepsilon(x) = \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 + \frac{\lambda}{4}\left(\varphi^2 - v^2\right)^2$$

для конфигурации, которая изображена на рис. V.6.1b. Вдали от x_0 мы находимся в одном из двух вакуумов, и плотность энергии отсутствует. Вблизи x_0 оба члена в $\varepsilon(x)$ вносят вклад в энергию или массу $M = \int dx\varepsilon(x)$: член «пространственной вариации» (при несколько вольном использовании терминологии часто называемый «кинетической энергией») $\int dx\frac{1}{2}(d\varphi/dx)^2 \sim l(v/l)^2 \sim v^2/l$ и член «потенциальной энергии» $\int dx\lambda\left(\varphi^2 - v^2\right)^2 \sim l\lambda v^4$. Чтобы минимизировать полную энергию, член пространственной вариации требует, чтобы масштаб l был большим, тогда как член потенциала требует, чтобы l был малым. Конкуренция $dM/dl = 0$ приводит к $v^2/l \sim l\lambda v^4$, фиксируя таким образом $l \sim (\lambda v^2)^{-1/2} \sim 1/\mu$. Масса получается равной $\sim \mu v^2 \sim \mu\left(\mu^2/\lambda\right)$.

У нас есть сгусток энергии, распределенный по области длины l, порядка комптоновской длины волны χ-мезона. Вследствие трансляционной инвариантности центр этого сгустка x_0 может находиться где угодно. Более того, поскольку эта теория является лоренц-инвариантной, всегда можно заставить сгусток двигаться с любой скоростью, какая нам нравится. По аналогии с известной репликой из анналов американской политики («Она ходит как утка, крякает как утка, так почему же, господин сенатор, вы не хотите называть ее уткой?»), в нашем случае есть частица, называемая кинком или солитоном, с массой

$\sim \mu(\mu^2/\lambda)$ и размером $\sim l$. Многие физики представляют солитон как большой тяжело двигающийся объект, возможно, из-за того, как он был открыт. Но как мы видели, размер солитона $l \sim 1/\mu$ можно сделать насколько угодно малым, увеличивая μ. Поэтому солитон может выглядеть как точечная частица. В главе VI.3 мы вернемся к этому вопросу при обсуждении дуальности.

Топологическая устойчивость

Хотя кинк и мезон имеют одинаковый размер, при малой λ кинк намного тяжелее, чем мезон. Тем не менее кинк не может распасться на мезоны, потому что, для того чтобы уничтожить кинк [например, «поднимая» $\varphi(x)$ выше потенциального барьера, чтобы изменить его от $+v$ до $-v$ для x от некоторой точки $\gtrsim x_0$ до $+\infty$], требуется неограниченно много энергии. Говорят, что кинк топологически устойчив.

Эта устойчивость формально гарантируется сохраняющимся током

$$J^\mu = \frac{1}{2\mu}\varepsilon^{\mu\nu}\partial_\nu\varphi$$

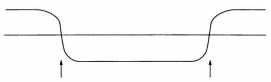
с зарядом

$$Q = \int\limits_{-\infty}^{+\infty} dx J^0(x) = \frac{1}{2v}[\varphi(+\infty) - \varphi(-\infty)]$$

Мезоны, которые представляют собой малые локализованные пакеты колебаний в поле, явно имеют $Q = 0$, тогда как кинк имеет $Q = 1$. Таким образом, кинк не может распасться на кучу мезонов. Кстати, плотность заряда $J^0 = (1/2v)(d\varphi/dx)$ локализована вокруг x_0, где, как можно ожидать, φ изменяется наиболее быстро.

Заметим, что равенство $\partial_\mu J^\mu = 0$ сразу же следует из антисимметричного символа $\varepsilon^{\mu\nu}$ и не зависит от уравнения движения. Ток J^μ известен как «топологический ток». Его существование следует не из теоремы Нётер (глава I.9), а из топологии.

Приведенные выше рассуждения также объясняют существование антикинка с $Q = -1$, который описывается конфигурацией с $\varphi(-\infty) = +v$ и $\varphi(+\infty) = -v$. Такое название оправданно, если рассмотреть конфигурацию, изображенную на рис. V.6.2, содержащую кинк и антикинк, расположенные друг от друга на большом расстоянии. Когда кинк и антикинк сближаются, они, очевидно, могут проаннигилировать с образованием мезонов, поскольку конфигурация, показанная на рис. V.6.2, и конфигурация вакуума с $\varphi(x) = +v$ повсюду разделены конечным количеством энергии.



Антикинк
Кинк
Рис. V.6.2

Непертурбативное явление

То, что масса кинка оказывается обратно пропорциональной константе связи λ, является явным признаком того, что теоретики-полевики могли бы до изнеможения проводить вычисления в теории возмущений по λ, так никогда не открыв кинк. Фейнмановские диаграммы не могли бы нам об этом сообщить.

Массу кинка можно вычислить, минимизируя функционал

$$M = \int dx \left[\frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 + \frac{\lambda}{4}\left(\varphi^2 - v^2\right)^2\right] = \left(\frac{\mu^2}{\lambda}\right)\mu \int dy \left[\frac{1}{2}\left(\frac{df}{dy}\right)^2 + \frac{1}{4}\left(f^2 - 1\right)^2\right]$$

где на последнем шаге выполнено очевидное масштабирование $\varphi(x) \to v f(y)$ и $y = \mu x$. Из масштабирования сразу же следует, что масса кинка имеет вид $M = a\,(\mu^2/\lambda)\,\mu$, где a - безразмерное число. Эвристическая оценка этой массы оказалась в высшей степени достоверной. Настоящую функцию $\varphi(x)$ и, следовательно, a можно вычислить с использованием стандартных вариационных методов.

Неравенство Богомольного

Следует обратить внимание на то, что плотность энергии (1) есть сумма двух квадратов. Используя $a^2 + b^2 \geqslant 2|ab|$, получаем

$$M \geqslant \int d \left(\frac{\lambda}{2}\right)^{1/2} \left| \left(\frac{d\varphi}{dx}\right) \left(\varphi^2 - v^2\right) \right| \geqslant \left(\frac{\lambda}{2}\right)^{1/2} \left| \left[\frac{1}{3}\varphi^3 - v^2\varphi\right]_{-\infty}^{+\infty} \right| =$$
$$= \left| \frac{4}{3\sqrt{2}} \mu \left(\frac{\mu^2}{\lambda}\right) Q \right|.$$

Приходим к изящному результату

$$M \geqslant |Q|$$

с массой M, измеряемой в единицах $(4/3\sqrt{2})\mu\,(\mu^2/\lambda)$. Это пример неравенства Богомольного, которое играет важную роль в теории струн.

Упражнения

V.6.1. Покажите, что если $\varphi(x)$ является решением уравнения движения, то решением также является $\varphi\left[(x - vt)/\sqrt{1 - v^2}\right]$.

V.6.2. Рассмотрите солитоны в так называемой теории синус-Гордона $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - g\cos(\beta\varphi)$. Найдите топологический ток. Является ли солитон $Q = 2$ устойчивым?

V.6.3. Получите массу кинка прямым вычислением. Удовлетворяет ли она неравенству Богомольного?

30.2.7 Zi.V.7 Vortices, Monopoles, and Instantons

Кинк является лишь простейшим примером большого класса топологических объектов в квантовой теории поля.

Рассмотрим теорию комплексного скалярного поля в $(2 + 1)$-мерном пространстве-времени $\mathcal{L} = \partial\varphi^\dagger\partial\varphi - \lambda\left(\varphi^\dagger\varphi - v^2\right)^2$ с теперь уже знакомым потенциалом в форме сомбреро. С небольшими изменениями в обозначениях она соответствует теории, которая используется для описания взаимодействующих бозонов и сверхтекучих жидкостей. (Рассмотрим релятивистский, а не нерелятивистский вариант теории; как будет видно, этот выбор не связан с теми вопросами, которые здесь обсуждаются.)

Есть ли в этой теории солитоны, то есть объекты, подобные кинку?

Имея некоторую не зависящую от времени конфигурацию $\varphi(x)$, посмотрим, какую она имеет массу или энергию:

$$M = \int d^2x \left[\partial_i\varphi^\dagger\partial_i\varphi + \lambda\left(\varphi^\dagger\varphi - v^2\right)^2\right]$$

Подынтегральное выражение является суммой двух квадратов, каждый из которых должен дать конечный вклад. В частности, чтобы вклад второго члена был конечным, величина φ должна стремиться к v при пространственной бесконечности.

Однако требование конечной энергии не фиксирует фазу φ. Используя полярные координаты (r,θ), рассмотрим анзац $\varphi \overset{r\to\varphi}{\longrightarrow} ve^{i\theta}$. Записывая $\varphi=\varphi_1+i\varphi_2$, легко видеть, что на бесконечности вектор $(\varphi_1,\varphi_2)=v(\cos\theta,\sin\theta)$ направлен радиально наружу. Вспомним приведенное в главе III.5 определение тока $J_i=i\left(\partial_i\varphi^\dagger\varphi-\varphi^\dagger\partial_i\varphi\right)$ в бозонной жидкости. На пространственной бесконечности поток закручивается, и поэтому такая конфигурация называется вихрем.

Используя явное дифференцирование или анализ размерностей, получаем $\partial_i\varphi\sim v(1/r)$ при $r\to\infty$. Теперь посмотрим на первый член в M и увидим, что энергия расходится логарифмически как $v^2\int d^2x\,(1/r^2)$.

Можно ли решить эту проблему? Можно, если изменить теорию.

Вихрь в трубках потока

Предположим, что мы калибруем теорию, заменяя $\partial_i\varphi$ на $D_i\varphi=\partial_i\varphi$ - - $ieA_i\varphi$. Теперь можно получить конечную энергию, требуя, чтобы два члена в $D_i\varphi$ сокращали друг друга, так что $D_i\varphi\overset{r\to\infty}{\longrightarrow}0$ быстрее, чем $1/r$. Другими словами, $A_i\overset{r\to\infty}{\longrightarrow}-(i/e)\,(1/|\varphi|^2)\,\varphi^\dagger\partial_i\varphi$. Сразу же получаем

$$\text{Flux}\ \equiv\int d^2xF_{12}=\oint_C dx_iA_i=\frac{2\pi}{e}$$

где C является бесконечно большой окружностью на пространственной бесконечности; здесь использована теорема Стокса. Таким образом, в $U(1)$-калибровочной теории вихрь несет магнитный поток, обратно пропорциональный заряду. Говоря «магнитный», я полагаю, что A представляет собой электромагнитное калибровочное поле. Рассматриваемый здесь вихрь появляется в виде трубки потока в так называемых сверхпроводниках II рода. Следует отметить, что эта квант потока (2) в литературе по физике конденсированных сред обычно записывается в неестественных единицах, в виде

$$\Phi_0=\frac{hc}{e}$$

очень удачно объединяя три фундаментальные константы Природы.

Гомотопические группы

Поскольку пространственная бесконечность в двухмерном пространстве соответствует топологически единичной окружности S^1 и поскольку конфигурация поля с $|\varphi|=v$ также образует окружность S^1, это граничное условие может быть представлено как отображение $S^1\to S^1$. Поскольку такое отображение нельзя гладко преобразовать к тривиальному отображению, в котором S^1 отображается на точку в S^1, соответствующая конфигурация поля действительно является топологически устойчивой. (Представьте, что виток струны оборачивается вокруг кольца.)

С математической точки зрения отображения S^n в многообразие M классифицируются гомотопической группой $\Pi_n(M)$, которая подсчитывает число топологически неэквивалентных отображений. Вы можете познакомиться с гомотопическими группами для различных многообразий по таблицам'. В частности, для $n\geqslant 1\Pi_n\left(S^n\right)=Z$, где Z - математическое

[1] См. таблицы 6.V и 6.VI в энциклопедическом словаре S. Iyanaga and Y. Kawada. eds.. Enclclopedic Dictionary of Mathematics. p. 1415. обозначение множества всех целых чисел. Простейший пример $\Pi_1\left(S^1\right)=Z$ доказывается почти сразу путем представления отображений в виде $\varphi\overset{r\to\infty}{\longrightarrow}ve^{im\theta}$, где m - любое целое число (положительное или отрицательное). Для удобства можно использовать наш контекст и обозначения. Ясно, что это отображение оборачивает одну окружность вокруг другой m раЗ.

Язык гомотопических групп используется не только для того, чтобы произвести на людей впечатление, но и служит унифицированным языком при обсуждении

топологических солитонов. Действительно, оглядываясь назад, вы теперь понимаете, что кинк является физическим проявлением $\Pi_0(S^0) = Z_2$ (поскольку 0-мерная сфера $S^0 = \{+1, -1\}$ состоит только из двух точек и топологически эквивалентна пространственной бесконечности в одномерном пространстве).

Ежи и монополи

Если все предыдущее усвоено, можно перейти к $(3+1)$-мерному пространству-времени. Пространственная бесконечность теперь топологически является S^2. Вы уже представляете, что если скалярное поле существует на многообразии M, то на бесконечности получаем отображение $S^2 \to M$. Простейший вариант заключается в том, чтобы взять S^2 для M. Поэтому перейдем к скалярным полям $\varphi^a (a = 1, 2, 3)$, которые преобразуются как вектор $\vec{\varphi}$ для группы внутренней симметрии $O(3)$ и описываются лагранжианом $\mathcal{L} = \frac{1}{2}\partial\vec{\varphi}\cdot\partial\vec{\varphi} - V(\vec{\varphi}\cdot\vec{\varphi})$. (У вас не должно возникнуть путаницы при использовании стрелки для обозначения вектора в группе внутренней симметрии.)

Выберем $V = \lambda\left(\vec{\varphi}^2 - v^2\right)^2$. Ход событий разворачивается здесь во многом так же, как в случае с вихрем. Чтобы масса независимой от времени конфигурации

$$M = \int d^3x \left[\frac{1}{2}(\partial\vec{\varphi})^2 + \lambda\left(\vec{\varphi}^2 - b^2\right)^2\right]$$

была конечной, необходимо, чтобы $|\vec{\varphi}| = v$ на пространственной бесконечности, так что $\vec{\varphi}(r = \infty)$ действительно существует на S^2.

Тождественное отображение $S^2 \to S^2$ указывает на то, что нужно рассмотреть такую конфитурацию, для которой

$$\varphi^a \overset{r\to\infty}{\longrightarrow} v\frac{x^a}{r}$$

На первый взгляд это уравнение выглядит несколько странно, поскольку в нем одновременно участвует индекс группы внутренней симметрии с ин- дексом пространственных координат (но фактически мы уже сталкивались с этим явлением в вихре). На пространственной бесконечности поле $\vec{\varphi}$ направлено радиально наружу, поэтому такую конфигурацию образно называют «еж». Нарисуйте ее, если не можете представить!

Как и в случае с вихрем, условие конечности первого члена в (4) заставляет ввести $O(3)$-калибровочное поле A^b_μ, с тем чтобы можно было заменить обычную производную $\partial_i\varphi^a$ на ковариантную производную $D_i\varphi^a = \partial_i\varphi^a + e\varepsilon^{abc}A^b_i\varphi^c$. Тогда можно сделать так, чтобы $D_i\varphi^a$ обращалось в нуль на бесконечности. Используя обычную арифметику, можно показать, что для уравнения (5) калибровочный потенциал должен вести себя как

$$A^b_i \overset{r\to\infty}{\longrightarrow} \frac{1}{e}\varepsilon^{bij}\frac{x^j}{r^2}$$

Представьте себя в лаборатории на пространственной бесконечности. Поле $\vec{\varphi}$ внутри этой довольно небольшой лаборатории ориентировано в разных точках приблизительно в одном направлении. Калибровочная группа $O(3)$ нарушена до $O(2) \simeq U(1)$. В этой лаборатории экспериментаторы наблюдают за связанным с $U(1)$ безмассовым калибровочным полем, которое они вполне могут назвать электромагнитным полем («крякает как утка»). Действительно, калибровочно-инвариантное тензорное поле

$$\mathcal{F}_{\mu\nu} \equiv \frac{F^a_{\mu\nu}\varphi^a}{|\varphi|} - \frac{\varepsilon^{abc}\varphi^a\left(D_\mu\varphi\right)^b\left(D_\nu\varphi\right)^c}{e|\varphi|^3}$$

можно отождествлять с электромагнитным полем (см. упражнение V.7.5).

Электрического поля нет, поскольку такая конфигурация не зависит от времени и $A^b_0 = 0$. Можно получить только магнитное поле $\vec{B}$, которое сразу вычисляется, поскольку

известен A_i^b, но уже в силу симметрии видно, что поле $\vec{B}$ может быть ориентировано только в радиальном направлении. Это и есть легендарный магнитный монополь, впервые постулированный Дираком!

Существование магнитных монополей в спонтанно нарушенной калибровочной теории было открыто 't Хоофтом и Поляковым. Если вычислить полный магнитный поток, исходящий из монополя $\int d\vec{S} \cdot \vec{B}$ (где $d\vec{S}$, как обычно, обозначает малый элемент поверхности на бесконечности, с нормалью, ориентированной наружу), можно убедиться, что он квантован в соответствующих единицах точно так, как должно быть по утверждению Дирака (вспомним главу IV.4). Можно вновь записать неравенство Богомольного для массы монополя

$$M = \int d^3x \left[\frac{1}{4} \left(\vec{F}_{ij} \right)^2 + \frac{1}{2} \left(D_i \vec{\varphi} \right)^2 + V(\vec{\varphi}) \right]$$

$\left[\vec{F}_{ij} \right.$ преобразуется как вектор относительно $O(3)$; вспомним выражение (IV.5.17).] Обратите внимание, что

$$\frac{1}{4} \left(\vec{F}_{ij} \right)^2 + \frac{1}{2} \left(D_i \vec{\varphi} \right)^2 = \frac{1}{4} \left(\vec{F}_{ij} \pm \varepsilon_{ijk} D_k \vec{\varphi} \right)^2 \mp \frac{1}{2} \varepsilon_{ijk} \vec{F}_{ij} \cdot D_k \vec{\varphi}.$$

Таким образом

$$M \geqslant \int d^3x \left[\mp \frac{1}{2} \varepsilon_{ijk} \vec{F}_{ij} \cdot D_k \vec{\varphi} + V(\vec{\varphi}) \right]$$

Далее заметим, что выражение

$$\int d^3x \frac{1}{2} \varepsilon_{ijk} \vec{F}_{ij} \cdot D_k \vec{\varphi} = \int d^3x \frac{1}{2} \varepsilon_{ijk} \partial_k \left(\vec{F}_{ij} \cdot \vec{\varphi} \right) = v \int d\vec{S} \cdot \vec{B} = 4\pi v g$$

имеет изящную интерпретацию в терминах магнитного заряда g монополя. Кроме того, если можно отбросить $V(\vec{\varphi})$, сохраняя при этом $|\vec{\varphi}| \stackrel{r \to \infty}{\longrightarrow} v$, то неравенство $M \geqslant 4\pi v |g|$ насыщается $\vec{F}_{ij} = \pm \varepsilon_{ijk} D_k \vec{\varphi}$. Решения этого уравнения известны как состояния Богомольного-Прасада-Зоммерфельда (или БПС-состояния).

Нетрудно построить электрически заряженный магнитный монополь, называемый дионом. Для этого просто берется $A_0^b = \left(x^b / r \right) f(r)$ с некоторой подходящей функцией $f(r)$.

Замечательной особенностью топологического монополя является то, что его масса оказывается равной $\sim M_W / \alpha \sim 137 M_W$ (упражнение V.7.11), где M_W обозначает массу промежуточного векторного бозона слабого взаимодействия. В главе VII.2 мы отчасти предвидели, что калибровочный бозон, который в главе IV.6 становится массивным вследствие механизма Андерсона - Хигтса, можно отождествить с промежуточным векторным бозоном. Это естественно объясняет, почему такой монополь еще не был обнаружен.

Инстантон

Рассмотрим неабелеву калибровочную теорию и осуществим поворот в интеграле по траекториям $Z = \int DA e^{-S(A)}$ в четырехмерное евклидово пространство. Мы могли бы определить Z в приближении наискорейшего спуска, для которого необходимо найти экстремум

$$S(A) = \int d^4x \frac{1}{2g^2} \operatorname{tr} F_{\mu\nu} F_{\mu\nu}$$

с конечным действием. Это означает, что на бесконечности $|x| = \infty$ множитель $F_{\mu\nu}$ должен убывать быстрее, чем $1/|x|^2$, и поэтому калибровочное поле A_μ должно быть чистой

калибровкой: $A = gdg^\dagger$ для элемента калибровочной группы g [см. выражение (IV.5.6)]. Конфигурации, для которых это условие имеет место, называют инстантонами. Мы видим, что инстантон является еще одним звеном в «великой цепи существования»: кинк-вихрь-монополь-инстантон.

Выберем для определенности калибровочную группу $SU(2)$. В параметризации $g = x_4 + i\vec{x}\cdot\vec{\sigma}$ по определению имеем $g^\dagger g = 1$ и $\det g = l$, что означает $x_4^2 + \vec{x}^2 = 1$. Мы знаем, что групповым многообразием $SU(2)$ является S^3. Таким образом, в инстантоне калибровочный потенциал на бесконечности $A \xrightarrow{r\to\infty} gdg^\dagger + O\left(1/|x|^2\right)$ определяет отображение $S^3 \to S^3$. Не кажется ли это знакомым? Действительно, вы уже знаете, какую роль в теории поля играют $S^0 \to S^0, S^1 \to S^1, S^2 \to S^2$. ственно

Напомним, что согласно главе IV.5 $\operatorname{tr} F^2 = d\operatorname{tr}\left(AdA + \frac{2}{3}A^3\right)$. Соответ-

$$\int \operatorname{tr} F^2 = \int\limits_{S^3} \operatorname{tr}\left(AdA + \frac{2}{3}A^3\right) = \int\limits_{S^3} \operatorname{tr}\left(AF - \frac{1}{3}A^3\right) = -\frac{1}{3}\int\limits_{S^3} \operatorname{tr}\left(gdg^\dagger\right)^3,$$

где использован тот факт, что F обращается в нуль на бесконечности. Это условие явно показывает, что $\int \operatorname{tr} F^2$ зависит только от гомотопии отображения $S^3 \to S^3$, определяемого g, и, следовательно, является топологической величиной. Кстати, $\int\limits_{S^3} \operatorname{tr}\left(gdg^\dagger\right)^3$ известен математикам как индекс Понтрягина (см. упражнение V.7.13).

В главе IV.7 упоминалось, что на киральную аномалию не влияют квантовые флуктуации более высоких порядков. Теперь у вас есть возможность привести изящное топологическое доказательство этого утверждения (упражнение V.7.13).

Переход Костерлица-Таулесса

Мы немного поспешили отказаться от вихря в некалиброванной теории $\mathcal{L} = \partial\varphi^\dagger\partial\varphi - \lambda\left(\varphi^\dagger\varphi - v^2\right)^2$ в $(2+1)$-мерном пространстве-времени. Вокруг вихря поле имеет вид $\varphi \sim ve^{i\theta}$, и, как мы уже отмечали, энергия одиночного вихря расходится логарифмически. Но что происходит с вихрем в паре с антивихрем?

Изобразим вихрь и антивихрь, разделенные расстоянием R, большим по сравнению с масштабами расстояний в теории. Вокруг антивихря поле имеет вид $\varphi \sim ve^{-i\theta}$. Поле φ наматывается вокруг вихря в одном направлении, а вокруг антивихря в другом направлении. Сделав рисунок, убедитесь в том, что на пространственной бесконечности поле φ вовсе не наматывается: оно лишь стремится к фиксированному значению. Наматывание в одном направлении компенсирует наматывание в другом направлении.

Таким образом, конфигурация, состоящая из пары вихрь-антивихрь, не требует затрат бесконечной энергии. Однако она требует затрат конечного количества энергии. На самом деле в области между вихрем и антивихрем поле φ наматывается примерно в два раза быстрее (что видно по рисунку). Следовательно, приблизительная оценка энергии имеет вид $v^2\int d^2x\,(1/r^2) \sim v^2\log(R/a)$ после интегрирования по области размером R, относящейся к физической шкале в задаче. (Чтобы задача имела смысл, разделим R на размер вихря a.) Вихрь и антивихрь притягиваются друг к другу логарифмическим потенциалом. Другими словами, конфигурация не может быть статической. Вихрь и антивихрь стремятся встретиться и «в пылком объятии» уничтожить друг друга, высвободив конечное количество энергии $v^2\log(R/a)$. (Отсюда название антивихрь.)

Все это происходит при нулевой температуре, но в физике конденсированных сред нас интересует свободная энергия $F = E - TS$ (с энтропией S) при некоторой температуре T, а не энергия E. Принимая это во внимание, Костерлиц и Таулесс открыли, что при росте температуры происходит фазовый переход. Рассмотрим газ вихрей и антивихрей при некоторой ненулевой температуре. Из-за теплового перемешивания движущиеся вокруг вихри и антивихри могут или найти, или не найти друг друга. Если находят - аннигилируют. Насколько высокой должна быть температура, чтобы это произошло?

Сделаем эвристическую оценку. Рассмотрим одиночный вихрь. Согласно Больцману, энтропией является логарифм «числа» способов, которыми мы можем поместить вихрь внутрь коробки размером L (устремим его в бесконечность). Таким образом, $S \sim \log(L/a)$. Энтропия S должна бороться с энергией $E \sim v^2 \log(L/a)$. Легко видеть, что свободная энергия $F \sim (v^2 - T)\log(L/a)$ стремится к бесконечности, если $T \lesssim v^2$, которую мы определяем, по существу, как критическую температуру $\widetilde{T}_c$.

Одиночный вихрь не может существовать при температуре ниже T_c. Вихри и антивихри тесно связаны при тсмпсратуре ниже T_c, но высвобождаются при температуре выше T_c.

Черная дыра

Открытие в 1970-е годы таких топологических объектов, которые нельзя увидеть в теории возмущений, потрясло поколение физиков, воспитанных на фейнмановских диаграммах и каноническом квантовании. Их (включая вашего покорного слугу) учили, что оператор поля $\varphi(x)$ является сингулярным квантовым оператором и как таковой он не имеет физического смысла, а также тому, что квантовая теория поля определяется по теории возмущений фейнмановскими диаграммами. Даже очень известные физики в недоумении вопрошали, что бы могла означать запись типа $\varphi \stackrel{r\to\infty}{\longrightarrow} ve^{i\theta}$? Научные дискуссии, которые велись в прошлом, теперь не имеют никакого значения. Как сказано во введении к главе V.6, я люблю называть этот исторический процесс «избавлением теоретиков-полевиков от оков фейнмановских диаграмм».

Следует упомянуть один аргумент, который в то время использовали физики, чтобы убедить себя в существовании солитонов. В конце концов, черная дыра Шварцшильда, определяемая метрикой $g_{\mu\nu}(x)$ (см. главу I.10), известна с 1916 года. Что собой представляют компоненты метрики $g_{\mu\nu}(x)$? Этими компонентами являются поля, точно так же как полями являются скалярное поле $\varphi(x)$ и калибровочное поле $A_\mu(x)$. В квантовой теории гравитации $g_{\mu\nu}$ надо заменить на квантовый оператор, такой как φ и A_μ. Итак, объекты, открытые в 1970-е годы, концептуально не отличаются от черной дыры, известной в 1910-е годы. Но в начале 1970-х годов большинство теоретиков, занимающихся физикой элементарных частиц, не были особенно сведующими в квантовой гравитации.

Упражнения

V.7.1. Объясните связь между математическим утверждением $\Pi_0(S^0) = Z_2$ и физическим результатом, согласно которому кинки с $|Q| \geqslant 2$ не существуют.

V.7.2. Изучите в вихре масштабы длин, на которых меняются поля φ и A. Оцените массу вихря.

V.7.3. Рассмотрите конфигурацию вихря, в которой $\varphi \stackrel{r\to\varphi}{\longrightarrow} ve^{i\nu\theta}$, где ν - целое число. Вычислите магнитный поток. Покажите, что магнитный поток, исходящий из антивихря (для которого $\nu = -1$), направлен противоположно магнитному потоку, исходящему из вихря.

V.7.4. Поскольку с математической точки зрения $g(\theta) \equiv e^{i\nu\theta}$ можно рассматривать как элемент группы $U(1)$, мы можем говорить об отображении S^1 (окружности на пространственной бесконечности) на группу $U(1)$. Вычислите $(i/2\pi)\int\limits_{S^1} gdg^\dagger$, показывая тем самым, что степень отображения определяется этим интегралом от 1-формы. V.7.5. Покажите, что в области, где φ^a постоянно, определенный в тексте $\mathcal{F}_{\mu\nu}$ является напряженностью электромагнитного поля. Вычислите $\vec{B}$ вдали от центра магнитного монополя и покажите, что имеет место квантование Дирака.

V.7.6. Постройте явно отображение $S^2 \to S^2$, которое дважды оборачивает одну сферу вокруг другой. Проверьте, что это отображение соответствует магнитному монополю с магнитным зарядом 2.

V.7.7. Запишите вариационные уравнения, которые минимизируют (8).

V.7.8. Найдите явно решение БПС.

V.7.9. Рассмотрите дионное решение. Получите его в БПС-пределе.

V.7.10. Проверьте явно, что магнитный монополь является инвариантным относительно вращений, несмотря на внешние различия. Под этим понимается, что все физические калибровочно-инвариантные величины, такие как $\vec{B}$, при вращении ковариантны. Калибровочно-зависимые величины, такие как A_i^b, при вращении могут и действительно изменяются. Выпишите генераторы вращения.

V.7.11. Покажите, что масса магнитного монополя составляет примерно $137 M_W$.

V.7.12. Вычислите $n \equiv -(1/24\pi^2) \int\limits_{S^3} \mathrm{tr}\left(g dg^\dagger\right)^3$ для отображения $g = e^{i\vec{\theta}\cdot\vec{\sigma}}$. [Указание: по симметрии надо вычислить подынтегральное выражение лишь в малой окрестности единичного элемента группы, т. е., что эквивалентно, в окрестности северного полюса S^3. Далее рассмотреть $g = e^{i(\theta_1\sigma_1+\theta_2\sigma_2+m\theta_3\sigma_3)}$ для целого числа m и убедиться, что m соответствует числу намоток S^3 вокруг S^3.] Сравните с упражнением V.7.4 и восхититесь красотой математики.

V.7.13. Докажите, что поправки более высокого порядка не меняют киральную аномалию $\partial_\mu J_5^\mu = [1/(4\pi)^2]\,\varepsilon^{\mu\nu\lambda\sigma}\,\mathrm{tr}\,F_{\mu\nu}F_{\lambda\sigma}$ (я изменил масштаб $A \to (1/g)A$). [Указание: проинтегрируйте по пространству-времени и покажите, что левая часть определяется числом правых фермионных мод минус число левых фермионных мод, так что обе части даются целыми числами.]

30.3 Zi.VI Field Theory and Condensed Media

30.3.1 Zi.VI.1 Fractional Statistics, Chern–Simons Term, and Topological Field Theory

Дробная статистика

Существование бозонов и фермионов является одним из наиболее фундаментальных свойств квантовой физики. Если поменять местами две тождественные квантовые частицы, у волновой функции появляется множитель, равный либо +1, либо -1. Лейнаас и Мирхейм, а затем Вильчек независимо друг от друга поняли, что в (2 + 1)-мерном пространстве-времени частицы могут тоже подчиняться так называемой дробной, или анионной, статистике, отличной от бозе- или ферми-статистики. Такие частицы теперь известны как анионы.

Чтобы поменять местами две частицы, можно переместить одну из них на пол-оборота вокруг другой, а затем сдвинуть их соответствующим образом. Если один анион повернуть на пол-оборота вокруг другого аниона против часовой стрелки, то волновая функция приобретет множитель $e^{i\theta}$, где θ - характерное для частицы действительное число. Для $\theta = 0$ получаем бозоны, а для $\theta = \pi$ - фермионы. Частицы с $\theta = \pi/2$ в середине между бозонами и фермионами известны как семионы.

После выхода статьи Вильчека многие именитые физики старшего поколения были очень озадачены. Рассуждая в терминах волновой функции Шредингера, они погрузились в бесконечные споры о том, должна ли эта волновая функция быть однозначной. Действительно, анионная статистика дает поразительный пример того, насколько формализм интеграла по траекториям в некоторых случаях оказывается прозрачнее. Понятие анионной статистики можно сформулировать с помощью волновых функций, но для этого необходимо хорошо представлять конфигурационное пространство, на котором определена волновая функция.

Рассмотрим две неразличимые частицы в положениях x_1^i и x_2^i в некоторый начальный момент времени, которые через время T оказываются в положениях x_1^f и x_2^f. В представлении интеграла по траектори- ям для $\left\langle x_1^f, x_2^f \left| e^{-iHT} \right| x_1^i, x_2^i \right\rangle$ суммировать нужно по всем траекториям. В пространстве-времени мировые линии этих двух частиц переплетаются (см. рис. VI.1.1). (Мы неявно допускаем, что частицы не могут проходить друг сквозь друга, что соответствует случаю сильного отталкивания между ними.)

Очевидно, что траектории можно разделить на топологически различные классы, характеризующиеся целым числом n, равным тому, сколько раз мировые линии двух частиц оплетают друг друга. Так как эти классы не могут деформироваться друг в друга, соответствующие амплитуды не могут интерферировать квантово-механически, но нам позволено в каждом классе ставить в соответствие этим амплитудам добавочный фазовый множитель $e^{i\alpha_n}$, помимо обычного множителя, связанного с действием.

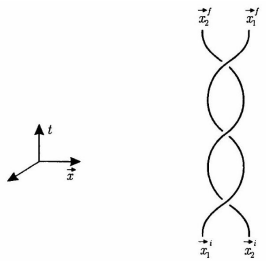


Рис. VI.1.1

Зависимость α_n от n определяется композицией квантовых амплитуд. Представим, что одна частица перемещается вокруг другой на угол $\Delta\varphi_1$; этому событию мы ставим в соответствие добавочный фазовый множитель $e^{if(\Delta\varphi_1)}$, где f - пока еще неизвестная функция. Допустим, что за этим событием следует другое, в котором та же частица перемещается вокруг другой на дополнительный угол $\Delta\varphi_2$. Фазовый множитель $e^{if(\Delta\varphi_1+\Delta\varphi_2)}$, который ставится в соответствие комбинированному событию, явно должен удовлетворять закону композиции $e^{if(\Delta\varphi_1+\Delta\varphi_2)} = e^{if(\Delta\varphi_1)}e^{if(\Delta\varphi_2)}$. Другими словами, $f(\Delta\varphi)$ должна быть линейной функцией своего аргумента.

Мы приходим к выводу, что в $(2+1)$-мерном пространстве-времени фазовый множитель $e^{i(\theta/\pi)\Delta\varphi}$, где θ - произвольный вещественный па- раметр, можно ассоциировать с квантовой амплитудой, соответствующей траекториям, по которым одна частица перемещается вокруг другой против часовой стрелки на угол $\Delta\varphi$. Заметим, что когда одна частица перемещается вокруг другой по часовой стрелке на угол $\Delta\varphi$, квантовая амплитуда приобретает фазовый множитель $e^{-i\frac{\theta}{\pi}\Delta\varphi}$.

Если менять местами два аниона, необходимо быть внимательными и следить за тем, выполнятся это перемещенис «против часовой стрслки» или «по часовой стрелке», поскольку при этом появляются множители $e^{i\theta}$ или $e^{-i\theta}$ соответственно. Такая ситуация сразу указывает на то, что нарушается четность P и инвариантность относительно обращения времени T.

Теория Черна-Саймонса

Следующий важный вопрос заключается в том, можно ли все вышесказанное включить в локальную квантовую теорию поля. Ответ был дан Вильчеком и Зи, которые показали, что дробная статистика может вытекать из эффекта взаимодействия с калибровочным полем. Важность формулировки теории поля заключается в том, насколько убедительно она демонстрирует, что понятие анионной статистики полностью совместимо с дорогими нашему сердцу принципами, которыми мы дорожим и которые являются элементами конструкции квантовой теории поля.

Имея лагранжиан $\mathcal{L}_0$ с сохраняющимся током j^μ, построим лагранжиан

$$\mathcal{L} = \mathcal{L}_0 + \gamma\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + a_\mu j^\mu.$$

Здесь $\varepsilon^{\mu\nu\lambda}$ обозначает полностью антисимметричный символ в $(2+1)$-мерном пространстве-времени, а γ является произвольным вещественным параметром. При калибровочном преобразовании $a_\mu \to a_\mu + \partial_\mu\Lambda$ член $\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$, известный как член Черна-Саймонса, изменяется следующим образом: $\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \to \varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \varepsilon^{\mu\nu\lambda}\partial_\mu\Lambda\partial_\nu a_\lambda$. Действие меняется на $\delta S = \gamma\int d^3x\varepsilon^{\mu\nu\lambda}\partial_\mu(\Lambda\partial_\nu a_\lambda)$, и поэтому, если можно отбросить

граничные члены (что мы полагаем здесь корректным), действие Черна-Саймонса калибровочно-инвариантно. Заметим в этой связи, что на языке дифференциальных форм, который известен нам из глав IV.5 и IV.6, член ЧернаСаймонса можно компактно записать как ada.

Теперь решим уравнение движения, полученное из (1):

$$2\gamma\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda = -j^\mu$$

для частицы, находящейся в состоянии покоя (так что $j_i = 0$). Интегрируя $\mu = 0$ компоненту уравнения (2), получаем

$$\int d^2x\,(\partial_1 a_2 - \partial_2 a_1) = -\frac{1}{2\gamma}\int d^2x j^0$$

Таким образом, член Черна-Саймонса выполняет в теории роль снабжения заряженных частиц потоком. (Здесь термин «заряженные частицы» означает частицы, которые взаимодействуют с калибровочным полем a_μ. В данном контексте понятно, что, ссылаясь на заряд и поток, мы, конечно, не имеем в виду заряд и поток, связанные с обычным электромагнитным полем. Мы лишь заимствуем полезную терминологию.)

В соответствии с эффектом Аронова - Бома (глава IV.4), когда одна из таких частиц движется вокруг другой, волновая функция приобретает фазу, наделяя таким образом частицы анионной статистикой с углом $\theta = 1/4\gamma$ (см. упражнение VI.1.5).

Строго говоря, термин «дробная статистика» несколько вводит в заблуждение. Во-первых, сделаем тривиальное замечание, согласно которому параметр статистики θ не должен быть дробным. Во-вторых, статистика непосредственно не связана с подсчетом того, сколько частиц можно поместить в одно состояние. Статистику анионов, пожалуй, лучше представить как дальнодействующее фазовое взаимодействие, обусловленное калибровочным полем a.

Появление $\varepsilon^{\mu\nu\lambda}$ в (1) свидетельствует о нарушении четности P и инвариантности относительно обращения времени T, что нам уже известно.

Член Хопфа

Альтернативный подход заключается в интегрировании по a в (1). В главе III.4 мы говорили, что, как и в любой калибровочной теории, оператор, обратный дифференциальному оператору $\varepsilon\partial$, не определен. Он имеет нулевую моду, так как $\left(\varepsilon^{\mu\nu\lambda}\partial_\nu\right)(\partial_\lambda F(x)) = 0$ для любой гладкой функции $F(x)$. Выберем калибровку Лоренца $\partial_\mu a^\mu = 0$. Тогда, используя фундаментальное тождество теории поля (см. приложение А), получаем нелокальный лагранжиан

$$\mathcal{L}_{\text{Hopf}} = \frac{1}{4\gamma}\left(j_\mu \frac{\varepsilon^{\mu\nu\lambda}}{\partial^2} j_\lambda\right)$$

известный как член Хопфа.

Чтобы определить статистический параметр θ, рассмотрим событие, когда одна частица перемещается на пол-оборота вокруг другой частицы, находящейся в состоянии покоя. Тогда ток j равен сумме двух слагаемых, описывающих две частицы. Подставляя в (4), вычисляем квантовую фазу $e^{iS} = e^{i\int d^3x L_{\text{Hopf}}}$ и получаем $\theta = \frac{1}{4\gamma}$.

Топологическая теория поля

В чистой теории Черна-Саймонса есть нечто концептуально новое.

$$S = \gamma \int\limits_M d^3x\varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$$

Она топологическая.

Напомним, что, как описано в главе I.10, теорию поля для плоского пространства-времсни можно немедленно обобщить до теории поля в искривленном пространстве-времени путем замены метрики Минковского $\eta^{\mu\nu}$ на метрику Эйнштейна $g^{\mu\nu}$ и включения множителя $\sqrt{-g}$ в меру интегрирования по пространству-времени.

Но в теории Черна-Саймонса $\eta^{\mu\nu}$ не появляется! Лоренцевские индексы сворачиваются с полностью антисимметричным символом $\varepsilon^{\mu\nu\lambda}$. Более того, покажем, что множитель $\sqrt{-g}$ не нужен. Вспомним также, что, как говорилось в главе I.10, векторное поле преобразуется как $a_\mu(x) = \left(\partial x'^\lambda/\partial x'^\mu\right) a'_\lambda(x')$, и поэтому для трех векторных полей

$$\varepsilon^{\mu\nu\lambda} a_\mu(x) b_\nu(x) c\lambda(x) = \varepsilon^{\mu\nu\lambda} \frac{\partial x'^\sigma}{\partial x^\mu} \frac{\partial x'^\tau}{\partial x^\nu} \frac{\partial x'^\rho}{\partial x^\lambda} a'_\sigma(x') b'_\tau(x') c'_\rho(x') =$$
$$= \det\left(\frac{\partial x'}{\partial x}\right) \varepsilon^{\sigma\tau\rho} a'_\sigma(x') b'_\tau(x') c'_\rho(x')$$

С другой стороны, $d^3x' = d^3x \det(\partial x'/\partial x)$. Тогда имеет место соотношение

$$d^3x \varepsilon^{\mu\nu\lambda} a_\mu(x) b_\nu(x) c_\lambda(x) = d^3x' \varepsilon^{\sigma\tau\rho} a'_\sigma(x') b'_\tau(x') c'_\rho(x')$$

которое инвариантно без использования множителя $\sqrt{-g}$.

Итак, действие Черна-Саймонса в (5) инвариантно относительно общего координатного преобразования - оно уже написано для искривленного пространства-времени. Метрика $g^{\mu\nu}$ никуда не входит. Для построения теории Черна-Саймонса не нужны часы и линейки! Ей известна лишь топология пространства-времени, поэтому ее правильно называют топологической теорией поля. Другими словами, когда интеграл в (5) вычисляется по замкнутому многообразию M, свойство теории поля $\int Da e^{iS(a)}$ зависит только от топологии многообразия, а совсем не от метрики, которое можно оснастить это многообразие.

Вырождение основного квантового состояния

Напомним приведенное в главе I.10 основное определение энергии и импульса. Тензор энергии-импульса определяется вариацией действия относительно $g^{\mu\nu}$, но смотрите-ка, здесь действие не зависит от $g_{\mu\nu}$. Тензор энергии-импульса, а значит, и гамильтониан тождественно равны нулю! Из этого следует только один вывод: чтобы определить гамильтониан, нам нужны часы и линейки.

Что означает для квантовой системы иметь гамильтониан $H = 0$? Когда после курса квантовой механики на экзамене надо было решить задачу нахождения спектра для нулевого гамильтониана, мы легко могли это сделать! Все состояния имеют энергию $E = 0$. Мы готовы сдавать экзамен!

Однако нетривиальная задача заключается в том, чтобы найти, сколько существует состояний. Их число известно как вырождение основного состояния и зависит только от топологии многообразия M.

Массивные дираковские фермионы и член Черна- Саймонса

Рассмотрим калибровочное поле a_μ, связанное с массивным дираковским фермионом в $(2+1)$-мерном пространстве-времени: $\mathcal{L} = \bar{\psi}(i\not{\partial} + \not{a}$ - - m). Решая ранее задачу из главы II.1, вы узнали о довольно удивительном явлении, что в $(2+1)$-мерном пространстве-времени дираковский массовый член нарушает P и T. Таким образом, мы рассчитываем сгенерировать нарушающий P и T член Черна-Саймонса $\varepsilon^{\mu\nu\lambda} a_\mu \partial_\lambda a_\nu$ интегрированием по фермионам и получить в результате член $\mathrm{tr}\log(\not{\partial} + \not{K} - m)$ в эффективном действии. Аналогичным образом мы поступали в главе IV.3.

В однопетлевом порядке есть диаграмма поляризации вакуума (графически подобная диаграмме из главы III.7, но содержащая на одно измерение меньше в пространстве-времени) с интегралом Фейнмана, пропорциональным

$$\int \frac{d^3p}{(2\pi)^3}\,\mathrm{tr}\left(\gamma^\nu \frac{1}{p\!\!\!/+q\!\!\!/-m}\gamma^\mu \frac{1}{p\!\!\!/-m}\right).$$

Как мы увидим, замена $4 \to 3$ все меняет в этом мире. Я предлагаю вам самостоятельно вычислить (6) (упражнение VI.1.7), лишь отмечу явно вы- раженные здесь особенности. Так как ∂_λ в члене Черна-Саймонса соответствует q_λ в импульсном пространстве, то для определения коэффициента члена Черна-Саймонса необходимо продифференцировать (6) по q_λ и устремить $q \to 0$:

$$\int \frac{d^3p}{(2\pi)^3}\,\mathrm{tr}\left(\gamma^\nu \frac{1}{p\!\!\!/-m}\;\gamma^\lambda \frac{1}{q\!\!\!/-m}\gamma^\mu \frac{1}{p\!\!\!/-m}\right) = \\ = \int \frac{d^3p}{(2\pi)^3}\frac{\mathrm{tr}\left[\gamma^\nu(p\!\!\!/+m)\gamma^\lambda(q\!\!\!/+m)\gamma^\mu(p\!\!\!/+m)\right]}{\left(p^2-m^2\right)^3}.$$

Я сосредоточу внимание только на одной части интеграла, которая обуславливается членом в следе, пропорциональном m^3:

$$\varepsilon^{\mu\nu\lambda}m^3\int\frac{d^3p}{\left(p^2-m^2\right)^3}$$

Как отмечено в упражнении II.1.11, в $(2+1)$-мерном пространстве-времени γ^μ являются тремя матрицами Паули, и поэтому след $\gamma^\nu\gamma^\lambda\gamma^\mu$ пропорционален $\varepsilon^{\mu\nu\lambda}$. Антисимметричный символ появляется, как и ожидалось, в результате нарушения P и T.

Из размерного анализа следует, что интеграл в (8) равен $1/m^3$ с точностью до безразмерной постоянной, и поэтому m сокращается.

Но будьте внимательны! Этот интеграл зависит только от m^2 и ничего не знает о знаке m. Корректным будет сказать, что он пропорционален $1/|m|^3$, а не $1/m^3$. Таким образом, коэффициент члена Черна-Саймонса равен $m^3/|m|^3 = m/|m| =$ знаку m с точностью до безразмерной постоянной. Это поучительный пример, демонстрирующий важность знака! Полученный результат имеет смысл, поскольку при P (или T) дираковское поле с массой m преобразуется в дираковское поле с массой $-m$. В инвариантной относительно четности теории с дублетом дираковских полей, имеющих массы m и $-m$, член Черна-Саймонса не должен генерироваться.

Упражнения

VI.1.1. Можно считать, что в нерелятивистской теории есть два разных члсна Черна-Саймонса - $\varepsilon_{ij}a_i\partial_0 a_j$ и $\varepsilon_{ij}a_i\partial_i a_j$. Покажите, что калибровочная инвариантность заставляет эти два члена собираться в один член Черна-Саймонса $\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$. Для члена Черна-Саймопса калибровочная инвариантность означает лоренц-инвариантность. В отличис от ттого, член Максвелла будет в общем случае нерелятивистским, состоящим из двух членов f_{0i}^2 и f_{ij}^2 с произвольным коэффициентом отношения между ними (с $f_{\mu\nu}=\partial_\mu a_\nu-\partial_\nu a_\mu$, как обычно).

VI.1.2. Исходя из соображений размерности, убедитесь, что на больших расстояниях член Черна-Саймонса доминирует над членом Максвелла. Это одна из причин, по которой специалисты в области релятивистской теории поля находят анионные жидкости столь привлекательными. Пока их интересует только физика больших расстояний, они могут не учитывать член Максвелла и заниматься релятивистской теорией (см. упражнение VI.1.1). Заметьте, что в такой ситуации $(2+1)$-мерное пространство-время выделяется как особое. В $(3+1)$-мерном пространстве-времени обобщение члена Черна-Саймонса $\varepsilon^{\mu\nu\lambda\sigma}f_{\mu\nu}f_{\lambda\sigma}$ имеет такую же размерность, как и член Максвелла f^2. В (4+1)-мерном пространстве на больших расстояниях член $\varepsilon^{\rho\mu\nu\lambda\sigma}a_\rho f_{\mu\nu}f_{\lambda\sigma}$ менее важен, чем член Максвелла f^2.

VI.1.3. Существует обобщение члена Черна-Саймонса на пространство-время с большей размерностью, которое отличается от приведенного в упражнении VI.1.2. Мы можем ввести

калибровочный потенциал p-формы (см. главу IV.4). Запишите обобщенный член Черна-Саймонса в $(2p+1)$-мерном пространстве-времени и обсудите полученную теорию.

VI.1.4. Рассмотрите лагранжиан $\mathcal{L} = \gamma\varepsilon a\partial a - (1/4g^2)\, f^2$. Вычислите пропагатор и покажите, что калибровочный бозон массивен. Некоторые физики, озадаченные дробной статистикой, приводили следующий довод: поскольку при наличии члена Максвелла калибровочный бозон является массивным и поэтому короткодействующим, он вряд ли может генерировать дробную статистику, которая явно является взаимодействием на бесконечно большом расстоянии. (Независимо от того, насколько удалены друг от друга меняющиеся местами две частицы, волновая функция все равно приобретает фазу.) Проблема разрешается исходя из того, что информация на самом деле распространяется на бесконечно большое расстояние через полюс $q = 0$, связанный с калибровочной степенью свободы. Именно этот явный парадокс оказался причиной замешательства многих физиков, когда они впервые услышали об эффекте Ааронова - Бома. Как может произвольно далекая от магнитного потока частица в области, где совсем нет магнитного поля, знать о существовании магнитного потока?

VI.1.5. Покажите, что $\theta = 1/4\gamma$. Здесь есть один хитрый множитель' 2. Поэтому не отчаивайтесь, если потеряете множитель 2. Попробуйте решить задачу еще раз.

VI.1.6. Найдите неабелеву версию члена Черна-Саймонса ada. [Указание: возможно, проще использовать дифферснциальныс формы, как в главс IV.6.]

VI.1.7. Используя канонический формализм, описанный в главе I.8, покажите, что лагранжиан Черна - Саймонса дает нулевой гамильтониан $H = 0$.

VI.1.8. Вычислите (6).

[1] X. G. Wen and A. Zee, J. de Physique, 50: 1623, 1989.

30.3.2 Zi.VI.2 Quantum Hall Liquids

Взаимное влияние двух областей физики

В последнее десятилетие появился интерес к изучению топологических квантовых жидкостей (примером которых является холловская жидкость). Квантовая система Холла состоит из электронов, движущихся в плоскости в присутствии внешнего магнитного поля B, перпендикулярного этой плоскости. Предполагается, что магнитное поле является достаточно сильным для того, чтобы все электроны имели спин, скажем, направленный вверх, и поэтому с ними можно обращаться как с бесспиновыми фермионами. Хорошо известно, что в этой, казалось бы, несущественной и простой физической ситуации содержится богатейшая физика, разъяснение которой привело к получению двух Нобелевских премий. Это необычайное богатство связано со взаимным влиянием двух основных областей физики. cTBO.

1. Хотя электрон является точечным, он занимает конечное простран-

В классическом представлении заряженная частица движется в магнитном поле по ларморовской окружности радиуса r, заданной уравнением $evB = mv^2/r$. В классическом представлении этот радиус не фиксирован, и частицы более высокой энергии движутся по большим окружностям, но если проквантовать угловой момент mvr так, чтобы $h = 2\pi$ (в единицах, где $\hbar = 1$), получим соотношение $eBr^2 \sim 2\pi$. Квантовый электрон занимает область порядка $\pi r^2 \sim 2\pi^2/eB$.

2. Электроны стремятся держаться подальше друг от друга.

Каждый электрон не только обязательно занимает конечное пространство, но должен иметь собственное пространство. Таким образом, квантовую задачу Холла можно описать как своего рода проблему заселения или как проблему предоставления площади служебных помещений в Институте тсоретической физики тем визитерам, которые не хотят делить свой офис с другими. Уже на этой стадии можно ожидать, что, когда числа электронов N_e как раз достаточно, чтобы полностью занять пространство, а именно когда область системы имеет вид $N_e\pi r^2 \sim N_e\,(2\pi^2/eB) \sim A$, происходит нечто особенное.

Уровни Ландау и целочисленный эффект Холла

Эти эвристические рассуждения, конечно же, могут быть более строгими. Задача из учебника об одном бесспиновом электроне в магнитном поле

$$-\left[(\partial_x - ieA_x)^2 + (\partial_y - ieA_y)^2\right]\psi = 2mE\psi$$

была решена Ландау десятки лет назад 1. Такие состояния электрона имеют место и являются вырожденными с энергией $E_n = \left(n+\frac{1}{2}\right)\frac{eB}{m}(n = 0,1,2,...)$, известной как n-й уровень Ландау. Каждый уровень Ландау имеет вырождение $BA/2\pi$, где A - площадь системы, отражающее тот факт, что ларморовские окружности можно поместить где угодно. Заметим, что один уровень Ландау отделен от следующего конечным количеством энергии (eB/m).

Представим, что один за другим вводятся невзаимодействующие электроны. По принципу запрета Паули каждый следующий электрон, который вводится, должен занимать другое состояние на уровне Ландау. Так как на каждом уровне Ландау может находиться $BA/2\pi$ электронов, коэффициент заполнения естественно определить как $\nu \equiv N_e/(BA/2\pi)$ (см. упражнение VI.2.1). Когда ν равно целому числу, первый уровень Ландау ν заполнен. Если вводится еще один электрон, он должен будет занять уровень Ландау $(\nu+1)$, что потребует больше энергии, чем потрачено на предшествующий электрон.

Таким образом, для ν, равного целому числу, холловская жидкость несжимаема. Любая попытка ее сжать уменьшает вырождение уровней Ландау (уменьшается эффективная область A и поэтому уменьшается вырождение $BA/2\pi$) и приводит к выталкиванию некоторых электронов на следующий уровень, что требует больших затрат энергии.

Электрическое поле E_y, наложенное на холловскую жидкость в направлении y, создает в направлении x ток $J_x = \sigma_{xy}E_y$, где $\sigma_{xy} = \nu$ (в единицах e^2/h). Это легко понять исходя из закона силы Лоренца, которому подчиняются электроны в присутствии магнитного поля. Удивительным экспериментальным открытием было то, что график зависимости

[1] В 1929 г. - Прим. ped. холловской проводимости σ_{xy} от B проходит через несколько горизонтальных участков кривой, о чем вам, возможно, известно. Чтобы понять появление таких плоских участков, нужно рассмотреть влияние примесей. Я затрону эту увлекательную тему о примесях и беспорядке в главе VI.8.

Итак, целочисленный квантовый эффект Холла понять довольно просТо.

Дробный эффект Холла

Экспериментальное открытие дробного эффекта Холла после целочисленного эффекта Холла, а именно то, что жидкость Холла также несжимаема при коэффициенте заполнения ν, равном простым дробям с нечетным знаменателем (таким, как $\frac{1}{3}$ и $\frac{1}{5}$), оказалось совершенно неожиданным для теоретиков. Для $\nu = \frac{1}{3}$ в первом уровне Ландау заполнена только одна треть состояний. Казалось бы, что добавление еще нескольких электронов не окажет столь большого воздействия на систему. Почему жидкость Холла с $\nu = \frac{1}{3}$ должна быть несжимаемой?

Оказалось, что критическим моментом является взаимодействие между электронами. Тот факт, что первый уровень Ландау на треть заполнен невзаимодействующими бесспиновыми электронами, не определяет уникальное многочастичное состояние: существует огромное вырождение, так как каждый из электронов может занять любое из доступных состояний $BA/2\pi$ только при условии соблюдения принципа запрета Паули. Но как только между электронами возникает взаимное отталкивание, на огромном пространстве вырожденных состояний, по-видимому, выбирается единственное основное состояние. Вен описал дробное состояние Холла как замысловатый танец электронов: каждый электрон не только занимает конечное пространство в танцевальном зале, но и по причине взаимного отталкивания должен заботиться о том, чтобы неожиданно не столкнуться с другим электроном. Этот танец должен быть тщательно поставлен и возможен только для специальных значений ν.

Примеси также играют существенную роль, но обсуждение примесей отложим до главы VI.6.

Чтобы понять дробный эффект Холла, учтем еще один важный момент. Напомним, что основная единица потока в главе V. 7 включает 2π, и поэтому число квантов потока, проходящих сквозь плоскость, равно $N_\phi = BA/2\pi$. Итак, загадка состоит в том, что, когда число квантов потока, приходящихся на электрон, $N_\phi/N_e = \nu^{-1}$, является нечетным цслым числом, происходит нечто особое. Я расположил главы так, чтобы то, что вы узнали из предыдущей главы, имело отношение к решению этой загадки. Предположим, что кванты потока ν^{-1} как-то связаны с каждым электроном. Когда две такие связанные системы меняются местами, появляется дополнительная фаза Ааронова-Бома, помимо фазы (-1) из статистики Ферми для электронов. Для нечетных ν^{-1} эти связанные системы эффективно подчиняются статистике Бозе и могут быть описаны комплексным скалярным полем φ. Оказывается, что конденсация поля φ отвечает за физику квантовой холловской жидкости.

Эффективная теория поля холловской жидкости

Мы хотим вывести эффективную теорию поля квантовой жидкости Холла, впервые полученную Кивелсоном, Хэнссоном и Чжаном. Для этого есть два альтернативных способа - длинный и короткий.

Если использовать длинный путь, начинают с лагранжиана, описывающего бесспиновые электроны в магнитном поле в формализме вторичного квантования

$$\mathcal{L} = \psi^+ i\left(\partial_0 - iA_0\right)\psi - \frac{1}{2m}\psi^+\left(\partial_i - iA_i\right)^2\psi - V\left(\psi^\dagger\psi\right)$$

и приводят его к необходимому виду. Из предыдущей главы известно, что путем введения калибровочного поля Черна-Саймонса можно преобразовать ψ в скалярное поле. Тогда можно воспользоваться дуальностью, о которой мы узнаем в следующей главе, чтобы представить фазовую степень свободы скалярного поля как калибровочное поле. После выполнения ряда шагов окажется, что эффективная теория холловской жидкости становится теорией Черна-Саймонса.

Вместо этого используем короткий путь. Будем рассуждать в духе: «чем еще это может быть» или, выражаясь изящнее, из общих принципов.

Начнем с перечисления того, что нам известно о системе Холла.

1. Мы живем в $(2+1)$-мерном пространстве-времени (потому что электроны ограничены плоскостью).

2. Электромагнитный ток J_μ сохраняется: $\partial_\mu J_\mu = 0$.

Очевидно, что эти два утверждения неоспоримы. Если их объединить, получим утверждение, согласно которому ток может быть записан как ротор векторного потенциала:

$$J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$$

Множитель $1/(2\pi)$ определяет нормировку a_μ. Нас учили в школе, что если в трехмерном пространстве-времени производная чего-то равна нулю, то это что-то является ротором чего-то еще. Именно это и означает уравнение (2). Единственная сложность здесь заключается в том, что то, чему нас учили в школе, применимо к пространству Минковского, так же как и к евклидовому пространству - это лишь вопрос нескольких знаков.

Калибровочное поле идет нас искать

Обратите внимание, что в результате преобразования a_μ к $a_\mu \to a_\mu$ - $-\partial_\mu\Lambda$ ток остается неизменным. Другими словами, a_μ является калибровочным полем.

Мы не искали калибровочное поле; калибровочное поле само идет нас искать! Прятаться некуда. Существование калибровочного поля следует из самых общих соображений.

3. Мы хотим теоретически описать поле системы с помощью эффективного локального лагранжиана.

4. Нас интересует только физика больших расстояний и больших времен, то есть при малом волновом числе и низкой частоте.

Действительно, описание физической системы в теории поля можно рассматривать как способ систематизации различных относящихся к делу аспектов физики по степени их важности на больших расстояниях и в соответствии с принципами симметрии. Члены в лагранжиане теории поля классифицируются по степеням производных, степеням полей и т. д. Общая схема классификации членов осуществляется в соответствии с их размерностями, как объяснено в главе III.2. Калибровочное поле a_μ имеет размерность 1, как и всегда в случае любого калибровочного поля, связанного с полями материи в соответствии с калибровочным принципом; и поэтому уравнение (2) не противоречит тому факту, что ток имеет размерность 2 в $(2+1)$-мерном пространстве-времени.

5. Четность и обращение времени нарушаются внешним магнитным полем.

Последнее утверждение так же неоспоримо, как утверждения 1 и 2. Экспериментатор создает магнитное поле, пропуская ток через катушку; при этом ток течет либо по часовой стрелке, либо против часовой стрелки.

Исходя из этих пяти общих утверждений можно получить выражение для эффективного лагранжиана.

Поскольку калибровочная инвариантность запрещает в лагранжиане член размерности $2a_\mu a^\mu$, простейшим возможным членом, по существу, является член Черна-Саймонса размерности $3\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$. Таким образом, лагранжиан имеет вид:

$$\mathcal{L} = \frac{k}{4\pi} a\epsilon\partial a + \ldots$$

где k - безразмерный параметр, подлежащий определению.

Мы сейчас ввели и будем отныне использовать компактную запись $\epsilon a\partial b \equiv \epsilon^{\mu\nu\lambda}a_\mu\partial_\nu b_\lambda = \epsilon b\partial a$ для двух векторных полей a_μ и b_μ.

Члены, обозначенные (...) в выражении (3), включают член Максвелла размерности $4\,(1/g^2)\left(f_{0i}^2 - f_{ij}^2\right)$ и другие члены более высоких размерностей. Важно отметить, что эти члены более высоких порядков на больших расстояниях играют менее важную роль. Физика больших расстояний определяется только членом Черна-Саймонса. В общем случае коэффициент k может быть нулем, и в этом случае физика определяется членами, соответствующими малым расстояниям, которые обозначены в (3) как). .). Иначе говоря, холловскую жидкость можно определить как двухмерную систему электронов, для которой коэффициент члена ЧернаСаймонса не равен нулю и, следовательно, таков, что физика больших расстояний этой системы в значительной степени не зависит от определяющих ее мелких деталей. Действительно, двухмерные системы электронов можно классифицировать соответственно тому, равен коэффициент k нулю или нет.

Связывая систему с «внешним» или «дополнительным» электромагнитным калибровочным потенциалом A_μ и используя (2), получаем (интегрируя по частям и опуская поверхностный член)

$$\mathcal{L} = \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu a_\lambda = \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}a_\mu\partial_\nu A_\lambda.$$

Заметим, что калибровочный потенциал магнитного поля, отвечающий за эффект Холла, не должен включаться в A_μ; он уже неявно содержится в коэффициенте k.

Понятие квазичастиц или «элементарных» возбуждений составляет основу физики конденсированных сред. Эффекты взаимодействия многих тел могут быть такими, что квазичастицы в системе перестают быть электронами. Здесь мы определяем квазичастицы как объекты, которые взаимодействуют с калибровочным полем, и поэтому записываем

$$\mathcal{L} = \frac{k}{4\pi} a\epsilon\partial a + a_\mu j^\mu - \frac{1}{2\pi}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu A_\mu ...$$

Определяя $\widetilde{j}_\mu \equiv j_\mu - (1/2\pi)\varepsilon_{\mu\nu\lambda}\partial^\nu A^\lambda$ и отынтегрировав калибровочное поле, получаем [см. уравнение (VI.1.4)]

$$\mathcal{L} = \frac{\pi}{k}\widetilde{j}_\mu \left(\frac{\epsilon^{\mu\nu\lambda}\partial_\nu}{\partial^2}\right)\widetilde{j}_\lambda.$$

Дробный заряд и статистика

Теперь из (6) мы можем легко считать физику явления. Лагранжиан содержит члены трех типов: AA, Aj и jj. Член AA имеет схематический вид $A\,(\epsilon\partial\epsilon\partial\epsilon\partial/\partial^2)\,A$. Используя $\epsilon\partial\epsilon\partial \sim \partial^2$ и сокращая множители в числителе и знаменателе, получаем

$$\mathcal{L} = \frac{1}{4\pi k}\epsilon A\partial A$$

Варьируя по A, определяем электромагнитный ток

$$J^\mu_{\rm em} = \frac{1}{4\pi k} e^{\mu\nu\lambda}\partial_\nu A_\lambda$$

Компонента этого уравнения с $\mu = 0$ указывает на то, что избыточная плотность δn электронов связана с локальной флуктуацией магнитного поля отношением $\delta n = (1/2\pi k)\delta B$. Поэтому коэффициент заполнения ν можно считать равным $1/k$, а из компонент $\mu = i$, которые создают электрическое поле, определить ток в поперечном направлении с $\sigma_{xy} = (1/k) = \nu$.

Схематически член Aj имеет вид $A\,(\epsilon\partial\epsilon\partial/\partial^2)\,j$. Сокращая дифференциальные операторы, находим

$$\mathcal{L} = \frac{1}{k} A_\mu j^\mu$$

Таким образом, квазичастица несет электрический заряд $1/k$.

В итоге квазичастицы взаимодействуют друг с другом следующим образом:

$$\mathcal{L} = \frac{\pi}{k} j^\mu \frac{\epsilon_{\mu\nu\lambda}\partial^\nu}{\partial^2} j^\lambda$$

Мы просто убираем в (6) тильду. Вспоминая главу VI.1, легко видеть, что квазичастицы подчиняются дробной статистике с

$$\frac{\theta}{\pi} = \frac{1}{k}$$

Хотя до сих пор все было ясно, вас, возможно, интересует, откуда следует, что ν^{-1} должно быть нечетным целым числом?

Теперь покажем, что электрон или дырка должны появиться где-то в спектре возбуждений. В конце концов, предполагается, что теория должна описывать систему электронов, а до сих пор наш довольно общий лагранжиан не содержит никаких ссылок на электрон!

Поищем дырку (или электрон). Заметим, что в (9) связанный объект, состоящий из k квазичастиц, будет иметь заряд, равный 1. Возможно, это и есть дырка! Чтобы это было так, k должно быть целым числом. Пока все хорошо, но k все еще не должно быть нечетным.

Какая статистика у этого связанного объекта? Переместим один из таких объектов на пол-оборота вокруг другого такого же связанного объекта, таким образом эффективно меняя их

местами. Движение одной квазичастицы вокруг другой дает фазу, определяемую $\theta/\pi = 1/k$ в соответствии с (11). Но у нас есть k квазичастиц, движущихся вокруг k квазичастиц, и поэтому получаем фазу

$$\frac{\theta}{\pi} = \frac{1}{k}k^2 = k$$

Чтобы дырка была фермионом, мы должны потребовать, чтобы θ/π было нечетным. Это условие фиксирует k нечетным.

Так как $\nu = 1/k$, получаем классические лафлиновские жидкости Холла с нечетным знаменателем и с коэффициентом заполнения $\nu = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, ...$ При этом получаем известный результат, что квазичастицы несут дробные заряд и статистику [см. (9) и (11)].

Действительно, впечатляет! Электроны вращаются в плоскости в магнитном поле, соответствующим $\nu = \frac{1}{3}$, и, подумать только, каждый электрон разделяется на три части, каждая из которых имеет заряд $\frac{1}{3}$ и дробную статистику $\frac{1}{3}$!

Новый вид порядка

Задача физики конденсированных сред заключается в том, чтобы понять различные состояния вещества. Состояния вещества характеризуются присутствием (или отсутствием) порядка: ферромагнетик становится упорядоченным ниже температуры фазового перехода. Как показано в главе V.3, в теории Гинзбурга-Ландау порядок связан со спонтанным нарушением симметрии, описываемой, естественно, теорией групп. Гирвин и Мак- дональд первыми заметили, что порядок в холловских жидкостях на самом деле не встраивается в схему Ландау-Гинзбурга: мы не нарушили никакую явную симметрию. Топологическое свойство холловских жидкостей дает ключ к разгадке происходящего. Как объясняется в предыдущей главе, вырождение основного состояния холловской жидкости зависит от топологии многообразия, на котором она живет; такую зависимость теория групп объяснить не может. Вен настоятельно подчеркивал, что изучение топологического порядка, или, в более общем случае, квантового порядка, может открыть широкую перспективу возможности существования новых состояний вещества.

Комментарии и обобщение

Эту главу я завершу несколькими комментариями, которые могут побудить вас к изучению многочисленных работ, посвященных холловской жидкости.

1. Появление целых чисел означает, что нами получен разумный результат. Можно привести прекрасный довод, используя замечание из предыдущей главы, что для построения члена Черна-Саймонса не нужны часы и линейки и, как следствие, он, вероятно, не может зависеть от микрофизики, например, рассеяния электронов на примесях, которое нельзя определить без часов и линеек. В противоположность этому, физика, которая не является частью топологической теории поля и описывается как (...) в (3), будет, конечно, зависеть от микрофизики.

2. Если пойти по длинному пути вывода эффективной теории поля холловской жидкости, можно увидеть, что квазичастица на самом деле является вихрем, построенным (как описано в главе V.7) из скалярного поля, представляющего электрон. Учитывая, что холловская жидкость несжимаема, фактически единственным возбуждением, которое можно себе представить, является вихрь с когерентно вращающимися вокруг него электронами.

3. В предыдущей главе говорилось, что член Черна-Саймонса калибровочно инвариантен только при исключении граничного члена. Но в лаборатории реальные холловские жидкости находятся в образцах, у которых есть границы. Так как же теория (3) может быть правильной? Примечательно, что этот явный «дефект» теории на самом

деле является одним из ее преимуществ! Допустим, что теория (3) определена на двумерном многообразии с границей, например, на диске. Тогда, как впервые было показано Веном, на границе должны существовать физические степени свободы, представленные действием, изменение которого при калибровочном преобразовании уничтожает изменение $\int d^3x(k/4\pi)a\epsilon\partial a$. С точки зрения физики ясно, что несжимаемая жидкость имеет краевые возбуждения [2], соответствующие волнам на ее границе.

4. Что произойдет, если мы откажемся вводить калибровочный потенциал? Поскольку ток J_μ имеет размерность 2, простейший член $J_\mu J^\mu$, построенный из токов, уже имеет размерность 4; он очевидно является членом Максвелла. Локальное взаимодействие с размерностью 3 нельзя построить непосредственно из токов. Чтобы понизить размерность, необходимо ввести величину, обратную производной, и схематически записать $J(1/\epsilon\partial)J$, что, конечно, является нелокальным членом Хопфа. Таким образом, на часто задаваемый вопрос «почему именно калибровочное поле?» можно частично ответить фразой: «Введение калибровочных полей позволяет избежать рассмотрения нелокальных взаимодействий».

5. Экспериментаторы построили двухслойные квантовые системы Холла с бесконечно малой амплитудой туннелирования электронов с одного слоя на другой. Полагая, что ток $J^\mu_I (I = 1, 2)$ в каждом слое сохраняется раздельно, введем два калибровочных потенциала, записывая ток $J^\mu_I = \frac{1}{2\pi}\epsilon^{\mu\nu\lambda}\partial_\nu a_{I\lambda}$ как в (2). Можно повторить общую схему вывода и получить в результате эффективный лагранжиан

$$\mathcal{L} = \sum_{I,J} \frac{K_{IJ}}{4\pi} a_I \epsilon\partial a_J + ...$$

Целое число k превратилось в матрицу K. В качестве упражнения вы можете вывести холловскую проводимость, дробный заряд и статистику квазичастиц. Вас не удивит, что всюду, где появляется $1/k$, вместо него теперь есть обратная матрица K^{-1}. Возникает интересный вопрос: что происходит, когда K имеет нулсвое собственное значение? Например, можно получить $K = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Тогда низкоэнергетическая динамика калибровочного поля $a_- \equiv a_1 - a_2$ определяется не членом Черна-Саймонса, а членом Максвелла в слагаемом (...) в (13). Мы получаем моду с линейной дисперсией и, таким образом, сверхтекучесть! Это поразительное предсказание [3] проверено экспериментально.

6. Наконец, интересно отметить, что в используемом формализме туннелирование электронов соответствует несохранению тока $J^\mu_- \equiv J^\mu_1 - J^\mu_2 =$

[2] Существование краевых токов в целочисленной холловской жидкости впервые было отмечено Гальпериным.

[3] X. G. Wen and A. Zee, Phys. Rev. Lett. 69: 1811, 1992. $= (1/2\pi)\epsilon^{\mu\nu\lambda}\partial_\nu a_{-\lambda}$. Разность чисел электронов в двух слоях $N_1 - N_2$ не сохраняется. Но как может $\partial_\mu J^\mu_- \neq 0$, несмотря на то, что J^μ_- является ротором $a_{-\lambda}$ (как я явно показал)? Вспоминая главу IV.4, вы, как проницательный читатель, скажите, что все дело в магнитных монополях! Туннелирование в двухслойной системе Холла в евклидовом пространстве-времени можно описать как газ монополей и антимонополей [4]. (Подумайте, почему монополей и антимонополей?) Заметим, конечно, что это монополи не в обычном электромагнитном калибровочном поле, а в калибровочном поле $a_{-\lambda}$.

Приведенный в этом разделе вывод эффективной теории холловской жидкости на больших расстояниях кажется, конечно, не очень убедительным. Кто-то скажет, что очень неубедительным. Вернемся к приведенным выше пяти основным утверждениям или принципам. Из них четыре, безусловно, неоспоримы. На самом деле наиболее

сомнительным является третье утверждение, которое рядовому читателю кажется наиболее несущественным. В общем случае эффективный лагранжиан для системы конденсированного вещества должен быть нелокальным. Мы неявно допускаем, что такая система не содержит безмассовое поле, обмен которым привел бы к нелокальному взаимодействию 5. Также неявным в (3) является предположение, что лагранжиан можно полностью выразить через калибровочное поле a. Мы, конечно, не знаем априори, что нет других релевантных степеней свободы. Дело в том, что пока эти степени свободы не являются бесщелевыми, по ним можно интегрировать без риска.

Упражнения

VI.2.1. Чтобы точно определить коэффициент заполнения, нужно рассмотреть квантовую систему Холла на сфере, а не на плоскости. Поместим магнитный монополь с зарядом G (который по Дираку может быть только полуцелым или целым числом) в центр единичной сферы. Поток через эту сферу равен $N_\phi = 2G$. Покажите, что энергия одного электрона имеет вид $E_l = \left(\frac{1}{2}\hbar\omega_c\right)\left[l(l+1) - G^2\right]/G$ с уровнями Ландау, соответствующими $l = G, G+1, G+2, ...$, и что вырождение уровня l равно $2l+1$. Покажите, что $N_\phi = \nu^{-1}N_e - \mathcal{S}$, когда L уровней Ландау заполнено невзаимодействующими электронами ($\nu = L$), где топологическую величину $\mathcal{S}$ называют сдвигом.

[4] X. G. Wen and A. Zee. Phys. Rev. B47: 2265, 1993.

[5] Техническое примечание. Вихри (т. е. квазичастицы), локализованные на примесях в холловской жидкости. могут порождать нелокальное во времени взаимодействие. VI.2.2. Попробуйте вывести эффективную теорию поля для холловских жидкостей с коэффициентом заполнения $\nu = m/k$, где k - нечетное целое число, например, $\nu = 2/5$. [Указание: нужно ввести m калибровочных потенциалов $a_{I\lambda}$ и обобщить (2) до вида $J^\mu = (1/2\pi)\epsilon^{\mu\nu\lambda}\partial_\nu \sum_{I=1}^{m} a_{I\lambda}$. Получается, что эффективная теория имеет вид

$$\mathcal{L} = \frac{1}{4\pi}\sum_{I,J=1}^{m} a_I K_{IL}\epsilon\partial a_J + \sum_{I=1}^{m} a_{I\mu}\tilde{j}^{I\mu} + ...,$$

где целое число k заменяется на матрицу K. Сравните с лагранжианом (13).

VI.2.3. Для лагранжиана (13) выведите аналоги (8), (9) и (11).

30.3.3 Zi.VI.3 Duality

Многообешающая концепция

Дуальность является глубокой и многообещающей концепцией [1] теоретической физики, истоки которой находятся в электромагнетизме и статистической механике. Появление в последние годы дуальности в нескольких областях современной физики, начиная от квантовых холловских жидкостей до теории струн, отражает тот факт, что мы стали лучше понимать квантовую теорию поля. Здесь я приведу один конкретный пример, только чтобы передать дух этого обширного предмета.

Вначале рассмотрим релятивистскую теорию поля, а потом, когда будет понятна ее суть, перейдем к рассмотрению нерелятивистской теории. Вполне логично, что интересная физика, которую описывает нерелятивистская теория, в релятивисткой формулировке отчасти отсутствует. Более высокий уровень симметрии накладывает больше ограничений. К тому же, релятивистскую теорию, как ни удивительно, намного проще понять хотя бы из-за простоты обозначений.

Вихри

Пусть скалярное поле (2+1)-размерности взаимодействует с внешним электромагнитным калибровочным полем, где в дальнейшем для удобства электрический заряд q обозначается явно:

$$\mathcal{L} = \frac{1}{2}\left|\left(\partial_\mu - iqA_\mu\right)\varphi\right|^2 - V\left(\varphi^\dagger\varphi\right)$$

Мы уже не раз изучали эту теорию, причем совсем недавно в главе V.7 применительно к вихрям. Как обычно, запишем $\varphi = |\varphi|e^{i\theta}$. Минимизация

[1] Для первого знакомства с дуальностью настоятельно рекомендую прочитать работу J. M. Figueroa-O'Farril. Electromagnetic Duality for Children. http://www.maths.ed.ac.uk/ jmf/Teaching/Lectures/F.DC.pdf. потенциала V при $|\varphi| = v$ определяет конфигурацию поля для основного состояния. Задавая $\varphi = ve^{i\theta}$ в (1), получаем лагранжиан

$$\mathcal{L} = \frac{1}{2}v^2\left(\partial_\mu\theta - qA_\mu\right)^2$$

который при включении θ в A путем калибровочного преобразования становится лагранжианом Мейсснера. Для дальнейшего удобства введем также альтернативную форму лагранжиана

$$\mathcal{L} = -\frac{1}{2v^2}\xi_\mu^2 + \xi^\mu\left(\partial_\mu\theta - qA_\mu\right)$$

Вновь получаем (2), исключая вспомогательное поле ξ^μ (см. приложение A).

Из главы V.7 известно, что спектр возбуждения включает вихри и антивихри, расположенные там, где исчезает $|\varphi|$. Если θ изменяется на 2π вблизи нуля $|\varphi|$, получаем вихрь. Вокруг антивихря $\Delta\theta = -2\pi$. Вспомним, что вокруг неподвижного вихря электромагнитное калибровочное поле должно вести себя на пространственной бесконечности как

$$qA_i \to \partial_i\theta$$

чтобы энергия вихря была конечной, как легко видеть из (2). Магнитный поток

$$\int d^2x\varepsilon_{ij}\partial_i A_j = \oint d\vec{x}\cdot\vec{A} = \frac{\Delta\theta_{\text{вихр.}}}{q} = \frac{2\pi}{q}$$

квантован в единицах $2\pi/q$.

Остановимся на минуту, чтобы физически осмыслить происходящее. На масштабе расстояний, большом по сравнению с размером вихря, вихри и антивихри представляют собой точки. Как обсуждалось в главе V.7, энергия взаимодействия вихря и антивихря, разделенных расстоянием R, задается просто путем подстановки в (2). Игнорируя пробное поле A_μ, которое можно считать сколь угодно слабым, получаем $\sim \int\limits_a^R drr(\nabla\theta)^2 \sim \log(R/a)$, где a - обрезание на малых расстояниях. Но вспомним, что кулоновское взаимодействие в двухмерном пространстве изменяется логарифмически, поскольку из размерного анализа следует $\int d^2k\left(e^{i\vec{k}\cdot\vec{x}}/k^2\right) \sim \log(|\vec{x}|/a)$ (где a^{-1} - некоторое ультрафиолетовое обрезание). Таким образом, газ вихрей и антивихрей представляет собой газ точечных «зарядов» с кулоновским взаимодействием между ними.

Вихрь как заряд в дуальной теории

Некоторые теоретики часто понимают под дуальностью раздел высокой математики, но на самом деле она происходит из чисто физической идеи. Принимая во внимание последний

параграф, нельзя ли переписать теорию так, чтобы вихри представляли собой точечные «заряды» некоторого еще неизвестного калибровочного поля? Другими словами, необходима дуальная теория, в которой фундаментальное поле создает и уничтожает вихри, а не кванты φ. В свое время мы объясним значение слова «дуальный».

Примечательно, что переписать эту теорию можно всего лишь в несколько простых шагов. Используя физический и эвристический подходы, изобразим фазовое поле θ как плавно флуктуирующее, исключая те места, где оно закручивается на 2π. Запишем $\partial_\mu\theta = \partial_\mu\theta_{\text{плавн.}} + \partial_\mu\theta_{\text{вихр.}}$. Подставляя это в (3), получим

$$\mathcal{L} = -\frac{1}{2v^2}\xi_\mu^2 + \xi^\mu\left(\partial_\mu\theta_{\text{плавн.}} + \partial_\mu\theta_{\text{вихр.}} - qA_\mu\right)$$

Проинтегрируем по $\theta_{\text{плавн.}}$ и получим связь $\partial_\mu\xi^\mu = 0$, которую можно разрешить, записав

$$\xi^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$$

Такой прием мы уже использовали в главе VI.2. Как и тогда, появляется ищущее нас калибровочное поле, так как замена $a_\lambda \to a_\lambda + \partial_\lambda\Lambda$ не меняет ξ^μ. Подставляя в (6), находим

$$\mathcal{L} = -\frac{1}{4v^2}f_{\mu\nu}^2 + \varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda\left(\partial_\mu\theta_{\text{вихр.}} - qA_\mu\right)$$

где $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$.

Наши рассуждения являются эвристическими, поскольку мы игнорируем тот факт, что $|\varphi|$ исчезает на вихрях. Физически мы считаем вихри почти точечными, так что «преимущественно» везде $|\varphi| = v$. Как говорилось в главе V.6, солитоны, вихри и т. д. можно сделать сколь угодно малыми, если соответствующим образом выбрать параметры. Другими словами, мы пренебрегаем взаимодействием между $\theta_{\text{вихр. и}}$ $|\varphi|$. Строгое рассмотрение потребовало бы соответствующего обрсзания малых расстояний путем помещения этой системы на решетку [2]. Но мы нс будем учитывать подоб-

[2] Например, M. P. A. Fisher, «Mott Insulators, Spin Liquids. and Quantum Disordered Superconductivity», cond-mat/9806164, приложение A. ного рода тонкости, поскольку пытаемся понять (и, несомненно, поймем) физику явления.

Заметим для дальнейшего использования, что электромагнитный ток J^μ, будучи коэффициентом $-A_\mu$ в (8), определяется через калибровочное поле a_λ в виде

$$J^\mu = q\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$$

Проинтегрируем член $\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda\partial_\mu\theta_{\text{вихр. в}}$ (8) по частям и получим $a_\lambda\varepsilon^{\lambda\mu\nu}\partial_\mu\partial_\nu\theta_{\text{вихр.}}$. Согласно Ньютону и Лейбницу ∂_μ коммутирует с ∂_ν, и потому мы явно получаем нуль. Но ∂_μ и ∂_ν коммутируют только тогда, когда действуют на глобально заданную функцию, но, ей-Богу, поле $\theta_{\text{вихр.}}$. не определено глобально, так как оно меняется на 2π, когда мы обходим вихрь. В частности, рассмотрим вихрь в состоянии покоя и посмотрим, с чем связана величина a_0 в (8), а именно в $\varepsilon^{ij}\partial_i\partial_j\theta_{\text{вихр.}} = \vec{\nabla}\times\left(\vec{\nabla}\theta_{\text{вихр.}}\right)$ в обозначениях общей физики. После интегрирования этого выражения по области, содержащей вихрь, получаем $\int d^2x\vec{\nabla}\times\left(\vec{\nabla}\theta_{\text{вихр.}}\right) = \oint d\vec{x}\cdot\vec{\nabla}\theta_{\text{вихр.}} =$

$= 2\pi$. Таким образом, легко видеть, что $(1/2\pi)\varepsilon^{ij}\partial_i\partial_j\theta_{\text{вихр.}}$ является плот-

ностью вихрей, временной компонентой некоторого вихревого тока $j^\lambda_{\text{вихр.}}$. Из лоренц-инвариантности следует, что $j^\lambda_{\text{вихр.}} = (1/2\pi)\varepsilon^{\lambda\mu\nu}\partial_\mu\partial_\nu\theta_{\text{вихр.}}$.

Теперь (8) можно записать в виде

$$\mathcal{L} = -\frac{1}{4v^2}f_{\mu\nu}^2 + (2\pi)a_\mu j^\mu_{\text{вихр.}} - A_\mu\left(q\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda\right)$$

Вот так, мы завершили то, что намеревались сделать. Мы переписали теорию так, чтобы вихрь представлял собой «электрический заряд» для калибровочного поля a_μ. Иногда ее

называют дуальной теорией, но, строго говоря, более точным названием будет «дуальное представление исходной теории (1)».

Введем комплексное скалярное поле Φ, которое назовем вихревым полем, которое будет создавать и уничтожать вихри и антивихри. Другими словами, мы достраиваем действие (10) до

$$\mathcal{L} = -\frac{1}{4v^2} f_{\mu\nu}^2 + \frac{1}{2} \left|(\partial_\mu - i(2\pi)a_\mu)\,\Phi\right|^2 - W(\Phi) - A_\mu \left(q\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda\right).$$

Потенциал $W(\Phi)$ содержит такие члены, как $\lambda\left(\Phi^\dagger\Phi\right)^2$, описывающие взаимодействие двух вихрей (или вихря и антивихря) на малом расстоянии. В принципе, если полностью понять всю физику малых расстояний, содержащуюся в исходной теории (1), то все эти члены будут определяться исходной теорией.

Вихрь вихря

Теперь мы подошли к наиболее увлекательной стороне дуального представления и причине, по которой вообще используется слово «дуальный». Вихревое поле Φ является таким же комплексным скалярным полем, как поле φ, с которого мы начинали. Таким образом, вполне можно сформировать вихрь из Φ, т. е. места, в окрестности которого Φ исчезает, а фаза Φ меняется на 2π. Занятно, что мы формируем, так сказать, вихрь вихря.

Так что же такое вихрь вихря?

Теорема дуальности утверждает, что вихрь вихря является не чем иным, как исходным зарядом, который описывается полем φ, с которого мы начинали! Отсюда и слово «дуальность».

Доказательство необыкновенно просто. В теории (11) вихрь несет «магнитный поток». Возвращаясь к (11), видим, что на пространственной бесконечности $2\pi a_i \to \partial_i\theta$. Используя такую же операцию, как в (5), получаем

$$2\pi \int d^2x \varepsilon_{ij}\partial_i a_j = 2\pi \oint d\vec{x}\cdot\vec{a} = 2\pi$$

Обратите внимание, что термин «магнитный поток» заключен в кавычки, поскольку очевидно, что речь идет о потоке, связанном с калибровочным полем a_μ, а не потоке, связанном с электромагнитным полем A_μ. Но вспомним, что согласно (9) электромагнитный ток имеет вид $J^\mu = q\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$ и, в частности, $J^0 = q\varepsilon^{ij}\partial_i a_j$. Следовательно, электрический заряд (заметьте, без кавычек) этого вихря вихря равен $\int d^2x J^0 = q$, т. е. заряду исходного комплексного скалярного поля φ. Это доказывает наше утверждение.

Здесь мы рассматривали вихри, но такого же рода дуальность применима и к монополям. Как говорилось в главе IV.4, дуальность позволяет получить представление о теориях поля в режиме сильной связи. Из главы V.7 известно, что некоторые спонтанно нарушенные неабелевы калибровочные теории в (3 + 1)-мерном пространстве-времени содержат магнитные монополи. Дуальную теорию можно записать в терминах поля монополей, из которого можно построить монополи. Монополь монополя оказывается не чем иным, как заряженными полями исходной калибровочной теории. Существование такой дуальности впервые предположили много лет назад Олив и Монтонен, и, как впоследствии было показано, дуальность реализуется в некоторых суперсимметричных калибровочных теориях ЗайбергаВиттена. Несколько лет назад изучение этой дуальности было «горячей» темой, что позволило по-новому и глубжс понять, как некоторые теории струн дуальны друг другу [3]. В противоположность этому, по мнению одного из моих коллег, выдающегося специалиста в области конденсированных сред, важное понятие дуальности все еще не получило достаточного признания сообществом физиков в области конденсированных сред.

Мейсснер порождает Максвелла и так далее

Завершим эту тему небольшим разъяснением дуальности в (2 + 1)-мерном пространстве-времени и того, какое он может иметь отношение к физике двумерных веществ. Рассмотрим лагранжиан $\mathcal{L}(a)$ второго порядка по векторному полю a_μ. Подключим внешний электромагнитный калибровочный потенциал A_μ к сохраняющемуся току $\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$:

$$\mathcal{L} = \mathcal{L}(a) + A_\mu \left(\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda\right)$$

Зададимся следующим вопросом: если при разных выборах лагранжиана $\mathcal{L}(a)$ отынтегрировать a, что будет представлять собой эффективный лагранжиан $\mathcal{L}(A)$, описывающий динамику A?

Если вы добрались до этого места в книге, то с легкостью осуществите интегрирование. Снова имеем дело с основным тождеством квантовой теории поля! Имея лагранжиан

$$\mathcal{L}(a) \sim aKa$$

получаем

$$\mathcal{L}(A) \sim (\varepsilon\partial A)\frac{1}{K}(\varepsilon\partial A) \sim A\left(\varepsilon\partial\frac{1}{K}\varepsilon\partial\right)A.$$

Можно выбрать из трех лагранжианов $\mathcal{L}(a)$, которым я дал известные имена:

$\mathcal{L}(a) \sim a^2$	Мейсснер
$\mathcal{L}(a) \sim a\varepsilon\partial a$	Черн-Саймонс
$\mathcal{L}(a) \sim f^2 \sim a\partial^2 a$	Максвелл

Поскольку мы стремимся к концептуальному пониманию, я не буду отслеживать индексы и не относящиеся к данному случаю общие константы. (Вы можете их проставить в качестве упражнения.) Например, имея $\mathcal{L}(a) = f_{\mu\nu}f^{\mu\nu}$, где $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$, можно записать $\mathcal{L}(a) \sim a\partial^2 a$, и поэтому $K = \partial^2$. Таким образом, эффективная динамика внешнего электромагнитного калибровочного потенциала определяется выражением (15)

[3] Например. D. I. Olive and P. C. West, eds., Duality and Supersymmetric Theories. как $\mathcal{L}(A) \sim A\left[\varepsilon\partial\left(1/\partial^2\right)\varepsilon\partial\right]A \sim A^2$, то есть мейсснеровским лагранжианом! С помощью такого «быстрого и грязного» способа заработать себе на жизнь мы задаем $\varepsilon\varepsilon \sim$ и сокращаем множители ∂ в числителе и знаменателе. Продолжая в том же духе, строим следующую таблицу:

Динамика a	K	Эффективный лагранжиан $\mathcal{L}(A) \sim A[\varepsilon\partial(1/K)\varepsilon\partial\|A$	Динамика внешнего пробного A
a^2 Мейсснера	1	$A(\varepsilon\partial\varepsilon\partial)A \sim A\partial^2 A$	F^2 Максвелла
$a\varepsilon\partial a$ Черна-Саймонса	$\varepsilon\partial$	$A\left(\varepsilon\partial\frac{1}{\varepsilon\partial}\varepsilon\partial\right)A \sim A\varepsilon\partial A$	$A\varepsilon\partial A$ Черна-Саймонса
$f^2 \sim a\partial^2 a$ Максвелла	∂^2	$A\left(\varepsilon\partial\frac{1}{0^2}\varepsilon\partial\right)A \sim AA$	A^2 Мейсснера

Мейсснер порождает Максвелла, Черн-Саймонс порождает ЧернаСаймонса и Максвелл порождает Мейсснера. Я нахожу это красивым и фундаментальным результатом, который очень убедительно представляет собой пример дуальности. Черн-Саймонс является самодуальным: он порождает сам себя.

Становясь нерелятивистским

Поучительно провести сравнение с нерелятивистским рассмотрением дуальности 4. Вернемся к лагранжиану сверхтекучей жидкости, приведенному в главе V.1:

$$\mathcal{L} = i\varphi^\dagger \partial_0 \varphi - \frac{1}{2m}\partial_i\varphi^\dagger\partial_i\varphi - g^2\left(\varphi^\dagger\varphi - \bar{\rho}\right)^2$$

Как и раньше, подставим $\varphi \equiv \sqrt{\rho}e^{i\theta}$ и получим лагранжиан

$$\mathcal{L} = -\rho\partial_0\theta - \frac{\rho}{2m}\left(\partial_i\theta\right)^2 - g^2(\rho-\bar{\rho})^2 + ...,$$

который перепишем в виде

$$\mathcal{L} = -\xi_\mu\partial^\mu\theta + \frac{m}{2\rho}\xi_i^2 - g^2(\rho-\bar{\rho})^2 + ...$$

В (19) мы отбросили член $\sim \left(\partial_i\rho^{1/2}\right)^2$. В (20) мы определили, что $\xi_0 \equiv \rho$. Интегрируя по ξ_i в (20), возвращаемся к (19).

[4] Приведенное здесь рассмотрение, по существу. вытекает из рассмотрения, данного М. П. А. Фишером и Д. Х. Ли. Делаем все, как и раньше. Записывая $\theta = \theta_{\text{глад.}} + \theta_{\text{вихр. и интегрируя}}$ по $\theta_{\text{глад., получаем связь}}$ $\partial^\mu\xi_\mu = 0$, разрешить которую можно, записав $\xi 6\mu = \epsilon^{\mu\nu\lambda}\widehat{a}_\lambda$. Шляпка над $\widehat{a}_\lambda$ понадобится нам для дальнейшего удобства. Заметим, что плотность

$$\xi_0 \equiv \rho = \epsilon_{ij}\partial_i\widehat{a}_j \equiv \widehat{f}$$

является напряженностью «магнитного» поля, тогда как

$$\xi_i = \epsilon_{ij}\left(\partial_0\widehat{a}_j - \partial_j\widehat{a}_0\right) \equiv \epsilon_{ij}\widehat{f}_{0j}$$

является напряженностью «электрического» поля.
Подставляя все это в (20), получаем

$$\mathcal{L} = \frac{m}{2\rho}\widehat{f}_{0i}^2 - g^2(\widehat{f}-\bar{\rho})^2 - 2\pi\widehat{a}_\mu j^\mu_{\text{вихр.}} + ...$$

Чтобы «вычесть» фоновое «магнитное» поле $\bar{\rho}$, очевидно, следует записать

$$\widehat{a}_\mu = \bar{a}_\mu + a_\mu$$

где фоновое калибровочное поле определяется через $\bar{a}_0 = 0, \partial_0\bar{a}_j = 0$ (фоновое «электрическое» поле отсутствует) и

$$\epsilon_{ij}\partial_i\bar{a}_j = \bar{\rho}$$

Тогда лагранжиан (23) приобретает более ясную форму

$$\mathcal{L} = \left(\frac{m}{2\bar{\rho}}f_{0i}^2 - g^2f^2\right) - 2\pi a_\mu j^\mu_{\text{вихр.}} - 2\pi\bar{a}_i j^i_{\text{вихр.}} + ...$$

Мы разложили $\rho \sim \bar{\rho}$ в первом члене. Как и в (10), первые два члена образуют максвелловский лагранжиан, а соотношение их коэффициентов определяет скорость распространения

$$c = \left(\frac{2g^2\bar{\rho}}{m}\right)^{1/2}$$

Используя единицы, в которых $c = 1$, получаем

$$\mathcal{L} = -\frac{m}{4\bar{\rho}} f_{\mu\nu} f^{\mu\nu} - 2\pi a_\mu j^\mu_{\text{вихр.}} - 2\pi \bar{a}_i j^i_{\text{вихр.}} + ...$$

Сравним этот лагранжиан с лагранжианом (10). При релятивистском рассмотрении не хватает только последнего члена в (28) по той простой причине, что мы не включили фон. Вспомним, что член типа $A_i J_i$ в обычном электромагнетизме означает, что движущаяся частица, связанная с током J_i, видит магнитное поле $\vec{\nabla} \times \vec{A}$. Таким образом, движущийся вихрь увидит «магнитное поле»

$$\epsilon_{ij} \partial_i (\bar{a} + a)_j = \bar{\rho} + \epsilon_{ij} \partial_i a_j$$

равное сумме плотности исходных бозонов $\bar{\rho}$ и флуктуирующего поля.

В кулоновской калибровке $\partial_i a_i = 0$ получаем выражение $(f_{0i})^2 = (\partial_0 a_i)^2 + (\partial_i a_0)^2$, где перекрестный член $(\partial_0 a_i)(\partial_i a_0)$ эффективно обращается в нуль при интегрировании по частям. Интегрируя по кулоновскому полю a_0, получаем

$$\mathcal{L} = -\frac{\bar{\rho}}{2m} (2\pi)^2 \iint d^2x d^2y \left[j_0(\vec{x}) \log \frac{|\vec{x} - \vec{y}|}{a} j_0(\vec{y}) \right] + \\ + \frac{m}{2\bar{\rho}} (\partial_0 a_i)^2 - g^2 f^2 + 2\pi (a_i + \bar{a}_i) j_i^{\text{вихр.}} .$$

Вихри отталкивают друг друга в результате логарифмического взаимодействия $\int d^2k \left(e^{i\vec{k}\cdot\vec{x}} / k^2 \right) \sim \log(|\vec{x}|/a)$, что нам уже давно известно.

Самодуальная теория

Интересно, что пространственная часть f^2 максвелловского лагранжиана появляется в результате отталкивания между исходными бозонами на малых расстояниях.

Если бы мы считали, что бозоны взаимодействуют через произвольный потенциал $V(x)$, то вместо последнего члена в (20) получили бы интеграл

$$\iint d^2x d^2y [\rho(x) - \bar{\rho}] V(x - y) [\rho(y) - \bar{\rho}]$$

Легко увидеть, что, по существу, все проделанные шаги аналогичны прежним, только теперь второй член лагранжиана (26) имеет вид

$$\iint d^2x d^2y f(x) V(x - y) f(y)$$

Таким образом, калибровочное поле распространяется в соответствии с дисперсионным соотношением

$$\omega^2 = (2\bar{\rho}/m) V(k) \vec{k}^2$$

где $V(k)$ является преобразованием Фурье $V(x)$. В частном случае $V(x) = g^2 \delta^{(2)}(x)$ мы восстанавливаем линейную дисперсию, приведенную в (27). Действительно, у нас есть линейная дисперсия $\omega \propto |\vec{k}|$, до тех пор пока потенциал $V(x)$ остается достаточно короткодействующим, чтобы $V(\vec{k} = 0)$ было конечным.

Интересная ситуация возникает, когда потенциал $V(x)$ является логарифмическим. Тогда $V(k)$ имеет вид $1/k^2$, и поэтому дисперсия ω постоянна. Калибровочное поле a_i становится массивным и исчезает. Низкоэнергетическая эффективная теория состоит из вихрей, с логарифмическим взаимодействием между ними. Таким образом, теория бозонов с логарифмическим отталкиванием является самодуальной в низкоэнергетическом пределе.

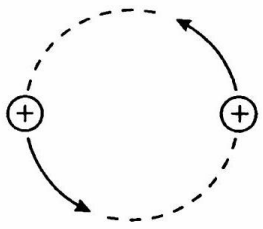
Танец вихрей и антивихрей

Завершив нерелятивистское рассмотрение дуальности, наградим себя тем, что выведем уравнение движения вихрей в жидкости. Пусть основная масса жидкости находится в состоянии покоя. В соответствии с (28) вихрь ведет себя как заряженная частица в фоновом магнитном поле $\bar{b}$, пропорциональном средней плотности жидкости $\bar{\rho}$. Таким образом, сила, действующая на вихрь, является обычной силой Лоренца $\vec{v} \times \vec{B}$, и тогда уравнение движения вихря в присутствии силы F имеет вид

$$\bar{\rho}\epsilon_{ij}\dot{x}_j = F_i$$

Хорошо известно, что при приложении силы вихрь движется в направлении, перпендикулярном этой силе.

Рассмотрим два вихря. В соответствии с (30) они отталкивают друг друга в результате логарифмического взаимодействия. Они перемещаются перпендикулярно силе и в итоге совершают круговое движение вокруг друг друга. В противоположность этому рассмотрим вихрь и антивихрь, которые притягивают друг друга. В результате притяжения они оба движутся в одном направлении, перпендикулярном соединяющей их прямой линии (см. рис. VI.3.1). Вихрь и антивихрь продолжают синхронно перемещаться, сохраняя между собой расстояние. В этом фактически состоит объяснение известного движения кольца дыма. Если разрезать кольцо дыма по центру и перпендикулярно плоскости, в которой оно находится, то для каждой части получим вихрь с антивихрем. Таким образом, все кольцо дыма движется в направлении, перпендикулярном плоскости, в которой оно находится.



(a)

(b)

Рис. VI.3.1

Все это можно понять исходя из общей физики. Важным наблюдением является тот факт, что вихри и антивихри приводят к образованию круговых потоков в жидкости, которые движутся, скажем, по часовой стрелке для вихрей и против часовой стрелки для антивихрей. Другим важным наблюдением является то, что если в жидкости есть локальный поток, то любой объект, будь то захваченный им вихрь или антивихрь, будет течь в том же направлении, что и локальный поток. Такая ситуация является следствием галилеевой инвариантности. Сделав простой рисунок, можно увидеть, что характер движения будет таким же, что рассмотренный выше.

30.3.4 Zi.VI.4 σ-Models as Effective Field Theories

Лагранжиан как мнемоника

Наша любимая квантовая теория поля прошла через два почти смертельных испытания. Первое началось где-то в середине 1930-х годов, когда оказалось, что физические величины бесконечны. Но в конце 1940-х - начале 1950-х годов она громогласно вернулась к жизни, благодаря усилиям поколения физиков, в число которых входили Фейнман, Швингер, Дайсон и другие. Второе испытание произошло в конце 1950-х годов. Как уже обсуждалось, квантовая теория поля казалась совершенно непригодной для описания сильного взаимодействия: оно было слишком сильным, чтобы от теории возмущения был какой-то толк. Многие физики, которых всех вместе называли школой S-матрицы, понимали, что теория поля не подходит для изучения сильного взаимодействия, и поддерживали программу исследований, согласно которой для получения результатов пытались использовать общие принципы без теории поля. Например, при выводе соотношения Голдбергера - Треймана можно было бы отказаться от любого упоминания теории поля и фейнмановских диаграмм.

В итоге, как реакция против такой тенденции, пришло понимание, что если какие-то результаты можно получить из общих принципов, таких как понятие спонтанного нарушения симметрии и т. д., то аналогичные результаты должен бы воспроизвести любой лагранжиан, содержащий в себе такие общие свойства. По меньшей мере, лагранжиан дает мнемонику для любого физического результата, полученного без использования квантовой теории поля. Таким образом, родилось понятие эффективной теории поля для больших расстояний или низких энергий, которая оказалась чрезвычайно полезной как в физике элементарных частиц, так и в физике конденсированных сред (как мы уже видели).

Сильное взаимодействие при низких энергиях

Одним из самых ранних примеров является σ-модель Гелл-Манна и Леви, которая описывает взаимодействие нуклонов и пионов. Теперь известно, что сильное взаимодействие должно описываться в терминах кварков и глюонов. Тем не менее на больших расстояниях степенями свободы являются два нуклона и три пиона. Протон и нейтрон преобразуются как спинор $\psi \equiv \begin{pmatrix} p \\ n \end{pmatrix}$ относительно $SU(2)$ изоспина. Рассмотрим член кинетической энергии $\bar{\psi} i\gamma\partial\psi = \bar{\psi}_L i\gamma\partial\psi_L + \bar{\psi}_R i\gamma\partial\psi_R$. Заметим, что этот член имеет более широкую симметрию $SU(2)_L \times SU(2)_R$, где левое поле ψ_L и правое поле ψ_R преобразуются как дублет относительно $SU(2)_L$ и $SU(2)_R$ соответственно. [SU(2) изоспина является диагональной подгруппой $SU(2)_L \times SU(2)_R$.] Можно записать $\psi_L \sim \left(\frac{1}{2}, 0\right)$ и $\psi_R \sim \left(0, \frac{1}{2}\right)$.

Теперь сразу видна проблема. Массовый член $m\bar{\psi}\psi = m\left(\bar{\psi}_L\psi_R + \text{э. с.}\right)$ не годится, поскольку $\bar{\psi}_L\psi_R \sim \left(\frac{1}{2}, \frac{1}{2}\right)$, 4-мерное представление $SU(2)_L \times SU(2)_R$ группы, локально изоморфной $SO(4)$.

На данном этапе уже меньше физиков могли бы сказать, в чем заключается проблема. Мы всегда знали, что сильное взаимодействие инвариантно только относительно $SU(2)$-группы изоспина, которую запишем как $SU(2)_I$. Билинейные формы, построенные из $\bar{\psi}_L$ и ψ_R, преобразуются относительно $SU(2)_I$ как $\frac{1}{2} \times \frac{1}{2} = 0 + 1$, синглет берется в виде $\bar{\psi}\psi$, а триплет в виде $\bar{\psi} i\gamma^5 \tau^a \psi$. Конечно, только лишь с $SU(2)_I$-симметрией можно включить массовый член $\bar{\psi}\psi$.

Другими словами, чтобы полностью связать все четыре билинейные формы, которые можно построить из $\bar{\psi}_L$ и ψ_R, а именно из $\bar{\psi}\psi$ и $\bar{\psi} i\gamma^5 \tau^a \psi$, нужны четыре мезонных поля, преобразующихся как векторное представление относительно $SO(4)$. Но известны только три пионных поля. Вроде ясно, что у нас есть лишь $SU(2)_I$-симметрия.

Тем не менее Гелл-Манн и Леви уверенно настаивали на более широкой симметрии $SU(2)_L \times SU(2)_R = SO(4)$ и просто постулировали присутствие дополнительного мезонного поля, которое они назвали σ, так что $(\sigma, \vec{\pi})$ образовывало четырехмерное представление.

Проверьте самостоятельно, что выражение $\bar{\psi}_L(\sigma + i\vec{\tau}\cdot\vec{\pi})\psi_R +$ э. с. $= \bar{\psi}\left(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5\right)\psi$ инвариантно. Следовательно, можно записать инвариантный лагранжиан

$$\mathcal{L} = \bar{\psi}\left[i\gamma\partial + g\left(\sigma + i\vec{\tau}\cdot\vec{\pi}\gamma_5\right)\right]\psi + \mathcal{L}(\sigma,\vec{\pi})$$

где часть, не включающая нуклоны, имеет вид

$$\mathcal{L}(\sigma,\vec{\pi}) = \frac{1}{2}(\partial\sigma)^2 + \frac{1}{2}(\partial\vec{\pi})^2 + \frac{\mu^2}{2}\left(\sigma^2+\vec{\pi}^2\right) - \frac{\lambda}{4}\left(\sigma^2+\vec{\pi}^2\right)^2$$

Это так называемая линейная σ-модель.

Когда σ-модель была предложена, большинству физиков она показалась довольно странной тем, что у нуклона нет массы и есть дополнительное мезонное поле. Итак, вы уже видите, что (2) соответствует лагранжиану (IV.1.2) (для $N = 4$), который мы рассматривали и который обнаруживает спонтанное нарушение симметрии. Четыре скалярных поля $(\varphi_1,\varphi_2,\varphi_3,\varphi_4)$ в лагранжиане (IV.1.2) соответствуют $(\sigma,\vec{\pi})$. Не теряя общности, можно сделать так, чтобы вакуумное среднее φ указывало первое направление, а именно вакуум, в котором $\langle 0|\sigma|0\rangle = \sqrt{\mu^2/\lambda} \equiv v$ и $\langle 0|\vec{\pi}|0\rangle = 0$. Разлагая выражение $\sigma = v + \sigma'$, сразу получаем, что нуклон имеет массу $M = gv$. Вас не должно удивлять, что пион оказывается безмассовым. У мезона, связанного с полем σ', который мы назовем σ-мезоном, нет причины быть безмассовым, и он, в самом деле, таким не является.

Может ли крайне важный параметр v иметь отношение к измеряемой величине? Несомненно, может. Вспомните из главы I.9, что аксиальный ток дается теоремой Нётер в виде $J^a_{\mu5} = \bar{\psi}\gamma_\mu\gamma_5\left(\tau^a/2\right)\psi + \pi^a\partial_\mu\sigma$ - $-\sigma\partial_\mu\pi^a$. После того как σ получает вакуумное среднее, $J^a_{\mu5}$ начинает содержать член $-v\partial_\mu\pi^a$. Этот член означает, что матричный элемент имеет вид $\left\langle 0\left|J^a_{\mu5}\right|\pi^b\right\rangle = ivk_\mu$, где k обозначает импульс пиона, и таким образом v пропорционален f, определенной в главе IV.2. В самом деле, мы видим, что соотношение для массы $M = gv$ является в точности соотношением Голдбергера-Треймана (IV.2.7), где $F(0) = 1$ (см. упражнение VI.4.4).

Нелинейная σ-модель

Со временем пришло понимание, что основное назначение существования потенциала в $\mathcal{L}(\sigma,\vec{\pi})$ заключается в том, чтобы заставлять вакуумные средние полей быть такими, какие они есть; поэтому этот потенциал можно заменить связью $\sigma^2 + \vec{\pi}^2 = v^2$. Более физическое рассмотрение можно представить так, что σ-мезон, если такой вообще существует, должен быть очень широким, поскольку при сильном взаимодействии он может распадаться на два пиона. σ-мезон также можно изгнать из низкоэнергетического спектра, сделав его массу большой. Из глав IV.1 и V.1 известно, что масса σ-мезона, а именно $\sqrt{2}\mu$, может быть устремлена к бесконечности при сохранении v фиксированным, предоставляя μ^2 и λ стремиться к бесконечности при фиксированном соотношении между ними.

Теперь сосредоточим внимание на $\mathcal{L}(\sigma,\vec{\pi})$. Вместо того чтобы рассматривать $\mathcal{L}(\sigma,\vec{\pi}) = \frac{1}{2}\left[(\partial\sigma)^2 + (\partial\vec{\pi})^2\right]$ со связью $\sigma^2+\vec{\pi}^2 = v^2$, можно решить это уравнение для связи и подставить решение $\sigma = \sqrt{v^2-\vec{\pi}^2}$ в лагранжиан, получая тем самым так называемую нелинейную σ-модель.

$$\mathcal{L} = \frac{1}{2}\left[(\partial\vec{\pi})^2 + \frac{(\vec{\pi}\cdot\partial\vec{\pi})^2}{f^2-\vec{\pi}^2}\right] = \frac{1}{2}(\partial\vec{\pi})^2 + \frac{1}{2f^2}(\vec{\pi}\cdot\partial\vec{\pi})^2 + \ldots$$

Заметим, что $\mathcal{L}$ можно записать в виде $\mathcal{L} = (\partial\pi^a)\,G^{ab}(\vec{\pi})\left(\partial\pi^b\right)$; некоторые предпочитают рассматривать G^{ab} как «метрику» в пространстве полей. [Кстати, вспомните, что ранее в главе I.3 мы ограничились простейшим членом кинетической энергии $\frac{1}{2}(\partial\varphi)^2$, отбрасывая такие варианты, как $U(\varphi)(\partial\varphi)^2$. Но вспомните также, что в главе IV. 3 говорилось, что такой член появляется при квантовых флуктуациях.]

В соответствии с рассуждениями, с которых начинается эта глава, любой лагранжиан, который обладает правильными свойствами симметрии, должен описывать одну и ту же физику низких энергий. Это значит, что кто угодно, включая вас, может ввести свою собственную параметризацию полей.

Нелинейная σ-модель на самом деле является примером широкого класса теорий поля, лагранжиан которых имеет простой вид, но появляющиеся в нем поля подчиняются некоторой нетривиальной связи. Примером является теория, определяющаяся лагранжианом

$$\mathcal{L}(U) = \frac{f^2}{4}\,\mathrm{tr}\left(\partial_\mu U^\dagger \cdot \partial^\mu U\right)$$

где $U(x)$ - матричнозначное поле и элемент $SU(2)$. Действительно, если записать $U = e^{(i/f)\vec{\pi}\cdot\vec{\tau}}$, то легко видеть, что лагранжиан $\mathcal{L}(U) = \frac{1}{2}(\partial\vec{\pi})^2 + + (1/2f^2)\,(\vec{\pi}\cdot\partial\vec{\pi})^2 + ...$, совпадает с (3) вплоть до указанных членов.

О нелинейных σ-моделях и их приложениях в физике элементарных частиц и конденсированных сред можно сказать значительно больше, но зсестороннее рассмотрение этой темы выходит за рамки настоящей книги. Поэтому некоторые свойства этих моделей я раскрою в задачах, а в слецующей главе опишу в общих чертах, как такие модели могут появляться в одном классе систем конденсированных сред.

Упражнения

VI.4.1. Покажите, что вакуумное среднее $(\sigma, \vec{\pi})$ действительно может имеет любое направление без изменения физики. На первый взгляд это утверждение кажется странным, так как за счет своего γ_5-взаимодействия с нуклоном пион является псевдоскалярным полем и не может иметь вакуумное среднее без нарушения четности. Но $(\sigma, \vec{\pi})$ всего лишь греческие буквы. Покажите, что при соответствующем преобразовании нуклонного поля четность сохраняется, как и должно быть при сильном взаимодействии.

VI.4.2. Вычислите амплитуду пион-пионного рассеяния до членов второго порядка по внешним импульсам, используя нелинейную σ-модель (3). [Указание: если нужна помощь, см. S. Weinberg, Phys. Rev. Lett. 17: 616, 1966.]

VI.4.3. Вычислите амплитуду пион-пионного рассеяния до членов второго порядка по внешним импульсам, используя линейную σ-модель (2). Не забудьте фейнмановскую диаграмму, включающую обмен σ-мезонами. Вы должны получить тот же результат, что и в упражнении VI.4.2.

VI.4.4. Покажите, что соотношение для массы $M = gv$ эквивалентно соотношению Голдбергера - Треймана.

30.3.5 Zi.VI.5 Ferromagnets and Antiferromagnets

Магнитные моменты

В главах IV.1 и V.3 обсуждалось происхождение понятия намбу-голдстоуновского бозона как спиновой волны в ферромагнитных или антиферромагнитных телах. Модельное описание таких тел представляет собой регулярную решетку, в каждом узле которой находится локальный магнитный момент, который обозначим единичным вектором $\vec{n}_j; j$ обозначает узел. В ферромагнетике магнитные моменты соседних узлов стремятся выстроиться в одном направлении, в то время как в антиферромагнетике магнитные моменты соседних узлов стремятся быть противоположно направленными. Другими словами, энергия имеет вид $\mathcal{H} = J\sum_{\langle ij\rangle}\vec{n}_i\cdot\vec{n}_j$, где i и j обозначают соседние узлы. Для антиферромагнетиков $J > 0$, а для ферромагнетиков $J < 0$.

При более микроскопическом рассмотрении следовало бы начать с гамильтониана (например, хаббардовского гамильтониана), описывающего перескоки электронов и взаимодействие между ними. В приближении среднего поля классическая переменная $\vec{n}_j$

может появиться как единичный вектор, указывающий направление $\left\langle c_j^\dagger \vec{\sigma} c_j \right\rangle$, где $c_j^\dagger$ и c_j - операторы рождения и уничтожения электронов соответственно. Но у вас в руках не учебник по физике твердого тела.

Первый или второй порядок по времени

В этом разделе мы бы хотели получить эффективное описание низкоэнергетических состояний ферромагнетика и антиферромагнетика в духе описания сигма-модели из предшествующей главы. Это описание будет существенно длиннее, чем стандартное, которое приводится в некоторых книгах по теории поля, но в нем есть небольшое преимущество - оно корректно. В том, какой член кинетической энергии надо добавить к $-\mathcal{H}$, чтобы образовать лагранжиан L, есть некоторая тонкость. Так как для единичного вектора $\vec{n}$ имеет место выражение $\vec{n} \cdot (d\vec{n}/dt) = (d(\vec{n} \cdot \vec{n})/dt) = 0$, одной производной по времени обойтись нельзя. С двумя производными можно образовать $(d\vec{n}/dt) \cdot (d\vec{n}/dt)$ и тогда

$$L_{\text{wrong}} = \frac{1}{2g^2} \sum_j \frac{\partial \vec{n}_j}{\partial t} \cdot \frac{\partial \vec{n}_j}{\partial t} - J \sum_{\langle ij \rangle} \vec{n}_i \cdot \vec{n}_j .$$

В типичном учебнике по теории поля далее записывается непрерывный предел и в результате получается плотность лагранжиана

$$\mathcal{L} = \frac{1}{2g^2} \left(\frac{\partial \vec{n}}{\partial t} \cdot \frac{\partial \vec{n}}{\partial t} - c_s^2 \sum_l \frac{\partial n}{\partial x^l} \cdot \frac{\partial \vec{n}}{\partial x^l} \right)$$

со связью $[\vec{n}(x,t)]^2 = 1$. Это еще один пример нелинейной сигма-модели. Как и в нелинейной сигма-модели, рассмотренной в главе VI.4, в этом случае лагранжиан выглядит свободным, но ограничение приводит к нетривиальной динамике. Постоянная c_s (которая определена через микроскопическую переменную J) является скоростью спиновой волны, что легко увидеть, если записать уравнение движения $(\partial^2/\partial t^2)\,\vec{n} - c_s^2 \nabla^2 \vec{n} = 0$.

Однако вам может показаться, что здесь что-то не так. Из курса квантовой механики известно, что динамика спиновой переменной $\vec{S}$ имеет первый порядок по времени. Рассмотрим наиболее типичный пример спина в постоянном магнитном поле, описываемом выражением $H = \mu \vec{S} \cdot \vec{B}$. Тогда $d\vec{S}/dt = i[H, \vec{S}] = \mu \vec{B} \times \vec{S}$. Кроме того, вы, возможно, помните из курса физики твердого тела, что в ферромагнетике дисперсионное соотношение спиновой волны имеет нерелятивистский вид $\omega \propto k^2$, а вовсе не релятивистский вид $\omega^2 \propto k^2$, что следует из выражения (2).

Разрешение этого явного парадокса основано на тождестве ПаулиХопфа. Имея единичный вектор $\vec{n}$, можно всегда записать $\vec{n} = z^\dagger \vec{\sigma} z$, где $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ состоит из двух комплексных чисел, таких что $z^\dagger z \equiv z_1^\dagger z_1 + + z_2^\dagger z_2 = 1$. Проверьте это! (Математическое отступление. Записывая z_1 и z_2 через действительные числа, находим, что это определяет так называемое отображение Хопфа $S^3 \to S^2$.) Несмотря на то что нельзя образовать член второго порядка по $\vec{n}$ и линейный по временной производной, можно записать член второго порядка по комплексному дублету z и линейный по временной производной. Не могли бы вы это проверить, прежде чем перейти к следующей строчке? Правильной версией (1) является

$$L_{\text{correct}} = i \sum_j z_j^\dagger \frac{\partial z_j}{\partial t} + \frac{1}{2g^2} \sum_j \frac{\partial \vec{n}_j}{\partial t} \cdot \frac{\partial \vec{n}_j}{\partial t} - J \sum_{\langle ij \rangle} \vec{n}_i \cdot \vec{n}_j .$$

Добавленный член известен как фаза Берри и имеет глубокий топологический смысл. Вам нужно вывести уравнение движения, используя тождество

$$\int dt\delta\left(z_j^\dagger\frac{\partial z_j}{\partial t}\right)=\frac{1}{2}i\int dt\delta\vec{n}_j\cdot\left(\vec{n}_j\times\frac{\partial\vec{n}_j}{\partial t}\right).$$

Примечательно, что, хотя член $z_j^\dagger\left(\partial z_j/\partial t\right)$ нельзя записать в терминах $\vec{n}_j$, это можно сделать для его вариации.

Низкоэнергетические моды в ферромагнетике и антиферромагнетике

В основном состоянии ферромагнетика все магнитные моменты имеют одно и то же направление, которое можно считать z-направлением. Разлагая уравнение движения по малым флуктуациям относительно этого основного состояния $\vec{n}_j=\widehat{e}_z+\delta\vec{n}_j$ (где $\widehat{e}_z$ означает соответствующий единичный вектор) и выполняя преобразование Фурье, получаем

$$\left(\begin{array}{cc}-\frac{\omega^2}{g^2}+h(k) & -\frac{1}{2}i\omega\\ \frac{1}{2}i\omega & -\frac{\omega^2}{g^2}+h(k)\end{array}\right)\left(\begin{array}{c}\delta n_x(k)\\ \delta n_y(k)\end{array}\right)=0$$

соотношение на две компоненты $\delta\vec{n}(k)$, т. е. $\delta n_x(k)$ и $\delta n_y(k)$. Условие $\vec{n}_j\cdot\vec{n}_j=1$ означает, что $\delta n_z(k)=0$. Здесь a - период решетки и $h(k)\equiv 4J[2-\cos\left(k_xa\right)-\cos\left(k_ya\right)]\simeq 2Ja^2k^2$ для малого k. (Я неявно работаю в двух пространственных измерениях, о чем свидетельствуют k_x и k_y.)

При низких частотах член Берри $i\omega$ доминирует над членом ω^2/g^2, который поэтому можно отбросить. Положив определитель матрицы равным нулю, получаем правильное квадратичное дисперсионное соотношение $\omega\propto k^2$.

Рассмотрение антиферромагнетика связано с интересной особенно-

стью. Так называемое состояние Нееля[1] для антиферромагнетика определе-

[1] Заметим, что. хотя состояние Нееля описывает конфигурацию классического антиферромагнетика с наименьшей энергией. оно не описывает основное состояние квантового антиферромагнетика. Члены $S_i^+S_j^-+S_i^-S_j^+$в гамильтониапе.J изменяют направление спинов вверх и вниз. ется выражением $\vec{n}_j=(-1)^j\widehat{e}_z$. Записывая $\vec{n}_j=(-1)^j\widehat{e}_z+\delta\vec{n}_j$, получаем выражение

$$\left(\begin{array}{cc}-\frac{\omega^2}{g^2}+f(k) & -\frac{1}{2}i\omega\\ \frac{1}{2}i\omega & -\frac{\omega^2}{g^2}+f(k)\end{array}\right)\left(\begin{array}{c}\delta n_x(k)\\ \delta n_y(k+Q)\end{array}\right)=0$$

связывающее δn_x и δn_y, зависящих от разных импульсов. Здесь $f(k)=4J\left[2+\cos\left(k_xa\right)+\cos\left(k_ya\right)\right]$ и $Q=[\pi/a,\pi/a]$. Появление Q связано с тем, что $(-1)^j=e^{iQaj}$. (Постарайтесь разобраться со слишком краткими обозначениями.) Антиферромагнитный множитель $(-1)^j$ явно нарушает трансляционную инвариантность и вставляет импульс Q, когда бы он ни появлялся. Аналогичное уравнение связывает $\delta n_y(k)$ и $\delta n_x(k+Q)$. Решая эти уравнения, находим, что существует высокочастотная ветвь, которая не представляет для нас интереса, и низкочастотная ветвь с линейной дисперсией $\omega\propto k$.

Таким образом, низкочастотную динамику антиферромагнетика можно описать нелинейной сигма-моделью (2), которую можно представить в релятивистском виде, когда скорость спиновой волны нормирована к единице:

$$\mathcal{L}=\frac{1}{2g^2}\partial_\mu\vec{n}\cdot\partial^\mu\vec{n}$$

Упражнения

VI.5.1. Получите две ветви спинового волнового спектра для случая ферромагнетика, обращая особое внимание на поляризацию.

VI.5.2. Проверьте, что в случае антиферромагнетика член фазы Берри только меняет скорость спиновой волны и не влияет качественно на спектр, как в случае ферромагнетика.

30.3.6 Zi.VI.6 Surface Growth and Field Theory

В этой главе рассматривается довольно необычная для учебника по теории поля тема, взятая из неравновесной статистической механики - одного из разделов теоретической физики, который последние несколько лет развивается наиболее активно. Я расскажу вам еще об одной области, где используются идеи теории поля.

Представим атомы, беспорядочно расположенные на некоторой поверхности. Буквально выражаясь, так выращиваются некоторые новые вещества. По мере роста высота этой поверхности $h(x,t)$ определяется уравнением Кардара-Паризи - Чжана

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\vec{x},t)$$

Это уравнение описывает обманчиво простой прототип неравновесной динамики и имеет поразительно широкую область применения.

Чтобы понять уравнение (1), рассмотрим различные слагаемые в правой его части. Смысл слагаемого $\nu\nabla^2 h$ (где $\nu > 0$) очевиден: положительное во впадинах h и отрицательное на пиках, оно стремится сгладить поверхность. При наличии единственного этого слагаемого задача была бы линейной и поэтому тривиальной. Нелинейное слагаемое $(\lambda/2)(\nabla h)^2$ делает задачу весьма нетривиальной и интересной. В качестве упражнения убедитесь в геометрическом происхождении этого слагаемого. Третье слагаемое описывает случайное появление атомов со случайной переменной $\eta(\vec{x},t)$, которая, как предполагается, подчиняется гауссовому распределению с нулевым средним значением [1] и коррелятором

$$\langle \eta(\vec{x},t)\eta(\vec{x}',t')\rangle = 2\sigma^2\delta^D(\vec{x}-\vec{x}')\,\delta(t-t').$$

Другими словами, распределение вероятности для реализации $\eta(\vec{x},t)$ определяется выражением

$$P(\eta) \propto e^{-\frac{1}{2\sigma^2}\int d''xdt\eta(\vec{x},t)^2}$$

[1] Здесь не нарушается общность. поскольку алиитивную постоянную в η можно убрать путем сдвига $h \to h + ct$. Здесь $\vec{x}$ представляет собой координаты в D-мерном пространстве. В эксперименте для описанной ситуации $D = 2$, однако теоретически можно рассматривать эту задачу для любой размерности D.

Специалистов в области физики конденсированных сред обычно интересует корреляции между высотой поверхности при двух различных положениях в пространстве и времени:

$$\left\langle [h(\vec{x},t) - h(\vec{x}',t')]^2 \right\rangle = |\vec{x}-\vec{x}'|^{2\chi} f\left(\frac{|\vec{x}-\vec{x}'|^z}{|t-t'|}\right).$$

Скобки $\langle ...\rangle$ здесь и в выражении (2) означают усреднение по разным реализациям случайной переменной $\eta(\vec{x},t)$. Правая часть (3) представляет собой выражение для динамического скейлинга, которое обычно постулируется в физике конденсированных сред, где χ и z - это так называемые экспонента шероховатости и динамическая экспонента. Итак, задача состоит в том, чтобы доказать корректность выражения для скейлинга, а также найти χ и z. Отметим, что из динамической экспоненты z (которая в общем случае не является целым числом) следует, грубо говоря, сколько степеней пространства стоят одной степени времени. (Для $\lambda = 0$ получаем простую диффузию, для которой $z = 2$.) Здесь f обозначает неизвестную функцию.

Я не буду вдаваться в прочие технические подробности. Для нас представляет интерес то, как эта задача, которая даже не включает квантовую механику, может быть преобразована в квантовую теорию поля. Начнем с выражения

$$Z \equiv \int \mathcal{D}h \int \mathcal{D}\eta e^{-\frac{1}{2\sigma^2}\int d^D x dt \eta(\vec{x},t)^2} \delta\left[\frac{\partial h}{\partial t} - \nu\nabla^2 h - \frac{\lambda}{2}(\nabla h)^2 - \eta(\vec{x},t)\right].$$

Интегрируя по η, получаем $Z = \int \mathcal{D}h e^{-S(h)}$, где действие имеет вид:

$$S(h) = \frac{1}{2\sigma^2}\int d^D\vec{x}dt\left[\frac{\partial h}{\partial t} - \nu\nabla^2 h - \frac{\lambda}{2}(\nabla h)^2\right]^2$$

Вы узнаете, что данное выражение описывает нерелятивистскую теорию скалярного поля $h(\vec{x},t)$. Тогда интересующая нас физическая величина определяется выражением

$$\left\langle \left[h(\vec{x},t) - h\left(\vec{x}',t'\right)\right]^2 \right\rangle = \frac{1}{Z}\int \mathcal{D}h e^{-S(h)}\left[h(\vec{x},t) - h\left(\vec{x}',t'\right)\right]^2.$$

Таким образом, задача определения экспоненты шероховатости и динамической экспоненты в статистической физике эквивалентна задаче определения пропагатора

$$D(\vec{x},t) \equiv \frac{1}{Z}\int \mathcal{D}h e^{-S(h)} h(\vec{x},t) h(\overrightarrow{0},0)$$

скалярного поля h.

Кстати, с учетом масштабирования $t \to t/\nu$ и $h \to \sqrt{\sigma^2/\nu}h$ действие можно записать в виде

$$S(h) = \frac{1}{2}\int d^D\vec{x}dt\left[\left(\frac{\partial}{\partial t} - \nabla^2\right)h - \frac{g}{2}(\nabla h)^2\right]^2$$

где $g^2 \equiv \lambda^2\sigma^2/\nu^3$. Раскладывая, как обычно, действие по степеням h
$S(h) = \frac{1}{2}\int d^D\vec{x}dt\left\{\left[\left(\frac{\partial}{\partial t} - \nabla^2\right)h\right]^2 - g(\nabla h)^2\left(\frac{\partial}{\partial t} - \nabla^2\right)h + \frac{g^2}{4}(\nabla h)^4\right\}$,
получаем знакомый квадратичный член, дающий довольно необычный пропагатор $1/\left(\omega^2 + k^4\right)$ для скалярного поля h, а также члены кубической и четвертой степени, описывающие взаимодействие. Как обычно, чтобы получить искомую физическую величину, вычисляем функциональный интеграл, или интеграл «по траекториям»,

$$Z = \int \mathcal{D}h e^{-S(h) + \int d^l x dt J(x,t)h(x,t)}$$

и затем функционально дифференцируем несколько раз по J.

Я хочу не столько обучить вас неравновесной статистической механике, сколько показать, как квантовая теория поля может возникать в разных физических ситуациях, в том числе тех, которые включают только классическую физику. Заметим, что «квантовые флуктуации» здесь появляются из случайного основного члена. Между хаотической динамикой и квантовой физикой явно существует тесная методологическая связь.

Упражнения

VI.6.1. Задача по элементарной геометрии: нарисуйте прямую линию под углом θ к горизонтали. Эта линия представляет сегмент поверхности в момент времени t. Теперь нарисуйте несколько окружностей диаметра d, касательных к этой линии сверху. А теперь нарисуйте другую линию под углом θ к горизонтали, которая лежит поверх окружностей, а именно касательную к ним. Эта новая линия представляет собой сегмент поверхности в более поздний момент времени (см. рис. VI.6.1). Обратите внимание, что

$\Delta h = d/\cos\theta \simeq d\left(1+\frac{1}{2}\theta^2\right)$. Покажите, что порождается нелинейный член $(\lambda/2)(\nabla h)^2$ в уравнении (1) Кардара - Паризи - Чжана. О применениях уравнения Кардара-Паризи - Чжана читайте, например, в T. Halpin-Healy and Y.-C. Zhang, Phys. Rep. 254: 215, 1995; A. L. Barabasi and H. E. Stanley, Fractal Concepts in Surface Grown.

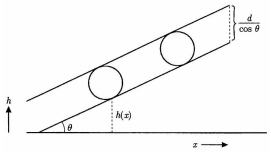


Рис. VI.6.1

VI.6.2. Покажите, что скалярное поле h имеет пропагатор $1/\left(\omega^2+k^4\right)$.

VI.6.3. Квантовая теория поля при заменах переменных может часто проявляться в довольно разных видах. Покажите, что, записав $U = e^{\frac{1}{2}gh}$, можно заменить действие (7) на

$$S = \frac{2}{g^2}\int d^D\vec{x}dt\left(U^{-1}\frac{\partial}{\partial t}U - U^{-1}\nabla^2 U\right)^2$$

то есть своего рода нелинейную σ-модель.

30.3.7 Zi.VI.7 Disorder: Replicas and Grassmann Symmetry

Примеси и случайный потенциал

Важной областью физики конденсированных сред являются неупорядоченные системы, изучению которых за последние несколько десятилетий было посвящено огромное число теоретических работ. Электроны в реальных материалах рассеиваются на неизбежно присутствующих примесях и эффективно движутся в случайном потенциале. Кратко опишу эту захватывающую тему и покажу, как подобного рода задачу можно отобразить в квантовой теории поля.

Прототипом является задача о квантовой частице, подчиняющейся уравнению Шредингера $H\psi = \left[-\nabla^2 + V(x)\right]\psi = E\psi$, где $V(x)$ - случайный потенциал (представляющий примеси), имеющий нормальное гауссово распределение $P(V) = \mathcal{N}e^{-\int d''x\left(1/2g^2\right)V(x)^2}$ с нормировочным множителем $\mathcal{N}$, который определяется выражением $\int DVP(V) = 1$. Параметр g измеряет количество примесей: чем больше g, тем сильнее разупорядочена система. Это, конечно, идеализация, в которой не учитываются взаимодействие между электронами и ряд других физических эффектов.

Как и в статистической механике, представим себе ансамбль систем, каждая из которых характеризуется конкретного вида функцией $V(x)$, взятой из распределения $P(V)$. Изучим усредненные или типичные свойства системы. В частности, нас интересует усредненная плотность состояний, определяемая выражением $\rho(E) = \langle \mathrm{tr}\,\delta(E-H)\rangle = \left\langle \sum_i \delta\left(E-E_i\right)\right\rangle$, где сумма берется по i-му собственному состоянию H с соответствующим собственным значением E_i. Обозначим среднее значение любого функционала $O(V)$ функции $V(x)$ как $\langle O(V)\rangle \equiv \int DVP(V)O(V)$. Ясно, что $E^* + \delta E \int\limits_{E^*} dE\rho(E)$ соответствует числу состояний в интервале от E^* до $E^*+ +\delta E$, которое является важным, например, в экспериментах по туннелированию.

Андерсоновская локализация

Другой важный физический вопрос заключается в том, распространяются ли волновые функции при определенной энергии E по всей системе или они локализованы в пределах

характеристического масштаба длин $\xi(E)$. Очевидно, что от этого зависит, является вещество проводником или изолятором. На первый взгляд создается впечатление, что следует изучить выражение

$$S(x,y;C) \equiv \left\langle \sum_i \delta\left(E-E_i\right)\psi_i^*(x)\psi_i(y) \right\rangle$$

из которого можно было бы узнать, как волновая функция в точке x коррелирована с волновой функцией в некоторой другой точке y. Однако выражение для S не подходит, потому что $\psi_i^*(x)\psi_i(y)$ имеет фазу, которая зависит от V. Таким образом, S должно обращаться в нуль при усреднении по беспорядку. Вместо S следует рассмотреть величину

$$K(x-y;E) \equiv \left\langle \sum_i \delta\left(E-E_i\right)\psi_i^*(x)\psi_i(y)\psi_i^*(y)\psi_i(x) \right\rangle$$

так как член $\psi_i^*(x)\psi_i(y)\psi_i^*(x)\psi_i(y)$ явно положительный. Заметим, что при усреднении по всем возможным $V(x)$ восстанавливается трансляционная инвариантность, так что K является функцией только расстояния $|x-y|$. Если $|x-y| \to \infty$, то при $K(x-y;E) \sim e^{-|x-y|/\xi(E)}$ волновые функции с энергией E локализованы на так называемой длине локализации $\xi(E)$. С другой стороны, если $K(x-y;E)$ убывает как степень от $|x-y|$, то говорят, что волновые функции не локализованы.

Андерсон с коллегами сделал поразительное открытие, заключающееся в том, что свойства локализации зависят от размерности пространства D, а не от детального вида $P(V)$ (пример понятия универсальности). Для $D = 1$ и 2 все волновые функции локализованы независимо от того, насколько слабым может быть примесный потенциал. Это утверждение в высшей степени нетривиально, поскольку априори можно прийти к заключению, что локализация или нелокализация волновых функций зависит от силы потенциала, как думали в свое время выдающиеся физики. В противоположность этому, для $D = 3$ волновые функции не локализованы для E в диапазоне $(-E_c, E_c)$. По мере того как E приближается сверху к энергии E_c (известной как предел подвижности), длина локализации $\xi(E)$ расходится как $\xi(E) \sim 1/\left(E-E_c\right)^{\mu}$ с некоторой критической экспонентой [1]μ. За эту и дру-

1Это пример квантового фазового перехода. Все обсуждение относится к нулевой температуре.

гие работы в теории конденсированных сред Андерсон получил Нобелевскую премию.

С физической точки зрения локализация обусловлена деструктивной интерференцией между квантовыми волнами, рассеивающимися на случайном потенциале.

Когда включено магнитное поле, перпендикулярное плоскости электронного газа с $D = 2$, ситуация кардинально меняется: при $E = 0$ появляется нелокализованная волновая функция. Для ненулевой E все волновые функции остаются локализованными, но с длиной локализации, расходящейся как $\xi(E) \sim 1/|E|^{\nu}$. Такая ситуация является причиной одного из наиболее поразительных особенностей квантового эффекта Холла (см. главу VI.2): холловская проводимость остается постоянной, тогда как энергия Ферми возрастает, но затем внезапно скачкообразно изменяется на дискретное значение вследствие вклада нелокализованного состояния, когда энергия Ферми переходит через $E = 0$. Количественная оценка такого поведения является сложной задачей для теоретиков в области физики конденсированных сред. Действительно, многие рассматривают аналитическое вычисление критической экспоненты ν как один из «священных Граалей» теории конденсированного вещества.

Формализм функций Грина

Слишком много информации для молниеносного знакомства с теорией локализации. Каким бы интересным ни был переход к локализации, какое отношение к этому может иметь квантовая теория поля? Все-таки это учебник по теории поля. Перед тем как продолжить, необходимо описать формализм. Рассмотрим так называемую функцию Грина

$G(z) \equiv \langle \mathrm{tr}[1/(z-H)] \rangle$ в комплексной z-плоскости. Так как $\mathrm{tr}[1/(z-H)] = \sum_i 1/(z-E_i)$, эта функция состоит из суммы полюсов при собственных значениях E_i. При усреднении полюсы сливаются в разрез. Используя стандартное тождество $\lim_{\varepsilon\to\infty} \mathrm{Im}[1/(x+i\varepsilon)] = -\pi\delta(x)$, легко видеть, что

$$\rho(E) = -\frac{1}{\pi} \lim_{\varepsilon\to 0} \mathrm{Im}\, G(E+i\varepsilon)$$

Поэтому если известна функция $G(z)$, то известна плотность состояний.

туре. В отличие от фазового перехода. рассмотренного в главе V.3, вместо температуры здесь меняются E.

Скверный знаменатель

Теперь можно объяснить, как квантовая теория поля возникает в этой задаче. Начнем с того, что возьмем логарифм тождества (A.12)

$$J^\dagger \cdot K^{-1} \cdot J = \log\left(\int D\varphi^\dagger D\varphi e^{-\varphi^\dagger \cdot K\cdot \varphi + J^\dagger\cdot\varphi + \varphi^\dagger \cdot J} \right)$$

(где, как обычно, отбрасывается несущественный член). Дифференцируя по $J^\dagger$ и J и затем задавая $J^\dagger$ и J равными 0, получаем интегральное представление для обратной эрмитовой матрицы:

$$\left(K^{-1}\right)_{ij} = \frac{\int D\varphi^\dagger D\varphi e^{-\varphi^\dagger\cdot K\cdot\varphi}\varphi_i\varphi_j^\dagger}{\int D\varphi^\dagger D\varphi e^{-\varphi^\dagger\cdot K\cdot\varphi}}$$

(Как видно, это выражение, по сути, связано с формулой (I.7.14) для пропагатора скалярного поля.) Теперь мы знаем, как представить функцию $1/(z$ - $-H)$. Надо взять ее след, что означает положить в уравнении (2) $i = j$ и просуммировать. В этой задаче $H = -\nabla^2 + V(x)$, а индекс i соответствует непрерывной переменной x, тогда как суммирование - интегрированию по пространству. Заменяя K на $i(z-H)$ (и позаботившись о соответствующей дельта-функции), получаем

$$\mathrm{tr}\,\frac{-i}{z-H} = \int d^D y \left\{ \frac{\int D\varphi^\dagger D\varphi e^{i\int d^D x\left\{\partial\varphi^\dagger\partial\varphi + [V(x)-z]\varphi^\dagger\varphi\right\}}\varphi(y)\varphi^\dagger(y)}{\int D\varphi^\dagger D\varphi e^{i\int d^I x\left\{\partial\varphi^\dagger\partial\varphi+[V(x)-z]\varphi^\dagger\varphi\right\}}} \right\}$$

Начинает походить на теорию скалярного поля в D-мерном евклидовом пространстве с действием $S = \int d^D x \left\{\partial\varphi^\dagger\partial\varphi + [V(x)-z]\varphi^\dagger\varphi\right\}$. [Заметим, что, для того чтобы уравнение (3) было хорошо определено, z должна находиться в нижней полуплоскости.]

Но теперь нужно усреднить по $V(x)$, то есть проинтегрировать по V с распределением вероятности $P(V)$. Сразу же возникает трудность, которая долгое время ставила теоретиков в тупик. Знаменатель в (3) не позволяет двигаться дальше. Если бы этого знаменателя не было, то функциональное интегрирование по $V(x)$ соответствовало бы вычислению гауссова интеграла, которым вы занимались много раз. Можно ли этот скверный знаменатель, если можно так выразиться, как-то перенести в числитель? Умные головы предложили два приема, известные соответственно как репличный и суперсимметричный. Если вы можете предложить другой прием, честь вам и хвала.

Реплики

Репличный подход основан на хорошо известном пределе $(1/x) = \lim_{n\to 0} x^{n-1}$, который позволяет записать этот столь неприятный знаменатель в виде

$$\lim_{n\to 0}\left(\int D\varphi^\dagger D\varphi e^{i\int d^l x\left\{\partial\varphi^\dagger\partial\varphi+[V(x)-z]\varphi^\dagger\varphi\right\}}\right)^{n-1}=$$
$$=\lim_{n\to 0}\int\prod_{a=2}^{n}D\varphi_a^\dagger D\varphi_a e^{i\int d^D x\sum\limits_{a=2}^{n}\left\{\partial\varphi_a^\dagger\partial\varphi_a+[V(x)-z]\varphi_a^\dagger\varphi_a\right\}}.$$

Тогда (3) приобретает вид

$$\operatorname{tr}\frac{1}{z-H}=\lim_{n\to 0}i\int d^D y\times$$
$$\times\int\left(\prod_{a=1}^{n}D\varphi_a^\dagger D\varphi_a\right)e^{i\int d^D x\sum\limits_{a=1}^{n}\left\{\partial\varphi_a^\dagger\partial\varphi_a+[V(x)-z]\varphi_a^\dagger\varphi_a\right\}}\varphi_1(y)\varphi_1^\dagger(y).$$

Заметим, что функциональный интеграл теперь берется по n комплексным скалярным полям φ_a. Реплика поля φ получена.

Усредняя по случайному потенциалу, восстанавливаем трансляционную инвариантность. Таким образом, подынтегральное выражение для $\int d^D y$ не зависит от y, а $\int d^D y$ просто дает объем системы $\mathcal{V}$. Используя (A.12), получаем

$$\left\langle\operatorname{tr}\frac{1}{z-H}\right\rangle=i\mathcal{V}\lim_{n\to 0}\int\left(\prod_{a=1}^{n}D\varphi_a^\dagger D\varphi_a\right)e^{i\int d^D x\,\mathcal{L}}\varphi_1(0)\varphi_1^\dagger(0),$$

где

$$\mathcal{L}(\varphi)\equiv\sum_{a=1}^{n}\left(\partial\varphi_a^\dagger\partial\varphi_a-z\varphi_a^\dagger\varphi_a\right)+\frac{ig^2}{2}\left(\sum_{a=1}^{n}\varphi_a^\dagger\varphi_a\right)^2$$

Мы получаем теорию поля (со специфическим множителем i) n скалярных полей со старым добрым взаимодействием φ^4, инвариантным относительно $O(n)$ (известной как репличная симметрия). Заметим целесообразность замены K на $i(z-H)$: если не включить i, то функциональный интеграл расходился бы при большом φ, что можно легко проверить. Для z в верхней полуплоскости заменим K на $-i(z-H)$. Величина, из которой можно извлечь искомую усредненную плотность состояний, определяется пропагатором скалярного поля. Кстати, в уравнении (5) $\varphi_1(0)\varphi_1^\dagger(0)$ можно заменить более симметричным выражением $(1/n)\sum\limits_{b=1}^{n}\varphi_b^\dagger\varphi_b$.

Убирая $\mathcal{V}$ и, таким образом, переходя к вычислению плотности состояний на единицу объема, находим

$$G(z)=i\lim_{n\to 0}\int\left(\prod_{a=1}^{n}D\varphi_a^\dagger D\varphi_a\right)e^{-S(\varphi)}\left(\frac{1}{n}\sum_{b=1}^{n}\varphi_b^\dagger(0)\varphi_b(0)\right).$$

Для любого целого числа n теория поля определяется однозначно, поэтому тонким моментом в репличном подходе является взятие предела $n\to 0$. Этому пределу посвящена интереснейшая литература. (Посмотрите книгу о спиновых стеклах.)

Некоторые теоретики в области физики элементарных частиц имели обыкновение пренебрежительно отзываться о физике конденсированных сред как о грязной физике. Влияние примесей и беспорядка на вещество действительно является одной из основных проблем современной физики конденсированных сред. Но, как видно из этого примера, между усреднением по беспорядочности и суммированием по квантовым флуктуациям во многих отношениях нет математического различия. В итоге получаем теорию поля φ^4, на изучение которой многие специалисты по физике элементарных частиц потратили в прошлом немало усилий. Кроме того, поразительный вывод Андерсона о том, что для

$D = 2$ любая степень беспорядка (сколь угодно малая) локализует все состояния, означает, что теорию поля, определяемую выражением (6), следует понимать в высшей степени нетривиально. Величина беспорядка проявляется как взаимодействие g^2, так что никакая теория возмущений по g^2 не может помочь нам объяснить локализацию. Андерсоновская локализация является существенно непертурбативным эффектом.

Грассманов подход

Как упоминалось выше, для борьбы со скверным знаменателем придуман не один, а два приема. Второй прием основан на интегрировании по грассмановым переменным, которое было рассмотрено в главе II.5. Пусть $\eta(x)$ и $\bar{\eta}(x)$ будут грассмановыми поля, тогда

$$\int D\eta D\bar{\eta} e^{-\int d^4x \bar{\eta} K \eta} = C \det K = \widetilde{C} \left(\int D\varphi D\varphi^\dagger e^{-\int d^4x \varphi^\dagger K \varphi} \right)^{-1},$$

где C и $\widetilde{C}$ - две несущественные константы, которые можно включить в определение $D\eta D\bar{\eta}$. С учетом этого тождества выражение (3) можно записать в виде

$$\operatorname{tr} \frac{1}{z-H} = i \int d^D y$$
$$\int D\varphi^\dagger D\varphi D\eta D\bar{\eta} e^{i\int d^D x \left\{ \left\{ \partial\varphi^\dagger \partial\varphi + \left[V(x) - z | \varphi^\dagger \varphi \right\} + \{ \partial\bar{\eta}\partial\eta + [V(x)-z]\bar{\eta}\eta \} \right\}} \varphi(y)\varphi^\dagger(y)$$

и тогда легко усреднить по беспорядку, чтобы получить

$$\left\langle \operatorname{tr} \frac{1}{z-H} \right\rangle = i \int D\varphi^\dagger D\varphi D\eta D\bar{\eta} e^{i\int d^I x \mathcal{L}\left(\bar{\eta},\eta,\varphi^\dagger,\varphi\right)} \varphi(x)\varphi^\dagger(x)$$

с лагранжианом

$$\mathcal{L}\left(\bar{\eta},\eta,\varphi^\dagger,\varphi\right) = \partial\varphi^\dagger\partial\varphi + \partial\bar{\eta}\partial\eta - z\left(\varphi^\dagger\varphi + \bar{\eta}\eta\right) + \frac{ig^2}{2}\left(\varphi^\dagger\varphi + \bar{\eta}\eta\right)^2$$

В итоге получаем теорию поля с бозонными (коммутирующими) полями $\varphi^\dagger$ и φ и фермионными (антикоммутирующими) полями η и $\bar{\eta}$, взаимодействие которых определяется беспорядком. Действие S обладает очевидной симметрией по отношению к вращению бозонных полей в фермионные поля и наоборот; и поэтому этот подход известен среди специалистов в области физики конденсированных сред как суперсимметричный. (Следует обратить внимание, что η и $\bar{\eta}$ не являются спинорными полями, что мы подчеркиваем, не записывая их в виде $\bar{\psi}$ и ψ. Суперсимметрия здесь, которую, пожалуй, лучше назвать грассмановой симметрией, крайне отличается от суперсимметрии в физике элементарных частиц, рассмотренной в глаВе VIII.4.)

У обоих подходов, как репличного, так и суперсимметричного, есть свои трудности; и я не шутил, когда говорил, что если вам удастся изобрести новый подход, в котором нет такого рода трудностей, он с восторгом будет принят специалистами по физике конденсированных сред.

Изучение локализации

Я показал, как вычислять усредненную плотность состояний $\rho(E)$. А как изучать локализацию? Попробуйте сами получить ответ на этот вопрос, решая упражнение в конце главы. Из предыдущего рассмотрения должно быть понятно, что объект, полученный из (3), нужно изучать путем замены $\varphi(y)\varphi^\dagger(y)$ на $\varphi\left(y'\right)\varphi^\dagger(y)\varphi(y)\varphi^\dagger\left(y'\right)$. Если сформулировать репличную теорию поля на языке физики элементарных частиц, т. е. как описание взаимодействия некоторого скалярного мезона, то можно увидеть, что плотность состояний будет определяться мезонным пропагатором, а локализация - мезон-мезонным рассеянием.

Упражнения

VI.7.1. Получите теорию поля, которая позволит изучать андерсоновскую локализацию. [Указание: рассмотрите объект

$$\left\langle \left(\frac{1}{z-H}\right)(x,y)\left(\frac{1}{w-H}\right)(y,x)\right\rangle$$

для двух комплексных чисел z и w. Необходимо ввести два набора репличных полей, обычно обозначаемых как φ_a^+и φ_a^-.] {Примечание: $[1/(z$ - $-H)](x,y)$ обозначает элемент xy матрицы, или оператора, $[1/(z-H)]$.

VI.7.2. В качестве другого примера рассмотрите задачу, взятую из литературы, посвященной беспорядку. Разместите случайным образом N точек в D-мерном евклидовом пространстве объемом V. Обозначьте положения этих точек как $\vec{x}_i (i = 1, ..., N)$. Пусть

$$f(\vec{x}) = (-)\int \frac{d^D k}{(2\pi)^D}\frac{e^{i\vec{k}\vec{x}}}{k^2+m^2}.$$

Рассмотрите $N \times N$-матрицу $H_{ij} = f(\vec{x}_i - \vec{x}_j)$. Вычислите плотность собственных значений $H\rho(E)$ как среднее по ансамблю матриц в пределе $N \to \infty, V \to \infty$, с фиксированной плотностью точек $\rho \equiv N/V$ (не надо путать с $\rho(E)$). [Указание: используйте репличный подход и получите действие теории поля

$$S(\varphi) = \int d^D x \left[\sum_{a=1}^{n}\left(|\nabla\varphi_a|^2 + m^2|\varphi_a|^2\right) - \rho e^{-(1/z)}\sum_{u=1}^{n}|\varphi_a|^2\right].$$

Эта задача не совсем тривиальна; если вам нужна помощь, смотрите M. Mzard et al., Nucl. Phys. B559: 689, 2000, cond-mat/9906135.

30.3.8 Zi.VI.8 RG Flow as a Natural Concept in High-Energy and Condensed Matter Physics

Таким образом, выводы, основанные на ренорм-групповом подходе... опасны и должны рассматриваться с должной осторожностью. То же относится ко всем выводам, полученным из локальных релятивистских теорий поля.
Дж. Бьеркен и С. Дрелл, 1965

Это не опасно

Ренорм-группа представляет собой наиболее важное концептуальное достижение в квантовой теории поля за последние три-четыре десятилетия. Основные идеи развивались одновременно как в сообществе специалистов в области физики высоких энергий, так и специалистов в области физики конденсированных сред, а в некоторых областях исследования ренорм-групповой поток стал частью рабочего языка.

Легко представить, насколько богат и многосторонен этот предмет; его можно рассматривать с разных точек зрения, а для полного описания потребовалась бы целая книга. К сожалению, рассмотрение этого предмета никогда не было достаточно основательным и всеобъемлющим. Рассмотрения, приведенные в некоторых ранее изданных книгах, только вводят читателей в заблуждение и являются беспорядочными, как, например, в хорошо известном учебнике, по которому я изучал квантовую теорию поля и из которого взята цитата к настоящей главе. Учитывая ограниченный объем книги, я попытаюсь изложить самую суть этого предмета, избегая, по возможности, технических подробностей. Вначале я подойду к рассмотрению с точки зрения физики высоких энергий, а затем с точки зрения физики конденсированных сред. Как и раньше, внимание будет сосредоточено на смысловом содержании, а не на вычислениях. Несмотря на порядок

изложения, вы поймете, что проще усвоить роль ренорм-группы в физике конденсированных сред, чем в физике высоких энергий.

Я заложил основу для рассмотрения ренорм-группы еще в главе III.1 я все планирую заранее! Вернемся к нашему другу-экспериментатору, с которым мы обсуждали теорию $\lambda\varphi^4$. По-прежнему будем считать, что наш мир описывается простой теорией $\lambda\varphi^4$ и приближения до порядка λ^2 достаточно.

На чем настаивают экспериментаторы

Нашего друга-экспериментатора не интересовала написанная на листе бумаги константа связи λ; для него это всего лишь греческая буква. Он настаивал на том, что признает такие величины, которые фактически можно измерить, пусть даже только в принципе экспериментально. В результате нашей с ним дискуссии понимание того, что такое константа связи, стало глубже, и мы усвоили, что надо определить физическую константу связи следующим образом [см. выражение (III.1.4)]:

$$\lambda_P(\mu) = \lambda - 3C\lambda^2 \log\left(\frac{\Lambda^2}{\mu^2}\right) + O\left(\lambda^3\right).$$

По его настоянию мы научились представлять результат для физических амплитуд через $\lambda_P(\mu)$, а не теоретически введенную λ. В частности, нужно записать амплитуду мезон-мезонного рассеяния как

$$\mathcal{M} = -i\lambda_P(\mu) + iC\lambda_P(\mu)^2\left[\log\left(\frac{\mu^2}{s}\right) + \log\left(\frac{\mu^2}{t}\right) + \log\left(\frac{\mu^2}{u}\right)\right] + \\ + O\left[\lambda_P(\mu)^3\right]$$

В чем заключается физический смысл $\lambda_P(\mu)$? Она, конечно же, измеряет силу взаимодействия между мезонами, как отражено в (2). Но почему нужно выбирать конкретное μ? Из (2) очевидно, что $\lambda_P(\mu)$ очень удобна для изучения физики в таком режиме, когда все кинематические переменные s, t и u имеют порядок μ^2. Амплитуда рассеяния имеет вид $-i\lambda_P(\mu)$ с малыми логарифмическими поправками. (Вспомним из сноски в главе III.3, что точка перенормировки $s_0 = t_0 = u_0 = \mu^2$ введена только для удобства теоретического рассмотрения и не может быть достигнута в реальных экспериментах. Для концептуального понимания здесь это не имеет отношения к делу.) Короче говоря, $\lambda_P(\mu)$ известна как константа связи, соответствующая физике на масштабе энергий μ.

В отличие от этого, если быть настолько глупыми, чтобы использовать константу связи $\lambda_P(\mu')$ при изучении физики в режиме с s, t и u, имеющими порядок μ^2, где μ очень отличается от μ', то амплитуда рассеяния имела бы вид

$$\mathcal{M} = -i\lambda_P(\mu') + iC\lambda_P(\mu')^2\left[\log\left(\frac{\mu'^2}{s}\right) + \log\left(\frac{\mu'^2}{t}\right) + \log\left(\frac{\mu'^2}{u}\right)\right] \\ + O\left[\lambda_P(\mu')^3\right]$$

где второе слагаемое [с большим $\log(\mu'^2/\mu^2)$] может быть одного порядка с первым членом или превышать его. Таким образом, для каждой энергетической шкалы μ есть «соответствующая» константа связи $\lambda_P(\mu)$.

Вычитая (2) из (3), можно легко связать $\lambda_P(\mu)$ и $\lambda_P(\mu')$ для $\mu \sim \mu'$:

$$\lambda_P(mu') = \lambda_P(\mu) + 3C\lambda_P(\mu)^2 \log\left(\frac{\mu'^2}{\mu^2}\right) + O\left[\lambda_P(\mu)^3\right]$$

Последнее выражение можно записать в виде дифференциального «уравнения потока»

$$\mu\frac{d}{d\mu}\lambda_P(\mu) = 6C\lambda_P(\mu)^2 + O\left(\lambda_P^3\right)$$

Вы уже неоднократно сталкивались с тем, что квантовая теория поля полна исторических ошибок в употреблении терминов. Описание того, как $\lambda_P(\mu)$ меняется в зависимости от μ, известно как ренорм-группа. Группа здесь появляется лишь как аддитивная [1] группа преобразования $\mu \to \mu + \delta\mu$.

Для концептуального рассмотрения материала, изложенного в главе III.1 и в этой главе, можно вовсе не знать, что собой представляет константа C. Если C окажется отрицательной, тогда константа связи $\lambda_P(\mu)$ будет убывать с увеличением масштаба энергий μ и возрастать, если C окажется положительной. (На самом деле она имеет положительный знак, так что, когда масштаб энергии увеличивается, λ_P растет относительно первоначального значения.)

[1] Группа преобразований - мультипликативна. инфинитезимальные преобразования аддитивны. - Прим. ped.

Поток электромагнитной константы связи

Рассмотренное поведение λ типично для констант связи в четырехмерных квантовых теориях поля. Например, в квантовой электродинамике константа связи e или, что эквивалентно, $\alpha = e^2/4\pi$, измеряет интенсивность электромагнитного взаимодействия. Здесь происходит то же самое, что и с теорией $\lambda\varphi^4$: нашего друга-экспериментатора не интересует латинская буква e, он хочет знать реальное взаимодействие, когда соответствующие возведенные в квадрат импульсы имеют порядок μ^2. Хорошо, что мы уже сделали вычисление: эффективную константу связи можно извлечь при таком значении импульса, который перенесен возведенным в квадрат $q^2 = \mu^2$ из уравнения (III.7.14):

$$e_P(\mu)^2 = e^2 \frac{1}{1 + e^2\Pi\left(\mu^2\right)} \simeq e^2\left[1 - e^2\Pi\left(\mu^2\right) + O\left(e^4\right)\right].$$

Возьмем значение μ, гораздо большее массы электрона m, но гораздо меньшее масштаба обрезания M. Тогда из выражения (III.7.13) получаем:

$$\mu\frac{d}{d\mu}e^P(\mu) = -\frac{1}{2}e^3\mu\frac{d}{d\mu}\Pi\left(\mu^2\right) + O\left(e^5\right) = +\frac{1}{12\pi^2}e_P^3 + O\left(e_P^5\right).$$

Видим, что электромагнитная константа связи возрастает при увеличении масштаба энергий. Электромагнетизм становится сильнее с переходом к более высоким энергиям или, что эквивалентно, к меньшим расстояниям.

В физическом смысле происхождение этого явления тесно связано с физикой диэлектриков. Рассмотрим фотон, взаимодействующий с электроном, который назовем пробным электроном, чтобы в последующем избежать путаницы. Как показано еще в главе I.1, вследствие квантовых флуктуаций пространство-время наполняется появляющимися и исчезающими электрон-позитронными парами. Вблизи пробного электрона электроны в таких виртуальных парах отталкиваются от него и поэтому стремятся от него удалиться, тогда как позитроны стремятся к нему приблизиться. Таким образом, на больших расстояниях заряд пробного электрона отчасти экранируется облаком позитронов, что приводит к более слабому взаимодействию с фотоном, в то время как на малых расстояниях взаимодействие с фотоном становится сильнее. Квантовый вакуум просто настолько же диэлектрик, как и реальный образец.

Сейчас вы уже, возможно, заметили, что само название «константа связи» является совершенно неправильным по той причине, что многое в истории физики делалось главным образом на одном масштабе энергии, а именно на «почти нулевом»! В частности, некоторые люди говорят о «постоянной» тонкой структуры $\alpha = 1/137$, и фантазеры продолжают попытки «вывести» число 137 из нумерологии или других еще более причудливых методов. На самом деле, α является просто «константой» связи электромагнитного взаимодействия при очень низких энергиях. Экспериментально установлено, что α, которую правильнее записать как $\alpha_P(\mu) \equiv e_P^2(\mu)/4\pi$, меняется вместе с рассматриваемым масштабом энергий μ. Но увы, название «константа связи», по-видимому, уже укоренилось.

Ренорм-групповой поток

В общем случае в квантовой теории поля с константой связи g имеет место уравнение ренорм-группового потока

$$\mu = \frac{dg}{d\mu} = \beta(g)$$

которое иногда записывают как $dg/dt = \beta(g)$, задавая $t \equiv \log(\mu/\mu_0)$. Теперь опустим подстрочный индекс P у физических констант связи. Если окажется, что теория имеет несколько констант связи $g_i, i = 1, ..., N$, то получаем уравнение

$$\frac{dg_i}{dt} = \beta_i(g_1, ..., g_N)$$

Можно представить $(g_1, ..., g_N)$ как координаты частицы в N-мерном пространстве, t как время и $\beta_i(g_1, ..., g_N)$ как зависящее от положения поле скоростей. Нас интересует, как движется или течет частица при увеличении μ или t. Для простоты обозначим все координаты $(g_1, ..., g_N)$ как g. Очевидно, что те связи, на которых $\beta_i(g^*)$ исчезает (для всех i), представляют особый интерес: g^* известна как фиксированная точка. Если поле скоростей вокруг фиксированной точки g^* такое, что частица движется по направлению к этой точке (и когда ее достигает, остается там, поскольку ее скорость теперь равна нулю), фиксированную точку называют притягивающей или стабильной. Таким образом, чтобы изучить асимптотическое поведение квантовой теории поля при высоких энергиях, надо «попросту» найти все ее притягивающие фиксированные точки относительно ренорм-группового потока. В данной теории мы, как правило, видим, что некоторые константы связи устремляются к большим значениям, тогда как другие стремятся к нулю.

К сожалению, это замечательное теоретическое представление сложно осуществить на практике, потому что, по сути, не существует способа вычисления функций $\beta_i(g)$. В частности, точка g^* вполне могла бы быть довольно большой, связанной с так называемой фиксированной точкой сильной связи, а теория возмущений и фейнмановские диаграммы в этом случае бесполезны при определении свойств теории. Действительно, структура фиксированной точки известна для очень немногих теорий.

К счастью, известна одна особенно простая фиксированная точка, а именно $g^* = 0$, для которой теория возмущений, безусловно, применима. Уравнение (8) всегда может быть вычислено пертурбативно: $dg_i/dt = c_i^{jk} g_j g_k + d_i^{jkl} g_j g_k g_l + ...$ (В некоторых теориях такая последовательность начинается с квадратичных членов, а в других - с кубических. Иногда присутствует также и линейный член.) Таким образом, как мы уже видели на паре примеров, асимптотическое или высокоэнергетическое поведение теории зависит от знака β_i в уравнении (8).

Давайте сейчас заглянем на съемки фильма «История физики». В конце 1960-х годов экспериментаторы, изучающие так называемое глубоко неупругое рассеяние электронов на протонах, обнаружили, что полученные результаты указывают на то, что после соударения с высокоэнергетическим электроном один из кварков внутри протона распространяется свободно, без сильного взаимодействия с другими кварками. Обычно все три кварка внутри протона связаны друг с другом сильным взаимодействием, образуя таким образом протон. В конце концов, некоторые теоретики поняли, что такая сложная ситуация объяснима, если теория сильного взаимодействия такова, что константа связи течет в сторону фиксированной точки $g^* = 0$. В этом случае сильное взаимодействие между кварками слабеет с ростом энергии.

Теперь, оглядываясь в прошлое, нам все кажется «очевидным». Но, дорогие студенты, не забывайте, что в то время теория поля считалась неприемлемой для молодых умов, а ренорм-группа рассматривалась как «вредная» даже для учебника по теории поля!

Теория сильного взаимодействия не была известна. Если бы хватило смелости принять опасные идеи ренорм-группы, то можно было даже построить теорию сильного взаимодействия путем поиска асимптотически свободных теорий, как стали нам известны теории с притягивающей фиксированной точкой при $g^* = 0$.

Асимптотически свободные теории, безусловно, удивительны. Их поведение при высоких энергиях можно изучать с использованием пертурбативных методов. Так и была открыта фундаментальная теория сильного взаимодействия, ныне известная как квантовая хромодинамика, о которой речь пойдет позже.

Рассмотрение физики на различных масштабах длин

Необходимость ренорм-групп для физики конденсированных сред очевидна. Вместо общих положений сосредоточимся на особо показательном примере, а именно поверхностном росте. Именно поэтому в главе VI.6 введено уравнение Кардара-Паризи - Жанга. Мы знаем, что для изучения поверхностного роста нужно вычислить интеграл по траекториям

$$Z(\Lambda) = \int \mathcal{D}h e^{-S(h)}$$

где, как вы помните,

$$S(h) = \frac{1}{2}\int d^D\vec{x}dt\left(\frac{\partial h}{\partial t} - \nabla^2 h - \frac{g}{2}(\nabla h)^2\right)^2.$$

Это выражение определяет теорию поля. Как в случае любой теории поля, надо ввести параметр обрезания Λ. Проинтегрируем только по тем конфигурациям полей $h(\vec{x},t)$, которые не содержат фурье-компонент с $\vec{k}$ и ω, большими, чем Λ. (В принципе, так как эта теория нерелятивистская, обрезания должны быть разными для $\vec{k}$ и ω, но в общем случае для простоты описания обозначим их одинако как Λ.) Введение параметра обрезания необходимо и совершенно обоснованно физически. По крайней мере на масштабах длин, как минимум сравнимых с размером соответствующих молекул, непрерывное описание системы в терминах поля $h(\vec{x},t)$ непримеНИМО.

С физической точки зрения можно ожидать, что в микроскопическом масштабе поверхность будет выглядеть очень неровной, как показано на рис. VI.8.1a, поскольку случайно выбранный основной член $\eta(\vec{x},t)$ соответствует белому шуму, то есть η при $\vec{x}$ и $\vec{x}'$ (а также при разных моментах времени) не коррелируют вообще. Однако предположим, что нас интересует не столько детальная микроскопическая структура, сколько поведение поверхности на больших масштабах. Другими словами, мы согласны надеть такие очки, через которые поверхность выглядит расплывчато, как на рисунке VI.8.1b. Это вполне естественный подход к изучению физической системы, который нам хорошо знаком. Нас может интересовать физика на некотором масштабе длин L, и не заботить, что происходит на масштабах длин, гораздо меньших L.

Ренорм-группа - это формализм, который позволяет нам связать физику на разных масштабах длин или, что эквивалентно, физику на разных масштабах энергии. В физике конденсированных сред есть тенденция рассматривать масштабы длин, тогда как в физике элементарных частиц -

(a)

(b)

Рис. VI.8.1

масштабы энергии. Современный подход к ренорм-группам появился в результате изучения критических явлений Кадановым, Фишером, Вильсоном и другими, как упоминалось в главе V.3. Рассмотрим, например, модель Изинга, в которой спины могут

быть направлены вверх или вниз, и с ферромагнитным взаимодействием между соседними спинами. При высоких температурах спины направлены беспорядочно вверх и вниз. Когда температура снижается до точки перехода в ферромагнитное состояние, начинают появляться острова спинов, направленных вверх (мы говорим «направленных вверх» для определенности, можно точно так же говорить о спинах, направленных вниз). Их становится все больше до тех пор, пока не будет достигнута критическая температура T_c, при которой все спины во всей системе направлены вверх. Характерный масштаб длин в физике при любой определенной температуре задается типичным размером таких островов. В физически обусловленном методе блок-спина Каданова и др. блоки направленного вверх спина рассматриваются как один эффективный направленный вверх спин, и аналогично для блоков направленных вниз спинов. Тогда понятие ренорм-группы является естественным для описания эффективных спинов эффективным гамильтонианом, соответствующим такому масштабу длин.

Более или менее ясно, как применить эту физическую идею изменения масштабов в функциональном интеграле (9). Надо проинтегрировать по тем $h(\vec{k},\omega)$, где $\vec{k}$ и ω меньше Λ. Предположим, мы делаем лишь часть того, что должны сделать. Проинтегрируем по тем $h(\vec{k},\omega)$, где $\vec{k}$ и ω больше $\Lambda-\delta\Lambda$, но меньше Λ. Именно это имеется в виду, когда мы говорим, что нас не заботят флуктуации $h(\vec{x},t)$ на масштабах длин и времени, меньших $(\Lambda$ - $-\delta\Lambda)^{-1}$.

Надеваем размывающие очки

Вернемся для простоты к нашей любимой $\lambda\varphi^4$-теории вместо того, чтобы решать задачу поверхностного роста. Вспомним из предыдущих глав важность евклидовой $\lambda\varphi^4$-теории в современной теории конденсированных сред. Итак, продолжим $\lambda\varphi^4$-теорию в евклидово пространство и проанализируем интеграл

$$Z(\Lambda)=\int\limits_{\Lambda}\mathcal{D}\varphi e^{-\int d^dx\mathcal{L}(\varphi)}$$

Обозначение $\int\limits_{\Lambda}$ указывает на то, что интегрировать необходимо только по таким конфигурациям поля $\varphi(x) = \int\left[d^dk/(2\pi)^d\right]e^{ikx}\varphi(x)$, при которых $\varphi(k) = 0$ для $|k| \equiv \left(\sum\limits_{i=1}^{d}k_i^2\right)^{1/2}$ больше, чем Λ. Как уже говорилось, подобного рода интегрирование эквивалентно надеванию размывающих очков с разрешением $L = 1/\Lambda$: мы не допускаем или не видим флуктуации с масштабами длин меньше L.

Очевидно, что $O(d)$ - инвариантность, а именно евклидовый эквивалент лоренц-инвариантности значительно облегчит нам жизнь. В противоположность этому, для задачи поверхностного роста необходимы специальные очки, которые по-разному размывают пространство и время [2].

Теперь мы готовы к тому, чтобы сделать очки еще более размытыми, позволяя $\Lambda\to\Lambda-\delta\Lambda($ с $\delta\Lambda>0)$. Запишем $\varphi=\varphi_s+\varphi_w(s$ означает «гладкий» (от англ. «smooth»), w означает «извилистый» (от англ. «wriggly»)), определенную так, что фурье-компоненты $\varphi_s(k)$ и $\varphi_w(k)$ не равны нулю только для $|k|\leqslant(\Lambda-\delta\Lambda)$ и $(\Lambda-\delta\Lambda)\leqslant|k|\leqslant\Lambda$ соответственно. (Очевидно, что обозначения «гладкий» и «извилистый» введены для удобства.) Подставляя в (11), можно записать

$$Z(\Lambda)=\int\limits_{\Lambda-\delta\Lambda}\mathcal{D}\varphi_s e^{-\int d^dx\mathcal{L}(\varphi_s)}\int\mathcal{D}\varphi_w e^{-\int d^dx\mathcal{L}_1(\varphi_s,\varphi_w)}$$

[2] В физике конденсированных сред эту разницу измеряет так называемая динамическая экспонента z. Более точно, в контексте задачи поверхностного роста коррелятор (введенный в главе VI.6) удовлетворяет типу динамическопо скейлинга, приведенному в уравнении (VI.6.3). Наивно, динамическая экспонента z должна быть равна 2. (Краткий обзор по данному

вопросу Iриведен в работе M. Kardar and A. Zee, Nucl. Phys. B464[FS]: 449. 1996, cond-mat/9507112.) где все члены в $\mathcal{L}_1(\varphi_s, \varphi_w)$ зависят от φ_w. (То, что здесь происходит, несколько напоминает то, что мы делали в главе IV.3.) Представим себе, что интеграл берется по φ_w. Вспомним результат

$$e^{-\int d^d x \delta\mathcal{L}(\varphi_s)} \equiv \int \mathcal{D}\varphi_w e^{-\int d^d x \mathcal{L}_1(\varphi_s, \varphi_w)}$$

и, таким образом, получим

$$Z(\Lambda) = \int\limits_{\Lambda - \delta\Lambda} \mathcal{D}\varphi_s e^{-\int d^d x [\mathcal{L}(\varphi_s) + \delta\mathcal{L}(\varphi_s)]}$$

Все, мы закончили! Мы переписали теорию в терминах «гладкого» поля φ_s.

Конечно, это только формальный подход, поскольку на практике интеграл по φ_w можно взять лишь пертурбативно, допуская, что соответствующие константы связи малы. Если бы интеграл по φ_w можно было взять строго, то также можно было бы вычислить интеграл по φ, и тогда эти ренорм-группы и все, что с ними связано, не были бы нужны.

В педагогических целях рассмотрим в более общем виде $\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 + + \sum\limits_n \lambda_n \varphi^n + ...$ (так что λ_2 является обычным $\frac{1}{2}m^2$, а λ_4 - обычным λ). Поскольку такие члены, как $\partial\varphi_s \partial\varphi_w$, зануляются при интегрировании, получаем уравнение

$$\int d^d x \mathcal{L}_1(\varphi_s, \varphi_w) = \int d^d x \left[\frac{1}{2}(\partial\varphi_w)^2 + \frac{1}{2}m^2\varphi_w^2 + ...\right]$$

которое описывает поле φ_w, взаимодействующее как с самим собой, так и с внешним полем $\varphi_s(x)$. Из соображений симметрии следует, что $\delta\mathcal{L}(\varphi_s)$ имеет тот же вид, что и $\mathcal{L}(\varphi_s)$, но с другими коэффициентами. Таким образом, добавление $\delta\mathcal{L}(\varphi_s)$ к $\mathcal{L}(\varphi_s)$ сдвигает [3] константу связи λ_n (и коэффициент $\frac{1}{2}(\partial\varphi_s)^2$). Эти сдвиги генерируют поток в пространстве констант связи, описанный выше.

Выражение (13) вполне можно было бы рассматривать как конечный результат. Но предположим, что нужно сравнить (13) с (11). Тогда хотелось бы заменить $\int\limits_{\Lambda-\delta\Lambda}$ в (13) на $\int\limits_{\Lambda}$. Для удобства введем действительное

[3] Такие члены, как $(\partial\varphi)^4$, также можно сгенерировать, поэтому я записал выражение для $\mathcal{L}(\varphi)$ с $(...),.,$ приложений эти члены несущественны в техническом смысле, который будет определен ниже. число $b < 1$ с помощью выражения $\Lambda - \delta\Lambda = b\Lambda$. В $\int\limits_{\Lambda-\delta\Lambda}$ необходимо интегрировать по полям с$|k| \leqslant b\Lambda$. Поэтому нужно только лишь выполнить тривиальную замену переменной: пусть $k = bk'$, так что $k = bk'$. Но тогда надо соответствующим образом изменить $x = x'/b$, чтобы $e^{ikx} = e^{ik'x'}$. Делая подстановку, находим

$$\int d^d x \mathcal{L}(\varphi_s) = \int d^d x' b^{-d} \left[\frac{1}{2}b^2(\partial'\varphi_s)^2 + \sum_n \lambda\varphi_s^n + ...\right]$$

(где $\partial' = \partial/\partial x' = (1/b)\partial/\partial x$). Определим φ' через $b^{2-d}(\partial'\varphi_s)^2 = (\partial'\varphi')^2$ или, другими словами, $\varphi' = b^{\frac{1}{2}(2-d)}\varphi_s$. Тогда интеграл (14) приобретает вид

$$\int d^d x' \left[\frac{1}{2}(\partial'\varphi')^2 + \sum_n \lambda_n b^{-d+(n/2)(d-2)}\varphi'^n + ...\right]$$

Таким образом, если определить коэффициент при φ'^n как λ'_n, то получим

$$\lambda'_n = b^{(n/2)(d-2)-d}\lambda_n$$

- важный результат в теории ренорм-группы.

Релевантные, иррелевантные и маргинальные

Постараемся понять, что это значит. (На некоторое время пренебрежем $\delta\mathcal{L}(\varphi_s)$, чтобы упростить наши рассуждения.) Когда мы надеваем размывающие очки или, другими словами, когда нас интересует физика на больших расстояниях, можно снова записать $Z(\Lambda)$, как в (11), с учетом того, что константы связи λ_n надо заменить на λ'_n. Так как $b<1$, то из (15) легко видеть, что λ_n при $(n/2)(d-2)-d>0$ становятся все меньше и меньше и в конечном счете ими можно пренебречь. Немного жаргона: соответствующие операторы φ^n (по историческим причинам перейдем на миг с языка функционального интеграла на язык операторов) называют иррелевантными. Это проигравшие. В противоположность им, победители, а именно φ^n, для которых $(n/2)(d-2)-d<0$, называют релевантными. Операторы, для которых $(n/2)(d-2)-d=0$, называют маргинальными.

Например, возьмем $n=2: m'^2=b^{-2}m^2$, и массовый член всегда является релевантным в любом измерении. С другой стороны, возьмем $n=4$, и мы видим, что $\lambda'=b^{d-4}\lambda$ и оператор φ^4 является релевантным для $d<4$, иррелевантным для $d>4$ и маргинальным для $d=4$. Аналогично $\lambda'_6=b^{2d-6}\lambda$, и оператор φ^6 является маргинальным для $d=3$ и становится иррелевантным для $d>3$.

Легко видеть, что случай $d=2$ особый: все φ^n являются релевантными. Все это может показаться знакомым, если вы добросовестно решали упражнения. В упражнении III.2.1 вы должны были доказать, что константа связи λ_n имеет размерность массы $[\lambda_n]=(n/2)(2-d)+d$. Таким образом, величина $(n/2)(d-2)-d$ есть размерность длины константы связи λ_n. Например, для $d=4\lambda_6$ имеет массовую размерность -2, и поэтому, как говорилось в главе III.2, взаимодействие φ^6 является неперенормируемым и, значит, плохо себя ведет при большой энергии. Однако специалистов в области физики конденсированных сред интересует предел больших расстояний, противоположный тому, который интересует специалистов в области физики элементарных частиц. Следовательно, такие неприятные операторы, как φ^6, становятся иррелевантными в пределе больших расстояний.

Еще немного жаргона: для заданной теории скалярного поля размерность d, при которой наиболее существенное взаимодействие становится маргинальным, известна в физике конденсированных сред как критическая размерность. Например, критической размерностью для теории φ^6 является 3. Приведение уравнения (15) к дифференциальному виду - всего лишь вопрос «арифметики уровня средней школы». Запишем $\lambda'_n=\lambda_n+\delta\lambda_n$; тогда из $b=1-(\delta\Lambda/\Lambda)$ получаем $\delta\lambda_n=-\left[\frac{n}{2}(d-2)-d\right]\lambda_n(\delta\Lambda/\Lambda)$.

Теперь будем особенно осторожными со знаками. Как я уже отметил, для $(n/2)(d-2)-d>0$ константа связи λ_n (которую для определенности представим положительной) становится меньше, как следует из (15). Но поскольку Λ уменьшается до $\Lambda-\delta\Lambda$, положительный член $\delta\Lambda$ фактически соответствует разрешающей способности наших размывающих очков $L=\Lambda^{-1}$, которая возрастает до $L+L(\delta\Lambda/\Lambda)$. Таким образом, получаем

$$L\frac{d\lambda_n}{dL}=-\left[-\frac{n}{2}(d-2)-d\right]\lambda_n$$

так что для $(n/2)(d-2)-d>0$ положительная λ_n будет убывать по мере возрастания L[4].

В частности, для $n=4$ имеет место $L(d\lambda/dL)=(4-d)\lambda$. В большинстве приложений физики конденсированных сред $d\leqslant 3$, и поэтому λ

4Заметим. что в правой части выражения фигурирует линейная размерность λ_n со знаком «минус», а не линейная размерность $(n/2)(d-2)-d$, как можно было бы наивно предположить. возрастает по мере возрастания масштаба длин рассматриваемой физики. Как говорилось выше, взаимодействие φ^4 является релевантным.

Производная $\delta\mathcal{L}(\varphi_s)$, которой мы временно пренебрегли, вносит в правую часть (16) добавочный член, который назовем динамическим, в отличие от геометрического или «тривиального» члена. Таким образом, в общем случае $L(d\lambda_n/dL)=-[(n/2)(d-2)-d]\lambda_n+K(d,n,...,\lambda_j,...)$, где динамический член K зависит не только от d и n, но также от других констант связи. (Например, в (5) «тривиальный» член

обращается в нуль, так как мы находимся в четырехмерном пространстве-времени; остается только динамический вклад.)

Из нашего обсуждения следует, что более образно ренорм-группу можно назвать «трюком для вычисления интеграла почти за один прием».

Использование симметрии

Чтобы определить ренорм-групповой поток для константы связи g, можно повторить те же вычисления, что и при определении потока для констант связи λ и e в двух предыдущих примерах; а именно используя язык физики элементарных частиц, вычислить амплитуду рассеяния $h-h$ в однопетлевом приближении. Однако вместо этого мы будем следовать физической картине Каданова и др. В $Z(\Lambda)=\int \mathcal{D}h e^{-S(h)}$ интегрируем только по тем $h(\vec{k},\omega)$, где $\vec{k}$ и ω больше $\Lambda-\delta\Lambda$, но меньше Λ.

Теперь покажу, как в этом случае использовать симметрию, чтобы свести к минимуму затраты времени. Решающим здесь является не изучение динамики поверхностного роста, а метод, который пригодится вам в других ситуациях. Я выбрал особо «сложную» нерелятивистскую задачу с неявными симметриями, так что если осилить ренорм-группу для этой задачи, можно быть готовым почти к чему угодно.

Представим, что мы выполнили интегрирование по частям и получили результат $\int \mathcal{D}h e^{-\widetilde{S}(h)}$. На данном этапе следует определить симметрии, фигурирующие в задаче, так, как это показано в упражнениях. После этого можно утверждать, что $\widetilde{S}(h)$ будет иметь вид

$$\widetilde{S}(h)=\frac{1}{2}\int d^Dx dt\left[\left(\alpha\frac{\partial}{\partial t}-\beta\nabla^2\right)h-\alpha\frac{g}{2}(\nabla h)^2\right]^2+...$$

зависящий от двух параметров - α и β. Многоточием (..) обозначены члены более высокого порядка по h и его производных. Для упрощения задачи заметим, что перед $\partial h/\partial t$ и $(q/2)(\nabla h)^2$ стоит один и тот же коэффициент α. Если известны α и β, то с помощью соответствующего изменения масштаба можно свести действие $\widetilde{S}(h)$ к тому же виду, что и $S(h)$, и таким образом найти изменение g. Следовательно, достаточно посмотреть на члены $(\partial h/\partial t)^2$ и $\left(\nabla^2 h\right)^2$ в действии или, что эквивалентно, на пропагатор, который вычислить гораздо проще. Как однажды сказал Рудольф Пайерлс молодому Гансу Бете: «Erst kommt das Denken, dann das Integral» [5]. (Что примерно звучит как: «Прежде чем брать интеграл, подумайте».) Мы опустим здесь вычисления. Достаточно отметить, что g имеет знакомую со средней школы размерность: (длина) $)^{\frac{1}{2}(D-2)}$ (см. упражнение VI.8.5). Таким образом, согласно нашему обсуждению, мы должны получить

$$L\frac{dg}{dL}=\frac{1}{2}(2-D)g+C_Dg^3+...$$

Детальный расчет необходим для того, чтобы определить коэффициент c_D, который явно зависит от размерности пространства D, поскольку фейнмановские интегралы зависят от D. Уравнение показывает, как изменяется в физике поверхностного роста эффективная мера нелинейности g с изменением масштаба длин L. Для справки: $c_D=\left[S(D)/4(2\pi)^D\right](2D$ - $-3)/D$, где $S(D)$ есть D-мерный телесный угол. Интересным множителем, несомненно, является $(2D-3)$, который меняет знак [6] между $D=1$ и 2.

Локализация

Как говорилось ранее, ренорм-групповой поток буквально стал частью языка физики конденсированных сред и высоких энергий. Приведу еще один пример, демонстрирующий мощь ренорм-группы. Вернемся к андерсоновской локализации (глава VI.7), которая в свое время впечатлила сообщество физиков. Удивление вызвала сильная зависимость поведения локализации от размерности пространства D, настолько сильная, что для $D=2$ все

состояния оказываются локализованными независимо от того, насколько мала степень беспорядка. Обычная физическая интуиция подсказала бы нам, что существует критическая степень беспорядка. Сейчас мы увидим, что обе эти особенности вполне естественно объясняются на языке ренормгруппы. Уже из выражения (18) следует, что D играет определяющую роль. Теперь я приведу эвристическое, но красивое (по крайней мере для меня) доказательство, полученное Абрахамсом, Андерсоном, Личчиардел-

[5] Когда я был студентом. Джон Уилер дал мне аналогичный совет: «Никогда не приступайте к вычислениям, если не знаете ответ заранее».

[6] Кстати. эта теория точно решается для случая $D = 1$ (с использованием методов. которые в этой книге не обсуждаются). ло и Рамакришнаном, которых впоследствии стали называть в сообществе физиков конденсированных сред «бандой четырех». Прежде всего, надо понять, в чем заключается разница между удельной проводимостью σ и кондактансом G на языке физики твердого тела. Удельная проводимость [7] определяется выражением $\vec{J} = \sigma\vec{E}$, где $\vec{J}$ измеряет число электронов, проходящих через единичную область за единицу времени. Кондактанс G является обратным к сопротивлению (мнемоническое правило: эти слова рифмуются). Сопротивленис R является свойством образца и определяется в учебнике по физике для средней школы как $V = IR$, где ток I служит мерой числа электронов, проходящих за единицу времени. Чтобы связать σ и G, рассмотрим массу вещества в виде куба размером L с падением напряжения V в поперечном направлении. Тогда $I = JL^2 = \sigma EL^2 = \sigma(V/L)L^2 = \sigma LV$ и, таким образом,$G(L) = 1/R = I/V = \sigma L$. Теперь перейдем к двум измерениям. Рассмотрим тонкий лист вещества длиной и шириной L, толщиной $a \ll L$. (Мы занимаемся всего лишь школьной физикой, не обращаясь к какому-то сложному представлению двумерного пространства в теории поля!) Допустим падение напряжения V вдоль длины L: $I = J(aL) = \sigma EaL = \sigma(V/L)aL = \sigma Va$, и тогда $G(L) = 1/R = I/V = \sigma a$. Попробуйте сами перейти к одному измерению: рассмотрите провод длиной L, шириной и толщиной a. Таким образом, получаем $G(L) \propto L^{D-2}$. Кстати, специалисты по физике конденсированных сред обычно определяют безразмерную проводимость как $g(L) \equiv \hbar G(L)/e^2$.

Нам также известно поведение $g(L)$, когда $g(L)$ мало или, другими словами, когда вещество является изолятором, для которого ожидается, что $g(L) \sim ce^{-L/\xi}$, где ξ - некоторая характеристическая длина вещества, которая определяется микроскопической физикой. Таким образом, для малого $g(L)$ имеет место $L(dg/dL) = -(L/\xi)g(L) = g(L)[\log g(L) - \log c]$, где для рассматриваемых условий постоянный логарифм $\log c$ пренебрежимо мал.

[7] На протяжении нескольких лет я обращался к специалистам по теории высоких энергий с вопросом, как можно получить уравнение $\vec{J} = \sigma\vec{E}$, которое явно нарушает инвариантность относительно обращения времени, если микроскопическая физика электронного рассеяния на примесном атоме полностью подчиняется инвариантности относительно обращения времени. Очень немногие знали ответ. Разрешение этого явного парадокса заключается в порядке пределов! Специалисты по теории конденсированных сред вычисляют удельную проводимость $\sigma(\omega, \vec{k})$. зависящую от частоты и волнового вектора. и затем берут предел $\omega, \vec{k} \to 0$ и $\vec{k}^2/\omega \to 0$. До того. как предел взят. инвариантность относительно обращения времени сохраняется. Время. которое требуется частице. чтобы оказаться в ящике размером порядка $1/k$, имеет порядок $1/(Dk^2)$ (где D - коэффициент диффузии). Физическая суть заключается в том. что это время должно быть значительно больше. чем время наблюдения $\sim 1/\omega$.

[8] Сэм Трейман сказал мне, что, когда он служил в армии США радистом, его учили, что есть три вида закона Ома: $V = IR.I = V/R$ и $R = V/I$. Во втором равенстве здесь используется четвертая форма. Объединяя все вместе, получаем

$$\beta(g) \equiv \frac{L}{g}\frac{dg}{dL} = \begin{cases} (D-2) + \ldots & \text{для больших } g \\ \log g + \ldots & \text{для малых } g. \end{cases}$$

Прежде всего, тривиальное замечание: в разных дисциплинах $\beta(g)$ определяется поразному (что, конечно, не влияет на физику). Как мы говорили, в теории локализации $\beta(g)$ традиционно определяется как $d\log g/d\log L$. Из графика для $\beta(g)$ на рис. VI.8.2 видно,

что для $D = 2$ (и $D = 1$) кондактанс $g(L)$ всегда стремится к нулю при переходе к большим расстояниям (макроскопические измерения макроскопических веществ) независимо от того, где мы начинаем. В отличие от этого, если g_0 (начальное значение g) больше, чем критическое g_c, то для $D = 3g(L)$ стремится к бесконечности (предположительно этот бег обрезается физикой, которую мы не включили), и вещество является металлом; тогда как, если $g_0 < g_c$, вещество является изолятором. Кстати, специалисты по теории конденсированных сред часто говорят о критической размерности D_c, при которой поведение системы на больших расстояниях радикально изменяется; в данном случае это $D_c = 2$.

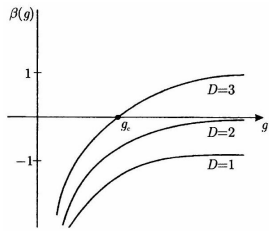


Рис. VI.8.2

Эффективное описание

В некотором смысле ренорм-группа берет начало от основного понятия физики, которое заключается в том, что эффективное описание может и должно изменяться по мере перехода от одного масштаба длин к другому. Например, в гидродинамике не надо детально отслеживать взаимодействие между молекулами воды. Таким же образом, когда ренорм-групповой поток используется для сильного взаимодействия, начиная с высоких энергий и двигаясь к низким энергиям, эффективное описание переходит от теории кварков и глюонов к теории нуклонов и мезонов. Тогда в этой более общей картине мы рассматриваем потоки не в пространстве констант связи, а в «пространстве гамильтонианов», о котором любят говорить некоторые теоретики в области физики конденсированных сред.

Упражнения

VI.8.1. Покажите, что решением $dg/dt = -bg^3 + \ldots$ будет

$$\frac{1}{\alpha(t)} = \frac{1}{\alpha(0)} + 8\pi bt + \ldots$$

где мы определили $\alpha(t) = g(t)^2/4\pi$.

VI.8.2. При рассмотрении ренорм-группы в $\lambda\varphi^4$-теории или в КЭД мы приняли для простоты, что масса m частицы гораздо меньше, чем μ, и поэтому приравняли m к нулю. Но в понятии ренорм-группы ничто не говорится о том, что нельзя переходить к масштабу массы меньше m. Действительно, в физике элементарных частиц масса t кварка m_t отличается от массы и кварка m_u на много порядков. Мы могли бы изучить, как изменяется константа связи сильного взаимодействия при переходе от некоторого масштаба массы, намного превышающего m_t, к некоторому масштабу массы μ, который намного меньше m_t, но все еще велик по сравнению с m_u. Довольно часто в качестве грубого приближения любую массу m, меньшую μ, приравнивают нулю, а любую массу m, большую μ, приравнивают к бесконечности (т. е. без вклада в ренорм-групповой поток). В действительности, по мере того как μ приближается m сверху, частица начинает давать меньший вклад и исчезает, когда μ становится меньше m. Изучите этот так называемый пороговый эффект, используя либо $\lambda\varphi^4$-теорию, либо КЭД.

VI.8.3. Покажите, что уравнение (10) является инвариантным относительно так называемого преобразования Галилея

$$h(\vec{x},t) \to h'(\vec{x},t) = h(\vec{x}+g\vec{u}t,t) + \vec{u}\cdot\vec{x} + \frac{g}{2}u^2 t.$$

Покажите, что в силу этой симметрии в (17) появляются только два параметра α и β.

VI.8.4. В выражении для $\widetilde{S}(h)$ могут появляться только производные поля h, а само поле не может. (Поскольку преобразованис имеет вид $h(\vec{x},t) \to h(\vec{x},t)+c$, где константа c соответствует тривиальному сдвигу точки, где измеряется высота поверхности, физический процесс должен быть инвариантным относительно этого преобразования.) Члены, содержащие h в первой степени, не могут появляться, так как все они являются полными производными. Таким образом, выражение $\widetilde{S}(h)$ должно начинаться с членов второй степени по h. Проверьте, что выражение $\widetilde{S}(h)$, приведенное в (17), действительно является наиболее общим. Член, пропорциональный $(\nabla h)^2$, также разрешен симметриями и действительно генерируется. Однако этот член можно устранить путем преобразования к движущейся системе координат $h \to h+ +ct$.

VI.8.5. Покажите, что g имеет размерность (длина $)^{\frac{1}{2}(D-2)}$, известную нам из школьного учебника по физике. [Указание: вид $S(h)$ означает, что t имеет размерность длины в квадрате, и поэтому h имеет размерность (длина) $^{\frac{1}{2}(2-D)}$. Сравнивая члены $\nabla^2 h$ и $\nabla^2 h$, определяем размерность g.]

VI.8.6. Вычислите пропагатор h в однопетлевом приближении. Извлеките коэффициенты членов ω^2 и k^4 в разложении по низким частотам и волновым числам обратного пропагатора и определите α и β.

VI.8.7. Изучите ренорм-групповой поток g для $D = 1, 2, 3$.

30.3.9 Nonequilibrium Field Theory in Matter: Introduction

(An Introduction to Kamenev's book, see note on kinetics for now about it)

30.4 Zi.VII Grand Unification

30.4.1 Zi.VII.1 Quantization of Yang–Mills Theory and Lattice Gauge Theory

Одна из причин, по которой теория Янга-Миллса не была сразу же принята физиками, заключается в том, что они не знали, как в ней проводить вычисления. По крайней мере, следует сформулировать правила Фейнмана и уметь проводить пертурбативные вычисления. Фейнман сам взялся за эту проблему и после рассмотрения различных диаграмм пришел к заключению, что, для того чтобы теория была согласованной, необходимо ввести дополнительные поля с духовыми свойствами. Теперь известно, как получить этот результат, используя более систематический подход.

Рассмотрим чистую теорию Янга-Миллса, а поля материи легко добавить позже. Воспользуемся нашими знаниями. Как обычно, разобьем лагранжиан $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ на две части (мы также переопределяем $A \to gA$):

$$\mathcal{L}_0 = -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2$$

и

$$\mathcal{L}_1 = -\frac{1}{2}g\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) f^{abc} A^{n\mu} A^{c\nu} - \frac{1}{4}g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{r\nu}.$$

Затем обращаем дифференциальный оператор в квадратичной части выражения (1), чтобы получить пропагатор. Эта часть аналогична соответствующей процедуре для квантовой электродинамики, за исключением того, что появляется индекс a. Так же, как в

электродинамике, обратного оператора в нашем случае не существует, поэтому необходимо фиксировать калибровку.

Для квантования квантовой электродинамики я использовал сложный метод Фаддеева-Попова, и, как уже отмсчалось, в рассматриваемом контексте это было несколько излишне. Однако теперь мы от этого только вы- играем. Вспомним из главы III.4, что метод Фаддеева-Попова ведет к результату

$$\mathcal{Z} = \int DAe^{iS(A)}\Delta(A)\delta[f(A)]$$

где $\Delta(A) \equiv \left\{\int Dg\delta\left[f\left(A_g\right)\right]\right\}^{-1}$ и $S(A) = \int d^4x\mathcal{L}$ является действием Янга-Миллса. (Как и в главе III. 4, $A_g \equiv gAg^{-1} - i(\partial g)g^{-1}$ обозначает калибровочное преобразование A. Здесь $g \equiv g(x)$ обозначает элемент группы, который определяет калибровочное преобразование в точке x, и его, конечно, нельзя путать с константой связи.)

Поскольку $\Delta(A)$ в (3) умножается на $\delta[f(A)]$, следует ожидать, что в интеграле по g при разумном выборе $f(a)$ вклад дадут только бесконечно малые g. Пусть $f(A) = \partial A - \sigma$. При инфинитезимальном преобразовании имеет место $A^a_\mu \to A^a_\mu - f^{abc}\theta^b A^c_\mu + \partial_\mu\theta^a$, следовательно:

$$\Delta(A) = \left\{\int D\theta\delta\left[\partial A^a - \sigma^a - \partial^\mu\left(f^{abc}\theta^b A^c_\mu - \partial_\mu\theta^a\right)\right]\right\}^{-1} \ll=$$
$$= \left\{\int D\theta\delta\left[\partial^\mu\left(f^{abc}\theta^b A^c_\mu - \partial_\mu\theta^a\right)\right]\right\}^{-1}$$

где присутствует «фактически знак равенства», поскольку $\Delta(A)$ потом умножается на $\delta[f(A)]$.

Запишем формально

$$\delta^\mu\left(f^{abc}\theta^b A^c_\mu - \delta_\mu\theta^a\right) = \int d^4y K^{ab}(x,y)\theta^b(y)$$

определяя таким образом оператор $K^{ab}(x,y) = \partial^\mu\left(f^{abc}A^c_\mu - \partial_\mu\delta^{ab}\right)\delta^{(4)}(x\text{ - }-y)$. Отметим, что здесь, в отличие от электродинамики, K зависит от калибровочного поля. Простой результат $\int d\theta\delta(K\theta) = 1/K$ для действительных чисел θ и K может быть сведен к выражению $\int d\theta\delta(K\theta) = 1/\det K$ для действительного вектора θ и невырожденной матрицы K. Рассматривая $K^{ab}(x,y)$ как матрицу, получаем $\Delta(A) = \det K$, но из главы II.5 известно, как представить детерминант в виде функционального интеграла по грассмановым переменным. Запишем уравнение $\Delta(A) = \int DcDc^\dagger e^{iS_{.1yx}\left(c^\dagger,c\right)}$, где

$$S_{\text{дух}}\left(c^\dagger,c\right) = \int d^4xd^4yc^\dagger_a(x)K^{ab}(x,y)c_b(y) =$$
$$= \int d^4x\left[\partial c^\dagger_a(x)\partial c_a(x) - \partial^\mu c^\dagger_a(x)f^{abc}A^c_\mu(x)c_b(x)\right] =$$
$$= \int d^4x\partial c^\dagger_a(x)Dc_a(x)$$

а D - ковариантная производная для присоединенного представления, к которому принадлежат поля c_a и $c^\dagger_a$, так же, как A^a_μ. Поля c_a и $c^\dagger_a$ называют духовыми полями, поскольку они нарушают связь спина со статистикой. Хотя эти поля скалярные, их рассматривают как антикоммутирующие. Такое «нарушение» приемлемо, поскольку духовые поля не связаны с физическими частицами и вводятся только для того, чтобы представить $\Delta(A)$ в удобном виде.

Таким образом, вопрос с множителем $\Delta(A)$ в уравнении (3) рсшен. Для множителя $\delta[f(A)]$ используем тот же прием, что и в главе III.4, и интегрируем $\mathcal{Z}$ по $\sigma^a(x)$ с гауссовым весом $e^{-(i/2\xi)\int d^4x\sigma^a(x)^2}$, так что $\delta[f(A)]$ заменяется на $e^{-(i/2\xi)\int d^4x(\partial A^a)^2}$.

Объединяя все вместе, получаем

$$\mathcal{Z} = \int DADcDc^\dagger e^{iS(A)-(i/2\xi)\int d^4x(\partial A)^2+iS_{\text{Iyx}}\left(c^\dagger,c\right)},$$

где ξ - параметр калибровки. Из сравнения с соответствующим выражением для абелевой калибровочной теории, приведенным в главе III.4, следует, что в неабелевых калибровочных теориях в дополнение к действию ЯнгаМиллса есть действие для духов $S_{\text{дух}}$. Таким образом, $\mathcal{L}_0$ и $\mathcal{L}_1$ изменяются на

$$\mathcal{L}_0 = -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 - \frac{1}{2\xi}\left(\partial^\mu A^a_\mu\right)^2 + \partial c^\dagger_a \partial c_a$$

и

$$\mathcal{L}_1 = -\frac{1}{2}g\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) f^{abc} A^{b\mu} A^{c\nu} + \frac{1}{4}g^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu} -$$
$$- \partial^\mu c^\dagger_a g f^{abc} A^c_\mu c_b$$

Теперь из (8) можно сразу получить пропагаторы для калибровочного бозона и духового поля. В частности, легко видеть, что за исключением группового индекса a квадратичные по калибровочному полю члены являются такими же, как и квадратичные по электромагнитному полю члены в уравнении (III.4.8). Таким образом, пропагатор калибровочного бозона имеет вид

$$\frac{(-i)}{k^2}\left[g_{\nu\lambda} - (1-\xi)\frac{k_\nu k_\lambda}{k^2}\right]\delta_{ab}.$$

Сравним с выражением (III.4.9). Из члсна $\partial c^\dagger_a(x)\partial c_a(x)$ в уравнении (8) находим, что духовой пропагатор имеет вид $(i/k^2)\,\delta_{ab}$. Из $\mathcal{L}_1$ следует, что между калибровочными бозонами существует взаимодействие третей и четвертой степени, а также взаимодействие между калибровочным бозоном и духовым полем, как показано на рис. VII.1.1. Взаимодействия третьей и четвертой степени можно легко представить в виде

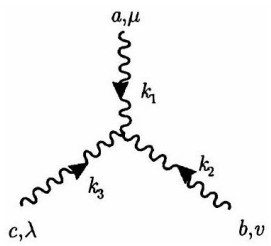
$$g f^{abc}\left[g_{\mu\nu}\left(k_1-k_2\right)_\lambda + g_{\nu\lambda}\left(k_2-k_3\right)_\mu + g_{\lambda\mu}\left(k_3-k_1\right)_\nu\right]$$

и

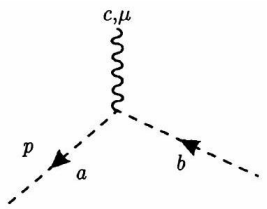
$$-ig^2\left[f^{abe}f^{cde}\left(g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}\right) + f^{ade}f^{cbe}\left(g_{\mu\lambda}g_{\nu\rho}-g_{\mu\nu}g_{\rho\lambda}\right) + \right.$$
$$\left. + f^{ace}f^{bde}\left(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\rho}g_{\nu\lambda}\right)\right]$$

соответственно. Взаимодействие с духовым полем имеет вид

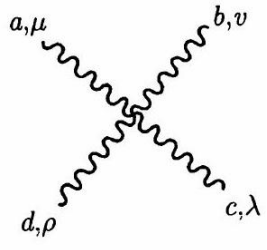
$$g f^{abc} p^\mu$$



(a)



(c)



(b)
Рис. VII.1.1

Очевидно, что для этой записи справедливы различные перестановочные симметрии. Например, второй член в (12) получается из первого путем пе- рестановки $\{c, \lambda\} \leftrightarrow \{d, \rho\}$, а третий и четвертый члены получаются из первого и второго путем перестановки $\{a, \mu\} \rightarrow \{c, \lambda\}$.

Неестественный шаг

В теории с высокой симметрией, такой как теория Янга-Миллса, теория возмущений, очевидно, является неестественным действием, так как оно включает грубое расщепление $\mathcal{L}$ на две части: квадратичную относительно полей и остальную часть. Рассмотрим, например, точно решаемую квантово-механическую задачу для одной частицы, такую как уравнение Шредингера с $V(x) = 1 - (1/\operatorname{ch} x)^2$. Представим себе запись $V(x) = \frac{1}{2}x^2 + + W(x)$ и рассмотрим $W(x)$ как возмущение гармонического осциллятора. Воспроизводить точный спектр было бы очень трудно, но именно так безжалостно мы обходимся с теорией Янга-Миллса в пертурбативном подходе. Мы взяли «целостный элемент» $\operatorname{tr} F_{\mu\nu}F^{\mu\nu}$ и разделили его на «гармонический осциллятор» $\operatorname{tr}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2$ и «возмущение».

Если когда-либо окажется, что теория Янга-Миллса точно решаема, то точно не в рамках пертурбативного подхода, не сохраняющего калибровочную инвариантность.

Калибровочная теория на решетке

Вильсон предложил выход из положения: нарушать лоренц-инвариантность, а не калибровочную инвариантность. Сформулируем теорию ЯнгаМиллса на гиперкубической решетке в четырехмерном евклидовом пространстве-времени. Поскольку период решетки $a \rightarrow 0$, мы надеемся восстановить инвариантность относительно четырехмерных поворотов и (путем виковского поворота) лоренц-инвариантность. Формулировку Вильсона, известную как калибровочная теория на решетке, понять легко, но ее запись несколько громоздкая из-за отсутствия вращательной инвариантности. Обозначим положение узлов решетки вектором x_i. Каждому ребру, скажем от x_i до ближайшего соседнего x_j, ставим в соответствие унитарную $N \times N$-матрицу U_{ij}. Рассмотрим квадрат, называемый плакетом, который ограничен четырьмя вершинами x_i, x_j, x_k и x_l (ближайшими соседними друг к другу). Посмотрим на рис. VII.1.2. Каждому плакету P поставлена в соответствие величина $S(P) = \operatorname{Re}\operatorname{tr} U_{ij}U_{jk}U_{kl}U_{li}$, построенная так, чтобы быть инвариантной относительно локального преобразования

$$U_{ij} \to V_i^\dagger U_{ij} V_j$$

Симметрия является локальной, поскольку каждому узлу x_i можно поставить в соответствие независимую V_i.

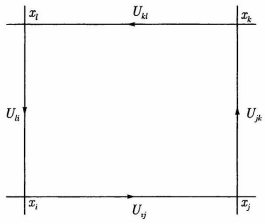


Рис. VII.1.2

Вильсон определил теорию Янга-Миллса в следующим виде:

$$\mathcal{Z} = \int \Pi dU^{\left(1/2f^2\right)\sum\limits_P S(P)}$$

где сумма берется по всем плакетам в решетке. Константа связи f определяет, насколько сильно флуктуируют унитарные матрицы U_{ij}. Для малых f предпочтительны большие значения $S(P)$, и поэтому все U_{ij} приближенно равны единичной матрице (с точностью до несущественного глобального преобразования).

Без всяких вычислений, из симметрийных соображений можно показать, что в непрерывном пределе $a \to 0$ должна появиться теория ЯнгаМиллса такая, какой мы ее знаем. Это действие явно инвариантно относительно локальных $SU(N)$-преобразований. Чтобы действительно это увидеть, определим поле $A_\mu(x)$ с $\mu = 1, 2, 3, 4$, живущим в четырехмерном евклидовом пространстве, где находится решетка, уравнением

$$U_{ij} = V_i^\dagger e^{iaA_\mu(x)} V_j$$

где $x = \frac{1}{2}(x_i + x_j)$ (это средняя точка ребра, где находится U_{ij}) и μ - направление, связывающее x_i с x_j (то есть $\widehat{\mu} \equiv (x_j - x_i)/a$ является единичным вектором в направлении μ). Параметры V отражают только калибровочную свободу в (14) и по построению, конечно, не входят в действие на плакете $S(P)$. Я вас попрошу показать в качестве упражнения, что

$$U_{ij}U_{jk}U_{kl}U_{li} = e^{ia^2F_{\mu\nu}+O\left(a^3\right)},$$

где $F_{\mu\nu}$ - напряженность поля Янга-Миллса, определенная в центре плакета. Мы действительно могли бы таким образом найти напряженность поля Янга-Миллса. Я надеюсь, что вы начинаете понимать глубокое геометрическое значение $F_{\mu\nu}$. Продолжая решать упражнение, вы найдете, что действие на каждом плакете имеет вид

$$\begin{aligned} S(P) &= \operatorname{Re}\operatorname{tr} e^{ia^2F_{\mu\nu}+O\left(a^3\right)} = \\ &= \operatorname{Re}\operatorname{tr}\left[1 + ia^2F_{\mu\nu} - \frac{1}{2}a^4F_{\mu\nu}F_{\mu\nu} + O\left(a^5\right)\right] = \operatorname{tr} 1 - \frac{1}{2}a^4 \operatorname{tr} F_{\mu\nu}F_{\mu\nu} + ..., \end{aligned}$$

и, таким образом, с точностью до несущественной аддитивной постоянной мы восстанавливаем в (15) действие Янга-Миллса в непрерывном пределе. Еще раз следует подчеркнуть, что без всяких вычислений можно зафиксировать в (18) член a^4 (с точностью до общего множителя) из анализа размерностей и калибровочной инвариантности [1].

Формулировка Вильсона хороша тем, что, для того чтобы (15) имело смысл, не нужна ни фиксация калибровки, ни детерминант Фаддеева-Попова, ни духовые поля и прочее. Вспоминая главу V.3, вы видите, что (15) определяет как одну из задач статистической механики. Вместо интегрирования, скажем, по некоторым спиновым переменным,

проинтегрируем по группе $SU(N)$ для каждого ребра. Важнее всего то, что формулировка калибровочной теории на решетке открывает возможность численного определения свойств сильно нетривиальной квантовой теории поля. Калибровочная теория на решетке является бурно развивающейся областью исследований. Попробуйте свои силы и попытайтесь включить фермионы в калибровочную теорию на решетке. Это весьма непростая задача, которая стоит на повестке дня, поскольку фермионы и спинорные поля естественным образом связаны с $SO(4)$, которая не очень хорошо переносится на решетку.

Петля Вильсона

Специалисты по теории поля обычно имеют дело с локальными наблюдаемыми, то есть наблюдаемыми, которые определены в точке x про-

[1] Знак можно легко проверить на соответствие абелеву случаю. странства-времени, такими как $J^\mu(x)$ или $\operatorname{tr} F_{\mu\nu}(x)F^{\mu\nu}(x)$. Но можно, конечно, также иметь дело с нелокальными наблюдаемыми, такими как $e^{i\oint dx^\mu A_\mu}$ в электродинамике, где интеграл вычисляется по замкнутой кривой C. Калибровочно-инвариантная величина в экспоненте равна электромагнитному потоку, проходящему через поверхность, ограниченную кривой C. (В самом деле, вспомните главу IV.4.)

Вильсон показал, что калибровочная теория на решетке содержит естественную калибровочно-инвариантную, но нелокальную наблюдаемую величину, равную $W(C) \equiv \operatorname{tr} U_{ij}U_{jk}...U_{nm}U_{mi}$, где набор звеньев, соединяющих x_i с x_j, x_j с x_k и т. д., и, наконец, с x_m и вновь с x_i, очерчивает петлю, называемую C. Возвращаясь к выражению (16), видим, что $W(C)$, называемая петлей Вильсона, является следом произведения многих множителей e^{iaA_μ}. Таким образом, в непрерывном пределе $a \to 0$ очевидным образом получаем

$$W(C) \equiv \operatorname{tr} P e^{i\oint dx^\mu A_\mu}$$

где C теперь является произвольной кривой в евклидовом пространстве-времени. Здесь P обозначает упорядоченность траектории, что, конечно, необходимо, поскольку A_μ, связанные с различными сегментами кривой C, будучи матрицами, не коммутируют друг с другом. [Действительно, P определяется заданием решетки $W(C)$.]

Чтобы понять физический смысл петли Вильсона, обратимся к главам I.4 и I.5. Для получения потенциальной энергии E между двумя противоположно заряженными тяжелыми частицами необходимо вычислить

$$\lim_{T\to\infty} \frac{1}{\mathcal{Z}} \int DA e^{iS_{\text{Maxwell}}(A) + i\int d^4x A_\mu J^\mu} = e^{iET}.$$

Для двух частиц, находящихся на расстоянии R друг от друга, подставляем

$$J^\mu(x) = \eta^{\mu 0}\left\{\delta^{(3)}(\vec{x}) - \delta^{(3)}[\vec{x} - (R,0,0)]\right\}$$

и видим, что на самом деле мы вычисляем вакуумное среднее $\left\langle e^{i\left(\int\limits_{C_1} dx^\mu A_\mu - \int\limits_{C_2} dx^\mu A_\mu\right)} \right\rangle$ во флуктуирующем электромагнитном поле, где C_1 и C_2 обозначают два прямолинейных сегмента при $\vec{x} = (0,0,0)$ и $\vec{x} = (R,0,0)$ соответственно. Удобно представить сближение двух частиц в далеком будущем (и аналогично в далеком прошлом). Тогда мы явно имеем дело с калибровочно-инвариантной величиной $\left\langle e^{i\oint dx^\mu A_\mu}\right\rangle$, где C является прямоугольником, как показано на рис. VII.1.3. Заметим, что для большого T имеет место $\log\left\langle e^{i\oint dx^\mu A_\mu}\right\rangle \sim iE(R)T$, который существенно пропорционален периметру прямоугольника C.

Как будет рассмотрено в главе VII.3 и как вы, несомненно, слышали, принятая в настоящее время теория сильного взаимодействия включает кварки, взаимодействующие с неабелевым калибровочным полем Янга-Миллса A_μ. Таким образом, чтобы определить

потенциальную энергию $E(R)$ между кварком и антикварком, удерживаемых на расстоянии R друг от друга, нужно просто вычислить вакуумное среднее петли Вильсона

$$\langle W(C)\rangle = \frac{1}{\mathcal{Z}}\int \Pi dU e^{-\left(1/2f^2\right)\sum_P S(P)} W(C).$$

В калибровочной теории на решетке можно численно определить $\log\langle W(C)\rangle$ для большого прямоугольника C на рис. VII.1.3 и извлечь $E(R)$. (Мы потеряли i, потому что в рассматриваемом случае находимся в евклидовом пространстве-времени.)

Рис. VII.1.3

Конфайнмент кварков

Я уверен, вы слышали, что, поскольку свободные кварки никто никогда не видел, полагают, что они постоянно удерживаются. В частности, предполагается, что потенциальная энергия между кварком и антикварком растет линейно с расстоянием $E(R) \sim \sigma R$. Представим себе струну, соединяющую кварк и антикварк с натяжением струны σ. Если это правильное представление, то соотношение $\log\langle W(C)\rangle \sim \sigma RT$ должно иметь место, когда область RT ограничивается петлей C. Вильсон называет такое поведение законом площадей, а не законом периметров, характерным для известных теорий, таких как электродинамика. Доказательство закона площадей в теории Янга-Миллса являстся одной из выдающихся нерешенных проблем теоретической физики.

Упражнения

VII.1.1. Выведите (17) и сопоставьте f с константой связи g в формулировке теории Янга-Миллса. [Указание: используйте формулу Бейкера-КэмпбеллаХаусдорфа

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]])+[B,(B,A]])+\cdots}.\Big]$$

VII.1.2. Рассмотрите калибровочную теорию на решетке в $(D+1)$-мерном пространстве с периодом решетки a в D-мерном пространстве и b в дополнительном измерении. Получите D-мерную теорию поля непрерывного спектра в пределе $a \to 0$ при фиксированном b.

VII.1.3. Изучите в выражении (2) альтернативный предел $b \to 0$ при фиксированном a и получите тем самым теорию на пространственной решетке, но с непрерывным временем.

VII.1.4. Покажите, что для калибровочной теории на решетке закон площадей Вильсона сохраняется в пределе сильной связи. [Указание: разложите выражение (20) по степеням f^{-2}.]

30.4.2 Zi.VII.2 Electroweak Unification

Бич безмассовых частиц со спином 1

Учитывая опыт прошлого, мы теперь знаем, что Природа предпочитает теорию Янга-Миллса. В конце 1960-х и начале 1970-х годов электромагнитное и слабое взаимодействия были объединены в электрослабое взаимодействие, описываемое неабелевой калибровочной теорией, которая основана на группе $SU(2) \otimes U(1)$. Несколько позже, в начале 1970 - х годов, стало понятно, что сильное взаимодействие можно описать, пользуясь неабелевой

калибровочной теорией, основанной на группе $SU(3)$. Природа буквально состоит из сети взаимодействующих полей Янга-Миллса.

Но когда в 1954 году эта теория была впервые предложена, она казалась совершенно несовместимой с наблюдениями в интерпретации того времени. Янг и Миллс сами показали в своей статье, что их теория содержит безмассовые частицы со спином 1, которые, конечно, не были экспериментально известны. Поэтому эта теория постепенно была предана забвению и в 1960-х годах не входила в обязательную университетскую программу по физике элементарных частиц. Исключение составляли только некоторые теоретики (Швингер, Глэшоу, Бладмен и др.), которые считали такую изящную математическую структуру привлекательной и были убеждены, что неабелева калибровочная теория должна иметь какое-то отношение к слабому взаимодействию.

И опять же, оглядываясь назад, мы, по-видимому, можем найти только два логических объяснения того, почему экспериментаторы не видят никаких безмассовых частиц со спином 1 за исключением фотона: (1) частицы Янга-Миллса каким-то образом приобретают массу, или (2) частицы Янга-Миллса действительно являются безмассовыми, но почему-то не могут наблюдаться в экспериментах. Теперь мы знаем, что первая возможность была реализована в электрослабом взаимодсйствии, а вторая - в сильном взаимодействии.

Построение теории электрослабых взаимодействий

Теперь рассмотрим электрослабое объединение. Возможно, с педагогической точки зрения следует мотивировать шаги, приводящие к построению этой теории. Но, как отмечено выше, это не учебник по физике элементарных частиц, и обсуждение этого предмета я вынужден свести к самому минимуму. В главе IV.2 дано краткое введение в структуру слабого взаимодействия. Как говорилось в главе II.1, существенным обстоятельством здесь является то, что слабое взаимодействие нарушает четность. В частности, левое поле электрона e_L и правое поле электрона e_R, которые переходят друг в друга при преобразовании четности, участвуют в слабом взаимодействии совсем по-разному.

Начнем со слабого распада мюона $\mu^- \to e^- + \bar{\nu} + \nu'$, где ν и ν' - электронное нейтрино и мюонное нейтрино соответственно. Подходящим членом в лагранжиане является $\bar{\nu}'_L \gamma^\mu \mu_L \bar{e}_L \gamma_\mu \nu_L$, где e_L - левое поле электрона, ν_L - поле электронного нейтрино (которое является левым) и т. д. Поле μ_L уничтожает мюон, поле $\bar{e}_L$ создает электрон и т. д. (Впредь будем опускать слово «поле».) Вам, возможно, известно, что элементарные составляющие материи образуют три семейства, первое из которых состоит из двух лептонов $-\nu, e$ и двух кварков - верхнего u и нижнего d, второе семейство - из двух лептонов ν', μ и двух кварков - очарованного c и странного s, и т. д. Для наших целей ограничимся рассмотрением первого семейства. Итак, начнем с $\bar{\nu}_L \gamma^\mu e_L \bar{e}_L \gamma_\mu \nu_L$.

Как говорилось в главе III.2, подобного рода взаимодействие Ферми может порождаться в результате обмена промежуточным векторным бозоном W^+с взаимодействием $W^+_\mu \bar{\nu}_L \gamma^\mu e_L + W^-_\mu \bar{e}_L \gamma_\mu \nu_L$.

Идея состоит в том, чтобы рассмотреть калибровочную теорию $SU(2)$ с триплетом калибровочных бозонов, которые обозначим W^a_μ, где $a = 1, 2, 3$. Поместим ν_L и e_L в дублетное представление, а правое поле электрона e_R в синглетное представление, тогда

$$\psi_L \equiv \begin{pmatrix} \nu \\ e \end{pmatrix}_L, \quad e_R$$

(В этой записи верхней компонентой ψ_L является ν_L, а нижней компонентой e_L.)

Поля ν_L и e_L, но не поле e_R взаимодействуют с калибровочными бозонами W^a_μ. Действительно, согласно (IV.5.21) лагранжиан содержит

$$W^a_\mu \bar{\psi}_L \tau^a \gamma^\mu \psi_L = \left(W^{1-i2}_\mu \bar{\psi}_L \frac{1}{2} \tau^{1+i2} \gamma^\mu \psi_L + \text{ э.с. } \right) + W^3_\mu \bar{\psi}_L \tau^3 \gamma^\mu \psi_L$$

где $W^{1-i2}_\mu \equiv W^1_\mu - iW^2_\mu$ и т. д. Мы видим, что $\tau^{1+i2} \equiv \tau^1 + i\tau^2$ является повышающим

оператором, а первые два слагаемых равны $\left(W_\mu^{1-i2}\bar{\nu}_L\gamma^\mu e_L+\ +\ \text{э. с.}\right)$, именно то, что нам нужно. По замыслу обмен полем $W_\mu^\pm$генерирует искомый член $\bar{\nu}_L\gamma^\mu e_L\bar{e}_L\gamma_\mu\nu_L$.

Нам нужно больше места

Мы надеялись, что бозон W^3, который было необходимо ввести, окажется фотоном, так что будет включен электромагнетизм. Но увы, W^3 взаимодействует с током $\bar{\psi}_L\tau^3\gamma^\mu\psi_L = (\bar{\nu}_L\gamma^\mu\nu_L - \bar{e}_L\gamma_\mu e_L)$, а не с электромагнитным током $-(\bar{e}_L\gamma_\mu e_L + \bar{e}_R\gamma_\mu e_R)$. Проблема!

Есть еще одна проблема. Чтобы генерировать массовый член для электрона, нужно иметь дублет полей Хигтса $\varphi \equiv \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix}$ для построения $SU(2)$-инвариантного члена $f\bar{\psi}_L\varphi e_R$ в лагранжиане, так что, когда φ приобретет вакуумное среднее $\begin{pmatrix} 0 \\ v \end{pmatrix}$, мы получим

$$\bar{\psi}_L\varphi e_R \to f(\bar{\nu},\bar{e})_L\begin{pmatrix} 0 \\ v \end{pmatrix}e_R = fv\bar{e}_L e_R$$

Но ни одно из преобразований $SU(2)$ не оставляет $\begin{pmatrix} 0 \\ v \end{pmatrix}$ инвариантным. Вакуумное среднее поля φ спонтанно нарушает симметрию $SU(2)$, оставляя массивными все три бозона W. В этой несовершенной теории нет места для фотона. Вот в чем дело!

Нам нужно больше места. Замечательно, что обе проблемы можно избежать путем расширения калибровочной симметрии до $SU(2)\otimes U(1)$. Обозначая генератор $U(1)$ как $\frac{1}{2}Y$ (называемый гиперзарядом), соответствующий калибровочный потенциал как B_μ (а их эквивалентные величины для $SU(2)$ как T^a и W_μ^a), получаем ковариантную производную $D_\mu = \partial_\mu$ - $-igW_\mu^aT^a - ig'B_\mu\frac{Y}{2}$. Имея четыре калибровочных бозона, можно надеяться, что один из них окажется фотоном.

Калибровочные поля нормируются с помощью соответствующих членов кинетической энергии в $\mathcal{L} = -\frac{1}{4}(B_{\mu\nu})^2 - \frac{1}{4}\left(W_{\mu\nu}^a\right)^2 + ...$, где $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ - напряженность абелева поля и $W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + +\varepsilon^{abc}W_\mu^bW_\nu^c$ - напряженность неабелева поля. Генераторы T^a, естественно, нормируются с помощью коммутационных соотношений, которые определяют $SU(2)$. В абелевой алгебре $U(1)$, напротив, нет коммутационного соотношения, чтобы зафиксировать нормировку генератора $\frac{1}{2}Y$. Пока она не зафиксирована, не зафиксирована и нормировка $U(1)$-калибровочного взаимодействия g'.

Как же зафиксировать нормировку генератора $\frac{1}{2}Y$? Мы хотим, чтобы, по построению, спонтанное нарушение симметрии оставляло линейную комбинацию T_3 и $\frac{1}{2}Y$ инвариантной и чтобы эта комбинация отождествлялась с генератором, взаимодействующим с безмассовым фотоном, а именно с оператором заряда Q. Итак, запишем

$$Q = T_3 + \frac{1}{2}Y$$

Если мы знаем T_3 и $\frac{1}{2}Y$ для произвольного поля, то из этого уравнения мы знаем его заряд. Например, $Q(\nu_L) = \frac{1}{2} + \frac{1}{2}Y(\nu_L)$ и $Q(e_L) = -\frac{1}{2} + \frac{1}{2}Y(e_L)$. В частности, легко видеть, что коэффициент при T_3 в уравнении (3) должен быть равен 1, поскольку заряды ν_L и e_L отличаются на 1. Соотношение (3) фиксирует нормировку $\frac{1}{2}Y$.

Чему равен гиперзаряд?

Следующий шаг заключается в определении гиперзаряда различных мультиплетов в теории, что в свою очередь определяет то, как с этими мультиплетами взаимодействует B_μ.

Рассмотрим дублет ψ_L. Чтобы получить заряд -1 для поля e_L, дублет ψ_L должен иметь $\frac{1}{2}Y = -\frac{1}{2}$. В противоположность этому поле e_R имеет $\frac{1}{2}Y = -1$, поскольку $T_3 = 0$ для e_R.

Имея гиперзаряд дублета ψ_L и поля e_R, легко видеть, что инвариантность члена $f\bar{\psi}_L\varphi e_R$ относительно $SU(2) \otimes U(1)$ заставляет поле Хиггса φ иметь $\frac{1}{2}Y = +\frac{1}{2}$. Таким образом, согласно (3) верхняя компонента поля φ имеет электрический заряд $Q = +\frac{1}{2} + \frac{1}{2} = +1$, а нижняя компонента $Q = -\frac{1}{2} + \frac{1}{2} = 0$. Итак, запишем $\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix}$. Напомним, что поле φ имеет вакуумное среднее $\begin{pmatrix} 0 \\ v \end{pmatrix}$. Утверждение о том, что электрически нейтральное поле φ^0 приобретает вакуумное среднее, а заряженное поле φ^+нет, является проверкой на согласованность теории.

Теория создается сама

Теперь, когда определены взаимодействия калибровочных бозонов с различными полями, в частности, с полем Хигтса, можно легко определить спектр масс калибровочных бозонов, что (позвольте напомнить) вы уже делали при решении упражнения IV.6.3!

При спонтанном нарушении симметрии $\varphi \to (1/\sqrt{2}) \begin{pmatrix} 0 \\ v \end{pmatrix}$ (обычная нормировка): просто подставим в

$$\mathcal{L} = (D_\mu\varphi)^\dagger (D^\mu\varphi) \to \frac{g^2v^2}{4} W_\mu^+ W^{-\mu} + \frac{v^2}{8} \left(gW_\mu^3 - g'B_\mu\right)^2.$$

Полагаю, именно такой результат вы и получили! Таким образом, линейная комбинация $gW_\mu^3 - g'B_\mu$ становится массивной, тогда как ортогональная комбинация остается безмассовой и ее можно отождествить с фотоном. Разумеется, угол θ удобно определить с помощью $\operatorname{tg}\theta = g'/g$. Тогда уравнение

$$Z_\mu = \cos\theta W_\mu^3 - \sin\theta B_\mu$$

описывает массивный калибровочный бозон, называемый Z-бозоном, а электромагнитное поле задается выражением $A_\mu = \sin\theta W_\mu^3 + \cos\theta B_\mu$. Объединим (4) и (5) и проверим, что возведенная в квадрат масса Z-бозона имеет вид $M_z^2 = v^2 (g^2 + g^2)/4$; таким образом, используя элементарную тригонометрию, получаем соотношение

$$M_W = M_Z \cos\theta.$$

Обмен W-бозоном порождает слабое взаимодействие Ферми

$$\mathcal{L} = -\frac{g^2}{2M_W^2} \bar{\nu}_L\gamma^\mu e_L \bar{e}_L\gamma_\mu\nu_L = -\frac{4G}{\sqrt{2}} \bar{\nu}_L\gamma^\mu e_L \bar{e}_L\gamma_\mu\nu_L$$

где второе равенство просто является историческим определением взаимодействия Ферми G. Таким образом,

$$\frac{G}{\sqrt{2}} = \frac{g^2}{8M_W^2}.$$

Далее запишем соответствующую часть ковариантной производной

$$gW_\mu^3 T^3 + {}'B_\mu \frac{Y}{2} = g\left(\cos\theta Z_\mu + \sin\theta A_\mu\right) T^3 + g'\left(-\sin\theta Z_\mu + \cos\theta A_\mu\right) \frac{Y}{2}$$

в терминах физически наблюдаемых Z и A. Коэффициент при A_μ преобразуется к виду $g\sin\theta T^3+g'\cos\theta(Y/2)=g\sin\theta\,(T^3+Y/2)$; тот факт, что возникает комбинация $Q=T^3+Y/2$, является прекрасной проверкой формализма. Кроме того, получаем

$$e=g\sin\theta$$

Между тем удобно записать $g\cos\theta T^3-g'\sin\theta(Y/2)$ - коэффициент при Z_μ в ковариантной производной - через физически знакомый электрический заряд Q, а не теоретический гиперзаряд Y. Таким образом,

$$g\cos\theta T^3-g'\sin\theta\left(Q-T^3\right)=\frac{g}{\cos\theta}\left(T^3-\sin^2\theta Q\right).$$

Другими словами, мы определили взаимодействие Z-бозона с произволь-

ным фермионным полем Ψ в теории:

$$\mathcal{L}=\frac{g}{\cos\theta}Z_\mu\bar{\Psi}\gamma^\mu\left(T^3-\sin^2\theta Q\right)\Psi$$

Используя (9), можно, например, сразу же описать взаимодействие Z-бозона с лептонами:

$$\mathcal{L}=\frac{g}{\cos\theta}Z_\mu\left[\frac{1}{2}\left(\bar{\nu}_L\gamma^\mu\nu_L-\bar{e}_L\gamma^\mu e_L\right)+\sin^2\theta\bar{e}\gamma^\mu e\right]$$

Включение кварков

Теперь становится очевидным, как включить адроны. Учитывая, что в слабом взаимодействии принимают участие только левые поля, поместим кварки первого поколения в мультиплеты $SU(2)\otimes U(1)$ следующим обра30м:

$$q_L^\alpha\equiv\begin{pmatrix}u^\alpha\\d^\alpha\end{pmatrix}_L,\quad u_R^\alpha,d_R^\alpha$$

гдс $\alpha=1,2,3$ обозначает цветовой индекс, который обсуждается в следующей главе. Правые кварки u_R^α и d_R^α помещены в синглеты, чтобы промежуточные W^a-бозоны с ними не взаимодействовали. Напомним, что верхний u-кварк и нижний d-кварк имеют электрические заряды $\frac{2}{3}$ и $-\frac{1}{3}$ соответственно. Возвращаясь к соотношению (3), мы видим, что $\frac{1}{2}Y=\frac{1}{6}$, $\frac{2}{3}$ и $-\frac{1}{3}$ для q_R^α, u_R^α и d_R^α соответственно. Из (9) можно сразу получить взаимодействие Z-бозона с кварками:

$$\mathcal{L}=\frac{g}{\cos\theta}Z_\mu\left[\frac{1}{2}\left(\bar{u}_L\gamma^\mu u_L-\bar{d}_L\gamma^\mu d_L\right)-\sin^2\theta J_{em}^\mu\right]$$

Наконец, постарайтесь самостоятельно проверить, что из четырех степеней свободы, заключенных в φ (поскольку φ^+и φ^0 комплексны), три «съедаются» W - и Z-бозонами, и остается одна физическая степень свободы H, относящаяся к неуловимой хиггсовской частице, которую экспериментаторы все еще ищут на момент написания этой книги.

Нейтральный ток

Благодаря изящно экономной структуре калибровочной группы, электрослабая $SU(2)\otimes U(1)$-теория Глэшоу, Салама и Вайнберга привела к последней великой эре предсказаний в теоретической физике элементарных частиц. Записывая (10) и (12) в виде

$$\mathcal{L}=\frac{g}{\cos\theta}Z_\mu\left(J_{\text{пегтоны}}^\mu+J_{\text{кварки}}^\mu\right)$$

и используя (6), мы видим, что обмен Z-бозоном порождает до сих пор неизвестное взаимодействие нейтральных токов

$$\mathcal{L}_{\text{нейтральный ток}} = -\frac{g^2}{2M_W^2}\left(J_{\text{лептоны}} + J_{\text{кварки}}\right)^{\mu}\left(J_{\text{лептоны}} + J_{\text{кварки}}\right)_{\mu}$$

между лептонами и кварками. Изучая различные процессы, описываемые нейтральными токами $\hat{C}_{\text{нейтр. ток}}$, можно определить слабый угол θ. Если угол θ определен, из уравнения (8) можно предсказать g. Если определена g, из (7) можно предсказать M_W. Если определена M_W, из (6) можно иредсказать M_Z.

Заключение

Как я упомянул, в Природе существуют три поколения лептонов и кварков, состоящие из (ν_e, e, u, d), (ν_μ, μ, c, s) и (ν_τ, τ, t, b) соответственно. Появление этой повторяющейся структуры поколений, о которой $SU(2) \otimes U(1)$-теория ничего не может сказать, представляет одну из важных нерешенных задач физики элементарных частиц. Эти три семейства просто встраиваются в теорию с соответствующими углами поворота между собой путем повторения того, что записано выше.

Более логический подход, по сравнению с приведенным выше, заключается в том, чтобы начать с $SU(2) \otimes U(1)$-теории с дублетным полем Хиггса, имеющим некоторый гиперзаряд, и сказать: «Заметьте, что при спонтанном нарушении симметрии одна линейная комбинация генераторов остается ненарушенной с соответствующим безмассовым калибровочным полем». Но я думаю, что наш квазиисторический подход яснее.

Как уже неоднократно упоминалось, теория слабых взаимодействий Ферми является неперенормируемой. В 1999 году 't Хоофт и Вельтман получили Нобелевскую премию за доказательство того, что электрослабая $SU(2) \otimes U(1)$-теория является перенормируемой, проложив тем самым путь к торжеству неабелевых калибровочных теорий при описании сильных, электромагнитных и слабых взаимодействий. Я не могу здесь вдаваться в подробности их доказательства, но хочу отметить, что ключ к нему лежит в следующем: надо начинать с неабелева аналога унитарной калибровки (вспомним главу IV.6) и перейти к калибровке R_ξ. При больших импульсах пропагаторы массивного калибровочного бозона в унитарной калибровке ведут себя как $\sim (k_\mu k_\nu/k^2)$, а в калибровке R_ξ - как $\sim (1/k^2)$. Тогда из подсчета степеней следует что теория перенормируема.

Упражнения

VII.2.1. К сожалению, масса неуловимой хиггсовской частицы H зависит от параметров двухъямного потенциала $V = -\mu^2\varphi^\dagger\varphi + \lambda\left(\varphi^\dagger\varphi\right)^2$, ответственного за спонтанное нарушение симметрии. Полагая, что H достаточно массивна, чтобы распасться на $W^+ + W^-$и $Z + Z$, определите скорость распада H на различные моды.

VII.2.2. Покажите, что можно оставаться с калибровочной группой $SU(2)$ и отождествлять W^3 с фотоном A за счет введения некоторых экспериментально ненаблюдаемых лептонных полей. Эта теория не описывает наш мир прежде всего потому, что в нее невозможно включить кварки. Покажите это! [Указание: поместите лептоны в триплет $SU(2)$ вместо дублета.]

30.4.3 Zi.VII.3 Quantum Chromodynamics

Кварки

Существуют шесть типов кварков - верхний, нижний, странный, очарованный, боттом и топ, которые обозначаются как u, d, s, c, b и t. Например, протон состоит из двух верхних кварков и нижнего кварка $\sim (uud)$, тогда как нейтральный пион соответствует $\sim (u\bar{u} - d\bar{d})/\sqrt{2}$. Детали можно найти в любом учебнике по физике элементарных частиц.

К концу 1960-х годов понятие кварков стало широко принятым, но существовали два независимых друг от друга свидетельства, указывающие на отсутствие важного элемента.

Когда установили, что адроны состоят из кварков, стало понятно, что волновая функция кварков в нуклоне не оказывается антисимметричной относительно перестановки любой пары кварков, как того требует принцип запрета Паули. Примерно в то же время также стало понятно, что в идеальном мире, который используется для получения соотношения Голдбергера-Треймана и где пион является безмассовым, можно вычислить скорость распада для процесса $\pi^0 \to \gamma + \gamma$, как упомянуто в главе IV.7. Довольно странно, что при этом вычисленная скорость оказалась в $9 = 3^2$ раз меньше экспериментальной.

Обе загадки можно было решить одним махом, наделяя кварки до этого неизвестной внутренней степенью свободы, которую Гелл-Манн назвал цветом. Кварк любого типа имеет один из трех цветов. Таким образом, верхний кварк может быть красным, синим или желтым. Тогда в нуклоне волновая функция трех кварков будет содержать множитель, относящийся к цвету, помимо множителей, относящихся к орбитальному движению, спину и т. д. Нужно только сделать цветовую часть волновой функции антисимметричной; на самом деле просто взять ее в виде $\varepsilon^{\alpha\beta\gamma}$, где α, β и γ обозначают цвета, которые несут три кварка. Имея кварки трех цветов, нужно умножить амплитуду π^0-распада на множитель 3, тем самым точно устраняя расхождение между теорией и экспериментом.

Асимптотическая свобода

Как говорилось в главе VI.6, в результате изучения глубоко неупругого рассеяния электронов на нуклонах был получен важный ключ к пониманию. Экспериментаторы сделали любопытное открытие, что при сильном ударе кварки в нуклонах ведут себя так, как будто они почти не взаимодействуют друг с другом; другими словами, как будто они свободны. С другой стороны, поскольку кварки никогда не наблюдаются как изолированные объекты, они оказываются тесно связанными друг с другом в нуклоне. Как я объяснил, такое загадочное и явно противоречивое поведение кварков можно понять, если константа связи сильного взаимодействия стремится к нулю в пределе большого импульса (ультрафиолетового) и к бесконечности или по крайней мере к некоторому большому значению в пределе малого импульса (инфракрасного). Ряд теоретиков предложили поискать теории, в которых связи стремились бы к нулю в ультрафиолетовом пределе. В настоящее время такие теории называют асимптотически свободными. Со временем Гросс, Вильчек и Политцер обнаружили, что теория ЯнгаМиллса является асимптотически свободной.

Этот результат прекрасно согласуется с представлением о том, что кварки несут цвет. Неабелево калибровочное преобразование меняет кварк одного цвета на кварк другого цвета. Таким образом, чтобы записать теорию сильного взаимодействия, нужно просто взять результат решения упражнения IV.5.6

$$\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} + \bar{q}\left(i\gamma^\mu D_\mu - m\right) q$$

с ковариантной производной $D_\mu = \partial_\mu - iA_\mu$. Калибровочной группой является $SU(3)$ с кварковым полем q в фундаментальном представлении. Другими словами, калибровочные поля $A_\mu = A^a_\mu T^a$, где $T^a (a = 1, ..., 8)$, являются бесследовыми эрмитовыми 3×3-матрицами. Очевидно, что $(A_\mu q)^\alpha = A^a_\mu \left(T^a\right)^\alpha_\beta q^\beta$, где $\alpha, \beta = 1, 2, 3$. Эта теория известна как квантовая хромодинамика или кратко КХД, а неабелевы калибровочные бозоны известны как глюоны. Чтобы учесть аромат кварка, для второго члена в (1) просто записываем $\sum_{j=1}^{f} \bar{q}_j \left(i\gamma^\mu D_\mu - m_j\right) q_j$, где индекс j определяет один из f аромат. Заметим, что кварки разных ароматов имеют разные массы.

Инфракрасная неволя

Обратной стороной асимптотической свободы является инфракрасная неволя. Мы не можем следовать за ренорм-групповым потоком все время до малого масштаба импульсов,

характерного для кварков, заключенных внутри адронов, поскольку константа связи g становится все больше и вычисление $\beta(g)$ с помощью теории возмущений уже неприемлемо. Тем не менее вероятно (хотя пока не доказано), что g стремится к бесконечности и что глюоны удерживают себя и кварки в постоянном конфайнменте. Петля Вильсона, описанная в главе VII.1, является параметром порядка для конфайнмента.

В элементарной физике силы уменьшаются с увеличением расстояния между взаимодействующими объектами, поэтому постоянный конфайнмент является довольно странным понятием. Существуют ли какие-либо другие примеры постоянного конфайнмента? Рассмотрим магнитный монополь в сверхпроводнике. Мы собираем воедино результаты, полученные в главах IV.4 и V.4 (и даже VI.2)! Из монополя исходит квантованное количество магнитного потока, но согласно эффекту Мейсснера сверхпроводник выталкивает магнитный поток. Таким образом, один магнитный монополь не может жить внутри сверхпроводника.

Теперь рассмотрим антимонополь, находящийся на расстоянии R (рис. VII.3.1). Магнитный поток, исходящий из монополя, может входить в антимонополь, образуя трубку, соединяющую монополь с антимонополем, и вынуждая сверхпроводник потерять свои сверхпроводящие свойства в области трубки потока. На языке, использованном в главе V.4, это означает, что полю или параметру порядка поля φ больше энергетически невыгодно быть всюду постоянным; вместо этого в области трубки потока оно исчезает. Затраты энергии на такую конфитурацию явно растут с увеличением расстояния R (в соответствии с законом площадей Вильсона).

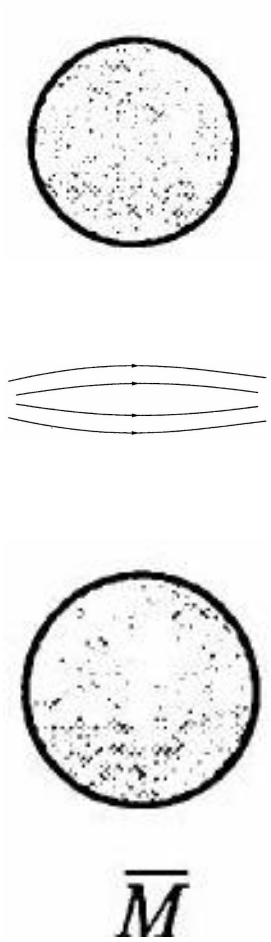


Рис. VII.3.1

Другими словами, экспериментатор, находящийся внутри сверхпроводника, обнаружил бы, что для разделения монополя и антимонополя требуется все больше и больше энергии. Такой конфайнмент монополей внутри сверхпроводника часто берется в качестве модели

конфайнмента кварков, который еще нужно доказать. Пользуясь электромагнитной дуальностью, можно представить магнитный, а не обычный электрический сверхпроводник. Внутри магнитного сверхпроводника электрические заряды будут постоянно удерживаться. Нашу Вселенную можно сравнить с цветным магнитным сверхпроводником, в котором удерживаются кварки (аналог электрических зарядов).

На масштабах расстояний, больших по сравнению с радиусом цветной трубки потока, соединяющей кварк с антикварком, трубка может рассматриваться как струна. Именно так исторически возникла теория струн. Задача, молодые люди, заключается в том, чтобы доказать, что основное состояние или вакуум в теории (1) является цветным магнитным сверхпроводником.

Симметрии сильного взаимодействия

Теперь, когда у нас есть теория сильного взаимодействия, можно понять происхождение симметрий сильного взаимодействия, а именно изоспиновой симметрии Гейзенберга и киральной симметрии, которая при спонтанном нарушении приводит к появлению пиона в виде намбу-голдстоуновского бозона (как обсуждалось в главах IV.2 и VI.4).

Рассмотрим мир с двумя ароматами; это все, что существенно для обсуждения пиона. Введем обозначения $u \equiv q_1, d \equiv q_2$ и $q = \begin{pmatrix} u \\ d \end{pmatrix}$ так, чтобы лагранжиан можно было записать в виде

$$\mathcal{L} = -\frac{1}{4g^2} F^a_{\mu\nu} F^{a\mu\nu} + \bar{q}\left(i\gamma^\mu D_\mu - m\right) q$$

с

$$m = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}$$

где m_u и m_d - массы верхнего и нижнего кварков соответственно. Если $m_u = m_d$, то лагранжиан является инвариантным относительно $q \to e^{i\theta\cdot\tau} q$, что соответствует изоспиновой симметрии Гейзенберга.

В пределе, когда m_u и m_d равны нулю, лагранжиан является инвариантным относительно $q \to e^{i\vec{\varphi}\cdot\tau\gamma_5} q$ и известен как киральная симметрия $SU(2)$; киральная потому, что правые кварки q_R и левые кварки q_L преобразуются по-разному. Если m_u и m_d намного меньше масштаба энергии сильного взаимодействия, киральная симметрия $SU(2)$ является приближенной.

Пион является намбу-голдстоуновским бозоном, связанным со спонтанным нарушением киральной симметрии $SU(2)$. Это, несомненно, является примером нарушения динамической симметрии, поскольку под рукой нет простейшего скалярного поля, чтобы получить вакуумное среднее. Вместо этого предполагается, что динамика сильных взаимодействий заставляет комплексные скалярные поля $\bar{u}u$ и $\bar{d}d$ «конденсироваться в вакуум» с тем, чтобы $\langle 0|\bar{u}u|0\rangle = \langle 0|\bar{d}d|0\rangle$ стали неисчезающими, где равенство между двумя вакуумными средними гарантирует, что изоспин Гейзенберга не нарушается спонтанно; это экспериментальный факт, так как соответствующих намбу-голдстоуновских бозонов не существует. Предполагается, что вакуум КХД, в терминах дублета q, должен быть таким, что $\langle 0|\bar{q}q|0\rangle \neq 0$, тогда как $\langle 0|\bar{q}\vec{\tau}q|0\rangle = 0$.

Ренорм-групповой поток

Ренорм-групповой поток для константы связи КХД определяется выражением

$$\frac{dg}{dt} = \beta(g) = -\frac{11}{3} T_2(G) \frac{g^3}{16\pi^2}$$

с обязательным знаком минус. Здесь

$$T(G)\delta^{ab} = f^{acd}f^{bcd}$$

Я не стану вычислять $\beta(g)$, но, прочитав главы VI.8 и VII.1, вы должны быть уверены, что сможете, если захотите, сделать это самостоятельно 1. Как минимум вы должны понять смысл множителя g^3 и $T_2(G)$, нарисовав соответствующие фейнмановские диаграммы.

Если добавить фермионы, то

$$\frac{dg}{dt} = \beta(g) = \left[-\frac{11}{3}T_2(G) + \frac{4}{3}T_2(F)\right]\frac{g^3}{16\pi^2}$$

где

$$T_2(F)\delta^{ab} = \mathrm{tr}\left[T^a(F)T^b(F)\right]$$

Я оставляю вам получить (4) из (2). Для группы $SU(N)$ для каждого фермиона в фундаментальном представлении $T_2(F) = \frac{1}{2}$. Обратите внимание, что асимптотическая свобода теряется, когда фермионов слишком много.

В упражнении VI.8.1 вы уже решали уравнение типа (4). По аналогии с квантовой электродинамикой определим $\alpha_S(\mu) \equiv g(\mu)^2/4\pi$ - сильное

[1] Подробное вычисление см.. например. С. Вайнберг: Квантовая теория поля. Т. 2. П. 18.7.

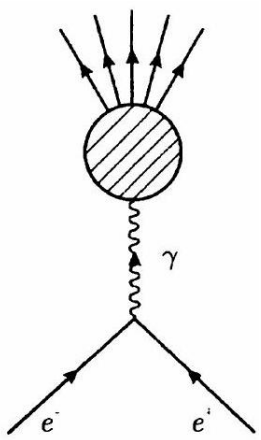


Рис. VII.3.2

взаимодействие на масштабе импульсов μ. Из (4) получаем выражение [2]

$$\alpha_S(Q) = \frac{\alpha_S(\mu)}{1 + (1/4\pi)\left(11 - \frac{2}{3}n_f\right)\alpha_S(\mu)\log\left(Q^2/\mu^2\right)}$$

откуда следует, что $\alpha_S(Q) \to 0$ логарифмически при $Q \to \infty$.

Электрон-позитронная аннигиляция

Объем книги позволяет мне продемонстрировать только одно физическое приложение. Экспериментаторы измерили сечение σ аннигиляции e^+e^-с образованием адронов как функцию общей энергии E в системе центра масс. Амплитуда показана на рис. VII.3.2. Чтобы вычислить сечение через амплитуду, нужно проделать то, что некоторые называют «скучной кинематикой»; например, все правильно нормировать, разделить на поток двух пучков и т. д. Вы, конечно, должны проделать такого рода вычисление, по крайней мере, один раз ради собственного блага. Поверьте, я делал это большее число раз, чем могу вспомнить. Но к счастью, этого неприятного занятия можно избежать, и сейчас я это продемонстрирую. Прежде всего, рассмотрим соотношение

$$R(E) \equiv \frac{\sigma\left(e^+e^- \to \text{ адроны }\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)}$$

2 Экспериментальные данные $\alpha_S(Q)$ можно найти на рис. 14.3 в работе F. Wilczek, in: V. Fitch et al., eds.. Critical Problems in Physics, p. 281. Кинематическая часть сокращается. На рис. VII.3.2 половина диаграммы, содержащая электрон-позитронные линии и фотонный пропагатор, появляется также в фейнмановской диаграмме $e^+e^- \to \mu^+\mu^-$(рис. VII.3.3) и поэтому исчезает в $R(E)$. Круг на рис. VII.3.2, за которым скрывается вся сложность сильного взаимодействия, дается выражением $\langle 0\,|J^\mu(0)|\,h\rangle$, где J^μ - электромагнитный ток, а состояние $|h\rangle$ может содержать любое число адронов. Чтобы получить сечение, нужно возвести в квадрат амплитуду, учесть δ-функцию, ответственную за сохранение импульса, и просуммировать по всем $|h\rangle$, получая в результате

$$\sum_h (2\pi)^4\delta^4\left(p_h - p_{e^+} - p_{e^-}\right)\langle 0\,|J^\mu(0)|\,h\rangle\,\langle h\,|J^\nu(0)|\,0\rangle$$

(где $q \equiv p_{e^+} + p_{e^-} = (E,\,\overrightarrow{0})\Big)$. Эту величину можно записать в виде

$$\int d^4x e^{iqx}\,\langle 0\,|J^\mu(x)J^\nu(0)|\,0\rangle = \int d^4x e^{iqx}\,\langle 0\,|[J^\mu(x), J^\nu(0)]|\,0\rangle = \\ = 2\,\mathrm{Im}\left(i\int d^4x e^{iqx}\,\langle 0\,|TJ^\mu(x)J^\nu(0)|\,0\rangle\right).$$

(Первое равенство следует из $E > 0$, а второе - из упражнения I.8.5). Чтобы определить эту величину, нужно вычислить бесконечное число фейнмановских диаграмм, включающих множество кварков и глюонов. Типичная диаграмма показана на рис. VII.3.4. Это совершенно безнадежно!

На помощь приходит асимптотическая свобода! Из главы VI.7 известно, что соответствующей силой взаимодействия для процесса с энергией E является $g(E)$. Но когда энергия E увеличивается, $g(E)$ постепенно уменьшается. Таким образом, все диаграммы из показанных на рис. VII.3.4, которые включают много степеней $g(E)$, исчезают; остаются диаграммы без степени $g(E)$ (рис. VII.3.5a) и с двумя степенями $g(E)$ (рис. VII.3.5b,c,d). Чтобы получить главный член в $R(E)$, никакие вычисления не нужны, поскольку диаграмма на рис. VII.3.5a аналогична той, что входит в $e^+e^- \to \mu^+\mu^-$. Нужно просто заменить кварковый пропагатор на мюонный пропагатор (массы кварка и мюона пренебрежимо малы по сравнению с E). При высокой энергии кварки свободны, а $R(E)$ просто подсчитывает квадрат заряда Q_a различных кварков, вносящих вклад в эту энергию. Можно ожидать, что

$$R(E) \underset{E\to\infty}{\longrightarrow} 3\sum_a Q_a^2$$

Множитель 3 учитывает цвет.

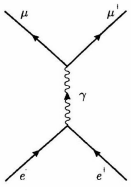
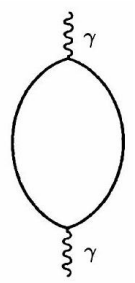
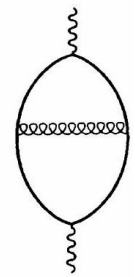


Рис. VII. 3.3



(a)



(b)

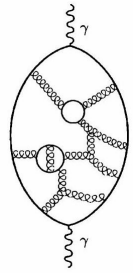
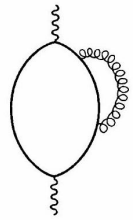
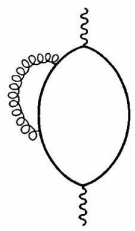


Рис. VII.3.4



(c)



(d)
Рис. VII.3.5

КХД не только отключает себя при высоких энергиях, но и указывает нам на то, как быстро она отключается. Таким образом, можно определить, как достигается предел в (8):

$$R(E) = \left(3\sum_a Q_a^2\right)\left(1 + C\frac{2}{\left(11 - \frac{2}{3}n_f\right)\log(E/\mu)} + \ldots\right)$$

Вычислите C самостоятельно.

Мечты о точном решении

Для многих специалистов в области теории поля аналитическое решение квантовой хромодинамики является своего рода «священным граалем» (граалем, за который обещан приз в миллион долларов: см. www.ams.org/claymath/). Многие специалисты по теории поля мечтали о точном решении, по крайней мере «чистой» КХД, то есть КХД без кварков. Если все-таки какую-то четырехмерную квантовую теорию поля удастся точно решить, то, вероятнее всего, это будет чистая теория Янга-Миллса со всеми ее удивительными симметриями. (Возможно даже, что более вероятным кандидатом на такое решение является суперсимметричная теория Янга-Миллса. Мы коснемся вопроса о суперсимметрии в главе VIII.4.)

Я объясню точнее, что означает решить КХД. Рассмотрим мир, где есть только верхние и нижние кварки, массы которых m_u и m_d берутся равными нулю; а именно мир, который описывается лагранжианом

$$\mathcal{L} = -\frac{1}{4g^2}F^a_{\mu\nu}F^{a\mu\nu} + \bar{q}i\gamma^\mu D_\mu q$$

Наша цель заключается в том, чтобы вычислить нечто, подобное отношению массы ρ-мезона m_ρ к массе протона m_P.

Для успешного решения физикам-теоретикам в таком случае, как правило, нужен малый параметр разложения, но при попытке немедленного решения (10) возникает трудность, связанная с отсутствием такого параметра. Можно было бы считать таким параметром g, но это неправильно. Ренорм-групповой анализ учит нас, что $g(\mu)$ является функцией масштаба энергии μ, на котором она измеряется. Таким образом, нет такого безразмерного числа, про которое можно сказать, что оно измеряет силу КХД. Поэтому самое лучшее, что можно сделать, это указать на значение μ, при котором $\left(g\left(\mu^2\right)/4\pi\right)$ становится порядка 1. Это есть энергия, известная как Λ_{QCD}, при которой сильное взаимодействие становится сильным, когда мы уходим вниз от высокой энергии (рис. VII.3.6). Но Λ_{QCD} попросту определяет шкалу, относительно которой должны измеряться другие величины. Другими словами, если вам удастся вычислить массу m_P, то хорошо бы, чтобы она была пропорциональна Λ_{QCD}, поскольку Λ_{QCD} является единственной величиной с размерностью массы. То же относится и к массе m_ρ. Скажем прямо, если вы опубликуете статью с формулой, определяющей m_ρ/m_P в виде отвлеченных чисел, таких как 2 и π, сообщество специалистов по теории поля назовет вас героем-победителем, сумевшим получить точное решение КХД.

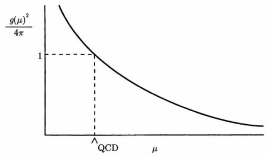


Рис. VII.3.6

Очевидный обмен безразмерной константы связи g на размерный массовый масштаб Λ_{QCD} известен как размерная трансмутация, - еще один пример который приведен в следующей главе.

Упражнения

VII.3.1. Вычислите C в (9). [Указание: за помощью обращайтесь к статьям T. Appelquist and H. Georgi, Phys. Rev. D8: 4000, 1973; и A. Zee, Phys. Rev. D8: 4038, 1973.]

VII.3.2. Вычислите (2).

30.4.4 Zi.VII.4 Large-N Expansion

Изобретение параметра разложения

Квантовая хромодинамика является теорией без параметров, поэтому трудно получить даже первое приближение. В отчаянии теоретики придумали параметр, по которому можно разлагать КХД. Допустим, что вместо трех цветов имеется N цветов. 'т Хоофт' заметил, что при $N \to \infty$ происходят поразительные упрощения. Идея состоит в том, что если вычислить m_ρ/m_P, например, в пределе больших N, то результат может быть близок к действительному значению. Иногда шутят, что специалисты в области физики элементарных частиц считают 3 большим числом, но на самом деле поправка к пределу больших N составляет обычно порядка $1/N^2$, что в реальном мире соответствует примерно 10%. Специалисты в области физики элементарных частиц были бы чрезвычайно рады, если бы смогли с такой степенью точности вычислить массы адронов.

Так же, как нарушение спонтанной симметрии и ряд других важных понятий, разложение по большим N пришло из физики конденсированных сред, но теперь оно регулярно используется в самых разных контекстах. Например, делались попытки использовать разложение по большим N для решения проблемы высокотемпературной сверхпроводимости и для того, чтобы свернуть РНК.

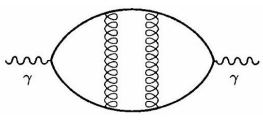
Масштабирование константы связи в КХЛ

Итак, пусть цветной группой будет $U(N)$, тогда лагранжиан запишется в виде

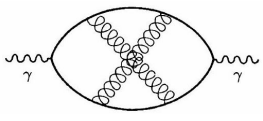
$$\mathcal{L} = -\frac{N^a}{2g^2}\,\mathrm{tr}\,F_{\mu\nu}F^{\mu\nu} + \bar{\psi}[i(\not{\partial} - i\not{\mathcal{A}}) - m]\psi$$

'G. 't Hooft, Under the Spell of the Gauge Principle, p. 378. Обратите внимание, что мы заменили g^2 на g^2/N^a. Для конечного N такое изменение не имеет существенного значения. Дело в том, что нужно выбрать степень a так, чтобы в пределе $N \to \infty$ происходили интересные упрощения при фиксированном g^2. Вершины взаимодействия третьего и четвертого порядка для глюонов пропорциональны N^a.С другой стороны, поскольку глюонный пропагатор является обратным к квадратичным членам в $\mathcal{L}$, он пропорционален $1/N^a$. Взаимодействие глюона с кварком не зависит от N.

Чтобы зафиксировать a, сосредоточим внимание на конкретном приложении - вычислении $\sigma\,(e^+e^- \to$ адроны) из предыдущей главы. Предположим, мы хотим вычислить это сечение при низких энергиях. Рассмотрим диаграммы с обменом двумя глюонами, показанные на рис. VII. 4.1a и b. Эти две диаграммы имеют порядок g^2, и нам нужно их вычислить. Заметим, что рис. VII.4.1b неплоский. Поскольку один глюон пересекает другой, эту диаграмму нельзя изобразить на плоскости, если мы считаем, что линии не могут проходить сквозь друг друга.



(a)



(b)

Рис. VII.4.1

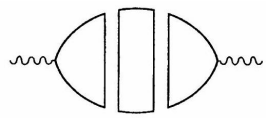
Теперь настала очередь формализма двойных линий, описанного в главе IV.5. Диаграммы, приведенные на рис. VII.4.1a и b, изображаются в этом формализме по-другому (рис. VII.4.2a и b). Два глюонных пропагатора, общие для обеих диаграмм, дают множитель $1/N^{2a}$. Теперь мы приблизились к кульминационному моменту. Просуммируем по трем независимым цветовым индексам на рис. VII.4.2a, получая таким образом множитель N^3. Возьмите несколько цветных карандашей и попытайтесь раскрасить каждую линию на рис. VII.4.2a разным цветом: вам понадобятся три цветных карандаша. Вместо этого просуммируем только по одному независимому индексу на рис. VII.4.2b, получая тем самым множитель N. Другими словами, рис. VII.4.2a доминирует над рис. VII.4.2b в N^2 раз. В пределе больших N рис. VII.4.2b можно отбросить.

Очевидно, что это правило должно связывать один множитель N с каждой петлей. Таким образом, диаграмма наинизшего порядка, показан- ная на рис. VII.4.2c, с присутствующими на ней различными цветами N, масштабируется как N; а диаграмма на рис. VII.4.2a масштабируется как N^3/N^{2a}. Мы хотим, чтобы диаграммы рис. VII.4.2a и c масштабировались одинаково, поэтому выбираем $a = 1$.

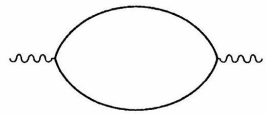
Рисуя еще больше диаграмм [например, рис. VII.4.2d, который масштабируется как $N\,(1/N^4)\,N^4$ с тремя множителями, обусловленными взаимодействием четвертого порядка, пропагаторами и суммой по цветам соответственно], можно убедиться в том, что планарные диаграммы доминируют в пределе больших N и все масштабируются как N. Для пробы сил попытайтесь это доказать. Очевидно, что за всем этим стоит определенная топологическая структура.

Приведение диаграмм к планарному виду является значительным упрощением, но при этом все еще остается бесконечное число диаграмм. На данном этапе владения теорией поля все еще невозможно решить КХД для больших N. (Когда я начал писать эту книгу, существовала надежда, основанная на интуиции и методах, разработанных в теории струн, что решение КХД для больших N уже близко. Но теперь, когда я просматриваю окончательную корректуру, надежда эта исчезла.)

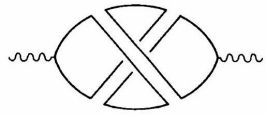
У формализма двойных линий есть естественное объяснение. С точки зрения теории групп калибровочное поле матрицы A^i_j преобразуется как $\bar{q}^i q_j$ (но мы, конечно, не считаем, что глюон является связанным состоянием кварк-антикварк). В этом случае можно считать, что две линии описы-



(a)



(c)



(b)

(d)

Рис. VII.4.2 вают распространяющиеся вдоль друг друга кварк и антикварк, а стрелки линий указывают направление, в котором течет цвет.

Теория случайных матриц

Есть теория, структурно подобная КХД для больших N, но более простая, которую, как ни удивительно, можно решить. Я имею в виду теорию случайных матриц.

Чуть преувеличивая, можно сказать, что квантовая механика заключается в написании матрицы, известной как гамильтониан, а затем нахождении ее собственных значений и собственных векторов. В начале 1950-х годов, столкнувшись с проблемой изучения свойств сложных атомных ядер, Ю. Вигнер разработал своего рода статистическую квантовую механику, в которой вместо решения истинного гамильтониана в некотором сомнительном приближении можно случайным образом создавать большие матрицы и изучать распределение собственных значений. С тех пор теория случайных матриц стала обширной и развивающейся темой с огромным и постоянно растущим числом публикаций; она применяется в многочисленных областях теоретической физики и даже чистой математики (таких как операторная алгебра и теория чисел) [2]. Эта теория имеет наглядные применения к разупорядоченным системам конденсированных сред и менее наглядные применения к случайным поверхностям и, следовательно, даже к теории струн. Здесь я ограничусь тем, что покажу, как наблюдение 't Хоофта, касающееся планарных диаграмм, работает в контексте теории случайных матриц.

Сгенерируем случайным образом эрмитовы $N \times N$-матрицы φ с вероятностью

$$P(\varphi) = \frac{1}{Z} e^{-N \operatorname{tr} V(\varphi)}$$

где $V(\varphi)$ - полином по φ. Например, пусть $V(\varphi) = \frac{1}{2} m^2 \varphi^2 + g\varphi^4$. Нормировка $\int D\varphi P(\varphi)$ фиксирует

$$Z = \int D\varphi e^{-N \operatorname{tr} V(\varphi)}$$

Везде подразумевается предел $N \to \infty$.

[2] Беглый просмотр математических публикаций см. в D. Voiculescu, ed., Free Probabilit?, Theory. Как и в главе VI.7, нас интересует $\rho(E)$ - плотность собственных значений φ. Чтобы вы действительно поняли, что это означает, я опишу, что нужно бы сделать для численного определения $\rho(E)$. Для некоторого большого целого числа N с помощью компьютера сгенерируем эрмитову матрицу φ с вероятностью $P(\varphi)$ и затем решим уравнение для собственных Значений $\varphi v = Ev$. После многократного повторения этой процедуры компьютер может построить распределение собственных значений в виде гистограммы, которая в итоге приближается к гладкой кривой, которая называется плотностью собственных значений $\rho(E)$.

Мы уже развили формализм для вычисления $\rho(E)$, в упражнении VI.7.1. Вычислим вещественно-аналитическую функцию $G(z) \equiv \langle (1/N) \operatorname{tr}[1/z - \varphi] \rangle$ и $\rho(E) = -(1/\pi) \lim_{\varepsilon \to 0} \operatorname{Im} G(E + i\varepsilon)$. Среднее $\langle ... \rangle$ взято с вероятностью $P(\varphi)$:

$$\langle O(\varphi) \rangle = \frac{1}{Z} \int D\varphi e^{-N \operatorname{tr} V(\varphi)} O(\varphi)$$

Вы понимаете, что мой выбор φ в качестве обозначения матрицы и $V(\varphi) = \frac{1}{2} m^2 \varphi^2 + g\varphi^4$ в качестве примера является провоцирующим. Вычисление Z подобно вычислению интеграла

по траекториям, но для действия $S(\varphi) = N \operatorname{tr} V(\varphi)$ оно не включает $\int d^d x$. Теорию случайных матриц можно представить как квантовую теорию поля в $(0 + 0)$-мерном пространстве-времени!

Разнообразные методы теории поля, такие как фейнмановские диаграммы, можно применить к теории случайных матриц. Но жить в $(0+ + 0)$-мерном пространстве-времени очень приятно: там нет пространства, времени, энергии и импульса, а значит нет интеграла, который нужно вычислять при нахождении фейнмановских диаграмм.

Полукруговый закон Вигнера

Давайте посмотрим, как все это работает в простом случае $V(\varphi) = \frac{1}{2}m^2\varphi^2$ (m всегда можно включить в φ, но мы не будем это делать). Вместо $G(z)$ несколько легче вычислить

$$G^i_j(z) \equiv \left\langle \left(\frac{1}{z - \varphi} \right)^i_j \right\rangle = \delta^i_j G(z).$$

I Iоследнее равенство следует из инвариантности относительно унитарных преобразований:

$$P(\varphi) = P\left(U^\dagger \varphi U\right)$$

Разложим функцию

$$G^i_j(z) = \sum_{n=0}^{\infty} \frac{1}{z^{2n+1}} \left\langle \left(\varphi^{2n}\right)^i_j \right\rangle$$

Возьмем гауссов интеграл

$$\frac{1}{Z}\int D\varphi e^{-N \operatorname{tr} \frac{1}{2}m^2\varphi^2} \varphi^i_k \varphi^l_j = \frac{1}{Z}\int D\varphi e^{-N\frac{1}{2}m^2 \sum\limits_{p,4} \varphi^p_q \varphi^q_p} \varphi^i_k \varphi^l_j = \delta^i_j \delta^l_k \frac{1}{Nm^2}$$

Полагая $k = l$ и суммируя, находим, что член $n = 1$ в уравнении (5) равен $(1/z^3)\,\delta^i_j\,(1/m^2)$

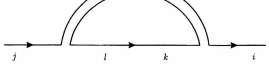


Рис. VII.4.3

Так же, как в любой теории поля, с каждым из членов в уравнении (5) можно сопоставить фейнмановскую диаграмму. Для члена $n = 1$ у нас есть рис. VII.4.3. Матричный характер поля φ естественным образом приводит к формализму двойных линий 'т Хоофта, так что мы с легкостью можем говорить о кварковом и глюонном пропагаторах. Правила Фейнмана приведены на рис. VII.4.4. Мы видим, что φ - это глюонное поле, а выражение (6) - глюонный пропагатор. В самом деле, нашу задачу можно сформулировать следующим образом: имея голый кварковый пропагатор $1/z$, надо вычислить истинный кварковый пропагатор $G(z)$ с учетом всех эффектов взаимодействия.

Теперь посмотрим на член $n = 2$ в (5) $1/z^5 \left\langle \varphi^i_h \varphi^h_k \varphi^k_l \varphi^l_j \right\rangle$, который представлен на рис. VII.4.5a. Немного подумав, можно понять, что индекс i можно свернуть с k, l или j, в результате чего появляются рис. VII.4.5b, c, d. Суммируя по цветовым индексам так же, как в КХД, видим, что планарные диаграммы на рис. VII.4.5b и рис. VII.4.5d доминируют над диаграммой рис. VII.4.5c в N^2 раз. Итак, мы соглашаемся с наблюдением 'т Хоофта,

что планарные диаграммы доминируют.

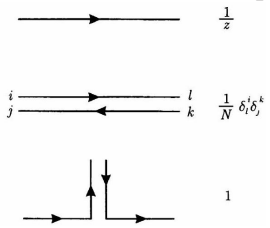


Рис. VII.4.4

Кстати, на этом примере понятно, какую важную роль играют большие N, позволяя избавиться от непланарных диаграмм. Ведь если я попрошу вычислить плотность собственных значений, скажем, для $N = 7$, вы, конечно, возразите и скажите, что для решения полиномиального уравнения 7-й степени неизвестно даже общее выражение.

Простой пример на рис. VII.4.5 указывает еще на то, как можно было бы построить всевозможные диаграммы. На рис. VII.4.5b повторяется тот же «элемент», тогда как на рис. VII.4.5d этот «элемент» вложен внутрь более базовой диаграммы. На рис. VII.4.5e показан более сложный пример. Можно убедиться, что для $N = \infty$ все диаграммы, которые вносят вклад в $G(z)$, могут быть образованы либо при помощи «вложения» существующих диаграмм внутрь образующего арку глюонного пропагатора, либо путем многократного «повторения» существующей структуры. Преобразуйте предыдущее предложение в два уравнения: «повторение» (см. рис. VII.4.6a),

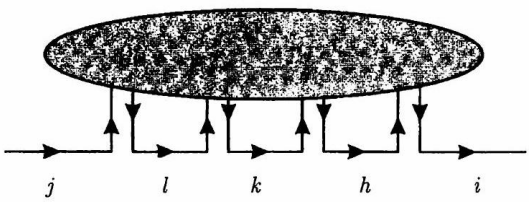
$$G(z) = \frac{1}{z} + \frac{1}{z}\Sigma(z)\frac{1}{z} + \frac{1}{z}\Sigma(z)\frac{1}{z}\Sigma(z)\frac{1}{z} + \ldots = \frac{1}{z - \Sigma(z)},$$

и «вложение» (см. рис. VII.4.6b),

$$\Sigma(z) = \frac{1}{m^2}G(z).$$

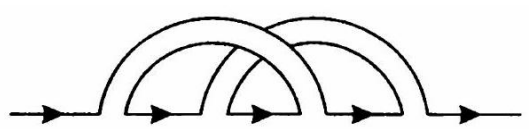
Объединяя эти два уравнения, получаем простое квадратное уравнение для $G(z)$, которое можно немедленно решить и получить

$$G(z) = \frac{m^2}{2}\left(z - \sqrt{z^2 - \frac{4}{m^2}}\right).$$

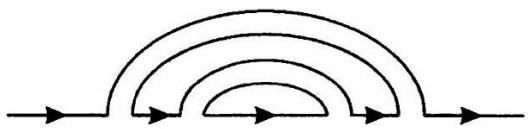


(a)

(b)



(c)

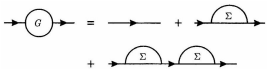


(d)

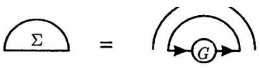
(e)
Рис. VII.4.5
(Из определения $G(z)$ следует, что для больших $zG(z) \to 1/z$, и поэтому выбираем отрицательный корень.) Сразу получаем

$$\rho(E) = \frac{2}{\pi a^2}\sqrt{a^2 - E^2}$$

где $a^2 = 4/m^2$. Это известный результат, называемый полукруговым законом Вигнера.



(a)



(b)
Рис. VII.4.6

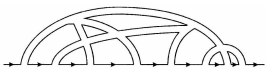


Рис. VII.4.7

Дайсоновский газ

Надеюсь, вы оценили элегантность подхода больших N и планарных диаграмм. Однако вы могли заметить, что глюоны не взаимодействуют. Это как если бы мы решили задачу квантовой электродинамики, когда нужно было решать задачу квантовой хромодинамики. А что, если нам приходится иметь дело с $V(\varphi) = \frac{1}{2}m^2\varphi^2 + g\varphi^4$? Слагаемое $g\varphi^4$ заставляет глюоны взаимодействовать друг с другом, образуя ужасные диаграммы такого вида, как показано на рис. VII.4.7. Ясно, что число диаграмм растет, и, насколько мне известно, никто никогда не смог вычислить $G(z)$, используя фейнмановские диаграммы. К счастью, $G(z)$ можно вычислить по-другому, используя так называемый метод дайсоновского газа. Для этого нужно записать выражение

$$\varphi = U^\dagger \Lambda U$$

где Λ обозначает диагональную $N \times N$-матрицу с диагональными элементами, равными λ_i, где $i = 1, ..., N$. Заменим переменную интегрирования в уравнении (3) с φ на U и Λ:

$$Z = \int dU \int (\Pi_i d\lambda_i)\, J e^{-N\sum_k V(\lambda_k)}$$

где J - якобиан. Поскольку подынтегральное выражение не зависит от U, интеграл по U можно отбросить. Получаем объем группы $SU(N)$. Не напоминает ли это вам главу VII.1? В самом деле, в уравнении (11) U соответствует степеням свободы нефизической калибровки: нужными степенями свободы являются собственные значения $\{\lambda_i\}$. В качестве упражнения вы можете использовать для вычисления J метод Фаддеева-Попова.

Мы же воспользуемся вместо этого более изящным способом определения J исходя из общих принципов. Замена переменных в (11) плохо определена, если произвольные два значения λ_i совпадают; в этом случае J должен занулиться. (Напомним, что переход от декартовых координат к сферическим плохо определен у Северного и Южного полюсов, и якобиан в $\sin\theta d\theta d\varphi$ действительно равен нулю при $\theta = 0$ и π.) значения λ_i эквивалентны, симметрия относительно перестановок требует, чтобы $J = \left[\Pi_{m>n}\left(\lambda_m - \lambda_n\right)\right]^\beta$. Степень β можно найти, используя анализ размерностей. CN^2-матричными элементами $d\varphi$ явно имеет размерность λ^{N^2}, тогда как $\left(\Pi_i d\lambda_i\right) J$ имеет размерность $\lambda^N \lambda^{\beta N(N-1)/2}$; таким образом, $\beta = 2$

Определив J, перепишем выражение (12) в виде

$$Z = \int \left(\Pi_i d\lambda_i\right) \left[\Pi_{m>n}\left(\lambda_m - \lambda_n\right)\right]^2 e^{-N\sum\limits_k V(\lambda_k)} =$$
$$= \int\limits_2 \left(\Pi_i d\lambda_i\right) e^{-N\sum\limits_k V(\lambda_k) + \sum\limits_{m>n} \log\left((\lambda_{mn}-\lambda_n)^2\right)}$$

Дайсон показал, что в таком виде $Z = \int \left(\Pi_i d\lambda_i\right) e^{-NE(\lambda_1,\ldots,\lambda_N)}$ является статистической суммой классического одномерного газа (вспомним главу V.2). Представьте вещественное число λ_i как положение i-й молекулы.

Энергия конфигурации

$$E\left(\lambda_1, \ldots, \lambda_N\right) = \sum_k V\left(\lambda_k\right) - \frac{1}{N} \sum_{m>n} \log\left(\lambda_m - \lambda_n\right)^2$$

состоит из двух членов с явным физическим смыслом. Газ заключен в потенциальной яме $V(x)$, и молекулы отталкивают [3] друг друга за счет двухчастичного потенциала $-(1/N)\log(x-y)^2$. Заметим, что эти два члена в E имеют одинаковый порядок по N, поскольку в каждой сумме учитывается степень N. В пределе больших N (N можно представить как обратную температуру) вычисляем Z методом наискорейшего спуска и минимизируя E, получая уравнение

$$V'\left(\lambda_k\right) = \frac{2}{N} \sum_{n\neq k} \frac{1}{\lambda_k - \lambda_n}$$

которое в непрерывном пределе, когда полюсы в (15) сливаются в разрез, имеет вид $V'(\lambda) = 2\mathcal{P}\int d\mu[\rho(\mu)/(\lambda-\mu)]$, где $\rho(\lambda)$ - неизвестная функция, для которой мы хотим найти решение, а $\mathcal{P}$ - главное значение.

Определяя, как и раньше, $G(\lambda) = \int d\mu[\rho(\mu)/(\lambda-\mu)]$, видим, что рассматриваемое уравнение для $\rho(\mu)$ можно записать в виде $\mathrm{Re}\, G(\lambda + i\varepsilon) = \frac{1}{2}V'(\lambda)$. Другими словами, $G(z)$ является вещественно-аналитической функцией с разрезами вдоль действительной оси. Нам дана действительная часть $G(z)$ на разрезе, и нужно получить решение для мнимой части. Брезан, Ициксон, Паризи и Зюбер предложили остроумное решение этой задачи. Предположим для простоты, что $V(z)$ является четным полиномом и что есть только один разрез (см. упражнение VII.4.7). Используем симметрию и постулируем следующую форму:

$$G(z) = \frac{1}{2}\left[V'(z) - P(z)\sqrt{z^2 - a^2}\right]$$

с неизвестным четным полиномом $P(z)$. Примечательно, что требование $G(z) \to 1/z$ для больших z полностью определяет $P(z)$. Разумеется, что с педагогической точки зрения следует обратиться к конкретному примеру, скажем $V(z) = \frac{1}{2}m^2z^2 + gz^4$. Поскольку $V'(z)$ является многочленом третьей степени по $z, P(z)$ должен быть квадратным (четным) многочленом по z. Беря предел $z \to \infty$ и требуя, чтобы коэффициенты при z^3 и z в $G(z)$

3Заметим, что это соответствует отталкиванию мсжду энергетическими уровнями в квантовой механике. исчезли, а коэффициент при l/z был равен 1, получаем три уравнения на три неизвестные [а именно a и двух неизвестных в $P(z)$]. Тогда плотность собственных значений определяется в виде $\rho(E) = (1/\pi)P(E)\sqrt{a^2 - E^2}$.

Я полагаю, что из этого можно извлечь урок, который заключается в том, что эффективность фейнмановских диаграмм несколько переоценивается, несмотря на их исторически важную роль в квантовой электродинамике и полезность для наглядного описания происходящего. Никто, конечно, и не думает, что КХД, даже КХД больших N, будет когда-нибудь решена путем суммирования фейнмановских диаграмм. Что необходимо, так это аналог метода дайсоновского газа для КХД больших N. С другой стороны, если моим читателям удастся вычислить $G(z)$ путем суммирования планарных диаграмм (ведь ответ известен!), то полученная при этом информация, возможно, могла бы быть полезной для понимания того, как обращаться с планарными диаграммами в КХД при больших N.

Теории поля в пределе больших N

Ряд теорий поля был также решен с использованием разложения по большим N. Я приведу в качестве примера модель Гросса-Невье отчасти потому, что она несколько напоминает КХД. Эта модель определяется выражением

$$S(\psi) = \int d^2x \left[\sum_{a=1}^{N} \bar{\psi}_a i\not{\partial} \psi_a + \frac{g^2}{2N} \left(\sum_{a=1}^{N} \bar{\psi}_a \psi_a \right)^2 \right].$$

Напомним, что согласно главе III.3 эта теория должна быть перенормируемой в $(1+1)$-мерном пространстве-времени. Для некоторого конечного N, скажем $N = 3$, наверняка окажется, что такую теорию решить не легче, чем любую другую теорию поля с полным взаимодействием. Но, как мы увидим, при $N \to \infty$ из модели можно извлечь интересную физику.

Используя тождество (A.14), теорию можно переписать в виде

$$S(\psi, \sigma) = \int d^2x \left[\sum_{a=1}^{N} \bar{\psi}_a (i\not{\partial} - \sigma) \psi_a - \frac{N}{2g^2}\sigma^2 \right].$$

Вводя скалярное поле $\sigma(x)$, получаем «незаконченное» четырехфермионное взаимодействие. (Вспомним, что аналогичный прием мы использовали в главе III.5.) Вы узнаете, что физика в этом случае подобна той, которая лежит в основе введения промежуточного бозона с целью генерирования взаимодействия Ферми. С учетом того что известно из глав II.5 и IV.3, можно сразу же проинтегрировать по фермионным полям и получить действие, записанное только в терминах поля σ.

$$S(\sigma) = -\int d^2x \frac{N}{2g^2}\sigma^2 - iN \operatorname{tr} \log(i\not{\partial} - \sigma).$$

Обратим внимание на множитель N перед логарифмическим членом следа, который получается в результате интегрирования по фермионным полям N. С заранее обдуманным намерением мы или, вернее, Гросс и Невье, ввели в (16) явный множитель $1/N$ в выражение для силы взаимодействия, с тем чтобы оба члена в (18) масштабировались как N. Таким

образом, интеграл по траектории $\mathcal{Z} = \int D\sigma e^{iS(\sigma)}$ можно вычислить методом перевала или стационарной фазы в пределе больших N. Мы просто находим экстремумы $S(\sigma)$.

Кстати, таким же образом можно объяснить разумность выбора $a = 1$ в КХД больших N. Интегрируя по кваркам в (1), получаем

$$S = -\int d^4x \frac{N}{2g^2} \operatorname{tr} F_{\mu\nu}F^{\mu\nu} + N \operatorname{tr} \log(i(\not{\partial} - iA) - m)$$

и, следовательно, оба пропорциональны N и могут уравновешивать друг друга. Увеличение числа степеней свободы должно компенсироваться ослаблением взаимодействия.

Чтобы изучить поведение основного состояния теории, ограничимся рассмотрением не зависящих от x конфигураций поля $\sigma(x)$. (Другими словами, мы не ожидаем, что трансляционная симметрия будет спонтанно нарушаться.) Можно сразу взять результат, полученный при решении упражнения IV.3.3, и записать эффективный потенциал в виде

$$\frac{1}{N}V(\sigma) = \frac{1}{2g(\mu)^2}\sigma^2 + \frac{1}{4\pi}\sigma^2\left(\log\frac{\sigma^2}{\mu^2} - 3\right).$$

Мы наложили условие $(1/N)\left[d^2V(\sigma)/d\sigma^2\right]|_{\sigma=\mu} = 1/g(\mu)^2$ как определение константы связи $g(\mu)$, зависящей от масштаба (сравните с уравнением (IV.3.18)). Утверждение, что $V(\sigma)$ не зависит от μ, сразу же приводит к выражению

$$\frac{1}{g(\mu)^2} - \frac{1}{g\left(\mu'\right)^2} = \frac{1}{\pi}\log\frac{\mu}{\mu'}$$

Если $\mu \to \infty$, то $g(\mu) \to 0$. Примечательно, что эта теория является асимптотически свободной, как и КХД. При желании можно двигаться в обратном направлении, чтобы найти уравнение ренорм-группы

$$\mu\frac{d}{d\mu}g(\mu) = -\frac{1}{2\pi}g(\mu)^3 + \ldots$$

Эта теория в различных воплощениях - (16), (17) и (18) - обладает дискретной симметрией Z_2, под действием которой $\psi_a \to \gamma^5\psi_a$ и $\sigma \to -\sigma$. Как показано в главе IV.3, эта симметрия динамически нарушается квантовыми флуктуациями. Минимум $V(\sigma)$ имеет место при $\sigma_{\min} = \mu e^{1-\pi/g(\mu)^2}$, и тогда в соответствии с уравнением (17) фермионы приобретают массу

$$m_F = \sigma_{\min} = \mu e^{1-\pi/g(\mu)^2}.$$

Заметим, что этот весьма нетривиальный результат едва ли можно увидеть, глядя на выражение (16), и мы никоим образом не можем его доказать для конечных N. Однако, как и при разложении по большим N, можно ожидать, что масса фермионов может иметь вид $m_F = \mu e^{1-\pi/g(\mu)^2} + O\left(1/N^2\right)$, так что уравнение (22) было бы хорошим приближением даже, скажем, для $N = 3$. Поскольку m_F - физически измеримая величина, она не должна зависеть от μ. Можете это проверить.

В этой теории также возникает размерная трансмутация, описанная в предыдущей главе. Мы стартуем с теории с безразмерной константой связи g и имеем в итоге теорию с размерной массой фермиона m_F. В самом деле, любая другая величина с размерностью массы должна быть пропорциональна m_F.

Динамически генерируемые кинки

Существование кинков и солитонов рассмотрено в главе V.1. Вы хорошо усвоили, что существование таких объектов следует из общих соображений симметрии и топологии, а не из конкретной динамики. Здесь мы имеем (1+1)-мерную теорию с дискретной симметрией Z_2, поэтому можно, несомненно, ожидать появление кинка, а именно не зависящей от

времени конфигурации $\sigma(x)$ (впредь x будет обозначать только пространственную координату и не будет определять общую точку в пространстве-времени) такую, что $\sigma(-\infty) = -\sigma_{\min}$ и $\sigma(+\infty) = \sigma_{\min}$. [Очевидно, что существует также антикинк с $\sigma(-\infty) = \sigma_{\min}$ и $\sigma(+\infty) = -\sigma_{\min}$.] На первый взгляд, определение точной формы кинка может показаться почти невозможным. По существу, нужно определить $\operatorname{tr}\log[i\not\partial - \sigma(x)]$ для произвольной функции $\sigma(x)$, так что $\sigma(+\infty) = -\sigma(-\infty)($ и, как говорится в главе IV.3, это включает в себя вычисление всех собственных значений оператора $i\not\partial - \sigma(x)$, суммирование по логарифму собственных значений), и затем следует проварьировать этот функционал от $\sigma(x)$, чтобы найти оптимальную форму кинка.

Примечательно, что эту форму действительно можно найти, опираясь на одно нетривиальное наблюдение [4]. По аналогии с шагами, приведшими к (IV.3.24), заметим, что

$$\operatorname{tr}\log[i\not\partial - \sigma(x)] = \operatorname{tr}\log\gamma^5[i\not\partial - \sigma(x)]\gamma^5 = \operatorname{tr}\log(-1)[i\not\partial + \sigma(x)]$$

и, следовательно, с точностью до несущественной аддитивной постоянной получаем

$$\operatorname{tr}\log(i\not\partial - \sigma(x)) = \frac{1}{2}\operatorname{tr}\log[[\not\partial - \sigma(x)][i\not\partial + \sigma(x)]] = \\ = \frac{1}{2}\operatorname{tr}\log\left\{-\partial^2 + i\gamma^1\sigma'(x) - [\sigma(x)]^2\right\}$$

Поскольку γ^1 имеет собственные значения $\pm i$, последнее выражение будет равно

$$\frac{1}{2}\left\{\operatorname{tr}\log\left\{-\partial^2 + \sigma'(x) - [\sigma(x)]^2\right\} + \operatorname{tr}\log\left\{-\partial^2 - \sigma'(x) - [\sigma(x)]^2\right\}\right\}$$

но в силу четности (пространственного отражения) эти два слагаемых равны и поэтому

$$\operatorname{tr}\log[i\not\partial - \sigma(x)] = \operatorname{tr}\log\left\{-\partial^2 - \sigma'(x) - [\sigma(x)]^2\right\}$$

Возвращаясь к выражению (18), видим, что $S(\sigma)$ является суммой двух членов - члена второго порядка по $\sigma(x)$ и члена, который зависит только от комбинации $\sigma'(x) + [\sigma(x)]^2$. Но нам известно, что $\sigma_{\min}$ минимизирует $S(\sigma)$. Таким образом, солитон определяется путем решения обычного дифференциального уравнения вида

$$\sigma'(x) + [\sigma(x)]^2 = \sigma_{\min}^2$$

[4] C. Callan, S. Coleman, D. Gross, and A. Zee. (неоиубликовано). See D. J. Gross. «Applications of the Renormalization Group to High-Energy Physics», in: R. Balian and J. Zinn-Justin. eds., Methods in Field Theory, p. 247. Кстати. я рекомснцую эту книгу студентам. занимающимся теорией поля. а именно $\sigma(x) = \sigma_{\min}\operatorname{th}\sigma_{\min}x$. Солитон наблюдался бы как объект размером $1/\sigma_{\min} = 1/m_F$. Я оставляю вам показать, что его масса дается выражением

$$m_S = \frac{N}{\pi}m_F$$

В точном соответствии с теоретическим рассмотрением, приведенным в последней главе, отношение m_S/m_F оказывается равным просто числу N/π, как и следовало ожидать.

Используя еще более удачный метод, для описания которого здесь нет места, Дашен, Хасслахер и Невье смогли исследовать зависящие от времени конфигурации σ и определить спектр масс этой модели.

Упражнения

VII.4.1. Поскольку число глюонов отличается только на единицу, обычно утверждают, что нет никакой разницы в том, чтобы изучать $U(N)$-теорию или $SU(N)$-теорию. Рассмотрите, чем глюонный пропагатор в $U(N)$-теории отличается от глюонного пропагатора в $SU(N)$-теории, и решите, какой из них проще.

VII.4.2. Попробуйте свои силы и решите КХД больших N в $(1+1)$-мерном пространстве-времени. [Указание: ключ к решению состоит в том, что в (1+ +1)-мерном пространстве-времени при выборе подходящей калибровки можно отынтегрировать калибровочное поле A_μ.] Если нужна помощь, см. 't Hooft, Under the Spell of the Gauge Principle, p. 443.

VII.4.3. Покажите, что если бы мы решили вычислить $G(z) \equiv \langle (1/N)\,\mathrm{tr}(1/z-\varphi)\rangle$, нужно было бы соединить два открытых конца кваркового пропагатора. Видим, что рис. VII.4.5b и d приводят к одной и той же диаграмме. Завершите таким способом вычисление $G(z)$.

VII.4.4. Предположим, что матрица случайных величин φ является вещественной симметричной, а не эрмитовой. Покажите, что правила Фейнмана в этом случае более сложные. Вычислите плотность собственных значений. [Указание: пропагатор, представленный двойной линией, может закручиваться.]

VII.4.5. Для эрмитовой случайной матрицы φ вычислите

$$G_c(z,w) \equiv \left\langle \frac{1}{N}\,\mathrm{tr}\,\frac{1}{z-\varphi}\,\frac{1}{N}\,\mathrm{tr}\,\frac{1}{w-\varphi}\right\rangle - \left\langle \frac{1}{N}\,\mathrm{tr}\,\frac{1}{z-\varphi}\right\rangle \left\langle \frac{1}{N}\,\mathrm{tr}\,\frac{1}{w-\varphi}\right\rangle$$

для $V(\varphi) = \frac{1}{2}m^2\varphi^2$, используя фейнмановские диаграммы. [Заметим, что этот объект гораздо проще для изучения, чем тот, который нам нужно изучить, чтобы узнать, что такое локализация (см. упражнение VI.6.1.).] По- кажите, что, выбирая подходящие мнимые части, можно получить корреляцию плотности собственных значений с собой. Если нужна помощь, см. E. Brzin and A. Zee, Phys. Rev. E51: p. 5442, 1995.

VII.4.6. Используйте метод Фаддеева-Попова, чтобы вычислить J методом дайсоновского газа.

VII.4.7. Найдите $\rho(E)$ для $V(\varphi) = \frac{1}{2}m^2\varphi^2 + g\varphi^4$. Для достаточно большой отрицательной m^2 (снова двухъямный потенциал) ожидается, что плотность собственных значений разделится на две части. Это следует из представления дайсоновского газа. Найдите критическое значение m_c^2. При $m^2 < m_c^2$ использованное в тексте предположение, что $G(z)$ имеет только один разрез, неверно. Покажите, как в этом случае вычислить $\rho(E)$.

VII.4.8. Вычислите массу солитона (25).

30.4.5 Zi.VII.5 Grand Unification

Острая необходимость в объединении

Калибровочная теория определяется группой и представлениями, которым принадлежат поля материи. Вернемся к главе VII.2 и составим каталог для теории $SU(3)\otimes SU(2)\otimes U(1)$. Например, левые верхний и нижний кварки образуют дублет $\begin{pmatrix} u^\alpha \\ d^\alpha \end{pmatrix}_L$ с гиперзарядом $\frac{1}{2}Y = \frac{1}{6}$. Обозначим его как $\left(3,2,\frac{1}{6}\right)_L$ с тремя числами, указывающими на то, как эти поля преобразуются относительно $SU(3)\otimes SU(2)\otimes U(1)$. Аналогичным образом обозначим правый кварк как $\left(3,1,\frac{2}{3}\right)_R$. Записывая все это, мы видим, что кварки и лептоны каждого семейства помещены в

$$\left(3,2,\frac{1}{6}\right)_L, \left(3,1,\frac{2}{3}\right)_R, \left(3,1,-\frac{1}{3}\right)_L, \left(1,2,-\frac{1}{2}\right)_L \quad \text{и} \quad (1,1,-1)_R.$$

По сути, этот пестрый набор представлений остро нуждается в дальнейшем объединении. Кто бы мог построить Вселенную, выбросив этот странный на вид список?

Что мы хотели бы получить, так это большую калибровочную группу G, содержащую $SU(3)\otimes SU(2)\otimes U(1)$, чтобы этот длинный список представлений объединился (в идеале) в одно внушительное большое представление. Калибровочные бозоны в G [но, конечно, не в $SU(3)\otimes SU(2)\otimes U(1)$] будут связывать друг с другом представления в (1).

Прежде чем начать поиск G, заметим, что, поскольку калибровочные преобразования коммутируют с группой Лоренца, искомые калибровочные преобразования не могут изменить левые поля на правые поля. Поэтому заменим все поля в (1) на левые поля. Напомним из

упражнения II.1.8, что зарядовые сопряжения изменяют левые поля на правые и наоборот. Таким образом, вместо (1) можно записать

$$\left(3,2,\frac{1}{6}\right),\left(3^*,1,-\frac{2}{3}\right),\left(3^*,1,\frac{1}{3}\right),\left(1,2,-\frac{1}{2}\right)\ \text{и}\ (1,1,1).$$

Теперь опустим подстрочные индексы L и R: все поля становятся левыми.

Идеальная подгонка

Самой малой группой, содержащей $SU(3)\otimes SU(2)\otimes U(1)$, является $SU(5)$. (Если вы недостаточно хорошо знаете теорию групп, ознакомьтесь для начала с приложением B). Напомним, что $SU(5)$ имеет 24 генератора ($5^2-1=24$). Эти генераторы явно представлены эрмитовыми бесследовыми 5×5-матрицами, действующими на пять объектов, которые обозначим как ψ^μ, где $\mu=1,2,...,5$. [Эти пять объектов образуют фундаментальное, или определяющее, представление $SU(5)$.

Теперь понятно, как можно поместить $SU(3)$ и $SU(2)$ в $SU(5)$. Из 24-х матриц, которые генерируют $SU(5)$, восемь имеют вид $\begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix}$, а три вид $\begin{pmatrix} 0 & 0\\ 0 & B\end{pmatrix}$, где A представляет собой эрмитовы бесследовые 3×3-матрицы (из которых $3^2-1=8$ - так называемые матрицы Гелл-Манна), а B представляет собой эрмитовы бесследовые 2×2-матрицы (из которых $2^2-1=3$ - матрицы Паули). Ясно, что матрицы первого типа генерируют $SU(3)$, а матрицы второго типа - SU(2). Кроме того, эрмитова бесследовая 5×5-матрица

$$\frac{1}{2}Y=\begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0\\ 0 & -\frac{1}{3} & 0 & 0 & 0\\ 0 & 0 & -\frac{1}{3} & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2} & 0\\ 0 & 0 & 0 & 0 & \frac{1}{2}\end{pmatrix}$$

генерирует $U(1)$. Без лишней скромности мы уже назвали эту матрицу гиперзарядом $\frac{1}{2}Y$.

Другими словами, если разбить индекс $\mu=\{\alpha,i\}$ с $\alpha=1,2,3$ и $i=4,5$, то $SU(3)$ будет действовать на индекс α, а $SU(2)$ будет действовать на индекс i. Таким образом, три объекта ψ^α преобразуются как трехмерное представление относительно $SU(3)$, и, следовательно, могут быть 3 или 3^*. Выберем ψ^α, преобразующиеся как 3 ; далее мы увидим, что это правильный выбор, если $Y/2$ задано в таком виде, как в (3). Три объекта ψ^α не преобразуются относительно $SU(2)$, и, следовательно, каждый из них принадлежит представлению 1 синглета. Кроме того, они несут гиперзаряд $-\frac{1}{3}$, как видно из (3). В итоге ψ^α преобразуются как $\left(3,1,-\frac{1}{3}\right)$ относительно $SU(3)\otimes SU(2)\otimes U(1)$. С другой стороны, два объекта ψ^i преобразуются как 1 относительно $SU(3)$, как 2 относительно $SU(2)$ и несут гиперзаряд $\frac{1}{2}$; таким образом, они преобразуются как $\left(1,2,\frac{1}{2}\right)$. Другими словами, мы вкладываем $SU(3)\otimes SU(2)\otimes U(1)$ в $SU(5)$, определяя тем самым, как фундаментальное представление $SU(5)$ разлагается на представления $SU(3)\otimes SU(2)\otimes U(1)$

$$5\to\left(3,1,-\frac{1}{3}\right)\oplus\left(1,2,\frac{1}{2}\right).$$

Сопрягая, видим, что

$$5^*\to\left(3^*,1,\frac{1}{3}\right)\oplus\left(1,2,-\frac{1}{2}\right)$$

Сравнивая с (2), видим, что в списке появляются $\left(3*,1,\frac{1}{3}\right)$ и $\left(1,2,-\frac{1}{2}\right)$. Мы на правильном пути! Поля в этих двух представлениях хорошо помещаются в 5^*.

Таким образом, учтены пять полей из содержащихся в (2); но остаются еще десять полей

$$\left(3,2,\frac{1}{6}\right),\left(3^*,1,-\frac{2}{3}\right) \text{ и } (1,1,1).$$

Рассмотрим следующее по размеру представление $SU(5)$, а именно представление антисимметричного тензора $\psi^{\mu\nu}$. Его размер равен $(5\times 4)/2 = 10$. Именно такое число нам и нужно, только если получаются квантовые числа относительно $SU(3)\otimes SU(2)\otimes U(1)$!

Поскольку известно, что $5 \to \left(3,1,-\frac{1}{3}\right)\oplus\left(1,2,\frac{1}{2}\right)$, нужно просто (см. приложение B!) найти антисимметричное произведение $\left(3,1,-\frac{1}{3}\right)\oplus\left(1,2,\frac{1}{2}\right)$ с самим, то есть найти прямую сумму вида (где $\otimes_A$ обозначает антисимметричное произведение)

$$\left(3,1,-\frac{1}{3}\right)\otimes_A\left(3,1,-\frac{1}{3}\right)=\left(3^*,1,-\frac{2}{3}\right)$$
$$\left(3,1,-\frac{1}{3}\right)\otimes_A\left(1,2,\frac{1}{2}\right)=\left(3,2,-\frac{1}{3}+\frac{1}{2}\right)=\left(3,2,\frac{1}{6}\right)$$

и

$$\left(1,2,\frac{1}{2}\right)\otimes_A\left(1,2,\frac{1}{2}\right)=(1,1,1).$$

[Я помогу вам разобраться с (7): в $SU(3)3\otimes_A 3 = 3^*$ (помните ε_{ijk} из приложения B?), в $SU(2)1\otimes_A 1 = 1$ и в $U(1)$ гиперзаряды складываются $-\frac{1}{3}-\frac{1}{3}=-\frac{2}{3}$.]

Посмотрите, эти представления $SU(3)\otimes SU(2)\otimes U(1)$ образуют такой же набор представлений в (6). Другими словами,

$$10\to\left(3,2,\frac{1}{6}\right)\oplus\left(3^*,1,-\frac{2}{3}\right)\oplus(1,1,1)$$

Известные кварковые и лептонные поля в данном поколении идеально укладываются в представления 5^* и 10 группы $SU(5)$!

Я только что описал $SU(5)$-теорию великого объединения ДжорджиГлэшоу. Несмотря на то что эта теория до сих пор прямо не проверена на эксперименте, мне и многим другим физикам очень трудно не поверить, что группа $SU(5)$ является по меньшей мере структурно правильной, учитывая идеальное теоретико-групповое вложение.

Часто удобно представить состав представлений 5^* и 10, используя названия, которые различные поля получили в прошлом. Запишем представление 5^* в виде вектор-столбца

$$\psi_\mu=\begin{pmatrix}\psi_\alpha\\ \psi_\beta\end{pmatrix}=\begin{pmatrix}\bar{d}_\alpha\\ \nu\\ e\end{pmatrix}$$

а представление 10 в виде антисимметричной матрицы

$$\psi^{\mu\nu}=\left\{\psi^{\alpha\beta},\psi^{\alpha i},\psi^{ij}\right\}=\begin{pmatrix}0 & \bar{u} & -\bar{u} & u & d\\ -\bar{u} & 0 & \bar{u} & u & d\\ \bar{u} & -\bar{u} & 0 & u & d\\ -u & -u & -u & 0 & \bar{e}\\ -d & -d & -d & -\bar{e} & 0\end{pmatrix}$$

(Цветовые индексы опущены.)

Углубляя наше понимание физики

Помимо эстетической привлекательности, великое объединение значительно углубляет наше понимание физики.

1. Вы когда-нибудь размышляли о том, почему электрический заряд квантуется? Почему мы не видим частиц с зарядом, в $\sqrt{\pi}$ раз большим заряда электрона? В квантовой электродинамике вполне можно было бы записать

$$\mathcal{L} = \bar{\psi}[i(\not{\partial} - iA) - m]\psi + \bar{\psi}'\left[i(\not{\partial} - i\sqrt{\pi}A) - m'\right]\psi' + ...$$

В теории великого объединения, наоборот, A_μ взаимодействует с генератором калибровочной группы великого объединения. А как вы знаете, генераторы любой простой группы, такой как $SU(N)$ (считается, что группа, не образующаяся путем прямого произведения $U(1)$ на другие группы, является простой), должны через нетривиальные коммутационные соотношения $[T_a, T_b] = if_{abc}T_c$ допускать квантованные значения. Например, собственные значения T_3 в $SU(2)$, которые, конечно, зависят от представления, должны быть кратными $\frac{1}{2}$. В пределах группы $SU(3) \otimes SU(2) \otimes U(1)$, которая не является простой, квантование заряда понять нельзя. Генератор $U(1)$ не квантуется. Но при великом объединении в $SU(5)$ (или, в более общем случае, в любую простую группу) электрический заряд квантуется.

Здесь этот результат глубоко связан с замечанием Дирака (см. главу IV.4), согласно которому электрический заряд квантуется, если существует магнитный монополь. Из главы V.8 известно, что спонтанно нарушенные неабелевы калибровочные теории, такие как теория $SU(5)$, содержат монополь.

2. Вы когда-нибудь размышляли о том, почему заряд протона точно равен по величине и противоположен по знаку заряда электрона? Этот важный факт позволяет построить Вселенную в известном нам виде. Чтобы работала обычная космология, атомы должны быть электрически нейтральны с фантастической степенью точности; иначе электростатические силы, действующие в макроскопическом веществе, разорвали бы Вселенную.

Этот поразительный факт прекрасно вписывается в $SU(5)$. Наблюдение за тем, как это происходит, доставляет истинное удовольствие. Из вычисления $\operatorname{tr} Q = 0$ по представлению 5^* следует, что $3Q_{\bar{d}} = -Q_{e^-}$ЗЗесь использовано то, что сильное взаимодействие коммутирует с электромагнетизмом, и поэтому кварки различного цвета имеют одинаковый заряд. Теперь вычислим заряд протона

$$Q_p = 2Q_u + Q_d = 2\left(Q_d + 1\right) + Q_d = 3Q_d + 2 = Q_{e^-} + 2$$

Если $Q_{e^-} = -1$, то $Q_P = -Q_{e^-}$, как и есть на самом деле!

3. Напомним, что в теории электрослабых взаимодействий мы определили $\operatorname{tg}\theta = g_1/g_2$ с взаимодействием калибровочных бозонов $g_2 A_\mu^a T_a +$ $+ g_1 B_\mu (Y/2)$. Поскольку нормировка A_μ^a и B_μ фиксирована их соответствующим членом кинетической энергии, относительная величина g_2 и g_1 определяется нормировкой $Y/2$ относительно T_3. Вычислим $\operatorname{tr} T_3$ и $\operatorname{tr}(Y/2)^2$ в фундаментальном представлении 5: $\operatorname{tr} T_3^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$ и $\operatorname{tr}(Y/2)^2 = \left(\frac{1}{3}\right)^2 3 + \left(\frac{1}{2}\right)^2 2 = \frac{5}{6}$.

Таким образом, T_3 и $\sqrt{3/5}(Y/2)$ нормированы одинаково. Итак, правильной комбинацией великого объединения является $A_\mu^a T_a + B_\mu\sqrt{3/5}(Y/2)$, и поэтому $\operatorname{tg}\theta = g_1/g_2 = \sqrt{3/5}$ или

$$\sin^2\theta = \frac{3}{8}$$

на масштабе великого объединения. Чтобы сравнить с экспериментальным значением $\sin^2\theta$, необходимо установить, как бегут константы взаимодействия g_2 и g_1 под действиcм ренорм-группы в направлении низких энергий. Отложим это рассмотрение до следующей главы.

Свобода от аномалий

Напомним, что, как описано в главе VII.2, ключом к доказательству перенормируемости неабелевой калибровочной теории является возможность свободного перехода между унитарной калибровкой и калибровкой R_ξ. Принципиальным элементом здесь является калибровочная инвариантность и получающиеся в результате тождества Уорда-Такахаши (см. главу II.7).

Внезапно у нас возникает беспокойство. Присутствует ли киральная аномалия? Существование такой аномалии означает, что некоторые тождества Уорда-Такахаши не могут выполняться. Чтобы наши теории имели смысл, им лучше не иметь аномалий. В главе IV.7 говорилось, что историческое название «аномалия» звучит так, как какая-то болезнь. В некотором смысле это так.

Теорию $SU(3)\otimes SU(2)\otimes U(1)$ пора уже было проверить на аномалии, но мы это не сделали. В качестве упражнения сделайте это самостоятельно. Здесь же я покажу, что теория $SU(5)$ безаномальна. Если в теории $SU(5)$ аномалии отсутствуют, то, значит, их также нет в теории $SU(3)\otimes SU(2)\otimes U(1)$

В главе IV.7 мы вычислили аномалию в абелевой теории, но, как там говорилось, все, что нужно сделать для обобщения до неабелевой теории, это ввести генератор T_a калибровочной группы для каждой вершины треугольной диаграммы на рис. IV.7.1. Суммируя по различным фермионам, бегущим по петле, легко видеть, что эта аномалия пропорциональна $A_{abc}(R) \equiv \operatorname{tr}(T_a\{T_b,T_c\})$, где R обозначает представление, которому принадлежат фермионы. Нужно просуммировать $A_{abc}(R)$ по всем представлениям в теории, не забывая связать противоположные знаки с левыми и правыми фермионными полями. (Полезно вспомнить о замечании 3 и упражнении 6 из главы IV.7.)

Теперь мы готовы проверить, не больна ли теория $SU(5)$. Во-первых, все фермионные поля в (2) левые. Во-вторых, можно проверить (просто представьте себе вычисление A_{abc} для всех возможных abc), что достаточно положить все T_a, T_b и T_c равными

$$T \equiv \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

мультиплету гиперзаряда. Теперь вычислим $\operatorname{tr} T^3$ в представлении 5^*:

$$\operatorname{tr} T^3\big|_{5^*} = 3(-2)^3 + 2(+3)^3 = 30$$

и в представлении 10:

$$\operatorname{tr} T^2\big|_{10} = 3(+4)^3 + 6(-1)^3 + (-6)^3 = -30.$$

Явное чудо! Аномалия сократилась.

Это поразительное сокращение сумм кубов из странного списка чисел убедительно свидетельствует о том, что группа $SU(5)$ - еще не конец истории. Кроме того, было бы хорошо, если бы представления 5^* и 10 можно было объединить в одно представление.

Упражнения

VII.5.1. Запишите оператор заряда Q, действующий на представление 5, т. е. определяющее представление ψ^μ. Найдите заряды для представления $10 = \psi^{\mu\nu}$ и идентифицируйте содержащиеся там различные поля.

VII.5.2. Покажите, что для любой теории великого объединения, если она основана на простой группе, на масштабе объединения получаем

$$\sin^2\theta = \frac{\sum T_3^2}{\sum Q^2},$$

где сумма берется по всем фермионам. VII.5.3. Проверьте, что в теории $SU(3)\otimes SU(2)\otimes U(1)$ отсутствует аномалия. [Указание: вычисление является более сложным, чем в $SU(5)$ из-за присутствия большего числа независимых генераторов. Сначала докажите, что достаточно определить $\mathrm{tr}\, Y\,\{T_a, T_b\}$ и $\mathrm{tr}\, Y^3$ с генераторами T_a и Y для $SU(2)$ и $U(1)$ соответственно.]

VII.5.4. Постройте теории великого объединения, основанные на $SU(6), SU(7), SU(8), ...$, пока вам не надоест это занятие. Обычно такое занятие увлекает. [Указание: нужно придумать фермионы, которые еще предстоит открыть экспериментально.]

30.4.6 Zi.VII.6 Protons Are Not Eternal (???)

(??? or maybe they are???)

Распад протона

Сохранение заряда гарантирует устойчивость электрона, но как же обстоит дело с устойчивостью протона? Сохранение заряда допускает распад $p \to \pi^0 + e^+$. Ни один фундаментальный принцип не гласит, что протон живет вечно, но, однако же, протон известен своей долговечностью. По существу, так было со времени возникновения Вселенной.

Стабильность протона должна была быть декларирована авторитетной личностью. Юджин Вигнер первым обнаружил закон сохранения барионного числа. Говорят, что когда Вигнера спросили, откуда он знает, что протон живет вечно, он усмехнулся: «Я чувствую это своими костями». Я трактую эту фразу следующим образом: из одного только факта, что мы не светимся в темноте, можно сделать приемлемый вывод о нижнем пределе продолжительности существования протона.

Приступая к великому объединению, мы начинаем беспокоиться. В общем, в моделях великого объединения мы помещаем кварки и лептоны в одно и то же представление некоторой калибровочной группы [см. (VII.5.11) и (VII.5.12)]. Такое смешение сразу же предполагает, что существуют калибровочные бозоны, преобразующие кварки в лептоны и наоборот. Комбинация из трех кварков, известная как протон, могла бы вполне превратиться в лептоны при обмене этими калибровочными бозонами. Другими словами, протон, будучи опорой, на которой построен наш мир, может оказаться не вечным! Таким образом, теория великого объединения рискует тотчас же оказаться несостоятельной.

Пусть в общем случае M_X обозначает массы тех калибровочных бозонов, которые превращают кварки в лептоны и наоборот. Тогда амплитуда распада протона имеет порядок g^2/M_X^2, где g - константа взаимодействия калибровочной группы великого объединения, а скорость распада протона Γ определяется выражением $(g^2/M_X^2)^2$, умноженным на фактор фазового пространства, который определяется, по сути, массой протона m_P, поскольку массы пиона и позитрона незначительны по сравнению с массой прото- на. С помощью анализа размерностей определяем, что $\Gamma \sim (g^2/M_X^2)^2\, m_P^5$. Поскольку известно, что протон живет по крайней мере 10^{31} лет, лучше, чтобы масса M_X была намного больше тех масштабов энергии, которые можно достичь экспериментально.

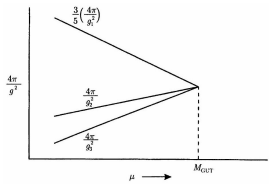


Рис. VII.6.1

Масса M_X имеет тот же порядок, что и массовый масштаб $M_{\rm GUT}$, на котором теория великого объединения спонтанно нарушается вплоть до $SU(3)\otimes SU(2)\otimes U(1)$. В частности, в теории $SU(5)$ поле Хиггса H^μ_ν, преобразующееся как сопряженное 24, с вакуумным средним $\langle H^\mu_\nu\rangle$, равным диагональной матрице с элементами $\left(-\frac{1}{3},-\frac{1}{3},-\frac{1}{3},\frac{1}{2},\frac{1}{2}\right)$, умноженными на некоторое v, может привести к необходимому результату, как обсуждалось в главе IV.6. Калибровочные бозоны в $SU(3)\otimes SU(2)\otimes U(1)$ остаются безмассовыми, тогда как другие калибровочные бозоны приобретают массу M_X порядка gv.

Чтобы определить $M_{\rm GUT}$, применим ренорм-групповой поток к g_3, g_2 и g_1 - константам взаимодействий в $SU(3), SU(2)$ и $U(1)$ соответственно. Идея заключается в том, что при движении вверх по масштабу массы или энергии μ две асимптотически свободные константы взаимодействия $g_3(\mu)$ и $g_2(\mu)$ уменьшаются, тогда как $g_1(\mu)$ увеличивается. Таким образом, при некотором массовом масштабе $M_{\rm GUT}$ они встретятся, и именно там $SU(3)\otimes SU(2)\otimes U(1)$ объединятся в $SU(5)$ (см. рис. VII.6.1). Из-за исключительно медленного логарифмического бега (который нужно бы называть продвижением или даже ползанием, но опять же по историческим причинам мы говорим о беге) константы связи, мы ожидаем, что массовый масштаб $M_{\rm GUT}$ объединения окажется намного больше любого масштаба, который использовался в физике элементарных частиц до великого объединения. На самом деле оказывается, что $M_{\rm GUT}$ имеет огромное значение порядка $10^{14}-10^{15}$Г эВ, и идея великого объединения преодолевает первое препятствие.

Устойчивость Вселенной предполагает слабость электромагнетизма

Используя результат решения упражнения VI.8.1, получаем (здесь $\alpha_S \equiv g_3^2/4\pi$ и $\alpha_{\rm GUT} = g^2/4\pi$ обозначают аналоги постоянной тонкой структуры α для случая сильного взаимодействия и великого объединения соответственно, а F - количество поколений)

$$\frac{4\pi}{[g_3(\mu)]^2}\equiv\frac{1}{\alpha_S(\mu)}=\frac{1}{\alpha_{\rm GUT}}+\frac{1}{6\pi}(4F-33)\log\frac{M_{\rm GUT}}{\mu},$$
$$\frac{4\pi}{[g_2(\mu)]^2}\equiv\frac{\sin^2\theta(\mu)}{\alpha_S(\mu)}=\frac{1}{\alpha_{\rm GUT}}+\frac{1}{6\pi}(4F-22)\log\frac{M_{\rm GUT}}{\mu},$$
$$\frac{3}{5}\frac{4\pi}{[g_1(\mu)]^2}\equiv\frac{3}{5}\frac{\cos^2\theta(\mu)}{\alpha_S(\mu)}=\frac{1}{\alpha_{\rm GUT}}+\frac{1}{6\pi}4F\log\frac{M_{\rm GUT}}{\mu}.$$

Под $\theta(\mu)$ понимается значение θ на масштабе μ. При $\mu = M_{\rm GUT}$ три константы взаимодействия связаны друг с другом через $SU(5)$.

Вычислим эти уравнения для некоторого экспериментально доступного значения μ, подставляя измеренные значения α_S и α. Имея три уравнения, можно не только определить шкалу объединения $M_{\rm GUT}$ и взаимодействие $\alpha_{\rm GUT}$, но также предсказать θ. Другими словами, если отношение g_1 к g_2 не является точно правильным, три линии на рис. VII.6.1 не пересекутся в одной точке.

Заметим, что число поколений фермионов F входит одинаково в уравнения (1), (2) и (3). Так и должно быть, поскольку при нашем вычислении фермионы эффективно считаются безмассовыми и не «знают», что объединяющая группа нарушена до $SU(3)\otimes SU(2)\otimes U(1)$. Эти уравнения выведены исходя из предположения, что все фермионные массы малы по сравнению с μ. Несколько преобразуя эти уравнения, находим

$$\sin^2\theta=\frac{1}{6}+\frac{5\alpha(\mu)}{9\alpha_S(\mu)},$$
$$\frac{\sin^2\theta}{\alpha(\mu)}=\frac{1}{\alpha_S(\mu)}+\frac{1}{6\pi}11\log\frac{M_{\rm GUT}}{\mu},$$
$$\frac{1}{\alpha(\mu)}=\frac{8}{3}\frac{1}{\alpha_{GUT}}+\frac{1}{6\pi}\left(\frac{32}{3}F-22\right)\log\frac{M_{\rm GUT}}{\mu}.$$

Уравнение (4) позволяет предсказать значение $\sin^2\theta(\mu)$, которое не зависит от $M_{\rm GUT}$ и числа поколений.

Заметим, что уравнение (5) дает ограничение на

$$\frac{1}{\alpha(\mu)} \geqslant \frac{1}{6\pi} 11 \log \frac{M_{\rm GUT}}{\mu}.$$

Нижняя граница времени жизни протона (и следовательно, шкалы $M_{\rm GUT}$) преобразуется в верхнюю границу для константы тонкой структуры. Занятно, что устойчивость Вселенной предполагает слабость электромагнетизма.

Как говорилось ранее, подставляя в измеренное значение α_S, получаем огромное значение для $M_{\rm GUT}$. Я считаю это торжеством великого объединения. Масштаб $M_{\rm GUT}$ мог бы получиться намного меньше, что сразу привело бы к противоречию с наблюдаемой стабильностью протона, но этого не произошло. Взглянем на ситуацию по-другому: если нам каким-то образом известны $M_{\rm GUT}$ и $\alpha_{\rm GUT}$, то великое объдинение фиксирует константы связи всех трех негравитационных взаимодействий! Дело не в том, что это простейшее испытание великого объединения не совсем согласуется с экспериментом. Удивительно то, что оно вообще работает.

Подробное рассмотрение того, как уравнения (4), (5) и (6) согласуются с экспериментом, выходит за рамки настоящей книги. Для глубокого феноменологического рассмотрения необходимо включить пороговые эффекты (см. упражнение VII.6.1), поправки более высокого порядка и т. д. Короче говоря, появление теории великого объединения вызвало огромное волненис в связи с возможностью распада протона. Увы, экспсримснтально полученная нижняя граница времени жизни протона в итоге оказалась выше предсказанной. Это, конечно, не означает кончину идеи великого обьдинения. В самом деле, как я отметил ранее, идеального вложения достаточно, чтобы убедить большинство теоретиков, занимающихся физикой элементарных частиц, в существенной правильности этой идеи. С годами люди предложили добавить к теории различные гипотетические частицы, чтобы увеличить продолжительность жизни протона. Идея состоит в том, что такие частицы воздействовали бы на ренорм-групповой поток и, следовательно, на $M_{\rm GUT}$. Время жизни протона на самом деле не является самым критическим вопросом. При более точных измерениях α_S и θ было установлено, что три взаимодействия не совсем пересекаются в одной точке. Действительно, вера сторонников суперсимметрии при малых энергиях частично основана на том, что при включении суперсимметричных частиц три константы связи в самом деле встречаются'. Но скептики, конечно, могут утверждать, что существует дополнительная свобода действий.

Относительные вероятности

Вы, должно быть, осознаете, что уравнения (1), (2) и (3) не являются специфичными для $SU(5)$: они сохраняются до тех пор, пока $SU(3) \otimes SU(2) \otimes U(1)$ объединяется в некоторую простую группу (настолько простую, что существует только одно калибровочное взаимодействие g).

Теперь сосредоточимся на $SU(5)$. Вспомним, что мы делим $SU(5)$-индекс μ, который может принимать пять значений, на два типа. Другими словами, индекс μ обозначается как $\{\alpha, i\}$, где α принимает три значения, а i принимает два значения. Калибровочные бозоны в $SU(5)$ соответствуют 24 независимым составляющим бесследового эрмитова поля A^μ_ν $(\mu, \nu = 1, 2, ..., 5)$, преобразующегося как присоединенное представление. Фокусируясь на теоретико-групповых свойствах $SU(5)$, опустим лоренцевы индексы, спинорные индексы и т. д. Ясно, что восемь калибровочных бозонов в $SU(3)$ преобразуют индекс типа α в индекс типа α, тогда как три калибровочных бозона в $SU(2)$ преобразуют индекс типа i в индекс типа i. Еще существует калибровочный бозон $U(1)$, который взаимодействует с гиперзарядом $\frac{1}{2}Y$. (Вы, конечно, понимаете, что я имею в виду, когда использую несколько вольный стиль изложения. Калибровочные бозоны $SU(3)$ преобразуют поля, несущие цветовой индекс, в поле, несущее цветовой индекс.)

Интересная ситуация возникает, когда калибровочные бозоны A_i^α и A_α^i преобразуют индекс α в индекс i и наоборот. Поскольку на α приходится три значения, а на i два значения, в итоге имеем $6+6=12$ таких калибровочных бозонов, учитывая, таким образом, все калибровочные бозоны в $SU(5)$. Другими словами, $24 \to (8,1)+(1,3)+(1,1)+(3,2)+(3^*,2)$. Теперь ясно видно, что обмен этими бозонами между кварками и лептонами ведет к распаду протона.

[1] См., например, F. Wilczek, in: V. Fitch et al., eds., Critical Problems in Physics, p. 297.

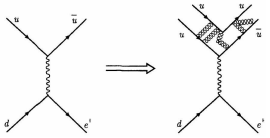


Рис. VII.6.2

Нужно просто записать в лагранжиане члены, включающие взаимодействие бозонов A_i^α и A_α^i с фермионами, и нарисовать соответствующие фейнмановские диаграммы. Я частично выполню теоретико-групповой анализ, но остальное оставлю за вами. Сворачивая индексы, легко видеть, что бозон A_ν^μ, действуя на ψ_μ, преобразует его к виду ψ_ν, а действуя на $\psi^{\nu\rho}$, преобразует его к виду $\psi^{\mu\rho}$. Используя полученное вами решение упражнения VII.5.1, посмотрим, что делает A_α^5. Он приводит к следующим переходам:

$$\psi_5 = e^- \to \psi_\alpha = \bar{d}$$
$$\psi^{\alpha\beta} = \bar{u} \to \psi^{5\beta} = u$$

К

$$\psi^{\alpha 4} = d \to \psi^{54} = e^+$$

Таким образом, обмен A_α^5 генерирует процесс (рис. VII.6.2) $u+d \to \bar{u}+e^+$, ведущий к распаду протона $p(uud) \to \pi^0(u\bar{u})+e^+$. Обратите внимание, что хотя распад $p \to \pi^0 + e^+$нарушает как барионное число B, так и лептонное число L, он сохраняет комбинацию $B-L$.

В упражнении VII.6.2 вы получите относительные вероятности для различных мод распада. Нерадивые экспериментаторы их еще не измерили.

Массы фермионов

Можно было надеяться, что с великим объединением у нас возникнет новое понимание масс кварков и лептонов. К сожалению, ситуация с массами фермионов в $SU(5)$ запутана, и до сегодняшнего дня никто не понимает происхождение масс кварков и лептонов. Вводя поле Хиггса φ^μ, преобразующееся как представление 5 (это видно из обозначения), можно записать взаимодействие

$$\psi_\mu C \psi^{\mu\nu} \varphi_\nu$$

К

$$\psi^{\mu\nu} C \psi^{\lambda\rho} \varphi^\sigma \varepsilon_{\mu\nu\lambda\rho\sigma}$$

(где φ_ν - сопряженное представление 5^*), отражающее теоретико-групповой факт (см. приложение С), что $5^* \otimes 10$ содержит представление 5, а $10 \otimes 10$ содержит представление 5^*.

Поскольку $5 \to \left(3,1,-\frac{1}{3}\right) \oplus \left(1,2,\frac{1}{2}\right)$, легко видеть, что поле Хиггса является просто естественным расширением $SU(2) \otimes U(1)$-дублета Хиггса $\left(1,2,\frac{1}{2}\right)$. Не желая нарушать электромагнетизм, мы допускаем, чтобы вакуумное среднее приобрела только электрически нейтральная четвертая компонента φ. Задавая $\langle\varphi^4\rangle = v$, получаем (с точностью до несущественных общих постоянных)

$$\psi_\alpha C \psi^{\alpha 4} + \psi_5 C \psi^{54} \Rightarrow m_d = m_e$$

K

$$\psi^{\alpha\beta}C\psi^{\gamma^5}\varepsilon_{\alpha\beta\gamma} \Rightarrow m_u \neq 0$$

Более высокая симметрия дает отношение масс $m_d = m_e$ на масштабе объединения; мы еще раз должны использовать ренорм-групповой поток. Следует отметить, что отношение масс $m_d = m_e$ возникает потому, что, когда речь идет о фермионах, $SU(5)$ нарушается только до $SU(4)$ из-за φ. Трудность в том, что мы получаем более или менее одинаковое отношение для каждого из трех поколений, поскольку большая часть пробега происходит между $M_{\rm GUT}$ масштаба объединения и массой верхнего кварка, так что пороговые эффекты дают только малую поправку. Подставляя числа, получаем примерно следующее:

$$\frac{m_b}{m_\tau} \sim \frac{m_s}{m_\mu} \sim \frac{m_d}{m_e} \sim 3$$

Используем полученный результат для предсказания масс кварков нижнего сектора через массы лептонов. Соотношение $m_b \sim 3m_\tau$ работает достаточно хорошо и косвенно доказывает, что могут существовать только три поколения, поскольку ренорм-групповой поток зависит от F. Соотношение $m_s \sim 3m_\mu$ более или менее верно в зависимости от того, какое взять «экспериментальное» значение для m_s. С другой стороны, выражение для m_d явно не входит ни в какие рамки. Одни приговаривают при этом, что первое поколение очень легкое, и поэтому важное значение могут иметь другие эффекты, такие как однопетлевые поправки. Другие, делая теорию более уродливой, придумывают также разные схемы, вводя большее число полей Хиггса, например 45, чтобы наделить фермионы массой.

Заметим, что в одном отношении $SU(5)$ не так экономна, как $SU(2)\otimes U(1)$, в которой поле Хигтса, которое дает массу калибровочным бозонам, также наделяет массой фермионы.

Вселенная не пуста, а почти пуста

Я расскажу об еще одном триумфе великого объединения: о его способности объяснить происхождение вещества во Вселенной. Физикам понадобилось много времени, чтобы понять два принципиальных факта, касающихся Вселенной: во-первых, Вселенная не пуста, и, во-вторых, Вселенная почти пуста. Для физиков первое означает, что во Вселенной нет симметрии между веществом и антивеществом, то есть барионное число N_B не равно нулю; а второй факт количественно выражается в поразительно малой наблюдаемой величине отношения числа барионов к числу фотонов $N_B/N_\gamma \sim 10^{-10}$.

Допустим, мы начинаем со Вселенной с равным количеством вещества и антивещества. Чтобы Вселенная развивалась в такую, где доминирует наблюдаемое вещество, должны соблюдаться три условия. (1) Законы Вселенной должны быть асимметричными между веществом и антивеществом. (2) Соответствующие физические процессы должны были быть неравновесными, чтобы была стрела времени. (3) Должно нарушаться барионное число.

Нам достоверно известно, что условия (1) и (2) действительно соблюдаются во Вселенной: в слабом взаимодействии есть нарушение комбинированной четности CP, и ранняя Вселенная расширялась быстро. Что касается условия (3), великое объединение естественным образом нарушает барионное число. Кроме того, хотя распад протонов (сдерживаемый множителем $1/M^2_{\rm GUT}$ в амплитуде) происходит с мучительно медленной скоростью (для тех, кто участвует в эксперименте по распаду протона!), в ранней Вселенной, когда X - и Y-бозонов образовывалось в изобилии, их быстрые распады легко могли приводить к нарушению барионного числа. Сдерживающий множитель $1/M^2_{\rm GUT}$ не появляется. Я не сомневаюсь, что в конечном счете число 10^{-10}, измеряющее «количество грязи во Вселенной» будет вычислено в какой-то теории великого объединения.

Иерархия

Я обещал, что явление Вайскопфа вернется и будет нас преследовать. То, что массовый масштаб великого объединения $M_{\rm GUT}$ естественным образом получается таким большим, считается триумфом, но это также создает проблему, известную как проблема иерархии.

Иерархия связана с огромным значением отношения $M_{\rm GUT}/M_{\rm EW}$, где $M_{\rm EW}$ обозначает шкалу электрослабого объединения порядка 10^2 ГэВ. Я опишу этот довольно туманный предмет в общих чертах. Посмотрим на поле Хиггса φ, ответственного за нарушение электрослабой теории. Его ренормированная или физическая масса точно неизвестна, но известно, что она имеет порядок $M_{\rm EW}$. Представим вычисление затравочного ряда возмущений в некой теории великого объединения (сама теория не обсуждается), начиная с некоторой затравочной массы μ_0 для φ. Из явления Вайскопфа следует, что квантовая поправка смещает μ_0^2 на огромную величину, квадратично зависящую от параметра обрезания, $\delta\mu_0^2 \sim f^2\Lambda^2 \sim f^2 M_{\rm GUT}^2$, где единственный естественный массовый масштаб заменяется на Λ, а именно $M_{\rm GUT}$, и где f обозначает некоторое безразмерное взаимодействие. Чтобы физическая масса в квадрате $\mu^2 = \mu_0^2 + \delta\mu_0^2$ имела порядок $M_{\rm EW}^2$, что примерно на 28 порядков меньше $M_{\rm GUT}^2$, потребовалось бы исключительно тонкое и весьма неестественное сокращение между μ_0^2 и $\delta\mu_0^2$. Как это может произойти «естественным образом» - серьезная задача для физиков-теоретиков.

Натуральность

Проблема иерархии тесно связана с дорогим сообществу физиков-теоретиков понятием натуральности. Мы, естественно, ожидаем, что безразмерные отношения параметров в наших теориях должны быть порядка единицы, где «порядок единицы» свободно трактуется между коллегами, скажем, где-то от 10^{-2} или 10^{-3} до 10^2 или 10^3. Малость безразмерного параметра η рассматривается как естественная, если только в пределе $\eta \to 0$ появляется симметрия. Таким образом, массы фермионов могли бы быть естественно малыми, поскольку, как вы помните из главы II.1, киральная симметрия появляется, когда масса фермиона полагается равной нулю. С другой стороны, никакая конкретная симметрия не появляется, когда мы задаем равной нулю либо затравочную, либо ренормированную массу скалярного поля. В этом состоит суть проблемы иерархии.

Упражнения

VII.6.1. Предположим, существуют новые поколения F' кварков и лептонов с массами порядка M'. Используя грубое приближение, описанное в упражнении VI.8.2 и заключающееся в пренебрежении этими поколениями для μ, меньших M', и рассмотрении M' как пренебрежимо малой для μ, больших M', примените ренорм-групповой поток и рассмотрите, как изменяются различные предсказания, такие как время жизни протона.

VII.6.2. Проведите все вычисления для распада протона. Выведите соотношения между следующими ширинами: $\Gamma\left(p \to \pi^0 e^+\right), \Gamma\left(p \to \pi^+ \bar{\nu}\right), \Gamma\left(n \to \pi^- e^+\right)$ и $\Gamma\left(n \to \pi^0 \bar{\nu}\right)$.

VII.6.3. Покажите, что $SU(5)$ сохраняет комбинацию $B - L$. Попробуйте свои силы и придумайте теорию великого объединения, которая нарушает $B - L$.

30.4.7 Zi.VII.7 $SO(10)$ Unification

Каждое поколение в одном представлении

В конце главы VII.5 мы полагали, что есть достаточно оснований считать, что объединение $SU(5)$ не окончательное. Зададимся вопросом, можно ли вложить представления 5 и 10^* в одно представление большей группы G, содержащей $SU(5)$?

Оказывается, что существует естественное вложение $SU(5)$ в ортогональную группу $SO(10)$[1], но чтобы это объяснить, я должен немного рассказать вам о теории групп. Начало может показаться несколько неожиданным. Вернемся к главе II.3, где описано, что группа Лоренца $SO(3,1)$ или ее евклидов родственник $SO(4)$ имеют спинорные представления. Теперь обобщим спиноры на d-мерное евклидово пространство. Я подробно опишу конструкцию для четных d, а спиноры в нечетных размерностях оставляю вам в качестве упражнения. Возможно, вам сейчас стоит также обратиться к приложению B.

Алгебра Клиффорда и спинорные представления

Начнем с утверждения. Мы утверждаем, что для любого целого числа n можно найти $2n$ эрмитовых матриц $\gamma_i (i = 1, 2, ..., 2n)$, которые удовлетворяют алгебре Клиффорда

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

Другими словами, чтобы доказать наше утверждение, нужно построить $2n$ эрмитовых матриц γ_i, которые антикоммутируют друг с другом и квадрат которых равен единичной матрице. Будем называть γ_i-матрицы как γ-матрицы для $SO(2n)$.

Для $n = 1$ задача проста: $\gamma_1 = \tau_1$ и $\gamma_2 = \tau_2$. Вот и все.

[1] Говард Джорджи говорил мне, что он фактически нашел $SO(10)$ раньше $SU(5)$. Теперь выполним итерацию: имея $2n\gamma$-матриц для $SO(2n)$, построим $(2n+2)\gamma$-матриц для $SO(2n+2)$ следующим образом:

$$\gamma_j^{(n+1)} = \gamma_j^{(n)} \otimes \tau_3 = \begin{pmatrix} \gamma_j^{(n)} & 0 \\ 0 & -\gamma_j^{(n)} \end{pmatrix}, \quad j = 1, 2, ..., 2n,$$

$$\gamma_{2n+1}^{(n+1)} = 1 \otimes \tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\gamma_{2n+2}^{(n+1)} = 1 \otimes \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

(В этой книге 1 обозначает единичную матрицу соответствующего размера.) Очевидно, верхний индекс в круглых скобках нужен, чтобы знать, о каком наборе γ-матриц идет речь. Проверим, что если матрицы $\gamma^{(n)}$ удовлетворяют алгебре Клиффорда, тогда то же относится к матрицам $\gamma^{(n+1)}$. Например,

$$\left\{\gamma_j^{(n+1)}, \gamma_{2n+1}^{(n+1)}\right\} = \left(\gamma_j^{(n)} \otimes \tau_3\right) \cdot (1 \otimes \tau_1) + (1 \otimes \tau_1) \cdot \left(\gamma_j^{(n)} \otimes \tau_3\right) =$$
$$= \gamma_j^{(n)} \otimes \{\tau_3, \tau_1\} = 0.$$

Эта итерационная конструкция дает для $SO(2n)\gamma$-матрицы

$$\gamma_{2k-1} = 1 \otimes 1 \otimes ... \otimes 1 \otimes \tau_1 \otimes \tau_3 \otimes \tau_3 \otimes ... \otimes \tau_3$$

и

$$\gamma_{2k} = 1 \otimes 1 \otimes ... \otimes 1 \otimes \tau_2 \otimes \tau_3 \otimes \tau_3 \otimes ... \otimes \tau_3$$

где 1 появляется $k-1$ раз, а τ_3 появляется $n-k$ раз. Очевидно, что γ-матрицы имеют размер $2^n \times 2^n$. Если вы чувствуете, что сбиваетесь с толку в каком-либо месте наших рассуждений, вам следует получить решения в явном виде для $SO(4), SO(6)$ и т. д.

По аналогии с группой Лоренца определяем $2n(2n-1)/2 = n(2n-1)$ эрмитовых матриц

$$\sigma_{ij} \equiv \frac{i}{2}[\gamma_i, \gamma_j]$$

Заметим, что σ_{ij} равна $i\gamma_i\gamma_j$ при $i \neq j$ и зануляется при $i = j$. Таким образом, легко получить коммутаторы σ друг с другом. Например,

$$[\sigma_{12}, \sigma_{23}] = -[\gamma_1\gamma_2, \gamma_2\gamma_3] = -\gamma_1\gamma_2\gamma_2\gamma_3 + \gamma_2\gamma_3\gamma_1\gamma_2 = -[\gamma_1, \gamma_3] = 2i\sigma_{13}$$

Грубо говоря, γ_2-матрицы в σ_{12} и σ_{23} выбивают друг друга. Таким образом, видно, что $\frac{1}{2}\sigma_{ij}$ удовлетворяют тем же коммутационным соотношениям, что и J^{ij}-генераторы $SO(2n)$ (как приведено в приложении B). Матрицы $\frac{1}{2}\sigma_{ij}$ представляют собой J^{ij}.

Как и $2^n \times 2^n$-матрицы, σ действуют на объект ψ с 2^n компонентами, который назовем спинором ψ. Рассмотрим унитарное преобразование $\psi \to e^{i\omega_{ij}\sigma_{ij}}\psi$, с набором действительных чисел $\omega_{ij} = -\omega_{ji}$. Тогда

$$\psi^\dagger \gamma_k \psi \to \psi^\dagger e^{-i\omega_{ij}\sigma_{ij}\sigma_{ij}}\gamma_k e^{i\omega_{ij}\sigma_{ij}}\psi = \psi^\dagger\gamma_k\psi - i\omega_{ij}\psi^\dagger\left[\sigma_{ij},\gamma_k\right]\psi + ...$$

для бесконечно малого ω_{ij}. Используя алгебру Клиффорда, легко вычислить коммутатор в виде $[\sigma_{ij},\gamma_k] = -2i\left(\delta_{ik}\gamma_j - \delta_{jk}\gamma_k\right)$. (Если k не равно i или j, то ясно, что γ_k коммутирует с σ_{ij}, а если k равно i или j, то используем $\gamma_k^2 = = 1$.) Видно, что набор объектов $v_k \equiv \psi^\dagger\gamma_k\psi, k = 1, ..., 2n$ преобразуется как вектор в $2n$-мерном пространстве с бесконечно малым углом поворота $4\omega_{ij}$ в плоскости ij:

$$v_k \to v_k - 2\left(\omega_{kj}v_j - \omega_{ik}v_i\right) = v_k - 4\omega_{kj}v_j$$

(что полностью аналогично $\bar{\psi}\gamma^\mu\psi$, преобразующемуся как вектор относительно группы Лоренца). Мы другим способом доказали, что $\frac{1}{2}\sigma_{ij}$ представляет собой генераторы $SO(2n)$.

Определим матрицу $\gamma^{\text{FIVE}} = (-i)^n\gamma_1\gamma_2...\gamma_{2n}$, которая в используемом нами базисе имеет явный вид

$$\gamma^{\text{FIVE}} = \tau_3 \otimes \tau_3 \otimes ...\tau_3$$

где τ_3 появляется n раз. По аналогии с группой Лоренца определяем «левый» спинор как $\psi_L \equiv \frac{1}{2}\left(1 - \gamma^{\text{FIVE}}\right)\psi$ и «правый» спинор как $\psi_R \equiv \frac{1}{2}(1+ \ +\gamma^{\text{FIVE}})\,\psi$, так что $\gamma^{\text{FIVE}}\psi_L = -\psi_L$ и $\gamma^{\text{FIVE}}\psi_R = \psi_R$. При $\psi \to e^{i\omega_{ij}\sigma_{ij}}\psi$ получаем $\psi_L \to e^{i\omega_{ij}\sigma_{ij}}\psi_L$ и $\psi_R \to e^{i\omega_{ij}\sigma_{ij}}\psi_R$, поскольку γ^{FIVE} коммутирует с σ_{ij}. Проекция на левый и правый спиноры делит число компонент пополам, и, таким образом, мы приходим к важному выводу, согласно которому два неприводимых спинорных представления $SO(2n)$ имеют размерность 2^{n-1}. (Убедитесь в том, что это представление нельзя еще более упростить.) В частности, спинорное представление $SO(10)$ является 16-мерным $\left(2^{10/2-1} = 2^4 = 16\right)$. Мы увидим, что представления 5^* и 10 группы $SU(5)$ можно поместить в 16 состояний группы $SO(10)$.

Вложение унитарных групп в ортогональные группы

Унитарная группа $SU(5)$ может быть естественным образом вложена в ортогональную группу $SO(10)$. В самом деле, я сейчас покажу, что вложить $SU(n)$ в $SO(2n)$ так же легко, как $z = x + iy$.

Рассмотрим $2n$-мерные действительные векторы $x = (x_1, ..., x_n, \ y_1, ..., y_n)$ и $x' = (x'_1, , ..., x'_n, y'_1, ..., y'_n)$. По определению, $SO(2n)$ состоит из линейных преобразований этих двух действительных векторов, оставляя их скалярное произведение $x'x = \sum\limits_{j=1}^{n}\left(x'_jx_j + y'_jy_j\right)$ инвариантным.

Теперь из этих двух действительных векторов можно построить два n-мерных комплексных вектора $z = (x_1 + iy_1, ..., x_n + iy_n)$ и $z' = (x'_1 + \ +iy'_1, ..., x'_n + iy'_n)$. Группа $U(n)$ состоит из преобразований двух n-мерных комплексных векторов z и z', оставляя инвариантным их скалярное произведение

$$(z')^*\,z = \sum_{j=1}^{n}\left(x'_j + iy'_j\right)^*\left(x_j + iy_j\right) = \sum_{j=1}^{n}\left(x'_jx_j + y'_jy_j\right) + i\sum_{j=1}^{n}\left(x'_jy_j - y'_jx_j\right).$$

Другими словами, $SO(2n)$ оставляет инвариантной $\sum\limits_{j=1}^{n}\left(x'_jx_j + y'_jy_j\right)$, но $U(n)$ состоит из подмножества таких преобразований в $SO(2n)$, что оставляет инвариантной не только $\sum\limits_{j=1}^{n}\left(x'_jx_j + y'_jy_j\right)$, но также $\sum\limits_{j=1}^{n}\left(x'_jy_j - \ -y'_jx_j\right)$.

Теперь, когда мы осознаем смысл естественного вложения $U(n)$ в $SO(2n)$, легко видеть, что определяющее или векторное представление $SO(2n)$, которое назовем просто $2n$, разлагается при ограничении на $U(n)$ на два фундаментальных представления $U(n)-n$ и n^*; таким образом,

$$2n \to n \oplus n^*$$

Другими словами, $(x_1, ..., x_n, y_1, ..., y_n)$ можно переписать в виде $(x_1 + \ + iy_1, ..., x_n + iy_n)$ и $(x_1 - iy_1, ..., x_n - iy_n)$. Заметим, что это есть аналог (VII.5.4) и, значит, фундаментальное представление $SU(5)$ разлагается на представления группы $SU(3) \otimes SU(2) \otimes U(1)$:

$$5 \to \left(3^*, 1, \frac{1}{3}\right) \oplus \left(1, 2, -\frac{1}{2}\right).$$

С законом разложения представлений (10) мы можем найти, как разлагаются другие представления $SO(2n)$, когда они ограничены естественной подгруппой $U(n)$. Тензорные представления $SO(2n)$ просто устроены, поскольку они построены из векторного представления. [Именно это мы делали, переходя от (VII.5.4) к (VII.5.7), (VII.5.8) и (VII.5.9).] Например, присоединенное представление $SO(2n)$, имеющее размерность $2n(2n-1)/2 = n(2n-1)$, преобразуется как антисимметричный тензор с двумя индексами $2n \otimes_A 2n$ и поэтому разлагается на

$$2n \otimes_A 2n \to (n \oplus n^*) \otimes_A (n \oplus n^*)$$

в соответствии с (10). Антисимметричное произведение $\otimes_A$ в правой части, конечно, должно браться по $U(n)$. Например, $n \otimes_A n$ является $n(n-1)/2$ представлением $U(n)$. Таким образом, видно, что

$$\begin{aligned} n(2n-1) \to & n^2 - 1 \text{ (присоединенное представление)} \oplus \\ & \oplus 1 \text{ (синглет)} \oplus \\ & \oplus n(n-1)/2 \oplus \\ & \oplus (n(n-1)/2)^*. \end{aligned}$$

Проверим: общая размерность представлений $U(n)$ в правой части составляет $(n^2-1) + 1 + 2n(n-1)/2 = n(2n-1)$. В частности, для $SO(10) \supset SU(5)$ имеем $45 \to 24 \oplus 1 \oplus 10 \oplus 10^*$ и, конечно, $24 + 1 + 10 + 10 = 45$.

Разложение спинора

Труднее представить, как спинорное представление $SO(2n)$ разлагается при ограничении до $U(n)$. Здесь приведено эвристическое рассмотрение, которое убеждает большинство физиков, но, конечно, не математиков. Я получу только $SO(10) \supset SU(5)$ и предоставлю вам возможность рассмотреть общий случай. Вопрос в том, как разлагаются 16 состояний. Непосредственно из нумерологии и знания размерностей меньших представлений $SU(5)(1, 5, 10, 15)$ следует, что существует очень много возможностей, некоторые из которых довольно маловероятны, например, разложение 16 на 16 единиц.

Изобразим 16-компонентный спинор группы $SO(10)$ распадающимся на набор представлений группы $SU(5)$. По определению, 45 генераторов $SO(10)$ смешивают все эти представления вместе. Спросим себя, что делают с этими представлениями различные части 45, а именно $24 \oplus 1 \oplus 10 \oplus 10^*$?

Из них 24 преобразуют каждое из представлений $SU(5)$ в само себя, конечно, потому что они являются 24 генераторами $SU(5)$, и это то, что должны делать генераторы. Генератор 1 может только умножить каждое из этих представлений на вещественное число. (Другими словами, соответствующий элемент группы умножает каждое из представлений на фазовый множитель.)

Что делает с этими представлениями 10, которое, как вы помните из главы VII.5, представлено как антисимметричный тензор с двумя верхними индексами и поэтому

известно также как [2]? Предположим, что набор представлений, на которые распадается S, содержит синтлет $[0] = 1$ группы $SU(5).10 = [2]$, действуя на $[0]$, дает $[2] = 10$. (Это настолько очевидно, что не требует объяснений! Антисимметричный тензор с двумя индексами, объединенный с тензором без индексов, является антисимметричным тензором с двумя индексами.) А что происходит с $10 = [2]$, действующим на $[2]$? Результатом является тензор с четырьмя верхними индексами. Он, конечно, содержит $[4]$, что эквивалентно $[1]^* = 5^*$. Но $1 \oplus 10 \oplus 5^*$ уже составляет в сумме 16. Таким образом, все учтено. Больше уже быть не может. Поэтому мы делаем следующий вывод:

$$S^+ \to [0] \oplus [2] \oplus [4] = 1 \oplus 10 \oplus 5^*$$

Представления 5^* и 10 группы $SU(5)$ вмещаются в 16^+группы $SO(10)$!

Дальше мы узнаем, что два спинорных представления $SO(10)$ сопряжены друг другу. Действительно, вы, возможно, заметили, что я незаметно убрал верхний индекс «плюс» у символа S. Сопряженный спинор S^-распадается на сопряженные представления в (14):

$$S^- \to [1] \oplus [3] \oplus [5] = 5 \oplus 10^* \oplus 1^*$$

Давно потерянное антинейтрино

Вложение было бы идеальным, если бы мы ввели еще одно поле, преобразующееся как 1, то есть синглет относительно $SU(5)$, и, следовательно, синглет относительно $SU(3) \otimes SU(2) \otimes U(1)$. Другими словами, это поле не участвует в сильном, слабом и электромагнитном взаимодействиях, или, попросту говоря, оно описывает лептон, не имеющий электрического заряда и не участвующий в известном слабом взаимодействии. Таким образом, это поле можно определить как поле «давно потерянного» антинейтрино ν_L^c. Оно не чувствует ни один из известных калибровочных бозонов.

Напомним, что мы используем правило, по которому все фермионные поля являются левыми, и поэтому мы записали ν_L^c. В соответствии с сопряженным преобразованием, как говорилось ранее, такая запись эквивалентна правому полю нейтрино ν_R. Поскольку ν_R является синглетом $SU(5)$, ему можно присвоить майорановскую массу M, не нарушая $SU(5)$. Поэтому можно ожидать, что M больше или порядка масштаба нарушения $SU(5)$, который, как следует из главы VII.5, намного больше масштабов, исследованных экспериментально. Это объясняет, почему ν_R не наблюдалось.

С другой стороны, в присутствии ν_R у нас может быть и дираковская масса $m\,(\bar{\nu}_L \nu_R +$ э. с.). Поскольку этот член нарушает $SU(2) \otimes U(1)$ подобно известным массовым членам для кварков и лептонов, ожидается, что m будет иметь тот же порядок величины [2], что и известные массы кварков и лептонов (которые по неизвестным причинам охватывают огромный диапазон).

Таким образом, в пространстве с базисом (ν, ν^c) есть матрица (майорановской) массы

$$\mathcal{M} = \begin{pmatrix} 0 & m \\ m & M \end{pmatrix},$$

где $M \gg m$. Поскольку следом и детерминантом $\mathcal{M}$ являются соответственно M и $-m^2$, $\mathcal{M}$ имеет большое собственное значение $\sim M$ и малое собственное значение $\sim m^2/M$. Крошечная масса $\sim m^2/M$, уменьшенная в m/M раз относительно обычных масс кварков и лептонов, естественным образом генерируется для (наблюдаемого) левого нейтрино. Этот довольно привлекательный сценарий, известный, по понятной причине, как механизм «качели» [3], был открыт независимо друг от друга Янагидой и Гелл-Манном, а также Рамоном и Сланским.

Опять-таки плотное вложение представлений 5^* и 10 группы $SU(5)$ в представление 16^+группы $SO(10)$ убедило многих физиков, что это, безусловно, правильно.

ДвоичнЬй Код для Вселенной

Для прямого произведения γ-матриц в (5) и (6) и, значит, матриц σ_{ij} можно записать состояния спинорных представлений в виде

$$|\varepsilon_1\varepsilon_2...\varepsilon_n\rangle$$

[2] Очевидно, что теперь, имея ν_R, в теорию $SU(5)\otimes U(1)$. рассмотренную в главе VII.2, можно добавить член $f'\widetilde{\varphi}\bar{\nu}_R\psi_L$. где $\widetilde{\varphi}\equiv\tau^2\varphi^\dagger$. Если нет каких-либо других указаний, можно предположить, что константа f' имеет тот же порядок величины, что и f, которая приводит к массе электрона.

[3] В русскоязычной литературе отсутствует общепринятый термин для «seesaw mechanism». Имеется в виду доска, уравновешенная в центре: когда одно собственное значение матрицы стремится к бесконечности. другое стремится к нулю. - Прим. ред. где каждое ε принимает значение \pm 1. Из (9) легко видеть, что

$$\gamma^{\text{FIVE}}\,|\varepsilon_1\varepsilon_2...\varepsilon_n\rangle=\left(\Pi_{j=1}^n\varepsilon_j\right)|\varepsilon_1\varepsilon_2...\varepsilon_n\rangle$$

Правый спинор S^+состоит из состояний $|\varepsilon_1\varepsilon_2...\varepsilon_n\rangle$ с $\left(\Pi_{j=1}^n\varepsilon_j\right)=+1$, а левый спинор S^-–из состояний с $\left(\Pi_{j=1}^n\varepsilon_j\right)=-1$. Действительно, спинорные представления имеют размерность $2n-1$.

Таким образом, в объединении $SO(10)$ фундаментальные кварки и лептоны описываются пятиразрядным двоичным кодом с такими состояниями, как $|++--+\rangle$ и $|-+---\rangle$. Лично я нахожу такую картину Вселенной довольно привлекательной.

Получим эти состояния в явном виде. При этом у меня будет возможность убедиться в том, что вы понимаете теорию групп, описанную в этой главе. Начнем с гораздо более простого случая $SO(4)$. Спинор S^+состоит из $|++\rangle$ и $|--\rangle$, а спинор S^-–из $|+-\rangle$и $|-+\rangle$. Как рассматривалось в главе II.3, $SO(4)$ содержит две различные подгруппы $SU(2)$. Удаляя из приведенного в главе II. 3 рассмотрения несколько множителей i, видим, что третий генератор $SU(2)$, назовем его σ_3, может иметь вид либо $\sigma_{12}\sigma_{34}$, либо $\sigma_{12}+\sigma_{34}$. Эти два варианта соответствуют двум разным подгруппам $SU(2)$. Выбираем (произвольно) $\sigma_3=\frac{1}{2}\left(\sigma_{12}-\sigma_{34}\right)$. Из (5) и (6) получаем $\sigma_{12}=i\gamma_1\gamma_2=i\left(\tau_1\otimes\tau_3\right)\left(\tau_2\otimes\tau_3\right)=-\tau_3\otimes 1$ и $\sigma_{34}=-1\otimes\tau_3$, тогда $\sigma_3=\frac{1}{2}(\ -\ -\tau_3\otimes 1+1\otimes\tau_3)$. Чтобы представить преобразование четырех состояний $|++\rangle,|--\rangle,|+-\rangle$ и $|-+\rangle$ относительно выбранной группы $SU(2)$, подействуем на них с помощью σ_3. Например,

$$\sigma_3|++\rangle=\frac{1}{2}\left(-\tau_3\otimes 1+1\otimes\tau_3\right)|++\rangle=\frac{1}{2}(-1+1)|++\rangle=0$$

и

$$\sigma_3|-+\rangle=\frac{1}{2}\left(-\tau_3\otimes 1+1\otimes\tau_3\right)|-+\rangle=\frac{1}{2}(1+1)|-+\rangle=|-+\rangle$$

Итак, $|++\rangle$ и $|--\rangle$ являются двумя синглетами относительно группы $SU(2)$, тогда как $|+-\rangle$ и $|-+\rangle$ образуют дублет.

Заметим, что полученный вывод согласуется с обобщением (14) и (15), а именно с тем, что при ограничении $SO(2n)$ на $U(n)$ спиноры разлагаются как

$$S^+\to[0]\oplus[2]\oplus...$$

И

$$S^-\to[1]\ ()\ [3|\ (\mathrm{D})...$$

Я не показал, чем заканчиваются эти две последовательности. Легко сообразить, что они зависят от того, четным или нечетным является n. В рассматриваемом примере $n=2$, и поэтому $2^+\to[0]\oplus[2]=1\oplus 1$ и $2^-\to[1]=2$. Аналогично для $n=3$ при ограничении $SO(6)$

на $U(3)$ $4^+ \to [0] \oplus [2] = 1 \oplus 3^*$ и $4^- \to [1] \oplus [3] = 3 \oplus 1$. (Какое триплетное представление $U(3)$ назвать 3, а какое 3^*, зависит от того, будет ли это соответствовать традиционному употреблению. Убедимся в этом позже.)

Теперь мы готовы идентифицировать каждое из 16 состояний, таких как $|++--+\rangle$, в $SO(10)$-объединении. Прежде всего, выражение (18) указывает на то, что в подгруппе $SO(4) \otimes SO(6)$ группы $SO(10)$ спинор 16^+ разлагается следующим образом (поскольку $\Pi_{j=1}^5 \varepsilon_j = +1$ предполагает, что $\varepsilon_1\varepsilon_2 = \varepsilon_3\varepsilon_4\varepsilon_5$):

$$16^+ \to \left(2^+, 4^+\right) \oplus \left(2^-, 4^-\right)$$

Определим естественную группу $SU(2)$ подгруппы $SO(4)$ как $SU(2)$ электрослабого взаимодействия, а естественную группу $SU(3)$ подгруппы $SO(6)$ как цветную $SU(3)$ сильного взаимодействия. Таким образом, согласно предыдущему рассмотрению $(2^+, 4^+)$являются $SU(2)$-синглетами стандартной модели $U(1) \otimes SU(2) \otimes SU(3)$, тогда как $(2^-, 4^-)$являются $SU(2)$-дублетами. Такое построение имеет вид (все поля при этом, как обычно, левые):

$SU(2)$-дублеты:

$$\begin{aligned}
\nu &= |-+--\!-\rangle, \\
e^- &= |+---\!-\rangle, \\
u &= |-+++\!-\rangle, |-++-\!+\rangle \text{ и } |-+-+\!+\rangle, \\
d &= |+-++\!-\rangle, |+-+-\!+\rangle \text{ и } |+--+\!+\rangle;
\end{aligned}$$

$SU(2)$-синглеты:

$$\begin{aligned}
\nu^c &= |+++\!+\!+\rangle, \\
e^+ &= |--++\!+\rangle, \\
u^c &= |+++\!--\rangle, |++-+\!-\rangle \text{ и } |++--\!+\rangle, \\
d^c &= |--+-\!-\rangle, |---+\!-\rangle \text{ и } |---\!-\!+\rangle.
\end{aligned}$$

Уверяю вас, что процесс получения этого очень занимателен, и советую вам восстановить таблицу, не глядя на нее. Если нужна помощь, ниже дано несколько указаний. Из рассмотрения $SU(2)$ мне известно, что $\nu = 1-+\ +\varepsilon_3\varepsilon_4\varepsilon_5\rangle$ и $e^- = |+-\varepsilon_3\varepsilon_4\varepsilon_5\rangle$, но откуда мне известно, что $\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 1$? Во-первых, я знаю, что $\varepsilon_3\varepsilon_4\varepsilon_5 = -1$. Мне также известно, что $4^- \to 3 \oplus 1$ при ограничении $SO(6)$ на цветную $SU(3)$. Итак, из четырех состояний $|--\!-\rangle, |+\!+\!-\rangle, |+-\!+\rangle$ и $|-++\rangle$ лишним явно является $|--\!-\rangle$. В соответствии с тем же эвристическим рассуждением, из 16 возможных состояний лишним является $|+++\!+\!+\rangle$ и таким же должно быть ν^c.

Существует множество проверок на согласованность. Например, если определить $\nu = 1-+--\!-\rangle, e^- = |+---\!-\rangle$и $\nu^c = |+++\!+\!+\rangle$, то можно представить электрический заряд Q, который, преобразовываясь как синтлет относительно цветовой группы $SU(3)$, должен иметь значение $Q = a\varepsilon_1 + b\varepsilon_2 + c\left(\varepsilon_3 + \varepsilon_4 + \varepsilon_5\right)$ при действии на состояние $|\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\varepsilon_5\rangle$. Константы a, b и c можно определить из трех уравнений $Q(\nu) = -a + b$ - $-3c = 0, Q\left(e^-\right) = -1$ и $Q\left(\nu^c\right) = 0$. Таким образом, $Q = -\frac{1}{2}\varepsilon_1 + \frac{1}{6}\left(\varepsilon_3 + \varepsilon_4 + \varepsilon_5\right)$.

Живя в век компьютеров, меня интригует тот факт, что фундаментальные составляющие вещества имеют пятиразрядный двоичный код. Можете сказать своим коллегам, занимающимся физикой конденсированных сред, что их любимый электрон состоит из двоичных цепочек $+\!-\!-$- и $-\!+\ +\!+\!+$. Занятная возможность [4] сама по себе предполагает, что кварки и лептоны могут состоять из пяти разных видов фундаментальных фермионных объектов. Мы строим составные объекты, записывая $+$, если такой вид присутствует, и -, если он отсутствует. Например, из приведенного выше выражения для Q следует, что разновидность 1 несет электрический заряд $-\frac{1}{2}$, разновидность 2 - нейтральный, а разновидности 3,4 и 5 несут заряд $\frac{1}{6}$. Можно даже представить более или менее конкретную модель, связывая эти фундаментальные фермионные объекты с магнитным монополем.

Размышление о происхождении поколений

Одной из фундаментальных нерешенных загадок физики элементарных частиц является проблема поколений. Почему кварки и лептоны появляются в трех поколениях $\{\nu_e, e, u, d\}, \{\nu_\mu, \mu, c, s\}$ и $\{\nu_\tau, \tau, t, b\}$? То, как этот экспериментальный факт включается в сегодняшнюю теорию, можно считать просто жалким. Мы повторяем три раза фермионную часть лагранжиана без всякого на то понимания. Три живущих вместе поколения порождают мучительную для нас проблему поколений.

Наше представление мира в двоичном коде наводит на мысль об исключительно спекулятивном подходе к проблеме поколений (возможно,

[4] Более подробную информацию можно найти в работе F. Wilczek and A. Zee. Phys. Rev. D25: 553, 1982, Section IV. слишком спекулятивном, чтобы его упоминать в учебнике?): мы добавляем больше битов. Для меня разумной возможностью является «гиперобъединение» в теорию $SO(18)$, когда все фермионы помещаются в одно спинорное представление $S^+ = 256^+$, которое при нарушении $SO(18)$ до $SO(10) \otimes SO(8)$ разлагается следующим образом:

$$256^+ \to \left(16^+, 8^+\right) \oplus \left(16^-, 8^-\right)$$

У нас есть много состояний 16^+.К сожалению, видно, что теория групп [см. также (21)] предписывает нам получение также целого букета нежелательных состояний 16^-. Одно из предположений заключается в том, что Природа может повторить этот прием. Она использует цветовую группу $SU(3)$, сильное взаимодействие которой удерживает поля, не являющиеся цветовыми синглетами (глава VII.3). Интересно, что можно использовать удивительную особенность группы $SO(8)$, которую некоторые считают самой красивой из всех групп. В частности, два спинорных представления $8^\pm$ имеют такую же размерность, как и векторное представление 8^v (уравнение $2^{n-1} = 2n$ имеет единственное решение $n = 4$). Существует преобразование, которое циклически вращает три представления $8^+, 8^-$и 8^v друг в друга (переходя на жаргон, скажу, что группа $SO(8)$ допускает внешний автоморфизм). Таким образом, существует подгруппа $SO(5)$ группы $SO(8)$ такая, что когда $SO(8)$ нарушается до $SO(5)$, то 8^+ведет себя как 8^v, а 8^- ведет себя как спинор, то есть

$$8^+ \to 5 \oplus 1 \oplus 1 \oplus 1 \quad \text{и} \quad 8^- \to 4 \oplus 4^*$$

Если назвать эту группу [5]$SO(5)$ гиперцветовой и предположить, что связанное с ней сильное взаимодействие удерживает все поля, которые не являются гиперцветовыми синглетами, то остаются только три состояния 16^+! К сожалению, поскольку соответствующий физический процесс происходит в режиме энергий, превышающих энергию великого объединения, наше знание динамики нарушения симметрии является слишком жалким, чтобы можно было делать еще какие-либо предположения.

Зарядовое сопряжение

Используемая здесь запись прямого произведения $\otimes$ позволяет в явном виде построить матрицу сопряжения C. По определению, $C^{-1}\sigma_{ij}^* C = -\sigma_{ij}$

5Читатель, знакомый с теорией групп, поймет. что $SO(5)$ является изоморфной симплектической группе $Sp(4)$ и что схема Дынкина для группы $SO(8)$ является самой симметричной из всех. (так что C изменяет $e^{i\theta_{ij}\sigma_{ij}}$ на комплексно сопряженный). Из (2), (3) и (4) легко видеть, что можно построить

$$C^{(n+1)} = \begin{cases} C^{(n)} \otimes \tau_1, & \text{если } n \text{ нечетно} \\ C^{(n)} \otimes \tau_2, & \text{если } n \text{ четно.} \end{cases}$$

Проверьте, что это дает $C^{-1}\gamma_j^* C = (-1)^n \gamma_j$, и, следовательно, искомый результат.

Очевидно, что C является прямым произведением знакопеременной последовательности τ_1 и τ_2, и мы получаем важное свойство. Действуя на $|\varepsilon_1\varepsilon_2 ... \varepsilon_n\rangle$, C изменяет знак всех ε. Таким образом, C изменяет знак $\left(\Pi_{j=1}^n \varepsilon_j\right)$ для нечетных n и не изменяет его для четных n.

Для нечетных n два спинорных представления S^+и S^-сопряжены друг с другом, тогда как для четных n они сопряжены сами с собой, или, другими словами, они являются вещественными. Это же можно увидеть непосредственно из $C^{-1}\gamma^{\rm FIVE}\,C = (-1)^n\gamma^{\rm FIVE}$. Убедитесь в этом на всех явных примерах, с которыми мы сталкивались: $SO(2), SO(4), SO(6), SO(8), SO(10)$ и $SO(18)$. См. также упражнение (3).

Аномалии

Что же происходит с аномалиями в $SO(2n)$ великом объединении? Согласно рассмотрению, приведенному в главе VII.5, необходимо вычислить $A^{ijklmn} \equiv \mathrm{tr}\left(J^{ij}\left\{J^{kl}, J^{mn}\right\}\right)$ по фермионному представлению. Используя $SO(2n)$-преобразование $J^{ij} \to O^T J^{ij} O$, легко видеть, что A^{ijklmn} является инвариантным тензором. Можно ли построить в $SO(2n)$ инвариантный тензор шестого ранга с соответствующими свойствами симметрии (например, $A^{ijklmn} = -A^{\rm jiklmn}$? Этого сделать нельзя, за исключением группы $SO(6)$, для которой есть ε^{ijklmn}. Таким образом, A^{ijklmn} исчезает, за исключением группы $SO(6)$, где этот тензор пропорционален ε^{ijklmn}. Таково элегантное и краткое доказательство того, что любая теория великого объединения, основанная на $SO(2n)$, если $n \neq 3$, свободна от аномалий!

Исчезновение аномалии между представлениями 5^* и 10 в конце главы VII.5 больше не кажется каким-то чудом. По мере углубления наших знаний чудеса, как правило, исчезают.

Забавно, что при обсуждении физического вопроса, а именно является ли калибровочная теория перенормируемой или нет, мы пришли к математическому результату. Какой особенностью отличается группа $SO(6)$? См. упражнение VII.7.5.

Упражнения

VII.7.1. Получите алгебру Клиффорда в d-мерном пространстве для нечетных d.

VII.7.2. Получите алгебру Клиффорда в d-мерном пространстве Минковского.

VII.7.3. Покажите, что алгебра Клиффорда для $d = 4k$ и для $d = 4k + 2$ имеет несколько различные свойства. (Если нужна помощь для решения этой и двух предыдущих задач, прочтите F. Wilczek and A. Zee, Phys. Rev. D25: 553, 1982.

VII.7.4. Рассмотрите хиггсовскую часть группы $SO(10)$. Что требуется, чтобы дать массу кварки и лептоны?

VII.7.5. Группа $SO(6)$ имеет 15 генераторов $[6(6-1)/2 = 15]$. Заметьте, что группа $SU(4)$ также имеет 15 генераторов $[4^2 - 1 = 15]$. Обоснуйте предположение, что $SU(4)$ и $SO(6)$ изоморфны. Отождествите представления малой размерности.

VII.7.6. Покажите, что (к сожалению) число поколений, которое мы получаем в $SO(18)$, зависит от того, какую подгруппу $SO(8)$ считать гиперцветовой.

VII.7.7. Если вы хотите стать специалистом в области теории струн, вы должны знать уравнение Дирака в различных размерностях, и особенно в размерности 10. Для разминки изучите уравнение Дирака в 2-мерном пространстве-времени. Затем перейдите к уравнению Дирака в 10-мерном пространстве-времени.

30.5 Частb VIII Gravity and Beyond

30.5.1 What Does It Mean, To Quantize Gravity?

(??? this question is left???)

30.5.2 Zi.VIII. Gravity as a Field Theory and the Kaluza–Klein Picture

Учитывая гравитацию

В книгах по теории поля, написанных поколение назад, как правило, даже не упоминается гравитация. Гравитационное взаимодействие настолько слабее трех других взаимодействий, что его не включали в учебные программы по физике элементарных частиц. Сейчас ситуация в корне изменилась: главной задачей теоретической физики высоких энергий стало объединение гравитации с тремя другими видами взаимодействия, а теория струн - главным претендентом на звание единой теории.

Из курса общей теории относительности вы знаете о гравитационном действии Гильберта-Эйнштейна

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R \equiv \int d^4x\sqrt{-g}M_P^2 R,$$

где $g = \det g_{\mu\nu}$ - детерминант криволинейной метрики $g_{\mu\nu}$ пространства-времени, R - скалярная кривизна, G - постоянная Ньютона. Напомню, что тензор кривизны Римана

$$R^{\lambda}_{\mu\nu\kappa} = \partial_\nu\Gamma^{\lambda}_{\mu\kappa} - \partial_\kappa\Gamma^{\lambda}_{\mu\nu} + \Gamma^{\sigma}_{\mu\kappa}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\lambda}_{\kappa\sigma}$$

выражается через символы Римана-Кристоффеля (см. главу I.10):

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}\left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\rho\nu} - \partial_\rho g_{\mu\nu}\right).$$

Тензор Риччи определяется как $R_{\mu\kappa} = R^{\nu}_{\mu\nu\kappa}$, а скалярная кривизна - как $R = g^{\mu\nu}R_{\mu\nu}$. Варьируя S, получаем [1] эйнштейновское уравнение поля

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}.$$

'См.. например. С. Вейнберг. Гравитация и космология. С. 364. Действие Гильберта-Эйнштейна однозначно определяется условием того, что действие координатно-инвариантно и содержит пространственно-временные производные второй степени. Из уравнений (2) и (3) видим, что скалярная кривизна R содержит производные второй степени и безразмерное поле $g_{\mu\nu}$, поэтому его размерность равна 2. Следовательно, размерность G^{-1} должна равняться 2. В современных работах по гравитации чаще всего используют вторую форму представления (1). (Модифицированная масса Планка $M_P \equiv 1/\sqrt{16\pi G}$ отличается от обычной массы Планка наличием тривиального множителя как в соотношении h и $\hbar$.)

Теория возникла из интуитивного представления Эйнштейна о кривизне пространства-времени и была сформулирована в геометрических терминах. Во многих учебниках теория Эйнштейна строится (и это справедливо) на чисто геометрических понятиях.

С другой стороны, гравитацию можно рассматривать в том же ключе, что и другие виды взаимодействия. В конце концов, гравитон можно считать еще одной элементарной частицей, подобной фотону. Действие (1), однако, не похоже ни на одну формулу из теории поля, с которыми мы до сих пор имели дело. Сейчас я продемонстрирую, что оно в самом деле имеет такую же структуру.

Гравитация как теория поля

Запишем $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, где $\eta_{\mu\nu}$ - плоская метрика Минковского, а $h_{\mu\nu}$ - отклонение от плоской метрики. Разложим действие в ряд по степеням $h_{\mu\nu}$. Чтобы не запутаться в лоренцевых индексах, для начала опустим их. Зная, что по определению скалярная кривизна R включает в себя две производные ∂, приходим к выводу, что разложение должно быть следующим (после исключения полных производных):

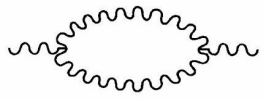
$$S = \int d^4x \frac{1}{16\pi G} \left(\partial h \partial h + h \partial h \partial h + h^2 \partial h \partial h + ...\right)$$

В главе I.10 говорилось, что поле $h^{\mu\nu}(x)$ описывает гравитон в плоском пространстве и должно рассматриваться как любое другое поле. Первый член $\partial h \partial h$ характеризует распространение гравитона и концептуально ничем не отличается от первого члена в действии для скалярного поля $\partial\varphi\partial\varphi$ или фотонного поля $\partial A \partial A$. Члены третьей и выше степеней по h описывают самодействие гравитации.

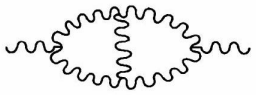
Действие Гильберта-Эйнштейна, записанное в виде разложения по слабому полю, напоминает своей структурой действие Янга-Миллса, ко- торое схематично можно представить в виде $S = \int d^4x \,(1/g^2)\,(\partial A \partial A + \ + A^2 \partial A + A^4)$. В главе IV.5 приводилось физическое обоснование самодействия бозонов Янга-Миллса: бозоны сами несут заряд, с которым они взаимодействуют. Самодействие гравитации имеет тот же смысл: гравитон взаимодействует со всем, что обладает зарядом и импульсом, при этом он сам несет заряд и импульс. Фотон, напротив, не взаимодействует с собой.

Говорят, что теории Янга-Миллса и Эйнштейна нелинейны, тогда как теория Максвелла линейна. Первые две теории сложны, а вторая проста.

Однако, в то время как действие Янга-Миллса содержит конечное число членов, действие Гильберта-Эйнштейна из-за наличия $\sqrt{-g}$ и обратной к $g_{\mu\nu}$ метрики представляет собой бесконечный ряд по гравитонному полю $h_{\mu\nu}$.



(a)



(b)

Рис. VIII. 1.1

Другим существенным отличием является тот факт, что теория ЯнгаМиллса перенормируема, тогда как хорошо известно, что гравитация неперенормируема, что следовало из анализа размерностей в главе III.2. Сейчас мы можем показать это явно. Рассмотрим собственно-энергетическую поправку к гравитонному пропагатору на рис. VIII.1.1a. Из второго члена в (5) мы видим, что взаимодействие трех гравитонов включает в себя импульс второй степени. Следовательно, фейнмановский интеграл будет иметь вид $\int d^4k\,(kkkk/k^2k^2)$, где четвертая степень по k в числителе обусловлена двумя вершинами, а четвертая степень по k в знаменателе - пропагаторами. Используя из них две степени для выделения коэффициента при $\partial h \partial h$, видим, что поправка к $1/G$ квадратично расходится. Поскольку взаимодействие явно содержит степени импульса, расходимость будет тем сильнее, чсм выше порядок. Сравните рис. VIII.1.1b с 1a: в нем больше на три пропагатора, что дает $\sim 1/k^6$, на одно интегрирование по петле $\int d^4k$ и на две вершины, дающие $\sim k^4$. Степень расходимости увеличивается на два. Все то мы уже знали из анализа размерностей.

Как говорилось в главе I.10, из фундаментального определения

$$T^{\mu\nu}(x) = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}(x)}$$

следует, что взаимодействие гравитона с материей (в пределе слабого поля) можно учесть, добавляя в действие член

$$-\int d^4x \frac{1}{2} h_{\mu\nu} T^{\mu\nu}$$

Здесь $T^{\mu\nu}$ - тензор энергии-импульса всех полей материи (в плоском пространстве-времени), причем, под полем материи понимается любое поле, отличное от гравитона. Таким образом, действие (5) с учетом материи выглядит схематически как [2]:

$$S = \int d^4x \left[\frac{1}{16\pi G} \left(\partial h \partial h + h \partial h \partial h + h^2 \partial h \partial h + ... \right) + \left(hT + ... \right) \right]$$

В главе IV.5 мы говорили, что теорию Янга-Миллса можно привести к виду, аналогичному теории Максвелла, если сделать простое масштабирование $A \to gA$. Аналогичным образом можно привести теорию Эйнштейна к такому же виду, если масштабировать гравитонное поле $h^{\mu\nu} \to \sqrt{16\pi G} h^{\mu\nu}$ и выразить действие как

$$S = \int d^4x \left(\partial h \partial h + \sqrt{16\pi G} h \partial h \partial h + 16\pi G h^2 \partial h \partial h + ... + \sqrt{16\pi G} hT \right)$$

Отсюда очевидно, что $\sqrt{16\pi G} = 1/M_P$ характеризует величину взаимодействия гравитона с собой и всеми другими полями. Поскольку значение M_P очень велико (например, по сравнению с масштабом сильных взаимодействий), сила гравитационного взаимодействия чрезвычайно мала.

Мы раскладывали $g_{\mu\nu}$ относительно плоской метрики, хотя могли бы разложить как $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, где $\bar{g}_{\mu\nu}$ - искривленная метрика, например, черной дыры (см. главу V.7).

Действие в пределе слабого поля

Теперь восстановим опущенные ранее индексы. Для нахождения первого члена $\partial h \partial h$ в (7), необходимого для вычисления гравитонного пропагатора, разложим действие $S \equiv M_P^2 \int d^4x \sqrt{-g} g^{\mu\nu} R_{\mu\nu}$ до членов второго порядка h^2 включительно. Из (2) и (3) следует, что тензор Риччи $R_{\mu\nu}$

[2] Если записать в разложении действия Гильберта-Эйнштейна члены третьей и четвертой степеней по h, то, строго говоря, мы должны также включить в него вклады с членами более высокого порядка по h, содержащиеся в выражении $T^{\mu\nu}(x) = -(2/\sqrt{-g}) \left(\delta S_M \right) / \left(\delta g_{\mu\nu}(x) \right)$. начинается с $O(h)$, поэтому достаточно вычислить $\sqrt{-g} g^{\mu\nu}$ с точностью до $O(h)$. Это несложно: из главы I.10 мы знаем, что $g = -\left[1 + \eta^{\mu\nu} h_{\mu\nu} + \ + O\left(h^2 \right) \right]$ и $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O\left(h^2 \right)$, так что $\sqrt{-g} g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \frac{1}{2} \eta^{\mu\nu} h + + O\left(h^2 \right)$, где определено $h \equiv \eta^{\mu\nu} h_{\mu\nu}$. Теперь необходимо вычислить $R_{\mu\nu}$ с точностью до $O\left(h^2 \right)$. Это довольно громоздкое, хотя и прямое вычисление, берущее начало с (2) и (3).

Чтобы не противоречить своей концепции избегать в книге громоздких вычислений, покажу, как избежать их и сейчас. Воспользуемся соображениями симметрии!

При общем координатном преобразовании $x^\mu \to x'^\mu = x^\mu - \varepsilon^\mu(x)$ изменение метрики имеет вид $g'^{\mu\nu} = \left(\partial x'^\mu / \partial x^\sigma \right) \left(\partial x'^\nu / \partial x^\tau \right) g^{\sigma\tau}$. Подставляя метрику в виде $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + ...$, опуская индексы (с помощью $\eta_{\mu\nu}$ в этом порядке) и учитывая, что $\left(\partial x'^\mu / \partial x^\sigma \right) = \delta^\mu_\sigma - \partial_\sigma \varepsilon^\mu$, находим, считая, что $\partial_\mu \varepsilon_\nu$ одного порядка с $h^{\mu\nu}$:

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$$

Мы считаем, что порядок $\partial_\mu \varepsilon_\nu$ совпадает с порядком $h^{\mu\nu}$. Обратите внимание на структурное подобие с электромагнитным калибровочным преобразованием $A'_\mu = A_\mu - \partial_\mu \Lambda$. Очень хорошо! В каком смысле гравитацию можно рассматривать как калибровочную теорию, мы обсудим позже.

Найдем члены второго порядка по h и ∂ в выражении для действия. Согласно лоренцевской инвариантности существует четыре возможных члена (чтобы убедиться в этом, запишите сначала члены с совпадающими индексами для двух ∂, а затем члены с совпадающими индексами для одного ∂ и одного h, и т. Д.):

$$S = \int d^4x \left(a \partial_\lambda h^{\mu\nu} \partial^\lambda h_{\mu\nu} + b \partial_\lambda h^\nu_\mu \partial^\lambda h^\nu_\nu + c \partial_\lambda h^{\lambda\nu} \partial^\mu h_{\mu\nu} + d h^\lambda_\lambda \partial^\mu \partial^\nu h_{\mu\nu} \right)$$

где a, b, c и d - четыре неизвестных коэффициента. Теперь проварьируем S по $\delta h_{\mu\nu} = \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu$, интегрируя, где нужно, по частям. Например,

$$\delta\left(\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu}\right) = 2\left[\partial_\lambda\left(2\partial^\mu\varepsilon^\nu\right)\right]\left(\partial^\lambda h_{\mu\nu}\right) = 4\varepsilon^\nu\partial^2\partial^\mu h_{\mu\nu}$$

[Iоскольку имеется три объекта, линейных по h и ε и кубических по ∂ (это $\varepsilon^\nu\partial^2\partial_\nu h, \varepsilon^\nu\partial_\nu\partial^\lambda\partial^\mu h_{\lambda\mu}$ и еще один, приведенный ранее), условие $\delta S = 0$ дает три уравнения, что достаточно для фиксирования действия с точностью до общего коэффициента, соответствующего постоянной Ньютона. Их инвариантная комбинация равна

$$\mathcal{J} \equiv \frac{1}{2}\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu} - \frac{1}{2}\partial_\lambda h^\mu_\mu \partial^\lambda h^\nu_\nu - \partial_\lambda h^{\lambda\nu}\partial^\mu h_{\mu\nu} + \partial^\nu h^\lambda_\lambda \partial^\mu h_{\mu\nu}$$

Таким образом, даже если вы никогда не слышали о действии Гильберта-Эйнштейна, вы сможете определить действие для гравитации в пределе слабого поля, если потребовать инвариантность действия относительно преобразования (8). В этом нет ничего удивительного, поскольку инвариантность относительно координатных преобразований и определяет действие Гильберта-Эйнштейна. К тому же, приятно ведь сформулировать теорию гравитации «с нуля».

Нам не нужно разлагать R до $O\left(h^2\right)$, чтобы записать с учетом (6) разложение S по слабому полю:

$$S_{\text{wfg}} = \int d^4x \left(\frac{1}{32\pi G}\mathcal{J} - \frac{1}{2}h_{\mu\nu}T^{\mu\nu}\right)$$

Коэффициент при $\mathcal{J}$ фиксируется требованием, что мы воспроизводим обычную ньютоновскую гравитацию (см. далее).

Гравитонный пропагатор

Как следует из (5), действие S_{wfg} действительно имеет такую же квадратичную структуру, как и все теории поля, которые мы изучали, поэтому гравитонный пропагатор является обратным к дифференциальному оператору. Но точно так же, как в теории Максвелла и теории Янга-Миллса, соответствующий дифференциальный оператор в теории Эйнштейна-Гильберта не имеет обратного вследствие «калибровочной инвариантности» (8).

Ничего страшного. С данной трудностью справляется метод ФаддееваПопова. Более того, для нашей ограниченной цели - построения гравитонного пропагатора в плоском пространстве-времени, нет необходимости использовать формализм Фаддеева-Попова в полном объеме, включая духи и т. д [3]. Действительно, из главы III.4 известно, что для фейнмановской калибровки ($\xi = 1$) мы просто добавляем $(\partial A)^2$ в инвариант

$$\frac{1}{2}F^{\mu\nu}F_{\mu\nu} = \partial^\mu A^\nu\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right) \ll= \; - A^\mu\eta_{\mu\nu}\partial^2 A^\nu - (\partial A)^2$$

в результате чего сокращается последний член. Обращая дифференциальный оператор $-\eta_{\mu\nu}\partial^2$, получаем фотонный пропагатор в фейнмановской

[3] Причина в том, что, как и в теории Максвелла. но в отличие от теории Янга-Миллса. уравнение (8) не содержит поле $h_{\mu\nu}$. Раз нам не надо вычислять петлевые диаграммы в квантовой гравитации, мы не будем использовать всю мощь метода Фаддеева-Попова. калибровке $-i\eta_{\mu\nu}/k^2$. Такой же «трюк» проделываем для гравитации. Начав с выражения

$$\mathcal{J} = \frac{1}{2}\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu} - \frac{1}{2}\partial_\lambda h^\nu_\mu \partial^\lambda h^n_\nu u - \partial_\lambda h^{\lambda\nu}\partial^\mu h_{\mu\nu} + \partial^\nu h^\lambda_\lambda \partial^\mu h_{\mu\nu}$$

и добавив в него член $\left(\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h^\lambda_\lambda\right)^2$, мы сокращаем последние два члена, так что S_{cnr} сводится к виду:

$$S_{\rm cnr} = \int d^4x \frac{1}{2}\left[\frac{1}{32\pi G}\left(\partial_\lambda h^{\mu\nu}\partial^\lambda h_{\mu\nu} - \frac{1}{2}\partial_\lambda h \partial^\lambda h\right) - h_{\mu\nu}T^{\mu\nu}\right].$$

Другими словами, свобода выбора $h_{\mu\nu}$ в (8) позволяет нам наложить так называемое условие гармонической калибровки

$$\partial_\mu h^\mu_\nu = \frac{1}{2}\partial_\nu h^\lambda_\lambda$$

(Это линеаризованная версия $\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\right) = 0$.)

Записав (10) в форме

$$S = \frac{1}{32\pi G}\int d^4x \left[h^{\mu\nu}K_{\mu\nu;\lambda\sigma}\left(-\partial^2\right)h^{\lambda\sigma} + O\left(h^3\right)\right]$$

видим, что должны обратить матрицу

$$K_{\mu\nu;\lambda\sigma} \equiv \frac{1}{2}\left(\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\sigma}\right)$$

в которой $\mu\nu$ и $\lambda\sigma$ - это два индекса. Обратите внимание, что мы должны сохранять симметрию $h^{\mu\nu}$. Иначе говоря, мы оперируем матрицами, действующими в линейном пространстве, образованном симметричными двухиндексными тензорами. Итак, единичная матрица равна

$$I_{\mu\nu;\lambda\sigma} \equiv \frac{1}{2}\left(\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda}\right)$$

Можете проверить, что $K_{\mu\nu;\lambda\sigma}K^{\lambda\sigma}_{;\rho\omega} = I_{\mu\nu;\rho\omega}$ и, следовательно, $K^{-1} = K$. Таким образом, в гармонической калибровке гравитонный пропагатор в плоском пространстве-времени (положив постоянную Ньютона равной единице) задается в виде

$$D_{\mu\nu,\lambda\sigma}(k) = \frac{1}{2}\frac{\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} - \eta_{\mu\nu}\eta_{\lambda\sigma}}{k^2 + i\varepsilon}$$

От Эйнштейна к Ньютону

Варьируя (10) по $h_{\mu\nu}$, получаем уравнение движения Эйлера-Лагранжа [4] $\frac{1}{32\pi G}\left(-2\partial^2 h_{\mu\nu} + \eta_{\mu\nu}\partial^2 h\right) - T_{\mu\nu} = 0$. Вычислив след, найдем $\partial^2 h = 16\pi G T$ (где $T \equiv \eta_{\mu\nu}T^{\mu\nu}$) и в конечном итоге получим [5]

$$\partial^2 h_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right).$$

В статическом пределе T_{00} является доминирующей компонентой [6] тензора энергии импульса, поэтому (13) сводится к виду $\vec{\nabla}^2\phi = 4\pi G T_{00}$, где $\phi \equiv \frac{1}{2}h_{00}$ - ньютоновский гравитационный потенциал (см. главу I.5). Мы только что вывели уравнение Пуасона для ϕ.

Кстати, есть еще один способ избежать трудоемкого разложения действия Гильберта-Эйнштейна (и, следовательно, R) до порядка $O\left(h^2\right)$ в случае, когда вы используете эйнштейновское уравнение поля (4). Необходимо разложить $R_{\mu\nu}$ лишь до порядка $O(h)$ и получить (13) из (4). Затем из (13) можно построить уравнение действия до порядка $O\left(h^2\right)$. Действительно, из выражений (2) и (3) следует, что

$R_{\mu\nu} = \frac{1}{2}\left(-\partial^2 h_{\mu\nu} + \partial_\mu\partial_\lambda h^\lambda_\nu + \partial_\nu\partial_\lambda h^\lambda_\mu - \partial_\mu\partial_\nu h^\lambda_\lambda\right) + O\left(h^2\right) \to -\frac{1}{2}\partial^2 h_{\mu\nu} + O\left(h^2\right)$.

Возможно дальнейшее упрощение в гармонической калибровке. Однако такой подход недостаточно хорош в связи с трудностью [7] вывода (4) из (1) (необходимо использовать тождество Палатини и все остальное).

Теория Эйнштейна и отклонение света

Рассмотрим две частицы с тензорами энергии импульса $T^{\mu\nu}_{(1)}$ и $T^{\mu\nu}_{(2)}$, взаимодействующие путем обмена гравитоном. Амплитуда рассеяния будет равна (с точностью до некоторой константы, несущественной для наших целей):

$$GT^{\mu\nu}_{(1)}D_{\mu\nu,\lambda\sigma}(k)T^{\lambda\sigma}_{(2)} = \frac{G}{2k^2}\left(2T^{\mu\nu}_{(1)}T_{(2)\mu\nu} - T_{(1)}T_{(2)}\right).$$

[4] Заметим, что сохранение энергии импульса в плоском пространстве-времени $\partial^\mu T_{\mu\nu} = 0$ вместе с уравнением движения означают, что $\partial^2\left(\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h\right) = 0$.

[5] Уравнение Эйнштейна в вакууме $R_{\mu\nu} = 0$ сводится к виду $\partial^2 h_{\mu\nu} = 0$; отсюда название «гармоническая».

[6] Заметим. что. в отличие от T_{00}, компонента h_{00} не является доминирующим среди остальных компонент $h_{\mu\nu}$.

[7] С. Вейнберг. Гравитаиия и космология. С. 290 и 364. Для нерелятивистской материи $T^{(00)}$ гораздо больше значений остальных компонент T^{0j} и T^{ij} (я уже говорил об этом), поэтому амплитуда рассеяния между двумя нерелятивистскими телами (скажем, Землей и вами) равна:

$$\frac{G}{2k^2}\left(2T^{00}_{(1)}T^{00}_{(2)} - T^{00}_{(1)}T^{00}_{(2)}\right) = \frac{G}{2k^2}T^{00}_{(1)}T^{00}_{(2)}$$

Как говорилось в главах I.4 и I.5, потенциал взаимодействия определяется фурье-преобразованием амплитуды рассеяния, а именно:

$$G\iint d^3x d^3x' T^{(1)(00)}(x)T^{(2)(00)}(x')\int d^3k e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}\frac{1}{\vec{k}^2}.$$

Таким образом, для двух удаленных объектов ньютоновский потенциал равен $GM_{(1)}M_{(2)}/r$.

Теперь вернемся к вопросу, сформулированному в конце главы I.5. Допустим, теоретик в области элементарных частиц, доктор Грэвити, решил построить теорию гравитации, альтернативную теории Эйнштейна. Доктор Г. утверждает, что гравитация обусловлена обменом частицей со спином 2 и малой массой m_G, связанной с тензором энергии импульса $T^{\mu\nu}$. В главе I.5 мы получили пропагатор массивной частицы со спином 2, он равен

$$D^{\text{cnhH}}_{\mu\nu,\lambda\sigma}\,2(k) = \frac{1}{2}\left(G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda} - \frac{2}{3}G_{\mu\nu}G_{\lambda\sigma}\right)/\left(k^2 - m_G^2 + i\varepsilon\right),$$

где $G_{\mu\nu} = \eta_{\mu\nu} - k_\mu k_\nu/m_G^2$ (после приведения в соответствие условных обозначений). Поскольку эта частица взаимодействует с сохраняющимся источником $k_\mu T^{\mu\nu} = 0$, можно заменить $G_{\mu\nu}$ на $\eta_{\mu\nu}$. В пределе $m_G \to 0$ пропагатор будет иметь вид:

$$D^{\text{cnuH2}}_{\mu\nu,\lambda\sigma}(k) = \frac{1}{2}\frac{\eta_{\mu\lambda}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\lambda} - \frac{2}{3}\eta_{\mu\nu}\eta_{\lambda\sigma}}{k^2 + i\varepsilon}$$

Сравните с выражением (12). Предложенный доктором Г. пропагатор отличается от пропагатора, полученного Эйнштейном, множителем $\frac{2}{3}$ вместо 1. Удивительно, но гравитация вовсе не порождается почти безмассовой ча-

стицей со спином 2. «Разницу в $\frac{2}{3}$» между пропагаторами (12) и (14) об-

наружили в 1970 году независимо друг от друга Ивасаки, Дам и Вельтман, а также Захаров.

В теории доктора . (с его собственной гравитационной константой связи G_G) взаимодействие двух частиц задается уравнением:

$$G_G T^{\mu\nu}_{(1)}D_{\mu\nu,\lambda\sigma}(k)T^{\lambda\sigma}_{(2)} = \frac{G_G}{2k^2}\left(2T^{\mu\nu}_{(1)}T_{(2)\mu\nu} - \frac{2}{3}T_{(1)}T_{(2)}\right)$$

Для случая двух объектов нерелятивистской материи уравнение сводится к виду:

$$\frac{G_G}{2k^2}\left(2T^{00}_{(1)}T^{00}_{(2)} - \frac{2}{3}T^{00}_{(1)}T^{00}_{(2)}\right) = \frac{4}{3}\frac{G_G}{2k^2}T^{00}_{(1)}T^{00}_{(2)}$$

Доктор Г. просто определяет $G_G = \frac{3}{4}G$, после чего его теория проходит все экспериментальные проверки.

Но подождите, хорошо известно наблюдение 1919 года, когда ученые зафиксировали отклонение света звезд Солнцем, хотя фотон не является нерелятивистским телом. Действительно, из главы I.10 (или из курса электромагнетизма) известно, что для фотона равен нулю след $T \equiv T^{\mu}_{\mu}$. Значит, принимая $T^{\mu\nu}_{(1)}$ и $T^{\mu\nu}_{(2)}$ за тензоры энергии импульса для Солнца и фотона соответственно, Эйнштейн получил бы амплитуду рассеяния $(G/2k^2)\,2T^{\mu\nu}_{(1)}T_{(2)\mu\nu}$, тогда как амплитуда рассеяния в теории доктора Г. оказалась бы равной $(G_G/2k^2)\,2T^{\mu\nu}_{(1)}T_{(2)\mu\nu} = \frac{3}{4}G/2k^2\Big)\,2T^{\mu\nu}_{(1)}T_{(2)\mu\nu}$. Доктор Г. оценил бы угол отклонения как $3GM/R$, вместо $4GM/R$ (здесь M и R - масса и радиус Солнца). Так в 1919 году на бразильском острове Собрал Эйнштейн победил доктора Г.

В главе I.5 говорилось, что массивная частица со спином 2 имеет 5 степеней свободы, тогда как безмассовый гравитон - всего 2 степени свободы. (В приложении 2 приводится анализ \pm 2 -спиральной структуры обмена одним гравитоном.) Пять степеней свободы можно рассматривать как состоящие из \pm 2 -спиральных степеней свободы, двух \pm 1 -спиральных степеней свободы и одной 0-спиральной степени свободы. Взаимодействие \pm 1 -спиральных степеней свободы исчезает вследствие условия $k_\mu T^{\mu\nu} = 0$. В результате остается дополнительное скалярное взаимодействие со следом $T \equiv T^{\mu}_{\nu}\eta_{\mu\nu}$ тензора энергии импульса; это расхождение сводится к отличию в последних членах выражений (12) и (14).

Если измерить отклонение света звезд, то придем к выводу, что физическая величина, характеризующая массу гравитона m_G, точно равна нулю, а вовсе не принимается меньшей некоторого малого значения. Этот парадокс разрешил в 1971 году Вайнштейн [8]. Он обнаружил, что в теории доктора Г. масштаб расстояний в гравитационном поле вокруг тела массой M равен

$$f_V = \left(\frac{GM}{m_G^4}\right)^{\frac{1}{5}}$$

[8] A. I. Vainshtein, Phys. Lett. 39B:393, 1977; см. также C. Defayet. G. Dvali. G. Gabadadze. and A. I. Vainshtein, Phys. Rev. D65:044026, 2002. 0-спиральная степень свободы становится эффективной лишь на масштабах расстояний $r \gg r_V$. В пределах радиуса Вайнштейна r_V гравитационное поле идентично полю в теории Эйнштейна, и результаты экспериментов для теорий Эйнштейна и доктора Г. ничем не отличаются между собой. Если использовать текущий астрофизический предел $m_G \ll (10^{24}\text{ cm})^{-1}$ и массу Солнца для параметра M, значение r_V окажется гораздо больше размера Солнечной системы. Иначе говоря, парадокс обуславливается заменой пределов: сначала можно устремить к бесконечности либо характеристическое

расстояние измерения $r_{\text{изм.}}$ (радиус Солнца при отклонении света звезд),

либо радиус Вайнштейна r_V.

Итак, все хорошо: теория доктора Г. не противоречит результатам измерений, если он выберет m_G достаточно малой. Что ему не нельзя делать в этой теории, так это использовать приближение одногравитонного обмена. Вместо этого ему следует решать массовный аналог эйнштейновского уравнения поля (4) вблизи массивного тела, например Солнца, что и проделал в 1972 году Вайнштейн. Данный подход эквивалентен разложению в ряд по гравитонному полю h до всех порядков и повторному суммированию: у нас есть бесконечное количество фейнмановских диаграмм, соответствующих испусканию Солнцем $1, 2, 3, ..., \infty$ гравитонов соответственно. Парадокс формально разрешается вследствие того, что при $m_G \to 0$ высшие порядки по h становятся все более сингулярными.

Гравитационное взаимодействие света

На данном этапе вы готовы к расчетам в пертурбативной квантовой гравитации: у вас есть гравитонный пропагатор (12) и вы можете найти взаимодействия между гравитонами из детальной версии (7)) и взаимодействия между гравитоном и любым другим полем, которое задается членом $-\frac{1}{2}h_{\mu\nu}T^{\mu\nu}$. Единственной трудностью может послужить «целое море индексов», среди которых вы легко запутаетесь, если не будете особо бдительными.

Мне известно одно вычисление (одно из моих любимых в теоретической физике), позволяющее запросто избавиться от индексов. Эйнштейн говорил, что свет отклоняется массивным объектом. Отсюда вопрос: происходит ли гравитационное отклонение света другим светом? Толман, Эренфест и Подольский пришли к выводу, что в пределе слабого поля два луча света, падающие в одном направлении, не взаимодействуют гравитационно, а два луча света, падающие в противоположных направлениях, взаимодействуют. Удивительно, не правда ли? Рассеяние двух фотонов $k_1 + k_2 \to p_2 + p_2$ за счет обмена гравитоном описывается фейнмановской диаграммой на рис. VIII.1.2, где $q \equiv p_1 - k_1$ - перенос импульса, и другой диаграммой, в которой p_1 и p_2 поменяны местами. Правило Фейнмана, описывающее взаимодействие гравитона с двумя фотонами, получается из уравнения

$$h^{\mu\nu}T_{\mu\nu} = -h^{\mu\nu}\left(F_{\mu\lambda}F^{\lambda}_{\nu} - \frac{1}{4}\eta_{\mu\nu}F_{\rho\lambda}F^{\rho\lambda}\right).$$

Нас интересует только то обстоятельство, что взаимодействие включает в себя пространственно-временные производные второй степени ∂, действующие на электромагнитное поле A_μ, так что вершина гравитон-фотон-фотон содержит импульс во второй степени, по одному на каждый фотон. Таким образом, амплитуда рассеяния (без учета лоренцевых индексов) схематично равна $\sim (k_1p_1)\, D\, (k_2p_2)$. Параметр η в гравитонном пропагаторе D связывает между собой индексы от (k_1p_1) и (k_2p_2). (Мы опустили векторы поляризации фотонов, поскольку они усредняются в выражении для квадрата амплитуды.) С учетом (12) видим, что амплитуда равна сумме трех членов: $\sim (k_1 \cdot p_1)(k_2 \cdot p_2)/q^2, \sim (k_1 \cdot k_2)(p_1 \cdot p_2)/q^2$ и $\sim (k_1 \cdot p_2)(k_2 \cdot p_1)/q^2$. Вследствие того что, согласно Фурье, составляющая потенциала взаимодействия, описывающая взаимодействие на длинных расстояниях, определяется поведением амплитуды рассеяния при малом значении q, достаточно найти эти члены в пределе $q \to 0$. От них почти ничего не остается. Смотрите:

$$k_1 \cdot p_1 \to k_1 \cdot k_1 = 0, \quad k_1 \cdot p_2 = k_1 \cdot (k_1 + k_2 - p_1) \to k_1 \cdot k_2.$$

Получаем амплитуду $\sim (k_1 \cdot k_2)(p_1 \cdot p_2)/q^2$.

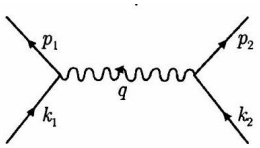


Рис. VIII. 1.2

Если направления k_1 и k_2 совпадают, то $k_1 \cdot k_2 \propto k_1 \cdot k_1 = 0$. Два фотона, движущихся в одном направлении, не взаимодействуют гравитационно.

К сожалению, полученный результат не представляет никакой практической пользы, поскольку электромагнитные эффекты в жизни имеют гораздо большее значение, но эта книга не является руководством для инженеров. В приложении 1 приводится другой вывод этого удивительного результата.

Компактификация Калуцы-Клейна

Вы, должно быть, читали, какое сильное впечатление произвело на Эйнштейна предложение Калуцы и Клейна расширить пространство-время до пяти измерений и тем самым объединить электромагнетизм с гравитацией. Пятое измерение считается компактифицированным на малую окружность с радиусом a, невидимым глазу экспериментатора; другими словами, x^5 есть угловая переменная, такая что $x^5 = x^5 + 2\pi a$. Я уверен, вы слышали, что теория струн, по крайней мере в оригинальной версии, построена на идее Калуцы-Клейна. Струны живут в 10-мерном пространстве-времени, шесть из которых компактифицированы.

Теперь продемонстрирую, как работает механизм Калуцы-Клейна. Начнем с действия в 5-мерном пространстве-времени

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g_5}R_5$$

Подстрочный индекс 5 служит для обозначения 5-мерных величин. Индексы A и B 5-мерной метрики g_{AB} могут принимать значения $0, 1, 2, 3, 5$.

Пусть метрика g_{AB} не зависит от x^5. Подставим ее в выражение для S, проинтегрируем по x^5 и вычислим эффективное 4-мерное действие. Поскольку R_5 и 4-мерная скалярная кривизна R содержат производную ∂ второй степени, а g_{AB} содержит $g_{\mu\nu}$, имеем (упражнение VIII.1.5) $R_5 = R + ...$. Таким образом, уравнение (15) описывает действие Гильберта-Эйнштейна с ньютоновской гравитационной константой $G \sim G_5/a$.

Что еще мы получаем? Нам далее не нужно делать никаких выкладок. Воспользуемся симметриями. Для 5-мерного координатного преобразования $x^A \to x^A = x^A + \varepsilon^A(x)$ справедливо [см. (8)] $h'_{AB} = h_{AB} - \partial_A\varepsilon_B$ - $-\partial_B\varepsilon_A$. Пусть $\varepsilon_\mu = 0$, а $\varepsilon_5(x)$ не зависит от x_5: мы чуть-чуть поворачиваем каждую из малых окружностей, прикрепленных ко всем точкам нашего пространства-времени. Итак, имеем $h'_{\mu\nu} = h_{\mu\nu}$ и $h'_{55} = h_{55}$, но $h'_{\mu5} = h_{\mu5}$ - $-\partial_\mu\varepsilon_5$. Если назвать лоренцев 4-вектор $h_{\mu5}$ и 4-скаляр ε_5 н A_μ и Λ, получим обычное электромагнитное калибровочное преобразование $A'_\mu = A_\mu - \partial_\mu\Lambda$.

Известно, что 5-мерное действие (15) инвариантно относительно преобразования $x^A \to x^A = x^A + \varepsilon^A(x)$. Отсюда следует, что 4-мерное действие должно быть инвариантно относительно $A_\mu \to A'_\mu = A_\mu - \partial_\mu\Lambda$ и потому содержать действие Максвелла. Еще раз обращаю ваше внимание на эффективность симметрийных аргументов, которая позволяет избежать громоздких вычислений.

Электромагнетизм возникает из гравитации!

Дифференциальная геометрия римановых многообразий

Ранее я указывал на глубокую связь между общим координатным преобразованием и калибровочным преобразованием. Поясним это утверждение на примере дифференциальной геометрии и гравитации. Здесь мы будем рассматривать локальные евклидовы пространства (а не пространства Минковского).

Дифференциальная геометрия риманова многообразия может быть изящно описана в терминах дифференциальньх форм. Рассмотрим риманово многообразие (например, сферу) с метрикой $g_{\mu\nu}(x)$. По определению, многообразие является локально евклидовым, т. е.

$$g_{\mu\nu}(x) = e^a_\mu(x)\delta_{ab}e^b_\nu(x)$$

где матрицу $e(x)$ можно рассматривать в качестве преобразования подобия, которое диагонализирует матрицу $g_{\mu\nu}$ и масштабирует ее до единичной матрицы. Следовательно, для D-мерного многообразия существует D «мировых векторов» $e^a_\mu(x)$, зависящих от x и обозначаемых индексами $a = 1, 2, ..., D$. Функции $e^a_\mu(x)$ называют «тетрадами» (многоножками понемецки) (тетрадами для $D = 4$, триадами для $D = 3$ и т. д.). В некотором смысле тетрады можно считать «квадратным корнем» из метрики.

Поясним на простом примере. Обычная 2-сфера (единичного радиуса) характеризуется линейным элементом [9]$ds^2 = d\theta^2 + \sin^2\theta d\varphi^2$. Из метрики $\left(g_{\theta\theta} = 1, g_{\varphi,\varphi}\sin^2\theta\right)$ следует, что $e^1_\theta = 1$ и $e^2_\varphi = \sin\theta$ (все остальные компоненты равны нулю). Теперь определим 1-формы $e^a = e^a_\mu dx^\mu$ в количестве D. (В нашем примере $e^1 = d\theta, e^2 = \sin\theta d\varphi$.)

На искривленном многообразии, при параллельном переносе, вектор в локальной евклидовой системе координат меняет свои координаты. (Нам уже знакомо утверждение, согласно которому на искривленном многообразии, подобном поверхности Земли, понятие вектора, направленного строго на север, является локальным: если продвинуться на бесконечно малое расстояние вдоль нашего «северного вектора», придерживаясь первоначального направления, то в результате мы повернемся на бесконечно малый угол относительно «северного вектора», заданного в точке, из которой мы только что вышли.) Этот инфинитезимальный поворот тетрад описывается уравнением:

$$de^a = -\omega^{ab}e^b$$

9 Замечу, что он представляет из себя квадрат бесконечно малопо элемента расстояния, а не элемент площади, т. е. величина $d\theta^2$ есть буквально квадрат $d\theta$, а не антисимметризованное произведение $d\theta d\theta$ (из главы IV.4), тождественно равное нулю. Обратите внимание, что ω является антисимметричной матрицей, поскольку осуществляет инфинитезимальный поворот: $\omega^{ab} = -\omega^{ba}$. Поскольку de^a есть 2-форма, то ω будет 1-формой, называемой связностью: она «связывает» локальные евклидовы системы координат в соседних точках. (Индексы a, b и т. д. ассоциируются с евклидовой метрикой δ_{ab}, поэтому мы не проводим различий между верхними и нижними индексами. Символы a, b и т. д. в качестве верхних или нижних индексов используются из соображений типографического удобства.) В простом примере со сферой $de^1 = 0$ и $de^2 = \cos\theta d\theta$, поэтому связность имеет только одну ненулевую компоненту $\omega^{12} = -\omega^{21} = -\cos\theta d\varphi$.

Мы можем свободно вращать тетрады в любой точке пространства. Если вы используете функции e^a_μ, я могу использовать вместо них другие функции e'^a_μ, так как мои функции связаны с вашими поворотом $e^a_\mu(x) = O^a_b(x)e'^b_\mu(x)$. [Убедитесь, что $g_{\mu\nu}(x) = e^a_\mu(x)\delta_{ab}e^b_\nu(x) = e'^a_\mu(x)\delta_{ab}e'^b_\nu(x)$ при условии, что $O^TO = 1$.] Связность ω' задается как $de'^a = -\omega'^{ab}e'^b$. Проверьте (опустив индексы), что

$$\omega = O\omega'O^T - (dO)O^T.$$

Локальная кривизна многообразия является мерой того, как меняется связность от точки к точке. Нам бы хотелось, чтобы кривизна была инвариантна относительно локального вращения O (или, по крайней мере, чтобы она преобразовывалась как тензор и, сворачиваясь с векторами, давала скаляр.) Этому условию удовлетворяет 2-форма $R^{ab} = d\omega^{ab} + \omega^{ac}\omega^{cb}$. Проверьте, что $R = OR'O^T$. (Для сферы имеем $R^{12} = d\omega^{12} + \omega^{1c}\omega^{c2} = \sin\theta d\theta d\varphi$.) Если записать в компонентах, получим $R^{ab} = R^{ab}_{\mu\nu}dx^\mu dx^\nu$. Попробуйте самостоятельно проверить, что $R^{ab}_{\mu\nu}\lambda^\lambda_a e^\sigma_b$ - обычный тензор кривизны Римана $R^{\lambda\sigma}_{\mu\nu}$, в котором e^λ_a - матрица, обратная матрице e^a_λ. В частности, $R^{ab}_{\mu\nu}e^\mu_a e^\nu_b$ есть скалярная кривизна, равная для сферы $+1$.

Таким образом, риманову геометрию можно элегантно описать двумя выражениями (снова опуская индексы):

$$de + \omega e = 0$$

Y

$$R = d\omega + \omega^2$$

Кажется знакомым? Вы, должно быть, поражены тому, что (20) выглядит так же, как напряженность поля в неабелевых калибровочных теориях $F = dA + A^2$. Замечу, что ω преобразовывается [см. выражение (20)] в точности так же, как калибровочное поле A. Но одно различие, а именно отсутствие в калибровочной теории аналога e, долго беспокоило

физиков-теоретиков (но признанное в конце концов большинством как несущественное).Также не забывайте, что теория Эйнштейна линейна по R, тогда как теория Янга-Миллса квадратична по F.

В завершение сделаю одно техническое замечание, касающееся взаимодействия гравитации и полей спина $\frac{1}{2}$. Сначала мы, естественно, должны осуществить виковский поворот, чтобы тетрада e^a_μ относилась не к евклидовой, а локальной системе координат Минковского. Индексы a, b и т. д. теперь сворачиваются с метрикой Минковского η_{ab}. Тонкость состоит в том, что дираковские гамма-матрицы γ^a ассоциируются с лоренцевым поворотом тетрады $e^a_\mu(x) = O^a_b(x)e'^b_\mu(x)$ и, следовательно, несут лоренцев индекс a, вместо «мирового» индекса μ. Таким образом, дираковское действие в плоском пространстве $\int d^4x\bar{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi$ должно быть обобщено до вида $\int d^4x\sqrt{-g}\bar{\psi}\left(i\gamma^a\eta_{ab}e^b_\mu g^{\mu\nu}\partial_\nu - m\right)\psi$. В отличие от действия для полей с целым спином в искривленном пространстве-времени (см. главу I.10), дираковское действие в искривленном пространстве-времени явным образом включает в себя тетраду.

Приложение 1. Свет на свете

Тензор энергии импульса $T^{\mu\nu}$ светового луча, движущегося в x-направлении, имеет четыре ненулевые компоненты: плотности энергии T^{00}, затем $T^{0x} = T^{00}$ вследствие того, что фотоны несут одну энергию и импульс, затем $T^{x0} = T^{0x}$ вследствие симметрии и, наконец, $T^{xx} = T^{00}$ вследствие того, что тензор энергии импульса электромагнитного поля дается матрицей с нулевым следом (глава I.10). Даже не решая уравнения Эйнштейна в пределе слабого поля (уравнение (13)), мы знаем, что $h^{00} = h^{0x} = h^{x0} = h^{xx} \equiv h$. Вокруг светового луча задается метрика $g_{00} = 1 + h, g_{0x} = g_{x0} = -h$ и $g_{xx} = -1 + h$ (и, естественно, $g_{yy} = g_{zz} = -1$; остальные компоненты равны нулю). Рассмотрим фотон, движущийся параллельно световому лучу. Его мировая линия определяется выражением (см. главу I.10):

$$\frac{d^2x^\rho}{d\zeta^2} = -\Gamma^\rho_{\mu\nu}\frac{dx^\mu}{d\zeta}\frac{dx^\nu}{d\zeta}$$

Вычислим $d^2y/d\zeta^2$ и $d^2z/d\zeta^2$ при условии $(dy/d\zeta), (dz/d\zeta) \ll (dt/d\zeta), (dx/d\zeta)$. Используя (3), находим (μ, ν принимают значения только $0, x$)

$$\begin{aligned}\frac{d^2y}{d\zeta^2} =& \frac{1}{2}\left(\partial_\nu g_{y\mu} + \partial_\mu g_{y\nu} - \partial_y g_{mu\nu}\frac{dx^\mu}{d\zeta}\frac{dx^\nu}{d\zeta}\right) = \\ &= -\frac{1}{2}\left(\partial_y h\right)\left[\left(\frac{dt}{d\zeta}\right)^2 + \left(\frac{dx}{d\zeta}\right)^2 - 2\frac{dt}{d\zeta}\frac{dx}{d\zeta}\right] = -\frac{1}{2}\left(\partial_y h\right)\left(\frac{dt}{d\zeta} - \frac{dx}{d\zeta}\right)^2\end{aligned}$$

Для фотона, движущегося в одном направлении с лучом света, $dt = dx$ и $d^2y/d\zeta^2 = d^2z/d\zeta^2 = 0$. Мы снова получили эффект Толмана-Эренфеста-Подольского. Обратите внимание, что мы не вычисляли при этом h.

Если вы сомневаетесь в справедливости $dt = dx$, то знайте, что условие $ds = 0$ для луча света, падающего в направлении x, сводится к виду $(1 + h)dt^2 - 2hdtdx$ - $-(1 - h)dx^2 = 0$. Делением на dt^2 получаем $-(1 + h) + 2hv + (1 - h)v^2 = 0$, где $v \equiv dx/dt$. У квадратного уравнения два корня $v = \mp(1 \pm h)/(a - h)$. Отрицательный корень дает $v = 1$, поэтому для фотона, движущегося в одном направлении с лучом света, $dx/dt = 1$. Положительный корень дает $v = -(1+ +h)/(1 - h)$, что соответствует случаю, когда фотон движется в противоположном направлении.

Приложение 2. Спиральная структура гравитации

Чтобы еще глубже понять разницу между теориями Эйнштейна и доктора Г., обратим внимание на спиральную структуру взаимодействия в двух случаях. Сначала рассмотрим взаимодействие между двумя сохраняющимися токами, обусловленное обменом частицей со

спином 1, импульсом k и массой m : $J^{\mu}_{(1)}J_{(2)\mu} = J^{0}_{(1)}J^{0}_{(2)}$ - - $J^{i}_{(1)}J^{i}_{(2)}$. Воспользуемся сохранением тока $k_\mu J^\mu = 0$, чтобы исключить $J^0 = k^i J^i/\omega$ (где $\omega \equiv k^0$). Получаем $\left(k^i k^j/\omega^2 - \delta^{ij}\right) J^i_{(1)} J^j_{(2)}$. Примем, что направление вектора $\vec{k}$ совпадает с направлением 3, выразим $\vec{k}^2 = \omega^2 - m^2$ и перепишем выражение в виде $-\left[\left(m^2/\omega^2\right) J^3_{(1)}J^3_{(2)} + J^1_{(1)}J^1_{(2)} + J^2_{(1)}J^2_{(2)}\right]$. Видим, что при $m \to 0$ продольная компонента тока J^3 действительно отщепляется, как было объяснено в главе II.7, и мы получаем $-\frac{1}{2}\left(J^{1+i2}_{(1)}J^{1-i2}_{(2)} + J^{1-i2}_{(1)}J^{1+i2}_{(2)}\right)$. Отсюда следует, что фотон имеет спиральность \pm 1. (Мы сделали очевидное обозначение: $J^{1+i2} \equiv J^1 + iJ^2$ и т. д.) Обратимся к гравитации. Рассмотрим взаимодействие $T^{\mu\nu}_{(1)}T_{(2)\mu\nu} - \xi T_{(1)}T_{(2)}$, где $\xi = \frac{1}{2}$ в теории Эйнштейна и $\xi = \frac{1}{3}$ в теории доктора Г. Для упрощения записи опустим индексы (1) и (2). Условие сохранения $k_\mu T^{\mu\nu} = 0$ позволяет нам исключить $T^{0i} = k^j T^{ji}/\omega$ и $T^{00} = k^j k^l T^{jl}/\omega^2$. Примем, что направление вектора $\vec{k}$ совпадает с направлением 3, тогда получим громоздкое выражение

$$\left(\frac{m}{\omega}\right)^4 T^{33}T^{33} + 2\left(\frac{m}{\omega}\right)^2 \left(T^{13}T^{13} + T^{23}T^{23}\right) + T^{11}T^{11} + T^{22}T^{22} + 2T^{12}T^{12} - \\ - \xi\left[\left(\frac{m}{\omega}\right)^2 T^{33} + T^{11} + T^{22}\right]\left[\left(\frac{m}{\omega}\right)^2 T^{33} + T^{11} + T^{22}\right]$$

которое в пределе $m \to 0$ упрощается до:

$$T^{11}T^{11} + T^{22}T^{22} + 2T^{12}T^{12} - \xi\left(T^{11} + T^{22}\right)\left(T^{11} + T^{22}\right).$$

В теории Эйнштейна $\xi = \frac{1}{2}$, поэтому выражение принимает вид:

$$\frac{1}{2}\left(T^{11} - T^{22}\right)\left(T^{11} - T^{22}\right) + 2T^{12}T^{12}$$

и в конечном итоге - вид $\frac{1}{2}\left(T^{1+i2,1+i2}T^{1-i2,1-i2} + T^{1-i2,1-i2}T^{1+i2,1+i2}\right)$, что доказывает следующее: гравитон действительно несет спиральность \pm 2.

Упражнения

VIII.1.1. Найдите $T^{\mu\nu}$ для скалярного поля. Нарисуйте фейнмановскую диаграмму, описывающую вклад одногравитонного обмена в рассеяние двух скалярных мезонов. Вычислите амплитуду и энергию взаимодействия между двумя мезонами, находящимися в состоянии покоя, получив тем самым ньютоновский закон гравитации.

VIII.1.2. Найдите $T^{\mu\nu}$ дя поля Янга-Миллса.

VIII.1.3. Покажите, что если $h_{\mu\nu}$ не удовлетворяет гармонической калибровке, всегда можно осуществить калибровочное преобразование при условии, что ε_ν определяется выражением $\partial^2\varepsilon_\nu = \partial_\mu h^\mu_\nu - \frac{1}{2}\partial_\nu h^\lambda_\lambda$. После этого $h_{\mu\nu}$ будет удовлетворять гармонической калибровке. Все это вам уже известно из курса электродинамики.

VIII.1.4. Подсчитайте количество степеней поляризации гравитона. [Указание: рассмотрите плоскую волну $h_{\mu\nu}(x) = h_{\mu\nu}(k)e^{ikx}$, поскольку работать в пространстве импульсов несколько легче. Симметричный тензор имеет 10 компонент, а гармоническая калибровка $k_\mu h^\mu_\nu = \frac{1}{2}k_\nu h^\lambda_\lambda$ накладывает 4 условия. Как так, у нас осталось 6 степеней свободы. Что происходит?] [Указание: продолжите калибровочное преобразование и останьтесь в гармонической калибровке. Гравитон должен иметь всего 2 степени поляризации.]

VIII.1.5. Результат Калуцы-Клейна, к которому мы пришли, из симметрийных соображений, можно получить явным вычислением. Предлагаю вам следующий план действий. Рассмотрите метрику

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu - a^2\left[d\theta + A_\mu(x)dx^\mu\right]^2$$

где θ - угловая переменная, $0 \leqslant \theta < 2\pi$. При $A_\mu = 0$ она соответствует метрике искривленного пространства-времени, к каждой точке которого прикреплена окружность радиуса a. Преобразование $\theta \to \theta + \Lambda(x)$ оставляет ds инвариантным при условии, что мы также преобразовываем $A_\mu(x) \to A_\mu(x) - \partial_\mu \Lambda(x)$. Вычислите 5-мерную скалярную кривизну R_5 и докажите, что $R_5 = R_4 - \frac{1}{4}a^2 F_{\mu\nu}F^{\mu\nu}$. Если не считать коэффициента $\frac{1}{4}$, полученный результат полностью совпадает с результатом, полученным на основе соображений симметрии и того факта, что R_5 включает две производные по 5-мерной метрике. После подходящего масштабирования получаем обычное действие для гравитации плюс электромагнетизм. Обратите внимание, что 5-мерная метрика в явном виде равна

$$g^5_{AB} = \begin{pmatrix} g_{\mu\nu} - a^2 A_\mu A_\nu & -a^2 A_\mu \\ -a^2 A_\nu & -a^2 \end{pmatrix}$$

VIII.1.6. Обобщите схему Калуцы-Клейна, заменив окружности сферами большей размерности. Покажите, что появляются поля Янга-Миллса.

VIII.1.7. Тетрады для пространства-времени с метрикой Минковского задаются в виде $g_{\mu\nu}(x) = e^a_\mu(x)\eta_{ab}e^b_\nu(x)$, где метрика Минковского η_{ab} используется вместо евклидовой метрики δ_{ab}. Индексы a и b следует сворачивать с η_{ab}. Например, $R^{ab} = d\omega^{ab} + \omega^{ac}\eta_{cd}\omega^{ab}$. Покажите, что все, как и ожидалось, работает.

30.5.3 Zi.VIII.2 The Cosmological Constant Problem and the Cosmic Coincidence Problem

Сила, которая слишком много знает

Понятие парадокса обесценено неправильным употреблением в физической литературе. Настоящий парадокс должен содержать существенное и ясное расхождение между теоретическим предсказанием и экспериментальными данными. Ультрфиолетовая катастрофа, например, является парадоксом, решение которого на заре двадцатого века привело к созданию квантовой физики. Теперь поговорим о самом большом парадоксе современной физики.

Электромагнитная сила знает о частицах, несущих заряд, сильное взаимодействие знает о частицах, несущих цвет. А как насчет гравитационной силы? Она знает всех! Точнее, всех, кто несет энергию и импульс.

С точки зрения физики элементарных частиц, а именно с этой точки зрения мы исследуем фундаментальную структуру физики, гравитон можно рассматривать как еще одну частицу. Стартуя с безмассовой частицы со спином 2, взаимодействующей с тензором энергии импульса, можно воссоздать теорию Эйнштейна.

Однако такое представление мира оставляет у нас неоднозначные чувства. Гравитация имеет дело с кривизной пространства-времени, ареной в которой живут все поля и частицы. Гравитон не просто еще одна частица.

В этом состоит суть и корень парадокса космологической постоянной. Гравитон не просто частица - он слишком много знает!

Космологическая постоянная

В отсутствие гравитации добавление константы Λ в лагранжиан $\mathcal{L} \to \mathcal{L} - \Lambda$ ни к чему не приводит. В классической физике уравнения движения Эйлера-Лагранжа зависят исключительно от вариации лагранжиана. В квантовой теории поля нужно вычислять интеграл по траекториям $Z = \int D\varphi e^{i\int d^4x \mathcal{L}(x)}$, который после добавления Λ просто умножается на множитель. Мы неоднократно видели, что мультипликативный фактор Z не участвует в вычислениях функции Грина и амплитуд рассеяния.

Гравитация, однако, знает о Λ. С физической точки зрения включение Λ соответствует сдвигу в гамильтониане $H \to H + \int d^3x \Lambda$. Следовательно, «космологическая постоянная» Λ

характеризует постоянную энергию или массу на единицу объема, проникающую во Вселенную, и, естественно, гравитация ее знает.

С технической точки зрения член $-\int d^4x\Lambda$ в действии, не инвариантен относительно координатного преобразования $x \to x'(x)$. Если учитывать гравитацию, то вследствие инвариантности относительно общего координатного преобразования член $-\int d^4x\Lambda$ в выражении для действия S должен модифицироваться к виду $-\int d^4x\sqrt{g}\Lambda$, как говорилось в главе I.10. Таким образом, гравитационное поле $g_{\mu\nu}$ знает о Λ, печально известной космологической постоянной, введенной Эйнштейном, который сам считал ее самой большой своей ошибкой. Но ссылаться на его слова будет само по себе ошибкой. Введение космологической постоянной не является ошибкой, она должна присутствовать.

Нарушение симметрии генерирует энергию вакуума

В нашем обсуждении спонтанного нарушения симметрии мы все время пренебрегали членом $\mu^4/4\lambda$, появляющимся в лагранжиане.

Физика элементарных частиц построена на ряде спонтанных нарушений симметрии. По мере охлаждения Вселенной происходит спонтанное нарушение симметрии великого объединения, за которой следует нарушение электрослабой симметрии, киральной симметрии (это только из тех видов симметрии, которые мы обсуждали). На каждой стадии в лагранжиане появляется член вида $\mu^4/4\lambda$, что не остается не замеченным гравитацией.

Насколько большой должна быть космологическая постоянная? Далее мы увидим, что для наших целей достаточно использовать самое грубое приближение. Пусть значение λ имеет порядок 1. Что касается параметра μ, то для трех упомянутых мною нарушений симметрии его значение будет порядка $10^{17}, 10^2$ и 1 ГэВ соответственно. Отсюда величину космологической постоянной Λ можно принять равной $\mu^4 = \mu/\left(-\mu^{-1}\right)^3$. Последняя форма записи μ^4 отражает тот факт, что Λ есть плотность массы или энергии: энергия порядка μ содержится в кубе размером μ^{-1}. Однако данная оценка слишком преувеличена, даже если использовать наименьшее значение для μ. Мы знаем, что плотность массы Вселенной вовсе не порядка 1 ГэВ на каждый куб, размером $1\,(\Gamma\ \text{В})^{-1}$. Нам не нужны реальные значения, чтобы увидеть, что между теоретическими оценками и экспериментальными данными существует огромная разница. Если вам нужны числа, скажу, что существующее экспериментальное ограничение сверху равно $\lesssim (10^{-3}\ \text{В})^4$. С масштабом великого объединения мы ошибаемся на $(17+9+3)\times 4 = 116$ порядков. В этом кроется причина всех расхождений!

С массой Планка $M_{Pl} \sim 10^{19}$ ГэВ, естественным масштабом гравитации и допуская гравитационное происхождение космологической постоянной, можно ожидать, что $\Lambda \sim M_{Pl}^4$. Теперь мы расходимся на 124 порядка. Мы говорим не об ординарном вычислении какого-нибудь несчастного теоретика, у которого расхождение с экспериментальными данными в 2 раза.

Мы можем представить, что, Вселенная, возникшая с отрицательным значением космологической постоянной, тонко подстраивается рядом спонтанных нарушений симметрии так, что космологическая постоянная зануляется. Либо должен существовать динамический механизм, который подстраивает космологическую постоянную к нулю.

Обратите внимание, что я говорю «к нулю», потому что задача о космологической постоянной обуславливается несоответствием между естественными единицами в космологии и единицами, естественными в физике элементарных частиц. Если измерять космологическую постоянную в ГэВ, то она окажется настолько крошечной, что специалисты в области физики элементарных частиц должны бы традиционно принять, что она должна быть равной нулю, и вели напрасные поиски правдоподобного механизма ее зануления. Одним из недостатков теории струн является неразрешимость с ее помощью проблемы космологической постоянной. При написании этой главы около 2000 г. появились сценарии бранных миров, которые дали нам надежду на решение проблемы. Грубо говоря, идея состоит в том, что гравитационная динамика большего пространства, куда вложена наша вселенная, может сократить эффект космологической постоянной.

Космическое совпадение

Природа приготовила нам большой сюрприз. В то время, когда теоретики пытались найти выход из затруднительного положения поиском аргументов почему $\Lambda = 0$, специалисты по наблюдательной космологии последовательно уточняли свои измерения и недавно заменили ограничение сверху на примерное значение

$$\Lambda \sim \left(10^{-3}\text{эВ}\right)^4 !!!$$

Парадокс космологической постоянной еще более усугубился. Теоретически нам легче объяснить, почему какая-то величина равна строго нулю, чем тот факт, почему она оказывается равной $\sim 10^{-124}$ в естественных (?) для этой проблемы единицах.

Хуже того, значение $(10^{-3}$ эВ $)^4$ имеет тот же порядок, что и плотность материи во Вселенной ρ_M сегодня. Это явление иногда называют проблемой космического совпадения.

На сегодняшний день космологическую постоянную Λ мы считаем параметром лагранжиана. С другой стороны, поскольку большая часть плотности Вселенной складывается из массы барионов, то по мере расширения Вселенной значение $\rho_M(t)$ уменьшается по закону $[1/R(t)]^3$, где R - масштаб Вселенной [1]. В далеком прошлом значение ρ_M было гораздо больше Λ, а в отдаленном будущем оно будет гораздо меньше, чем Λ. Так уж сложилось, что мы с вами живем в эпоху такой Вселенной, когда $\rho_M \sim \Lambda$. Или менее антропоцентрично, скажу, что эпоха $\rho_M \sim \Lambda$ совпадает по времени с завершающей стадией образования галактики.

Очень странно!

В отчаянии некоторые теоретики вынуждены даже опираться на антропный принцип.

'Простое введение в космологию читайте в книге А.Зи (A.Zee. Unity of Forces in the Universe, vol.II, chap.10).

[2] См. недавний обзор (A. Vilenkin, hep-th/0106083).

30.5.4 Zi.VIII.3 Effective Field Theory as an Approach to Understanding Nature

Низкоэнергетическое проявление

Основоположники квантовой теории поля, например Дирак, склонны были рассматривать теорию поля как внутренне замкнутое фундаментальное описание Природы. Я уже неоднократно упоминал, что в 1950-х гг. после успеха квантовой электродинамики многие ведущие физики в области элементарных частиц отказались от квантовой теории поля, как неспособной описать сильное и слабое взаимодействия, не говоря уж о гравитации. Затем в начале 1970-х гг. наступил великий триумф теории поля. Но после того как теоретики достали теорию поля из мусорного пакета теоретческой физики, они осознали, что теории поля, которые они изучали, могут быть всего лишь низкоэнергетическим проявлением более глубокой структуры, которую сначала отождествляли с теорией великого объединения и позже с теорией струн. Так была развита точка зрения, известная как эффективная теория поля, с позволения Дирака.

В основе подхода лежит идея, согласно которой мы можем использовать теорию поля для описания, скажем, физики низких энергий или, что тоже самое, физики больших расстояний даже в том случае, если не имеем представлений о всеобщей теории, будь то теория струн или любая другая неизвестная теория. Важным следствием этого изменения парадигмы явился тот факт, что неперенормируемые теории теперь стали приемлемыми. Я проиллюстрирую эти соображения на частных примерах.

Появление философии эффективной теории поля, развитой Вильсоном, стало еще одним примером «перекрестного опыления» между физикой конденсированного состояния и физикой элементарных частиц.

К концу 1960-х гг. Вильсон совместно с другими учеными разработали мощный подход к пониманию критических явлений, основанный на эффективной теории поля. За свой труд

он был удостоен Нобелевской премии. Ситуация в физике конденсированных сред во многом противопо- ложна ситуации в физике элементарных частиц, по крайней мере так, как она понималась в 1960-е годы. Физики, занимающиеся конденсированным состоянием, знают физику на малых расстояниях - квантовую механику электронов и ионов. Но, конечно, уравнение Шредингера для электронов и ионов не помогает решить задачу. Напротив, хотелось бы иметь эффективное описание того, как система откликнется на пробные воздействия с малой частотой и волновым вектором. Ярким примером является эффективная теория квантовой холловской жидкости (глава VI.2). Релевантная степень свободы - калибровочное поле совсем не похоже на лежащий в основе электрон. Следует отметить, как и при σ-модельном описаний квантовой хромодинамики (глава VI.4), без экспериментальной поддержки теоретику было бы ужасно тяжело выяснить, какие будут релевантные низкоэнергетические степени свободы. Вы видели и другие многочисленные примеры в физике конденсированного состояния, начиная с теории сверхпроводимости Гинзбурга-Ландау и заканчивая неустойчивостью Пайерлса.

Граница неведения

В нашем обсуждении перенормировок моя философия состояла в том, что квантовая теория поля обеспечивает эффективное описание физических явлений до определенного масштаба энергии Λ-границы неведения. В неперенормируемой теории многие физические величины, которые мы бы хотели вычислить, оказываются зависимыми от Λ. Это означает, что для понимания физики низких энергий, которую мы исследуем, существенной в этом случае оказывается физика за пределами масштаба Λ. Недостатком неперенормируемых теорий является их неполная предсказательность, тем не менее они могут оказаться полезными. В конце концов, теория слабых взаимодействий Ферми смогла описать результаты экспериментов и даже предсказать свою собственную гибель.

В перенормируемой теории физические величины оказываются не зависящими от Λ, если выразить результаты вычислений через физические константы связи и физические массы, а не в терминах ничего не значащих затравочных констант связи и затравочных масс. Низкоэнергетическая физика не чувствительна к тому, что происходит на высоких энергиях, поэтому мы параметризуем наше отсутствие знаний в отношении высоэнергетической физики с помощью нескольких физических констант.

С конца 1960-х гг. до 1970-х гг. одной из главных задач фундаментальной физики была классификация и исследование перенормируемых теорий. Мы знаем, что ее итоги оказались «более чем впечатляющими». Она позволила нам установить теорию сильных, слабых и электромагнитных взаимодействий.

Ренорм-групповой поток и анализ размерностей

Философия эффективной теории поля неразрывно связана ренормгрупповым потоком. Для данной теории поля, если мы движемся в сторону низких энергий, одни константы связи стремятся к нулю, а другие нет (если они при этом стремятся к бесконечности, как в КХД, нам не удается построить эффективную теорию без использования экспериментальных данных). Таким образом, на первом шаге необходимо вычислить ренорм-групповой поток. Простой пример дается в упражнении VIII.3.1.

Во многих случаях мы можем просто воспользоваться анализом размерностей. Ранее, при обсуждении теории перенормировок, я говорил, что константы связи с отрицательной размерностью массы не важны в низкоэнергетическом пределе. Для определенности добавим член $g\varphi^6$ в теорию $\lambda\varphi^4$. Константа связи g имеет размерность, обратную квадрату массы. Определим $M^2 \equiv 1/g$. При низких энергиях вклад члена $g\varphi^6$ подавляется фактором $(E/M)^2$.

Как мы должны понимать вычисление Швингера аномального магнитного момента электрона в эффективной теории поля?

Сначала отвечу на вопрос с традиционной (довильсоновской) точки зрения. Допустим, студент спрашивает: «Профессор Швингер, почему вы не включили в лагранжиан член $(1/M)\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$?»

Профессор отвечает, что иначе мы не могли бы предсказать величину аномального магнитного момента, поскольку она бы зависела от M. Вспомним, что $[\psi] = \frac{3}{2}$ и $[A_\mu] = 1$. Следовательно, размерность $\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$ равна $\frac{3}{2} + \frac{3}{2} + 1 + 1 = 5 > 4$. Условие перенормируемости, при котором лагранжиан должен содержать лишь операторы размерности 4 или ниже, позволяет обоснованно исключить этот член.

На самом деле сам Швингер, наверное, не стал бы отвечать на поставленный вопрос. Всем известно, что тогда студентам было запрещено задавать вопросы. Швингер просто проигнорировал бы поднятую руку студента. И после занятий тоже не было возможности задать вопрос. Как только Швингер произносил последнее предложение своей неизменно блестяще подготовленной лекции, он величественно покидал аудиторию. Дирак иначе обходился с вопросами. Я был слишком молод и не мог быть тому свидетелем, но история гласит, что на фразы типа «Профессор Дирак, я не понимаю...» он отвечал: «Это не вопрос, а утверждение». Современное изложение сюжета об аномальном магнитном моменте сильно отличается. Сейчас мы рассматриваем лагранжиан квантовой электродинамики как эффективный лагранжиан, который должен включать в себя бесконечную последовательность членов все более высоких порядков с коэффициентами, параметризующими границу нашего неведения. Член $(1/M)\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$ тоже присутствует в лагранжиане с неизвестным коэффициентом M, имеющим размерность массы. Таким образом, результат Швингера, согласно которому квантовые флуктуации порождают член $(\alpha/2\pi)\,(1/2m_e)\,\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$, интерпретируется следующим образом: предсказанное значение аномального магнитного момента электрона равно $[(\alpha/2\pi)\,(1/2m_e) + 1/M]$. Хорошее совпадение $(\alpha/2\pi)\,(1/2m_e)$ с экспериментальным значением аномального магнитного момента позволяет установить ограничение снизу на $M \gg (4\pi/\alpha)m_e$.

Иначе, результат Швингера предсказывает аномальный магнитный момент электрона, если у нас есть независимые основания считать, что M гораздо больше $[(\alpha/2\pi)\,(1/2m_e)]^{-1}$. Особо обращаю ваше внимание, что все это имеет глубокий физический смысл. Например, если вы считаете, что электрон имеет конечный размер a, то можно ожидать, что $M \sim 1/a$. Вычисление аномального магнитного момента электрона дает верхнюю границу для a, т. е. электрон должен выглядеть как точка до некоторого малого масштаба. В качестве альтернативы мы можем иметь независимые основания полагать, например, из рассеяния электронов, что а должна быть меньше определенной длины. Отсюда следует ограничение снизу на M.

Чтобы подчеркнуть это место, представим, что в 1948 году мы, следуя Швингеру, быстро вычисляем аномальный магнитный момент протона. Мы можем сделать это буквально за 3 секунды, заменив в лагранжиане m_e на m_p и получив тем самым выражение $(\alpha/2\pi)\,(1/2m_p)\,\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$, которое в корне противоречит эксперименту. Такое противоречие говорит нам, что мы не в полной мере учли физику рассматриваемого явления, а именно что протон не точечный и сильно взаимодействует. Действительно, мы сейчас знаем, что в аномальный магнитный момент протона вносят вклады аномальные магнитные моменты кварков и их орбитальное движение внутри протона.

Эффективная теория распада протона

Может показаться, что в подходе на основе эффективной теории поля мы теряем предсказательную способность. Эффективные теории поля моIут также на удивление хорошо предсказывать результаты. Приведу один ример. Допустим, нам ничего не известно о теории великого объединения. Все, что мы знаем, - это $SU(3) \otimes SU(2) \otimes U(1)$-теория. Экспериментатор нам сообщил, что планирует проверить, распадается ли протон.

Без малейших представлений о том, что могло бы привести к распаду протона, мы все равно сможем записать теорию поля, описывающую его распад. Лагранжиан $\mathcal{L}$ должен содержать поля кварка q и лептона l, а также удовлетворять известным нам симметриям. Три кварка исчезают, так что схематично записываем qqq, но три спинора не образуют лоренцевый скаляр. Мы должны включить поле лептона и записать qqql.

Поскольку задействованы четыре фермионных поля, размерность члена $qqql$ равна 6 и в лагранжиан он входит в виде $(1/M^2)\,qqql$, где M некоторая масса, соответствующая масштабу

физических процессов, ответственных за распад протона. Экспериментальное ограничение снизу на время жизни протона устанавливает ограничение снизу на M.

Предположим, мы проводим анализ распада протона с помощью эффективной теории поля в докварковую эпоху. В этом случае мы построим лагранжиан из имеющихся полей, т. е. протонного поля p, электронного поля e и пионного поля π. В итоге получим что-то вроде $g\pi^-\bar{e}p$, где g - некоторая константа. Обратите внимание, что размерность $\pi^-\bar{e}p$ равна 4 и, следовательно, g безразмерна. Поскольку $\pi^-\bar{e}p$ нарушает изоспиновую инвариантность, можно предположить, что g порядка некоторой величины изоспинового нарушения, то есть порядка электромагнитной константы связи α, но которая, однако, дает неприемлемо короткое время жизни протона. Нам придется тогда положить g равным очень малому числу, что будет казаться неестественным. Таким образом, по крайней мере задним числом мы можем сказать, что исключительно большое время жизни протона почти указывает на существование кварков. (Можно ли решить подобным образом загадку космологической постоянной?)

Иначе говоря, $SU(3) \otimes SU(2) \otimes U(1)$-теория и ее перенормируемость позволяет предсказать одно из самых поразительных явлений Вселенной стабильность протона. Старая пион-нуклонная теория, напротив, очевидно, не смогла объяснить этот экспериментальный факт.

В соответствии с нашей философией, лагранжиан $\mathcal{L}$ должен быть инвариантен относительно группы $SU(3) \otimes SU(2) \otimes U(1)$, при которой поля кварка и лептона преобразуются довольно специфически (см. главу VII.5). Чтобы построить лагранжиан, нам придется сесть и выписать все лоренц-инвариантные члены $SU(3) \otimes SU(2) \otimes U(1)$ вида $qqql$.

За этим занятием мы обнаружим, что если для простоты рассмотреть только одно поколение кварков и лептонов, то распад протона будет описываться всего четырьмя членами, которые я приведу здесь для полноты: $\left(\tilde{l}_L C q_L\right)(u_R C d_R)$, $(e_R C u_R)\left(\widetilde{q}_L C q_L\right)$, $\left(\widetilde{l}_L C q_L\right)\left(\widetilde{q}_L C q_L\right)$ и $(e_R C u_R)(u_R C d_R)$. Здесь $l_L = \begin{pmatrix} \nu \\ e \end{pmatrix}_L$ и $q_L = \begin{pmatrix} u \\ d \end{pmatrix}_L$ обозначают дублеты лептона и кварка для $SU(2) \otimes U(1)$, знак тильды определяется как $\widetilde{l^j} = l_i \varepsilon^{ij}$ с $SU(2)$-индексами $i, j = 1, 2$ (см. приложение B), а C обозначает зарядово-сопряженную матрицу. Цветовые индексы кварковых полей сворачиваются единственно возможным способом. Таким образом, эффективный лагранжиан дается суммой этих четырех членов с четырьмя неизвестными коэффициентами.

Эффективная теория поля сообщает нам, что все возможные распады с нарушением барионного числа можно выразить через четыре неизвестные. Предполагаемая точность наших предсказаний будет порядка $(M_W/M)^2$. (Если M_W равны нулю, то $SU(3)\otimes SU(2)\otimes U(1)$ является точной.)

Конечно, можно усилить наши предсказательные способности за счет дополнительных допущений. Например, если считать, что распад протона происходит за счет обмена векторной частицей, как в универсальной теории великого объединения, то разрешены только два первых члена из вышеприведенного списка. В конкретной теории великого объединения, такой как $SU(5)$, два неизвестных коэффициента выражаются через константу связи великого объединения и массу X-бозона.

Чтобы оценить эффективную теорию поля с точки зрения способности к предсказанию, проанализируем четыре возможных оператора. Сразу можно сказать, что, хотя распад протона нарушает как барионное число B, так и лептонное число L, он сохраняет комбинацию $B - L$. Замечу, что это вовсе не очевидно без специального анализа. Могли бы вы заранее сказать экспериментатору, какую из двух возможных мод $n \to e^+\pi^-$или $n \to e^-\pi^+$ сму следует ожидать? Априори могла бы сохраняться комбинация $B + L$.

Обратите внимание, что теорию слабых взаимодействий Ферми сегодня мы бы назвали эффективной теорией поля. Конечно, бета-распад, в отличие от распада протона, можно наблюдать явным образом, поэтому результаты предсказания на основе такого сорта симметрийного анализа, а именно предсказания существования нейтрино, триумфально подтвердились.

Аналогичным образом можно построить эффективную теорию поля для масс нейтрино. Безусловно, одним из самых ярких экспериментальных открытий в области физики элементарных частиц за последние годы послужило открытие того, что нейтрино не безмассовы. Построим эффективную теорию, инвариантную относительно $SU(2) \otimes U(1)$. Поскольку ν_L входит в состав l_L, то даже без какого-либо детального анализа мы видим псобходимость включения 5-мерного оператора: схематично член $l_L l_L$ содержит требуемую нейтринную билинейность, но несет гиперзаряд $Y/2 = -1$; с другой стороны, дублет Хиггса φ несет гиперзаряд $\frac{1}{2}$, так что оператор низшей размерности имеет форму $ll\varphi\varphi$ и размерность $\frac{3}{2} + \frac{3}{2} + +1 + 1 = 5$. Следовательно, эффективный лагранжиан $\mathcal{L}$ должен включать член $(1/M)ll\varphi\varphi$, где M - массовый масштаб новой физики, ответственной за массу нейтрино. Значит, с помощью анализа размерностей мы можем оценить $m_\nu \sim m_l^2/M$, где m_l - некоторая типичная масса заряженного

лептона. Если взять m_l равной массе мюона $\sim 10^8$ МэВ, а $m_\nu \sim 10^{-1}$ эВ,

то получим $M \sim (10^2\text{МВ})^2 / 10^{-1} (10^{-6}\text{МВ}) = 10^8$ ГэВ.

Философия эффективных теорий поля, справедливых до определенного масштаба энергии Λ, кажется сегодня настолько очевидной, что с трудом верится, что когда-то многие великие физики ждали от квантовой теории поля гораздо большего: чтобы она была фундаментальной до произвольно высоких энергий.

Упражнения

VIII.3.1. Рассмотрим лагранжиан

$$\mathcal{L} = \frac{1}{2}\left[(\partial\varphi_1)^2 + (\partial\varphi_2)^2\right] - \lambda\left(\varphi_1^4 + \varphi_2^4\right) - g\varphi_1^2\varphi_2^2.$$

Мы взяли $O(2)$-теорию из главы I.9 и явно нарушили симметрию. Найдите ренорм-групповой поток на плоскости $(\lambda - g)$ и сделайте собственные выводы.

VIII.3.2. Предполагая, что правого нейтрино ν_R не существует (т. е. минимальный состав частиц в стандартной модели), запишите все члены, инвариантные относительно $SU(2) \otimes U(1)$, которые изменяют лептонное число на 2, и таким образом постройте эффективную теорию поля для массы нейтрино. Конечно, с помощью конкретной теории можно получить более сильные предсказания. Из произведения $l_L l_L$ можно построить лоренцев скаляр, преобразующийся относительно группы преобразований $SU(2)$ как синглет или триплет. Рассмотрите синглетный случай и сформулируйте для него теорию. [Указание: за помощью обращайтесь A. Zee, Phys. Lett. 93B: p. 389, 1980.

VIII.3.3. Пусть A, B, C, D - четыре поля со спином $\frac{1}{2}$. Обозначим их киральность подстрочным индексом: $\gamma^5 A_h = hA_h$ при $h = \pm 1$. То есть A_+−правое поле, A_-−левое поле и т. д. Покажите, что

$$(A_h B_h)(C_{-h} D_{-h}) = -\frac{1}{2}(A_h \gamma^\mu D_{-h})(C_{-h}\gamma_\mu B_h).$$

Это одно из многочисленных тождеств Фирца (некоторые из них нам понадобятся при обсуждении суперсимметрии). Покажите, что в том слу- чае, когда распад протона в низшем порядке идет за счет обмена векторной частицей, в лагранжиане разрешены лишь члены $\left(\widetilde{l}_L C q_L\right)(u_R C d_R)$ и $(e_R C u_R)(\widetilde{q}_L C q_L)$.

VIII.3.4. С учетом выводов из предыдущего упражнения покажите, что ширины распадов для процессов $p \to \pi^+ + \bar{\nu}, p \to \pi^0 + e^+, n \to \pi^0 + \bar{\nu}$ и $n \to \pi^- + +e^+$пропорциональны друг другу, а коэффициенты пропорциональности определяются единственной неизвестной константой [равной отношению коэффициентов для членов $\left(\widetilde{l}_L C q_L\right)(u_R C d_R)$ и $(e_R C u_R)(\widetilde{q}_L C q_L)$].

За помощью при решении последних трех упражнений обращайтесь к работам Вайнберга (S. Weinberg, Phys. Rev. Lett. 43, page 1566, 1979), Вильчека и Зи (F. Wilczek and A. Zee, ibid. p.1571), Велдона и Зи (H. A. Weldon and A. Zee, Nucl. Phys. B173:269, 1980).

30.5.5 Zi.VIII.4 Supersymmetry: A Very Brief Introduction

Объединяя бозоны и фермионы

Начну с изложения некоторых мотиваций к открытию суперсимметрии. Во-первых, все экспериментально установленные симметрии связывают бозоны с бозонами, а фермионы с фермионами. Хорошо бы, если бы некая суперсимметрия связывала бозоны с фермионами. Во-вторых, для фермионов естественно не иметь массы (вспомните главу VII.6), тогда как бозоны имеют ее. Возможно, объединяя в пару поле Хиггса с фермионным полем, получится разрешить проблему иерархий, упомянутую в главе VII.6. В-третьих, из главы II.5 известно, что фермионы создают отрицательный вклад в энергию вакуума, поэтому нельзя ли за счет фермионного вклада компенсировать вклад бозонов в энергию вакуума и решить тем самым задачу о космологической постоянной?

К сожалению, уже более 30 лет [1] никто не может опытным путем получить явное подтверждение суперсимметрии (первую суперсимметричную теорию поля построили в 1971 году Гольфанд и Лихтман). Все существующие суперсимметричные теории связывают известные бозоны с неизвестными фермионами, а известные фермионы с неизвестными бозонами. Суперсимметрия должна нарушаться на некотором масштабе массы M, находящемся вне уже исследованной экспериментально области. Тогда можно ожидать, что космологическая постоянная будет порядка M^4 (см. главу VIII.2).

Как бы то ни было, суперсимметричные теории поля обладают множеством хороших свойств (что не удивительно, поскольку симметрия таких теорий гораздо больше). Таким образом, суперсимметрия привлекает к себе внимание множества поклонников. Постараюсь как можно более кратко познакомить вас с понятием суперсимметрии, избегая упоминания о ее

[1] О ранней истории суперсимметрии читайте в книге Кейна и Шифмана (G. Kane and M. Shifman, eds., The Supersymmetric World: The Beginning of the Theory.) тонкостях и спорных моментах. Надеюсь, мое введение окажется для студентов полезным, прежде чем они возьмутся за изучение соответствующих монографий.

Открытие суперсимметрии

Представьте, что однажды вы просыпаетесь с желанием построить теорию поля с симметрией, связывающей бозоны с фермионами. Первое, что вам понадобится, - это одинаковое число фермионных и бозонных степеней свободы. Простейшим фермионным полем является двухкомпонентный спинор Вейля ψ. Теперь у вас есть одна комплексная степень свободы [2], так что вы должны включить комплексное скалярное поле φ. Далее вы можете действовать методом проб и ошибок: запишите лагранжиан, включая в него все члены с размерностью не выше четвертой, и подгоните значения различных параметров лагранжиана так, чтобы он удовлетворял желаемому условию симметрии. Например, можно так подобрать значение μ в массовых членах $\mu^2\varphi^\dagger\varphi + m(\psi\psi + \bar{\psi}\bar{\psi})$, чтобы бозон и фермион имели одинаковую массу и теория стала бы более симметричной.

Если бы вы решили использовать дираковский спинор Ψ и комплексный скаляр φ, то потерпели бы неудачу на первых шагах, поскольку получили бы в два раза больше фермионных степеней свободы, чем бозонных степеней свободы. Хочется верить, что открытие суперсимметрии откладывалось так долго исключительно из-за того, что до начала 1970-х гг. большинство теоретиков-полевиков, выросших на дираковских спинорах, имели лишь небольшое представление о вейлевских спинорах. Теперь для вас настало время познакомиться с пунктирными и непунктирными спинорами из приложения Е. Для дальнейшего изучения этой главы вы должны хорошо понимать обозначения.

Суперсимметричная алгебра

Достаточно просто построить суперсимметричную теорию поля, известную как модель Весса-Зумино, методом проб и ошибок. Однако сначала я продемонстрирую вам, как работает один изящный, но более абстрактный подход, именуемый суперпространственный

и суперполевой формализм. Этот формализм изобрели Салам и Стратди. Нам понадобится достаточно сложный формальный аппарат. Все супер в этой главе.

2 Одна комплексная степень свободы на массовой поверхности и две комплексные степени свободы вне массовой поверхности. См. ниже суериюлевой формализм. Запишем генератор суперсимметрии, переводящий φ в ψ_α, как Q_α (известный как суперзаряд). Тот факт, что Q_α преобразуется как вейлевский спинор, означает $[J^{\mu\nu}, Q_\alpha] = -i\left(\sigma^{\mu\nu}\right)_\alpha{}^\beta Q_\beta$, где $J^{\mu\nu}$ - генераторы группы Лоренца. Очевидно, что $[P^\mu, Q_\alpha] = 0$ вследствие независимости Q_α от координат пространства-времени. Обозначим величину, сопряженную Q_α, как $\bar{Q}_{\dot\alpha}$ (см. приложение Е) и получим $\left[J^{\mu\nu}, \bar{Q}^{\dot\alpha}\right] = -i\left(\bar\sigma^{\mu\nu}\right)^{\dot\alpha}{}_{\dot\beta'}\bar{Q}^{\dot\beta}$.

Запишем антикоммутационное соотношение между грассмановыми объектами Q_α и $\bar{Q}_{\dot\beta}$, при этом в полной мере учтем материал приложения Е. Алгебра суперсимметрий задается выражением

$$\left\{Q_\alpha, \bar{Q}_{\dot\beta}\right\} = 2\left(\sigma^\mu\right)_{\alpha\dot\beta} P_\mu.$$

Мы воспользовались методом «чем еще это может быть»? Правая часть выражения должна идти с индексами α и $\dot\beta$, а мы знаем, что единственным объектом, несущим эти индексы, может быть σ^μ. Лоренцев индекс μ нужно свернуть и единственный подходящий вектор - это P_μ. Множитель 2 фиксирует нормировку Q.

Аналогичным образом получаем $\left\{Q_\alpha, Q^\beta\right\} = c_1\left(\sigma^{\mu\nu}\right)_\alpha{}^\beta J_{\mu\nu} + c_2\delta_\alpha^\beta$. Коммутируя с P^λ, видим, что константа c_1 равна нулю. С учетом $Q_\gamma = \varepsilon_{\gamma\beta}Q^\beta$ приходим $\kappa\left\{Q_\alpha, Q_\gamma\right\} = c_2\varepsilon_{\gamma\alpha}$; вследствие симметричности левой части последнего выражения относительно α и γ получаем $c_2 = 0$. Таким образом, $\{Q_\alpha, Q_\beta\} = 0$ и $\left\{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\right\} = 0$ (см. упражнение VIII.4.2).

Основная теорема

Из соотношения (1) следует важный физический факт. Сворачивая с $\left(\bar\sigma^\nu\right)^{\dot\beta\alpha}$, получаем

$$4P^\nu = \left(\bar\sigma\right)^{\dot\beta\alpha}\left\{Q_\alpha, \bar{Q}_{\dot\beta}\right\}.$$

В частности, временная компонента говорит о том, что гамильтониан равен:

$$4H = \sum_\alpha\left\{Q_\alpha, \bar{Q}_{\dot\alpha}\right\} = \sum_\alpha\left\{Q_\alpha, \bar{Q}_\alpha^\dagger\right\} = \sum_\alpha\left(Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha\right)$$

Приходим к важной теореме, которая гласит, что в суперсимметричной теории поля любое физическое состояние $|S\rangle$ должно иметь неотрицательную энергию:

$$\langle S|H|S\rangle = \frac{1}{2}\sum_\alpha\sum_{S'}\left|\langle S'\left|Q_\alpha\right|\left|S\right\rangle\right|^2 \geqslant 0.$$

Суперпространство

Теперь, когда мы построили суперсимметричную алгебру, обратимся к нашей цели построить суперсимметричные теории поля. Для этого мы должны выяснить, как преобразуются поля при действии суперсимметричной алгебры. Нам придется использовать формализмом, необходимость которого станет очевидной позднее.

Представьте, что мы пытаемся построить формализм суперпространства. Начнем с основного соотношения (1) $\left\{Q_\alpha, \bar{Q}_{\dot\beta}\right\} = 2\left(\sigma^\mu\right)_{\alpha\dot\beta}P_\mu$. Суперсимметричное преобразование Q, вместе с $\bar{Q}$, генерирует трансляцию P_μ. Гм, давайте посмотрим, $P_\mu \equiv i\left(\partial/\partial x^\mu\right)$ генерирует трансляцию в x^μ, значит, Q_α, возможно, будучи грассмановым, генерирует трансляцию в некоторой абстрактной грассмановой координате θ^α. (Аналогично $\bar{Q}_{\dot\beta}$ генерирует трансляцию в $\bar\theta^{\dot\beta}$.)

Салам и Стратди ввели понятие суперпространства с бозонными и фермионными координатами $\left\{x^\mu, \theta^\alpha \cdot \bar{\theta}^{\dot{\beta}}\right\}$ с алгеброй суперсимметрий, представленной трансляциями в этом пространстве.

Пусть Q_α и $\bar{Q}_{\dot{\beta}}$ будут чем-то типа $\partial/\partial\theta^\alpha$ и $\partial/\partial\bar{\theta}^{\dot{\beta}}$ соответственно. Тогда $\left\{Q_\alpha, \bar{Q}_{\dot{\beta}}\right\} = 0$ и мы не получим соотношение (1). Продолжим наши игры и модифицируем Q_α и $\bar{Q}_{\dot{\beta}}$. Вы, возможно, уже знаете, что мы ищем. Если добавить в $\bar{Q}_{\dot{\beta}}$ член вида $\theta\sigma^\mu\partial_\mu$, то $\partial/\partial\theta^\alpha$, действующий в Q_α на $\theta\sigma^\mu\partial_\mu$, приведет к выражению, напоминающему правую часть уравнения (1). Аналогичным образом мы решаем добавить в Q_α член вида $\bar{\theta}\sigma^\mu\partial_\mu$. (Напоминаю, что система пунктирных обозначений фиксирует, что мы должны записывать, а именно $(\sigma^\mu)_{\alpha\dot{\alpha}}\,\bar{\theta}^{\dot{\alpha}}\partial_\mu$, так что сворачиваемые индексы подчиняются правилу «от юго-запада к северо-востоку».) Итак, суперзаряды мы представляем как

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\,(\sigma^\mu)_{\alpha\dot{\alpha}}\,\bar{\theta}^{\dot{\alpha}}\partial_\mu$$

Н

$$\bar{Q}_{\dot{\beta}} = -\frac{\partial}{\partial\theta^{\dot{\beta}}} + i\theta^\beta\,(\sigma^\mu)_{\beta\dot{\beta}}\,\partial_\mu$$

Геперь уравнение (1) имеет место. Интсресно, что, когда мы сдвигаем в фермионном направлении, приходится также слегка сдвигать в бозонном паправлении.

Суперполе

Суперполе $\Phi\left(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\beta}}\right)$, как следует из названия, есть поле, живущее в суперпространстве. Инфинитезимальное преобразование суперсимметрии сводится к виду:

$$\Phi \to \Phi' = \left(1 + i\xi^\alpha Q_\alpha + i\bar{\xi}_{\dot{\alpha}}\bar{Q}_{\dot{\alpha}}\right)\Phi$$

где ξ и $\bar{\xi}$ - два грассмановых параметра.

Оказывается, можно наложить на Φ условие и тем самым несколько ограничить его широкое определение. Пристально вглядевшись в (5) и (6), вы обнаружите, что можно определить другие два объекта:

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\,(\sigma^\mu)_{\alpha\dot{\alpha}}\,\bar{\theta}^{\dot{\alpha}}\partial_\mu$$

и

$$\bar{D}_{\dot{\beta}} = -\left[\frac{\partial}{\partial\theta^{\dot{\beta}}} + i\theta^\beta\,(\sigma^\mu)_{\beta\dot{\beta}}\,\partial_\mu\right],$$

которые являются комбинациями, ортогональными к Q_α и $\bar{Q}_{\dot{\beta}}$. Очевидно, что D_α и $\bar{D}_{\dot{\beta}}$ антикоммутируют с Q_α и $\bar{Q}_{\dot{\beta}}$. Следовательно, если наложить на суперполе Φ условие $\tilde{D}_{\dot{\beta}}\Phi = 0$, то его преобразованная форма Φ' также будет удовлетворять этому условию.

Суперполе Φ, удовлетворяющее условию $\bar{D}_{\dot{\beta}}\Phi = 0$, известно как киральное суперполе. На самом деле условие легко выполнить [3]. Смотрите, если мы определим $y^\mu \equiv \left(x^\mu + i\theta^\alpha\,(\sigma^\mu)_{\alpha\dot{\alpha}}\,\bar{\theta}^{\dot{\alpha}}\right)$ (обратите внимание, что мы складываем две бозонные величины), то

$$\bar{D}_{\dot{\beta}}y^\mu = -\left[\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}} + i\theta^\beta\,(\sigma^\nu)_{\beta\dot{\beta}}\,\partial_\nu\right]y^\mu = -\left[-i\theta^\alpha\,(\sigma^\mu)_{\alpha\dot{\beta}} + i\theta^\beta\,(\sigma^\mu)_{\beta\dot{\beta}}\right] = 0.$$

Таким образом, суперполе $\Phi(y,\theta)$, зависящее только от y и θ, является киральным суперполем.

Разложим Φ в ряд по степеням θ при фиксированном y. Вспомним, что θ содержит две компоненты (θ^1,θ^2). Значит, можно построить объект с θ максимум второй степени, т. е. $\theta\theta$, что вы проделали в упражнении E.3.

З'Эо аналогично задаче нахождения функции $f(x,y)$, удовлетворяющей условию $Lf=0$, где $L\equiv[x(\partial/\partial y)-y(\partial/\partial x)]$. Тогда любая f, зависящая только от r, будет удовлетворять заданному условию. Как обычно, степенной ряд по грассмановым переменным обрывается и мы получаем:

$$\Phi(y.\theta)=\varphi(y)+\sqrt{2}\theta\psi(y)+\theta\theta F(y)$$

где $\varphi(y),\psi(y)$ и $F(y)$ - пока просто коэффициенты ряда. Можно осуществить повторное разложение в ряд Тейлора в x:

$$\begin{aligned}\Phi(y,\theta)=\varphi(x)+\sqrt{2}\theta\psi(x)+\theta\theta F(x)+\\ +i\theta\sigma^\mu\bar\theta\partial_\mu\varphi(x)-\frac{1}{2}\theta\sigma^\mu\bar\theta\theta\sigma^\nu\bar\theta\partial_\mu\partial_\nu\varphi(x)+\sqrt{2}\theta i\theta\sigma^\mu\bar\theta\partial_\mu\psi(x).\end{aligned}$$

Видим, что киральное суперполе Φ включает в себя вейлевское фермионное поле ψ и два комплексных скалярных поля φ и F.

Нахождение полной производной

Сделаем небольшой анализ размерностей для пользы и удовольствия. Известно, что P_μ имеет размерность массы и, применяя обозначения из главы III.2, мы можем записать $[P_\mu]=1$. Тогда из выражений (1), (5) и (6) следует, что $[Q]=[\bar Q]=\frac{1}{2}$ и $[\theta]=[\bar\theta]=-\frac{1}{2}$. Известно, что $[\varphi]=1$, тогда из уравнения (8) следует, что $[\psi]=\frac{3}{2}$ (это мы уже знаем) и $[F]=2$ (этого мы не знали). В действительности мы никогда не встречали лоренцево скалярное поле с размерностью 2. Как может кинетический член для F в лагранжиане $\mathcal{L}$ иметь размерность 4? Никак. Член $F^\dagger F$ уже имеет размерность 4, а любая производная может лишь увеличить ее. Кроме того, разве мы не говорили, что с помощью φ и ψ уравниваем количество бозонных и фермионных степеней свободы?

С полем $F(x)$ определенно связано что-то странное. Что оно делает в нашей теории?

При инфинитезимальном суперсимметричном преобразовании суперполе изменяется на величину $\delta\Phi=i(\xi Q+\bar\xi\bar Q)\Phi$. Используя уравнения (8), (5) и (6), вы можете определить, как именно преобразуются поля φ,ψ и F (см. упражнение 5). Но пойдем более длинным путем, применяя принцип симметрии и анализ размерностей. Например, функция δF линейна относительно ξ или $\bar\xi$, поэтому вследствие $[F]=2$ и $[\xi]=[\bar\xi]=-\frac{1}{2}$ она должна умножаться на что-то с размерностью $\left[\frac{5}{2}\right]$. Единственным объектом с размсрностью $\left[\frac{5}{2}\right]$ является $\partial_\mu\psi$ с непунктирным индексом. Обратите внимание, что в качестве этого объекта не может выступать $\partial_\mu\bar\psi$, поскольку Φ не содержит поля $\bar\psi$. Согласно лоренцевской инвариантности необходимо найти такой объект, который имеет индекс μ. Это может быть только $(\sigma^\mu)_{\alpha\dot\alpha}$. Пунктирный индекс, имеющийся у $(\sigma^\mu)_{\alpha\dot\alpha}$, можно свернуть только с $\bar\xi$. Таким образом, все параметры фиксированы, за исключением общей константы:

$$\delta F\sim\partial_\mu\psi^\alpha\,(\sigma^\mu)_{\alpha\dot\alpha}\,\bar\xi^{\dot\alpha}$$

Аналогичным образом можно легко показать, что $\delta\varphi\sim\xi\psi$ и $\delta\psi\sim\xi F+{}+\partial_\mu\varphi\sigma^\mu\bar\xi$

Главное здесь - это не общая константа (9), а тот факт, что δF является полной производной.

Обозначим через $[\Phi]_F$ коэффициент при $\theta\theta$ в разложении произвольного суперполя Φ (как в (8)). Мы уже знаем, что при суперсимметричном преобразовании $\delta\left([\Phi]_F\right)$ есть полная производная, поэтому $\int d^4x[\Phi]_F$ инвариантно относительно преобразования суперсимметрии.

Наше следующее наблюдение состоит в том, что если $\bar{D}_{\dot{\beta}}\Phi = 0$, то $\bar{D}_{\dot{\beta}}\Phi^2 = 0$. Иначе говоря, если Φ есть киральное суперполе, то Φ^2 также является киральным суперполем (а еще Φ^3, Φ^4 и т. д.).

Суперсимметричное действие

Чего бы мы хотели в любом случае? Мы бы хотели получить действие, инвариантное относительно суперсимметрии.

Наконец, мы получили готовый формализм. Из материала последних двух параграфов почти очевидно, что выражение $\int d^4x \left[\frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3 + \ +...\right]_F$ инвариантно относительно преобразования суперсимметрии. Если возвести в квадрат (8) и взять из получившегося выражения коэффициент при $\theta\theta$, то увидим, что $[\Phi^2]_F = (2F\varphi - \psi\psi)$. Аналогично $[\Phi^3]_F = 3\,(F\varphi^2 - \ -\varphi\psi\psi)$. Теперь выполните упражнение VIII.4.6.

Создается впечатление, что мы построили массовый член для вейлевского фермиона ψ и нашли его взаимодействие со скалярным полем φ. Где же члены кинетической энергии, такие как $\bar{\psi}_{\dot{\alpha}}\left(\bar{\sigma}^{\mu}\right)^{\dot{\alpha}\alpha}\partial_{\mu}\psi_{\alpha}$?

Векторное суперполе

Члены кинетической энергии содержат $\bar{\psi}_{\dot{\alpha}}$, которого нет в Φ. Чтобы получить сопряженное поле $\bar{\psi}_{\dot{\alpha}}$, мы, очевидно, должны использовать член $\Phi^{\dagger}\Phi$. Еще немного формализма! Суперполе $V(x,\theta,\theta)$ мы называем векторным, если $V = V^{\dagger}$. Так, $\Phi^{\dagger}\Phi$ есть векторное суперполе.

Представим, что мы раскладываем $\Phi^{\dagger}\Phi = \varphi\,\dagger\,\varphi + ...$ или любое векторное суперполе V в ряд по степеням θ и $\bar{\theta}$. Старшей степенью однозначно является $\bar{\theta}\bar{\theta}\theta\theta$, так как единственный объект, который мы можем построить с учетом свойств грассмановых переменных, - это $\bar{\theta}^{\dot{1}}\bar{\theta}^{\dot{2}}\theta_1\theta_2$. Любой объект квадратичный по θ и $\bar{\theta}$, такой как $\left(\theta\sigma^{\mu}\bar{\theta}\right)\left(\theta\sigma_{\mu}\bar{\theta}\right)$, можно свести к виду $\bar{\theta}\bar{\theta}\theta\theta$, если воспользоваться одним из тождеств, которые вы нашли в упражнениях из приложения Е. Пусть $[V]_D$ обозначает коэффициент при $\bar{\theta}\bar{\theta}\theta\theta$ в разложении V.

Снова воспользуемся анализом размерностей. Если размерность V равна n, то размерность $[V]_D$ равна $n+2$, поскольку размерности θ и $\bar{\theta}$ равны $-\frac{1}{2}$. Посмотрим, как изменяется $[V]_D$ при инфинитезимальном преобразовании суперсимметрии $\delta V = i(\xi Q + \bar{\xi}\bar{Q})V$. Функция $\delta\left([V]_D\right)$ линейна относительно ξ и $\bar{\xi}$, поэтому она должна умножаться на что-то с размерностью $n + \frac{5}{2}$, ибо $[\xi] = [\bar{\xi}] = -\frac{1}{2}$. В качестве подходящего объекта может выступать лишь производная ∂ с размерностью $n + \frac{3}{2}$, а именно коэффициенты при $\bar{\theta}\bar{\theta}\theta$ и $\bar{\theta}\theta\theta$ в разложении V. Заключаем, что $\delta\left([V]_D\right)$ должна иметь форму $\partial_{\mu}(...)$. Отсюда функция $\delta\left([V]_D\right)$ есть полная производная.

Следовательно, действие $\int d^4x \left[\Phi^{\dagger}\Phi\right]_D$ инвариантно относительно преобразования суперсимметрии.

Посмотрим еще раз на уравнение (8), которое я приведу еще раз для вашего удобства:

$$\Phi(y,\theta) = \varphi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) + \\ + i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\varphi(x) - \frac{1}{2}\theta\sigma^{\mu}\bar{\theta}\theta\sigma^{\nu}\bar{\theta}\partial_{\mu}\partial_{\nu}\varphi(x) + \sqrt{2}\theta i\theta\sigma^{\mu}\bar{\theta}\partial_{\mu}\psi(x).$$

Видим, что $\int d^4x \left[\Phi^{\dagger}\Phi\right]$ содержит $\int d^4x\varphi^{\dagger}\partial^2\varphi$ (если умножить первый член в $\Phi^{\dagger}$ на пятый член в Φ), $\int d^4x\partial\varphi^{\dagger}\partial\varphi$ (если умножить четвертый член в $\Phi^{\dagger}$ на четвертый член в Φ), $\int d^4x\bar{\psi}\bar{\sigma}^{\mu}\partial_{\mu}\psi$ (если умножить второй член в $\Phi^{\dagger}$ на шестой член в Φ) и, наконец, $\int d^4xF^{\dagger}F$ (если умножить третий член в $\Phi^{\dagger}$ на третий член в Φ). Поражает, как появляются производные полей в суперсимметричных теориях поля. Обратите внимание, что действие $\int d^4x \left[\Phi^{\dagger}\Phi\right]_D$ не включает в себя явные производные.

Подведем итог. Для заданного суперполя Φ мы построили суперсимметричное действие

$$S = \int d^4x \left\{ \left[\Phi^\dagger \Phi\right]_D - \left([W(\Phi)]_F + \text{ э. с. }\right)\right\}$$

Выбирая $W(\Phi) = \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3$, получаем в явном виде

$$S = \int d^4x \{\partial\varphi^\dagger\partial\varphi + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^\dagger F - \\ - \left(mF\varphi - \frac{1}{2}m\psi\psi + gF\varphi^2 - gF\varphi^2 - g\varphi\psi\psi + \text{ э. с. }\right)\Big\}.$$

Вспомогательное поле

С самого начала поле F казалось нам странным. Из условия $[F] = 2$ мы сделали вывод, что оно не может иметь члена кинетической энергии с размерностью 4, и это действительно так. Мы знаем, что оно не является динамическим полем, которое распространяется - оно является вспомогательным полем (в точности как σ в главе III.5 и ξ_μ в главе VI.3) и может быть отынтегрировано в интеграле по траекториям $\int DF^\dagger DF e^{iS}$. В самом деле, в выражении для S сгруппируем члены, зависящие от F:

$$F^\dagger F - F\left(m\varphi + g\varphi^2\right) - F^\dagger\left(m\varphi^\dagger + g\varphi^{\dagger 2}\right) = \left|F - \left(m\varphi + g\varphi^2\right)^\dagger\right|^2 - \left|m\varphi + g\varphi^2\right|^2.$$

Проинтегрируем по F и $F^\dagger$:

$$S = \int d^4x \left\{\partial\varphi^\dagger\partial\varphi + i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi - \left|m\varphi + g\varphi^2\right|^2 + \left(\frac{1}{2}m\psi\psi - g\varphi\psi\psi + \text{ э.с. }\right)\right\}$$

Замечу, что в соответствии с (4) скалярный потенциал равен $V\left(\varphi^\dagger, \varphi\right) = \left|m\varphi + g\varphi^2\right|^2 \geqslant 0$, в минимуме он равен нулю, что обеспечивает нулевую космологическую постоянную. Обратите внимание, что мы больше не можем добавить в $V\left(\varphi^\dagger, \varphi\right)$ произвольную константу, как в несуперсимметричных теориях поля.

Как и следовало ожидать, суперсимметричные теории оказались гораздо более ограниченными, по сравнению с обычными теориями поля, но гораздо более симметричными. Используемый для их описания формализм можно обобщить до суперсимметричной теории Янга-Миллса. Ранее я говорил, что если какая-нибудь нетривиальная 4-мерная квантовая теория поля окажется точно решаемой, то это, скорее всего, будет $N = 4$ суперсимметричная теория Янга-Миллса (упражнение VIII.4.3).

Надеюсь, что это краткое введение передало вам дух суперсимметрии и поможет вам перейти к более серьзным трудам по этому предмету.

Упражнения

VIII.4.1. Постройте лагранжиан Весса-Зумино методом проб и ошибок.

VIII.4.2. В общем случае могут существовать N суперзарядов $Q^I_\alpha, I = 1, ..., N$. Покажите, что можно получить $\left\{Q^I_\alpha, Q^J_\beta\right\} = \varepsilon_{\alpha\beta}Z^{IJ}$, где Z^{IJ} – c-числа, называемые центральными зарядами.

VIII.4.3. С учетом того что мы не знаем, как записывать непротиворечивые квантовые теория поля со спином, большим двух, докажите, что значение N из предыдущего упражнения не превышает 8. Теории с суперсимметрией $N = 8$ называют максимально суперсимметричными. Докажите, что, если не включать гравитацию, значение N не должно превышать 4. Суперсимметричная теория Янга-Миллса при $N = 4$ обладает многими замечательными свойствами.

VIII.4.4. Покажите, что $\partial\theta_\alpha/\partial\theta^\beta = \varepsilon_{\alpha\beta}$.

VIII.4.5. Найдите $\delta\varphi, \delta\psi$ и δF из вариации $\delta\Phi = i\left(\xi^\alpha Q_\alpha + \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}\right)\Phi$.

VIII.4.6. Для любого многочлена $W(\Phi)$ покажите, что $[W(\Phi)]_F = F[dW(\varphi)/d\varphi] +$ + члены, не содержащие F. Покажите, что для теории (11) потенциальная энергия равна $V\left(\varphi^\dagger, \varphi\right) = |\partial W(\varphi)/\partial\varphi|^2$.

VIII.4.7. Постройте теорию поля, в которой суперсимметрия спонтанно нарушена. [Указание: вам понадобятся как минимум три киральных суперполя.]

VIII.4.8. Если мы можем построить суперсимметричную квантовую теорию поля, то определенно сможем построить суперсимметричную квантовую механику. Действительно, рассмотрим $Q_1 \equiv \frac{1}{2}\left[\sigma_1 P + \sigma_2 W(x)\right]$ и $Q_2 \equiv \frac{1}{2}\left[\sigma_2 P - -\sigma_1 W(x)\right]$, где, как обычно, $P = -i(d/dx)$ - оператор импульса. Определите $Q \equiv Q_1 + iQ_2$. Изучите свойства гамильтониана H, определяемого как $\left\{Q, Q^\dagger\right\} = 2H$.

30.5.6 Zi.VIII.5 A Bit About String Theory as a 2D Field Theory

Геометрическое действие для бозонной струны

В этой заключительной главе я постараюсь дать вам небольшое представление о теории струн. Разумеется, вы получите лишь минимальное представление о предмете. К счастью, существуют действительно замечательные книги и, я надеюсь, что эта книга подготовила вас для их изучения. Главной своей целью я считаю показать вам, что теория струн естественно формулируется в терминах 2-мерной теории поля.

В главе I.10 мы описывали точечную частицу, с мировой линией $X^\mu(\tau)$ в D-мерном пространстве-времени. Вспомним, что действие дается геометрически длиной мировой линии:

$$S = -m\int d\tau\sqrt{\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}}$$

При репараметризации $\tau \to \tau'(\tau)$ действие остается неизменным. Вспомним также, что в классическом случае S эквивалентно действию

$$S_{\text{imp}} = -\frac{1}{2}\int d\tau\left(\frac{1}{\gamma}\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau} + \gamma m^2\right).$$

Теперь рассмотрим струну, заметающую мировой лист $X^\mu(\tau,\sigma)$ в D-мерном пространстве-времени, что мы уже проделывали в главе IV.4, когда изучали дифференциальные формы. По аналогии с (1) Намбу и Гото предложили задавать действие геометрически как площадь мирового листа:

$$S_{NG} = T\int d\tau d\sigma\sqrt{\det\left(\partial_a X^\mu \partial_b X_\mu\right)}$$

где $\partial_1 X^\mu \equiv \partial X^\mu/\partial\tau, \partial_2 X^\mu \equiv \partial X^\mu/\partial\sigma$ и $(\partial_a X^\mu \partial_b X_\mu)$ обозначают элемент ab матрицы 2×2. Здесь (как и в (1)) μ может принимать D значений: $0, 1, ..., D-1$. Константа T ($\equiv 1/2\pi\alpha'$, где α' - наклон реджевской траектории в феноменологии элементарных частиц) соответствует натяжению струны, поскольку растяжение струны для увеличения мирового листа требует дополнительного действия, пропорционального T.

Точно так же, как и для точечной частицы, желательно избавиться от квадратного корня и взамен использовать действие

$$S = \frac{1}{2}T\int d\tau d\sigma\gamma^{\frac{1}{2}}\gamma^{ab}\left(\partial_a X^\mu \partial_b X_\mu\right)$$

где $\gamma = \det\gamma_{ab}$ в интеграле по траекториям при квантовании струны. Мы сейчас покажем, что классически S эквивалентно S_{NG}.

Как и для (2), проварьируем S по вспомогательной переменной γ_{ab}, которую затем исключим. Для матрицы M имеем $\delta M^{-1} = -M^{-1}(\delta M)M^{-1}$ и

$\delta \det M = \delta e^{\operatorname{tr}\log M} = e^{\operatorname{tr}\log M} \operatorname{tr} M^{-1}\delta M = (\det M) \operatorname{tr} M^{-1}\delta M$. Следовательно, $\delta\gamma^{ab} = -\gamma^{ac}\delta\gamma_{cd}\gamma^{db}$ и $\delta\gamma = \gamma\gamma^{ba}\delta\gamma_{ab}$. Для упрощения записи введем обозначение $h_{ab} \equiv \partial_a X^\mu \partial_b X_\mu$. Тогда варьирование подынтегрального выражения в (4) даст в результате:

$$\delta\left[\gamma^{\frac{1}{2}}\gamma^{ab}h_{ab}\right] = \gamma^{\frac{1}{2}}\left[\frac{1}{2}\gamma^{dc}\delta\gamma_{cd}\left(\gamma^{ab}h_{ab}\right) - \gamma^{ac}\delta\gamma_{cd}\gamma^{db}h_{ab}\right]$$

Полагая коэффициент при $\delta\gamma_{cd}$ равным нулю, получаем:

$$h_{cd} = \frac{1}{2}\gamma_{cd}\left(\gamma^{ab}h_{ab}\right)$$

где индексы h поднимаются и опускаются метрикой γ. Умножая (5) на h^{dc} (и суммируя по повторяющимся индексам), получаем $\gamma^{ab}h_{ab} = 2$ и, следовательно, $\gamma_{cd} = h_{cd}$. Подставляя полученное выражение в (4), находим, что $S = T\int d\tau d\sigma (\det h)^{\frac{1}{2}}$. Значит, действия S и S_{NG} действительно эквивалентны в классическом случае. Действие (4), впервые полученное Бринком, Ди Веккиа и Хау, а также Дезером и Зумино, известно как действие Полякова.

Обратите внимание, что (5) определяет γ_{ab} с точностью до произвольного локального изменения масштаба, известного как преобразование Вейля:

$$\gamma_{ab}(\tau,\sigma) \to e^{2\omega(\tau,\sigma)}\gamma_{ab}(\tau,\sigma)$$

31 Other QFT Methods

(если тут окажется что-то типичное, что очень важно знать, то перенесу в 2ю часть. пока тут, потому что скорее всего это только в специфических вопросах понадобится.)

31.0.1 Deformation Quantization

(?????)

31.0.2 Deformation Quantization: Applications

(?????)

31.0.3 C^*-Algebras and Quantization

(?????)

31.0.4 C^*-Algebras and Quantization: Applications

(?????)

31.1 Wilson Loops (???)

(??)

31.2 Wilson Loops: Applications (???)

(??)

В одномерной гравитации

В теории Черна Саймонса

Так сказал Володя

31.3 On QFT Methods in Waveguides (???)

(пока в отдельной статье пишу)

31.4 QFT and String Theory by Polyakov

(тут много крутых методов)

31.4.1 Asymptotic Freedom and the Renormalization Group

Главные киральные поля

Модель n-поля

Неабелевы калибровочные поля при $D = 4$

31.4.2 Strong-Coupling Approximation

Модель Изинга

Непрерывные глобальные симметрии

Калибровочные симметрии

31.4.3 Instantons in Abelian Systems

Инстантоны в квантовой механике и в модели Изинга

Инстантоны в модели с глобальной $O(2)$-симметрией

Компактная КЭД ($O(2)$-калибровочная модель)

31.4.4 Quark Confinement, Superfluidity, Elasticity: Criteria and Analogies

31.4.5 Exactly Solvable Models (??)

Обсудим подробно инстантоны, кинки, солитоны в КТП

Инстантоны для $D = 2, N = 3n$-полей

Инстантоны в неабелевых калибровочных теориях

Качественные эффекты инстантонов

Инстантоны по Шварцу

(там тоже есть теория, мб добавлю потом)

Число инстантониых параметров

(там тоже есть теория, мб добавлю потом)

31.4.6 Topological Phenomena in QFT by Schwartz

(мб добавлю элементы этой главы в первые главы)
(сделаю основы - буду разбираться.)

обзор топологических явлений

Топологически стабилыные дефекты

Топологические интегральт движения Двумерная модель

Абрикосовские вихри

Монополи Полякова-Хоофта

Топологические интегралы движения в калибровочных теориях

Частицы в калибровочных теориях

Магнитный заряд

Общие формулы для злектромагнитной напряженности и магнитного заряда в калибровочньтх теориях

Экстремумы симметричных функниоиалов

31.4.7 Other Topics of Topology in QFT

Симметричные калибровочные поля

Оценка знергии магнитного монополя

Топологически нетривиальные нити (????)

Частицы в присутствии нити

(? интересно кст)

Нелинейные поля

Многозначные функциональт действия

Функциональные интегралы Применение функииональных интегралов в квантовой теории

Квантование калибровочных теорий

Эллиптические операторы

Свойства эллиптических операторов

Индекс эллиптического оператора

Детерминанть эллиптических операторов

Квантовые аномалии

(выше вроде это должно быть уже?)

32 Other Constructions of QFT

32.1 Quantum Field Theory on the Line by Tong Gauge theories

(нужно будет - начну с выгрузок.)

32.1.1 Electromagnetism in Two Dimensions (!!!??)

332 7.1.1 The Theta Angle 334 7.1.2 The Theta Angle is a Background Electric Field 336 7.2

32.1.2 The Abelian-Higgs Model

337 7.2.1 Vortices 339 7.2.2 The Wilson Loop 342 7.3

32.1.3 The CPN-1 Model

344 7.3.1 A Mass Gap 346 7.3.2 Confinement 348 7.3.3 Instantons 349 7.4

32.1.4 Fermions in Two Dimensions

351 7.4.1 The Gross-Neveu Model 352 7.4.2 Kinks in the Gross-Neveu Model 355 7.4.3 The Chiral Gross-Neveu Model 357 7.4.4 Back to Basics: Quantising Fermions in 2d 361 7.5

32.1.5 Bosonization in Two Dimensions

362 7.5.1 T-Duality 364 7.5.2 Canonical Quantisation of the Boson 366 7.5.3 The Bosonization Dictionary 370 7.5.4 The Allowed Operators: Is the Boson Really a Fermion? 372 7.5.5 Massive Thirring = Sine-Gordon 373 7.5.6 QED2: The Schwinger Model 376 7.6

32.1.6 Non-Abelian Bosonization

The Wess-Zumino-Witten Term

8.

32.2 Quantum Field Theory on the Plane by Tong Gauge theories

8.1

32.2.1 Electromagnetism in Three Dimensions

387 8.1.1 Monopole Operators 388 8.2 The Abelian-Higgs Model 390 8.2.1 Particle-Vortex Duality 392 8.3 Confinement in d = 2 + 1 Electromagnetism 397 8.3.1 Monopoles as Instantons 398 8.3.2 Confinement 401 8.4

32.2.2 Chern-Simons Theory

403 8.4.1 Quantisation of the Chern-Simons level 403 8.4.2 A Topological Phase of Matter 406 8.4.3 Non-Abelian Chern-Simons Theories 410 8.5

32.2.3 Fermions and Chern-Simons Terms

412 8.5.1 Integrating out Massive Fermions 413 8.5.2 Massless Fermions and the Parity Anomaly 416 8.6

32.2.4 3d Bosonization

418 8.6.1 Flux Attachment 418 8.6.2 A Bosonization Duality 420 8.6.3 The Beginning of a Duality Web 424 8.6.4 Particle-Vortex Duality Revisited 426 8.6.5 Fermionic Particle-Vortex Duality 427

32.3 L14 Quantum Anomalies .

32.3.1 1. Introduction

В этой лекции мы рассматриваем безмассовую электродинамику ($m = 0$). В безмассовом случае имеются две глобальные симметрии: дираковское действие инвариантно относительно двух непрерывных глобальных преобразований:

$$\psi = e^{i\alpha}\psi', \bar{\psi} = \bar{\psi}' e^{-i\alpha} \tag{1}$$

$$\psi = e^{i\beta\gamma^5}\psi', \bar{\psi} = \bar{\psi}' e^{i\beta\gamma^5} \tag{2}$$

Здесь α, β - вещественные числа.

Если в пункте 2 лекции 10 $\partial_\mu \alpha = 0$, то преобразование (3 Л10) сводится к преобразованию (1). Согласно теореме Нётер в классической теории поля каждой непрерывной симметрии соответствует сохраняющийся ток. Таким образом, в безмассовой классической электродинамике имеется два сохраняющихся тока: векторный и аксиально-векторный:

$$\partial_\mu J^\mu = 0, \tag{3}$$
$$J^\mu = \bar{\psi}\gamma^\mu\psi \tag{4}$$
$$\partial_\mu J^{5\mu} = 0, \quad J^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi$$

В квантовой теории поля теорема Нётер не может оставаться автоматически справедливой вследствие расходимостей. Действительно, рассмотрим амплитуду перехода Z из вакуума в вакуум. Для получения, например, закона сохранения (3), в континуальном интеграле для Z следует сделать бесконечно близкую к тождественной замену переменных вида (3 Л10) и провести все те манипуляции, которые привели к равенству (5 Л10). Если затем положить $\eta = \bar{\eta} = 0$, то из (5 Л10) следует закон сохранения

$$\partial_\mu \langle J^\mu(x)\rangle = 0 \tag{5}$$

Чрезвычайно важно, что равенство (5) получается не в четырёхмерном, а в комплексном d-мерном пространстве, в котором амплитуда перехода регуляризована и все те свойства теории, на которых основывается получение равенства (5), остаются справедливыми при комплексной размерности пространства. Поэтому после получения равенства (5) мы можем перейти к пределу $d \to 4$ и провести перенормировку. Тем самым закон сохранения (5) остаётся справедливым в уже перенормированной теории.

Аналогичный подход для получения закона сохранения аксиально-векторного тока оказывается несостоятельным. Это видно, в частности, из того, что в предположении аналитичности относительно размерности пространства d и справедливости формул (61 Л9) имеем:

$$\operatorname{tr}\left(\gamma^5\gamma^{\mu_1}...\gamma^{\mu_{2n}}\right) = 0 \tag{6}$$

для всех n. Это входит в противоречие с (68 Л8). Равенство (6) означает, что размерная регуляризация неудовлетворительна, когда исследуются аксиальные преобразования (2). Более того, не существует такой регуляризации в рамках теории S-матрицы, которая сохраняет аксиальную инвариантность (инвариантность относительно преобразования (2)). Поэтому аксиально-векторный ток не сохраняется в квантовой безмассовой электродинамике, что далее показывается.

Для доказательства равенства (6) вставим под знак следа тождество $\gamma_\mu\gamma^\mu = d$, затем "протащим"по циклу одну из матриц γ^μ под знаком следа, используя правила антикоммутации (5 Л8). В результате получим:

$$(d-2n)\operatorname{tr}\left(\gamma^5\gamma^{\mu_1}...\gamma^{\mu_{2n}}\right) + ... = 0 \tag{7}$$

Здесь многоточие означает сумму таких же следов с $(2n-2)$ дираковскими матрицами. В частности, для $n = 0$ имеем $d\operatorname{tr}\gamma^5 = 0$, т.е. $\operatorname{tr}\gamma^5 = 0$. Теперь равенство (6) следует из равенства (7) по индукции.

Покажем, что если бы не было ультрафиолетовых расходимостей, то аксиальная аномалия (несохранение аксиально-векторного тока) также отсутствовала бы. Для наших целей достаточно рассмотреть распространение квантованного дираковского поля на фоне неквантованного, но произвольного электромагнитного поля. На языке функционального интеграла такая ситуация реализуется, если сначала вычисляется функциональный интеграл по дираковскому полю, и лишь после этого вычисляется интеграл по электромагнитному полю. Тогда внутренний фермионный интеграл, который обозначим $Z\{A\}$, и описывает амплитуду перехода дираковского поля в заданном электромагнитном поле. Заметим, что согласно (26 Л9)

$$Z\{A\} = \det\left(i\gamma^\mu\nabla_\mu\right) \tag{8}$$

Формула (8) верна, если внешние источники равны нулю. В противном случае эта формула должна быть исправлена за счёт соответствующей экспоненты.

Рассмотрим эту же задачу на операторном языке.

Пусть $\{\psi_N(t,\mathbf{x})\}$ образует в каждый момент времени t полный ортонормированный набор дираковских мод, т.е.

$$\int \mathrm{d}^{(3)}x\psi^\dagger_M(t,\mathbf{x})\psi_N(t,\mathbf{x}) = \delta_{M,N} \quad \longleftrightarrow \sum_N \psi_N(t,\mathbf{x})\psi^\dagger_N(t,\mathbf{y}) = \delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{9}$$

Для упрощения записи мы ввели дискретный индекс N, нумерующий моды. Кроме того предполагается, что эти моды являются решениями уравнения Дирака в заданном электромагнитном поле:

$$i\gamma^\mu\nabla_\mu\psi_N = 0, \quad i\nabla_\mu\bar{\psi}_N\gamma^\mu = 0 \tag{10}$$

Удобно считать, что в момент времени $t = t_0$ набор мод $\{\psi_N\}$ является полным ортонормированным набором следующего эрмитового оператора:

$$\left(-i\alpha^i\nabla_i\right)\big|_{t=t_0}\psi_N = \epsilon_N\psi_N \tag{11}$$

Уравнение Дирака (10) может быть записано в виде

$$i\frac{\partial}{\partial t}\psi_N = \mathcal{H}_\psi\psi_N, \quad \mathcal{H}_\psi = -i\alpha^i\nabla_i + eA_0 \tag{12}$$

Так как оператор $\mathcal{H}_\psi$ является эрмитовым (сравни с (33 Л8)), то условия (9)-(11) совместны. Если в (11) $\epsilon_N > 0\,(\epsilon_N < 0)$, то соответствующая мода называется положительно-(отрицательно-) частотной и обозначается $\psi^{(+)}_N\left(\psi^{(-)}_N\right)$. Имеется взаимно однозначное соответствие между положительнои отрицательно-частотными модами, так как преобразование $\psi_N \to \gamma^0\gamma^5\psi_N$ меняет их местами. Поэтому будем считать, что индекс N нумерует лишь положительно-частотные моды. Представим квантованное дираковское поле в виде (сравни с (46 Л8))

$$\begin{aligned}\psi(t,\mathbf{x}) &= \sum_N\left(a_N\psi^{(+)}_N(t,\mathbf{x}) + b^\dagger_N\psi^{(-)}_N(t,\mathbf{x})\right)\\ \psi^\dagger(t,\mathbf{x}) &= \sum_N\left(b_N\psi^{(-)\dagger}_N(t,\mathbf{x}) + a^\dagger_N\psi^{(+)\dagger}_N(t,\mathbf{x})\right)\end{aligned} \tag{13}$$

Чтобы дираковские поля удовлетворяли каноническому антикоммутационному соотношению (52 Л8) должны выполняться следующие антикоммутационные соотношения (как обычно, выписываются лишь ненулевые соотношения):

$$\left\{a_M, a^\dagger_N\right\} = \left\{b_M, b^\dagger_N\right\} = \delta_{M,N} \tag{14}$$

Подчеркнём, что операторы рождения и уничтожения в разложениях (13) не зависят от времени. Поэтому квантованные дираковские поля удовлетворяют уравнению Дирака:

$$i\gamma^\mu\nabla_\mu\psi = 0, \quad i\nabla_\mu\bar{\psi}\gamma^\mu = 0 \tag{15}$$

и, таким образом, они являются гейзенберговыми операторами во внешнем электромагнитном поле. Вакуумное состояние естественно определить как

$$a_N|0\rangle = b_N|0\rangle = 0 \tag{16}$$

Составим из квантованных полей векторный $J^\mu = \bar{\psi}\gamma^\mu\psi$ и аксиально-векторный $J^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi$ токи. Формально в силу уравнений Дирака-Гейзенберга (15) оба эти тока сохраняются, т.е. они удовлетворяют уравнениям (3) и (4), соответственно. Однако в

данном случае мы имеем дело с неопределённостями, которые требуют тщательного рассмотрения. Выпишем вакуумные средние рассматриваемых токов:

$$\langle 0\,|J^{\mu}(x)|\,0\rangle = -i\,\mathrm{tr}\,S(x,x+\varepsilon)\gamma^{\mu}, \quad \varepsilon^{\mu}=(\varepsilon,0,0,0), \quad \varepsilon\to+0 \tag{17}$$

$$\left\langle 0\,\middle|J^{5\mu}(x)\right|\,0\rangle = -i\,\mathrm{tr}\,S(x,x+\varepsilon)\gamma^{\mu}\gamma^{5}, \tag{18}$$

$$S(x,y)\equiv -i\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle, \quad -iS(x,y)|_{y^0>x^0}=\sum_N \psi_N^{(-)}(x)\bar{\psi}_N^{(-)}(y) \tag{19}$$

Из (19) видно, что средние (17), (18) сингулярны. Поэтому непосредственное вычисление дивергенций правых частей в (17), (18) невозможно до проведения регуляризации. Регуляризация, в свою очередь, должна быть согласована с условиями квантования теории. В КЭД регуляризация должна

(i) сохранять возможность виковского поворота в евклидово пространство;

(ii) сохранять калибровочную инвариантность.

Мы видели, что условие (i) означает, что за бесконечное время вакуум переходит в вакуум и вакуумы при $t\to\pm\infty$ различаются не более чем фазовым множителем. Нарушение условия (ii) означало бы нарушение свойства перенормируемости КЭД, поскольку свойство перенормируемости основывается, в частности, на тождествах Уорда, отражающих калибровочную инвариантность.

Например, регуляризация по энергиям, заключающаяся в отбрасывании высокочастотных мод с такими N, для которых $|\epsilon_N|>\Lambda\to\infty$, удовлетворяет условию (ii), но не удовлетворяет условию (i). Действительно, оператор Дирака в евклидовом пространстве относится к операторам эллиптического типа первого порядка. Поэтому все решения уравнения Дирака (10) в евклидовом пространстве растут на пространственной бесконечности. Так как при регуляризации по энергии после поворота Вика уравнения Дирака в евклидовом пространстве для регуляризованных полей по прежнему имеют место, то мы видим, что в данном случае поворот Вика приводит к неприемлемым с физической точки зрения решениям. Поэтому поворот Вика несовместен с регуляризацией по энергии. Стоит, однако, заметить, что при регуляризации по энергии оба закона сохранения (3) и (4) выполняются.

Размерностная регуляризация сохраняет оба требования (i) и (ii). Однако вследствие парадокса (6) использовать этот метод для вычисления дивергенции аксиально-векторного тока затруднительно. Поэтому мы будем использовать другие подходы.

32.3.2 2. Diagrammatic Method of Computing the Axial Anomaly

Легко увидеть, что пропагатор (19) удовлетворяет уравнению

$$i\gamma^{\mu}\nabla_{\mu}S(x,y)=\delta^{(4)}(x-y) \tag{20}$$

Действительно, если $x^0\neq y^0$, то в силу уравнения (15) имеем $i\gamma^{\mu}\nabla_{\mu}S=0$. При $x^0=y^0$ пропагатор испытывает разрыв

$$[S(x+\varepsilon,y)-S(x-\varepsilon,y)]|_{\varepsilon\to+0}=-i\left\{\psi\left(x^0,\mathbf{x}\right),\bar{\psi}\left(x^0,\mathbf{y}\right)\right\}=-i\gamma^0\delta^{(3)}(\mathbf{x}-\mathbf{y}) \tag{21}$$

Поэтому производная $\partial/\partial x^0$ в (20) даёт дополнительный вклад, равный $i\gamma^0\delta\left(x^0-y^0\right)\times($ разрыв пропагатора). Таким образом уравнение (20) проверено.

Заметим, что пропагатор свободного безмассового дираковского поля удовлетворяет уравнению (ср. с (49 Л9))

$$i\gamma^{\mu}\partial_{\mu}S_c(x-y)=\delta^{(4)}(x-y) \tag{22}$$

Поэтому

$$S(x,y) = \left(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu\right)^{-1}(x,y) = S_c(x-y) + \int \mathrm{d}^{(4)}z_1 S_c\left(x-z_1\right)\left(e\gamma^\mu A_\mu\left(z_1\right)\right)S_c\left(z_1-y\right) +$$
$$+ \int \mathrm{d}^{(4)}z_1\ \mathrm{d}^{(4)}z_2 S_c\left(x-z_1\right)\left(e\gamma^\mu A_\mu\left(z_1\right)\right)S_c\left(z_1-z_2\right)\left(e\gamma^\nu A_\nu\left(z_2\right)\right)S_c\left(z_2-y\right) + \ldots \tag{23}$$

Далее мы изучаем среднее от аксиально-векторного тока (18). Так как $\operatorname{tr}\gamma^5\gamma^\mu\gamma^\nu = 0$, то первое слагаемое в правой части (23) не даёт вклада в (18). Второе слагаемое в правой части (23) также не даёт вклада в (18), поскольку оно, согласно (68 Л8) и (49 Л9), пропорционально $\varepsilon_{\mu\nu\lambda\rho}\left(x-z_1\right)^\lambda\left(x-z_1\right)^\rho = 0$. Поэтому начнём с изучения третьего слагаемого в правой части (23), которое перепишем через Фурье-образы соответствующих величин:

$$e^2\int\frac{\mathrm{d}^{(4)}p\ \mathrm{d}^{(4)}q}{(2\pi)^8}\int\frac{\mathrm{d}^{(4)}r}{(2\pi)^4}A_\nu(p)A_\lambda(q)e^{ir(y-x)}e^{-ipx-iqy}S_c(r+p)\gamma^\nu S_c(r)\gamma^\lambda S_c(r-q) \tag{24}$$

Подставим (24) в (18):

$$\left\langle 0\left|J^{5\mu}(x)\right|0\right\rangle = \frac{1}{2}\int\frac{\mathrm{d}^{(4)}p\ \mathrm{d}^{(4)}q}{(2\pi)^8}e^{-i(p+q)x}A_\nu(p)A_\lambda(q)\times$$
$$\times\left\{ie^2\int\frac{\mathrm{d}^{(4)}r}{(2\pi)^4}e^{i\varepsilon r^0}\operatorname{tr}\gamma^5\gamma^\mu\left[S_c(r+p)\gamma^\nu S_c(r)\gamma^\lambda S_c(r-q) + S_c(r+q)\gamma^\lambda S_c(r)\gamma^\nu S_c(r-p)\right]\right\} + \ldots \tag{25}$$

В правой части множитель $1/2$ перед интегралом появился вследствие бозе-симметризации квадратной скобки в (25): лишь симметричная часть квадратной скобки в (25) относительно перестановки $(p,\nu) \longleftrightarrow (q,\lambda)$ даёт вклад в интеграл. Далее будем изучать выражение в фигурной скобке в (25), которое изображается суммой двух диаграмм на рис. 30.

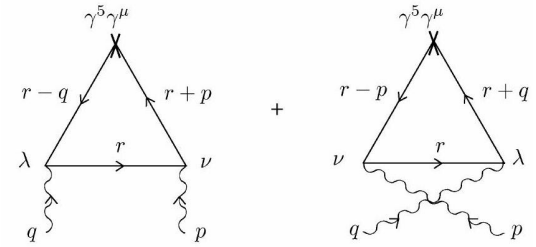


Рис. 30

Крестик в вершинах диаграмм означает, что в этих вершинах вместо множителя $(-ie\gamma^\mu)$ имеется множитель $(\gamma^5\gamma^\mu)$. В (25) можно положить $\varepsilon = 0$ ещё до вычисления интеграла, так как это не влияет на результат (те величины, которые зависят от порядка устремления ε к нулю, не дают вклада в изучаемое среднее).

Обозначим через $T^{\mu\nu\lambda}(p,q)$ величину в фигурной скобке в (25) и

$$T^{\mu\nu\lambda}(p,q) = \Gamma^{\mu\nu\lambda}(p,q) + \Gamma^{\mu\lambda\nu}(q,p) \tag{26}$$

где $\Gamma^{\mu\nu\lambda}(p,q)$ и $\Gamma^{\mu\lambda\nu}(q,p)$ изображаются первой и второй диаграммами на рис. 30, соответственно.

Диаграммы на рис. 30 расходятся линейно. Поэтому сдвиг переменной интегрирования изменяет эти диаграммы. При изучении диаграмм, встречавшихся в предыдущих лекциях, такого рода неопределённости устранялись условиями нормировки. В настоящем же случае никаких условий, кроме условия (ii), мы не имеем (условие (i) выполнено автоматически, так

как используются фейнмановские пропагаторы). Таким образом, мы должны рассмотреть семейство величин $\Gamma^{\mu\nu\lambda}(p,q|a)$, зависящих от сдвигового параметра 4 -вектора a^μ:

$$\Gamma^{\mu\nu\lambda}(p,q|a) = \tilde{\Gamma}^{\mu\nu\lambda}(p,q) + \Delta^{\mu\nu\lambda}(p,q|a) \tag{27}$$

$$\tilde{\Gamma}^{\mu\nu\lambda}(p,q) = ie^2 \int \frac{\mathrm{d}^{(4)}r}{(2\pi)^4} \operatorname{tr} \gamma^5\gamma^\mu S_c(r+p)\gamma^\nu S_c(r)\gamma^\lambda S_c(r-q) \tag{28}$$

$$\Delta^{\mu\nu\lambda}(p,q|a) = ie^2 \int \frac{\mathrm{d}^{(4)}r}{(2\pi)^4} \operatorname{tr} \gamma^5\gamma^\mu \times$$

$$\times \left\{ S_c(r+a+p)\gamma^\nu S_c(r+a)\gamma^\lambda S_c(r+a-q) - S_c(r+p)\gamma^\nu S_c(r)\gamma^\lambda S_c(r-q) \right\} \tag{29}$$

Для прояснения ситуации рассмотрим одномерный интеграл

$$\Delta(a) = \int\limits_{-\infty}^{+\infty} \mathrm{d}x [f(x+a) - f(x)] \tag{30}$$

Если $f(x)$ - интегрируемая функция на всей прямой, то при помощи сдвига переменной интегрирования устанавливается, что $\Delta(a) = 0$. Теперь разложим подынтегральное выражение:

$$\Delta(a) = \int\limits_{-\infty}^{+\infty} \mathrm{d}x \left[af'(x) + \frac{a^2}{2} f''(x) + ... \right] = a[f(+\infty) - f(-\infty)] + \frac{a^2}{2} \left[f'(+\infty) - f'(-\infty) \right] + ...$$

Отсюда видно, что если $f(x)$ стремится к разным пределам при $x \to +\infty$ и $x \to -\infty$, то

$$\Delta(a) = a[f(+\infty) - f(-\infty)] \neq 0 \tag{31}$$

Аналогично вычисляется в задаче 1 величина (29):

$$\Delta^{\mu\nu\lambda}(p,q|a) = \frac{ie^2}{8\pi^2} \varepsilon^{\mu\nu\lambda\rho} a_\rho, \quad a_\rho = (a+b)p_\rho + bq_\rho \tag{32}$$

Мы ограничили класс векторов a_ρ тем требованием, чтобы вектор a_ρ зависел лишь от векторов p, q, уже имеющихся в задаче. В (32) a и b - некие числа. Тогда наиболее общий вид величины (26) таков:

$$T^{\mu\nu\lambda}(p,q) = \left(\tilde{\Gamma}^{\mu\nu\lambda}(p,q) + \tilde{\Gamma}^{\mu\lambda\nu}(q,p) \right) + a \frac{ie^2}{8\pi^2} \varepsilon^{\mu\nu\lambda\rho} (p_\rho - q_\rho) \tag{33}$$

Комбинируя (25) и (26) и вычисляя дивергенцию, находим:

$$\left\langle 0 \left| \partial_\mu J^{5\mu}(x) \right| 0 \right\rangle = -\frac{i}{2} \int \frac{\mathrm{d}^{(4)}p\ \mathrm{d}^{(4)}q}{(2\pi)^8} e^{-i(p+q)x} A_\nu(p) A_\lambda(q) \left[(p+q)_\mu T^{\mu\nu\lambda}(p,q) \right] + ... \tag{34}$$

При помощи явной формулы (28) получаем:

$$(p+q)_\mu \tilde{\Gamma}^{\mu\nu\lambda}(p,q) \sim \int \frac{\mathrm{d}^{(4)}r}{(2\pi)^4} \operatorname{tr} \gamma^5 \left[(\not{r}+\not{p}) S_c(r+p)\gamma^\nu S_c(r)\gamma^\lambda S_c(r-q) + \right.$$

$$\left. + S_c(r+p)\gamma^\nu S_c(r)\gamma^\lambda S_c(r-q)(\not{r}-\not{q}) \right] =$$

$$= \int \frac{\mathrm{d}^{(4)}r}{(2\pi)^4} \operatorname{tr} \gamma^5 \left[\gamma^\nu S_c(r)\gamma^\lambda S_c(r-q) + S_c(r+p)\gamma^\nu S_c(r)\gamma^\lambda \right] = 0 \tag{35}$$

Здесь в последнем интеграле каждое из слагаемых является двухиндексным псевдотензором в 4мерном пространстве, составленным при помощи одного единственного вектора (q или p), что невозможно. Отсюда следует обращение в нуль величины (35). Поэтому согласно (33) и (35)

$$(p+q)_\mu T^{\mu\nu\lambda}(p,q) = -a\frac{ie^2}{4\pi^2}\varepsilon^{\nu\lambda\mu\rho}p_\mu q_\rho \tag{36}$$

Далее следует учесть, что калибровочная инвариантность не нарушена. Очевидно, что оба изучаемых тока (17) и (18) инвариантны относительно калибровочных преобразований. Поэтому так же, как в лекции 10 при выводе тождества Уорда, находим тождество

$$\frac{\partial}{\partial y^\nu}\left\langle 0\left|TJ^{5\mu}(x)J^\nu(y)J^\lambda(z)\right|0\right\rangle = 0$$

из которого в наинизшем приближении следует равенство

$$p_\nu T^{\mu\nu\lambda}(p,q) = 0 \tag{37}$$

Из (28) находим:

$$p_\nu\tilde\Gamma^{\mu\nu\lambda}(p,q) = ie^2\int\frac{\mathrm{d}^{(4)}r}{(2\pi)^4}\,\mathrm{tr}\,\gamma^5\gamma^\mu S_c(r+p)[(\not r+\not p)-\not x]S_c(r)\gamma^\lambda S_c(r-q) =$$
$$= ie^2\int\frac{\mathrm{d}^{(4)}r}{(2\pi)^4}\,\mathrm{tr}\,\gamma^5\gamma^\mu\left[S_c(r)\gamma^\lambda S_c(r-q) - S_c(r+p)\gamma^\lambda S_c(r-q)\right] \tag{38}$$

Первый интеграл в правой части (38) исчезает, так как невозможно построить псевдотензор с двумя индексами из одного единственного 4 -вектора. Аналогичное вычисление даёт:

$$p_\nu\tilde\Gamma^{\mu\lambda\nu}(q,p) = ie^2\int\frac{\mathrm{d}^{(4)}r}{(2\pi)^4}\,\mathrm{tr}\,\gamma^5\gamma^\mu S_c(r+q)\gamma^\lambda S_c(r-p) \tag{39}$$

Нам нужна сумма интегралов (38) и (39):

$$p_\nu\left(\tilde\Gamma^{\mu\nu\lambda}(p,q)+\tilde\Gamma^{\mu\lambda\nu}(q,p)\right)\equiv\Delta^{\mu\lambda}(p,q) =$$
$$= ie^2\int\frac{\mathrm{d}^{(4)}r}{(2\pi)^4}\,\mathrm{tr}\,\gamma^5\gamma^\mu\left[S_c(r+p+a)\gamma^\lambda S_c(r-q+a) - S_c(r+p)\gamma^\lambda S_c(r-q)\right],\quad a=q-p \tag{40}$$

В задаче 2 эта величина вычислена:

$$\Delta^{\mu\lambda}(p,q) = \frac{ie^2}{4\pi^2}\varepsilon^{\mu\lambda\nu\rho}p_\nu q_\rho \tag{41}$$

При помощи (26), (27), (32) и (41) находим:

$$p_\nu T^{\mu\nu\lambda}(p,q) = \frac{ie^2}{4\pi^2}\varepsilon^{\mu\lambda\nu\rho}p_\nu q_\rho\left(1+\frac{a}{2}\right) \tag{42}$$

Отсюда видно, что для выполнения калибровочной инвариантности теории необходимо

$$a = -2 \tag{43}$$

Согласно (36) такое значение параметра a означает несохранение аксиально-векторного тока. С другой стороны, если потребовать сохранение аксиально-векторного тока, для чего

необходимо положить $a = 0$, то нарушается калибровочная инвариантность и векторный ток не сохраняется. Поэтому теория самосогласована лишь при выполнении равенства (43).

Важно, что многоточие в (34) может быть опущено, так как все слагаемые, обозначенные многоточием, не содержат ультрафиолетовых расходимостей и потому, как это было указано во введении, не могут дать вклад в аномальную дивергенцию каких либо величин. Этот факт даёт возможность переписать с учётом равенств (36) и (43) соотношение (34) как точное равенство вида

$$\left\langle 0 \left| \partial_\mu J^{5\mu}(x) \right| 0 \right\rangle = -\frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\lambda\rho} \int \frac{\mathrm{d}^{(4)}p\ \mathrm{d}^{(4)}q}{(2\pi)^8} e^{-i(p+q)x} \left[p_\mu A_\nu(p) - p_\nu A_\mu(p)\right] \left[q_\lambda A_\rho(q) - q_\rho A_\lambda(q)\right] =$$
$$= \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\lambda\rho} F_{\mu\nu}(x) F_{\lambda\rho}(x) \tag{44}$$

Равенство (44) было получено Ю. Швингером (1951) и затем ещё раз открыто С. Л. Адлером (1969) и Ю. С. Беллом и Р. Джекивом (1969).

32.3.3 3. Non-Invariance of the Functional Measure as the Source of Anomaly

Рассмотрим задачу об аксиальной аномалии непосредственно в евклидовом пространстве. Тем самым условие (i) автоматически выполнено.

Формулировка фотодинамики в евклидовом пространстве очевидна. Поэтому мы лишь прокомментируем формулировку теории Дирака в евклидовом пространстве размерности d.

Пусть $\{\gamma^\mu\}, \mu = 1, ..., d$ - набор эрмитовых матриц Дирака таких, что

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta^{\mu\nu}, \quad \gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4 = \left(\gamma^5\right)^\dagger \quad (d = 4) \tag{45}$$

где $\delta^{\mu\nu}$ - символ Кронекера. Дираковское действие в безмассовом случае имеет вид

$$\mathcal{S}_\psi = -\int \mathrm{d}^{(d)}x \bar\psi i\gamma^\mu \nabla_\mu \psi, \quad \nabla_\mu = \frac{\partial}{\partial x^\mu} + ieA_\mu \tag{46}$$

Здесь поля ψ и $\bar\psi$ взаимно эрмитовски сопряжены и поэтому действие (46) эрмитово. Рассмотрим фермионный функциональный интеграл (аналог величины (8))

$$Z\{A_\mu\} = \int D\bar\psi D\psi \exp\left(-\mathcal{S}_\psi\right) \tag{47}$$

В функциональном интеграле (47) переменные $\{\psi, \bar\psi\}$ рассматриваются как независимые.

Определим математически строго функциональную меру в интеграле (47) следующим образом. Предположим, во-первых, что x-пространство является компактным множеством (например, это может быть тор, на котором для полей заданы периодические граничные условия). Пусть $\{\psi_N\}$ - полный ортонормированный набор собственных функций оператора Дирака:

$$i\gamma^\mu\nabla_\mu\psi_N = \epsilon_N\psi_N \tag{48}$$

Заметим, что $(i\gamma^\mu\nabla_\mu)$ - эрмитов оператор и индекс N вследствие компактности x-пространства пробегает счётное неограниченное множество. Разложим фермионные поля по этому полному набору функций:

$$\psi(x) = \sum_N{}' \eta_N\psi_N(x), \quad \bar\psi(x) = \sum_N{}' \bar\eta_N\bar\psi_N(x) \tag{49}$$

где $\{\eta, \bar{\eta}\}$ - множество независимых грассмановых переменных и переменные $\eta_N, \bar{\eta}_N$ взаимно комплексно сопряжены. В разложениях (49) штрих над знаком суммы означает, что суммирование ограничено теми значениями N, для которых

$$|\epsilon_N| < \Lambda \to \infty \tag{50}$$

где Λ - импульс обрезания. Таким образом фермионные поля оказываются регуляризованными и при этом они разложены по конечному числу мод.

Обычно подразумевается, что континуальная фермионная мера в интеграле (47) представляется в виде

$$D\bar{\psi}D\psi = \prod_{x,a} \mathrm{d}\bar{\psi}_a(x)\mathrm{d}\psi_a(x) = \prod_N \mathrm{d}\bar{\eta}_N \, \mathrm{d}\eta_N \tag{51}$$

Здесь все равенства являются формальными, поскольку произведение континуального или счётного бесконечного числа дифференциалов не является корректно и однозначно определённой величиной. Поэтому определим регуляризованную фермионную меру при помощи следующей формулы:

$$(D\bar{\psi}D\psi)_{reg} = \prod_N{}' \mathrm{d}\bar{\eta}_N \, \mathrm{d}\eta_N = \prod_{|\epsilon_N|<\Lambda} \mathrm{d}\bar{\eta}_N \, \mathrm{d}\eta_N \tag{52}$$

Определение (52) является вторым моментом в определении регуляризованной дираковской континуальной меры. Мера (52) является калибровочно инвариантной. Действительно, при калибровочных преобразованиях

$$\psi_N(x) \to e^{-ie\alpha(x)}\psi_N(x), \quad \psi(x) \to e^{-ie\alpha(x)}\psi(x), \quad A_\mu(x) \to A_\mu(x) + \partial_\mu\alpha(x)$$

собственные значения ϵ_N, а также грассмановы переменные $\{\eta, \bar{\eta}\}$ остаются инвариантными.

Далее мы используем регуляризованную меру (52), которая определена математически строго. Таким образом условие квантования (ii) также выполнено.

Совершим бесконечно малое аксиальное преобразование фермионных полей:

$$\psi(x) \to \tilde{\psi}(x) = \psi(x) + i\alpha(x)\gamma^5\psi(x)$$

$$\bar{\psi}(x) \to \tilde{\bar{\psi}}(x) = \bar{\psi}(x) + i\alpha(x)\bar{\psi}(x)\gamma^5 \tag{53}$$

В терминах переменных η_N и $\bar{\eta}_N$ преобразование (53) переписывается следующим образом:

$$\eta_N \to \tilde{\eta}_N = \eta_N + i\sum_M{}' \eta_M \int \mathrm{d}^{(d)}x\alpha(x)\bar{\psi}_N(x)\gamma^5\psi_M(x)$$
$$\bar{\eta}_N \to \tilde{\bar{\eta}}_N = \bar{\eta}_N + i\sum_M{}' \bar{\eta}_M \int \mathrm{d}^{(d)}x\alpha(x)\bar{\psi}_M(x)\gamma^5\psi_N(x) \tag{54}$$

Преобразование (54) изменяет меру (52) в первом порядке по α (см. (23 Л9)):

$$\prod_N{}' \mathrm{d}\bar{\eta}_N \, \mathrm{d}\eta_N = \left(1 + 2i\int \mathrm{d}^{(d)}x\alpha \sum_M{}' \bar{\psi}_M\gamma^5\psi_M\right) \prod_N{}' \mathrm{d}\tilde{\eta}_N \, \mathrm{d}\tilde{\eta}_N \tag{55}$$

Используя в выражении (47) регуляризованную и калибровочно инвариантную фермионную меру (52), произведём в этом функциональном интеграле замену переменных согласно формулам (53), (54):

$$\prod_N{}' \mathrm{d}\bar{\eta}_N \; \mathrm{d}\eta_N e^{-\mathcal{S}_\psi} = \int \prod_N{}' \mathrm{d}\tilde{\eta}_N \; \mathrm{d}\tilde{\eta}_N e^{-\tilde{\mathcal{S}}_\psi}$$
$$\left[1 + 2i \int \mathrm{d}^{(d)}x \alpha \sum_M{}' \bar{\psi}_M \gamma^5 \psi_M + i \int \mathrm{d}^{(d)}x \alpha \partial_\mu \tilde{J}^{5\mu}\right] \tag{56}$$

Здесь $J^{5\mu} = i\bar{\psi}^\mu \gamma^\mu \gamma^5 \psi$ - эрмитовский аксиальный ток. Все функции или функционалы, над которыми стоит волна, выражены через переменные с волной по обычным правилам. Полагая в (56) $\alpha(x) \sim \delta^{(d)}(x-y)$, получаем:

$$\left\langle \partial_\mu J^{5\mu} \right\rangle = -2 \sum_N{}' \bar{\psi}_N \gamma^5 \psi_N \tag{57}$$

В уравнении (57)... означает усреднение по фермионному вакууму. Правая часть в уравнении (57) возникла вследствие неинвариантности регуляризованной фермионной меры (52) относительно преобразований (54). Правая часть в (57) легко вычисляется, если обрезание в сумме сделать "мягким", то есть вместо обрезанной суммы $\sum_N{}'$ использовать регуляризованную сумму $\sum_N \exp\left(-\tau\epsilon_N^2\right), \tau \to +0$. Таким образом, перепишем равенство (57) в виде

$$\left\langle \partial_\mu J^{5\mu}(x) \right\rangle = -2 \operatorname{tr} \gamma^5 K(x,x|\tau), \quad \tau \to +0 \tag{58}$$
$$K(x,y|\tau) = \theta(\tau) \sum_N e^{-\tau\epsilon_N^2} \psi_N(x) \bar{\psi}_N(y) = \theta(\tau) \left\{ \exp\left[-\tau \left(i\gamma^\mu \nabla_\mu\right)^2\right] \right\}_{xy} \tag{59}$$

Для оператора K имеем уравнение:

$$\left[\frac{\partial}{\partial\tau} + \left(i\gamma^\mu \nabla_\mu\right)^2\right] K(x,y|\tau) = \delta(\tau)\delta^{(d)}(x-y) \tag{60}$$

Воспользуемся равенством

$$\gamma^\mu \gamma^\nu = \delta^{\mu\nu} + 2\sigma^{\mu\nu}, \quad \sigma^{\mu\nu} = \frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right]$$

и запишем оператор $\left(i\gamma^\mu \nabla_\mu\right)^2$ в виде

$$\left(i\gamma^\mu \nabla_\mu\right)^2 = -\nabla_\mu^2 - ie\sigma^{\mu\nu} F_{\mu\nu} \tag{61}$$

Теперь уравнение (60) принимает вид

$$\left[\frac{\partial}{\partial\tau} - \nabla_\mu^2 - ie\sigma^{\mu\nu} F_{\mu\nu}\right] K(x,y|\tau) = \delta(\tau)\delta^{(d)}(x-y) \tag{62}$$

Будем вычислять оператор $K(x,y|\tau)$ в пределе $\tau \to +0$ при помощи ТВ. В нулевом приближении оператор $K^{(0)}(x,y|\tau)$ удовлетворяет уравнению

$$\left[\frac{\partial}{\partial\tau} - \partial_\mu^2\right] K^{(0)}(x,y|\tau) = \delta(\tau)\delta^{(d)}(x-y) \tag{63}$$

и он равен при $d = 4$

$$K^{(0)}(x,y|\tau)=K^{(0)}(x-y|\tau)=\frac{\theta(\tau)}{16\pi^2\tau^2}\exp\left[-\frac{(x-y)^2}{4\tau}\right] \tag{64}$$

Это выражение проще всего получить, переходя в импульсное представление, в котором

$$K^{(0)}(k|\tau)=\theta(\tau)e^{-\tau k^2},\quad K^{(0)}(x-y|\tau)=\int\frac{\mathrm{d}^{(4)}k}{(2\pi)^4}e^{ik(x-y)}K^{(0)}(k|\tau)$$

Таким образом имеем:

$$\begin{aligned}K(x,x|\tau)=K^{(0)}(0|\tau)+\int\limits_0^\tau \mathrm{d}\tau_1\int \mathrm{d}^{(4)}yK^{(0)}\left(x-y|\tau-\tau_1\right)\left[ie\sigma^{\mu\nu}F_{\mu\nu}(y)\right]K^{(0)}\left(y-x|\tau_1\right)+\\ +\int\limits_0^\tau \mathrm{d}\tau_1\int\limits_0^{\tau_1}\mathrm{d}\tau_2\int \mathrm{d}^{(4)}y\int \mathrm{d}^{(4)}zK^{(0)}\left(x-y|\tau-\tau_1\right)\left[ie\sigma^{\mu\nu}F_{\mu\nu}(y)\right]\\ \cdot K^{(0)}\left(y-z|\tau_1-\tau_2\right)\left[ie\sigma^{\lambda\rho}F_{\lambda\rho}(z)\right]K^{(0)}\left(z-x|\tau_2\right)+\ldots\end{aligned} \tag{65}$$

Невыписанные здесь слагаемые, обозначенные многоточием, либо исчезают под знаком следа в (58), либо исчезают в пределе $\tau\to+0$. Так как

$$\operatorname{tr}\gamma^5=0,\quad \operatorname{tr}\gamma^5\gamma^\mu\gamma^\nu=0,\quad \operatorname{tr}\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho=4\varepsilon^{\mu\nu\lambda\rho} \tag{66}$$

то вклад в (58) даёт лишь третье слагаемое в (65). Из (64) видно, что при $\tau\to+0$ интегралы по y и z насыщаются при $y\to x, z\to x$ и потому гладкие величины $F_{\mu\nu}(y)$ и $F_{\lambda\rho}(z)$ могут быть вынесены из под знака интеграла и взяты в точке x. После этого все свёртки между функциями $K^{(0)}$ следует вычислять при помощи тривиальной формулы

$$\int \mathrm{d}^{(4)}zK^{(0)}\left(x-z|\tau-\tau_1\right)K^{(0)}\left(z-y|\tau_1\right)=K^{(0)}(x-y|\tau)$$

Таким образом находим, что третье слагаемое в (65) равно

$$-\frac{e^2}{16\pi^2}\sigma^{\mu\nu}\sigma^{\lambda\rho}F_{\mu\nu}(x)F_{\lambda\rho}(x)\left(\frac{1}{\tau^2}\int\limits_0^\tau \mathrm{d}\tau_1\int\limits_0^{\tau_1}\mathrm{d}\tau_2\right) \tag{67}$$

Из проведенного вычисления видно, что если бы мы учли более высокие члены разложения в (65), то они были бы пропорциональны, например (в следующем порядке)

$$\frac{1}{\tau^2}\int\limits_0^\tau \mathrm{d}\tau_1\int\limits_0^{\tau_1}\mathrm{d}\tau_2\int\limits_0^{\tau_2}\mathrm{d}\tau_3=\frac{\tau}{6}\to 0$$

Все такие слагаемые стремятся к нулю в пределе $\tau\to+0$. Поэтому, подставляя (67) в (58) и используя (66), окончательно находим:

$$\left\langle\partial_\mu J^{5\mu}(x)\right\rangle=\frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\lambda\rho}F_{\mu\nu}(x)F_{\lambda\rho}(x) \tag{68}$$

Замечание. Строго говоря, преобразованные поля (53) должны быть спроектированы на регуляризованное пространство фермионных полей (49) с базисом мод $\{\psi_N\},|\epsilon_N|<\Lambda$. Иными словами, для преобразованных полей также должны иметь место разложения (49):

$$\tilde{\psi}(x) = \sum_N{}' \tilde{\eta}_N \psi_N(x), \quad \tilde{\bar{\psi}}(x) = \sum_N{}' \tilde{\bar{\eta}}_N \bar{\psi}_N(x) \tag{69}$$

В противном случае действие

$$\tilde{\mathcal{S}}_\psi = -\int \mathrm{d}^{(d)}x \tilde{\bar{\psi}} i\gamma^\mu \nabla_\mu \tilde{\psi}$$

в (56) как функция переменных $\{\tilde{\bar{\eta}}_M, \tilde{\eta}_M\}$ оказалось бы отличным от действия $\mathcal{S}$ в (47) как функции переменных $\{\bar{\eta}_M, \eta_M\}$, и из (56) не следовало бы соотношение (57). Однако вследствие того, что моды в разложениях (49) и (69) являются собственными для оператора Дирака $i\gamma^\mu\nabla_\mu$, мы имеем:

$$\tilde{\mathcal{S}}_\psi = \sum_{|\epsilon_N|<\Lambda} \epsilon_N \tilde{\bar{\eta}}_N \tilde{\eta}_N + \sum_{|\epsilon_N|>\Lambda} \epsilon_N \tilde{\bar{\eta}}_N \tilde{\eta}_N$$

Здесь второе слагаемое имеет порядок $O(\alpha^2)$ и потому не играет роли в приведенном выше вычислении. Мы видим, что в фермионном действии проектирование преобразованных фермионных полей на регуляризованное фермионное подпространство происходит автоматически. Подчеркнём, что этот факт является следствием использования собственных функций оператора Дирака для регуляризации фермионных полей. Поэтому корректность перехода от равенства (56) между регуляризованными фермионными интегралами к соотношению (57) очевидна.

Вычисление аномальной расходимости аксиально-векторного тока путём учёта неинвариантности фермионной меры впервые было проделано С. Н. Вергелесом (1978). Вскоре этот же метод был открыт К. Фуджикавой (1979).

Вывод: Не все сохраняющиеся величины в классической теории поля остаются сохраняющимися в квантовой теории. Этому препятствуют квантовые аномалии. В частности, в безмассовой КЭД векторнъй ток сохраняется, а аксиалъно-векторный ток не сохраняется вследствие квантовой аномалии.

Задачи

1. Вычислить интеграл (29).

   Ответ:

$$\Delta^{\mu\nu\lambda}(p,q|a) = \frac{ie^2}{8\pi^2}\varepsilon^{\mu\nu\lambda\rho}a_\rho$$

2. Вычислить интеграл (40).

Ответ:

$$\Delta^{\mu\lambda}(p,q) = \frac{ie^2}{4\pi^2}\varepsilon^{\mu\lambda\nu\rho}p_\nu q_\rho$$

3. Сформулировать и выписать действие для двумерной электродинамики (электромагнитное + дираковское поля). В безмассовом случае решить задачу также и в евклидовом пространстве. Вычислить аномалию в дивергенции аксиально-векторного тока в безмассовой евклидовой теории.

4*. Вычислить в двумерной безмассовой электродинамике (модель Швингера) величину $\ln Z\{A\}$.

Указание. Действовать в евклидовом пространстве и принять во внимание следующие факты:
а)

$$\delta_A \ln Z\{A\} = -e \int \mathrm{d}^{(2)}x \delta A_\mu \langle J^\mu \rangle$$

где $J^\mu = \bar{\psi}\gamma^\mu\psi$ - векторный ток.
б) Кинематическое соотношение

$$J^\mu = \varepsilon_{\mu\nu}J^{5\nu}, \quad J^{5\mu} = -\varepsilon_{\mu\nu}J^\nu, \quad J^{5\mu} = i\bar{\psi}\gamma^\mu\gamma^5\psi$$

в) Равенства $\partial_\mu \langle J^\mu \rangle = 0$ и $\partial_\mu \langle J^5\mu \rangle = -\frac{e}{2\pi}\varepsilon_{\mu\nu}F_{\mu\nu}$.

32.4 L15 Spontaneous Symmetry Breaking

32.4.1 1. Spontaneous Symmetry Breaking in Classical Theory

Рассмотрим теорию скалярного вещественного поля с лагранжианом

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1}$$

Гамильтониан такой теории

$$\mathcal{H} = \int \mathrm{d}^{(d-1)}x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 - \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right] \tag{2}$$

В теории (1) - (2) имеется дискретная симметрия $\phi \to -\phi$.

Энергия классической системы (2) принимает минимальное значение в том случае, когда $\phi = \text{const}$ и потенциал

$$V(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{3}$$

минимален, что имеет место при

$$\phi_0 = \pm v = \pm\sqrt{\frac{6}{\lambda}}\mu \neq 0 \tag{4}$$

Константа v называется вакуумным средним поля ϕ.

Предположим, что система находится вблизи одного из минимумов потенциала, совершая вблизи него малые колебания. Поэтому естественно разложить поле ϕ вблизи вакуумного среднего:

$$\phi = \phi_0 + \sigma \tag{5}$$

Подставим (5) в лагранжиан (1):

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{(2\mu^2)}{2}\sigma^2 - \frac{\lambda\phi_0}{3!}\sigma^3 - \frac{\lambda}{4!}\sigma^4 \tag{6}$$

Мы учли, что линейное слагаемое относительно σ автоматически исчезает в (6) и опустили $\mathcal{L}(\phi_0) = \text{const}$. Лагранжиан (6) описывает скалярное поле σ с массой $m = \sqrt{2}\mu$. Симметрия $\phi \to -\phi$ теперь проявляется лишь в соотношении между тремя коэффициентми теории с лагранжианом (6).

Теперь рассмотрим линейную сигма-модель (см. задачу 3 к лекции 3) с лагранжианом

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi^i\right)^2 + \frac{\mu^2}{2}\left(\phi^i\right)^2 - \frac{\lambda}{4}\left[\left(\phi^i\right)^2\right]^2,$$
$$i = 1, ..., n, \quad \left(\phi^i\right)^2 \equiv \left(\phi^1\right)^2 + \left(\phi^2\right)^2 + ... \tag{7}$$

Лагранжиан (7) инвариантен относительно преобразований

$$\phi^i \to \mathcal{O}^i_j \phi^j, \quad \mathcal{O}\mathcal{O}^t = 1 \tag{8}$$

Таким образом $\mathcal{O}$ - элемент ортогональной группы вращений n-мерного евклидового пространства.

Также как и в вышеприведенном случае классическая конфигурация с минимальной энергией есть постоянное поле ϕ^i_0, минимизирующее потенциал

$$V\left(\phi^i\right) = -\frac{\mu^2}{2}\left(\phi^i\right)^2 + \frac{\lambda}{4}\left[\left(\phi^i\right)^2\right]^2 \tag{9}$$

Этот минимум достигается, если

$$\left(\phi^i_0\right)^2 = \frac{\mu^2}{\lambda} \tag{10}$$

Условие (10) определяет лишь длину вектора ϕ^i_0, но не его направление. Для удобства будем считать, что

$$\phi^i_0 = (0, ..., v), \quad v = \frac{\mu}{\sqrt{\lambda}} \tag{11}$$

Аналогично (5) теперь имеем:

$$\phi^i = \left(\pi^k(x), v + \sigma(x)\right), \quad k = 1, ..., n-1 \tag{12}$$

Перепишем лагранжиан (7) в терминах новых полей:

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \pi^k\right)^2 + \frac{1}{2}\left(\partial_\mu \sigma\right)^2 - \frac{1}{2}\left(2\mu^2\right)\sigma^2 - \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu\left(\pi^k\right)^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}\left(\pi^k\right)^2\sigma^2 - \frac{\lambda}{4}\left(\left(\pi^k\right)^2\right)^2 \tag{13}$$

Лагранжиан (13) описывает динамику массивного поля σ и $(n-1)$ безмассовых полей π^k. Исходная $O(n)$ симметрия оказывается нарушенной, от неё остаётся подгруппа $O(n-1)$ вращений полей π в своём пространстве. Колебания поля σ происходят вдоль длины вектора ϕ^i, изменяя эту длину. Поэтому вторая производная лагранжиана по σ отлична от нуля и масса этого поля не равна нулю. Масса π-частиц равна нулю, так как π-частицы деформируют поле ϕ^i в $(n-1)$ направлениях, ортогональных вакуумному среднему.

32.4.2 2. Goldstone's Theorem

Заметим, что группа симметрии исходного лагранжиана (7) инвариантна относительно группы Ли $O(n)$, размерность которой $n(n-1)/2$ (число различных двумерных плоскостей, в которых можно делать независимые друг от друга повороты). Когда исходная симметрия нарушена до симметрии, содержащейся в лагранжиане (13), остаточная симметрия описывается группой $O(n-1)$ с $(n-1)(n-2)/2$ независимыми преобразованиями. Таким образом, утрачены

$$\frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} = (n-1)$$

преобразований симметрии и вместе с тем появились $(n-1)$ безмассовые частицы, которые называются голдстоуновскими бозонами.

Теорема Голдстоуна утверждает, что для каждой спонтанно нарушенной непрерывной симметрии в теории содержится безмассовая частица.

Доказательство теоремы Голдстоуна проведём в самом общем случае.

Рассмотрим теорию, содержащую несколько полей ϕ^i с лагранжианом

$$\mathcal{L} = (\text{ члены с производными }) - V(\phi). \tag{14}$$

Пусть ϕ_0^i - постоянное поле, минимизирующее V:

$$\frac{\partial}{\partial \phi^i} V \Big|_{\phi^2(x)=\phi_0^2} = 0 \tag{15}$$

Разложим V в окрестности этого минимума:

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi-\phi_0)^i (\phi-\phi_0)^j \left(\frac{\partial^2}{\partial\phi^i \partial\phi^j} V\right)_{\phi_0} + \ldots \tag{16}$$

Коэффициент при квадратичном слагаемом

$$m_{ij}^2 = \left(\frac{\partial^2}{\partial\phi^i \partial\phi^j} V\right)_{\phi_0} \tag{17}$$

является симметричной матрицей, собственные значения которой дают массы полей. Они неотрицательны, так как V минимален при $\phi = \phi_0$. Для доказательства теоремы Голдстоуна необходимо показать, что всякая непрерывная симметрия лагранжиана (14), не являющаяся симметрией (т.е. изменяющая) поле ϕ_0, приводит к нулевому собственному значению матрицы (17).

Произвольное преобразование непрерывной симметрии имеет вид

$$\phi^i \longrightarrow \phi^i + \alpha \Delta^i(\phi), \quad \alpha \to 0 \tag{18}$$

где $\Delta^i(\phi)$ - некие функции поля ϕ, определяющие данное преобразование симметрии. Достаточно рассмотреть поля, постоянные в пространстве-времени. Тогда потенциал V инвариантен сам по себе относительно преобразования (18):

$$V\left(\phi^i\right) = V\left(\phi^i + \alpha\Delta^i(\phi)\right) \longrightarrow \Delta^i(\phi)\frac{\partial}{\partial\phi^i} V(\phi) = 0 \longrightarrow$$

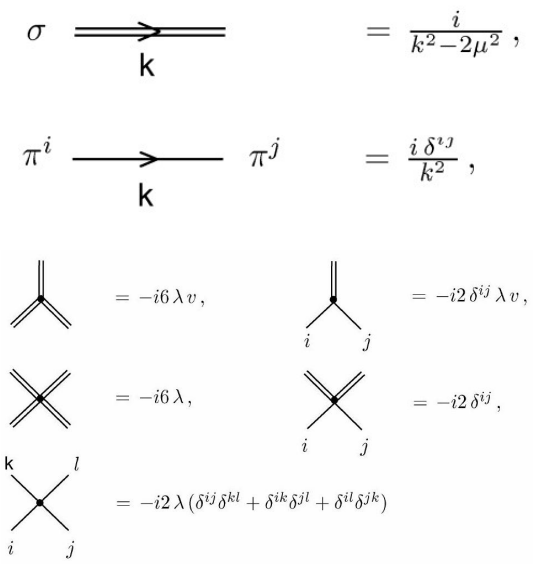
$$\frac{\partial}{\partial\phi^j}\left[\Delta^i(\phi)\frac{\partial}{\partial\phi^i}V(\phi)\right]\Big|_{\phi=\phi_0} = \frac{\partial\Delta^i}{\partial\phi^j}\Big|_{\phi=\phi_0} \frac{\partial V}{\partial\phi^i}\Big|_{\phi=\phi_0} + \Delta^i(\phi_0)\, m_{ij}^2 = 0$$

Вследствие (15)

$$m_{ij}^2 \Delta^j(\phi_0) = 0 \tag{19}$$

Заметим, что если $\Delta^i(\phi_0) = 0$, то $\phi_0^i + \Delta^i(\phi_0) = \phi_0^i$. Это означает, что вакуумное среднее инвариантно относительно относительно рассматриваемого преобразования. Наоборот, если $\Delta^i(\phi_0) \neq 0$, то мы имеем преобразование, изменяющее ϕ_0^i. Соответствующая симметрия отсутствует в лагранжиане вида (13). Но равенство (19) означает, что такому преобразованию соответствует нулевая масса. Тем самым теорема Голдстоуна доказана.

Вернёмся к линейной σ-модели с лагранжианом (13) и построим соответствующую квантовую теорию поля. Обозначим пропагаторы σ - и π-частиц соответственно двойной и одинарной линиями. Тогда элементы диаграммной техники Фейнмана представлены на рис. 31:



$$= -i2\lambda \left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right)$$

Рис. 31

Очевидно, что теория с указанной на рис. 31 диаграммной техникой перенормируема, так как перенормируема теория с лагранжианом (7), а ультрафиолетовые расходимости не могут зависеть от того, как "растасованы"и обозначены отдельные поля теории. Однако имеют место гораздо более сильные утверждения:

а) Для устранения ультрафиолетовых расходимостей в теории (13) с диаграммной техникой, представленной на рис. 31, достаточно введение контрчленов того же вида, что и в лагранжиане (13).

б) Теорема Голдстоуна остаётся справедливой с учётом квантовых поправок.

Ввиду громоздкости мы не доказываем здесь этих утверждений. Укажем лишь удобные условия перенормировки, фиксирующие все константы теории:

$$CBA = \quad = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}k^2}\,(= CB = |_{k^2=0} = 0 \tag{20}$$

32.4.3 3. Effective Action

Рассмотрим для некоторой скалярной теории поля с лагранжианом $\mathcal{L}(\phi)$ производящий функционал (см. (16 ЛЗ))

$$e^{-iE(\eta)} \equiv Z(\eta) = \int D\phi \exp\left[i \int \mathrm{d}^{(d)}x(\mathcal{L}(\phi) + \eta\phi)\right] \tag{21}$$

Имеем:

$$\frac{\delta}{\delta\eta(x)}E(\eta) = i\frac{\delta}{\delta\eta(x)}\ln Z = -\frac{\int D\phi e^{i\int(\mathcal{L}+\eta\phi)}\phi(x)}{\int D\phi e^{i\int(\mathcal{L}+\eta\phi)}} \equiv -\langle\phi(x)\rangle_\eta \tag{22}$$

Здесь правая часть - среднее по вакууму в присутствии внешнего источника $\eta(x)$. Определим классическое поле

$$\phi_{cl}(x) \equiv \langle\phi(x)\rangle_\eta \tag{23}$$

и эффективное действие

$$\Gamma\left(\phi_{cl}\right) \equiv -E(\eta) - \int \mathrm{d}^{(d)} y \eta(y) \phi_{cl}(y) \tag{24}$$

Тот факт, что эффективное действие зависит лишь от классического поля, но не от внешнего источника, устанавливается вычислением полной вариации величины (24):

$$\delta\Gamma = -\int \frac{\delta E}{\delta\eta(y)} \delta\eta(y) \mathrm{d}^{(d)} y - \int \phi_{cl}(y) \delta\eta(y) \mathrm{d}^{(d)} y - \int \eta(y) \delta\phi_{cl}(y) \mathrm{d}^{(d)} y$$

С учётом (22) здесь первые два слагаемых взаимно сокращаются. Поэтому

$$\frac{\delta\Gamma\left(\phi_{cl}\right)}{\delta\phi_{cl}(x)} = -\eta(x) \tag{25}$$

Если внешний источник обратить в нуль, то из (25) получаем:

$$\frac{\delta\Gamma\left(\phi_{cl}\right)}{\delta\phi_{cl}(x)} = 0 \tag{26}$$

Решения последнего уравнения являются значениями $\langle\phi(x)\rangle$ в стабильных квантовых состояниях теории. Для трансляционно-инвариантного вакуума ϕ_{cl} также не зависит от x. Именно такой случай далее мы будем рассматривать. Заметим лишь, что в случае трансляционно- и лоренцинвариантного вакуума иногда могут быть дополнительные решения уравнения (26), нарушающие симметрию вакуума и соответствующие локализованным сгусткам поля, удерживаемым за счёт самодействия и топологических факторов. В случае, когда действие таких сгустков конечно, они называются инстантонами, а если конечен гамильтониан, то они называются солитонами. Солитоны являются частицами, которые отсутствуют при наивном изучении теории с помощью теории возмущений.

Если $\phi_{cl} = \mathrm{const}$, то поскольку $\Gamma\left(\phi_{cl}\right)$ есть экстенсивная величина - она пропорциональна объёму пространства-времени - мы имеем:

$$\Gamma\left(\phi_{cl}\right) = -(VT) V_{eff}\left(\phi_{cl}\right) \tag{27}$$

Здесь T - временная протяжённость пространства-времени, а V - объём. $V_{eff}\left(\phi_{cl}\right)$ в (27) называется эффективным потенциалом. В терминах эффективного потенциала уравнение (26) записывается как

$$\frac{\partial V_{eff}\left(\phi_{cl}\right)}{\partial\phi_{cl}} = 0 \tag{28}$$

Так как $\Gamma = -E$ при $\eta = 0$, то V_{eff}, вычисленное в точке (28), является плотностью энергии в этом состоянии.

Будем вычислять V_{eff} при помощи ТВ. Разобьём лагранжиан, как обычно, на слагаемое, зависящее от перенормированных параметров, и слагаемое, содержащее лиишь контрчлены:

$$\mathcal{L} = \mathcal{L}_1 + \delta\mathcal{L} \tag{29}$$

Наша цель - вычисление функции $\Gamma\left(\phi_{cl}\right)$, в то время как $Z(\eta)$ зависит от ϕ_{cl} косвенно через источник η. Попробуем найти, хотя бы неявно, связь между η и ϕ_{cl}. В наинизшем порядке ТВ эта связь задаётся уравнением (см. (25))

$$\left.\frac{\delta \int \mathrm{d}^{(d)} y \mathcal{L}}{\delta\phi(x)}\right|_{\phi=\phi_{cl}} = -\eta(x) \quad \text{(в низшем порядке)} \tag{30}$$

Тогда уравнение

$$\left.\frac{\delta\int \mathrm{d}^{(d)}y\mathcal{L}_\infty}{\delta\phi(x)}\right|_{\phi=\phi_{cl}} = -\eta_1(x) \tag{30}$$

определяет "перенормированный"источник $\eta_1(x)$, так что

$$\eta(x) = \eta_1(x) + \delta\eta(x) \tag{32}$$

"Контрчлен" $\delta\eta(x)$ определяется последовательно при помощи ТВ и уравнений (23), (31), (32). Перепишем формулу (21) следующим образом:

$$e^{-iE(\eta)} = \int D\phi \left\{\exp\left[i\int \mathrm{d}^{(d)}x\left(\mathcal{L}_1(\phi)+\eta_1\phi\right)\right]\right\}\left\{\exp\left[i\int \mathrm{d}^{(d)}x(\delta\mathcal{L}(\phi)+\delta\eta\phi)\right]\right\} \tag{33}$$

В первой экспоненте справа разложим показатель вблизи ϕ_{cl} : $\phi(x) = \phi_{cl}(x)+\rho(x)$. Имеем:

$$\begin{aligned}\int \mathrm{d}^{(d)}x\left(\mathcal{L}_1(\phi)+\eta_1\phi\right) = \int \mathrm{d}^{(d)}x\left(\mathcal{L}_1\left(\phi_{cl}\right)+\eta_1\phi_{cl}\right) + \int \mathrm{d}^{(d)}x\rho(x)\left(\left.\frac{\delta\mathcal{L}_1}{\delta\phi}\right|_{\phi=\phi_{cl}}+\eta_1\right)+\\ +\frac{1}{2}\int \mathrm{d}^{(d)}x\ \mathrm{d}^{(d)}y\rho(x)\rho(y)\left.\frac{\delta^2\mathcal{L}_1}{\delta\phi(x)\delta\phi(y)}\right|_{\phi=\phi_{cl}}+\\ +\frac{1}{3!}\int \mathrm{d}^{(d)}x\ \mathrm{d}^{(d)}y\ \mathrm{d}^{(d)}z\rho(x)\rho(y)\rho(z)\left.\frac{\delta^3\mathcal{L}_1}{\delta\phi(x)\delta\phi(y)\delta\phi(z)}\right|_{\phi=\phi_{cl}}+\ldots\end{aligned} \tag{34}$$

Вследствие (30) линейное по ρ слагаемое в (34) исчезает. Таким образом, следует вычислить гауссов интеграл по полю ρ, а нелинейности учесть при помощи ТВ в соответствии с разработанной схемой.

В низшем приближении вторая экспонента справа в (33) не даёт вклада, а первая экспонента даёт гауссов интеграл:

$$\begin{aligned}\int D\rho\exp\left\{i\left[\int\left(\mathcal{L}_1\left(\phi_{cl}\right)+\eta_1\phi_{cl}\right)+\frac{1}{2}\int\rho\frac{\delta^2\mathcal{L}_1}{\delta\phi\delta\phi}\rho\right]\right\} =\\ =\exp\left\{i\int\left(\mathcal{L}_1\left(\phi_{cl}\right)+\eta_1\phi_{cl}\right)\right\}\cdot\left(\det\left[-\frac{\delta^2\mathcal{L}_1}{\delta\phi\delta\phi}\right]\right)^{-1/2}\end{aligned} \tag{35}$$

Отсюда видно, что при дальнейшем учёте квантовых поправок возникают фейнмановские диаграммы, в которых в качестве пропагатора используется оператор

$$-i\left(\frac{\delta^2\mathcal{L}_1}{\delta\phi\delta\phi}\right)^{-1} \tag{36}$$

а вершинами являются третья и т.д. функциональные производные $\mathcal{L}_1$.

Вторая экспонента в (33) также разлагается вблизи ϕ_{cl} по полю ρ и даёт контрчлены. Как обычно в аналогичных вычислениях (см. (14 Л6)), получаем:

$$\begin{aligned}-iE(\eta) = i\int \mathrm{d}^{(d)}x\left(\mathcal{L}_1\left(\phi_{cl}\right)+\eta_1\phi_{cl}\right) - \frac{1}{2}\ln\det\left(-\frac{\delta^2\mathcal{L}_1}{\delta\phi\delta\phi}\right)+\\ +(\text{ связные диаграммы }) + i\int \mathrm{d}^{(d)}x\left(\delta\mathcal{L}\left(\phi_{cl}\right)+\delta\eta\cdot\phi_{cl}\right)\end{aligned} \tag{37}$$

Здесь последнее слагаемое является нулевым приближением при разложении по ρ показателя второй экспоненты в (33). Сопоставляя (24) и (37), находим:

$$\Gamma\left(\phi_{cl}\right)=\int \mathrm{d}^{(d)}x\mathcal{L}_1\left(\phi_{cl}\right)+\frac{i}{2}\ln\det\left(-\frac{\delta^2\mathcal{L}_1}{\delta\phi\delta\phi}\right)-i(\text{ связные диаграммы })+\int \mathrm{d}^{(d)}x\delta\mathcal{L}\left(\phi_{cl}\right) \quad (38)$$

Здесь все слагаемые зависят явно от ϕ_{cl}. Важно также то, что каждая связная диаграмма в (38) не имеет внешних линий и потому является не ниже чем двухпетлевой. Таким образом однопетлевой вклад содержится во втором слагаемом в (38).

Отметим, что контрчлен $\delta\eta$ определяется из того требования, чтобы во всех порядках было $\langle\phi\rangle=\phi_{cl}$: если в процессе вычислений возникает ненулевой вклад в $\langle\rho\rangle$, то его "убивают", добавляя соответствующий вклад в $\delta\eta$.

Вычислим эффективное действие (38) для линейной сигма-модели в однопетлевом приближении, т.е. учитывая в (38) лишь первые два слагаемых. При помощи (7) мы находим:

$$\left.\frac{\delta^2\mathcal{L}_1}{\delta\phi^i\delta\phi^j}\right|_{\phi=\phi_{cl}}=\delta^{ij}\left(-\partial^2+\mu^2\right)-\lambda\left[\left(\phi_{cl}^k\right)^2\delta^{ij}+2\phi_{cl}^i\phi_{cl}^j\right] \quad (39)$$

Предположим, что $\phi_{cl}^i=\text{const}$ и

$$\phi_{cl}^i=(0,0,...,0,\phi_{cl}) \quad (40)$$

Тогда оператор (39) совпадает с диагональной матрицей $n\times n$, на диагонали которой находятся операторы Клейна-Гордона-Фока $(-\partial^2-m_i^2)$, где

$$m_i^2=\left\{\begin{array}{ll}\lambda\phi_{cl}^2-\mu^2 & \text{при действии на } \rho^1,...,\rho^{n-1}\\ 3\lambda\phi_{cl}^2-\mu^2 & \text{при действии на } \rho^n\end{array}\right. . \quad (41)$$

Поэтому детерминант в (38) равен произведению детерминантов этих операторов:

$$\det\left(-\frac{\delta^2\mathcal{L}_1}{\delta\phi^i\delta\phi^j}\right)=\left\{\det\left[\partial^2+\left(\lambda\phi_{cl}^2-\mu^2\right)\right]\right\}^{n-1}\cdot\det\left[\partial^2+\left(3\lambda\phi_{cl}^2-\mu^2\right)\right] \quad (42)$$

Воспользуемся известной формулой

$$\ln\det A=\mathrm{tr}\ln A$$

где A - линейный оператор. Имеем:

$$\ln\det\left(\partial^2+m^2\right)=\mathrm{tr}\ln\left(\partial^2+m^2\right) \quad (43)$$

Сделаем поворот Вика в Евклидово пространство. При помощи (7), (38), (42), (43) получаем:

$$V_{eff}\left(\phi_{cl}\right)=-\frac{1}{2}\mu^2\phi_{cl}^2+\frac{\lambda}{4}\phi_{cl}^4+\frac{1}{VT}\cdot\frac{1}{2}\cdot\sum_{i=1}^{n}\mathrm{tr}\ln\left(-\partial^2+m_i^2\right)+\frac{1}{2}\delta_\mu\phi_{cl}^2+\frac{1}{4}\delta_\lambda\phi_{cl}^4 \quad (44)$$

Здесь набор масс m_i задаётся согласно (41), $\partial^2=\sum\limits_{\mu=1}^{d}\left(\partial_\mu\right)^2$, а два последние слагаемые - возможные контрчлены. Задача свелась к вычислению величины $\mathrm{tr}\ln\left(-\partial^2+m^2\right)$. В ортонормированном базисе $\left\{e^{ikx}\right\}$ матричные элементы оператора $\ln\left(-\partial^2+m^2\right)$ равны

$$\left\langle p\left|\ln\left(-\partial^2+m^2\right)\right|k\right\rangle=\int\limits_E \mathrm{d}^{(d)}xe^{i(k-p)x}\ln\left(k_E^2+m^2\right)$$

Поэтому

$$\operatorname{tr}\ln\left(-\partial^2+m^2\right)=(VT)\int\limits_E\frac{\mathrm{d}^{(d)}k}{(2\pi)^d}\ln\left(k_E^2+m^2\right) \tag{45}$$

Но

$$\begin{aligned}\int\limits_E\frac{\mathrm{d}^{(d)}k}{(2\pi)^d}\ln\left(k_E^2+m^2\right)&=-\frac{\partial}{\partial\alpha}\int\limits_E\frac{\mathrm{d}^{(d)}k}{(2\pi)^d}\frac{1}{\left(k_E^2+m^2\right)^\alpha}\Bigg|_{\alpha=0}=\\ =-\frac{\partial}{\partial\alpha}\left[\alpha\cdot\frac{\left(m^2\right)^{d/2-\alpha}}{(4\pi)^{d/2}}\cdot\frac{\Gamma(\alpha-d/2)}{\Gamma(1+\alpha)}\right]\Bigg|_{\alpha=0}&=-\frac{\left(m^2\right)^{d/2}}{(4\pi)^{d/2}}\Gamma(-d/2)\end{aligned} \tag{46}$$

Теперь при помощи (41), (42), (44), (46) получаем:

$$\begin{aligned}V_{eff}\left(\phi_{cl}\right)=-\frac{1}{2}\mu^2\phi_{cl}^2+\frac{\lambda}{4}\left(\phi_{cl}^2\right)^2-\frac{1}{2}\cdot\frac{\Gamma(-d/2)}{(4\pi)^{d/2}}\left\{(n-1)\left(\lambda\phi_{cl}^2-\mu^2\right)^{d/2}+\left(3\lambda\phi_{cl}^2-\mu^2\right)^{d/2}\right\}+\\ +\frac{1}{2}\delta_\mu\phi_{cl}^2+\frac{1}{4}\delta_\lambda\left(\phi_{cl}^2\right)^2\end{aligned} \tag{47}$$

Здесь $\phi_{cl}^2=\left(\phi_{cl}^i\right)^2$. Эффективное действие (47), конечно, $O(n)$-инвариантно. Выражение (47) конечно при $d\to 4$ (т.е. конечны коэффициенты при при степенях ϕ_{cl}), если

$$\begin{aligned}\delta_\mu&=-\lambda\mu^2(n+2)\frac{\Gamma(2-d/2)}{(4\pi)^2}+\text{ конечные члены}\\ \delta_\lambda&=\lambda^2(n+8)\frac{\Gamma(2-d/2)}{(4\pi)^2}+\text{ конечные члены}\end{aligned} \tag{48}$$

Конечные члены в (48) регулируются условиями нормировки. Подберём их так, чтобы

$$\begin{aligned}V_{eff}\left(\phi_{cl}\right)=-\frac{1}{2}\mu^2\phi_{cl}^2+\frac{\lambda}{4}\left(\phi_{cl}^2\right)^2+\\ +\frac{1}{4(4\pi)^2}\left\{(n-1)\left(\lambda\phi_{cl}^2-\mu^2\right)^2\ln\frac{\lambda\phi_{cl}^2-\mu^2}{M^2}+\left(3\lambda\phi_{cl}^2-\mu^2\right)^2\ln\frac{3\lambda\phi_{cl}^2-\mu^2}{M^2}\right\}\end{aligned} \tag{49}$$

Здесь M - точка нормировки. На рис. 32 пунктирной линией изображён график классического
эффективного потенциала, а сплошной линией - потенциала с квантовой поправкой (49). Сплошная линия рвётся там, где поправка не определена. Из (49) видно, что при $\mu^2>0$ симметрия спонтанно нарушена, при $\mu^2=-m^2=-M^2<0$ имеется один тривиальный минимум потенциала при $\phi_{cl}=0$.

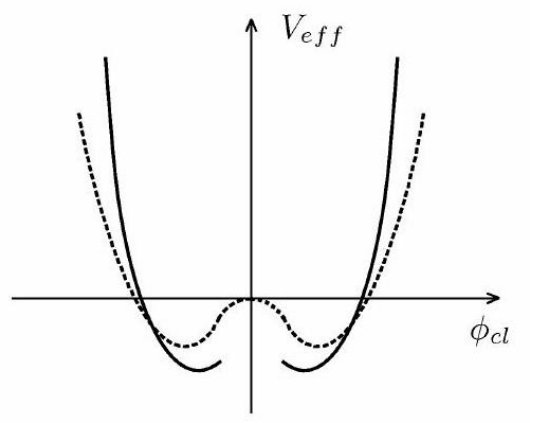


Рис. 32.

Рассмотрим $V_{eff}(49)$ в случае $\mu^2 = 0$:

$$V_{eff}\left(\phi_{cl}\right) = \frac{1}{4}\left(\phi_{cl}^2\right)^2 \left\{\lambda + \frac{\lambda^2}{(4\pi)^2}\left[(n+8)\ln\frac{\lambda\phi_{cl}^2}{M^2} + 9\ln 3\right]\right\} \tag{50}$$

Интересен вопрос: имеет ли потенциал (50) нетривиальный минимум? Легко найти, что $\mathrm{d}V_{eff}/\mathrm{d}\phi_{cl}^2 = 0$ при

$$\phi_{cl}^2 \sim \frac{M^2}{\lambda}\exp\left[-\frac{(4\pi)^2}{(n+8)\lambda}\right] \neq 0 \tag{51}$$

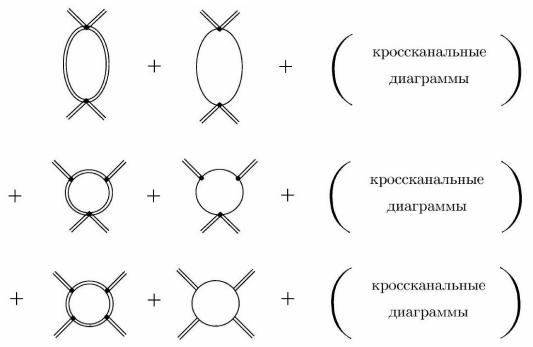
Однако при таком значении ϕ_{cl}^2 второе слагаемое в фигурной скобке в (50) почти сокращает первое слагаемое, т.е. поправки имеют тот же порядок, что и нулевое приближение. Это означает, что что для решения поставленной задачи ТВ непригодна.

Задачи

1. Нарисовать все однопетлевые диаграммы, дающие поправки к вершине туду рассеяния двух σ-частиц в линейной сигма-модели.

, дающей ампли-

Ответ:



2. Имеет место равенство

$$\frac{\delta^n \Gamma\left(\phi_{cl}\right)}{\delta\phi_{cl}\left(x_1\right)...\delta\phi_{cl}\left(x_n\right)} = -i\left\langle T\phi\left(x_1\right)...\phi\left(x_n\right)\right\rangle_{CC} \tag{52}$$

где индекс CC означает, что этот коррелятор сильно связан и его ноги ампутированы.
а) Доказать, что

$$\frac{\delta^2 \Gamma\left(\phi_{cl}\right)}{\delta\phi_{cl}(x)\delta\phi_{cl}(y)} = -[D(x,y)]^{-1}$$

гДе

$$-D(x,y) \equiv \frac{\delta^2 E(\eta)}{\delta\eta(x)\delta\eta(y)} = -i\left[\langle T\phi(x)\phi(y)\rangle - \phi_{cl}(x)\phi_{cl}(y)\right] \equiv -i\langle T\phi(x)\phi(y)\rangle_{\text{свзн}}$$

Индекс "связн"означает, что коррелятор является связным. Если вакуум трансляционно инвариантен, то согласно (13 Л5) и (22 Л5)

$$D(x,y) = \int \frac{\mathrm{d}^{(4)}k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{-k^2 + m^2 + \Sigma\left(k^2\right)}, \quad -D^{-1}(k) = k^2 - m^2 - \Sigma\left(k^2\right)$$

где $\Sigma\left(k^2\right)$ - сумма сильно связных диаграмм с двумя концами.
б) Доказать формулу (52) для $n = 3$.

32.5 K-l.VIII: Anomalies

The gauge invariance leads to numerous relations between various operators and their vacuum averages, i.e., the Green functions. We have already come across such relations called the Ward or the Slavnov-Taylor identities. They are the consequences of the gauge symmetry of the classical theory. In case when one has divergences in a theory and is bound to use some regularization, the validity of these identities depends on invariance of the regularization. However, one can always perform the subtraction of divergences in such a way that the finite parts obey these relations.

The exception from this rule is the so-called *anomalies*. By anomalies one usually means the violation in quantum theory of some relation, for instance, the conservation of the current or the Ward identity following from the symmetry properties of a classical theory. The well-known examples of quantum anomalies is the anomaly of the trace of the energy-momentum tensor or the axial anomaly. The characteristic feature of the anomaly is the impossibility of its removing by the redefinition of any quantities or parameters.

32.5.1 The axial anomaly

Consider quantum electrodynamics. Let us define the vector and the axial vector currents

$$j_\mu = \bar{\psi}\gamma^\mu\psi, \quad j^5_\mu = \bar{\psi}\gamma^\mu\gamma^5\psi. \tag{32.1}$$

In classical theory the equations of motion lead to the conservation or partial conservation of the current

$$\partial_\mu j_\mu = 0, \quad \partial_\mu j^5_\mu = 2imj^5, \tag{32.2}$$

where $j^5 = \bar{\psi}\gamma^5\psi$.

On the other hand, as a consequence of the gauge invariance, the vector and the axial vertices obey the Ward identities

$$(p-p')^\mu\Gamma_\mu(p,p') = S^{-1}(p) - S^{-1}(p'), \tag{32.3}$$

$$(p-p')^\mu\Gamma^5_\mu(p,p') = S^{-1}(p)\gamma^5 + \gamma^5 S^{-1}(p') + 2m\Gamma^5(p,p'), \tag{32.4}$$

where Γ_μ, Γ^5_μ and Γ^5 are the vector, axial and pseudoscalar vertices, respectively, and S is the fermion propagator.

If one looks how the identities (32.3,32.4) are fulfilled in perturbation theory, one first of all has to introduce some regularization due to the presence of the ultraviolet divergences. If the regularization is gauge invariant, then the vector Ward identity is satisfied in any order of PT. For the axial identity there are two types of diagrams: in the first one the axial current is in the outgoing fermion line, and in the second one the axial current is in the internal loop (see Fig.32.1). For the first type of a diagram the identity (32.4) is satisfied, and for the second type

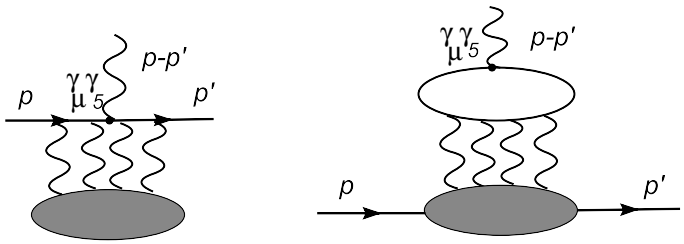


Figure 32.1: The diagrams with the axial current in external and internal fermion lines

there exists one famous triangle diagram (see Fig.32.2) where it is violated due to the ultraviolet divergence of the integral.

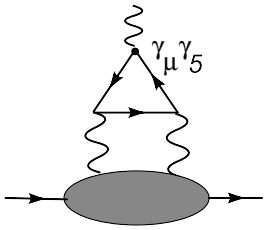
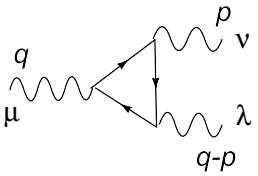


Figure 32.2: The anomalous triangle diagram for the axial current

Indeed, the corresponding integral in momentum space looks like



$$= (-)(-ie)^2 \int \frac{d^4k}{(2\pi)^4} Tr\left[\gamma^\mu\gamma^5 \frac{i\hat{k}}{k^2}\gamma^\nu \frac{i(\hat{k}+\hat{p})}{(p+k)^2}\gamma^\lambda \frac{i(\hat{k}+\hat{q})}{(q+k)^2}\right] \tag{32.5}$$

and is formally divergent requiring the regularization.

To preserve the conservation of the gauge invariance, it is useful to introduce the dimensional regularization; however, here we for the first time face a problem since the γ^5 matrix has no natural and consistent continuation to non-integer dimension. Two properties of the γ^5 matrix, namely, the anticommutation with all γ^μ, $\mu = 0, 1, 2, 3$ and the property of the trace $Tr(\gamma^5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\epsilon^{\mu\nu\rho\sigma}$ are in contradiction if the dimension is noninteger. To calculate the axial anomaly, we use the following trick: we use the formula for the trace but reject the property of anticommutativity of γ^5. This allows one to perform al the calculations in a consistent and unambiguous way.

The divergence of the axial current can be obtained by multiplication of (32.5) by iq^μ which gives

$$e^2 \int \frac{d^4k}{(2\pi)^4} \frac{Tr\left[\hat{q}\gamma^5\hat{k}\gamma^\nu(\hat{k}+\hat{p})\gamma^\lambda(\hat{k}+\hat{q})\right]}{k^2(k+p)^2(k+q)^2} \tag{32.6}$$

Using the cyclic property of the trace we move $\hat{q}$ to the right and write it as $\hat{q} = (\hat{q}+\hat{k}) - \hat{k}$. Then the first term multiplied by $\hat{k}+\hat{q}$ gives $(k+q)^2$ and cancels with the denominator. As a result, one gets the integral

$$\int \frac{d^4k}{(2\pi)^4} \frac{Tr\left[\hat{q}\gamma^5\hat{k}\gamma^\nu(\hat{k}+\hat{p})\gamma^\lambda\right]}{k^2(k+p)^2},$$

which depends only on p and after the integration turns to zero due to the antisymmetry of the trace with the γ^5 matrix.

In the second term we will drag $\hat{k}$ to the left until it is multiplied by $\hat{k}$ giving k^2. As a result, at each step we always get the trace of four γ-matrices with γ^5 for which we have the formula with the ϵ-tensor. We obtain in the numerator

$$-4i\epsilon^{\alpha\nu\beta\lambda}k^\alpha(k+p)^\beta[(k+q)^2-q^2] + 8i\epsilon^{\alpha\nu\beta\rho}k^\alpha(k+p)^\beta q^\rho k^\lambda - 4i\epsilon^{\alpha\nu\lambda\rho}k^\alpha q^\rho[(k+p)^2-p^2]$$

$$-4i\epsilon^{\nu\alpha\lambda\rho}p^\alpha q^\rho k^2 + 8i\epsilon^{\alpha\beta\lambda\rho}k^\alpha p^\beta q^\rho k^\nu.$$

Despite the fact that the integral is formally divergent, using a dimensional regularization and collecting all terms together we finally get the finite answer equal to

$$-\frac{e^2}{4\pi^2}\epsilon^{\mu\nu\rho\lambda}p^\mu q^\rho = -\frac{e^2}{4\pi^2}\epsilon^{\mu\nu\rho\lambda}p^\mu(q-p)^\rho, \tag{32.7}$$

One has to add to this expression the same diagram but with the replacement $p \leftrightarrow q-p, \nu \leftrightarrow \lambda$ and take the sum, but the answer is already invariant with respect to this replacement. Multiplying (32.7) by $A_\nu(p)A_\lambda((q-p)$ and transforming to the coordinate representation, one gets

$$\partial_\mu j^5_\mu = \frac{e^2}{4\pi^2}\epsilon^{\mu\nu\rho\lambda}\partial_\mu A_\nu \partial_\rho A_\lambda = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda}. \tag{32.8}$$

As a result one has the following modification of equations for the divergence of the axial current and the axial vertex

$$\partial_\mu j^5_\mu = 2imj^5 + \frac{\alpha}{4\pi}F_{\mu\nu}F_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma}, \tag{32.9}$$

$$(p-p')^\mu\Gamma^5_\mu(p,p') = S^{-1}(p)\gamma^5 + \gamma^5 S^{-1}(p') + 2m\Gamma^5(p,p') - i\frac{\alpha}{4\pi}F(p,p'), \tag{32.10}$$

where $F(p,p')$ is the vertex with insertion of the operator $F\tilde{F}$. The appearance of the r.h.s in these equations is called *anomaly* known as the Adler-Bell-Jackiw or triangle anomaly.

The most essential here is not the violation of the Ward identity but the fact that subtracting the anomaly and restoring the "normal" Ward identity for the axial vertex we violate the conservation of the vector current. In other words, it is impossible to satisfy the conservation of axial and vector currents *simultaneously*.

Notice that the violation of the conservation of the axial current preserving the conservation of the vector current (32.9) can be obtained by accurately calculating the matrix element for the divergence of the axial current in x-space splitting the arguments of the field operators. Consider the vacuum average of the divergence of the axial current, and to avoid the singularity for the product of two operators at coinciding points, split the arguments. Then to preserve the gauge invariance, we have to insert between the operators the exponent of the Wilson line. The axial current then takes the form

$$j^5_\mu(x) = \lim_{\varepsilon\to 0}\{\bar\psi(x+\varepsilon/2)\gamma^\mu\gamma^5 \exp[-ie\int\limits_{x-\varepsilon/2}^{x+\varepsilon/2} dz^\nu A_\nu(z)]\psi(x-\varepsilon/2)\}, \tag{32.11}$$

and for the divergence we get

$$\begin{aligned}
\partial_\mu j^5_\mu(x) &= \lim_{\varepsilon\to 0}\{\partial_\mu\bar\psi(x+\varepsilon/2)\gamma^\mu\gamma^5 \exp[-ie\int\limits_{x-\varepsilon/2}^{x+\varepsilon/2} dz^\nu A_\nu(z)]\psi(x-\varepsilon/2) \\
&+ \bar\psi(x+\varepsilon/2)\gamma^\mu\gamma^5 \exp[-ie\int\limits_{x-\varepsilon/2}^{x+\varepsilon/2} dz^\nu A_\nu(z)]\partial_\mu\psi(x-\varepsilon/2) \\
&+ \bar\psi(x+\varepsilon/2)\gamma^\mu\gamma^5[-ie\varepsilon^\nu\partial_\mu A_\nu(x)] \exp[-ie\int\limits_{x-\varepsilon/2}^{x+\varepsilon/2} dz^\nu A_\nu(z)]\psi(x-\varepsilon/2)\}.
\end{aligned} \tag{32.12}$$

Using the equations of motion

$$\gamma^\mu\partial_\mu\psi = -ie\hat{A}\psi, \quad \partial_\mu\bar\psi\gamma^\mu = ie\bar\psi\hat{A}$$

and keeping the terms of the order of ε we find

$$\begin{aligned}
\partial_\mu j^5_\mu(x) &= \lim_{\varepsilon\to 0}\{\partial_\mu\bar\psi(x+\varepsilon/2)[-ie\hat{A}(x+\varepsilon/2) - ie\hat{A}(x-\varepsilon/2) \\
&\qquad -ie\varepsilon^\nu\gamma^\mu\partial_\mu A_\nu(x)]\gamma^5\psi(x-\varepsilon/2)\} \\
&= \lim_{\varepsilon\to 0}\{\bar\psi(x+\varepsilon/2)[-ie\varepsilon^\nu\gamma^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu)]\gamma^5\psi(x-\varepsilon/2)\}
\end{aligned} \tag{32.13}$$

Now we have to calculate the vacuum average over the fermion vacuum (the photon field is assumed to be external) which means that we have to permute the fermion operators. The permutation function of the fermion operators is singular and this is the reason for appearance of a nonzero term similarly to the appearance of triangle anomaly due to divergency of the integral. Indeed, calculating the propagator of the fermion in external field and keeping the terms linear in the photon field, we get

$$S(y-z) = \int\frac{d^4k}{(2\pi)^4}e^{ik(y-z)}\frac{i\hat{k}}{k^2} + \int\frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}e^{i(k+p)y}e^{-ikz}\frac{i(\hat{k}+\hat{p})}{(k+p)^2}(-ie\hat{A}(p)\frac{i\hat{k}}{k^2} + ... \tag{32.14}$$

The propagator (32.14) is singular as $y \to z$; however, the first term does not give a contribution to the divergence, while the second one leads to

$$\begin{aligned}
&\langle\bar\psi(x+\varepsilon/2)\gamma^\mu\gamma^5\psi(x-\varepsilon/2)\rangle = \\
&= \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}e^{ipx}e^{-ik\varepsilon}Tr[\frac{i(\hat k+\hat p)}{(k+p)^2}(-ie\hat A(p))\frac{i\hat k}{k^2}\gamma^\mu\gamma^5] \\
&= \int \frac{d^4k}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}e^{ipx}e^{-ik\varepsilon}\frac{4e\epsilon^{\mu\nu\rho\sigma}(k+p)_\nu A_\rho(p)k_\sigma}{(k+p)^2k^2}.
\end{aligned} \tag{32.15}$$

To find the limit as $\varepsilon \to 0$, one can expand the integrand for large k, which gives

$$\begin{aligned}
\langle\bar\psi(x+\varepsilon/2)\gamma^\mu\gamma^5\psi(x-\varepsilon/2)\rangle &= 4e\epsilon^{\mu\nu\rho\sigma}\int\frac{d^4p}{(2\pi)^4}e^{ipx}p_\nu A_\rho(p)\int\frac{d^4k}{(2\pi)^4}e^{-ik\varepsilon}\frac{k_\sigma}{k^4} \\
&= -4e\epsilon^{\mu\nu\rho\sigma}i\partial_\nu A_\rho(x)\frac{2\varepsilon_\sigma}{16\pi^2\varepsilon^2} = -e\epsilon^{\mu\nu\rho\sigma}iF_{\nu\rho}(x)\frac{\varepsilon_\sigma}{4\pi^2\varepsilon^2},
\end{aligned} \tag{32.16}$$

Substituting this expression into (32.13) we find

$$\partial_\mu j^5_\mu = \lim_{\varepsilon\to0}\{-e\epsilon^{\mu\nu\rho\sigma}iF_{\nu\rho}(x)\frac{\varepsilon_\sigma}{4\pi^2\varepsilon^2}(-ie\varepsilon^\tau F_{\mu\tau})\} = \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\nu\rho}F_{\sigma\mu}, \tag{32.17}$$

that coincides with (32.9).

The axial anomaly has one very important property: the obtained formulas (32.9) and (32.10) are *exact* in all orders of perturbation theory, i.e., have no radiative corrections. More rigorous statement is: there exists such a renormalization scheme (and it was constructed explicitly) that the radiative corrections to the axial anomaly are absent. This statement is the subject of the Adler-Bardeen theorem. Graphically, this means the cancellation of the contributions of the diagrams shown in Fig.32.3, which was checked by explicit calculation.

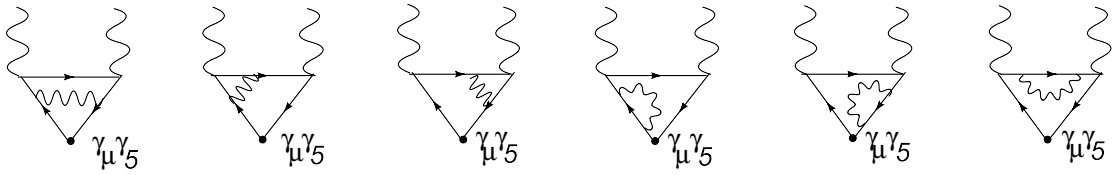


Figure 32.3: Cancellation of radiative corrections to the axial anomaly

The Adler-Bardeen theorem is valid also in non-Abelian theories. It has important consequences: if the anomaly is compensated in the lowest order, it will not appear further.

32.5.2 Consequences of the axial anomaly

Let us ask the question what are the consequences of the axial anomaly? Here one has to distinguish two cases: when the operator of the axial current is an external operator with respect to the Lagrangian and when it is present in the interaction Lagrangian.

In the first case, the presence of anomaly does not lead to any troubles and even may be useful. Thus, for instance, in the current algebra which describes the low energy hadron interactions, the axial anomaly is responsible for the neutral pion decay $\pi^0 \to 2\gamma$ and is in agreement with the experiment.

In the second case, the triangle anomaly leads to that the ultraviolet renormalizations of the vector vertex do not remove all divergences from the axial vertex. This has destructive consequences for the renormalizability of the whole theory. To see this, compare the two processes of

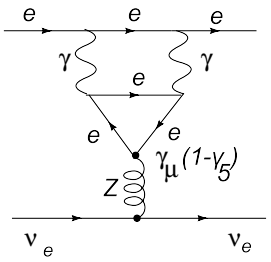


Figure 32.4: The anomaly in the process of lepton scattering in the Standard Model

the elastic scattering of leptons: $\nu_e + e \to \nu_e + e$ and $\nu_\mu + e \to \nu_e + \mu$ in the Standard Model. Graphically, in the lowest order they differ by one diagram containing the triangle anomaly (See Fig.32.4).

As a result, after the renormalization the amplitude of $\nu_\mu e$-scattering has finite radiative corrections, while that of ν_e-scattering is divergent. This led to nonrenormalizability of the theory and was a serious problem for the left-right nonsymmetric model with $SU_L(2) \times U(1)$ symmetry before the introduction of the

$$-quark. Remarkably, the$$

-quark introduced by Glashow, Iliopoulos and Maiani for suppression of the neutral current changing strangeness leads to the compensation of the contributions of quarks and leptons to triangle anomaly and restores the renormalizability of the theory.

In the Standard Model due to its left-right asymmetry the presence of the axial currents for quarks and leptons leads to several kinds of triangle anomalies where all three gauge fields may be in the vertices of the triangle. However, not all of them lead to anomalies. In general, the anomaly is proportional to the trace

$$Tr\ T^a\{T^b, T^c\},$$

where the matrix T^a is the generator of the corresponding gauge group in the representation corresponding to the fields that run inside the triangle. The necessary condition of the existence of anomaly is the presence of the complex representations and the nontrivial anticommutator of the generators of the group. Among the simple Lie groups which satisfy this requirement, only the groups $SU(n)$, $SO(4n+2)$ and E_6 have complex representations and out of them only the $SU(n), n > 2$ and $SO(6)$ groups have a symmetric invariant needed for the construction of the anomaly. The gauge theories built on other groups are free from anomalies.

The non-vanishing anomalies corresponding to the symmetry group of the Standard Model $SU_c(3) \times SU_L(2) \times U_Y(1)$ are presented in Fig.32.5 where the gauge fields adjusted to the groups $U(1)$ and $SU(2)$ are shown prior to mixing. The particles that run over the triangle can be either left or right quarks and leptons. Particles of different helicity give the opposite sign contribution to the axial anomaly.

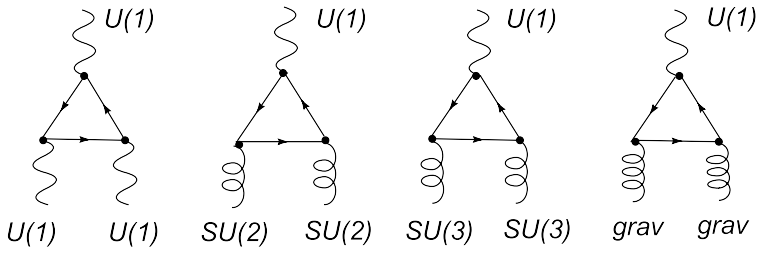


Figure 32.5: The triangle anomaly in the Standard Model

In the first case, the anomaly is proportional to the trace of the cube of hypercharge $TrY^3 = TrY_L^3 - TrY_R^3$ and its absence is achieved by the cancellation of the contributions of quarks and

leptons in each generation

$$\begin{aligned} TrY^3 = \; = \; & 3\left[(\frac{1}{3})^3+(\frac{1}{3})^3-(\frac{4}{3})^3-(-\frac{2}{3})^3\right]+(-1)^3+(-1)^3-(-2)^3=0. \\ & \uparrow \quad \uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow \\ & colour \;\; u_L \quad d_L \quad u_R \quad d_R \quad \nu_L \quad e_L \quad e_R. \end{aligned} \tag{32.18}$$

In further diagrams the anomaly is proportional to, respectively,

$$\begin{aligned} TrY_L &= 3\left(\frac{1}{3}+\frac{1}{3}\right)-1-1=0, \\ TrY_q &= 3\left(\frac{1}{3}+\frac{1}{3}-\frac{4}{3}-(-\frac{2}{3})\right)=0, \\ TrY &= 3\left(\frac{1}{3}+\frac{1}{3}-\frac{4}{3}-(-\frac{2}{3})\right)-1-1-(-2)=0. \end{aligned} \tag{32.19}$$

This way the anomaly is miraculously canceled in all the cases and does not break the renormalizability of the SM.

32.5.3 The conformal anomaly

Another example of quantum anomaly is the conformal anomaly or the anomaly of the trace of the energy-momentum tensor. The requirement of conformal (scale) invariance means the invariance of the action with respect to the transformation

$$x_\mu \to x_\mu e^{-\sigma}, \quad \phi(xe^{-\sigma}) \to e^{\Delta\sigma}\phi(x), \tag{32.20}$$

where Δ is the dimension of a field. This condition is fulfilled in the classical Lagrangian if it has no dimensional parameters. In this case, according to the Noether theorem, there exists a conserved current called the dilatation current $D^\mu = \Theta^{\mu\nu}x_\nu$, so that

$$\partial_\mu D^\mu = \Theta^\mu_\mu,$$

where Θ^μ_ν is the symmetric energy-momentum tensor.

The easiest way to see it is to define the energy-momentum tensor as a variation of the action of the matter fields with respect to the space-time metric in the external gravitational filed

$$\Theta^{\mu\nu} = 2\frac{\delta}{\delta g_{\mu\nu}}\int d^4x\ \mathcal{L}(x). \tag{32.21}$$

The scale transformation can be realized as a variation of the metric

$$g_{\mu\nu}(x) \to e^{2\sigma}g_{\mu\nu}(x). \tag{32.22}$$

This means that the variation of the Lagrangian under this transformation is the trace of $\Theta^{\mu\nu}$. The deviation of the trace of the energy-momentum tensor from zero indicates the violation of the scale (and hence conformal) invariance.

In the quantum case, due the presence of the ultraviolet divergences the new scale appears. This is the same phenomenon of dimensional transmutation discussed above. Therefore, the scale invariance of the action is violated.

Since the coupling constant becomes scale dependent, its variation with the scale (32.20) takes the form

$$\delta g = \sigma\mu\frac{dg}{d\mu} = \sigma\beta(g). \tag{32.23}$$

Hence, for the variation of the Lagrangian we get

$$\delta\ \mathcal{L} = \sigma \frac{\delta \mathcal{L}}{\delta g_i} \beta_i(\{g\}), \tag{32.24}$$

i.e.,

$$\partial_\mu D^\mu = \Theta^\mu_\mu = \frac{\delta \mathcal{L}}{\delta g_i} \beta_i(\{g\}). \tag{32.25}$$

This relation is known as *the trace anomaly* of the energy-momentum tensor.

Similarly to the axial anomaly, relation (32.25) can be checked by perturbation theory. However, in this case the result is defined by the full β-function calculated in all orders of PT.

32.6 K-l.IX: Infrared Divergences

One more problem that we encounter on the way of calculating the finite expressions for the probabilities of physical processes is the presence of the so-called *infrared divergences*. They appear when calculating the matrix elements of the scattering matrix on shell, i.e., when the squares of external momenta are equal to the corresponding masses squared and the theory contains massless particles like photons or gluons. The infrared divergences can be of two types: the divergences for small values of momenta (the genuine infrared divergences) and the divergences at parallel momenta (the collinear divergences). Contrary to the ultraviolet divergences, the infrared divergences have a clear physical meaning: a massless particle with a very small momentum can not be registered and with momentum parallel to another particle cannot be distinguished. For this reason in the theories with massless particles one has to define the physical process to be evaluated in a proper way.

32.6.1 The double logarithmic asymptotics

For illustration consider the process of creation of a muon pair in the e^+e^- annihilation. The leading diagrams for this process are shown in Fig.32.6.

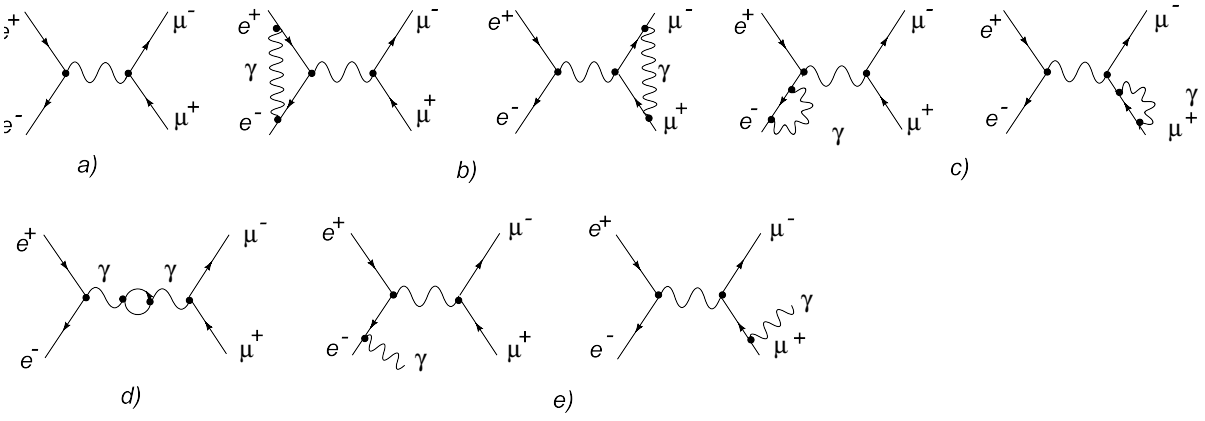


Figure 32.6: The diagrams contributing to the process $e^+e^- \to \mu^+\mu^-$ in QED: a) the leading order, b)- d) the virtual corrections of the order of α, e) the real corrections of the order of α.

The first diagram is the tree amplitude, it gives the contribution in the leading order. The radiative corrections due to emission of virtual photons (Fig.32.6 b)) are the corrections to the vertex function considered above (see (18.40)). It is easy to see that if one puts in this formula all fermion momenta on mass shell, i.e. $p^2 = (p-q)^2 = m^2$, then in the second integral in the denominator one gets $[-m^2x^2 + q^2y(x-y)]$. Performing the change of variables $y \to yx$ so that

all the integrations are performed within the limits [0,1], we get $[-m^2x^2 + q^2x^2y(1-y)]$, and the integral (with account of the Jacobian = x) is logarithmically divergent as $x \to 0$.

The appeared divergence has the infrared nature. Like the ultraviolet one it can be regularized, for instance, by introducing the nonzero photon mass or cutting the integral over momenta at the lower limit, or with the help of dimensional regularization but it cannot be removed by any renormalization.

Let us calculate this diagram on mass shell introducing the nonzero photon mass m_{ph} into the virtual photon line. This will not break the gauge invariance since, as it will be clear later, after the cancellation of the IR divergences one can put the mass of a photon equal to zero.

Let us go back to eq.(18.40), remove the UV divergence by the minimal subtraction and go to the mass shell for the fermion fields taking into account that the external fermion operators obey the Dirac equation $(\hat{p} - m)u(p) = 0$ and $\bar{u}(p-q)(\hat{p} - \hat{q} - m) = 0$. Then after some exercise we obtain for the vertex function the following expression:

$$\Gamma_1^R(p,q) = ie\left[F_1(q^2)\gamma^\mu + iF_2(q^2)\frac{\sigma^{\mu\nu}q^\nu}{2m}\right], \qquad \sigma^{\mu\nu} \equiv i\frac{\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu}{2}, \tag{32.26}$$

where the form-factors $F_i(q^2)$ have the form

$$\begin{aligned} F_1(q^2) &= \frac{e^2}{16\pi^2}\left[-2 - 2\int\limits_0^1 dx\int\limits_0^1 dy\ x\ \log\left(\frac{-m^2x^2 + q^2x^2y(1-y)}{-\mu^2}\right)\right. \\ &+ \left.\int\limits_0^1 dx\int\limits_0^1 dy\ x\ \frac{2m^2(2-2x-x^2) - 2q^2(1-xy)(1-x+xy)}{-m^2x^2 + q^2x^2y(1-y) - m_{ph}^2(1-x)}\right], \end{aligned} \tag{32.27}$$

$$F_2(q^2) = \frac{e^2}{16\pi^2}\left[\int\limits_0^1 dx\int\limits_0^1 dy\ x\ \frac{-4m^2x(1-x)}{-m^2x^2 + q^2x^2y(1-y) - m_{ph}^2(1-x)}\right]. \tag{32.28}$$

The form factor F_2 is IR convergent and does not need any regularization. Substituting $m_{ph} = 0$, we get

$$F_2(q^2) = \frac{\alpha}{4\pi}\int\limits_0^1 dy\frac{2m^2}{m^2 - q^2y(1-y)}. \tag{32.29}$$

For $q^2 = 0$ it can be easily calculated and equals

$$F_2(q^2 = 0) = \frac{\alpha}{2\pi}, \tag{32.30}$$

which is nothing else but the first correction to the g-factor, which is called the anomalous magnetic moment of electron (muon).

As for the form factor F_1, it is IR divergent. We calculate its divergent part in the limit $m_{ph} \to 0$. It comes only from the second integral in (32.27). To simplify the integration, we notice that the divergence is defined by the region of the parameter $x \sim 0$. Therefore, we put $x = 0$ everywhere in the numerator and in the coefficient of m_{ph} in the denominator. Then one gets

$$F_1(q^2) \simeq \frac{e^2}{16\pi^2}\int\limits_0^1 dy\int\limits_0^1 xdx\frac{2(2m^2 - q^2)}{[-m^2 + q^2y(1-y)]x^2 - m_{ph}^2}. \tag{32.31}$$

The integral over x is now easily evaluated

$$F_1(q^2) \simeq \frac{\alpha}{4\pi}\int\limits_0^1 dy\frac{2m^2 - q^2}{[-m^2 + q^2y(1-y)]}\log\left(\frac{-m^2 + q^2y(1-y) - m_{ph}^2}{-m_{ph}^2}\right). \tag{32.32}$$

The remaining integral over y is also simple. We calculate it in the limit $-q^2 \to \infty$. Then it takes the form

$$F_1(q^2) \simeq -\frac{\alpha}{4\pi}\int\limits_0^1 dy \frac{q^2}{[-m^2+q^2y(1-y)]}\log\left(\frac{-q^2}{m_{ph}^2}\right) \simeq -\frac{\alpha}{2\pi}\log\left(\frac{-q^2}{m^2}\right)\log\left(\frac{-q^2}{m_{ph}^2}\right). \qquad (32.33)$$

The obtained double logarithmic behaviour of the form-factor is called *the Sudakov* double logarithm. It contains the infrared cutoff in the form of the photon mass. In the amplitude of creation of the muon pair there are two of such form factors for the electron and the muon vertices, respectively. The corrections to the fermion and the photon propagators do not contain the IR divergences. Thus, the cross-section of the process $e^+e^- \to \mu^+\mu^-$ is logarithmically divergent. In order to understand the reason of appearance of the IR divergence and to find the method of its elimination, consider the process of creation of the muon pair from the point of view of an observer.

32.6.2 The soft photon emission

During the process of electron-positron annihilation the muon pair is created with momenta that satisfy the conservation law and can be measured. However, they are registered with some accuracy, and momentum smaller than some value which depends on a particular detector is not registered. Therefore, if besides the muon pair the photon with momentum smaller than this value is created, then this process with emission of the "soft" γ-quantum $e^+e^- \to \mu^+\mu^-\gamma$ is experimentally indistinguishable from the initial process $e^+e^- \to \mu^+\mu^-$. The diagrams corresponding to the process $e^+e^- \to \mu^+\mu^-\gamma$ are shown in Fig.32.6 e). They contain an additional vertex and hence additional coupling, but being squared give a correction to the main process of the order of α, exactly as the radiative corrections due to the virtual photon.

Let us compare the differential cross-sections of the precess $e^+e^- \to \mu^+\mu^-$ in the one-loop approximation and $e^+e^- \to \mu^+\mu^-\gamma$ in the tree approximation. We have, respectively,

$$\frac{d\sigma}{d\Omega}(e^+e^- \to \mu^+\mu^-) = \left(\frac{d\sigma}{d\Omega}\right)_0\left[1-\frac{\alpha}{\pi}\log\left(\frac{-q^2}{m_{e,\mu}^2}\right)\log\left(\frac{-q^2}{m_{ph}^2}\right)+...+\mathcal{O}(\alpha^2)\right] \qquad (32.34)$$

$$\frac{d\sigma}{d\Omega}(e^+e^- \to \mu^+\mu^-\gamma) = \left(\frac{d\sigma}{d\Omega}\right)_0\left[+\frac{\alpha}{\pi}\log\left(\frac{-q^2}{m_{e,\mu}^2}\right)\log\left(\frac{-q^2}{m_{ph}^2}\right)+...+\mathcal{O}(\alpha^2)\right] \qquad (32.35)$$

where the second cross-section is written down without derivation which we will perform later. As follows from eqs.(32.34,32.35), each of these cross-sections is IR divergent, but in the sum the divergences cancel and one gets the finite answer.

What is observable after all? In fact, neither the first nor the second process is observable separately. In a real detector with limited sensitivity one observes the process of creation of the muon pair plus an arbitrary number of soft photons with the total energy below the sensitivity threshold. In a given order of perturbation theory we have to sum the cross-sections of the two processes in order to get the observed cross-section

$$\left(\frac{d\sigma}{d\Omega}\right)_{observable} = \left(\frac{d\sigma}{d\Omega}\right)(e^+e^- \to \mu^+\mu^-) + \left(\frac{d\sigma}{d\Omega}\right)(e^+e^- \to \mu^+\mu^-\gamma, E < E_{min}). \qquad (32.36)$$

The latter cross-section is given by the same formula (32.35) with the replacement in the second logarithm of the photon energy by E_{min}. Thus, we get

$$\left(\frac{d\sigma}{d\Omega}\right)_{observable} = \left(\frac{d\sigma}{d\Omega}\right)_0\left[1-\frac{\alpha}{\pi}\log\left(\frac{-q^2}{m_{e,\mu}^2}\right)\log\left(\frac{-q^2}{E_{min}^2}\right)+...+\mathcal{O}(\alpha^2)\right]. \qquad (32.37)$$

As one can see, for the proper statement of the problem the cross-section of the observable process is finite and does not depend on the IR regulator. At the same time, it depends on the

sensitivity of the detector E_{min} and for improved sensitivity tends to infinity. However, this infinity also is not physical and is the artefact of perturbation theory: when the logarithm becomes large we go beyond the scope of applicability of perturbation theory and it is necessary to perform the summation of these corrections by analogy with what happens with the ultraviolet logarithms within the renormalization group method.

Thus, the IR divergences appear due to the contributions of the photons with "soft" momenta: real with the energy smaller than E_{min} and virtual with momenta $k^2 < E^2_{min}$. What is important is that the momenta of fermions are on mass shell, otherwise the singularities in the propagator do not arise. The typical diagram of higher order contains a big amount of real and virtual photon lines (see Fig.32.7).

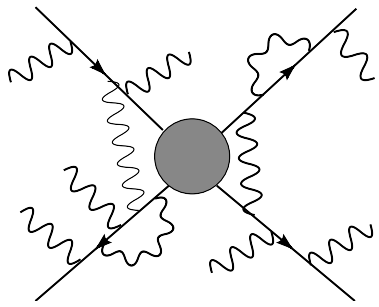


Figure 32.7: The hard process with creation of the soft photons

Let us try to sum up the contributions of these soft photons. Consider first the external fermion line with the outgoing photons (real and virtual).

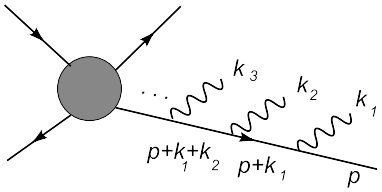


Figure 32.8: The emission of the soft photons from the fermion line

It corresponds to the following expression:

$$\bar{u}(p)\ (-ie\gamma^{\mu_1})\ \frac{i(\hat{p}+\hat{k}_1+m)}{2pk_1}\ (-ie\gamma^{\mu_2})\ \frac{i(\hat{p}+\hat{k}_1+\hat{k}_2+m)}{2p(k_1+k_2)+O(k^2)}\ \cdots \tag{32.38}$$
$$\cdots\ (-ie\gamma^{\mu_n})\ \frac{i(\hat{p}+\hat{k}_1+\cdots+\hat{k}_n+m)}{2p(k_1+\cdots k_n)+O(k^2)}\ \ iM_{hard}.$$

We use now the fact that the operator $\bar{u}(p)$ obeys the Dirac equation $\bar{u}(p)(\hat{p}-m)=0$ and omit the momenta $k_i \ll p$ in the numerator. Then we get

$$\bar{u}(p)\gamma^{\mu_1}(\hat{p}+m)\gamma^{\mu_2}(\hat{p}+m)\cdots = \bar{u}(p)2p^{\mu_1}\gamma^{\mu_2}(\hat{p}+m)\cdots = \bar{u}(p)2p^{\mu_1}2p^{\mu_2}\cdots . \tag{32.39}$$

Hence, eq.(32.38) takes the form

$$\bar{u}(p)\ (e\frac{p^{\mu_1}}{pk_1})(e\frac{p^{\mu_2}}{p(k_1+k_2)})\cdots(e\frac{p^{\mu_n}}{p(k_1+\cdots+k_n)}). \tag{32.40}$$

The next step is the summation over all the permutations of the photon lines and the permutations of momenta k_i. (So far we have not distinguished between the real and virtual photons, we will

do it later.) This operation is non-trivial but leads to the simple result. One has

$$\sum_{permutations} \frac{1}{pk_1}\,\frac{1}{p(k_1+k_2)}\,\cdots\,\frac{1}{p(k_1+k_2+\cdots+k_n)} = \frac{1}{pk_1}\,\frac{1}{pk_2}\,\cdots\,\frac{1}{pk_n}. \tag{32.41}$$

The same procedure can be applied to the incoming fermion line. The difference is that the fermion momentum has the opposite direction which leads to the replacement of $(p+k_i)^2$ to $(p-k_i)^2$ in the propagator, i.e., the change of the sign $p \to -p$ in the denominator. Collecting both factors together we get the following expression for the amplitude of emission of soft photons from arbitrary points of the incoming and the outgoing line (Fig.32.9):

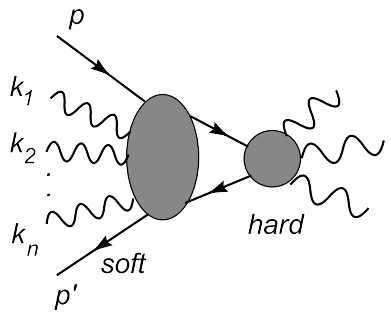


Figure 32.9: The emission of soft photons from arbitrary points of the incoming and the outgoing lines

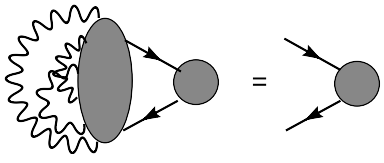
$$\mathcal{M} = \bar{u}(p')\; i\mathcal{M}_{hard}\; u(p)\; e\left(\frac{p'^{\mu_1}}{p'k_1} - \frac{p^{\mu_1}}{pk_1}\right) e\left(\frac{p'^{\mu_2}}{p'k_2} - \frac{p^{\mu_2}}{pk_2}\right)\cdots e\left(\frac{p'^{\mu_n}}{p'k_n} - \frac{p^{\mu_n}}{pk_n}\right). \tag{32.42}$$

Now we have to decide which photons are real and which are virtual. The virtual photon can be obtained by joining the two photon momenta k_i and k_j, taking $k_i = -k_j = k$, multiplying by the photon propagator and integrating over k. In this way for any virtual photon we get the expression:

$$\frac{e^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{-i}{k^2}\left(\frac{p'}{p'k} - \frac{p}{pk}\right)\left(\frac{p'}{-p'k} - \frac{p}{-pk}\right), \tag{32.43}$$

where the factor $1/2$ compensates the double counting due to permutation of k_i and k_j. The obtained integral is nothing else but the vertex function in the one-loop approximation, i.e., the form factor $F_1(q^2)$.

If the number of virtual photons equals n, one gets the product of n expressions like (32.43) and the factor $1/n!$ taking into account the permutations which do not change the result. The full answer is obtained with the help of summation over the soft virtual photons, which gives



$$\times \sum_{n=0}^{\infty} \frac{F_1^n}{n!} = \bar{u}(p')\; i\mathcal{M}_{hard}\; u(p)\exp(F_1). \tag{32.44}$$

At the same time, if the real photon is emitted, then instead of the propagator one has to multiply the amplitude by the polarization operator, sum up over all polarizations and integrate the square of the matrix element over the photon phase space. In this case, one gets the following expression:

$$I(q^2) = e^2\int \frac{d^3k}{(2\pi)^3}\frac{-g^{\mu\nu}}{2|k|}\left(\frac{p'^{\mu}}{p'k} - \frac{p^{\mu}}{pk}\right)\left(\frac{p'^{\nu}}{p'k} - \frac{p^{\nu}}{pk}\right), \tag{32.45}$$

which is the element of the cross-section of the process $e^+e^- \to \mu^+\mu^-\gamma$. The integration over the modulus of the three-vector $\vec{k}$ has to be performed within the limits (m_{ph}, E_{min}). Contracting the indices one gets

$$I(q^2) = -\frac{e^2}{(2\pi)^3}\int \frac{d^3k}{2|k|}\left(\frac{p'^2}{(p'k)^2} - 2\frac{p'p}{(pk)(p'k)} + \frac{p^2}{(pk)^2}\right). \tag{32.46}$$

The first and the last integrals are equal to each other. Let us consider the last one and choose the frame where $\vec{p} = 0$. This gives

$$I_1 = -\frac{e^2}{(2\pi)^3}4\pi \int\limits_{m_{ph}}^{E_{min}} \frac{k^2dk}{2k}\frac{m^2}{(mk)^2} = -\frac{\alpha}{2\pi}\log(\frac{E^2_{min}}{m^2_{ph}}). \tag{32.47}$$

As for the second integral, we proceed in the following way: first we also choose the frame $\vec{p} = 0$, and then we covariantize the answer. One has

$$\begin{aligned} I_2 &= \frac{e^2}{(2\pi)^3}2\pi \int\limits_{m_{ph}}^{E_{min}} \frac{k^2dk}{k}\int\limits_{-1}^{1} d\cos\theta \frac{m\sqrt{\vec{p'}^2+m^2}}{(mk)(\sqrt{\vec{p'}^2+m^2}k - |\vec{p'}|k\cos\theta)} \\ &= \frac{\alpha}{2\pi}\log(\frac{E^2_{min}}{m^2_{ph}})\frac{\sqrt{\vec{p'}^2+m^2}}{|\vec{p'}|}\log\left(\frac{\sqrt{\vec{p'}^2+m^2}-|\vec{p'}|}{\sqrt{\vec{p'}^2+m^2}+|\vec{p'}|}\right). \end{aligned} \tag{32.48}$$

Covariantizing this answer and having in mind that $q = p - p'$, $p^2 = p'^2 = m^2$ and, hence, $q^2 = 2m^2 - 2m\sqrt{\vec{p'}^2+m^2}$ one gets

$$I_2(q^2) = \frac{\alpha}{2\pi}\log(\frac{E^2_{min}}{m^2_{ph}})\frac{2m^2-q^2}{\sqrt{-q^2(4m^2-q^2)}}\log\left(\frac{2m^2-q^2-\sqrt{-q^2(4m^2-q^2}}{2m^2-q^2+\sqrt{-q^2(4m^2-q^2}}\right). \tag{32.49}$$

Thus,

$$I(q^2) = \frac{\alpha}{2\pi}\log(\frac{E^2_{min}}{m^2_{ph}})\left[\frac{2m^2-q^2}{\sqrt{-q^2(4m^2-q^2)}}\log\left(\frac{2m^2-q^2-\sqrt{-q^2(4m^2-q^2}}{2m^2-q^2+\sqrt{-q^2(4m^2-q^2}}\right)-2\right]. \tag{32.50}$$

In the limit $-q^2 \to \infty$ we get the desired answer

$$I(q^2) \to \frac{\alpha}{\pi}\log(\frac{E^2_{min}}{m^2_{ph}})\log(\frac{-q^2}{m^2}), \tag{32.51}$$

coinciding with (32.35).

If there are n real photons, there are n such contributions and the symmetry factor $1/n!$ taking into account the identity of the final particles. The cross-section of the process with emission of an arbitrary number of photons with the energy smaller than E_{min} hence equals

$$\sum_{n=0}^{\infty}\frac{d\sigma}{d\Omega}(e^+e^-\to\mu^+\mu^-+n\gamma) = \frac{d\sigma}{d\Omega}(e^+e^-\to\mu^+\mu^-)\times\sum_{n=0}^{\infty}\frac{I^n}{n!} = \frac{d\sigma}{d\Omega}(e^+e^-\to\mu^+\mu^-)e^I. \tag{32.52}$$

Combining the results for the real and virtual photons one gets the final expression for the observable cross-section with emission of an arbitrary number of photons with the energy smaller

than E_{min}

$$\left(\frac{d\sigma}{d\Omega}\right)_{observable} = \left(\frac{d\sigma}{d\Omega}\right)_0 \times \exp\left(2F_1\right) \times \exp\left(I\right)$$
$$= \left(\frac{d\sigma}{d\Omega}\right)_0 \exp\left[-\frac{\alpha}{\pi}\log(\frac{-q^2}{m_{e,\mu}^2})\log(\frac{-q^2}{m_{ph}^2})\right] \exp\left[\frac{\alpha}{\pi}\log(\frac{-q^2}{m_{e,\mu}^2})\log(\frac{E_{min}^2}{m_{ph}^2})\right]$$
$$= \left(\frac{d\sigma}{d\Omega}\right)_0 \exp\left[-\frac{\alpha}{\pi}\log(\frac{-q^2}{m_{e,\mu}^2})\log(\frac{-q^2}{E_{min}^2})\right]. \quad (32.53)$$

The obtained expression is valid in all orders of perturbation theory. The exponential factor does not depend on the IR cutoff but on the sensitivity of the detector. It is called the *Sudakov form factor*. When E_{min} tends to zero, the form factor decreases and in the limit $E_{min} \to 0$ vanishes. This is the manifestation of the statement that he amplitude of creation of the fermion pair without accompanying soft photons indeed vanishes: the charged particle inevitably emits the low frequency electromagnetic waves. This means that the cross-section of elastic electron scattering without inclusion of emission of bremsstrahlung quanta should vanish, precisely as it follows from eq.(32.53).

Let us estimate the value of the Sudakov form factor for some real process. A good example is the cross-section of e^+e^- annihilation into hadrons which in the leading order in the fine structure constant is described by one diagram with Z-boson exchange in the s-channel. The cross-section has a maximum in the Z-boson peak where it is described by the Breit-Wigner resonance formula. The energy is equal to the Z-boson mass M_Z and the energy resolution is defined by the Z-boson width Γ_Z. Substituting the values $M_Z = 91.187$ GeV, $\Gamma_Z = 2.496$ GeV, $m_e = 0.5$ MeV, $\alpha = 1/128$ into the form factor (32.53) we get

$$\exp\left[-\frac{\alpha}{\pi}\log(\frac{M_Z^2}{m_e^2})\log(\frac{M_Z^2}{\Gamma_Z^2})\right] \approx 0.648.$$

As one can see, the form factor, despite the smallness of the fine structure constant, considerably departs from unity and has to be taken into account when analysing the experimental data.

32.6.3 The cancellation of the infrared divergences

The considered example is typical of the QED and one can make the general statement concerning the infrared divergences for the elements of the S-matrix.

The infrared divergences in radiative corrections to the cross-section of any physical process in QED are cancelled in every order of perturbation theory if to the cross-section of the elastic process one adds the inelastic cross-section of the process with emission of an arbitrary number of additional photons integrated over the phase space with the requirement that the total photon energy does not exceed some value E_{min}.

This statement is also valid for the cross-sections of the processes in non-Abelian gauge theories like the electroweak theory and some processes in QCD, though in this case, due to the self-interaction of the non-Abelian gauge fields, there is no full factorization with the exponentiation, and the proof of this statement presents some problem. Nevertheless, for many processes the result has the same form. Thus, for example, the electromagnetic form-factor in QCD has the same Sudakov form (32.53) but with the replacement $\alpha \to C_F\alpha_s$.

Thus, one can say that the problem of obtaining the ultraviolet and the infrared finite radiative corrections to the cross-sections of the physical processes is solved in two steps: first, with the help of the renormalization procedure one gets rid of the ultraviolet divergences, which is under full control in renormalizable theories; second, defining the correct physical process including the emission of the soft quanta, the cancellation of the infrared divergences takes place.

As we will see below, this is not sufficient in non-Abelian gauge theories with massless gauge fields. They contain additional divergences which require some ads-inn to the described procedure. We will consider this question in the last lecture.

32.7 K-l.X: Collinear Divergences

32.7.1 The collinear divergences in massless theory

The obtained result (32.53) for the cross-section of creation of the muon pair in the process of e^+e^--annihilation with emission of additional soft photons is typical of the theories with a massive fermion and massless photons. It can be generalized to non-Abelian theories with massless gluon, though the gluon interactions cause some problems in proving the cancellation of the IR divergences. Note, however, that eq. (32.53) contains the logarithmic singularity with respect to the fermion mass, and if the latter tends to zero, one has the new divergence. This would not cause any problem since all the fermions are massive but the masses of the electron and the light quarks are so small compared to the characteristic energies of the scattering process that with good precision it is reasonable to neglect them. As for the QCD, considering the processes with gluons in initial states due to the self-interaction of the gluons we face this problem for the gluon amplitudes.

Let us analyse what is the reason for the appearance of the new divergence after the IR divergence at small photon momenta if regularized by introducing the photon mass. Consider for this purpose eq. (32.42) for the contribution of the real or virtual photons. The difference is that in one case the integration goes over the four-momentum of the virtual photon; and in the other case, over the three-momentum of the real photon, but what is essential that for the massless electron its propagator takes the form

$$\frac{1}{2pk} = \frac{1}{2(p^0k^0 - \vec{p}\vec{k})} \simeq \frac{1}{2(|\vec{p}||\vec{k}| - |\vec{p}||\vec{k}|\cos\theta)} = \frac{1}{2|\vec{p}||\vec{k}|(1-\cos\theta)}, \tag{32.54}$$

where θ is the angle between the electron and photon momenta. (In the case of a virtual photon we use the fact that the contribution to the singularity comes from the region of photon momentum close to the mass shell.)

Thus, the divergence appearing in the massless case comes from the integration over the angles and not over the modulus, as in the case of the IR divergence, and is related to the collinearity of momenta of two particles. For this reason it is called *the collinear divergence*. To get rid of these divergences, one can introduce the angular sensitivity of the detector analogously to the IR divergence. This would reflect the fact that two massless particles having almost parallel momenta are not distinguishable from a single particle with the same total momentum. Hence, the observed cross-section should include besides the main process the process of emission of the soft photons and the process of emission of the collinear photons with the kinematically allowed absolute values of momenta.

However, in real life the quarks and leptons are massive though their masses are very small; therefore, the problem of collinear divergences occurs for the processes with the gluon fields. Since the gluons are not free particles but exist inside hadrons, any process with the gluons has a similar process with quarks and it is reasonable to consider them together. For this reason, one usually speaks about *the inclusive* processes where besides the particles of the main process one includes the creation of all kinematically allowed particles, in particular the gluons. In this case, we do not impose any restriction on the gluon energy, we do not introduce any detector sensitivity to the energy or the angle, but sum over all the possibilities. It happens, however, that this is not sufficient to get the finite answer. It is necessary to take into account the possibility of existence of collinear gluons in the initial state, and only after this one can get the finite answer for the cross-section of the observable process.

The multiloop analysis in this case is much more complicated and is the subject of the Kinoshita-Lee-Nauenberg theorem which states:

The infrared and collinear divergences in a massless theory are cancelled in the cross-section of any process if one takes into account the existence in the initial and final states of an arbitrary number of the soft quanta as well as the particles having the parallel momenta with the same total momentum. The probabilities of these processes integrated over the phase space of these additional soft (collinear) quanta in the initial and final states should be added to the probability

of the initial process.

As an illustration we consider the model example of the electron-proton (quark) scattering and put all the masses equal to zero. We will be interested in the radiative corrections in the first order with respect to the strong coupling α_s. The corresponding diagrams are shown in Fig.32.10.

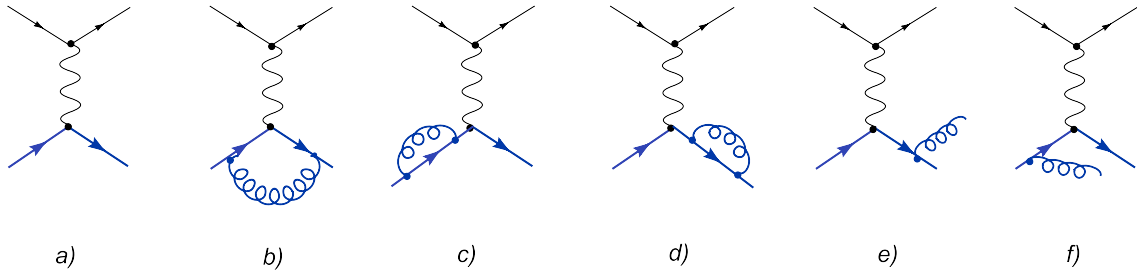


Figure 32.10: The process of electron-quark scattering in the first order in α_s:) the Born diagram, b)-d) the corrections due to the virtual gluons, e)-f) the corrections due to the real gluons

We have already calculated the matrix elements corresponding to these diagrams, but now we proceed in a different way. Since the ultraviolet divergences which appear in the diagrams b)-d) are compensated due to the Ward identity in QED ($Z_1 = Z_2$), all the arising divergences are solely infrared and collinear. To extract them we will use the dimensional regularization. Then both the divergences are manifested in the form of the poles over ε and, since we have both of them, there will be poles of the first and the second order.

We start with the virtual corrections. The diagrams of self-energy c) and d) in the massless case are identically zero due to the above-mentioned property of a massless integral depending on one argument equal to zero ($p^2 = 0$ on the mass shell). As we explained, here one has the cancellation of the UV and the IR divergences. Therefore, all divergences in the vertex diagram b) may be considered as infrared. (The UV divergences should cancel with the UV ones from the self-energy diagrams and the latter in their turn cancel with the IR). The integral for the vertex part is defined by two form factors $F_1(q^2)$ and $F_2(q^2)$ (32.26). Taking the expression for the vertex function (18.39) as the starting point, we put $m = 0$ and go to the mass shell. The result is

$$F_1(q^2) = -C_F \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{-q^2}\right)^{\varepsilon} \left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 8\right), \tag{32.55}$$

$$F_2(q^2) = 0, \tag{32.56}$$

where instead of the logarithm of the photon mass as the IR regulator we have the pole over ε. In order to avoid the transcendental numbers, we used the helpful definition of the angular measure in the space of $4 - 2\varepsilon$ dimensions and multiplied the standard expression by $\Gamma(1-\varepsilon)/(4\pi)^{\varepsilon}$. Then the constants like γ_E, $log(4\pi)$ and $\zeta(2)$ disappear from the intermediate expressions. Due to the cancellation of divergences in the final expressions, this redefinition does not influence the answer.

Thus, the cross-section for the diagrams with virtual gluon has the form

$$\left(\frac{d\sigma}{d\Omega}\right)_{virt} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left[1 - 2C_F \frac{\alpha_s}{4\pi} \left(\frac{\mu^2}{-t}\right)^{\varepsilon} \left(\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + 8\right)\right], \tag{32.57}$$

where the differential cross-section in the Born approximation is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\alpha^2}{2E^2} \left(\frac{s^2 + u^2 - \varepsilon t^2}{t^2}\right) \left(\frac{\mu^2}{s}\right)^{\varepsilon}. \tag{32.58}$$

In the c.m. frame $s = E^2, t = -E^2/2(1 - \cos\theta), u = -E^2/2(1 + \cos\theta)$, where the angle θ is the electron scattering angle.

Consider now the diagrams with the emission of the real gluons e) and f). Besides the squares of each of the diagrams one should also take into account the interference term. The calculation in fact repeats that in QED but instead of the photon mass we again use the dimensional regularization and do not restrict the integration region over the momentum of additional gluon. The calculation is a bit tedious, after contracting all the indices the phase integral takes the form

$$
\begin{aligned}
d\sigma_{2\to 3} &= \frac{1}{2\pi E^2}\int d^D p_3 \delta^+(p_3^2)\int \frac{d^D k}{(2\pi)^D}\delta^+(k^2)\delta^+((p_4-k)^2)|M|^2_{p_4=p_1+p_2-p_3} \qquad (32.59)\\
|M|^2 &= \frac{e^4 g^2}{4} 8\frac{M_0+\epsilon M_1+\epsilon^2 M_2}{t(s+t+u)},\\
M_0 &= 4s-8p_1k-4p_2k+\frac{-8(p_1k)^2+4(2s+t)p_1k-(3s^2+t^2+u^2+2st)}{p_2k},\\
M_1 &= -4(s+u)+8p_1k+8p_2k+\frac{8(p_1k)^2-4(s+t+u)p_1k+2)s+t+u)^2-2(u+s)t}{p_2k}\\
M_2 &= 4(s+t+u)-4p_2k-\frac{(s+t+u)^2}{p_2k}=-\frac{(s+t+u+2p_2k)^2}{p_2k}.
\end{aligned}
$$

It is useful to pass to the spherical coordinates and use the c.m. frame. After the integration over the phase volume the result can be represented in the form

$$
\left(\frac{d\sigma}{d\Omega}\right)_{real}=\left(\frac{d\sigma}{d\Omega}\right)_0\left[2C_F\frac{\alpha_s}{4\pi}\left(\frac{\mu^2}{-t}\right)^{\varepsilon}\left(\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+8\right)\right]+C_F\frac{\alpha^2}{E^2}\frac{\alpha_s}{4\pi}\left(\frac{\mu^2}{s}\right)^{\varepsilon}\left(\frac{\mu^2}{-t}\right)^{\varepsilon}\left(\frac{f_1}{\varepsilon}+f_2\right), \qquad (32.60)
$$

where the functions f_1 and f_2 in the c.m. frame are ($x=\cos\theta$)

$$
\begin{aligned}
f_1 &= -2\frac{(1-x)(x^3+5x^2-3x+5)\log(\frac{1-x}{2})-(x-1)^2(x+1)(x-11)/4}{(1-x)^2(1+x)^2}, \qquad (32.61)\\
f_2 &= -\frac{1}{(1-x)^2(1+x)^2}\left[(1-x)(x^3+5x^2-3x+5)\log^2(\frac{1-x}{2})\right.\\
&\quad +\frac{1}{2}(1-x)(3x^3+15x^2+77x-31)\log(\frac{1-x}{2})+(1+x)^2(x^2+5x+3)\pi^2\\
&\quad \left.-12(9x^2+2x+5)Li_2(\frac{1+x}{2})+\frac{1}{2}(1-x)(1+x)(5x^2-42x-23)\right]. \qquad (32.62)
\end{aligned}
$$

As one can see from the comparison of the cross-sections of the processes with the virtual (32.57) and the real gluons (32.60), in the sum the second order poles cancel. However, the total cancellation of divergences does not happen. The remaining divergences in the form of a single pole have a collinear nature. As was already mentioned, for their cancellation one has to define properly the initial states. The point is that the massless quark can emit the collinear gluon which will carry part of the initial momentum and in this case, it is impossible to distinguish one particle propagating with the speed of light from the two flying parallel.

32.7.2 The quark distributions and the splitting functions

To take into account this possibility, let us come back to the scattering process and assume that the initial quark has emitted the parallel gluon (see Fig.32.11). The two particles can be almost parallel with small relative transverse momentum. The three four-momenta can be chosen in the form:

$$
p=(p;0,0,p),\quad q\approx(zp;p_\perp,0,zp),\quad k\approx((1-z)p;-p_\perp,0,(1-z)p),
$$

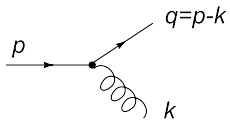


Figure 32.11: The diagram corresponding to the splitting of the quark into the quark and the gluon

so that all of them obey the condition $p^2 = q^2 = k^2 = 0$ with the accuracy up to $p_\perp^2$. It is helpful, however, to use another method, namely to choose the momenta in such a way that they obey the mass shell condition with the accuracy up to $p_\perp^4$, but to give up the energy conservation in the order of $p_\perp^2$. The advantage of this approach consists in the use of formulas for the spinors and the polarization vectors on mass shell. Therefore, we choose the momenta as follows:

$$p = (p; 0, 0, p),\ q \approx (zp + \frac{p_\perp^2}{2zp}; p_\perp, 0, zp),\ k \approx ((1-z)p + \frac{p_\perp^2}{2(1-z)p}; -p_\perp, 0, (1-z)p).$$

The square of the matrix element corresponding to the process of splitting on mass shell in this case can be written in the standard form

$$|M(q \to qG)|^2 = \frac{g^2}{2}_F Tr(\gamma^\mu \hat{p} \gamma^\nu \hat{q}) \sum_{pol} \epsilon^{*\mu} \epsilon^\nu, \tag{32.63}$$

where the factor $1/2$ comes from the averaging over the spin states. Here we must take into account the physical polarizations of the gluon only, i.e.

$$\sum_{pol} \epsilon^{*\mu} \epsilon^\nu \to \delta^{ij} - \frac{k^i k^j}{(\vec{k})^2},$$

which gives

$$|M(q \to qG)|^2 = 4 g_F^2 \left[p^0 q^0 - \frac{(\vec{p}\vec{k})(\vec{q}\vec{k})}{(\vec{k})^2} \right], \tag{32.64}$$

or, substituting the values of momenta,

$$|M(q \to qG)|^2 = C_F \frac{2g^2 p_\perp^2}{z(1-z)} \frac{1+z^2}{1-z}, \qquad z < 1. \tag{32.65}$$

The obtained expression does not depend on the choice of momenta and has a universal character.

Now one can calculate the cross-section of the process of interest. Graphically, it will be the same diagram Fig.32.10); however, the additional gluon will be referred not to the final state but to the initial one. Here we use the standard Feynman rules when the energy conservation law is not violated, but the massless particle is slightly off shell. Since in the case of interest the quark with momentum q is virtual, it is useful to choose the momenta like

$$p = (p; 0, 0, p), q \approx (zp - \frac{p_\perp^2}{2(1-z)p}; p_\perp, 0, zp), k \approx ((1-z)p + \frac{p_\perp^2}{2(1-z)p}; -p_\perp, 0, (1-z)p).$$

In this case,

$$q^2 = -\frac{p_\perp^2}{1-z}. \tag{32.66}$$

Then the cross-section of the process can be written in the factorized form

$$d\sigma(p) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2k^0}|M_{q\to qG}|^2(\frac{1}{q^2})^2(\frac{p^0z}{p^0})d\sigma(pz), \tag{32.67}$$

where the factor $(\frac{p^0z}{p^0})$ is due to fact that the cross-section is normalized to the energy of initial particles, and we have replaced the quark with the energy p^0 by the quark with the energy zp^0.

Rewriting the differential d^3k in terms of the new variables

$$d^3k = pdzd^2p_\perp = pdz\pi dp_\perp^2,$$

and substituting the value of the matrix element (32.65) and q^2 from (32.66), we get

$$\begin{aligned} d\sigma(p) &= C_F\frac{^2}{16\pi^2}\int \frac{pdzdp_\perp^2}{(1-z)p}\frac{(1-z)^2}{p_\perp^4}\frac{2p_\perp^2}{z(1-z)}\frac{1+z^2}{1-z}zd\sigma(pz) \\ &= C_F\frac{\alpha_s}{2\pi}\int \frac{dzdp_\perp^2}{p_\perp^2}\frac{1+z^2}{1-z}d\sigma(pz). \end{aligned} \tag{32.68}$$

The integral over the transverse momentum is divergent at zero and this is nothing else but the manifestation of the collinear divergence. The upper limit is not of great importance, it is restricted by kinematic considerations. We assume that the integration over $p_\perp^2$ goes from zero to some scale Q^2. Later, we will see that one can change this scale analogously to the change of the ultraviolet scale μ^2.

To extract the divergence we use the dimensional regularization. Changing the dimension of transverse integration from 2 to $2-2\varepsilon$ one gets

$$\begin{aligned} d\sigma(p) &= C_F\frac{\alpha_s}{2\pi}\int\limits_0^1 dz\frac{1+z^2}{1-z}\int\limits_0^{Q^2}\frac{(p_\perp^2)^{-\varepsilon}(-\mu^2)^\varepsilon dp_\perp^2}{p_\perp^2}d\sigma(pz) \\ &= C_F\frac{\alpha_s}{2\pi}\int\limits_0^1 dz\frac{1+z^2}{1-z}\,\frac{1}{\varepsilon}\left(-\frac{\mu^2}{Q^2}\right)^\varepsilon d\sigma(pz). \end{aligned} \tag{32.69}$$

At first sight the obtained expression still contains the pole in the integrand as $z\to 1$. However, it only looks like a singularity. It came from the matrix element (32.65), which we have calculated only for $z<1$ and it needs to be redefined for $z\to 1$. We will come back to this question below and, at first, discuss the interpretation of relation (32.69).

Let us introduce the notion of distribution of the initial quark with respect to the fraction of the carried momentum z: $q(z)$. Then the initial distribution corresponds to $q(z)=\delta(1-z)$, and the emission of a gluon leads to the splitting: the quark carries the fraction of momentum equal z, while the gluon - $(1-z)$. The probability of this event is given by the so-called *splitting functions* $P_{qq}(z)$ and $P_{qG}(1-z)$. In the lowest order of perturbation theory in α_s the quark and gluon distributions can be written in the form

$$q(z,Q^2) = \delta(1-z)+\frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}\left(\frac{\mu^2}{Q^2}\right)^\varepsilon P_{qq}(z), \tag{32.70}$$

$$G(z,Q^2) = \frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}\left(\frac{\mu^2}{Q^2}\right)^\varepsilon P_{qG}(1-z), \tag{32.71}$$

where the splitting functions are defined by the corresponding matrix elements one of which for $P_{qq}(z)$ has been calculated in the leading order in α_s earlier (see (32.65)). The result has the

following form:

$$P_{qq}(z) = C_F\left(\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right), \tag{32.72}$$

$$P_{qG}(z) = \frac{z^2+(1-z)^2}{2}. \tag{32.73}$$

Note that eq. (32.72) contains the redefinition of the function $P_{qq}(z)$ at the point $z=1$ mentioned above, namely the sign " $+$ " should be understood as the following integration rule:

$$\int\limits_0^1 dz\frac{f(z)}{(1-z)_+} \equiv \int\limits_0^1 dz\frac{f(z)-f(1)}{(1-z)},$$

and the coefficient of the δ-function is defined from the requirement of conservation of the number of quarks

$$\int\limits_0^1 q(x,Q^2)dz = 1 \quad \Rightarrow \int\limits_0^1 P_{qq}(z)dz = 0.$$

Thus, eq. (32.69) together with the Born diagram can be written as

$$d\sigma(p) = \int\limits_0^1 dz\ q(z,Q^2)\ d\sigma(pz), \tag{32.74}$$

where the quark distribution $q(z,Q^2)$ is given by (32.70).

It seems strange at first sight that the answer depends on the scale Q^2 which defines the quark distribution. However, it has the physical interpretation. This is the measure of collinearity of the emitted gluons that can be distinguished, i.e., it refers to the definition of the initial state. In fact, in the massless case one cannot define the initial state that contains just the quark, it exists together with the set of collinear gluons. (The same is true for the massless electron with collinear photons.) This scale is sometimes called the factorization scale, at this scale the scattering cross-section (32.74) takes the factorized form. The factorization scale can be varied. The dependence of the quark and the gluon distributions on the scale is governed by the so-called DGLAP equations well known in QCD.

32.7.3 The finite answers

Thus, besides the two contributions to the cross-section from the virtual and the real gluons there is one more contribution related to the splitted initial state (32.69). In the lowest order of perturbation theory in α_s it can be written as

$$\left(\frac{d\sigma}{d\Omega}\right)_{split} = \frac{1}{\varepsilon}\frac{\alpha_s}{2\pi}\int\limits_0^1 dz\left(\frac{\mu^2}{Q_f^2}\right)^{\varepsilon} P_{qq}(z)\frac{d\sigma_0}{d\Omega}(pz), \tag{32.75}$$

where the Born cross-section is given by (32.58) with the replacement of the initial quark momentum p by pz, and the factorization scale Q_f^2 is an arbitrary quantity associated with the quark distribution function. Note that the scale Q_f^2 may depend on z. It is quite natural to choose the factorization scale equal to the characteristic scale of the process of interest. Thus, in our case this

choice corresponds to $Q_f^2 = -\hat{t}$, where $\hat{t}$ is the Mandelstam parameter t for the process where p is replaced by pz. One has $\hat{t} = t\frac{2z}{(z+1)+(z-1)x}$. This leads to the following result:

$$\left(\frac{d\sigma}{d\Omega}\right)_{split} = C_F \frac{\alpha^2}{2E^2}\frac{\alpha_s}{2\pi}\left(\frac{\mu^2}{s}\right)^{\varepsilon}\left(\frac{\mu^2}{-t}\right)^{\varepsilon}(-\frac{f_1}{\varepsilon} + f_3), \tag{32.76}$$

where f_1 is given by (32.61) and

$$\begin{aligned} f_3 &= -\frac{1}{(1-x)^2(1+x)^2}\left[2(1-x)(x^3+x^2-33x+7)\log(\frac{1-x}{2})\right. \\ &+12(9x^2+2x+5)Li_2(\frac{1+x}{2}) - (1+x)^2(x^2+5x+3)\pi^2 \\ &\left.-\frac{1}{2}(1-x)(1+x)(11x^2-19)\right]. \end{aligned} \tag{32.77}$$

Comparing the obtained expression with (32.57) and (32.60) we see that the last divergence cancels and the final expression for the cross-section of the electron-quark scattering with account of possible creation of the gluon in the initial and final states takes the form ($x = \cos\theta$)

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right) &= \left(\frac{d\sigma}{d\Omega}\right)_{virt} + \left(\frac{d\sigma}{d\Omega}\right)_{real} + \left(\frac{d\sigma}{d\Omega}\right)_{split} \\ &= \frac{\alpha^2}{2E^2}\left\{\frac{x^2+2x+5}{(1-x)^2} - \frac{\alpha_s}{2\pi}\frac{C_F}{(1-x)(1+x)^2}\left[(x^3+5x^2-3x+5)\log^2\frac{1-x}{2}\right.\right. \\ &\left.\left.+\frac{1}{2}(7x^3+19x^2-55x-3)\log\frac{1-x}{2} - (1+x)(3x^2+21x+2)\right]\right\}. \end{aligned} \tag{32.78}$$

This expression is our final answer for the cross-section of the physical process of electron-quark scattering where the initial and the final state include the soft and collinear gluons. It includes also the definition of the initial state and can be recalculated for the alternative choice of the factorization scale similar to what happens to the ultraviolet scale which defines the coupling constant. Thus, we practically deal with the scattering not of individual particles but rather with coherent states with a fixed total momentum. Only this process has a physical meaning.

In Fig.34.169, we show the differential cross-section of this process as a function of the electron scattering angle: $\frac{E^2}{\alpha^2}\frac{d\sigma}{d\Omega}(\cos\theta)$. We have chosen here the strong coupling $\alpha_s = 0.2$, and $C_F = 4/3$. As one can see, the inclusion of the radiative correction $\sim \alpha_s$ practically does not change the result, the difference from the Born approximation is less than a per cent, that justifies the use of perturbation theory.

Let us stress once more that the obtained answer for the cross-section of the observable process depends on: a) the ultraviolet subtraction scheme that manifests itself, in particular, in the appearance of the ultraviolet scale μ^2 (canceled in our case in the lowest order of perturbation theory) and b) the definition of the initial coherent state, which manifests itself in the appearance of the factorization scale Q_f^2. The universality in the description of the physical processes is based on the fact that choosing the UV and the IR scale one way or another and fitting the experimental data of some process, one can then recalculate the obtained values of the running coupling and of the quark (lepton) distribution for any other choice of the scales. This way the result for the observable quantities does not depend on a particular choice of these scales and is universal.

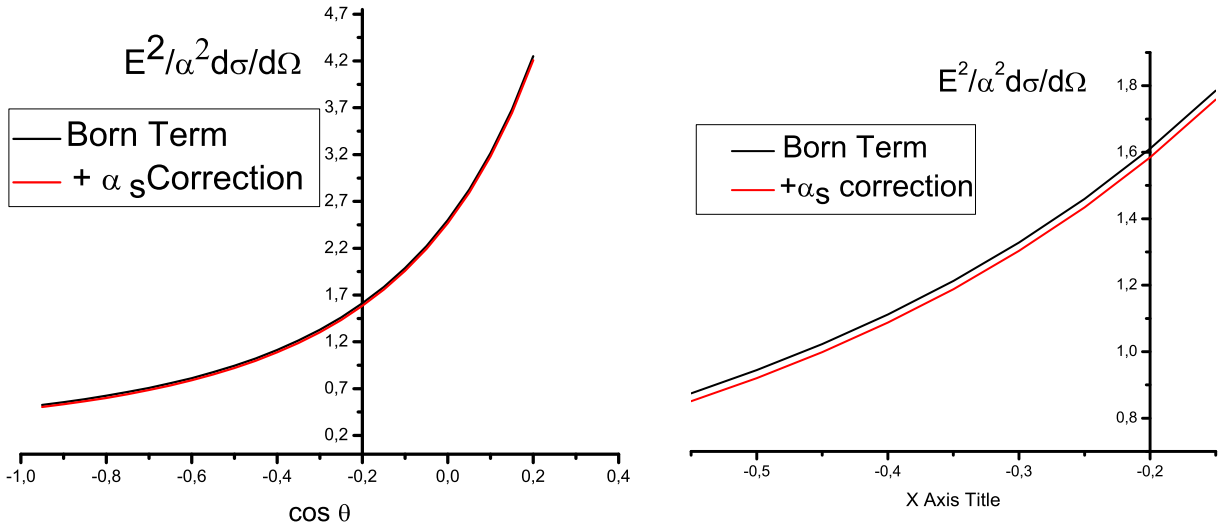


Figure 32.12: The differential cross-section of *eq* scattering in the Born approximation and with allowance for the α_s correction. On the right plane the same plot is shown in the bigger scale

33 Quantum Chromodynamics

или элетродинамика адронов
(хромо - цвет)
(пока из ЛЛ4 выгрзуки)

33.1 Fundamentals of Quantum Chromodynamics

33.1.1 Essence of QCD

(думаю, это правильнее тут писать, супер не скоро в 1ю часть буду переносить, потому что не до этого)

33.1.2 Overview of Applications of QCD

(тоже важно это выделить)

33.1.3 Popular Overview of QCD

(такое тоже есть, много видео, пропишу потом, много разных аналогий.)

33.2 QCD by Landau

33.2.1 Electromagnetic Form Factors of Hadrons

(лл4 пока что, потом буду думать и разбирать)

Теория (??)

До сих пор в этой книге речь шла о квантовой электродинамике частиц, не способных к сильным взаимодействиям, - электронов, позитронов и мюонов. Существует также большое число частиц, участвующих в сильных взаимодействиях; их называют адронами).

Адронами являются, например, протоны и нейтроны, имеющие спин $1/2, \pi$ -мезоны со спином 0 и другие частицы. Адронами, разумеется, являются и атомные ядра, так как они состоят из протонов и нейтронов.

Построение исчерпывающей электродинамики адронов в рамках существующей теории невозможно. Ясно, что нельзя составить уравнений, определяющих электромагнитные взаимодействия адронов без учета значительно более интенсивных сильных взаимодействий.

В частности, без учета последних нельзя установить и явный вид адронного тока, с помощью которого должны описываться взаимодействия в квантовой электродинамике.

В этой ситуации адронный ток вводится как феноменологическая величина, структура которой устанавливается лишь исходя из обших кинематических требований, не связанных с какимилибо предположениями о динамике взаимодействий [2]). Оператор же электромагнитного взаимодействия будет иметь по-прежнему вид

$$e(\widehat{J}\widehat{A})$$

где теперь ток обозначен прописной буквой J (в отличие от электронного тока j). Поскольку порядок величины этого взаимодействия задается тем же элементарным зарядом e, можно попрежнему пользоваться методами теории возмущений [3]).

Установим вид тока перехода между двумя состояниями свободно движущегося адрона (не сопровождающегося каким-либо

превращением самого адрона). Этот ток входит в «треххвостку»

которая сама может входить как часть в какую-либо более сложную диаграмму (например, упругого рассеяния электрона на адроне). Штриховая линия в диаграмме (138.2) изображает виртуальный фотон; она не может отвечать реальному фотону, так как свободная частица не может поглотить (или испустить) такой фотон. При этом $q^2 = (p_2 - p_1)^2 < 0$.

Рассмотрим сначала адрон со спином 0. Пусть u_1 и u_2 - волновые амплитуды начального и конечного состояний адрона, в которых он имеет 4-импульсы p_1 и p_2; для частицы со спином 0 эти амплитуды - скаляры (или псевдоскаляры)). Адронный ток перехода J_{fi} между этими двумя состояниями должен быть билинеен по u_1, и u_2^*. Запишем его в виде

$$J_{fi} = u_2^* \Gamma u_1$$

где 4-вектор Γ - неизвестный вершинный оператор (кружок на диаграмме (138.2)). Если положить $u_1 = u_2 = 1$, то будет просто $J_{fi} = \Gamma$ Универсальным свойством тока в электродинамике, связанным с калибровочной инвариантностью теории, является его сохранение. В импульсном представлении оно выражается ортогональностью тока перехода 4-импульсу фотона $q = p_2 - p_1$:

$$qJ_{fi} = 0$$

В данном случае это значит, что оператор Γ должен иметь вид

$$\Gamma = PF\left(q^2\right)$$

где $P = p_1 + p_2, F\left(q^2\right)$ - скалярная функция единственной инвариантной независимой переменной - квадрата q^2. Поскольку род адрона при переходе не меняется, то $p_1^2 = p_2^2 = M^2$(M - масса адрона), и потому $Pq = 0$.

Матричные элементы (138.3) сΓ из (138.5)(а с ними и сам оператор J) - истинные 4-векторы. Поэтому оператор взаимодействия (138.1) - истинный скаляр. Таким образом, электромагнитное взаимодействие адронов со спином 0 оказывается P -инвариантным автоматически. Оно оказывается также и T -инвариантным. Действительно, обращение времени, во-первых, переставляет начальный и конечный 4-импульсы; при этом сумма

$P = p_1 + p_2$ не меняется. Во-вторых, обращение времени меняет знак пространственных компонент 4-импульсов, не меняя их временных компонент; но таким же образом преобразуются и компоненты 4-потенциала A, так что произведение $\widehat{J}\widehat{A}$ не меняется.

Инвариантную функцию $F(q^2)$ называют электромагитны.м формфактором адрона. В рамках феноменологической теории ее вид, разумеется, не может быть установлен. Можно, однако, утверждать, что эта функция вещественна (в рассматриваемой области $q^2 < 0$). Это следует из тех же соображений, которые были применены в \$116 к формфакторам электрона: при $q^2 < 0$ во всяком случае отсутствуют промежуточные состояния, которые могли бы фигурировать в правой стороне соотношения унитарности; поэтому матрица M_{fi}, а с нею и J_{fi} оказываются эрмитовыми.

При $q = 0$ начальное и конечное состояния совпадают, так что J_{fi} становится диагональным матричным элементом. В частности, $e(J^0)_{ii}/2\varepsilon_i = eF(0)$ есть плотность заряда, совпадающая

(нормировка на одну частицу в единичном объеме!) с полным зарядом частицы Ze.

Для электрически нейтральной частицы $F(0) = 0$. Подчеркнем, однако, что это отнюдь не означает еще истинной нейтральности частицы. Если частица истинно нейтральна и обладает определенной зарядовой четностью, то $F(q^2) \equiv 0$ при всех q^2: так как оператор тока зарядово-нечетен (см. §13), его матричные элементы между двумя состояниями одного и того же адрона равны нулю).

Перейдем к адронам со спином 1/2. В этом случае волновые амплитуды u_1, u_2 - биспиноры и адронный ток имеет вид

$$J_{fi} = \bar{u}_2 \Gamma u_1$$

Из билинейных комбинаций $\bar{u}_2$ и u_1 и 4 -векторов p_1, p_2 можно составить как истинные 4-векторные, так и псевдовекторные величины (удовлетворяющие условию (138.4)). Поэтому условие P -инвариантности взаимодействия не удовлся автоматически и должно быть поставлено дополнительно). Как было показано в пар 116, при этом условии вершинный оператор содержит два независимых вещественных (при $q^2 < 0$) формфактора.

Запишем его теперь в виде

$$\Gamma^\mu = 2M(F_e - F_m)\frac{P^\mu}{P^2} + F_m\gamma^\mu = 2M\left(F_e - \frac{q^2}{4M^2}F_m\right)\frac{P^\mu}{P^2} - \frac{F_m}{2M}\sigma^{\mu\nu}q_\nu =$$
$$= (4M^2F_e - q^2F_m)\frac{\gamma^\mu}{P^2} + \frac{2M}{P^2}(F_e - F_m)\sigma^{\mu\nu}q_\nu, \quad (138.7)$$

где $F_e(q^2)$ и $F_m(q^2)$ - инвариантные формфакторы (M - масса адрона); в эквивалентности трех написанных выражений легко убедиться с помощью равенств $P^2 + q^2 = 4M^2$ и $(116.5)^2$).

Электромагнитные формфакторы относятся к категории инвариантных амплитуд, понятие о которых было введено в §70. Их можно рассматривать как амплитуды «реакции», представляющей собой (в своем аннигиляционном канале) распад виртуального фотона на адрон и антиадрон. Виртуальный фотон - «частица» со спином 1. В том, что ее распад на две частицы со спином 1/2 должен описываться двумя независимыми амплитудами,

легко убедиться и подсчетом соответствующих спиральных амплитуд $\langle \lambda_b\lambda_c | S^J | \lambda_a \rangle$ (см. § 69). Действительно, в силу P -инвариантности четыре отличных от нуля элемента S -матрицы попарно равны друг другу:

$$\langle 1/21/2 | S^1 | 1 \rangle = \langle -1/2 - 1/2 | S^1 | -1 \rangle$$
$$\langle 1/2 - 1/2 | S^1 | 0 \rangle = \langle -1/2^1/2 | S^1 | 0 \rangle$$

Требование T -инвариантности)или C -инвариантности -в аннигиляционном канале) не добавляет новых связей между эти-

ми элементами. С этим обстоятельством связан тот факт, что взаимодействие, описываемое вершинным оператором (138.7), автоматически оказывается также и T -инвариантным (такая ситуация, однако, не имеет уже места для частиц с более высокими спинами).

При $q \to 0$ члены нулевого и первого (по q) порядка в (138.7):

$$\Gamma^{\mu} = F_e(0)\gamma^{\mu} - \frac{1}{2M}\left[F_m(0) - F_e(0)\right]\sigma^{\mu\nu}q_{\nu}$$

Отсюда видно (см. §116), что $F_e(0) \equiv Z$ - электрический заряд частицы (в единицах e), а $F_m(0) - F_e(0)$ - ее аномальный магнитный момент (в единицах $e/2M$)).

До сих пор мы пользовались только формфакторами в импульсном пространстве. Этого, разумеется, достаточно для описания наблюдаемых явлений. С чисто иллюстративной целью, однако, можно дать формфакторам и несколько более наглядную интерпретацию, рассматривая их как фурье-образы некоторых функций от координат.

Для этого удобно выбрать систему отсчета, в которой $\mathbf{P} =$ (так называемая система Брейта); это всегда возможно, поскольку $P^2 > 4M^2 > 0$. В этой системе $\varepsilon_1 = \varepsilon_2 \equiv \varepsilon$, так что $P^0 = 2\varepsilon$, а составляющие 4-вектора q равны $q^0 = 0, \mathbf{q} = 2\mathbf{p}_2 = -2\mathbf{p}_1$ Для адрона со спином 0 ток перехода принимает в системе

Брейта особенно простую форму:

$$\frac{J^0_{fi}}{2\varepsilon} = F\left(-\mathbf{q}^2\right), \quad \mathbf{J} = 0$$

Отсюда видно, что $F\left(-\mathbf{q}^2\right)$ можно истолковать как фурье-образ статического распределения зарядов с плотностью

$$e\rho(\mathbf{r}) = e\frac{1}{(2\pi)^3}\int F\left(-\mathbf{q}^2\right)e^{i\mathbf{q}\mathbf{r}}d^3q$$

В этом смысле говорят о пространственной электромагнитной структуре частицы: при $F =$ const $= Z$ было бы $\rho(\mathbf{r}) = Z\delta(\mathbf{r})$; зависимость же формфактора от q интерпретируется как отклонение распределения заряда от точечного. Подчеркнем, однако, что этой интерпретации не следует придавать буквального смысла. Функция $\rho(\mathbf{r})$ вообще не относится к какой-либо определенной системе отсчета, так как каждому значению q отвечает своя система.

Лишь в нерелятивистском пределе $\mathbf{q}^2 \ll M^2$, когда изменением энергии частицы при рассеянии можно пренебречь, система Брейта совпадает с системой покоя частицы и не зависит от q.

Начальные и конечные состояния частицы в этом приближении одинаковы, так что ток перехода становится диагональным матричным элементом и функция ρ(r) приобретает реальный смысл пространственного распределения зарядов. Для элементарных частиц, однако, характерные значения |q|, на которых существенно меняются формфакторы, лишь немногим меньше M. Поэтому в нерелятивистском пределе для них можно вообще заменить $F\left(-q^2\right)$ на $F(0)$, т. е. рассматривать частицу как точечную. Иная ситуация для ядер. Масса ядра M пропорциональна числу A нуклонов в нем, а характерное значение $|\mathbf{q}| \sim 1/R$, т. е. пропорционально $A^{-1/3}$(R - радиус ядра). Поэтому для достаточно тяжелых ядер характерные $\mathbf{q}^2 \ll M^2$, и, таким образом, нерелятивистское рассмотрение допустимо во всем существенном интервале; тем самым понятие электромагнитной структуры ядра приобретает вполне определенный смысл.

Для частицы со спином 1/2 из (138.7) получим в системе Брейта

$$J^0_{fi} = (F_e - F_m)\,\tfrac{M}{\varepsilon}\,(\bar{u}_2 u_1) + F_m\,(\bar{u}_2\gamma^0 u_1) = F_e\,(\bar{u}_2\gamma^0 u_1)$$
$$\mathbf{J}_{fi} = -\tfrac{1}{2M}F_m\,[i\mathbf{q}\,(\bar{u}_2\boldsymbol{\Sigma}u_1)$$

где $\boldsymbol{\Sigma}$ -трехмерный оператор (матрица) спина (21.21), а в (138.10) использовано равенство $\varepsilon\,(\bar{u}_2\gamma^0 u_1) = M\,(\bar{u}_2 u_1)$, которое легко проверить с помощью уравнений Дирака для u_1 и $\bar{u}_2$ при

$\mathbf{p}_1 = -\mathbf{p}_2$ Временная компонента тока перехода (138.10) отличается от выражения для «точечной частицы»-электрона множителем $F_e\,(-\mathbf{q}^2)$. Поэтому можно сказать, что

формфактор F_e (его называют зарядовым) описывает «пространственное распределение заряда» согласно (138.9).

Аналогичным образом трехмерному вектору (138.11) можно привести в соответствие «пространственное распределение» плотности токов $e\mathbf{j}(\mathbf{r}) = \operatorname{rot} \boldsymbol{\mu}(\mathbf{r})$, где

$$\boldsymbol{\mu}(\mathbf{r}) = \frac{e}{2M}\boldsymbol{\Sigma} \int F_m\left(-\mathbf{q}^2\right) e^{i\mathbf{qr}} d^3q$$

представляет собой «плотность магнитного момента». Таким образом, формфактор F_m (его называют магнитны. м) можно интерпретировать как плотность пространственного распределения магнитного момента - разумеется, с теми же оговорками, которые были сделаны выше по поводу распределения заряда. При этом F_m включает в себя как «нормальный» дираковский магнитный момент, так и специфический для адрона «аномальный» момент; «плотности» последнего отвечает разность $F_m - F_e$.

Естественно считать, что особые точки адронных электромагнитных формфакторов, как и электронных, лежат при вещественных положительных значениях аргумента $t = q^2 = -\mathbf{q}^2$. Это позволяет сделать определенные заключения об асимптотическом поведении распределения $\rho(r)$ (и $\boldsymbol{\mu}(r)$) при $r \to \infty$. Именно, такое же преобразование интеграла (138. 9), которое было применено в §114 для перехода от (114. 3) к (114.4), приведет к результату, что при больших r будет $\rho(r) \propto e^{-x_0 r}$, где x_0^2 - абсцисса первой особой точки формфактора $F\left(q^2\right)$ (ср. также примеч. на с. 564). Если ближайшая особенность дается порогом образования виртуальным фотоном пары адронов (массы M_0 каждый), то $\mathcal{X}_0 = 2M_0$

33.2.2 Scattering of Electrons by Hadrons

обзор и мотивация

Theory

Применим полученные в предыдущем параграфе формулы к упругому рассеянию электрона на адроне. Обозначим начальный и конечный 4-импульсы адрона через p_h и p'_h, а 4-импульсы электрона через p_e и p'_e; при этом

$$p_e + p_h = p'_e + p'_h$$

Рассматриваемый процесс изображается диаграммой

(тут рис)

Испусканию виртуального фотона электроном отвечает обычный вершинный оператор γ; поглощению его адроном - операТор Γ

Рассмотрим наибилее интересный случай адрона со спином 1/2 (например, рассеяние электрона протоном или нейтроном). Диаграмме (139.2) соответствует амплитуда рассеяния

$$M_{fi} = -4\pi e^2 \frac{1}{q^2}\left(\bar{u}'_e \gamma^\mu u_e\right)\left(\bar{u}'_h \Gamma_\mu u_h\right)$$

(в этой главе заряд электрона есть - e!). Вычисление сечения по этой амплитуде не представляет принципиальных отличий от произведенных в §81 вычислений; при том оператор Γ удобно писать в виде первого из выражений (138.7).

Для рассеяния неполяризованных частиц получается следующий результат:

$$d\sigma = \frac{\pi\alpha^2 dt}{[s-(M+m)^2][s-(M-m)^2]t^2(1-t/4M^2)} \times$$
$$\times \left\{F_e^2\left[(s-u)^2 + (4M^2-t)\,t\right] - \frac{t}{4M^2}F_m^2\left[(s-u)^2 - (4M^2-t)\left(4m^2+t\right)\right]\right\}$$

Здесь M - масса адрона, m - масса электрона,

$$s = (p_e + p_h)^2\,, \quad t = q^2 = (p_e - p'_e)^2\,, \quad u = (p_e - p'_h)^2$$
$$s + t + u = 2m^2 + 2M^2$$

Рассмотрим некоторые предельные случаи.

Для рассеяния электронов на тяжелом ядре представляет интерес случай, когда передача импульса электроном ядру $|\mathbf{q}|$ мала по сравнению с массой ядра,

но не мала по сравнению с $1/R$(R - радиус ядра), так что ядро нельзя рассматривать как точечное.

В таком случае система центра инерции приближенно совпадает с системой покоя ядра, отдачей ядра можно пренебречь и энергия электрона не меняется. При этом

$$-t = \mathbf{q}^2 \ll M^2, \quad \pi|dt| = \mathbf{p}_e^2 do'_e, \quad s - M^2 \approx M^2 - u \approx 2M\varepsilon_e$$

и формула (139.4) принимает вид

$$d\sigma = \frac{\alpha^2 do'_e}{q^4}\left(4\varepsilon_e^2 - q^2\right) F_e^2\left(-\mathbf{q}^2\right)$$

В этом приближении в сечении остается лишь член с электрическим формфактором и (139.5) соответствует формуле (80.5),

справедливой для рассеяния электрона на статическом распределении зарядов.

При рассеянии электрона на неподвижном нейтроне в том же предельном случае $\varepsilon_e \ll M$(M - масса нейтрона) формфакторы можно заменить их значениями при $\mathbf{q} = 0$, поскольку,

как уже отмечалось, для отдельного нуклона характерный «радиус» распределения зарядов сравним с $1/M$[1]). В силу электрической нейтральности нейтрона $F_e(0) = 0$, и сечение принимает вид

$$d\sigma = \alpha\mu^2\left[\frac{4\left(\varepsilon_e^2 - m^2\right)}{\mathbf{q}^2} + 1\right] do'_e = \alpha\mu^2\left(\frac{1}{\sin^2(\vartheta/2)} + 1\right) do'_e$$

где $\mu = \frac{e}{2M}F_m(0)$ - магнитный момент нейтрона, ϑ - угол рассеяния.

Эта формула отвечает рассеянию электрона на неподвижном точечном магнитном моменте.

Наконец, напишем формулу для сечения рассеяния ультрарелятивистского электрона на нуклоне при $|\mathbf{q}| \gg m$.

Под $\mathbf{q}^2$ мы попрежнему понимаем квадрат передачи импульса в системе центра инерции, так что инвариант $t = -\mathbf{q}^2$. В системе же покоя начального нуклона (лабораторная система) имеем

$$-t \approx 2\left(p_e p'_e\right) = 2\varepsilon_e\varepsilon'_e(1 - \cos\vartheta)$$

где $\varepsilon_e, \varepsilon'_e$ - начальная и конечная энергия электрона, а ϑ - угол рассеяния в этой системе.

В ультрарелятивистском случае ε'_e связана с ϑ той же формулой, что и при рассеянии фотона (ср. (86.8)):

$$\frac{1}{\varepsilon'_e} - \frac{1}{\varepsilon_e} = \frac{1}{M}(1 - \cos\vartheta)$$

Поэтому имеем

$$-t = \frac{4\varepsilon_e^2 \sin^2\frac{\vartheta}{2}}{1 + \frac{2\varepsilon_e}{M}\sin^2\frac{\vartheta}{2}}$$

$$\pi d|t| = \frac{\varepsilon_e^2 do'_e}{\left(1 + \frac{2\varepsilon_e}{M}\sin^2\frac{\vartheta}{2}\right)^2}$$

где $do'_e = 2\pi\sin\vartheta d\vartheta$. В формуле (139.4) можно везде опустить массу электрона m; выразив все величины через t и $s - M^2 =$

$$d\sigma = \frac{\pi\alpha^2 d|t|}{\varepsilon_e^2 t^2}\left\{F_e^2(t)\left[\frac{(4M\varepsilon_e + t)^2}{4M^2 - t} + t\right] - \frac{t}{4M^2}F_m^2(t)\left[\frac{(4M\varepsilon_e + t)^2}{4M^2 - t} - t\right]\right\},$$

или, используя что-то выше

$$d\sigma = do'_e \frac{\alpha^2}{4\varepsilon_e^2} \frac{\cos^2\frac{\vartheta}{2}}{\sin^4\frac{\vartheta}{2}} \frac{1}{1+\frac{2\varepsilon_e}{M}\sin^2\frac{\vartheta}{2}} \left\{ \frac{F_e^2 - \frac{t}{4M^2}F_m^2}{1-\frac{t}{4M^2}} - \frac{t}{2M^2}F_m^2 \operatorname{tg}^2\frac{\vartheta}{2} \right\}$$

(норм формула, да?)

Обратим внимания на то, что формфакторы F_e и F_m дают независимые вклады в сечение, интерференционные члены отсутствуют. Это оправдывает целесообразность сделанного выбора формфакторов.

ЛЛ4-? Найти сечение рассеяния электрона на адроне со спином 0

(зачем она?)

Решение следующее:

Используя (138.5), имеем вместо (139.3)

$$M_{fi} = -\frac{4\pi e^2}{q^2} \left(\bar{u}'_e \left(\gamma P_h\right) u_e\right) F\left(q^2\right)$$

Для сечения получаем

$$d\sigma = \frac{\pi\alpha^2 dt \left[(s-u)^2 + (4M^2 - t)\, t\right]}{\left[s-(M+m)^2\right]\left[s-(M-m)^2\right] t^2} F^2(t)$$

(обозначения те же, что и в (139.4)). При $|t| \gg m^2$

$$d\sigma = do'_e \frac{\alpha^2}{4\varepsilon_e^2} \frac{\cos^2\frac{\vartheta}{2}}{\sin^4\frac{\vartheta}{2}} \frac{F^2(t)}{1+\frac{2\varepsilon_e}{M}\sin^2\frac{\vartheta}{2}}$$

(обозначения те же, что и в (139.10)).

33.2.3 Low-Energy Theorem for Bremsstrahlung

обзор и мотивация

Theory

В $ 98 был исследован процесс испускания фотона при столкновении частиц в пределе, когда частота фотона стремится к нулю. Оказалось, что амплитуда процесса обратно пропорциональна ω и простым образом выражается через амплитуду того же столкновения без испускания мягкого фотона (об этой последней мы будем снова говорить условно как об амплитуде «упругого» рассеяния и обозначать ее как $M_{fi}^{\text{упр}}$) .В следующем по ω приближении будет

$$M_{fi} = M_{fi}^{(-1)} + M_{fi}^{(0)}$$

где к главному члену ($\propto \omega^{-1}$) добавляется не зависящий от ω ($\propto \omega^0$) поправочный член. Мы увидим, что и этот поправочный (как и главный) член может быть выражен через $M_{fi}^{(\text{упр})}$, причем независимо от деталей электромагнитной структуры адрона. Это утверждение называют низкоэнергетической теоремой для тормозного излучения (*F.E.Low*, 1958).

Мы видели в §98, что основной вклад в амплитуду испускания мягкого фотона (отвечающий первому члену в (140.1)) возникает от диаграмм, в которых фотон излучается непосредственно начальной или конечной частицей. Это - диаграммы вида

(тут диаграммы с кругом в центре)

в противоположность диаграммам вида
(тут диаграмма с кругом в центре)
которых фотонная линия выходит из внутренних частей диа- диаграммы. Для графиков А40.2) характерно, что они могут быть рассечены на две части путем пересечения одной линии вирту- виртуального адрона (начального или конечного). Другими словами,
они иллюстрируют существенное в данном аспекте свойство: на- наличие одночастичного промежуточного состояния с одним ад- роном. Мы видели в пар 79, что в силу требований унитарности это свойство уже само по себе приводит к появлению полюсной особенности в амплитуде.

Предположим для простоты, что из двух сталкивающихся ад- ронов электрически заряжен (и потому может излучать) лишь один (первый) и что оба адрона не имеют спина. Волновые амплитуды и таких адронов — скаляры, которые полагаем рав- равными 1.

Тогда вклад в амплитуду от полюсной части диаграммы А40.2,а) имеет вид
Тогда вклад в амплитуду от полюсной части диаграммы $(140.2, a)$ имеет вид

$$iM_{fi}^{(a)} = \sqrt{4\pi} e_\mu^* \left(2p_1^\mu - k^\mu\right) eF \frac{1}{\left(p_1 - k\right)^2 - M^2} i\Gamma$$

Первый множитель отвечает фотону k (e_μ— его 4-вектор поляризации). Второй множитель отвечает электромагнитной адронной вершине (жирная точка на диаграмме); она записана в форме (138.5), F - формфактор адрона. Третий множитель - пропагатор виртуального адрона $p_1 - k$(M - его масса). Наконец, множитель i Γ обозначает весь остальной блок. Последний отличается от амплитуды упругого процесса

заменой реального адрона p_1 виртуальным $p_1 - k$. Среди первых членов разложения выражения (140.4) по степеням ω будут члены: 1) обратно пропорциональные $\omega, 2)$ не зависяшие от ω, но зависящие от направления $\mathbf{k}, 3)$ не зависящие от ω, k вовсе. Члены третьего (и только такого) рода возникнут также от «неособых» диаграмм - диаграмм вида (140.3),
не содержащих полюсной особенности, и от неполюсных частей диаграмм (140.2). Мы увидим, что все такие члены вместе однозначно определяются по членам первых двух типов условием калибровочной инвариантности и потому не требуют специального вычисления.

Амплитуда упругого процесса (140.5) зависит лишь от двух инвариантных переменных:

$$s = \left(p_1 + p_2\right)^2 = \left(p_1' + p_2'\right)^2, \quad t = \left(p_2' - p_2\right)^2$$

Замена p_1 на $p_1 - k$ не только превращает s в $\left(p_1 - k + p_2\right)^2$, но вводит еще и зависимость от новой переменной

$$\left(p_1 - k\right)^2 - M^2 = -2\left(p_1 k\right)$$

характеризующей «нефизичность» импульса $p_1 - k$. Но уже пер- вый член разложения по этой новой (малой) переменной устраняет особенность в амплитуде (140.4) и потому может дать в ней лишь не зависящие от k члены, которые, согласно сказанному выше, нас пока не интересуют. Таким образом, мы приходим к важному заключению, что вместо величины Γ в (140.4) можно подставить физическую амплитуду $M_{fi}^{(\text{упр})}(s,t)$, лишь заменив в ней

$$s \to \left(p_1 + p_2 - k\right)^2 = s - 2k\left(p_1 + p_2\right)$$

Первые члены ее разложения:

$$\Gamma \to M_{fi}^{(\text{упр})}(s,t) - 2\left(kp_1 + kp_2\right)\left(\frac{\partial M_{fi}^{(\text{упр})}}{\partial s}\right)_t$$

По такой же причине несуществен тот факт, что электромагнитный формфактор F относится здесь к вершине, в которой из двух адронных концов (p_1 и $p_1 - k$) лишь один физический. Его можно поэтому заменить рассмотренным в §138 формфактором вершины

с двумя физическими концами, а поскольку фотон k в данном случае реальный, то $F(k^2) = F(0) = Z_1$, где eZ_1 - заряд адрона. Таким образом, находим из (140.8)

$$M_{fi}^a = Z_1 \quad e\sqrt{4\pi}\frac{2(e^*p_1)}{-2(kp_1)}M_{fi}^{(\text{упр})} - \\ -Z_1 e\sqrt{4\pi}2\,(e^*p_1)\,\frac{1}{-2(kp_1)}\cdot 2\,(p_2k)\,\frac{\partial M_{fi}^{(\text{упр})}}{\partial s} + \ldots$$

где многоточие означает члены, не зависящие от k вовсе (между тем как второй член в (140.8) зависит от направления $\mathbf{k}$). Аналогичным образом найдем, что вклад в M_{fi} диаграммы (140.2,6)

отличается от (140.8) заменой p_1, p_2, k на $p_1', p_2', -k$. Для главного члена разложения получим в результате известное уже нам выражение

$$M_{fi}^{(-1)} = Z_1 e\sqrt{4\pi}\left(\frac{(p_1'e^*)}{(p_1'k)} - \frac{(p_1e^*)}{(p_1k)}\right)M_{fi}^{(\text{угр})}$$

(ср.(98.5)) Не зависящие же от k члены можно определить из требования, чтобы амплитуда в целом была калибровочно-инвариантна. Именно, она не должна меняться при замене $e^* \to e^* + \text{const}\cdot k$, т. е. должна иметь вид $M_{fi} = e_\mu^* J^\mu$, причем $k_\mu J^\mu = 0$. Легко видеть, что для этого нужно добавить к (140.8) не зависящий от k член

$$-2Z_1 e\sqrt{4\pi}\,(p_2e^*)$$

и аналогично для диаграммы $(140.2, 6)$. В результате получим окончательно

$$M_{fi}^{(0)} = 2Z_1 e\sqrt{4\pi}e_\mu^*\left[p_1^\mu\frac{(p_2k)}{(p_1k)} - p_2^\mu + p_1^\mu\frac{(p_2'k)}{(p_1'k)} - p_2'^\mu\right]\frac{\partial M_{fi}^{\text{упр}}}{\partial s}$$

Эта формула решает поставленную задачу. Ее можно представить в более компактном виде, заменив тождественно

$$2p_{2\nu}\left(\frac{\partial}{\partial s}\right)_t = \left(\frac{\partial}{\partial p_1^\nu}\right)_{p_1', p_2, p_2'}$$

(и аналогично для $\partial/\partial p_1'$) и введя дифференциальные операторы

$$\widehat{d}_{1\mu} = \frac{p_{1\mu}}{(p_1k)}k^\nu\frac{\partial}{\partial p_1^\nu} - \frac{\partial}{\partial p_1^\mu}$$

(и аналогично $\widehat{d}_{1\mu}'$). Тогда

$$M_{fi}^{(0)} = Z_1 e\sqrt{4\pi}e_\mu^*\left(\widehat{d}_1^\mu + \widehat{d}_1'^\mu\right)M_{fi}^{(\text{упр})}$$

Сечение определяется квадратом $|M_{fi}|^2$ с требуемой точноСТЬЮ

$$|M_{fi}|^2 = \left|M_{fi}^{(-1)}\right|^2 + 2\,\text{Re}\left(M_{fi}^{(-1)}M_{fi}^{(0)*}\right)$$

Второй член дает искомую поправку к сечению излучения. Просуммировав по поляризациям фотона, получим для этой поправки следующее выражение:

$$-4\pi\,(Z_1e)^2\left(\frac{p'}{(p'k)} - \frac{p}{(pk)}\right)^\mu\left(\widehat{d}_1' + \widehat{d}_1\right)_\mu\left|M_{fi}^{(\text{упр})}\right|^2$$

Таким образом, поправка к сечению излучения выражается через сечение упругого процесса и его производную по s.

Если заряженный адрон имеет спин 1/2, то вся принципи- принципиальная сторона вычислений остается прежней. Меняется лишь конкретный вид вершин и пропагаторов. При этом оказывается,

что после усреднения по поляризациям адронов и фотона оста- остается справедливой формула A40.14) (T. N. Burnett, N. M. Kroll,

1968).

33.2.4 Low-Energy Theorem for Photon-Hadron Scattering

обзор и мотивация

Theory

В пределе малых частот сечение рассеяния фотона на всякой неподвижной заряженной частице стремится к своему классическому значению, даваемому формулой Томсона.

Этому пределу соответствует не зависящая от частоты фотона ω амплитуда, которую обозначим через $M_{fi}^{(0)}$. Оказывается, однако, что и для рассеяния фотона (как и для рассмотренного в предыдущем параграфе тормозного излучения) от деталей электромагнитной структуры адрона не зависит не только этот первый, но и следующий член разложения амплитуды по степеням ω:

$$M_{fi} = M_{fi}^{(0)} + M_{fi}^{(1)}$$

где $M^{(1)} \sim \omega$(*F.E.* Low, 1954; M. Gell-Mann, *M.* L. Goldberger, 1954) Рассматриваемый процесс изображается диаграммами трех видов:

(диаграммы)

из которых первые две снова характеризуются наличием одноча- стичного промежуточного состояния и потому обладают полюс- полюсной особенностью.

Аргументация и принципиальная сторона вычислений оста- остаются теми же, что и в § 140. Достаточно фактически вычис- вычислить лишь вклад от полюсных частей диаграмм A41.2,а—б"),

причем электромагнитные вершины в них выражаются через статические формфакторы (заряд Ze и аномальный магнитный момент μ).

Однако, в отличие от случая тормозного излучения, интере- интересующие нас теперь поправки к сечению комптон-эффекта суще- существуют лишь для частиц со спином. Дело в том, что в случае тормозного излучения кроме поправок, связанных со спином,

имеются также поправки, связанные с энергетической зависимо- зависимостью амплитуды «упругого» процесса. Но в данном случае роль последней играют формфакторы, которые для «физических кон- концов» сводятся к постоянным и от энергии не зависят. Поэтому для рассеяния фотона поправки возникают только за счет маг- магнитного момента, отсутствующего у частиц без спина. Ниже мы рассмотрим рассеяние фотона на адроне со спином 1/2.

Понимая под M_{fi} вклад в амплитуду рассеяния от полюсных диаграмм, имеем (ср. (86.3),(86.4))

$$M_{fi} = -4\pi(Ze)^2 e'^*_\mu e_\nu \,(\bar{u}' Q^{\mu\nu} u)$$

где

$$Q^{\mu\nu} = (\gamma^\mu + S'^\mu)\,\tfrac{\gamma p+\gamma k+M}{s-M^2}\,(\gamma^\nu - S^\nu) + \\ + (\gamma^\nu - S^\nu)\,\tfrac{\gamma p-\gamma k'+M}{u-M^2}\,(\gamma^\mu + S'^\mu)$$

$$s = (p+k)^2 = (p'+k')^2\,, \quad u = (p-k')^2 = (p'-k)^2$$

и для краткости введены обозначения

$$\mu_{\text{ан}}\sigma^{\mu\lambda}k_\lambda = ZeS^\mu, \quad \mu_{\text{ан}}\sigma^{\mu\lambda}k'_\lambda = ZeS'^\mu$$

Переставляя операторы $\gamma p + M$ и учитывая уравнения

$$\bar{u}'(\gamma p' - M) = (\gamma p - M)u = 0$$

можно преобразовать выражение (141.4) к виду

$$\begin{aligned} Q^{\mu\nu} = &\left[(\gamma^\mu + S'^\mu)\frac{(\gamma k)\gamma^\nu + 2p^\nu}{2(pk)} + \frac{\gamma^\nu(\gamma k) - 2p'^\nu}{2(p'k)}(\gamma^\mu + S'^\mu)\right] - \\ &- \left[\frac{\gamma^\mu(\gamma k') + 2p'^\mu}{2(p'k')}S^\nu + S^\nu\frac{\gamma^\mu(\gamma k') - 2p^\mu}{2(pk')}\right] - \\ &- \left[S'^\mu\frac{\gamma p + \gamma k + M}{2(pk)}S^\nu - S^\nu\frac{\gamma p - \gamma k' + M}{2(pk)'}S'^\mu\right] \end{aligned}$$

Такая форма записи (и аналогичная с переставленными k и k') делает очевидной калибровочную инвариантность выражения (141.3), условием которой являются равенства

$$k'_\mu(\bar{u}'Q^{\mu\nu}u) = (\bar{u}'Q^{\mu\nu}u)k_\nu = 0$$

(при проверке надо помнить, что $(\gamma k)(\gamma k) = 0, kS = k'S' = 0$).

Поскольку полюсная часть амплитуды рассеяния оказывается, таким образом, калибровочно-инвариантной уже сама по себе, должна быть инвариантной сама по себе также и регулярная часть амплитуды, включающая в себя и вклад диаграммы (141.2, 6). Отсюда в свою очередь следует, что разложение этой части по степеням k и k' должно начинаться с квадратичных членов (ср. аналогичное замечание в связи с условием (127.5)). Другими словами, регулярная часть амплитуды содержит лишь члены, начиная с пропорциональных $\omega\omega'$ т. е. не дает никакого вклада в интересующие нас члены, пропорциональные ω^0 и ω^1. Все последние содержатся, следовательно, в выражении (141.3) Для их фактического вычисления выбираем лабораторную систему отсчета, в которой покоится начальный адрон. Для фотонов же выбираем трехмерно поперечную калибровку, в которой $e_0 = e'_0 = 0$. Тогда $(pe) = 0, (p'e'^*) \sim |\mathbf{p}'| \sim \omega$, и из (141.6) видно, что первые члены разложения M_{fi} будут пропорциональны ω^0, а члены, содержащие $\mu_{\text{ан}}$, дадут вклад лишь в члены пропорциональные ω^1.

Волновые амплитуды начального и конечного адронов в лабораторной системе отсчета с нужной точностью имеют вид

$$u = \sqrt{2M}\begin{pmatrix} w \\ 0 \end{pmatrix}, \quad \bar{u}' = \sqrt{2M}\left(w'^*, -\frac{w'^*}{2M}(\mathbf{k} - \mathbf{k}')\sigma\right)$$

где w, w' – 3 -спиноры. Прямое вычисление приводит к следующему результату:

$$\begin{gathered} M_{fi}^{(0)} = -8\pi(Ze)^2(\mathbf{e}'^*\mathbf{e})(w'^*w) \\ M_{fi}^{(1)} = -16\pi i M\mu_{\text{aH}}^2\omega(w'^*\sigma w)[[\mathbf{n}'\mathbf{e}'^*][\mathbf{ne}]] - \\ -4\pi i Ze\mu_{\text{ar}}\omega(w'^*\boldsymbol{\sigma}w)\{\mathbf{n}([\mathbf{ne}]\mathbf{e}'^*) + [\mathbf{ne}](\mathbf{ne}'^*) - \\ -\mathbf{n}'([\mathbf{n}'\mathbf{e}'^*]\mathbf{e}) - [\mathbf{n}'\mathbf{e}'^*](\mathbf{ne}) - 2[\mathbf{e}'^*\mathbf{e}]\}, \end{gathered}$$

где $\mathbf{n} = \mathbf{k}/\omega, \mathbf{n}' = \mathbf{k}'/\omega'$ Сечение рассеяния

$$d\sigma = \frac{1}{64\pi^2}|M_{fi}|^2\frac{\omega'^2}{M^2\omega^2}do'$$

(см. (64.19)). Для рассеяния на заряженной частице отличны от нуля как $M_{fi}^{(1)}$, так и $M_{fi}^{(0)}$. Принятая точность допускает при этом сохранение в квадрате $|M_{fi}|^2$ членов $\left|M_{fi}^{(0)}\right|^2$ и $\text{Re}\left(M_{fi}^{(0)}M_{fi}^{(1)*}\right)$. Первый дает томсоновское сечение. Второй же обращается в нуль

при усреднении по поляризациям фотонов и адронов. Поэтому при рассеянии на заряженном адроне рассматриваемые поправки проявляются только в поляризационных эффектах.

Для рассеяния же на электрически нейтральном адроне $M_{fi}^{(0)} = 0$ и сечение определяется квадратом $\left|M_{fi}^{(1)}\right|^2$. После усреднения по поляризациям начальных и суммирования по поляризациям конечных частиц оно оказывается равным (в обычных единицах)

$$d\sigma = \frac{2\mu^4\omega^2}{\hbar^2 c^4}\left(2 + \sin^2\vartheta\right) do'$$

где ϑ - угол рассеяния фотона, а аномальный магнитный момент совпадает с полным моментом μ. Отметим, что по своей угловой зависимости это сечение соответствует случаю антисимметрического рассеяния (см. задачу 2 к §60).

33.2.5 Multipole Moments of Hadrons

обзор и мотивация

Theory

Рассмотрим теперь ток перехода, соответствующий такой же как A38. 2), диаграмме, в которой, однако, линии p_1 и p_2 отвечают разным частицам (массы M_1 и M_2); фотонную линию $k = p_1 - p_2$ удобнее представлять Здесь исходящей из вершины. При этом фотон может быть теперь как виртуальным, так и реальным: должно быть лишь $k^2 < (M_1 - M_2)^2$, так что значение $k^2 = 0$ допустимо. Таким образом, применения рассматриваемой диаграммы включают в себя, в частности, процессы испускания фотона при превращениях частиц, в том числе ядер (в последнем случае начальной и конечной частицами является ядро в различных состояниях). В связи с поставленным вопросом наиболее интересен случай, когда длина волны фотона велика по сравнению с характерными «размерами» частицы (т. е. размерами, входящими в ее формфакторы; для ядра они совпадают, конечно, с его «радиусом»). Тогда ток перехода может быть разложен по степеням k^1). Отметим прежде всего, что должно быть

$$J_{fi} = 0 \quad \text{при} \quad k = 0$$

Действительно, пределу $k \to 0$ отвечает постоянный в пространстве и времени потенциал. Но такой потенциал не имеет физического значения и не может являться причиной каких-либо реальных процессов. К этому же выводу можно подойти и с более формальной точки зрения: рассмотренные в §138 токи были отличны от нуля при $k = 0$ за счет членов, пропорциональных 4-вектору $P = p_1 + p_2$; но при $M_1 \neq M_2$ произведение $(Pk) \neq 0$, так что такие члены запрещены условием поперечности тока. Запишем условие поперечности тока $J_{fi} = (\rho_{fi}, \mathbf{J}_{fi})$ в трехмерном виде:

$$\mathbf{k}\mathbf{J}_{fi} = \omega\rho_{fi}$$

Этому условию можно удовлетворить двумя способами:

$$\mathbf{J}_{fi} = \omega\mathbf{v}(\mathbf{k},\omega), \quad \rho_{fi} = \mathbf{k}\mathbf{v}(\mathbf{k},\omega)$$

или

$$\mathbf{J}_{fi} = [\mathbf{k}\mathbf{a}(\mathbf{k},\omega)], \quad \rho_{fi} = 0$$

Здесь $\mathbf{v}$ - некоторый полярный, а а - аксиальный векторы. В первом случае говорят о токе электрического, а во втором - магнитного типа. Согласно (142. 2) $\mathbf{v}$ и а при $\mathbf{k}, \omega \to 0$ остаются конечными или обращаются в нуль.

Пусть энергия фотона $\omega \ll M_1$. Тогда можно пренебречь эффектом отдачи и считать покоящейся (в системе покоя частицы M_1) также и конечную частицу M_2; при этом ω

становится заданной величиной: $\omega = M_1 - M_2$. Состояния покоящихся частиц M_1 и M_2 характеризуются трехмерными спинорами w_1 и w_2 рангов $2s_1$ и $2s_2$, где s_1 и s_2 - спины частиц. Ток перехода должен быть билинейной комбинацией w_1 и w_2^*. Из произведений компонент этих спиноров можно составить неприводимые тензоры рангов $l = s_1 + s_2, ..., |s_1 - s_2|$ (при заданном l это будет истинный или псевдотензор в зависимости от внутренних четностей частиц M_1 и M_2). Кроме этих тензоров в нашем распоряжении имеется только вектор $\mathbf{k}$. Чтобы построить первый член разложения тока по степеням $\mathbf{k}$, надо с помощью этих величин составить вектор как можно более низкой степени по k. Мы достигнем этой цели взяв тензор наименьшего ранга и умножив его скалярно $l-1$ раз на вектор к. Это и будет полярный вектор $\mathbf{v}$ или аксиальный вектор a.

Пусть Q_{lm} - сферические компоненты тензора, составленного из волновых амплитуд частиц. Сферические же компоненты тензора ранга $l-1$, составленного из компонент $\mathbf{k}$, равны $|\mathbf{k}|^{l-1}Y_{l-1,m}(\mathbf{n})$, где $\mathbf{n} = \mathbf{k}/\omega$. По общему правилу сложения сфе- рических тензоров (см. III, (107,3)) сферические компоненты,

вектора $\mathbf{v}$ можно написать в виде

$$\mathbf{v}_\lambda = (-1)^{\lambda+1} i^l \frac{\sqrt{4\pi}}{(2l-1)!!}\sqrt{\frac{2l+1}{l}}|\mathbf{k}|^{l-1}\times \\ \times \sum_m \begin{pmatrix} l-1 & 1 & l \\ \lambda+m & -\lambda & -m \end{pmatrix} Q_{l,-m} Y_{l-1,\lambda+m}(\mathbf{n}),$$

где λ пробегает значения 0,±1 (о выборе общего множителя см. ниже). Используя формулы (7.16), можно выразить $\mathbf{v}$ через шаровые векторы:

$$\mathbf{v} = i^l \frac{\sqrt{4\pi}|\mathbf{k}|^{l-1}}{(2l-1)!!\sqrt{l(2l+1)}} \sum_m ((-1)^{l-m} Q_{l,-m}\times \\ \times \left[\sqrt{l+1}\mathbf{Y}^{(\ni)}_{lm}(\mathbf{n}) + \sqrt{l}\mathbf{Y}^{(\mathrm{n})}_{lm}(\mathbf{n})\right].$$

Подставив в (142. 4), найдем El-ток перехода: $\mathbf{J}_{fi} = i^l \frac{\sqrt{4\pi}\omega|\mathbf{k}|^{l-1}}{(2l-1)!!\sqrt{l(2l+1)}} \sum_m (-1)^{l-m} Q^{(\ni)}_{l,-m}\times$

$$\times \left[\sqrt{l+1}\mathbf{Y}^{(\ni)}_{lm}(\mathbf{n}) + \sqrt{l}\mathbf{Y}^{(\mathrm{m})}_{lm}(\mathbf{n})\right]$$

Подставив в (142. 4), найдем El-ток перехода:

$$\mathbf{J}_{fi} = i^l \frac{\sqrt{4\pi}\omega|\mathbf{k}|^{l-1}}{(2l-1)!!\sqrt{l(2l+1)}} \sum_m (-1)^{l-m} Q^{(\ni)}_{l,-m}\times \\ \times \left[\sqrt{l+1}\mathbf{Y}^{(\ni)}_{lm}(\mathbf{n}) + \sqrt{l}\mathbf{Y}^{(\mathrm{n})}_{lm}(\mathbf{n})\right] \\ \rho_{fi} = i^l \frac{\sqrt{4\pi|\mathbf{k}|^l}}{(2l-1)!!\sqrt{l(2l+1)}} \sum_m (-1)^{l-m} Q^{(9)}_{l,-m} Y_{lm}(\mathbf{n})\Big]$$

(мы различаем везде $|\mathbf{k}|$ и ω, имея в виду возможные применения как к реальным, так и к виртуальным фотонам, для которых эти величины не совпадают).

В (142. 7),(142. 8) подразумевается, что сферический тензор Q_{lm} (обозначенный здесь $Q^{(\ni)}_{lm}\Big)$ - истинный тензор. Если же это псевдотензор (в таком случае обозначим его $Q^{(\mathrm{M})}_{\mathrm{lm}}$), то формула (142. 6) определит псевдовектор a. Подстановка в (142. 5) дает тогда Ml -ток перехода:

$$\mathbf{J}_{fi} = i^l \frac{\sqrt{4\pi}}{(2l-1)!!}\sqrt{\frac{l+1}{l(2l+1)}}|\mathbf{k}|^l \sum_m (-1)^{l-m} Q^{(\mathrm{M})}_{l,-m}\mathbf{Y}^{(\mathrm{M})}_{lm}(\mathbf{n}), \\ \rho_{fi} = 0$$

Величины $Q_{lm}^{()}$ и $Q_{lm}^{(\mathrm{M})}$ представляют собой адронные электрические и магнитные мультипольные моменты перехода. Их роль в электродинамике адронов вполне аналогична роли соответствующих величин в электродинамике электронов. В то время, однако,

как для электронных систем эти моменты могут быть, в принципе, вычислены по волновым функциям (как матричные элементы соответствующих операторов), в электродинамике адронов они выступают как феноменологические величины, значения которых находятся из опыта.

Нормировка этих величин в (142.7) – (142.9) выбрана в соответствии с их определением в §46. В этом можно убедиться, рассматривая токи (142.7) – (142.9) как компоненты Фурье тока перехода в координатном представлении. Так, разложив множитель $e^{-i\mathbf{kr}}$ в интеграле

$$\rho_{fi}(\mathbf{k}) = \int \rho_{fi}(\mathbf{r}) e^{-i\mathbf{kr}} d^3x$$

с помощью формулы (46. 3), получим

$$\rho_{fi}(\mathbf{k}) = 4\pi i^l \sum_{l,m} Y_{lm}(\mathbf{n}) \int \rho_{fi}(\mathbf{r}) Y_{lm}^* \left(\frac{\mathbf{r}}{r}\right) g_l(|\mathbf{k}|r) d^3x$$

Оставив здесь член с наименьшим l, для которого интеграл отличен от нуля, и заменив функцию $g_l(|\mathbf{k}|r)$ при $|\mathbf{k}|r \ll 1$ ее первым членом разложения (46.5), мы вернемся к формуле (142.9), причем

$$Q_{lm}^{(\ni)} = \sqrt{\frac{4\pi}{2l+1}} \int r^l \rho_{fi}(\mathbf{r}) Y_{lm} \left(\frac{\mathbf{r}}{r}\right) d^3x$$

в соответствии с определением (46. 7). Покажем также, что при применении к испусканию реального фотона полученные формулы приводят к уже известным нам результатам.

Амплитуда перехода с испусканием фотона с импульсом $\mathbf{k} = \omega\mathbf{n}$ и поляризацией $e = (0, \mathbf{e})$:

$$M_{fi} = -e\sqrt{4\pi}\mathbf{e}^* \mathbf{J}_{fi}$$

Если в начальном и конечном состояниях ядро обладает определенным значением проекции момента (M_i и M_f), то в сумме по m в (142.7) – (142.9) остается лишь по одному члену: $m = M_i$ - $-M_f$. Поскольку согласно (16. 23) произведения $\mathbf{Y}_{lm}^{(\ni)}\mathbf{e}^{(\lambda)*}$ или $\mathbf{Y}_{lm}^{(\mathrm{M})}\mathbf{e}^{(\lambda)*}$ ($\lambda = \pm 1-$ спиральность фотона, $e^{(\lambda)} \perp \mathbf{n}$) пропорци- ональны $D_{\lambda m}^l$, мы возвращаемся к формулам, рассмотренным в $48

Дифференциальная вероятность излучения [1])

$$dw = 2\pi\delta\left[\omega - (E_i - E_f)\right] |M_{fi}|^2 \frac{d^3k}{2\omega(2\pi)^3}$$

($E_i, E_f -$ начальная и конечная энергия ядра). Полная вероятность получится суммированием по поляризациям и интегрированием no d^3k. Подставив (142. 7) или (142. 9) в (142. 12) и затем в (142. 13) и произведя указанные действия, мы вернемся к формуле (46.9)(или (47. 2)). Формулы (142.7) – (142.9) включают в себя все случаи, которые могут иметь место для испускания реального фотона. Для виртуальных же фотонов возможен еще и другой случай, не описываемый этими формулами (*R.* H. Fowler, 1930).

Если спины и четности начального и конечного состояний ядра одинаковы, то из их волновых амплитуд можно составить скаляр Q_0, а с его помощью - ток перехода вида

$$\rho_{fi} = Q_0\mathbf{k}^2, \quad \mathbf{J}_{fi} = Q_0\omega\mathbf{k}$$

Величину Q_0 называют монопольны. м ($E0$) моментом перехода. Для испускания реального фотона соответствующая амплитуда перехода обращается в нуль (так как $\mathbf{e}^*\mathbf{k} = 0$).

Монопольный ток, однако, может быть источником переходов, связанных с испусканием виртуального фотона. Более того, он является единственным таким источником при $s_1 = s_2 = 0$, когда все мульти-

По своей зависимости от ω и $\mathbf{k}$ монопольный ток (142. 14) аналогичен электрическому квадрупольному. Соответственно и момент Q_0 представляет собой величину того же порядка, что и квадрупольный момент. К этому заключению можно прийти также и путем истолкования (142. 14) как компоненты Фурье тока в координатном представлении. Разложив в (142. 10) множитель $e^{-i\mathbf{kr}}$ по степеням kr и положив функцию $\rho_{fi}(\mathbf{r})$ сфсрическисимметричной, получим

$$\rho_{fi}(\mathbf{k}) = -\frac{1}{6}\mathbf{k}^2 \int \rho_{fi}(r) r^2 d^3x$$

Сравнив с (142. 14), найдем

$$Q_0 = -\frac{1}{6} \int \rho_{fi}(r) r^2 d^3x$$

Сходство этой величины с квадрупольным моментом очевидно.

ЛЛ4-? вероятность ионизации атома из K -оболочки при ядерном Ml -переходе

Найти вероятность ионизации атома из K -оболочки за счет энергии возбуждения ядра ω (так называемая внутренняя конверсия γ -излучения) при ядерном Ml -переходе в пренебрежении энергией связи электрона в атоме и влиянием поля ядра на его волновые функции).

Процесс описывается диаграммой где p_1 и p_2 относятся к неподвижному ядру в различных состояниях, а $p = (m, 0)$ и $p' = (m + \omega, \mathbf{p}') - 4$ -импульсы начального и конечного электронов. Этой диаграмме отвечает амплитуда

$$M_{fi} = -e^2 \frac{4\pi}{q^2} \bar{u}\left(p'\right) \left(\gamma J_{fi}\right) u(p)$$

где J_{fi} - ток перехода ядра. После суммирования по конечным и усреднения по начальным поляризациям электрона получим

$$\frac{1}{2} \sum_{\text{поляр}} |M_{fi}|^2 = e^4 \frac{16\pi^2}{\left(q^2\right)^2} \left\{ q^2 \left(J_{fi} J_{fi}^*\right) + 4 \left(J_{fi} p\right) \left(J_{fi}^* p\right) \right\}$$

(использовано, что $J_{fi} q = 0$ и поэтому $J_{fi} p = J_{fi} p'$) . Вероятность конверсии вычисляется как

$$dw_{\text{конв}} = 2 \left|\psi_i(0)\right|^2 \left(\frac{|\mathbf{p}|}{m} d\sigma \right)_{\mathbf{p} \to 0}$$

(ср. примеч. на с. 710). Для Ml -переходов ток J_{fi} надо взять из (142.9). Интегрирование $dw_{\text{конв}}$ по $d\varepsilon'$ устраняет δ -функцию, а ингегрирование по do' обращает квадрат $\left|\mathbf{Y}_{lm}^{(\mathrm{m})}\right|^2$ в 1. В результате вероятность конверсии окажется выраженной через квадрат $\left|Q_{l,-m}^{(\mathrm{M})}\right|^2$. Но через эту же величину выражается согласно (46. 9) вероятность w_γ спонтанного излучения фотона при том же ядерном переходе. Окончательно получается

$$\frac{w_{\text{конв}}}{w_\gamma} = 2\alpha (Z\alpha)^3 \frac{m}{\omega} \left(1 + \frac{2m}{\omega}\right)^{l+1/2}$$

(это отношение называют коэффициентом конверсии).

ЛЛ4-? То же для ядерного *El* -перехода.

Тем же способом с током перехода из (142.7),(142.8) получается

$$\frac{w_{\text{конв}}}{w_\gamma} = 2\alpha(Z\alpha)^3 \left(1 + \frac{4l}{l+1}\frac{m^2}{\omega^2}\right)^{l-1/2} \frac{m}{\omega}$$

ЛЛ4-? То же для монопольного перехода ядра.

С током перехода из (142.14) получается

$$w_{\text{конв}} = 4\alpha^2(Z\alpha)^3 m^3\omega^2 \left(1 + \frac{2m}{\omega}\right)^{3/2} |Q_0|^2$$

Поскольку монопольное испускание фотона невозможно, исключить отсюда $|Q_0|^2$ нельзя

33.2.6 Inelastic Scattering of Electrons by Hadrons

обзор и мотивация

Theory

В §139 было рассмотрено упругое рассеяние электронов адронами. Аналогичным образом может быть поставлена задача о неупругом рассеянии. Отличие состоит в том, что конечное адронное состояние будет теперь отвечать другому адрону или же совокупности адронов. Закон сохранения импульса (139. 1) останется в силе, если под p'_h подразумевать 4-импульс конечного адрона или суммарный 4-импульс всей образовавшейся в процессе рассеяния совокупности адронов. Таким образом, теперь $p_h'^2 \neq p_h^2 \neq M^2$, где M - масса начального адрона.

С этим отличием процесс неупругого рассеяния описывается той же диаграммой (139. 2). Нижнюю вершину этой диаграммы мы обозначим через J_{fi}, как это делалось в §138. Однако в отличие от (138. 3) или (138. 6) мы не будем выражать ток перехода через вершинный оператор и амплитуды состояний, чтобы не фиксировать заранее характер конечного адронного состояния. Теперь можно записать амплитуду рассеяния в аналогичном (139. 3) виде

$$M_{fi} = -\frac{4\pi e^2}{(p_e - p'_e)^2} (\bar{u}'_e \gamma_\mu u_e) J^\mu_{fi}$$

(такая амплитуда уже использовалась в задаче 1 к §142, где рассматривалась передача энергии электрону; аналогичную структуру имеет амплитуда в задаче о возбуждении ядер электронами)

Будем считать энергию начального электрона достаточно большой, чтобы в конечном состоянии могло образоваться боль- большое число адронов. Мы будем интересоваться так называемым инклюзивным сечением, отвечающим тому,

что в конечном состоянии фиксируется только импульс электрона, а по всем адронным состояниям произведено суммирование.

Такое дифференциальное сечение запишем, в соответствии формулами §64, в виде

$$d\sigma = \frac{d^3p'_e}{4I(2\pi)^3 2\varepsilon'_e} \sum_f (2\pi)^4 \delta^{(4)} (p_h + p'_h - p_e - p'_e) |M_{fi}|^2$$

Инклюзивное сечение может зависеть лишь от трех кинематических инвариантов, которые могут быть определены путем измерений, производимых только над электронами. Таких инвариантов существует три:

$$t = q^2 \equiv (p_e - p'_e)^2, \quad s = (p_e + p_h)^2$$

и $p_h'^2$. Необходимость учета третьего инварианта связана с тем, что в отличие от упругого рассеяния p'^2 - «масса» конечного адронного состояния - теперь не задана. Вместо $p_h'^2$ удобно, однако, пользоваться инвариантом

$$\nu = qp_h$$

Связь между ν и ${p'^2}_h$ следует из равенства $p_h' = p_h + q$:

$$p_h'^2 = M^2 + t + 2\nu$$

Если начальный адрон стабилен (например, протон), то энергия

электрона в начальном и конечном состояниях ε_e и ε_e и угол рас сеяния θ. Ниже будем считать электрон ультрарелятивистским ($\varepsilon_e \gg m, \varepsilon_e' \gg m$) и пренебрегать его массой. Тогда в системе покоя начального адрона (лабораторная система) получим

$$t = -4\varepsilon_e\varepsilon_e' \sin^2(\theta/2), \quad \nu = M\left(\varepsilon_e - \varepsilon_e'\right), \quad s - M^2 = 2M\varepsilon$$

Подставив (143.1) в (143.2) и выполнив обычным образом суммирование по поляризациям электронов, получим сечение рассеяния неполяризованных электронов. Запишем его в виде

$$d\sigma = \frac{\alpha^2}{(q^2)^2}\frac{d^3p_e'}{(2\pi)^3 \cdot 8M\varepsilon_e\varepsilon_e'} w_{\mu\nu}W^{\mu\nu}$$

или

$$d\sigma = \frac{\alpha^2}{(q^2)^2}\frac{dtd\nu}{4\left(p_hp_e\right)^2} w_{\mu\nu}W^{\mu\nu}$$

где

$$w_{\mu\nu} = 4p_{e\mu}p_{e\nu}' - 2\left(p_{e\mu}q_\nu + p_{e\nu}q_\mu\right) + q^2g_{\mu\nu}$$
$$W^{\mu\nu} = \sum_f (2\pi)^4\delta^{(4)}\left(p_h' - p_h - q\right) J_{fi}^\mu J_{fi}^{\nu *}$$

Тензор $W^{\mu\nu}$, конечно, существенно зависит от свойств адронных токов, и мы можем в общем случае только поставить задачу о его феноменологической структуре, аналогичную задаче о формфакторах адронов. Прежде всего, воспользуемся тем, что тензорная структура $W^{\mu\nu}$ должна определяться только 4-векторами, имеющими отношение к нижней вершине диаграммы (139.2), т. е. p_h и q. Из них (а также метрического тензора $g_{\mu\nu}$) можно составить всего пять независимых тензоров. Требование инвариантности относительно обращения времени сводится к требованию симметричности тензора; таких тензоров можно построить четыре. Наконец, условие сохранения тока, т. е.

$$W^{\mu\nu}q_\nu = 0, \quad W^{\mu\nu}q_\mu = 0$$

сводит число независимых тензоров к двум. Их можно выбрать в виде

$$\tau_{\mu\nu}^{(1)} = \frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}, \quad \tau_{\mu\nu}^{(2)} = \left(p_{h\mu} - \frac{\nu}{t}q_\mu\right)\left(p_{h\nu} - \frac{\nu}{t}q_\nu\right)$$

и записать $W_{\mu\nu}$ как

$$W_{\mu\nu} = 4\pi MW_1\tau_{\mu\nu}^{(1)} + \frac{4\pi}{M}W_2\tau_{\mu\nu}^{(2)}$$

Подставив в (143.8) выражения (143.10) и (143.13), представим сечение в виде

$$d\sigma = \left(W_2 + 2W_1\,\mathrm{tg}^2\,\frac{\theta}{2}\right)d\varepsilon_e'd\sigma_{\text{упр}}$$

где

$$d\sigma_{\text{упр}} = \frac{\alpha^2}{4\varepsilon_e^2}\frac{\cos^2(\vartheta/2)}{\sin^4(\vartheta/2)}do'$$

- сечение рассеяния ультрарелятивистского электрона в кулоновом поле)ср. (80. 7))

Мы видим, что сечение определяется двумя структурными функциями, зависящими от двух инвариантов t и ν. Если при больших энергиях физика адронов не содержит характерных величин размерности массы (гипотеза масштабной инвариантно*сти*), то можно ожидать, что структурные функции будут зависеть при больших энергиях от единственного безразмерного параметра t/ν. Тогда функции W_1, W_2 должны иметь вид функций одной переменной:

$$W_1 = \frac{M}{\nu} F_1\left(\frac{t}{\nu}\right), \quad W_2 = \frac{M}{\nu} F_2\left(\frac{t}{\nu}\right)$$

(заметим, что отношение M/ν не зависит от M).

33.2.7 Electron-Positron Pair Conversion into Hadrons

обзор и мотивация

(а что, мне интересно почитать это.)

Theory

Рассмотрим теперь процесс превращения электрон-позитронной пары в адроны. Обозначим 4-импульсы электрона и позитрона через p_- и p_+, а 4 -импульс (суммарный) совокупности образующихся адронов через p_h; при этом $p_- + p_+ = p_h$. Процесс изображается диаграммой

Нижней вершине этой диаграммы отвечает ток перехода из ва- куума в некоторое адронное состояние $|n\rangle$, который обозначим, как это делалось в §104, через $\langle n|J|0\rangle$. Диаграмме (144.1) соответствует амплитуда рассеяния

$$M_n = -\frac{4\pi\alpha}{q^2} \bar{u}\left(-p_+\right) \gamma_\mu u\left(p_-\right) \langle n \left|J^\mu\right| 0\rangle$$

Мы будем интересоваться полным сечением аннигиляции в адроны σ_h, т. е. просуммируем по всем конечным состояниям $|n\rangle$. Тогда, в соответствии с (64.18),

$$\sigma_h = \frac{1}{4I} \sum_n \left|M_n\right|^2 (2\pi)^4 \delta^{(4)}\left(p_h - q\right)$$

где $q = p_- + p_+$. В дальнейшем будем пренебрегать массой электрона; тогда $q^2 = 2\left(p_- p_+\right), I = q^2/2$. Аналогично тому, как мы поступали в §143, запишем сечение в виде

$$\sigma_h = \frac{(4\pi)^2}{2t^3} w^{\mu\nu} W_{\mu\nu}$$

где

$$w^{\mu\nu} = \alpha\left(p_-^\mu q^\nu + p_-^\nu q^\mu - 2p_-^\mu p_-^\nu - 1/2 q^2 g^{\mu\nu}\right)$$
$$W_{\mu\nu} = \alpha \sum_n (2\pi)^4 \delta^{(4)}\left(p_h - q\right) \langle 0 \left|J_\nu\right| n\rangle \langle n \left|J_\mu\right| 0\rangle$$

и $t = q^2 > 0$ Заметим, что t является единственным кинематическим инвариантом рассматриваемой задачи («треххвостой» диаграммы (144.1)) и q - единственным 4 -вектором, от которого может зависеть $W_{\mu\nu}$. Поэтому с учетом требования сохранения тока тензор $W_{\mu\nu}$ можно представить в виде

$$W_{\mu\nu} = \frac{1}{2} \rho_h(t) \left(\frac{q_\mu q_\nu}{q^2} - g_{\mu\nu}\right)$$

где $\rho_h(t)$ - единственная инвариантная функция, зависяшая от свойств адронного тока и определяющая сечение аннигиляции. Подставив (144.5) – (144.7) в (144.4), получим

$$\sigma_h = \frac{4\pi^2\alpha}{t^2}\rho_h(t)$$

Обратим внимание на то, что функция $\rho_h(t) = -2W^\mu_\mu/3$ в точности совпадает с определенной в (104. 9) функцией $\rho(t)$, если в последней формуле понимать под токами адронные токи. Напомним также, что $\rho(t)$ является спектральной плотностью фотонной собственно-энергетической функции $\Pi(t)$: $\operatorname{Im}\Pi(t) = -\pi\rho(t)$. В рассматриваемом низшем приближении по α функция Π совпадает с поляризационным оператором $\mathcal{P}$. В этом приближении, следовательно, $\rho_h(t)$ является также и спектральной плотностью адронного вклада в поляризационный оператор:

$$\operatorname{Im}\mathcal{P}_h(t) = -\pi\rho_h(t)$$

Использовав дисперсионное соотношение (111.13) и выразив ρ_h через σ_h согласно (144.8), получим формулу

$$\mathcal{P}_h(t) = -\frac{t^2}{4\pi^2\alpha}\int\limits_0^\infty \frac{\sigma_h(t')\,dt'}{t'-t-i0}$$

выражающую адронный вклад в поляризацию вакуума через измеряемое на опыте сечение аннигиляции в адроны.

Заметим, что таким же точно способом можно было бы решить задачу об аннигиляции электрон-позитронной пары в мюонную пару (в первом приближении по α может образоваться только одна такая пара). Аналогично результату (144.8) мы получили бы

$$\sigma_\mu = \frac{4\pi^2\alpha}{t^2}\rho_\mu(t)$$

где $\rho_\mu(t)$ - спектральная плотность мюонной поляризации вакуума. Она отличается от электронной поляризации лишь заменой массы электрона m массой мюона μ и согласно (113.8) дается выражением

$$\rho_\mu(t) = \frac{\alpha}{3\pi}\left(t+2\mu^2\right)\sqrt{1-\frac{4\mu^2}{t}}$$

Подставив его в (144.11), мы воспроизведем результат, полученный уже в задаче 8 к §81.

33.3 Some Experiments for Testing QCD

(отдельную неделю потом подумаю про эксперименты!)

33.4 Other Topics of QCD

(наверняка еще что-то про это есть.)

34 Lectures on the Infrared Structure of Gravity and Gauge Theory

This is a redacted transcript of a course given by the author at Harvard in spring semester 2016. It contains a pedagogical overview of recent developments connecting the subjects of soft theorems, the memory effect and asymptotic symmetries in four-dimensional QED, nonabelian gauge theory and gravity with applications to black holes.

34.1 Introduction

34.1.1 The Infrared Triangle

These lectures concern a triangular equivalence relation that governs the infrared (IR) dynamics of all physical theories with massless particles. Each of the three corners of the triangle, illustrated in figure 34.1, represents an old and central subject in physics on which hundreds or even thousands (in the case of soft theorems) of papers have been written. Over the past few years we have learned that these three seemingly unrelated subjects are actually the same subject, arrived at from very different starting points and expressed in very different notations.

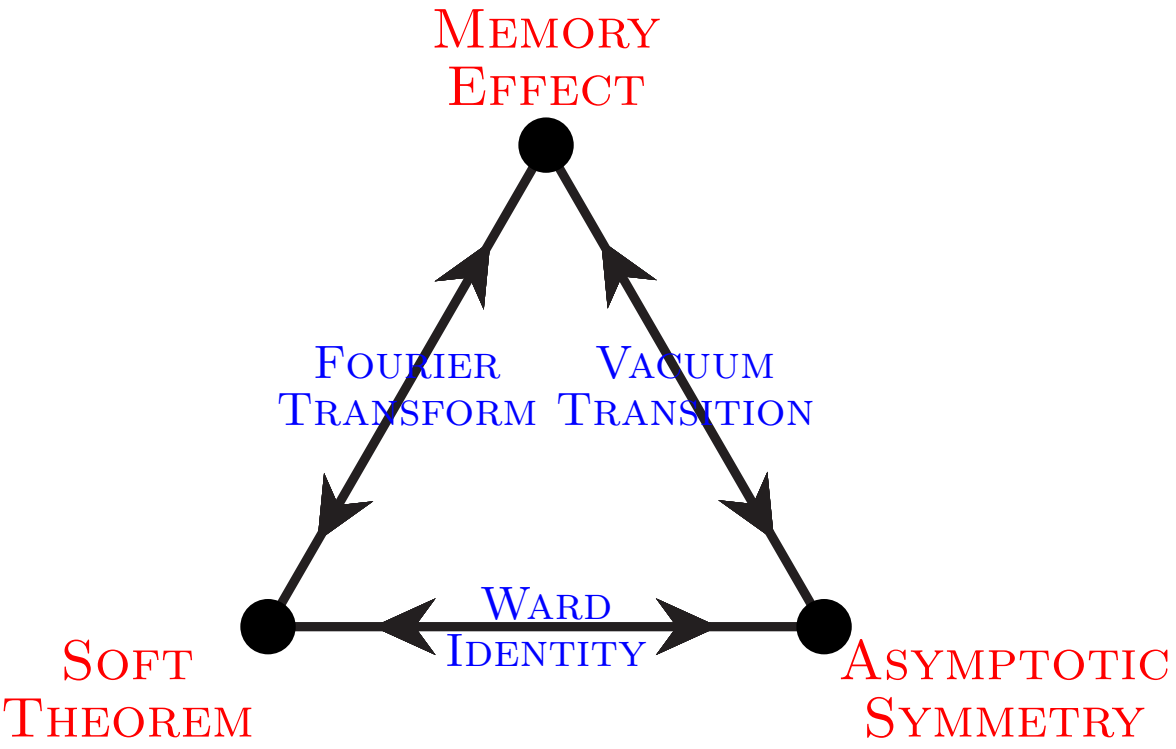


Figure 34.1: **The infrared triangle.**

The first corner is the topic of soft theorems. These originated in quantum electrodynamics (QED) in 1937 with the work of Bloch and Nordsieck [1], were significantly developed in 1958 by Low and others[2, 3, 4, 5**?**], and were generalized to gravity in 1965 by Weinberg [7]. Soft theorems characterize universal properties of Feynman diagrams and scattering amplitudes when a massless external particle becomes *soft* (i.e., its energy is taken to zero). These theorems tell us that a surprisingly large — in fact, infinite — number of soft particles are produced in any physical process, but in a highly controlled manner that is central to the consistency of quantum field theory.

The second corner is the subject of asymptotic symmetries. This is the study of the nontrivial exact symmetries or conserved charges of any system with an asymptotic region or boundary. One of the earliest examples appears in the pioneering work of Bondi, van der Burg, Metzner, and Sachs (BMS) [8, 9], who sought to recover the Poincaré group of special relativity as the symmetry group of asymptotically flat spacetimes in general relativity (GR). Instead, in a spectacular failure of their original program, they discovered the infinite-dimensional BMS group whose deep implications are still being unravelled today. In contrast, analogous asymptotic symmetries in QED and nonabelian gauge theory were discovered only recently [10, 11, 12, 13, 14] and are a subject of ongoing research [15, 16, 17, 18, 19, 20, 21, 22].

The third corner of the triangle is the memory effect, investigated in the context of gravitational physics in 1974 by Zel'dovich and Polnarev [23], and significantly developed by Christodoulou and others [24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. This is a subtle DC effect, in which the passage of gravitational waves produces a permanent shift in the relative positions of a pair of inertial detectors. Detection of the memory effect has been proposed at LIGO [34] or via a pulsar timing array [35, 36]. It is an exciting experimental prospect for the coming decades. Again, the gauge theory analog appeared only recently [37, 38, 39, 40]. This memory corner of the triangle provides an important physical realization of the more abstractly formulated results of the other two corners, giving direct observational consequences of the infinite number of symmetries and conservation laws.

Figure 34.1 also depicts the mathematical equivalence relations connecting the three corners. Perhaps the simplest is the connection between the soft theorem and the memory effect [41]. The former is a statement about momentum space poles in scattering amplitudes, while the latter concerns a DC shift in asymptotic data between late and early times. These are the same thing, because the Fourier transform of a pole in frequency space is a step function in time. The step function in turn can be understood as a domain wall connecting two inequivalent vacua that are related by an asymptotic symmetry [41]. Hence the memory effect both physically manifests and directly measures the action of the asymptotic symmetries. The triangle in figure 34.1 is closed by noting that every symmetry has a Ward identity that equates scattering amplitudes of symmetry-related states.[184] These Ward identities turn out to be nothing but the soft theorems — which relate amplitudes with and without soft particles — in disguise [42, 10, 11, 43].

It is a testimony to the unity of physics that these three seemingly disparate avenues of investigation led to the same mathematical structures. We will see in section 34.5 that Weinberg's 1965 soft graviton theorem reproduced the 1962 results of BMS, but in the wildly different language and notation of Feynman diagrams as opposed to asymptotic structures at null infinity. We will further see in section 34.6 that the 1987 Braginsky-Thorne formula for the gravitational memory effect is a Fourier transform of the Weinberg soft graviton theorem. Of course, Weinberg was scattering elementary particles, while Braginsky and Thorne were scattering black holes, but this distinction is irrelevant in the deep infrared!

The bigger picture emerging from the triangle is that deep IR physics is extremely rich, perhaps richer than previously appreciated. Every time we breathe, an infinite number of soft photons and gravitons are produced. Quantum field theory analyses tend to treat this as a technical problem to be overcome by resorting to carefully constructed inclusive cross sections. IR regulators are often used that explicitly break the symmetries, and it is difficult to see them emerge as the regulator is removed.[185] GR analyses tend to treat the BMS discovery that the deep IR limit of GR is not special relativity as a problem to be avoided by the ad hoc imposition of extra boundary conditions at infinity. In fact, BMS discovered the classical version of infinite graviton production, an equivalent indication of the richness of the deep IR. Here we will see that, far from being technicalities, these IR phenomena are associated with fundamental symmetries of nature whose fascinating consequences we are just beginning to unravel.

34.1.2 New Developments

New developments in this area go beyond demonstrating the equivalence of previously known phenomena. The three-dimensional perspective afforded by the triangle has led to better conceptual understanding of each old corner, discovery of new corners, and new research programs searching for and exploring new triangles. We now mention a few of the many new results that have emerged in the past few years.

Until recently, one corner of the triangle — asymptotic symmetries — was essentially unknown for any example in four-dimensional Minkowski space. The best understood case was the BMS action in gravity. BMS showed that infinite-dimensional subgroups $\mathrm{BMS}^{\pm}$ of the diffeomorphism

[184] Alternately, these identities can be derived as the vanishing of all matrix elements of the associated charge conservation laws.

[185] BMS_3 has been recovered as the large radius limit of gravity in AdS_3[44, 45, 46], but efforts [45] to recover BMS_4 in the large radius limit of AdS_4 have so far not succeeded.

group act nontrivially at future and past null infinity $\mathcal{I}^{\pm}$, transforming one set of Cauchy data to a physically inequivalent one. The full group $\text{BMS}^+ \times \text{BMS}^-$, however, is clearly not a symmetry of gravitational scattering. For example, it includes elements that boost the past but not the future. It was widely held (in part due to misconceptions about asymptotic falloffs cleared up only recently by Christodoulou and Klainerman [47]) that there is no nontrivial infinite-dimensional symmetry of gravitational scattering; see [48] for further discussion. However, the equivalence to the soft graviton theorem was used in [42] to show that, in a generic physical setting, a certain 'antipodal' subgroup of $\text{BMS}^+ \times \text{BMS}^-$ is an exact asymptotic symmetry of gravitational scattering. This discovery is detailed in section 34.5.2.

The triangular perspective further led to the conclusion that the vacuum in GR (as well as in gauge theory) is not unique, as it transforms nontrivially under the antipodal symmetries.[186] In particular, distinct vacua are found to have different angular momenta [51]. Ignoring the vacuum degeneracy leads to the false conclusion that angular momentum is not conserved (see figure 34.9)! This degeneracy resolves the so-called problem of angular momentum in GR. The electromagnetic analog is discussed in section 34.2.11, and the gravity version in section 34.5.

Another new development from the triangle involves a conjectured extension of the BMS group involving *superrotations*, a Virasoro-like symmetry that acts on the sphere at $\mathcal{I}$ [52, 53, 54, 55]. Because of certain singularities in the generators, standard asymptotic symmetry analyses could not establish whether these are bona fide symmetries. However, the result was established [56, 57] using Feynman diagrams and proving the equivalent soft theorem, as discussed in section 34.5.3.

In abelian and nonabelian gauge theories, various soft theorems have long been known. New symmetries in gauge theory, precise analogs of the antipodal subgroup of BMS in gravity, have been discovered using the triangular equivalence, as discussed in sections 34.2 and 34.4.

Given one corner of a triangle, others can be systematically determined. Insights from gravity can be applied to gauge theory and vice versa. One or more triangles potentially exists for every type of massless particle. Let us now discuss a few of these triangular variations.

34.1.3 Echoing Triangles

The IR triangle in figure 34.1 has many copies, which echo throughout much of physics, as illustrated in figure 34.2. Here are some examples.

QED, Yang-Mills Theory, Gravity, Massless Particles, ...

The simplest cases of the triangle involve massless particles associated with local gauge symmetries, such as the photon, which will be our first example. However, whenever there is a massless particle, one can consider the soft limit of the scattering amplitude. A full triangle, including symmetry and memory, is often found to emerge, even when no underlying local gauge symmetry exists. A simple example of this discussed here is the soft photino theorem [58], which lies at the corner of a triangle including an infinite number of asymptotic fermionic symmetries. Potentially, a triangle exists for every type of massless particle, many of which have yet to be studied from this perspective.

Leading, Subleading, Sub-Subleading, ...

Soft theorems are statements about the behavior of an amplitude at zero or low energy. In an expansion in powers of the soft energy, universal behavior may appear in the leading as well as the subleading or even sub-subleading orders. Indeed, Low [2, 4, 59] already discussed a subleading soft theorem in QED. Recent research derived the symmetry based on Low's subleading theorem [60, 61, 62]. The associated memory effect remains to be analyzed. In gravity, a new subleading soft theorem was proven at tree-level [63, 64, 65, 56] and was shown to imply [57, 66, 67, 68] superrotation symmetry [52, 53, 54, 55], a recently conjectured asymptotic symmetry. The third corner of the triangle, the spin memory effect, was derived in [69]. A proposal for measuring it using

[186] This result was partially anticipated in prescient work of Ashtekar [49]; see also Balachandran and Vaidya [50].

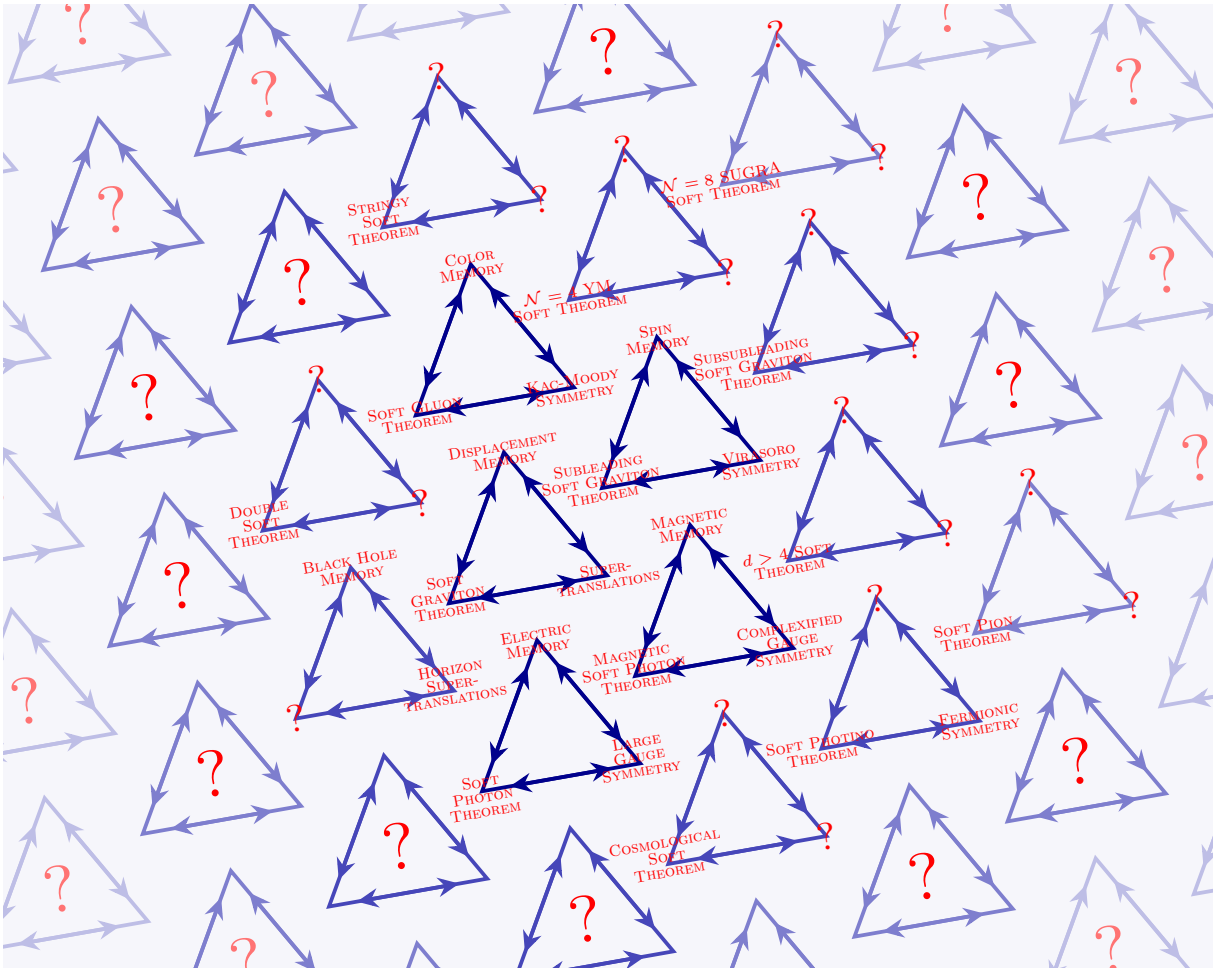


Figure 34.2: **The infrared triangle echoes throughout disparate areas of physics.**

the Einstein Telescope appears in Nichols [**?**]. A gravitational sub-subleading soft theorem has also been proven [56, 71, 72, 73, 74, 75, 76, 77], and the associated symmetry has been discussed in [78]. A complete classification of all possible soft theorems at various orders of the soft expansion has not appeared, although some discussion of it exists [79, 80]. New soft theorems at various orders have recently been discovered in nonabelian gauge theories [81, 82, 20, 77], supersymmetric theories [83, 84, 85], nonlinear sigma models [86], string theories [87, 88, 89, 90, 91, 92, 76, 93], and in other contexts [94, 95, 96, 97, 98].

Double Soft Theorems, ...

Interesting universal structures are known for limits when two or more external particles are taken to be soft. For example, for soft moduli in $\mathcal{N}=8$ supergravity, the structure constants of the hidden $E_{7(7)}$ symmetry algebra appear [99]. In recent years, double-soft limits have been further examined in gravity [100, 101], gauge theories [102, 103, 100, 104, 105], supersymmetric theories [83, 106, 103, 100], nonlinear sigma models [104, 86, 107], and string theory [103, 108, 93]. This behavior is presumably at the corner of yet another unexplored triangle and is perhaps related to structure constants or central charges of an asymptotic symmetry group.

Classical, Quantum, ...

There is a purely classical triangle involving tree-level soft theorems and classical symmetries. This regime is where these lectures begin. In the simplest cases, the soft theorem is uncorrected, and the quantum triangle is similar to the classical one. In general, however, an interesting story involving anomalies may appear [109, 110, 111, 112, 113, 114, 115, 116, 117].

Minkowski, $\boldsymbol{D < 4}$*,* $\boldsymbol{D > 4}$*, Cosmology, ...*

In these lectures, we consider only the physical case of four-dimensional Minkowski space. However, the triangle also appears in both higher dimensions [118, 119, 95] and in cosmology, where there are also soft theorems [120, 121, 122] and asymptotic symmetries [123]. The sub-subleading soft graviton theorem has been explored in arbitrary dimensions [72, 73]. The case of three-dimensional gravity, where there is generally better control, is especially interesting[44, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135]. However, since three-dimensional gravitons do not propagate, this case has no soft theorem. This feature gives the problem a rather different flavor, and I do not review it in these lectures.

Supersymmetry, $\boldsymbol{\mathcal{N} = 1, 2, ...}$

For many reasons it is also of interest to consider supersymmetric theories. The local $\mathcal{N} = 1$ case has been worked out [136, 137], as has the global one in [58]. New features should appear with soft scalars and R-symmetries for $\mathcal{N} = 4$ Yang-Mills and $\mathcal{N} = 8$ supergravity. Soft theorems in various supersymmetric theories have been worked out [109, 84, 116] and sometimes give rise to recursion relations [138]. At present, even the asymptotic symmetry group for these interesting higher-$\mathcal{N}$ cases is unknown.

Every one of the combined possibilities mentioned here can potentially result in a new triangle. So far, each story has its own unique and surprising features, which need to be worked out case by case. Most have yet to be explored. This subject is still in its infancy, which makes it a lot of fun to talk about!

34.1.4 Motivation and Applications

Why is this interesting? What are the motivations and applications? While the development of this subject heavily borrows some insights from modern string theory, it directly pertains to the real world. It has interwoven motivations and applications, which we now discuss.

Connecting Disparate Subjects

Given a concrete mathematical relation between two different subjects, results in one can be translated into what are often new results in the other. New methods of calculation, new physical phenomena and insights, and new avenues of investigation are often suggested. The current context is no exception. Soft theorems, memory effects, and asymptotic symmetries are three different ways of characterizing the behavior of the universe around us at very long distance scales. This equivalence has led to the realization that physics in the deep infrared is much richer, more subtle, and certainly much less understood than we previously believed. Notably, looking at IR structure from all three corners of the triangle has led to the realization that the vacuum in all gauge and gravitational theories is best described as infinitely degenerate. Prior to the past several years, there was no case in which all three corners were understood. Now that we know about these triangles, once we find one corner of one triangle, we can work out the other two. As a result, numerous surprising discoveries have been made, some of which are covered in these lectures.

Flat Space Holography

A central motivation for these IR investigations is to understand the holographic structure of quantum gravity in four-dimensional asymptotically flat spacetimes, which is a good approximation to the real world. This is how I came into the subject. A very beautiful story has unfolded over the past twenty years concerning the holographic structure of quantum gravity in anti-de Sitter space. The story begins [139] with the identification of the symmetries in anti-de Sitter space with those of its proposed holographic dual. Following this successful example, the very first question we should ask in attempting a holographic formulation of flat space quantum gravity is: "What are the symmetries?" Up until three years ago, the answer to this question was unknown. At the very least we now know [42] that the symmetry group is infinite dimensional and includes a certain subgroup of, but not all of, the BMS group on past and future null infinity. In addition, good evidence [57, 67, 117] indicates a conjectured Virasoro (or superrotation) symmetry [52, 53, 54, 55] that acts on the sphere at null infinity, referred to as the celestial sphere $\mathcal{CS}^2$. The Lorentz group $SL(2,\mathbb{C})$ acts as the global conformal group on this sphere, and the Virasoro action enhances the global to the local conformal group. This suggests that the holographic dual of four-dimensional flat-space quantum gravity might be realized as an exotic two-dimensional conformal field theory (CFT) on $\mathcal{CS}^2$ [12, 140, 141, 142, 143, 144].

The Gauge Theory $\mathcal{S}$-Matrix

The study of soft particles and infrared divergences in QED, Yang-Mills theory, and collider physics is a large and central enterprise in quantum field theory. Notable recent progress includes the development of soft collinear effective theory (SCET) [145], in which infinite-dimensional symmetries likely related to those discussed here make an appearance [146]. SCET has important practical applications for understanding jet structure and interpreting LHC and other collider events. From a more formal point of view, it is disturbing that generically no IR finite $\mathcal{S}$-matrix[187] exists in gauge theories, even if suitable inclusive cross sections are finite [151]. While this does not pose an obstacle for collider predictions, it is a big elephant in the room for mathematical quantum field theory. Surely the central object of study in these theories must somehow be mathematically well defined! Our hope is that the discovery of new symmetries in the deep IR will enable us to formulate the soft part of the $\mathcal{S}$-matrix in a more satisfactory way. Moreover, the symmetries might be exploited to develop more efficient computational methods, perhaps via a connection with the SCET program.

Miracles in $\mathcal{N} = 4$ Yang-Mills

Scattering amplitudes in $\mathcal{N} = 4$ Yang-Mills have been a very active subject for more than 10 years e.g., [152, 153, 154, 155, 99, 156, 157, 158, 159]. Extensive computations have revealed miraculous cancellations that reduce very complicated expressions to simple ones. When many cancellations occur it is natural to suspect that they are required by an underlying symmetry. It is possible that the new IR symmetries provide an explanation of those miracles or, even better, a derivation of some of the incredibly beautiful formulas for scattering amplitudes in $\mathcal{N} = 4$ Yang-Mills theory.

Black Holes

Although I did not start this IR project with black holes in mind, as usual, all roads lead to black holes [41, 160, 48]. The IR structure has important implications for the information paradox[161]. This paradox is intertwined with the deep IR because an infinite number of soft gravitons and soft photons are produced in the process of black hole formation and evaporation. These soft

[187] In massive abelian gauge theories, the Faddeev-Kulish construction [147] yields some IR finite amplitudes. An analogous construction for nonabelian gauge theory was given in [148, 149] and references therein. It was recently revisited for gravity [150] and for QED [22]. However, Faddeev-Kulish states are not complete: for example, they exclude an incoming positron-electron pair with no incoming radiation. See section 34.2.15 for further discussion.

particles carry information with a very low energy cost. They must be carefully tracked to follow the flow of information. Tracking is hard to do without a definition of the $\mathcal{S}$-matrix! Moreover, their production is highly constrained by an infinite number of exact quantum conservation laws that correlate them with energetic hard particles and also with the quantum state of the black hole itself [162]. Thus black holes must carry an infinite number of conserved charges, described as "soft hair" in a recent collaboration with Hawking and Perry [160, 48]. The information paradox cannot be clearly stated [163], let alone solved, without accounting for soft particles. The implications of soft hair recently have been discussed [164, 165, 166, 128, 167, 168, 163, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178, 179, 180], for example.

These lectures both review the old material on soft theorems, asymptotic symmetries and memory as well as the new connections among them. Unsolved problems are pointed out along the way. Some exciting developments in the past year whose proper perspective has not yet been clarified are omitted. These lectures are not meant to be an exhaustive review; see other sources in the References [50, 94, 181, 182, 45, 183, 184, 185, 186, 187, 188, 51, 189, 190, 191] for interesting related developments.

34.2 QED

Since this is a book about the interconnections among many different subjects, we might start at many different places — gauge theory or gravity, soft theorems, symmetries or memories — and weave our way through the connections. Here I choose the simplest possible starting point: classical, nineteenth-century electromagnetism. Beginning from this point, we will trace out these interconnections for any theory with an abelian gauge symmetry — including, of course, QED.

We begin in section 2.1 by recalling the Liénard-Wiechert solutions for a collection of charges and noting that these solutions are discontinuous at the boundary of Minkowski space. Characterizing this discontinuity requires a careful description of the boundary, which is provided in section 2.2. In section 2.3, we show that the Liénard-Wiechert field obeys a boundary matching condition equating its values at the far past of $\mathcal{I}^+$ ($\mathcal{I}^+_-$) with those at antipodal points of the far future of $\mathcal{I}^-$ ($\mathcal{I}^-_+$). Section 2.4 describes the electromagnetic Cauchy data at $\mathcal{I}^\pm$. Using only the matching condition, an infinity of conservation laws is derived in section 2.5. Section 2.6 presents the covariant canonical formulations of the physical phase spaces at $\mathcal{I}^\pm$. In section 2.7, I show for massless fields, that the conserved charges generate (via the Dirac bracket) nontrivial large gauge transformations associated with antipodally matched angle-dependent gauge parameters that do not die off at $\mathcal{I}^\pm$. A Ward identity of this symmetry is derived in section 2.8 and shown to be equivalent to the soft photon theorem. For completeness, the standard diagrammatic derivation of the soft theorems for both photons and gravitons is reviewed in section 2.9. In section 2.10, I adapt the BMS-type analysis of asymptotic symmetries employed from studies of gravity to electromagnetism and show that it reproduces the antipodally matched large gauge symmetries. In section 2.11, it is noted that the large gauge symmetry is spontaneously broken, resulting in an infinite vacuum degeneracy with soft photons as the Goldstone bosons. The generalization to massive fields, and hence QED, is in section 2.12. This extension employs a hyperbolic slicing of Minkowski space and heavily borrows results from investigations of the anti-de Sitter/conformal field theory correspondence (AdS/CFT). In 2.13, a magnetically corrected soft photon theorem is presented and conjectured to be nonperturbatively exact. I derive a second infinity of conserved charges and symmetries arising from large magnetic gauge transformations. Section 2.14 shows that for $\mathcal{N} = 1$ supersymmetry, the soft photino theorem leads to an infinity of fermionic conservation laws and symmetries. These are surprising, as they are unrelated to any previously discussed local symmetry. In section 2.15, IR divergences are discussed and shown to disappear for some and possibly all fully charge-conserving amplitudes.

34.2.1 Liénard-Wiechert Solution

Maxwell's theory of electromagnetism is described by the action

$$S = -\frac{1}{4e^2}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu} + S_M \ , \tag{34.1}$$

where $\mu,\nu = 0,1,2,3$ and S_M denotes a general matter action. In form notation, the field strength F is related to the gauge field A by

$$F = \mathrm{d}A \ . \tag{34.2}$$

The equation of motion is

$$\mathrm{d}*F = e^2 * j \quad \Longrightarrow \quad \nabla^\mu F_{\mu\nu} = e^2 j_\nu \ , \tag{34.3}$$

where $*$ is the Hodge dual, and the charge current is

$$j^\nu = -\frac{\delta S_M}{\delta A_\nu} \ . \tag{34.4}$$

The electric charge inside a two-sphere (S^2) at infinity is defined by

$$Q_E = \frac{1}{e^2}\int_{S^2} *F = \int_\Sigma *j \in \mathbb{Z} \ . \tag{34.5}$$

The second equality is obtained using integration by parts and the constraint component of the equation of motion. It says that if we take Σ to be any slice with the S^2 boundary, the electric charge is just the normal component of the charge current integrated over that slice. For convenience, we adopt here a convention in which Q_E is an integer in the quantum theory. For example, the electron has charge $Q_E = 1$ (not e). The magnetic charge $\frac{1}{2\pi}\int F$ is also integral, and it has an associated conservation law [192]. We return to the interesting story of magnetic charges in section 2.13, but for now, assume they are absent.

The theory (34.1) is invariant under the infinitesimal gauge transformations

$$\delta_\varepsilon A = \mathrm{d}\varepsilon \ , \qquad \delta_\varepsilon \Phi_k = i\varepsilon Q_k \Phi_k \ , \tag{34.6}$$

where $\varepsilon \sim \varepsilon + 2\pi$, and Φ_k is a matter field with charge Q_k. Since $Q_k \in \mathbb{Z}$, the finite transformation

$$\Phi_k \to e^{iQ_k\varepsilon}\Phi_k \tag{34.7}$$

is invariant under $\varepsilon \to \varepsilon + 2\pi$.

Now let us solve the equation of motion. Consider a source j corresponding to n particles, each moving with constant four-velocity $U_k^\mu = \gamma_k\left(1,\vec{\beta}_k\right)$, where $U_k^2 = -1$ and $\gamma_k^2 = \frac{1}{1-\beta_k^2}$. Then we have

$$j_\mu(x) = \sum_{k=1}^n Q_k \int d\tau \ U_{k\mu}\delta^4(x^\nu - U_k^\nu\tau) \ . \tag{34.8}$$

Each particle is a point source with a worldline parametrized by τ, so that $x_k^\mu(\tau) = U_k^\mu\tau$.

Equation (34.3) determines the electromagnetic field due to this source. Liénard and Wiechert solved this problem in 1898. The solution for the radial component of the electric field is

$$F_{rt}(\vec{x},t) = \frac{e^2}{4\pi}\sum_{k=1}^n \frac{Q_k\gamma_k\left(r - t\hat{x}\cdot\vec{\beta}_k\right)}{\left|\gamma_k^2\left(t - r\hat{x}\cdot\vec{\beta}_k\right)^2 - t^2 + r^2\right|^{3/2}} \ , \tag{34.9}$$

where

$$r^2 = \vec{x} \cdot \vec{x} \ , \quad \vec{x} = r\hat{x} \ . \tag{34.10}$$

This formula is valid for either the advanced or the retarded potential. It has the peculiar property that it is not single valued and has a discontinuity at $r = \infty$. Although this fact was presumably well known to many since the nineteenth century, this basic point surprised me when I became aware of it only recently. It is crucial for all that follows, so we will go through how this happens in pedantic detail. To fully explain it, we must first digress and review the structure of infinity in Minkowski space.

34.2.2 Minkowski Space Penrose Diagram

To talk about infinity, it is useful to introduce Penrose diagrams. The Penrose diagram for four-dimensional Minkowski space is shown in figure 34.3. In this figure all of Minkowski space is pulled

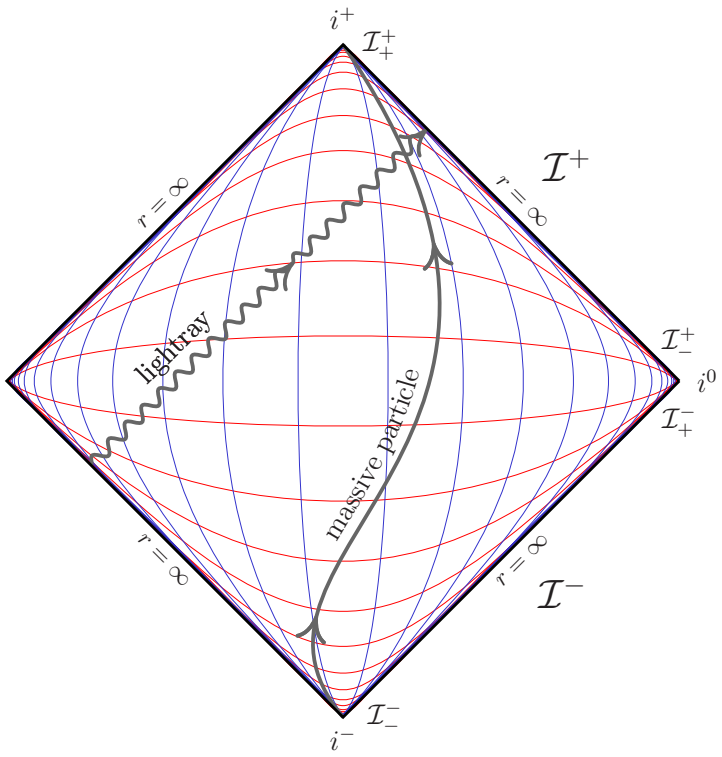


Figure 34.3: **Penrose diagram of Minkowski space. Red lines represent surfaces of constant t, while blue lines represent surfaces of constant r. The thick gray line is the worldline of a massive particle moving at constant velocity, and the thick wavy gray line is a light ray. Every two-sphere of constant $(r > 0, t)$ is represented by two points, one on the left and one on the right, which are exchanged by the antipodal map. Past and future null infinities are labelled by $\mathcal{I}^{\pm}$, and their four S^2 boundary components by $\mathcal{I}^{\pm}_{\pm}$. The points $i^{\pm}$ are past and future timelike infinity, while the point i^0 is spatial infinity.**

into a finite region by a conformal transformation that diverges at the boundaries. Distances are not faithfully represented, but the causal structure is unaffected by the conformal transformation and is clearly indicated by the rule that light rays always propagate at 45 degrees. Every left-right pair of points at the same (r, t) in figure 34.3 corresponds to an S^2 (except $r = 0$, which is a point), and the pair is interchanged by the antipodal map on S^2. Sometimes the picture is drawn differently, as in figure 34.4. Every point on this diagram represents an S^2 except for $r = 0$, which

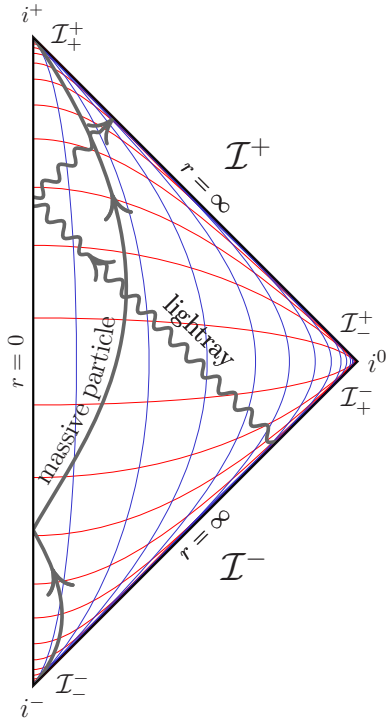


Figure 34.4: **Alternative Penrose diagram of Minkowski space in which every point except for $r = 0$ is an S^2.**

is the origin. If you have a sequence of two-spheres that are getting smaller and smaller and you add a point in the middle, you get $\mathbb{R}^3$. For our purposes (which include a study of the antipodal map), figure 34.3 will be more useful.

We want to study the electric field of n charges moving at constant velocity. The charges have worldlines that look like the thick gray line in figure 34.3. They cannot escape to $r = \infty$, because light rays move at 45 degrees on Penrose diagrams; every light ray will eventually catch up to a particle moving with constant velocity (less than c) and cross it to larger radius. As a result, massive charges always end up at the uppermost point, "called future timelike infinity" and denoted i^+ in figure 34.3. Similarly, the worldlines of (massive) charges start at past timelike infinity (i^-). Likewise, we call the place where light rays go to "future null infinity" ($\mathcal{I}^+$) and the place where light rays come from "past null infinity" ($\mathcal{I}^-$). We will be especially concerned about behavior near spatial infinity, denoted i^0. This behavior turns out to be very subtle. As we will see, almost anything, even Maxwell electromagnetism with one point particle, is singular at this point. $\mathcal{I}^+$ is the product of S^2 with a null line. It will be important to distinguish between the past of $\mathcal{I}^+$, which we call $\mathcal{I}^+_-$, and the future of $\mathcal{I}^-$, which we call $\mathcal{I}^-_+$. Both these points are near, but distinct from, the point i^0. Figures 34.3 and 34.4 also illustrate $\mathcal{I}^-_-$, which is an S^2 near but distinct from the point i^-. Likewise, near the point i^+, we have the S^2 denoted $\mathcal{I}^+_+$.

Now let us discuss the problem of scattering in Minkowski space, a central topic of this book. Whether in electrodynamics or gravity, classical or quantum, one starts by specifying initial data at $\mathcal{I}^-$. If there are stable massive particles, initial data at past timelike infinity i^- are also needed, but we suppress that for now and return to it in section 34.2.12. In the far past, fields and particles disperse, and so the theory is weakly interacting. Incoming particles or wavepackets evolve towards each other, interact in some very complicated way, and ultimately come out on $\mathcal{I}^+$. The classical version of the scattering problem is to find a classical map from the phase space defined on $\mathcal{I}^-$ to the phase space defined on $\mathcal{I}^+$. The quantum version of the scattering problem is to find the $\mathcal{S}$-matrix that evolves incoming states defined on $\mathcal{I}^-$ to outgoing states defined on $\mathcal{I}^+$.

Clearly in this setting, what goes on near i^0 is important, because we have to know how to

match on final data on $\mathcal{I}^+$ given initial data on $\mathcal{I}^-$. Specifically, we need to understand what the matching conditions are that relate the fields at $\mathcal{I}^-_+$ to the fields at $\mathcal{I}^+_-$. One might think that the fields should simply be taken to be equal, but this turns out to be completely wrong. In fact, such a matching condition is not even Lorentz invariant! We will soon see it is also not obeyed by the Liénard-Wiechert solution.

To describe $\mathcal{I}^-$, it is useful to introduce advanced time:

$$v = t + r \ . \tag{34.11}$$

To describe $\mathcal{I}^+$, we introduce retarded time:

$$u = t - r \ . \tag{34.12}$$

$\mathcal{I}^-$ is a three-dimensional surface, so four coordinates are not needed. We could try to use the usual $(t, r, \hat{x})$, $\hat{x}$ being a unit vector labeling a point on the sphere, but this choice is awkward, because t and r are both infinite on $\mathcal{I}$. However, if we follow null rays backward (in time), $t + r$ is finite, and if we follow them forward, $t - r$ is finite. So $\mathcal{I}^+$ is naturally parametrized by $(u, \hat{x})$, and $\mathcal{I}^-$ is naturally parametrized by $(v, \hat{x})$.

34.2.3 Antipodal Matching Condition

A peculiar property of the Liénard-Wiechert solution is that if we start at a point in the bulk, take the limit first to $\mathcal{I}^+$ and then to i^0, we get a different answer than if we take the limit to $\mathcal{I}^-$ and then to i^0. In fact, the Liénard-Wiechert solution takes different values at fixed angles (or $\hat{x}$) on $\mathcal{I}^+_-$ and $\mathcal{I}^-_+$, but obeys an *antipodal* matching condition.

To see this, first rewrite (34.9) in retarded coordinates $u = t - r$:

$$F_{rt} = F_{ru} = \frac{e^2}{4\pi} \sum_{k=1}^{n} \frac{Q_k \gamma_k \left(r - (u + r)\hat{x} \cdot \vec{\beta}_k \right)}{\left| \gamma_k^2 \left(u + r - r\hat{x} \cdot \vec{\beta}_k \right)^2 - (u + r)^2 + r^2 \right|^{3/2}} \ . \tag{34.13}$$

To reach $\mathcal{I}^+$, hold u fixed and take the limit $r \to \infty$:

$$F_{rt}\big|_{\mathcal{I}^+} = \frac{e^2}{4\pi r^2} \sum_{k=1}^{n} \frac{Q_k}{\gamma_k^2 (1 - \hat{x} \cdot \vec{\beta}_k)^2} \ . \tag{34.14}$$

To reach $\mathcal{I}^+_-$, we must further take $u \to -\infty$, but since the expression is u-independent, (34.14) is the final answer. Note, if we take the velocity $\vec{\beta}_k = 0$, we recover the usual Coulomb field for a static charged particle.

The leading $\frac{1}{r^2}$ component of the electric field due to moving charges is not a constant: it depends on the angle of the sphere at infinity. In standard electrodynamics texts, such as Jackson [193], one often studies the multipole expansion of static configurations. The $\frac{1}{r^2}$ component is then constant and proportional to the total electric charge, while the $\frac{1}{r^3}$ term comes from the static electric dipole moment with angular momentum $\ell = 1$. In contrast, for the case of a single charge moving at constant velocity, no electric dipole is in the picture, but there is a dipole moment in the $\frac{1}{r^2}$ term, in the sense that the $\ell = 1$ mode of the distribution over the sphere is nonzero. This $\ell = 1$ mode dipole moment of the $\frac{1}{r^2}$ term is not to be confused with what is usually called the the electric dipole moment in classical electrodynamics!

To find the electric field at $\mathcal{I}^-$, we work in advanced coordinates $v = t + r$:

$$F_{rt} = F_{rv} = \frac{e^2}{4\pi} \sum_{k=1}^{n} \frac{Q_k \gamma_k \left(r - (v - r)\hat{x} \cdot \vec{\beta}_k \right)}{\left| \gamma_k^2 \left(v - r - r\hat{x} \cdot \vec{\beta}_k \right)^2 - (v - r)^2 + r^2 \right|^{3/2}} \ . \tag{34.15}$$

To reach $\mathcal{I}^-$, we hold v fixed and take the limit $r \to \infty$:

$$F_{rt}\big|_{\mathcal{I}^-} = \frac{e^2}{4\pi r^2} \sum_{k=1}^{n} \frac{Q_k}{\gamma_k^2 (1 + \hat{x} \cdot \vec{\beta}_k)^2} \ . \tag{34.16}$$

Once again, since the expression is v-independent, this is also the value of F_{rt} at $\mathcal{I}_+^-$. Comparing equation (34.14) and equation (34.16), we find that the values of the fields do not match. Even for the case of one moving charge in Minkowski space, the radial component of the electric field is not continuous at spatial infinity. Its value depends on how one approaches spatial infinity, and it is discontinuous there. The field takes a definite value if one specifies a way of approaching i^0. Going down from $\mathcal{I}^+$, or up from $\mathcal{I}^-$, one gets a definite answer, but they are not equal as long as $\vec{\beta}_k \neq 0$.

Since the solution is Lorentz covariant and can be obtained by boosting a static charge, the discontinuity is dictated by Lorentz invariance. Indeed, Lorentz transformations themselves are not smooth at spatial infinity, because the vector fields that generate them are singular at i^0. A boost toward the north pole in the future is a boost away from the north pole in the past. The point i^0 is the dividing point of these two different behaviors.

Although the fields $F_{rt}\big|_{\mathcal{I}_+^-}$ and $F_{rt}\big|_{\mathcal{I}_-^+}$ are not equal, they have an important antipodal relation: taking $\hat{x}$ to $-\hat{x}$ takes $F_{rt}\big|_{\mathcal{I}_-^+}$ to $F_{rt}\big|_{\mathcal{I}_+^-}$. Comparison of equation (34.14) and equation (34.16) reveals the matching condition,

$$\lim_{r\to\infty} r^2 F_{ru}(\hat{x})\big|_{\mathcal{I}_-^+} = \lim_{r\to\infty} r^2 F_{rv}(-\hat{x})\big|_{\mathcal{I}_+^-} \ , \tag{34.17}$$

where we have used the relation $F_{ru} = F_{rv} = F_{rt}$. That is, the leading term in the radial electric field for a collection of n particles at any point on $\mathcal{I}_-^+$ will be equal to the value of the field at the antipodal point on $\mathcal{I}_+^-$.

So far we have been considering a relatively simple nonstatic situation, in which nothing is interacting. However, it is not hard to check (see exercise 1) that equation (34.17) holds in basically all reasonable circumstances. If we add electromagnetic waves on top of the n-particle solution (34.9), the field strengths at $\mathcal{I}^+$ and $\mathcal{I}^-$ would no longer be constant. Electromagnetic waves are radiative as opposed to Coulombic modes of the electromagnetic field. If we want them to have finite energy, they must die off in the infinite future and the infinite past. For finite energy configurations, adding radiative modes will give more complicated expressions (for example the field strength on $\mathcal{I}^+$ will depend on u) but will not affect this basic behavior of the Coulombic, deep IR part of the electric field near spatial infinity, which is what we have been studying. It is also the case that letting the charges interact or scatter off one another preserves (34.17). If one uses the retarded Liénard-Wiechert potential, one obtains equation (34.9) in a neighborhood of i^0 with $\vec{\beta}_k$ the outgoing velocities, while the advanced Liénard-Wiechert gives (34.9) in a neighborhood of i^0 with $\vec{\beta}_k$ the incoming velocities. These solutions are different, but both obey antipodal matching. In section 2.9, we will also see that in quantum theory, Feynman diagrammatics implicitly assume equation (34.17) in the standard weak field expansion. That might have been guessed from the fact that the matching condition is Lorentz and CPT invariant.

Another way of thinking about the matching condition, using the conformal invariance of electromagnetism, is to map or "conformally compactify" Minkowski space to the cylinder using a conformal rescaling of the metric (figure 34.5). This mapping is especially simple in electromagnetism, which is invariant under such rescalings, but it is also very useful in gravity, where it formed the basis of Penrose's analysis of null infinity [194, 195]. The cross sections of the cylinder are S^3, and Minkowski space is represented by the blue diamond. Null infinity $\mathcal{I}$ is the lightcone of spatial infinity i^0. The future lightcone wraps around the cylinder and meets itself at i^+, while the past lightcone meets itself at i^-. The Minkowski diamond is wrapped on the cylinder and touches itself at i^0. The null generators of $\mathcal{I}$ start at past null infinity $\mathcal{I}^-$ and pass through spatial infinity i^0 to future null infinity $\mathcal{I}^+$. The matching condition (34.17) then states that the fields are continuous along the generators of $\mathcal{I}$, including when they cross i^0, even though it is generically a singular point. It is singular, because, for example, even just one charge in the interior needs

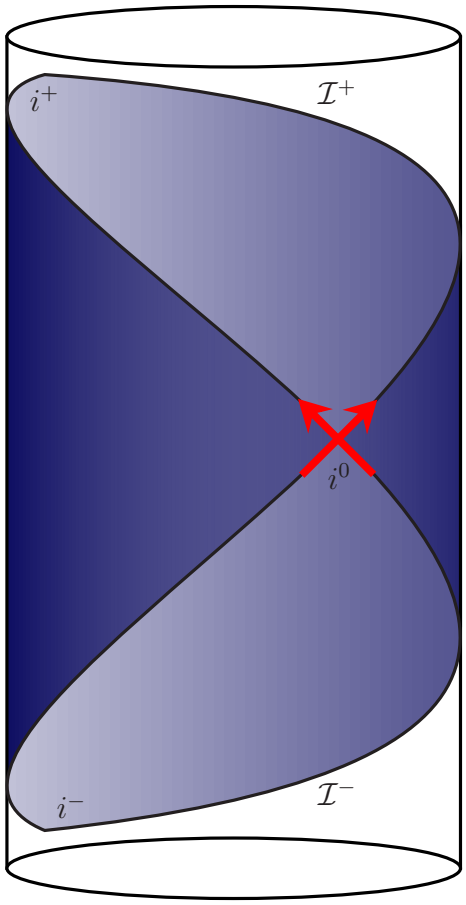


Figure 34.5: **The blue diamond represents Minkowski space conformally compactified onto the $S^3 \times \mathbb{R}$ Einstein static universe. The red arrows indicate how the generators of null infinity pass through spatial infinity.**

an image charge at i^0, which will cause the electric field to diverge there. One certainly cannot demand that the fields be smooth at spatial infinity. In the presence of multiple moving charges, the singularity can become very complicated and in general requires arbitrarily many parameters to describe. However, we are still able to consistently require, without violating Lorentz invariance (which is a conformal symmetry of the compactified geometry), that the fields are continuous along the null generators passing through i^0 as prescribed by equation (34.17).

34.2.4 Asymptotic Expansions

Before going any further, it is convenient to introduce specific advanced and retarded coordinates, illustrated in figure 34.6, and to be more precise about our large-r expansions around $\mathcal{I}^{\pm}$.

In retarded coordinates $(r, u, z, \bar{z})$, the Minkowski line element is

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \ . \tag{34.18}$$

These coordinates will be used in the neighborhood of $\mathcal{I}^+$. Here, $u = t - r$ is the retarded time coordinate introduced in equation (34.12), r is the radial coordinate, and z is a complex coordinate on the unit sphere with metric

$$\gamma_{z\bar{z}} = \frac{2}{(1 + z\bar{z})^2} \ . \tag{34.19}$$

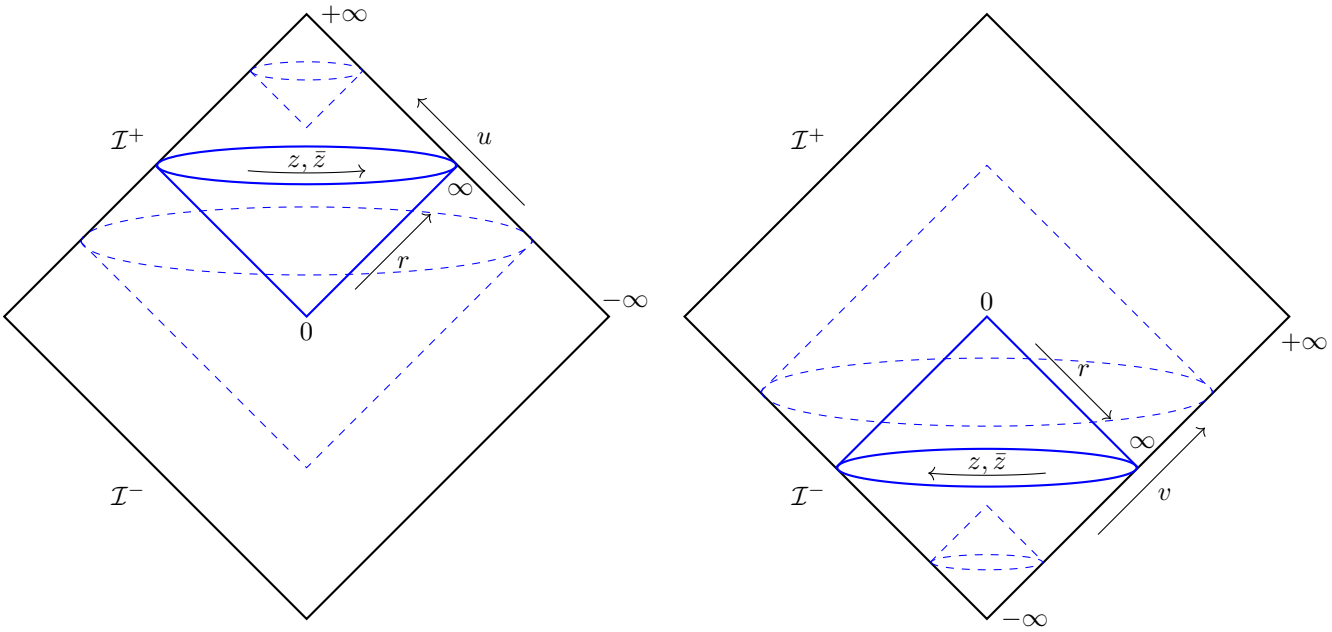


Figure 34.6: **In the left diagram, $\mathcal{I}^+$ is parametrized by retarded time u and spherical coordinates $(z, \bar{z})$ in retarded Bondi coordinates, while in the right diagram, $\mathcal{I}^-$ is parametrized by advanced time v and spherical coordinates $(z, \bar{z})$ in advanced Bondi coordinates. The advanced and retarded $(z, \bar{z})$ coordinates are chosen so that they are related by an antipodal map on spheres of constant (u, r).**

If we keep $(u, z, \bar{z})$ fixed and take the limit $r \to \infty$, we move out along a null line to $\mathcal{I}^+$. One can see that this is a null line, because along this path, $du = dz = d\bar{z} = 0$, which implies $ds^2 = 0$. The standard metric on Minkowski space,

$$ds^2 = -dt^2 + (d\vec{x})^2 \ , \tag{34.20}$$

is related to the metric in equation (34.18) by the coordinate transformations

$$(\vec{x})^2 = r^2 \ , \quad t = u + r \ , \quad x^1 + ix^2 = \frac{2rz}{1 + z\bar{z}} \ , \quad x^3 = r\frac{1 - z\bar{z}}{1 + z\bar{z}} \ . \tag{34.21}$$

The inverse transformation is

$$r^2 = (\vec{x})^2 \ , \quad u = t - r \ , \quad z = \frac{x^1 + ix^2}{x^3 + r}. \tag{34.22}$$

Here, z runs over the entire complex plane; $z = 0$ is the north pole, $z = \infty$ is the south pole, $z\bar{z} = 1$ is the equator, and $z \to -1/\bar{z}$ is the antipodal map. This is a convenient coordinate system to work in near $\mathcal{I}^+$ because, as we will see, everything falls off near $\mathcal{I}^+$, so we can expand fields in powers of $\frac{1}{r}$. However, we cannot easily use these coordinates near $\mathcal{I}^-$, because $u = -\infty$ there. To work in a neighborhood of $\mathcal{I}^-$, we introduce advanced coordinates. The advanced line element is

$$ds^2 = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \ . \tag{34.23}$$

This metric can be obtained from the standard Cartesian metric on Minkowski space by means of the coordinate transformations

$$(\vec{x})^2 = r^2 \ , \quad t = v - r \ , \quad x^1 + ix^2 = -\frac{2rz}{1 + z\bar{z}} \ , \quad x^3 = -r\frac{1 - z\bar{z}}{1 + z\bar{z}} \ . \tag{34.24}$$

Crucial minus signs introduced into the last two terms of (34.24) imply that the z in the advanced coordinates denotes the antipodal point on the sphere to the z in the retarded coordinates (the sign reverses under $z \to -1/\bar{z}$). If we take a light ray which crosses Minkowski space, then the value of z at which it starts in advanced coordinates will be the same as the value of z at which it ends in retarded coordinates. Moreover, z is constant along the null generators of $\mathcal{I}^-$ as they pass through i^0 to $\mathcal{I}^+$. This perhaps odd-seeming choice of notation simplifies subsequent formulas.

Now we wish to expand around $\mathcal{I}^+$. Given any field—say, the z-component of the vector potential—we can write an expansion for it as a sum

$$A_z(u, r, z, \bar{z}) = \sum_{n=0}^{\infty} \frac{A_z^{(n)}(u, z, \bar{z})}{r^n} \ , \tag{34.25}$$

where the coefficients depend only on the coordinates $(u, z, \bar{z})$ parametrizing $\mathcal{I}^+$. In exercise 2, it is shown that $A_z^{(0)}$ is the local Cauchy data. The superscript (n) will be used to denote the order in the expansion (the power of $1/r$) about $r = \infty$. In general, we are going to be expanding many fields about both $\mathcal{I}^+$ and $\mathcal{I}^-$. Using our new notation, we can finally rewrite the matching condition (34.17) as

$$F_{ru}^{(2)}(z, \bar{z})\Big|_{\mathcal{I}_-^+} = F_{rv}^{(2)}(z, \bar{z})\Big|_{\mathcal{I}_+^-} \ , \tag{34.26}$$

where $F_{ru}^{(2)}$ is the $\frac{1}{r^2}$ term in the expansion of the ru component of the field strength around $\mathcal{I}^+$. Evaluating it at $\mathcal{I}_-^+$ is equivalent to taking $u = -\infty$:

$$F_{ru}^{(2)}(z, \bar{z})\Big|_{\mathcal{I}_-^+} = F_{ru}^{(2)}(-\infty, z, \bar{z}) \ . \tag{34.27}$$

The simplicity of expression (34.26) motivated our definition of the z values on $\mathcal{I}^+$ as antipodally related to those on $\mathcal{I}^-$.

34.2.5 An Infinity of Conserved Charges

In this section we will show that the matching condition (34.26) immediately implies, without further ado, the existence of an infinite number of conserved charges in all electromagnetic theories in Minkowski space.

Consider any function ε on Minkowski space obeying the boundary condition

$$\varepsilon(z, \bar{z})|_{\mathcal{I}_-^+} = \varepsilon(z, \bar{z})|_{\mathcal{I}_+^-} \ . \tag{34.28}$$

Note that $\varepsilon(z, \bar{z})$ is not smooth near spatial infinity, but instead is antipodally identified. Now we define future and past charges:

$$Q_\varepsilon^+ = \frac{1}{e^2} \int_{\mathcal{I}_-^+} \varepsilon * F, \quad Q_\varepsilon^- = \frac{1}{e^2} \int_{\mathcal{I}_+^-} \varepsilon * F \ . \tag{34.29}$$

It then immediately follows from (34.26) that any theory involving electromagnetism has an infinite number of conservation laws, one for every function ε:

$$Q_\varepsilon^+ = Q_\varepsilon^- . \tag{34.30}$$

For example, we could take $\varepsilon|_{\mathcal{I}_+^-} = Y_{\ell m}$ to be a spherical harmonic, in which case we have one conservation law for every value of the angular momentum (ℓ, m). One way of stating this conservation law is that all the incoming multipole moments of the electromagnetic field are equal to

the antipodal outgoing multipole moments. All of these moments are nonzero in a generic time-dependent situation. In this basis, the antipodal map acts as $(-)^\ell$. The $\ell = 0$ mode is of course total charge conservation, which equates the net incoming to the net outgoing charge. Here we discover an infinite family of higher harmonics of this familiar global conservation law.

The conservation laws look even more interesting and nontrivial when we use the Gauss law to write the surface integral expression for the electric field as a volume integral expression of the incoming and outgoing fields. At this point, we make the simplifying assumption that all stable charged particles are massless, deferring the more intricate discussion of the (more realistic) massive case to section 34.2.12. We then find

$$Q^+_\varepsilon = \frac{1}{e^2}\int_{\mathcal{I}^+} \mathrm{d}\varepsilon \wedge *F + \int_{\mathcal{I}^+} \varepsilon * j + \frac{1}{e^2}\cancelto{0}{\int_{\mathcal{I}^+_+} \varepsilon * F} \,, \tag{34.31}$$

where we recall j is the charged matter current. This last term arises because $\mathcal{I}^+$ has two boundaries: we began in (34.29) with an integral over $\mathcal{I}^+_-$ and used integration by parts to get a volume integral over $\mathcal{I}^+$ plus a surface integral over $\mathcal{I}^+_+$. However, we can set this last surface integral to zero under our assumption that there are no massive charged particles, which implies that the electric field will vanish at $\mathcal{I}^+_+$. Likewise, for $\mathcal{I}^-$,

$$Q^-_\varepsilon = \frac{1}{e^2}\int_{\mathcal{I}^-} \mathrm{d}\varepsilon \wedge *F + \int_{\mathcal{I}^-} \varepsilon * j + \frac{1}{e^2}\cancelto{0}{\int_{\mathcal{I}^-_-} \varepsilon * F} \,. \tag{34.32}$$

Charge conservation, equation (34.30), is true for any ε obeying equation (34.28). However, we restrict to the special case in which

$$\partial_u \varepsilon = \partial_v \varepsilon = 0 \,. \tag{34.33}$$

Our formulas are all true for the more general case, but consideration of those cases turns out not to yield anything new. A particular case of equation (34.28) is $\varepsilon =$ constant, in which case the first term on the RHS of the $Q^\pm_\varepsilon$ equation vanishes. Equation (34.30) reduces to, for m (n) incoming (outgoing) particles with charges Q^{in}_k (Q^{out}_k)

$$\sum_{k=1}^m Q^{\text{in}}_k = \sum_{k=1}^n Q^{\text{out}}_k \,. \tag{34.34}$$

This is just the familiar statement that the sum of all the incoming charges must be equal to the sum of all the outgoing charges.

Now suppose ε is not constant. Then we get an infinite number of conserved charges that are on exactly the same footing as the usual conservation law (34.34) but with both terms in (34.32) and (34.31) contributing. The second term is still a sum over electric charges but is now weighted by an arbitrary function of the angle. The first is an additional mysterious term linear in the electromagnetic field F. We will call this a *soft photon term*, because, as we show later, when promoted to an operator it creates and annihilates zero-energy photons; for now it is just a classical integral of F. Unlike the constant ε case, whose associated charge involves only incoming matter fields, these more general conservation laws mix the charged matter and the electromagnetic field. We have an exact statement that relates various integrals, weighted by ε, of the incoming initial data to the outgoing final data. The conservation law says that, for any ε, the number obtained from the incoming integral exactly equals the number obtained from the outgoing integral.

Thus far, we have written the charges in form notation in (34.31) and (34.32). For many computations, explicit coordinate representations are useful. In retarded coordinates (34.18),

$$Q^+_\varepsilon = \frac{1}{e^2}\int_{\mathcal{I}^+_-} \varepsilon * F = \frac{1}{e^2}\int_{\mathcal{I}^+_-} d^2z \gamma_{z\bar{z}} \varepsilon F^{(2)}_{ru} \,. \tag{34.35}$$

Note there is an r^2 in the Hodge dual, which cancels the $1/r^2$ that comes with the $F^{(2)}_{ru}$ term, resulting in an integral which is finite for $r \to \infty$. This is equal, by virtue of the boundary condition (34.28), to

$$Q^-_\varepsilon = \frac{1}{e^2}\int_{\mathcal{I}^-_+} d^2z\gamma_{z\bar{z}}\varepsilon F^{(2)}_{rv} \ . \tag{34.36}$$

To integrate these formulas by parts, we need to use the constraint equations on the null surfaces $\mathcal{I}^\pm$. The constraint equation near $\mathcal{I}^+$ has an expansion in powers of $1/r$, with leading term

$$\partial_u F^{(2)}_{ru} + D^z F^{(0)}_{uz} + D^{\bar{z}} F^{(0)}_{u\bar{z}} + e^2 j^{(2)}_u = 0 \ . \tag{34.37}$$

D_z is the covariant derivative with respect to the unit round S^2 metric $\gamma_{z\bar{z}}$, and we also define $D^z = \gamma^{z\bar{z}} D_{\bar{z}}$. Choosing $\partial_u \varepsilon|_{\mathcal{I}^+} = 0$, integrating the boundary expression for Q^+_ε by parts, and using the constraint equation gives two terms,

$$Q^+_\varepsilon = \underbrace{-\frac{1}{e^2}\int_{\mathcal{I}^+} du d^2z \left(\partial_z \varepsilon F^{(0)}_{u\bar{z}} + \partial_{\bar{z}}\varepsilon F^{(0)}_{uz}\right)}_{Q^+_S} + \underbrace{\int_{\mathcal{I}^+} du d^2z \varepsilon \gamma_{z\bar{z}} j^{(2)}_u}_{Q^+_H} \ , \tag{34.38}$$

where we have defined a "soft" charge Q^+_S, which is linear in the electromagnetic field, and a "hard" charge Q^+_H, which is typically linear in the charge current but quadratic or higher in the charged matter fields. We have already seen the equation in form notation in (34.31). The term "soft" refers to objects with zero energy, while "hard" denotes energetic excitations. Q^+_H is referred to as the hard charge, because j is the matter current for energy-carrying matter fields. If ε is constant, then Q^+_S vanishes, and Q^+_ε collapses to the total charge flux through future null infinity. If ε is not constant, then the hard term is still there, but it has an unfamiliar form, because the charges that pierce null infinity are weighted by an arbitrary function that depends on the angle at which they exit the spacetime. However, we still have an overall conserved quantity, provided we compensate by including the soft term. The soft charge involves a term of the form

$$\int_{-\infty}^{\infty} du \ F^{(0)}_{uz} \equiv N_z \ , \tag{34.39}$$

convoluted with $\partial_{\bar{z}}\varepsilon$ and integrated over the sphere. Equation (34.39) is the $\omega \to 0$ limit of

$$\int_{-\infty}^{\infty} du \ F^{(0)}_{uz} e^{i\omega u} \ , \tag{34.40}$$

which is a Fourier component with nonzero energy and frequency ω of the electromagnetic field. Moreover, it is a component of the electromagnetic field that is transverse to $\mathcal{I}^+$, as the z index is transverse to a light ray passing through $\mathcal{I}^+$. Promoted to a quantum operator, (34.40) creates and annihilates outgoing photons of energy ω. However, the expression we are interested in appearing in (34.38) has no ω; rather it is an $\omega \to 0$ limit. This term creates and annihilates soft particles with zero energy, which is why we call it the soft term. Equation (34.38) indicates that these soft photons have polarization $\partial_{\bar{z}}\varepsilon$.

The soft photon mode N_z is simply related to gauge transformations at $\mathcal{I}^+_\pm$. To see this, consider the curl

$$\begin{aligned} \partial_{\bar{z}} N_z - \partial_z N_{\bar{z}} &= \int_{-\infty}^{\infty} du \ \left[\partial_{\bar{z}} F^{(0)}_{uz} - \partial_z F^{(0)}_{u\bar{z}}\right] \\ &= -\int_{-\infty}^{\infty} du \ \partial_u F^{(0)}_{z\bar{z}} = -F^{(0)}_{z\bar{z}}\Big|^{\mathcal{I}^+_+}_{\mathcal{I}^+_-} \ , \end{aligned} \tag{34.41}$$

where we have used the Bianchi identity in the second line. At this point we assume that there are no asymptotic states with magnetic charges (i.e., no magnetic monopoles), and no long-range magnetic fields, so that $F_{z\bar{z}}\big|_{\mathcal{I}^+_\pm} = 0$. Then the curl (34.41) vanishes, and we can define a real scalar N by

$$N_z \equiv e^2 \partial_z N \ . \tag{34.42}$$

Imposing the gauge condition $A^{(0)}_u = 0$, it follows that

$$e^2 \partial_z N = \int_{-\infty}^{\infty} du F^{(0)}_{uz} = A^{(0)}_z\big|_{\mathcal{I}^+_+} - A^{(0)}_z\big|_{\mathcal{I}^+_-} \ . \tag{34.43}$$

The soft photon mode (34.39) is the difference between the z-component of the gauge field at the future of $\mathcal{I}^+$ and the past of $\mathcal{I}^+$. If we want to have finite energy, the gauge field better be pure gauge at both the beginning and the end of $\mathcal{I}^+$, and the relative shift better also be pure gauge. Indeed, we have just seen that the shift is a gauge transformation with parameter $e^2 N$. The charge has a simple expression in terms of the soft photon mode:

$$Q^+_\varepsilon = 2 \int d^2 z N \partial_z \partial_{\bar{z}} \varepsilon + \int_{\mathcal{I}^+} du d^2 z \varepsilon \gamma_{z\bar{z}} j^{(2)}_u \ . \tag{34.44}$$

34.2.6 Canonical Electrodynamics at $\mathcal{I}$

We have found an infinity of conserved charges in abelian gauge theories. One expects an associated infinity of symmetries. In a canonical Hamiltonian formalism, these symmetries are easily identified as the Dirac bracket action of the charges on the phase space. (To go the other way, from symmetries to charges, one uses Noether's theorem.) In this subsection we develop this formalism, and in the next we determine the symmetry.

In a Hamiltonian formulation, one has a phase space Γ with coordinates $x^I = \{q^i, p_j\}$, where $i, j = 1, \ldots, N/2$, and N is the (even) dimension of the phase space. For example, the theory of a particle in three dimensions has a six-dimensional phase space with $i, j = 1, 2, 3$. In field theory, the index I becomes continuous, and the phase space is infinite-dimensional. The symplectic two-form Ω,

$$\Omega = \frac{1}{2} \Omega_{IJ} dx^I \wedge dx^J \ , \tag{34.45}$$

is needed to define the Hamiltonian dynamics. Ω_{IJ} is antisymmetric and must be invertible. Once the symplectic form is known, quantum commutators are constructed as

$$[A, B] = i \Omega^{IJ} \partial_I A \partial_J B \ . \tag{34.46}$$

These are related to classical Poisson brackets by a factor of i. Much in these lectures is relevant at the purely classical level, but we will always use quantum commutators to avoid writing two sets of formulas that differ only by a factor of i.

Symplectic Form

The classical phase space in electromagnetism can be defined as the allowed initial data on any Cauchy surface Σ. For a Cauchy surface Σ in Minkowski space, the symplectic form for free electrodynamics can be written as (see exercise 2 and [196, 49, 197, 198])

$$\Omega_\Sigma = -\frac{1}{e^2} \int_\Sigma \delta(*F) \wedge \delta A \ . \tag{34.47}$$

Here, δA is an on-shell variation of the gauge field A and also a 1-form on the infinite-dimensional phase space Γ on which $\wedge$ denotes the wedge product. $\delta(*F)\wedge\delta A$ is a closed 3-form in spacetime, and its integral Ω_Σ is a 2-form on Γ. Equation (34.47) gives the same value for any pair of 3-surfaces Σ with the same boundary. In particular, Σ can be chosen either to be a spacelike slice or pushed out to a null surface at $\mathcal{I}^\pm$. For Ω_Σ to be invertible, it must have no zero modes. This requires that δA be restricted by gauge conditions, constraints, and boundary conditions. Specifying all of these can be quite subtle (and may entail boundary corrections to (34.47)). Writing out the indices,

$$\Omega_\Sigma = \frac{1}{e^2}\int_\Sigma d\Sigma^\mu \delta F_{\mu\nu}\wedge\delta A^\nu \ , \tag{34.48}$$

where $d\Sigma^\mu$ is the induced measure times the unit normal vector to Σ.

Now we take $\Sigma=\mathcal{I}^+$. This choice is very nice, because at $\mathcal{I}^+$ everything spreads out, becomes very weakly interacting, and essentially reduces to free field theory. The symplectic form becomes, very simply,

$$\Omega_{\mathcal{I}^+} = \frac{1}{e^2}\int du d^2z \left(\delta F^{(0)}_{uz}\wedge\delta A^{(0)}_{\bar z} + \delta F^{(0)}_{u\bar z}\wedge\delta A^{(0)}_z\right) \ . \tag{34.49}$$

This expression typically remains exact even in interacting theories (as long as the IR theory is free), because interactions are weak near $\mathcal{I}^+$.

Now we come to a tricky point, which was incorrectly treated in some of the literature until recently. We need to be very careful about what is happening at the boundaries of $\mathcal{I}^+$ and $\mathcal{I}^-$. We begin by separating out the constant (u-independent) part of $A^{(0)}_z$ [11, 12],

$$A^{(0)}_z(u,z,\bar z) = \hat A_z(u,z,\bar z) + \partial_z\phi(z,\bar z) \ , \tag{34.50}$$

where

$$\partial_z\phi \equiv \frac{1}{2}\left[A^{(0)}_z\big|_{\mathcal{I}^+_+} + A^{(0)}_z\big|_{\mathcal{I}^+_-}\right] \ . \tag{34.51}$$

The constant piece $\partial_z\phi$ is pure gauge, because we are demanding that the magnetic field vanish at the boundaries $\mathcal{I}^+_\pm$.[188] This boundary condition is crucial in the following: without it, the charges below would fail by a factor of 2 to generate the proper symmetries. Now we can substitute (34.50) into the expression for the symplectic form and find

$$\Omega_{\mathcal{I}^+} = \frac{2}{e^2}\int du d^2z\ \partial_u\delta\hat A_z\wedge\delta\hat A_{\bar z} - 2\int d^2z\ \partial_z\delta\phi\wedge\partial_{\bar z}\delta N \ , \tag{34.52}$$

where N was defined in (34.42). We see that the radiative components of $A^{(0)}_z$ are paired with each other in the symplectic form,[189] and the soft photon mode is paired with the field ϕ, which is the sum of the boundary values.

Note that the very existence of a Hamiltonian formulation requires that we keep the mode ϕ, which is absent in a standard Fourier decomposition. Without it, there is no symplectic partner for the soft photon zero-mode N.

Commutators

The next thing to discuss is the commutators. As we see from equation (34.46), to derive the commutators one must invert the symplectic form. Since it is a sum of terms, one of which only

[188] The symplectic form in the presence of magnetic charges and fields has so far not been constructed.

[189] Since $\hat A_z$ has no constant term, the sum of its boundary values vanishes, and N therefore does not appear in the first term.

involves $\hat{A}$ and the other which only involves the boundary fields N and ϕ, we can invert them separately. Consider the first one. It implies the commutator

$$\left[\partial_u \hat{A}_z\left(u,z,\bar{z}\right),\hat{A}_{\bar{w}}\left(u',w,\bar{w}\right)\right]=-\frac{ie^2}{2}\delta\left(u-u'\right)\delta^2\left(z-w\right)\ . \tag{34.53}$$

Integrating with respect to u, we find

$$\left[\hat{A}_z\left(u,z,\bar{z}\right),\hat{A}_{\bar{w}}\left(u',w,\bar{w}\right)\right]=-\frac{ie^2}{4}\Theta\left(u-u'\right)\delta^2\left(z-w\right)\ , \tag{34.54}$$

where

$$\Theta(u)=\frac{1}{\pi i}\int\frac{d\omega}{\omega}e^{i\omega u}\ , \tag{34.55}$$

with $\Theta(u<0)=-1$ and $\Theta(u>0)=1$, and a potential integration constant in (34.54) is fixed by antisymmetry. Equation (34.54) is a standard lightcone commutator. We can also compute the commutator between the boundary fields ϕ and N, which does not depend on u:

$$[\phi(z,\bar{z}),N(w,\bar{w})]=-\frac{i}{4\pi}\log|z-w|^2+f(z,\bar{z})+g(w,\bar{w})\ . \tag{34.56}$$

At various points, one must be careful about the integration functions f and g, but we are going to ignore them for the present as the computations below require only

$$[\partial_z\phi(z,\bar{z}),\partial_{\bar{w}}N(w,\bar{w})]=-\frac{i}{4\pi}\partial_z\partial_{\bar{w}}\log|z-w|^2=\frac{i}{2}\delta^2(z-w)\ , \tag{34.57}$$

which follows from

$$\partial_{\bar{z}}\frac{1}{z-w}=2\pi\delta^2(z-w)\ . \tag{34.58}$$

34.2.7 Large Gauge Symmetry

Now we can easily compute the commutator action of Q^+_ε. Q^+_ε has a term involving the matter field and a linear term involving $\partial_z N$ and $\partial_{\bar{z}}N$ (i.e., the soft photon terms). The soft terms do not commute with $A^{(0)}_z$. We find

$$\left[Q^+_\varepsilon,A^{(0)}_z(u,z,\bar{z})\right]=i\partial_z\varepsilon(z,\bar{z})\ , \tag{34.59}$$

while a similar calculation on $\mathcal{I}^-$ yields

$$\left[Q^-_\varepsilon,A^{(0)}_z(v,z,\bar{z})\right]=i\partial_z\varepsilon(z,\bar{z})\ . \tag{34.60}$$

Some other useful commutators are

$$\left[Q^+_\varepsilon,N(z,\bar{z})\right]=0\ ,\quad\left[Q^+_\varepsilon,\hat{A}_z(u,z,\bar{z})\right]=0\ ,\quad\left[Q^+_\varepsilon,\phi(z,\bar{z})\right]=i\varepsilon(z,\bar{z})\ . \tag{34.61}$$

This leads to a remarkable conclusion in the free Maxwell theory: *the infinite number of symmetries generated by the conserved charges Q^+_ε in a canonical formalism are just gauge transformations with parameter ε!* Of course, we have long known that for constant ε, the total charge $Q^+_{\varepsilon=1}$ generates constant gauge transformations on all the fields. Here, we see that even when ε is an arbitrary function, Q^+_ε transforms A_z by a non-trivial, "large" gauge transformation that does not die off at infinity. Rather, the gauge parameter goes to an angle-dependent, but u-independent,

function at $\mathcal{I}^+$. At $\mathcal{I}^-$, Q^-_ε generates a transformation for which the gauge parameter approaches the antipodally transformed function. These are pretty weird gauge transformations! The fact that the condition $A_z = 0$ is not invariant under these symmetries implies that they are spontaneously broken and the vacuum is infinitely degenerate, but more on that story in Section 2.11.

Now suppose our theory includes charged matter. The electromagnetic field commutators above are unchanged as long as the coupling approaches a constant in the infrared. We showed that the commutator of the soft charge with the gauge field itself is a large gauge transformation. With matter added, we must check that Q^+_ε properly generates the gauge transformations on the matter fields. This is the role of the hard term. The charged matter current j_u is the conserved Noether current associated with the global U(1) matter symmetry. By the Noether construction, j_u canonically generates this symmetry on $\mathcal{I}^+$:[190]

$$\left[j^{(2)}_u(u', w, \bar{w}), \Phi_k(u, z, \bar{z})\right] = -Q_k \Phi_k(u, z, \bar{z}) \gamma^{z\bar{z}} \delta^2(z - w) \delta(u - u') \ , \tag{34.62}$$

where Φ_k is a matter field of charge Q_k. This commutator implies

$$\left[\int_{\mathcal{I}^+} \varepsilon * j, \Phi_k(u, z, \bar{z})\right] = -Q_k \varepsilon(z, \bar{z}) \Phi_k(u, z, \bar{z}) = i\delta_\varepsilon \Phi_k(u, z, \bar{z}) \tag{34.63}$$

and

$$\left[Q^+_\varepsilon, \Phi_k(u, z, \bar{z})\right] = i\delta_\varepsilon \Phi_k(u, z, \bar{z}) \ . \tag{34.64}$$

Hence, we conclude that the sum of the hard and soft term properly generates local angle-dependent gauge transformations on $\mathcal{I}^+$.

Historically, the story ran the other way around. First it was discovered [11], through an asymptotic analysis, that QED had an infinite number of nontrivially acting gauge symmetries generated by the antipodally matched functions ε. Then the generators were constructed via the Noether method and identified as conserved charges. However, the asymptotic symmetry analysis — which we will see some of in Section 2.10 — requires assumptions about asymptotic falloffs near $\mathcal{I}$. Such assumptions are unnecessary in the treatment here starting from the conserved charges. As we have seen, their existence follows directly from nineteenth-century electromagnetism.

To summarize, so far we have found that there are an infinite number of conserved charges in electromagnetism. We have further showed that these charges canonically generate an infinite number of large gauge symmetries that act nontrivially on the physical phase space.

34.2.8 Ward Identity

In this subsection we derive Ward identities, which relate quantum scattering amplitudes. These identities express the dynamical consequences of the fact that the conserved charges commute with the Hamiltonian, or equivalently, with the $\mathcal{S}$-matrix, since $\mathcal{S} \sim \exp(iHT)$ for $T \to \infty$.

Symmetries of the $\mathcal{S}$-matrix

Quantum scattering amplitudes can be written in the form

$$\langle \text{out} | \mathcal{S} | \text{in} \rangle \ . \tag{34.65}$$

Charge conservation is then

$$\langle \text{out} | \left(Q^+_\varepsilon \mathcal{S} - \mathcal{S} Q^-_\varepsilon \right) | \text{in} \rangle = 0 \ . \tag{34.66}$$

[190] Here as elsewhere in this section, we make the assumption that the charged fields are massless, so that the charge flux all goes through $\mathcal{I}^+$. It remains true in the massive case that Q^+_ε generates gauge transformations, but there are very interesting new twists; see section 34.2.12.

By the matching condition, Q^+_ε is equal to Q^-_ε, but we use Q^+_ε when acting on out-states and Q^-_ε when acting on in-states, because they are expressed in variables appropriate for acting on the out- and in-states, respectively. Exponentiating the charge to generate a finite symmetry, (34.66) becomes the statement: given an in-state X that evolves to an out-state Y, a large-gauge-transformed in-state X evolves to a large-gauge-transformed out-state Y.

The action of Q^-_ε on the in-state is

$$Q^-_\varepsilon|\text{in}\rangle = -2\int d^2z\partial_{\bar z}\varepsilon\partial_z N^-(z,\bar z)|\text{in}\rangle + \sum_{k=1}^{m} Q_k^{\text{in}}\varepsilon(z_k^{\text{in}},\bar z_k^{\text{in}})|\text{in}\rangle\ , \tag{34.67}$$

where $N^-(z,\bar z)$ denotes the incoming soft photon field on $\mathcal{I}^-$. We have assumed that the in-state can be described by m hard particles that are coming in at points on the asymptotic sphere denoted by z_k^{in}. The first term is the action of the soft charge, and the second term is the action of the hard charge. Similarly, the action of Q^+_ε on the out-state takes the form

$$\langle\text{out}|Q^+_\varepsilon = 2\int d^2z\partial_z\partial_{\bar z}\varepsilon\langle\text{out}|N(z,\bar z) + \sum_{k=1}^{n} Q_k^{\text{out}}\varepsilon(z_k^{\text{out}},\bar z_k^{\text{out}})\langle\text{out}|\ . \tag{34.68}$$

Finally, we can write the Ward identity as

$$\begin{aligned} 2\int d^2z\partial_z\partial_{\bar z}\varepsilon\langle\text{out}|\left(N(z,\bar z)\mathcal{S}-\mathcal{S}N^-(z,\bar z)\right)|\text{in}\rangle \\ = \left[\sum_{k=1}^{m} Q_k^{\text{in}}\varepsilon(z_k^{\text{in}},\bar z_k^{\text{in}}) - \sum_{k=1}^{n} Q_k^{\text{out}}\varepsilon(z_k^{\text{out}},\bar z_k^{\text{out}})\right]\langle\text{out}|\mathcal{S}|\text{in}\rangle\ . \end{aligned} \tag{34.69}$$

Equation (34.69) is an infinite number of Ward identities, one for every function ε on the sphere. They relate any $\mathcal{S}$-matrix element between any pair of incoming and outgoing states multiplied by the soft factor in square brackets, to the same $\mathcal{S}$-matrix element with the insertion of certain soft photon modes. This is a very general relation. It would be incredibly surprising if, after 90 years of studying QED, we discovered an infinite number of new relationships between scattering processes. We have not done that. Rather, we have rediscovered some relations that have been known for a long time but were derived using very different methods, namely, Feynman diagrams and mode expansions, and go under the name of "soft theorems." What we will do next is show that these identities, which we derived from charge conservation, are precisely the same thing as the well-known soft theorems in abelian gauge theories.

Before proceeding with this demonstration, as a little teaser for section 34.4, it is interesting to consider the special case in which ε is taken to be

$$\varepsilon(w,\bar w) = \frac{1}{z-w}\ . \tag{34.70}$$

Using the fact that

$$\partial_{\bar z}\frac{1}{z-w} = 2\pi\delta^2(z-w) \tag{34.71}$$

to easily perform the integral on the LHS of equation (34.69), we can rewrite the Ward identity as

$$4\pi\langle\text{out}|\left(\partial_z N\mathcal{S}-\mathcal{S}\partial_z N^-\right)|\text{in}\rangle = \left[\sum_{k=1}^{m}\frac{Q_k^{\text{in}}}{z-z_k^{\text{in}}} - \sum_{k=1}^{n}\frac{Q_k^{\text{out}}}{z-z_k^{\text{out}}}\right]\langle\text{out}|\mathcal{S}|\text{in}\rangle\ . \tag{34.72}$$

In two-dimensional CFT with a current algebra, when one inserts a $U(1)$ Kac-Moody current into a correlation function of operators of charge Q_k, one gets exactly this kind of Ward identity. More generally, it was recently shown [199] that the entire soft factor of the $\mathcal{S}$-matrix is computed by a current algebra with level determined by the cusp anomalous dimension. We will see in section 34.4, in the more general nonabelian setting, that this is in fact not a coincidence and that the large gauge symmetry is equivalent to a $U(1)$ Kac-Moody symmetry on the sphere at null infinity.

Mode Expansions

To show that the Ward identity (34.69) is in fact a soft theorem, we have to transcribe notation. Until this section, the presentation has been rather different than what is found in most quantum field theory textbooks. We have characterized particles by the points at which they came in at null infinity, used advanced and retarded coordinates, and derived conservation laws from antipodal matching conditions.

In traditional quantum field theory, one works in a basis of plane waves. To compare notations we must rewrite (34.69) in terms of a plane wave basis, using a conventional mode expansion for A_z. It is not manifestly obvious that the commutation relations (34.54) and (34.56) are equivalent to the conventional ones. One way to see that they must nevertheless be the same is to note that both follow from the slice-independent covariant symplectic form (34.47). We evaluated it by pushing the slice up to $\mathcal{I}^+$, as opposed to the conventional use of the $t = 0$ slice and a plane wave basis. Such bases employ the standard Cartesian coordinates for Minkowski space

$$ds^2 = -dt^2 + d\vec{x} \cdot d\vec{x} \ . \tag{34.73}$$

Near $\mathcal{I}^+$, A_ν has the on-shell outgoing plane wave mode expansion

$$A_\nu(x) = e \sum_{\alpha=\pm} \int \frac{d^3q}{(2\pi)^3} \frac{1}{2\omega} \left[\varepsilon_\nu^{*\alpha}(\vec{q}) a_\alpha^{\text{out}}(\vec{q}) e^{iq\cdot x} + \varepsilon_\nu^{\alpha}(\vec{q}) a_\alpha^{\text{out}}(\vec{q})^\dagger e^{-iq\cdot x} \right] \ , \tag{34.74}$$

where $q^2 = 0$, the two polarization vectors satisfy a normalization condition $\varepsilon_\alpha^\nu \varepsilon^*_{\beta\nu} = \delta_{\alpha\beta}$, and

$$\left[a_\alpha^{\text{out}}(\vec{q}), a_\beta^{\text{out}}(\vec{q}')^\dagger \right] = \delta_{\alpha\beta} (2\pi)^3 (2\omega_q) \delta^3 \left(\vec{q} - \vec{q}' \right) \ . \tag{34.75}$$

This is the standard textbook formula for commutators of modes of the free electromagnetic field.

The next step is to rewrite asymptotic quantities on $\mathcal{I}^+$ in terms of the familiar creation and annihilation operators appearing in (34.74). Near $\mathcal{I}^+$ it is convenient to use retarded coordinates, in which the metric takes the form

$$ds^2 = -du^2 - 2dudr + 2r^2 \gamma_{z\bar{z}} dz d\bar{z} \ . \tag{34.76}$$

The transformation to retarded from Cartesian coordinates was given in (34.22). A null vector q^μ, satisfying $q^\mu q_\mu = 0$, is labelled by a point on the sphere, up to its overall magnitude. Hence, there is a natural map from null vectors q^μ to points $(z, \bar{z})$ on the sphere toward which the null vector is directed. We can write this as

$$q^\mu = \frac{\omega}{1 + z\bar{z}} \left(1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z} \right) = (\omega, q^1, q^2, q^3) \ . \tag{34.77}$$

As an example of this, let us suppose that z is taken to be the north pole at $z = 0$. Then we find $q^\mu = \omega(1, 0, 0, 1)$ (i.e., a null vector pointing along the x^3-axis). We may further choose the polarization vectors orthogonal to q^μ as

$$\varepsilon^{+\mu}(\vec{q}) = \frac{1}{\sqrt{2}} \left(\bar{z}, 1, -i, -\bar{z} \right) \ , \quad \varepsilon^{-\mu}(\vec{q}) = \frac{1}{\sqrt{2}} \left(z, 1, i, -z \right) \ , \quad q_\mu \varepsilon^{\pm\mu}(\vec{q}) = 0 \ , \quad \varepsilon_\alpha^\mu \varepsilon^*_{\beta\mu} = \delta_{\alpha\beta} \ . \tag{34.78}$$

Now let us consider the $\mathcal{I}^+$ field $A_z^{(0)}(u, z, \bar{z})$. By definition,

$$A_z^{(0)}(u, z, \bar{z}) = \lim_{r\to\infty} A_z(u, r, z, \bar{z}) \ . \tag{34.79}$$

Using the above formulas, we can take all the q^μs in the mode expansion and rewrite them in terms of points on the asymptotic sphere. Since $A_z(u, r, z, \bar{z})$ has an expansion in terms of creation and

annihilation operators, $A_z^{(0)}(u,z,\bar{z})$ must also have such an expansion. In particular, we expect $A_z^{(0)}(u,z,\bar{z})$ to create and annihilate photons that land at the point $(z,\bar{z})$, since it is an operator that is localized at that point on the asymptotic sphere. Moreover, rotating about the point $(z,\bar{z})$, $A_z^{(0)}$ gets one phase, while $A_{\bar{z}}^{(0)}$ gets the opposite phase. Hence we expect $A_z^{(0)}$ to create one photon helicity and annihilate the other, while $A_{\bar{z}}$ does the opposite. These expectations are realized when one evaluates (34.79) in a large-r saddle-point approximation (see exercise 3):

$$A_z^{(0)}(u,z,\bar{z}) = -\frac{i}{8\pi^2}\frac{\sqrt{2}e}{1+z\bar{z}}\int_0^\infty d\omega\left[a_+^{\text{out}}(\omega\hat{x})e^{-i\omega u} - a_-^{\text{out}}(\omega\hat{x})^\dagger e^{i\omega u}\right] , \qquad \hat{x}=\hat{x}(z,\bar{z}) . \tag{34.80}$$

The "out" creation and annihilation operators involve the three-momentum $\omega\hat{x}$; $\hat{x}$ is a unit vector that points to $(z,\bar{z})$ on the sphere, with the relationship between $\hat{x}_1$, $\hat{x}_2$, and $\hat{x}_3$ and the point $(z,\bar{z})$ as in equation (34.77). So $A_z^{(0)}(u,z,\bar{z})$ annihilates a positive helicity photon that is headed to the point $\hat{x}(z,\bar{z})$ and creates a negative helicity photon that is headed to the same point. It can create and annihilate photons of all different frequencies. Equation (34.80) is the basic relationship between the out fields in the $1/r$ expansion at $\mathcal{I}^+$ and the standard outgoing creation and annihilation operators in the plane wave expansion usually employed in quantum field theory.

The Ward identity involves $\partial_z N$, so we need to determine its mode expansion. To be precise about the zero-momentum limit, define

$$\partial_z N = \frac{1}{2e^2}\lim_{\omega\to 0^+}\int_{-\infty}^{\infty} du\left(e^{i\omega u}+e^{-i\omega u}\right)F_{uz}^{(0)} . \tag{34.81}$$

This definition ensures that $\partial_z\partial_{\bar{z}}N$ is Hermitian. Had our definition only involved either $e^{i\omega u}$ or $e^{-i\omega u}$, this would not be the case. Using (34.80), we find

$$\partial_z N = -\frac{1}{8\pi e}\frac{\sqrt{2}}{1+z\bar{z}}\lim_{\omega\to 0^+}\left[\omega a_+^{\text{out}}(\omega\hat{x}) + \omega a_-^{\text{out}}(\omega\hat{x})^\dagger\right] . \tag{34.82}$$

One might wonder why we care about this operator, since we are taking the limit $\omega\to 0$ while multiplying by ω. One might think it is just identically zero. However, if it were zero, the Ward identity would not make sense, because it would imply that the RHS of equation (34.72) would vanish. The reconciliation is that the $\mathcal{S}$-matrix elements with insertions of these operators have poles that cancel the explicit factor of ω appearing in equation (34.82).

There is a similar formula for $\partial_z N^-$:

$$\partial_z N^- = -\frac{1}{8\pi e}\frac{\sqrt{2}}{1+z\bar{z}}\lim_{\omega\to 0^+}\left[\omega a_+^{\text{in}}(\omega\hat{x}) + \omega a_-^{\text{in}}(\omega\hat{x})^\dagger\right] . \tag{34.83}$$

The Ward identity in the form (34.72) can be reexpressed as

$$\begin{aligned}\lim_{\omega\to 0}\Big[\omega\langle\text{out}|\left(a_+^{\text{out}}(\omega\hat{x})\mathcal{S} - \mathcal{S}a_-^{\text{in}}(\omega\hat{x})^\dagger\right)|\text{in}\rangle\Big] \\ = \sqrt{2}e(1+z\bar{z})\left[\sum_{k=1}^{n}\frac{Q_k^{\text{out}}}{z-z_k^{\text{out}}} - \sum_{k=1}^{m}\frac{Q_k^{\text{in}}}{z-z_k^{\text{in}}}\right]\langle\text{out}|\mathcal{S}|\text{in}\rangle .\end{aligned} \tag{34.84}$$

This is beginning to look very much like the standard soft photon theorem. There is a soft pole in the matrix element on the LHS, which we have rendered finite via explicit multiplication by ω. There are also collinear poles on the RHS for $z\to z_k$.

Soft Theorems

Now in principle to show that (34.84) is exactly equivalent to the standard soft theorem, I could take the RHS of the equation and rewrite the zs and z_ks in terms of the momenta of the particles. However, it is easier and equivalent to proceed in reverse. That is, we begin with the textbook formula for the soft photon theorem and show that it is equivalent to (34.84) by going from momentum space to points on a sphere. For the reader's benefit, the next section reviews the standard textbook derivation of the soft photon theorem. The soft theorem states

$$\begin{aligned}\lim_{\omega\to 0}\left[\omega\langle \text{out}|a_+^{\text{out}}(\vec{q})\mathcal{S}|\text{in}\rangle\right] &= e\lim_{\omega\to 0}\left[\sum_{k=1}^{m}\frac{\omega Q_k^{\text{out}} p_k^{\text{out}}\cdot\varepsilon^+}{p_k^{\text{out}}\cdot q}-\sum_{k=1}^{n}\frac{\omega Q_k^{\text{in}} p_k^{\text{in}}\cdot\varepsilon^+}{p_k^{\text{in}}\cdot q}\right]\langle \text{out}|\mathcal{S}|\text{in}\rangle \\ &= -\lim_{\omega\to 0}\left[\omega\langle \text{out}|\mathcal{S}a_-^{\text{in}\dagger}(\vec{q})|\text{in}\rangle\right] ,\end{aligned} \tag{34.85}$$

where $q^\mu=(\omega,\vec{q})$ is the momentum of the soft photon, and we are taking the in-state and out-state in plane wave bases:

$$|\text{in}\rangle = |p_1^{\text{in}},\dots,p_n^{\text{in}}\rangle , \qquad \langle\text{out}| = \langle p_1^{\text{out}},\dots,p_m^{\text{out}}| . \tag{34.86}$$

The equality of the matrix elements involving in and out soft photons is a consequence of $\mathcal{CPT}$ invariance. Equation (34.85) is not quite the standard form, as we have multiplied both sides of the equation by a factor ω in order to have a finite limit. In the limit where $\omega\to 0$ (the soft limit), the ratios appearing on the RHS are finite and depend only on the direction of $\vec{q}$, but not its magnitude. When one writes the formula without this factor of ω, there is a pole as $\omega\to 0$, which is called the "soft pole" or "Weinberg pole."

To make the comparison between equation (34.84) and equation (34.85), we note that in addition to the $\omega\to 0$ pole, there is a collinear pole, since q^μ and p_k^μ are both null vectors. When p_k^μ is proportional to q^μ, $p_k\cdot q=0$, and $\frac{1}{p_k\cdot q}$ diverges. In (34.84), collinearity implies $z_k=z$, which again gives a pole. Hence the pole structure of equation (34.84) and equation (34.85) agrees.

Now we want to turn equation (34.85) into equation (34.84) by writing the momentum of each hard particle in terms of the energy E_k and a point on the sphere $(z_k,\bar{z}_k)$. For example,

$$(p_k^{\text{in}})^\mu = E_k^{\text{in}}\left(1,\frac{z_k^{\text{in}}+\bar{z}_k^{\text{in}}}{1+z_k^{\text{in}}\bar{z}_k^{\text{in}}},\frac{-i\left(z_k^{\text{in}}-\bar{z}_k^{\text{in}}\right)}{1+z_k^{\text{in}}\bar{z}_k^{\text{in}}},\frac{1-z_k^{\text{in}}\bar{z}_k^{\text{in}}}{1+z_k^{\text{in}}\bar{z}_k^{\text{in}}}\right) . \tag{34.87}$$

Similar formulas apply to the soft photon momentum q^μ (see (34.77)) and outgoing hard particle momenta p_k^{out}. In other words, knowing that the massless charged particle is going to land at the point $(z_k,\bar{z}_k)$ determines what its momentum must be. This enables us to replace all momenta in equation (34.85) with their expressions in terms of energies and points on a sphere. Summing the contribution from an outgoing positive helicity soft photon and an incoming negative helicity soft photon then exactly reproduces equation (34.84). The straightforward but somewhat intricate algebra is left to exercise 5. The standard soft photon formula reduces to the Ward identity following from invariance under large gauge symmetries, or equivalently, the matrix element of the charge conservation law. This establishes the central result connecting the soft photon theorem to the large gauge symmetry of electromagnetism.

The story might have been told backward. Starting with the soft photon theorem, we could reverse engineer and deduce the fact that abelian gauge theories have an infinite number of conservation laws associated to antipodally identified large gauge transformations. They are mathematically equivalent statements.

34.2.9 Feynman Diagrammatics

In this section, we review the standard field theory derivation of the leading photon and graviton soft theorems in the form given by Weinberg [7, 151].

Soft Photons

The soft photon theorem states that any $\mathcal{S}$-matrix element with an additional soft ($q^\mu \to 0$) photon is equal to the original matrix element multiplied by the soft factor plus corrections of order q^0:

$$\langle \text{out}|a_+^{\text{out}}(\vec{q})\mathcal{S}|\text{in}\rangle = e\left[\sum_{k=1}^{m}\frac{Q_k^{\text{out}}p_k^{\text{out}}\cdot\varepsilon^+}{p_k^{\text{out}}\cdot q} - \sum_{k=1}^{n}\frac{Q_k^{\text{in}}p_k^{\text{in}}\cdot\varepsilon^+}{p_k^{\text{in}}\cdot q}\right]\langle \text{out}|\mathcal{S}|\text{in}\rangle + \mathcal{O}(q^0)\ . \tag{34.88}$$

The leading order term in the soft expansion is a pole.

To derive this formula, let us take any scattering process with n incoming and m outgoing particles and then consider adding to it one outgoing photon, denoted by a wavy line in figure 34.7, with momentum q. (The derivation for an incoming photon is similar.) In the soft limit, we

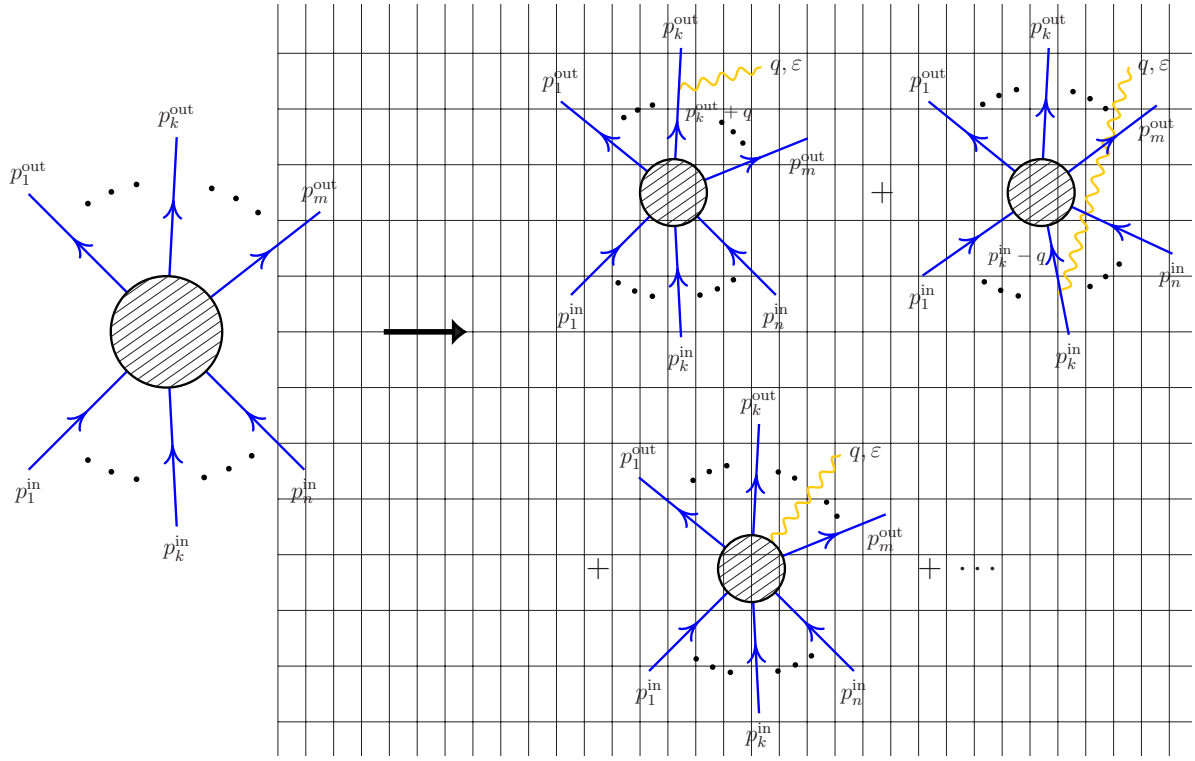


Figure 34.7: **On the left is a Feynman diagram representing $n \to m$ scattering. On the right the effect of adding an outgoing soft photon (or graviton) with momentum q and polarization ε is illustrated. In the upper diagrams the soft particle attaches to an external propagator, while in the lower one it attaches to an internal propagator.**

can write the amplitude as a sum of two types of terms, ones in which the soft photon attaches to an external line and others in which the soft photon attaches to an internal line. The soft photon can attach to any one of the $n+m$ external lines, so we must include a sum over all such terms. The full amplitude has a Laurent expansion in q with an infinite number of terms whose detailed form depends on what theory we are talking about. For the pole we need not specify what theory we are studying except that it has a photon. That is one of the beauties of this formula.

The LSZ rule for computing scattering amplitudes starts out by computing the time-ordered Green's functions using the Feynman $i\epsilon$ prescription and then amputating the external legs. The Feynman diagrams have factors for vertices and propagators. What happens when we attach the extra photon to an external leg is, since external legs are amputated, we need only add a vertex

and propagator for the particle to whose external leg the photon is added. The difference between the diagram with and without the attached external soft photon is just the vertex and propagator.

Now I have to say a little bit about the interaction vertex. Let us take the interaction to be

$$\mathcal{L}_{\text{int}} = -A^\mu j_\mu \ . \tag{34.89}$$

For a scalar field of charge Q, the charge current is

$$j_\mu = iQ(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) \ . \tag{34.90}$$

For a plane wave, this is just

$$j_\mu \sim 2Qp_\mu \ , \tag{34.91}$$

where we have used the normalization for single-particle states

$$\langle p|p'\rangle = 2\omega_p(2\pi)^3\delta^3(p-p') \ . \tag{34.92}$$

This is the electromagnetic current associated to a scalar field of charge Q, meaning that under gauge transformations, it acquires a phase $e^{iQ\varepsilon}$. Moreover, we take the polarization of the photon to obey $\varepsilon^\mu q_\mu = 0$, where q^μ is the momentum of the photon. The propagator for a scalar particle of mass m gives a factor

$$\frac{-i}{(p+q)^2+m^2} = \frac{-i}{p^2+2p\cdot q+q^2+m^2} = \frac{-i}{2p\cdot q} \ , \tag{34.93}$$

where we have used the fact that in a scattering amplitude, all the external lines must be on-shell, so $q^2 = 0$ and $p^2 = -m^2$. The vertex factor is, up to $\mathcal{O}(q)$ corrections,

$$ie\varepsilon^\mu 2Qp_\mu \ , \tag{34.94}$$

where ε^μ comes from A^μ and $2Qp_\mu$ comes from j_μ. The total contribution is

$$ie\varepsilon^\mu(2Qp_\mu)\frac{-i}{(p+q)^2+m^2} \to \frac{eQ\varepsilon\cdot p}{q\cdot p} \ . \tag{34.95}$$

There is one such term for every outgoing particle, while for the incoming particles there is an additional minus sign. Altogether these give

$$\sum_{k=1}^{m}\frac{eQ_k^{\text{out}}p_k^{\text{out}}\cdot\varepsilon}{p_k^{\text{out}}\cdot q} - \sum_{k=1}^{n}\frac{eQ_k^{\text{in}}p_k^{\text{in}}\cdot\varepsilon}{p_k^{\text{in}}\cdot q} \ . \tag{34.96}$$

At the end of the day, if we just consider the diagrams in which the photon attaches to an external leg, we simply multiply the $\mathcal{S}$-matrix element by the factor in equation (34.96), sometimes called the "soft factor", which you may recognize from equation (34.85).

We have not yet considered the terms coming from the photon attaching to an internal leg. The key point is that the internal propagators are never on-shell (i.e., they never have $p^2 = -m^2$). In the propagator, one then never has the cancellation between p^2 and m^2, so if we take $q^\mu \to 0$, the difference between p^2 and m^2 will dominate, and we will not get a pole. These types of diagrams are most certainly nonzero, but they do not contribute to the pole, so we can forget about them in the soft limit. This is an extremely simple derivation. Up to some signs, one finds the same thing for a soft incoming photon.

Now we note an important feature of this formula. The condition $\varepsilon^\mu q_\mu = 0$ defines the polarization vector only up to shifts of ε^μ by q^μ, because $q^2 = 0$. The physical amplitude with the soft photon should be invariant if we shift ε^μ by any multiple of q^μ. Now it is interesting to see what happens to the soft factor (34.96). If we shift ε^μ by q^μ, it shifts by

$$\sum_{k=1}^{m} eQ_k^{\text{out}} - \sum_{k=1}^{n} eQ_k^{\text{in}} = 0 \ . \tag{34.97}$$

In other words, global charge conservation guarantees that this soft factor is gauge invariant. This observation was in fact the basis of Low's derivation of the soft formula in 1958 [4].

We have only worked out the soft theorem for the case of a scalar. For a fermion or some other kind of charged particle, it is a little more complicated, but it works out to the same expression [151]. One way of seeing that this must be so is that (34.96) is the only formula with the right dimensions that is invariant under shifts $\varepsilon^\mu \to \varepsilon^\mu + q^\mu$.

Soft Gravitons

So far we have largely deferred any discussion of gravity. However, having just derived the soft formulas for quantum electrodynamics, the generalization to gravity is short and simple, so we do it now. We do not need to redraw diagrams, because we can just imagine that the wavy lines are gravitons of momentum q^μ rather than photons. We do need to specify the interaction

$$\mathcal{L}_{\rm int} = \sqrt{8\pi G} h^{\mu\nu} T_{\mu\nu} \ , \tag{34.98}$$

which couples the graviton field $h^{\mu\nu}$ in the expansion $g^{\mu\nu} = \eta^{\mu\nu} - \sqrt{32\pi G} h^{\mu\nu}$ to matter. This field is normalized so that there are no factors of G in the kinetic term. The graviton has polarization tensor $\varepsilon_{\mu\nu}$ satisfying

$$\varepsilon_{\mu\nu} q^\mu = 0 \ , \quad \varepsilon^{\mu\nu}\eta_{\mu\nu} = 0 \ . \tag{34.99}$$

For a scalar field,

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\phi\partial_\rho\phi \ . \tag{34.100}$$

Now, we apply the same argument, which is that we multiply the $\mathcal{S}$-matrix element by a factor of the propagator and the vertex. The propagator is the same,

$$\frac{-i}{2p\cdot q} \ . \tag{34.101}$$

Whether one has added an external photon or graviton, these poles, which become the holomorphic poles on the sphere at null infinity, are ubiquitous in soft limits. For a scalar field the interaction is

$$T_{\mu\nu} \sim 2p_\mu p_\nu \ , \tag{34.102}$$

where the term in $T_{\mu\nu}$ proportional to $\eta_{\mu\nu}$ does not contribute due to $\varepsilon_{\mu\nu}$ being traceless. In total, the product of the vertex and propagator gives a factor

$$i\sqrt{32\pi G}\varepsilon^{\mu\nu}p_\mu p_\nu \frac{-i}{(p+q)^2+m^2} \to \sqrt{8\pi G}\frac{\varepsilon^{\mu\nu}p_\mu p_\nu}{p\cdot q} \ . \tag{34.103}$$

This is just the result for one external particle. More generally, if we have m outgoing and n incoming particles, we have a soft factor

$$\sqrt{8\pi G}\sum_{k=1}^{m}\frac{\varepsilon^{\mu\nu}p^{\rm out}_{k\mu}p^{\rm out}_{k\nu}}{p^{\rm out}_k\cdot q} - \sqrt{8\pi G}\sum_{k=1}^{n}\frac{\varepsilon^{\mu\nu}p^{\rm in}_{k\mu}p^{\rm in}_{k\nu}}{p^{\rm in}_k\cdot q} \ . \tag{34.104}$$

We get a very similar-looking formula to the gauge theory case with p_k replacing Q_k. Just as in the gauge theory case, we cannot get poles by coupling the soft graviton to internal lines.

The soft factor must be invariant under the shift $\varepsilon^{\mu\nu} \to \varepsilon^{\mu\nu} + \Lambda^\mu(q)q^\nu$, the analog of gauge invariance in the gravity case. Performing such a shift, we can pull the Λ^μ out from under the sum and find that the soft factor shifts by

$$\Lambda^\mu\left[\sum_{k=1}^{m}p^{\rm out}_{k\mu} - \sum_{k=1}^{n}p^{\rm in}_{k\mu}\right] = 0 \ , \tag{34.105}$$

due to global energy-momentum conservation. We saw above that global charge conservation was required for the consistency and gauge invariance of the soft photon theorem. Here we see that global energy-momentum conservation is required for the consistency and gauge invariance of the soft graviton theorem. So far we have not used global angular momentum conservation, which we will see in section 34.5.3 is responsible for a new soft theorem and set of asymptotic symmetries called "superrotations".

34.2.10 Asymptotic Symmetries

Our discussion so far began with the derivation of conserved charges. These were then shown, via a canonical formalism, to generate asymptotic symmetries. We might have instead begun (as was the case historically) with a direct analysis of the asymptotic symmetries. Such analyses are more of an art than a science, in part because of ambiguities in the choice of asymptotic falloffs and gauge conditions, which are often only a posteriori justified. However, despite the lack of rigor, this approach has been extraordinarily fruitful and the source of many physical insights. Once one has an idea of how it should go, more rigorous methods can then be developed. The fact that it is an art and not a science is not a derogatory statement. In fact, maybe art is better than science!

An asymptotic symmetry group (ASG) is defined as

$$\text{ASG} = \frac{\text{allowed gauge symmetries}}{\text{trivial gauge symmetries}} \ . \tag{34.106}$$

One studies the theory on some spacetime and imposes boundary conditions that describe how the field components behave near the boundary. The boundary conditions should be weak enough so that all physically reasonable solutions are allowed, but strong enough so that relevant charges are finite and well defined. The *allowed gauge symmetries* are any ones that respect the boundary conditions. The *trivial gauge symmetries* are the ones that act trivially on the physical data of the theory.

It can be quite subtle to decide what kind of behavior is allowed at infinity or which data is physical. Indeed, this problem remains unresolved in full generality for asymptotically flat spacetimes in GR. In many cases, part of the definition of the theory is specifying the boundary conditions. Moreover, we shall encounter examples in sections 34.2.13 and 34.2.14 for which the ASG is not even a subset of the local symmetry group, indicating that the definition (34.106) is too narrow. Nevertheless, despite the vagaries of this procedure, it has proven extraordinarily useful.

Oddly, asymptotic symmetry analyses were first carried out in the context of gravity rather than gauge theory. An early application was the seminal work of Bondi, van der Burg, Metzner and Sachs [9, 8] (BMS), who wanted to find the subgroup of diffeomorphisms of asymptotically flat spacetimes that act nontrivially on the asymptotic data. It seemed intuitively obvious that if one has an isolated system that is flat near infinity, it should be acted on non-trivially by the Poincaré group. For an asymptotically flat black hole, a boost is a diffeomorphism that should certainly be allowed but must be nontrivial, because it changes the energy. Similarly, translations move the black hole to a different place. BMS expected the asymptotic symmetry analysis would reproduce the isometries of flat spacetime itself, namely, the Poincaré group. To everyone's great surprise and consternation, what they got instead was an infinite-dimensional group, now called the BMS group. This contains the finite-dimensional Poincaré group as a subgroup but has an additional infinity of generators known as "supertranslations." This result implies that GR does *not* reduce to special relativity for weak fields and long distances, as naïvely expected. As we will see below, it is now realized that the BMS group is not the whole story: it is both too big and too small, but much more on this in section 5.

Another famous asymptotic symmetry analysis appeared in the work of Brown and Henneaux in the late 1980s on AdS_3 [200]. They found a result that no one was expecting: the asymptotic symmetry algebra is two copies of the Virasoro algebra with a computable central charge — a harbinger of AdS/CFT. This beautifully demonstrated the power of the method. It is not just a way of confirming what we already know.

What would we expect for the asymptotic symmetry group of electrodynamics? This question has been asked only in the past few years [10, 11]. In Minkowski space electromagnetism, we

are interested in boundary conditions at $\mathcal{I}^+$ or $\mathcal{I}^-$. At these null boundaries, one does not have the same freedom in choosing the boundary conditions as one does, for example, for the timelike boundary of a box. Rather one must derive the asymptotic behavior from the field equations. If we consider a sphere at large r, its surface area grows like r^2, so for the energy flux at any moment to be finite, $T_{uu} \sim \mathcal{O}(1/r^2)$. Note that we are not talking about the integrated energy flux, whose finiteness requires that fields fall off in a certain way at late and early times on $\mathcal{I}^\pm$. In electromagnetism,

$$T_{uu} \sim F_{uz} F_{u\bar{z}} \frac{\gamma^{z\bar{z}}}{r^2} + \dots \ , \tag{34.107}$$

where the factor of $\frac{1}{r^2}$ comes from inverting the metric on the sphere of radius r. This suggests that $F_{uz} \sim \mathcal{O}(1)$. However, F_{ru} is the long-range electric field, so it should go as $\mathcal{O}(1/r^2)$ to have finite charge configurations. Similarly, $F_{rz} \sim \mathcal{O}(1/r^2)$. To summarize:

$$F_{uz} \sim \mathcal{O}(1), \quad F_{ur}, F_{zr} \sim \mathcal{O}\left(\frac{1}{r^2}\right) \ . \tag{34.108}$$

This suggests we choose the boundary falloff conditions for the gauge fields to be

$$A_z \sim \mathcal{O}(1) \ , \quad A_r \sim \mathcal{O}\left(\frac{1}{r^2}\right) \ , \quad A_u \sim \mathcal{O}\left(\frac{1}{r}\right) \ . \tag{34.109}$$

If we try to impose stronger boundary conditions for A_μ, we would teleologically disallow some physically reasonable initial data configurations.

Now we ask: what kind of gauge transformations do these falloffs allow? Generic gauge transformations take the form

$$\delta A_z = \partial_z \varepsilon \ , \quad \delta A_u = \partial_u \varepsilon \ , \quad \delta A_r = \partial_r \varepsilon \ . \tag{34.110}$$

These boundary conditions are consistent with

$$\varepsilon = \varepsilon(z, \bar{z}) + \mathcal{O}\left(\frac{1}{r}\right) \ . \tag{34.111}$$

Note, the $\mathcal{O}(1)$ piece of ε cannot depend on u without violating the boundary conditions (34.109). The transformations (34.111) are the exact electrodynamic analog of the BMS transformations found much earlier for gravity. The additional necessity of an antipodal matching condition for transformations on $\mathcal{I}^+$ and $\mathcal{I}^-$ may be deduced from the need for a well-defined Lorentz invariant scattering problem.

There are several ways to see that the transformations (34.111) are nontrivial as well as allowed. One is to show that their Ward identities have nontrivial physical content and are equivalent to the soft photon theorem. Another is to note that they impart distinct phases to charged particles at different positions on $\mathcal{I}^+$. This leads to an Aharonov-Bohm type interference pattern, which could be observed for electrons stationed near $\mathcal{I}^+$. Details of how this "electromagnetic memory effect" can be observed are discussed in the literature [201, 38, 39]. We will discuss the similar gravitational analog in detail in section 34.6.

Equation (34.111) together with antipodal matching are of course exactly the large gauge symmetries discussed in section 34.2.7. This very simple asymptotic analysis quickly leads to the result that we previously arrived at in a lengthier manner starting from conserved charges. In fact, this is how the symmetries were first discovered [10]. However, if one thinks about it a little bit harder, one might ask, for example, why we did not allow A_us that are $\mathcal{O}(r^0)$ but pure gauge, since they would maintain the falloff behavior of the field strengths. Indeed, interesting new developments [61, 202, 62] suggest that pure gauge modes for which (34.108) is preserved but (34.109) is not provide an effective way to understand the subleading soft theorem [2, 4] and associated symmetries [60]. So while some basics of asymptotic symmetry analyses are firmly understood, much remains to be learned.

34.2.11 Spontaneous Symmetry Breaking, Vacuum Degeneracy, and Goldstone Bosons

As we shall now elucidate, the preceding results imply that abelian gauge theories have an infinite vacuum degeneracy and undergo spontaneous symmetry breaking. However, because of the unusual angular dependence of the symmetry parameters, the nature of the symmetry breaking and relation of the degenerate vacua are different than what we are used to.[191] Let us begin by recalling the standard case of spontaneous breaking of a global symmetry.

A classic example of spontaneous symmetry breaking is a scalar field theory with the "Mexican hat" potential

$$V(\Phi) = -m^2\Phi^*\Phi + \lambda\big(\Phi^*\Phi\big)^2 \ , \tag{34.112}$$

illustrated in figure 34.8. This theory has a global symmetry, under which

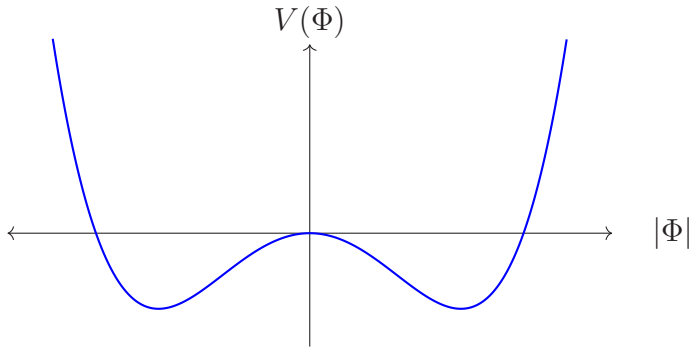


Figure 34.8: **The Mexican hat potential.**

$$\Phi \to e^{i\theta}\Phi \ , \qquad \Phi^* \to e^{-i\theta}\Phi^* \ . \tag{34.113}$$

The minimum of the potential occurs when

$$\Phi_{\rm vac}(x) = \frac{m}{\sqrt{2\lambda}}e^{i\rho} \ , \tag{34.114}$$

for arbitrary constant ρ. If we want to study excitations of this theory, we expand around $\Phi = \Phi_{\rm vac}$, not $\Phi = 0$. In the quantum theory, a particular choice $|\,\rho = \rho_0\,\rangle$ of vacuum state is made. Clearly, this state is not invariant under the global symmetry transformations (34.113), that is, if Q_θ generates (34.113), then it acts on the vacuum state as

$$e^{iQ_\theta}|\,\rho_0\,\rangle \sim |\,\rho_0 + \theta\,\rangle \neq 0 \ . \tag{34.115}$$

Hence we have a symmetry of the Lagrangian but not a symmetry of the vacuum state. In this case, we say that the symmetry (34.113) is spontaneously broken.

Goldstone's theorem states that whenever a continuous global symmetry is spontaneously broken, there exists a massless excitation about the spontaneously broken vacuum. Decomposing

[191] Large gauge symmetries are unlike any previously discussed symmetry both in their asymptotic angle dependence and in the fact that the action is described at null, not spatial, infinity. Phrases like "spontaneous symmetry breaking," "Goldstone boson," "superselection sector," and even "conservation law" are used with slightly different meanings in different physical contexts. In importing those words to the present context, I have necessarily adapted and refined their meanings. I have done so in the way I thought most natural, but other adaptations might be possible.

$\Phi(x) = |\Phi(x)|e^{i\rho(x)}$, ρ transforms as $\rho(x) \to \rho(x) + \theta$. Hence the Lagrangian can depend on ρ only via the derivative $\partial_\mu \rho$; there cannot be any mass term for ρ, and it is a massless field. ρ — identified as the field that transforms inhomogeneously under the broken symmetry — is referred to as the Goldstone boson.

Let us now relate this elementary example to our discussion in the previous sections. We have a charge Q^+_ε that generates a symmetry of the Lagrangian of any abelian gauge theory. However, this charge does not annihilate the vacuum. Instead, it creates an extra soft photon mode ϕ, which, according to (34.61), transforms inhomogeneously under the broken symmetry. Hence the soft photons are the Goldstone bosons of spontaneously broken large gauge symmetry. There is an infinite vacuum degeneracy, since we can add any number of soft photons to any vacuum state and obtain another vacuum state with the same zero energy. Classically, the infinite-dimensional space of vacua can be labeled by flat abelian connections $\partial_z \varepsilon(z, \bar{z})$ on the sphere.

There is a crucial difference between the usual Mexican hat story of spontaneous global symmetry breaking and the spontaneous breaking of the large gauge symmetries. In the usual story, the different vacua form superselection sectors (i.e., no physical finite energy operator exists that can move us from one vacuum to another). In other words, the charge Q_θ that would in principle move us between various vacua is not a normalizable operator in the Hilbert space. To see this, consider a system in a vacuum ρ_0. Now, suppose we have a bubble with a different value of ρ, say ρ_1, in the interior. Then along the surface of this bubble, Φ must have nonzero gradient. Hence the bubble carries energy proportional to its area. To change the vacuum from ρ_0 to ρ_1 everywhere, we need this bubble to grow in size and cover the entire volume of the system. In infinite volume systems, this process will take an infinite amount of energy and therefore cannot be achieved. Thus, there are no finite-energy physical processes that change the vacuum state in systems of infinite size, and the Q_θ appearing in (34.115) is not a finite energy operator on the Hilbert space. This argument relied on the area of the bubble growing with its volume. This is only true in more than one space dimension. In two spacetime dimensions the bubble wall is a point, and the area does not grow. Finite energy processes may then give rise to a change in the vacuum. In spacetime dimensions $d > 2$, scattering processes break up into superselection sectors, but in $d = 2$ they do not.

Such superselection sectors clearly do not arise for the large gauge symmetry. The vacuum state is changed by soft photon creation, which occurs in nearly all scattering processes.[192] The $\mathcal{S}$-matrix elements do not factorize into superselection sectors. One way to understand this is that, even though we are working in $d = 4$, the symmetry action in some regards mimics the $d = 2$ case. The parameter that generates the spontaneously broken symmetry is an arbitrary local function on the sphere — one parameter for every point — as opposed to the constant parameter in (34.113). Hence vacuum-to-vacuum transitions need not occur simultaneously everywhere at once on the sphere at infinity. Restricting to just one point, finite energy transitions are possible.

At the classical level, the degenerate electromagnetic vacua connected by large gauge transformations all have vanishing $F_{\mu\nu}$ and hence carry no large gauge charge. This is because the large gauge group is abelian, and hence its action commutes with the charge. As discussed in section 34.5, an analogous classical vacuum degeneracy appears in gravity. However in this case, the asymptotic symmetry group is nonabelian, and the different vacua are classically distinguished by the charges they carry (e.g., angular momentum $\vec{J}$). The existence of classically degenerate vacua is then more obvious, as it can then be deduced from charge conservation. Consider, for example, the situation depicted in figure 34.9, in which an initial configuration with angular momentum $\vec{J}_{initial}$ radiates angular momentum $\vec{J}_{rad}$ and decays to a vacuum spacetime with vanishing Riemann tensor but angular momentum $\vec{J}_{vac}$.[193] Angular momentum conservation implies

$$\vec{J}_{initial} = \vec{J}_{rad} + \vec{J}_{vac} \ . \tag{34.116}$$

Insisting on vacuum uniqueness in gravity leads to the nonsensical conclusion that angular momentum is not conserved. This is sometimes referred to as the problem of angular momentum in

[192] In the gravitational version of this story, the change in vacuum is classically measured by the gravitational memory effect, which one hopes to measure at LIGO [34] or elsewhere.

[193] Explicit solutions of this general type are studied below in section 34.7.2. Formulas for the infinitesimal change in the vacuum angular momentum under a supertranslation are in [203], and the finite case is treated in [51]. Vacua with nonzero $\vec{J}$ are also discussed in [204].

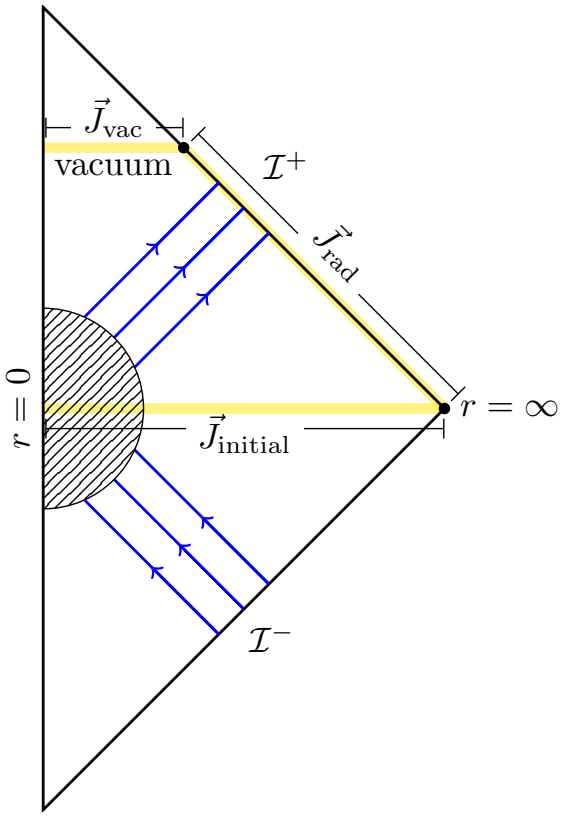


Figure 34.9: **The existence of degenerate vacua in general relativity can be deduced from angular momentum conservation. According to the Einstein equation, an initial configuration with angular momentum $\vec{J}_{initial}$ can decay to flat space while emitting total angular momentum flux $\vec{J}_{rad} \neq \vec{J}_{initial}$. Angular momentum is nevertheless conserved, because the final vacuum carries angular momentum.**

GR, but it is neatly resolved by the existence of degenerate vacua with nonzero $\vec{J}$.

34.2.12 Massive QED

In the previous sections, we have seen that massless electrodynamics has infinitely many conserved charges, and moreover that the quantum matrix elements of the conservation laws are the soft photon theorem for amplitudes with massless charged external states. Of course, in the real world, charged particles like the electron have mass. In this subsection we generalize the previous results to massive particles.

The basic obstacle is that our analysis so far has been based on $\mathcal{I}$, but massive particles never reach $\mathcal{I}$ (for discussion, see [205]). For theories that only have massless particles, $\mathcal{I}$ forms a Cauchy surface — that is, specification of initial (or final) data on $\mathcal{I}^-$ (or $\mathcal{I}^+$) is sufficient to obtain the complete evolution of the field into the interior. This no longer works for massive particles, since they do not arrive at $\mathcal{I}$. Rather, they arrive from i^- and asymptote to i^+. Interestingly, resolving this problem essentially forces one to the use of hyperbolic slices of Minkowski space employed for holographic reduction by de Boer and Solodukhin [52]. The massive formulas are a bit more complicated than the massless ones, but they have a beautiful structure and make suggestive connections to AdS/CFT.

The first step is to understand how the symmetries act near $i^\pm$. This has been discussed in two papers [13, 14]. In the first paper, the authors present a gauge invariant description of the action of the asymptotic symmetries on outgoing massive particles and show that the Ward identities

correctly imply the soft theorem. In the second paper, the authors worked in Lorenz gauge. Despite making a specific gauge choice, the discussion of the second paper is more illuminating. We now review it.

Most of the requisite machinery has already been set up in the massless analysis. The main step left is to understand how the large gauge symmetries act on massive states. Given this knowledge, we know the action of the hard part of the charge on asymptotic massive states. With this extra term in hand, we can proceed as before by transforming into momentum space to reproduce the soft theorem.

For this purpose, a prescription is needed to extend the asymptotic gauge parameter $\varepsilon(z,\bar{z})$ into the bulk interior of Minkowski space and $i^{\pm}$. While this bulk extension can be done in many different ways, due to gauge invariance, any extension should give rise to the same final result. It turns out to be very convenient to extend the gauge parameter from the boundary into the bulk using Lorenz gauge

$$\nabla^{\mu} A_{\mu} = 0 \ , \tag{34.117}$$

which implies that the gauge parameter must satisfy

$$\Box \varepsilon = 0 \ . \tag{34.118}$$

We need to solve this equation with the boundary condition that it asymptote to a specific function $\varepsilon(z,\bar{z})$ on $\mathcal{I}^{+}$. We might try to solve it by Fourier transforming. However, this gives scattering solutions that fall off like $\frac{1}{r}$ near $\mathcal{I}^{+}$ and therefore cannot satisfy our boundary condition. Fortunately, (34.118) also has solutions that have no Fourier transform and do not die off at $\mathcal{I}^{+}$ (see exercise 6). The solution that approaches $\varepsilon(z,\bar{z})$ can be written as

$$\varepsilon(x) = \int d^2\hat{q}\, G\big(x,\hat{q}\big)\varepsilon(\hat{q}) \ , \tag{34.119}$$

where we have parametrized the point on the asymptotic sphere by a unit vector $\hat{q}$ that points toward $(z,\bar{z})$, and G is a Green's function that must satisfy

$$\Box G(x,\hat{q}) = 0 \ , \qquad \lim_{\substack{r\to\infty \\ u \text{ fixed}}} G(x,\hat{q}) = \delta^2(\hat{x}-\hat{q}) \ . \tag{34.120}$$

This has the solution

$$G\big(x,\hat{q}\big) = -\frac{\sqrt{\gamma(\hat{q})}}{4\pi}\frac{x^{\mu}x_{\mu}}{(q\cdot x)^2} \ , \tag{34.121}$$

where $q^{\mu} = \big(1,\hat{q}\big)$. The first property is easy to show:

$$\begin{aligned} \Box G\big(x,\hat{q}\big) &= -\frac{\sqrt{\gamma(\hat{q})}}{4\pi}\partial_{\nu}\left[\frac{2x^{\nu}}{(q\cdot x)^2} - \frac{2x^{\mu}x_{\mu}}{(q\cdot x)^3}q^{\nu}\right] \\ &= -\frac{\sqrt{\gamma(\hat{q})}}{4\pi}\left[\frac{8}{(q\cdot x)^2} - \frac{4}{(q\cdot x)^2} - \frac{4}{(q\cdot x)^2} + \frac{6x^{\mu}x_{\mu}}{(q\cdot x)^4}q^2\right] = 0 \ . \end{aligned} \tag{34.122}$$

The second property is exhibited by writing G in retarded coordinates:

$$G\big(x,\hat{q}\big) = \frac{\sqrt{\gamma(\hat{q})}}{4\pi}\frac{u(u+2r)}{\big[u+r\,(1-\hat{q}\cdot\hat{x})\,\big]^2} \tag{34.123}$$

which at large r vanishes if $\hat{q}\neq\hat{x}$ and diverges if $\hat{q}=\hat{x}$. The overall normalization of the delta function can then be determined by integrating over $\hat{q}$.

Another important property of G can be seen by considering the limit toward $\mathcal{I}^{-}$, $r\to\infty$, keeping $v = u+2r$ constant. In this limit, G localizes to the point $\hat{q} = -\hat{x}$. Thus, we find that in the $\mathcal{I}^{-}$ limit, the large gauge parameter is antipodally related to the one obtained in the $\mathcal{I}^{+}$ limit. In other words, G "knows" about the antipodal map, which is a very convenient feature of Lorenz gauge. In hindsight, this may not be altogether surprising, since the gauge chosen here is Lorentz invariant, and the antipodal map is required by Lorentz invariance.

Hyberbolic Slices

Now we are in a position to find the limit of this bulk gauge parameter onto i^+. For this purpose, we introduce a new set of coordinates. The retarded $(u, r, z, \bar{z})$ or advanced $(v, r, z, \bar{z})$ coordinates that we have employed so far are natural when one is interested in null infinity. However, they are not good near $i^{\pm}$. To discuss these boundaries, consider a hyperbolic slicing of Minkowski space, with slices labeled by the coordinate

$$\tau^2 = -x^\mu x_\mu = t^2 - r^2 \ . \tag{34.124}$$

Hypersurfaces corresponding to constant τ are hyperbolic spaces. For $\tau^2 > 0$, the hypersurfaces are $\mathbb{H}_3$ (or Euclidean AdS_3), whereas for $\tau^2 < 0$ the hypersurfaces are three-dimensional de Sitter space dS_3, as illustrated in figure 34.10. Here we are interested in resolving the structure of $i^{\pm}$,

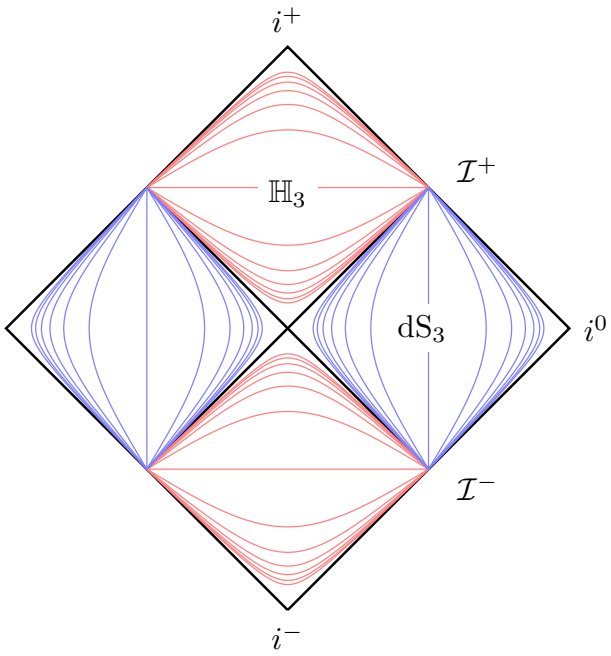


Figure 34.10: **Hyperbolic slicing of Minkowski space. The slices correspond to constant $\tau^2 = t^2 - r^2$ surfaces. The red lines correspond to $\mathbb{H}_3$ slices and have $\tau^2 > 0$, whereas the blue lines correspond to the dS_3 slices with $\tau^2 < 0$.**

for which we focus on the $\mathbb{H}_3$ slices. The dS_3 slices were used to resolve the structure of spatial infinity i^0 by Ashtekar and Romano [206] in a related context.

On the $\mathbb{H}_3$ slices, we introduce the coordinate ρ:

$$\rho = \frac{r}{\sqrt{t^2 - r^2}} \ . \tag{34.125}$$

The Minkowski metric then takes the form

$$ds^2 = -d\tau^2 + \tau^2 \left[\frac{d\rho^2}{1+\rho^2} + \rho^2 d\Omega_2^2 \right] \ . \tag{34.126}$$

The metric on the $\tau =$ constant hypersurfaces is that of $\mathbb{H}_3$, whose isometry group is $SL(2, \mathbb{C})$. These are Lorentz transformations, which in Minkowski space map each of these slices into themselves. Similarly, $SL(2, \mathbb{C})$ maps the dS_3 slices into themselves.

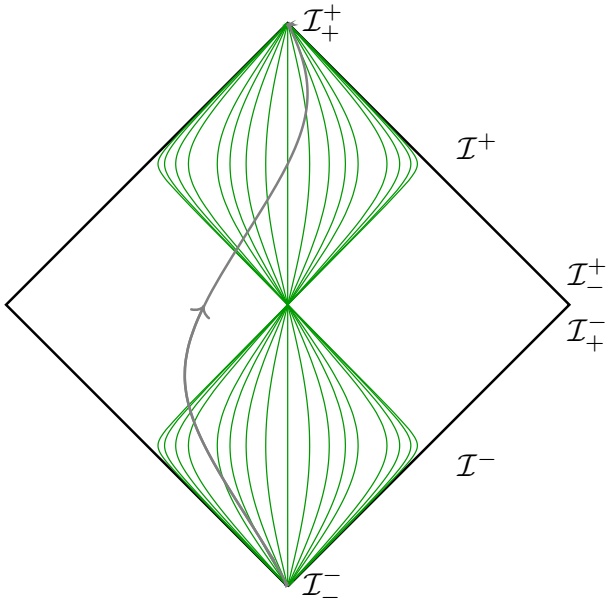


Figure 34.11: **Hypersurfaces of constant ρ are shown in green. The grey line is the worldline of a massive particle moving at constant velocity. The worldline of a particle asymptotes to a surface of constant ρ as $\tau \to \infty$.**

The Green's function (34.121) in these coordinates takes the form

$$G(\tau, \rho, \hat{x}; \hat{q}) = \frac{\sqrt{\gamma(\hat{q})}}{4\pi \left[\sqrt{1+\rho^2} - \rho \hat{q} \cdot \hat{x}\right]^2} \,. \tag{34.127}$$

Note that τ has completely dropped out of this equation (i.e., $\partial_\tau G = 0$). This implies that the gauge parameter (34.119) is independent of τ. Viewed as a three-dimensional Green's function, (34.127) has been studied extensively in the context of AdS_3 and is known as the bulk-to-boundary propagator for a massless scalar, which relates quantities on the boundary to those in the bulk. We are beginning to see the holographic structure of AdS_3 echoing in Minkowski space. Of course, this Green's function seems quite special since it does not depend on τ at all, but in other applications to flat space holography [142, 143], $\mathbb{H}_3$ Green's functions (or bulk-to-boundary propagators) for fields of any conformal weight can arise and are accompanied by τ-dependence in the full Minkowski space solutions.

Now consider a massive particle moving with constant momentum and following the trajectory $\vec{r} = \frac{1}{E}\vec{p}t + \vec{r}_0$ for some fixed $\vec{r}_0$, as illustrated in figure 34.11. $(\tau, \rho, \hat{x})$ are given by

$$\begin{aligned}
\rho &= \frac{r}{\sqrt{t^2 - r^2}} = \frac{|\frac{1}{E}\vec{p}t + \vec{r}_0|}{\sqrt{t^2 - \left(\frac{1}{E}\vec{p}t + \vec{r}_0\right)^2}} \,, \\
\tau &= \sqrt{t^2 - r^2} = \sqrt{t^2 - \left(\frac{1}{E}\vec{p}t + \vec{r}_0\right)^2} \,, \\
\hat{x} &= \frac{\vec{r}}{r} = \frac{\vec{p}t + E\vec{r}_0}{|\vec{p}t + E\vec{r}_0|} \,.
\end{aligned} \tag{34.128}$$

At late times, $t \to \infty$,

$$\rho \to \frac{|\vec{p}|}{m} , \qquad \tau \to \frac{m}{E} t , \qquad \hat{x} \to \hat{p} . \tag{34.129}$$

Hence ρ, and $\hat{x}$ asymptote to constants, and a massive particle approaches a fixed location on the unit hyperboloid at i^+. Hence in this resolution, i^+ is the hyperbola $\mathbb{H}_3$, much as $\mathcal{I}^+$ is $\mathbb{R} \times S^2$. The symplectic form for massive particles on $\mathbb{H}_3$ is worked out in exercise 7.

This construction readily gives the large gauge symmetry action on such a one-particle state at asymptotically late times. As ε is independent of the parameter τ labelling the hypersurfaces, we simply evaluate it at (34.129). Writing the hard part of the charge in a vicinity of i^+ as an integral over the asymptotic late-time $\mathbb{H}_3$, one concludes that

$$Q_\varepsilon^{+H} |\, \vec{p}\,\rangle = Q\varepsilon \left(\frac{|\vec{p}|}{m}, \hat{p} \right) |\, \vec{p}\,\rangle , \tag{34.130}$$

where ε is given by (34.119). To summarize, the action of the large gauge charge on a massive particle is proportional to the Lorenz gauge value of the gauge parameter at the $\mathbb{H}_3$ point to which the massive particle asymptotes.

Soft Theorem

We are now set to use this result to derive the quantum Ward identity. The derivation is essentially identical to the massless case. Equations (34.67) and (34.68) now include additional terms on the RHS from massive states:[194]

$$\begin{aligned} \langle \text{ out } | Q_\varepsilon^+ = -2 \int d^2w \partial_{\bar{w}} \varepsilon \langle \text{ out } | \partial_w N \\ + \sum_{k \in \text{massless}} Q_k \varepsilon(z_k, \bar{z}_k) \langle \text{ out } | + \sum_{k \in \text{massive}} Q_k \varepsilon \left(\frac{|\vec{p}_k|}{m_k}, \hat{p}_k \right) \langle \text{ out } | . \end{aligned} \tag{34.131}$$

The Ward identity becomes

$$\begin{aligned} -2 \int d^2w \partial_{\bar{w}} \varepsilon \partial_w \langle \text{ out } | \big[N(w,\bar{w}) \mathcal{S} - \mathcal{S} N^-(w,\bar{w}) \big] | \text{ in } \rangle \\ = - \left[\sum_{k \in \text{massless}} Q_k \varepsilon(z_k, \bar{z}_k) + \sum_{k \in \text{massive}} Q_k \varepsilon \left(\frac{|\vec{p}_k|}{m_k}, \hat{p}_k \right) \right] \langle \text{ out } | \mathcal{S} | \text{ in } \rangle . \end{aligned} \tag{34.132}$$

To verify that this is the soft photon theorem, set, as before, $\varepsilon(w,\bar{w}) = \frac{1}{z-w}$ and obtain

$$\begin{aligned} \frac{\sqrt{2}}{1+z\bar{z}} \lim_{\omega \to 0^+} \big[\omega \langle \text{ out } | a_+\big(\omega \hat{x}(z,\bar{z})\big) \mathcal{S} | \text{ in } \rangle \big] \\ = e \left[\sum_{k \in \text{massless}} \frac{Q_k}{z - z_k} + \sum_{k \in \text{massive}} Q_k \varepsilon \left(\frac{|\vec{p}_k|}{m_k}, \hat{p}_k \right) \right] \langle \text{ out } | \mathcal{S} | \text{ in } \rangle . \end{aligned} \tag{34.133}$$

[194] For notational brevity, we introduce here a simplified notation for n-point amplitudes with m outgoing particles and $n-m$ incoming particles. We drop the superscripts "in" and "out" on the momenta and charges and simply take the particles labeled $k = 1, \ldots, m$ to be outgoing and particles labelled $k = m+1, \ldots, n$ to be incoming. We further adopt a convention in which an incoming negative helicity photon of energy p^0 is described as a outgoing positive helicity photon with energy $-p^0$. Similarly, an incoming matter state of helicity h, charge Q, and energy p^0 is described as an outgoing matter state of helicity $-h$, charge $-Q$, and energy $-p^0$.

Using the property

$$G\left(\frac{|\vec{p_k}|}{m},\hat{p}_k;w,\bar{w}\right)=\frac{1}{2\pi}\partial_{\bar{w}}\left[\frac{\sqrt{2}}{1+w\bar{w}}\frac{p_k\cdot\varepsilon^+}{p_k\cdot\hat{q}}\right]\ , \tag{34.134}$$

we find

$$\varepsilon\left(\frac{|\vec{p_k}|}{m},\hat{p}_k\right)=\int d^2wG\left(\frac{|\vec{p_k}|}{m},\hat{p}_k;w,\bar{w}\right)\frac{1}{z-w}=\frac{\sqrt{2}}{1+z\bar{z}}\frac{p_k\cdot\varepsilon^+(z,\bar{z})}{p_k\cdot\hat{q}(z,\bar{z})}\ . \tag{34.135}$$

Plugging this into (34.133) and multiplying both sides by $\frac{1+z\bar{z}}{\sqrt{2}}$, we reproduce the soft photon theorem for massive charged particles in the standard momentum space form (34.88). It is rather satisfying to see this detailed equivalence emerge from the properties of bulk-to-boundary propagators on $\mathbb{H}_3$!

34.2.13 Magnetic Charges

In this section, we derive a new nonperturbative soft theorem and generalize the analysis to include magnetic charges [192].

In perturbation theory, it has been proven using Feynman diagrams [151] that the soft photon theorem is tree-level exact (i.e., it does not receive any loop corrections). It is natural to ask whether there can be nonperturbative corrections. To even talk about nonperturbative contributions to the soft theorem, one first needs a theory that exists nonperturbatively. QED — the theory of photons and electrons — does not exist due to the Landau pole. It must be embedded in some bigger theory — maybe one that is asymptotically free — that does exist nonperturbatively. To the best of my knowledge, all examples of such bigger theories contain magnetic monopoles.

This leads us to ask whether the soft photon theorem is corrected when magnetic monopoles are present in the asymptotic states. It is not hard to see that there must indeed be such a correction. In low-energy effective field theory, a magnetic monopole is represented by a point particle with magnetic charge. A soft photon will certainly couple to such an object, and the pole due to the nearly on-shell propagator (34.93) is still present. The only thing that changes is the coupling of the magnetic particle to the gauge field (34.94). This coupling gives rise to a different soft factor, but the pole remains.

To derive the nonperturbative corrections to the soft theorem, we begin with the perturbative electric soft factor

$$S_0^\alpha=\sum_k\frac{eQ_kp_k\cdot\varepsilon^\alpha}{q\cdot p_k}\ , \tag{34.136}$$

for a soft photon of polarization ε_α. Consider an amplitude involving asymptotic particles with integral magnetic charges M_k defined by

$$M_k=\frac{1}{2\pi}\int_{S_k^2}F\ , \tag{34.137}$$

where S_k^2 is a sphere surrounding the kth particle. Since the pole in the soft factor is unchanged, we need to simply determine the vertex factor. A trick for this purpose is to transform to dual variables

$$\tilde{F}=-\frac{2\pi}{e^2}*F\ ,\qquad \tilde{e}=\frac{2\pi}{e}\ ,\qquad \tilde{Q}_k=\frac{1}{\tilde{e}^2}\int_{S_k^2}*\tilde{F}=M_k\ ,\qquad \tilde{M}_k=\frac{1}{2\pi}\int_{S_k^2}\tilde{F}=-Q_k\ . \tag{34.138}$$

This duality interchanges the electric and magnetic charges as $(\tilde{Q}_k,\tilde{M}_k)=(M_k,-Q_k)$. Note that we are not assuming here that the theory has any kind of duality symmetry; we are simply rewriting

it in terms of the dual field strength as a shortcut to derive magnetic terms in the soft factor. It is useful to define a dual gauge potential that couples to magnetic charges in the same way that the usual gauge potential couples to electric charges:

$$\tilde{F} = \mathrm{d}\tilde{A} = -\frac{2\pi}{e^2} * \mathrm{d}A \ , \tag{34.139}$$

which determines $\tilde{A}$ in terms of A (up to magnetic gauge transformations). In the bulk four dimensions, this is a highly nonlocal relationship, but as we shall see, it becomes local on $\mathcal{I}^+$. Note that (34.139) also relates the polarization ε_α to its dual $\tilde{\varepsilon}_\alpha$ as defined by

$$A_\mu(q) = e\varepsilon^\alpha_\mu(q) \ , \qquad \tilde{A}_\mu(q) = \tilde{e}\tilde{\varepsilon}^\alpha_\mu(q) \ . \tag{34.140}$$

Since $\tilde{A}$ couples to magnetic particles in exactly the same way that A couples to electric particles, we immediately conclude that magetic charges must correct the soft formula to

$$S^\alpha_0 = \sum_k \frac{p_k \cdot (Q_k e\varepsilon^\alpha + M_k\tilde{e}\tilde{\varepsilon}^\alpha)}{q \cdot p_k} \ . \tag{34.141}$$

Electric and magnetic gauge invariance implies that the amplitude should remain invariant under separate shifts of ε^α and $\tilde{\varepsilon}^\alpha$. This indeed follows from the fact that electric and magnetic charge are each conserved. It is easy to check that this formula is invariant under electric-magnetic duality. It is conjectured to be nonperturbatively exact [192].

It is illuminating to describe the duality transformations (34.138) in retarded coordinates near $\mathcal{I}^+$. Expanding near $\mathcal{I}^+$, they simplify to

$$\tilde{F}^{(0)}_{z\bar{z}} = \frac{2\pi i}{e^2}\gamma_{z\bar{z}}F^{(2)}_{ru} \ , \qquad \tilde{F}^{(2)}_{ru} = \frac{2\pi i}{e^2}\gamma^{z\bar{z}}F^{(0)}_{z\bar{z}} \ , \qquad \tilde{F}^{(0)}_{uz} = \frac{2\pi i}{e^2}F^{(0)}_{uz} \ . \tag{34.142}$$

Integrating the last equation of (34.142) and choosing the integration constant to get a linear relation between A_z and $\tilde{A}_z$, we find

$$\tilde{A}^{(0)}_z = \frac{2\pi i}{e^2}A^{(0)}_z \ . \tag{34.143}$$

This formula is quite interesting. It states that the duality transformation is simply multiplication by i. This is unsurprising, since F_{uz} corresponds to the radiative mode of the electromagnetic field with one polarization. A multiplication by i corresponds to a phase shift by $\frac{\pi}{2}$, which is equivalent to exchanging electric and magnetic fields. The soft factor may then finally be written in the simple form for positive helicity

$$S^\alpha_0 = \sum_k \frac{(eQ_k + \frac{2\pi i}{e}M_k)p_k \cdot \varepsilon^+}{q \cdot p_k} \ , \tag{34.144}$$

while the combination $eQ_k - \frac{2\pi i}{e}M_k$ appears for the opposite helicity.

In the presence of magnetic charges, there is a second infinity of conserved charges. Let us define an outgoing charge by

$$\tilde{Q}^+_\varepsilon = \frac{1}{2\pi}\int_{\mathcal{I}^+_-} \varepsilon F = \frac{i}{2\pi}\int_{\mathcal{I}^+_-} d^2z\varepsilon F^{(0)}_{z\bar{z}} . \tag{34.145}$$

The Lorentz-invariant matching condition

$$F^{(0)}_{z\bar{z}}|_{\mathcal{I}^+_-} = -F^{(0)}_{z\bar{z}}|_{\mathcal{I}^-_+} \tag{34.146}$$

then implies that this outgoing magnetic charge is equal to the incoming magnetic charge on $\mathcal{I}^-$:

$$\tilde{Q}^+_\varepsilon = \tilde{Q}^-_\varepsilon = -\frac{i}{2\pi}\int_{\mathcal{I}^-_+} d^2z\varepsilon F^{(0)}_{z\bar{z}} \ . \tag{34.147}$$

The extra minus sign in (34.146) is present because the $(z,\bar{z})$ coordinates used on $\mathcal{I}^-$ are antipodally mapped to the ones on $\mathcal{I}^+$, reversing the orientation of the sphere.

To what symmetry is this new charge associated? In the case of the electric charges, this question was answered by constructing the asymptotic Dirac brackets and working out the action of the charge on the fields. In doing so, we assumed that $F^{(0)}_{z\bar{z}}$ vanished on the boundaries of $\mathcal{I}^+_\pm$. This boundary condition is consistent in the absence of magnetically charged particles but obviously cannot be imposed here. In fact, the correct boundary conditions and Dirac brackets in the presence of both electric and magnetic charges have not yet been worked out. It remains an important open problem.

However, without going through this long procedure, duality covariance gives an obvious guess for the final result: $\tilde{Q}^\pm_\varepsilon$ should generate *magnetic* gauge transformations on the dual gauge fields, namely

$$\tilde{\delta}_\varepsilon \tilde{A}^{(0)}_z = \partial_z\varepsilon \ . \tag{34.148}$$

Using (34.143), this can be written in terms of the gauge field A_z as

$$\tilde{\delta}_\varepsilon A^{(0)}_z = -\frac{ie^2}{2\pi}\partial_z\varepsilon \ . \tag{34.149}$$

These may be interpreted as electric gauge transformations but with an imaginary gauge parameter $\tilde{\varepsilon} = -\frac{ie^2}{2\pi}\varepsilon$. The original real $U(1)$ symmetry is now enhanced to a complex $U(1)$. This complexification enables us to simultaneously and *locally* realize both the electric and magnetic gauge symmetry on the asymptotic fields! From this point forward, showing that the Ward identity of this complexified large gauge symmetry is precisely the full nonperturbative electric+magnetic soft photon theorem follows through exactly as before. We will not work through the derivation here.

It is instructive to note here that in the standard electric presentation of abelian gauge theory, the large electric symmetries are manifest as a nontrivial subgroup of the gauge symmetry, whereas magnetic symmetries are not. In the dual magnetic description, the opposite is true. The lesson here is that not all asymptotic symmetries can be thought of as subgroups of some bulk gauge symmetry. As truly physical symmetries with measurable dynamical implications, they have a more fundamental status. Since trivial gauge symmetries correspond to redundant degrees of freedom, one may have many different presentations of a theory in which different redundant degrees of freedom appear. However, since the asymptotic symmetries are physical, they must exist in each presentation, although not necessarily describable as a subgroup of a local gauge symmetry. Indeed, we will see in the next section that the soft photino theorem in $\mathcal{N}=1$ QED implies the existence of infinitely many fermionic symmetries. These are not a subgroup of a local gauge symmetry in any known presentation of the theory.

34.2.14 Supersymmetry

In supersymmetric theories, every particle has a superpartner. The partner of the photon is called the photino. One might expect a soft photino theorem as a superpartner of the soft photon theorem. This is indeed the case. For every soft theorem, one expects an asymptotic symmetry. Furthermore, since the soft theorem can be thought of as one relationship between two $\mathcal{S}$-matrix amplitudes for every direction $(z,\bar{z})$ of the soft particle, the associated asymptotic symmetry should be infinite dimensional. The soft photino theorem is thus expected to give rise to an infinite number of fermionic symmetries of the $\mathcal{S}$-matrix.

This simple conclusion is quite surprising as, in the simplest case of $\mathcal{N} = 1$, the global supersymmetry is generated by four real fermionic parameters. Clearly, the infinitely many fermionic symmetries alluded to above cannot possibly be a subgroup of these finitely many supersymmetries! This is therefore another example, following the magnetic one, of an asymptotic symmetry that is not realized as a subgroup of any manifest local symmetry of the full bulk theory. However, in the magnetic case, we were able to construct a reformulation of the theory in terms of dual variables in which the magnetic symmetry was manifest as a subgroup of the local gauge symmetry of the dual magnetic potentials. In contrast, there is no known description of the fermionic symmetries in $\mathcal{N} = 1$ theories in which a nontrivial infinite-dimensional symmetry is manifest.[195] Hence, the fermionic symmetries that we uncover here are completely new and different.

In this section, we present the soft photino theorem and rewrite it as the matrix element of an infinite set of fermionic conservation laws, with some details referred to the original work [58]. In section 34.2.14, we review the soft photino theorem. Section 34.2.14 derives the equivalent fermionic conservation laws. Further directions involving supersymmetric asymptotic symmetries are discussed in section 34.2.14.

Soft Photino Theorem

First we review the soft photino theorem. The Lagrangian for $\mathcal{N} = 1$ supersymmetric QED with photino Λ and generic matter supercurrent $\mathcal{K}$ (the superpartner of the charged matter current j_μ) is

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \frac{i}{e^2}\bar{\Lambda}\bar{\sigma}^\mu D_\mu\Lambda + i\bar{\mathcal{K}}\bar{\Lambda} - i\Lambda\mathcal{K} + \mathcal{L}_{\text{matter}} + \ldots, \tag{34.150}$$

where we follow the spinor conventions of Wess and Bagger [207]. Here, $\mathcal{L}_{\text{matter}}$ is the full interacting matter Lagrangian and "..." denotes any higher derivative interactions. Our discussion of the soft photino theorem will not depend on the precise nature of these terms. The only important term is the linear coupling of the supercurrent to the photino. The structure of this interaction is fixed completely by gauge invariance and supersymmetry.

For clarity, we work with a specific example, but the discussion is completely general and applies to any supersymmetric abelian gauge theory. Consider the case where the matter content of this theory contains a single, massless charged chiral multiplet with charge Q. This multiplet contains a complex scalar field Φ and its superpartner Ψ_α. The fermionic current is

$$\mathcal{K}_\alpha = \sqrt{2}Q\bar{\Phi}\Psi_\alpha, \qquad \bar{\mathcal{K}}_{\dot{\alpha}} = \sqrt{2}Q\Phi\bar{\Psi}_{\dot{\alpha}}. \tag{34.151}$$

To describe the soft theorem, consider an amplitude with n particles with momenta p_k, $k = 1, \ldots, n$, and a photino with momentum q. Since all momenta are null, we can write them in terms of spinors as

$$p_{k\mu}\sigma^\mu_{\alpha\dot{\alpha}} = \eta_{k\alpha}\bar{\eta}_{k\dot{\alpha}}, \qquad q_\mu\sigma^\mu_{\alpha\dot{\alpha}} = \eta_\alpha\bar{\eta}_{\dot{\alpha}}. \tag{34.152}$$

The soft photino theorem (a derivation is in [58]) is then simply

$$\lim_{\eta\to 0}\langle \text{ out } |a_+(\eta)\mathcal{S}| \text{ in } \rangle = \sqrt{2}ie\sum_k \frac{1}{\eta\cdot\eta_k}\langle \text{ out } |\mathscr{F}_k\mathcal{S}| \text{ in } \rangle, \tag{34.153}$$

where $\mathscr{F}$ is the operator whose action on states is given by

$$\langle\, \Phi\,|\mathscr{F} = -Q\langle\, \Psi\,|, \qquad \langle\, \bar{\Psi}\,|\mathscr{F} = Q\langle\, \bar{\Phi}\,|, \qquad \langle\, \bar{\Phi}\,|\mathscr{F} = \langle\, \Psi\,|\mathscr{F} = 0. \tag{34.154}$$

[195] In superspace formalisms, the gauge parameter sometimes has a fermionic superpartner that is gauge fixed to zero. However, the fermionic symmetries we find here are unrelated to these superpartners. It is of course possible that alternate superspace formulations exist in which our fermionic symmetries are superpartners of gauge symmetries.

Note that the soft theorem here takes on a qualitatively different form than the examples considered so far. In the soft photon theorem, the soft factors were all simply numbers in a plane wave basis, whereas here the soft factor is an *operator* that changes external bosons into external fermions and vice versa. Going forward, we will encounter many more examples in which the soft factor is an operator acting on the external states.

Note that $\mathscr{F}$ is a fermionic operator (i.e., it takes bosonic states into fermionic states and vice versa). This is to be expected, as all nonvanishing amplitudes must have an even number of external fermions. If the LHS of (34.153) is nonvanishing, the $\mathcal{S}$-matrix $\langle \text{out} | \mathcal{S} | \text{in} \rangle$ must itself vanish. Note also that $\mathscr{F}$ only acts on charged states, and even then it does not act on all charged states, only on $\langle \Phi |$ and on $\langle \bar{\Psi} |$. This action is quite distinct from that of the supercharge itself, which acts on all states; see Exercise 8.

We will not derive the soft photino theorem in detail, but we can understand qualitatively why the soft photino theorem must involve the operator $\mathscr{F}$ as follows. The derivation of the soft theorem for the photino follows through in exactly the same way as for the photon, with the $\frac{1}{p_k \cdot q}$ pole coming from the untruncated, nearly on-shell propagator. The only new element is the coupling of the photino with the chiral fields. For an outgoing positive helicity photino, the relevant interaction term is $\bar{\mathcal{K}}_{\dot{\alpha}} = \sqrt{2} Q \Phi \bar{\Psi}_{\dot{\alpha}}$. The factor of Q indicates that one only gets a contribution to the soft factor from particles with charge, as seen in (34.153). Moreover, one only gets a nonzero soft factor from states created by fields $\bar{\Phi}$ and Ψ_α, which are precisely the states on which the operator $\mathscr{F}$ has a nonzero action.[196]

Fermionic Conservation Laws

Following the theme of these lectures, we strongly suspect that (34.153) can be recast as the quantum matrix element of a charge conservation law. There are not too many sensible possibilities for what that charge could be. Our approach will be to simply motivate an Ansatz and then establish the conservation law by checking that the corresponding Ward identity is equivalent to (34.153).

Note that the generic structure of this fermionic charge, $\mathscr{F}[\chi]$ (χ is an arbitrary angle-dependent fermionic parameter), has to be similar to the structure of the large gauge charge — namely, that there is a soft charge linear in the photino field and a hard charge containing the matter particles. Thus, we guess that the new infinity of conserved charges $\mathscr{F}[\chi]$ loosely takes the form of a $\mathcal{I}_-^+$ integral:

$$\mathscr{F}[\chi] \sim \int \chi \Lambda + \int \chi \mathcal{K}. \tag{34.155}$$

To refine this guess, we require that somehow supersymmetry should play a role in this story. This is because the soft photino theorem itself was obtained as a supersymmetrization of the soft photon theorem. Supersymmetry commutes with all gauge transformations. Thus, it cannot be possible to obtain $\mathscr{F}[\chi]$ directly by the action of supersymmetry on Q_ε^+. However, the reverse might be (and as we shall see, is) true: the action of supersymmetry on $\mathscr{F}[\chi]$ could give us Q_ε^+. Another motivation for suspecting such a relation is that it implies $\mathscr{F}[\chi]$ as an operator has mass dimension $-\frac{1}{2}$, consistent with the mass dimension of $\mathscr{F}$ as defined in (34.154).

To implement this guess, note that the supersymmetry transformations of the photino field are

$$\delta_\alpha^{\text{susy}} \Lambda_\beta = -F_{\alpha\beta} + \ldots. \tag{34.156}$$

Here, $\delta_\alpha^{\text{susy}}$ is a supersymmetry transformation with a spinor index α, and $F_{\alpha\beta} = \sigma_{\alpha\beta}^{\mu\nu} F_{\mu\nu}$. The corrections "..." depend on the precise form of the Lagrangian and of the theory under consideration. In the off-shell superfield formalism, these corrections are simply $\varepsilon_{\alpha\beta} D$, where D is an auxiliary field. In any case, these terms will not matter for our purposes, since they die off rapidly near

[196] The notation used here is that the ket state $|\Phi\rangle$ is created by the field Φ. Thus, its adjoint $\langle \Phi |$ is created by the field $\bar{\Phi}$.

$\mathcal{I}^+$. Equation (34.156) then suggests that for the supersymmetry transformation of the fermionic charge,

$$\zeta^\alpha \delta^{\text{susy}}_\alpha \mathscr{F}[\chi] = Q^+_{\zeta\chi}, \qquad \bar\zeta^{\dot\alpha} \delta^{\text{susy}}_{\dot\alpha} \mathscr{F}[\chi] = 0. \tag{34.157}$$

This equation can be used to determine the precise form of $\mathscr{F}[\chi]$. One might anticipate at this point that it is just a $\mathcal{I}^+_-$ integral of $\chi\Lambda$. This is almost right, except that Λ has a term that grows in the far past of $\mathcal{I}^+$, which we must take some care to project out.

To do so explicitly, let us describe in more detail the structure of spinors near $\mathcal{I}^+$. Define a spinor basis by

$$\sigma_z{}^z \xi^{(\pm)} = \pm \frac{1}{2} \xi^{(\pm)}, \qquad \xi^{(+)} \xi^{(-)} = 1. \tag{34.158}$$

Near $\mathcal{I}^+$, we can then expand the photino and fermionic current as

$$\begin{aligned} \Lambda_\alpha &= \frac{1}{r} \lambda_{(+)} \xi^{(+)}_\alpha + \frac{1}{r^2} \lambda_{(-)} \xi^{(-)}_\alpha + \mathcal{O}\big(r^{-3}\big), \\ \bar{\mathcal{K}}_{\dot\alpha} &= \frac{1}{r^2} \bar k_{(-)} \bar\xi^{(-)}_{\dot\alpha} + \frac{1}{r^3} \bar k_{(+)} \bar\xi^{(+)}_{\dot\alpha} + \mathcal{O}\big(r^{-4}\big). \end{aligned} \tag{34.159}$$

The constraint equation on $\mathcal{I}^+$ is

$$(1 + z\bar z) D_{\bar z} \lambda_{(+)} - 2\partial_u \lambda_{(-)} = e^2 \bar k_{(-)}. \tag{34.160}$$

This is the superpartner of the leading-order Maxwell constraint equation (34.37), and it will play the same role as that constraint in converting boundary expressions for charges into bulk ones. The supersymmetry transformation of $\lambda_{(\pm)}$ can be determined by taking the $\mathcal{I}^+$ limits of (34.156). It is convenient to define the following supersymmetry transformation

$$\delta_{(-)} = \xi^\alpha_{(-)} \delta^{\text{susy}}_\alpha. \tag{34.161}$$

Then, near $\mathcal{I}^+$, (34.156) simplifies to

$$\delta_{(-)} \lambda_{(+)} = -2(1 + z\bar z) F^{(0)}_{uz}, \qquad \delta_{(-)} \lambda_{(-)} = F^{(2)}_{ur} - \gamma^{z\bar z} F^{(0)}_{z\bar z} + \ldots, \tag{34.162}$$

where "…" are the D-term contributions. In general, $\lambda_{(+)}$ goes to a finite value at large $|u|$. Equation (34.160) then implies that $\lambda_{(-)}$ may diverge linearly in u. Thus, to define a fermionic charge as an integral over $\mathcal{I}^+_-$ of $\lambda_{(-)}$, we must project out the term linear in u:

$$\mathscr{F}[\chi] = \frac{1}{e^2} \int d^2z \gamma_{z\bar z} \chi \xi^{(-)} \left(1 - u\partial_u\right) \lambda_{(-)} \big|_{\mathcal{I}^+_-}. \tag{34.163}$$

The supersymmetry transformation (34.162) then immediately implies

$$\zeta^\alpha \delta^{\text{susy}}_\alpha \mathscr{F}[\chi] = Q^+_{\zeta^\alpha \chi_\alpha}. \tag{34.164}$$

This fermionic charge satisfies all the properties discussed above. Appropriate photino matching conditions imply that it commutes with the $\mathcal{S}$-matrix. The corresponding Ward identity states

$$\langle \text{ out} \,| \left(\mathscr{F}[\chi] \mathcal{S} - \mathcal{S} \mathscr{F}^-[\chi] \right) | \,\text{in} \rangle = 0. \tag{34.165}$$

Then, using formulas from the previous sections for rewriting boundary expressions in a plane wave basis, we find that this is indeed equivalent to the soft photino theorem (34.153).

The symmetry corresponding to this charge follows with the help of the symplectic form for fermions on $\mathcal{I}^+$. The action of the charge on the photino is found to be

$$\left\{ \mathscr{F}[\chi], \lambda_{(+)}(u, z, \bar z) \right\} = 0, \qquad \left\{ \mathscr{F}[\chi], \bar\lambda_{(-)}(u, z, \bar z) \right\} = -\partial_{\bar z} \chi. \tag{34.166}$$

We see that the photino gets an inhomogeneous shift, implying a new kind of "fermionic gauge transformation." This new and infinite set of symmetries exists for all supersymmetric abelian gauge theories, yet is not a subset of any previously considered local fermionic symmetry.

$\mathcal{N} = 4, 8$

It is an important and outstanding problem to extend this construction to gauge theories with $\mathcal{N} = 4$ supersymmetry or gravity theories with $\mathcal{N} = 8$ supersymmetry. Clearly, conceptually new features will arise from supermultiplets of soft theorems with nonabelian R-symmetry groups. Multiple soft limits are known not to commute in this more general context[99]. At this time, there is not even a conjecture for what the asymptotic symmetry group should be!

The fact that the action of supersymmetry on the fermionic charges/symmetries gives the large gauge charges/symmetries is closely related to an observation of Larkoski [94] that the leading and subleading soft theorems in QED are related by conformal symmetry. Going forward to higher supersymmetry groups, we may encounter a rich structure in which hierarchies of asymptotic symmetries related to leading, subleading, and sub-subleading soft theorems merge into a unified infinite-dimensional symmetry group. This will be interesting indeed to explore.

In Barnich et al. [125], a charge-weighted vacuum partition function that encodes the BMS-degeneracy of the vacuum in three-dimensional gravity was constructed. It would be of interest in supersymmetric theories to define and compute charge-weighted vacuum indices characterizing the degeneracy.

34.2.15 Infrared Divergences

A salient and much-studied feature of QED in the deep infrared is the appearance of IR divergences in the integral over momenta in internal loops [151]. These divergences set all conventional Fock-basis $\mathcal{S}$-matrix elements to zero. Often they are dealt with by restricting to inclusive cross sections in which physically unmeasurable photons below some IR cutoff are traced over. The trace gives a divergence, which offsets the zero and yields a finite result for the physical measurement [6, 208, 209, 7]. While this is adequate for most experimental applications, for some purposes it is nice to have an $\mathcal{S}$-matrix. For example, discussions of unitarity or asymptotic symmetries require an $\mathcal{S}$-matrix.

It is natural to ask whether the considerations of this book bear on the issue of IR divergences. We shall see that they indeed give a satisfying picture: imposing the infinity of conservation laws on amplitudes eliminates some and possibly all IR divergences. To see this, consider Bhabha scattering as illustrated in figure 34.12, in which an incoming electron-positron pair scatters to an outgoing electron-positron pair, with no incoming or outgoing radiation. To restrict attention to $\mathcal{I}$ (rather than $i^{\pm}$), we take the charged particles to be massless.[197] In this case the constraint equations on $\mathcal{I}^+$ imply (see (34.37))

$$\partial_u F^{(2)}_{ru} + \cancelto{0}{D^z F^{(0)}_{uz}} + \cancelto{0}{D^{\bar z} F^{(0)}_{u\bar z}} = -e^2 j^{(2)}_u, \tag{34.167}$$

while on $\mathcal{I}^-$,

$$\partial_v F^{(2)}_{rv} - \cancelto{0}{D^z F^{(0)}_{vz}} - \cancelto{0}{D^{\bar z} F^{(0)}_{v\bar z}} = e^2 j^{(2)}_v. \tag{34.168}$$

Here we have set $F^{(0)}_{uz} = F^{(0)}_{vz} = 0$, as this process involves no radiative modes. $j^{(2)}_v$ and $j^{(2)}_u$ are the electron and positron charge fluxes at $\mathcal{I}$. For example, an outgoing positron at $(u_0, z_0, \bar z_0)$ creates a flux

$$j^{(2)}_u = \gamma^{z\bar z}\delta(u - u_0)\delta^2(z - z_0). \tag{34.169}$$

Since the incoming (outgoing) state is taken to be the vacuum in the far past (future), we should solve (34.168) ((34.167)) with the boundary condition that the Coulomb field vanishes in the far past (future). One then finds that, as long as the scattering angle is nontrivial, the fields cannot obey the matching conditions (34.26)

$$F^{(2)}_{ru}(z, \bar z)\Big|_{\mathcal{I}^+_-} = F^{(2)}_{rv}(z, \bar z)\Big|_{\mathcal{I}^-_+} \tag{34.170}$$

[197] We suppress here the effects of loop corrections in the massless case, which make the coupling run in the deep IR. The qualitatively similar and more realistic massive case is treated in [210].

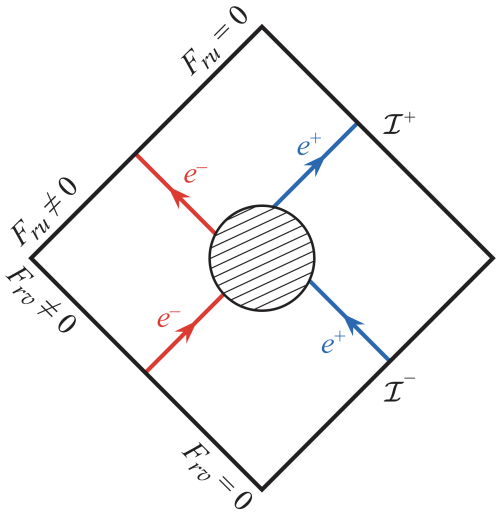


Figure 34.12: **Bhabha scattering, $e^+e^- \to e^+e^-$, involves incoming and outgoing electron-positron pairs unaccompanied by radiative photons. In the massless case depicted, the Maxwell equation implies Coulomb fields on $\mathcal{I}$ to the future of the incoming pair and the past of the outgoing pair. If the scattering angle is nonzero, these Coulomb fields cannot obey the required matching conditions where $\mathcal{I}^+$ and $\mathcal{I}^-$ meet. The amplitude for Bhabha scattering therefore vanishes identically.**

at the boundary of $\mathcal{I}$ near i^0. Therefore Bhabha scattering must vanish identically.

This may sound bizarre at first, but in fact the result is well known and is usually attributed to IR divergences. As depicted in figure 34.13, the tree level one-photon exchange is finite, but the one loop correction is infinite due to IR divergences in pairwise exchanges of soft photons between charged external legs. The source of this divergence is the $\omega \to 0$ pole in the coupling of a soft photon to a charged particle. Including arbitrary numbers of soft photon exchanges and resumming, this divergence exponentiates with a minus sign and sets the full amplitude to zero. Here we have provided a simple physical interpretation of that well-known result. IR divergences are not "real", rather they are a clever trick employed by the Feynman rules to set conservation-law-violating amplitudes to zero!

Finding a new way of computing zero, however, is not enough: we would like to use our insights to construct finite amplitudes. The problem above was the mismatched Coulomb fields produced

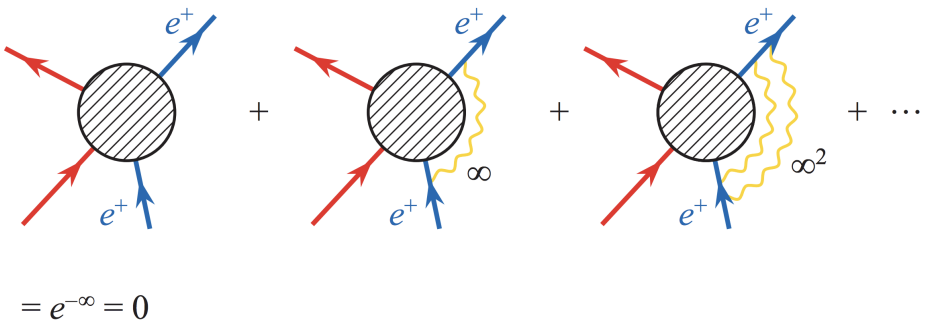


Figure 34.13: **The resummation of IR divergent processes, in which pairs of external charged legs exchange soft photons, sets the amplitude for Bhabha scattering to zero.**

by the asymptotic charge fluxes. This mismatch can be avoided by using radiative, rather than the Coulombic, modes to satisfy the constraints:

$$\begin{aligned} \cancelto{0}{\partial_u F^{(2)}_{ru}} + D^z F^{(0)}_{uz} + D^{\bar z} F^{(0)}_{u\bar z} &= -e^2 j^{(2)}_u, \\ \cancelto{0}{\partial_v F^{(2)}_{rv}} - D^z F^{(0)}_{vz} - D^{\bar z} F^{(0)}_{v\bar z} &= e^2 j^{(2)}_v. \end{aligned} \tag{34.171}$$

For a single outgoing positron, this has the radiative shock wave solution[198]

$$A^{(0)}_z = -\frac{e^2}{4\pi(z-z_0)}\theta(u-u_0). \tag{34.172}$$

The field strength has support only at at $u = u_0$, but the gauge potential is shifted by the shock wave, indicating a transition between the degenerate vacua. Its frequency space Fourier transform has the signature $\omega \to 0$ pole. The corresponding "dressed" coherent quantum state for the outgoing positron,

$$|j_0\rangle_{\text{dressed}} = e^{(\frac{i}{2\pi}\int \frac{d^2z}{z_0-z} A^{(0)}_{\bar z}(u_0,z,\bar z) - h.c.)}|j_0\rangle \ , \tag{34.173}$$

is annihilated by the $\mathcal{I}^+$ constraint equation (34.171) (with the fields promoted to operators). This state can be described as a positron surrounded by a cloud of soft photons. Dressing all the positrons and electrons in this manner, one finds for "dressed" Bhabha scattering[199]

$$F^{(2)}_{ru}(z,\bar z)\Big|_{\mathcal{I}^+_-} = 0 = F^{(2)}_{rv}(z,\bar z)\Big|_{\mathcal{I}^-_+} \ , \tag{34.174}$$

and the matching conditions (34.170) are trivially satisfied. Of course, there are many other ways to satisfy the matching condition, but the example (34.173) is particularly simple and illustrates the main point.

Since the conservation laws are satisfied, there is no reason for dressed Bhabha amplitudes to vanish and no need for IR divergences. In fact, these amplitudes are known to be completely IR finite![200] The dressed state here is a special case of the ones shown by Kulish and Faddeev in 1970 [147] to have IR finite scattering amplitudes. As shown in figure 34.14, there is a pairwise cancellation of IR divergences in soft photon exchanges between pairs of external charges and external charges and soft clouds.[201] We can now understand the role of Faddeev-Kulish clouds as inserting radiative photons to satisfy the conservation laws.

This is progress, but Faddeev-Kulish states are highly nongeneric. When the net charge is zero, they all have vanishing leading radial electric fields near spatial infinity as in (34.174). When it is not zero, the leading electric field is nonzero but is independent of the angle, because the soft photon cloud shields all the angle-dependent components. Such states are unphysical. When pairs of protons are thrown at one another at the LHC, they are not followed up with a finely tuned cloud of soft photons to shield the angle-dependent part of the electric field. The $\mathcal{S}$-matrix $\mathcal{S}_{FK}$ restricted to Faddeev Kulish states is not unitary: rather $\mathcal{S}^\dagger_{FK}\mathcal{S}_{FK} = \mathcal{P}_{FK}$ is a projection operator projecting onto such shielded states. $\mathcal{P}_{FK}$ projects out the states we scatter in accelerator experiments.

It is desirable to have IR finite scattering amplitudes for all physical states. The discussion here suggests that IR divergences arise only to enforce charge conservation. Accordingly, it was conjectured in [210] that all amplitudes allowed by charge conservation are free of IR divergences.[202] As depicted in figure 34.15, a leading order analysis corroborates this conjecture. Let us move the

[198] This solution is singular at $z = \infty$, but in the case of Bhabha scattering with zero net global charge the singularity is cancelled.

[199] When there is net global charge, this procedure sets the leading radial component of the electric field to a nonzero but angle-independent constant.

[200] There are UV divergences in this particular example, as the positrons, electrons, and shock waves are all infinitely localized. These divergences can be eliminated by smearing but are not our interest here.

[201] More generally, cloud-cloud exchanges also participate in the cancellations.

[202] This conjecture was recently proven by R. Akhoury and S. Choi [211].

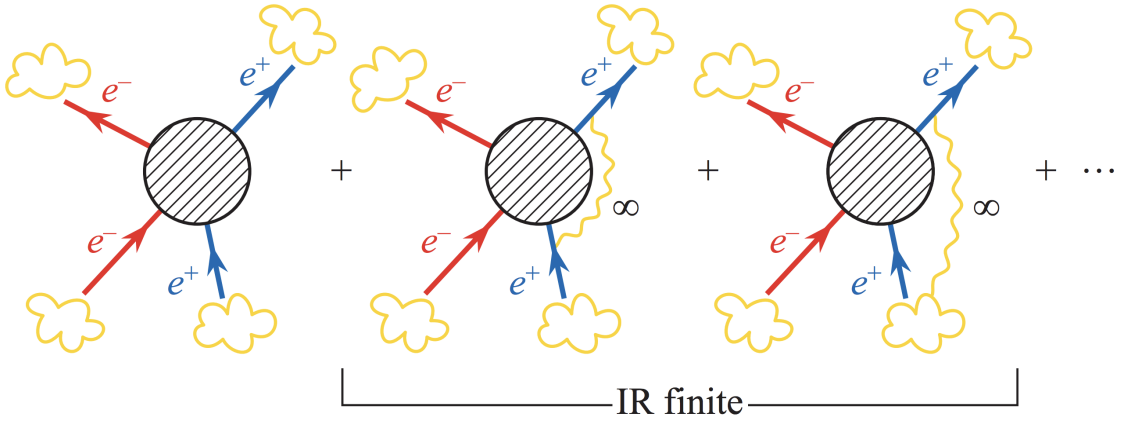


Figure 34.14: **The Faddeev-Kulish mechanism. IR divergences from soft exchanges between external charges are pairwise cancelled by exchanges between external charges and soft photon clouds surrounding each particle.**

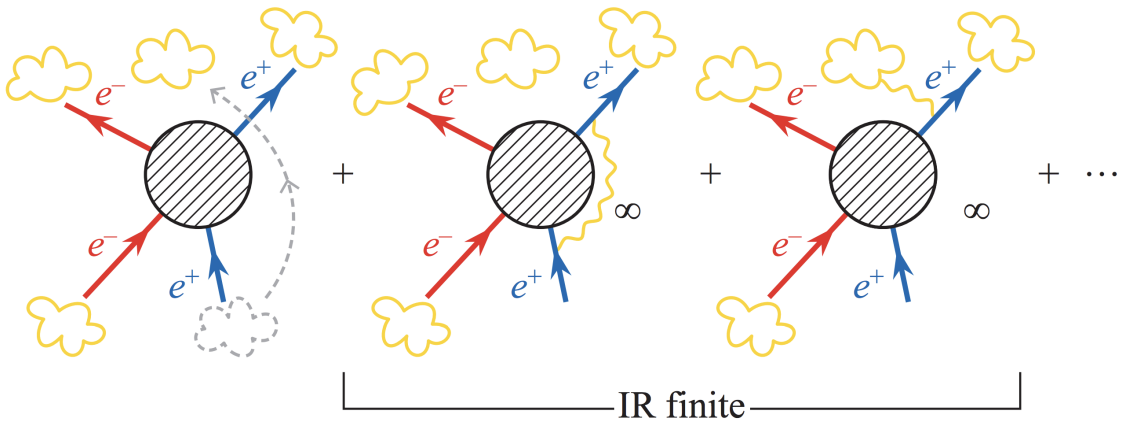


Figure 34.15: **IR divergences from soft exchanges between external charges are pairwise cancelled by exchanges between external charges and soft photon clouds associated with, but not necessarily surrounding, each particle.**

soft cloud surrounding an incoming positron to an antipodal outgoing position in a manner such that all the constraints on $\mathcal{I}$ are still satisfied. This can be done by omitting the incoming positron dressing operator and acting with its hermitian conjugate on the out state. When we move the radiative cloud from $\mathcal{I}^-$ to $\mathcal{I}^+$, the constraints imply nontrivial but matched Coulomb fields will be created near i^0. The original IR divergence from soft photon exchange between the outgoing positron and the soft cloud around the incoming positron is traded for a divergence from exchange between the outgoing positron and the new antipodally positioned outgoing cloud. The soft photon theorem, which takes the same form (up to a sign) for ingoing and outgoing photons, implies that the IR divergent parts of these diagrams are equal, and hence the pairwise cancellation of these leading divergences is still in effect with the repositioned cloud [210].

It is possible to antipodally reposition all the incoming clouds to outgoing ones while maintaining the leading Faddeev-Kulish cancellation of IR divergences. Physical scattering processes at accelerators are described by such incoming charges with no incoming soft clouds. The resulting outgoing soft clouds are then simply identified as the soft radiation inevitably produced in the scattering process.

34.3 The $\mathcal{S}$-matrix as a Celestial Correlator

So far, the framework for our discussion has been the conventional one of an $\mathcal{S}$-matrix that maps an incoming to an outgoing Hilbert space. Multiparticle states in the in- and out-Hilbert spaces are described as asymptotic, noninteracting energy-momentum eigenstates. In this section, we present an alternate description, which for some purposes is both computationally and conceptually more convenient.

The alternate description of scattering [12] is as a type of correlation function on a sphere, depicted in figure 34.16. In the massless case, "in" and "out" massless particles are labelled by

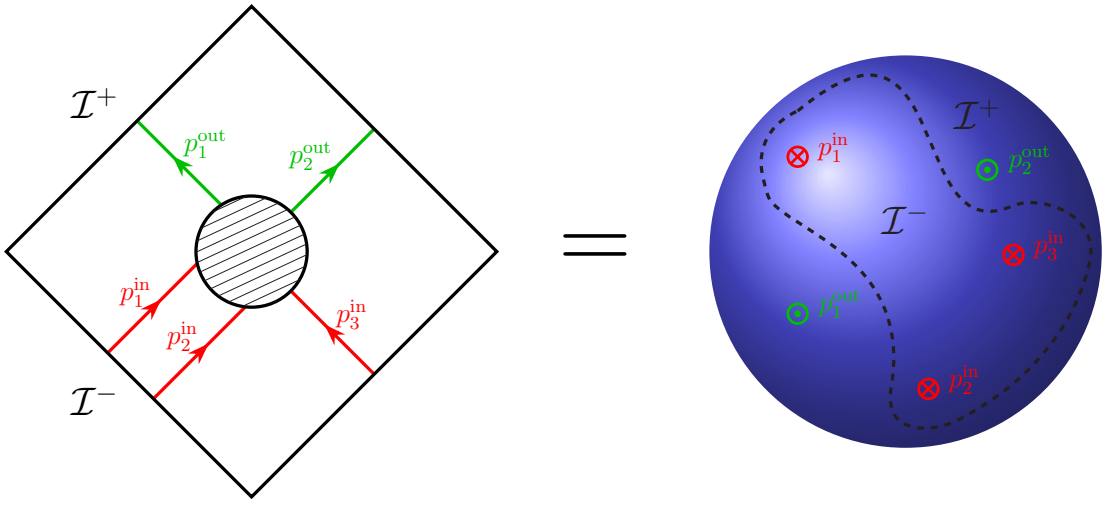


Figure 34.16: **The four-dimensional Minkowski $\mathcal{S}$-matrix for any theory can be rewritten as a two-dimensional correlator on the celestial sphere $\mathcal{CS}^2$, which is parametrized by the asymptotic angle $(z, \bar{z})$ on $\mathcal{I}$. Incoming or outgoing massless particles are represented by operators at the location where they enter $\mathcal{I}^-$ or exit $\mathcal{I}^+$, labeled by quantum numbers such as the energy or boost charge. The angles on $\mathcal{I}^+$ and $\mathcal{I}^-$ are antipodally identified so that a free massless particle enters and exits at the same point on $\mathcal{CS}^2$. The dashed line on $\mathcal{CS}^2$ separates the regions associated with incoming and outgoing particles. The four-dimensional $SL(2,\mathbb{C})$ Lorentz invariance acts as the global two-dimensional conformal group on $\mathcal{CS}^2$.**

operators denoted

$$O_k(z,\bar{z}). \tag{34.175}$$

Here,

$$z = \frac{x^1 + ix^2}{r + x^3} \tag{34.176}$$

denotes the point on the sphere at past or future null infinity where the particle of type k enters or exits the spacetime. This sphere is referred to as the celestial sphere and is denoted $\mathcal{CS}^2$. O_k may depend on other quantum numbers, such as the angular momentum, boost weight, energy, or electromagnetic charge, but we often suppress these labels. The asymptotic angle $(z,\bar{z})$ and either energy or boost weight replace the conventional three-momentum $\vec{p}$ characterizing a massless particle.

Massive particles in energy-momentum eigenstates, in contrast, do not enter or exit spacetime at a definite point on $\mathcal{CS}^2$. Rather, as discussed in section 34.2.12, they enter or exit $i^\pm$ at a definite point on the hyperboloid $\mathbb{H}_3$ in (34.129). They correspond to operators that are smeared on $\mathcal{CS}^2$ with the weighting given by (34.127). Alternately, boost-eigenstate wave functions for massive particles associated to a point on $\mathcal{CS}^2$ can be constructed using the bulk-to-boundary propagator on $\mathbb{H}_3$ [142].

It is then natural to express the n-particle scattering amplitudes in the form of a celestial correlator on $\mathcal{CS}^2$:

$$\langle \text{ out } |\mathcal{S}| \text{ in } \rangle \to \langle\, \mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_n(z_n,\bar{z}_n)\,\rangle. \tag{34.177}$$

Note that no assumptions are being made here; we are simply rewriting the $\mathcal{S}$-matrix in a different notation.

We are interested in Lorentz invariant theories. The four-dimensional Lorentz symmetry is $SL(2,\mathbb{C})$, which acts on $\mathcal{CS}^2$ as

$$z \to \frac{az+b}{cz+d}, \quad ad-bc=1. \tag{34.178}$$

This formula is very familiar to anyone who has studied two-dimensional CFT, because $SL(2,\mathbb{C})$ is the global conformal group of the two-sphere. In quantum field theory, scattering amplitudes must transform covariantly under the Lorentz group, so these correlation functions must transform covariantly under the global conformal group. Hence they must look a lot like the familiar correlation functions in a two-dimensional CFT on the sphere. This connection becomes manifest when the external particle wave functions are taken to be $SL(2,\mathbb{C})$ primaries labeled by their conformal dimensions and location on $\mathcal{CS}^2$ rather than the traditional plane waves [68, 142, 144].

This connection becomes even more interesting later on when we get to quantum gravity. We will see that — up to currently unresolved issues involving IR finite quantum anomalies — the global conformal group gets enhanced to the full local conformal group $z \to w(z)$ that appears in two-dimensional CFT. There is even an operator constructed from soft gravitons [67, 68, 117] whose Ward identities are those of a two-dimensional stress tensor. This likely places powerful constraints on the structure of scattering amplitudes that are currently under investigation [142, 144]. More ambitiously, one may attempt to find a two-dimensional CFT whose correlators on $\mathcal{CS}^2$ reproduce the $\mathcal{S}$-matrix of a four-dimensional quantum theory of gravity. This would provide a microscopic realization of the holographic principle in four-dimensional flat space quantum gravity, or flat space holography.

At this point in the lectures, we are still considering QFT and have not started to discuss gravity in any depth. Nevertheless, if the QFT can be coupled to gravity, it suggests that the celestial correlators (34.177) are those of a two-dimensional CFT, possibly in unusual representations.[203] Indeed, as a simple example we will see in the next section that tree-level celestial correlators involving soft gluons are those of a nonabelian two-dimensional Kac-Moody algebra. A brief discussion of the abelian case already appeared around equation (34.72).

[203] The work of Pasterski et al. and Cardona and Huang [140, 141, 142, 144] suggests that the principal series plays an important role.

34.4 Nonabelian Gauge Theory

Now we turn to another type of soft theorem, namely, the soft gluon theorem. This theorem applies to theories with a nonabelian gauge group $\mathcal{G}$ which is neither confined nor higgsed. Otherwise, there are no soft gluons! Unlike the (leading) soft photon theorem, the soft gluon theorem has corrections at the loop level. These are related to the running of the gauge coupling in the infrared, which affects the coupling constant appearing in front of the hard part of the charges. However, this is not the whole story of quantum corrections. Even in $\mathcal{N} = 4$ Yang-Mills, where the coupling does not run, there are still corrections to the soft gluon theorem that are one-loop exact [110, 111, 112, 113]. I have a little bit to say about this below, but the consequences of these anomalies for the asymptotic symmetries remain to be analyzed. A related issue in gravity was recently addressed in He et al.[117].

Soft theorems are essential for controlling IR divergences and getting IR finite inclusive cross sections. The absence of an uncorrected quantum soft theorem in the nonabelian case is intertwined with the fact that there is no known unitary $\mathcal{S}$-matrix for quantum nonabelian gauge theories, although a large class of IR finite dressed states were constructed in [148, 149] and earlier references therein. For a general state, the best one can do is to define finite inclusive cross sections with an IR cutoff. This is perfectly acceptable for most or all experimental applications, which in practice have an IR cutoff from limits on detector sensitivity. However, it is a problem, for example, for the theoretical studies of $\mathcal{N} = 4$ Yang-Mills scattering amplitudes, which do not actually exist. It is hard to analyze unitarity or symmetries of the $\mathcal{S}$-matrix without an $\mathcal{S}$-matrix! The lack of an $\mathcal{S}$-matrix for these theories is an elephant in the room. We hope that understanding the IR symmetries will enable the construction of a finite unitary $\mathcal{S}$-matrix or a suitable replacement thereof.

In this section, which largely reviews He et al. [12], we sidestep these issues simply by restricting the discussion to tree level. The presentation will proceed a bit differently than that for the abelian case. Rather than discuss a canonical formalism with an in- and out-Hilbert space, it is more efficient to illustrate the physics in the language of celestial correlators discussed in section 34.3. In this language, the scattering of a soft gluon becomes the insertion of a current into a correlation function on $\mathcal{CS}^2$. In section 34.4.1, it is shown that this current obeys the Ward identities of a $\mathcal{G}$-current algebra, which provides a useful alternate representation of the asymptotic symmetry group. In section 34.4.2, we construct the infinity of nonabelian conserved charges.

34.4.1 $\mathcal{G}$-Kac-Moody Algebra

Following the discussion around expression (34.177), we adopt the language of celestial correlators, in which asymptotic particles are represented by operators $\mathcal{O}_k$ on $\mathcal{CS}^2$:

$$\langle \text{ out } | \mathcal{S} | \text{ in } \rangle = \langle \mathcal{O}_1(E_1, z_1, \bar{z}_1) \cdots \mathcal{O}_n(E_n, z_n, \bar{z}_n) \rangle. \tag{34.179}$$

The operators $\mathcal{O}_k$ are taken to be in the kth representation of the nonabelian gauge group $\mathcal{G}$, whose generators satisfy

$$\big[T^a_k, T^b_k\big] = i f^{abc} T^c_k, \tag{34.180}$$

where the f^{abc} are the real, completely antisymmetric structure constants normalized so that

$$f^{acd} f^{bcd} = \delta^{ab} = \mathrm{tr}\big[T^a T^b\big], \tag{34.181}$$

with T^a the generators in the adjoint representation, and the trace is over the suppressed color index. The field strength is constructed from the gauge field $\mathcal{A}_\mu = \mathcal{A}^a_\mu T^a$ as

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - i\big[\mathcal{A}_\mu, \mathcal{A}_\nu\big] = \mathcal{F}^a_{\mu\nu} T^a \tag{34.182}$$

and obeys the equations of motion

$$\nabla^\nu \mathcal{F}_{\nu\mu} - i\big[\mathcal{A}^\nu, \mathcal{F}_{\nu\mu}\big] = g_{YM}^2 j_\mu^M. \tag{34.183}$$

Here g_{YM} is the gauge coupling, and j^M is the matter color current. The nonabelian gauge transformations act on the gauge and matter fields as

$$\delta_\varepsilon \mathcal{A}_\mu = \partial_\mu \varepsilon - i\big[\mathcal{A}_\mu, \varepsilon\big] \ , \qquad \delta_\varepsilon \phi_k = i\varepsilon^a T_k^a \phi_k \ , \qquad \delta_\varepsilon j_\mu^M = -i\big[j_\mu^M, \varepsilon\big]. \tag{34.184}$$

Near $\mathcal{I}^+$, the gauge field has a large-r expansion as in the abelian case:

$$\begin{aligned} \mathcal{A}_z(u,r,z,\bar z) &= A_z(u,z,\bar z) + \mathcal{O}\left(\frac{1}{r}\right), \\ \mathcal{A}_r(u,r,z,\bar z) &= \frac{1}{r^2} A_r(u,z,\bar z) + \mathcal{O}\left(\frac{1}{r^3}\right), \\ \mathcal{A}_u(u,r,z,\bar z) &= \frac{1}{r} A_u(u,z,\bar z) + \mathcal{O}\left(\frac{1}{r^2}\right). \end{aligned} \tag{34.185}$$

The field strength then has the expansion

$$\mathcal{F}_{ur} = \frac{1}{r^2} F_{ur} + \mathcal{O}\left(\frac{1}{r^3}\right), \qquad \mathcal{F}_{uz} = F_{uz} + \mathcal{O}\left(\frac{1}{r}\right), \qquad \mathcal{F}_{z\bar z} = F_{z\bar z} + \mathcal{O}\left(\frac{1}{r}\right), \tag{34.186}$$

where the leading components are given by

$$F_{ur} = \partial_u A_r + A_u, \qquad F_{uz} = \partial_u A_z, \qquad F_{z\bar z} = \partial_z A_{\bar z} - \partial_{\bar z} A_z - i\big[A_z, A_{\bar z}\big]. \tag{34.187}$$

The asymptotics (34.185) allow large gauge transformations infinitesimally generated by

$$\delta_\varepsilon A_z(u,z,\bar z) = D_z \varepsilon(z,\bar z). \tag{34.188}$$

A finite transformation yields an arbitrary flat $\mathcal{G}$-connection over $\mathcal{CS}^2$. These flat connections label the inequivalent vacua.

In classical Yang-Mills theory, the scattering problem is to find a map from initial data on $\mathcal{I}^-$ to final data on $\mathcal{I}^+$. These data include the connection A_z, as the gauge field explicitly appears in the equations of motion and therefore must be specified as part of the initial Cauchy data on $\mathcal{I}^-$. However, the maximal Cauchy development of incoming data on $\mathcal{I}^-$ via the field equations cannot by itself fully determine the outgoing data on $\mathcal{I}^+$, because given any one solution, another can always be obtained by acting with a large gauge transformation on $\mathcal{I}^+$. As in the abelian case, when there are no long-range magnetic fields,[204] the ambiguity is eliminated, and the scattering problem becomes well defined if we impose the antipodal boundary condition

$$A_z\big|_{\mathcal{I}^+_-} = A_z\big|_{\mathcal{I}^-_+}, \tag{34.189}$$

along with $F_{ru}|_{\mathcal{I}^+_-} = F_{rv}|_{\mathcal{I}^-_+}$. This boundary condition is preserved by large gauge transformations provided the parameter on $\mathcal{I}^+$ is related to the one on $\mathcal{I}^-$ by

$$\varepsilon(z,\bar z)\big|_{\mathcal{I}^+_-} = \varepsilon(z,\bar z)\big|_{\mathcal{I}^-_+}. \tag{34.190}$$

[204] Although it has not been carefully worked out, a generalization that drops this restriction presumably exists, as it is not required in the derivation of the Kac-Moody symmetry which follows.

Hence given any one solution of the scattering problem, infinitely more may be generated by acting on both the in- and the out-data with such a transformation.

Note that the boundary condition (34.189) is required simply for the existence of a well-defined classical scattering problem. The maximal Cauchy development of data on $\mathcal{I}^-$ gives the data on $\mathcal{I}^+$ only up to large gauge transformations. Without an additional condition such as (34.189), we cannot even determine whether the final state is a color singlet. As in the abelian case, the infinity of conservation laws is an immediate consequence of the need for a matching condition, although in this section, we adopt a different approach based on celestial correlators.

The tree-level nonabelian soft theorem can be written

$$\langle\, \mathcal{O}_1(p_1)\cdots\mathcal{O}_n(p_n)\mathcal{O}^a(q,\varepsilon)\,\rangle_{U=1} = g_{YM}\sum_{k=1}^{n}\frac{p_k\cdot\varepsilon}{p_k\cdot q}\langle\, \mathcal{O}_1(p_1)\cdots T^a_k\mathcal{O}_k(p_k)\cdots\mathcal{O}_n(p_n)\,\rangle_{U=1} + \mathcal{O}(q^0). \tag{34.191}$$

Here, $\mathcal{O}^a(q,\varepsilon)$ is the soft gluon operator of momentum q and polarization ε with color index a displayed. In textbook QFT, a state with a red gluon at the south pole and an anti-red gluon at the north pole is usually said to have zero total color charge. This implicitly assumes that the color connection $A_z = U^{-1}\partial_z U$ with $U\in G$ on the S^2 is trivial ($U=1$ or constant) so that the gluon does not change color under parallel transport from the north to the south pole. This assumption is made explicit in (34.191) by the $U=1$ subscript.

In fact, given the boundary condition (34.185), it is not possible in all contexts to choose the color frame $U=1$ everywhere. Consider a generic classical solution with an initially trivial flat connection $A_z=0$ with $U=1$ and a pulse of radiation passing through $\mathcal{I}^+$. The final connection will be flat if the system decays to zero energy, but the equations of motion imply that the final flat connection will not be trivial: $A_z = U^{-1}\partial_z U \neq 0$.[205] A pair of quarks near $\mathcal{I}^+$ initially in a color singlet will generically not be a singlet at late times, due to a change in the color frame. This is the "color memory" effect. An explicit formula for the color memory in terms of a Green's function convoluted with the color flux at $\mathcal{I}^+$, analogous to the one derived for gravity in [41], is given in [40]. As in QED, the nonabelian gauge theory vacua are infinitely degenerate and labeled classically by flat connections.

We define the soft gluon operator at $\mathcal{I}^+$ just as in the abelian case:

$$N_z = \int_{-\infty}^{\infty} du F_{uz} = A_z\big|_{\mathcal{I}^+_+} - A_z\big|_{\mathcal{I}^+_-}. \tag{34.192}$$

Similarly, at $\mathcal{I}^-$,

$$N^-_z = \int_{-\infty}^{\infty} dv F^-_{vz} = A^-_z\big|_{\mathcal{I}^-_+} - A^-_z\big|_{\mathcal{I}^-_-}. \tag{34.193}$$

Finally, we define the soft gluon current

$$J_z = -\frac{4\pi}{g^2_{YM}}\left(N_z - N^-_z\right). \tag{34.194}$$

Trading the operator momenta p_k in (34.191) for z_k and q for z, the soft theorem takes the suggestive form

$$\langle\, J^a_z\mathcal{O}_1(z_1,\bar z_1)\cdots\mathcal{O}_n(z_n,\bar z_n)\,\rangle_{U=1} = \sum_{k=1}^{n}\frac{1}{z-z_k}\langle\, \mathcal{O}_1(z_1,\bar z_1)\cdots T^a_k\mathcal{O}_k(z_k,\bar z_k)\cdots\mathcal{O}_n(z_n,\bar z_n)\,\rangle_{U=1}. \tag{34.195}$$

[205] The boundary condition (34.185) prevents us from gauging this away with a time dependent gauge transformation.

This is a very familiar formula in two-dimensional CFT. It is the Ward identity of a holomorphic Kac-Moody symmetry.[206]

Consider now the weighted integral of the current around a contour $\mathcal{C}$ on $\mathcal{CS}^2$

$$J_{\mathcal{C}}[\varepsilon] = \oint_{\mathcal{C}} \frac{dz}{2\pi i} \mathrm{tr}\left[\varepsilon J_z\right], \tag{34.196}$$

where $\varepsilon(z)$ is any holomorphic function in the interior of $\mathcal{C}$. Insertions of these operators obey

$$\langle\, J_{\mathcal{C}}[\varepsilon]\mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_{U=1} = \sum_{k\in\mathcal{C}} \langle\, \mathcal{O}_1 \cdots \varepsilon^a(z_k) T^a_k \mathcal{O}_k \cdots \mathcal{O}_n \,\rangle_{U=1}. \tag{34.197}$$

That is, they generate gauge transformations ε inside $\mathcal{C}$.

So far, all the amplitudes have been constructed with the trivial connection $U = 1$ on $\mathcal{CS}^2$. Large gauge transformations change this connection, so we would like to understand the more general case. The Ward identity (34.197) relates $U = 1$ amplitudes to the more general case. Consider, for instance, an infinitesimal change in flat connection

$$\delta U(z, \bar{z}) = i\varepsilon(z, \bar{z}) + \dots . \tag{34.198}$$

Suppose we choose ε so that $\varepsilon = 0$ outside $\mathcal{C}$ and is nonzero and holomorphic in $\mathcal{C}$. Then, the change in the correlator under this shift is simply given by a large gauge transformation of the operators themselves:

$$\begin{aligned} \delta_\varepsilon \langle\, \mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_{U=1} &\equiv \langle\, \mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_{U=1+i\varepsilon} - \langle\, \mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_{U=1} \\ &= i \sum_{k\in\mathcal{C}} \langle\, \mathcal{O}_1 \cdots \varepsilon^a(z_k) T^a_k \mathcal{O}_k \cdots \mathcal{O}_n \,\rangle_{U=1} \\ &= i \langle\, J_{\mathcal{C}}[\varepsilon]\mathcal{O}_1 \cdots \mathcal{O}_n \,\rangle_{U=1}. \end{aligned} \tag{34.199}$$

We can characterize this relation by

$$J_{\mathcal{C}}[\varepsilon] = \int_{R_{\mathcal{C}}} d^2z \gamma_{z\bar{z}} \varepsilon^a \frac{\delta}{\delta U^a}, \tag{34.200}$$

where $R_{\mathcal{C}}$ is the region inside the contour $\mathcal{C}$. In other words, insertions of $J_{\mathcal{C}}[\varepsilon]$ generate locally holomorphic large gauge transformations on the boundary that are characterized by a general flat connection $A_z = U^{-1}\partial_z U$.

34.4.2 Conserved Charges

In section 34.4.1, I presented the soft gluon theorem as a Kac-Moody Ward identity. It is also possible to formulate the entire problem as an infinity of charge conservation laws. The nonabelian analogue of (34.29) is[207]

$$Q^+_\varepsilon = \frac{1}{g^2_{YM}} \int_{\mathcal{I}^+_-} \mathrm{tr}\left[\varepsilon * F\right]. \tag{34.201}$$

[206] There is also an antiholomorphic current $J_{\bar{z}}$, whose insertions are related to negative helicity gluons. One may consider the double soft limits involving either $J_z J_w$ or $J_z J_{\bar{w}}$. It turns out that the former is unambiguous, whereas the latter depends on the order of limits [12]: see exercise 9. This was not the case with photons and begs for interpretation.

[207] As in section 34.2.13, the magnetic generalization is obtained by replacing $*F$ with F in (34.201). It would be interesting to understand the action of S-duality on the electric and magnetic charges.

As before, by integrating by parts and using the nonabelian constraint equations, this can be written as an integral over all of $\mathcal{I}^+$. The final expression contains a soft term Q_ε^{+S} and a hard term Q_ε^{+H}. Similar constructions apply on $\mathcal{I}^-$. The soft term is, of course, related to the current defined here. The precise relationship turns out to be

$$J_{\mathcal{C}}[\varepsilon] = Q_\varepsilon^{-S} - Q_\varepsilon^{+S}. \tag{34.202}$$

The left-hand side depends only on the value of ε on the curve $\mathcal{C}$. This has a meromorphic extension to $\mathcal{CS}^2$. On the right-hand side, we take ε to be given by this extension inside $\mathcal{C}$ and zero outside $\mathcal{C}$. Actually, this procedure gives two possible expressions, depending on which side of $\mathcal{C}$ we regard as the interior. It is illuminating to check that the equivalence of these two expressions is a consequence of the soft theorem.

I close with a comment on loop corrections. Rather than using the soft theorem, the nonabelian conservation law $Q_\varepsilon^+ = Q_\varepsilon^-$ can be derived directly from the nonabelian matching conditions (34.189), as we did for the abelian case. As discussed, such matching conditions are classically required to determine the future color frame from the past one, and without them the scattering problem makes no sense. This suggests that the conservation laws somehow survive the one-loop quantum corrections [110, 111, 112, 113] to the soft theorem, perhaps in a modified or shifted form along the lines discussed for gravity in [117].

34.5 Gravity

Finally we turn to gravity, where the implications of the IR structure become even more far reaching. Oddly, many of the ideas herein were first understood in the gravitational rather than the seemingly simpler gauge-theoretic context. These lectures have presented the material in ahistorical order.

Section 34.5.1 reviews the basics of asymptotically flat spacetimes and Bondi coordinates. Section 34.5.2 begins with a review of the BMS analysis of asymptotic data and the discovery of supertranslations of $\mathcal{I}^+$ and $\mathcal{I}^-$. Then I present the antipodally matched subgroup of past and future supertranslations, which is a symmetry of gravitational scattering. Following the abelian gauge theory case, the conserved charges are derived from the matching condition, shown to generate the symmetry and to imply the leading soft graviton theorem. In section 34.5.3, conserved charges are derived from a matching condition for subleading metric components and their relation to superrotation symmetry (an extension of BMS symmetry), and a new subleading soft theorem are discussed.

34.5.1 Asymptotically Flat Spacetimes

In the previous sections, flat Minkowski space in retarded coordinates near $\mathcal{I}^+$ was described by the metric

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z}. \tag{34.203}$$

We would now like to study gravitational theories in which the metric is asymptotic to, but not exactly equal to, the flat metric. We will work in Bondi coordinates $(u, r, z, \bar{z})$, and we abbreviate $\Theta^A = (z, \bar{z})$. In this gauge, the most general four-dimensional metric takes the form

$$ds^2 = -Udu^2 - 2e^{2\beta}dudr + g_{AB}\left(d\Theta^A + \frac{1}{2}U^A du\right)\left(d\Theta^B + \frac{1}{2}U^B du\right), \tag{34.204}$$

where

$$\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0. \tag{34.205}$$

Equation (34.205) implies that r is the luminosity distance. Note that we have completely fixed here the local diffeomorphism invariance by the conditions $g_{rr} = g_{rA} = 0$ together with (34.205). We adopt these coordinates largely because they are used in most of the literature on asymptotically flat spacetimes. However, important insights have arisen in other gauges, such as the harmonic gauge [183, 184], which may ultimately prove more useful.

So far, we have not imposed any sort of asymptotic flatness condition. Any geometry can be described locally by the metric (34.204). Imposing asymptotic flatness at large r with fixed $(u, z, \bar{z})$ leads to falloff conditions on the metric components. There is no a priori preferred method of determining what these falloffs must be, and the literature discusses various options. They are typically chosen to be weak enough to allow for all interesting solutions, including gravity waves, but strong enough to rule out unphysical solutions, such as those with infinite energy. For the natural choice made by Bondi, van der Burg, Metzner, and Sachs (BMS) [9, 8], the large-r structure of the metric is constrained to be (see exercise 10)

$$\begin{aligned} ds^2 = &-du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \\ &+ \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + D^zC_{zz}dudz + D^{\bar{z}}C_{\bar{z}\bar{z}}dud\bar{z} \\ &+ \frac{1}{r}\left(\frac{4}{3}(N_z + u\partial_z m_B) - \frac{1}{4}\partial_z(C_{zz}C^{zz})\right)dudz + c.c. + ..., \end{aligned} \tag{34.206}$$

where D_z is the covariant derivative with respect to $\gamma_{z\bar{z}}$; C_{zz}, m_B and N_z depend on $(u, z, \bar{z})$ but not on r. The first three terms in (34.206) are simply the flat Minkowski metric, and the remaining terms are the leading corrections. The ellipsis "..." involves further subleading terms at large r, whose precise structure we will not need. Near $\mathcal{I}^+$, spacetime is flat to leading order, and we can relate the retarded Bondi coordinates to the standard Minkowski coordinates $u = t - r + ...$. Equation (34.206) corresponds to the large r falloffs:

$$\begin{aligned} &g_{uu} = -1 + \mathcal{O}\left(\frac{1}{r}\right), \qquad g_{ur} = -1 + \mathcal{O}\left(\frac{1}{r^2}\right), \qquad g_{uz} = \mathcal{O}(1), \\ &g_{zz} = \mathcal{O}(r), \qquad g_{z\bar{z}} = r^2\gamma_{z\bar{z}} + \mathcal{O}(1)\ , \qquad g_{rr} = g_{rz} = 0. \end{aligned} \tag{34.207}$$

So far, no mention of Einstein's equations has been made. (34.206) is simply a geometric constraint that defines the class of spacetimes that we are interested in studying.

The quantity m_B is known as the Bondi mass aspect. For Kerr spacetimes, $m_B = GM$ is constant and proportional to the mass. However, in a generic spacetime, it will depend on both the retarded time u and on the angle $(z, \bar{z})$. Its integral over the sphere is the total Bondi mass. N_z is known as the angular momentum aspect, because its integrals over the sphere, contracted with a rotational vector field, give the total angular momentum. Our definition here of this term is shifted relative to some of the literature to simplify some formulas below. C_{zz} describes gravitational waves. It is $\frac{1}{r}$-suppressed relative to the leading metric component and is transverse to $\mathcal{I}^+$. The "Bondi news tensor" is defined by

$$N_{zz} = \partial_u C_{zz}. \tag{34.208}$$

It is the gravitational analogue of the field strength $F_{uz} = \partial_u A_z$, and its square is proportional to the energy flux across $\mathcal{I}^+$.

Eventually we will need to supplement (34.207) with additional boundary conditions near the boundaries $\mathcal{I}^+_\pm$ of $\mathcal{I}^+$. If N_{zz} does not fall off fast enough for $u \to \pm\infty$, the solution will have infinite energy. We discuss this in Section 34.5.2.

34.5.2 Supertranslations

We now turn to the asymptotic symmetries of gravitational theories in asymptotically flat spacetimes. As argued by BMS [9, 8], these are generated by diffeomorphisms that preserve both

the Bondi gauge (34.204) as well as the boundary falloffs (34.207). Since researchers were looking for symmetries that act in the asymptotic region where spacetime is almost flat, it was expected that one would reproduce the isometries of flat spacetime itself, namely, the Poincaré group. Had this been the case, GR would reduce to special relativity at large-distances and weak fields. Surprisingly, as we now review, what they got instead was an infinite-dimensional group, now called the BMS group. This contains as a subgroup the finite-dimensional Poincaré group. However, the four global translations are elevated to a whole function's worth of "supertranslations" that act independently on each point of the asymptotic sphere. Moreover, as we shall see, GR does not reduce to special relativity at large distances and weak fields. Instead, a large space of degenerate vacua remains.

BMS Analysis

We will not reproduce here the entire calculation of BMS [9, 8], which is a bit lengthy. We instead make a simplifying assumption that eliminates six Lorentz generators. Namely, we restrict consideration to diffeomorphisms that have the large-r falloffs:

$$\xi^u, \xi^r \sim \mathcal{O}(1) \ , \qquad \xi^z, \xi^{\bar{z}} \sim \mathcal{O}\left(\frac{1}{r}\right) . \tag{34.209}$$

This condition is equivalent to the statement that the vector field is $\mathcal{O}(1)$ at large r in an orthonormal frame, thereby eliminating boosts and rotations that grow with r at infinity. We return to a discussion of these in section 34.5.3.

The Lie derivative of the metric components at large r are then

$$\begin{aligned}
\mathcal{L}_\zeta g_{ur} &= -\partial_u \zeta^u + \mathcal{O}\left(\frac{1}{r}\right) , \\
\mathcal{L}_\zeta g_{zr} &= r^2 \gamma_{z\bar{z}} \partial_r \zeta^{\bar{z}} - \partial_z \zeta^u + \mathcal{O}\left(\frac{1}{r}\right) , \\
\mathcal{L}_\zeta g_{z\bar{z}} &= r \gamma_{z\bar{z}} \left[2\zeta^r + r D_z \zeta^z + r D_{\bar{z}} \zeta^{\bar{z}}\right] + \mathcal{O}(1), \\
\mathcal{L}_\zeta g_{uu} &= -2\partial_u \zeta^u - 2\partial_u \zeta^r + \mathcal{O}\left(\frac{1}{r}\right) .
\end{aligned} \tag{34.210}$$

Then, requiring that the Bondi gauge conditions (34.204) and falloffs (34.207) are both preserved implies that at large r,

$$\zeta = f\partial_u - \frac{1}{r}\left(D^z f \partial_z + D^{\bar{z}} f \partial_{\bar{z}}\right) + D^z D_z f \partial_r + \ldots, \tag{34.211}$$

where $f(z, \bar{z})$ is any function of $(z, \bar{z})$. It turns out that (34.211) also preserves all the remaining conditions. We note here that the last term in (34.211) is derived from the condition that $g_{ur} = -1 + \mathcal{O}\left(\frac{1}{r^2}\right)$. In other formulations of asymptotically flat spacetimes (for instance, the one considered by Newman and Unti [212], where $g_{ur} = -1$), this term takes on a different form. The first two terms are universal and are directly measured by the gravitational memory effect, which we discuss in Section 34.6. The transformations generated by (34.211) are called *supertranslations*, depicted in figure 34.17. They are generalizations of the four translations in Minkowski space. For instance, for $f(z, \bar{z}) =$ constant, (34.211) generates u-translations. What is less obvious, though, is that if $f(z, \bar{z})$ is taken to be the $\ell = 1$ harmonic on the sphere, we retrieve the three spatial translations. The most general f effectively allows separate translations along every null generator of $\mathcal{I}^+$.

Supertranslations transform one geometry into a new, physically inequivalent geometry, despite the fact that they are diffeomorphisms. To see this, consider a solution where an outgoing pulse of gravitational or electromagnetic waves crosses the south pole of $\mathcal{I}^+$, and another pulse crosses

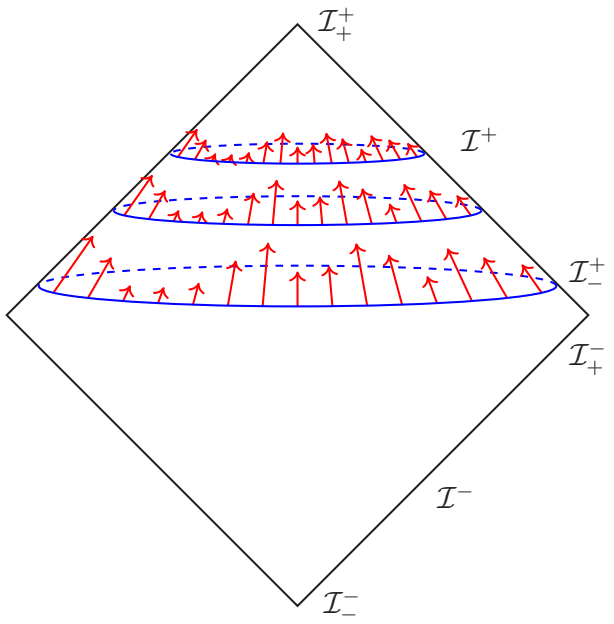


Figure 34.17: **Under a supertranslation, retarded time u is shifted independently at every angle on $\mathcal{I}$.**

the north pole of $\mathcal{I}^+$, both at retarded time $u = 100$. Now supertranslate this solution with a function $f(z, \bar{z})$ that has the property that f(south pole)=100 and f(north pole)=0. The new solution now has one outgoing pulse at the north pole only at $u = 100$ and one at the south pole only at $u = 200$. The outgoing data are measurably changed by the supertranslation.

This simple example shows that the effect of a supertranslation on a solution can be discerned even at the classical level. While the structure here follows the abelian gauge theory case quite closely, the large abelian gauge transformations only modified phases of states and therefore are somewhat quantum in nature. For this reason, it can be less confusing to discuss asymptotic symmetries in the case of gravity. Although supertranslations were not understood to be a symmetry of gravitational scattering until recently [42, 43], the fact that they act nontrivially on the phase spaces at $\mathcal{I}^\pm$ was understood fifty years ago by BMS [8, 9]. It is odd that this symmetry of gravity was at least partially understood fifty years ago, whereas the precisely analogous symmetries in electromagnetism were only understood in the past few years [10, 11, 13, 14]. This is perhaps in part due to the quantum nature of the electromagnetic symmetries.

The action of supertranslations on the $\mathcal{I}^+$ data m_B, C_{zz}, and N_{zz} can be determined by computing the Lie derivative of the appropriate component of the metric and then extracting the appropriate coefficient in the large-r expansion. This process gives (see exercise 11)

$$\begin{aligned} \mathcal{L}_f N_{zz} &= f\partial_u N_{zz}, \\ \mathcal{L}_f m_B &= f\partial_u m_B + \frac{1}{4}\left[N^{zz}D_z^2 f + 2D_z N^{zz} D_z f + c.c.\right], \\ \mathcal{L}_f C_{zz} &= f\partial_u C_{zz} - 2D_z^2 f. \end{aligned} \tag{34.212}$$

The last equation above is especially interesting. Suppose we supertranslate flat Minkowski spacetime described by $m_B = N_{zz} = C_{zz} = 0$. Equation (34.212) implies that the supertranslated spacetime will still have zero Bondi mass and Bondi news and vanishing Riemann tensor. This is consistent with the fact that a diffeomorphism cannot change the physical mass squared or create gravitational waves. However, the supertranslated spacetime does have a nonzero C_{zz}. One may check that the vanishing of the curvature in fact requires

$$C_{zz} = -2D_z^2 C, \tag{34.213}$$

for some function $C(z,\bar{z})$. Under a supertranslation,

$$\mathcal{L}_f C = f. \tag{34.214}$$

Hence C is the Goldstone boson of spontaneously broken supertranslation invariance. It parametrizes the classically inequivalent gravitational vacua. Since the $\ell = 0, 1$ modes of C are annihilated by D_z^2, the four rigid spacetime translations are not broken.

If one drops the overly restrictive falloffs (34.209) on ζ, one obtains the larger BMS$^+$ group which is a semidirect product of supertranslations with Lorentz transformations on $\mathcal{I}^+$. The four spacetime translations are an ideal of BMS$^+$ and so may be canonically identified in any BMS$^+$ frame. In general, there is no preferred Poincaré subgroup of BMS$^+$. Different subgroups are transformed into one another via conjugation by supertranslations. In particular, this implies that there is no BMS$^+$ invariant definition of angular momentum. This is sometimes referred to as "the problem of angular momentum in general relativity," although I view it as a feature, not a bug. However, given any particular classical choice of vacuum, there is a unique Poincaré subgroup of BMS$^+$ under which it is invariant.[208]

So far, we have not utilized any equations of motion. Assume that the geometry is governed by the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T^M_{\mu\nu}. \tag{34.215}$$

Since we are here interested in the structure of null infinity, we assume that $T^M_{\mu\nu}$ is a matter stress tensor corresponding to massless modes.[209] Plugging in the explicit form of the metric (34.206) and expanding in large r, we find that the leading uu component of Einstein's equations is

$$\partial_u m_B = \frac{1}{4}\left[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right] - T_{uu}, \tag{34.216}$$

where

$$T_{uu} = \frac{1}{4} N_{zz} N^{zz} + 4\pi G \lim_{r\to\infty}\left[r^2 T^M_{uu}\right]. \tag{34.217}$$

Equations (34.216) and (34.217) constrain the leading data at $\mathcal{I}^+$. There is an additional constraint involving N_z from the uz component of the Einstein equation. However, (34.216) suffices for our current discussion on supertranslations.

The traceless Bondi news $N_{zz}(u,z,\bar{z})$ comprises two unconstrained real functions on $\mathcal{I}^+$ as expected for the two helicities of the massless graviton. We assume that near the past and future boundaries of $\mathcal{I}^+$, $\mathcal{I}^+_-$ and $\mathcal{I}^+_+$, the news falls off faster than $\frac{1}{|u|}$. These (and stronger) asymptotic boundary conditions were proven by Christodoulou and Klainerman [47] to hold in a finite neighborhood of flat space: here we shall consider spacetimes with this asymptotic behavior but do not require them to be near flat space in the deep interior.[210] The news then trivially determines C_{zz} up to an integration function by integrating (34.208). Moreover, the vanishing of the Weyl tensor at $\mathcal{I}^+_-$ requires $C_{zz}|_{\mathcal{I}^+_-} = -2D_z^2 C|_{\mathcal{I}^+_-}$[42]. Therefore we may take the integration function to be $C|_{\mathcal{I}^+_-}$. Given the news tensor and these initial data at $\mathcal{I}^+_-$, the constraints may be integrated to give the mass m_B and N_z everywhere on $\mathcal{I}^+$. Hence the Cauchy data include

$$\{N_{zz}(u,z,\bar{z}), C(z,\bar{z})|_{\mathcal{I}^+_-}, m_B(z,\bar{z})|_{\mathcal{I}^+_-}\}. \tag{34.218}$$

[208] Thus, classically, there is always an unbroken Poincaré subgroup of BMS$^+$ in flat space. In the quantum theory, a generic superposition of vacua will have only the four global translations as an unbroken symmetry.

[209] The massive case is treated by Campiglia and Laddha [183, 184].

[210] If the news is only required to fall off slowly enough to make the total energy finite, C_{zz} is not well defined at the boundaries $\mathcal{I}^+_\pm$. This is one of the reasons it was thought impossible to match the BMS frames of $\mathcal{I}^\pm$. However, the results of [47] demonstrate that the region of phase space in which the boundary values of C_{zz} are well defined and the matching (given below) is possible is a physically reasonable one.

At higher orders in $\frac{1}{r}$, more data are needed, including $N_z|_{\mathcal{I}^+_-}$. This will be important when we consider superrotations below, but is suppressed for now.

There is an analogous story at $\mathcal{I}^-$. The metric in advanced Bondi coordinates $(v, r, z, \bar{z})$ has the expansion

$$ds^2 = -dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + \frac{2m_B}{r}dv^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 + \dots, \tag{34.219}$$

where m_B and C_{zz} depend on $(v, z, \bar{z})$ rather than on $(u, z, \bar{z})$. The angular $(z, \bar{z})$ coordinates on $\mathcal{I}^-$ are as usual antipodal to the ones on $\mathcal{I}^+$, and $v = t + r + \dots$. Supertranslations act on $\mathcal{I}^-$ as

$$\mathcal{L}_f N_{zz} = f\partial_v N_{zz}, \qquad \mathcal{L}_f C_{zz} = f\partial_v C_{zz} + 2D_z^2 f, \tag{34.220}$$

which may be enlarged with boosts and rotations to give the action of BMS$^-$ on $\mathcal{I}^-$. The constraint equation takes the form

$$\partial_v m_B = \frac{1}{4}\left(D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right) + T_{vv}, \qquad T_{vv} = \frac{1}{4}N_{zz}N^{zz} + 4\pi G \lim_{r\to\infty}\left[r^2 T^M_{vv}\right]. \tag{34.221}$$

Defining $C_{zz}|_{\mathcal{I}^-_+} = 2D_z^2 C|_{\mathcal{I}^-_+}$, the analog of the Cauchy data (34.218) is

$$\{N_{zz}(v, z, \bar{z}), C(z, \bar{z})|_{\mathcal{I}^-_+}, m_B(z, \bar{z})|_{\mathcal{I}^-_+}\}. \tag{34.222}$$

The scattering problem

The scattering problem in classical general relativity is, roughly speaking, to find the map from Cauchy data on $\mathcal{I}^-$ to that on $\mathcal{I}^+$.[211] Such a map is not even formally determined from the maximal Cauchy development of the $\mathcal{I}^-$ data (34.218) with the Einstein equation. This determines the data on $\mathcal{I}^+$ at most up to a supertranslation, or more generally, a BMS$^+$ frame. A prescription is needed to attach $\mathcal{I}^+$, choose a future BMS$^+$ frame, and determine the initial values for integrating m_B (and N_z) along $\mathcal{I}^+$ using the constraints. *Without such a prescription, the scattering problem in GR is not defined.* In [42] it was proposed that the BMS$^+$ frame should be determined by the Lorentz- and CPT-invariant matching conditions

$$C(z, \bar{z})|_{\mathcal{I}^+_-} = C(z, \bar{z})|_{\mathcal{I}^-_+}, \quad m_B(z, \bar{z})|_{\mathcal{I}^+_-} = m_B(z, \bar{z})|_{\mathcal{I}^-_+}. \tag{34.223}$$

The matching condition for N_z is discussed below. Conditions (34.223) break the combined BMS$^+ \times$ BMS$^-$ action on $\mathcal{I}^+$ and $\mathcal{I}^-$ down to the diagonal subgroup that preserves these conditions, namely,

$$f(z, \bar{z})|_{\mathcal{I}^+_-} = f(z, \bar{z})|_{\mathcal{I}^-_+}\ . \tag{34.224}$$

This condition fixes the BMS$^+$ frame in terms of the BMS$^-$ frame. The diagonal subgroup generated by (34.224) remains as a symmetry of gravitational scattering. Given one solution of the scattering problem, this group generates infinitely many more.

With our conventions, (34.223) *antipodally* equates past and future fields near spatial infinity. We expect that it is the only Lorentz and CPT invariant choice, and it is implicit in most or all GR computations in asymptotically flat spacetimes. The matching condition (34.223) was proven [43] to be implicit to all orders in standard weak field perturbation theory by demonstrating its equivalence to Weinberg's soft graviton theorem [7]. This perturbative analysis motivates the proposal that (34.223) is part of the definition of the scattering problem whenever the fields are sufficiently weak near spatial infinity, even if the interior contains a black hole.

[211] Of course, if a black hole is formed, we need Cauchy data on $\mathcal{I}^+ \cup \mathcal{H}$, where $\mathcal{H}$ is the future horizon, but for now, we assume that black holes are absent.

Conserved Charges

The mere existence of infinitely many matching conditions, one for every point on the celestial sphere, implies an infinite number of conserved charges. Following closely the gauge theory discussion, the supertranslation charges are

$$\begin{aligned} Q_f^+ &= \frac{1}{4\pi G}\int_{\mathcal{I}^+_-} d^2z\gamma_{z\bar{z}} f m_B, \\ Q_f^- &= \frac{1}{4\pi G}\int_{\mathcal{I}^-_+} d^2z\gamma_{z\bar{z}} f m_B. \end{aligned} \tag{34.225}$$

The matching conditions (34.223) immediately imply the conservation law

$$Q_f^+ = Q_f^- \ . \tag{34.226}$$

Integrating by parts, using the constraint equation, and assuming m_B decays to zero in the far future, we can write

$$\begin{aligned} Q_f^+ &= \frac{1}{4\pi G}\int_{\mathcal{I}^+} du d^2z\gamma_{z\bar{z}} f\left[T_{uu} - \frac{1}{4}\left(D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right)\right], \\ Q_f^- &= \frac{1}{4\pi G}\int_{\mathcal{I}^-} dv d^2z\gamma_{z\bar{z}} f\left[T_{vv} + \frac{1}{4}\left(D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right)\right]. \end{aligned} \tag{34.227}$$

To understand the conservation law better, suppose we take $f(z,\bar{z}) = \delta^2(z-w)$. Then (34.226) implies that the integrated (over u) energy flux at a point w on $\mathcal{I}^+$ is equal to the integrated energy flux at the antipodal point w on $\mathcal{I}^-$:

$$\int_{\mathcal{I}^+} du\gamma_{z\bar{z}}\left[T_{uu} - \frac{1}{4}\left(D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right)\right] = \int_{\mathcal{I}^-} dv\gamma_{z\bar{z}}\left[T_{vv} + \frac{1}{4}\left(D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}}\right)\right]. \tag{34.228}$$

Here, the local energy at a point includes not only the usual term involving the stress tensor, but also a crucial extra term that is linear in the gravitational field and is also a total u derivative (i.e., it is a soft graviton contribution to the local energy). Thus, the conservation law (34.226) is the statement that *energy is conserved at every angle.*

In the quantum theory, conserved charges commute with the $\mathcal{S}$-matrix:

$$Q_f^+\mathcal{S} - \mathcal{S}Q_f^- = 0. \tag{34.229}$$

We can now sandwich this between in- and out-states. The resulting Ward identity is of course equivalent to Weinberg's soft graviton theorem,

$$\langle \text{ out } | a_+\mathcal{S} | \text{ in } \rangle = \sqrt{8\pi G}\sum_k \frac{\varepsilon^{+\mu\nu} p_{k\mu} p_{k\nu}}{q\cdot p_k}\langle \text{ out } | \mathcal{S} | \text{ in } \rangle, \tag{34.230}$$

where a_+ annihilates a helicity $+$ graviton. We already derived this formula using Feynman diagrams in section 34.2.9. The proof [43] that (34.230) and (34.228) are equivalent is almost exactly the same as in the gauge theory case and is not repeated here.

One expects that, in a Hamiltonian formulation, supertranslation symmetries corresponding to f are generated via Dirac brackets or commutators with Q_f^+:

$$[Q_f^+, ...] = i\delta_f, \qquad [Q_f^+, H] = 0, \qquad H = Q^+_{f=1}. \tag{34.231}$$

This guess turns out to be correct, but, as in the gauge theory case, verification is quite subtle [43]. It requires computing the terms in the symplectic form involving the Goldstone mode C

and the soft graviton mode N. On the physical phase space one must impose the vanishing of the Weyl tensor as a constraint at the boundaries $\mathcal{I}^+_\pm$ of $\mathcal{I}^+$. This changes various commutators by important factors of 2, yielding exactly (34.231). These factors of 2 are, as in the gauge case, related to the fact that soft gravitons and Goldstone modes are both described by a single real scalar field, despite the existence of two polarizations of the former [43].

The existence of these conserved charges is in principle experimentally verifiable. Indeed, proposed tests of the gravitational memory effect [34, 35, 36], although not initially recognized as such, are tests of supertranslation charge conservation [41].

34.5.3 Superrotations

Our analysis so far has centered around C_{zz} and the Bondi mass m_B, which are the first nontrivial corrections to the flat metric near $\mathcal{I}$. Proceeding in the large-r expansion, at the next order we encounter the angular momentum aspect N_z, which bears the same relation to the total angular momentum that the m_B bears to the total mass. In this subsection, we see that the matching condition for N_z leads to conserved superrotation charges, just as the matching condition for m_B led to conserved supertranslation charges.

Conserved Charges

The angular momentum aspect N_z is subject to the constraint equation $G_{uz} = 8\pi G T^M_{uz}$. The leading uz component of the Einstein equations is (compare to (34.216) for the analogous constraint on m_B)

$$\partial_u N_z = \frac{1}{4}\partial_z \left(D_z^2 C^{zz} - D_{\bar{z}}^2 C^{\bar{z}\bar{z}}\right) - u\partial_u \partial_z m_B - T_{uz}. \tag{34.232}$$

Here we have introduced the (rescaled) momentum density in the z-direction of the gravitational field (compare to (34.217) for T_{uu}):

$$T_{uz} \equiv 8\pi G \lim_{r\to\infty}\left[r^2 T^M_{uz}\right] - \frac{1}{4}\partial_z\left(C_{zz}N^{zz}\right) - \frac{1}{2}C_{zz}D_z N^{zz}. \tag{34.233}$$

N_z is constrained in relation to a momentum density T_{uz}, in contrast to m_B, which was constrained in relation to the energy density T_{uu}. Once m_B and C_{zz} are fully specified, then $\partial_u N_z$ is also specified by (34.232). This fixes N_z up to an integration function. We fix this function by the matching condition [57]

$$N_z(z,\bar{z})|_{\mathcal{I}^+_-} = N_z(z,\bar{z})|_{\mathcal{I}^-_+}, \tag{34.234}$$

which is just like the matching condition for m_B. A new subleading soft graviton theorem was proven [56, 213, 214], to all orders using Feynman tree diagrams,[212] and then shown to imply (34.234) [57, 215]. Motivated by this perturbative analysis, we propose that (34.223) and (34.234) are part of the definition of the scattering problem whenever the fields are sufficiently weak near spatial infinity, even if the interior contains a black hole.

Condition (34.234) implies a second infinity of conserved charges, which can be constructed from an arbitrary vector field Y^z on the sphere. Using (34.234), we find

$$Q^+_Y = \frac{1}{8\pi G}\int_{\mathcal{I}^+_-} d^2z (Y_{\bar{z}}N_z + Y_z N_{\bar{z}}) = \frac{1}{8\pi G}\int_{\mathcal{I}^-_+} d^2z (Y_{\bar{z}}N_z + Y_z N_{\bar{z}}) = Q^-_Y. \tag{34.235}$$

[212] Quantum mechanically, (34.234) is possibly deformed by an anomaly at one-loop [110, 112, 111, 113, 79, 114]. Since *some* matching relation of the form (34.234) must exist for gravitational scattering to be defined, this reasoning suggests that these one-loop corrections deform rather than eliminate the conserved charges. This is an important open problem on which recent progress has been made [67, 117, 210].

Equation (34.235) expresses conservation of superrotation charge. The special cases for which Y^z is one of the 6 global conformal Killing vectors on S^2 are conservation of ADM angular momentum and boost charge, sometimes referred to as the BORT (Beig-O'Murchada-Regge-Teitelboim) [216, 217] center-of-mass. Choosing the vector field to be a delta function, these new conservation laws equate net "in" and "out" angular momentum flux for every angle. A recent proposal to test superrotation charge conservation via the gravitational spin memory effect [69] was made by Nichols [70].

Symmetries

Given an infinite number of conserved charges that generalizes angular momentum and boost charges, one naturally expects an infinite number of symmetries generalizing Lorentz transformations. Indeed, conjectures that such a superrotation symmetry exists have been made from several different perspectives, as reviewed in this subsection. We will see that subtleties arise in part because a finite symmetry action changes the asymptotics, the consequences of which remain to be analyzed.

Aspects of superrotation symmetry were implicit in early work of Penrose [218], but were first conjectured in their modern form in the prescient work of de Boer and Solodukhin [52], who were trying to understand holography in Minkowski space. These authors considered the hyperbolic slicing of Minkowski space, which is discussed in section 34.2.12, and argued that the parameter τ that labels the slices should be viewed as the coordinate of an internal space. This describes four-dimensional Minkowski space as a "noncompactification" to the three-dimensional hyperbolas $\mathbb{H}_3$. Because the internal space is noncompact, the effective field theory on $\mathbb{H}_3$ contains a continuum of representations of the Lorentz isometry group. De Boer and Solodukhin [52] conjecture that standard holographic arguments nevertheless apply. In particular, the asymptotic analysis of Brown and Henneaux for AdS_3[200] implies that Lorentz symmetry should be enhanced to the Virasoro symmetry, or equivalently, to superrotations.

A second line of reasoning that leads to superrotations arose while revisiting the original BMS analysis. It was suggested by Banks [53] in a footnote, and independently in greater detail by Barnich and Troessaert [54, 55, 219, 203], that some of the falloff assumptions in BMS are overly restrictive. We now review this illuminating perspective.

In the course of the derivation of the supertranslations in section 34.5.2, an important restriction was imposed: we required that the components of the vector field ζ be bounded in an orthonormal frame as the latter approaches $\mathcal{I}^+$. However, in general this is clearly too strong an assumption, because it rules out both boosts and rotations. One should (and BMS did [8, 9]) do an analysis without that assumption. We now summarize the salient points of this analysis and show along the way how the supertranslations naturally generalize to superrotations.

Lorentz Killing vectors are of the general form

$$\zeta_Y = \left(1+\frac{u}{2r}\right) Y^z \partial_z - \frac{u}{2r} D^{\bar z} D_z Y^z \partial_{\bar z} - \frac{1}{2}(u+r) D_z Y^z \partial_r + \frac{u}{2} D_z Y^z \partial_u + c.c., \tag{34.236}$$

where $(Y^z, Y^{\bar z})$ is a two-dimensional vector field on $\mathcal{CS}^2$. At null infinity ζ_Y simplifies to

$$\zeta_Y|_{\mathcal{I}^+} = Y^z \partial_z + \frac{u}{2} D_z Y^z \partial_u + c.c.. \tag{34.237}$$

The claim is (see exercise 12) that for flat Minkowski space, described by the first three terms of (34.206), if we take

$$Y^z = 1, z, z^2, i, iz, iz^2, \tag{34.238}$$

then the six real vector fields ζ_Y generate the Lorentz transformations.

Instead of restricting ourselves to $Y^z \sim 1, z, z^2$, let us instead compute, for a general Y^z, the Lie derivative with respect to ζ_Y (denoted by $\mathcal{L}_Y$) of various components of the metric. We find

that

$$\mathcal{L}_Y g_{ur} = \mathcal{O}\left(\frac{1}{r^2}\right), \tag{34.239}$$

$$\mathcal{L}_Y g_{zr} = \mathcal{O}\left(\frac{1}{r}\right), \tag{34.240}$$

$$\mathcal{L}_Y g_{z\bar{z}} = \mathcal{O}(r), \tag{34.241}$$

$$\mathcal{L}_Y g_{uu} = \mathcal{O}\left(\frac{1}{r}\right), \tag{34.242}$$

$$\mathcal{L}_Y g_{\bar{z}\bar{z}} = 2r^2\gamma_{z\bar{z}}\partial_{\bar{z}}Y^z + \mathcal{O}(r). \tag{34.243}$$

Let us focus now on the last equation, which is the most important one. The falloff conditions (34.207) are obeyed if the first $\mathcal{O}(r^2)$ term vanishes. This requires that Y^z be a holomorphic vector field. Locally this is solved by $Y^z = z^n$ for any integer n. However, only the restricted choice $Y^z \sim 1, z, z^2$ leads to globally defined vector fields, the six global conformal Killing vector fields of the sphere. BMS discarded all the rest on the grounds that equation (34.243) is violated at isolated points in the more general meromorphic case. For example, if $Y^z = \frac{1}{z-w}$, then

$$\partial_{\bar{z}}Y^z = 2\pi\delta^2(z-w) \neq 0, \tag{34.244}$$

and the falloff condition is violated at $z = w$.[213] The Lie bracket algebra of $Y^z = z^n$ for any n is the centerless Virasoro algebra.

Interestingly, exactly the same equation (34.243) was encountered some 20 years later by Belavin, Polyakov, and Zamolodchikov [221] in analyzing the symmetries of two-dimensional CFTs. However Belavin et al. chose to allow Y^z to be any meromorphic function on the sphere, allowing for analytic singularities. This was a very good idea. While every practitioner of two-dimensional CFT will agree that this is the right thing to do, different explanations are given to support it. One is that the symmetries can be analyzed in a local patch, and singularities outside the patch do not affect the local identities obtained with the symmetry. Another is that the Ward identities of these symmetries provide quick derivations of relations among correlators that can be independently derived using other, more tedious methods. This justification readily adapts to four-dimensional quantum gravity where, as shown in the next subsection, the Ward identities are equivalent to the new subleading soft theorem [56, 213, 214], as well as to the superrotation conservation laws of the previous subsection.

In any case, since allowing meromorphic vector fields was such a good idea in two-dimensional CFT, perhaps we should go beyond the six global vector fields that were retained by BMS in their original analysis and instead consider the whole infinite-dimensional local conformal group generated by $Y^z = z^n$ for all n. This is a tantalizing conjecture, because we have learned an incredible amount about two-dimensional CFTs simply by unraveling the powerful implications of the local conformal = Virasoro group. One may hope for similarly powerful insights into four-dimensional quantum gravity. Perhaps we have even gotten our foot in the door of the long-sought holographic description of quantum gravity in Minkowski space, in which bulk quantum gravity in Minkowski space is dual to an exotic CFT on $\mathcal{CS}^2$. The conjectured superrotation symmetry will be verified at linearized and tree level in section 34.5.3.

Canonical Formalism

It is natural to ask whether superrotation charges generate superrotation symmetries in a canonical formalism. We will see that the answer to this question is yes, but only at linearized order. As discussed at the end of this subsection, singularities prevent the exponentiation of the infinitesimal transformations.

[213] This violation can be physically interpreted as due to cosmic strings [220].

As with supertranslations, we begin by investigating how the boundary data specifying the geometry change under superrotations; see Barnich and Troessaert [203] for further details. The Lie derivative with respect to Y (denoted δ_Y) of the C_{zz} component of the metric is given by

$$\delta_Y C_{zz} = \frac{u}{2} D \cdot Y N_{zz} + Y \cdot D C_{zz} - \frac{1}{2} D \cdot Y C_{zz} + 2 D_z Y^z C_{zz} - u D_z^3 Y^z. \tag{34.245}$$

Taking the u derivative yields

$$\delta_Y N_{zz} = \frac{u}{2} D \cdot Y \partial_u N_{zz} + Y \cdot D N_{zz} + 2 D_z Y^z N_{zz} - D_z^3 Y^z. \tag{34.246}$$

Note that if we sit at $u = 0$, then the last three terms in (34.246) are exactly the infinitesimal transformation law of a stress tensor in a two-dimensional CFT (i.e., the linearization of the Schwarzian derivative).

Now consider the conserved superrotation charge

$$Q_Y^+ = \frac{1}{8\pi G} \int_{\mathcal{I}_-^+} d^2 z [Y_{\bar{z}} N_z + Y_z N_{\bar{z}}] \ . \tag{34.247}$$

Integrating by parts and using the constraints (34.232) gives

$$\begin{aligned} Q_Y^+ &= Q_H^+ + Q_S^+, \\ Q_S^+ &= -\frac{1}{16\pi G} \int_{\mathcal{I}^+} du d^2 z [D_z^3 Y^z u N^z{}_{\bar{z}} + D_{\bar{z}}^3 Y^{\bar{z}} u N^{\bar{z}}{}_z], \\ Q_H^+ &= \frac{1}{8\pi G} \int_{\mathcal{I}^+} du d^2 z \left(Y_{\bar{z}} T_{uz} + Y_z T_{u\bar{z}} + u \partial_z Y_{\bar{z}} T_{uu} + u \partial_{\bar{z}} Y_z T_{uu} \right), \end{aligned} \tag{34.248}$$

where the graviton stress-energy component T_{uz} was defined in (34.233). The soft charges are linear in the metric flutuation C_{zz}, while the hard charge is quadratic. To compute commutators, we need [222, 223, 224, 49]

$$[N_{\bar{z}\bar{z}}(u, z, \bar{z}), C_{ww}(u', w, \bar{w})] = 16\pi G i \gamma_{z\bar{z}} \delta^2(z - w) \delta(u - u'). \tag{34.249}$$

It follows that

$$\left[Q_S^+, C_{zz}\right] = -i u D_z^3 Y^z, \tag{34.250}$$

$$\left[Q_H^+, C_{zz}\right] = \frac{iu}{2} D \cdot Y N_{zz} + i Y \cdot D C_{zz} - \frac{i}{2} D \cdot Y C_{zz} + 2 i D_z Y^z C_{zz}. \tag{34.251}$$

Putting this together, we conclude that

$$\left[Q_Y^+, ...\right] = i \delta_Y. \tag{34.252}$$

The conserved charge generates the symmetry as expected.

However, something is not quite right. Commutators with Q_Y^+ shift the news by a function that goes to a constant at $\mathcal{I}_\pm^+$, while C_{zz} diverges linearly. Such behavior violates the boundary conditions we started off with and hence maps points in the initially defined phase space to points outside that phase space. To make good sense of this behavior, we need to define a larger phase space. This is related to the observation [220] that singularities produced by superrotations correspond to cosmic strings, which destroy asymptotic flatness. Hence, to fully understand superrotations, we will at a minimum need to consider a larger phase space that allows for such defects. This is a topic for future investigations.

Subleading Soft Theorem

Given the infinities of conserved superrotation charges and symmetries, one expects an associated soft theorem. There is of course Weinberg's soft graviton theorem, but we have already seen that this is equivalent to supertranslations. Invoking the IR triangle (Section 34.1.1) leads to the conclusion that there must be a second soft theorem in gravity. Indeed, motivated by this observation, a new subleading soft theorem was proven diagrammatically [56, 63, 64, 65] and shown [57, 66] to be equivalent to the superrotation charge conservation equation (34.235). This logic can be turned around and the subleading soft theorem used to prove (34.235).

At the level of the quantum $\mathcal{S}$-matrix, superrotation charge conservation states

$$\langle \text{out}| \left(Q_Y^+ \mathcal{S} - \mathcal{S} Q_Y^-\right) |\text{in}\rangle = 0. \tag{34.253}$$

From the decomposition (34.248) it is clear that this identity equates a soft graviton insertion to a hard term. Equation (34.253) is most directly expressed in terms of the $(z_k, \bar{z}_k)$ coordinates at which asymptotic particles pierce $\mathcal{CS}^2$. After quite a lengthy calculation [57] using relations such as (34.87) and setting $Y^z = \frac{1}{z-(q^1+iq^2/q^0+q^3)}$, one finds this equation can be reexpressed in momentum space as

$$\begin{aligned} \lim_{\omega\to 0} (1+\omega\partial_\omega) \langle p_{n+1}, p_{n+2}, ...| \, a_-(q)\mathcal{S} \, |p_1, p_2, ...\rangle = \\ \sqrt{8\pi G} S^{(1)-} \langle p_{n+1}, p_{n+2}, ...| \, \mathcal{S} \, |p_1, p_2, ...\rangle \, , \end{aligned} \tag{34.254}$$

where $a_-(q)$ is the annihilation operator for a negative helicity graviton of four-momentum $q = \omega(1, \hat{q})$, and the subleading soft factor is

$$S^{(1)-} = -i \sum_k \frac{p_{k\mu} \varepsilon^{-\mu\nu} q^\lambda J_{k\lambda\nu}}{p_k \cdot q}, \quad J_{k\mu\nu} \equiv L_{k\mu\nu} + S_{k\mu\nu}, \tag{34.255}$$

with $L_{k\mu\nu}$ the orbital angular momentum and $S_{k\mu\nu}$ the helicity of the internal spin of the kth particle. In reference [56] it was directly shown, through some diagrammatics far more tedious than Weinberg's, that this subleading expression is valid at tree level. This confirms the classical existence of an infinite number of conserved superrotation charges.

Comments about this soft relation are now in order. Recall from our discussion of supertranslations that the addition of a soft graviton to a scattering amplitude produces a Weinberg pole proportional to $1/\omega$. But here, the prefactor $(1+\omega\partial_\omega)$ projects out this pole, eliminates the leading divergence and leaves the finite, subleading $\mathcal{O}(1)$ piece (note the factors of q in both the numerator and denominator of (34.255)). We see there is a universal form not just for the leading but also for the subleading part of the amplitude in the soft limit. Although we have not discussed it, a similar (and much older) story exists for QED: there is also a subleading soft theorem discovered in 1958 by Low [4]. Following the triangle, it is also related to a new symmetry in gauge theory, as discussed in [60, 61].

The subleading soft factor $S^{(1)-}$ can be obtained from the leading soft factor by replacing p_ν with $q^\mu J_{k\mu\nu}$, in other words, by replacing translations with rotations about q. This makes sense, because the first term in the hard piece Q_H^+ is $Y^z T_{uz}$, which generates a rotation around the axis through which the particle is emerging, whereas the hard part of the supertranslation-generating charge involves fT_{uu}. This relation can be made more precise by looking at details of geodesics with nonzero angular momentum, for which it can be shown that the replacement $fT_{uu} \to Y^z T_{uz}$ is akin to the replacement $p_\nu \to q^\mu J_{k\mu\nu}$, with q the direction around which the rotation is performed. Indeed, this is how the soft formula was first guessed before it was properly derived.

While I do not reproduce the diagrammatic proof of the subleading soft graviton theorem here, I present a simple but important consistency check. Recall that the Weinberg soft graviton theorem involves the leading soft factor

$$\sum_k \frac{\varepsilon^{\mu\nu} p_{k\mu} p_{k\nu}}{p_k \cdot q}, \tag{34.256}$$

which for consistency had better vanish for pure gauge gravitons, that is, when

$$\varepsilon^{\mu\nu} = \Lambda^\mu q^\nu. \tag{34.257}$$

Indeed, with this choice, the soft factor (34.256) becomes

$$\Lambda^\mu \sum_k \frac{p_{k\mu} p_k \cdot q}{p_k \cdot q} = \Lambda^\mu \sum_k p_{k\mu} = 0, \tag{34.258}$$

where the last equality follows from energy-momentum conservation.

Consistency demands that a similar property hold for our new expression for $S^{(1)}(\epsilon_\Lambda)$. Since it is not automatically symmetric, let us now insert

$$\varepsilon_\Lambda^{\mu\nu} = q^\mu \Lambda^\nu + q^\nu \Lambda^\mu \tag{34.259}$$

and hence find that

$$iS^{(1)}(\varepsilon_\Lambda) = q^\mu \Lambda^\nu \sum_k J_{k\mu\nu} + \sum_k \frac{p_k \cdot \Lambda q^\mu q^\nu J_{k\mu\nu}}{p_k \cdot q} = 0, \tag{34.260}$$

as required. In the last step, the second term vanishes just because $J_{k\mu\nu}$ is antisymmetric while $q^\mu q^\nu$ is symmetric. Meanwhile, the first term vanishes by angular momentum conservation. We again see that angular momentum is to superrotations as energy-momentum is to supertranslations.

A powerful reformulation of superrotational invariance has recently been achieved in the celestial language of section 34.3. References [67, 68] construct a four-dimensional soft operator whose amplitudes are precisely those of a two-dimensional CFT stress tensor.

34.6 The Memory Effect

In this section, we describe the third corner of the IR triangle, the memory effect. In the case of gravity, the memory effect characterizes a pair of inertial detectors stationed near $\mathcal{I}^+$ in a region with no Bondi news at both late and early times. At intermediate times, gravity waves may pass through, causing oscillating distortions in their relative separations, which we denote $(s^z, s^{\bar{z}})$ and depict in figure 34.18. The equation of geodesic deviation implies (see exercise 13)

$$r^2 \gamma_{z\bar{z}} \partial_u^2 s^{\bar{z}} = -R_{uzuz} s^z \ , \tag{34.261}$$

where

$$R_{uzuz} = -\frac{r}{2} \partial_u^2 C_{zz} \ . \tag{34.262}$$

Integrating this equation reveals a DC effect. Namely, the initial and final separations differ by (in retarded coordinates)

$$\Delta s^{\bar{z}} = \frac{\gamma^{z\bar{z}}}{2r} \Delta C_{zz} s^z \ . \tag{34.263}$$

This is the gravitational memory effect [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33]. The difference ΔC_{zz} between initial and final transverse metric components need not vanish, as flatness does not require $C_{zz} = 0$. Proposals to measure the gravitational memory effect with a variety of methods are given in Lasky et al. and van Haasteren and Levin [34, 35].

The effect is harder to see than gravity waves themselves but has a decent chance of being measured in the coming decades. The connection of the memory effect with basic symmetries of gravity makes this an exciting prospect.

A quick way [41] to see the equivalence of gravitational memory to the soft graviton theorem is to compare formulas in the original literature. An excerpt with the central formula for the outgoing metric fluctuation sourced by elementary particle collisions from Weinberg [7] is

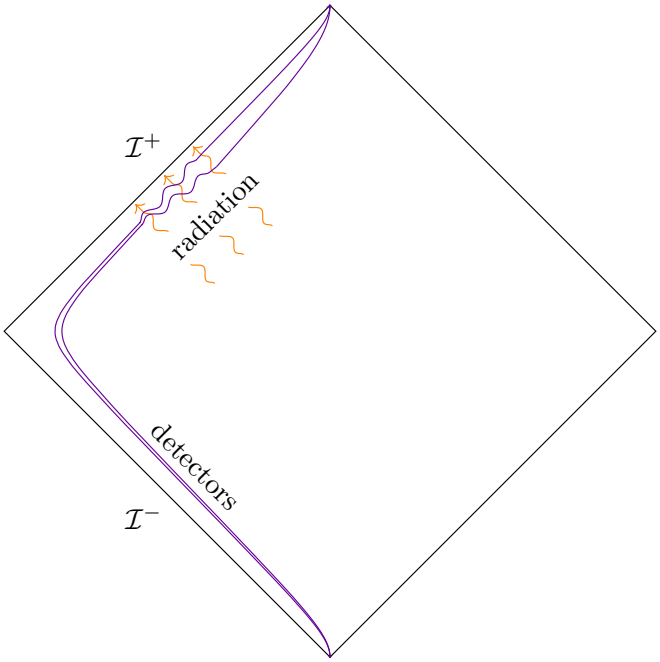


Figure 34.18: **The memory effect. The passage of gravitational radiation past a pair of inertial detectors stationed near $\mathcal{I}^+$ causes a temporary oscillation in their relative positions followed by a permanent displacement which "remembers" certain moments of the energy flux.**

The dominance of the $1/(p\cdot q)$ pole in (2.5) implies that the effect of attaching one soft-graviton line to an arbitrary diagram is to supply a factor equal to the sum of (2.5) over all external lines in the diagram

$$(8\pi G)^{1/2}\sum_n \eta_n p_n{}^\mu p_n{}^\nu/[p_n\cdot q - i\eta_n\epsilon]. \qquad (2.7)$$

It is interesting to compare this to the central equation for the DC shift in the outgoing metric sourced by black hole or neutron star collisons in Braginsky and Thorne [25]:

permanent change in the gravitational-wave field (the burst's memory) δh_{ij}^{TT} is equal to the 'transverse, traceless (TT) part'[36] of the time-independent, Coulomb-type, $1/r$ field of the final system minus that of the initial system. If $\mathbf{P}^A$ is the 4-momentum of mass A of the system and P_i^A is a spatial component of that 4-momentum in the rest frame of the distant observer, and if $\mathbf{k}$ is the past-directed null 4-vector from observer to source, then δh_{ij}^{TT} has the following form:

$$\delta h_{ij}^{TT} = \delta\left(\sum_A \frac{4P_i^A P_j^A}{\mathbf{k}\cdot\mathbf{P}^A}\right)^{TT} \qquad (1)$$

Here we use units with $G = c = 1$. In the observer's local Car-

A glance reveals that the two equations are indeed strikingly similar! In fact, to see that they are identical, not just similar, requires a conversion of notation and a Fourier transform, as follows [41]:

- Replace the four-momenta P_i^A of colliding stars or black holes with the four-momenta p_n^μ of elementary particles.
- Account for the different conventions for Newton's constant G and normalization.
- Substitute the graviton momentum q with its energy ω times the unit null vector k via $q = \omega k$.

- Act with a Fourier transform $\int dt e^{i\omega t}$ on the Weinberg momentum-space formula to obtain the Braginsky-Thorne formula for the difference between the initial and final transverse metric components.

The fact that identical formulas were independently derived, using wildly different methods, for both the collisions of black holes and the collisions of elementary particles is a testimony to the universality of the IR phenomena discussed in these lectures. At very long distances, astrophysical black holes and elementary particles are both effectively pointlike!

Universal phenomena are often related to symmetries. Since the memory effect is a Fourier transform of the soft theorem, and the soft theorem is the Ward identity of supertranslation symmetry, there should be a direct connection between memory and supertranslations that does not involve soft theorems. To derive this connection, note that if there is no energy flux or retarded time dependence of the asymptotic data at late and early times, C_{zz} must vanish up to a supertranslation:

$$C_{zz}^{\text{early}} = -2D_z^2 C^{\text{early}}, \qquad C_{zz}^{\text{late}} = -2D_z^2 C^{\text{late}} \ , \tag{34.264}$$

where $\partial_u C^{\text{early}} = \partial_u C^{\text{late}} = 0$. The early and late geometries are related by a supertranslation

$$\Delta C = C^{\text{late}} - C^{\text{early}} \ . \tag{34.265}$$

Hence we can think of a pulse of radiation passing through $\mathcal{I}^+$ as a domain wall separating diffeomorphic but BMS-inequivalent vacua. Using the constraint equation (34.216), we find

$$D_z^2 \Delta C^{zz} = 2\Delta m_B + 2\int du T_{uu} \ , \tag{34.266}$$

where the u-integral extends from the early to the late regions. Solving this differential equation, it follows that

$$\Delta C(z,\bar{z}) = -\int d^2w \gamma_{w\bar{w}} G(z,\bar{z};w,\bar{w}) \left(\int du T_{uu}(w,\bar{w}) + \Delta m_B \right) \ , \tag{34.267}$$

where the Green's function is

$$G(z,\bar{z};z',\bar{z}') = \frac{1}{\pi} \sin^2 \frac{\Delta\Theta}{2} \log \sin^2 \frac{\Delta\Theta}{2} \ , \tag{34.268}$$

with $\Delta\Theta$ the angle on $\mathcal{CS}^2$ between $(z,\bar{z})$ and $(z',\bar{z}')$. This is an explicit formula for the supertranslation induced by waves (gravity or otherwise) passing through $\mathcal{I}^+$. One of its unusual characteristics is that it is highly nonlocal on $\mathcal{CS}^2$. Indeed, if a gravity wave passes through the north pole, the effect vanishes both there and at the south pole but is large near the equator.

The relative displacements (34.263) of inertial detectors have a simple explanation in this framework. Imagine tiling $\mathcal{CS}^2$ with initially evenly spaced inertial detectors. Passage of a gravity wave will adjust the relative $(z,\bar{z})$ positions by amounts of order $\frac{1}{r}$, defining a diffeomorphism on $\mathcal{CS}^2$, as depicted in figure 34.19. This was shown [41] to be exactly the angular diffeomorphism appearing in the subleading term of the supertranslation vector field (34.211). It would be exciting indeed to observe these supertranslations in the sky!

The gravitational memory effect imparts a physical meaning to the soft graviton theorem. Soft gravitons may seem a bit unphysical, because it takes longer and longer to measure them as the energy goes to zero. Surprisingly, despite this, the memory effect can be measured in a finite time, because the Fourier transform of the Weinberg pole is a step function in retarded time.

This completes the third leg of the triangle containing Weinberg's soft graviton theorem and asymptotic BMS symmetries. Memory is connected to the soft theorem by a Fourier transform and measures transitions between BMS-inequivalent vacua. There are other kinds of memory effects associated with other triangles. Taking the Fourier transform of the subleading soft theorem led to a new gravitational memory effect called spin memory [69], which gives relative time delays to

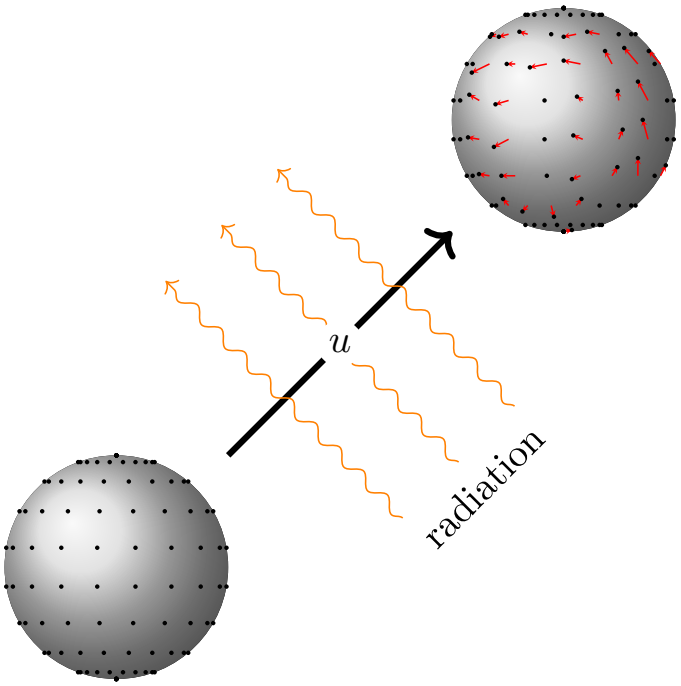


Figure 34.19: **An array of evenly spaced inertial detectors (black dots) on the sphere near $\mathcal{I}^+$ will be permanently displaced (red arrows) by the passage of gravitational radiation. The displacements are a measurement of the BMS diffeomorphism, which relates the vacua before and after the passage of the radiation.**

counterorbiting objects and might be measurable [70]. In (unconfined and unhiggsed) nonabelian gauge theory, the color memory effect rotates the relative colors of nearby quarks [40]. If a pulse of gluons passes a pair of initially singlet quarks, it will generically no longer be in a singlet. In abelian gauge theories, such as QED, the electromagnetic memory effect gives relative phases to adjacent charged particles, which can be measured by quantum interference or other experiments, as recently discussed in the literature [37, 201, 38, 39].

34.7 Black Holes

So far, black holes have barely been mentioned. It turns out that the IR triangle is highly relevant to the behavior of black holes and to the famous information paradox [161]. While many of the implications of the IR symmetries for black hole physics are topics of current investigation and remain to be fully understood, others follow straightforwardly from the groundwork laid in these lectures. Some of the latter already well understood material is described in this section, with occasional pointers to open problems.

We begin in section 34.7.1 with a lightning review of the information paradox. In section 34.7.2, it is shown that the infinity of conservation laws requires that black holes must carry corresponding "soft hair", invalidating Hawking's argument [161] that information is destroyed. In section 34.7.2, the properties of soft hair are perturbatively described at the classical level. Section 34.7.2 constructs the horizon contribution to the linearized supertranslation charge and shows that this charge generates horizon supertranslations. Quantum aspects of soft hair and its relation to soft gravitons on the horizon are discussed in section 34.7.2. The electromagnetic version involving soft photons is in section 34.7.2. Future outlook and the status of the information paradox are discussed in section 34.7.3.

34.7.1 The Information Paradox

There are many ways to characterize the information paradox, but perhaps the most striking is in the formation and total evaporation of a black hole in an asymptotically flat space. We begin with the process of black hole formation, which is classically described by the Penrose diagram in figure 34.20. One could consider a more generic collapsing geometry, but it is useful to be specific.

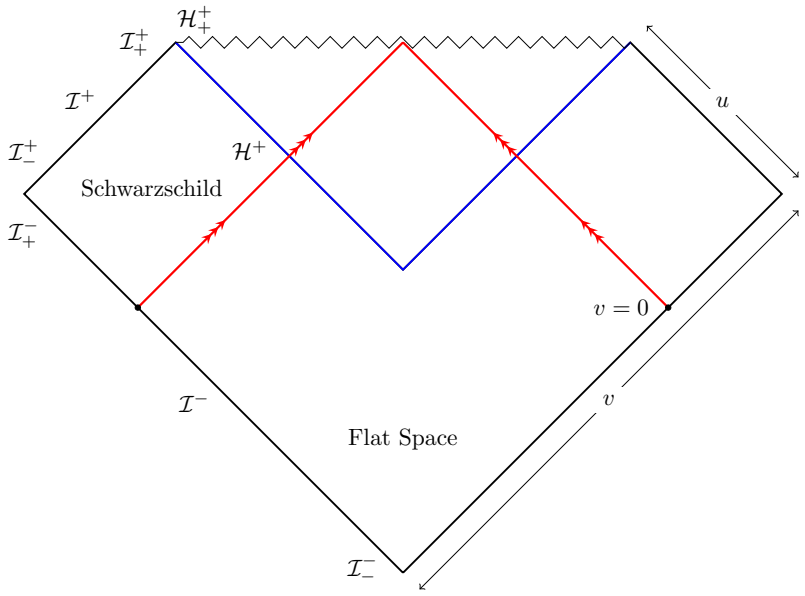


Figure 34.20: **Vaidya geometry. The red lines denote a null spherical shockwave imploding at advanced time $v = 0$. When the shockwave crosses its Schwarzschild radius, the (blue) event horizon forms. The causal future of the event horizon is the black hole and ends at the jagged singularity.**

In this picture, the far past is empty flat space. Then at advanced time $v = 0$, a null shockwave with total energy M is sent in. This infinitely thin collapsing shell of matter eventually forms a Schwarzschild black hole of mass M. The resulting spacetime geometry can be described by the Vaidya metric, which in advanced coordinates takes the form

$$ds^2 = -\left(1 - \frac{2MG\theta(v)}{r}\right) dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \ , \tag{34.269}$$

where θ denotes the step function

$$\theta(v) = \begin{cases} 0 & \text{if } v < 0 \ , \\ 1 & \text{if } v \geq 0 \ . \end{cases} \tag{34.270}$$

This metric evidently describes flat space before the passing of the shockwave and a Schwarzschild black hole after. The two regions are glued along the shockwave of infalling matter, whose collapse results in the creation of an event horizon, that is, a region from which light rays cannot escape to infinity (depicted in blue). The spacelike singularity is hidden behind the horizon.

More generally, the collapsing matter may have arbitrary multipole moments and may not be spherically symmetric. However, according to the usual interpretation (which we question below) of the no-hair theorem [225], regardless of how it is formed, after a rather short time scale (roughly speaking, the light-crossing time of the black hole), it will settle down to a stationary state that is characterized by only ten conserved Poincaré charges: the energy-momentum $(E, \vec{P})$, the total

angular momentum $\vec{J}$ and the boost charge or BORT center of mass $\vec{K}$. The standard statement is that a black hole in Einstein gravity has ten hairs, or equivalently, the phase space is ten-dimensional.[214] Of course, in the presence of gauge fields, there may be more, such as the total electric charge, but we ignore these for the moment.

Quantum mechanically, the situation differs radically. If we wait a long time, the black hole will eventually evaporate [226]. This leads to the qualitatively different picture in figure 34.21. As

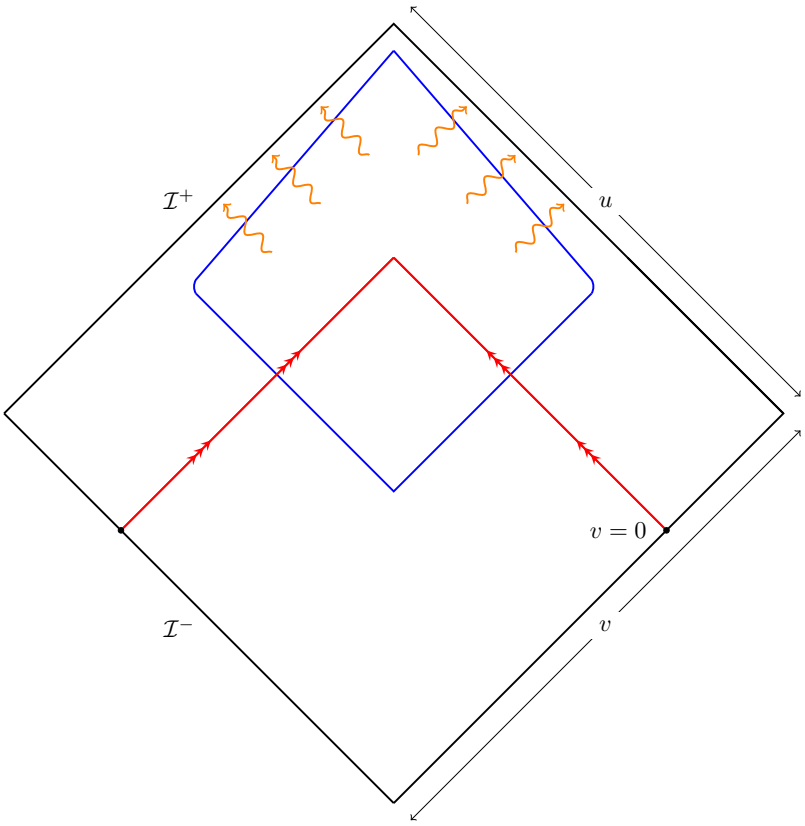


Figure 34.21: **Evaporating black hole. The red collapsing shell makes a black hole, which at early times, looks just like the classical black hole of figure 34.20. However, at late times the black hole evaporates. This process causes the horizon, depicted in blue, to become timelike, and the black hole eventually disappears. In the far future the system reverts to a vacuum state.**

before, we first throw in a shockwave to create the black hole. Then, after a very long time, the black hole begins to evaporate. Since classical black holes do not evaporate, this must take a period of time that goes to infinity as $\hbar \to 0$. As long as $\hbar > 0$, however, the black hole eventually starts to Hawking evaporate, becoming smaller and smaller until (assuming no remnants) it ultimately disappears altogether, as illustrated at the top of the diagram.

Hawking gave a semiclassical derivation of the spectrum of radiated particles in his famous 1975 paper [226]. He found a blackbody spectrum with a temperature given by the Hawking temperature,

$$T_H = \frac{\hbar}{8\pi G M} \ . \tag{34.271}$$

[214] Of these, only two — the magnitude of the intrinsic spin J and mass $M = \sqrt{E^2 - \vec{P}^2}$ — are invariant under Poincaré transformations.

Moreover, he argued that the outgoing Hawking radiation does not carry any information about what formed the black hole. The radiation results from a pair production process outside the horizon, in which a negative energy particle goes into the black hole and is correlated with a positive energy particle going out to infinity. This process is insensitive to all details concerning the interior of the black hole. The no-hair theorem is invoked in this argument to conclude that the exterior contains no information about the formation process. We could have made the black hole of red, green, or blue matter, but since the final geometry is the same in all cases, so must be the outgoing Hawking radiation. Finally, the black hole is presumed to eventually decay to the vacuum, traditionally assumed to be unique (also questioned below) and incapable of storing information. Hence, Hawking argued [161], all information falling into the black hole has no place to go and is destroyed in the formation/evaporation process.

Both in Hawking's argument and in figure 34.21, a divide is placed between the periods of black hole formation and evaporation. It is assumed that quantum effects are negligible during formation and that during evaporation the black hole can be approximated as stationary. These assumptions are justified if the mass M is large enough in Planck units. Ignoring constants, the Hawking temperature (34.271) varies like

$$T \sim \frac{\hbar}{GM} \ . \tag{34.272}$$

Thus, $T \to 0$ as $\hbar \to 0$ or as the black hole gets bigger. The Stefan-Boltzmann law then implies that the power radiated by the black hole varies as

$$P \sim \frac{AT^4}{\hbar^3} \sim \frac{\hbar}{G^2M^2} \ , \tag{34.273}$$

where A is the area of the horizon. It follows that the rate of mass loss also obeys

$$\frac{dM}{dt} \sim \frac{\hbar}{G^2M^2} \ . \tag{34.274}$$

The typical time for the black hole to lose a small fraction, say 1%, of its total mass is

$$t_{\rm typ} \sim .01\frac{G^2M^3}{\hbar} \ . \tag{34.275}$$

So we have to wait a long time of order $G^2M^3/\hbar$ to see the geometry change appreciably due to Hawking emission. Any classical transient effects that occur during the matter shell infall, such as those associated with emission of radiation to $\mathcal{I}^+$, will all clear out and settle down in timescales that cannot involve $\hbar$. Hence we have a clean division between the classical black hole formation phase and the onset of Hawking evaporation. The argument for information loss [161] applies only when M is large and such a clean division can be made.

We now revisit this argument in light of the new insights into the IR structure of gravity.

34.7.2 Soft Hair

In the standard analysis, it is *not* claimed that the Hawking radiation is *completely* uncorrelated with the quantum state of the black hole. For example, even in the semiclassical regime, it has always been presumed that the total mass is exactly conserved. If the black hole emits a quantum of Hawking radiation of energy ΔE, then it must necessarily decrease its mass by an amount exactly equal to ΔE. This holds at the level of quantum states and has never been thought to be a probabilistic statement. Instead there should exist an exact correlation for all conserved quantum numbers (such as the energy) of the black hole and the conserved quantum numbers of the outgoing radiation. This correlation is not manifest in the leading-order Hawking calculation. It must be enforced by hand when computing the backreaction of the outgoing Hawking quanta

on the geometry. The result will be exact correlations between the quantum states of the outgoing Hawking radiation and of the black hole. Not much attention has been paid to these correlations, even if they do modify the thermal spectrum and imply that some information is carried out by the radiation. The reason is they are too few in number — one for each of the ten black hole hairs — to rescue the macroscopic quantities of lost information.

This story is, however, significantly modified [41, 160, 48] by the infinite number of conserved charges discussed in these lectures. First, it is a common misconception, originating in a misinterpretation of the no-hair theorem, that black holes (in the absence of gauge fields) have only ten hairs, corresponding to the ten conserved Poincaré charges. The no-hair theorem states that all stationary solutions are diffeomorphic to Kerr spacetimes. While this statement is mathematically correct, these diffeomorphisms, if they are within BMS, need not act trivially on the black hole spacetime. Generically they map one black hole spacetime to a second, physically inequivalent one. A simple example is a boost that changes both the energy and momentum and so clearly changes the quantum state. However, we have learned that there is no canonically preferred "boost" element of the asymptotic BMS symmetry group. For the same reason that two black hole spacetimes that differ by a boost are physically inequivalent, two black hole spacetimes which differ by any BMS transformation (except Killing time translations) are also physically inequivalent. This simple observation implies that there are not merely ten, but an infinite[215] number of "soft hairs" [160] on classical black holes. We will see below that superrotation charges serve as a diagnostic distinguishing the different black hole "hairdos" and that a supertranslated Schwarzschild black hole generically carries nonzero angular momentum.

The presence of a lush head of soft hair on black holes is required for supertranslation and superrotation charge conservation. Since the conservation laws follow from the long-distance behavior of fields near spatial infinity, they should not be affected by the presence of black holes — classical or quantum. The conserved charges can be expressed as bulk integrals over any Cauchy surface. In the presence of a classical black hole, $\mathcal{I}^+$ is no longer a Cauchy surface. One must include a contribution from the future event horizon $\mathcal{H}^+$, which is the black hole contribution to the conserved charge. In the quantum theory, black holes evaporate, but one may still choose to consider a Cauchy surface — such as the green one in figure 34.22 — which crosses the spacetime before the black hole has disappeared. The integrand on such a surface will contain a contribution both from the black hole and from $\mathcal{I}^+$.

A generic massless Hawking quantum from an evaporating black hole will carry nonzero amounts of all the supertranslation, superrotation, and (if it is charged) large-gauge charges across $\mathcal{I}^+$. Exact conservation of these charges requires that the black hole decrease its supertranslation, superrotation, and large-gauge charges by an exactly compensating amount. This enforces an infinite number of correlations between the quantum state of the black hole and the quantum state of the outgoing Hawking radiation. A mere ten constraints from Poincaré charges forcing correlations cannot have a significant effect on how macroscopic quantities of information flow in and out of the black hole; an infinite number of constraints gives a very different story. It potentially significantly modifies the information flow, or equivalently, the entanglement, between the portions of the bulk quantum state on $\mathcal{I}^+$ and on the horizon. Motivated by this observation, in the next few subsections we begin a detailed perturbative analysis of soft hair. We note that it has been suggested that soft hair is an alternate description of the edge modes discussed in [227, 228, 229, 230, 231].

Classical Hair

The first and simplest thing to understand about soft hair, and the subject of this section, is classical, linearized supertranslation hair on a Schwarzschild black hole.

Our first problem is to extend the action of supertranslations from a neighborhood of $\mathcal{I}$ to the entire spacetime. The extension of an asymptotic gauge symmetry into the interior is gauge dependent. In a general time-dependent situation, there is unlikely to be a canonical choice of gauge. Quantum fluctuations further diminish the utility of specific choices. For quantum gravity

[215] In the quantum theory, it may not be sensible to discuss hairs localized to a region smaller than the Planck length, which would render the number of hairs finite[160].

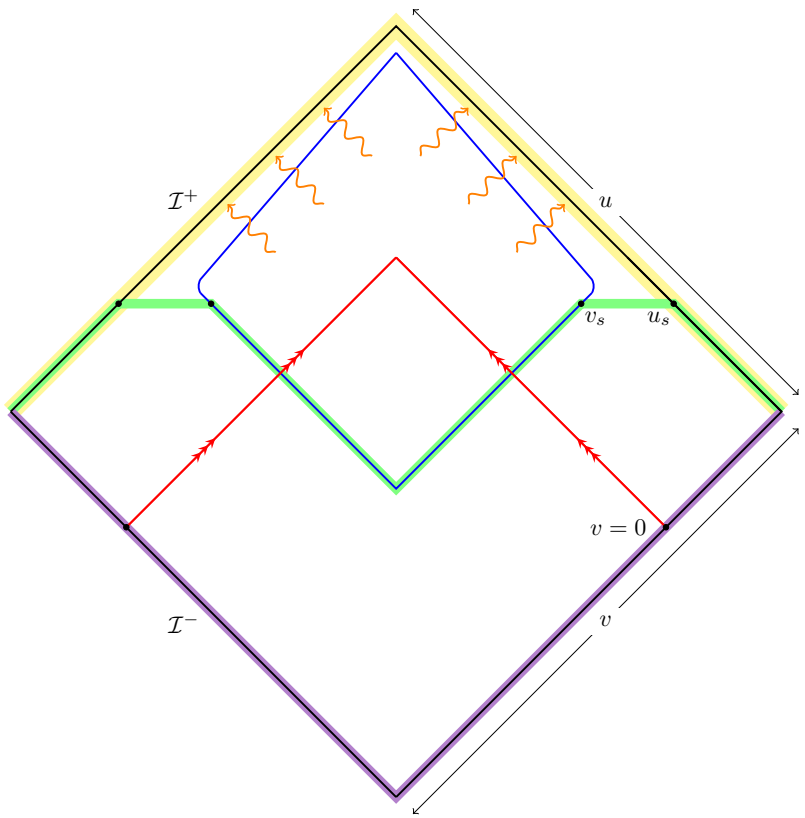


Figure 34.22: **An infinite number of conserved charges can be defined as surface integrals at $\mathcal{I}^+_-$ or $\mathcal{I}^-_+$. Integrating by parts and using the constraints, they can be written as an outgoing integral over $\mathcal{I}^+$ (yellow surface), an incoming integral over $\mathcal{I}^-$ (purple surface), or an intermediate time integral over the green surface that includes a black hole contribution. The equality of these three expressions is possible only if the black hole itself can carry an infinite number of charges, or soft hair.**

in asymptotically flat spacetimes, we expect the only fully well-defined observables are supported at the boundary at infinity. It is nevertheless sometimes possible, armed with a gauge choice, to define interior quantities, such as local gravitational energy densities, at first nontrivial order in perturbation theory around the Schwarzschild solution. This can be useful in developing a picture and intuition for the behavior of the spacetime away from its boundary. For example, one may show at leading order in perturbation theory that, at both the classical and quantum levels, the total energy comprised of linearized perturbations plus the mass of the black hole itself is conserved. Moreover, this perturbative conservation law is the linearization of an exact, nonperturbative conservation law, which can only be exactly phrased in terms of asymptotic quantities. It is in this spirit that we study the linearized action of supertranslations in the Schwarzschild solution.

Supertranslated Schwarzschild Solution. In advanced Bondi coordinates, the Schwarzschild metric is

$$ds^2 = -V dv^2 + 2dvdr + r^2 \gamma_{AB} d\Theta^A d\Theta^B \ , \quad V \equiv 1 - \frac{2m_B}{r} \ , \tag{34.276}$$

where $m_B = GM$. It is not hard to show that, for the Schwarzschild geometry, the supertranslation vector field,

$$\zeta_f = f\partial_v + \frac{1}{r} D^A f \partial_A - \frac{1}{2} D^2 f \partial_r \ , \quad f = f(z, \bar{z}) \ , \tag{34.277}$$

preserves Bondi gauge for all r, not just to leading order. Taking the Lie derivative, we find

$$\begin{aligned} ds^2 + \mathcal{L}_f ds^2 &= -\left(V - \frac{m_B D^2 f}{r^2}\right) dv^2 + 2dvdr - dvd\Theta^A D_A(2Vf + D^2 f) \\ &\quad + (r^2 \gamma_{AB} + 2r D_A D_B f - r\gamma_{AB} D^2 f) d\Theta^A d\Theta^B \ . \end{aligned} \tag{34.278}$$

The event horizon is at $r = 2m_B + \frac{1}{2}D^2 f$. This geometry describes a black hole with linearized supertranslation hair. Horizon supertranslations have been studied in other gauges [232, 233, 234].

Superrotation Charges. Supertranslating a black hole does not add supertranslation charges to the black hole, just as an ordinary translation of a black hole does not add momentum. This result follows from the fact that the supertranslation group is abelian and may also be seen at linear order directly, because m_B is unshifted in (34.278). However, as supertranslations and superrotations do not commute, a supertranslated black hole can and does carry superrotation charges, already at the classical level. In (34.235), we derived the conserved superrotation charges

$$Q_Y^- = \frac{1}{8\pi G} \int_{\mathcal{I}_+^-} d^2\Theta \sqrt{\gamma} Y^A N_A \ , \tag{34.279}$$

where Y^A is any smooth vector field on the sphere. We are interested in the differential superrotation charges carried by an infinitesimally supertranslated Schwarzschild black hole of the type considered in the previous paragraph. As seen from (34.278), under a supertranslation $\delta_f g_{\mu\nu} = \mathcal{L}_f g_{\mu\nu}$ of the Schwarzschild geometry,

$$\delta_f N_A = -3 m_B \partial_A f \ . \tag{34.280}$$

It follows immediately that [164, 48]

$$\hat{Q}_Y^-(g, h = \delta_f g) = -\frac{3}{8\pi} \int_{\mathcal{I}_+^-} d^2\Theta \sqrt{\gamma} Y^A M \partial_A f \ . \tag{34.281}$$

The hat here on $\hat{Q}_Y^-$ and on other quantities below are to remind the reader that we are doing a linearized analysis. Equation (34.281) is nonzero for a generic vector field Y^A and supertranslation f. An infinite number of superrotation charges can be independently added to the black hole by different choices of f. Hence superrotation charges classically distinguish differently supertranslated black holes. Classical black holes sport an infinite head of "supertranslation hair."

Hair implants. In the previous paragraphs we described a supertranslated eternal Schwarzschild black hole. To be certain such objects really exist, in this section we describe how one physically implants hair, or rearranges the hairdo, by throwing something into a Schwarzschild black hole. Similar conclusions were reached using the membrane paradigm [166, 169].

At advanced time v_0 in the Schwarzschild geometry, we send in a linearized shockwave with energy-momentum density

$$\hat{T}_{vv} = \frac{\mu + \hat{T}(\Theta)}{4\pi r^2} \delta(v - v_0) \tag{34.282}$$

near $\mathcal{I}^-$. Here μ is the total mass of the shockwave, and we take $\hat{T}$ to have only $\ell > 2$ components for simplicity. We wish to solve for the linearized metric in such a way that the solution is diffeomorphic to the Schwarzschild solution both before and after the shockwave. Stress-energy conservation, $\nabla_\mu \hat{T}^{\mu\nu} = 0$, then mandates subleading-in-$\frac{1}{r}$ corrections to the stress tensor for shockwaves that are not spherically symmetric. Defining

$$\hat{C}(\Theta) \equiv G \int d^2\Theta' G(\Theta, \Theta') \hat{T}(\Theta'), \tag{34.283}$$

where G is the Green's function (34.268) arising in the memory effect, the solution is

$$\hat{T}_{vv} = \frac{1}{4\pi r^2}\left[\mu + \frac{1}{4G}D^2\left(D^2+2\right)\hat{C} - \frac{3M}{2r}D^2\hat{C}\right]\delta\left(v-v_0\right) \ ,$$
$$\hat{T}_{vA} = -\frac{3M}{8\pi r^2}D_A\hat{C}\delta\left(v-v_0\right) \ . \tag{34.284}$$

Solving the linearized Einstein equation with this shockwave source then gives (see exercise 15)

$$\begin{aligned} h_{vv} &= \theta(v-v_0)\left(\frac{2G\mu}{r} - \frac{m_B D^2\hat{C}}{r^2}\right) \ , \\ h_{AB} &= -2r\theta(v-v_0)\left(D_A D_B\hat{C} - \frac{1}{2}\gamma_{AB}D^2\hat{C}\right) \ , \\ h_{vA} &= \theta(v-v_0)\partial_A\left(1 - \frac{2m_B}{r} + \frac{1}{2}D^2\right)\hat{C} \ . \end{aligned} \tag{34.285}$$

Comparing with the formula (34.278) for a supertranslation of the Schwarzschild metric, we find that (34.285) can be written

$$h_{\mu\nu} = \theta(v-v_0)\left(\mathcal{L}_{f=-\hat{C}}g_{\mu\nu} + \frac{2G\mu}{r}\delta^v_\mu\delta^v_\nu\right) \ . \tag{34.286}$$

Hence the shockwave is a domain wall interpolating between two BMS-inequivalent Schwarzschild vacua, whose mass parameters differ by μ.

The shockwave induces a shift in the transverse components of the metric perturbation on the horizon. Integrating over a null generator of the horizon, we have

$$\Delta h_{AB}(r=2m_B,\Theta) = \int_{-\infty}^{\infty} dv\partial_v h_{AB}(r=2m_B,v,\Theta) = -4m_B\left(D_A D_B\hat{C}(\Theta) - \frac{1}{2}\gamma_{AB}D^2\hat{C}(\Theta)\right) \ . \tag{34.287}$$

Hence the shockwave induces a supertranslation on the Schwarzschild horizon, which is registered by a nonzero value for the zero mode of the transverse, traceless fluctuations of the horizon metric. In the physical phase space, two black hole spacetimes that differ by any element of BMS correspond to different points. For the case of boosts, the two spacetimes have different energies. For supertranslations, they are energetically degenerate but carry different angular momenta and other superrotation charges. An important difference between boosts and supertranslations is that, as discussed in section 34.5.2, the latter act nontrivially on all the zero-energy vacua as well, imparting angular momentum and superrotation charges at quadratic order [51]. Hence the phase space of asymptotically flat geometries with nonzero energy and four Killing vectors is not a simple product of vacuum and black hole phase spaces.[216]

The formula (34.281) of superrotation charges requires only the asymptotic behavior of the black hole and hence pertains to any geometry with nonzero mass. We would like to understand how a classical black hole compares in this regard to a configuration with the same mass and many internal states, such as a star, or an elementary particle with few or no internal states. Suppose we send the supertranslating shockwave into a star or a collection of stars instead of into a black hole. The wave will excite and rearrange the interior structure of the star and, in the case of multiple stars, their relative motions. Generically gravitational radiation will carry some, but not all, of the superrotation charge back out to infinity, while some will be retained by the star(s). It is unsurprising that a star or a collection of stars (which has many internal degrees of freedom

[216] We consider here supertranslations that act in unison on the horizon and null infinity. As discussed at the end of this subsection, one may try to consider separate actions, but we do not do so here.

and possible interior states) can carry many superrotation charges. There is no no-hair theorem for a star! Now consider instead replacing the black hole by a massive, stable, "bald" elementary particle with no internal degrees of freedom. Such an object cannot carry arbitrary superrotation charges: the pre- and post-superrotation charges are generically the same (except for the $l = 1$ component). To leading order, the supertranslating shockwave will simply be reflected through the origin and scatter back up to future null infinity. The elementary particle has no mechanism to absorb all the superrotation charges. The outgoing wave will cancel the superrotation charges induced by the ingoing wave and, in the far future, the superrotation charges will revert to their initial incoming values.

So we see that, in its ability to absorb superrotation charges, a black hole acts more like a complex "hairy" star with many internal degrees of freedom than a massive "bald" elementary particle. The observer at infinity can confirm this by sending in shockwaves and watching what comes out. This is a classical signal that black holes are not bald.

However, there are also intriguing differences, yet to be fully understood, between the supertranslation hair on a black hole as described above and the internal states of a star. Bondi gauge supertranslations change the metric of the spacetime everywhere, including both at the event horizon and at infinity, while internal states of a star can be changed with possibly[217] no effect at infinity. It is interesting to ask whether horizon supertranslations can be untethered from supertranslations of null infinity. There has been discussion of this point [128, 166, 48, 170, 173], but as it remains unsettled, I do not go into detail here. A related point is that we do not know how to describe supertranslation hair (one proposal is edge modes [227, 228, 229]) on a classical stationary black hole in AdS, whose asymptotic boundary does not support supertranslation symmetry. In contrast, a star in AdS clearly retains many internal degrees of freedom, which, however, AdS/CFT tells us are recorded on the boundary. Evidently this issue goes to the heart of the holographic structure of spacetime. We will not settle it here. The next subsection sheds some light on it by constructing the intrinsic horizon contribution to the supertranslation charge without reference to infinity.

Horizon Charges

In the absence of eternal black holes or massive fields, we saw in section 34.5.2 that the linearized supertranslation charges $\hat{Q}_f^+$ can be written as volume integrals of local operators over $\mathcal{I}^+$. However, for the Schwarzschild geometry this is clearly impossible, as $\mathcal{I}^+$ is not a Cauchy surface. Instead, for massless fields, $\mathcal{I}^+ \cup \mathcal{H}^+$ is a Cauchy surface. Hence one expects a relation of the form

$$\hat{Q}_f^+ = \hat{Q}_f^{\mathcal{I}^+} + \hat{Q}_f^{\mathcal{H}^+}. \tag{34.288}$$

The second term, $\hat{Q}_f^{\mathcal{H}^+}$, is the horizon supertranslation charge. In general (for example, with Hawking radiation), supertranslation charge can be exchanged between the horizon and $\mathcal{I}^+$, and one does not expect them to be separately conserved.

The literature [223, 196, 235, 197, 198, 236, 237] on the covariant canonical formalism, symplectic forms, and conserved charges in GR contains general expressions for the charges associated with diffeomorphisms of a surface Σ as an integral over its boundary $\partial\Sigma$. In our case, $\mathcal{H}^+$ has past and future boundaries, $\mathcal{H}^+_\pm$, which is the key reason it is possible to have a nontrivial horizon charge.[218] The precise form of the horizon contribution $\hat{Q}_f^{\mathcal{H}^+}$ will depend on the coordinate choice used to extend the supertranslations in from infinity to the horizon. Adopting the Bondi gauge,

[217] I add the caveat "possibly" because in full quantum gravity, the holographic principle may ultimately imply that the internal states of the star are recorded at the boundary at infinity. However, ignoring this, it is at least true semiclassically.

[218] In the quantum theory, the black hole evaporates, and the horizon (however it is defined) may not have a boundary. This suggests there cannot be a quantum exact notion of horizon charges independently of those at infinity.

one finds the boundary expression [48]

$$\hat{Q}_f^{\mathcal{H}^+} = \frac{M}{8\pi}\int d^2\Theta\sqrt{\gamma}f\left[D^A\partial_r h_{Av} + 2h_{vv} + D^2 h_{vr}\right]_{\mathcal{H}_-^+}^{\mathcal{H}_+^+} . \tag{34.289}$$

Integrating by parts and using the constraints gives the horizon integral

$$\begin{aligned}\hat{Q}_f^{\mathcal{H}^+} = \frac{1}{8\pi G}\int_{\mathcal{H}^+} d^2\Theta\sqrt{\gamma}dvf\Bigg(&\frac{1}{4m_B}D^AD^B\partial_v h_{BA}\\ &+\frac{1}{2}D^2h_{vv} + m_BD^2\partial_r h_{vv} + \frac{1}{2}D^2h_{vr} - \frac{1}{4m_B}D^Ah_{Av}\Bigg) .\end{aligned} \tag{34.290}$$

As a check, we would like to see that (34.290) generates supertranslations of the horizon components of the metric via the inverted symplectic form. However, this is still not quite possible, because the Bondi gauge allows residual trivial diffeomorphisms that vanish at the boundaries $\mathcal{H}_\pm^+$ of the horizon and are zero modes of the symplectic form. We must first eliminate these modes and reduce to the physical horizon phase space $\Gamma_{\mathcal{H}^+}$, which is parametrized by h_{AB}. After some effort [48], $\hat{Q}_f^{\mathcal{H}^+}$ reduces to

$$\hat{Q}_f^{\mathcal{H}^+} = \frac{1}{32\pi G m_B}\int_{\mathcal{H}^+} dvd^2\Theta\sqrt{\gamma}D^AD^Bf\partial_v h_{AB} . \tag{34.291}$$

This charge is nothing but a soft graviton on $\mathcal{H}^+$, just as the linearized charge $\hat{Q}_f^{\mathcal{I}^+}$ is a soft graviton on $\mathcal{I}^+$. The symplectic form can now be inverted, and the commutator is

$$\left[\hat{Q}_f^{\mathcal{H}^+}, h_{AB}\right] = i2m_B(2D_AD_Bf - \gamma_{AB}D^2f) . \tag{34.292}$$

This relation is the desired linearized supertranslation action on the horizon metric fluctuations.

This construction makes sense only in leading-order perturbation theory: in the general case the classical horizon is defined only nonlocally, and in the quantum case it evaporates altogether. In the presence of interactions it is unlikely that a clean separation can be made between the two terms on the right-hand side of (34.288). Nevertheless, it provides a starting point for an analysis of horizon supertranslations.

We note that the construction requires the charges $\hat{Q}_f^{\mathcal{H}^+}$ (i.e., the soft graviton modes) to be incorporated as symplectic partners of the v-independent part of h_{AB} in the physical phase space $\Gamma_{\mathcal{H}^+}$. Moreover, the analysis is fully intrinsic to the horizon and does not require any properties of asymptotic infinity. Because the classical horizon has boundaries, it is possible for pure diffeomorphisms that do not vanish at these boundaries to be nonzero eigenvectors of the symplectic form. This is the basic reason that supertranslations act nontrivially on the black hole horizon.

Quantum Hair

Let $|M\rangle$ denote a quantum state corresponding to a Schwarzschild black hole. Then an infinitesimally supertranslated black hole corresponds to the quantum state

$$|M_f\rangle = |M\rangle + iQ_f^+|M\rangle . \tag{34.293}$$

In section 34.7.2 it was noted that, because the supertranslation group is abelian, a classically supertranslated black hole does not carry classical supertranslation hair. The quantum version of this statement is that to linear order in f

$$\langle M_f|Q_{f'}^+|M_f\rangle = 0 \tag{34.294}$$

for any non-constant f'. However, it does not imply that the supertranslation charge Q_f^+ annihilates $|M\rangle$. Instead, Q_f^+ adds a soft graviton to $|M\rangle$, leaving its energy unchanged but adding angular momentum and more general superrotation charges. According to the decomposition (34.288), the soft part of the charge has two terms, one that acts on $\mathcal{H}^+$ and the other on $\mathcal{I}^+$. Hence the soft graviton may be either a soft mode on $\mathcal{H}^+$ or a soft mode on $\mathcal{I}^+$.

Similar statements apply to the quantum vacuum. The supertranslated vacuum carries no classical supertranslation charge, as $m_B = 0$. However, the soft part Q_f^{+S} acts nontrivially on the vacuum, turning it into an orthogonal vacuum with an extra soft graviton and different superrotation charges.

Electric Hair

In this section, we repeat the discussion in the context of abelian gauge theories, replacing supertranslations with large gauge transformations and soft gravitons with soft photons. Technically, the discussion is much simpler, both because there are fewer indices and because there is less ambiguity in the extension of boundary quantities into the bulk. However, conceptually it is more subtle, because the action of large gauge transformations on charged particles is only a phase, which is most naturally detected by quantum interference experiments. The quantum nature of the large gauge symmetry action contrasts with that of supertranslations, which has a clear classical interpretation as advancing or delaying the arrival times of massless particles at $\mathcal{I}^+$. Moreover, in the gauge theory case, there is no analog of superrotation charge that provides a classical measurement of the supertranslation hairdo on the black hole. Because the large gauge symmetry is abelian, classical large gauge charges of large gauge transformed black holes must vanish. It is for these conceptual reasons that we discussed the gravity case before the gauge theory one. However, the gauge theory case is also illuminating, as it displays many of the basic concepts in simpler forms that are uncluttered by proliferating indices.

To begin, let us briefly recall the argument for conservation of the total electric charge. The total charge may be written as a surface integral at infinity,

$$\hat{Q}^+_{\varepsilon=1} = \frac{1}{e^2}\int_{\mathcal{I}^+_-} *F \ , \tag{34.295}$$

where again the hat on $\hat{Q}$ serves to remind the reader that in this section, we linearize around a fixed background. Equivalently, using integration by parts, we may write it as an integral of the charged matter current on a bulk slice Σ that ends at $\mathcal{I}^+_-$:

$$\hat{Q}^+_{\varepsilon=1} = \int_\Sigma *j_M \ . \tag{34.296}$$

The result must be the same regardless of which slice Σ through the bulk we choose, as illustrated in the context of black hole formation and evaporation depicted in figure 34.22. For instance, we could compute $\hat{Q}^+_{\varepsilon=1}$ by integrating over the purple slice hugging $\mathcal{I}^-$ at the bottom of the diagram before we create the black hole. Alternately, we could use the yellow slice at the top of the diagram after the black hole has evaporated or the green slice that runs into the horizon of the black hole. In the latter case, we can either continue the slice along the horizon as depicted, or we could end the slice at the horizon with an extra boundary term that tells us about the charge carried by the black hole itself. Either way, the black hole typically contributes to the electric charge. This contribution is required to ensure that the total $\hat{Q}^+_{\varepsilon=1}$ is conserved throughout the entire bulk evolution.

This argument may be applied to the infinite number of such conserved electric charges that generalize the total charge. That is, given any function ε on $\mathcal{CS}^2$, there is a conserved charge (34.29),

$$\hat{Q}^+_\varepsilon = \frac{1}{e^2}\int_{\mathcal{I}^+_-} \varepsilon * F = \frac{1}{e^2}\int_\Sigma \left(\mathrm{d}\varepsilon \wedge *F + e^2 \varepsilon * j_M\right) \ , \tag{34.297}$$

where Σ again is an arbitrary three-dimensional slice of the bulk ending on $\mathcal{I}^+_-$, and ε on the slice is any function with the prescribed boundary value at $\mathcal{I}^+_-$. Clearly, as we vary this slice and let it end at different surfaces on the black hole horizon, there will be different amounts of large gauge charge that pass through the horizon portion of the slice. This amount will vary in a complicated way as we change the time at which the slice meets the horizon. Therefore, to ensure charge conservation, it must be the case that black holes are capable of storing $\hat{Q}^+_\varepsilon$ charge, just as they store global charge. That is, the black hole must carry an infinite amount of soft electric hair.

We wish to analyze the linearized charges in perturbation theory about a fixed black hole background. For the case of supertranslations, we linearized around the eternal Schwarzschild black hole. It should also be possible to do a linearized analysis around the more physical geometry of a black hole formed by gravitational collapse. In practice, for supertranslations, the equations become unwieldy in a time-dependent background. However, for the large gauge transformations, the algebra simplifies and such an analysis is both instructive and possible, as we shall now see.

Consider a Vaidya black hole formed by a neutral null incoming shockwave at $v = 0$. Then at some later time v_0, we send in a linearized null shockwave with an asymmetric null charge current j, so that the higher large gauge charges are excited. This situation is depicted in figure 34.23. To

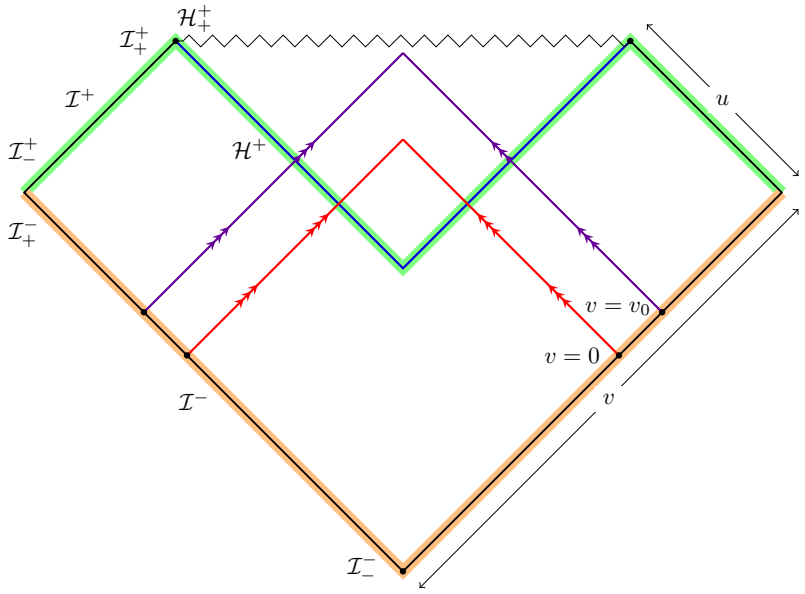


Figure 34.23: **A black hole formed by the spherically symmetric red shockwave at $v = 0$ is classically supertranslated at a later time $v = v_0$ by the asymmetric purple shockwave. The solution is diffeomorphic to the Schwarzschild geometry or vacuum away from the shockwaves but generically carries arbitrary superrotation charges in the region $v > v_0$. The charges computed on the orange and green surfaces must agree, but the latter adds contributions from the horizon and null infinity.**

be explicit, let us take

$$j_v = \frac{Y_{\ell m}(z, \bar{z})}{r^2} \delta(v - v_0) \ . \tag{34.298}$$

It takes only one line to check that this current is conserved. We are going to examine the effects of this shell of charge in linearized perturbation theory, neglecting its backreaction on the geometry. Note that since it has an angular profile $Y_{\ell m}$, the net charge is zero. If the net charge were nonzero, we would need to discuss a transition from a Schwarzschild to a Reissner-Nördstrom black hole instead of just a Schwarzschild to a Schwarzschild one.

Now we solve the sourced Maxwell equation. The leading constraint equation at $\mathcal{I}^-$ (34.37) says that

$$\partial_v F^{(2)}_{rv} + \gamma^{z\bar{z}} \left(\partial_z F^{(0)}_{\bar{z}v} + \partial_{\bar{z}} F^{(0)}_{zv} \right) = e^2 Y_{\ell m}(z,\bar{z}) \delta(v - v_0) \ . \tag{34.299}$$

There are many different ways of solving this constraint equation, as it is one equation for the four components of the vector potential. The one that most simply illustrates the physics is obtained by stipulating that the radial electric field vanish on $\mathcal{I}^-$,

$$F^{(2)}_{vr} = 0 \ , \tag{34.300}$$

and the initial connection is $A_z|_{\mathcal{I}^-_-} = 0$. These initial data on $\mathcal{I}^-$ then determine the full solution. Explicitly in $A_v = 0$ gauge, we have

$$A_z = \partial_z \left[\theta(v - v_0) \frac{e^2}{\ell(\ell+1)} Y_{\ell m}(z,\bar{z}) \right] \ , \tag{34.301}$$

as in equation (34.172). It is easy to check that this potential is indeed a solution by using $\partial_v \theta(v - v_0) = \delta(v - v_0)$ and

$$D^2 Y_{\ell m}(z,\bar{z}) = -\ell(\ell+1) Y_{\ell m}(z,\bar{z}) \ . \tag{34.302}$$

If we ignore the factor $\theta(v - v_0)$, A_z is manifestly just a pure gauge transformation, so F is identically zero both before and after the shockwave. We sent in a complicated angle-dependent charge distribution but dressed it with photons in just such a way that no electric field was produced. After the shockwave passes, the entire spacetime (including both the black hole and null infinity) is acted on by a gauge transformation $A \to A + \mathrm{d}\varepsilon$, with gauge parameter

$$\varepsilon = \frac{e^2}{\ell(\ell+1)} Y_{\ell m}(z,\bar{z}) \ . \tag{34.303}$$

Hence, the effect of this shockwave is to induce a large gauge transformation. It is a domain wall connecting a black hole spacetime to a large gauge transformation thereof.

The large gauge transformation (34.303) depends only on the data of the charge current and not the further choice of electric field data (34.300). In Hawking et al. [160], for instance, it was not assumed that $F^{(2)}_{vr} = 0$, but the same conclusion was reached. To see why, integrate the constraint along the horizon:

$$\int_{\mathcal{H}^+} dv \left[\partial_v F^{(2)}_{rv} + \gamma^{z\bar{z}} \left(\partial_z F^{(0)}_{\bar{z}v} + \partial_{\bar{z}} F^{(0)}_{zv} \right) \right] = e^2 Y_{\ell m}(z,\bar{z}) \ . \tag{34.304}$$

If the radial electric field eventually decays near $\mathcal{H}^+_+$, $\int_{\mathcal{H}^+} dv \partial_v F^{(2)}_{vr} = 0$. In the more general case, the gauge field is a Heun function and does not have a simple analytic expression. We nevertheless learn that the zero-modes of the transverse components of the field strength are always excited when charge is asymmetrically thrown in.

Equation (34.304) can be rewritten as

$$\int_{\mathcal{H}^+} dv d^2z Y_{\ell' m'} \left(\partial_z F^{(0)}_{\bar{z}v} + \partial_{\bar{z}} F^{(0)}_{zv} \right) = e^2 \delta_{\ell,\ell'} \delta_{m,m'} \ . \tag{34.305}$$

This expresses the fact that the classical soft photon mode is excited by the asymmetric charge shell or, for that matter, by any charged object that is thrown into the black hole. In the quantum theory, the quantum state of the soft photons on the horizon will be a coherent state in which the

expectation value of the transverse electric field obeys (34.305). We note that, in contrast, horizon soft photons cannot be excited by throwing in photons from $\mathcal{I}^-$, because the relevant greybody factors vanish at zero momentum. Nevertheless, it follows from (34.305) that soft photons on the horizon can be excited by throwing in charges and are needed for a perturbative description of black hole dynamics.

The incoming charge $\hat{Q}^-_\varepsilon$ can be written as an integral over the orange slice hugging $\mathcal{I}^-$ in figure 34.23. This slice is punctured by i^-, past timelike infinity. In general, one must be careful about contributions from the i^- puncture, but for the present case of massless fields they vanish. This conclusion follows from the mathematical fact that $\mathcal{I}^-$ by itself constitutes a complete Cauchy surface for massless fields. Physically, it means massless fields decay before reaching timelike infinity.

Likewise, the integral for the charge $\hat{Q}^+_\varepsilon$ can be extended up along $\mathcal{I}^+$, giving an integral $\hat{Q}^{\mathcal{I}^+}_\varepsilon$ along $\mathcal{I}^+$. Clearly, this term is no longer on its own equal to $\hat{Q}^-_\varepsilon$, because $\mathcal{I}^+$ no longer constitutes a Cauchy surface. To apply Gauss's law, we need a complete Cauchy slice. We may obtain one by following the green slice hugging the event horizon $\mathcal{H}^+$. This additional portion of the surface gives the black hole contribution to the charge. Again if we do not include massive particles, $\mathcal{H}^+$ and null infinity $\mathcal{I}^+$ together form a Cauchy surface for massless fields, so there should be no contributions from i^+.

The conservation law thus takes the form

$$\hat{Q}^-_\varepsilon = \hat{Q}^{\mathcal{I}^+}_\varepsilon + \hat{Q}^{\mathcal{H}^+}_\varepsilon \ . \tag{34.306}$$

This equation says that the large gauge charge computed along the green slice has two pieces: one from null infinity (the outgoing charge $\hat{Q}^{\mathcal{I}^+}_\varepsilon$) and one from the black hole (the horizon charge $\hat{Q}^{\mathcal{H}^+}_\varepsilon$). Moreover, the sum of these two charges equals the single term obtained by computing $\hat{Q}^-_\varepsilon$ on the single incoming orange slice.

We have already seen in section 34.2.6 that in a canonical formalism, $\hat{Q}^{\mathcal{I}^+}_\varepsilon$ generates large gauge transformations on the data at $\mathcal{I}^+$. It is not hard to see that $\hat{Q}^{\mathcal{H}^+}_\varepsilon$ similarly generates large gauge transformations on the data at $\mathcal{H}^+$. Integrating by parts, we find

$$\hat{Q}^{\mathcal{H}^+}_\varepsilon = \frac{1}{e^2}\int_{\mathcal{H}^+_+} \varepsilon * F = \frac{1}{e^2}\int_{\mathcal{H}^+} \left(\mathrm{d}\varepsilon \wedge *F + e^2 \varepsilon * j_M\right) \ . \tag{34.307}$$

In exercise 14, the interested reader is invited to show, following the same steps as in section 34.2.6, that in advanced gauge $A_v = 0$,[219]

$$\left[\hat{Q}^{\mathcal{H}^+}_\varepsilon, A_z\right]_{\mathcal{H}^+} = i\partial_z \varepsilon|_{\mathcal{H}^+} \ , \tag{34.308}$$

which are large gauge transformations on the horizon. This equation also states that the soft part of the horizon charge — the soft photons — are symplectically paired with the Goldstone bosons of spontaneously broken large gauge transformations on the horizon.

In general, a necessary condition for a gauge mode to be physical is that it have a nonvanishing symplectic form. From the general formula for the symplectic form, this is possible only if the surface has a boundary. The boundary of the future horizon $\mathcal{H}^+$ is $\mathcal{H}^+_+$,[220] and v-independent gauge transformations are indeed nonvanishing there.

Note that classically, in an asymptotically stationary situation, $\hat{Q}^{\mathcal{H}^+}_\varepsilon$ always vanishes, because the electromagnetic fields (except for the $\ell = 0$ mode) go to zero at late times. We already saw this in our linearized analysis: equation (34.304) says that the hard and soft part of the charge

[219] $A_v = 0$ gauge is natural for analyzing how excitations from $\mathcal{I}^-$, such as our shockwave, affect $\mathcal{H}^+$, because advanced Bondi coordinates cover both regions. However, they do not cover $\mathcal{I}^+$. For this reason, it would be interesting to analyze the problem in harmonic gauge, as considered by Campiglia and Laddha [215].

[220] For eternal black holes there is also a past boundary $\mathcal{H}^+_-$.

are classically equal and opposite. However, it is not zero as an operator, as we have just seen its commutators create large gauge transformations. Moreover, when we pass to the quantum theory and expand in creation/annihilation operators, the associated charges will create and annihilate soft photons on the horizon. Acting on the horizon Hilbert space with the operator $\hat{Q}_\varepsilon^{\mathcal{H}^+}$ will not change the energy, since it commutes with the Hamiltonian (as we know, because we can write the charge as a boundary integral at infinity), but it nonetheless inserts a soft photon, which will change the angular momentum of the black hole by one unit. Thus, a black hole with a soft photon is different from a black hole without one. The action of this infinitesimal symmetry is nontrivial:

$$\hat{Q}_\varepsilon^{\mathcal{H}^+} |M\rangle = \left(\frac{1}{e^2} \int_{\mathcal{H}^+} \mathrm{d}\varepsilon \wedge *F \right) |M\rangle \neq 0 \ , \tag{34.309}$$

where we use $|M\rangle$ to denote the portion of the black hole quantum state on the null surface $\mathcal{H}^+$. The classical vanishing of the linearized horizon charge holds at the level of expectation values:

$$\langle M| \hat{Q}_\varepsilon^{\mathcal{H}^+} |M\rangle = 0 \ . \tag{34.310}$$

A similar situation was already encountered in the electromagnetic vacuum in the absence of black holes. Since the field strength vanishes, the charges $\hat{Q}_\varepsilon^+$ are all classically zero. However, they are not zero as operators. Instead, they create and annihilate photons, mapping one classical vacuum to another with different angular momenta.

In conclusion, black holes have an infinite electromagnetic degeneracy that can be described at both the classical and quantum levels. At the classical level, the degeneracy arises from nontrivially-acting gauge transformations. At the quantum level, there is an alternate perturbative description in terms of zero-energy soft photons on the horizon.

34.7.3 Discussion

The no-hair theorem tells us that Einstein-Maxwell gravity has a three-parameter[221] family of black hole solutions labeled by the mass M, charge Q, and total angular momentum J, up to diffeomorphisms and gauge transformations. It has long been accepted that a few diffeomorphisms (for example, boosts or translations) act nontrivially on the physical phase space and impart a sparse head of hair on black holes. We have shown that there are in fact an infinite number of diffeomorphisms, including an antipodal subgroup of $\mathrm{BMS}^+ \times \mathrm{BMS}^-$, that act nontrivially and thereby impart a lush head of soft hair on black holes. The soft hairdo is classically measured by conserved superrotation charges at infinity. More hair is supplied by large gauge transformations. The hairdo is rearranged whenever anything is thrown into the black hole, and it thereby stores partial information about how the black hole is formed. Exactly how much information is stored, and how it is transmitted to the outgoing Hawking radiation, is presently unknown. At the very least, understanding black hole information requires understanding the properties of soft hair and more generally keeping careful track of how much information flows into or is stored in the deep infrared [163].

Our analysis reveals two subtly flawed assumptions underlying the argument given in Hawking [161] that there is an information paradox:

- The adage “black holes have no hair" is misleading at best. In reality they carry an infinite number of conserved charges which generalize the ten Poincaré and one electric charges. These provide some memory of how the black hole was formed. Moreover, since these charges are exactly conserved, they must affect the quantum state of the outgoing Hawking radiation, correlating it with both the quantum state of the black hole and outgoing soft radiation [162]. The argument in Hawking [161] incorrectly assumed such correlations do not exist.

[221] We suppress possible magnetic charge.

- "When the black hole evaporates completely, there is no place where the information can be stored" is a second incorrect assumption in Hawking [161]. We have seen that the vacuum itself is infinitely degenerate and hence can store information. In particular, from what is understood so far, it is a logical possibility that the final vacuum state and or the outgoing soft quanta could be correlated with the outgoing Hawking radiation in such a way as to maintain quantum purity.

Of course, it is always possible that these assumptions are ultimately inessential and the paradox can be resolved without taking them into account. This does not seem to be the case so far, but much remains to be understood.

To phrase the issue more generally, soft gravitons are produced in every scattering process. The IR pole in the soft theorem says that their production is more ubiquitous than might have been expected. In fact, infinitely many are produced in any physical process. The soft modes are correlated highly with the hard modes [162, 238] and can store information at little or no cost in energy. Many are made in any process of black hole formation/evaporation in a manner that is highly regulated by an infinite number of conservation laws. It strikes me as implausible that we could solve the information paradox in asymptotically flat spacetime without a good understanding of these modes.

What would it actually mean to solve the black hole information paradox? First, a convincing solution must microscopically reproduce the Bekenstein-Hawking entropy $S_{BH} = \frac{\text{Area}}{4G}$ with the right factor of 1/4 in front, not just a squiggly line. As we now understand them, soft modes on the horizon are both too many and too few for this purpose (although see Donnay et al. [170]). They are too many because there is no UV cutoff. Presumably, modes that are localized to shorter than the Planck length on the horizon should be thrown out, but we do not know how to do this systematically. They are too few because of a variant of the flavor problem: there are many different ways of throwing things into the black hole that produce the same soft hairdo. Second, a satisfactory understanding of black hole information should not be merely verbal but should enable explicit computations. In particular, one would like an algorithm for computing the rate at which the information is returned to infinity [239].

The last 40 plus years of contemplating the wonders of quantum black holes have been truly amazing. It has led us to unexpected and fundamental new insights about the structure of classical spacetime, exotic quantum states of matter, quantum chaos, nonperturbative string theory, and a potentially revolutionary holographic reformulation of the laws of nature! At the same time, clearly much more is to come, as the basic paradox in Hawking [161] remains unresolved, while the current literature is replete with promising ideas and directions. The adventure continues!

34.8 Exercises *with Prahar Mitra and Monica Pate*

This section contains exercises. We set $G = 1$ in all exercises and solutions.

1. Exercise 1. (Section 2)

   Consider a scattering process in which an incoming null shockwave of the form

   $$j_v(v, r, z, \bar{z}) = \frac{1}{r^2}\delta(v)Y_{\ell m}(z, \bar{z}) \ , \qquad j_r = j_z = 0 \ , \tag{34.311}$$

   scatters at the origin $u = v = 0$ into an outgoing null shockwave

   $$j_u(u, r, z, \bar{z}) = \frac{1}{r^2}\delta(u)Y_{\ell' m'}(z, \bar{z}) \ , \qquad j_r = j_z = 0 \ , \tag{34.312}$$

   where $Y_{\ell m}\big(z, \bar{z}\big)$ are the usual spherical harmonics on S^2.

   (a) Find a solution for the field strength sourced by the null shockwave described above, assuming that

   $$F_{vr}^{(2)} = \lim_{r\to\infty}\big[r^2 F_{vr}\big] = 0 \ , \tag{34.313}$$

and that $F_{vz}^{(0)} = \lim_{r\to\infty} F_{vz}$ only has support at the source $v = 0$.

(b) Using the result of part (a), compute $A_z^{(0)}$ on $\mathcal{I}^+$ and $\mathcal{I}^-$ assuming $A_z^{-(0)}\big|_{\mathcal{I}_-^-} = 0$ and $A_z^{(0)}\big|_{\mathcal{I}_-^+} = A_z^{-(0)}\big|_{\mathcal{I}_+^-}$.

2. Exercise 2. (Section 2)

Working in Lorenz gauge $\nabla^\mu A_\mu = 0$ and, using a large-r expansion, show that in pure abelian gauge theory, the four-dimensional gauge field $A_\mu(u, r, z, \bar{z})$ is determined, up to u-independent integration constants by the equations of motion with boundary data (initial conditions)

$$A_z^{(0)}(u, z, \bar{z}) = \lim_{r\to\infty} A_z(u, r, z, \bar{z}) \ . \tag{34.314}$$

Hint: Use the residual gauge freedom to set $A_u^{(1)} = 0$.

3. Exercise 3. (Section 2)

(a) Starting from the usual action for free abelian gauge theories

$$S = -\frac{1}{2e^2}\int F \wedge *F \ , \tag{34.315}$$

use the formalism in [198] to derive the symplectic form (34.47).

(b) Work out the symplectic form explicitly on $\Sigma = \mathcal{I}^+$, and derive the commutators (34.54) and (34.56).

4. Exercise 4. (Section 2)

Starting from the free field mode expansion

$$A_\mu(x) = e\sum_{\alpha=\pm}\int \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}\left[\varepsilon_\mu^{\alpha *}(\vec{q})a_\alpha(\vec{q})e^{iq\cdot x} + \varepsilon_\mu^{\alpha}(\vec{q})a_\alpha^\dagger(\vec{q})e^{-iq\cdot x}\right] , \tag{34.316}$$

show that

$$e^2\partial_z N(z, \bar{z}) = -\frac{1}{8\pi}\frac{\sqrt{2}e}{1 + z\bar{z}}\lim_{\omega\to 0^+}\left[\omega a_+(\omega\hat{x}) + \omega a_-(\omega\hat{x})^\dagger\right] \ , \tag{34.317}$$

where $\hat{x}$ is the unit three-vector

$$\hat{x} = \left(\frac{z + \bar{z}}{1 + z\bar{z}}, \frac{-i(z - \bar{z})}{1 + z\bar{z}}, \frac{1 - z\bar{z}}{1 + z\bar{z}}\right) \ . \tag{34.318}$$

5. Exercise 5. (Section 2)

In this problem, you will show that the soft photon theorem for a single outgoing photon,

$$\langle \text{out}|\, a_\pm^{\text{out}}(\vec{q})\mathcal{S}\,|\text{in}\rangle \to e\left(\sum_{k=1}^m Q_k^{\text{out}}\frac{p_k^{\text{out}}\cdot\varepsilon_\pm}{p_k^{\text{out}}\cdot q} - \sum_{k=1}^n Q_k^{\text{in}}\frac{p_k^{\text{in}}\cdot\varepsilon_\pm}{p_k^{\text{in}}\cdot q}\right)\langle \text{out}|\,\mathcal{S}\,|\text{in}\rangle \, , \tag{34.319}$$

is equivalent to the Ward identity for large gauge transformations.

(a) For massless momenta, use parameterizations of the type (34.77), namely,

$$p^\mu = E\left(1\,,\frac{z+\bar{z}}{1+z\bar{z}}\,,\frac{-i(z-\bar{z})}{1+z\bar{z}}\,,\frac{1-z\bar{z}}{1+z\bar{z}}\right)\,, \tag{34.320}$$

and rewrite the soft theorem (34.319) in terms of z, $z_k^{\rm in}$, $z_k^{\rm out}$, and N_z.

(b) For massive particles of mass m, use the momentum parameterization (34.129),

$$\vec{p} = m\rho\hat{x}\,, \tag{34.321}$$

while keeping the parameterization $q = \omega(1,\hat{q})$ for the photon momentum, and rewrite the soft theorem for a scattering amplitude of massive particles in the $(\rho,\hat{x})$ coordinates.

6. Exercise 6. (Section 2)

In this problem, we study large gauge transformations in Lorenz gauge, $\nabla^\mu A_\mu = 0$. These are gauge transformations that are nonzero on $\mathcal{I}^\pm$ and satisfy $\nabla^\mu\nabla_\mu\varepsilon = 0$. We will work in coordinates (ν, r, θ, ϕ), where $\nu = \frac{t}{r}$.

(a) On a Penrose diagram, draw lines of constant ν and lines of constant r.

(b) Determine the scalar wave equation in these coordinates.

(c) Find the most general non singular r-independent solution that is of the form $\varepsilon_{\ell m}(\nu,\theta,\phi) = A_\ell(\nu)Y^\ell_m(\theta,\phi)$. Be sure to demand that the solution is continuous and finite at both $\nu = \pm 1$ and the origin $\nu = \pm\infty$ (by choosing appropriate branch cuts at $\nu = \pm 1$).

(d) Determine the near-$\mathcal{I}^+$ expansion of $\varepsilon_{2m}(u, r, \theta, \phi)$ to $\mathcal{O}(r^{-1})$.

(e) Verify from a large-r expansion around $\mathcal{I}^+$ (in retarded radial coordinates) that the form of the subleading behavior obtained in part (d) is required for any solution of the wave equation whose limit on $\mathcal{I}^+$ is an arbitrary nonvanishing and nonsingular function on S^2.

7. Exercise 7. (Section 2)

In this problem, you will derive the symplectic form for massive scalar fields near future timelike infinity. To resolve i^+, work in the coordinates (34.126) [14],

$$\tau = \sqrt{t^2 - r^2}\,, \qquad \rho = \frac{r}{\sqrt{t^2-r^2}}\,. \tag{34.322}$$

(a) Start with the free field mode expansion for a real massive scalar

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left[a(\vec{p})e^{ip\cdot x} + a^\dagger(\vec{p})e^{-ip\cdot x}\right]\,, \tag{34.323}$$

and determine the leading order term in the asymptotic expansion of the field in the limit $\tau \to \infty$.

(b) Determine the symplectic form on a Cauchy surface Σ for a free massive real scalar, governed by the action

$$S = -\frac{1}{2}\int d^4x\sqrt{-g}\left[(\partial_\mu\phi)^2 + m^2\phi^2\right]\,. \tag{34.324}$$

(c) Now take Σ to be a constant-τ slice in the coordinates defined in (34.322), and determine the symplectic form in the limit $\tau \to \infty$.

(d) Verify that the commutation relations determined by this symplectic form are the standard free field commutation relations:

$$[a(\vec{p}), a^\dagger(\vec{p}')] = (2\pi)^3(2\omega_p)\delta^3(\vec{p}-\vec{p}') \ . \tag{34.325}$$

8. Exercise 8. (Section 2)

Consider the hard part $\mathscr{F}^h[\chi]$ of the fermionic charge that generates the soft-photino theorem,

$$\mathscr{F}^h[\chi] = \frac{1}{4}\int du d^2z \gamma_{z\bar{z}}\chi(z,\bar{z})u\partial_u \bar{k}_- \ , \tag{34.326}$$

where $\bar{k}_-$ is the large-r limit of the fermionic current. For minimally coupled $\mathcal{N}=1$ supersymmetric QED, $\bar{k}_- = \sqrt{2}Q\phi\bar{\psi}_-$, where ϕ and $\bar{\psi}_-$ are the boundary values of the bulk fields Φ and $\bar{\Psi}_{\dot{\alpha}}$, respectively:

$$\begin{aligned} \Phi(u,r,z,\bar{z}) &= \frac{1}{r}\phi(u,z,\bar{z}) + \mathcal{O}(r^{-2}) \ , \\ \bar{\Psi}_{\dot{\alpha}}(u,r,z,\bar{z}) &= \frac{1}{r}\bar{\psi}_-(u,z,\bar{z})\bar{\xi}^-_{\dot{\alpha}} + \mathcal{O}(r^{-2}) \ . \end{aligned} \tag{34.327}$$

$\xi^\pm_\alpha$ and $\bar{\xi}^\mp_{\dot{\alpha}}$ are commuting two-component helicity basis spinors (i.e., they are eigenspinors of ${\sigma_z}^z$)

$$({\sigma_z}^z\xi^\pm)_\alpha = \pm\frac{1}{2}\xi^\pm_\alpha \ , \qquad \left(\xi^\pm_\alpha\right)^* = \bar{\xi}^\mp_{\dot{\alpha}} \ . \tag{34.328}$$

Using the form above, derive the action of $\mathscr{F}^h[\chi]$ on the radiative modes $\partial_u\bar{\phi}$ and ψ_+. Note that the action of $\mathscr{F}^h[\chi]$ mimics that of the superconformal generators S_α except that it acts only on charged matter.

Hint: You can either derive the symplectic form on $\mathcal{I}^+$ for scalars and fermions or rewrite the fields on $\mathcal{I}^+$ in terms of the usual creation and annihilation operators.

9. Exercise 9. (Section 4)

(a) Consider an amplitude $\mathcal{A}^{a_1,a_2}_{n+2}(q_1,\varepsilon_1,q_2,\varepsilon_2;p_1,\cdots,p_n)$ containing two outgoing gluons with color, momentum, and polarization (a_1,q_1,ε_1) and (a_2,q_2,ε_2), and n particles charged under the gauge group. Derive the following double-soft limit

$$\left[\lim_{q_1\to 0}, \lim_{q_2\to 0}\right] \mathcal{A}^{a_1,a_2}_{n+2}(q_1,\varepsilon_1,q_2,\varepsilon_2;p_1,\cdots,p_n) \ . \tag{34.329}$$

(b) Show that when the two gluons have the same helicity the above quantity vanishes.

10. Exercise 10. (Section 5)

Any four-dimensional Lorentzian metric can be written in Bondi gauge as

$$ds^2 = -Udu^2 - 2e^{2\beta}dudr + g_{AB}\left(dx^A + \frac{1}{2}U^A du\right)\left(dx^B + \frac{1}{2}U^B du\right) \ . \tag{34.330}$$

Asymptotically flat spacetimes are defined by the following falloff conditions for the Weyl tensor:

$$C_{rzrz} \sim \mathcal{O}(r^{-3}) \ , \qquad C_{rurz}, C_{rur\bar{z}} \sim \mathcal{O}(r^{-3}) \qquad \text{at large } r, \tag{34.331}$$

along with some other assumptions of uniform smoothness that will not be considered here. In (34.206), asymptotically flat spacetimes are described by the large-r expansion

$$ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + U_z dudz + c.c. + \cdots \ . \quad (34.332)$$

Show that this metric satisfies the first equation of (34.331). Use the second equation to derive a constraint on U_z and $U_{\bar{z}}$.

11. Exercise 11. (Section 5)

In retarded Bondi gauge, a four-dimensional Lorentzian spacetime can be written as

$$ds^2 = -Udu^2 - 2e^{2\beta}dudr + g_{AB}\left(dx^A + \frac{1}{2}U^A du\right)\left(dx^B + \frac{1}{2}U^B du\right) \ , \quad (34.333)$$

where $\det g = r^4 \det \gamma$, and γ_{AB} is the two-dimensional metric on the round sphere, S^2. As discussed in section 34.5.1, the asymptotic falloff conditions for the component fields are

$$\begin{aligned} U &= 1 - \frac{2m_B}{r} + \mathcal{O}(r^{-2}) \ , \qquad \beta = \mathcal{O}(r^{-2}) \ , \\ U_A &= \frac{1}{r^2}D^B C_{BA} + \mathcal{O}(r^{-3}) \ , \qquad g_{AB} = r^2\gamma_{AB} + rC_{AB} + \mathcal{O}(1) \ , \end{aligned} \quad (34.334)$$

where γ_{AB} is the round metric on S^2, D_A is its covariant derivative, and all indices $(A, B, C, \cdots)$ are raised and lowered with respect to γ_{AB}.

(a) Show that the coordinate condition $\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0$ implies that C_{AB} is traceless.

(b) Find the most general diffeomorphism ξ that preserves the form of this metric and that satisfies the asymptotic falloffs $\xi^u, \xi^r \sim \mathcal{O}(1)$ and $\xi^z, \xi^{\bar{z}} \sim \mathcal{O}(r^{-1})$ at large r. *Be sure to keep track of the leading and the first subleading terms in the large-r expansion.*

(c) The diffeomorphism in part (b) is parametrized by a function $f(x^A)$ on the sphere. Show that, to leading order, the Lie algebra of supertranslations is abelian:

$$[\xi(f_1), \xi(f_2)] = \mathcal{O}(r^{-1}) \ . \quad (34.335)$$

(d) Show that the action of $\xi(f)$ on m_B, C_{AB}, and $N_{AB} = \partial_u C_{AB}$ is

$$\begin{aligned} \mathcal{L}_f C_{AB} &= f\partial_u C_{AB} - 2D_A D_B f + \gamma_{AB} D^2 f \ , \\ \mathcal{L}_f N_{AB} &= f\partial_u N_{AB} \ , \\ \mathcal{L}_f m_B &= f\partial_u m_B + \frac{1}{4}\left(N^{AB}D_A D_B f + 2D_A N^{AB} D_B f\right) \ . \end{aligned} \quad (34.336)$$

Note that the traceless condition on C_{AB} is preserved under supertranslations.

(e) The function $f(x^A)$ can be expanded in spherical harmonics on the sphere. Show that the $\ell = 0$ and $\ell = 1$ modes in this expansion correspond to the standard global translations in Minkowski spacetime. Find the action of these modes on C_{AB}, N_{AB}, and m_B.

12. Exercise 12. (Section 5)

In addition to the supertranslations, the four-dimensional BMS algebra also contains generators of an (extended) Lorentz group. We will study these in this problem.

(a) Determine the Killing vector ζ that generates Lorentz transformations in flat spacetime in the $(u, r, z, \bar{z})$ coordinates. Show that it can be parametrized in terms of a global CKV Y^A on S^2, and find $\zeta(Y)$.[222]

(b) Show that $[\zeta(Y_1), \zeta(Y_2)] = \zeta([Y_1, Y_2])$. This shows that the Lorentz algebra is isomorphic to $SL(2, \mathbb{C})$, the algebra of global CKVs of S^2.

(c) Show that $\zeta(Y)$ preserves the asymptotic form of the metric in asymptotically flat spacetimes even when Y is a local CKV of S^2, except at possibly isolated analytic singularities. Find the action of $\zeta(Y)$ on C_{AB}. These diffeomorphisms are called superrotations.

13. Exercise 13. (Section 6)

In this problem, we study the gravitational memory effect experienced by inertial observers due to supertranslations.

(a) Working in retarded Bondi coordinates, determine the first subleading correction to the component of the four-velocity of an inertial time-like observer that is stationary with respect to retarded time at leading order (i.e., $v^\mu = \delta^\mu_u + \mathcal{O}(r^{-1})$). Then determine the trajectory with this four-velocity that passes through the point $(u_0, r_0, z_0, \bar{z}_0)$ and is valid for $|u - u_0| \ll r_0$. Be sure to determine any subleading corrections needed to reproduce the subleading corrections to the four-velocity.

(b) Now consider a nearby inertial observer, who at $u = u_0$ is located at the same radius r_0 but at a different point $(z_1, \bar{z}_1)$ on the asymptotic S^2, where $|z_1 - z_0| \sim \mathcal{O}(r_0^{-1})$. Determine the proper distance between the observers at retarded time $u > u_0$ to leading order in the large-r_0 limit. Assume $u - u_0 \ll r_0$, so you can use the trajectory determined in part (a).

(c) A burst of radiation of the following form passes the observers:

$$T^M_{uu}(u, r, z, \bar{z}) = \frac{\mu}{4\pi r^2}\delta(u - u_{\rm rad})\gamma^{z\bar{z}}\delta^2(z - z_{\rm rad}) \ , \tag{34.337}$$

where

$$u_{\rm rad} > u_0 \ , \qquad \frac{|u_{\rm rad} - u_0|}{r_0} \ll 1 \ . \tag{34.338}$$

Determine the change in proper distance between the observers as a result of the passage of this radiation. You will need to work consistently to an order at which this change is nonzero. Also, assume that the Bondi mass m_B is independent of $(z, \bar{z})$.

(d) Find the supertranslation f that gives rise to the same change in proper distance as calculated in part (c).

14. Exercise 14. (Section 7)

Consider a Schwarzschild black hole with mass M perturbed by a null shockwave at $v = v_0$ described by the stress tensor

$$\begin{aligned} T_{vv} &= \frac{1}{4\pi r^2}\left[\mu + \frac{1}{4}D^2\left(D^2 + 2\right)f\right]\delta\left(v - v_0\right) - \frac{3M}{8\pi r^3}D^2 f\delta\left(v - v_0\right) \ , \\ T_{vA} &= -\frac{3M}{8\pi r^2}D_A f\delta\left(v - v_0\right) \ . \end{aligned} \tag{34.339}$$

(a) Solve the linearized Einstein's equations about the initial Schwarzschild solution in Bondi gauge. Assume that only the zero mode of the Bondi mass aspect m_B on $\mathcal{I}^-$ changes under this perturbation.

[222] A conformal Killing vector (CKV) on S^2 is one that satisfies $D_A Y_B + D_B Y_A = \gamma_{AB} D_C Y^C$. In the $(z, \bar{z})$ coordinates, this equation reads $\partial_{\bar{z}} Y^z = \partial_z Y^{\bar{z}} = 0$, implying that any holomorphic vector field $Y^z(z)$ is locally a CKV. The ones that are defined globally on the sphere S^2 take the special form $Y^z(z) = a + bz + cz^2$ for $a, b, c \in \mathbb{C}$.

(b) Show that the final black hole (after $v > v_0$) can be obtained instead by sending a spherically symmetric null shockwave with stress tensor

$$T_{vv} = \frac{\mu}{4\pi r^2}\delta\left(v - v_0\right) \ , \tag{34.340}$$

followed by an infinitesimal supertranslation.

15. Exercise 15. (Section 7)

Consider the Vaidya black hole created by a spherically symmetric null shockwave at $v = 0$. We define the horizon large gauge charge as

$$Q_\varepsilon^{\mathcal{H}^+} \equiv \frac{1}{e^2}\int_{\mathcal{H}_+^+} \varepsilon * F \ , \tag{34.341}$$

where $\mathcal{H}_+^+$ is the future of the horizon of the Vaidya black hole.

(a) Using the constraints derived from Maxwell's equations on the horizon $\mathcal{H}^+$, rewrite $Q_\varepsilon^{\mathcal{H}^+}$ as an integral over $\mathcal{H}^+$, and determine the soft and hard parts of this charge. Write the soft charge explicitly in the $(v, r, z, \bar{z})$ coordinates.

(b) Determine the symplectic form for the gauge fields on $\mathcal{H}^+$. (Hint: You can use the symplectic form Ω_Σ derived in exercise 3 and set $\Sigma = \mathcal{H}^+$). Write it out explicitly in $(v, r, z, \bar{z})$ coordinates.

(c) By using the explicit forms of the charge and symplectic form derived in parts (a) and (b), show that $\hat{Q}_\varepsilon^{\mathcal{H}^+}$ generates large gauge transformations on the horizon, that is,

$$\left[\hat{Q}_\varepsilon^{\mathcal{H}^+}, A_z\right]_{\mathcal{H}^+} = i\partial_z\varepsilon\big|_{\mathcal{H}^+} \ . \tag{34.342}$$

34.9 Solutions *with Prahar Mitra and Monica Pate*

1(a). First consider the electric field produced by an incoming null shockwave. We work in the $(v, r, z, \bar{z})$ coordinates. Maxwell's equations take the explicit form

$$\begin{aligned} \left(\partial_v + \partial_r\right)\left(r^2 F_{rv}\right) - \gamma^{z\bar{z}}\left(\partial_z F_{v\bar{z}} + \partial_{\bar{z}} F_{vz}\right) &= e^2\delta(v) Y_{\ell m} \ , \\ \partial_r\left(r^2 F_{rv}\right) + \gamma^{z\bar{z}}\left(\partial_z F_{r\bar{z}} + \partial_{\bar{z}} F_{rz}\right) &= 0 \ , \\ r^2\left(\partial_v + \partial_r\right) F_{rz} + r^2\partial_r F_{vz} - \partial_z\left(\gamma^{z\bar{z}} F_{z\bar{z}}\right) &= 0 \ . \end{aligned} \tag{34.343}$$

We also have the Bianchi identities, namely,

$$\begin{aligned} \partial_v F_{z\bar{z}} + \partial_{\bar{z}} F_{vz} - \partial_z F_{v\bar{z}} &= 0 \ , \\ \partial_r F_{z\bar{z}} + \partial_{\bar{z}} F_{rz} - \partial_z F_{r\bar{z}} &= 0 \ , \\ \partial_v F_{rz} - \partial_z F_{rv} - \partial_r F_{vz} &= 0 \ . \end{aligned} \tag{34.344}$$

A simple solution satisfying the boundary conditions is

$$F_{vz} = -e^2\delta(v)\partial_z\varepsilon_{\ell m} \ , \qquad F_{vr} = F_{rz} = F_{z\bar{z}} = 0 \ , \tag{34.345}$$

where

$$\varepsilon_{\ell m} = \begin{cases} \log\left(1 + z\bar{z}\right) Y_{00} \ , & \ell = 0 \ , \\ -\frac{1}{\ell(\ell+1)} Y_{\ell m} \ , & \ell \neq 0 \ , \end{cases} \tag{34.346}$$

and $Y_{\ell m}$ satisfies

$$2\gamma^{z\bar{z}}\partial_z\partial_{\bar{z}}Y_{\ell m} = -\ell(\ell+1)Y_{\ell m} \ . \tag{34.347}$$

This implies

$$2\gamma^{z\bar{z}}\partial_z\partial_{\bar{z}}\varepsilon_{\ell m} = Y_{\ell m} \ . \tag{34.348}$$

Any other solution differs from this one by a solution to the homogeneous Maxwell equation. Similarly, for the outgoing null shockwave, the solution is

$$F_{uz} = -e^2\delta(u)\partial_z\varepsilon_{\ell' m'} \ , \qquad F_{ur} = F_{rz} = F_{z\bar{z}} = 0 \ . \tag{34.349}$$

1(b). We have $F^{(0)}_{vz} = \partial_v A_z^{-(0)}$ and the boundary condition $A_z^{-(0)}\big|_{\mathcal{I}^-_-} = 0$, we find

$$A_z^{-(0)}(v,z,\bar{z}) = -e^2\theta(v)\partial_z\varepsilon_{\ell m}(z,\bar{z}) \ , \tag{34.350}$$

where θ is the Heaviside step function. Then, using the antipodal matching condition, we find

$$A_z^{(0)}\big|_{\mathcal{I}^+_-} = -e^2\partial_z\varepsilon_{\ell m}(z,\bar{z}) \ . \tag{34.351}$$

Finally, integrating $F^{(0)}_{uz} = \partial_u A^{(0)}_z$ and using the above boundary condition, we find

$$A^{(0)}_z(u,z,\bar{z}) = -e^2\partial_z\varepsilon_{\ell m}(z,\bar{z}) - e^2\theta(u)\partial_z\varepsilon_{\ell' m'}(z,\bar{z}) \ . \tag{34.352}$$

2(a). In the $(u,r,z,\bar{z})$ coordinates, the Lorenz gauge condition and Maxwell equations take the form

$$\begin{aligned}
(\partial_u - \partial_r)\left(r^2F_{ru}\right) + \gamma^{z\bar{z}}\left(\partial_z F_{u\bar{z}} + \partial_{\bar{z}}F_{uz}\right) &= 0 \ , \\
-\partial_r\left(r^2F_{ru}\right) + \gamma^{z\bar{z}}\left(\partial_z F_{r\bar{z}} + \partial_{\bar{z}}F_{rz}\right) &= 0 \ , \\
r^2\left(\partial_u - \partial_r\right)F_{rz} + r^2\partial_r F_{uz} + \partial_z\left(\gamma^{z\bar{z}}F_{z\bar{z}}\right) &= 0 \ , \\
-\partial_u\big(r^2A_r\big) - \partial_r\big(r^2A_u\big) + \partial_r\big(r^2A_r\big) + \gamma^{z\bar{z}}\left(\partial_z A_{\bar{z}} + \partial_{\bar{z}}A_z\right) &= 0 \ .
\end{aligned} \tag{34.353}$$

To impose the Bianchi identity, express the Maxwell equations in terms of the gauge field A_μ, and in addition, substitute the Lorenz gauge condition into each Maxwell equation:

$$\begin{aligned}
-2r\partial_r(r\partial_u A_u) + \partial_r(r^2\partial_r A_u) + 2\gamma^{z\bar{z}}\partial_z\partial_{\bar{z}}A_u &= 0 \ , \\
-2\partial_r(rA_u) - 2\partial_u\partial_r(r^2A_r) + \partial_r^2(r^2A_r) + 2\gamma^{z\bar{z}}\partial_z\partial_{\bar{z}}A_r &= 0 \ , \\
-2r^2\partial_u\partial_r A_z + r^2\partial_r^2 A_z + 2r\partial_z(A_r - A_u) + 2\partial_z(\gamma^{z\bar{z}}\partial_{\bar{z}}A_z) &= 0 \ .
\end{aligned} \tag{34.354}$$

The gauge field can be expanded at large r as

$$A_u = \sum_{n=2}^{\infty}\frac{A_u^{(n)}}{r^n} \ , \qquad A_r = \sum_{n=2}^{\infty}\frac{A_r^{(n)}}{r^n} \ , \qquad A_z = \sum_{n=0}^{\infty}\frac{A_z^{(n)}}{r^n} \ . \tag{34.355}$$

Note that A_u has a slightly different expansion than what was given in section 34.2, namely, $A_u^{(1)} = 0$. The Lorenz gauge condition leaves unfixed a set of residual gauge transformations satisfying $\Box\varepsilon = 0$. Among these are a set of trivial gauge transformations with falloff behavior $\mathcal{O}(1/r)$ near $\mathcal{I}^+$. This set of residual trivial gauge transformations has precisely the degrees

of freedom of a massless scalar field with the usual falloff near $\mathcal{I}^+$, and thus, the residual trivial gauge symmetry can be used to set $A_u^{(1)} = 0$. Then the Lorenz gauge condition has the following expansion:

$$\begin{aligned}
\partial_u A_r^{(2)} &= \gamma^{z\bar{z}} \left(\partial_z A_{\bar{z}}^{(0)} + \partial_{\bar{z}} A_z^{(0)} \right) \ , \\
\partial_u A_r^{(3)} &= \gamma^{z\bar{z}} \left(\partial_z A_{\bar{z}}^{(1)} + \partial_{\bar{z}} A_z^{(1)} \right) \ , \\
\partial_u A_r^{(n+2)} &= (n-1)(A_u^{(n+1)} - A_r^{(n+1)}) + \gamma^{z\bar{z}} \left(\partial_z A_{\bar{z}}^{(n)} + \partial_{\bar{z}} A_z^{(n)} \right) \ , \qquad n > 1 \ .
\end{aligned} \tag{34.356}$$

Using the leading order equation, one can determine $A_r^{(2)}$ up to integration constants in terms of the boundary data $A_z^{(0)}$. The first equation in (34.354) has the following expansion:

$$\begin{aligned}
\partial_u A_u^{(2)} &= 0 \ , \\
\partial_u A_u^{(n+1)} &= -\frac{1}{2n} \left[n(n-1) + 2\gamma^{z\bar{z}} \partial_z \partial_{\bar{z}} \right] A_u^{(n)} \ , \qquad n > 1 \ .
\end{aligned} \tag{34.357}$$

Hence, we can determine all $A_u^{(n)}$ up to integration constants. The second equation in (34.354) has the following expansion:

$$\begin{aligned}
\partial_u A_r^{(3)} &= -A_u^{(2)} - \gamma^{z\bar{z}} \partial_z \partial_{\bar{z}} A_r^{(2)} \ , \\
\partial_u A_r^{(n+1)} &= -A_u^{(n)} - \left[\frac{1}{2}(n-2) + \frac{1}{n-1} \gamma^{z\bar{z}} \partial_z \partial_{\bar{z}} \right] A_r^{(n)} \ , \qquad n > 2 \ .
\end{aligned} \tag{34.358}$$

Hence, with the previously determined set of $A_u^{(n)}$ and $A_r^{(2)}$, we can determine all $A_r^{(n)}$ up to integration constants. Finally, the third equation in (34.354) has the following expansion:

$$\begin{aligned}
\partial_u A_z^{(1)} &= -\partial_z (\gamma^{z\bar{z}} \partial_{\bar{z}} A_z^{(0)}) \ , \\
\partial_u A_z^{(n+1)} &= \frac{1}{n+1} \left[-\partial_z A_r^{(n+1)} + \partial_z A_u^{(n+1)} - \frac{1}{2} n(n+1) A_z^{(n)} - \partial_z (\gamma^{z\bar{z}} \partial_{\bar{z}} A_z^{(n)}) \right] ,
\end{aligned} \tag{34.359}$$

for $n > 0$. With the previously determined sets of $A_u^{(n)}$ and $A_r^{(n)}$ plus the boundary data $A_z^{(0)}$, we can determine all $A_z^{(n)}$, up to integration constants. Therefore, we have determined the full four-dimensional gauge field entirely in terms of $A_z^{(0)}$ up to integration constants. Note the remaining (seemingly unused) equations in (34.357) give rise to additional constraints on the integration constants.

3(a). Varying the action, we find

$$\delta S = -\frac{1}{e^2} \int d\delta A \wedge *F = \frac{1}{e^2} \int (d * F) \wedge \delta A - \frac{1}{e^2} \int d \left(*F \wedge \delta A \right) \ . \tag{34.360}$$

The equations of motion are $d*F = 0$. Following [198], the (pre) symplectic potential current density can be read off as

$$\boldsymbol{\Theta} \left(A, \delta A \right) = -\frac{1}{e^2} \left(*F \right) \wedge \delta A \ . \tag{34.361}$$

The symplectic form is then

$$\Omega_\Sigma = \int_\Sigma \boldsymbol{\omega} \left(A, \delta_1 A, \delta_2 A \right) = -\frac{1}{e^2} \int_\Sigma \left[\delta_1 \left(*F \right) \wedge \delta_2 A - \delta_2 \left(*F \right) \wedge \delta_1 A \right] \ . \tag{34.362}$$

In terms of a wedge product of differential forms on the symplectic manifold, this can be written as

$$\Omega_\Sigma = -\frac{1}{e^2}\int_\Sigma \delta\,(*F)\wedge\delta A\ . \tag{34.363}$$

3(b). For $\Sigma = \mathcal{I}^+$, we are interested in the $uz\bar{z}$ component of the above three-form. We find

$$\begin{aligned}\boldsymbol{\omega}_{uz\bar{z}}\big|_{\mathcal{I}^+} &= -\frac{1}{e^2}\left[\delta(*F)_{uz}\wedge\delta A_{\bar{z}} + \delta(*F)_{\bar{z}u}\wedge\delta A_z + \delta(*F)_{z\bar{z}}\wedge\delta A_u\right]_{\mathcal{I}^+}\\ &= -\frac{i}{e^2}\left[\delta F^{(0)}_{uz}\wedge\delta A^{(0)}_{\bar{z}} + \delta F^{(0)}_{u\bar{z}}\wedge\delta A^{(0)}_z\right]\ ,\end{aligned} \tag{34.364}$$

where

$$(*F)_{uz} = \varepsilon_{uzr\bar{z}}F^{r\bar{z}} = i\,(F_{uz}-F_{rz})\ ,\qquad (*F)_{z\bar{z}} = ir^2\gamma_{z\bar{z}}F_{ru}\ . \tag{34.365}$$

This implies

$$\Omega_{\mathcal{I}^+} = \frac{1}{e^2}\int du d^2z\left[\delta F^{(0)}_{uz}\wedge\delta A^{(0)}_{\bar{z}} + \delta F^{(0)}_{u\bar{z}}\wedge\delta A^{(0)}_z\right]\ . \tag{34.366}$$

We now use this to write everything in terms of $\hat{A}_z$, ϕ, and N. To do this, we write

$$A^{(0)}_z = \hat{A}_z + \partial_z\phi\ ,\qquad \int_{-\infty}^{\infty} du F^{(0)}_{uz} = e^2\partial_z N\ , \tag{34.367}$$

which gives

$$\Omega_{\mathcal{I}^+} = \frac{2}{e^2}\int du d^2z\partial_u\delta\hat{A}_z\wedge\delta\hat{A}_{\bar{z}} - 2\int d^2z\delta\partial_z\phi\wedge\partial_{\bar{z}}\delta N\ . \tag{34.368}$$

We can then derive the brackets

$$\begin{aligned}-\frac{2}{e^2}\left[\partial_u\hat{A}_z(u,z,\bar{z}),\hat{A}_{\bar{w}}(u',w,\bar{w})\right] &= i\delta(u-u')\delta^2(z-w)\ ,\\ 2\left[\partial_z\phi(z,\bar{z}),\partial_{\bar{w}}N(w,\bar{w})\right] &= i\delta^2(z-w)\ .\end{aligned} \tag{34.369}$$

Integrating the above, we find

$$\begin{aligned}\left[\hat{A}_z(u,z,\bar{z}),\hat{A}_{\bar{w}}(u',w,\bar{w})\right] &= -\frac{ie^2}{4}\Theta(u-u')\delta^2(z-w)\ ,\\ [\phi(z,\bar{z}),N(w,\bar{w})] &= -\frac{i}{4\pi}\log|z-w|^2 + f(z,\bar{z}) + g(w,\bar{w})\ .\end{aligned} \tag{34.370}$$

4. The free field mode expansion can be written as

$$\begin{aligned}A_\mu(x) &= e\sum_{\alpha=\pm}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}\left[\varepsilon^{*\alpha}_\mu(\vec{q})a_\alpha(\vec{q})e^{-i\omega_q u - i\omega_q r(1-\hat{q}\cdot\hat{x})}\right.\\ &\qquad\left.+\varepsilon^\alpha_\mu(\vec{q})a^\dagger_\alpha(\vec{q})e^{i\omega_q u + i\omega_q r(1-\hat{q}\cdot\hat{x})}\right]\\ &= \frac{e}{8\pi^2}\sum_{\alpha=\pm}\int_0^\infty d\omega_q\omega_q\int_0^\pi d\theta\sin\theta\left[\varepsilon^{*\alpha}_\mu(\vec{q})a_\alpha(\vec{q})e^{-i\omega_q u - i\omega_q r(1-\cos\theta)}\right.\\ &\qquad\left.+\varepsilon^\alpha_\mu(\vec{q})a^\dagger_\alpha(\vec{q})e^{i\omega_q u + i\omega_q r(1-\cos\theta)}\right]\ .\end{aligned} \tag{34.371}$$

At large r, we use the saddle point approximation. The exponent is stationary at $\theta = 0, \pi$. The saddle point at $\theta = \pi$ does not contribute due to the Riemann-Lebesgue lemma (the assumptions of the lemma hold here as operator statements due to the cluster decomposition principle that all $\mathcal{S}$-matrices satisfy). Then, expanding the integrand around $\theta = 0$, we find

$$\begin{aligned} A_\mu(x) &= \frac{e}{8\pi^2}\sum_{\alpha=\pm}\int_0^\infty d\omega_q \omega_q \varepsilon_\mu^{*\alpha}(\omega_q\hat{x}) a_\alpha(\omega_q\hat{x}) e^{-i\omega_q u}\int_0^\pi d\theta\theta e^{-i\omega_q r\theta^2/2} + c.c. + \mathcal{O}(r^{-2}) \\ &= -\frac{ie}{8\pi^2 r}\sum_{\alpha=\pm}\int_0^\infty d\omega_q \left[\varepsilon_\mu^{*\alpha}(\omega_q\hat{x}) a_\alpha(\omega_q\hat{x}) e^{-i\omega_q u} - c.c.\right] + \mathcal{O}(r^{-2}) \ . \end{aligned} \tag{34.372}$$

We are interested first in $A_z = \partial_z x^\mu A_\mu$. To determine this, note that

$$\partial_z x^\mu \varepsilon_\mu^+(\omega_q\hat{x}) = 0 \ , \qquad \partial_z x^\mu \varepsilon_\mu^-(\omega_q\hat{x}) = \frac{\sqrt{2}r}{1+z\bar{z}} \ . \tag{34.373}$$

Using this, we find

$$A_z = -\frac{i}{8\pi^2}\frac{\sqrt{2}e}{1+z\bar{z}}\int_0^\infty d\omega_q \left[a_+(\omega_q\hat{x})e^{-i\omega_q u} - a_-^\dagger(\omega_q\hat{x})e^{i\omega_q u}\right] + \mathcal{O}(r^{-1}) \ . \tag{34.374}$$

Finally, we can immediately read off from this equation the large-r mode:

$$A_z^{(0)} = -\frac{i}{8\pi^2}\frac{\sqrt{2}e}{1+z\bar{z}}\int_0^\infty d\omega_q \left[a_+(\omega_q\hat{x})e^{-i\omega_q u} - a_-^\dagger(\omega_q\hat{x})e^{i\omega_q u}\right] \ . \tag{34.375}$$

From this, we find

$$\begin{aligned} e^2\partial_z N &= \frac{1}{2}\lim_{\omega\to 0^+}\int_{-\infty}^\infty du \left(e^{i\omega u} + e^{-i\omega u}\right)\partial_u A_z^{(0)} \\ &= -\frac{1}{8\pi}\frac{\sqrt{2}e}{1+z\bar{z}}\lim_{\omega\to 0^+}\left[\omega a_+(\omega\hat{x}) + \omega a_-^\dagger(\omega\hat{x})\right] \ . \end{aligned} \tag{34.376}$$

5(a). Rewriting expression (34.319) is a simple problem of algebra. We are required to substitute

$$\begin{aligned} q^\mu &= \frac{\omega}{1+z\bar{z}}\left(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}\right) \ , \\ p_k^\mu &= \frac{E_k}{1+z_k\bar{z}_k}\left(1+z_k\bar{z}_k, z_k+\bar{z}_k, -i(z_k-\bar{z}_k), 1-z_k\bar{z}_k\right) \ , \\ \varepsilon_+^\mu(q) &= \frac{1}{\sqrt{2}}\left(\bar{z}, 1, -i, -\bar{z}\right) \ , \\ \varepsilon_-^\mu(q) &= \frac{1}{\sqrt{2}}\left(z, 1, i, -z\right) \ . \end{aligned} \tag{34.377}$$

To simplify the algebra a little, note that the polarizations satisfy the property

$$\varepsilon_+^\mu(q) = \frac{1}{\sqrt{2}\omega}\partial_z\left[(1+z\bar{z})\,q^\mu\right] \ , \qquad \varepsilon_-^\mu(q) = \frac{1}{\sqrt{2}\omega}\partial_{\bar{z}}\left[(1+z\bar{z})\,q^\mu\right] \ . \tag{34.378}$$

Then the soft factor for the kth particle takes the form

$$\begin{aligned}\frac{p_k\cdot\varepsilon_+}{p_k\cdot q}&=\frac{1}{\sqrt{2}\omega}\frac{\partial_z\left[(1+z\bar{z})\,p_k\cdot q\right]}{p_k\cdot q}\\&=\frac{1}{\sqrt{2}\omega}\,(1+z\bar{z})\,\partial_z\log\left[(1+z\bar{z})\,p_k\cdot q\right]\ .\end{aligned}\tag{34.379}$$

A similar formula for negative helicity holds with $\partial_z\to\partial_{\bar{z}}$. Finally, compute

$$p_k\cdot q=-\frac{2\omega E_k}{(1+z\bar{z})\,(1+z_k\bar{z}_k)}|z-z_k|^2\ .\tag{34.380}$$

Then the soft factor is

$$\begin{aligned}\frac{p_k\cdot\varepsilon_+}{p_k\cdot q}&=\frac{1}{\sqrt{2}\omega}\,(1+z\bar{z})\,\partial_z\log|z-z_k|^2=\frac{1}{\sqrt{2}\omega}\frac{1+z\bar{z}}{z-z_k}\ ,\\\frac{p_k\cdot\varepsilon_-}{p_k\cdot q}&=\frac{1}{\sqrt{2}\omega}\,(1+z\bar{z})\,\partial_{\bar{z}}\log|z-z_k|^2=\frac{1}{\sqrt{2}\omega}\frac{1+z\bar{z}}{\bar{z}-\bar{z}_k}\ .\end{aligned}\tag{34.381}$$

Thus, the soft-theorem can be written as

$$\begin{aligned}&\lim_{\omega\to0^+}\left[\omega\,\langle\text{out}|\,a_+^{\text{out}}(\omega\hat{x})\mathcal{S}\,|\text{in}\rangle\right]\\&\qquad=\frac{e}{\sqrt{2}}\,(1+z\bar{z})\left[\sum_{k\in\text{out}}\frac{Q_k}{z-z_k}-\sum_{k\in\text{in}}\frac{Q_k}{z-z_k}\right]\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle\ .\end{aligned}\tag{34.382}$$

A similar formula is true for the negative helicity photon. Finally, note that we can write the limit in terms of an insertion of N_z, by using formula (34.317) derived in exercise 4, namely,

$$e^2\partial_z N=-\frac{1}{8\pi^2}\frac{\sqrt{2}e}{1+z\bar{z}}\lim_{\omega\to0^+}\left[\omega a_+\left(\omega\hat{x}\right)+\omega a_-^\dagger\left(\omega\hat{x}\right)\right]\ .\tag{34.383}$$

This gives for an insertion of N_z,

$$\langle\text{out}|\,\partial_z N\ \mathcal{S}\,|\text{in}\rangle=-\frac{1}{8\pi^2}\left[\sum_{k\in\text{out}}\frac{Q_k}{z-z_k}-\sum_{k\in\text{in}}\frac{Q_k}{z-z_k}\right]\langle\text{out}|\,\mathcal{S}\,|\text{in}\rangle\ .\tag{34.384}$$

5(b). Again, it will be convenient to write the soft factor

$$\frac{p_k\cdot\varepsilon_+}{p_k\cdot q}=-\frac{1}{\sqrt{2}\omega}\,(1+z\bar{z})\,\partial_z\log\frac{(1+z\bar{z})^{-1}}{p_k\cdot q}\ .\tag{34.385}$$

However, now we parametrize $p_k^\mu=m_k\left(\sqrt{1+\rho_k^2}\ ,\rho_k\hat{x}_k\right)$ and $q^\mu=\omega\,(1,\hat{q})$. Then, we find

$$\frac{p_k\cdot\varepsilon_+}{p_k\cdot q}=-\frac{1}{2\sqrt{2}\omega}\,(1+z\bar{z})\,\partial_z\log\frac{\gamma_{z\bar{z}}}{\left(\sqrt{1+\rho_k^2}-\rho_k\hat{q}\cdot\hat{x}_k\right)^2}\ .\tag{34.386}$$

Note that the mass m_k drops out of the formula. Now, recall the scalar propagator (34.127):

$$G\left(\rho,\hat{x};\hat{q}\right)=\frac{1}{4\pi}\frac{\gamma_{z\bar{z}}}{\left(\sqrt{1+\rho^2}-\rho\hat{q}\cdot\hat{x}\right)^2}\ .\tag{34.387}$$

This is precisely the term in the logarithm in the soft factor. Thus, we find

$$\frac{p_k \cdot \varepsilon_+}{p_k \cdot q} = -\frac{1}{2\sqrt{2}\omega}\left(1 + z\bar{z}\right)\partial_z \log G\left(\rho_k, \hat{x}_k; \hat{q}\right) \ . \tag{34.388}$$

Then the soft theorem can be written as

$$\langle \text{out}| \, \partial_z N \, \mathcal{S} \, |\text{in}\rangle = \langle \text{out}| \, \mathcal{S} \, |\text{in}\rangle \left[\sum_{k\in\text{out}} - \sum_{k\in\text{in}}\right] \frac{Q_k}{16\pi^2}\partial_z \log G\left(\rho_k, \hat{x}_k; \hat{q}\right) \ . \tag{34.389}$$

6(a). The compactified coordinates on the Penrose diagram of Minkowski space are given by

$$\begin{aligned} T &= \tan^{-1}\left(t + r\right) + \tan^{-1}\left(t - r\right) \ , \\ R &= \tan^{-1}\left(t + r\right) - \tan^{-1}\left(t - r\right) \ , \end{aligned} \tag{34.390}$$

with the coordinate range $R + |T| < \pi$ and $R \geq 0$. Setting $t = \nu r$, we find

$$\begin{aligned} T &= \tan^{-1}\left[r\left(\nu + 1\right)\right] + \tan^{-1}\left[r\left(\nu - 1\right)\right] \ , \\ R &= \tan^{-1}\left[r\left(\nu + 1\right)\right] - \tan^{-1}\left[r\left(\nu - 1\right)\right] \ . \end{aligned} \tag{34.391}$$

Lines of constant ν and constant r are shown in figure 34.24 .

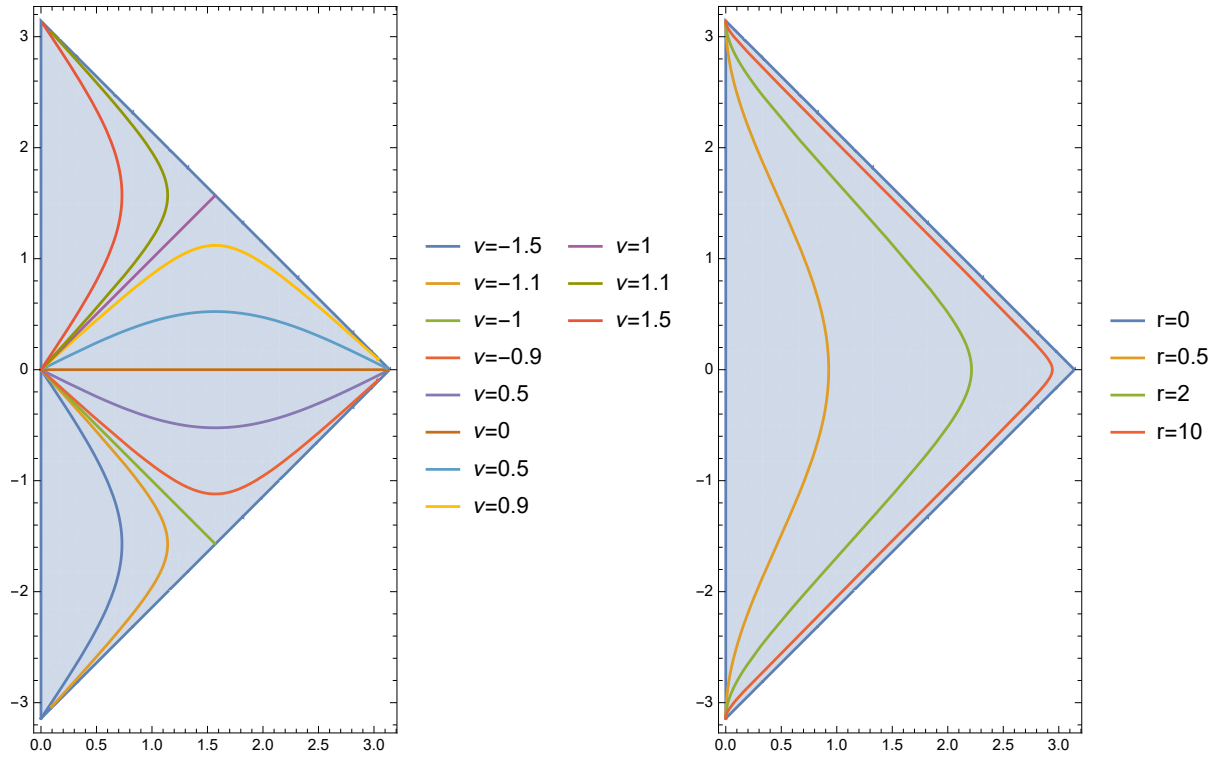


Figure 34.24: Lines of constant ν (left) and constant r (right).

6(b). Let us first determine the metric in these coordinates. We start from the metric of Minkowski space in the standard spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 \ . \tag{34.392}$$

Substituting in $t = \nu r$, we find

$$\begin{aligned} ds^2 &= -(rd\nu + \nu dr)^2 + dr^2 + r^2 d\Omega_2^2 \\ &= -r^2 d\nu^2 - 2\nu r d\nu dr + \left(1-\nu^2\right) dr^2 + r^2 d\Omega_2^2 \ . \end{aligned} \tag{34.393}$$

The inverse metric is

$$g^{\mu\rho} = \frac{1}{r^2} \begin{pmatrix} -\left(1-\nu^2\right) & -r\nu & 0 \\ -\nu r & r^2 & 0 \\ 0 & 0 & \gamma^{AB} \end{pmatrix} \ , \tag{34.394}$$

where $x^A = (\theta, \phi)$, and γ^{AB} is the inverse metric on the unit S^2. Note that $\sqrt{-g} = r^3\sqrt{\gamma}$. The scalar wave equation is then

$$\begin{aligned} 0 = D^2\varepsilon &= \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\rho} \partial_\rho \varepsilon\right) \\ &= -\frac{1}{r^2}\left(1-\nu^2\right)\partial_\nu^2 \varepsilon - \frac{2\nu}{r}\partial_\nu\partial_r\varepsilon + \frac{2}{r}\partial_r\varepsilon + \partial_r^2\varepsilon + \frac{1}{r^2} D^2\varepsilon \ , \end{aligned} \tag{34.395}$$

where D^2 is the Laplacian operator on S^2.

6(c). For r-independent solutions, the scalar wave equation reduces to

$$\left(1-\nu^2\right)\partial_\nu^2\varepsilon = D^2\varepsilon \ . \tag{34.396}$$

Now consider solutions of the form $\varepsilon_{\ell m}(\nu, x^A) = A_\ell(\nu) Y_m^\ell(x^A)$. By noting that $Y_m^\ell(x^A)$ satisfies $D^2 Y_m^\ell = -\ell\left(\ell+1\right) Y_m^\ell$, we find the following equation for $A_\ell(\nu)$:

$$\left(1-\nu^2\right)\partial_\nu^2 A_\ell(\nu) = -\ell\left(\ell+1\right) A_\ell(\nu) \ . \tag{34.397}$$

When $\ell = 0$, the most general solution is $A_0(\nu) = d_1\nu + d_2$. Requiring that the function be well defined at the origin, $\nu \to \pm\infty$ requires $d_1 = 0$. We also normalize the solution so that $A_\ell(1) = 1$. Thus, we must have $d_2 = 1$. So, for $\ell = 0$, the solution is

$$A_0(\nu) = 1 \ . \tag{34.398}$$

We now come to the less trivial case $\ell > 0$, which will be our assumption for the rest of this problem. The first step is to note that by taking a derivative of the equation above we arrive at

$$\left(1-\nu^2\right) A_\ell'''(\nu) - 2\nu A_\ell''(\nu) + \ell\left(\ell+1\right) A_\ell'(\nu) = 0 \ , \tag{34.399}$$

which is the Legendre equation for $A_\ell'(\nu)$. Thus, we must have

$$A_\ell'(\nu) = c_1 P_\ell(\nu) + c_2 Q_\ell(\nu) \ . \tag{34.400}$$

Integrating this equation yields

$$A_\ell(\nu) = \frac{c_1}{2\ell+1}\left[P_{\ell+1}(\nu) - P_{\ell-1}(\nu)\right] + \frac{c_2}{2\ell+1}\left[Q_{\ell+1}(\nu) - Q_{\ell-1}(\nu)\right] + c_3 \ , \tag{34.401}$$

where we have used the formula

$$\int d\nu P_\ell(\nu) = \frac{P_{\ell+1}(\nu) - P_{\ell-1}(\nu)}{2\ell+1} \ , \tag{34.402}$$

and similarly for Q_ℓ. Note that we originally started out with a second order differential equation, whereas our final answer for $A_\ell(\nu)$ contains three unknown constants. This is because we took a derivative of the original differential equation (34.397) in the process. To fix one of the constants, let us plug in our ansatz for the solution into the original differential equation. To do this, we use the property

$$\begin{aligned}\left[\left(1-\nu^2\right)\partial_\nu^2+\ell\left(\ell+1\right)\right]\left[P_{\ell+1}(\nu)-P_{\ell-1}(\nu)\right]=0\,,\\ \left[\left(1-\nu^2\right)\partial_\nu^2+\ell\left(\ell+1\right)\right]\left[Q_{\ell+1}(\nu)-Q_{\ell-1}(\nu)\right]=0\,.\end{aligned} \tag{34.403}$$

Thus, for (34.401) to satisfy the differential equation, we must have $c_3 = 0$. So, we find the solution

$$A_\ell(\nu)=c_1\left[P_{\ell+1}(\nu)-P_{\ell-1}(\nu)\right]+c_2\left[Q_{\ell+1}(\nu)-Q_{\ell-1}(\nu)\right]\,. \tag{34.404}$$

We are interested in studying the properties of this function at $\nu = \pm 1$ and $\nu = \pm\infty$. For this purpose, a more convenient representation of $Q_\ell(\nu)$ is

$$Q_\ell(\nu)=\frac{1}{2}P_\ell(\nu)\log\frac{1+\nu}{1-\nu}+F_\ell(\nu)\,,\qquad |\nu|<1\,, \tag{34.405}$$

where

$$F_\ell(\nu)=-\sum_{k=0}^{\lfloor\frac{\ell-1}{2}\rfloor}\frac{2\ell-4k-1}{(2k+1)\left(\ell-k\right)}P_{\ell-2k-1}(\nu)\,. \tag{34.406}$$

Note that $F_\ell(\nu)$ is a polynomial in ν. Then we have

$$A_\ell(\nu)=\left[c_1+\frac{c_2}{2}\log\frac{1+\nu}{1-\nu}\right]\left[P_{\ell+1}(\nu)-P_{\ell-1}(\nu)\right]+c_2\left[F_{\ell+1}(\nu)-F_{\ell-1}(\nu)\right], \tag{34.407}$$

for $|\nu| < 1$. Due to the logarithm in the parenthesis above, $A_\ell(\nu)$ has a branch point at $\nu = \pm 1$. Thus, we must choose an appropriate branch cut. We would like to choose the branch cut so that $A_\ell(\nu)$ is defined for all $\nu \in \mathbb{R}$ and so that it does not diverge at both $\nu \to \pm\infty$. As a result, the branch cuts must be chosen as follows. First, we choose an arbitrary branch cut at $\nu = +1$. This determines a particular way of defining $A_\ell(\nu)$ for $\nu > 1$ and in the process also specifies the asymptotic behavior of $A_\ell(\nu)$ at $\nu \to \infty$. Requiring that this is finite will fix c_1 in terms of c_2. Once this has been done, the choice of branch at $\nu = -1$ is fixed by requiring that the same solution (namely, for the same choice of c_1 and c_2) also be finite at $\nu \to -\infty$. We choose the branch cut at $\nu = 1$ so that it extends along $\nu = 1 + it$ for $t > 0$. With this choice of the branch cut, the extension of $A_\ell(\nu)$ to $\nu > 1$ is fixed as

$$A_\ell(\nu)=\left[c_1-\frac{ic_2\pi}{2}+\frac{c_2}{2}\log\frac{\nu+1}{\nu-1}\right]\left[P_{\ell+1}(\nu)-P_{\ell-1}(\nu)\right]+c_2\left[F_{\ell+1}(\nu)-F_{\ell-1}(\nu)\right]\,. \tag{34.408}$$

This extension can be seen by writing $\nu = 1 - re^{i\theta}$ near the branch point. Note that as θ goes from 0 to π, we move from $\nu < 1$ to $\nu > 1$ without crossing the branch cut. Plugging $\nu = 1 - re^{i\theta}$ into (34.407), then setting $\theta = \pi$, and finally setting $r = \nu - 1$, we reproduce (34.408). We can now study the asymptotics of this function at large ν. The leading term in the large ν expansion arises from $P_{\ell+1}(\nu)$, which goes like

$$P_{\ell+1}(\nu)\quad\to\quad\frac{2^{-\ell-1}(2\ell+2)!}{(\ell+1)!^2}\nu^{\ell+1}\qquad\text{as}\qquad\nu\to\infty\,. \tag{34.409}$$

Then we find

$$A_\ell(\nu) \to \left(c_1 - \frac{ic_2\pi}{2}\right) \frac{2^{-\ell-1}(2\ell+2)!}{(\ell+1)!^2}\nu^{\ell+1} + \cdots \ . \tag{34.410}$$

The quantity above is divergent unless $c_1 = \frac{ic_2\pi}{2}$. A proper check requires that we also verify that the subleading divergent powers also do not contribute in this limit. The subleading divergent powers do not contribute because

$$\frac{1}{2}\log\frac{\nu+1}{\nu-1}P_\ell(\nu) + F_\ell(\nu) = \mathcal{O}\left(\nu^{-\ell-1}\right) \qquad \text{at large } \nu \ . \tag{34.411}$$

Thus, we have determined that for the choice $c_1 = \frac{ic_2\pi}{2}$, $A_\ell(\nu)$ is completely finite at large ν. We now consider the extension of $A_\ell(\nu)$ to $\nu < -1$. Since the asymptotics of the Legendre polynomials is the same at $\nu \to -\infty$ as it is for $\nu \to +\infty$, it is clear that a branch cut must be chosen so that $A_\ell(\nu)$ takes exactly the form (34.408) for $\nu < -1$. This requires choosing a branch that starts at $\nu = -1$ and extends along $\nu = -1 + it$ for $t > 0$. The branch cuts for the function in the complex ν plane are shown in figure 34.25. In summary, we have so far

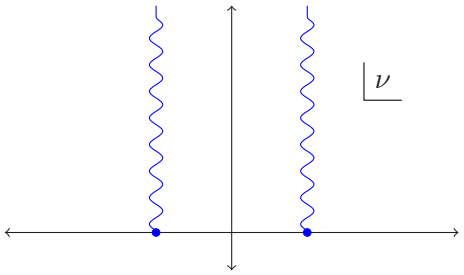


Figure 34.25: Branch cuts for $A_\ell(\nu)$.

constructed branch cuts so that the function is well defined at $\nu = \pm\infty$. In doing so, we have defined the function everywhere except at $\nu = \pm 1$. Indeed, due to the presence of the branch point, it is not a priori evident that the function can be defined at these points. However, note that this is not the case. The only divergence at $\nu = \pm 1$ arises from the logarithm. This is multiplied by $P_{\ell+1}(\nu) - P_{\ell-1}(\nu)$, which vanishes polynomially near $\nu = \pm 1$. Thus, $A_\ell(\pm 1)$ is finite and well defined. In fact, the function is also continuous at this point, since the only nonvanishing term in this limit arises from $F_\ell(\nu)$, which is a polynomial and continuous everywhere. Explicitly, we have

$$A_\ell(\pm 1) = \lim_{h\to 0^\pm} A_\ell(\pm 1 + h) = c_2\left[F_{\ell+1}(\pm 1) - F_{\ell-1}(\pm 1)\right] = -(\pm 1)^\ell \frac{c_2\,(2\ell+1)}{\ell\,(\ell+1)} \ . \tag{34.412}$$

As before, we normalize the function so that $A_\ell(1) = 1$. This fixes

$$c_2 = -\frac{\ell(\ell+1)}{2\ell+1} \ . \tag{34.413}$$

In summary, for $|\nu| < 1$, we have

$$A_\ell(\nu) = -\frac{\ell(\ell+1)}{2\ell+1}\left[\frac{1}{2}\log e^{i\pi}\frac{1+\nu}{1-\nu}\left[P_{\ell+1}(\nu) - P_{\ell-1}(\nu)\right] + F_{\ell+1}(\nu) - F_{\ell-1}(\nu)\right] \ , \tag{34.414}$$

whereas for $|\nu| > 1$, we have

$$A_\ell(\nu) = -\frac{\ell(\ell+1)}{2\ell+1}\left[\frac{1}{2}\left[P_{\ell+1}(\nu) - P_{\ell-1}(\nu)\right]\log\frac{\nu+1}{\nu-1} + F_{\ell+1}(\nu) - F_{\ell-1}(\nu)\right] \ . \tag{34.415}$$

6(d). For $\ell = 2$, the solution in part (c) (when $\nu > 1$) takes the form

$$A_2(\nu) = 3\nu^2 - 2 + \frac{3}{2}\nu\left(1-\nu^2\right)\log\frac{\nu+1}{\nu-1}\ . \tag{34.416}$$

Plugging in $\nu = \frac{t}{r} = 1 + \frac{u}{r}$ and expanding in powers of r, we find

$$A_2\left(1+\frac{u}{r}\right) = 1 + \frac{3u}{r}\left(2+\log\frac{u}{2r}\right) + \frac{3u^2}{2r^2}\left(1+3\log\frac{u}{2r}\right) + \mathcal{O}\big(r^{-3}\big)\ . \tag{34.417}$$

This implies that the expansion of the full solution is

$$\begin{aligned}\varepsilon_{2m}(u,r,\theta,\phi) = Y^2_m(\theta,\phi)\Bigg[1 + \frac{3u}{r}\left(2+\log\frac{u}{2r}\right)\\ +\frac{3u^2}{2r^2}\left(1+3\log\frac{u}{2r}\right) + \mathcal{O}\big(r^{-3}\big)\Bigg]\ .\end{aligned} \tag{34.418}$$

6(e). We now solve the scalar wave equation at large r in the retarded radial coordinates. The answer to part (d) of the problem shows that we must have a large-r falloff of the form

$$\begin{aligned}\varepsilon(u,r,z,\bar z) = \varepsilon^{(0)}(u,z,\bar z) + \frac{1}{r}\varepsilon^{(1)}(u,z,\bar z) + \frac{1}{r}f^{(1)}(u,z,\bar z)\log\frac{u}{2r}\\ + \frac{1}{r^2}\varepsilon^{(2)}(u,z,\bar z) + \frac{1}{r^2}f^{(2)}(u,z,\bar z)\log\frac{u}{2r} + \mathcal{O}\big(r^{-3}\big)\ .\end{aligned} \tag{34.419}$$

The scalar wave equation in the retarded radial coordinates is

$$\left[\partial_r^2 - 2\partial_u\partial_r + \frac{2}{r}\left(\partial_r - \partial_u\right) + \frac{2}{r^2}\gamma^{z\bar z}\partial_z\partial_{\bar z}\right]\varepsilon(u,r,z,\bar z) = 0\ . \tag{34.420}$$

Plugging in the expansion (34.419) and expanding in large r, we find the following equations order-by-order in large r:

$$\begin{aligned}&\partial_u\varepsilon^{(0)} = 0\ , \qquad \partial_u f^{(1)} + \frac{1}{2}D^2\varepsilon^{(0)} = 0\ , \qquad \partial_u f^{(2)} + \frac{1}{2}D^2 f^{(1)} = 0\ ,\\ &\partial_u\varepsilon^{(2)} = \frac{1}{2}D^2 f^{(1)} - \frac{1}{2}D^2\varepsilon^{(1)} - \frac{1}{2}f^{(1)} - \frac{1}{u}f^{(2)}\ .\end{aligned} \tag{34.421}$$

The solutions to these equations are

$$\begin{aligned}f^{(1)} &= -\frac{u}{2}D^2\varepsilon^{(0)} + g^{(1)}\ ,\\ f^{(2)} &= \frac{u^2}{8}\big(D^2\big)^2\varepsilon^{(0)} - \frac{u}{2}D^2 g^{(1)} + g^{(2)}\ ,\\ \varepsilon^{(2)} &= -\frac{3u^2}{16}\big(D^2\big)^2\varepsilon^{(0)} + \frac{u^2}{8}D^2\varepsilon^{(0)} + uD^2 g^{(1)} - \frac{u}{2}g^{(1)}\\ &\qquad - g^{(2)}\log u - \frac{1}{2}D^2\int du\varepsilon^{(1)} + h^{(2)}\ ,\end{aligned} \tag{34.422}$$

where all functions on the RHS are functions of $(z,\bar z)$, except for $\varepsilon^{(1)}$, which is an arbitrary function of $(u,z,\bar z)$. The solution is indeed the form of the solution ε_{2m} obtained in part (d). We can see this by choosing

$$\varepsilon^{(0)} = Y^2_m\ , \qquad \varepsilon^{(1)} = 6uY^2_m\ , \qquad g^{(1)} = g^{(2)} = h^{(2)} = 0\ . \tag{34.423}$$

In this case, we have

$$f^{(1)} = 3uY_m^2 \ , \qquad f^{(2)} = \frac{9u^2}{2}Y_m^2 \ , \qquad \varepsilon^{(2)} = \frac{3u^2}{2}Y_m^2 \ , \tag{34.424}$$

which is precisely the expansion (34.418).

7(a). Inverting the coordinates, we have $r = \rho\tau$ and $t = \tau\sqrt{1+\rho^2}$. We parametrize the massive momentum as $p^\mu = (\omega_p, \vec{p})$. Then we have

$$p \cdot x = \tau \left[-\sqrt{p^2+m^2}\sqrt{1+\rho^2} + \rho\vec{p}\cdot\hat{x} \right] = -\tau f(\vec{p}) \ . \tag{34.425}$$

In the mode expansion, we perform an integral over $\vec{p}_i$. In the limit of large τ, the value of the integral is dominated by the integrand at the minimum of the function $f(\vec{p})$:

$$\frac{\partial f}{\partial p_i} = 0 \qquad \Longrightarrow \qquad \frac{\vec{p}}{\sqrt{p^2+m^2}} = \frac{\rho}{\sqrt{1+\rho^2}}\hat{x} \ . \tag{34.426}$$

Let us simplify this expression a little more. Squaring both sides, we find

$$\frac{p^2}{p^2+m^2} = \frac{\rho^2}{1+\rho^2} \qquad \Longrightarrow \qquad \rho = \frac{|\vec{p}|}{m} \qquad \Longrightarrow \qquad \omega_p = m\sqrt{1+\rho^2} \ . \tag{34.427}$$

Then we find that the saddle point is at

$$\vec{p} = m\rho\hat{x} \ . \tag{34.428}$$

Next, in the saddle point approximation, we must evaluate the function f and its second derivative at this value. We find

$$f(m\rho\hat{x}) = m \ , \qquad \frac{\partial}{\partial p_i}\frac{\partial}{\partial p_j} f\Big|_{\vec{p}=m\rho\hat{x}} = \frac{1}{m}\left[\delta_{ij} - \frac{\rho^2}{1+\rho^2}\hat{x}_i\hat{x}_j \right] = \mathcal{P}_{ij} \ . \tag{34.429}$$

Then around the saddle point, expand

$$p \cdot x = -m\tau - \frac{\tau}{2}\mathcal{P}_{ij}(p_i - m\rho\hat{x}_i)(p_j - m\rho\hat{x}_j) + \mathcal{O}(p^3) \ . \tag{34.430}$$

At large τ, we can then write the mode expansion as

$$\begin{aligned} \phi(x) &= \frac{1}{2m(2\pi)^3\sqrt{1+\rho^2}} \Bigg[a(m\rho\hat{x})e^{-im\tau} \int d^3p e^{-i\frac{\tau}{2}\mathcal{P}_{ij}p_ip_j} \\ &\qquad\qquad + a^\dagger(m\rho\hat{x})e^{im\tau} \int d^3p e^{i\frac{\tau}{2}\mathcal{P}_{ij}p_ip_j} \Bigg] \\ &= \frac{\sqrt{\frac{(2\pi)^3}{\tau^3 \det\mathcal{P}}}}{2m(2\pi)^3\sqrt{1+\rho^2}} \left[a(m\rho\hat{x})e^{-im\tau}e^{-3i\pi/4} + a^\dagger(m\rho\hat{x})e^{im\tau}e^{3i\pi/4} \right] \ , \end{aligned} \tag{34.431}$$

where we used

$$\int_{-\infty}^{\infty} dx \ e^{ikx^2} = \sqrt{\frac{\pi}{k}}e^{i\pi/4} \ . \tag{34.432}$$

Next we must evaluate $\det \mathcal{P}$. An explicit computation shows

$$\det \mathcal{P} = \frac{1}{m^3 \left(1+\rho^2\right)} \ . \tag{34.433}$$

Using equation (34.433), we find

$$\phi(\tau, \rho, \hat{x}) = \frac{\sqrt{m}}{2(2\pi\tau)^{3/2}} \left[a(m\rho\hat{x}) e^{-im\tau} e^{-3i\pi/4} + a^\dagger(m\rho\hat{x}) e^{im\tau} e^{3i\pi/4}\right] \ . \tag{34.434}$$

7(b). We start with the action

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} \left[(\partial_\mu \phi)^2 + m^2 \phi^2\right] \ . \tag{34.435}$$

Varying (34.435) with respect to ϕ, we find

$$\delta S = \int d^4x \sqrt{-g} \left(\Box - m^2\right) \phi \delta\phi - \int d^4x \sqrt{-g} \nabla_\mu \left(\nabla^\mu \phi \delta\phi\right) \ . \tag{34.436}$$

The first term gives the equations of motion. The canonical one-form (symplectic potential current density) can be read off as $\Theta^\mu(\phi, \delta\phi) = -\nabla^\mu \phi \delta\phi$. The symplectic form is then

$$\begin{aligned} \Omega_\Sigma &= \int d\Sigma_\mu \omega^\mu = \int d\Sigma_\mu \left[\delta_1 \Theta^\mu \left(\phi, \delta_2 \phi\right) - 1 \leftrightarrow 2\right] \\ &= -\int d\Sigma_\mu \left[\nabla^\mu \delta_1 \phi \delta_2 \phi - \nabla^\mu \delta_2 \phi \delta_1 \phi\right] \ . \end{aligned} \tag{34.437}$$

7(c). The Minkowskian metric in $(\rho, \tau, \theta, \phi)$ coordinates is

$$ds^2 = -d\tau^2 + \tau^2 \left[\frac{d\rho^2}{1+\rho^2} + \rho^2 \gamma_{AB} dx^A dx^B\right] \ . \tag{34.438}$$

We consider the hypersurface Σ to be $\tau =$ constant. Then the unit normal is

$$n_\mu = \partial_\mu \tau = \delta^\tau_\mu \ . \tag{34.439}$$

The volume element is

$$d\Sigma_\mu = d^3V \tau^3 \delta^\tau_\mu \ , \tag{34.440}$$

where d^3V is the volume element on the unit hyperboloid. Using this volume element, we find

$$\Omega_\tau = \int d^3V \tau^3 \partial_\tau \delta\phi \wedge \delta\phi \ . \tag{34.441}$$

To simplify this, we can now insert the large-τ expression for ϕ. We then find at $\tau = \infty$ that

$$\Omega_{\tau\to\infty} = -\frac{im^2}{2(2\pi)^3} \int d^3V \ \delta a(m\rho\hat{x}) \wedge \delta a^\dagger(m\rho\hat{x}) \ . \tag{34.442}$$

7(d). The commutation relation obtained from equation (34.442) is

$$\left[a(m\rho\hat{x}), a^\dagger(m\rho'\hat{x}')\right] = \frac{2\,(2\pi)^3}{m^2}\delta\left(\rho-\rho'\right)\delta^2\left(\hat{x}-\hat{x}'\right)\ , \tag{34.443}$$

where the δ-functions are normalized so that

$$\int d^3V\delta\left(\rho-\rho'\right)\delta^2\left(\hat{x}-\hat{x}'\right) = 1\ . \tag{34.444}$$

To match (34.443) to the standard commutation relations in momentum space, we have to transform the δ-functions above to the standard three-vector notation. We do this as follows. Recall that we are parameterizing

$$\vec{p} = m\rho\hat{x} = \frac{m\rho}{1+z\bar{z}}\left(z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}\right)\ . \tag{34.445}$$

Then the standard volume form in momentum space is $dp^1\wedge dp^2\wedge dp^3$. Using the parameterization above, we find

$$dp^1\wedge dp^2\wedge dp^3 = im^3\rho^2\gamma_{z\bar{z}}d\rho\wedge dz\wedge d\bar{z}\ . \tag{34.446}$$

From equation (34.446), we find

$$\begin{aligned} 1 &= \int d^3p\delta^3\left(\vec{p}-\vec{p'}\right) \\ &= \int d\rho d^2z m^3\rho^2\gamma_{z\bar{z}}\delta^3\left(\vec{p}-\vec{p'}\right) \\ &= \int d^3V m^2\omega_p\delta^3\left(\vec{p}-\vec{p'}\right)\ . \end{aligned} \tag{34.447}$$

From equation (34.447), we can read off

$$m^2\omega_p\delta^3\left(\vec{p}-\vec{p'}\right) = \delta\left(\rho-\rho'\right)\delta^2\left(\hat{x}-\hat{x}'\right)\ . \tag{34.448}$$

Plugging this form into our commutation relations, we find

$$\left[a(m\rho\hat{x}), a^\dagger(m\rho'\hat{x}')\right] = (2\pi)^3\left(2\omega_p\right)\delta^3\left(\vec{p}-\vec{p'}\right)\ . \tag{34.449}$$

8. To determine the action of the charge, we first determine the symplectic form on $\mathcal{I}^+$. We do so by determining the boundary fields ϕ and ψ_+ on $\mathcal{I}^+$ using large-r limits of the bulk mode expansions. For this, we will use the saddle point approximation. We start by determining the boundary field of a scalar. Bulk scalar fields have mode expansions

$$\Phi(x) = \int\frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}\left[a(\vec{q})e^{iq\cdot x}+\tilde{a}^\dagger(\vec{q})e^{-iq\cdot x}\right]\ . \tag{34.450}$$

Here a and $\tilde{a}$ satisfy the usual creation-annihilation commutation relations. Then, taking a large-r limit and doing the usual saddle point procedure, we find

$$\phi(u,z,\bar{z}) = -\frac{i}{8\pi^2}\int_0^\infty d\omega\left[a(\omega\hat{x})e^{-i\omega u}-\tilde{a}^\dagger(\omega\hat{x})e^{i\omega u}\right]\ . \tag{34.451}$$

Using the standard commutators, we find

$$\left[\phi(u,z,\bar{z}),\bar{\phi}(u',z',\bar{z}')\right] = -\frac{i}{4}\gamma^{z\bar{z}}\Theta(u-u')\delta^2(z-z') \ . \tag{34.452}$$

Bulk spinors[223] have the mode expansion

$$\Psi_\alpha(x) = \int \frac{d^3q}{(2\pi)^3}\frac{1}{2\omega_q}\eta_\alpha(\vec{q})\left[b(\vec{q})e^{iq\cdot x} + \tilde{b}^\dagger(\vec{q})e^{-iq\cdot x}\right] \ , \tag{34.453}$$

where $\eta_\alpha(\vec{p})\bar{\eta}_{\dot{\alpha}}(\vec{p}) = p_\mu(\sigma^\mu)_{\alpha\dot{\alpha}}$, and b and $\tilde{b}$ satisfy the standard fermionic creation-annihilation commutation relations. To perform the large-r expansion, we will first have to convert the conventions of Wess and Bagger [207] to the $(u,r,z,\bar{z})$ coordinate system. First, we determine the basis spinors. In the $(u,r,z,\bar{z})$ coordinates used here, we have

$$\sigma_z{}^z = \frac{1}{r^2}\gamma^{z\bar{z}}\partial_z x^\mu \partial_{\bar{z}} x^\nu \sigma_{\mu\nu} = \begin{pmatrix} \frac{1-z\bar{z}}{2(1+z\bar{z})} & \frac{\bar{z}}{1+z\bar{z}} \\ \frac{z}{1+z\bar{z}} & -\frac{1-z\bar{z}}{2(1+z\bar{z})} \end{pmatrix} \ . \tag{34.454}$$

The eigenvalues of this matrix are $\pm\frac{1}{2}$ with eigenvectors

$$\xi^{(+)}_\alpha = \sqrt{\frac{1}{1+z\bar{z}}}\begin{pmatrix}1\\ z\end{pmatrix} \ , \qquad \xi^{(-)}_\alpha = \sqrt{\frac{1}{1+z\bar{z}}}\begin{pmatrix}\bar{z}\\ -1\end{pmatrix} \ . \tag{34.455}$$

The normalization of the eigenvectors is chosen so that $\xi^{(+)\alpha}\xi^{(-)}_\alpha = 1$. Since $\eta_\alpha(\vec{p})$ is a two-component spinor, it can be expanded in this basis. If we parametrize the momentum as

$$p^\mu = \frac{\omega}{1+z\bar{z}}\left(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}\right) \ , \tag{34.456}$$

then $\eta_\alpha(p) = \sqrt{2\omega}\xi^{(+)}_\alpha$. With all the explicit formulas determined, we can go ahead and take the large-r expansion of Ψ_α:

$$\psi_{(+)}(u,z,\bar{z}) = -\frac{i}{8\pi^2}\int_0^\infty d\omega_q\sqrt{2\omega_q}\left[b(\omega_q\hat{x})e^{-i\omega_q u} - \tilde{b}^\dagger(\omega_q\hat{x})e^{i\omega_q u}\right] \ . \tag{34.457}$$

Using the commutation relations for b and $\tilde{b}$, we find

$$\left\{\psi_{(+)}(u,z,\bar{z}),\bar{\psi}_{(-)}(u',z',\bar{z}')\right\} = \gamma^{z\bar{z}}\delta(u-u')\delta^2(z-z') \ . \tag{34.458}$$

Now we are ready to determine the action of the hard charge on the fields. The hard charge is

$$\mathscr{F}^h[\chi] = \frac{Q}{2\sqrt{2}}\int du d^2z\gamma_{z\bar{z}}\chi u\partial_u\left(\phi\bar{\psi}_{(-)}\right) \ . \tag{34.459}$$

It is clear that this acts trivially on ϕ and $\bar{\psi}_{(-)}$. However, we have

$$\begin{aligned}
\left[\mathscr{F}^h[\chi],\partial_u\bar{\phi}(u,z,\bar{z})\right] &= \frac{Q}{2\sqrt{2}}\int du'd^2z'\gamma_{z'\bar{z}'}\chi u'\partial_{u'}\partial_u\left(\left[\phi(u',z',\bar{z}'),\bar{\phi}(u,z,\bar{z})\right]\bar{\psi}_{(-)}\right) \\
&= -i\frac{Q}{4\sqrt{2}}\chi(z,\bar{z})\bar{\psi}_{(-)}(z,\bar{z}) \ , \\
\left\{\mathscr{F}^h[\chi],\psi_{(+)}(u,z,\bar{z})\right\} &= \frac{Q}{2\sqrt{2}}\int du'd^2z'\gamma_{z'\bar{z}'}\chi u'\partial_{u'}\left(\phi\left\{\bar{\psi}_{(-)},\psi_{(+)}(u,z,\bar{z})\right\}\right) \\
&= -\frac{Q}{2\sqrt{2}}\chi(z,\bar{z})\phi(u,z,\bar{z}) \ ,
\end{aligned} \tag{34.460}$$

which determines the action of the hard charge on the fields.

[223] Spinor conventions of Wess and Bagger [207] are used here.

9(a). We consider an $(n+2)$-point amplitude $\mathcal{A}_{n+2}^{a_1,a_2}(q_1,\varepsilon_1;q_2,\varepsilon_2)$. The amplitude also depends on the other external momenta, but we suppress the dependence for convenience. Taking first the limit $q_2 \to 0$ and using the soft gluon theorem, we find

$$\begin{aligned}\lim_{q_2\to 0}\mathcal{A}_{n+2}^{a_1,a_2}(q_1,\varepsilon_1;q_2,\varepsilon_2) \to g_{YM}\sum_{k=1}^{n}\frac{p_k\cdot\varepsilon_2}{p_k\cdot q_2}T_k^{a_2}\mathcal{A}_{n+1}^{a_1}(q_1,\varepsilon_1)\\ + g_{YM}\frac{q_1\cdot\varepsilon_2}{q_1\cdot q_2}\left(T_{\text{adj.}}^{a_2}\right)^{a_1 b}\mathcal{A}_{n+1}^{b}(q_1,\varepsilon_1)\ ,\end{aligned} \tag{34.461}$$

where we have used the fact that the gluon transforms in the adjoint representation. The matrix elements of the adjoint representations are $\left(T_{\text{adj.}}^{a_2}\right)^{a_1 b} = -if^{a_2 a_1 b} = if^{a_1 a_2 b}$. Using this, we find

$$\begin{aligned}\lim_{q_2\to 0}\mathcal{A}_{n+2}^{a_1,a_2}(q_1,\varepsilon_1;q_2,\varepsilon_2) \to g_{YM}\sum_{k=1}^{n}\frac{p_k\cdot\varepsilon_2}{p_k\cdot q_2}T_k^{a_2}\mathcal{A}_{n+1}^{a_1}(q_1,\varepsilon_1)\\ + ig_{YM}f^{a_1 a_2 b}\frac{q_1\cdot\varepsilon_2}{q_1\cdot q_2}\mathcal{A}_{n+1}^{b}(q_1,\varepsilon_1)\ .\end{aligned} \tag{34.462}$$

Let us now take the second limit, $q_1 \to 0$:

$$\begin{aligned}\lim_{q_1\to 0}\lim_{q_2\to 0}\mathcal{A}_{n+2}^{a_1,a_2}(q_1,\varepsilon_1;q_2,\varepsilon_2) \to g_{YM}^2\sum_{k=1}^{n}\sum_{k'=1}^{n}\frac{p_k\cdot\varepsilon_2}{p_k\cdot q_2}\frac{p_{k'}\cdot\varepsilon_1}{p_{k'}\cdot q_1}T_k^{a_2}T_{k'}^{a_1}\mathcal{A}_n\\ + ig_{YM}^2 f^{a_1 a_2 b}\frac{q_1\cdot\varepsilon_2}{q_1\cdot q_2}\sum_{k=1}^{n}\frac{p_k\cdot\varepsilon_1}{p_k\cdot q_1}T_k^{b}\mathcal{A}_n\ .\end{aligned} \tag{34.463}$$

When the limits are taken in the opposite order, we find

$$\begin{aligned}\lim_{q_2\to 0}\lim_{q_1\to 0}\mathcal{A}_{n+2}^{a_1,a_2}(q_1,\varepsilon_1;q_2,\varepsilon_2) \to g_{YM}^2\sum_{k=1}^{n}\sum_{k'=1}^{n}\frac{p_k\cdot\varepsilon_2}{p_k\cdot q_2}\frac{p_{k'}\cdot\varepsilon_1}{p_{k'}\cdot q_1}T_{k'}^{a_1}T_k^{a_2}\mathcal{A}_n\\ - ig_{YM}^2 f^{a_1 a_2 b}\frac{q_2\cdot\varepsilon_1}{q_2\cdot q_1}\sum_{k=1}^{n}\frac{p_k\cdot\varepsilon_2}{p_k\cdot q_2}T_k^{b}\mathcal{A}_n\ .\end{aligned} \tag{34.464}$$

Now subtract the two. Note that in the double sum, when $k \neq k'$ the two quantities are equal, since the generators T_k are acting on different sets of indices. The only nontrivial contribution to the difference occurs when $k = k'$. Thus, we find

$$\begin{aligned}\left[\lim_{q_1\to 0},\lim_{q_2\to 0}\right]\mathcal{A}_{n+2}^{a_1,a_2}(q_1,\varepsilon_1;q_2,\varepsilon_2) &\to g_{YM}^2\sum_{k=1}^{n}\frac{p_k\cdot\varepsilon_2}{p_k\cdot q_2}\frac{p_k\cdot\varepsilon_1}{p_k\cdot q_1}\left[T_k^{a_2},T_k^{a_1}\right]\mathcal{A}_n\\ &\quad + ig_{YM}^2 f^{a_1 a_2 b}\frac{q_1\cdot\varepsilon_2}{q_1\cdot q_2}\sum_{k=1}^{n}\frac{p_k\cdot\varepsilon_1}{p_k\cdot q_1}T_k^{b}\mathcal{A}_n\\ &\quad + ig_{YM}^2 f^{a_1 a_2 b}\frac{q_2\cdot\varepsilon_1}{q_2\cdot q_1}\sum_{k=1}^{n}\frac{p_k\cdot\varepsilon_2}{p_k\cdot q_2}T_k^{b}\mathcal{A}_n\\ &\to ig_{YM}^2 f^{a_1 a_2 b}\sum_{k=1}^{n}\left(\frac{q_1\cdot\varepsilon_2}{q_1\cdot q_2}\frac{p_k\cdot\varepsilon_1}{p_k\cdot q_1}\right.\\ &\quad \left.+\frac{q_2\cdot\varepsilon_1}{q_2\cdot q_1}\frac{p_k\cdot\varepsilon_2}{p_k\cdot q_2} - \frac{p_k\cdot\varepsilon_2}{p_k\cdot q_2}\frac{p_k\cdot\varepsilon_1}{p_k\cdot q_1}\right)T_k^{b}\mathcal{A}_n\ .\end{aligned} \tag{34.465}$$

This can be written in a more concise way by adding to it a fourth term that is identically zero:

$$-ig_{YM}^2 f^{a_1 a_2 b} \sum_{k=1}^{n} \frac{q_1\cdot\varepsilon_2}{q_1\cdot q_2}\frac{q_2\cdot\varepsilon_1}{q_2\cdot q_1} T_k^b \mathcal{A}_n = -ig_{YM}^2 f^{a_1 a_2 b} \frac{q_1\cdot\varepsilon_2}{q_1\cdot q_2}\frac{q_2\cdot\varepsilon_1}{q_2\cdot q_1} \sum_{k=1}^{n} T_k^b \mathcal{A}_n \ . \tag{34.466}$$

The last term vanishes due to global color charge conservation. Adding this, we can write the double-soft limit as

$$\begin{aligned}\left[\lim_{q_1\to 0}\ ,\ \lim_{q_2\to 0}\right] &\mathcal{A}_{n+2}^{a_1,a_2}(q_1,\varepsilon_1;q_2,\varepsilon_2) \\ &\to ig_{YM}^2 f^{a_1 a_2 b} \sum_{k=1}^{n} \left(\frac{p_k\cdot\varepsilon_2}{p_k\cdot q_2} - \frac{q_1\cdot\varepsilon_2}{q_1\cdot q_2}\right)\left(\frac{q_2\cdot\varepsilon_1}{q_2\cdot q_1} - \frac{p_k\cdot\varepsilon_1}{p_k\cdot q_1}\right) T_k^b \mathcal{A}_n \ .\end{aligned} \tag{34.467}$$

9(b). All we have to do here is to plug in the explicit forms of the momenta in the z parametrization. Recall that when the helicity is positive,

$$\frac{p_k\cdot\varepsilon}{p_k\cdot q} \propto \frac{1}{z-z_k} \ . \tag{34.468}$$

Then the soft factor is

$$\begin{aligned}\left(\frac{p_k\cdot\varepsilon_2}{p_k\cdot q_2} - \frac{q_1\cdot\varepsilon_2}{q_1\cdot q_2}\right)\left(\frac{q_2\cdot\varepsilon_1}{q_2\cdot q_1} - \frac{p_k\cdot\varepsilon_1}{p_k\cdot q_1}\right) &\propto \left(\frac{1}{w-z_k} - \frac{1}{w-z}\right)\left(\frac{1}{z-w} - \frac{1}{z-z_k}\right) \\ &= \frac{z_k - z}{(w-z_k)(w-z)}\frac{w-z_k}{(z-w)(z-z_k)} \\ &= \frac{1}{(z-w)^2} \ .\end{aligned} \tag{34.469}$$

This factor does not depend on z_k at all. Thus, we can pull this factor out of the sum to get

$$\left[\lim_{q_1\to 0}\ ,\ \lim_{q_2\to 0}\right] \mathcal{A}_{n+2}^{a_1,a_2}(q_1,\varepsilon_1;q_2,\varepsilon_2) \propto \frac{ig_{YM}^2 f^{a_1 a_2 b}}{(z-w)^2} \sum_{k=1}^{n} T_k^b \mathcal{A}_n = 0 \ . \tag{34.470}$$

The argument follows through in the same way when the helicity is negative (just take all z's to $\bar{z}$'s).

10. We take the metric to be of the form

$$\begin{aligned}ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + \frac{2m_B}{r}du^2 + rC_{zz}dz^2 + rC_{\bar{z}\bar{z}}d\bar{z}^2 \\ + U_z dudz + U_{\bar{z}}dud\bar{z} + \cdots \ .\end{aligned} \tag{34.471}$$

The four-dimensional Weyl tensor of a spacetime is given by

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{2}\left(g_{\nu\rho}R_{\sigma\mu} + g_{\mu\sigma}R_{\rho\nu} - g_{\nu\sigma}R_{\rho\mu} - g_{\mu\rho}R_{\sigma\nu}\right) + \frac{1}{6}R\left(g_{\mu\rho}g_{\sigma\nu} - g_{\mu\sigma}g_{\rho\nu}\right) \ . \tag{34.472}$$

We are interested in computing the following components:

$$C_{rzrz} = R_{rzrz} - \frac{1}{2}g_{zz}R_{rr} \ , \qquad C_{rurz} = R_{rurz} + \frac{1}{2}\left(g_{ur}R_{rz} - g_{uz}R_{rr}\right) \ . \tag{34.473}$$

For the metric of interest, we then find

$$C_{rzrz} = \mathcal{O}(r^{-3}) \ , \qquad C_{rurz} = -\frac{1}{4r^2}\left(U_z - D^z C_{zz}\right) + \mathcal{O}(r^{-3}) \ , \tag{34.474}$$

where D_z is the covariant derivative with respect to $\gamma_{z\bar{z}}$. Thus, to have an asymptotically flat spacetime, we must have

$$U_z = D^z C_{zz} \ . \tag{34.475}$$

11(a). We will show that $\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0$ implies that C_{AB} is traceless.

$$\begin{aligned} \det\left(\frac{g_{AB}}{r^2}\right) &= \det\left(\gamma_{AB} + \frac{C_{AB}}{r} + \mathcal{O}(r^{-2})\right) \\ &= \det\gamma\left(1 + \frac{C^A{}_A}{r} + \mathcal{O}(r^{-2})\right) \ . \end{aligned} \tag{34.476}$$

Hence,

$$\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = -\det\gamma\ \frac{C^A{}_A}{r^2} + \mathcal{O}(r^{-3}) \ . \tag{34.477}$$

Therefore,

$$\partial_r \det\left(\frac{g_{AB}}{r^2}\right) = 0 \qquad \Longrightarrow \qquad C^A{}_A = 0 \ . \tag{34.478}$$

11(b). First, we must determine the variation of the metric under a diffeomorphism generated by the vector field ξ, where $\xi^u, \xi^r \sim \mathcal{O}(1)$ and $\xi^A \sim \mathcal{O}(r^{-1})$. The variation of the metric under a diffeomorphism is given by

$$\mathcal{L}_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho}\partial_\nu \xi^\rho + g_{\nu\rho}\partial_\mu \xi^\rho \ . \tag{34.479}$$

By definition, the asymptotic symmetries must preserve the falloff conditions for the fields:

$$\begin{aligned} g_{uu} &= -U + \frac{1}{4} g_{AB} U^A U^B = -1 + \frac{2m_B}{r} + \mathcal{O}(r^{-2}) \ , \\ g_{ur} &= -e^{2\beta} = -1 + \mathcal{O}(r^{-2}) \ , \\ g_{uA} &= \frac{1}{2} g_{AB} U^B = \frac{1}{2} D^B C_{BA} + \mathcal{O}(r^{-1}) \ , \\ g_{rA} &= g_{rr} = 0 \ , \\ g_{AB} &= r^2 \gamma_{AB} + r C_{AB} + h^{(0)}_{AB} + \mathcal{O}(r^{-1}) \ . \end{aligned} \tag{34.480}$$

In Bondi gauge,

$$\mathcal{L}_\xi g_{rr} = 2\partial_r \xi^u g_{ur} \ , \tag{34.481}$$

which implies ξ^u must be independent of r

$$\xi^u = \xi^u(u, x^A) \ . \tag{34.482}$$

To leading order in the asymptotic expansion, we have

$$\mathcal{L}_\xi g_{ur} = \partial_u \xi^u g_{ur} + \mathcal{O}(r^{-1}) \ . \tag{34.483}$$

Hence, ξ^u must also be independent of u:

$$\xi^u = f(x^A) \ . \tag{34.484}$$

Then the remaining components have the following variations under ξ, with $\xi^u = f$:

$$\begin{aligned}
\mathcal{L}_\xi g_{uu} &= f\partial_u g_{uu} + \xi^r \partial_r g_{uu} + \xi^A \partial_A g_{uu} + 2\partial_u \xi^r g_{ru} + 2\partial_u \xi^A g_{Au} \ , \\
\mathcal{L}_\xi g_{rA} &= \partial_r \xi^C g_{CA} + g_{ur}\partial_A f \ , \\
\mathcal{L}_\xi g_{AB} &= f\partial_u g_{AB} + \xi^r \partial_r g_{AB} + \xi^C \partial_C g_{AB} + g_{uB}\partial_A f + \partial_A \xi^C g_{CB} \\
&\qquad + g_{uA}\partial_B f + \partial_B \xi^C g_{CA} \ .
\end{aligned} \tag{34.485}$$

Let us consider the asymptotic expansion of the vector field

$$\xi = f\partial_u + \sum_{n=0}^{\infty} \frac{\xi^{r(n)}}{r^n}\partial_r + \sum_{n=1}^{\infty} \frac{\xi^{A(n)}}{r^n}\partial_A \ . \tag{34.486}$$

The Lie derivative of the metric is then

$$\begin{aligned}
\mathcal{L}_\xi g_{uu} &= -2\partial_u \xi^{r(0)} + \frac{-2\partial_u \xi^{r(1)} + 2f\partial_u m_B + \partial_u \xi^{A(1)} D^B C_{BA}}{r} + \mathcal{O}(r^{-2}) \ , \\
\mathcal{L}_\xi g_{rA} &= -\gamma_{AB}\left(\xi^{B(1)} + D^B f\right) - \frac{2\gamma_{AB}\xi^{B(2)} + C_{AB}\xi^{B(1)}}{r} + \mathcal{O}(r^{-2}) \ , \\
\mathcal{L}_\xi g_{AB} &= r\Big(f\partial_u C_{AB} + 2\gamma_{AB}\xi^{r(0)} + D_A \xi_B^{(1)} + D_B \xi_A^{(1)}\Big) \\
&\quad + f\partial_u h_{AB}^{(0)} + C_{AB}\xi^{r(0)} + 2\gamma_{AB}\xi^{r(1)} + \frac{1}{2}D^C C_{CA} D_B f \\
&\quad + \frac{1}{2}D^C C_{CB} D_A f + \xi^{C(1)} D_C C_{AB} + D_A \xi_B^{(2)} \\
&\quad + C_{BC} D_A \xi^{C(1)} + D_B \xi_A^{(2)} + C_{AC} D_B \xi^{C(1)} + \mathcal{O}(r^{-1}) \ .
\end{aligned} \tag{34.487}$$

From $\mathcal{L}_\xi g_{rA}$ at $\mathcal{O}(1)$, we find

$$\xi^{A(1)} = -D^A f \ . \tag{34.488}$$

From $\mathcal{L}_\xi g_{rA}$ at $\mathcal{O}(r^{-1})$, we find

$$\xi^{A(2)} = \frac{1}{2}C^{AB} D_B f \ . \tag{34.489}$$

We require that the $\mathcal{O}(r)$ term of $\mathcal{L}_\xi g_{AB}$ be traceless, which implies

$$\xi^{r(0)} = \frac{1}{2} D_A D^A f \ . \tag{34.490}$$

Finally, to determine the $\mathcal{O}(r^{-2})$ piece of ξ^r, we need to consider a higher order expansion of g_{AB}:

$$g_{AB} = r^2 \gamma_{AB} + r C_{AB} + h_{AB}^{(0)} + \mathcal{O}(r^{-1}). \tag{34.491}$$

Then

$$\begin{aligned}
\det\left(\frac{g_{AB}}{r^2}\right) &= \det\gamma \exp \mathrm{tr}\left[\log\left(\delta_A^B + \frac{1}{r}C_A{}^B + \frac{1}{r^2}h_A^{(0)B} + \mathcal{O}(r^{-3})\right)\right] \\
&= \det\gamma\left[1 + \frac{1}{r}C_A{}^A \right. \\
&\qquad \left. + \frac{1}{r^2}\left(h_A^{(0)A} - \frac{1}{2}C_A{}^B C_B{}^A + \frac{1}{2}C_A{}^A C_B{}^B\right) + \mathcal{O}(r^{-3})\right] \ .
\end{aligned} \tag{34.492}$$

Assuming that $C_A{}^A = 0$ to satisfy the $\mathcal{O}(r^{-1})$ constraint, we additionally find that

$$h_A^{(0)A} = \frac{1}{2} C_{AB} C^{AB} \ . \tag{34.493}$$

This imposes the following constraint on the variations of the metric components

$$\delta h_A^{(0)A} = C_{AB} \delta C^{AB} \ . \tag{34.494}$$

An explicit computation reveals

$$\begin{aligned} \delta h_A^{(0)A} &= f \partial_u h_A^{(0)A} + 4\xi^{r(1)} + 2 D_A f D_B C^{AB} - C^{AB} D_A D_B f \ , \\ C_{AB} \delta C^{AB} &= \frac{f}{2} \partial_u (C^{AB} C_{AB}) - 2 C^{AB} D_A D_B f \ . \end{aligned} \tag{34.495}$$

Equating these terms, we find

$$\xi^{r(1)} = -\frac{1}{2} D_A f D_B C^{AB} - \frac{1}{4} C^{AB} D_A D_B f \ . \tag{34.496}$$

Therefore, the most general diffeomorphism ξ that preserves the metric to leading order is given by

$$\begin{aligned} \xi(f) = f \partial_u &+ \left[-\frac{D^A f}{r} + \frac{\frac{1}{2} C^{AB} D_B f}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \right] \partial_A \\ &+ \left[\frac{1}{2} D^2 f + \frac{-\frac{1}{2} D_A f D_B C^{AB} - \frac{1}{4} C^{AB} D_A D_B f}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \right] \partial_r \ , \end{aligned} \tag{34.497}$$

where $f = f(x^A)$.

11(c). We will show that the Lie algebra of the supertranslations at leading order is abelian, that is,

$$[\xi(f_1), \xi(f_2)] = \mathcal{O}(r^{-1}) \ . \tag{34.498}$$

Using the definition of the Lie bracket, we find

$$\begin{aligned} [\xi(f_1), \xi(f_2)]^u &= \xi(f_1)^\mu \partial_\mu f_2 - \xi(f_2)^\mu \partial_\mu f_1 \\ &= \frac{-D^A f_1 D_A f_2}{r} + \frac{\frac{1}{2} C^{AB} D_A f_1 D_B f_2}{r^2} - (f_1 \leftrightarrow f_2) \\ &= 0 \ , \\ [\xi(f_1), \xi(f_2)]^r &= \xi(f_1)^\mu \partial_\mu \left(\frac{1}{2} D^2 f_2 \right) - (f_1 \leftrightarrow f_2) + \mathcal{O}(r^{-2}) \\ &= \frac{1}{2r} D_A \left(-D^A f_1 D^2 f_2 + D^A f_2 D^2 f_1 \right) + \mathcal{O}(r^{-2}) \\ &= \mathcal{O}(r^{-1}) \ , \\ [\xi(f_1), \xi(f_2)]^A &= \xi(f_1)^\mu \partial_\mu \left[-\frac{D^A f_2}{r} \right] - (f_1 \leftrightarrow f_2) + \mathcal{O}(r^{-3}) \\ &= \frac{1}{2} \frac{D^2 f_1 D^A f_2}{r^2} + \frac{D^B f_1 D_B D^A f_2}{r^2} - 1 \leftrightarrow 2 + \mathcal{O}(r^{-3}) \\ &= \mathcal{O}(r^{-2}) \ . \end{aligned} \tag{34.499}$$

11(d). Recall in part (b), Eq.(34.487), we found

$$\begin{aligned}\mathcal{L}_\xi g_{uu} &= -2\partial_u \xi^{r(0)} + \frac{-2\partial_u \xi^{r(1)} + 2f\partial_u m_B + \partial_u \xi^{A(1)} D^B C_{BA}}{r} + \mathcal{O}(r^{-2}) \ , \\ \mathcal{L}_\xi g_{AB} &= r\Big(f\partial_u C_{AB} + 2\gamma_{AB}\xi^{r(0)} + D_A \xi^{(1)}_B + D_B \xi^{(1)}_A\Big) + \mathcal{O}(1) \ .\end{aligned} \tag{34.500}$$

Using the solution for part (b) and the variations above, we can calculate the action of $\xi(f)$ on m_B, C_{AB} and N_{AB}:

$$\begin{aligned}\mathcal{L}_f C_{AB} &= f\partial_u C_{AB} + \gamma_{AB} D^2 f - 2D_A D_B f \ , \\ \mathcal{L}_f N_{AB} &= f\partial_u N_{AB} \ , \\ \mathcal{L}_f m_B &= f\partial_u m_B + \frac{1}{4}\Big(N^{AB} D_A D_B f + 2D_A f D_B N^{AB}\Big) \ .\end{aligned} \tag{34.501}$$

Note that $\gamma^{AB}\mathcal{L}_f C_{AB} = 0$.

11(e). Let us now evaluate the vector field $\xi(f)$ on the $\ell = 0$ and $\ell = 1$ spherical harmonics.

$$\begin{aligned}\xi(Y_0^0) &= Y_0^0 \partial_u \ , \\ \xi(Y_1^m) &= Y_1^m \partial_u - \frac{\gamma^{AB}\partial_B Y_1^m}{r}\partial_A + \frac{1}{2}D^2 Y_1^m \partial_r \ .\end{aligned} \tag{34.502}$$

Consider the following normalization for the spherical harmonics:

$$Y_0^0 = 1, \qquad Y_1^1 = \frac{z}{1+z\bar{z}} \ , \qquad Y_1^0 = \frac{1-z\bar{z}}{1+z\bar{z}} \ , \qquad Y_1^{-1} = \frac{\bar{z}}{1+z\bar{z}} \ . \tag{34.503}$$

Then we have

$$\begin{aligned}\xi(Y_0^0) &= \partial_u \ , \\ \xi(Y_1^1) &= \frac{z}{1+z\bar{z}}(\partial_u - \partial_r) + \frac{z^2}{2r}\partial_z - \frac{1}{2r}\partial_{\bar{z}} \ , \\ \xi(Y_1^0) &= \frac{1-z\bar{z}}{1+z\bar{z}}(\partial_u - \partial_r) + \frac{z}{r}\partial_z + \frac{\bar{z}}{r}\partial_{\bar{z}} \ , \\ \xi(Y_1^{-1}) &= \frac{\bar{z}}{1+z\bar{z}}(\partial_u - \partial_r) - \frac{1}{2r}\partial_z + \frac{\bar{z}^2}{2r}\partial_{\bar{z}} \ .\end{aligned} \tag{34.504}$$

To compare with the standard global translations in Minkowski space, we begin with these translations in Cartesian coordinates:

$$T = \partial_t \ , \qquad X_i = \partial_i \ . \tag{34.505}$$

In Bondi coordinates, the global translations take the following form:

$$\begin{aligned}T &= \partial_u \ , \\ X_1 &= -\frac{z+\bar{z}}{1+z\bar{z}}\partial_u + \frac{z+\bar{z}}{1+z\bar{z}}\partial_r + \frac{1-z^2}{2r}\partial_z + \frac{1-\bar{z}^2}{2r}\partial_{\bar{z}} \ , \\ X_2 &= \frac{i(z-\bar{z})}{1+z\bar{z}}\partial_u - \frac{i(z-\bar{z})}{1+z\bar{z}}\partial_r + \frac{i(1+z^2)}{2r}\partial_z - \frac{i(1+\bar{z}^2)}{2r}\partial_{\bar{z}} \ , \\ X_3 &= -\frac{1-z\bar{z}}{1+z\bar{z}}\partial_u + \frac{1-z\bar{z}}{1+z\bar{z}}\partial_r - \frac{z}{r}\partial_z - \frac{\bar{z}}{r}\partial_{\bar{z}} \ .\end{aligned} \tag{34.506}$$

We then find

$$\begin{aligned}\xi(Y_0^0) &= T \ , \\ \xi(Y_1^1) &= -\frac{1}{2}(X_1 + iX_2) \ , \\ \xi(Y_1^0) &= -X_3 \ , \\ \xi(Y_1^{-1}) &= -\frac{1}{2}(X_1 - iX_2) \ .\end{aligned} \tag{34.507}$$

The action of the $\ell = 0$ mode on the fields is given by

$$\begin{aligned}\mathcal{L}_{\xi(Y_0^0)} C_{AB} &= Y_0^0 \partial_u C_{AB} \ , \\ \mathcal{L}_{\xi(Y_0^0)} N_{AB} &= Y_0^0 \partial_u N_{AB} \ , \\ \mathcal{L}_{\xi(Y_0^0)} m_B &= Y_0^0 \partial_u m_B \ .\end{aligned} \tag{34.508}$$

The action of the $\ell = 1$ modes on the fields is given by

$$\begin{aligned}\mathcal{L}_{\xi(Y_1^m)} C_{AB} &= Y_1^m \partial_u C_{AB} \ , \qquad \mathcal{L}_{\xi(Y_1^m)} N_{AB} = Y_1^m \partial_u N_{AB}, \\ \mathcal{L}_{\xi(Y_1^1)} m_B &= Y_1^1 \partial_u m_B + \frac{1}{4}\Big(D^{\bar{z}} N_{\bar{z}\bar{z}} - z^2 D^z N_{zz} \Big) \ , \\ \mathcal{L}_{\xi(Y_1^0)} m_B &= Y_1^0 \partial_u m_B - \frac{1}{2}\Big(z D^z N_{zz} + \bar{z} D^{\bar{z}} N_{\bar{z}\bar{z}} \Big) \ , \\ \mathcal{L}_{\xi(Y_1^{-1})} m_B &= Y_1^{-1} \partial_u m_B + \frac{1}{4}\Big(D^z N_{zz} - \bar{z}^2 D^{\bar{z}} N_{\bar{z}\bar{z}} \Big) \ ,\end{aligned} \tag{34.509}$$

where we have used the fact that $D_z D_z Y_1^m = 0$.

12(a). The Lorentz transformations in Cartesian coordinates are given by

$$K_i = x^0 \partial_i + x^i \partial_0, \qquad J_{ij} = x^i \partial_j - x^j \partial_i \ , \tag{34.510}$$

where K_i are the boosts and J_{ij} are the rotations. The Cartesian coordinates are related to the $(u, r, z, \bar{z})$ coordinates by the coordinate transformations

$$u = t - \sqrt{x^i x_i} \ , \qquad r = \sqrt{x^i x_i} \ , \qquad z = \frac{x^1 + i x^2}{x^3 + \sqrt{x^i x_i}} \ . \tag{34.511}$$

In these coordinates, the boosts and rotations take the following form:

$$\begin{aligned}K_1 &= -\frac{u(z+\bar{z})}{1+z\bar{z}} \partial_u + \frac{(r+u)(z+\bar{z})}{1+z\bar{z}} \partial_r \\ &\qquad - \frac{(z^2-1)\,(r+u)}{2r} \partial_z - \frac{(\bar{z}^2-1)\,(r+u)}{2r} \partial_{\bar{z}} \ , \\ K_2 &= \frac{iu(z-\bar{z})}{1+z\bar{z}} \partial_u - \frac{i(r+u)(z-\bar{z})}{1+z\bar{z}} \partial_r \\ &\qquad + \frac{i\,(z^2+1)\,(r+u)}{2r} \partial_z - \frac{i\,(\bar{z}^2+1)\,(r+u)}{2r} \partial_{\bar{z}} \ , \\ K_3 &= -\frac{u(1-z\bar{z})}{1+z\bar{z}} \partial_u + \frac{(r+u)(1-z\bar{z})}{1+z\bar{z}} \partial_r - \frac{z(r+u)}{r} \partial_z - \frac{\bar{z}(r+u)}{r} \partial_{\bar{z}} \ , \\ J_{12} &= iz\partial_z - i\bar{z}\partial_{\bar{z}} \ , \\ J_{23} &= \frac{1}{2} i \left(z^2-1\right) \partial_z - \frac{1}{2} i \left(\bar{z}^2-1\right) \partial_{\bar{z}} \ , \\ J_{31} &= \frac{1}{2} \left(z^2+1\right) \partial_z + \frac{1}{2} \left(\bar{z}^2+1\right) \partial_{\bar{z}} \ .\end{aligned} \tag{34.512}$$

Then in Bondi gauge, a generic Lorentz transformation is generated by the vector field ζ, with components

$$\begin{aligned}\zeta^u &= \frac{u}{2}\,\frac{2z(c-a^*)+2\bar{z}(c^*-a)+(b+b^*)(1-z\bar{z})}{1+z\bar{z}}\,,\\ \zeta^r &= -\frac{u+r}{2}\,\frac{2z(c-a^*)+2\bar{z}(c^*-a)+(b+b^*)(1-z\bar{z})}{1+z\bar{z}}\,,\\ \zeta^z &= a+bz+cz^2+\frac{u}{2r}\left((a-c^*)+(b+b^*)z+(c-a^*)z^2\right)\,.\end{aligned} \tag{34.513}$$

In particular, the individual rotations and boosts are reproduced for specific choices of a, b, and c:

$$\begin{aligned} J_{12} &= \zeta|_{\substack{a=0\\b=i\\c=0}}\,, & J_{23} &= \zeta|_{\substack{a=-\frac{i}{2}\\b=0\\c=\frac{i}{2}}}\,, & J_{31} &= \zeta|_{\substack{a=\frac{1}{2}\\b=0\\c=\frac{1}{2}}}\,,\\ K_1 &= \zeta|_{\substack{a=\frac{1}{2}\\b=0\\c=-\frac{1}{2}}}\,, & K_2 &= \zeta|_{\substack{a=\frac{i}{2}\\b=0\\c=\frac{i}{2}}}\,, & K_3 &= \zeta|_{\substack{a=0\\b=-1\\c=0}}\,. \end{aligned} \tag{34.514}$$

We now express this vector in terms of the global CKV on the sphere $Y^z = a + bz + cz^2$. Using the identities

$$\begin{aligned} D_z Y^z + D_{\bar{z}} Y^{\bar{z}} &= \frac{2z(c-a^*)+2\bar{z}(c^*-a)+(b+b^*)(1-z\bar{z})}{1+z\bar{z}}\,,\\ D^z\left(D_z Y^z + D_{\bar{z}} Y^{\bar{z}}\right) &= -\left((a-c^*)+(b+b^*)z+(c-a^*)z^2\right)\,, \end{aligned} \tag{34.515}$$

we find

$$\zeta(Y) = \frac{u}{2} D_A Y^A \partial_u - \frac{u+r}{2} D_A Y^A \partial_r + \left[Y^A - \frac{u}{2r} D^A D_B Y^B\right]\partial_A\,. \tag{34.516}$$

Comparing (34.515) with (34.503), we have $D\cdot Y \sim Y_1^m$. Noting that $D^z D_z Y^z = -Y^z$, we can write this vector field as presented in section 34.5.3

$$\begin{aligned}\zeta(Y) = &\frac{u}{2}\left(D_z Y^z + D_{\bar{z}} Y^{\bar{z}}\right)\partial_u - \frac{u+r}{2}\left(D_z Y^z + D_{\bar{z}} Y^{\bar{z}}\right)\partial_r\\ &+\left[Y^z + \frac{u}{2r}\left(Y^z - D^z D_{\bar{z}} Y^{\bar{z}}\right)\right]\partial_z\\ &+\left[Y^{\bar{z}} + \frac{u}{2r}\left(Y^{\bar{z}} - D^{\bar{z}} D_z Y^z\right)\right]\partial_{\bar{z}}\,.\end{aligned} \tag{34.517}$$

12(b). We show that the Lorentz algebra is isomorphic to the algebra of the global CKVs of S^2. First note that, since the r and u components of ζ are linear in r and u, the terms of the form $\zeta^\mu \partial_\mu \zeta^\nu$ will always cancel when $\mu, \nu = u, r$:

$$\begin{aligned}[\zeta(Y_1), \zeta(Y_2)]^u &= \zeta(Y_1)^A D_A \zeta(Y_2)^u - \zeta(Y_2)^A D_A \zeta(Y_1)^u\\ &= \frac{u}{2}\left[Y_1^A D_A D_B Y_2^B - Y_2^A D_A D_B Y_1^B\right]\\ &= \frac{u}{2} D_B\left[Y_1^A D_A Y_2^B - Y_2^A D_A Y_1^B\right]\\ &= \frac{u}{2} D_A\left[Y_1, Y_2\right]^A\,.\end{aligned} \tag{34.518}$$

In a similar way, we have

$$\begin{aligned}
[\zeta(Y_1),\zeta(Y_2)]^r &= \zeta(Y_1)^A D_A \zeta(Y_2)^r - \zeta(Y_2)^A D_A \zeta(Y_1)^r \\
&= -\frac{u+r}{2}\left[Y_1^A D_A D_B Y_2^B - Y_2^A D_A D_B Y_1^B\right] \\
&= -\frac{u+r}{2} D_A \left[Y_1, Y_2\right]^A \ , \\
[\zeta(Y_1),\zeta(Y_2)]^A &= [Y_1,Y_2]^A + \frac{u^2}{4r^2}\Big[D\cdot Y_2 D^A D\cdot Y_1 - D\cdot Y_1 D^A D\cdot Y_2 \\
&\qquad + D^C D\cdot Y_1 D_C D^A D\cdot Y_2 - D^C D\cdot Y_2 D_C D^A D\cdot Y_1\Big] \\
&\qquad - \frac{u}{2r}\Big[D^A D\cdot [Y_1,Y_2] - (D^A Y_1^C + D^C Y_1^A) D_C D\cdot Y_2 \\
&\qquad\quad + (D^A Y_2^C + D^C Y_2^A) D_C D\cdot Y_1 + D\cdot Y_1 D^A D\cdot Y_2 \\
&\qquad\quad - D\cdot Y_2 D^A D\cdot Y_1\Big] \\
&= [Y_1,Y_2]^A - \frac{u}{2r} D^A D\cdot [Y_1,Y_2] \ ,
\end{aligned} \tag{34.519}$$

where we have used the identities $D_A Y_B + D_B Y_A = \gamma_{AB} D\cdot Y$ and $D_A D_B (D\cdot Y) = \frac{1}{2}\gamma_{AB} D^2 D\cdot Y = -\gamma_{AB} D\cdot Y$.

12(c). The diffeomorphism generated by ζ induces the following variations in the metric components:

$$\begin{aligned}
\mathcal{L}_\zeta g_{uu} &= \frac{D\cdot Y(u\partial_u m_B + 3m_B) + 2Y^A\partial_A m_B - \frac{1}{2}D^B C_{BA} D^A D\cdot Y}{r} + \mathcal{O}(r^{-2}) \ , \\
\mathcal{L}_\zeta g_{ur} &= \mathcal{O}(r^{-2}) \ , \\
\mathcal{L}_\zeta g_{rr} &= 0 \ , \\
\mathcal{L}_\zeta g_{uA} &= \mathcal{O}(1) \ , \\
\mathcal{L}_\zeta g_{rA} &= \frac{u}{2}\left(\frac{g_{AB}\gamma^{BC}}{r^2} + g_{ur}\delta_A^C\right) D_C D\cdot Y \ , \\
\mathcal{L}_\zeta g_{AB} &= \frac{u}{2}\left(D\cdot Y\partial_u g_{AB} - D\cdot Y\partial_r g_{AB} + g_{uB} D_A D\cdot Y + g_{uA} D_B D\cdot Y\right) \\
&\quad - \frac{u}{2r}\left[D^C D\cdot Y D_C g_{AB} + g_{CB} D_A D^C D\cdot Y + g_{CA} D_B D^C D\cdot Y\right] \\
&\quad - \frac{r}{2} D\cdot Y\partial_r g_{AB} + \mathcal{L}_Y g_{AB} \ .
\end{aligned} \tag{34.520}$$

The first four equations in (34.520) indicate that ζ preserves the asymptotic form of the components g_{uu}, g_{rr}, g_{ur}, and g_{uA}. We now check the leading and subleading terms in the variations of g_{rA} and g_{AB}, using the asymptotic expansion $g_{AB} = r^2\gamma_{AB} + rC_{AB} + \mathcal{O}(1)$:

$$\begin{aligned}
\mathcal{L}_\zeta g_{rA} &= \frac{u}{2}\left(\gamma_{AB}\gamma^{BC} - \delta_A^C + \mathcal{O}(r^{-1})\right) D_C D\cdot Y \ , \\
\mathcal{L}_\zeta g_{AB} &= r^2\left(\mathcal{L}_Y\gamma_{AB} - \gamma_{AB} D\cdot Y\right) + r\Big[\frac{u}{2}\left(D\cdot Y\partial_u C_{AB} - 2\gamma_{AB} D\cdot Y\right. \\
&\qquad \left. -2D_A D_B D\cdot Y\right) - \frac{1}{2} C_{AB} D\cdot Y + \mathcal{L}_Y C_{AB}\Big] + \mathcal{O}(1) \ .
\end{aligned} \tag{34.521}$$

At order $\mathcal{O}(r^2)$, $\mathcal{L}_\zeta g_{AB}$ vanishes, due to the conformal Killing equation $\mathcal{L}_Y \gamma_{AB} = D_A Y_B + D_B Y_A = \gamma_{AB} D \cdot Y$. The $\mathcal{O}(r)$ term gives the variation of C_{AB}:

$$\begin{aligned}\delta C_{AB} = \frac{u}{2}\left(D\cdot Y \partial_u C_{AB} - 2\gamma_{AB} D\cdot Y - 2D_A D_B D\cdot Y\right) \\ - \frac{1}{2} C_{AB} D\cdot Y + \mathcal{L}_Y C_{AB}\ .\end{aligned} \tag{34.522}$$

Note that

$$\gamma^{AB}\delta C_{AB} = \frac{u}{2}\left(-4D\cdot Y - 2D^2 D\cdot Y\right) + C^{AB}\left(D_A Y_B + D_B Y_A\right) = 0\ , \tag{34.523}$$

where $D^2 D\cdot Y = -2D\cdot Y$, and $C^{AB}\left(D_A Y_B + D_B Y_A\right) = C^{AB}\gamma_{AB} D\cdot Y$. Hence, C_{AB} remains traceless under the diffeomorphism generated by ζ.

13(a). In the limit $r\to\infty$, the metric in Bondi gauge is flat, and $v^\mu = (1,0,0,0)$ is a solution for the four-velocity of a timelike observer. At finite but large r, an inertial observer's four-velocity must have subleading corrections, generically of the form

$$\begin{aligned} v^u &= 1 + \frac{1}{r} v^{u(1)} + \mathcal{O}(r^{-2})\ , \\ v^r &= \frac{1}{r} v^{r(1)} + \mathcal{O}(r^{-2})\ , \\ v^z &= \frac{1}{r} v^{z(1)} + \frac{1}{r^2} v^{z(2)} + \mathcal{O}(r^{-3})\ , \\ v^{\bar z} &= \frac{1}{r} v^{\bar z(1)} + \frac{1}{r^2} v^{\bar z(2)} + \mathcal{O}(r^{-3})\ . \end{aligned} \tag{34.524}$$

To compute the norm, we use the asymptotic form of the metric:

$$\begin{aligned} ds^2 = -du^2 - 2dudr + 2r^2\gamma_{z\bar z}dzd\bar z + \frac{2m_B}{r}du^2 \\ + D^z C_{zz} dudz + D^{\bar z} C_{\bar z\bar z} dud\bar z + rC_{zz}dz^2 + rC_{\bar z\bar z}d\bar z^2 + \cdots\ . \end{aligned} \tag{34.525}$$

By requiring that $v^\mu v_\mu = -1 + \mathcal{O}(r^{-2})$, we find

$$v^{z(1)} = v^{\bar z(1)} = 0\ , \qquad m_B = v^{u(1)} + v^{r(1)}\ . \tag{34.526}$$

Hence, we will study a four-velocity of the asymptotic form

$$\begin{aligned} v^u &= 1 + \frac{m_0(u,z,\bar z)}{r} + \mathcal{O}(r^{-2})\ , \\ v^r &= \frac{m_B(u,z,\bar z) - m_0(u,z,\bar z)}{r} + \mathcal{O}(r^{-2})\ , \\ v^z &= \frac{1}{r^2} v^{z(2)} + \mathcal{O}(r^{-3})\ , \\ v^{\bar z} &= \frac{1}{r^2} v^{\bar z(2)} + \mathcal{O}(r^{-3})\ . \end{aligned} \tag{34.527}$$

We require that $v^\mu \nabla_\mu v^{u,r} = \mathcal{O}(r^{-2})$ and $v^\mu \nabla_\mu v^{z,\bar z} = \mathcal{O}(r^{-3})$. Explicitly, these conditions are given by

$$\begin{aligned} \frac{\partial_u m_0}{r} + \Gamma^u_{\alpha\beta} v^\alpha v^\beta = \mathcal{O}(r^{-2})\ , \\ \frac{\partial_u m_B - \partial_u m_0}{r} + \Gamma^r_{\alpha\beta} v^\alpha v^\beta = \mathcal{O}(r^{-2})\ , \\ \frac{\partial_u v^{z(2)}}{r^2} + \Gamma^z_{\alpha\beta} v^\alpha v^\beta = \mathcal{O}(r^{-3})\ . \end{aligned} \tag{34.528}$$

Next, we need to compute the large-r expansion of the Christoffel connection. The only nonzero components at this order are

$$\Gamma^r_{uu} = -\frac{\partial_u m_B}{r} + \mathcal{O}(r^{-2}) \ , \qquad \Gamma^z_{uu} = \frac{D_z N^{zz}}{2r^2} + \mathcal{O}(r^{-3}) \ . \tag{34.529}$$

Substituting (34.529) into the geodesic equation, we find the following constraints:

$$\partial_u m_0 = 0 \ , \qquad \partial_u v^{z(2)} = -\tfrac{1}{2} D_z N^{zz} \ . \tag{34.530}$$

Integrating the second condition, we find

$$v^{z(2)}(u,z,\bar{z}) = -\tfrac{1}{2}\left(D_z C^{zz}(u,z,\bar{z}) - D_z C^{zz}(u_0,z,\bar{z})\right) \ . \tag{34.531}$$

Moreover, since m_0 is simply a function of $(z,\bar{z})$, we can also interpret it as a boundary condition. Hence, we have

$$\begin{aligned} v^\mu = \Bigg(1 + \tfrac{m_B(u_0,z,\bar{z})}{r} + \mathcal{O}(r^{-2}) \ , \ \tfrac{\Delta m_B(u,z,\bar{z})}{r} + \mathcal{O}(r^{-2}) \ , \\ - \tfrac{D_z \Delta C^{zz}(u,z,\bar{z})}{2r^2} + \mathcal{O}(r^{-3}), - \tfrac{D_{\bar{z}} \Delta C^{\bar{z}\bar{z}}(u,z,\bar{z})}{2r^2} + \mathcal{O}(r^{-3})\Bigg) \ , \end{aligned} \tag{34.532}$$

where

$$\begin{aligned} \Delta m_B(u,z,\bar{z}) &\equiv m_B(u,z,\bar{z}) - m_B(u_0,z,\bar{z}) \ , \\ \Delta C^{zz}(u,z,\bar{z}) &\equiv C^{zz}(u,z,\bar{z}) - C^{zz}(u_0,z,\bar{z}) \ . \end{aligned} \tag{34.533}$$

To leading order in r_0 and for small λ, the trajectory of an observer with this four-velocity is

$$X^\mu(\lambda) = (u_0 + \lambda, r_0, z_0, \bar{z}_0) \ . \tag{34.534}$$

The first correction to the four-velocity may now be approximated as

$$\begin{aligned} v^\mu(\lambda) = \Bigg(1 + \frac{m_B(u_0,z,\bar{z})}{r_0} \ , \ \frac{\lambda \partial_u m_B(u_0,z,\bar{z})}{r_0} \ , \\ - \frac{\lambda D_z N^{zz}(u_0,z,\bar{z})}{2r_0^2} \ , \ - \frac{\lambda D_{\bar{z}} N^{\bar{z}\bar{z}}(u_0,z,\bar{z})}{2r_0^2}\Bigg) \ . \end{aligned} \tag{34.535}$$

The trajectory with these subleading corrections is

$$\begin{aligned} X^\mu(\lambda) &= \Bigg(u_0 + \lambda\left(1 + \frac{m_B(u_0,z,\bar{z})}{r_0}\right), r_0 + \frac{\lambda^2 \partial_u m_B(u_0,z,\bar{z})}{2r_0}, \\ &\quad z_0 - \frac{\lambda^2 D_z N^{zz}(u_0,z,\bar{z})}{4r_0^2}, \bar{z}_0 - \frac{\lambda^2 D_{\bar{z}} N^{\bar{z}\bar{z}}(u_0,z,\bar{z})}{4r_0^2}\Bigg) + \cdots \\ &= \Bigg(u, r_0\left(1 + \frac{(u-u_0)^2 \partial_u m_B(u_0,z,\bar{z})}{2r_0^2}\right), \\ &\quad z_0 - \frac{(u-u_0)^2 D_z N^{zz}(u_0,z,\bar{z})}{4r_0^2}, \bar{z}_0 - \frac{(u-u_0)^2 D_{\bar{z}} N^{\bar{z}\bar{z}}(u_0,z,\bar{z})}{4r_0^2}\Bigg) + \cdots \ . \end{aligned} \tag{34.536}$$

13(b). Since both inertial observers pass through the point (u_0, r_0), we can use the trajectory found in part (a), and simply modify the angular dependence. To leading order in the large-r_0 limit, assuming $|u - u_0| \ll r_0$, we find

$$\Delta X^\mu \equiv X_1^\mu - X_0^\mu = (0, 0, z_1 - z_0, \bar{z}_1 - \bar{z}_0) + \cdots \ . \tag{34.537}$$

The leading order proper distance between the observers is then

$$L \approx \sqrt{2r_0^2 \gamma_{z_0\bar{z}_0} \delta z \delta \bar{z}} = \frac{2r_0|\delta z|}{1 + z_0\bar{z}_0} \ , \qquad \delta z \equiv z_1 - z_0 \ . \tag{34.538}$$

13(c). Due to the burst of radiation, the metric undergoes a transition. Inverting the leading order constraint equation, we find that the leading order change in the zz-component is given by

$$\begin{aligned} \Delta C_{zz}(z, \bar{z}) &\equiv C_{zz}(z, \bar{z})\big|_{u=u_f} - C_{zz}(z, \bar{z})\big|_{u=u_0} \\ &= 2 \int d^2 z' \ \gamma_{z'\bar{z}'} D_z^2 G(z, \bar{z}; z', \bar{z}') \left[\int_{u_0}^{u_f} du \ T_{uu}(u, z', \bar{z}') + \Delta m_B(z', \bar{z}') \right] , \end{aligned} \tag{34.539}$$

where $u_f > u_{\rm rad}$, $\frac{|u_f - u_0|}{r_0} \ll 1$, and

$$\begin{aligned} G(z, \bar{z}; z', \bar{z}') &= \frac{1}{\pi} \frac{|z - z'|^2}{(1 + z\bar{z})(1 + z'\bar{z}')} \log \left(\frac{|z - z'|^2}{(1 + z\bar{z})(1 + z'\bar{z}')} \right) \ , \\ \Delta m_B &= m_B(u_f, z, \bar{z}) - m_B(u_0, z, \bar{z}) \ . \end{aligned} \tag{34.540}$$

Henceforth we assume m_B is independent of $(z, \bar{z})$. To check this, note that

$$D_{\bar{z}}^2 D_z^2 G(z, \bar{z}; z', \bar{z}') = \gamma_{z\bar{z}} \delta^2(z - z') + \cdots \ . \tag{34.541}$$

To determine Δm_B, we used the constraint equation

$$\partial_u m_B = \frac{1}{4} \left[D_z^2 N^{zz} + D_{\bar{z}}^2 N^{\bar{z}\bar{z}} \right] - T_{uu} \ . \tag{34.542}$$

We can extract the zero mode of this equation by integrating over the two-sphere

$$\partial_u m_B = -\frac{1}{4\pi} \int d^2 z \ \gamma_{z\bar{z}} T_{uu} \ . \tag{34.543}$$

Integrating over u, we find

$$\Delta m_B = -\frac{1}{4\pi} \int_{u_0}^{u_f} \int d^2 z \ \gamma_{z\bar{z}} \ T_{uu} \ . \tag{34.544}$$

Then the change in C_{zz} takes the modified form

$$\begin{aligned} \Delta C_{zz}(z, \bar{z}) = 2 \int d^2 z' \ \gamma_{z'\bar{z}'} D_z^2 G(z, \bar{z}; z', \bar{z}') \\ \times \left[\int_{u_0}^{u_f} du \ \left(T_{uu}(u, z', \bar{z}') - \frac{1}{4\pi} \int d^2 z'' \gamma_{z''\bar{z}''} T_{uu}(u, z'', \bar{z}'') \right) \right] \ . \end{aligned} \tag{34.545}$$

T_{uu} due to the burst of radiation is given by

$$T_{uu}(u, z, \bar{z}) = \mu \delta(u - u_{\rm rad}) \frac{\delta^2(z - z_{\rm rad})}{\gamma_{z\bar{z}}} \ . \tag{34.546}$$

The change in C_{zz} due to the burst in radiation is then as follows:

$$\begin{aligned}\Delta C_{zz}(z,\bar{z}) &= 2\mu \int d^2z'\ D_z^2 G(z,\bar{z};z',\bar{z}')\left[\delta^2(z'-z_{\rm rad}) - \frac{\gamma_{z'\bar{z}'}}{4\pi}\right] \\ &= 2\mu D_z^2 G(z,\bar{z};z_{\rm rad},\bar{z}_{\rm rad}) \\ &\quad - \frac{\mu}{2\pi}\int d^2z'\ \gamma_{z'\bar{z}'} D_z^2 G(z,\bar{z};z',\bar{z}')\ .\end{aligned} \tag{34.547}$$

Since this component of the metric never contributed to the proper distance between inertial observers at leading order, to leading order the separation between the observers after the burst of radiation remains unchanged:

$$L \approx \frac{2r_0|\delta z|}{1+z_0\bar{z}_0} \equiv L_0\ . \tag{34.548}$$

However, this component of the metric does play a role in subleading corrections. Note that to calculate the first subleading correction to L, we can safely neglect any corrections to ΔX, since they will not contribute at this order. Hence, the first subleading correction comes from the metric used to compute the inner product of ΔX^μ. At a given time u, the distance is given by

$$\begin{aligned}L &\approx \sqrt{2r_0^2\gamma_{z_0\bar{z}_0}\delta z\delta\bar{z} + r_0 C_{zz}(u,z_0,\bar{z}_0)\delta z^2 + r_0 C_{\bar{z}\bar{z}}(u,z_0,\bar{z}_0)\delta\bar{z}^2} \\ &= L_0\sqrt{1+\frac{r_0}{L_0^2}\left[C_{zz}(u,z_0,\bar{z}_0)\delta z^2 + C_{\bar{z}\bar{z}}(u,z_0,\bar{z}_0)\delta\bar{z}^2\right]} \\ &\approx L_0 + \frac{r_0}{2L_0}\left[C_{zz}(u,z_0,\bar{z}_0)\delta z^2 + C_{\bar{z}\bar{z}}(u,z_0,\bar{z}_0)\delta\bar{z}^2\right]\ .\end{aligned} \tag{34.549}$$

Hence, the change in distance between u_0 and u_f is given by the difference in the subleading term at u_f and u_0:

$$\Delta L = \frac{r_0}{2L_0}\left[\Delta C_{zz}(z_0,\bar{z}_0)\delta z^2 + \Delta C_{\bar{z}\bar{z}}(z_0,\bar{z}_0)\delta\bar{z}^2\right]\ . \tag{34.550}$$

13(d). To determine the supertranslation that would give rise to the same change in proper distance as equation (34.550), recall that under a supertranslation parametrized by a function on the sphere f, we have

$$\mathcal{L}_f C_{zz} = f N_{zz} - 2D_z^2 f\ . \tag{34.551}$$

The change in the metric is related to the Lie derivative

$$\Delta C_{zz} = -\mathcal{L}_f C_{zz} = 2D_z^2 f\ , \tag{34.552}$$

where N_{zz} has been set to zero, since the region of $\mathcal{I}^+$ before and after the radiation has passed is in a vacuum state. Then, using equation (34.547), we find

$$f(z,\bar{z}) = \mu G(z,\bar{z};z_{\rm rad},\bar{z}_{\rm rad}) - \frac{\mu}{4\pi}\int d^2z'\ \gamma_{z'\bar{z}'} G(z,\bar{z};z',\bar{z}')\ . \tag{34.553}$$

14(a). To determine the linearized Einstein's equation, note that

$$\begin{aligned}R_{\mu\nu}[g+h] &= R_{\mu\nu}[g] + \frac{1}{2}\left(\nabla_\rho\nabla_\mu h_\nu{}^\rho + \nabla_\rho\nabla_\nu h_\mu{}^\rho - \nabla^2 h_{\mu\nu} - \nabla_\mu\nabla_\nu h\right) \\ &\quad + \mathcal{O}\left(h^2\right)\ , \\ R[g+h] &= R[g] - R_{\mu\nu}[g]h^{\mu\nu} + \nabla_\mu\nabla_\nu h^{\mu\nu} - \nabla^2 h + \mathcal{O}\left(h^2\right)\ ,\end{aligned} \tag{34.554}$$

where ∇_μ is the covariant derivative w.r.t. $g_{\mu\nu}$. Using this, we find

$$\begin{aligned} G_{\mu\nu}[g+h] = G_{\mu\nu}[g] &+ \frac{1}{2}\left(\nabla_\rho\nabla_\mu h_\nu{}^\rho + \nabla_\rho\nabla_\nu h_\mu{}^\rho - \nabla^2 h_{\mu\nu} - \nabla_\mu\nabla_\nu h\right) \\ &+ \frac{1}{2}g_{\mu\nu}h^{\rho\sigma}R_{\rho\sigma}[g] - \frac{1}{2}g_{\mu\nu}\nabla_\rho\nabla_\sigma h^{\rho\sigma} + \frac{1}{2}g_{\mu\nu}\nabla^2 h \\ &- \frac{1}{2}h_{\mu\nu}R[g] + \mathcal{O}\left(h^2\right) \ . \end{aligned} \tag{34.555}$$

We are interested in the linearized equations about the Schwarzschild background. In this case, $R_{\mu\nu}[g] = R[g] = 0$, which implies that the linearized Einstein equations are

$$\nabla_\rho\nabla_\mu h_\nu{}^\rho + \nabla_\rho\nabla_\nu h_\mu{}^\rho - \nabla^2 h_{\mu\nu} - \nabla_\mu\nabla_\nu h - g_{\mu\nu}\nabla_\rho\nabla_\sigma h^{\rho\sigma} + g_{\mu\nu}\nabla^2 h = 0 \ . \tag{34.556}$$

To solve the linearized Einstein's equations about the initial Schwarzschild solution in Bondi gauge, we consider a line element of the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} + h_{\mu\nu}dx^\mu dx^\nu \ , \tag{34.557}$$

where $h_{\mu\nu}$ is a small perturbation of the metric. We require that $h_{\mu\nu}$ preserve the Bondi gauge condition and the Bondi mass only shifts by a constant. In particular, since the metric describes a spacetime that is perturbed by a null shockwave at $v = v_0$, we can take $h_{\mu\nu}$ to be of the following form:

$$\begin{aligned} h_{vv}(v,r,z,\bar{z}) &= \theta(v-v_0)\left(\frac{2\delta m}{r} + \frac{1}{r^2}\tilde{h}_{vv}(r,z,\bar{z})\right) \ , \\ h_{vr}(v,r,z,\bar{z}) &= \frac{\theta(v-v_0)}{r^2}\tilde{h}_{vr}(r,z,\bar{z}) \ , \\ h_{vz}(v,r,z,\bar{z}) &= \theta(v-v_0)\tilde{h}_{vz}(r,z,\bar{z}) \ , \\ h_{zz}(v,r,z,\bar{z}) &= r\theta(v-v_0)\tilde{h}_{zz}(r,z,\bar{z}) \ , \\ h_{rr} &= h_{rz} = h_{z\bar{z}} = 0 \ , \end{aligned} \tag{34.558}$$

where $\tilde{h}_{\mu\nu}(r,z,\bar{z})$ must approach a finite function on the two-sphere as $r \to \infty$. Note that at linear order, the Bondi condition $\partial_r \det \frac{g_{AB}}{r^2}$ implies $\gamma^{AB}h_{AB} = 0$. Consider the rr-component of Einstein's equation,

$$G_{rr} = \frac{2\theta(v-v_0)\left(r\partial_r\tilde{h}_{vr} - 2\tilde{h}_{vr}\right)}{r^4} = 0 \tag{34.559}$$

which implies that

$$\tilde{h}_{vr} = r^2 c_{1vr}(z,\bar{z}) \ . \tag{34.560}$$

However, to satisfy the boundary condition that $\tilde{h}_{vr}$ is finite as $r \to \infty$, we must have $c_{1vr}(z,\bar{z}) = 0$. Using this result, next consider the zz-component of Einstein's equations:

$$\begin{aligned} G_{zz} = &-r\delta(v-v_0)\partial_r\tilde{h}_{zz} + \frac{1}{2r^2}\theta(v-v_0)\Big(2r^2\partial_r D_z\tilde{h}_{vz} - 2Mr\partial_r\tilde{h}_{zz} \\ &+r^2(2M-r)\partial_r^2\tilde{h}_{zz} + 2M\tilde{h}_{zz}\Big) \\ = &\ 0 \ , \end{aligned} \tag{34.561}$$

where D_z is the covariant derivative w.r.t. the two-sphere. From the first term, we find that $\tilde{h}_{zz}$ must be r-independent. Using these results, we have

$$G_{rz} = -\theta(v-v_0)\left(\frac{-2r^2\partial_r^2\tilde{h}_{vz} + 4\tilde{h}_{vz} + 2D^z\tilde{h}_{zz}}{4r^2}\right) = 0 \ . \tag{34.562}$$

If we take a derivative of the numerator with respect to r, we find a differential equations for only $\tilde{h}_{vz}$ with the general solution

$$\tilde{h}_{vz}(r,z,\bar{z}) = \frac{c_{1vz}(z,\bar{z})}{r} + c_{2vz}(z,\bar{z}) + c_{3vz}(z,\bar{z})r^2 \ . \tag{34.563}$$

The boundary conditions force $c_{3vz}(z,\bar{z}) = 0$. Moreover, substituting back into (34.562) implies that $c_{2vz} = -\frac{1}{2}D^z\tilde{h}_{zz}$. From the second term in (34.561), we learn that $\tilde{h}_{zz} = \frac{1}{M}D_z c_{1vz}$. Using these results, we have

$$G_{z\bar{z}} = \theta(v-v_0)\left(\frac{2D^z c_{1vz} + 2D^{\bar{z}}c_{1v\bar{z}} - 2r^2\partial_r^2\tilde{h}_{vv} + 4r\partial_r\tilde{h}_{vv} - 4\tilde{h}_{vv}}{2r^2(1+z\bar{z})^2}\right) = 0 \ . \tag{34.564}$$

Again, taking an r-derivative of the numerator, we can solve for the r-dependence of $\tilde{h}_{vv}$. The most general solution consistent with the boundary conditions is

$$\tilde{h}_{vv}(r,z,\bar{z}) = \frac{1}{2}\left(D^z c_{1vz} + D^{\bar{z}}c_{1v\bar{z}}\right). \tag{34.565}$$

Using these results, the vz-component is of the form

$$G_{vz} = \frac{3\delta(v-v_0)c_{1vz}}{2r^2} + \theta(v-v_0)G'_{vz}(r,z,\bar{z}) \ , \tag{34.566}$$

where $G'_{vz}(r,z,\bar{z})$ is defined to be the coefficient of the $\theta(v-v_0)$ term in G_{vz}. Using the Einstein's equations, we find

$$c_{1vz} = -2MD_z f \ . \tag{34.567}$$

Finally, we can compute

$$G_{vv} = \delta(v-v_0)\left(\frac{2\delta m}{r^2} + \frac{D^2(D^2+2)f}{2r^2} - \frac{3MD^2 f}{r^3}\right) \ . \tag{34.568}$$

Using Einstein's equations, we find $\delta m = \mu$. At this point, we have completely determined the metric to be

$$\begin{aligned}
g_{vv} &= -1 + \frac{2(M+\mu\theta(v-v_0))}{r} - \frac{\theta(v-v_0)MD^2 f}{r^2} \ , \\
g_{vr} &= 1 \ , \\
g_{vz} &= \theta(v-v_0)D_z\left[\left(1-\frac{2M}{r}\right) + \frac{1}{2}D^2\right] f \ , \\
g_{zz} &= -2r\theta(v-v_0)D_z^2 f \ , \\
g_{z\bar{z}} &= r^2\gamma_{z\bar{z}} \ , \\
g_{rr} &= g_{rz} = 0 \ .
\end{aligned} \tag{34.569}$$

One can check that this metric satisfies $G_{vr} = 0$, as desired.

14(b). Note that a spherically symmetric shockwave with stress tensor

$$T_{vv} = \frac{\mu\delta(v-v_0)}{4\pi r^2} \ , \tag{34.570}$$

is a particular case of equation (34.568) part (a) with $f = 0$, so that for $v > v_0$ the metric will take the form

$$ds^2 = -\left(1 - \frac{2(M+\mu)}{r}\right) dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \ . \tag{34.571}$$

The exact form of the generator of supertranslations in a Schwarzschild background is given by

$$\xi = f\partial_v + \frac{1}{r}D^A f\partial_A - \frac{1}{2}D^2 f\partial_r \ , \tag{34.572}$$

where $f = f(z,\bar{z})$. Under the supertranslation, the components of the metric have the following variations:

$$\begin{aligned} &\mathcal{L}_f g_{vv} = \frac{D^2 fM}{r^2} \ , \qquad \mathcal{L}_f g_{vr} = \mathcal{L}_f g_{rr} = \mathcal{L}_f g_{rz} = \mathcal{L}_f g_{z\bar{z}} = 0 \ , \\ &\mathcal{L}_f g_{vz} = -\frac{1}{2}D_z D^2 f - D_z f + \frac{2D_z fM}{r} \ , \qquad \mathcal{L}_f g_{zz} = 2rD_z^2 f \ . \end{aligned} \tag{34.573}$$

The change in the metric is related to the Lie derivative by $\delta g_{\mu\nu} = -\mathcal{L}_f g_{\mu\nu}$. Hence, the resulting metric is given by

$$\begin{aligned} g_{vv} &= -1 + \frac{2(M+\mu)}{r} - \frac{D^2 fM}{r^2} \ , \\ g_{vr} &= 1 \ , \\ g_{vz} &= \frac{1}{2}D_z D^2 f + D_z f - \frac{2D_z fM}{r} \ , \\ g_{zz} &= -2rD_z^2 f \ , \\ g_{z\bar{z}} &= r^2\gamma_{z\bar{z}} \ , \\ g_{rr} &= g_{rz} = 0 \ , \end{aligned} \tag{34.574}$$

which matches the metric found in part (a) when $v > v_0$.

15(a). Maxwell's equations are

$$d * F = e^2 * j \ . \tag{34.575}$$

Then, using Stokes' law, we can write

$$\hat{Q}^{\mathcal{H}}_{\varepsilon} = \frac{1}{e^2}\int_{\mathcal{H}} d\left[\varepsilon * F\right] = \frac{1}{e^2}\int_{\mathcal{H}} d\varepsilon \wedge *F + \int_{\mathcal{H}} \varepsilon * j \ . \tag{34.576}$$

The first term in this equation is the soft charge. To determine its explicit form, we need to determine the projection of the two-form $*F$ onto $\mathcal{H}$. To do this, note that the Vaidya metric takes the form

$$ds^2 = -\left[1 - \frac{2M\theta(v)}{r}\right] dv^2 + 2dvdr + 2r^2\gamma_{z\bar{z}}dzd\bar{z} \ . \tag{34.577}$$

Using this form, we can determine $*F$ as

$$\begin{aligned} *F = iF^z{}_z dv \wedge dr - ir^2 \gamma_{z\bar{z}} F_{vr} dz \wedge d\bar{z} \\ + \left[iF_{rz} dr \wedge dz - i \left[F_{vz} + \left[1 - \frac{2M}{r} \theta(v) \right] F_{rz} \right] dv \wedge dz + c.c. \right] . \end{aligned} \tag{34.578}$$

This is derived using

$$(*F)_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \ . \tag{34.579}$$

We must now project equation (34.579) onto $\mathcal{H}$. To do so, we note that the event horizon of the Vaidya shockwave metric is located at

$$r_H(v) = \begin{cases} \frac{v}{2} + 2M \ , & v \leq 0 \ , \\ 2M \ , & v > 0 \ . \end{cases} \tag{34.580}$$

Then, setting $r = r_H(v)$ in (34.578), we find

$$(*F) \big|_{\mathcal{H}} = -ir_H^2 \gamma_{z\bar{z}} F^{(0)}_{vr} dz \wedge d\bar{z} + \left\{ -i \left[F^{(0)}_{vz} + \frac{1}{2} \theta(-v) F^{(0)}_{rz} \right] dv \wedge dz + c.c. \right\} \ , \tag{34.581}$$

where here the superscript (0) indicates the projection of the corresponding component of the field strength onto $\mathcal{H}$. Now we are finally ready to compute the charge. We take ε to be extended onto $\mathcal{H}$ so that it is constant along null generators (i.e., $\partial_v \varepsilon = 0$). Then we find

$$\hat{Q}^{S,\mathcal{H}}_{\varepsilon} = \frac{1}{e^2} \int d^2z \varepsilon \left(\partial_z N_{\bar{z}} + \partial_{\bar{z}} N_z \right) \ , \tag{34.582}$$

where

$$N_z = \int dv \left(F^{(0)}_{vz} + \frac{1}{2} \theta(-v) F^{(0)}_{rz} \right) \ . \tag{34.583}$$

Here N_z is the analog of the zero mode on $\mathcal{I}^+$. Just like on $\mathcal{I}^+$, it is also a boundary term and only gets contributions from $\mathcal{H}^+$. To see this, we move to the advanced radial gauge:

$$A_r = 0 \ , \qquad A_v \big|_{\mathcal{H}} = 0 \ . \tag{34.584}$$

In this gauge, we have

$$F^{(0)}_{vz} + \frac{1}{2} \theta(-v) F^{(0)}_{rz} = \left(\partial_v + \frac{1}{2} \theta(-v) \partial_r \right) A_z \big|_{r=r_H} = \partial_v A^{(0)}_z \ . \tag{34.585}$$

In the second equality, we first take the appropriate derivatives and *then* set $r = r_H$ in A_z. By the chain rule, this is equal to first setting $r = r_H$ in A_z and then taking a v-derivative. Thus, we have

$$N_z = \int dv \partial_v A^{(0)}_z = A^+_z - A^-_z \ . \tag{34.586}$$

The past boundary of $\mathcal{H}$ is really a point at $v = -4M, r = 0$, where there is a coordinate singularity. To regulate this, we take the boundary to be at $v = -4M + \epsilon$ and then take $\epsilon \to 0$.

15(b). We recall the symplectic form determined in exercise 3:

$$\Omega_\Sigma = -\frac{1}{e^2}\int_\Sigma \delta(*F)\wedge\delta A\ . \tag{34.587}$$

Plugging in the explicit forms of $*F$ and A in the advanced radial gauge, we find

$$\Omega_{\mathcal{H}} = \frac{1}{e^2}\int_{\mathcal{H}} dvd^2z\left(\partial_v\delta A_z^{(0)}\wedge\delta A_{\bar z}^{(0)} + \partial_v\delta A_{\bar z}^{(0)}\wedge\delta A_z^{(0)}\right)\ . \tag{34.588}$$

Note that the symplectic form as well as the soft charge take exactly the same form as on $\mathcal{I}^+$. We can therefore proceed as we did there. Define

$$\hat A_z = A_z^{(0)} - \frac{1}{2}\left(A_z^+ + A_z^-\right) = A_z^{(0)} - \partial_z\phi\ . \tag{34.589}$$

Using this, we find

$$\Omega_{\mathcal{H}} = \frac{2}{e^2}\int dvd^2z\partial_v\delta\hat A_z\wedge\delta\hat A_{\bar z} - \frac{1}{e^2}\int d^2z\left[\delta\partial_z\phi\wedge\delta N_{\bar z} + \delta\partial_{\bar z}\phi\wedge\delta N_z\right]\ . \tag{34.590}$$

15(c). Just as on $\mathcal{I}^+$, we can write out the commutators from (34.590) as

$$\begin{aligned}\left[\hat A_z(v),\hat A_{\bar w}(v')\right] &= -\frac{ie^2}{4}\Theta(v-v')\delta^2(z-w)\ ,\\ [\partial_z\phi, N_{\bar w}] &= ie^2\delta^2(z-w)\ .\end{aligned} \tag{34.591}$$

Using this, we find

$$\left[Q_\varepsilon^{S,\mathcal{H}},\hat A_z\right] = \left[Q_\varepsilon^{S,\mathcal{H}},N_z\right] = 0\ ,\qquad \left[Q_\varepsilon^{S,\mathcal{H}},\partial_z\phi\right] = i\partial_z\varepsilon\ . \tag{34.592}$$

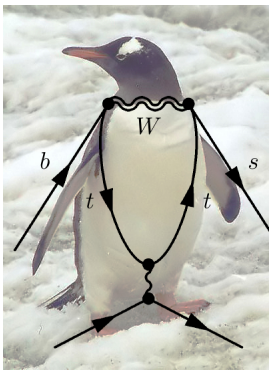
35 Other Topics of QFT

35.1 Analysis of some special famous diagrams

35.1.1 Penguin diagram

(есть такая известная диаграмма в СМ, но это специфика, заниматься пока не было ей времени, но пару статей про неё открывал)

Main idea



Theory

35.2 Introduction to Supersymmetry

(см. мою запись по отдельным теориям поля пока.)

35.2.1 Overview of Applications of Formalizm by Red'kov

(??? a very important general question, I'll write many examples her)

35.2.2 Particle Fields in Riemannian Space by Red'kov 1

(I'll write it first in a special note, then only here! it is too mathematical and too far from physics)

уравнение Вейля и Дирака, метод спинорных коэффициентов

Рецепт Тетроде - Вейля - Фока - Иваненко

(выгружу, это начинал смотреть, на этом и закончил.)

О нахождении спинорного преобразования в (3 + 1) -расщеплении 4-мерной матрицы Лоренца

(не актуально, но интересно)

Спинорное преобразование и (2 + 2) -расщепление

(не актуально, но интересно)

Примеры калибровочных спинорных преобразований

(круто это прописать бы!)

О биспинорных вращениях в произвольном базисе

О параметризации группы $SL(2C)$

(не актуально, но интересно)

Нерелятивистский предел в уравнении Дирака

(проходили же?)

О калибровочной симметрии уравнения Паули

(интересно)

Определение коэффициентов Ньюмана - Пенроуза, спинорный подход

(не актуально, но интересно)

35.2.3 Particle Fields in Riemannian Space by Red'kov 2

(I'll write it first in a special note, then only here! it is too mathematical and too far from physics)

Калибровочные преобразования

(не актуально, но интересно)

Уравнение Клейна - Фока - Гордона во внешних гравитационном и электромагнитном полях

Уравнение Клейна - Фока - Гордона в римановом пространстве - времени приводится к специальной форме, удобной для исследования нерелятивистского приближения в этом уравнении.

Уравнение для скалярной частицы в пространстве-времени с метрикой $dS^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ выберем в виде

$$\left[\left(i\hbar\nabla_\alpha + \frac{e}{c}A_\alpha\right) g^{\alpha\beta}(x) \left(i\hbar\nabla_\beta + \frac{e}{c}A_\beta\right) - \frac{\hbar^2}{6}R - m^2c^2\right]\Psi(x) = 0$$

На время проведения вычислений, чтобы не загромождать формулы, будем использовать сокращенные обозначения

$$\frac{e}{\hbar c} \Longrightarrow e, \quad \frac{mc}{h} \Longrightarrow m;$$

тогда основное уравнение примет вид

$$\left[\left(i\nabla_\alpha + eA_\alpha\right) g^{\alpha\beta}(x) \left(i\nabla_\beta + eA_\beta\right) - \frac{1}{6}R - m^2\right] \quad \Psi(x) = 0$$

Представляем его как

$$\left[i^2\nabla_\alpha g^{\alpha\beta}(x)\nabla_\beta + ie\left(\nabla_\alpha g^{\alpha\beta}(x)A_\beta\right) + 2ieA_\alpha g^{\alpha\beta}(x)\nabla_\beta + \right.$$
$$\left. e^2 A_\alpha g^{\alpha\beta}(x)A_\beta - \tfrac{1}{6}R - m^2\right]\Psi(x) = 0$$

Пользуясь известными формулами для общековариатных оператора Лапласа и дивергенции:

$$\nabla_\alpha g^{\alpha\beta}(x)\nabla_\beta\Phi = \tfrac{1}{\sqrt{-g}}\tfrac{\partial}{\partial x^\alpha}\sqrt{-g}g^{\alpha\beta}\tfrac{\partial}{\partial x^\beta}\Psi$$
$$\nabla_\alpha g^{\alpha\beta}A_\beta = \tfrac{1}{\sqrt{-g}}\tfrac{\partial}{\partial x^\alpha}\sqrt{-g}g^{\alpha\beta}A_\beta, \quad g = \det\left(g_{\alpha\beta}\right)$$

предыдущее уравнение переписываем в виде

$$\left[i^2\tfrac{1}{\sqrt{-g}}\tfrac{\partial}{\partial x^\alpha}\sqrt{-g}g^{\alpha\beta}(x)\tfrac{\partial}{\partial x^\beta} + \tfrac{1}{\sqrt{-g}}\tfrac{\partial}{\partial x^\alpha}\sqrt{-g}g^{\alpha\beta}(x)A_\beta + \right.$$
$$\left. 2\text{ie}\ \ A_\alpha g^{\alpha\beta}\tfrac{\partial}{\partial_\beta} + e^2 A_\alpha g^{\alpha\beta}(x)A_\beta - \tfrac{1}{6}R - m^2\right]\Psi(x) = 0$$

или

$$\left[\frac{1}{\sqrt{-g}}\left(i\frac{\partial}{\partial x^\alpha} + eA_\alpha\right)\sqrt{-g}g^{\alpha\beta}(x)\left(i\frac{\partial}{\partial x^\beta} + eA_\beta\right) - \frac{1}{6}R - m^2\right]\Psi(x) = 0$$

В обычных единицах это уравнение записывается как

$$\left[\frac{1}{\sqrt{-g}}\left(i\hbar\frac{\partial}{\partial x^\alpha} + \frac{e}{c}A_\alpha\right)\sqrt{-g}g^{\alpha\beta}(x)\left(i\hbar\frac{\partial}{\partial x^\beta} + \frac{e}{c}A_\beta\right) - \frac{\hbar^2}{6}R - m^2c^2\right]\Psi(x) = 0$$

Спиновые коэффициенты в сферической тетраде

Спинорный формализм и ортогональная группа

Уравнение Дирака в ортогональных координатах и тетраде

Уравнение Дирака и коэффициенты вращения Риччи

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Калибровочные свойства векторов $B_a(x)$ и $C_a(x)$

Связь с формализмом Ньюмана - Пенроуза

Майорановское спинорное поле в римановом пространстве

О структуре базиса Майораны

наверное, перекину к спинорам

35.2.4 Introduction to QFT in Curved Space by Hollands, Wald

Обсудим основные идеи полей в искривленном пространстве. Эти идеи отдельно развиваются в соответствующей части и скорее всего очень большими будут, поэтому выгодно выделить главу с основами.

(из Quantum fields in curved spacetime см литературу)

1.

The nature of quantum field theory in curved spacetime Quantum field theory in curved spacetime (QFTCS) is the theory of quantum fields propagating in a background, classical, curved spacetime $(\mathscr{M}, g)$. On account of its classical treatment of the metric, QFTCS cannot be a fundamental theory of nature. However, QFTCS is expected to provide an accurate description of quantum phenomena in a regime where the effects of curved spacetime may be significant, but effects of quantum gravity itself may be neglected. In particular, it is expected that QFTCS should be applicable to the description of quantum phenomena occurring in the early universe and near (and inside of black holes-provided that one does not attempt to describe phenomena occurring so near to singularities that curvatures reach Planckian scales and the quantum nature of the spacetime metric would have to be taken into account. It should be possible to derive QFTCS by taking a suitable limit of a more fundamental theory wherein the spacetime metric is treated in accord with the principles of quantum theory. However, this has not been done - except in formal and/or heuristic ways - simply because no present quantum theory of gravity has been developed to the point where such a well defined limit can be taken. Rather, the framework of QFTCS that we shall describe in this review has been obtained by suitably merging basic principles of classical general relativity with the basic principles of quantum field theory in Minkowski spacetime. As we shall explain further below, the basic principles of classical general relativity are relatively easy to identify and adhere to, but it is far less clear what to identify as the "basic principles" of quantum field theory in Minkowski spacetime. Indeed, many of the concepts normally viewed as fundamental to quantum field theory in Minkowski spacetime, such as Poincare invariance, do not even make sense in the context of curved spacetime, and therefore cannot be considered as "fundamental" from the viewpoint of QFTCS. By forcing one to re-think basic concepts, such as the notions of "vacuum state" and "particles", QFTCS has led to deep insights into the nature of quantum field theory-and one may hope that it will provide significant guidance towards the development of quantum gravity itself.

The fundamental ideas upon which classical general relativity is based are that (i) all aspects of spacetime structure are described by the topological and differential (i.e., manifold) properties of events together with a Lorentz signature metric g, and (ii) the metric and matter fields are

dynamical; furthermore their evolution is locally determined. More precisely, the metric and the tensor (and/or spinor) fields describing matter satisfy partial differential equations - namely, Einstein's equation together with the equations of motion for the matter fields - that have a well posed initial value formulation, so that these fields are uniquely determined (up to "gauge") from their initial data within a suitable domain of dependence. In particular, in classical general relativity, there is no non-dynamical, "background structure" in the laws of physics apart from the manifold structure of events. This lack of background structure in classical general relativity is usually referred to as the "covariance" or "coordinate invariance" of the theory; a "preferred set of coordinates" defined independently of the metric would provide non-dynamical, background structure.

It is much more difficult to identify the fundamental ideas upon which quantum field theory in Minkowski spacetime is based. One can attempt to formulate the quantum theory of a field in Minkowski spacetime by decomposing the field into modes and applying the rules of quantum mechanics to each mode. For a free field, each mode is an independent harmonic oscillator and one can obtain a mathematically sensible quantum field theory in this manner, although even here one encounters infinite expressions for quantities that are nonlinear in the fields. A well known example of this general phenomenon is that one obtains an infinite expression for the total energy (and energy density) of the field, as can be seen by adding the zero-point energies and/or energy densities of the infinite number of modes. The situation is considerably worse for interacting (i.e., nonlinear) fields, wherein one immediately encounters ill defined and/or infinite expressions in the calculation of essentially all physical quantities, arising from the fact that modes of arbitrarily high energies seemingly contribute to low energy processes. Historically, it appears to have been generally assumed in its earliest days that the quantum field theory description of nature would break down at, say, the energy scale of elementary particles, and there was no reason to presume that it was a mathematically consistent theory. However, starting from the early 1950 s, it was gradually understood how to give mathematically consistent rules to produce well defined expressions for physical quantities to all orders in perturbation theory for renormalizable theories such as quantum electrodynamics. This process culminated in the works of Bogoliubov, Parasiuk and of Hepp and Zimmermann, with important practical improvements (dimensional regularization) being given later by 't Hooft and Veltmann. It was also seen that the predictions of quantum field theory give truly excellent agreement with experiment-as they have continued to do through the present, LHC era. In the 1950 s and 1960 s, major progress was made towards putting quantum field on a mathematically sound footing via the development of the axiomatic [73], algebraic [40], and constructive [39] approaches. Nevertheless, the prevailing attitude towards quantum field theory today is not very different from what it was in its earliest days, namely, that it is not a fundamental theory but merely a valid description of quantum field modes up to some cutoff in energy (now assumed to be at a much higher energy scale than would have been assumed in its earliest days). At the present time, relatively little attention is generally paid to the issue of whether quantum field theory can be given a mathematically precise and consistent formulation-as compared with such issues as the "fine tuning" that would be necessary to give small values to the cosmological constant and Higgs mass if one views quantum field theory as the quantum theory of the modes of fields lying below some energy cutoff.

Our view is that it is very important to determine if quantum field theory can be given a mathematically precise and consistent formulation as a theory in its own right-and to provide such a formulation if it can be given. This is not because we believe that quantum field theory should be a "final" theory of nature; indeed, we do not believe that a quantum theory of the spacetime metric can be formulated within the existing framework of quantum field theory. However, even if quantum field theory has only a limited domain of validity, it is important to understand precisely what questions are well posed within its framework and how the answers to these questions are to be obtained. In this way, the predictions of quantum field theory can be made with clarity and precision, and hints may be provided for some of the features that might be expected to survive in a more fundamental theory that supersedes quantum field theory.

What are the "basic principles" of quantum field theory in Minkowski spacetime? The observables of the theory are the tensor fields representing the fundamental constituents of matter, together with "composite fields", such as the stress-energy tensor, derived from these matter fields. A key basic principle of quantum field theory is that each observable field, $\Phi(x)$, at each spacetime

point x should be represented as an operator. These operators will satisfy nontrivial algebraic relations, such as commutation relations. However, there are two important caveats to this statement that $\Phi(x)$ is represented as an operator.

The first caveat is that, even in the case of a free Klein-Gordon scalar field ϕ - where, as already mentioned above, a quantum field theory can be formulated by ordinary quantization of the independent modes - it can be seen that one cannot make mathematical sense of $\phi(x)$ as an operator at a sharply defined point x, since modes of arbitrarily high frequency and short wavelength contribute to $\phi(x)$. However, $\phi(x)$ does make sense as a distribution, i.e., by "averaging" $\phi(x)$ with a smooth function of compact support, $f(x)$, one effectively eliminates the arbitrarily high frequency and short wavelength oscillations and thereby obtains a well defined expression for the quantum field. Thus, quantum fields are operator-valued distributions. The distributional nature of quantum fields is the source of most, if not all, of the mathematical difficulties arising in quantum field theory. Nonlinear operations involving distributions are intrinsically ill defined, and one will typically get infinite answers if one attempts to evaluate nonlinear functions of a distribution via mode expansions or other procedures.

The second caveat is that the word "operator" presumes that there is some unique underlying Hilbert space of states on which this operator will act. However, even for a free field, there are an infinite number of unitarily inequivalent representations of the fundamental commutation relations. (This contrasts sharply with the situation for a quantum mechanical system with a finite number of degrees of freedom, where the Stone-von Neumann theorem asserts that, under mild additional assumptions, all such representations are unitarily equivalent.) In Minkowski spacetime, a preferred representation normally can be chosen based upon the additional requirement that the representation contain a Poincare invariant state ("the vacuum"). However, no criterion analogous to this can be applied in a general curved spacetime. As discussed further at the end of this section, it would therefore seem much more natural to view the algebraic relations satisfied by the field observables - rather than the choice of representation - to be fundamental. Thus, in quantum field theory, we will take as a basic principle that the quantum fields $\Phi(x)$ are distributions valued in an algebra.

Another basic principle of quantum field theory in Minkowski spacetime that the fields should "transform covariantly" under Poincare transformations. The Poincare group is the isometry group of the metric, η, of Minkowski spacetime, but a general, curved spacetime will not admit any isometries. Nevertheless, "Poincare invariance" may be viewed as a special relativistic version of the above general relativistic requirement of "covariance", i.e., that quantum field theory in curved spacetime be constructed out of the classical spacetime metric g and the fundamental quantum fields ϕ, without any additional "background structure". Furthermore, this construction should be local in nature. We will take as a basic principle that quantum field theory should be locally and covariantly constructed. We will give precise meaning to this statement in Section 3.

An additional basic principle of quantum field theory in Minkowski spacetime is the requirement of positivity of energy. Since the notion an energy operator for a quantum field theory in Minkowski spacetime is normally defined in terms of the transformation properties of the field under time translations, this requirement cannot be straightforwardly generalized to QFTCS. Nevertheless, one can formulate local conditions on the quantum field theory - known as microlocal spectral conditions - that correspond to positivity of energy in Minkowski spacetime and make sense in curved spacetime. We will take as a basic principle that the quantum field theory should satisfy suitable microlocal spectral conditions. We will give precise meaning to this statement in the Appendix.

Finally, an additional principle of quantum field theory in Minkowski spacetime that is usually taken to be fundamental is the existence of a unique, Poincare invariant state. However, this condition has no analog in a general curved spacetime as a condition on existence or uniqueness of states, [2] and we will not attempt to impose any condition of this nature.

Thus, we seek a formulation of quantum field theory in curved spacetime that implements the three basic principles written in italics above. In the remainder of this section, we briefly describe some of the standard approaches that have been used to formulate quantum field theory in Minkowski spacetime and explain why they do not appear suitable for the formulation of QFTCS. We will then describe the approach that we shall adopt.

Many discussions of quantum field theory are based upon a notion of "particles", and focus

almost entirely on the calculation of the S-matrix, describing the scattering of particles. For a free field in Minkowski spacetime, a notion of "vacuum state" and "particles" can be defined in a natural and precise manner. If an interacting field behaves like a free field in the asymptotic past and future, one can define asymptotic particle states. The S-matrix provides the relationship between the "in" and "out" particle descriptions of the states, and thereby directly yields the dynamical information about the interacting field that is most relevant to laboratory experiments in high energy physics. However, the use of an S -matrix description as a fundamental ingredient in the formulation of QFTCS is unsuitable for the following reasons. Although natural notions of "vacuum state" and "particles" can be defined for a free field in stationary spacetimes, no such natural notions exist in a general curved spacetime. The difficulty is not that a notion of "particles" cannot be defined at all in a general curved spacetime but rather that many notions exist and none appears preferred. Although it may be possible (and useful) to define an S-matrix in spacetimes that become asymptotically stationary in a suitable manner in the past and future, many of the spacetimes of greatest interest in QFTCS are cosmological spacetimes or spacetimes describing gravitational collapse, where singularities occur in the asymptotic past and/or future. If one wishes to apply QFTCS to such spacetimes, it clearly would be preferable to formulate it in a manner that does not require one to define a notion of "particles" near singularities before one can even pose a well defined question. Furthermore, even if the spacetime of interest is suitably asymptotically stationary in the past and future, many of the most interesting physical questions are concerned with the local dynamical behavior of the fields at finite times rather than the particle-like description of states at asymptotically early and late times. For example, one may wish to know the expected stress-energy tensor of a quantum field in order to estimate the "back reaction" effects of the quantum field on the dynamics of the spacetime. An S-matrix would not be useful for such a calculation.

In many discussions of quantum field theory in Minkowski spacetime, Euclidean methods play an important role in both the formulation of the theory and in calculational techniques. Minkowski spacetime can be viewed as a real 4 -dimensional section of a complex 4-dimensional manifold with complex metric, which contains another 4 -real-dimensional section (of "imaginary time") on which the metric is positive definite. If one can define a quantum field theory in a suitable manner on this "Euclidean section", a quantum field theory on Minkowski spacetime can then be obtained via analytic continuation. Since it is much easier to make sense of formal expressions in the Euclidean setting than in the Lorentzian setting, Euclidean methods have been employed in most of the attempts to rigorously define interacting quantum field theories and in most of the methods employed to regularize and renormalize quantities in perturbative quantum field theory [39]. Euclidean methods can be generalized so as to apply to static, curved spacetimes, where the transformation " $t \rightarrow$ it" takes one from a static Lorentzian spacetime to a Riemannian space. However, a general curved spacetime will not be a real section of a complex manifold that also contains a real section on which the metric is Riemannian. Thus, although it should be possible to define "Euclidean quantum field theory" on curved Riemannian spaces [62], there is no obvious way to connect such a theory with quantum field theory on Lorentzian spacetimes. Thus, if one's goal is to define quantum field theory on general Lorentzian spacetimes, it does not appear fruitful to attempt to formulate the theory via a Euclidean approach.

Finally, by far the most prevalent approach taken towards the formulation of quantum field theory in Minkowski spacetime is to write down a formal functional integral expression for an effective action. Suitable functional derivatives of this expression are then interpreted as providing the correlation functions of the quantum field in its vacuum state. Thus, one will have defined the quantum field theory if one can make sense of this functional integral and its functional derivatives. The difficulty with using a functional integral approach to formulate QFTCS is that, in effect, it requires one to single out a preferred state in order to define the theory-namely, the state for which the correlation functions are being given. This is not a difficulty in Minkowski spacetime, where Poincare invariance naturally selects a preferred state and, furthermore, Euclidean methods are available to make sense of the functional integral for this preferred state. However, as previously indicated above, no analogous notion of a preferred state exists in a general curved spacetime without symmetries. As in the above discussion of S -matrix approaches to the formulation of QFTCS, we do not believe that it will be fruitful to formulate QFTCS via an approach that requires one to define a preferred state in order to define the theory. [3]

For the above reasons, we shall adopt the "algebraic viewpoint" for the formulation of QFTCS. The basic idea of this approach is to take the relations satisfied by the quantum fields - such as commutation relations and field equations - as the fundamental starting point of the theory. To define the theory, one must specify the complete set of algebraic relations satisfied by the fundamental field and composite fields. As we shall see, this can be done for a free field. In addition, we can naturally define time ordered products, enabling one to give a perturbative construction of interacting quantum field theory. However, it is far less clear as to how to define appropriate algebraic relations so as to give a non-perturbative definition of interacting quantum field theory; we will return to this issue in Section 4.1. Once the algebraic relations have been given, states are defined to be positive linear functions on the algebra of quantum fields. The GNS construction then shows that every state in this sense arises as a vector in a Hilbert space that carries a representation of the field algebra, thus connecting the algebraic notion of states with usual notions of states in quantum theory. The key point is that one can formulate QFTCS via the algebraic approach in a manner that does not require one to single out a preferred state in order to define the theory. We begin the next section by formulating QFTCS for a free scalar field, taking into account only the fundamental field observables. We also discuss some key applications, including the Hawking and Unruh effects and quantum effects arising from inflation. In Section 3, we show how a wide class of nonlinear field observables can be defined for a free scalar field.

We then describe the perturbative construction of QFTCS for interacting fields. We also discuss the construction of QFTCS for gauge fields. Finally, in Section 4 we discuss ideas that may lead towards a nonperturbative formulation of interacting QFTCS and we make some remarks on the formulation of quantum gravity from the perspective of QFTCS. 2. Free quantum fields In this section, we provide a precise formulation of the theory of a free Klein-Gordon field in curved spacetime, insofar as the fundamental field observable, ϕ, is concerned. (Observables that are nonlinear in ϕ - i.e., "composite fields" - will be introduced in the next section.) We then discuss some key applications of the theory, namely, the Unruh effect, quantum field theory in deSitter spacetime, the Hawking effect, and cosmological perturbations.

Formulation of linear QFTCS via the algebraic approach (without nonlinear observables)

2.1.

We now describe how to define the quantum field theory of a real, linear Klein-Gordon field ϕ on a d -dimensional, curved, Lorentzian spacetime $(\mathscr{M}, g)$ along the lines sketched in the previous section. We begin with the classical Klein-Gordon field, which satisfies

$$\left(\Box_g - m^2\right)\phi = 0$$

where $\Box_g = g^{\mu v}\nabla_\mu\nabla_v$ is the D'Alembertian operator associated with g. In order that (1) have a well-posed initial value formulation, we restrict consideration - here and throughout this article - to globally hyperbolic spacetimes. By definition, a globally hyperbolic spacetime is a time oriented spacetime that possesses a "Cauchy surface", Σ, i.e., a smoothly embedded $(d-1)$-dimensional spacelike submanifold with the property that if $\gamma : \mathbb{R} \to \mathscr{M}$ is any inextendible causal curve, then γ intersects Σ precisely once. [4] The classical Klein-Gordon equation with source j $(\Box_g - m^2)\,\phi = j$ (where j is an arbitrary, fixed, smooth function on spacetime) has a well posed initial value formulation on globally hyperbolic spacetimes in the following sense. Let n denote the unit normal to Σ. Then, given any pair (f_0, f_1) of smooth functions on Σ, there exists a unique solution ϕ to the Klein-Gordon equation (2) such that $\phi|_\Sigma = f_0, \quad n^\mu\nabla_\mu\phi|_\Sigma = f_1$ Furthermore, solutions to the initial value problem have a causal dependence upon the initial data and the source in the sense that if $x \in J^+(\Sigma)$, then the solution ϕ to the above initial value problem will not change at x if we change the initial data (f_0, f_1) outside $J^-(x) \cap \Sigma$, or if we change the source outside of $J^-(x) \cap J^+(\Sigma)$. Here

$J^\pm(S) \equiv \{x \in \mathscr{M} \quad | \exists$ causal future/past directed curve from $y \in S$ to $x\}$ denotes the "causal future/past" of a set $S \subset \mathscr{M}$. Similarly, if $x \in J^-(\Sigma)$, then $\phi(x)$ will not change if we change the initial data (f_0, f_1) outside $J^+(x) \cap \Sigma$, or if we change the source outside of $J^+(x) \cap J^-(\Sigma)$. Finally, the solution ϕ depends continuously on (f_0, f_1) and j in a suitable sense.

We can define the retarded and advanced propagators $E^{\pm}$ of the Klein-Gordon equation as follows. For $j \in C_0^\infty(\mathscr{M})$ the advanced solution $E^- j$ is the unique solution to the Klein-Gordon equation with source j such that $E^- f(x) = 0$ for all $x \notin J^-(\mathrm{supp}\, j)$, with the opposite definition for the retarded solution $E^+ j$. The propagators may be viewed as maps $E^{\pm} : C_0^\infty(\mathscr{M}) \to C^\infty(\mathscr{M})$ or alternatively as distributional kernels on $\mathscr{M} \times \mathscr{M}$. As distributions, $E^{\pm}$ satisfy the differential equation $(\Box_g - m^2)\, E^{\pm}(x,y) = \delta(x,y)$ where the Klein-Gordon operator acts on the first variable, x, in the sense of distributions. The support properties are $\mathrm{supp}\, E^{\pm} \subset \{(y,x) \in \mathscr{M} \times \mathscr{M} | x \in J^{\pm}(y)\}$ The advanced and retarded propagators are related by exchanging x with y. The anti-symmetric combination $E = E^+ - E^-$ is called "commutator function". As described in the Introduction, we will formulate QFTCS for a Klein-Gordon field by defining a suitable algebra, $\mathscr{A}(\mathscr{M}, g)$ of quantum observables. In this section, we will consider only the algebra of observables generated by the funda-

mental field ϕ. An enlarged algebra that includes "composite fields" will be defined in the next section. The construction of $\mathscr{A}(\mathscr{M}, g)$ will take into account: (1) the distributional nature of the field $\phi(x)$, (2) the field equation, (3) the real character of ϕ and (4) the symplectic structure of the classical phase space of this theory. We construct $\mathscr{A}(\mathscr{M}, g)$ by starting with the free * -algebra generated by a unit 1 and elements $\phi(f)$, with $f \in C_0^\infty(\mathscr{M})$, and factoring by the following relations: (1) Linearity: $\phi(c_1 f_1 + c_2 f_2) = c_1 \phi(f_1) + c_2 \phi(f_2)$ for all $c_1, c_2 \in \mathbb{C}$, (2) Field equation: $\phi((\Box_g - m^2) f) = 0$ (3) Hermitian field: $\phi(f)^* = \phi(\bar{f})$, (4) Commutator: $[\phi(f_1), \phi(f_2)] = iE(f_1, f_2)\, 1$. Item (1) incorporates the distributional character of the field; informally we write

$$\phi(f) = \int_{\mathscr{H}} \phi(x) f(x) dv_g$$

and we think of $\phi(x)$ as an $\mathscr{A}$ -valued distribution. We refer to the algebra $\mathscr{A}$ as "abstract", because no reference has been made to any representation. In fact, we are not supposed to think, a priori, of the elements of $\mathscr{A}$ as operators on a particular Hilbert space, in the same way as an abstract Lie-algebra is defined irrespective of a particular representation. (2) incorporates the field equation in a distributional sense. (3) expresses that the field is real, and (4) implements the usual quantum mechanical relationship between classical Poisson-brackets and commutators. It also incorporates Einstein causality, because $E(x,y) = 0$ if x, y are spacelike related.

Rather than working with fields $\phi(f)$ that are smeared with test functions f on $\mathscr{M}$, we can also equivalently view ϕ as being "symplectically smeared" with solutions, F, having initial data of compact support on some Cauchy surface Σ. The correspondence is the following. If $f \in C_0^\infty(\mathscr{M})$, then $F = Ef$ is a source free solution having initial data (3) of compact support. Conversely, given a solution F with initial data of compact support, there exists a test function f - unique up to addition of $(\Box_g - m^2)\, h$ for $h \in C_0^\infty$ - such that $F = Ef$. Defining $\phi[F] = \phi(f)$ under this correspondence between F and f, we can then informally write the field as

$$\phi[F] = \int_\Sigma (F \nabla_\mu \phi - (\nabla_\mu F)\, \phi)\, n^\mu dS$$

We can think of $\pi = n^\mu \nabla_\mu \phi|_\Sigma, \varphi = \phi|_\Sigma$ as canonically conjugate variables and the commutation relation (4) then corresponds to $[\pi(x), \varphi(y)] = i\delta_\Sigma(x,y)\mathbf{1}$; for details, see, e.g. [23] or lemma 3.2.1 of [79].

A physical state, ω, is simply an "expectation value functional", i.e., a linear map $\omega : \mathscr{A}(\mathscr{M}, g) \to \mathbb{C}$ satisfying the normalization condition $\omega(\mathbf{1}) = 1$, and positivity, $\omega(a^* a) \geq 0$ for all $a \in \mathscr{A}$. Any state is, by construction, specified by the collection $(W_n)_{n \geq 1}$ of its "n-point functions",

$$W_n(f_1, ..., f_n) \equiv \omega(\phi(f_1) \cdots \phi(f_n))$$

The KG-equation, condition (2), is translated into the fact that W_n is a distributional solution in each entry. Condition (1) implies that W_n is an n -times multilinear functional on $C_0^\infty(\mathscr{M})$, which one normally requires to be distributional (i.e. continuous in the appropriate sense). The commutator condition (4) is translated into a linear condition which in the simplest case $n = 2$

is $W_2(x,y) - W_2(y,x) = iE(x,y)$ Positivity is translated into a rather complicated hierarchy of conditions on the multi-linear functionals W_n, the simplest of which is

$$W_2(\bar{f}, f) \geq 0 \text{ for all } f \in C_0^\infty$$

Given two states ω, ω', one can form a new state by forming any convex linear combination $\lambda\omega + (1-\lambda)\omega'$, where $0 \leq \lambda \leq 1$. A state which cannot be written as a non-trivial convex linear combination of others is called pure.

The notion of algebraic state is in principle sufficient to answer all physical questions about the field observables. In particular, the specification of a state ω directly yields the expected values of all powers of $\phi(f)$ for all, say real, test functions f. It follows from the classical "Hamburger moment problem" (see e.g. [67]) that there is a probability measure $dv(\lambda)$ such that

$$\int_{\mathbb{R}} \lambda^n dv(\lambda) = \omega(\underbrace{\phi(f)...\phi(f)}_{n}) \equiv m_n \text{ for all } n \geq 1.$$

The probability that an observation of $\phi(f)$ will yield a value within $[a,b]$ when the field is in state ω is then given by $P_{a,b} = \int_{[a,b]} \lambda dv(\lambda)$ The relationship between states in the algebraic sense as defined above and the usual notion of states as vectors in a Hilbert space can be seen as follows: First, assume that we have a representation $\pi : \mathscr{A}(\mathscr{M}, g) \to \mathscr{H}$ on a Hilbert space with dense invariant domain $\mathscr{D} \subset \mathscr{H}$. If Ψ is a non-zero vector in this domain, then

$$\omega_\Psi(a) = \frac{(\Psi, \pi(a)\Psi)}{(\Psi, \Psi)}$$

defines a state in the algebraic sense. More generally, any sufficiently regular (with respect to $\mathscr{D}$) density matrix on $\mathscr{H}$ - i.e., a non-negative, trace-class operator $\rho \in \mathscr{I}_1(\mathscr{H})$ - defines an algebraic state ω_ρ. Conversely, given any algebraic state ω, there is a simple construction - known as the GNS-construction, or Wightman reconstruction argument - that yields a Hilbert space $\mathscr{H}$, a representation π of $\mathscr{A}$ on $\mathscr{H}$ with invariant domain $\mathscr{D} \subset \mathscr{H}$, and a vector $\Omega \in \mathscr{D}$ such that the algebraic state corresponding to Ω is ω. As we shall explain further below, for the case of a Gaussian state, the GNS construction yields a Fock representation of $\mathscr{A}$, with Ω being the vacuum vector of this Fock space.

The above correspondences show that the algebraic and Hilbert space formulations of QFTCS are essentially equivalent. However, there is one important difference: There are many unitarily inequivalent representations of the field algebra $\mathscr{A}$. (Two representations are said to be unitarily equivalent if there is an isometry $U : \mathscr{H} \to \mathscr{H}'$ such that $U\pi(a)U^* = \pi'(a)$ for all $a \in \mathscr{A}$.) The Hilbert space formulation requires one to chose a "preferred representation" at the outset, while the algebraic formulation does not. For this reason, we feel that the formulation of QFTCS via the algebraic approach is conceptually superior.

The algebra $\mathscr{A}$ admits states with rather pathological properties of their n -point functions W_n. However, there is a natural criterion to select physically reasonable states that is motivated from a variety of closely related considerations, including that (i) the ultra-high-frequency modes of the field should be essentially in their ground state, (ii) the short distance singular structure of the n -point functions W_n should be similar to that of the n -point functions of the vacuum state in Minkowski spacetime, and (iii) the singular structure of the W_n 's should be of "positive frequency type". A mathematically precise implementation of these requirements in a general curved spacetime can be formulated in the language of wave front sets discussed in the Appendix, as first proposed in [65,66,15]. The criterion is that the 2 -point function have a wavefront set of the form

$$\mathrm{WF}\left(W_2\right) \subset \left\{(x_1, k_1; x_2, k_2) \in T^*\mathscr{M}^2 \backslash 0 | k_1 \in \dot{V}^+, k_2 \in \dot{V}^-, k_1 \sim -k_2\right\}$$

and that each "connected n -point function" $\omega_n^c\left(\phi\left(x_1\right), ..., \phi\left(x_n\right)\right)$ for $n \neq 2$ (defined below in Eq. (19)) is smooth. Here $V^+ \subset T^*\mathscr{M}$ is the collection of all non-zero, future directed time-like

or null co-vectors (and similarly V^-), and the relation $\sim$ holds between two covectors if they are tangent to a null geodesic in $(\mathscr{M}, g)$ and are parallel transported into each other. Such states are called "Hadamard states". As we will see in the next section, it will be necessary to restrict to Hadamard states in order to extend the action of the state to an enlarged algebra of field observables that includes all polynomials in the field and its derivatives at the same spacetime point. In particular, it will be necessary to restrict to Hadamard states when considering the perturbative expansion of interacting QFTCS.

One can prove [65,66] that the above definition of a Hadamard state is equivalent to the following local condition [58] on the 2 -point function together with the requirement that, globally, there be no singularities at spacelike separations: For every convex, normal neighborhood $U \subset \mathscr{M}$, the two-point function has the form (in $d = 4$ dimensions; similar expressions hold in arbitrary d):

$$W_2 = \frac{1}{4\pi^2}\left[\frac{u}{\sigma + i0^+ t} + \left(\sum_{n=0}^{N} v_n \sigma^n\right)\log\left(\sigma + i0^+ t\right)\right] + f \equiv H_N + R_{N,\omega}.$$

f - some N -times continuously differentiable function $R_{N,\omega}$.

Here, the spacetime arguments are understood as $(x, y) \in U \times U$, and we mean that the above formula should hold for every N, with (different) remainders $R_{N,\omega} \in C^N(U \times U)$.$\sigma$ is the signed squared geodesic distance, and $t = T(x) - T(y)$ for some (in fact, any) global time function $T : \mathscr{M} \to \mathbb{R}$. The functions u, v_n are determined by certain local "transport equations" [19] and also appear in similar local forms of the advanced and retarded Green's function for the operator $\Box_g - m^2$. In particular, H_N is locally and covariantly defined in terms of the metric, and is hence the same for any Hadamard state. By contrast, the remainder $R_{N,\omega}$ depends on the state.

The Hadamard condition does not single out a particular state, but a class of states. Existence of a large class of Hadamard states on any globally hyperbolic spacetime can be established by a deformation argument [37] combined with microlocal techniques, or by methods from the theory of pseudo-differential operators [57,38].

We now discuss two important classes of states: Gaussian states and thermal states. Gaussian States: Gaussian states (also called "quasi-free states") are defined by the condition that the "connected n -point functions" $\omega_n^c\left(\phi\left(f_1\right), ..., \phi\left(f_n\right)\right)$ vanish for all $n > 2$, where $\omega_n^c : \mathscr{A} \times \cdots \times \mathscr{A} \to \mathbb{C}$ is defined by [8]

$$\omega_n^c\left(a_1, ..., a_n\right) \equiv \frac{\partial^n}{\partial t_1 ... \partial t_n} \log\left\{\omega\left(e^{t_1 a_1} \cdots e^{t_n a_n}\right)\right\}\Big|_{t_i = 0}, \quad a_i \in \mathscr{A}.$$

Thus, the n -point functions of Gaussian states can be expressed entirely in terms of their 1 - and 2 -point functions. For a Gaussian state, positivity will hold if and only if (13) is satisfied. Thus, there exists a wide class of Gaussian states.

Any Gaussian state, ω, can be expressed as the "vacuum state" in a Fock representation of $\mathscr{A}$. To see this explicitly, let W_2 be the 2 -point function of ω. On the complex linear space $C_0^\infty(\mathscr{M}, \mathbb{C})$ of smooth complex-valued functions on $\mathscr{M}$ of compact support, define the inner product $\langle f | h \rangle = W_2(\bar{f}, h)$. This is hermitian and positive, $\langle f | f \rangle \geq 0$, but contains degenerate vectors, such as elements of the form $f = (\Box_g - m^2)\, h$. Let $\mathfrak{h}$ be the factor space of $C_0^\infty(\mathscr{M}, \overline{\mathbb{C}})$, divided by the degenerate vectors. The elements in this space can be identified more concretely with a subspace of complex valued smooth solutions to the KG-equation, corresponding to "positive frequency modes", see below. The completion of $\mathfrak{h}$, denoted by the same symbol, is a Hilbert space, usually referred to as the "1-particle space". Let $\mathscr{H}$ be the bosonic Fock space over $\mathfrak{h}$, $\mathscr{H} = \mathbb{C} \oplus \bigoplus_{n \geq 1} \underbrace{(\mathfrak{h} \otimes_S \cdots \otimes_S \mathfrak{h})}_{n}$, where $\otimes_s$ is the symmetrized tensor product. A representation, π, of $\mathscr{A}$ on $\mathscr{H}$ can then be defined by $\pi[\phi(f)] = a([f])^\dagger + a([f])$ where $a([f])$ is the annihilation operator associated with the equivalence class of f in $\mathfrak{h}$. The vacuum vector Ω given by the element $|1, 0, 0, ...\rangle$ in Fock-space then corresponds to ω, and, as already mentioned above, $(\mathscr{H}, \pi, \Omega)$ is precisely the GNS triple arising from the GNS construction.

A closely related construction often used in practice to construct Gaussian pure states is the following. Suppose that we have a set of smooth, complex-valued "mode functions" $u_\xi(x)$ that are solutions to Eq. (1), and which are labeled by $\xi \in X$ in some measure space $(X, d\mu)$. We assume that for each $f \in C_0^\infty(\mathscr{M}, \mathbb{R})$, the map

$$X \ni \xi \mapsto Kf(\xi) \equiv \int_{\mathscr{M}} \overline{u_\xi(x)} f(x) dv_g \in \mathbb{C}$$

is in $L^2(X, d\mu)$, and in fact that the (real linear) span of such vectors is dense in $L^2(X, d\mu)$. We also assume that the mode functions are such that

$$\mathrm{Im}\, \langle Kf_1 | Kf_2 \rangle_{L^2(X,d\mu)} = \frac{1}{2} E(f_1, f_2) \text{ for all } f_1, f_2 \in C_0^\infty(\mathscr{M})$$

(These properties are equivalent to the statement that the collection of modes $(u_\xi)_{\xi \in X}$ is "complete in the KG-norm".) Then, clearly

$$W_2(f_1, f_2) = \langle Kf_1 | Kf_2 \rangle_{L^2(X,d\mu)} \to W_2(x, y) = \int_X d\mu(\xi) u_\xi(x) \overline{u_\xi(y)}$$

defines the 2 -point function of a Gaussian state, which can be shown to be pure. Its GNS-representation is thus constructed as above. K is clearly well-defined on the equivalence classes $[f]$, and it can also be shown that it provides a bounded isomorphism $K : \mathfrak{h} \to L^2(X, d\mu)$. Hence, in this case, we may consider $\mathscr{H}$ as the bosonic Fock-space over $L^2(X, d\mu)$, and we may informally write the representative of the field on this Fock space as

$$\pi(\phi(x)) = \int_X d\mu(\xi) \left[u_\xi(x) a_\xi + \overline{u_\xi(x)} a_\xi^\dagger \right]$$

This is the usual form of the field in the vacuum representation on Minkowski space, where $\xi = \mathbf{k} \in \mathbb{R}^3 = X, d\mu(\mathbf{k}) = d^3\mathbf{k}/(2\pi)^3$, and $u_\mathbf{k}(\mathbf{x}, t) = e^{-i\omega_\mathbf{k} t + i\mathbf{l}\mathbf{x}}/\sqrt{2\omega_\mathbf{k}}$. In this example, the mode functions have positive frequency with respect to the global time translations, and similar constructions are available also on other spacetimes with a globally defined time-like isometry.

Given two pure Gaussian states ω and ω' with 2 -point functions W_2 respectively W_2' and vanishing 1 -point functions, one may ask when their associated GNS-representations are unitarily equivalent. A necessary condition is that there be a constant c such that

$$c^{-1} \mu(f, f) \leq \mu'(f, f) \leq c\mu(f, f)$$

for all $f \in C_0^\infty(\mathscr{M}, \mathbb{R})$, where μ and μ' denote the symmetric parts of W_2 and W_2'. Since it is easy to construct states violating this condition, one sees that there is in general a large class of inequivalent representations. Suppose the condition is satisfied. It immediately follows that the positive, sesquilinear [9] forms μ, μ' induce the same topology on the complex vector space $\mathcal{K}$ which is defined as $C_0^\infty(\mathscr{M}, \mathbb{C})$ modulo elements of the form $f = (\Box_g - m^2) h.\mathcal{K}$ may be thought of as the space of complex valued, spatially compact smooth solutions to the Klein-Gordon equation, and the bilinear forms μ and μ' are well-defined on $\mathcal{K}$ and positive definite. Furthermore, it is easy to see that the closure of this space (relative to either μ or μ') can be decomposed into "positive and negative frequency solutions" as $\mathcal{K} \cong \mathfrak{h} \oplus \overline{\mathfrak{h}} \cong \mathfrak{h}' \oplus \overline{\mathfrak{h}'}$, where an overbar means the complex conjugate Hilbert space, and where $\mathfrak{h}$ respectively $\mathfrak{h}'$ are the 1 -particle spaces corresponding to W_2 respectively W_2'. The Riesz representation theorem guarantees that there is a bounded map S on $\mathscr{K}$ such that $\mu(F_1, F_2) = \mu'(F_1, SF_2)$ for all $F_1, F_2 \in \mathscr{K}$. A necessary and sufficient criterion that the two states give rise to unitarily equivalent representations is [76] $\mathrm{tr}_\mathcal{K} |1 - S| < \infty$ where $|1 - S|$ is the absolute value of the operator defined using μ'. The operator $1 - S$ characterizes the

difference between the 2 -point functions. It is therefore plausible that if their difference is smooth - as happens e.g. if both states are Hadamard - and the manifold $\mathscr{M}$ has compact slices, then (27) should hold, and the representations should be unitarily equivalent. This can indeed be shown [10] [77]. On the other hand, for the case of a non-compact Cauchy surface, the representations can be unitarily inequivalent if the 2 -point functions have sufficiently different long-range behavior. Similarly, the representations can fail to be unitarily inequivalent if the states are not Hadamard and have a sufficiently different short range behavior.

(и там дальше еще целая куча всего, просто пока что все равно это выгрузки, так что нет смысла это писать, начало разобрать бы.)

(из Quantum fields in curved spacetime см литературу)

35.3 Integrable Models of QFT

У лашкевича аж полно лекций про это, чисто теорфизная такая заморочка, мб займусь этим тоже потом.

35.4 QFT on Lattices

(отдельный важный раздел, про который много докладов. соберу потом)

35.4.1 Basics of QFT on Lattices

Основы описания

(там много инфы, не до нее)

Пример вычисления интеграла Фейнмана по решетке

(напишу потом, давно про это вопрос есть.)

35.5 Other Special Topics in a Nutshell

35.5.1 Visualization of Quantum Fields (??)

(это большой вопрос, потому что внятного ответа я не находил. Пока нет времени разбираться в этом.)

35.5.2 Quantum Fields in Curved Space in a Nutshell

пока хз, квантовую механику там бы сперва изучить.

35.6 Overview of Attempts of Quantization of Gravity

(all about this is in this part of the note)

35.7 Some Special Applications of QFT

(пока раздел мелкий, потом мб увеличу, если будет нужно.)

35.7.1 Quantum Field Theory in Economics

я слышал, что есть такая книга, на самом деле никогда не занимался этим.

35.7.2 Applications of QFT in Kinetics

(Там Пригожин и другие что-то многое писали про это, пока не до этого. maybe Kamenev's book it exactly about it)

35.7.3 On Applications of QFT to Life Problems (??)

??? пока не знаю вообще.

Part IX
Adds

Обсудим используемые в КТП основы и отдельные конструкции других разделов физики.

A Introduction and Overview of QFT

A.1 Motivation for QFT

Обсудим, что нас мотивирует изучать классическую теорию поля.

A.1.1 Extra Motivation

Фундаментальные законы природы раскрываются в квантовой теории поля

Часто можно слышать, что на многие принципиальные вопросы физики вместо ответа идет отсылка к КТП.

И вот, наконец-то мы на этом уровне! Мы можем раскрыть самую настоящую суть природы. Вот в первую очередь зачем КТП изучать круто.

Многие величайшие физики человечества специализировались именно на КТП

Типа Фейнман, типа Ландау. Именно КТП и его приложениями они и занимались.

Круто владеть диаграммами

тоже аргумент изучать ее, но слабоватый конечно.

Открывает понимание квантовой гравитации и теории струн

Типа они есть использование КТП в более единых теориях.

Много связей с более простыми предметами

квантовая механика и теория поля при многих рассуждениях переходят в КТП
так что если там не знаешь основы, то тупиковое будет дело.

даже как-то другие предметы не назвать выученными, потому что задаешь вопрос о связях с КТП, и если этот вопрос не отвечен, то сама природа явлений не ясна.

Используются продвинутые математические методы

Поэтому, изучая правильно КТП, появится углубленная подготовка по математике, а также сам стиль работы станет заметно более профессиональным.

Верная ли вообще квантовая электродинамика (?!?!?!?!??!?)

(приведу видео из Ютуба, где доказывается, что КЭД это плохо!?!??! и что вот делать?)

Что такое уровень теормина, если КЭД не работал в 1948м году, а Ландау его с 33го года проводил?

Мы знаем, что КЭД - важная часть теормина. Видно, тогда не было КЭД в теормине.

A.1.2 Phenomena Deeply Understood from the Point of View of QFT

Углубленное понимание света

КТП (? по идее, пока хз) сполна ответит на вопросы о том, что есть свет и как он работает? Отражения, преломления. См. на Ютубе видео чуть что. Пока этих ответов я не нашел.

A.1.3 Overview of QFT Applications

неплохо бы выписать вопросы, которые перед нами появляются, когда мы начинаем КТП заниматься.

Физика элементарных частиц

вроде там она используется, но дорасти нужно сперва до этого.

когда КТП не нужно

(важный вопрос)
в квантмехе ввели бы как-то взаимодействие частиц, да и все.
решали бы квантовомеханические задачи про частицы.
зачем к полям переходить?

характеристики квантовых явлений, в которых не хватает квантовой механики

да хз вот...

A.1.4 Amazing Facts

(??? collect them???)

A.1.5 Results of QFT

объяснен аномальный магнитный момент

Швингер это сделал.
где-то у меня про это будет, Зи хорошо про это написал.

A.1.6 The Physical World with QFT

обсудим, как решаются вопросы, которые без КТП были не решенными.

Сравнение методов квантования

понятия частиц или квантов полей

супер частый вопрос, КТП на многообразиях изучу - впишу.
Кванты поля обладают всеми физическими свойствами самого поля.
да?
(тут в том числе выжимка Киселева про то, что квант это именно частица)

усовершенствование квантовой механики теорией поля

конкретные простые примеры эффективности подхода квантовых полей

решение проблемы принципа причинности

(как он решается?)

квантовая механика это КТП во временном пространстве (??)

это пока не пойму!
но там уравнение Клейна напоминает как раз осциллятор...
(???? не пойму вообще это)

A.2 How Professionals in QFT Think?

(это все еще раз перепишу с опытом! (мб и не раз))

A.2.1 What is Quantum Field Theory in General? (?!)

(напишу нормальный ответ для меня! пока как-то не готов писать это. формулы тоже добавлю.)

Суть квантовой теории поля

(?? как на этот вопрос ответить? в чем она?)

QFT in a nutshell is just a little number of basic fields and formalism for their interaction, which gives matrix elements

(допишу подробнее, как видеть КТП, чтобы не парится про тысячи страниц непоятно какой теории, потому что так или иначе, общая картина происходящего очень небольшая. так профессионалы и видят КТП, ничего особо там нет сложного, а практически все темы можно ине понимать, просто по потребности можно их дочитывать. Поэтому вообще ничего и нет сложного, хотя, казалось бы, сотни страниц уходят на обсуждение формализма.)

квантовые поля - это примененный аппарат квантования к классическим полям

пока думаю так.
тут вот интересно, что все-таки а почему это так?
там же тонна методов отдельных, а также как-то даже аппарат квантовой механики сперва отдельно выделить нужно, чтобы им пользоваться.
в общем, позже можно подумать, но пока останавливаюсь на этом.

О бесконечностях в КТП

(пока мелкий раздел, потом мб буду расширять)

О ответах на принципиально сложные вопросы квантовой теории поля

(На начальных этапах это лучше отложить. Тут обсуждение, когда ими стоит заниматься и сколько внимания уделять.)

КТП - суть всего материального мира (??)

По сути на квантовом уровне у нас первичны поля, они же проявляют себя как частицы. (?)
тоже пока что далеко не настолько я зрел, чтобы сказать про это точно.
так что они принципиально важны для понимания природы.
однако мб это абсолютно не так, пока еще предстоит разобраться в этом.

Map of QFT

(add picture and discussion about it!!!)

What is Quantum Field Theory?

(с опытом напишу, тут пока ответ Тонга.)

Having told you why QFT is necessary, I should really tell you what it is. The clue is in the name: it is the quantization of a classical field, the most familiar example of which is the electromagnetic field. In standard quantum mechanics, we're taught to take the classical degrees of freedom and promote them to operators acting on a Hilbert space. The rules for quantizing a field are no different. Thus the basic degrees of freedom in quantum field theory are operator valued functions of space and time. This means that we are dealing with an infinite number of degrees of freedom - at least one for every point in space. This infinity will come back to bite on several occasions.

It will turn out that the possible interactions in quantum field theory are governed by a few basic principles: locality, symmetry and renormalization group flow (the decoupling of short distance phenomena from physics at larger scales). These ideas make QFT a very robust framework: given a set of fields there is very often an almost unique way to couple them together.

What is Quantum Field Theory Good For?

(с опытом напишу, Тонг писал, что для всего, но вряд ли это так. потом когда-то свое мнение появится.)

Появление КТП в нашей картине мира

Суть КТП за 2 часа

Обсудим, что из себя представляет предмет наиболее кратко, выделяя самую суть.

обзор квантовой хромодинамики

(??)

Обзор лагранжианов КТП

(тут будет список лагранжианов с комментариями. не такой важный раздел, но прикольно было бы его иметь.)

Что за непертубативные эффекты? (????????)

(пока не дошел до них, потом дойду.)

Основные методы КТП за 2 часа

Наиболее прикладные темы КТП

обсудим, какие темы на самом деле самые важные тут.

обзор теоретических подходов

такие-то есть, такие полезные, такие - нет.

Нетипичные применения КТП

Обсудим применения, которые скорее всего не встретятся, однако видеть их очень полезно

применения в математической статистике

очень актуально кстати.
по идее мы когда там считаем разные усреднения и поправки, там же это делается.

применения в экономике

такие тоже есть, но до этого абсолютно не дойти, пока сам формализм не будет досконально понят.
хотя мб тут именно континуальный интеграл будет применяться, а не сами методы КТП.
пока рано писать.

Обзор дальнейших развитий КТП

пока не готов писать

хромодинамика

теория струн

Список физических величин

Посмотрим, какие величины и зачем нужно нам помнить?
масса планка

A.2.2 Attitude Toward QFT

(сомнительное название раздела, потом перенозову или удалю)

Стоит ли заниматься КТП?

Стоит ли заниматься КТП обычным физикам?

Однозначно, да, но ограничить себя по объему работы, иначе увязнешь! Потому что если говорить про фундаментальные взаимодействия, нано масштаб (?? какой конкретно масштаб???), то там если не знать КТП, то и сказать не будет чего в принципе.

Стоит ли заниматься КТП тем, кто специализируется в другой области физики (!?!)

Думаю, да, но если у человека нормальный подход в принципе к науке, а также нет чего-то горящего. Просто без проблем можно раз в пару недель досматривать формализм или прорешивать модели, это было бы совершенно не сложно. Самое главное тут иметь общий уровень развития и подход к физике, если они есть - нет проблем спокойно досматривать КТП, это было бы очень интересно и полезно. Но если нет развития - тут слишком много будет проблем, так что возможности получить пользу будет мало, так что особой важности в занятиях КТП нету.

A.2.3 Which Problems of QFT Are Important And Which Are Not?

(there are a lot of problems, but many of them looks useless. the discussion of this will be here. which problems solving is a good idea and which is just a waste of time? Maybe Kiselev doesn't know this)

A.2.4 Is QFT Really a Good Theory After All?

(с опытом напишу, важная часть обзора. вообще, мб и вряд ли, но пока рано это писать)

О строгости КТП

говорят, она не строгая математически, это плохо. да?
не знаю.
есть мнение, что для математиков это вызов, КТП можно сделать более строгой.
То есть ничего страшного, просто потом будут новые формулировки ее, более четкие.
А с этим появятся новые идеи.

A.2.5 “So Many Formulas, What is Going On?”

Типичный вопрос изучающего КТП это вопрос в духе “Столько формул, я вообще не понимаю, что с чем связано. Какие-то методы, какие-то параметры, что-то, что я даже не понимаю, к чему относится. Можно упрощенно понять, что вообще происходит?” Это типичный вопрос и часто нужна просто пара лет, чтобы все уложилось в голове. Тем не менее, я попробую на него ответить, потому что такой ответ очень может помочь все уложить в голове, в то же время моя первая часть тоже об этом.

(напишу просто очень упрощенно)

О том, что мы хотим найти и какие физические параметры у нас итоговые

(сечения рассеяния, ширины распадов.)
(мб что-то еще напишу, пока нет потребности в этом.)

Еще раз быстрый вывод амплитуды рассеяния простого процесса?

(в теории про это подробно, в 1й части - указаний много хороших, а тут - чисто для распутывания запутавшегося. просто иллюстрация метода. но мб это же в 1ю часть добавлю, просто там идея в нормальных указаниях, а не в указаниях для распутывания)

A.2.6 Which Method to Chose for a Given Problem?

(про это много вопросов, тут все это обсуждение. когда это полезно, а когда нет?)

Мотивация работать с интегралом по траекториям

Сохранение Лоренц инвариантности

Минусы работы с интегралом по траекториям

с амплитудой могут быть проблемы

A.2.7 How to Study of QFT Effectively?

Особая важность хорошего понимания основ (!)

В других предметах основы интуитивно понятны, но не в КТП. тут нужно очень ответственно подойти к тому, чтобы понять, откуда появляются диаграммы и как ими пользоваться!!! если в этом налажать, то весь предмет не зайдет. раскрою эту мысль подробнее.

Необходимые темы

теория поля

потом пропишу связь

квантовая механика

потом пропишу связь

минимальная теория групп

потому что в этом ориентироваться прямо обязательно!
представления, разные переходы!
позже всё пропишу.

Дополнительные темы

да вроде особо ничего и не нужно, там по ходу разобраться можно.
потом напишу с точки зрения мышления, пока хз.

физика элементарных частиц

потому что там приложений много

дифференциальная геометрия

потому что КТП на многообразиях мы любим смотреть.

теория струн

Теория струн напрямую связана с КТП и по сути обобщает ее.

Особенности эффективного изучения

КТП изучается по сути так же как и другой предмет, однако есть некоторые особенности, сильно увеличивающие эффективность его изучения.

мышление для КТП играет первостепенную роль

да уж, тут порой приходится полгода непонятно что учить, подготовительные предметы улучшать, и только ради того, чтобы создать работающее мышление, с которым изучение будет проходить не сложно.

по сути предмет простой, разве что необходимые темы не должны вызывать принципиальные вопросы, а также необходимо иметь силы и отдельное время на предмет.

и всё, далее всё получится. особенно если пользоваться методами этого раздела.

Очень хорошо было бы найти профессионала в КТП и приятного в общении, это ускорит обучение раз в 10

Это будет огромным плюсом, который будет стоить того, даже если этот человек будет немного фигни нести и приколы иметь в плане своего видения мира.

Как найти? Не знаю, потому что у нас полно своих дел, нам не до бегания и поиска их. Знаю, что просто идя вперед по жизни такого и встретишь, сразу будет видно, что ему особо интересно про КТП общаться и он понимает задачи и подход к ней. Так или иначе, двигаясь в науке, однажды и найдешь такого человека, вот и общайся с ним периодически.

Конечно же, не все профессионалы хороши в общении, очень много ни о чем людей. Обычный фильтр от не тех людей решает полностью эту проблему.

Обход типичных тупиков при изучении

Посмотрим на изучение с точки зрения выхода из тупиков, порой это куда полезнее, чем рассуждения о правильном подходе.
(пропишиу позже)

если книга не понятна - возьми новую книгу

Большинство книг написаны очень неплохо, в то же время если на данном этапе она для тебя непонятная - закрывай и открой новую!!
это решит полностью остановку и даст кучу сил идти вперед.
просто начинаешь заново и все.

О разных подходах к изучению КТП

(потом обсужу, лень писать пока, не нужно это почти)

О строгом подходе к КТП по типу Вайнберга или Боголюбова (???)

тоже есть такой

A.2.8 Is QFT Overcomplicated and Should Be Used Only if There Is No Way Without It?

(this is a big discussion, that sometimes yes, it is abstract theoretical mathematical and useless, but sometimes no, it is exactly what should be used. it is an important question, so a big discussion will be here)

A.2.9 Ways to Guess All the Main Ideas

незаменимая часть нормального понимания предмета.

Глубокое понимание концепции частиц

(что типа все квантутеся. так эйнштейн свою модель теплоемкости предложил. и так мы не все смотрим, что каждому полю обязательно должна соответствовать частица. это важный вывод из КТП, который пропишу тут. пока его не понимаю. но это важная концепция, которую точно нужно прописать нормально.)

В чем были основные ошибки и непонимания ученых в прошлом, когда не было КТП и до этих методов не могли догадаться? (!?!??!??!?)

(для профессионального уровня очень важно ответить на этот вопрос подробно!!)

A.2.10 How not to Turning into Pseudoscience by Doing QFT?

(они особо важны для КТП, потому что фигня тут какя-то часто происходит, и большая потребность в четких критериях, как отличить фигню от обоснованной теории.)

A.3 Acknowledgements

Currently, no one except me has worked on the sections of this note (with the exception of sections taken from books).

A.4 Literature on QFT

A.4.1 Main literature

Основная обучающая

[1] Tong D. Quantum Field Theory Part I, II
Небольшие лекции про основы квантовых полей, не многие темы раскрыты, тем не менее, много важных идей.

[2] Kevin Zhou Lecture Notes on Quantum Field Theory https://knzhou.github.io/
Хороший конспект сути в КТП.

[3] Зи Э. Квантовая теория поля в двух словах
Замечательная книга, очень многие вопросы обсуждаются в примерно свободном формате.

[4] Mandl F., Shaw G. Quantum Field Theory
Хорошая книга, много тем разобрано, много формул написано, но понять КТП по ней невозможно, можно разве что посмотреть примеры и найти местами хорошее изложение.

[5] Вергелес С. Н. Лекции по квантовой электродинамике
Хорошие лекции, очень важные темы разобраны, многому можно научиться, но многое написано вообще непонятно.

David ATKINSON Porter Wear JOHNSON "Exercises in Quantum Field Theory A Self-Contained Book of Questions and Answers Volume 4"
A really great book with problems and solutions. Covers QM, field theory, QFT, Standard Model. Top 5 book to read in free time for one who knows QFT.

Основная профессиональная

[6] Matthew D. Schwartz Quantum Field Theory and the Standard Model
Очень крутая книга, очень много неповторимо хорошо написано. Можно было бы считать ее основной обучающей, просто она очень большая.

[7] М. Е. Пескин, Д. В. Шредер Введение в квантовую теорию поля
Очень большая и хорошая книга, где очень подробно многое обсуждается. Не все, тем не менее, в основном по ней семестр учиться квантовой теории поля - очень хорошая идея. Можно было бы считать ее основной обучающей, просто она очень большая.

Zhong-Zhi Xianyu A Complete Solution to Problems in "An Introduction to Quantum Field Theory" by Peskin and Schroeder
Greate and very useful solutions.

Collection of books, which have many problems with solutions

Some of them will be also mentioned in another lists of literature. They are selected here to make them easier to be found.

David ATKINSON Porter Wear JOHNSON “Exercises in Quantum Field Theory A Self-Contained Book of Questions and Answers Volume 4”
A really great book with problems and solutions. Covers QM, field theory, QFT, Standard Model. Top 5 book to read in free time for one who knows QFT.

https://old.mccme.ru/ium//f13/f13-belavin-parhomenko.html Лекции и задачи с лекций Белавина
Очень крутые задачи, разберу потом!!!

A.4.2 Additional and professional literature

Дополнительная обучающая

[8] Казаков Д. И. Радиационные поправки, расходимости, ренормализация и все такое в квантовой теории поля
Хорошая книга про эти темы, много формул и примеров, по ней можно очень хорошо тренироваться. Примерно эти лекции и составляют типичную КТП 2 (из 3). Просто есть книги лучше, проще, подробнее, а также эти темы не первые, которые следует тренировать и они не так часто нужны.

[9] Dreiner, Herbi K. and Haber, Howard E. and Martin, Stephen P. Two-component spinor techniques and Feynman rules for quantum field theory and supersymmetry
A very good article with clear explanation and many examples.

[10] Киселев В. В. Квантовая теория поля элементарные основы (advanced)
Неплохая книга на тренировку, но считать ее основным учебником не вижу смысла, слишком особая теория, слишком в сыром виде все написано и слишком перегружено.

[11] Andrey Grozin Lectures on QED and QCD https://arxiv.org/abs/hep-ph/0508242
Выглядят очень крутыми лекции по КТП, много формул, 1ю часть скорее всего по ним буду создавать, очень вероятно, что неделю потренируюсь по этим лекциям.

Brian Hatfield Quantum Field Theory of Point Particles and Strings
Maybe, it is the best book to make a transition from QFT to string theory? Looks like there are a lot of detailed explanations of basics.

Hugh Osborn Advanced Quantum Field Theory (typeset by Steffen Gielen in 2007, revised by Carl Turner in 2013)
Можно подумать про темы и мелочи из этих лекций в течение недели, это будет полезно и интересно. В качестве основного лекционного учебника - и близко не тянет.

Matthew Schwartz Lecture “III-2: Vacuum polarization”, 2012
Выглядят как очень хорошо написанные лекции про поляризацию вакуума. Скорее всего это одна из его лекций в Гарварде, в интернете этот файл качается с сайта Гарварда, но откуда он - неизвестно.

[12] Robert Harlander Feynman diagrams From complexity to simplicity and back
Небольшая методичка про то, что я скорее всего уже знаю, обзор диаграмм Фейнмана. Будет свободное время - прочитаю, по идее тоже несколько полезных мыслей должно быть.

[13] Владимиров А. А. Введение в КТП
Компактная книга, которая мало что объясняет, но интересно посмотреть на взгляд автора, очень многое большое в пару страниц написал. Учиться по ней в принципе невозможно, просто когда нормальные книги уже прочитаны, можно посмотреть пару утверждений и подумать про них.

Michel Bellac Thermal field theory
Here, there is a discussion of thermal field theory, as is written in the title. Maybe 2 weeks will be enough to understand basics, how it works? QFT is needed a lot for it.

Углубленная и другая профессиональная

[14] Ландау Л. Д. Том 4, Квантовая электродинамика
Много тем КТП, акцент на электродинамику. Хорошая книга для доработки своего понимания, что есть некоторые разделы. Пока про адроны из нее у меня многое написано.

[15] Боголюбов Н. Н., Широков Д. В. Квантовые поля, 1980.
Многие ее очень много изучают, потому что говорят, что хорошо многое там написано. Поизучаю потом тоже, наверное, пока книги есть интереснее и проще.

[16] Боголюбов Ширков Введение в квантовую теорию поля 1984 link
Еще одна книга Боголюбова, которая многое скажет о КТП. Наверное, минимум на 2 недели тренировок в ней, чтобы усвоить ее.

[17] Oliver Consa The Unpublished Feynman Diagram IIc
Статья про то, что ассистенты Фейнмана неправильно посчитали диаграмму, так это или нет, так и не понял?

[18] Steven Weinberg What is Quantum Field Theory, and What Did We Think It Is?
Мировой эксперт обсуждает, что вообще такое КТП, очень интересно на досуге познакомиться с его мыслями.

[19] Aoyama, Tatsumi and Hayakawa, Masashi and Kinoshita, Toichiro and Nio, Makiko Complete Tenth-Order QED Contribution to the Muon
Еще одна статья про какие-то диаграммы и их вычисления, очень интересно, всего 5 страниц. Когда-то раз подумаю про это!
Thomas Borne, Georges Lochak, Harald Stumpf Nonperturbative Quantum Field Theory and the Structure of Matter
A discussion of special constructions in QFT and field theory. Very special methods. I don't know, if they are important or now, but specialists in QFT, maybe, know some of them.

[20] Каку М. Введение в теорию суперструн
Много для КТП деталей объяснено, да и скорее всего крутая книга, так что во многом добавлю.

Поляков А. М. Калибровочные поля и струны
Книга профессионала, поэтому есть смысл на нее и равняться.

[21] Вайнберг Квантовая теория поля 1
когда сделаю основы - дополнять буду его методами, ибо это профессионализм. Там даны подробно основы представлений группы Лоренца, так что во многом по нему буду писать. Ну и вообще, много деталей раскрыто.

[?] Вайнберг Квантовая теория поля 2

[22] Райдер Л. Квантовая теория поля
Пара хороших моделей, схожих со Стандартной моделью описывается. мб и нормальная книга, просто мб это же и в других дается, тут не знаю...

Волошин М. Б., Тер-Мартиросяи К. А. Теория калибровочных взаимодействий элементарных частиц. М.: Энергоатомиздат, 1984, 296 с.

Книга профессионалов и для профессионалов, очень концентрированная, минимум 2 месяца нужно на понимание основных ее тем, для минимального уровня можно и не открывать. Качается с либ-мипта.

[23] Шварц А. С. Квантовая теория поля и топология
интересная книга, пока не до нее. Пройду сперва основы, вдали раздел по топологическим явлениям во многом по ней создавать буду.

[24] Завьялов О. И. Перенормированные диаграммы Фейнмана
Перенормировки, возможно, буду по ней смотреть, вроде интересно написано. пока не до этого просто.

Фейнман Р. Квантовая электродинамика
Типичная книга, которую все читают. Когда-то тоже пройду, мб узнаю подход Фейнмана лучше, его образ мышления. Пока не актуально и не до этого.

Shvarts A S Matematicheskie osnovy kvantovoi teorii polya
Нет желания открывать, но судя по названию, очень хорошо там основы будут рассказаны.

Швебер ктп.pdf
Мб когда-то открыл и закрыл, еще какая-то известная книга.

Гриб А. А., Мамаев С. Г., Мостепаненко В. М. Вакуумные квантовые эффекты в сильных полях. — 2-е изд., перераб. и доп. — М.: Энергоатомиздат, 1988.—288 с—ISBN 5-283-03955-2.
Книга профессионалов про вакуум, очень известная. Чего только там не происходит! Если начну изучать, отдельные 3 недели на эти явления уйдут, в минимум знаний они не входят. Одним словом, черт ногу сломит разбираться, что там за явления в вакууме.

[?] Вайнберг Квантовая теория поля 3 суперсимметрия
Возможно, тоже хорошая книга, но не думаю, что это нужно кому-то, кроме энтузиастов.

[25] Volovik, Grigory The Universe in a Helium Droplet
Более 480 страниц (более 530 страниц с цитатами) большой теории и размышлений про вакуум, квантовые жидкости, всякие явления с ними. Очень интересная книга, в то же время это специфика, никак не для обязательного изучения.

Литература про важные особенности математики для КТП

[26] Jegerlehner, F. Facts of life with γ_5
Обсуждаются ее особенности, всяких много интересных мыслей, когда-то продумаю.

Про разные процессы в физике элементарных частиц в рамках КТП и вычисления диаграмм

[27] Sergey Volkov Calculation of lepton magnetic moments in quantum electrodynamics: a justification of the flexible divergence elimination method
Статья про вычисление магнитного момента. Когда-то почитаю, разберу хотя бы одну диаграмму!!!! Проверю, могу ли???

Про КТП в физике конденсированных сред

(тоже ее много, потом обсудим.)
[28] Левитов Шитов Функция Грина. Задачи и решения.
крутая книга с примерами КТП к твердотельным явлениям. Очень скоро буду его добирать и вписывать, очень важная книга.

Абрикосов Горьков Дзялошинский
Известная тоже книга, многими считающаяся одной из лучших.

Moritz Helias Lecture notes: Statistical physics 2018
Лекции из Ахена про всякие сложные темы. Выглядят крутыми, содержательными, много диаграмм. Отдельный курс, месяц примерно изучать только их нужно.

По квантовой механике и теории поля в помощь для КТП

[29] Тахтаджян Л. А. Квантовая механика для математиков
прекраснейшая книга, где прекрасные определения, которыми я и пользуюсь (буду пользоваться) просто шикарная структура.

[30] Зинн-Жюстен, Жан Континуальный интеграл в квантовой механике
самая прекрасная книга, по которой и буду создавать изначально запись. книга настолько прекрасна, что ничего другого перед ней читать не хочу! и уже многее выгрузки по ней сделаны!

A.4.3 Selected articles about applications and special phenomena

Статьи о проверке типичных явлений

[31] Levi, M. E. and Blocker, C. A. and Strait, J. and others Weak Neutral Currents in e^+e^- Collisions at $\sqrt{s} = 29$ GeV
Маленькая статья, где хороший график приведен сравнения сечений. Когда-то научусь его выводить.

О других темах и особых редких методах КТП

(потом группировку мб сделаю)
Mukhanov Quant Effects
многим известная книга, до которой пока мне нет дела.
[32] Schwinger, Julian Quantum Electrodynamics. I. A Covariant Formulation
Известная статья Швингера 1948 года, крутое развлечение на вечер, когда будет сделана работа - попробовать прочесть и понять все там. По идее на современном уровне 2 дня хватит на нее. Но не актуально пока что.

Ronald Kleiss Quantum Field Theory A Diagrammatic Approach https://agenda.irmp.ucl.ac.be/event/3340/contributions/4015/attachments/2087/2506/qft.pdf
Странная книга со многими диаграммами, странными формулами, пока не думаю, что мне такое нужно, ну и точно про нее много задуматься придется. Но выглядит она, конечно, круто очень.

[33] Stefan Hollands, Robert M. Waldb Quantum fields in curved spacetime
Интересная статья, как описывать квантовые поля в искривленном пространстве времени. (в неискривленном бы научиться сперва!) Много добавлений из нее выгружено, потом буду разбирать

[34] Редьков В. М. Поля частиц в римановом пространстве и группа Лоренца
прекрасно про искривленное пространство дана тонна теории. сделаю основы - буду дополнять ей. очень уж крутые и интересные темы!!! скорее бы собрать основы! Может, потом и другие его книги добавлю, не до него сейчас.

[35] Robert Karplus and Maurice. Neuman The Scattering of Light by Light

9 страниц про рассеяние света на свете. Такое чувство, что менее, чем за 3 дня это не пойму, так что отложил, интересно, но не актуально. Доступна через lib.mipt, в интернете в открытом доступе нет ее.

Kiryl Piasotski Mikhail Pletyukhov Diagrammatic approach to scattering of multiphoton states in waveguide QED
Применили КТП к волноводам, написали кучу сложных диаграммных формул. Скоро пойму, как это связано с обычной теорией, на первый взгляд кажется, что нужно долго подумать, прежде, чем это станет понятно.
(???? про теорию струн неплохо тоже написать будет потом книгу, пока не так особо читал это.)

Статьи о разных редких эффектах и явлениях

Про суперсимметрию в КТП

A.5 Description of This Note

A.5.1 List of QFT Parameters and Functions

(уже это все написано и в теории и в 1й части, но тут именно с обзорной стороны, чтобы точно было усвоено, что от чего зависит)

***W* эффективный производящий функционал (!!!)**

***J* токи (!!!)**

Коррелятор (!??!??!?)

(про него важные формулы, с ними и работаем ведь!)

Г дифференциальный оператор

$\hat{Z}(j)$ производящий функционал

замкнутым петлям в диаграммах соответствуют интеграции по *dp*

.

О пропагаторах

Пропагатор фотона (массивного векторного мезона) в двух словах (!??!?)

$$D_{\nu\lambda}(k) = \frac{-g_{\nu\lambda} + k_\nu k_\lambda / m^2}{k^2 - m^2}$$

(тут много важных формул будет!)

смысл пропагаторов и их применения

вкратце, зачем они нужны и как используются?
их суть тут запишу.
(пока не очень представляю, как конкретно они нужны...)

О двухточечной корреляционной функции

свойства

Двухточечная корреляционная функция имеет вид:

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle \quad \text{или} \quad \langle\Omega|T\psi(x)\bar{\psi}(y)|\Omega\rangle.$$

$$\int d^4x e^{ip\cdot x}\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle \underset{p^2\to m^2}{\sim} \frac{iZ}{p^2-m^2+i\varepsilon}.$$

В свободной теории поля двухточечная функция $\langle 0|T\phi(x)\phi(y)|0\rangle$ имеет простую интерпретацию.

Она является амплитудой вероятности того, что частица распространяется из точки y в точку x.

В какой степени такая интерпретация является справедливой В теории с взаимодействием?

примеры

связь с другими параметрами

двухточечная корреляционная функция в случае свободного скалярного поля

раздел 2.4 ПШ

Точное выражение для двухточечной корреляционной функции в случае свободного скалярного поля:

$$\int d^4x e^{ip\cdot x}\langle 0|T\phi(x)\phi(0)|0\rangle = \frac{i}{p^2-m^2+i\varepsilon}$$

константа связи

разберем, что есть она и зачем нужна.

***Z* производящий функционал (!!!)**

О многоточечной корреляционной функции

$$\int d^4x e^{ip\cdot x}\left\langle\Omega\left|T\left\{\phi(x)\phi\left(z_1\right)\phi\left(z_2\right)\cdots\right\}\right|\Omega\right\rangle$$

$$\int dx^0 = \int_{T_+}^{\infty} dx^0 + \int_{T_-}^{T_+} dx^0 + \int_{-\infty}^{T_-} dx^0,$$

$$\int_{T_+}^{\infty} dx^0 \int d^3x e^{ip^0x^0}e^{-i\mathbf{p}\cdot\mathbf{x}}\sum_\lambda \int \frac{d^3q}{(2\pi)^3}\frac{1}{2E_{\mathbf{q}}(\lambda)}\left\langle\Omega|\phi(x)|\lambda_{\mathbf{q}}\right\rangle\left\langle\lambda_{\mathbf{q}}\left|T\left\{\phi\left(z_1\right)\phi\left(z_2\right)\cdots\right\}\right|\Omega\right\rangle$$

ведь $\mathbf{1} = \sum_\lambda \int \frac{d^3q}{(2\pi)^3}\frac{1}{2E_{\mathbf{q}}(\lambda)}\left|\lambda_{\mathbf{q}}\right\rangle\left\langle\lambda_{\mathbf{q}}\right|$.

$$\langle\Omega|\phi(x)|\lambda_{\mathbf{q}}\rangle = \langle\Omega|\phi(0)|\lambda_0\rangle\, e^{-iq\cdot x}\big|_{q^0=E_{\mathbf{q}}(\lambda)}$$

$$\int d^4x e^{ip\cdot x}\left\langle\Omega\left|T\left\{\phi(x)\phi\left(z_1\right)\cdots\right\}\right|\Omega\right\rangle \underset{p^0\to -E_{\mathbf{p}}}{\sim} \left\langle\Omega\left|T\left\{\phi\left(z_1\right)\cdots\right\}\right|-\mathbf{p}\right\rangle\sqrt{Z}\frac{i}{p^2-m^2+i\varepsilon}.$$

$$\sum_\lambda \int \tfrac{d^3k}{(2\pi)^3}\varphi(\mathbf{k})\tfrac{1}{2E_{\mathbf{k}}(\lambda)}\tfrac{i}{p^0-E_{\mathbf{k}}(\lambda)+i\varepsilon}\left\langle\Omega|\phi(0)|\lambda_0\right\rangle\left\langle\lambda_{\mathbf{k}}\left|T\left\{\phi\left(z_1\right)\cdots\right\}\right|\Omega\right\rangle \underset{p^0\to +E_{\mathbf{p}}}{\sim}$$
$$\underset{p^0\to +E_{\mathbf{p}}}{\sim} \int \tfrac{d^3k}{(2\pi)^3}\varphi(\mathbf{k})\tfrac{i}{\tilde{p}^2-m^2+i\varepsilon}\sqrt{Z}\left\langle\mathbf{k}\left|T\left\{\phi\left(z_1\right)\cdots\right\}\right|\Omega\right\rangle,$$

$$\left(\prod_i\int\frac{d^3k_i}{(2\pi)^3}\int d^4x_i e^{i\tilde{p}_i\cdot x_i}\varphi_i\left(\mathbf{k}_i\right)\right)\left\langle\Omega\left|T\left\{\phi\left(x_1\right)\phi\left(x_2\right)\cdots\right\}\right|\Omega\right\rangle.$$

$$\left(\prod_{i=1,2}\int\frac{d^3k_i}{(2\pi)^3}\varphi_i\left(\mathbf{k}_i\right)\frac{i}{\tilde{p}_i^2-m^2+i\varepsilon}\cdot\sqrt{Z}\right)\left\langle\mathbf{k}_1\mathbf{k}_2\left|T\left\{\phi\left(x_3\right)\cdots\right\}\right|\Omega\right\rangle$$

Наиболее сингулярное слагаемое:

$$\left(\prod_{i=1,2}\frac{i}{p_i^2-m^2+i\varepsilon}\cdot\sqrt{Z}\right)\left(\prod_{i=3,\ldots}\frac{i}{p_i^2-m^2+i\varepsilon}\cdot\sqrt{Z}\right)_{\text{out}}\left(\mathbf{p}_1\mathbf{p}_2\left|-\mathbf{p}_3\cdots\right\rangle_{\text{in}}\right.\cdot$$

Итак, мы показали, что элемент S-матрицы можно получить, рассматривая соответствующее вакуумное среднее по волновым пакетам полей, выделяя главные сингулярности по энергиям p_i^0 и затем переходя к пределу, в котором волновые пакеты становятся дельта-функциями от импульсов.

A.5.2 Features of This Note

описание глав и разделов

описание записи в целом

(когда-то очень не скоро напишу)

первая часть

(когда-то очень не скоро напишу)

приложения

какие вообще приложения я разбирал?
(когда-то очень не скоро напишу)

Особенности структуры по КТП

(когда-то очень не скоро напишу)

Важность раздела в дополнениях с физическими основами (!!!)

(потом подробнее опишу, насколько он важен)

Именно эти знания предполагаются известными для начала изучения КТП

обозначения

В отличие от многих других разделов физики, в КТП есть много соврешенно непривычных обозначений, в которых можно запутаться, так что приведем их сводку, чтобы не останавливаться на этом.

цифровые индексы

хз, но такое есть. вроде, обозначают номер координаты, но я не уверен.

Константы

Системы единиц

Натуральная система единиц

Иногда в этой записи постоянная Планка $\hbar$ и скорость света c существенно загромождают формулы, поэтому я использую систему единиц, в которой $\hbar = c = 1$.

в ней нет отличий между импульсом и энергией, частотойи энергией, массой и импульсом и т.п.

....

A.6 Puzzles of QFT for Different Cases in Life

(особенно из Зи добавлю многое!!)

A.6.1 Puzzles for Motivation to Study Fundamental Physics

Раздел вдали, потому что не всегда занятие фундаментальной физикой приведет к результатам, так что не всегда полезно иметь мотивацию к ней. Однако, тем не менее, убирание этого полностью приведет к абсолютному непониманию происходящего, что очень не круто.

(в многом из Зи это будет)

Как так, что один и тот же метод представлений группы Пуанкаре - и такие принципиально разные поля, стоит только спин поменять?

(не знаю, вопрос о глубоком понимании пространства-времени, не мой уровень, нужно несколько лет думать про это, чтобы понять это, не актуально такое.)

Как вообще проквантовать поля?

(вопрос номер 1 из КТП. хз, мб я уже это ответил, пока пусть повесит вопрос)

когда электрон считаем полем, а когда частицей?

самый актуальный вопрос.

почему без слабого взаимодействия мы бы испарялись в полной темноте?

Если бы не было слабого взаимодействия, регулирующего горение звезд, то вы бы проваливались или воспаряли в полной темноте.

почему без связи спина со статистикой метрия бы коллапсировала?

Без связи спина со статистикой электрон не подчинялся бы принципу Паули. Материя просто коллапсировала бы.

(Зи говорит, это так)

мб посмотрю Доказательство устойчивости материи, данное Дайсоном и Ленардом, которое существенно опирается на принцип Паули.

почему если частица это поле,то на трековых детекторах видим только треки частиц?

A.6.2 Puzzles for Pleasant Conversation About Fundamental Physics

Тут то, что прикольно, но скорее для приятного проведения времени, подумать об интересном, а не для того, чтобы всерьез потом пойти и изучать это. Еще раз уточню, что имеется в виду именно случай, когда рядом есть собеседник, которому интересно и который что-то изучал про это.

m-th.1 В чем состоит истинная природа физического мира? (????)

(все-таки по факту именно КТП из типичной физике наиболее близка к ответу на этот вопрос. тут укажу коротко вопрос и суть ответа, указав на темы из теории. так-то можно много писать про это.)

m-th.1 Что на самом деле такое фотон?

. Вот и напишу. Укажу, что отговорки - не ответ.

(пока что остановился на том, что вообще неизвестно, но много разных свойств в разных случаях можно указать)

m-th.1 Верна ли КТП в принципе и можно ли избавляться так просто от бесконечностей?

Как так получается, что у нас теория предсказывает бесконечности, что является признаком ее провала, но в то же время есть многие задачи, которые решаются ей?

(потом распишу подробнее. так или иначе, это очень мотивирует все-таки понять, в чем суть метода?)

m-th.1 Почему в физике конденсированных сред могут применяться те же методы, которые в физике элементарных частиц используются?

Казалось бы, два разных раздела.

(распишу потом. формулы добавлю для полноты.)

Благодаря электромагнетизму мы не проваливаемся сквозь пол, это так?

(Зи)

Иногда говорят, что благодаря электромагнетизму вы не проваливаетесь сквозь пол

почему?

О квантовании массы

Вопрос: квантуется ли масса
(вроде никому это не интересно, типа ну квантуется, у каждого кванта своя масса, и что? мб что-то тут разное есть, не знаю.)

A.6.3 Puzzles for Motivation to Study QFT Methods

Тут больше про явления и приложения, где нужны методы.

Если увеличивать энергию электрона, то представляя его как волновую функцию, как объяснить, что распад возможен?

(задача из зи)
Запишем теперь уравнение Шредингера для рассеяния электрона на протоне.
Уравнение описывает волновую функцию одного электрона, и, как бы вы ни упражнялись с теорией дифференциальных уравнений в частных производных, электрон, за которым ведется наблюдение, всегда остается одним-единственным электроном.
Однако, согласно специальной теории относительности, энергия может быть преобразована в материю: если электрон обладает достаточной энергией, могут родиться электрон и позитрон («антиэлектрон»).
Уравнение Шредингера просто не в состоянии описать это явление, что свидетельствует о несостоятельности нерелятивистской квантовой механики.
Давайте рассуждать эвристически. В квантовой механике есть принцип неопределенности, который гласит, что энергия может испытывать резкие флуктуации за малый промежуток времени. Согласно же специальной теории относительности, энергию можно преобразовать в массу и наоборот.
Если объединить принципы квантовой механики и специальной теории относительности, то придем к выводу, что флуктуирующая энергия может превращаться в массу, то есть в ранее не существовавшие частицы.

Пусть два электрона летят друг на друга, как объяснить их взаимодействие с точки зрения КТП?

с точки зрения классической электродинамики все понятно (нужно только это сделать мне еще, пока осознанно не делал)
а вот с точки зрения КТП?
взаимодействие дираковских полей включаем что ли?
и почему оно внезапно электромагнитное?

Если я хорошо знаю релятивистскую механику, можно ли с помощью нее посчитать сечение рассеяния с образованием виртуальной частицы? Будет ли оно совпадать с сечением, расчитанным для этого процесса методами КТП? (?????)

(не знаю, но интересно будет поисследовать.)

А вихревое электрическое поле квантуется так же, как плоское?

Хочу на примере посмотреть.
хз

Почему электрон стабилен?

(?? напишу с формулами, пока не знаю.)

A.6.4 Selected Technical Problems

Приведу задачи для того, чтобы потренироваться/проверить уровень собеседника/продемонстрировать свой уровень.

Покажите, что если в формализме теории рассеяния положить $\hbar = 0$, то получим классическую механику?

(хороший вопрос)

как вывести аномальный магнитный момент?

что за перенормировка заряда?

а я вот сам не знаю еще, хотя и читал.

О типичных интегралах КТП

КТП известно тем, что там любят давать диаграммы и сказать, как быстро их считать? даже развлекаться этим можно.
так что соберем эти вопросы.
(см книгу казакова)

посчитать диаграмму “арбуз”

посчитать диаграмму “закат”

посчитать диаграмму “рыба”

тут во Казаков быстрый методы какой-то сказал, я в него не въехал.

посчитать диаграмму “треугольник”

там типа что-то такое быстрое не прокатывает, тогда другое быстрое приходит.

посчитать диаграмму “пингвин”

(?? как такую задачу задавать?? так и не научился такое считать.)

О типичных распадах КТП

(мб подподглавой сделаю)

О лептонном распаде пи-мезона

(типичная задача, посмотрю семинар Киселева, пропишу сюда суть её. чисто на проверку знаний.)

A.6.5 Puzzles for Those Claiming to Know QFT

A.6.6 Puzzles for Mastering Typical Concepts

Допустим, видим человека, который в принципе хотел бы понять суть, опишем, какие в какой последовательности будем задавать, чтобы в типичном случае ему было бы интереснее въезжать. В частности это могут быть младшие студенты или редко школьники.

Головоломки для освоения КТП уровня теормина

Головоломки для знакомства с понятием диаграмм (!??!??!??!!?)

(самое важное, что может быть, потому что КТП очень часто чисто в диаграммы и упирается.)

Головоломки для освоения других особенностей КТП

A.6.7 Puzzles About Fantastic Phenomena

Как проквантовать Вселенную? (??)

см. работу Розена "Квантовая механика мини Вселенной"
Под вселенной понимаем метрику Фридмана.

A.7 Connection of QFT and Other Parts of Science

A.7.1 QFT, Mechanics and Quantum Mechanics

Явные формулы сравнения законов механики и КТП (????)

(в частности формулы для задач по механике и задачи по КТП. Хочу написать просто как список: там такая формула - тут аналог - такая формула. это не так полезно, просто для более глубокого понимания.)
(!!! a very good question, I'll write it very very later!)

Явное обобщение квантмеха на КТП (??????)

(ну вот пропишу это потом, пока оба предмета доделывать нужно.)

A.7.2 QFT and Field Theory

(I think, it might be a good idea to write this connection explicitly here. it is not obvious!!! and in theory, we are not so much focused on this general and useless for application overview. because it is a comparison of theories!)

A.7.3 QFT and Solid State, Condensed Matter

Очень много применений КТП находит в нем.
(порешаю - пропишу)

A.7.4 QFT and Cosmology

Подсчет обратной реакции рождения частиц в черных дырах

ты не сможешь посчитать обратную реакцию рождения частиц в черных дырах или в расширяющейся вселенной без перенормировок квантовополевого ТЭИ
получить: космологический фактор и плотности энергии материи и пр во Вселенной

A.7.5 QFT and Special Parts of Mathematics

Связь с вычислительной математикой

(мы вычисляем что-то многое разное)

Про КТП на решетках

(такое есть, потом подумаю, хз)

A.7.6 QFT and Hypothetical High Energy Physics Theories

Связи тут большие.
пропишу позже, когда по теории струн разовьюсь.

A.8 Short Historical Overview

Посмотрим вкратце, как развивалось КТП и сделаем нужные выводы из этого.

О попытках подумать про КТП до 20го века

(если такие были, то напишу сюда)

О Фейнмане

Биография в двух словах

Основные выводы из биографии

О Швингере

Биография в двух словах

Основные выводы из биографии

B Physics and Mathematics for QFT

Обсудим коротко некоторые важные для КТП темы других разделов физики.
(??? I'll write it later!! only what is really additional, but still used sometimes)

B.1 Physics and Mathematics for Basic QFT

B.1.1 Some Mathematics for Path Integral

Лемма о трюке Швингера (???)

Lemma

$$G\left(\frac{\partial}{\partial \underline{b}}\right) F(\underline{b}) = F\left(\frac{\partial}{\partial \underline{x}}\right) G(\underline{x}) e^{\underline{x}\cdot\underline{b}}\Big|_{\underline{x}=\underline{0}}.$$

Proof We will show this for a special case: Let $G(\underline{x}) = e^{\underline{x}\cdot\underline{\alpha}}, F(\underline{b}) = e^{\underline{\beta}\,\underline{\underline{b}}}$. Then the left-hand side is

$$G\left(\frac{\partial}{\partial \underline{b}}\right) F(\underline{b}) = e^{\frac{\partial}{\partial \underline{b}}\cdot\underline{\alpha}} F(\underline{b}) = F(\underline{b}+\underline{\alpha}) = e^{\underline{\beta}\cdot(\underline{b}+\underline{\alpha})}.$$

Let us attempt to consider the right-hand side:

$$F\left(\frac{\partial}{\partial \underline{x}}\right) G(\underline{x}) e^{\underline{x}\cdot\underline{b}}\Big|_{\underline{x}=\underline{0}} = e^{\underline{\beta}\cdot\frac{\partial}{\partial \underline{x}}} e^{\underline{x}\cdot(\underline{\alpha}+\underline{b})}\Big|_{\underline{x}=\underline{0}} = e^{(\underline{x}+\underline{\beta})\cdot(\underline{\alpha}+\underline{b})}\big|_{\underline{x}=\underline{0}} = e^{\underline{\beta}\cdot(\underline{\alpha}+\underline{b})}.$$

The result is then true for any F and G as one may express F and G as a Fourier series.

B.1.2 Some Special Functions

(I'll read LL4 and write it here after it)

B.1.3 Some Methods for Matrices

(I'll read LL4 and write it here after it)

B.1.4 Some Numerical Methods

(maybe I'll make many sections here about it)

B.2 Physics and Mathematics for Exotic QFT Models

B.2.1 Some Special Algebra

(in algebra, there are some important ideas, which will be also good to mention)

B.2.2 Mathematics for Supersymmetry

(ok, I'll add also some this later)

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