

Superconductive Contacts, Part I

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This note is not intended for distribution.

Superconductive contacts are discussed in detail.

My Theory
problems

Theory of Typical Junctions
Old, Fundamental Theories
Main Theories of Multiterminal Junctions
Special Effects: Selected Useful Theories
Important Experiments

Contents

1	Preface and main motivation	2
I	Superconductive Contacts in a Nutshell	3
2	Overview of well-known formulas of JJ	3
2.1	Main general formulas	3
3	Special constructions in a nutshell	3
3.1	Models of multiterminal junctions	3
3.1.1	Multiterminal S-TS-S junctions by Zazunov et. al.	3
3.2	Other models	7
3.2.1	S-TS, S-TS-S junctions with a Majorana wires by Zazunov, Egger	7
3.2.2	S-TS, S-spinful TS, S-QD-TS, S-TS-S junctions by Zazunov et. al.	11
3.2.3	JE in a junction coupled to an electron reservoir by Beenakker	18
3.2.4	JC, Noise at a S-QSHI-S Junction by Fu, Kane	20
II	My Theory of Superconductive Contacts	22
4	Fundamentals of JJ	22
4.0.1	Basics	22
III	Problems	23
5	Типичные задачи	23
5.1	Общие вопросы	23
5.1.1	Вопросы на понимание сути предмета	23
5.1.2	Типичные вопросы и задачи на проверку знаний	23
5.1.3	Вопросы на понимание типичных деталей	23
5.2	Задачи типичных разделов	23
6	Другие задачи	23
IV	Typical Contacts: Well-Known, Important Theories	24

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7	Ballistic JC through an asymmetric SNS by Chang, Bagwell	24
7.1	VI. Conclusions	24
7.2	I. Introduction	25
7.3	II. Supercurrent through an $\mathbf{S_1NS_2}$ Junction	25
7.4	III. Point-contact $\mathbf{S_1NS_2}$ Junction	28
7.5	IV. Long $\mathbf{S_1NS_2}$ Junction	31
7.6	V. Finite temperature	33
7.7	Appendix A: Continuum current $I_3(\phi)$	34
7.8	Appendix B: Continuum current $I_2(\phi)$	37
7.9	Appendix C: Discrete current $I_1(\phi)$	39
7.10	Appendix D: Equivalence Of The Two Current Operators	40
8	Superconducting weak links by Likharev	42
8.1	II. JOSEPHSON EFFECT AND WEAK LINKS	43
8.1.1	A. Josephson effect	43
8.1.2	B. Tunnel junctions	46
8.1.3	C. Weak links	50
8.2	III. STATIONARY (DC) EFFECTS	53
8.2.1	A. Aslamazov-Larkin theory	53
8.2.2	B. Arbitrary temperatures	56
8.2.3	C. Effect of finite length	61
8.2.4	D. Clean structures	68
8.2.5	E. $I_S(\varphi)$ relationship and Josephson effect	73
8.3	IV. TYPES OF WEAK LINKS	79
8.3.1	A. Nonlinear processes in the electrodes	79
8.3.2	B. Structures without current concentration	80
8.3.3	C. Dayem bridges	83
8.3.4	D. Structures with strong current concentration	85
8.3.5	E. Controllable weak links	89
8.4	V. JOSEPHSON OSCILLATIONS AND $I-V$ CURVE	89
8.4.1	A. Influence of intrinsic capacitance	90
8.4.2	B. General pattern of $I - V$ curve	91
8.4.3	C. Description by time-dependent Ginzburg-Landau (TDGL) equations	97
8.4.4	D. Nonequilibrium quasiparticles	104
8.5	VI. OTHER NONSTATIONARY PROCESSES	109
8.5.1	A. High-frequency impedance	110
8.5.2	B. $I_A(\omega)$ function and current steps on $I - V$ curves	114
8.5.3	C. Microwave enhancement of superconductivity	119
8.5.4	D. Fluctuations	120
8.6	VII. MORE COMPLICATED WEAKLY LINKED STRUCTURES	122
8.6.1	A. Multiterminal weak links	123
8.6.2	B. Connections of weak links	124
8.6.3	C. Distributed structures	127
8.7	VIII. CONCLUSIONS	133
8.7.1	A. Basic results	133
8.7.2	B. Unsolved problems	135
8.7.3	C. Concluding remarks	135
9	Current-phase relation in Josephson junctions by Golubov, Kupriyanov, Il'ichev	136
9.1	IX. SUMMARY AND OUTLOOK	136
9.2	I. INTRODUCTION	137
9.3	II. DC JOSEPHSON EFFECT	138
9.3.1	A. General properties of the Josephson CPR	140
9.3.2	B. Basic Josephson structures.	142

9.4	III. JOSEPHSON EFFECT IN POINT CONTACTS	144
9.4.1	A. Aslamazov-Larkin model	144
9.4.2	B. Kulik-Omelyanchuk model, dirty limit (KO-1)	144
9.4.3	C. Kulik-Omelyanchuk model, clean limit (KO-2)	145
9.4.4	D. Point contact, the general case	146
9.5	IV. JOSEPHSON JUNCTIONS WITH TUNNEL TYPE OF CONDUCTIVITY	149
9.5.1	A. Ideal tunnel junctions	149
9.5.2	B. Tunnel junctions with extended barriers	150
9.5.3	C. Tunneling via localized states (LS)	151
9.5.4	D. SIS structures with high transparency: depairing by current	154
9.6	V. SNS JUNCTIONS.	155
9.6.1	A. The limit of high temperature $T \approx T_C$	155
9.6.2	1. Weak depairing in the electrodes (rigid boundary conditions)	156
9.6.3	2. Depairing in the S electrodes by the proximity effect	157
9.6.4	3. Depairing in the S electrodes by the supercurrent	158
9.6.5	4. Nonlinear phase shift in the electrodes	159
9.6.6	5. Influence of the critical temperature of weak link material on $I_C R_N$	160
9.6.7	B. Dirty limit, arbitrary temperature.	161
9.6.8	1. Rigid boundary conditions	161
9.6.9	2. SNS junction at arbitrary T . Depairing by the proximity effect	165
9.6.10	3. $SS'S$ junction at arbitrary T . Depairing by the supercurrent	166
9.6.11	C. Clean SNS junctions	167
9.7	VI. DOUBLE BARRIER SINIS JUNCTIONS	170
9.7.1	A. SINIS junctions, clean limit	170
9.7.2	1. The general case	170
9.7.3	2. Coherent regime (broad resonances)	172
9.7.4	3. Incoherent regime (narrow resonances)	173
9.7.5	B. SINIS junctions, dirty limit	174
9.7.6	1. The general case	174
9.7.7	2. Limit of high temperatures	175
9.7.8	3. Limit of low temperatures	175
9.7.9	4. Limit of small γ_{eff}	176
9.7.10	5. Limit of large γ_{eff}	176
9.8	VII. SFS JOSEPHSON JUNCTIONS	177
9.8.1	A. Proximity effect in SF bilayer	178
9.8.2	1. The formalism	178
9.8.3	2. Complex coherence length	179
9.8.4	3. Phase variation at the SF interface	180
9.8.5	B. CPR in SFS junctions: simple geometry	180
9.8.6	1. $0 - \pi$ transitions due to oscillating order parameter	181
9.8.7	2. $0 - \pi$ transitions due to phase jumps at the SF interfaces	183
9.8.8	C. Asymmetric case: SIFS tunnel junction	185
9.8.9	D. Clean SFcFS point contact	186
9.8.10	E. Diffusive SFcFS point contact	187
9.8.11	F. Double barrier SIFIS junction	188
9.9	VIII. EXPERIMENTAL METHOD OF MEASURING $I_S(\varphi)$ AND ITS APPLICATION	190
9.9.1	A. Description of the experimental method	190
9.9.2	B. The range of validity of the method	192
9.9.3	C. Measuring technique and calibration	193
9.9.4	D. Measurements of small critical currents	195
9.9.5	E. Measurement of asymmetry of barrier transparency in SINIS junctions	197
9.9.6	F. CPR in hybrid S-2DEG-S junctions	198
9.9.7	G. CPR anomalies in HTS junctions	199

9.9.8	1. d-wave effects in HTS	199
9.9.9	2. Detection of the second CPR harmonic	200
9.9.10	3. Manifestation of MGS in the <i>CPR</i>	202
9.9.11	4. c-axis HTS junctions	204
9.9.12	5. General remarks	206
9.10	XI. APPENDIX: MICROSCOPIC THEORY OF SUPERCONDUCTIVITY	206
9.10.1	A. Green's functions	206
9.10.2	B. The quasiclassical approximation	207
9.10.3	C. Ginzburg-Landau equations	210
10	Josephson and Andreev transport through quantum dots by Martín-Rodero and A. Levy Yeyati	211
10.1	Introduction	212
10.2	Basic models and formalism	214
10.3	Equilibrium properties of quantum dots with superconducting leads	217
10.3.1	Cotunneling approach	219
10.3.2	Mean field and variational methods	220
10.3.3	Diagrammatic approaches	223
10.3.4	Diagonalization by numerical methods	228
10.3.5	Functional renormalization group	232
10.3.6	Quantum Monte-Carlo	233
10.3.7	Experimental results	233
10.4	Quantum dots with normal and superconducting leads	236
10.4.1	Effect of interactions (linear regime)	236
10.4.2	Non-linear regime	240
10.4.3	Experimental results	242
10.5	Voltage biased S-QD-S systems	244
10.5.1	Effect of Coulomb interactions	249
10.5.2	Summary of Experimental results	252
10.6	Beyond the single level model: multidot, multilevel and multiterminal systems	254
10.6.1	Josephson effect through multidot systems	255
10.6.2	Multilevel quantum dots	260
10.6.3	Multidot-multiterminal systems with normal and superconducting leads	264
10.7	Concluding remarks	269
11	Josephson Current through a Semiconductor Nanowire: effect of strong spin-orbit coupling and Zeeman splitting Cheng, Lutchyn	269
11.1	Introduction	270
11.2	Theoretical Model	271
11.3	Short-Junction Results	273
11.3.1	Zeeman-field dominated regime	273
11.3.2	SOC dominated regime	274
11.3.3	General case	275
11.4	Long-junction results	277
11.5	Local Density of States at the junction	277
11.6	Conclusions and Discussions	280
11.7	Derivation of the scattering matrix $S(\epsilon)$	281
11.8	Validity of the short-junction approximation	282
12	Josephson effect in junctions of SC and TSC by Zazunov et al.	285
12.1	Introduction	285
12.2	S-QD-TS junction	287
12.2.1	Model	287
12.2.2	Cotunneling regime	290

12.2.3	Mean-field approximation	291
12.3	Spinful nanowire model for the TS	294
12.3.1	Model	294
12.3.2	S-TS junction	296
12.3.3	S-QD-TS junction with spinful TS wire: Mean-field theory	297
12.4	S-TS-S junctions: Switching the parity of a superconducting atomic contact	298
12.4.1	Model	298
12.4.2	Atomic limit	302
12.4.3	Another interpretation	304
12.4.4	Conventional midgap level	304
12.5	Conclusion	305
V	Old, Fundamental Theory	306
13	Macroscopic quantization and the proximity effect in SNS junctions by Kulik	306
13.0.1	1. Introduction	306
13.0.2	2. Wave functions and energy levels of single-particle excitations	307
13.0.3	3. Superconducting currents in a spatially inhomogeneous system	311
13.0.4	4. Calculation of the Josephson current	314
14	Quantum point contacts in magnetic fields by Ando	319
14.0.1	I. INTRODUCTION	319
14.0.2	II. CONDUCTANCE CALCULATION IN A LATTICE MODEL	320
14.0.3	A. Hamiltonian	320
14.0.4	B. Ideal wire	321
14.0.5	C. Scattering problem	322
14.0.6	III. SINGLE POINT CONTACT	324
14.0.7	A. Model	324
14.0.8	B. Numerical results	325
14.0.9	IV. SERIES OF TWO POINT CONTACTS	329
14.0.10	A. Model	329
14.0.11	B. Numerical results	331
14.0.12	V. SUMMARY AND CONCLUSION	335
15	Quasiclassical method in the theory of SC.tex by LARKIN, OVCHINNIKOV	336
16	Three “universal” mesoscopic Josephson effects by Beenakker	346
16.1	Introduction	347
16.2	Supercurrent from Excitation Spectrum	349
16.3	Excitation spectrum from scattering matrix	351
16.4	Short-Junction Limit	354
16.5	Universal Josephson Effects	355
16.5.1	Quantum Point Contact	355
16.5.2	Quantum Dot	356
16.5.3	Disordered Point Contact	358
17	Quantum dynamics of a superconducting tunnel junction by Eckern	360
17.1	I. INTRODUCTION	360
17.2	II. QUANTUM STATISTICS OF A SUPERCONDUCTING TUNNEL JUNCTION	361
17.3	III. REAL-TIME DESCRIPTION	368
17.4	IV. RESULTS, COMPARISON, AND DISCUSSION	373
17.4.1	A. Summary	373
17.4.2	B. Quantum Langevin equations	374
17.4.3	C. Macroscopic quantum phenomena in Josephson junctions	378

VI	Multiterminal Junctions: Well-known, Important Theories	381
18	Josephson Effect in multiterminal topological junctions by Zazunov et al.	381
18.0.1	Introduction	381
18.1	Low-energy approach	384
18.1.1	Model	384
18.1.2	Parallel configuration	385
18.1.3	Serial configuration	388
18.1.4	Atomic limit	390
18.2	Boundary GF approach for spinful nanowires	390
18.2.1	Derivation of the bGF	391
18.2.2	Spin structure of MBSs	393
18.2.3	CPR across the topological transition	394
18.3	Conclusions	396
18.4	On the boundary GF of the spinful model	396
19	Quantum circuits with multiterminal JA junctions by Matute-Cañadas, et al.	397
19.1	Conclusions	398
19.2	Introduction	398
19.3	Modeling, phase biased configuration and gauge choice	401
19.3.1	The model	401
19.3.2	Noise protection of the Andreev qubit	403
19.3.3	Topological properties	405
19.4	Charge islands' configuration	406
19.4.1	One terminal	408
19.4.2	Two terminals	409
19.4.3	Three terminals	411
19.4.4	Discussion	412
19.5	Loop configuration	413
19.5.1	Two terminals forming one loop	414
19.5.2	Three terminals: one loop and one island	415
19.6	Cotunneling limit	416
19.6.1	Trijunction in an island configuration	417
19.6.2	Trijunction with two terminals in a loop and one island configuration	417
19.7	General Discussion and Overview	418
19.7.1	Poisoning	419
19.7.2	Relaxation and dephasing	420
19.8	Effective Hamiltonian in phase biased conditions	422
19.9	Adiabatic approximation	424
19.10	Simultaneous diagonalization in the tunnel limit	424
19.11	Two level island limit	425
20	Multi-terminal JJ as topological materials by Riwar et al.	426
20.1	Supplementary information Multi-terminal Josephson junctions as topological materials	431
20.1.1	Derivation of the Weyl Hamiltonian	431
20.1.2	Derivation of the current	433
20.1.3	Comment on Andreev bound states reaching the gap edge	433
20.1.4	Estimation of the current noise	434
20.1.5	Occurence of Weyl points for 4-terminal junctions	436
VII	Special Effects: Selected Useful Theories	437

21 Josephson radiation in a superconductor-quantum dot-superconductor junction by Lamic, Meyer, Houzet	437
21.1 Theory	437
21.2 Supplemental material	444
21.2.1 Evaluation of the current noise spectrum	444
21.2.2 Equivalent model at $p_\pi = 1$	446
22 Phase-Controlled Force, Magnetization Oscillations in SC Ballistic NW	447
23 Josephson effect in a junction coupled to an electron reservoir by Beenakker	453
24 Casimir-Josephson force in JJ by Beenakker	458
24.0.1 Introduction	458
24.0.2 Scattering formulation	460
24.0.3 Ballistic point contact	461
24.0.4 Point contact with a tunnel barrier	462
24.0.5 Conclusion	463
24.0.6 Short-junction limit of the force integral	464
VIII Important Experiments	466
25 Microwave susceptibility observation of interacting many-body Andreev states by Fatemi et al.	466
26 Spin-degeneracy breaking, parity transitions in three-terminal JJ by Coraiola et al.	474
26.1 Experiment and results	475
26.2 Supplementary Material: Spin-degeneracy breaking and parity transitions in three-terminal Josephson junctions	483
26.2.1 Materials and methods	483
26.2.2 Tunnel-gate voltage dependence of differential conductance	484
26.2.3 Dependence of Andreev spectrum on individual phases	485
26.2.4 Voltage bias dependence of constant-energy cut planes	485
26.2.5 Effect of the switch junction	486
26.2.6 Effect of V_L , V_M and V_R	487
26.2.7 Remapping of flux-line-current space to flux space	489
26.2.8 Results for additional orientations of the in-plane magnetic field	490
26.2.9 Results for Device 2	493
27 Theory of microwave spectroscopy of Andreev bound states with a Josephson junction by LBretheau, Girit et al	494
27.1 Introduction	495
27.2 Model	497
27.2.1 Setup	497
27.2.2 Hamiltonian of the circuit	498
27.2.3 Andreev spin and plasma boson	498
27.2.4 Incoherent Cooper pair current	499
27.3 spectra calculation	500
27.3.1 Numerical resolution	500
27.3.2 Incorporating line broadening	502
27.3.3 Comparison with experiment	502
27.4 Andreev-plasma mode hybridization	502
27.5 Conclusions and perspectives	506

28 The multi-terminal Josephson effect by Pankratova, Lee et al.	506
29 Bibliography	514

1 Preface and main motivation

Let's discuss some minimum knowledge and motivation that would be good to understand for studying the subject.

Results

Let's discuss what useful and important humanity has gained by understanding this subject.

Some best general puzzles for motivation

Some best technical puzzles for motivation

Part I

Superconductive Contacts in a Nutshell

2 Overview of well-known formulas of JJ

2.1 Main general formulas

(see note on SC for it)

3 Special constructions in a nutshell

3.1 Models of multiterminal junctions

3.1.1 Multiterminal S-TS-S junctions by Zazunov et. al.

$$I_L + I_R + I_S = 0, \quad (3.1)$$

$$\phi = \phi_L - \phi_R, \quad \tilde{\phi} = \phi_L + \phi_R. \quad (3.2)$$

Model

$$\begin{aligned} G_{j=L/R}(\tau) &= -\langle \mathcal{T} \Phi_j(\tau) \Phi_j^\dagger(0) \rangle_0 = T \sum_{\omega} e^{-i\omega\tau} G_j(\omega), \\ G_j(\omega) &= \frac{1}{i\omega} \left(\sqrt{\omega^2 + \Delta_j^2} \sigma_0 + \Delta_j \sigma_x \right), \end{aligned} \quad (3.3)$$

$$\Psi = \left(\psi_{\uparrow}, \psi_{\downarrow} \right)^T.$$

$$\begin{aligned} G_S(\tau) &= -\langle \mathcal{T} \Psi(\tau) \Psi^\dagger(0) \rangle_0 = T \sum_{\omega} e^{-i\omega\tau} G_S(\omega), \\ G_S(\omega) &= -\frac{i\omega\sigma_0 + \Delta_S \sigma_x}{\sqrt{\omega^2 + \Delta_S^2}}. \end{aligned} \quad (3.4)$$

$$H_t = \sum_{j=L/R} \lambda_j e^{i\phi_j/2} \left(\alpha_j \psi_{\uparrow}^\dagger + \beta_j \psi_{\downarrow}^\dagger \right) c_j + \lambda_{LR} e^{-i\phi/2} c_L^\dagger c_R + \text{h.c.}, \quad (3.5)$$

$$\begin{aligned} H_t &= \frac{1}{2} \Phi_L^\dagger W_{LR} \Phi_R + \sum_{j=L/R} \Psi^\dagger W_j \Phi_j + \text{h.c.}, \\ W_{LR} &= \lambda_{LR} e^{-i\sigma_z \phi/2} \sigma_z, \quad W_j = \lambda_j e^{i\sigma_z \phi_j/2} \text{diag}(\alpha_j, -\beta_j^*). \end{aligned} \quad (3.6)$$

$$I_j = -iT \operatorname{tr} \sum_{\omega} \sigma_z [\Sigma(\omega) \mathcal{G}(\omega)]_{jj}. \quad (3.7)$$

$$\mathcal{G}^{-1}(\omega) = G^{-1}(\omega) - \Sigma(\omega), \quad (3.8)$$

$$\begin{aligned} \Sigma_{LL/RR}(\omega) &= -\frac{i\omega\lambda_{L/R}^2}{\sqrt{\omega^2 + \Delta_S^2}} \sigma_0, \\ \Sigma_{LR}(\omega) &= \Sigma_{RL}^\dagger(-\omega) = W_{LR} - \frac{\lambda_L \lambda_R}{\sqrt{\omega^2 + \Delta_S^2}} \begin{pmatrix} i\omega \cos(\theta/2) e^{-i\phi/2} & \sin(\theta/2) \Delta_S e^{-i\tilde{\phi}/2} \\ -\sin(\theta/2) \Delta_S e^{i\tilde{\phi}/2} & i\omega \cos(\theta/2) e^{i\phi/2} \end{pmatrix}, \end{aligned}$$

Parallel S-TS-S junction

Supercurrent $I_L = I_R$ vs phase $\tilde{\phi} = \phi_L + \phi_R$ for the obtained numerically from $I_j = -iT \operatorname{tr} \sum_{\omega} \sigma_z [\Sigma(\omega) \mathcal{G}(\omega)]_{jj}$ for $\lambda_L = \lambda_R = 1$, $\Delta_S = \Delta$, $T = 0.02\Delta$. From top to bottom, the spin angle is given by $4\theta/\pi = 4, 3, 2, 1, 0$. The upper (lower) panel is for $\lambda_{LR} = 0$ ($\lambda_{LR} = 1$). The red dashed curve in the upper panel is given by $I_{L/R}^{(\Delta_S \rightarrow \infty)}(\tilde{\phi}) = -\frac{\partial E_A}{\partial \tilde{\phi}} \tanh(E_A/2T)$ for $\Delta_S \rightarrow \infty$, $\theta = \pi$. The blue curve for $\theta = \pi$ in the upper panel is also known analytically by $I_{L/R}^{(\Delta_S = \Delta)}(\tilde{\phi}) = -\sum_{\pm} \frac{E_A^{(\pm)}}{\partial \tilde{\phi}} \tanh\left(\frac{E_A^{(\pm)}}{2T}\right)$

Parallel S-TS-S junction without direct TS-TS coupling

$$I_{L/R}^{(\Delta_S \rightarrow \infty)}(\tilde{\phi}) = -\frac{\partial E_A}{\partial \tilde{\phi}} \tanh(E_A/2T), \quad (3.9)$$

$$E_A(\tilde{\phi}) = \sqrt{\tau(\theta)} \Delta \cos(\tilde{\phi}/2).$$

$$\tau(\theta) = \frac{4\lambda^4 \sin^2(\theta/2)}{[1 + \lambda^4 \sin^2(\theta/2)]^2}, \quad (3.10)$$

$$\theta = \pi,$$

$$I_{L/R}^{(\Delta_S = \Delta)}(\tilde{\phi}) = -\sum_{\pm} \frac{E_A^{(\pm)}}{\partial \tilde{\phi}} \tanh\left(\frac{E_A^{(\pm)}}{2T}\right), \quad (3.11)$$

$$E_A^{(\pm)}(\tilde{\phi}) = \frac{\Delta}{\sqrt{2}} \left(1 \pm \sqrt{1 - \tau^2 \cos^2(\tilde{\phi}/2)}\right)^{1/2}, \quad (3.12)$$

where $\tau = 4\lambda^2/(1 + \lambda^2)^2$.

TS-TS supercurrent vs phase difference $\phi = \phi_L - \phi_R$ in the serial case $\tilde{\phi} = 0$, see $I_S = 0$ and $I_L = -I_R$. The S lead acts as currentless probe. The upper panel illustrates the supercurrent suppression with increasing spin angle θ for a normal probe ($\Delta_S = 0$), see Eq. (18.14), for $4\theta/\pi = 0, 1, 2, 3, 4$. The lower panel shows how current is suppressed with increasing Δ_S for $\theta = 0$, taking $\Delta_S/\Delta = 0, 1, 2, 4$.

Parallel S-TS-S junction with direct TS-TS coupling

$$I_{L/R}^{(\Delta \rightarrow \infty)} = \frac{T}{2} \sum_{\omega} \left(\lambda^2 \sin \tilde{\phi} - 2\lambda_{LR} \sqrt{1 + \frac{\omega^2}{\Delta_S^2}} \sin(\tilde{\phi}/2) \right) \frac{\lambda^2 \Delta_S^2}{\lambda^4 \omega^2 + [\lambda_{LR} \sqrt{\omega^2 + \Delta_S^2} - \lambda^2 \Delta_S \cos(\tilde{\phi}/2)]^2}$$

$$\lambda_L = \lambda_R = \lambda.$$

Serial S-TS-S junction

Serial S-TS-S junction without direct TS-TS coupling

$$I_L^{(\Delta_S=0)}(\phi) = -T \sum_{\omega} \frac{2E_A \partial_{\phi} E_A}{\omega^2 + E_A^2 + \gamma |\omega| \sqrt{\omega^2 + \Delta^2}}, \quad (3.13)$$

$$E_A(\phi) = \tilde{\Delta} \sqrt{1 - \tilde{\tau} \sin^2(\phi/2)},$$

$$\begin{aligned} \tilde{\Delta} &= \frac{\Delta}{\sqrt{1+y^2}}, \quad \tilde{\tau} = \cos^2(\theta/2), \\ y &= \frac{1 + \lambda^4 \sin^2(\theta/2)}{2\lambda^2}. \end{aligned} \quad (3.14)$$

TS-TS supercurrent vs ϕ in the serial configuration ($\tilde{\phi} = 0$) obtained from $I_j = -iT \text{tr} \sum_{\omega} \sigma_z [\Sigma(\omega) \mathcal{G}(\omega)]_{jj}$ for direct TS-TS coupling $\lambda_{LR} = 1$ and several values of the spin canting angle, $4\theta/\pi = 0, 1, 2, 4$.

$$I_L^{(\Delta_S=\Delta)}(\phi) = -\frac{\partial E_A}{\partial \phi} \tanh \left(\frac{E_A}{2T} \right), \quad (3.15)$$

Serial S-TS-S junction with direct TS-TS coupling

$$I_L^{(\Delta \rightarrow \infty)} = \frac{T}{2} \sum_{\omega} \left(\lambda_{LR} \sin \phi - \frac{2\lambda^2 \Delta_S}{\sqrt{\omega^2 + \Delta_S^2}} \sin(\phi/2) \right) \frac{\lambda_{LR}(\omega^2 + \Delta_S^2)}{\lambda^4 \omega^2 + [\lambda_{LR} \sqrt{\omega^2 + \Delta_S^2} \cos(\phi/2) - \lambda^2 \Delta_S]^2}$$

$$\lambda_L = \lambda_R = \lambda.$$

Atomic limit of S-TS-S junction

$$\Sigma_{LR} = \begin{pmatrix} \lambda_{LR} e^{-i\phi/2} & -\lambda_L \lambda_R \sin(\theta/2) e^{-i\tilde{\phi}/2} \\ \lambda_L \lambda_R \sin(\theta/2) e^{i\tilde{\phi}/2} & -\lambda_{LR} e^{i\phi/2} \end{pmatrix}, \quad (3.16)$$

$$\Phi_L \rightarrow \sqrt{\Delta} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \gamma_L, \quad \Phi_R \rightarrow i\sqrt{\Delta} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \gamma_R. \quad (3.17)$$

$$\begin{aligned}
H_{at} &= \frac{1}{2} \Phi_L^\dagger \Sigma_{LR} \Phi_R + \text{h.c.} = i E_A (\phi, \tilde{\phi}) \gamma_L \gamma_R, \\
E_A &= 2\Delta \left(\lambda_{LR} \cos(\phi/2) + \lambda_L \lambda_R \sin(\theta/2) \cos(\tilde{\phi}/2) \right).
\end{aligned} \tag{3.18}$$

$$I_L(\phi) = -I_R = \frac{e \lambda_{LR} \Delta}{\hbar} \sin(\phi/2) \text{sgn} [\lambda_{LR} \cos(\phi/2) + \lambda_L \lambda_R \sin(\theta/2)]. \tag{3.19}$$

Derivation of the bGF

A spatially discretized version of the model in Refs. [6, 7] is given by

$$\begin{aligned}
H_{\text{wire}} &= \frac{1}{2} \sum_j \left[\psi_j^\dagger \hat{h} \psi_j + \left(\psi_j^\dagger \hat{t} \psi_{j+1} + \text{h.c.} \right) \right], \\
\hat{h} &= (2t - \mu) \sigma_z \tau_0 + V_x \sigma_0 \tau_x + \Delta \sigma_x \tau_0, \\
\hat{t} &= -t \sigma_z \tau_0 + i \alpha \sigma_z \tau_z,
\end{aligned} \tag{3.20}$$

Figure 3.1: Local density of states $\rho(E)$ (in units of $1/\Delta$) at the wire boundary vs energy E (in units of Δ), using the spinful wire model (18.21). We have used a broadening parameter $\gamma_0 = 0.01\Delta$ to visualize the subgap structure. We put $\mu = 5$ meV and vary the Zeeman scale V_x from 0 to 6 meV, with all other parameters specified in the main text. The topological phase is realized for $V_x > V_c = \sqrt{\mu^2 + \Delta^2} \simeq 5.004$ meV [6, 7]. For $V_x = 6$ meV, the zero-energy Majorana peak is observed. The dashed curve gives the corresponding result obtained from the Kitaev model in Sec. 18.1, with an effective proximity gap and an effective normal density of states.

$$\begin{aligned}
\hat{G}^{(0)}(k, \omega) &= \left[\omega - \hat{h} - \hat{V}(k) \right]^{-1}, \\
\hat{V}(k) &= -2t \cos(ka) \sigma_z \tau_0 + 2\alpha \sin(ka) \sigma_z \tau_z.
\end{aligned} \tag{3.21}$$

$$\hat{G}_{lm}^{(0)}(\omega) = \oint_{|z|=1} \frac{dz}{iaz} \hat{G}^{(0)}(k, \omega) z^{(l-m)}, \quad z = e^{ika}. \tag{3.22}$$

$$\begin{aligned}
\hat{G}_L &= \hat{G}_{00}^{(0)} - \hat{G}_{01}^{(0)} \left(\hat{G}_{00}^{(0)} \right)^{-1} \hat{G}_{10}^{(0)}, \\
\hat{G}_R &= \hat{G}_{00}^{(0)} - \hat{G}_{10}^{(0)} \left(\hat{G}_{00}^{(0)} \right)^{-1} \hat{G}_{01}^{(0)},
\end{aligned} \tag{3.23}$$

$$\rho(E) = -(1/\pi) \text{Im}[\hat{G}_L(E)]_{11},$$

Spin structure of MBSs

Figure 3.2: Spin canting angle $\theta = \theta_L - \theta_R$ vs Zeeman scale V_x obtained from Eq. (18.26) for different values of μ . The shown results are within the topological regime, $V_x > V_c$. All other parameters of the spinful TS wire model (18.21) are specified in the main text.

$$S_{\nu,L/R}(\omega) = \frac{1}{2\pi i} \text{Tr} \left[(1 + \sigma_z) \tau_\nu (G_{L/R}^a(\omega) - G_{L/R}^r(\omega)) \right], \quad (3.24)$$

$$\tan \theta_{L/R} = \lim_{\omega \rightarrow 0} \frac{S_{y,L/R}(\omega)}{S_{x,L/R}(\omega)}. \quad (3.25)$$

CPR across the topological transition

CPR for the parallel configuration from the spinful model without direct TS-TS coupling, $\lambda_{LR} = 0$, $V_c \simeq 5.004$ meV, $\mu = \lambda_L = \lambda_R = 5$ meV. The dashed curve - CPR for $\lambda_R = 0$ and $V_x = 5.4$ meV.

CPR for the parallel configuration from the spinful model with a direct TS-TS coupling, $\lambda_{LR} \neq 0$. lower: $\lambda_{LR} = 5$ meV, upper: $\lambda_{LR} = -5$ meV, also the lower curves have been shifted by -0.06 for clarity.

CPR for the serial configuration from the spinful model for several values of Δ_S , $\mu = \lambda_L = \lambda_R = 5$ meV, $V_x = 7$ meV, and $\lambda_{LR} = 0$.

3.2 Other models

3.2.1 S-TS, S-TS-S junctions with a Majorana wires by Zazunov, Egger

$$I(\varphi) = I_c \cos \varphi, \quad (3.26)$$

Black dots - MBSs. Charging energy $E_c = e^2/(2C)$.

S-TS junction

$$H_L = \sum_{\mathbf{k}} \Psi_{\mathbf{k}}^\dagger (\xi_k \sigma_z + \Delta \sigma_x e^{i\varphi_L \sigma_z}) \Psi_{\mathbf{k}}, \quad (3.27)$$

$$\begin{aligned} \Psi_{\mathbf{k}} &= (c_{\mathbf{k},\uparrow}, c_{-\mathbf{k},\downarrow}^\dagger)^T. \\ c_{\mathbf{k},\sigma}^\dagger \\ \gamma_{j=L/R} &= \gamma_j^\dagger, \end{aligned}$$

$$\{\gamma_j, \gamma_{j'}\} = \delta_{jj'}. \quad (3.28)$$

$$d = (\gamma_L + i\gamma_R)/\sqrt{2}. \quad (3.29)$$

$$H_w = \sum_q \begin{pmatrix} f_{q,r}^\dagger \\ f_{-q,l} \end{pmatrix}^T \begin{pmatrix} v_F q & \Delta_w e^{-i\chi} \\ \Delta_w e^{i\chi} & -v_F q \end{pmatrix} \begin{pmatrix} f_{q,r} \\ f_{-q,l}^\dagger \end{pmatrix}. \quad (3.30)$$

$$H_t = \sum_{\mathbf{k}, \sigma \sigma'} \int dx c_{\mathbf{k}, \sigma}^\dagger \mathcal{T}_{\mathbf{k}, \sigma \sigma'}(x) \psi_{\sigma'}(x) + \text{H.c.} \quad (3.31)$$

$$H_t = \sum_{\mathbf{k}\sigma} c_{\mathbf{k},\sigma}^\dagger \psi_{\mathbf{k},\sigma} + \text{H.c.}, \quad (3.32)$$

$$\psi_{\mathbf{k},\sigma} = \lambda_{\mathbf{k},\sigma} \gamma_L + \sum_{q,\alpha} t_{\mathbf{k},\sigma;q,\alpha} f_{q,\alpha}.$$

$$I(\varphi) = (2e/\hbar) \partial_\varphi F(\varphi), \quad (3.33)$$

$$Z(\varphi) = \text{Tre}^{-\beta H} = e^{-\beta F}, \quad (3.34)$$

$$\varphi = \varphi_L - \chi.$$

$$\beta \mathcal{F}_s = \int_0^\beta d\tau_1 d\tau_2 \sum_{\mathbf{k}} \Phi_{\mathbf{k}}^\dagger(\tau_1) G_k(\tau_1 - \tau_2) \Phi_{\mathbf{k}}(\tau_2), \quad (3.35)$$

$$\Phi_{\mathbf{k}} = (\psi_{\mathbf{k},\uparrow}, \psi_{-\mathbf{k},\downarrow}^\dagger)^T$$

$$G_k(\omega) = \frac{\Delta}{\omega^2 + \xi_k^2 + \Delta^2} \sigma_x e^{i\varphi \sigma_z}. \quad (3.36)$$

$$\psi_{-\mathbf{k},\downarrow} = \zeta_{\mathbf{k}} \psi_{\mathbf{k},\uparrow}, \quad (3.37)$$

$$\lambda_{\mathbf{k},\sigma} \rightarrow \lambda_\sigma, \quad t_{\mathbf{k},\sigma;q,\alpha} \rightarrow t_\sigma, \quad (3.38)$$

S-TS-S junction

$$H_c = E_c \left(2\hat{N} + \hat{n}_d - N_0 + \sum_{q;i=1,2} \hat{n}_{q,i}^{(qp)} \right)^2. \quad (3.39)$$

$$\hat{n}_{q,i}^{(qp)} = \eta_{q,i}^\dagger \eta_{q,i},$$

$$H_w = \sum_q \sum_{i=1,2} E_q \eta_{q,i}^\dagger \eta_{q,i}, \quad (3.40)$$

$$E_q = \sqrt{v_F^2 q^2 + \Delta_w^2}. \quad (3.41)$$

$$\begin{pmatrix} f_{q,r} \\ e^{-i\chi} f_{-q,l}^\dagger \end{pmatrix} = \begin{pmatrix} u_q & -v_q \\ v_q & u_q \end{pmatrix} \begin{pmatrix} \eta_{q,1} \\ e^{-i\chi} \eta_{q,2}^\dagger \end{pmatrix}, \quad (3.42)$$

$$u_q = v_{-q} = \sqrt{(1 + v_F q / E_q) / 2}. \quad (3.43)$$

$$H = H_L + H_R + H_w + H_c + H_t, \quad (3.44)$$

$$H_t = H_t^{(\gamma)} + H_t^{(qp)}. \quad (3.45)$$

$$H_t^{(\gamma)} = \frac{1}{\sqrt{2}} \sum_{\mathbf{k}} \left[\lambda_{L,\sigma} c_{\mathbf{k},L,\sigma}^\dagger (d + e^{-i\chi} d^\dagger) - i \lambda_{R,\sigma} c_{\mathbf{k},R,\sigma}^\dagger (d - e^{-i\chi} d^\dagger) \right] + \text{H.c.} \quad (3.46)$$

$$\begin{aligned} H_t^{(qp)} &= \sum_{\mathbf{k},j,\sigma} t_{j,\sigma} c_{\mathbf{k},j,\sigma}^\dagger \eta_w + \text{H.c.}, \\ \eta_w &= \sum_q \left[u_q (\eta_{q,1} + \eta_{q,2}) + v_q e^{-i\chi} (\eta_{q,1}^\dagger - \eta_{q,2}^\dagger) \right]. \end{aligned} \quad (3.47)$$

Hybridization scales:

$$\begin{aligned} \Gamma_j &= 2\pi\nu_0 \sum_{\sigma} \sigma \lambda_{j,\sigma} t_{j,-\sigma}^*, & \tilde{\Gamma}_j &= 2\pi\nu_0 \sum_{\sigma} \lambda_{j,\sigma} t_{j,\sigma}^*, \\ \Gamma_{\lambda,j} &= 2\pi\nu_0 \sum_{\sigma} |\lambda_{j,\sigma}|^2, & \Gamma_t &= 2\pi\nu_0 \sum_{j,\sigma} |t_{j,\sigma}|^2. \end{aligned} \quad (3.48)$$

$$Z(\varphi) = \text{Tr} \left(e^{-\beta(H_w + H_c)} \mathcal{T} e^{-(S_M + S_{qp} + S_{\text{int}})} \right), \quad (3.49)$$

$$f(\tau) = \int \frac{d\omega}{2\pi} f_\omega e^{-i\omega\tau}, \quad f_\omega = \frac{1}{\sqrt{\omega^2 + \Delta^2}}, \quad (3.50)$$

$$S_M = \int d\tau_1 d\tau_2 \dot{f}(\tau_1 - \tau_2) \sum_{j,j'} \Lambda_{j,j'}(\chi_1, \chi_2) \gamma_j(\tau_1) \gamma_{j'}(\tau_2), \quad (3.51)$$

$$\begin{aligned}
\Lambda_{j,j} &= \sum_{j'} \frac{\Gamma_{\lambda,j'}}{8} [1 + \cos(\delta\chi) + \sigma_{jj'}(\cos\chi_1 + \cos\chi_2)], \\
\Lambda_{j,-j} &= \pm \sum_{j'} \frac{\Gamma_{\lambda,j'}}{8} [\sin(\delta\chi) - \sigma_{jj'}(\sin\chi_1 + \sin\chi_2)], \\
S_{qp} &= \Gamma_t \int d\tau_1 d\tau_2 \dot{f}(\tau_1 - \tau_2) \bar{\eta}_w(\tau_1) \eta_w(\tau_2),
\end{aligned} \tag{3.52}$$

$$S_{\text{int}} = \int \frac{d\tau_1 d\tau_2}{2\sqrt{2}} \sum_j \gamma_j(\tau_1) L_j(\tau_1, \tau_2) \eta_w(\tau_2) + \text{H.c.}, \tag{3.53}$$

$$(j = L/R = +/ -)$$

$$\begin{aligned}
L_j(\tau_1, \tau_2) &= \tilde{\Gamma}_j^* \dot{f}(\tau_1 - \tau_2) [1 + e^{-i\chi_1} \pm i(1 - e^{-i\chi_1})] \\
&\quad + i\Delta \Gamma_j^* f(\tau_1 - \tau_2) [ie^{i\varphi_j}(1 + e^{i\chi_1}) \pm e^{i\varphi-j}(1 - e^{i\chi_1})].
\end{aligned}$$

Noninteracting limit

$$I(\varphi) = -\frac{2e}{\hbar} \int_0^\infty \frac{d\omega}{2\pi} \partial_\varphi \ln \det M(\omega), \tag{3.54}$$

$$X_\omega = \left(\sum_q \frac{1}{\omega^2 + E_q^2} \right)^{-1} + \omega^2 f_\omega \Gamma_t, \tag{3.55}$$

$$M_{jj} = 1 + f_\omega \Gamma_{\lambda,j} + \frac{f_\omega^2}{X_\omega} \left(\Delta^2 |\Gamma_j|^2 - \omega^2 |\tilde{\Gamma}_j|^2 \right). \tag{3.56}$$

$$M_{j,-j} = \frac{f_\omega^2}{X_\omega} \text{Re} \left[\Delta^2 \Gamma_j^* \Gamma_{-j} e^{\pm i\varphi} - \omega^2 \tilde{\Gamma}_j^* \tilde{\Gamma}_{-j} + i\omega \Delta \left(\Gamma_j^* \tilde{\Gamma}_{-j} e^{i\varphi_j} + \tilde{\Gamma}_j^* \Gamma_{-j} e^{-i\varphi-j} \right) \right]. \tag{3.57}$$

$$I(\varphi) = -\frac{e\Delta}{\hbar} |\pi\nu_w \Gamma^2 / \Delta_w|^2 \mathcal{G}(\Delta_w / \Delta) \sin(2\varphi) \tag{3.58}$$

$$\mathcal{G}(x) = \frac{x(2+x)}{4(1+x)^2}. \tag{3.59}$$

$$I_c \sim -|\Gamma/\Delta_w|^4.$$

Weak tunneling limit

$$E_n = E_c(2N - N_0 + n)^2, \quad n \in \mathbb{Z}, \tag{3.60}$$

Figure 3.3: Eq. (??) vs $\delta = 2N - N_0$ in the cotunneling regime of an S-TS-S junction. I_c is computed from Eq. (??) for $\Delta_w/\Delta = 4$ and several E_c/Δ . Note that δ can be changed by varying a backgate, see Fig. 26.1(b). $I_c(\delta)$ is periodic; we show one fundamental interval only.

Q_1	Q_2	Q_3
$+(\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_1)$	$-(\epsilon_1, 0, \hat{\epsilon}_{-1})$	$+(\epsilon_1, 0, \hat{\epsilon}_1)$
$-(\hat{\epsilon}_{-1}, \hat{\epsilon}_2, \hat{\epsilon}_{-1})$	$+(\epsilon_3, 0, \hat{\epsilon}_1)$	$-(\epsilon_3, 0, \hat{\epsilon}_{-1})$
$+(\epsilon_1, \hat{\epsilon}_2, \epsilon_1)$	$-(\hat{\epsilon}_{-1}, 0, \epsilon_1)$	$+(\hat{\epsilon}_1, 0, \epsilon_1)$
$-(\epsilon_3, \hat{\epsilon}_2, \epsilon_3)$	$+(\hat{\epsilon}_1, 0, \epsilon_3)$	$-(\hat{\epsilon}_{-1}, 0, \epsilon_3)$
$-(\epsilon_1, \hat{\epsilon}_2, \hat{\epsilon}_1)$	$+(\epsilon_1, \Delta_w, \hat{\epsilon}_{-1})$	$-(\hat{\epsilon}_{-1}, \Delta_w, \hat{\epsilon}_{-1})$
$+(\epsilon_3, \hat{\epsilon}_2, \hat{\epsilon}_{-1})$	$-(\epsilon_3, \hat{\epsilon}_4, \hat{\epsilon}_1)$	$+(\hat{\epsilon}_1, \hat{\epsilon}_4, \hat{\epsilon}_1)$
$-(\hat{\epsilon}_1, \hat{\epsilon}_2, \epsilon_1)$	$+(\hat{\epsilon}_{-1}, \Delta_w, \epsilon_1)$	$-(\epsilon_1, \Delta_w, \epsilon_1)$
$+(\hat{\epsilon}_{-1}, \hat{\epsilon}_2, \epsilon_3)$	$-(\hat{\epsilon}_1, \hat{\epsilon}_4, \epsilon_3)$	$+(\epsilon_3, \hat{\epsilon}_4, \epsilon_3)$

Table 1: All combinations of signs $\sigma_p = \pm$ and energy arguments (E_a, E_b, E_c) summed over in Eq. (??) for the even-parity state, where $\hat{\epsilon}_n = \epsilon_n + \Delta_w$ and $\epsilon_n = E_n - E_0$.

$$I_c = \frac{e\Delta}{\hbar} \frac{\Delta|\Gamma|^2}{2} \sum_{p=1}^{24} \sigma_p Q_{i_p} (E_{a_p}, E_{b_p}, E_{c_p}). \quad (3.61)$$

$$Q_{i_p} = \int \frac{d\omega d\omega'}{(2\pi)^2} \frac{f_\omega f_{\omega'}}{i\omega + E_a} \times \begin{cases} \frac{1-\delta_{E_b,0}}{E_b(i\omega'+E_c)}, & i_p = 1, \\ \frac{1}{(i\omega+E_c)(i\omega+i\omega'+E_b)}, & i_p = 2, \\ \frac{1}{(i\omega'+E_c)(i\omega+i\omega'+E_b)}, & i_p = 3. \end{cases}$$

3.2.2 S-TS, S-spinful TS, S-QD-TS, S-TS-S junctions by Zazunov et. al.

S-QD-TS junction

Model of S-QD-TS junction

$$H = H_S + H_{TS} + H_{QD} + H_{\text{tun}}, \quad (3.62)$$

$$H_{QD} = \sum_{\sigma=\uparrow,\downarrow} \epsilon_0 \left(n_\sigma - \frac{1}{2} \right) + U n_\uparrow n_\downarrow - \mathbf{B} \cdot \mathbf{S}, \quad \mathbf{S}_{i=x,y,z} = \sum_{\sigma,\sigma'} d_\sigma^\dagger (\sigma_i)_{\sigma\sigma'} d_{\sigma'}, \quad (3.63)$$

$$H_{\text{tun}} = \lambda_S \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger d_\sigma + \lambda_{TS} e^{-i\phi/2} \psi^\dagger d_\uparrow + \text{h.c.}, \quad (3.64)$$

$$\hat{I} = \frac{2e}{\hbar} \partial_\phi H_{\text{tun}}. \quad (3.65)$$

$$g(\tau) = -\langle \mathcal{T}_\tau \Psi_S(\tau) \Psi_S^\dagger(0) \rangle_0 = \beta^{-1} \sum_\omega e^{-i\omega\tau} g(\omega), \quad \Psi_S = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} \quad (3.66)$$

$$g(\omega) = -\frac{i\omega\tau_0 + \Delta\tau_x}{\sqrt{\omega^2 + \Delta^2}}, \quad (3.67)$$

$$\begin{aligned}
G(\tau) &= -\langle \mathcal{T}_\tau \Psi_{\text{TS}}(\tau) \Psi_{\text{TS}}^\dagger(0) \rangle_0, \quad \Psi_{\text{TS}} = \begin{pmatrix} \psi \\ \psi^\dagger \end{pmatrix}, \\
G(\omega) &= \frac{1}{i\omega} \left(\sqrt{\omega^2 + \Delta_p^2} \tau_0 + \Delta_p \tau_x \right).
\end{aligned} \tag{3.68}$$

$$n_\uparrow + n_\downarrow = 1, \tag{3.69}$$

$$H_{\text{eff}} = H_0 + H_{\text{int}}, \quad H_0 = H_S + H_{\text{TS}} - \mathbf{B} \cdot \mathbf{S}, \tag{3.70}$$

$$\begin{aligned}
H_{\text{int}} &= -\frac{2}{U} \sum_{\sigma, \sigma'} \left(\eta_\sigma^\dagger d_\sigma d_{\sigma'}^\dagger \eta_{\sigma'} + \text{h.c.} \right) \\
&= \frac{2}{U} \sum_{\sigma=\uparrow/\downarrow=\pm} \left(\sigma S_z \eta_\sigma^\dagger \eta_\sigma + S_\sigma \eta_{-\sigma}^\dagger \eta_\sigma \right) + \frac{2}{U} \delta n \sum_{\sigma} \eta_\sigma^\dagger \eta_\sigma - \frac{2\Lambda}{U} (\delta n + 1),
\end{aligned}$$

$$S_\pm := S_x \pm iS_y; \quad \delta n = \sum_{\sigma} n_{\sigma} - 1. \quad \Lambda = [\eta_\sigma, \eta_\sigma^\dagger]_+, \quad \eta_\sigma = \lambda_S \psi_\sigma + \delta_{\sigma, \uparrow} \lambda_{TS} e^{i\phi/2} \psi.$$

$$H_{\text{int}} = \frac{4}{U} \sum_{\sigma, \sigma'} \mathcal{Q}_{\sigma\sigma'} \eta_\sigma^\dagger \eta_{\sigma'}, \tag{3.71}$$

$$\mathcal{Q}_{\sigma\sigma} = \frac{\sigma}{2} S_z, \quad \mathcal{Q}_{\sigma, -\sigma} = \frac{1}{2} S_{-\sigma}.$$

$$Z = \text{Tr} \Big|_{\delta n=0} \left(e^{-\beta H_0} \mathcal{T}_\tau e^{-\int_0^\beta d\tau H_{\text{int}}(\tau)} \right), \tag{3.72}$$

$$H_{\text{int}}(\tau) = e^{\tau H_0} H_{\text{int}} e^{-\tau H_0}$$

$$Z = Z_0 \left\langle \mathcal{T}_\tau e^{-\beta \hat{W}} \right\rangle_0 = e^{-\beta F}, \tag{3.73}$$

$$\hat{W} = \beta^{-1} \int_0^\beta d\tau H_{\text{int}}(\tau),$$

$$Z_0 = \text{Tr} \Big|_{\delta n=0} e^{-\beta H_0} = e^{-\beta F_0},$$

Cotunneling regime in S-QD-TS junction

$$\lambda_S \lambda_{TS} \ll \min\{\Delta, \Delta_p, U\}, \tag{3.74}$$

$$F - F_0 = W_0 - \frac{\beta}{2} \left(\left\langle \hat{W}^2 \right\rangle_0 - W_0^2 \right) + \mathcal{O}(W^3). \tag{3.75}$$

$$\langle \mathcal{T}_\tau \eta_\sigma(\tau) \eta_{\sigma'}^\dagger(0) \rangle_0 = \delta_{\sigma\sigma'} \left[\lambda_S^2 \langle \mathcal{T}_\tau \psi_\sigma(\tau) \psi_\sigma^\dagger(0) \rangle_0 + \delta_{\sigma, \uparrow} \lambda_{TS}^2 \langle \mathcal{T}_\tau \psi(\tau) \psi^\dagger(0) \rangle_0 \right]$$

$$\langle \mathcal{T}_\tau \eta_\sigma(\tau) \eta_{\sigma'}(0) \rangle_0 = \delta_{\sigma,-\sigma'} \lambda_S^2 \langle \mathcal{T}_\tau \psi_\sigma(\tau) \psi_{-\sigma}(0) \rangle_0 + e^{i\phi} \delta_{\sigma\sigma'} \delta_{\sigma,\uparrow} \lambda_{TS}^2 \langle \mathcal{T}_\tau \psi(\tau) \psi(0) \rangle_0.$$

$$\begin{aligned} I(\phi) &= -\beta^{-1} \partial_\phi \int_0^\beta d\tau_1 d\tau_2 \langle \mathcal{T}_\tau H_{\text{int}}(\tau_1) H_{\text{int}}(\tau_2) \rangle_0 \\ &= -\frac{\kappa^2}{\beta} \int_0^\beta d\tau_1 d\tau_2 g_{12}(\tau_1 - \tau_2) G_{12}(\tau_1 - \tau_2) \cdot i e^{i\phi} \sum_\sigma \sigma \langle \mathcal{T}_\tau \mathcal{Q}_{\sigma,\uparrow}(\tau_1) \mathcal{Q}_{-\sigma,\uparrow}(\tau_2) \rangle_0 + \text{h.c.}, \end{aligned}$$

$$\kappa = \frac{4\lambda_S \lambda_{TS}}{U} \ll 1. \quad (3.76)$$

$$g_{12}(\tau) = -\frac{\Delta}{\beta} \sum_\omega \frac{\cos(\omega\tau)}{\sqrt{\omega^2 + \Delta^2}}, \quad (3.77)$$

$$G_{12}(\tau) = -\frac{\Delta_p}{\beta} \sum_\omega \frac{\sin(\omega\tau)}{\omega} \simeq -\frac{\Delta_p}{2} \text{sgn}(\tau).$$

$$\lim_{\tau_1 \rightarrow \tau_2} \langle \mathcal{T}_\tau \mathcal{Q}_{\sigma,\uparrow}(\tau_1) \mathcal{Q}_{-\sigma,\uparrow}(\tau_2) \rangle_0 = \frac{\sigma}{4} \text{sgn}(\tau_1 - \tau_2) \langle S_+(\tau_1) \rangle_0. \quad (3.78)$$

$$I(\phi) = I_x \sin \phi + I_y \cos \phi, \quad (3.79)$$

$$I_{x,y} = \frac{e\kappa^2 \Delta_p}{2\hbar} \frac{B_{x,y}}{B} \tanh(\beta B),$$

Mean-field approximation in S-QD-TS junction

$$G_d(\tau) = -\langle \mathcal{T}_\tau \Psi_d(\tau) \Psi_d^\dagger(0) \rangle, \quad \Psi_d^\dagger = (d_\uparrow^\dagger, d_\downarrow^\dagger, d_\downarrow^\dagger, -d_\uparrow^\dagger)^T. \quad (3.80)$$

$$\begin{aligned} \mathcal{H}_{\text{MF}} &= \begin{pmatrix} \epsilon_\uparrow & \Delta_d & \alpha_d & 0 \\ \Delta_d^* & -\epsilon_\downarrow & 0 & \alpha_d \\ \alpha_d^* & 0 & \epsilon_\downarrow & \Delta_d \\ 0 & \alpha_d^* & \Delta_d^* & -\epsilon_\uparrow \end{pmatrix}, \\ \epsilon_\uparrow &= \epsilon_0 - B_z + U \langle n_\downarrow \rangle, \quad \epsilon_\downarrow = \epsilon_0 + B_z + U \langle n_\uparrow \rangle, \\ \alpha_d &= B_x + iB_y - U \langle d_\downarrow^\dagger d_\uparrow \rangle, \quad \Delta_d = U \langle d_\downarrow d_\uparrow \rangle. \end{aligned} \quad (3.81)$$

$$\langle n_\uparrow \rangle = \frac{1}{\beta} \sum_\omega G_{d,11}(\omega), \quad \langle n_\downarrow \rangle = \frac{1}{\beta} \sum_\omega G_{d,33}(\omega), \quad (3.82)$$

$$\langle d_\downarrow^\dagger d_\uparrow \rangle = \frac{1}{\beta} \sum_\omega G_{d,13}(\omega), \quad \langle d_\downarrow d_\uparrow \rangle = \frac{1}{\beta} \sum_\omega G_{d,21}(\omega),$$

$$G_d(\omega) = [i\omega - \mathcal{H}_{\text{MF}} - \Sigma_S(\omega) - \Sigma_{TS}(\omega)]^{-1}. \quad (3.83)$$

$$\Sigma_S = \Gamma_S \begin{pmatrix} g_{11} & -g_{12} & 0 & 0 \\ -g_{21} & g_{22} & 0 & 0 \\ 0 & 0 & g_{11} & -g_{12} \\ 0 & 0 & -g_{21} & g_{22} \end{pmatrix}, \quad \Sigma_{TS} = \Gamma_{TS} \begin{pmatrix} G_{11} & 0 & 0 & -G_{12}e^{-i\phi} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -G_{21}e^{i\phi} & 0 & 0 & G_{22} \end{pmatrix} \quad (3.84)$$

$$\Gamma_{S/TS} = \lambda_{S/TS}^2.$$

$$I(\phi) = -\frac{e}{\hbar\beta} \sum_{\omega} \frac{\partial_{\phi} \det [G_d^{-1}(\omega)]}{\det [G_d^{-1}(\omega)]}. \quad (3.85)$$

$$\Sigma \simeq \begin{pmatrix} \frac{2\Delta}{i\omega} \Gamma_{TS} & -\Gamma_S & 0 & -\frac{2\Delta}{i\omega} \Gamma_{TS} e^{-i\phi} \\ -\Gamma_S & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Gamma_S \\ -\frac{2\Delta}{i\omega} \Gamma_{TS} e^{i\phi} & 0 & -\Gamma_S & \frac{2\Delta}{i\omega} \Gamma_{TS} \end{pmatrix}. \quad (3.86)$$

$$i\omega \equiv E_A^{(\sigma_1=\pm, \sigma_2=\pm)} = \sigma_1 \sqrt{\frac{b_0 + \sigma_2 \sqrt{b_0^2 + 4c_0}}{2}}, \quad (3.87)$$

$$b_0 = \epsilon_{\downarrow}^2 + \epsilon_{\uparrow}^2 + 4\Gamma_{TS}\Delta + 2\Gamma_S^2 + 2|\alpha_d|^2,$$

$$c_0 = -4\Gamma_{TS}\Delta (\epsilon_{\downarrow}^2 + \Gamma_S^2 + |\alpha_d|^2) - \epsilon_{\uparrow}^2 \epsilon_{\downarrow}^2 - (|\alpha_d|^2 - \Gamma_S^2) (|\alpha_d|^2 - \Gamma_S^2 - 2\epsilon_{\uparrow}\epsilon_{\downarrow}) + 8\Delta\Gamma_S\Gamma_{TS}\text{Re}(\alpha_d e^{i\phi})$$

S-QD-TS: $U = 0$, here TS is low-energy limit of a Kitaev chain, $B_y = 0$, $B_x = B_z = B/\sqrt{2}$, $\epsilon_0 = 0$, $\Delta_p = \Delta$, $\Gamma_S = \Gamma_{TS} = \Gamma$. Left: $\Gamma = 0.045\Delta$, right: $B = 0.1\Delta$ ($\Gamma = B = 0.5\Delta$). Solid curves: numerical evaluation of $I(\phi) = -\frac{e}{\hbar\beta} \sum_{\omega} \frac{\partial_{\phi} \det [G_d^{-1}(\omega)]}{\det [G_d^{-1}(\omega)]}$. Dashed curves: analytical prediction $E_A^{(\sigma_1=\pm, \sigma_2=\pm)} = \sigma_1 \sqrt{\frac{b_0 + \sigma_2 \sqrt{b_0^2 + 4c_0}}{2}}$ (and rescaled by the factor $1 + \Gamma/\Delta$).

S-QD-TS, $U > 0$, $\epsilon_0 = -U/2$, $\Delta_p = \Delta$, $B_y = 0$, and $\Gamma_S = \Gamma_{TS} = \Gamma$. Left: $U = \Delta$, $\Gamma = 0.5\Delta$, $B_x = B_z = B/\sqrt{2}$, $B = 0.5\Delta$; right: $U = 10\Delta$, $\Gamma = 4.5\Delta$, $B_x = 15\Delta$, $B_z = 0$.

S-QD-TS, $\Delta_p = \Delta$, $U = 10\Delta$, $\epsilon_0 = -U/2$, $\Gamma_S = \Gamma_{TS} = \Gamma$, $B = 15\Delta$, and $B_z = 0$; main panel: $B_x = -B$, $B_y = 0$; inset: $B_y = -B_x = B/\sqrt{2}$:

S-QD-TS in mean-field: $U = 10\Delta$, $\epsilon_0 = -U/2$, $B_{y,z} = 0$, from left to right: $\Gamma/\Delta = 4.5, 8, 10, 12.5$; inset: $\mathbf{B} = B(\sin \theta_B, 0, \cos \theta_B)$, $B = 15\Delta$:

Spinful TS nanowire model

$$H_{\text{TS}} = \frac{1}{2} \sum_j \left[\psi_j^\dagger \hat{h} \psi_j + \left(\psi_j^\dagger \hat{t} \psi_{j+1} + \text{h.c.} \right) \right], \quad (3.88)$$

$$\hat{h} = (2t - \mu)\tau_z\sigma_0 + V_x\tau_0\sigma_x + \Delta_p\tau_x\sigma_0,$$

$$\hat{t} = -t\tau_z\sigma_0 + i\alpha\tau_z\sigma_z,$$

$$\psi_j = \left(c_{j\uparrow}, c_{j\downarrow}, c_{j\downarrow}^\dagger, -c_{j\uparrow}^\dagger \right)^T.$$

$$V_x > V_x^c = \sqrt{\mu^2 + \Delta_p^2}, \quad (3.89)$$

S-spinful TS junction

$$\mathcal{T} = \frac{4(\lambda/t)^2}{[1 + (\lambda/t)^2]^2}, \quad (3.90)$$

S-spinful TS: V_x -Zeeman energy, $\Delta_p = \Delta = 0.2$ meV, $\mu = 5$ meV; inset: decomposition of I_c for $\mathcal{T} = 1$ into ABS (dotted-dashed) and continuum (dashed) contributions

S-TS junction: $\mathcal{T} = 1$, V_x - bulk Zeeman fields (in meV) near $V_x^c = 5.004$ meV.

S-QD-spinful TS: Mean-field theory

S-QD-spinful TS junctions in mean field: V_x - Zeeman energy, $U = 10\Delta$, $\epsilon_0 = -U/2$, $\Delta_p = \Delta = 0.2$ meV, $\Gamma_S = 2\Gamma_{TS} = 9\Delta$, and $\mathbf{B} = (V_x, 0, 0)$. Inset: $\mu = 4$ meV, decomposition of I_c into the ABS (dotted-dashed) and the continuum (dashed) contribution.

S-QD-TS junctions: $\Gamma_S = \Gamma_{TS} = \Gamma$, $\mu = 4$ meV, the bulk Zeeman field $V_x = 5$ meV ($V_x > V_x^c$ for our parameters) is applied to the spinful TS wire, the QD is subject to the local $\mathbf{B} = B_z \hat{e}_z$. Inset: Decomposition of I_c into ABS (dotted-dashed) and continuum (dashed) contributions for $\Gamma = 1.6$ meV:

S-TS-S junctions: Switching the parity of a superconducting atomic contact

Model

$$\Psi_{S,j=1,2} = \begin{pmatrix} \psi_{j,\uparrow} \\ \psi_{j,\downarrow}^\dagger \end{pmatrix}, \quad (3.91)$$

$$g_j^{-1}(\omega) = g^{-1}(\omega) + b_j\tau_0. \quad (3.92)$$

$$\gamma = (f_{\uparrow} + f_{\uparrow}^{\dagger})/\sqrt{2}. \quad (3.93)$$

$$\gamma' = -i(f_{\uparrow} - f_{\uparrow}^{\dagger})/\sqrt{2},$$

$$S_0 = \sum_{j=1,2} \int_0^{\beta} d\tau d\tau' \bar{\Psi}_{S,j}(\tau) g_j^{-1}(\tau - \tau') \Psi_{S,j}(\tau') + \frac{1}{2} \int_0^{\beta} d\tau \gamma(\tau) \partial_{\tau} \gamma(\tau).$$

$$H_{\text{tun}} = t_0 \left(\Psi_{S,1}^{\dagger} \tau_z e^{i\tau_z \phi/2} \Psi_{S,2} + \text{h.c.} \right) + \sum_{j=1,2} \frac{\lambda_j}{\sqrt{2}} \left(\psi_{j,\uparrow}^{\dagger} e^{i\phi_j/2} - \text{h.c.} \right) \gamma. \quad (3.94)$$

$$\mathcal{T} = \frac{4t_0^2}{(1 + t_0^2)^2}. \quad (3.95)$$

$$\begin{aligned} S_{\text{eff}} = \int_0^{\beta} d\tau d\tau' & \left\{ \bar{\Psi}(\tau) K^{-1}(\tau - \tau') \Psi(\tau') + \right. \\ & + \Phi^T(\tau) \left[\frac{1}{2} \delta(\tau - \tau') \partial_{\tau'} - \lambda_2^2 P_{\uparrow} g_2(\tau - \tau') P_{\uparrow} \right] \Phi(\tau') \\ & \left. + \left[\bar{\Psi}(\tau) \left(\lambda_1 e^{i\phi_1/2} \delta(\tau - \tau') - \lambda_2 e^{i\phi_2/2} t_0 \tau_z e^{i\tau_z \phi/2} g_2(\tau - \tau') \right) P_{\uparrow} \Phi(\tau') + \text{h.c.} \right] \right\}, \\ K^{-1}(\omega) &= g_1^{-1}(\omega) - t_0^2 \tau_z e^{i\tau_z \phi/2} g_2(\omega) e^{-i\tau_z \phi/2} \tau_z, \end{aligned} \quad (3.96)$$

$$\Phi(\tau) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \gamma(\tau) = \tau_x \Phi^*(\tau). \quad (3.97)$$

$$Z = \int \mathcal{D}[\bar{\Psi}, \Psi, \gamma] e^{-S_{\text{eff}}} \equiv e^{-\beta F(\phi_1, \phi_2)}. \quad (3.98)$$

$$I_{\text{TS}} = -(I_1 + I_2), \quad (3.99)$$

Atomic limit of S-TS-S

$$S_{\text{at}} = \int_0^{\beta} d\tau \left\{ \frac{1}{2} \gamma \partial_{\tau} \gamma + \bar{\Psi} \left[\partial_{\tau} + \Delta \cos(\phi/2) \tau_x + r \Delta \sin(\phi/2) \tau_y + B_z \tau_0 \right] \Psi + \frac{1}{\sqrt{2}} \sum_{\sigma=\uparrow, \downarrow} (\lambda_{\sigma} \psi_{\sigma}^{\dagger} - \text{h.c.}) \gamma \right\},$$

$$\begin{aligned} \lambda_{\uparrow} &= \lambda_1 \sqrt{(1+r)\Delta/2} e^{i\phi_1/2}, \\ \lambda_{\downarrow} &= -\lambda_2 \sqrt{(1-r)\Delta/2} e^{-i\phi_2/2}, \\ B_z &= \left(\frac{1+r}{2} b_1 + \frac{1-r}{2} b_2 \right) \Delta. \end{aligned} \quad (3.100)$$

$$H_{\text{at}} = \sum_{\sigma=\uparrow, \downarrow, \pm} \sigma B_z \psi_{\sigma}^{\dagger} \psi_{\sigma} + \left(\delta_A \psi_{\uparrow}^{\dagger} \psi_{\downarrow}^{\dagger} + \text{h.c.} \right) + \frac{1}{\sqrt{2}} \sum_{\sigma} (\lambda_{\sigma} \psi_{\sigma}^{\dagger} - \text{h.c.}) \gamma, \quad (3.101)$$

$$\delta_A(\phi) = \Delta [\cos(\phi/2) - ir \sin(\phi/2)]. \quad (3.102)$$

$$E_A(\phi) = |\delta_A| = \Delta \sqrt{1 - \mathcal{T} \sin^2(\phi/2)}. \quad (3.103)$$

$$\mathcal{P}_{\text{tot}} = (-1)^{n_f + n_{\uparrow} + n_{\downarrow}} = \pm 1, \quad (3.104)$$

$$|n_{\uparrow}, n_{\downarrow}, n_f\rangle = \left(\psi_{\uparrow}^{\dagger}\right)^{n_{\uparrow}} \left(\psi_{\downarrow}^{\dagger}\right)^{n_{\downarrow}} \left(f_{\uparrow}^{\dagger}\right)^{n_f} |0\rangle, \quad (3.105)$$

$$(n_{\uparrow}, n_{\downarrow}, n_f) \in \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$$

$$\mathcal{H}_{\text{at}}(\phi_1, \phi_2) = \begin{pmatrix} 0 & \delta_A^* & \lambda_{\uparrow}^*/2 & \lambda_{\downarrow}^*/2 \\ \delta_A & 0 & \lambda_{\downarrow}/2 & -\lambda_{\uparrow}/2 \\ \lambda_{\uparrow}/2 & \lambda_{\downarrow}^*/2 & B_z & 0 \\ \lambda_{\downarrow}/2 & -\lambda_{\uparrow}^*/2 & 0 & -B_z \end{pmatrix}. \quad (3.106)$$

$$E_G^{(\text{e})} = \min(\varepsilon),$$

$$\det(\mathcal{H}_{\text{at}} - \varepsilon) = 0. \quad (3.107)$$

$$\varepsilon_{\pm} = \frac{1}{\sqrt{2}} \left(E_A^2 + B_z^2 + \frac{1}{2} |\lambda_{\uparrow}|^2 \pm \sqrt{(E_A^2 - B_z^2)^2 + |\lambda_{\uparrow}|^2 (E_A^2 + B_z^2)} \right)^{1/2}, \quad (3.108)$$

$$|\lambda_{\uparrow}|^2 = \lambda_1^2 (1 + r) \Delta / 2,$$

$$I_1 = -I_2 = \frac{2e}{\hbar} \partial_{\phi} E_G^{(\text{e})} = -\frac{2e}{\hbar} \partial_{\phi} \varepsilon_{+}. \quad (3.109)$$

$$|E_A^2 - B_z^2| \ll |\lambda_{\uparrow}|^2 \ll E_A^2 + B_z^2,$$

$$\varepsilon_{\pm} \simeq \frac{1}{\sqrt{2}} \left(E_A^2 + B_z^2 \pm \lambda_1 \sqrt{2(1+r)\Delta(E_A^2 + B_z^2)} \right)^{1/2}, \quad (3.110)$$

Conventional midgap level of S-TS-S

$$\tilde{H}_{\text{at}} = \sum_{\sigma=\uparrow,\downarrow=\pm} \sigma B_z \psi_{\sigma}^{\dagger} \psi_{\sigma} + \left(\delta_A \psi_{\uparrow}^{\dagger} \psi_{\downarrow}^{\dagger} + \text{h.c.} \right) + \sum_{\sigma} (\lambda_{\sigma} \psi_{\sigma}^{\dagger} d + \text{h.c.}). \quad (3.111)$$

$$\tilde{\mathcal{H}}_{\text{at}}(\phi_1, \phi_2) = \begin{pmatrix} 0 & \delta_A^* & 0 & 0 \\ \delta_A & 0 & \lambda_{\downarrow} & -\lambda_{\uparrow} \\ 0 & \lambda_{\downarrow}^* & B_z & 0 \\ 0 & -\lambda_{\uparrow}^* & 0 & -B_z \end{pmatrix}. \quad (3.112)$$

$$|\lambda_{\uparrow}| = |\lambda_{\downarrow}| \equiv \lambda,$$

$$\varepsilon_{\pm} = \frac{1}{\sqrt{2}} \left(E_A^2 + B_z^2 + 2\lambda^2 \pm \sqrt{(E_A^2 - B_z^2)^2 + 4\lambda^2 (E_A^2 + B_z^2 + \lambda^2)} \right)^{1/2}. \quad (3.113)$$

3.2.3 JE in a junction coupled to an electron reservoir by Beenakker

$$\rho(\varepsilon) = -\frac{1}{\pi} \text{Im} \frac{d}{d\varepsilon} \ln \det[1 - R_A(\varepsilon + i0^+) S_N(\varepsilon + i0^+)] + \rho_0(\varepsilon), \quad (3.114)$$

$$S_N(\varepsilon) = \begin{pmatrix} s_0(\varepsilon) & 0 \\ 0 & s_0^*(-\varepsilon) \end{pmatrix}, \quad (3.115)$$

$$R_A(\varepsilon) = i\alpha(\varepsilon) \begin{pmatrix} 0 & r_A \\ r_A^* & 0 \end{pmatrix}, \quad r_A = \begin{pmatrix} e^{i\phi/2} \sigma_y & 0 \\ 0 & e^{-i\phi/2} \sigma_y \end{pmatrix},$$

$$\alpha(\varepsilon) = e^{-i \arccos(\varepsilon/\Delta_0)} = \varepsilon/\Delta_0 - i\sqrt{1 - \varepsilon^2/\Delta_0^2}. \quad (3.116)$$

$$\rho(\varepsilon) = -\frac{1}{\pi} \text{Im} \frac{d}{d\varepsilon} \ln \det[1 - M(\varepsilon + i0^+)] + \rho_0(\varepsilon), \quad (3.117)$$

$$M(\varepsilon) = -\alpha(\varepsilon)^2 r_A^* s_0^*(-\varepsilon) r_A s_0(\varepsilon). \quad (3.118)$$

$$F = -k_B T \int_0^\infty d\varepsilon \rho(\varepsilon) \ln[2 \cosh(\varepsilon/2k_B T)]. \quad (3.119)$$

$$I = \frac{2e}{\hbar} \frac{dF}{d\phi}. \quad (3.120)$$

$$F = \frac{i}{4\pi} \int_{-\infty}^\infty d\varepsilon \tanh(\varepsilon/2k_B T) \ln \det[1 - M(\varepsilon + i0^+)] + F_0, \quad (3.121)$$

Opening up the system —

$$\rho(\varepsilon) = -\frac{1}{\pi} \text{Im} \frac{d}{d\varepsilon} \ln \det[1 - M(\varepsilon + i\gamma)] + \rho_0(\varepsilon). \quad (3.122)$$

$$F = \frac{i}{4\pi} \int_{-\infty}^\infty d\varepsilon \tanh(\varepsilon/2k_B T) \ln \det[1 - M(\varepsilon + i\gamma)] + F_0. \quad (3.123)$$

Quantum dot Josephson junction —

$$s_0(\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{i}{\varepsilon - \varepsilon_R + \frac{1}{2}i\Gamma} \begin{pmatrix} \Gamma_1 & \sqrt{\Gamma_1 \Gamma_2} \\ \sqrt{\Gamma_1 \Gamma_2} & \Gamma_2 \end{pmatrix}, \quad (3.124)$$

$$T_{\text{BW}} = \frac{\Gamma_1 \Gamma_2}{\varepsilon_R^2 + \frac{1}{4}\Gamma^2}. \quad (3.125)$$

$$\alpha(\varepsilon) \rightarrow \alpha(0) = -i.$$

$$\ln \det[1 - M(\varepsilon)] = g \ln [1 - (\varepsilon_A/\varepsilon)^2], \quad (3.126)$$

$$\varepsilon_A = \Delta_{\text{eff}} \sqrt{1 - T_{\text{BW}} \sin^2(\phi/2)}, \quad \Delta_{\text{eff}} = \sqrt{\varepsilon_{\text{R}}^2 + \frac{1}{4}\Gamma^2}, \quad (3.127)$$

$$F = \frac{gi}{4\pi} \int_{-\infty}^{\infty} d\varepsilon \tanh(\varepsilon/2k_{\text{B}}T) \ln \left[1 - \frac{\varepsilon_A^2}{(\varepsilon + i\gamma)^2} \right] + F_0. \quad (3.128)$$

$$\begin{aligned} I &= I_0 \int_{-\infty}^{\infty} \frac{d\varepsilon \varepsilon_A \tanh(\varepsilon/2k_{\text{B}}T)}{i\pi (\varepsilon_A)^2 - (\varepsilon + i\gamma)^2} \\ &= I_0 \frac{2}{\pi} \text{Im} \psi \left(\frac{1}{2} + \frac{i\varepsilon_A + \gamma}{2\pi k_{\text{B}}T} \right), \end{aligned} \quad (3.129)$$

$$I_0 = -\frac{ge}{\hbar} \frac{d\varepsilon_A}{d\phi} \quad (3.130)$$

$$\lim_{T \rightarrow 0} I = I_0 \frac{2}{\pi} \arctan(\varepsilon_A/\gamma), \quad (3.131)$$

Point contact Josephson junction —
 $\sigma_y S_{\text{N}}^* \sigma_y = S_{\text{N}}^\dagger.$

$$\varepsilon_n = \Delta_0 \sqrt{1 - T_n \sin^2(\phi/2)}, \quad n = 1, 2, \dots, N, \quad (3.132)$$

$$\ln \det[1 - M(\varepsilon)] = g \sum_{n=1}^N \ln [1 - (\varepsilon_n/\varepsilon)^2], \quad (3.133)$$

$$I = \sum_{n=1}^N I_n \frac{2}{\pi} \text{Im} \psi \left(\frac{i\varepsilon_n + \gamma + \pi k_{\text{B}}T}{2\pi k_{\text{B}}T} \right), \quad (3.134)$$

$$\lim_{T \rightarrow 0} I = \sum_{n=1}^N I_n \frac{2}{\pi} \arctan(\varepsilon_n/\gamma), \quad (3.135)$$

$I_n = -(ge/\hbar)d\varepsilon_n/d\phi$ the zero-temperature supercurrent in the n -th mode of the closed system.

Long SNS junction —

$$F = -k_{\text{B}}T \sum_{p=0}^{\infty} \ln \det[1 - M(i\omega_p + i\gamma)], \quad (3.136)$$

$$\omega_p = (2p + 1)\pi k_{\text{B}}T.$$

$$\lim_{T \rightarrow 0} F = -\frac{1}{2\pi} \int_0^{\infty} d\omega \ln \det[1 - M(i\omega + i\gamma)]. \quad (3.137)$$

$$s_0(\varepsilon) = e^{-ik(\varepsilon)L} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (3.138)$$

$$\begin{aligned} \ln \det[1 - M(\varepsilon)] &= \\ &= g \ln [1 + \alpha^4 e^{4i\varepsilon L/\hbar v_F} - 2\alpha^2 e^{2i\varepsilon L/\hbar v_F} \cos \phi]. \end{aligned} \quad (3.139)$$

$$\ln \det[1 - M(i\omega)] = g \ln |1 + e^{i\phi} e^{-2\omega L/\hbar v_F}|, \quad (3.140)$$

$$F = \frac{g\hbar v_F}{2\pi L} \operatorname{Re} \operatorname{Li}_2(-e^{i\phi-2\gamma L/\hbar v_F}). \quad (3.141)$$

$$I = \frac{gev_F}{\pi L} \arctan \left[\frac{\sin \phi}{e^{2\gamma L/\hbar v_F} + \cos \phi} \right], \quad (3.142)$$

$$\lim_{\gamma \rightarrow 0} I = \frac{gev_F}{\pi L} \arctan \tan(\phi/2) = \frac{gev_F}{\pi L} \phi, \quad -\pi < \phi < \pi. \quad (3.143)$$

Figure 3.4: Same as Fig. 23.2, but now for a single-mode ballistic SNS junction in the long junction limit (length L large compared to the superconducting coherence length $\xi_0 = \hbar v_F/\Delta_0$). The plot is computed from Eq. (23.29).

Connection to the non-Hermitian Josephson effect —

$$\mathcal{H} = \begin{pmatrix} \varepsilon_R - i\gamma & \frac{1}{2}e^{i\phi/2}\Gamma_1 + \frac{1}{2}e^{-i\phi/2}\Gamma_2 \\ \frac{1}{2}e^{-i\phi/2}\Gamma_1 + \frac{1}{2}e^{i\phi/2}\Gamma_2 & -\varepsilon_R - i\gamma \end{pmatrix}. \quad (3.144)$$

$$I = -\frac{2e}{\pi\hbar} \frac{d}{d\phi} \operatorname{Im} \operatorname{Tr}(\mathcal{H} \ln \mathcal{H}), \quad (3.145)$$

3.2.4 JC, Noise at a S-QSHI-S Junction by Fu, Kane

Liang Fu and C.L. Kane

We study junctions between superconductors mediated by the edge states of a quantum spin Hall insulator. We show that such junctions exhibit a fractional Josephson effect, in which the current phase relation has a 4π , rather than a 2π periodicity. This effect is a consequence of the conservation of fermion parity - the number of electrons modulo 2 - in a superconducting junction, and is closely related to the Z_2 topological structure of the quantum spin Hall insulator. Inelastic processes, which violate the conservation of fermion parity, lead to telegraph noise in the equilibrium supercurrent. We predict that the low frequency noise due to these processes diverges exponentially with temperature T as $T \rightarrow 0$. Possible experiments on HgCdTe quantum wells will be discussed.

$H = \Psi^\dagger \mathcal{H} \Psi / 2$, where $\Psi = [(\psi_\uparrow, \psi_\downarrow), (\psi_\downarrow^\dagger, -\psi_\uparrow^\dagger)]$

$$\mathcal{H} = -iv\tau_z\sigma_z\partial_x - \mu\tau_z + \Delta_0(\cos\phi\tau_x + \sin\phi\tau_y). \quad (3.146)$$

$$V_Z = M\psi^\dagger\sigma_x\psi = M\Psi^\dagger\sigma_x\Psi/2. \quad (3.147)$$

Figure 3.5: A S-QSHI-S junction in an RF SQUID geometry where the QSHI forms a Corbino disk.

$$\begin{aligned}\Delta(x) &= \Delta_0 [\theta(-x - L/2) + e^{i\phi}\theta(x - L/2)] \\ M(x) &= M_0\theta(x + L/2)\theta(-x + L/2).\end{aligned}\tag{3.148}$$

Figure 3.6: Spectrum of Andreev bound states in the junction as a function of phase difference ϕ for parameters indicated in each panel. L is in units of v/Δ_0 and M_0 and μ are in units of Δ_0 . (a) and (c) are independent of μ .

$$D = 1/[1 + (M_0 \sinh(\kappa L)/\kappa)^2], \text{ with } \kappa = \sqrt{M_0^2 - \mu^2}.$$

$$\epsilon_0(\phi) = \sqrt{D}\Delta_0 \cos(\phi/2).\tag{3.149}$$

$$\Gamma_{0+} = \Gamma_{0-}^\dagger \equiv \Gamma_0.$$

$$H = \epsilon_0(\phi)(\Gamma_0^\dagger \Gamma_0 - 1/2) = 2i\epsilon_0(\phi)\gamma_1\gamma_2,\tag{3.150}$$

$$\gamma_1 = (\Gamma + \Gamma^\dagger)/2, \gamma_2 = -i(\Gamma - \Gamma^\dagger)/2.$$

$$I_0(\phi) = \frac{1}{2}\sqrt{D}e\Delta_0 \sin \phi/2.\tag{3.151}$$

$$\langle I(\phi) \rangle = I_0(\phi) \tanh \epsilon_0(\phi)/2T.\tag{3.152}$$

$$\begin{aligned}H &= \epsilon_0\Gamma_0^\dagger\Gamma_0 + \sum_n E_n c_n^\dagger c_n + \sum_m \omega_m b_m^\dagger b_m \\ &+ \sum_{mn} [(V_{nm}^1 c_n^\dagger b_m + V_{nm}^2 c_n b_m^\dagger)\Gamma_0 + h.c.].\end{aligned}\tag{3.153}$$

$$\tau_\pm^{-1} = e^{\mp\epsilon_0/2T} (w_1(T)e^{\epsilon_0/2T} + w_2(T)e^{-\epsilon_0/2T}),\tag{3.154}$$

$$w_{1,2}(T) = 2\pi \sum_{n,m} e^{-E_n/T} |V_{nm}^{1,2}|^2 \delta(E_n - \omega_m).\tag{3.155}$$

$$w(T) \propto \begin{cases} e^{-\Delta_0/T} & \text{quasiparticles,} \\ e^{-(T_0/T)^{1/3}} & \text{hopping.} \end{cases}\tag{3.156}$$

$$\begin{aligned}dp/dt &= -(p - \bar{p})/\tau, \text{ where } \tau^{-1} = \tau_+^{-1} + \tau_-^{-1} = 4w \cosh^2 \epsilon_0/2T. \quad \bar{p} = (1 + \exp \epsilon_0/T)^{-1} \\ S(\omega) &= 2 \int_{-\infty}^{\infty} e^{i\omega t} \langle I(t)I(0) \rangle\end{aligned}$$

$$S(\omega) = \frac{4I_0^2}{\cosh^2 \epsilon_0(\phi)/2T} \frac{\tau}{1 + \omega^2 \tau^2}.\tag{3.157}$$

$$S(\omega \rightarrow 0) = \frac{I_0^2}{w(T) \cosh^4 \epsilon_0(\phi)/2T}.\tag{3.158}$$

Part II

My Theory of Superconductive Contacts

4 Fundamentals of JJ

(I'll write later)

4.0.1 Basics

(???)

Part III

Problems

5 Типичные задачи

5.1 Общие вопросы

5.1.1 Вопросы на понимание сути предмета

5.1.2 Типичные вопросы и задачи на проверку знаний

5.1.3 Вопросы на понимание типичных деталей

5.2 Задачи типичных разделов

6 Другие задачи

Part IV

Typical Contacts: Well-Known, Important Theories

7 Ballistic Josephson-current flow through an asymmetric superconductor-normal-metal-superconductor junction by Chang, Bagwell

Abstract

We study the Josephson current through a ballistic, normal metal, mesoscopic, one-dimensional quantum channel in contact with two superconducting electrodes. The two superconductors have different energy gaps Δ_1 and Δ_2 . Increasing either energy gap increases the critical current I_c . This enhancement of I_c results from altering the mesoscopic processes which control the Josephson current flow, rather than through the proximity effect. The effect is strongest in junctions shorter than the size of a Cooper pair. When the asymmetry is large, such that $\Delta_2 \gg \Delta_1$, we find the critical current can be several times larger than its bulk value in the weaker superconductor, namely, $I_c > e\Delta_1/\hbar$.

7.1 VI. Conclusions

We have solved the BdG equations to obtain the ballistic Josephson current through an asymmetric S_1NS_2 junction, where the two superconductors have different energy gaps Δ_1 and Δ_2 . Increasing the energy gap of either superconductor, i.e., by making $\Delta_2 > \Delta_1$, increases the critical current I_c . For very asymmetric point contacts at low temperature, the enhancement of I_c by a stronger superconductor can be several times the critical current of the weaker superconductor, namely several $e\Delta_1/\hbar$. A smaller critical current enhancement occurs in long S_1NS_2 junctions.

Resonant quasiparticle energy levels control the Josephson current flow in the S_1NS_2 junction, so that the current-phase relation is determined by mesoscopic phenomena. The leaky Andreev levels for energies $|E| \geq \Delta_1$ are sharpened by increasing the strength of the either superconductor, increasing the Josephson current. This enhancement of the critical current I_c therefore results from altering the mesoscopic processes which control the Josephson current flow, rather than through the proximity effect.

Including the proximity effect will reduce the critical current by rounding the shape of the pairing potential $\Delta(x)$ near the NS boundaries. If the Bardeen-CooperSchrieffer¹ coupling constant $g(x)$ is zero in the normal metal, including the proximity effect will only make the "effective" junction length L slightly longer, reducing the critical current. This should overcome the objections raised against non-self-consistent calculations in Ref. 9. In any case, the dominant factor controlling the critical current is mesoscopic processes occurring near the junction, rather than the proximity effect.

Changing the junction asymmetry also changes the current-phase relation. For a point contact, the current-phase relation evolves from $I(\phi) = I_c \sin(\phi/2)$ to $I(\phi) \simeq I'_c \sin(\phi)$ as the asymmetry increases. The critical current $I'_c \simeq (e\Delta_1/\hbar) (2/\pi) \ln[(2\Delta_2/\Delta_1) - 1]$ is almost immune to the effects of finite temperature, since it depends on the large energy scale Δ_2 . For a long S_1NS_2 junction, the current-phase relation is triangular for all values of the asymmetry. At very high temperatures, energy averaging forces the current-phase relation of all S_1NS_2 junctions into the usual Josephson form $I_c'' \sin(\phi)$, and suppresses the critical current.

7.2 I. Introduction

Mesoscopic physics and superconductivity are intimately related through the Bogoliubov-de Gennes (BdG) equations.^{1,2} It should therefore not be surprising that the wave interference of quasiparticles controls both the current-phase relationship of a Josephson junction³⁻⁹ and the current-voltage characteristic of a normal-metal-superconductor (NS) tunnel junction.^{10,11} Both the Josephson effect and Giaever tunneling are mesoscopic phenomena.

In a superconductor-normal-metal-superconductor (SNS) junction, for example, Andreev reflections from the NS boundaries confine the quasiparticles to the normal region. The resulting energy levels are sensitive to the superconducting phase difference across the junction.³ Since motion of the quasiparticles is phase coherent, placing impurities or other scattering centers in the N region alters the quasiparticle wave interference pattern,^{7,8} changing the Josephson current. By the same reasoning, altering the energy gap of either superconductor will also change the Josephson current.

The resonant quasiparticle energy levels controlling the Josephson-current flow are sharpened in an asymmetric S_1NS_2 junction, where the two superconductors have different energy gaps Δ_1 and Δ_2 . In a symmetric junction, where $\Delta_1 = \Delta_2$, the Josephson current is carried both by the bound Andreev levels having energies $|E| \leq \Delta_1$, and by leaky Andreev levels in the "continuum," where $|E| \geq \Delta_1$. The leakage rate of these continuum levels is reduced by increasing the strength of the second superconductor so that $\Delta_2 \geq \Delta_1$, enhancing the Josephson current. This critical current enhancement is therefore not caused by the proximity effect, which we neglect in our calculation but arises by altering the mesoscopic processes that control the Josephson-current flow.

Although previous works on the Josephson effect in SNS junctions considered mainly symmetric structures,³⁻⁸ a few have partially explored the consequences of choosing two different superconductors

to make up the Josephson junction.¹²⁻¹⁶ Kulik and Omel'yanchuk^{12,13} obtained the current-phase relation of an asymmetric "point-contact" Josephson junction at finite temperature for weak asymmetry, while Kuplevakhskii and Fal'ko¹⁴⁻¹⁶ calculated the bound energy levels of quasiparticles trapped inside the normal region of a S_1NS_2 point contact.

In this paper we solve the BdG equations to obtain the Josephson current for junctions both shorter and longer than the size of a Cooper pair, for both weak and strong asymmetry, and at both zero and finite temperatures. We show that a large asymmetry has the greatest effect on a point contact junction. For the short S_1NS_2 junction, the large asymmetry forces a large current flow in the continuum above $|E| \geq \Delta_1$, significantly enhancing the critical current and altering the current-phase relation of the junction.

7.3 II. Supercurrent through an S_1NS_2 Junction

The motion of quasiparticles in the S_1NS_2 junction is determined from the BdG equation¹

$$\begin{pmatrix} H(x) - \mu & \Delta(x) \\ \Delta^*(x) & -[H^*(x) - \mu] \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = E \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}, \quad (1)$$

where the one-electron Hamiltonian is $H(x) = -(\hbar^2/2m)(d^2/dx^2)$. We model the S_1NS_2 junction by the asymmetric change in the pairing potential shown in Fig. 1, namely,

$$\Delta(x) = \begin{cases} \Delta_1 e^{i\phi_1}, & x < 0 \\ 0, & 0 < x < L \\ \Delta_2 e^{i\phi_2}, & x > L \end{cases} \quad (2)$$

The two superconducting order parameter reservoirs are described by different magnitudes and phases of $\Delta(x)$.

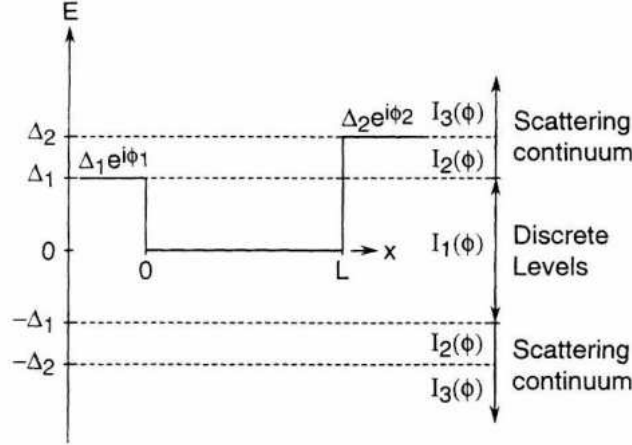


FIG. 1. Scattering potential of an asymmetric Josephson junction. The current $I_1(\phi)$ is carried through discrete energy levels, while $I_2(\phi)$ and $I_3(\phi)$ flow through broadened levels.

For definiteness, we will have $\Delta_2 > \Delta_1$ throughout this paper. By neglecting the self-consistency condition¹ for $\Delta(x)$ we neglect the proximity effect, although we feel it is important to remove this restriction in a future work.⁹

We compute the Josephson current using the electrical current density^{10,11}

$$J_Q(x) = e \sum_n f_n \{J_{u_n}(x) + J_{v_n}(x)\} \quad (3)$$

Here n is the quantum number and f_n the equilibrium Fermi occupation factor

$$f_n = f(E_n) = \frac{1}{1 + e^{E_n/k_B T}}. \quad (4)$$

The $J_{u_n}(x)$ and $J_{v_n}(x)$ in Eq. (3) are Schrödinger currents carried by the waves $u_n(x)$ and $v_n(x)$, namely,

$$J_{u_n}(x) = \frac{\hbar}{m} \text{Im} \{u_n^*(x) [\nabla u_n(x)]\} \quad (5)$$

and

$$J_{v_n}(x) = \frac{\hbar}{m} \text{Im} \{v_n^*(x) [\nabla v_n(x)]\} \quad (6)$$

In an asymmetric Josephson junction, the supercurrent flow must be analyzed in the three different energy regimes shown in Fig. 1. The total Josephson current is a sum of the currents flowing in each region as

$$I(\phi) = I_1(\phi) + I_2(\phi) + I_3(\phi) \quad (7)$$

We apply Eq. (3) to compute $I_3(\phi)$ in Appendix A, $I_2(\phi)$ in Appendix B, and $I_1(\phi)$ in Appendix C. Appendix D establishes contact with the alternative formula for the electrical current used in Refs. 4, 5, and 9.

For $|E| < \Delta_1$, the current $I_1(\phi)$ flows through discrete "Andreev levels".³ Our calculation for the energy levels $E_n^\pm(\phi)$ in Appendix A agrees with the result of Kuplevakhsii and Fal'ko,¹⁴ namely, for $E \geq 0$

$$-\cos^{-1}\left(\frac{E}{\Delta_1}\right) - \cos^{-1}\left(\frac{E}{\Delta_2}\right) + \left(\frac{2LE}{\hbar v_F}\right) \pm \phi = 2\pi n \quad (8)$$

Here $\phi = \phi_2 - \phi_1$ is the superconducting phase difference, the Fermi velocity is $v_F = \sqrt{2\mu/m}$, and $n = 0, \pm 1, \pm 2, \dots$. Equation (8) can be interpreted as the Bohr-Sommerfeld quantization condition for the periodic orbits in the normal region.^{7,14,17}

The contribution $I_1(\phi)$ from region 1, where each energy level $E_n^\pm(\phi)$ carries a certain electrical current $I_n^\pm(\phi)$, is

$$I_1(\phi) = \sum_n \{I_n^+(\phi)f(E_n^+(\phi)) + I_n^-(\phi)f(E_n^-(\phi))\} \quad (9)$$

The current $I_n^\pm(\phi)$ carried by each Andreev level we obtain in Appendix C as

$$I_n^\pm(\phi) = \mp \frac{ev_F}{L + \xi_1(E_n^\pm(\phi)) + \xi_2(E_n^\pm(\phi))} \quad (10)$$

Here $\xi_{1,2}(E)$ is the energy-dependent healing length¹⁰

$$\xi_{1,2}(E) = \xi_{1,2} \left(\frac{\Delta_{1,2}}{\sqrt{\Delta_{1,2}^2 - E^2}} \right) \quad (11)$$

with $\xi_{1,2} = \hbar v_F / 2 |\Delta_{1,2}|$ and $E_n^\pm(\phi)$ found from Eq. (8).

For energies outside the superconducting gap, the pair potential discontinuity at an NS interface is less effective at confining the quasiparticles, producing broadened energy levels. The solutions of Eq. (8) for energies $|E| > \Delta_1$ therefore become complex, so that the Andreev levels acquire a finite lifetime. The contributions $I_2(\phi)$ and $I_3(\phi)$ to the Josephson current, flowing in the energy ranges $\Delta_2 > |E| > \Delta_1$ and $|E| > \Delta_2$, respectively, must then be integrated over the continuous energy spectrum as

$$I_2(\phi) = \left(\int_{-\Delta_2}^{-\Delta_1} + \int_{\Delta_1}^{\Delta_2} \right) I_2(E, \phi) f(E) dE \quad (12)$$

and

$$I_3(\phi) = \left(\int_{-\infty}^{-\Delta_2} + \int_{\Delta_2}^{\infty} \right) I_3(E, \phi) f(E) dE \quad (13)$$

For energies $\Delta_2 > |E| > \Delta_1$, only the weaker superconductor allows the quasiparticles to leak away from the normal region. When $|E| > \Delta_2$, both superconductors allow quasiparticles to escape the normal region.

We employ the quasiparticle transmission formalism of van Wees, Lenssen, and Harmans⁶ to calculate the continuum electrical current densities $I_2(E, \phi)$ and $I_3(E, \phi)$. The computation for $I_3(\phi)$ in Appendix A proceeds as in previous works,^{6,7} yielding

$$I_3(\phi) = \frac{e}{h} \left(\int_{-\infty}^{-\Delta_2} + \int_{\Delta_2}^{\infty} \right) (|u_1^2 - v_1^2| + |u_2^2 - v_2^2|) \left(\frac{1}{D'(E, -\phi)} - \frac{1}{D'(E, \phi)} \right) f(E) dE \quad (14)$$

where the function $D'(E, \phi)$ is

$$D'(E, \phi) = u_1^2 u_2^2 + v_1^2 v_2^2 - 2u_1 u_2 v_1 v_2 \cos \left[\left(\frac{|E|}{\Delta_1} \right) \left(\frac{L}{\xi_1} \right) + \phi \right] \quad (15)$$

Here $u_{1,2}$ and $v_{1,2}$ are the coherence factors

$$2u_{1,2}^2 = 1 + \frac{\sqrt{E^2 - \Delta_{1,2}^2}}{E} \quad (16)$$

and

$$2v_{1,2}^2 = 1 - \frac{\sqrt{E^2 - \Delta_{1,2}^2}}{E} \quad (17)$$

Equations (14) and (15) show that $I_3(\phi) = 0$ for a point contact junction where $L \ll \xi_1$ and $L \ll \xi_2$, since $D'(E, \phi) = D'(E, -\phi)$ when $L = 0$. Applying the same formalism ⁶ in Appendix B for the current $I_2(\phi)$, we obtain

$$I_2(\phi) = \frac{2e}{h} \left(\int_{\Delta_1}^{\Delta_2} - \int_{-\Delta_2}^{-\Delta_1} \right) |u_1^2 - v_1^2| \left(\frac{1}{D(E, -\phi)} - \frac{1}{D(E, \phi)} \right) f(E) dE \quad (18)$$

where the function $D(E, \phi)$ is

$$D(E, \phi) = u_1^2 + v_1^2 - 2u_1v_1 \cos \left[\left(\frac{|E|}{\Delta_1} \right) \left(\frac{L}{\xi_1} \right) + \phi - \cos^{-1} \left(\frac{|E|}{\Delta_2} \right) \right]. \quad (19)$$

Equations (14), (15), (18), and (19) indicate that the electrical current densities $I_2(E, \phi)$ and $I_3(E, \phi)$ are significant only for energies within a few Δ_1 of the Fermi level, since the product u_1v_1 goes to zero outside this energy range.

7.4 III. Point-contact S_1NS_2 Junction

When $L \ll \xi_1$ and $L \ll \xi_2$, we have a "short" junction or a "point contact." In this limit, the Andreev energy level spectrum is obtained from Eq. (8) as ¹⁴

$$E^\pm = \pm \frac{\Delta_1 \Delta_2 \sin(\phi)}{\sqrt{\Delta_1^2 + \Delta_2^2 - 2\Delta_1 \Delta_2 \cos(\phi)}} \quad (20)$$

with the restriction $\cos(\phi) \leq \Delta_1/\Delta_2$. The energy levels from Eq. (20) are shown in Fig. 2. The level E^+ corresponds to a left-going electron, while E^- describes a right-moving one. The energy levels in Eq. (20) predict a current discontinuity ΔI at $\phi = \pi$ of

$$\frac{\Delta I}{2} = I_1(\phi = \pi) = \frac{2e}{h} \left(\frac{\Delta_1 \Delta_2}{\Delta_1 + \Delta_2} \right) \quad (21)$$

since $I_2(\phi = \pi) = 0$ and $I_3(\phi) = 0$.

We plot the total current $I(\phi) = I_1(\phi) + I_2(\phi)$ (solid) at zero temperature in Fig. 3, for both (a) weak and (b) strong asymmetry. ¹⁸ The discrete component $I_1(\phi)$ (short dashes) carries most of the current for weak asymmetry, while $I_2(\phi)$ (long dashes) is the dominant current (for most values of the phase) when the asymmetry is strong. For weak asymmetry the current is well described

by $I(\phi) \simeq I_c \sin(\phi/2)$, similar to the symmetric SNS junction but with $I_c \simeq (\Delta I/2)$.

In the limit of strong asymmetry, shown in Fig. 3(b), the current $I_2(\phi)$ depends roughly sinusoidally on the phase

$$I_2(\phi) \simeq \left(\frac{e\Delta_1}{h} \right) \left(\frac{2}{\pi} \right) R(\phi) \sin(\phi) \quad (22)$$

as shown in Appendix B. The factor $R(\phi)$ depends only weakly on the phase for large asymmetry, tending to

$$R(\phi) \simeq \ln \left(\frac{2\Delta_2}{\Delta_1} \right) - 1 \quad (23)$$

as $(\Delta_2/\Delta_1) \gg 1$.

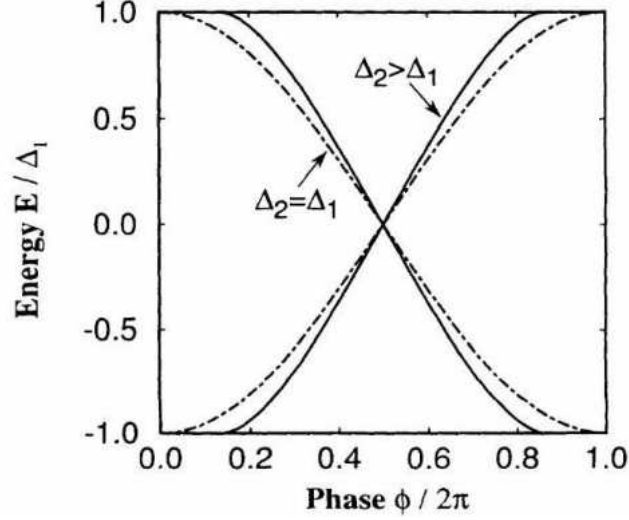


FIG. 2. Discrete energy levels vs superconducting phase difference ϕ in an asymmetric Josephson junction having $\Delta_2/\Delta_1 = 1.5$ (solid). The dashed line indicates the energy levels for the symmetric junction where $\Delta_2/\Delta_1 = 1.0$.

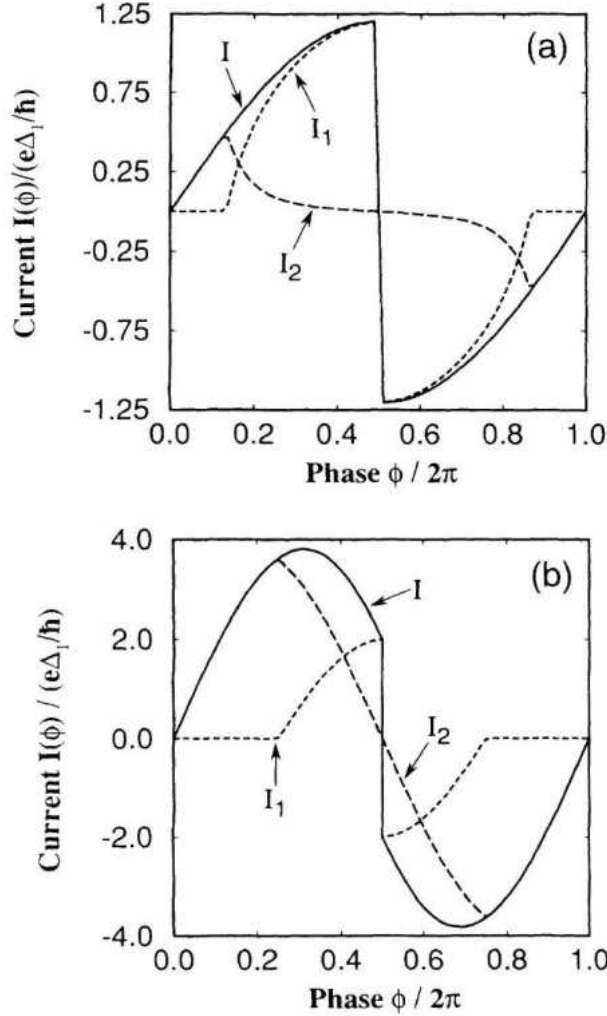


FIG. 3. Total supercurrent $I(\phi)$ (solid) through a point-contact junction at $T = 0$ when the asymmetry is (a) $\Delta_2/\Delta_1 = 1.5$ and (b) $\Delta_2/\Delta_1 = 200$. The critical current in (b) is greatly enhanced over (a), and the current-phase relation is changed. Both effects arise from sharpening the continuum Andreev levels as the strength of the second superconductor increases and are

unrelated to the proximity effect.

The current component $I_1(\phi)$ can also be obtained in the limit of strong asymmetry. When $(\Delta_2/\Delta_1) \gg 1$, discrete energy levels exist only between $\pi/2 \leq \phi \leq 3\pi/2$, given by $E^\pm = \pm \Delta_1 \sin(\phi)$. They produce a current $I_1(\phi) = (2e\Delta_1/\hbar) \cos(\phi) \text{sign}(\pi - \phi)$. The critical current is set, however, by $I_c \simeq I_2(\phi = \pi/2) \simeq (2e\Delta_1/\pi\hbar) R(\pi/2)$. For large asymmetry we then have

$$I_c \simeq \left(\frac{e\Delta_1}{\hbar} \right) \left(\frac{2}{\pi} \right) \left[\ln \left(\frac{2\Delta_2}{\Delta_1} \right) - 1 \right] \quad (24)$$

which can be much larger than the critical current of a symmetric point contact, $e\Delta_1/\hbar$.

Figure 4 shows the actual critical current calculated from Eq. (9) and (18) (solid), compared with the logarithmic approximation in Eq. (24) (dot dashed). Equation (24) captures the logarithmically increasing trend of the current, but does not become an accurate approximation until the asymmetry is very large. Realistic asymmetry values range from 19 for an Al–Nb₃Ge junction, to almost 200 for more exotic combinations of superconductors such as Cd–YBaCuO. The dotted line in Fig. 4 shows the critical current saturates with increasing asymmetry

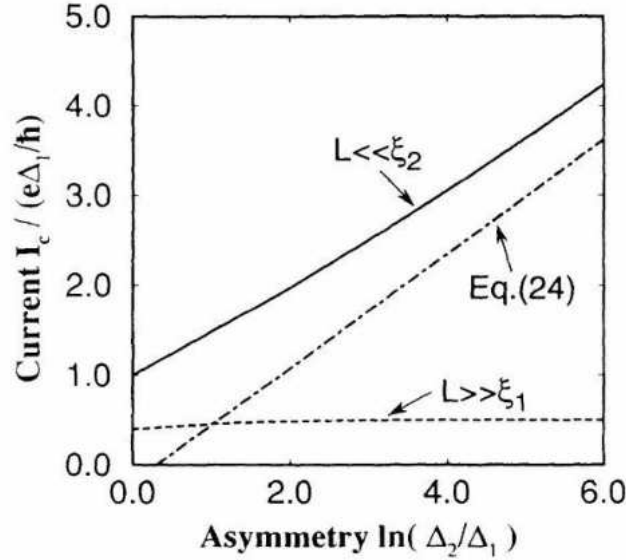


FIG. 4. Critical current of a short, asymmetric S_1NS_2 junction (solid) increases as the energy gap of either superconductor increases. This increase is approximately logarithmic, following the law (dot-dashed) $I_c \simeq (e\Delta_1/\hbar) (2/\pi) \ln [(2\Delta_2/\Delta_1) - 1]$ for large asymmetry. The critical current saturates with large asymmetry in a long S_1NS_2 junction (dashed).

in a long junction.

The decrease in current in Fig. 3(b), as the driving phase difference increases between $\pi/2 < \phi < \pi$, can be understood by the movement of the energy level from region II in Fig. 1 ($\Delta_1 < |E| < \Delta_2$) into region I ($|E| < \Delta_1$) near $\phi \simeq \pi/2$. When the energy level is located in region II, it carries more current than when it moves into region I, causing the drop in current with increasing phase difference. Reference 7 pointed out that such current-phase relations are analogous to negative differential resistance in a normal conductor, and invoked the formation of standing waves near $\phi = \pi$ in a disordered SNS junction to explain the current decrease. In this paper the electronic motion is ballistic, so the formation of standing waves cannot explain the current-phase relation in Fig. 3(b). The decrease in current with increasing phase difference results solely from the difference in currents carried by a ballistic electron confined to the Andreev energy level.

7.5 IV. Long S_1NS_2 Junction

A "long" Josephson junction satisfies $L \gg \xi_1$ and $L \gg \xi_2$, allowing more Andreev levels to become bound in the pairing potential well. Figure 5 shows the energy levels from Eq. (8) for a long junction having $L \simeq 8\xi_1$ and $\Delta_2/\Delta_1 = 1.5$. The spectrum is similar to that of a symmetric long and clean junction, and is analogous ⁷ to the spectrum for electrons in a spatially periodic potential or electrons confined to a mesoscopic normal metal ring. The motion of coupled electronlike and holelike quasiparticles trapped in the normal region of an SNS junction is periodic in time, even though the geometry is not periodic in space, producing the energy spectrum of Fig. 5.

In addition to the formation of more bound Andreev levels, a further difference with the point contact S_1NS_2

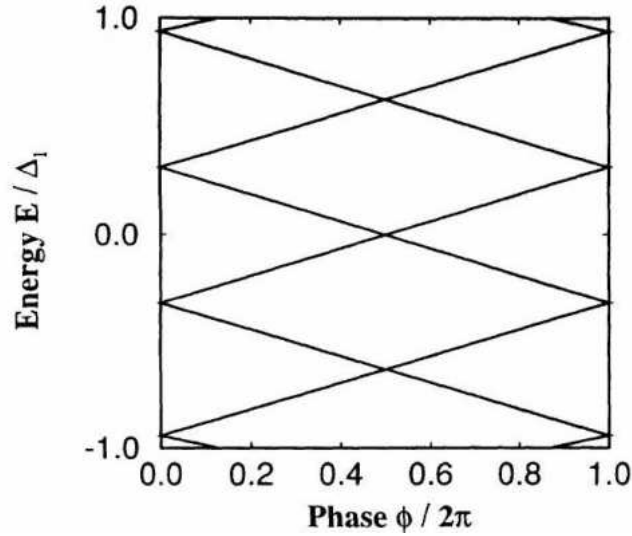


FIG. 5. Discrete energy levels for a long S_1NS_2 junction having $L \gg \xi_{1,2}$. The asymmetry is $\Delta_2/\Delta_1 = 1.5$. The long junction supports several Andreev bound levels, so that several competing left- and right-moving states exist below the Fermi level.

junction is that the current $I_3(\phi)$ begins to flow when $L \gg \xi_{1,2}$. The situation is analogous to transmission over a wide potential well in ordinary quantum mechanics, where transmission resonances in the continuum states above the well are very weak if the well is too narrow. $I_1(\phi)$ and $I_2(\phi)$ still contribute to the Josephson current as before.

Figure 6 shows the zero-temperature Josephson cur-

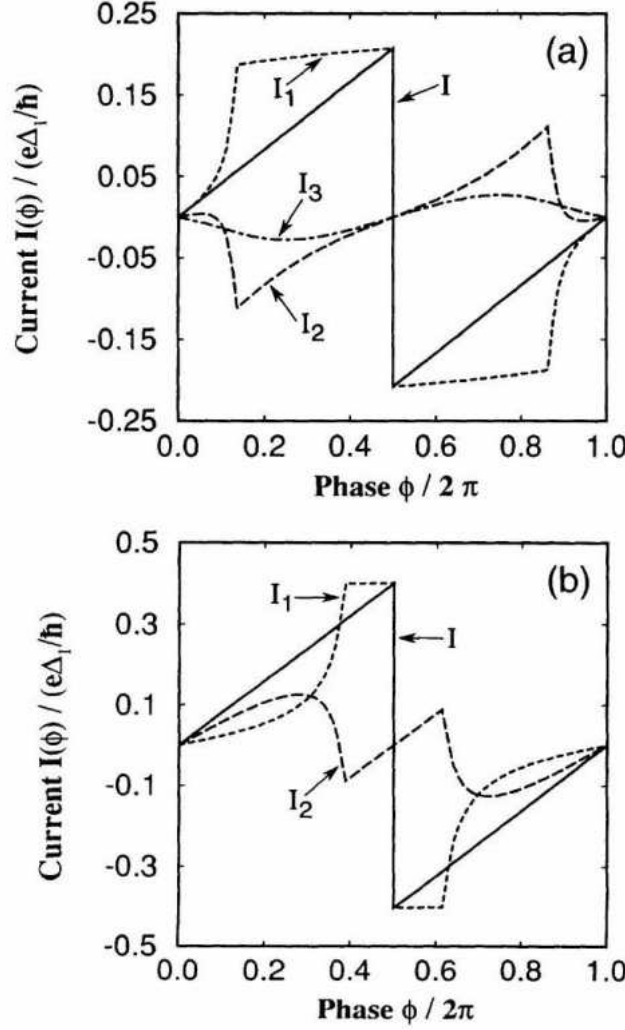


FIG. 6. Zero-temperature Josephson current $I(\phi)$ (solid) in a long S_1NS_2 junction. A triangular current phase relation holds for both (a) weak ($\Delta_2/\Delta_1 = 1.5$) and (b) strong ($\Delta_2/\Delta_1 = 200$) asymmetry.

rent in a long S_1NS_2 junction, for both (a) weak and (b) strong asymmetry. The discrete current $I_1(\phi)$ (short dashes), continuum current $I_2(\phi)$ (long dashes), and continuum current $I_3(\phi)$ (dot dash) combine to give the total supercurrent $I(\phi)$ (solid). The triangular current phase relation obtained for $I(\phi)$ is similar to the symmetric long and clean SNS junction at zero temperature.^{6,7} The current $I_3(\phi) \rightarrow 0$ as $\Delta_2/\Delta_1 \rightarrow \infty$, since the weaker superconductor cannot confine the quasiparticles at high energies above Δ_2 . For this reason $I_3(\phi) \simeq 0$ in Fig. 6(b), and is therefore not shown.

The critical current is solely contributed by $I_1(\phi)$, and is given by

$$I_c = \frac{ev_F}{L + \xi_1(0) + \xi_2(0)} = \frac{ev_F}{L + \left(\frac{\hbar v_F}{2\Delta_1}\right) + \left(\frac{\hbar v_F}{2\Delta_2}\right)} \quad (25)$$

since Eqs. (14) and (18) show that $I_3(\phi = \pm\pi) = 0$ and $I_2(\phi = \pm\pi) = 0$. Increasing the junction length suppresses the critical current. Equation (25) shows that the critical current saturates to $I_c \simeq (ev_F/L) / [1 + \{\xi_1(0)/L\}]$ as the asymmetry $\Delta_2/\Delta_1 \rightarrow \infty$ with Δ_1 fixed, as shown in Fig. 4.

Finally, we comment on the current-phase relation for an asymmetric Josephson junction of "intermediate" length, where $L \ll \xi_1$ but $L \gg \xi_2$. Not surprisingly, $I(\phi)$ itself is intermediate between the long junction of Fig. 6 and short junction of Fig. 3. Starting with a long junction, where $L \gg \xi_1$ and $L \gg \xi_2$, the linear current phase relation gradually deforms into the results

of Fig. 3 and the critical current increases as L is reduced.

7.6 V. Finite temperature

The temperature dependence of the Josephson current is mainly controlled ⁷ by thermally "energy averaging" states above the Fermi level ($E > 0$) with those below ($E < 0$). Therefore, introducing the new energy scale Δ_2 will make it more difficult for a finite temperature to suppress the currents $I_2(\phi)$ and $I_3(\phi)$ than $I_1(\phi)$, since $I_1(\phi)$ is controlled by the smaller energy scale Δ_1 . The energy gaps $\Delta_1(T)$ and $\Delta_2(T)$ will themselves also decrease with temperature, reducing the Josephson current. But the energy averaging effects from finite temperature are also important to understand the temperature dependence of $I(\phi)$.

The finite-temperature Josephson current through a point contact is shown in Fig. 7, corresponding to Fig. 3 at zero temperature. The finite temperature Josephson current in a long junction is given in Fig. 8, corresponding to Fig. 6 at zero temperature. In Figs. 7 and 8 the temperature is $T = 0.25T_{c1}$, while the asymmetry is either (a) weak or (b) strong. Note that $I_2(\phi)$ and $I_3(\phi)$ are not greatly affected by the finite temperature, while $I_1(\phi)$ is suppressed.

The total current (solid) eventually approaches $I(\phi) \simeq I_c \sin(\phi)$ at some temperature, regardless of the asymmetry or junction length. This must occur, since the current discontinuity ΔI near $\phi = \pi$ present at $T = 0$ arises from

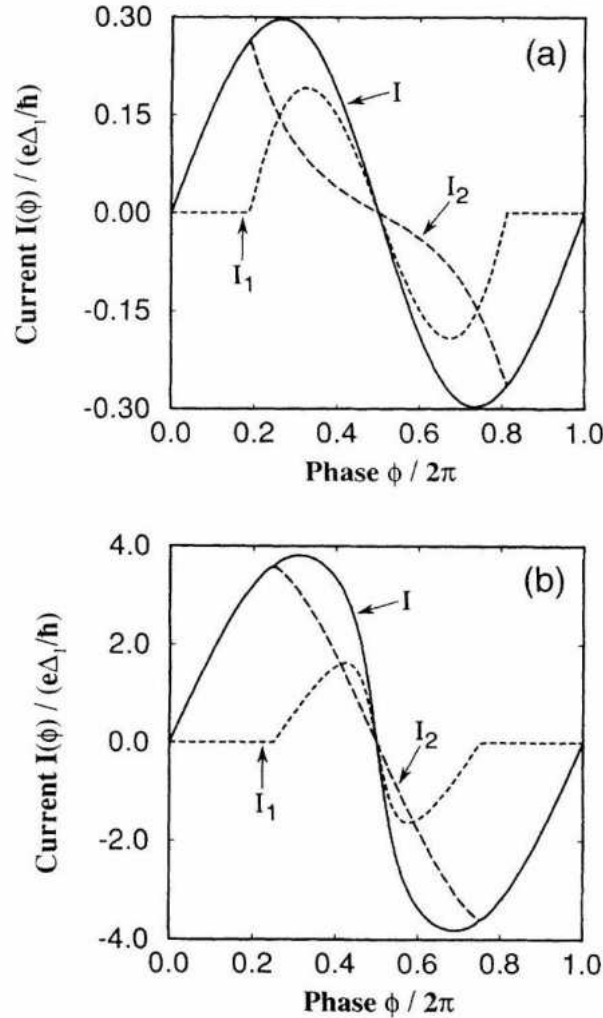


FIG. 7. Total supercurrent $I(\phi)$ (solid) through a point-contact junction at finite temperature ($T = 0.25T_{c1}$) approaches $I(\phi) \simeq I_c \sin(\phi)$. The asymmetry is (a) $\Delta_2/\Delta_1 = 1.5$ and (b) $\Delta_2/\Delta_1 = 200$, as in Fig. 3. The critical current in (a) is greatly suppressed, while in (b) is hardly affected by the finite temperature due to the large energy scale Δ_2 .

a very small energy bandwidth of states near the Fermi level. This current discontinuity is therefore thermally averaged away by any finite temperature, so that $I(\phi)$ looks increasingly sinusoidal.

The temperature variation of the critical current I_c also depends on the junction asymmetry. The critical current decreases strongly with temperature in all junctions except the very asymmetric short junction, where the critical current is $I_c \simeq I_2(\phi = \pi/2)$. Comparing Fig. 3 to Fig. 7 for the short junction, the critical current is greatly reduced in (a), yet hardly affected in (b). In a long junction, comparing Fig. 6 with Fig. 8, the reduction of the critical current by finite temperature is smaller in (b), where the asymmetry is strong. A large energy gap asymmetry makes the critical current relatively more insensitive to a finite temperature.

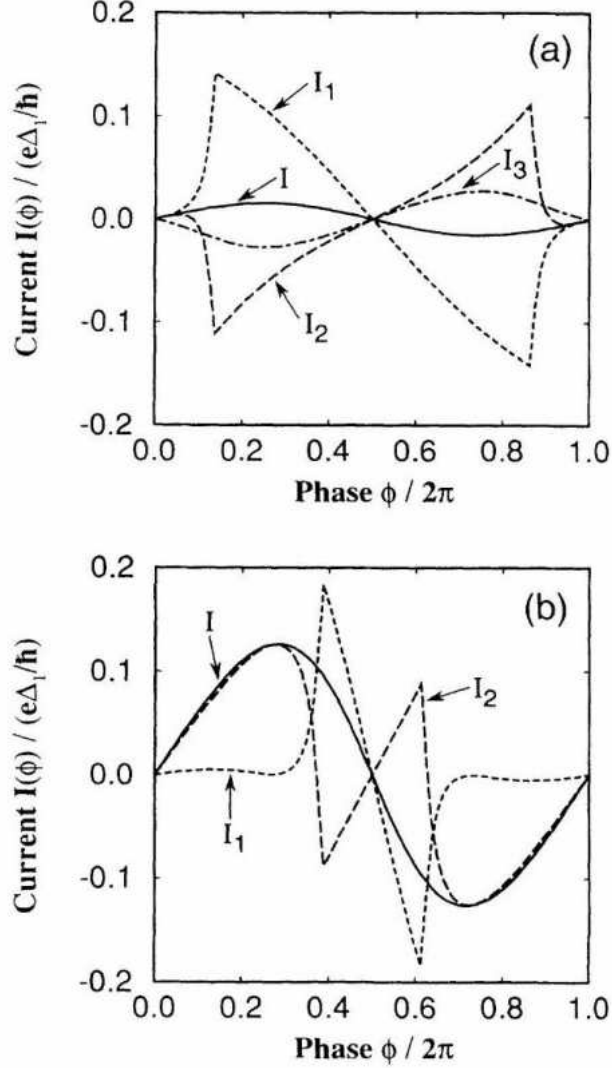


FIG. 8. Finite-temperature Josephson current $I(\phi)$ (solid) in a long S_1NS_2 junction for (a) weak ($\Delta_2/\Delta_1 = 1.5$) and (b) strong ($\Delta_2/\Delta_1 = 200$) asymmetry, corresponding to Fig. 6. Finite temperature ($T = 0.25T_{c1}$) rounds the triangular $I(\phi)$ into a sinusoidal shape through energy averaging. The critical current is suppressed by a finite temperature more strongly than in a short junction.

7.7 Appendix A: Continuum current $I_3(\phi)$

We calculate the electrical current carried in the energy region $E > \Delta_2$. An electron-like quasiparticle, normalized so that the injected quasiparticle density is unity, is injected from the

left contact. The resulting electrical current density is proportional to the electrical current transmission coefficient for the quasiparticle across the SNS junction. The complex energy poles of this transmission coefficient determine the discrete Andreev energy levels from Eq. (8). We then consider the remaining electrons and holes injected from both contacts to obtain $I_3(\phi)$ in Eq. (14).

The energy dispersion relation for a uniform superconductor, namely,

$$E^2 = \left(\frac{\hbar^2 k^2}{2m} - \mu \right)^2 + \Delta_{1,2}^* \Delta_{1,2} \quad (\text{A1})$$

is embodied in Eq. (1). We graph Eq. (A1) for $E > 0$ in Fig. 9. The dispersion relation is also described by the wave vectors $k_{e1,2}$ and $k_{h1,2}$, where

$$\frac{\hbar^2 k_{e1,2}^2}{2m} - \mu = \sqrt{E^2 - \Delta_{1,2}^2} \quad (\text{A2})$$

and

$$\frac{\hbar^2 k_{h1,2}^2}{2m} - \mu = -\sqrt{E^2 - \Delta_{1,2}^2} \quad (\text{A3})$$

We take $k_{e1,2} \geq 0$ and $k_{h1,2} \geq 0$.

The electronlike quasiparticle incident from the left

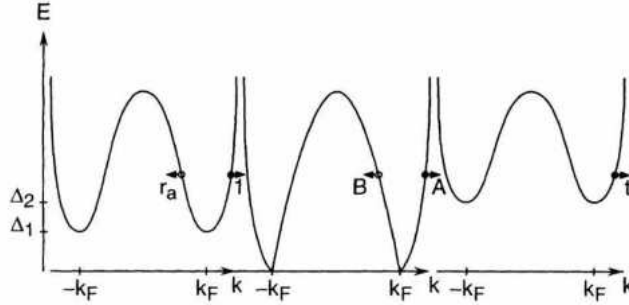


FIG. 9. Dispersion diagram showing the scattering states used to compute the portion of the Josephson current carried by electronlike quasiparticles injected from the left superconducting reservoir with energies above the Fermi level ($E > 0$).

contact and Andreev reflected holelike quasiparticle, shown in Fig. 9, have the wave function

$$\begin{pmatrix} u_1 e^{i\phi_1} \\ v_1 \end{pmatrix} e^{ik_{e1}x} + r_a \begin{pmatrix} v_1 e^{i\phi_1} \\ u_1 \end{pmatrix} e^{ik_{h1}x} \quad (x < 0) \quad (\text{A4})$$

In the normal region, the electron and Andreev reflected hole are expressed as

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\tilde{k}_e x} + B \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\tilde{k}_h x} \quad (0 < x < L) \quad (\text{A5})$$

The transmitted electronlike quasiparticle in the right contact is

$$t \begin{pmatrix} u_2 e^{i\phi_2} \\ v_2 \end{pmatrix} e^{ik_{e2}(x-L)} \quad (x > L) \quad (\text{A6})$$

The r_a , A , B , and t are complex numbers. $\tilde{k}_e$ and $\tilde{k}_h$ are obtained from Eqs. (A2) and (A3) when $\Delta_{1,2} = 0$. The incident "source" quasiparticle generates the other particles through both transmission and Andreev reflection.

By enforcing continuity of the wave function and derivative at $x = 0$ and $x = L$, we obtain the unknown factors r_a , A , B , and t . We assume that $\mu \gg \Delta_{2s}$ and $\mu \gg \Delta_2$, so that

Andreev's approximation ($\tilde{k}_e \simeq \tilde{k}_h \simeq k_e \simeq k_h$ except in an exponent) is valid. In Andreev's approximation, we obtain

$$t = \frac{u_1^2 - v_1^2}{u_1 u_2 e^{i\phi} e^{-i\tilde{k}_e L} - v_1 v_2 e^{-i\tilde{k}_h L}} \quad (\text{A7})$$

Evaluating the electrical current operator $J_Q = J_u + J_v$ from Eq. (3) for $0 < x < L$, using Eq. (A5), yields

$$J_Q^e = e \left(\frac{\hbar \tilde{k}_e}{m} \right) t t^* \quad (\text{A8})$$

The electrical current transmission amplitude for the electronlike quasiparticle incident from the left contact for $E > \Delta_2$ is $T_{L \rightarrow R}^e(E, \phi) \equiv t t^*$. Thus Eq. (A8) can also be written as

$$\begin{aligned} J_Q^e &= e \left(\frac{\hbar \tilde{k}_e}{m} \right) T_{L \rightarrow R}^e(E, \phi) \\ &= e \left(\frac{\hbar \tilde{k}_e}{m} \right) \frac{|u_1^2 - v_1^2|^2}{D'(E, -\phi)}. \end{aligned} \quad (\text{A9})$$

$D'(E, -\phi)$ is given in Eq. (15).

The electrical current per unit energy carried by the injected electronlike quasiparticle is

$$I_{L \rightarrow R}^e(E, \phi) = e \left(\frac{\hbar \tilde{k}_e}{m} \right) T_{L \rightarrow R}^e(E, \phi) N_S^+(E) \quad (\text{A10})$$

$N_S^+(E)$ is the one-dimensional superconducting density of states for the right-moving quasiparticle, and is related to the normal-metal density of states $N_N^+(E)$ by

$$N_S^+(E) = \frac{1}{|u_1^2 - v_1^2|} N_N^+(E). \quad (\text{A11})$$

The identity $(\hbar \tilde{k}_e / m) N_N^+(E) = 1/h$ then gives

$$I_{L \rightarrow R}^e(E, \phi) = \frac{e}{h} \frac{1}{|u_1^2 - v_1^2|} T_{L \rightarrow R}^e(E, \phi). \quad (\text{A12})$$

To obtain the current $I_3^e(\phi)$ carried by electronlike quasiparticles, we must repeat the above calculation for injection from the right superconductor. When this is done, we find

$$\begin{aligned} I_3^e(\phi) &= \frac{e}{h} \left(\int_{-\infty}^{-\Delta_2} + \int_{\Delta_2}^{\infty} \right) \\ &\quad \times \left(\frac{|u_1^2 - v_1^2|}{D'(E, -\phi)} - \frac{|u_2^2 - v_2^2|}{D'(E, \phi)} \right) f(E) dE \end{aligned} \quad (\text{A13})$$

where $f(E)$ is Fermi occupation factor. A similar calculation must also be done for holelike quasiparticles injected from either contact, yielding a current

$$\begin{aligned} I_3^h(\phi) &= \frac{e}{h} \left(\int_{-\infty}^{-\Delta_2} + \int_{\Delta_2}^{\infty} \right) \\ &\quad \times \left(\frac{|u_2^2 - v_2^2|}{D'(E, -\phi)} - \frac{|u_1^2 - v_1^2|}{D'(E, \phi)} \right) f(E) dE \end{aligned} \quad (\text{A14})$$

Equations (A13) and (A14) also indicate the results when the same calculations are repeated for the energy range $E < -\Delta_2$. The total continuum current $I_3(\phi) = I_3^e(\phi) + I_3^h(\phi)$ is therefore given by Eq. (14).

Equations (A13) and (A14) show that the electronlike and holelike components of the electrical current do not individually balance at zero phase difference, namely, $I_3^e(\phi = 0) \neq 0$ and $I_3^h(\phi = 0) \neq 0$. However, the total electrical current balances so that $I_3(\phi = 0) = 0$. This result shows that one must also inject the holelike quasiparticles from both contacts to obtain physically reasonable expressions for the Josephson current. In a symmetric Josephson junction, the electronlike and holelike components of the current individually balance and are equal to each other.⁷ But, in general, one must also inject the holelike quasiparticles from either contact.

Equation (14) can be written as an integral solely over the "excitation spectrum" ($E > \Delta_2$) as

$$I_3(\phi) = \frac{e}{h} \int_{\Delta_2}^{\infty} (|u_1^2 - v_1^2| + |u_2^2 - v_2^2|) \times \left(\frac{1}{D'(E, -\phi)} - \frac{1}{D'(E, \phi)} \right) [2f(E) - 1] dE. \quad (\text{A15})$$

The factor $2f(E) - 1 = -\tanh(E/2kT)$ accounts for the energy averaging of states above with those below the Fermi level, suppressing the Josephson current.

The poles of the transmission amplitude t [Eq. (A7)], namely,

$$1 = \left(\frac{v_1 v_2}{u_1 u_2} \right) e^{i(\tilde{k}_e - \tilde{k}_h)L} e^{-i\phi} \quad (\text{A16})$$

define the bound and quasibound energy levels. Since

$$\frac{v_1 v_2}{u_1 u_2} = e^{-i[\cos^{-1}(E/\Delta_1) + \cos^{-1}(E/\Delta_2)]} \quad (\text{A17})$$

and $\tilde{k}_e - \tilde{k}_h \simeq |E|/\Delta_1 \xi_1$, combining Eqs. (A16) and (A17) yields the "-" branch of Eq. (8). Obtaining the transmission amplitude for a holelike excitation, we would find the "+" branch of Eq. (8). Thus, Eq. (8) determines the Andreev energy levels.

7.8 Appendix B: Continuum current $I_2(\phi)$

We calculate the supercurrent $I_2(\phi)$ flowing in the energy range $\Delta_1 < E < \Delta_2$ in this appendix. The wave function for the injected electronlike quasiparticle is still given by Eq. (A4), and the wave function in the normal metal by Eq. (A5). However, in the right contact, we take the transmitted electronlike quasiparticle wave function to be

$$\frac{t}{\sqrt{|u_2|^2 + |v_2|^2}} \begin{pmatrix} u_2 e^{i\phi_2} \\ v_2 \end{pmatrix} e^{ik_{e2}(x-L)} \quad (x > L). \quad (\text{B1})$$

We have inserted a normalization factor $1/\sqrt{|u_2|^2 + |v_2|^2}$.

The electrical current operator is

$$J_Q(x) = e \left(\frac{\hbar \tilde{k}_e}{m} \right) T_{L \rightarrow R}^e(E, \phi), \quad 0 < x < L \quad (\text{B2})$$

where $T_{L \rightarrow R}^e(E, \phi) = tt^*$ is the current transmission amplitude for electronlike quasiparticles. Matching the boundary conditions in Andreev's approximation, we obtain

$$T_{L \rightarrow R}^e(E, \phi) = \frac{2(u_1^2 - v_1^2)^2}{D(E, -\phi)} \quad (\text{B3})$$

$D(E, -\phi)$ is given by Eq. (19). To obtain Eq. (B3), we have used the property that $|u_2|^2 = |v_2|^2 = u_2 v_2$ in the energy range $|E| < \Delta_2$. In this energy range, the coherence factors u_2 and v_2 become complex, namely,

$$2u_2^2 = 1 + i \frac{\sqrt{\Delta_2^2 - E^2}}{E}, \quad (\text{B4})$$

and

$$2v_2^2 = 1 - i \frac{\sqrt{\Delta_2^2 - E^2}}{E} \quad (\text{B5})$$

The component of the electrical current carried by the electronlike quasiparticles, $I_2^e(\phi)$, is then

$$\begin{aligned} I_2^e(\phi) = & \frac{e}{h} \left(\int_{\Delta_1}^{\Delta_2} - \int_{-\Delta_2}^{-\Delta_1} \right) \\ & \times 2 |u_1^2 - v_1^2| \left(\frac{1}{D(E, -\phi)} \right) f(E) dE \end{aligned} \quad (\text{B6})$$

A similar calculation for the holelike current component $I_2^h(\phi)$ gives

$$\begin{aligned} I_2^h(\phi) = & -\frac{e}{h} \left(\int_{\Delta_1}^{\Delta_2} - \int_{-\Delta_2}^{-\Delta_1} \right) \\ & \times 2 |u_1^2 - v_1^2| \left(\frac{1}{D(E, \phi)} \right) f(E) dE. \end{aligned} \quad (\text{B7})$$

Adding Eqs. (B6) and (B7) to form $I_2(\phi) = I_2^e(\phi) + I_2^h(\phi)$ thus gives Eqs. (18) and (19) of Sec. II. As in Appendix A, the electronlike component $I_2^e(\phi)$ and holelike component $I_2^h(\phi)$ of the current do not individually balance when $\phi = 0$, yet the total current balances so that $I_2(\phi = 0) = 0$. Writing Eq. (18) solely over the positive energy levels gives

$$\begin{aligned} I_2(\phi) = & \frac{2e}{h} \int_{\Delta_1}^{\Delta_2} |u_1^2 - v_1^2| \\ & \times \left(\frac{1}{D(E, -\phi)} - \frac{1}{D(E, \phi)} \right) [2f(E) - 1] dE. \end{aligned} \quad (\text{B8})$$

We wish to evaluate Eq. (18) in the limit of a very asymmetric point-contact junction at zero temperature. Taking $L = 0$ and restricting the integral over energy to only those occupied states where $E < 0$, Eq. (18) becomes

$$I_2(\phi) = \frac{4e}{h} \Delta_1 \sin(\phi) R(\phi) \quad (\text{B9})$$

where $R(\phi)$ is

$$R(\phi) = \int_{-\Delta_2}^{-\Delta_1} \frac{\sqrt{(\Delta_2^2 - E^2)(E^2 - \Delta_1^2)}}{E^2 [1 + (\Delta_1/\Delta_2)^2 - 2(\Delta_1/\Delta_2) \cos(\phi)] - \Delta_1^2 \sin^2(\phi)} \frac{dE}{\Delta_2}. \quad (\text{B10})$$

For large asymmetry, we neglect terms with $(\Delta_1/\Delta_2) \ll 1$ in the denominator of Eq. (B10). Further, we are interested in the maximum value of $R(\phi)$, which occurs at $\phi = \pi/2$. Therefore,

$$R_{\max} \equiv R(\phi = \pi/2) \simeq \int_{-\Delta_2}^{-\Delta_1} \frac{\sqrt{(\Delta_2^2 - E^2)} dE}{\sqrt{E^2 - \Delta_1^2} \Delta_2}. \quad (\text{B11})$$

Since $E^2 \gg \Delta_1^2$ for most of the integration range, we neglect the Δ_1^2 term in the denominator of Eq. (B11). This approximation will slightly underestimate the value of $R_{\max}$. Doing the integral in Eq. (B11) in this approximation, we find

$$R_{\max} \geq \ln \left(\frac{2\Delta_2}{\Delta_1} \right) - 1 \quad (\text{B12})$$

leading to Eq. (24).

7.9 Appendix C: Discrete current $I_1(\phi)$

The needed wave functions to calculate the discrete current $I_1(\phi)$ are

$$\frac{1}{a\sqrt{|u_1|^2 + |v_1|^2}} \begin{pmatrix} v_1 e^{i\phi_1} \\ u_1 \end{pmatrix} e^{ik_{h1}x} \quad (x < 0) \quad (\text{C1})$$

$$\text{and} \quad \frac{1}{b\sqrt{|u_2|^2 + |v_2|^2}} \begin{pmatrix} u_2 e^{i\phi_2} \\ v_2 \end{pmatrix} e^{ik_{e2}(x-L)} \quad (x > L).$$

Eq. (A5) still applies in the normal metal ($0 < x < L$). The a and b are complex numbers. Matching the boundary conditions at $x = 0$ and $x = L$, we find

$$|a|^2 = |b|^2 \quad (\text{C3})$$

The electrical current is

$$J_Q = e \left(\frac{\hbar \tilde{k}_e}{m} \right) \frac{1}{|a|^2} \quad (\text{C4})$$

To find $|a|^2$, we use the normalization condition

$$\int \{|u(x)|^2 + |v(x)|^2\} dx = 1. \quad (\text{C5})$$

The integrals inside the superconductor are simplified using the approximation

$$k_{e1,2} \simeq k_F + i \frac{1}{2\xi_{1,2}(E)} \quad (\text{C6})$$

and

$$k_{h1,2} \simeq k_F - i \frac{1}{2\xi_{1,2}(E)} \quad (\text{C7})$$

Carrying out the integration in Eq. (C5) gives

$$\frac{1}{|a|^2} [L + \xi_1 (E_n^-(\phi)) + \xi_2 (E_n^-(\phi))] = 1 \quad (\text{C8})$$

so that

$$I_n^-(\phi) = \frac{ev_F}{L + \xi_1(E_n^-(\phi)) + \xi_2(E_n^-(\phi))} \quad (C9)$$

which is half of Eq. (10). The same calculation for the " + " branch give the other half of Eq. (10). The same current given in Eqs. (9) and (10) can also be found using the Helmholtz free energy,^{5,7,19} since

$$I_n^\pm(\phi) = \frac{2e}{\hbar} \frac{dE_n^\pm(\phi)}{d\phi} \quad (C10)$$

is the same current as given in Eq. (10).

7.10 Appendix D: Equivalence Of The Two Current Operators

References 10 and 11 argue for the electrical current operator

$$J_Q(x) = e \sum_n f_n \{J_{u_n}(x) + J_{v_n}(x)\}, \quad (D1)$$

which we have used in Appendixes A-C. However Refs. 4, 5, and 9 use the slightly different electrical current operator

$$J'_Q(x) = e \sum_n \{f_n J_{u_n}(x) - (1 - f_n) J_{v_n}(x)\} \quad (D2)$$

The electrical current $J'_Q(x)$ from Eq. (D2) and $J_Q(x)$ from Eq. (D1) are the same if

$$\sum_n J_{v_n} = 0. \quad (D3)$$

Equation (D3) can be divided into a summation over the " + " energy levels and "-" levels as

$$\sum_n^+ J_{v_n} + \sum_n^- J_{v_n} = 0 \quad (D4)$$

We show in this appendix that Eq. (D4) holds in each energy range ($|E| < \Delta_1$, $\Delta_1 < |E| < \Delta_2$, and $|E| > \Delta_2$), so that either $J_Q(x)$ or $J'_Q(x)$ can be used to compute the Josephson current in this problem. We caution that we have no general proof for Eq. (D3), only that it holds for our particular choice of pairing potential $\Delta(x)$ and electrostatic potential $V(x)$.

In the energy region $|E| < \Delta_1$, the summation over the "-" states in Eq. (8) is

$$\sum_n^- J_{v_n} = \sum_n \frac{ev_F/2}{L + \xi_1(E_n^-(\phi)) + \xi_2(E_n^-(\phi))} \quad (D5)$$

Similarly, we sum the J_{v_n} over the " + " states in Eq. (8) to find

$$\sum_n^+ J_{v_n} = \sum_n \frac{-ev_F/2}{L + \xi_1(E_n^+(\phi)) + \xi_2(E_n^+(\phi))} \quad (D6)$$

We use the identity $-E_n^+(\phi) = E_p^-(\phi)$, where p is some integer and the property $\xi(E) = \xi(-E)$. With these substitutions, Eq. (D6) becomes

$$\sum_n^+ J_{v_n} = \sum_p \frac{-ev_F/2}{L + \xi_1 (E_p^-(\phi)) + \xi_2 (E_p^-(\phi))} \quad (D7)$$

which is simply $-\sum_n^- J_{v_n}$. Therefore $(\sum_n^- + \sum_n^+) J_{v_n} = 0$.

In the energy region $\Delta_1 < |E| < \Delta_2$, the summation of J_{v_n} over the injected particles for energies $E > 0$ (" + ") is

$$\begin{aligned} \sum_n^+ J_{v_n} &= \frac{e}{h} \int_{\Delta_1}^{\Delta_2} |u_1^2 - v_1^2| \left(\frac{\Delta_2}{2|E|} \right) \\ &\quad \times \left(\frac{1}{D(E, -\phi)} - \frac{1}{D(E, \phi)} \right) dE \end{aligned} \quad (D8)$$

while the summation over all injected particles having energies $E < 0$ ("-") is

$$\begin{aligned} \sum_n^- J_{v_n} &= \frac{e}{h} \int_{\Delta_1}^{\Delta_2} |u_1^2 - v_1^2| \left(\frac{\Delta_2}{2|E|} \right) \\ &\quad \times \left(\frac{1}{D(E, \phi)} - \frac{1}{D(E, -\phi)} \right) dE \end{aligned} \quad (D9)$$

Adding Eqs. (D8) and (D9) gives zero.

In the energy region $|E| > \Delta_2$, we must account for quasiparticles injection from both the left and the right superconducting reservoirs. The positive energy (" + ") states give

$$\begin{aligned} \sum_n^+ J_{v_n} &= \frac{e}{h} \left(\int_{\Delta_1}^{\infty} \frac{v_2^2 (u_1^2 - v_1^2) + u_1^2 (u_2^2 - v_2^2)}{D(E, -\phi)} dE \right. \\ &\quad \left. - \int_{\Delta_1}^{\infty} \frac{v_1^2 (u_2^2 - v_2^2) + u_2^2 (u_1^2 - v_1^2)}{D(E, \phi)} dE \right) \end{aligned} \quad (D10)$$

while the negative energy ("-") states produce

$$\begin{aligned} \sum_n^- J_{v_n} &= \frac{e}{h} \left(\int_{\Delta_1}^{\infty} \frac{v_2^2 (u_1^2 - v_1^2) + u_1^2 (u_2^2 - v_2^2)}{D(E, \phi)} dE \right. \\ &\quad \left. - \int_{\Delta_1}^{\infty} \frac{v_1^2 (u_2^2 - v_2^2) + u_2^2 (u_1^2 - v_1^2)}{D(E, -\phi)} dE \right) \end{aligned} \quad (D11)$$

Adding Eqs. (D10) and (D11) again gives zero, completing the proof of Eq. (D3).

¹⁸ In this calculation the mass is $m = 0.067m_e$ (where m_e is the free-electron mass) and the Fermi energy is $\mu = 10$ meV. Figures 2, 3(a), and 7(a) use $L = 0, \Delta_1 = 0.2$ meV, $\Delta_2 = 0.3$ meV, so that $\xi_1 = 3770\text{\AA}$, and $\xi_2 = 2513\text{\AA}$. Figures 3(b) and 7(b) use $L = 0, \Delta_1 = 0.001$ meV, $\Delta_2 = 0.2$ meV, so that $\xi_1 = 754000\text{\AA}$ and $\xi_2 = 3770\text{\AA}$. Figures 5, 6(a), and 8(a) use the same parameters as Fig. 3(a), except that $L = 3.0\mu\text{ m}$. Figures 6(b) and 8(b) use the same parameters as Fig. 3(b), except that $L = 300.0\mu\text{ m}$. Figure 4 uses $\Delta_1 = 0.001$ meV and varies Δ_2 . The long junction in Fig. 4 takes $L = 225.0\mu\text{ m} = 3\xi_1$.

8 Superconducting weak links by Likharev

Some readers may feel that the review is more theoretical than experimental. But the author would like to assert that only relatively simple mathematical tools have been employed throughout. A knowledge thereof is essential for any experimentalist working in such a field as weak superconductivity, where a large number of new unexpected phenomena have been and probably will be discovered and where really good rather than "satisfactory" agreement between theory and experiment is the rule. The review presupposes, on the part of the reader, a knowledge of the basic principles of superconductivity, as laid down, say, in the monograph by De Gennes (1966), or in the excellent textbook by Tinkham (1975).

The review proper begins with Sec. II, which outlines the main types of junctions where the Josephson effect is observed—tunnel junctions and weak links. The specific properties of tunnel junctions are reviewed, and their advantages and disadvantages from the viewpoint of applications. Then a qualitative analysis is made of how the various types of weak links differ from tunnel junctions and from one another.

Section III is devoted to the most explored class of processes in superconducting weak links—stationary (dc) effects in which a constant current I , less than the critical current I_C , flows through the junction. Under these conditions, there is no voltage across the junction, nor does any normal current flow through it. Even within the framework of these processes, it is possible to follow the transition from the Josephson effect to ordinary superconductivity as the length of the weak link increases. For this purpose, a simple model of weak links, the "One-Dimensional Structure with Electrodes in Equilibrium" (ODSEE) model, is introduced and widely used in further discussion.

In Sec. IV we consider in more detail the specific features of various real types of weak links paying particular attention to the extent to which weak links correspond to the ODSEE model. The applications of weak links in various devices based on the Josephson effect are briefly touched upon here. This analysis shows that essentially the same types of weak links are best suited both for physical investigations and for applications.

The next sections (Sec. V and VI) are devoted to the less widely studied nonstationary (ac) processes. Section V is concerned with situations that are the simplest from an experimental standpoint, where a constant current greater than the critical one flows through the weak link. Section VI deals with some effects which are more complicated for experimental investigation but oftentimes more useful for understanding the physical nature of the processes.

All the foregoing pertains to weak links interconnecting two electrodes. In recent years, weakly linked systems consisting of several electrodes have been investigated intensively. Examples of such systems are a single weak link that simultaneously connects more than two electrodes, or a system of several weak links connected in series or in parallel or in any other complex manner. Radically new phenomena are likely to occur in such weak link systems. These are analyzed in Sec. VII, which also deals with distributed ("wide") Josephson junctions, junctions which may be regarded as a parallel connection of a large number of lumped weak links; the importance of these weak link systems for practical applications is discussed at the end of the section.

In conclusion, Sec. VIII outlines the present status of our knowledge of superconducting weak links and suggests the most urgent problems for further investigation.

8.1 II. JOSEPHSON EFFECT AND WEAK LINKS

8.1.1 A. Josephson effect

1. General description

The Josephson effect is one of the macroscopic quantum phenomena of superconductivity. In it the current through a weak contact between two relatively bulky superconducting electrodes may contain a component I_S (supercurrent), which is dependent not on the voltage V across the electrodes but on the phase difference

$$\varphi = \chi_2 - \chi_1, \quad (1)$$

where χ_1 and χ_2 are the phases of the order parameter, Δ , in the electrodes

$$\Delta_{1,2} = |\Delta_{1,2}| \exp(i\chi_{1,2}). \quad (2)$$

In the simplest ("classical") case, the relationship $I_s(\varphi)$ is sinusoidal

$$I_S = I_C \sin \varphi, \quad (3)$$

where I_C is called the supercurrent amplitude or the critical current.

The phase φ is related to the voltage across the electrodes V by the following expression

$$\frac{d\varphi}{dt} = \frac{2e}{\hbar} v. \quad (4)$$

There is a significant difference between Eqs. (3) and (4) which qualitatively describe the Josephson effect. Equation (3) is an approximate one, and various kinds of deviations of supercurrent from this dependence may be observed in a superconducting weak link of any type. Moreover, the constant I_C involved in Eq. (3) is essentially dependent on the geometry and material of the weak link, the electrode material, temperature, and other factors.

In contrast, Eq. (4) is derived solely from the main principles of quantum mechanics and contains only fundamental constants. That this relation holds to a high degree of accuracy has been experimentally demonstrated (see, for instance, Clarke, 1968; Dan Bracken and Hamilton, 1972; Macfarlane, 1973). This does not mean, certainly, that Eq. (4) is universally applicable. If V is defined as an electrochemical potential difference measured by an ordinary voltmeter, then certain corrections, say, due to rotation of the superconductors (Zimmerman and Mercereau, 1965) have to be introduced into Eq. (4). Besides, Eq. (4) is violated when the equilibrium stationary state of the electrodes is disturbed by the flow of a normal current component; this flow may occur in the neighborhood of a normal specimen (Reiger et al., 1971) or due to heat flow (Ginzburg, 1944; Zavaritskii, 1974; Aronov and Gal'perin, 1974. See also Aronov and Spivak, 1975; Volkov and Zaitzev,

1976; Zaitzev, 1976). We shall disregard these effects and assume Eq. (4) to be exact, which is the case in most experiments on weak links. Thus our task is reduced to establishing a relationship between the current flowing through the weak link and the function $\psi(t)$.

2. dc and ac effects

All the electrodynamic phenomena taking place at the Josephson contacts are generally divided into stationary (dc) and nonstationary (ac) effects, depending on whether the variables, including the phase difference φ , change with time or not.

If the phase ψ remains constant (dc effects), as directly follows from Eq. (4), the voltage across the junction is zero. At the same time, a nonzero supercurrent (3) can flow through the junction. In the dc case, this current must be constant in time and less than the critical current

$$V = 0, \varphi(t) = \text{const}, I = I_S(\varphi), |I| \leq I_C. \quad (5)$$

Among the nonstationary (ac) effects occurring when the phase changes with time, the most important are Josephson oscillations which inevitably take place if the voltage V at the contact has a dc component $\bar{V}$. As follows from Eq. (4), in this case the phase φ contains a component that increases linearly with time at a rate

$$\omega_V = \frac{\overline{d\varphi}}{dt} = \frac{2e}{\hbar} \bar{V}, f_V = \frac{\omega_V}{2\pi} = \frac{2e}{h} \bar{V}. \quad (6)$$

The supercurrent, therefore, oscillates in time at a frequency ω_V directly proportional to $\bar{V}$. The proportionality factor ($f_V/\bar{V} \approx 500\text{MHz}/\mu\text{V}$) is rather high; consequently, Josephson junctions are highly sensitive to the electromagnetic field.

In ac processes, not only the supercurrent I_s , but other current components can flow through the Josephson junction. Moreover, the supercurrent may depend not only on the instantaneous value of the phase difference, but also on the preceding variations of φ . Therefore the ac effects are always far more complicated than the dc effects. Thus we shall begin our discussion with the dc case.

3. Tunnel junctions and weak links

In the history of prediction and discovery, as well as at the initial stages (from 1962 to 1967 or so) of Josephson effect research, a prominent role was played by tunnel junctions in which two superconducting electrodes are separated by a thin insulating layer [Fig. 1(a)]. Finite conductivity is exhibited in such structures only due to tunneling of electrons through the potential barrier created by an insulator. A finite supercurrent may flow in them owing to the correlated tunneling of the electrons forming a pair. These are the tunnel junctions for which the Josephson effect was theoretically predicted (Josephson, 1962) and its main consequences were observed experimentally: the flow of supercurrent without any voltage drop (Anderson and Rowell, 1963) and the specific magnetic field dependence of the supercurrent (Rowell, 1963). Josephson oscillations were detected both indirectly (Shapiro, 1963; Giaever, 1965) and directly (Yanson et al., 1965) with the help of the same

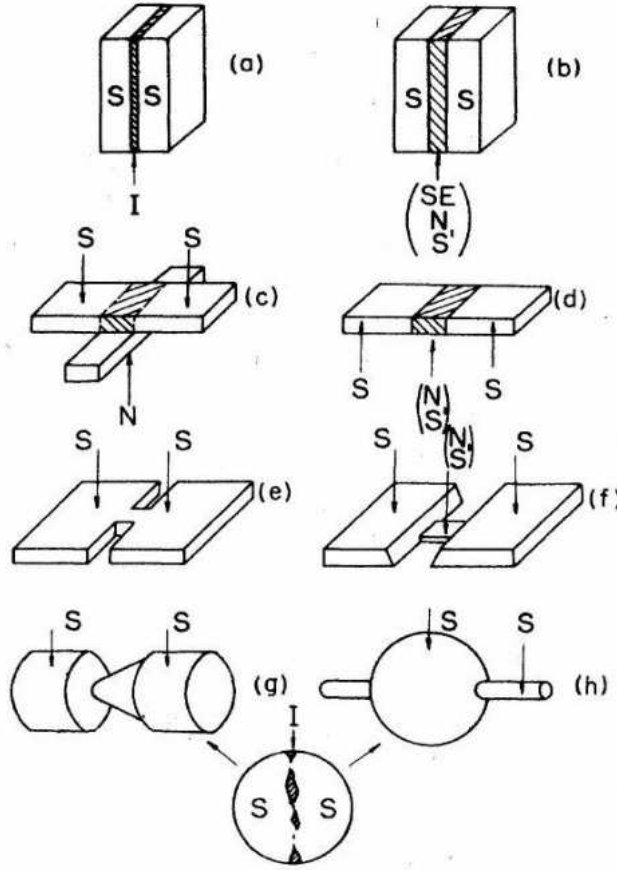


FIG. 1. Different types of structures where the Josephson effect can take place. (a) tunnel junction, i.e., $S-I-S$ sandwich. All others are different weak links, i.e., structures with direct (non-tunnel-type) conductivity; (b)-sandwich; (c)-proximity effect bridge; (d)-ion-implanted bridge; (e)-Dayem bridge; (f)-variable-thickness bridge; (g)-point contact; (h)-blob-type junction. The conducting regions of the two last types of weak links are shown schematically in the circle on the bottom. S stands for superconductor, S' for the superconductor with reduced critical parameters, N for normal metal or alloy, SE for semiconductor (usually highly doped), I stands for insulator. Thin-film structures are shown without the substrate.

The outstanding role of tunnel junctions has greatly affected further research in the field. Some workers were led to believe that this effect could take place only in tunnel junctions, although the Josephson effect was observed in structures with non-tunnel conductivity as early as 1964 (Anderson and Dayem, 1964; Lambe et al., 1964), and by 1966 all the main processes associated with this effect had already been detected in these structures (weak links) (Zimmerman and Silver, 1964; 1966a, b; Grimes et al., 1966; Dayem and Grimes, 1966). Overestimation of the significance of tunnel junctions has led to deplorable consequences. Firstly, considerable attention has been given to complex and specific processes in tunnel structures which are of minor theoretical and practical importance. Secondly, after the main effects had been experimentally observed and a fairly satisfactory theory of the Josephson effect had been developed (Werthamer, 1966; Larkin and Ovchinnikov, 1966), some investigators got the wrong impression that "everything is quite clear concerning the Josephson effect," and some prominent workers were thus unfortunately misled to abandon further research in the field.⁴

Therefore, although the use of weak links for practical purposes was begun in 1966-67, the physics of weak links has only become a subject of intensive study since 1974. So far these

⁰⁴ This is reflected in the literature too. Let alone the early surveys, in the existing monographs (Kulik and Yanson, 1970; Solymar, 1972) and even in some recent reviews on the Josephson effect, attention is focused on tunnel junctions.

investigations have produced more questions than answers, and the most significant progress achieved to date is, probably, the recognition of the fact that the Josephson effect is up to now far from being a clearly understood phenomenon, and that tunnel junctions are merely a particular case of Josephson junctions.

This review is devoted to weak links with non-tunnel conductivity; and yet we shall begin with a brief survey of the well known properties of tunnel junctions. This is done, first, for the sake of comparison and, second, because there are cases where the properties of weak links are similar to those of tunnel junctions.

8.1.2 B. Tunnel junctions

1. Basic equations

In most cases the properties of tunnel junctions are adequately described by the existing theory,⁵ which is based on two facts:

(i) The thickness of the energy barrier (insulating layer) is of the order of 10^{-7} cm, and is negligibly small compared to all other characteristic lengths in the superconductor, in particular, the mean free path of electrons ℓ . This means that the barrier may be and is described integrally, without introducing the coordinate dependences of any superconductivity parameters (e.g., Δ) inside the barrier.

(ii) The barrier transparency factor T is in most cases so small that the critical current through the junction (usually of order $1 - 10^2$ A/cm²) is far less than the critical current of the electrodes ($10^5 - 10^7$ A/cm²). This makes it possible to calculate the characteristics of the junctions using perturbation theory with respect to the small parameter $T \ll 1$, i.e., to disregard disturbances in the electrodes' state caused by tunnel current.

These facts greatly simplify theoretical analysis; as a result of which as far back as 1963 a complete theory was formulated for the dc case ($V = 0$) (Ambegaokar and Baratoff, 1963), and in 1966 a theory for arbitrary $V(t)$ was completed (Werthamer, 1966; Larkin and Ovchinnikov, 1966). A discussion of the theory can be found in the monograph by Kulik and Yanson (1970) or in the report by Poulsen (1973); here we shall quote the main results only.

Consider a junction of such small area that the phase difference and the voltage across the junction are independent of the coordinates in the plane of the barrier, and are interrelated by Eq. (4). Let the voltage contain components at a frequency ω and its harmonics, and possibly a frequency of the Josephson oscillations ω_V [Eq. (6)]. In this case $\exp(i\varphi/2)$ can be expressed in the form of a Fourier series

$$\exp(i\varphi/2) = \sum_n A_n \exp[i(\omega_V/2 + n\omega)t] \quad (7)$$

⁰⁵An exception is the problem of the sign of the "interference" and the "fourth" components of the current (see Sec. V).

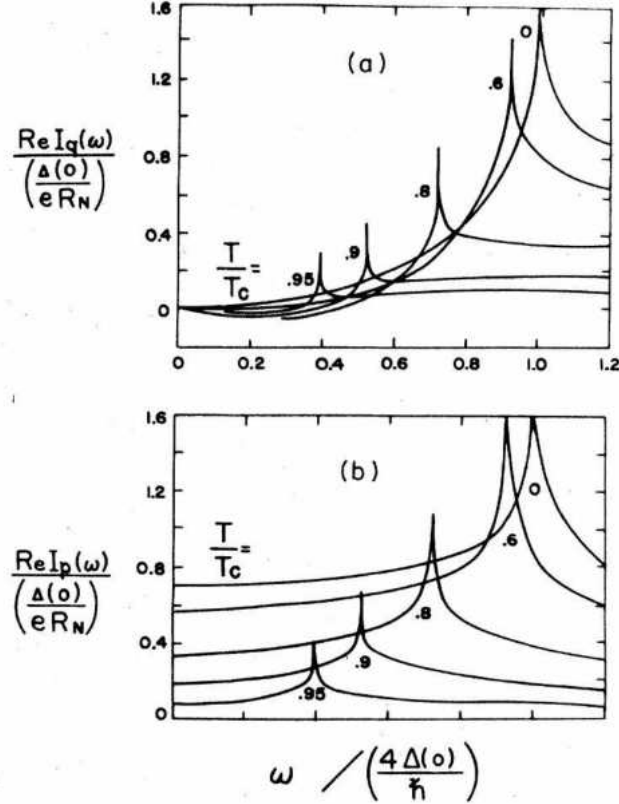


FIG. 2. The amplitudes of supercurrent (I_p) and normal electron (quasiparticle) current (I_q), which are used in tunnel theory [Eq. (8)], for different temperatures T/T_C is the critical temperature of the electrodes, which are taken to be identical. $\Delta(T)$ is the order parameter modulus in the electrodes. R_N is the normal junction resistance. Real parts of amplitudes are even, and imaginary parts are odd functions of the argument.

In this case the current through the junction is of the form (Poulsen, 1973):

$$I(t) = \text{Im} \left\{ \exp(i\varphi/2) \sum_n A_n^* \exp[-i(\omega_V/2 + n\omega)t] I_q(\omega_V + 2n\omega) + A_n \exp[i(\omega_V/2 + n\omega)t] I_p(\omega_V + 2n\omega) \right\} \quad (8)$$

where I_p and I_q are complex functions which describe the currents of Cooper pairs and unpaired electrons, respectively.⁶ These functions are shown in Fig. 2 for the case where the electrodes are made of the same material. Some curves for electrodes with different critical temperatures T_C are plotted in the paper of Harris (1974).

2. Basic properties of tunnel junctions

For the dc (zero-voltage) case, we have to change $\omega_V t \rightarrow \varphi$ in Eqs. (7) and (8); thus we get

$$I = I_C \sin \varphi, I_C = \text{Re } I_p(0) \quad (9)$$

which shows that the $I_S(\varphi)$ relationship is always sinusoidal for tunnel junctions. The critical current is given by $\text{Re}_p(0)$ which, for identical electrodes, is

⁶ As the properties of the unpaired electrons in a superconductor differ slightly from their properties in normal metals, they are generally referred to as "electron-like quasiparticles," "electron-like excitations," and others. For the sake of simplicity, however, we shall use the term "electrons" everywhere, bearing in mind the distinction mentioned above.

$$I_C = V_C/k_N, V_C = \frac{\pi}{2} \frac{\Delta(T)}{e} \tanh \frac{\Delta(T)}{2k_B T} \quad (10)$$

where $\Delta(T)$ is the equilibrium value of the energy gap, i.e., the modulus of Δ in electrodes, and R_N is the resistance of the junction in a normal state. An important result is that the product $I_C R_N$, which is called the characteristic voltage of the junction V_C , does not depend on any parameters of the barrier, and is determined solely by the operating (T) and critical (T_C) temperatures through the well known function $\Delta(T) = k_B T_C f(T/T_C)$. We recall here that in the microscopic theory (Bardeen et al., 1957) the quantity $\Delta(T)$ is almost constant at low temperatures

$$\Delta(T) \approx \Delta(0) \left\{ 1 - \left(\frac{2\pi k_B T}{\Delta(0)} \right) \exp \left[-\frac{\Delta(0)}{k_B T} \right] \right\}, \text{ at } T \lesssim 0.6 T_C \quad (11)$$

$$\Delta(0) = (\pi/\gamma) k_B T_C, \gamma \approx 1.78$$

and falls off rapidly as $T \rightarrow T_C$

$$\Delta^2(T) \approx \frac{8\pi^2}{7\zeta(3)} k_B^2 T_C (T_C - T), \text{ at } T_C - T \lesssim 0.1 T_C, \quad (12)$$

$$\zeta(3) \approx 1.202$$

Therefore the critical temperature determines the maximum possible value for the junction voltage ($T \rightarrow 0$)

$$\max_T (eV_C) = \frac{\pi}{2} \Delta(0) = \frac{\pi^2}{2\gamma} k_B T_C, \max_T (V_C) [\mu V] \approx 240 T_C [K] \quad (13)$$

Near T_C , the characteristic voltage is a linear function of temperature with a universal slope

$$V_C = \frac{\pi}{4} \frac{\Delta^2}{e k_B T} = \alpha (T_C - T), \quad \alpha = \frac{2\pi^3 k_B}{7\zeta(3)e} \approx 635 \frac{\mu V}{K} \quad (14)$$

If the junction voltage is nonzero, then other components also contribute to the current. For example, for constant $V = \bar{V}$, Eq. (8) directly yields

$$I = \text{Re } I_p(\omega_V) \sin \omega_V t + \text{Im } I_p(\omega_V) \cos \omega_V t + \text{Im } I_q(\omega_V) \quad (15)$$

which shows that the current through the junction is the sum of the oscillating Josephson supercurrent with an amplitude $I_p(\omega_V)$ and a constant normal current $\text{Im } I_q(\omega_V)$.

It is obvious from Fig. 2 that at low voltages [$e\bar{V} \leq 2\Delta(T)$] the amplitude of the current oscillations is practically constant and equal to the critical current. At a frequency $\omega_V = 4\Delta/\hbar$, i.e., at a voltage

$$\bar{V} = 2\Delta(T)/e \quad (16)$$

the amplitude has a weak (logarithmic) singularity (Riedel, 1964), and the electron current has a step of finite height. This behavior results from the following process: an electron pair from one electrode breaks and one of the electrons passes to another electrode, absorbing an electrostatic energy eV . Equation (16) follows precisely from the condition that this energy should be equal to the pair binding energy 2Δ .

At higher voltages, the electron current $\text{Im } I_q$ tends to assume its value in a normal junction, while the amplitude of the supercurrent oscillations falls off approximately as ω_V^{-1} . If the

contact junction voltage changes with time, a fourth component $\text{Re } I_q$ also contributes to the current.

A large number of phenomena occurring at tunnel junctions may be described by Eqs. (7) and (8) if the real geometry of the junction, its capacitance, possible phase variations in its plane, and several other factors are taken into account. These effects do not fall within the scope of our review since most of them do not take place in weak links.

3. Capacitance and current density

As regards the use of tunnel junctions as Josephson elements in various applications, an analysis of the dynamics of devices based on such elements shows the following. Almost all tunnel junctions prepared by existing techniques have parameters which are quite suitable for practical applications with the important exception of the large intrinsic capacitance of the junction C , a parasitic factor for almost all applications (see, for example, the analysis of Likharev and Ulrich, 1978).

As follows from circuit analysis, the criterion for the negligibly small influence of the capacitance on junction processes is the smallness (compared with unity) of the following dimensionless parameter ⁷

$$\beta = \omega_C R_N C = (2e/\hbar) I_C R_N^2 C, \quad (17)$$

where ω_C is the characteristic frequency of the junction, which is related to its characteristic voltage V_C by the Josephson relation (4)

$$\omega_C = \frac{2e}{\hbar} V_C = \frac{2e}{\hbar} I_C R_N \quad (18)$$

It is clear from Eq. (17) that β is independent of the junction area S , and that β shows whether or not the capacitance shunts the conductance at frequencies $\sim \omega_C$.

In a typical Josephson junction the density of the critical current is $\sim 10^2$ A/cm², and the critical temperature of the electrodes is about 10 K, while the electrical thickness of the insulating layer t/ϵ is $\sim 1\text{\AA}$. Hence the following estimates of the parameters

$$C/S = \epsilon/4\pi t \approx 10^{-5} \text{ F/cm}^2, V_C \approx 2 \times 10^{-3} \text{ V}, \omega_C \approx 6 \times 10^{12} \text{ s}^{-1},$$

$$R_N S = V_C/j_C \approx 2 \times 10^{-5} \Omega \times \text{cm}^2, \beta \approx 10^3 \gg 1.$$

Thus, in a typical tunnel junction, the capacitance effectively shunts the tunnel conductance. This situation has an adverse effect not only on microwave devices based on the Josephson effect, but also on such applications as magnetometry or computer devices, because the $I - V$ curve, under the effect of capacitance (see, for

example, Likharev and Ulrich, 1978), becomes hysteretic.

From Eq. (17) it is clear that β can be reduced in two different ways: either by increasing the current density j_C (decreasing R_N) or by increasing the electrical thickness of the barrier t/ϵ , the current density being kept the same. Each of these ways gives rise to its own difficulties in the technology of fabrication of tunnel junctions.

Since the publication of Giaver's classic papers (1960), tunnel junctions are usually prepared between two thin superconducting films as follows: first the lower film is evaporated, then it is oxidized to form an insulating layer, and finally the upper superconducting layer is deposited. Even relatively small current densities ($\sim 10^2$ A/cm²) are attained with an oxide layer as thin as $10^\circ - 20\text{\AA}$ at which, it is, indeed, a problem to provide layer uniformity and stability with respect to diffusion. In recent years considerable progress has been made in the technology of fabrication of stable oxide layers, using glow discharge oxidation and suitable electrode materials (Schroen, 1968; Greiner, 1971, 1974; Lahiri, 1976). However, only in a few cases has a current

⁰⁷ For applications at very high frequencies ($\omega \gg \omega_C$) β should be small as compared to $\omega_c/\omega \ll 1$.

density of $\sim 10^4$ A/cm² been obtained (see, for instance, Broom et al., 1975), β being still as large as several tens.⁸

Another method is to increase the thickness of the potential barrier, keeping the current density at the same level (or even increasing it) by using a semiconductor interlayer which provides a lower energy barrier height (see the review by Cardinne et al., 1974, and the papers by Barone et al., 1974; Barone and Russo, 1974; Simmonds, 1974). Recently, junctions having a current density $\sim 10^4$ A/cm² and $\beta \sim 1$ have been successfully prepared, using silicon single-crystal membranes ~ 1000 Å thick as an interlayer (Huang and Van Duzer, 1974, 1975; Schyfter et al., 1977). It should, however, be noted that this rather impressive result was obtained with the help of a semiconductor highly doped with boron to a concentration of $\sim 10^{20}$ cm⁻³, and hence practically degenerate. The conductivity of such material is of metallic type even at helium temperature. Thus these junctions, which will be discussed in detail below, are one of the types of weak links, namely, $S - N - S$ sandwiches.

As a result of the above-mentioned difficulties in manufacturing tunnel junctions with low capacitance, Josephson weak links, i.e., structures with direct (non-tunnel-type) conductivity, are now used for most applications. The most important exception is in digital device applications (see Footnote 2 for references) where some important development programs are based on the use of tunnel junctions. However, large junction capacitances create some hard problems here as well. The personal point of view of the author is that the real advantage of tunnel junctions in this field is the somewhat more advanced technology of their fabrication, and that the situation can change with time.

8.1.3 C. Weak links

1. Main types of weak links

The current in weak links, unlike that in tunnel junctions, flows along a conducting, either normal or superconducting, material. The relative weakness of the link, in other words, the low value of its critical current, is attained differently in different structures whose main types are shown in Fig. 1(b)-(h).

If a layer of normal metal, usually 0.1 to 10 μ m thick, is interposed between two superconductors, we obtain an " $S - N - S$ sandwich" junction [Fig. 1(b)]. A finite supercurrent can flow through such a junction due to the well known proximity effect (see, for example, De Gennes, 1966). The effect lies in the fact that if a normal metal and a superconducting metal are brought into proper electric contact, some Cooper pairs will penetrate into the normal metal from the superconductor. Thus in the normal metal there arises a nonzero order parameter Δ , which exponentially decreases within the metal over a distance of the order of "normal coherence length" or "decay length" ξ_N . On the other hand, the values of Δ in the superconductor over distances of the order of coherence length ξ from the boundary will become less than the equilibrium value. Hence, if the thickness of the normal interlayer is not very large, the order parameter will be different from zero throughout the normal metal, and a finite supercurrent may flow through the interlayer.

As a normal interlayer we may use either a "true normal" metal which does not suffer superconducting transition at any temperature (hereafter denoted N), or a superconductor having a critical temperature $0 < T_C < T$ (denoted by N'). Moreover, a sandwich material may also be superconducting ($T_C > T$), in which case the weakness of the link is to be provided by keeping its critical current small compared to that of the electrode material S . In this case the junction is called the $S - S' - S$ sandwich.

⁸ Recently a report has been published (Niemeyer and Kose, 1976) on the successful production of junctions with $j_C \approx 2 \times 10^5$ A/cm² and $\beta \sim 1$. Nonetheless, so far it is not quite clear whether those junctions possess tunnel-type conductivity.

A situation closely resembling the sandwich phenomenon can be created in proximity effect bridges⁹ [Fig. 1(c)] or in ion-implanted bridges [Fig. 1(d)]. In both cases a section of low critical current, usually of about a few microns long, is made in a narrow strip of superconducting film (about $10^3 \text{ \AA}$ thick). In the first case this section is formed by the proximity effect with a normal metal underlayer, and in the second case by the implantation of ions into this region (or, in some rare cases, into the main bulk of the film, i.e., into the electrodes). In practice, the only distinction between these structures and sandwiches is that their cross section can be made much smaller because one of the dimensions is formed by thin-film evaporation.

An entirely different method of forming weak links is realized in a constant-thickness bridge¹⁰ [Fig. 1(e)] or in a variable-thickness bridge [Fig. 1(f)]. Low critical current results here from the fact that the width (in the first case) or the width and the thickness (in the second case) of the "span" are far less than those of the electrodes ("banks"). The typical thickness of the bank film

is again $10^3 \text{ \AA}$, while the thickness of the span may be several hundred angstroms. The span length is kept as low as possible, say, a few tenths of a micron.

Of course, both of these methods of producing weak links may be used in combination. For instance, in the bridges with current concentration [Fig. 1(e), (f)], the span and the electrodes may be made of different materials (N, N' , or S'). From a practical viewpoint this is more natural in the more sophisticated variable-thickness bridges, because incorporation of a second material would greatly complicate fabrication of Dayem bridges.

The geometry of weak links is more complicated in the two other types of Josephson junctions, i.e., in point contacts [Fig. 1(g)] and in blob-type junctions¹¹ [Fig. 1(h)]. In the first case the electrical contact is created by weak touching of the two superconducting electrodes, and in the second case by forming a pellet of solder (superconducting at low temperatures) over a length of a superconducting wire.

When the pressure between the electrodes in a point contact increases or when the metal in a blob-type junction solidifies, the oxide layer, which inevitably coats the electrodes, is usually interrupted at many places, thereby forming several metallic shorts between electrodes (Fig. 1, sketch in circle). The typical crosssection area (10^{-13} to 10^{-10} cm^2) of such shorts is generally of the order of the square of the thickness of the oxide layer ($\sim 10^{-6} \text{ cm}$), and is much less than the area of mechanical contact between the electrodes. Thus the link weakness in this case is also attained due to the strong current concentration.¹²

Such weak links, especially point contacts, have found wide application due to the simplicity of their fabrication. A significant disadvantage of these structures as subjects for physical research is their poorly defined geometry. Their demerit in practical applications is their irreproducibility.

2. Length and effective length

Any weak link, in fact, is a partition between two superconducting electrodes. It is natural, therefore, that its basic geometrical dimension is its length L . By length we mean the electrode spacing, in other words, the dimension of the weak link in the direction of current flow. In sandwiches, for instance, this length [Fig. 1(a), (b)] is merely the thickness of an interlayer, while in ion-implanted bridges (Fig. 1d) this is the length of a section with changed properties.

However, it is an effective length $L_{\text{eff}} \geq L$ rather than the geometric length that plays a significant role in the processes under discussion. By effective length is meant that length within which the nonlinear effects in a weak link are localized. The point is that nonlinear electrodynamic processes almost inevitably extend somehow or other into the adjacent sections

⁹ These are sometimes termed "Notarys-Mercereau bridges".

¹⁰ These are generally known as "Dayem bridges."

of the electrodes, if only because of the proximity effect: variations in the

order parameter in weak links caused by the nonlinear effects lead to changes in $|\Delta|$ in the banks. Of course, the degree of involvement of the electrodes in these nonlinear processes depends strongly on the type of weak link. The lower the conductivity of the material and the greater the current concentration in a weak link, the lesser is the degree of involvement (and consequently, the closer the L_{eff} is to L) (for details, see Sec. IV).

Lack of distinct localization of the nonlinear effects in some structures is a serious drawback from the viewpoint of both physical research and practical applications. From the first standpoint, these effects introduce a noticeable amount of indeterminacy into the geometry of weak links, i.e., they complicate theoretical analysis and interpretation of experimental results. From the second standpoint, extension of the nonlinear effects to the banks prevents realization of weak links whose effective length is less than the coherence length, and, as we shall demonstrate below, the Josephson effect is observed precisely in structures with $L_{\text{eff}} \lesssim \xi$. For this reason, in studying the relationship between the properties of weak links and their length, we shall focus our attention mainly on those weak links for which $L \approx L_{\text{eff}}$ and only in Sec. IV shall we briefly discuss the consequences of the absence of distinct localization of nonlinear effects.

3. Classification of weak links

A reasonable classification of weak links is based on a comparison of the effective length L_{eff} with the characteristic lengths of the material forming the weak link, the coherence length ξ and the mean free path ℓ being most important among them.¹³

Junctions with $L_{\text{eff}} \ll \xi$ will be called "short" to distinguish them from "long" weak links for which $L_{\text{eff}} \gtrsim \xi$. We shall demonstrate that it is in short weak links where the "ideal" Josephson effect (see Sec. III for definition of the term) is observed, and an increase in this effective length causes considerable deviation from the ideality. For this reason we shall center our attention on the case $L_{\text{eff}} \ll \xi$.

Even in short weak links, however, the processes may differ depending on the mean free path ℓ . From experimental and practical viewpoints, the bridge-type weak links [Fig. 1 (c)-(f)] with exactly known geometry, sufficient normal resistance (say, $R_N \gtrsim 0.1\Omega$), and the possibility of reproducible fabrication are of particular importance. As a rule, the mean free path ℓ in bridges is limited by diffuse scattering on the film surface to a length on the order of a few hundred angstroms, which is far less than L_{eff} (from one tenth of a micron to a few microns). We shall mainly consider such "dirty" structures for which

$$\ell \ll L_{\text{eff}} \quad (20)$$

although at the end of Sec. III we shall briefly touch upon "clean" structures with a large mean free path $\ell \gtrsim L_{\text{eff}}$.

The terms "dirty" and "clean" for weak links are not to be confused with the well known dirty and clean limits in the general theory of superconductivity, where they are used to denote the relation between values of ℓ and the BCS coherence length ξ_0 .

⁰¹¹ Sometimes they are referred to as Clarke blobs.

¹² When the resistance of point contact is fairly high ($R_N \gtrsim 10^2\Omega$) the tunnel current through the oxide layer may be quite significant.

⁰¹³ A third important length for superconductors, the magnetic field penetration depth λ , which determines the force of current self-limitation (Meissner effect), is not so significant for weak links, and is, generally, compared only with the width of the structure W , i.e., with the characteristic size of the cross section.

8.2 III. STATIONARY (DC) EFFECTS

First we shall examine stationary phenomena in weak links where the phase difference ψ of the order parameter in the electrodes is constant in time. In this case a constant supercurrent I_S less than I_C flows through the weak link, and there is no electric field or normal electron current, these electrons being in equilibrium. The stationary theory of superconductivity may be applied, and a relationship between the supercurrent I_s and the phase difference ψ has been found for many important cases.

8.2.1 A. Aslamazov-Larkin theory

The basic principles of how the Josephson effect takes place in weak links with strong current concentration have been explained by Aslamazov and Larkin (1969), referred to below as AL. We shall, therefore, examine their theory in greater detail, although it is directly applicable only under rather stringent assumptions with regard to the geometry and the material of weak links.

1. Ginsburg-Landau equations

Let the temperature T be sufficiently close to the electrode critical temperature T_C . Then the modulus of the order parameter at every point in the structure is relatively small ($|\Delta| \ll k_B T_C$), so that the simple and convenient Ginsburg-Landau (GL) equations (1950) may be applied (see also De Gennes, 1966; Tinkham, 1975). These equations may be expressed as

$$\xi^2 \left(\nabla - i \frac{2e}{\hbar c} \mathbf{A} \right)^2 \Delta + \left[\pm 1 - \frac{|\Delta|^2}{\Delta_0^2} \right] \Delta = 0, \quad (21a)$$

$$\mathbf{j}_s = C_j \text{Im} \left[\Delta^* \left(\nabla - i \frac{2e}{\hbar c} \mathbf{A} \right) \Delta \right] \quad (21b)$$

where $\mathbf{A}$ is the vector potential, and the coherence length is given by the following expression in the dirty limit ($\ell \ll \xi_0$):

$$\xi^2(T) = \frac{\pi \hbar D}{8 k_B (T_C - T)}, \quad \xi(T) \approx 0.852 (\xi_0 \ell)^{1/2} \left(\frac{|T_C - T|}{T} \right)^{-1/2} \quad (22)$$

where $D = v_F \ell / 3$ is the diffusion coefficient, and ξ_0 is the size of the Cooper pair

$$\xi_0 = \frac{\hbar v_F}{\pi \Delta(0)} \approx 0.180 \frac{\hbar v_F}{k_B T_C}. \quad (23)$$

Δ_0 stands for some characteristic value of $|\Delta|$, which for $T < T_C$ is equal to the equilibrium value (12). For $T > T_C$ (GL equations hold true for this case as well) we may again use Eq. (12) for Δ_0 , taking the modulus of $(T_C - T)$; here also we take the minus sign before unity in Eq. (21a). The coefficient C_j in the dirty limit is proportional to normal conductivity σ_N

$$C_j = \frac{c^2 \hbar}{8 \pi e \lambda^2 \Delta_0^2} = \frac{\pi}{4} \frac{\sigma_N}{e k_B T}, \quad \sigma_N = 2 e^2 N(0) D, \quad (24)$$

where $N(0)$ is the density of states on the Fermi surface.

Equation (21) follows from the microscopic theory of superconductivity under the following restrictions in addition to the requirement that $|\Delta|$ be small (Gor'kov, 1959; see also, De

Gennes, 1966).

- (i) The order parameter varies slowly over the range of the self-consistency radius, equal approximately to $(\xi_0 \ell)^{1/2}$ in the dirty limit.
- (ii) The current density varies slowly over the range of the radius of integral coupling between the field and the current; it is of the order of ℓ in the dirty limit.

We shall use GL equations to describe a weak link of characteristic size L_{eff} inside which both Δ and j_S undergo significant changes in the general case, and therefore, the condition of the dirty weak link.

$$\ell \ll L_{\text{eff}} \quad (25)$$

should necessarily be satisfied.

2. Linearization of the equation for Δ

Aslamazov and Larkin considered a case in which, although condition (25) holds, nevertheless the length of the weak link is sufficiently small both compared with ξ

$$L_{\text{eff}} \ll \xi(T), \quad (26)$$

and compared with magnetic field penetration depth

$$L_{\text{eff}} \ll \lambda(T) \quad (27)$$

By virtue of Eq. (26), we can discard all terms except the first ("gradient") term in Eq. (21a). Indeed, the sum of all the discarded terms is not greater than Δ , while the gradient term is of the order of $\Delta(\xi/L)^2$. Inequality (27) shows that the influence of self-shielding, which is responsible for the nonuniform distribution of current over the cross section, is insignificant in the weak link. Formally, this inequality makes it possible to neglect the contribution made by the weak link current to the vector potential $\mathbf{A}$ which in principle has to be determined from the Maxwell equation. The remaining part ($\mathbf{A}_0$) is associated with the remote field sources ($r \gg L_{\text{eff}}$), and it can be represented in the form of a gradient of some scalar function.¹⁴ Since $\mathbf{A}$ is always chosen up to the gradient of an arbitrary function, we may take the gauge to be

$$\mathbf{A} = \mathbf{A}_0 = 0. \quad (28)$$

This means that the magnetic field influence is accounted for by the order parameter phase, which is in this case gauge invariant, i.e., a uniquely defined quantity. It is precisely under this gauge that we examined the Josephson effect in Sec. II.

Hence, under the constraints (26) and (27), Eq. (21a) is reduced to a simple linear Laplace equation

$$\nabla^2 \Delta = 0 \quad (29)$$

which makes it possible to solve the problem for an arbitrary weak link geometry. It has to be solved under the following boundary conditions. The order parameter inside the electrodes should tend to the following values

$$\Delta = \begin{cases} \Delta_1 \exp(i\chi_1) & \text{for electrode 1} \\ \Delta_2 \exp(i\chi_2) & \text{for electrode 2} \end{cases} \quad (30)$$

¹⁴ The only exception is the case where the external magnetic fields are strong: $B \sim \phi_0/L_{\text{eff}}$, where $\phi_0 = ch/2e$ is the magnetic flux quantum. But, by virtue of Eq. (26), these fields are far stronger than the field $H_{C2} = \phi_0/2\pi\xi^2$ which suppresses the superconductivity in the electrodes.

The current through the surface σ with the outer space (vacuum or an insulator) should be zero. Hence, by virtue of (21b), we obtain the condition

$$\partial\Delta/\partial n|_{\sigma} = 0, \quad (31)$$

where $\mathbf{n}$ is normal to the surface.

3. Josephson effect as the interference of wave functions

Aslamazov and Larkin pointed out that the (unique) solution of the problem may be expressed as

$$\Delta = \Delta_1 \exp(i\chi_1) f + \Delta_2 \exp(i\chi_2) (1 - f), \quad (32)$$

where $f(\mathbf{r})$ is a real function of the coordinates satisfying the following boundary-value problem

$$\begin{aligned} \nabla^2 f &= 0, \partial f / \partial n|_{\sigma} = 0, \\ f &\begin{cases} 1 & \text{for electrode 1,} \\ 0 & \text{for electrode 2.} \end{cases} \end{aligned} \quad (33)$$

Indeed, by substituting Eq. (32) into Eqs. (29)-(31), we find that this solution satisfies both the equation and the boundary conditions. For the time being, supposing that the function f is known, and after substituting Eq. (32) into Eq. (21b), since $A = 0$, we obtain

$$\mathbf{j}_s(r) = C_j \nabla f(r) \Delta_1 \Delta_2 \sin \varphi \quad (34)$$

According to (33), the function f does not depend on φ ; the current at each point of the weak link is, therefore, proportional to $\sin \varphi$. Consequently the total current is also proportional to $\sin \varphi$. Thus we have already arrived at an important result: if the conditions of the AL theory are satisfied, then the Josephson effect occurs in weak links, i.e., the supercurrent and the phase are governed by Eq. (3).

Now we return to Eq. (32), which gives the distribution of the order parameter in the bulk of the weak link. It shows that inside the weak link ($f \neq 0, 1$) the order parameter is a linear superposition of two terms, each of which is proportional to Δ at one of the electrodes and to the coordinate factor. The latter gradually diminishes as we recede from the electrode into the depths of the weak link. Thus inside the weak link there occurs simple interference of two wave functions whose sources are the condensates of Cooper pairs in the superconducting electrodes. Such interference gives rise to current [Eq. (34)] sinusoidally dependent on the phase difference of the sources; consequently, it is the direct cause of the Josephson effect. So, the AL theory not only demonstrates the existence of the Josephson effect in weak links, but also gives one a clear picture of its physical origin.

4. Magnitude of the critical current

It is impossible to find an analytical expression for the function f for an arbitrarily shaped weak link. To find the critical current I_C (and the characteristic voltage V_C), however, there is no need to determine f . To demonstrate this statement, we shall find the current flowing through the same contact in a normal state (for example, at $T > T_C$). From the obvious relationships $\mathbf{j}_N = \sigma_N \mathbf{E}$ and $\text{div } \mathbf{j}_N = 0$, and the definition of the scalar potential ($E = -\nabla\mu$), again we arrive at the Laplace equation.

$$\nabla^2 \mu = 0 \quad (35)$$

with the boundary conditions

$$\begin{aligned} \mu &= \begin{cases} \mu_1 & \text{for electrode 1,} \\ \mu_2 & \text{for electrode 2,} \end{cases} \\ \partial\mu/\partial n|_\sigma &= 0, \mu_1 - \mu_2 = V \end{aligned} \quad (36)$$

For this problem we can easily express the unique solution as follows

$$\mu = \mu_1 f + \mu_2 (1 - f), \quad (37)$$

where the function $f(\mathbf{r})$ is the same as in the case of Eq. (32). On calculating the current, we find that

$$\mathbf{j}_N = \sigma_N \mathbf{E} = -\sigma_N \nabla \mu = \sigma_N V \nabla f(\mathbf{r}). \quad (38)$$

By comparing Eqs. (34) and (38) we find that both in the normal and in the superconducting states the current is distributed identically over the bulk of the structure ($\mathbf{j} \propto \nabla f$). The total currents I_S and I_N , therefore, are in the same relation as their densities

$$\frac{I_S}{I_N} = \frac{j_S}{j_N} = \frac{C_j \Delta_1 \Delta_2 \sin \varphi}{\sigma_N V}. \quad (39)$$

But the total current in the normal state is evidently equal to V/R_N , and thus from Eq. (39) we have

$$I_S = I_C \sin \varphi, \quad I_C = \frac{C_j \Delta_1 \Delta_2}{\sigma_N R_N} \quad (40)$$

For the real case of the dirty limit ($\ell \ll \xi_0$) with the help of Eq. (24) we obtain an especially simple expression¹⁵

$$V_C = I_C R_N = (\pi/4) \Delta_1 \Delta_2 / e k_B T \quad (41)$$

For electrodes made of identical material ($\Delta_1 = \Delta_2 = \Delta$), on comparing Eqs. (41) and (14) we find that as $T \rightarrow T_C$, the characteristic voltages V_C for small-size weak links and tunnel junctions are identical, hence the derivative $\alpha = |dV_C/dT|$ is again equal to 635μ V/K

[Eq. (14)].

In deriving Eq. (41) it should be noted that nowhere were the weak link and the electrodes assumed to be of the same material. Hence the characteristic voltage V_C at the "ideal" Josephson effect is not dependent on the material of the weak link. The material only affects the normal resistance R_N of the contact and consequently the critical current, which is inversely proportional to the resistance.

8.2.2 B. Arbitrary temperatures

The fact that the expressions for $I_S(\psi)$ and V_C in small-size weak links are the same as the corresponding expressions for tunnel junctions when $T \rightarrow T_C$ suggests that the dc Josephson effect may also take place in an identical manner at arbitrary temperatures in these two types of structures. Kulik and Omelyanchuk (1975) in an article hereafter referred to as K0-1, have, however, demonstrated the contrary.

⁰¹⁵ The factor of 4 in the denominator was missing in the original AL paper.

1. General properties of the current-phase relationship

Here we may mention that the phase dependence of the supercurrent has certain general properties which do not depend on the geometry of weak links or the equations that describe them.

Firstly, a change of 2π in the phase χ of the order parameter in any of the electrodes does not change the physical state of the system, since

$$\Delta(\mathbf{r}, t) = |\Delta|e^{i\chi} = |\Delta|e^{i(\chi+2\pi)} \quad (42)$$

Therefore the function $I_S(\varphi)$ is always 2π periodic

$$I_S(\varphi + 2\pi) = I_S(\varphi) \quad (43)$$

Secondly, supercurrent may flow only when there is a gradient of the order parameter phase. Therefore there are no currents across the weak links in the two situations depicted in Fig. 3 (b) and (c). In the first case, the phase shift is zero, while in the second case it is π . As a result, the curve $I_S(\varphi)$ intersects the horizontal axis at $\varphi = \pi n$:

$$I_S(\pi n) = 0 \quad (44)$$

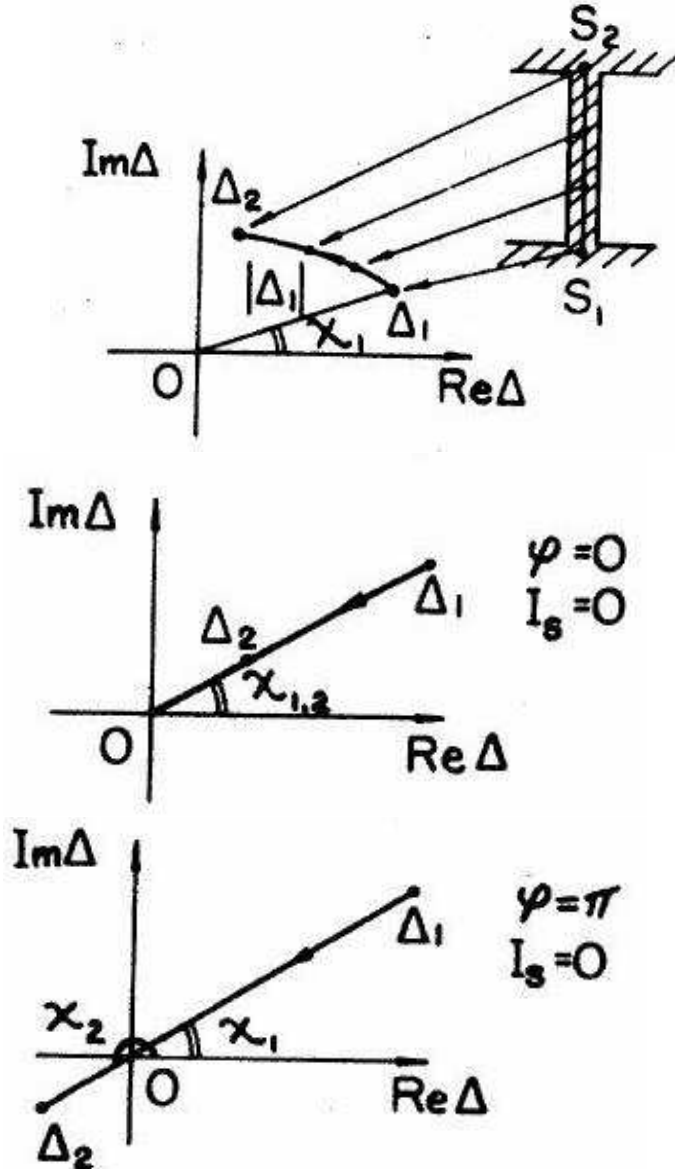
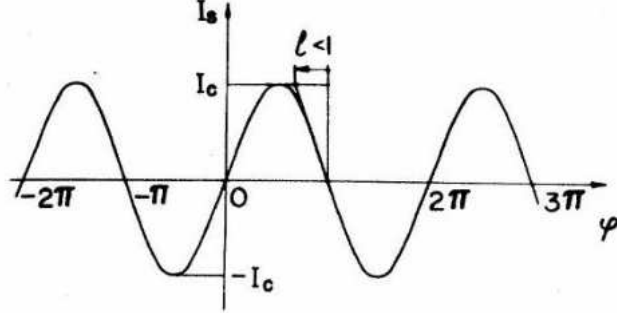
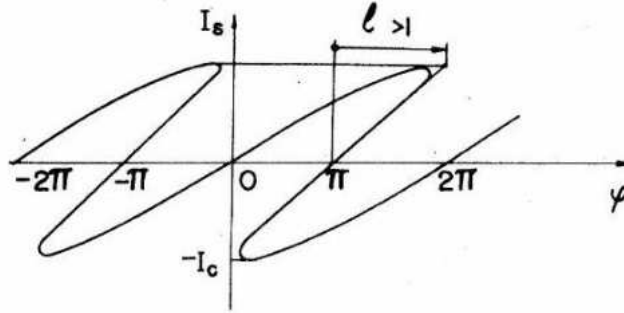


FIG. 3. The representation of process in a weak link on the phase plane of the order parameter Δ (a). Each point on the plane shows a value of Δ in the definite point of the weak link, and finite points of trajectory correspond to the states the electrodes. Two possible types of phase trajectories corresponding to absence of supercurrent through weak link are shown below: (b) $\varphi = 0 + 2\pi n$; (c) $\varphi = \pi + 2\pi n$.



(a)



(b)

FIG. 4. Possible $I_S(\varphi)$ relationships in weak links (schematically). Function $I_S(\varphi)$ is always odd, and its plot passes through the points $[I_S = 0, \varphi = \pi n]$. This function can be either single-valued (a), or multivalued (b). In the first case the parameter $l = I_C (dI_S/d\varphi)_{\varphi=\pi}^{-1}$ of the kinetic inductance is less than unity, and in the second case l is greater than unity.

Moreover, in the superconductor the opposite currents are equivalent;¹⁶ therefore, the function $I_S(\varphi)$ is symmetrical with respect to the origin

$$I_S(-\varphi) = -I_S(\varphi) \quad (45)$$

Typical examples of $I_S(\varphi)$ functions are schematically shown in Fig. 4. In order to plot the whole curve it suffices to determine only one of its segments, say, the segment joining the points $\varphi = 0$ and $\varphi = \pi, I_S = 0$.

It is convenient to characterize the relationship $I_S(\varphi)$ in terms of two constants, namely, the critical current

$$I_C = \max [I_S(\varphi)], \quad (46)$$

and a dimensionless parameter

$$l = I_C \left(\frac{dI_S}{d\varphi} \right)_{\varphi=\pi}^{-1} + 1 \quad (47)$$

whose geometrical meaning is evident from Fig. 4. For a sinusoidal $I_S(\varphi)$, $l = 0$; in the case where the curve is inclined towards the right, l increases and becomes more than unity if the function $I_S(\varphi)$ becomes multivalued [Fig. 4(b)].

The quantity l has a simple physical meaning. A function $I_S(\varphi)$ of the type shown in Fig. 4(b) may roughly be described by the expressions

$$\psi = \psi_1 + \frac{2e}{\hbar c^2} I_S \mathcal{L}(I_S), \quad I_S \approx I_C \sin \varphi_1. \quad (48)$$

By virtue of the Josephson relation (4), Eq. (48) shows that a weak link may be represented in the form of series connection of a "classical" Josephson junction (i.e., junction with $I_S \propto \sin \varphi$) and a nonlinear inductance $\mathcal{L}(I_S)$. Inductance $\mathcal{L}$ is due not to the magnetic field of the current I_S (which is eliminated as we have taken $A = 0$), but to the kinetic energy of the superconducting condensate.

It is quite evident that this "kinetic" inductance (Sass and Stewart, 1968; Meservey and Tedrow, 1969; Likharev, 1971a) should be particularly great in a very long weak link ($\mathcal{L} \propto L$). Substituting Eq. (48) into Eq. (47), we get

$$\ell = \frac{2e}{\hbar c^2} I_C \mathcal{L}(0) \quad (49)$$

Therefore ℓ has the meaning of kinetic inductance for a small current, normalized in the manner usual for Josephson junctions (see, for example, Likharev and Ulrich, 1978).

2. Usadel equations

Calculations of the stationary properties of weak links at arbitrary temperatures have become possible primarily because of some previous success in the general theory of superconductivity. Eilenberger (1968) has shown that the highly complicated general equations of stationary superconductivity (Gor'kov, 1958, 1959; see also Abrikosov et al., 1962) may be expressed in a fairly simple form. Usadel (1970) reduced the Eilenberger equations for the dirty limit ($\ell \ll \xi_0$) to even simpler equations for the complex functions F which only depend on the point ($\mathbf{r}$) and the "Matsubara frequency"

$$\hbar\omega = \pi k_B T (2n + 1) \quad (50)$$

The Usadel equations for F may be expressed in a very simple form if another variable $G = (1 - FF^*)^{1/2}$ is used

$$2\omega F - D(\nabla - i\frac{2e}{\hbar c}\mathbf{A})[G(\nabla - i\frac{2e}{\hbar c}\mathbf{A})F - F\nabla G] = 2\hbar^{-1}\Delta G.$$

All the measurable quantities are derived from the functions F by summing up with respect to ω , i.e., with respect to the integer n (Eq. 50). In particular, the expressions for the order parameter $\Delta(\mathbf{r})$ and the density $j_s(\mathbf{r})$ of the supercurrent are of the form

$$\ln(T'_c/T) = \Delta \ln(T_C/T) = 2\pi k_B T \sum_{\omega>0} [(\Delta/\hbar\omega) - F] \quad (52)$$

$$\mathbf{j}_s = \sigma_N (2\pi k_B T/e) \sum_{\omega>0} \text{Im} \left[F^* \left(\nabla - i\frac{2e}{\hbar c} \mathbf{A} \right) F \right] \quad (53)$$

From these expressions it is clear that the use of the Usadel equations is only slightly more complicated (involves one more additional summation) than the use of GL equations (21). Thus, for example, for a uniform specimen with no current ($\nabla = \mathbf{A} = 0$), from (51) we get

$$\Delta = |\Delta|, F = \Delta / [(\bar{n}\omega)^2 + \Delta^2]^{1/2} \quad (54)$$

⁰¹⁶ Strictly speaking, only in the absence of a strong (see footnote 14) magnetic field produced by the current.

and the self-consistency equation for Δ (52) takes the form:

$$2\pi k_B T \sum_{\omega>0} \left[\frac{1}{\hbar\omega} - \frac{1}{[(\hbar\omega)^2 + \Delta^2]^{1/2}} \right] = \ln \frac{T_C}{T} \quad (55)$$

Its solution gives the well known function $\Delta(T)$ and the asymptotes (11) and (12), in particular.

For $T \approx T_C$, Eqs. (50) and (53) may be transformed into GL equations (21). In so doing, we must recall that $|F| \rightarrow 0$, $|\Delta| \rightarrow 0$, and $G \approx 1 - FF^*/2$ as $T \rightarrow T_C$. The space scale of Δ variations has become relatively large: $\nabla \sim \xi^{-1}(T) \ll (\hbar D/\omega)^{-1/2}$, and therefore the gradient term in (51) is small. Hence from Eq. (51) we get

$$\frac{\Delta}{\hbar\omega} - F \approx -\frac{\hbar D}{2(\hbar\omega)^2} \left(\nabla - i\frac{2e}{\hbar c} \mathbf{A} \right)^2 \Delta + \frac{\Delta|\Delta|^2}{2(\hbar\omega)^3} \quad (56)$$

After substituting this expression into Eqs. (52) and (53) and summing up with respect to ω :

$$\sum_{\omega>0} (\hbar\omega)^{-2} = \frac{\pi^2}{2(2\pi k_B T)^2}, \quad \sum_{\omega>0} (\hbar\omega)^{-3} = \frac{7\zeta(3)}{(2\pi k_B T)^3} \quad (57)$$

we obtain GL equations (21) with parameters $\xi(T)$ and C_j corresponding to the dirty limit.

3. Kulik-Omelyanchuk (KO-1) theory

If a weak link is sufficiently small, the gradient term alone can be retained in Eq. (51), as was assumed in the AL theory with GL equations. Kulik and Omelyanchuk (1975) have thus simplified it for the case of one-dimensional geometry [$F = F(x)$], but we shall find the solution for the arbitrary shape of a weak link. Let us introduce (following M. Yu. Kupriyanov) the function $\phi(r, \omega)$ in such a way that

$$F = \phi [(\hbar\omega)^2 + \phi\phi^*]^{-1/2}, \quad G = \hbar\omega [(\hbar\omega)^2 + \phi\phi^*]^{-1/2} \quad (58)$$

Equating the gradient term in (51) to zero means in terms of ϕ

$$\nabla (G^2 \nabla \phi) = 0 \quad (59)$$

According to Eqs. (54) and (58), the value of ϕ in the electrodes is equal to Δ . Therefore the boundary conditions for ϕ are the same as for Δ in the AL theory [Eqs. (30) and (31)].

Now we find by direct substitution that for $\Delta_1 = \Delta_2 = \Delta$ the solution of our problem is

$$\phi = \exp [(\chi_1 + \chi_2)/2] \left\{ \Delta \cos \frac{\varphi}{2} + i\delta \tan \times \left[(1 - 2f) \arctan \frac{\Delta \sin(\varphi/2)}{\delta} \right] \right\} \quad (60)$$

$$\delta^2 = \Delta^2 \cos^2(\varphi/2) + (\hbar\omega)^2,$$

where f is the same as in the AL theory [Eq. (33)]. After calculating the current density $\mathbf{j}_s$ by Eq. (53), and then comparing it with the current density in the normal state (38), we obtain the expression for the current

$$I_S(\varphi) R_N = \frac{2\pi k_B T}{e} \sum_{\omega>0} \frac{2\Delta \cos(\varphi/2)}{\delta} \arctan \frac{\Delta \sin(\varphi/2)}{\delta} \quad (61)$$

For $T \rightarrow T_C$, this function is reduced to Eq. (40) of the AL theory, i.e., $I_S(\varphi)$ is a sinusoidal function, while V_C depends linearly on $(T_C - T)$ (14). But at lower temperatures the function is somewhat distorted (Fig. 5). In particular, the derivative $(dI_S/d\varphi)_\pi$ tends logarithmically to infinity as $T \rightarrow 0$. The characteristic voltage in the KO-1 theory depends on temperature as shown in Fig. 6, and attains its maximum as $T \rightarrow 0$

$$\max_T [eV_C] \approx 2.07\Delta(0), \max_T V_C [\mu V] \approx 316T_C [K] \quad (62)$$

which is 32% greater than the maximum value of V_c for

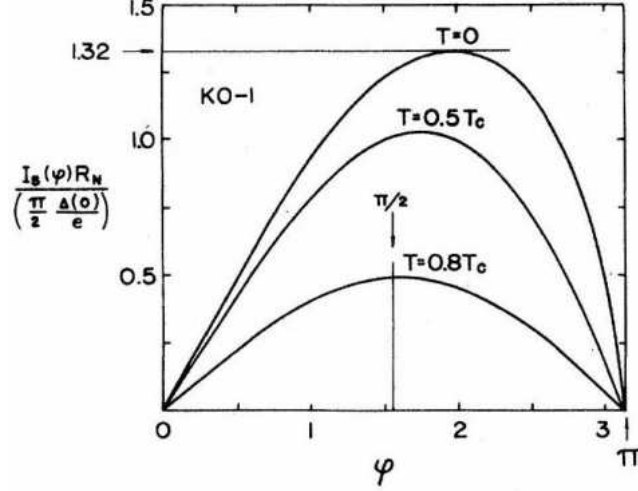


FIG. 5. $I_S(\varphi)$ relationship for a short dirty weak link ($l \ll L \ll (l\xi_0)^{1/2}$) according to the KO-1 theory at various temperatures. Because of the 2π -periodicity of the $I_S(\varphi)$ function it is shown only on the $[0, \pi]$ segment. Temperature dependence of critical current is shown in Fig. 6, and $l(T)$ is shown in Fig. 9 (b) by the line labeled $L/\xi_N = 0$. tunnel junctions [Eq. (13)].

The fact that even the simplest dc properties of smallsize weak links differ from those of tunnel junctions clearly shows the significant distinction between these two structures. If the mean free path of electrons ℓ is much shorter than the effective weak link length L_{eff} , as is assumed in the KO-1 theory, then electrons are strongly scattered inside a weak link and are sensitive to the "field" (Δ) distribution inside the weak link. As a result, a self-consistent distribution of the order parameter $^{17}\Delta$ is established inside the link. In the case of tunnel structures, however, scattering of the electrons inside the insulating layer is small, and it is impossible to introduce local values of the order parameter inside the layer. This is the difference in the conductivity mechanism that is responsible for the specific behavior of supercurrent at lower temperatures.

8.2.3 C. Effect of finite length

Now we shall examine the effects of increased length of the weak link, assuming that its cross section as before is negligibly small. For such an analysis we have to use a particular weak link model.

1. One-dimensional structure with electrodes in equilibrium (ODSEE) model

The simplest and the most natural model for the weak link is the "One-Dimensional Structure with Electrodes in Equilibrium" (ODSEE) model in which the following two conditions are assumed to be valid:

- (i) The cross section S of a weak link is small ($\ll \lambda^2$) and constant in the direction of current flow throughout

¹⁷ In the KO-1 theory the order parameter is equal to

$$\Delta(r) = \Delta \exp [i (\chi_1 + \chi_2) / 2] \left\{ \cos \frac{\varphi}{2} + i [1 - 2f(x)] \sin \frac{\varphi}{2} \right\}$$

only if the critical temperature T'_c of the weak link material is not very close to zero: $T'_c/T_c \gtrsim \xi_d/L^2$.

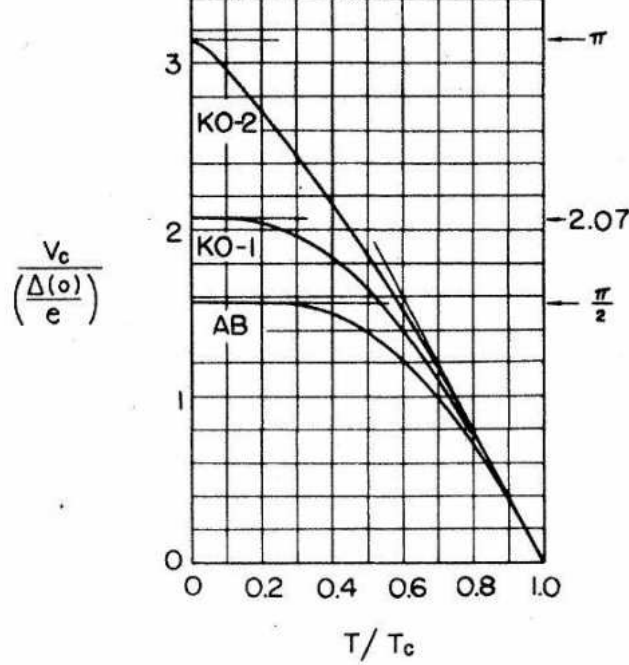


FIG. 6. Temperature dependences of critical current for different small weak links. (AB) tunnel theory (Ambegaokar and Baratoff, 1963); (KO-1) dirty case ($l \ll L < \xi$) (Kulik and Omelyanchuk, 1975); (KO-2) clean case ($L \ll l, \xi$) (Kulik and Omelyanchuk, 1977). the length L of the structure. As a result, all the variables may be taken to be constants along the cross section, i.e., they are functions of the coordinate x in the current direction alone

$$f = f(x), \text{ at } 0 < x < L. \quad (63)$$

(ii) Nonlinear processes are localized over the length L , i.e., the modulus of the order parameter at the boundary of the weak link is equal to its equilibrium value in the electrodes ("banks")

$$|\Delta(0)| = \Delta_1, |\Delta(L)| = \Delta_2. \quad (64)$$

In other words, the effective link length L_{eff} is equal to its geometrical length L .

The first of these conditions, as can be seen from Fig. 1, is satisfied in all weak links except for Dayem bridges [Fig. 1(e)] and for structures with random geometry, say, point contacts and blob-type junctions.

As regards the second condition, we may note that it is a rather more stringent one. It is easier to satisfy this condition in structures with strong current concentration, say, in variable-thickness bridges [Fig. 1(f)]. In Sec. VI we shall show that for any material of the bridge and banks (electrodes) this condition holds true if the span film thickness is sufficiently small. In contrast, for the second condition to hold true in structures without current concentration [Fig. 1(b)-(d)], certain significant conditions must be satisfied with regard to the link and electrode materials. Roughly speaking, it is necessary that the normal conductivity σ_N of the link material be small compared to the normal conductivity of the electrodes (see Sec. VI. A). Therefore results derived within the framework of the ODSEE model have limited application for structures without strong current concentration.

The ODSEE model has, in fact, been applied in many theoretical works without specifically mentioning the conditions under which the model holds valid. This has led to a number of errors in the interpretation of experimental data, although the quantitative results obtained with the help of this model are valid, of course, for any weak link if L_{eff} is substituted for L .

2. Temperatures close to the critical temperature

We shall begin with a case for which the GL equations (21) are applicable (Likharev and Yakobson, 1975a). This naturally restricts us to a region in which the critical temperatures of both the weak link material (T'_c) and the electrode material (T_c) are close to the physical temperature and, consequently, to each other

$$(T_c - T), |T'_c - T| \ll T \quad (65)$$

The sign of $(T'_c - T)$ indicates whether the bridge material is superconducting ($S', T'_c > T$), or normal ($V', T'_c < T$). GL equations are applicable in both cases, and for the ODSEE model they can be simplified as follows

$$\psi'' + \psi (\pm 1 - |\psi|^2) = 0, \quad (66a)$$

$$J = \text{Im}(\psi^* \psi'), \quad (66b)$$

where ψ is the order parameter normalized with respect to its characteristic value Δ_0 in the weak link, x is normalized to $\xi(T)$ (22), and the current is normalized to $I_0 A^{-2}$. Here I_0 is just the critical current which the weak link would have had at $L = \xi$ if the "classical" Josephson effect had taken place at this length (i.e., if the AL theory were valid)

$$I_0 = \frac{\pi}{4} \frac{\Delta^2}{k_B T e R_N} \cdot \frac{L}{\xi} = \frac{c \phi_0 S}{8 \pi^2 \lambda^2 \xi} A^2, \quad R_N = \frac{L}{\sigma_N S}. \quad (67)$$

In $I_0 A^{-2}$ units the depairing current (Ginzburg, 1958; see also De Gennes, 1966) is equal to $2/3\sqrt{3} \approx 0.385$.

The parameter A is defined by the expression

$$A^2 = (T_c - T) / |T'_c - T| \quad (68)$$

and, according to Eq. (12), it represents the ratio of the equilibrium value of $|\Delta|$ at the electrodes to its characteristic value (Δ_0) in weak link material. If the materials are the same, then $A = 1$. Using this notation, the boundary conditions [Eq. (64)] can be written as follows (for $\Delta_1 = \Delta_2 = \Delta$)

$$\psi(0) = A \exp(i\chi_1), \quad \psi(L/\xi) = A \exp(i\chi_2). \quad (69)$$

The solution of Eq. (66) may be expressed in terms of special functions (Mamaladze and Cheishvili, 1966; Baratoff et al., 1970; Christiansen et al., 1971), but only by recourse to numerical methods may we determine the constants contained in these functions. It is therefore more convenient (especially for $T_c > T$) to solve these equations numerically, using the analytical expression available for the first integral. We shall cite the main results obtained from the work of Likharev and Yakobson (1975a).

If the weak link length L is sufficiently small

$$L \ll \xi, \xi/A, \quad (70)$$

the solution of Eq. (66) is close to solution (32) derived in the AL model. Since the function $f(\mathbf{r})$ for this geometry is $(1 - x/L)$, the AL solution takes the form

$$\psi_{\text{AL}} = A [(1 - x/L) \exp(i\chi_1) + (x/L) \exp(i\chi_2)] \quad (71)$$

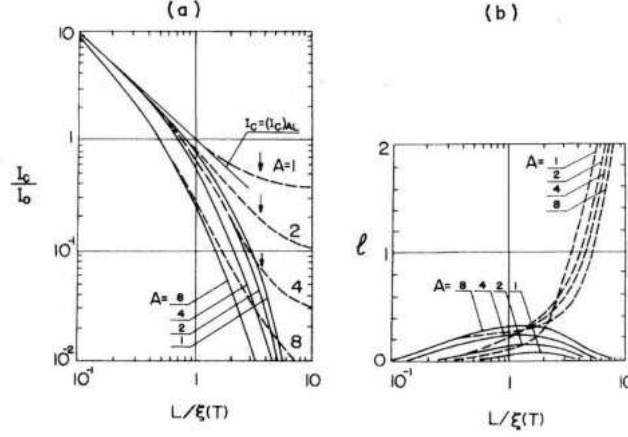


FIG. 7. Critical current (a) and parameter l (b) as a function of weak link length in the framework of the ODSEE model, as obtained from GL equations (Likharev and Yakobson, 1975a). Solid lines correspond to $S - N' - S$ structures ($T > T'_C$), and dashed lines to $S - S' - S$ structures ($T < T'_C$). Parameter A is determined by Eq. (68) and is equal to unity if the materials of the electrodes and weak link are the same. Thin line shows I_C , following from the AL theory.

Assuming Eq. (70), it is easy to find the first approximations in L for ψ and then for the current [signs are to be taken according to Eq. (21a)]

$$I_S = I_0 \frac{\xi}{L} \left\{ \left[1 \pm \frac{1}{6} \left(\frac{L}{\xi} \right)^2 - \frac{1}{10} \left(\frac{AL}{\xi} \right)^2 \right] \sin \varphi - \frac{1}{30} \left(\frac{L}{\xi} \right)^2 \sin 2\varphi \right\}. \quad (72)$$

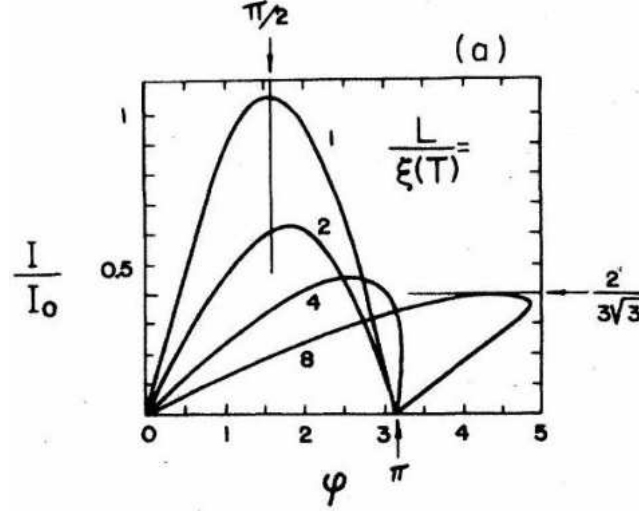
This equation clearly shows that the corrections for Eq. (40), which describes the "classical" Josephson effect, are small even when L is of the order of ξ or ξ/A . Since the normal resistance of a weak link R_N is proportional to its length L , the characteristic voltage V_C in the region defined by Eq. (70) is approximately constant and is equal to Eq. (14). A further increase in the link length will give rise to effects radically different in $S - N' - S$ and $S - S' - S$ structures. The variations in the critical current I_C and in the factor l with changing length are shown in Fig. 7 by solid lines ($T > T'_C$) and dashed lines ($T < T'_C$), respectively.

For $T > T'_C$ ($S - N' - S$ structure) an increase in L above a few ξ entails a drop in the critical current, the function $I_S(\varphi)$ remaining almost sinusoidal ($l \approx 0$). For $L \gg \xi$ from Eq. (66) we readily obtain the following analytical expression

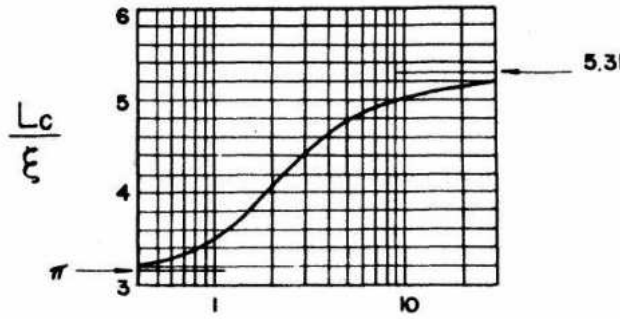
$$I_S = I_0 8 \left[1 + (1 + A^2/2)^{1/2} \right]^{-2} \exp(-L/\xi) \sin \varphi. \quad (73)$$

This drop results from the fact that in the proximity effect the order parameter in a normal substance decreases exponentially with the distance from its interface with the superconductor. Exponential decrease in the critical current as $L/\xi \rightarrow \infty$ is typical of $S - N - S$ structures (see De Gennes, 1966; Aslamazov et al., 1968; Clarke, 1969), but the exact value of the factor before the exponent depends significantly on the boundary conditions.

If $T < T'_C$, we are dealing with a weak link of the



(b)



$$A = \Delta/\Delta_0$$

FIG. 8. The deformation of the $I_S(\varphi)$ relationship as the $S-S'-S$ weak link length increases. This deformation shows the transition from the "ideal" Josephson effect to the depairing effect (a). It happens at the length $L = L_C$, when the $I_S(\varphi)$ relationship becomes multivalent. L_C is of the order of the coherence length of weak link material $\xi(T)$ for any A ; $L \approx 3.49\xi(T)$ at $A = 1$. After Likharev et al., 1974; Likharev and Yakobson, 1975a.

$S-S'-S$ type. For this structure, with increasing weak link length the critical current at first diminishes, then when $L \gtrsim 10\xi$ it tends to a constant value. Here radical changes take place in the shape of the $I_S(\psi)$ function, as shown in Fig. 8. The parameter of kinetic inductance ℓ grows with increasing length L , and for a critical length L_C it becomes greater than unity, i.e., the function $I_S(\varphi)$ becomes multivalued [see Fig. 4(b)].

For long lengths ($L \ll \xi$), the shape of this function is similar to that of a cubic parabola

$$I_s = \frac{I_0}{A^2} \left\{ \frac{\varphi}{(L/\xi)} - \left[\frac{\varphi}{(L/\xi)} \right]^3 \right\} \quad (74)$$

and the critical current is almost constant

$$I_C = \frac{2}{3\sqrt{3}} I_0 A^{-2} \quad (75)$$

This means that here we are dealing with the well known depairing effect due to diminution in the equilibrium number of Cooper pairs (i.e., $|\Delta|^2$), with the current growth taking place uniformly throughout the entire length of the weak link. ¹⁸ The unstable reverse branch of the

$I_S(\varphi)$ function joining the point $I_S = I_C$ to the point $I_S = 0, \psi = \pi$ corresponds to the sharp drop of $|\Delta|$ in the

middle of the weak link, i.e., to the nucleation of the "phase-slip center" of a length $\sim \xi(T)$. In long weak links with small cross-section inhomogeneities the location of this center will be determined primarily by the narrowest part of the structure.

In this way, with an increase in the length of a superconducting weak link (superconducting "filament" or "channel"), the Josephson effect becomes the depairing effect. It is the length $L = L_C$, where $I_S(\varphi)$ becomes multivalued, that may be assumed as the cutoff point of the transformation. In the succeeding pages we shall show that this critical length is, indeed, highly significant for the processes taking place in the weak link. Its dependence on the parameter A is shown in Fig. 8(b), from which it is clear that L_C is always of the order of several coherence lengths of the weak link material.

Before we proceed further, we shall make one important remark. If the temperature tends to the critical temperature of the weak link, then, according to Eq. (22), $\xi \rightarrow \infty$ for any length $L/\xi \rightarrow 0$, and as may be imagined from Fig. 7(a), the "classical" Josephson effect should be observed. However, when $T \approx T'_C$, A tends to infinity as $(T'_C - T)^{-1/2}$ [Eq. (68)], and it can easily be verified that the critical current does not exhibit any singularity. This corresponds to the fact that, for $T \approx T'_C$, it is the condition $L \lesssim \xi/A$ that limits the validity of the AL theory. From Eqs. (22) and (68), it is obvious that ξ/A is the coherence length of the weak link material formally taken at a temperature equal to $T - (T_C - T'_C)$.

3. S-N-S structure at arbitrary temperatures

Likharev (1976) has evaluated the effect of the length on the function $I_S(\psi)$ at arbitrary temperatures for a particularly important case where the weak link is made of "ideal" normal material, having a critical temperature $T'_C = 0$ ($S - N - S$ structure). According to Eq. (50) here $|\Delta| = 0$ for $0 < x < L$, so we are dealing with gapless superconductivity (De Gennes, 1966). The right-hand side of the Usadel equation (51) vanishes, making it possible to find analytically the first integral of this equation, and to solve the problem numerically within the framework of the ODSEE model in a relatively simple manner.

Figure 9 shows the results of such a calculation. The link length has been normalized with respect to the coherence length of the weak link material that naturally follows from Eq. (51)

$$\xi_N(T) = \left(\frac{\hbar D}{2\pi k_B T} \right)^{1/2} \approx 0.58 \left(\frac{\hbar v_F}{2\pi k_B T} \ell \right)^{1/2} \quad (76)$$

taken again at a temperature equal to the critical temperature of the banks.

The plots for $L/\xi_N = 0$ show the results of the KO-1 theory. From Fig. 9(a), it is evident that this theory is satisfactory for lengths $^{19}L \lesssim \xi_N(T_C)$. When the bridge length is increased, the critical current decreases-exponentially at $T \approx T_c$

⁰¹⁸ Note that the Josephson supercurrent in weak links may be much greater than the depairing current (Fig. 7).

⁰¹⁹ But not for $L_{\text{eff}} < \xi_N(T)$, as was stated in the original KO-1 paper.

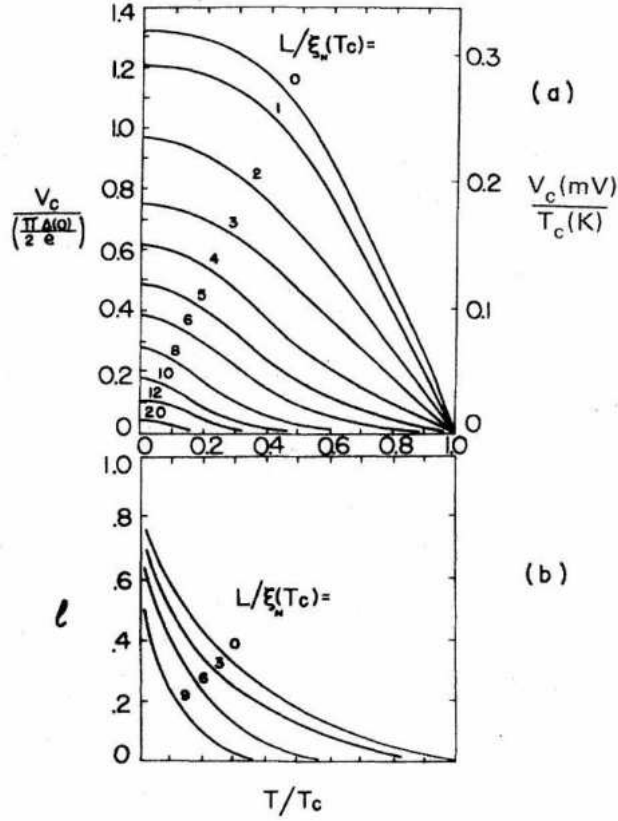


FIG. 9. Temperature dependences of critical current (a) and kinetic inductance parameter l (b) for an $S - N - S$ structure, with zero critical temperature for the weak link material ($T'_C = 0$). Plots are calculated from Usadel equations [(51) and (53)] within the framework of the ODSEE model. $\xi_N(T_C)$ is the decay length of weak link material, if taken at the critical temperature of the electrodes. After Likharev, 1976.

$$I_S R_N = \frac{\pi \Delta^2}{4 e k_B T} \sum_{n \geq 0} \frac{8}{\pi^2 (2n+1)^2 \sinh \ell_n} \ell_n \sin \varphi,$$

$$\ell_n = (2n+1)^{1/2} \frac{L}{\xi_N(T)},$$

$$I_S R_N = \frac{4}{\pi} \frac{\Delta^2}{e k_B T} (L/\xi_N) \exp(-L/\xi_N) \sin \varphi, \text{ at } L/\xi_N \rightarrow \infty,$$

and more slowly at $T \rightarrow 0$

$$I_C R_N \propto L^{-2}, \text{ at } \xi_N(T_C) \ll L \ll \xi_N(T) \quad (78)$$

As is obvious from Fig. 9(b), the function $I_S(\varphi)$ is always single-valued ($\ell < 1$) and is significantly nonsinusoidal only at very low temperatures.

4. Dependence of $I_S(\psi)$ on length-discussion

Let us use the term "ideal Josephson effect" for the case where the $I_S(\varphi)$ relationship is single-valued and V_C is comparable with its value in the tunnel junction (10). As we could see, in "dirty" weak links the "ideal" Josephson effect is conserved up to a certain characteristic length of the weak link. With the help of the calculations carried out for $T'_C \approx T$ and $T'_C = 0$, the condition for such an "ideal" effect may be formulated generally as follows

$$L \leq (2 - 4) \min [\xi(T), \xi(T_C)] \quad (79)$$

where ξ is the coherence length of the weak link material. This relationship shows that this material merely fulfills certain "transport" functions, thereby aiding the transfer of the wave functions of the superconducting condensate of the electrodes deep inside the weak link where their interference-Josephson effect-occurs. The parameter ξ characterizing these transport properties is affected by velocity on the Fermi surface v_F of the material alone, under a fixed l . v_F is almost the same for the majority of the known conducting metals, both superconductors and normal materials.

Therefore, for the Josephson effect to be observed in a certain weak link, the link material need not necessarily possess any special "superconducting" properties in the sense of having a finite critical temperature T'_C . Superconducting properties should be exhibited only by the electrode material, because $|\Delta|$ at the boundaries of the weak link is proportional to T_C , and an increase in this parameter entails an increase in V_C .

8.2.4 D. Clean structures

Now we shall briefly examine the properties of "clean" weak links in which the mean free path of electrons ℓ is greater than or comparable with the length l of the weak link. Such a relationship is more often encountered in microshorts of point contacts than in bridge-type structures, i.e., in those cases where the exact shape of the weak link is unknown. We shall, therefore, start our discussion with the most important case of a sufficiently small weak link ($L_{\text{eff}} \ll \ell, \xi_0$) of arbitrary shape. Recently Kulik and Omelyanchuk (1977) advanced a theory, referred to as KO-2, to describe the Josephson effect in such structures.

1. Eilenberger equations

The GL equations or the Usadel equations can no longer be used here, and thus we need to employ more general Eilenberger equations (1968). These equations have been derived for complex functions f and g , each of which depends not only on the point ($\mathbf{r}$) and the Matsubara frequency ω , but also on the spatial direction which is formally represented by a vector $\mathbf{v}_F$. These functions are interrelated by the equality

$$f^\dagger f + g^2 = 1 \quad (80)$$

where, for the sake of convenience, a new function $f^\dagger$ is introduced, defined by the following expression

$$f^\dagger(\mathbf{r}, \mathbf{v}_F, \omega) = f^*(\mathbf{r}, -\mathbf{v}_F, \omega) \quad (81)$$

The Eilenberger equation for f looks quite simple

$$\left[2\omega + \mathbf{v}_F \left(\nabla - i \frac{2e}{\hbar c} \right) \mathbf{A} \right] f = 2\hbar^{-1} \Delta g + \tau^{-1} (g \langle f \rangle - f \langle g \rangle) \quad (82)$$

where the order parameter Δ depends only on $\mathbf{r}$, the angular brackets stand for averaging over all the directions of the vector $\mathbf{v}_F$, and $\tau = \ell/v_F$ is the time of the electron mean free path. For the system of equations to be closed, Eqs. (80)-(82) have to be supplemented with a self-consistency equation for $\Delta(\mathbf{r})$

$$\Delta \ln \frac{T_C}{T} = 2\pi k_B T \sum_{\omega > 0} \left(\frac{\Delta}{\hbar \omega} - \langle f \rangle \right) \quad (83)$$

and an equation for the supercurrent (in principle, jointly with the Maxwell equations)

$$\mathbf{j}_S = -i2eN(0)2\pi k_B T \sum_{\omega>0} \langle \mathbf{v}_F g \rangle. \quad (84)$$

The system of Eqs. (80)-(84) is valid for any relationship between ξ_0 and ℓ . In the "dirty limit," f, g , and Δ vary only at distances $\sim \xi \approx (\ell\xi_0)^{1/2} \gg \ell$, and in addition are almost isotropic. In this situation the Eilenberger equations are satisfied by a solution of the following type:

$$f(\mathbf{r}, \mathbf{v}_F, \omega) = F(\mathbf{r}, \omega) + \mathbf{v}_F \left(\nabla - i \frac{2e}{\hbar c} \mathbf{A} \right) F_1(\mathbf{r}, \omega), \quad (85)$$

in which the second term is small. Direct substitution of Eq. (85) into Eqs. (80)-(84) gives the Usadel equations (51)-(53).

For a uniform superconductor carrying no current ($\nabla = \mathbf{A} = 0$), at any ratio between ℓ and ξ_0 , we obtain Eqs. (54) and (55) for F and Δ , respectively, which give the usual results of the BCS theory.

2. Weak links of small dimensions

In this case we may examine two regions separately: parts of the electrodes of size $r \sim \xi_0 l$, surrounding the weak link, and a region of size $r \sim L_{\text{eff}}$ around the weak link. As will be shown below, in the electrodes the quantities $\Delta, \langle f \rangle$, and $\langle g \rangle$ may be taken to be constant and equal to their equilibrium values

$$\begin{aligned} \langle f \rangle &= \langle f^+ \rangle^* = (\Delta/E) \exp(i\chi_{1,2}), \quad \langle g \rangle = \hbar\omega/E, \\ \Delta &= \Delta_{1,2} \exp(i\chi_{1,2}), \quad E^2 = (\hbar\omega)^2 + \Delta^2, \end{aligned} \quad (86)$$

as they vary only over distances $\sim L_{\text{eff}}$. So Eq. (82) for f and the equations for $f^\dagger$ and g derived from Eqs. (80)-(82) form a linear system (just as in Sec. II. A and B, we assume that $L_{\text{eff}} \ll \lambda$ and therefore $A = 0$)

$$\begin{aligned} (2\hbar\omega + \hbar\tau^{-1}\langle g \rangle + \hbar v_F \partial/\partial s) f - \tilde{\Delta} g &= 0, \\ (2\hbar\omega + \hbar\tau^{-1}\langle g \rangle - \hbar v_F \partial/\partial s) f^\dagger - \tilde{\Delta} g^* &= 0, \\ \tilde{\Delta}^* f - \tilde{\Delta} f^+ - (\hbar v_F \partial/\partial s) g &= 0, \\ 2\tilde{\Delta} &= 2\Delta + \hbar\tau^{-1}\langle f \rangle, \end{aligned} \quad (87)$$

where s is the coordinate along the "trajectory," i.e., a line in the direction of $\mathbf{v}_F$. The solution to this system is easily found

$$f = \sum_{\kappa} f_{\kappa} e^{\kappa s}, \quad g = \sum_{\kappa} g_{\kappa} e^{\kappa s}, \quad (88)$$

where for κ we have three values:

$$\kappa = \kappa_0, \pm\kappa_1, \quad \kappa_0 = 0, \quad \kappa_1 = (E + \hbar\tau^{-1}/2) / \hbar v_F. \quad (89)$$

This solution describes the relaxation of f and g towards their equilibrium values [Eq. (86)] at distances $\sim \min(\xi_0, l) \gg L_{\text{eff}}$. Along a trajectory, f_{κ} and g_{κ} are constants, but these constants vary on different trajectories. For the trajectories which do not pass through the weak link (Fig. 10) we may take f and g to be constants, i.e., only the terms with $\kappa = 0$ are

to be included in the sum [Eq. (88)]. Then it follows directly from Eqs. (87) that on these trajectories f and g are equal to their equilibrium values [Eq. (86)].

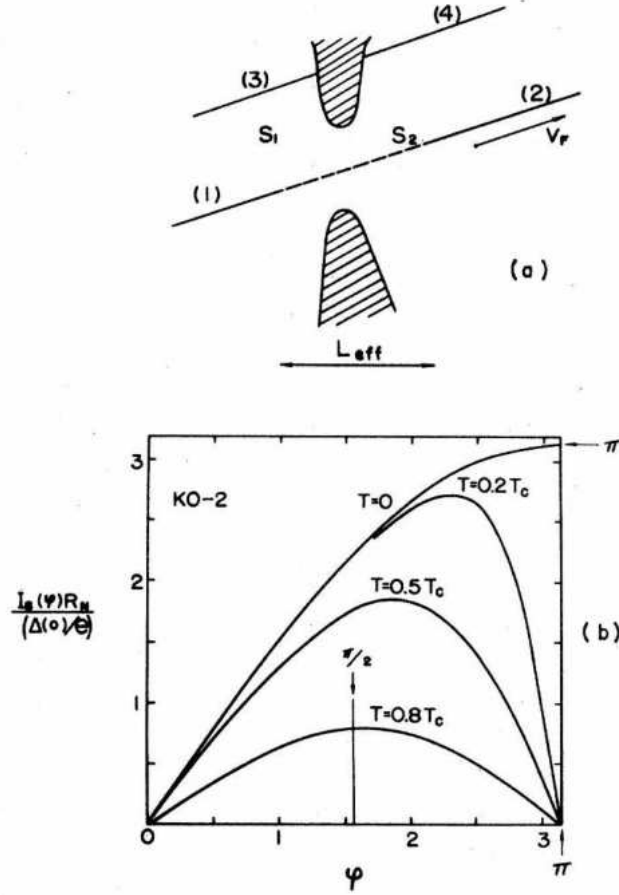


FIG. 10. The KO-2 theory for clean small weak links ($L_{\text{eff}} \ll l, \xi_0$). (a) Two types of electron trajectories in the weak link region: (1) "incoming" part of through trajectory; (2) "outgoing" part of this trajectory; (3) and (4) nonthrough trajectories. (b) Current-phase relationship for different temperatures. Temperature dependence of critical current is shown in Fig. 6.

Along those trajectories passing through the weak link, f and g cannot be constant, since for $\chi_1 \neq \chi_2$ the values of f in the electrodes are different. Since f and g are finite as $s \rightarrow \pm\infty$, therefore for the "outgoing" part of a through trajectory we have

$$f_o = \langle f \rangle + \tilde{f}_o \exp(-\kappa_1 s), \quad g_o = \langle g \rangle + \tilde{g}_o \exp(-\kappa_1 s) \quad (90)$$

and for the "incoming" part of this trajectory

$$f_i = \langle f \rangle^* + \tilde{f}_i \exp(+\kappa_1 s), \quad g_i = \langle g \rangle + \tilde{g}_i \exp(+\kappa_1 s). \quad (91)$$

What remains now is to find $\tilde{f}$ and $\tilde{g}$. For this purpose first we have to substitute Eqs. (90) and (91) into Eqs. (87), and thus derive a relationship between $\tilde{f}$ and $\tilde{g}$:

$$f_o = g_o \frac{2\Delta + \hbar\tau^{-1}\langle f \rangle}{2\hbar\omega - \hbar\tau^{-1}\langle g \rangle - \hbar v_F \kappa_1} \quad (92)$$

for f_i and g_i the sign of the term containing κ_1 has to be reversed. Then, we consider a region of size $\sim L_{\text{eff}}$. In this region the main term in Eq. (82) is the gradient one, because it alone is of the order of v_F/L_{eff} , while the remaining terms are $\sim v_F/\xi_0$ or $v_F/\ell(f, g \sim 1)$. Thus we obtain a simple equation:

$$\mathbf{v}_F \nabla f = 0, \quad (93)$$

which shows that f (and consequently g) remains constant on a trajectory at a distance of the order of L_{eff} . Hence in Eqs. (90) and (91) the values of f and g in the weak link region ($s = 0$) may be regarded as being the same, both on the incoming and on the outgoing parts of the trajectory. This gives

$$\langle f \rangle + \tilde{f}_o = \langle f \rangle^* + \tilde{f}_i, \tilde{g}_o = \tilde{g}_i \quad (94)$$

From Eqs. (92) and (94) we get, taking $\chi_1 = -\chi_2 = -\varphi/2$

$$\tilde{g} = i \sin \frac{\varphi}{2} \frac{\Delta^2}{E[\mathbf{E} \cos(\varphi/2) + i\hbar\omega \sin(\varphi/2)]} \quad (95)$$

Now substituting this value into Eq. (84), and using the expression for the resistance of a structure in the normal state (Omelyanchuk et al., 1977),

$$R_N^{-1} = e^2 v_F N(0) S, \quad (96)$$

we finally obtain the result

$$I_S R_N = \frac{\pi \Delta}{e} \sin(\varphi/2) \tanh \frac{\Delta \cos(\varphi/2)}{2k_B T} \quad (97)$$

which does not depend on the ratio between ℓ and ξ_0 .

It now only remains for us to verify the validity of our initial assumption that $\langle f \rangle$ and $\langle g \rangle$ are constants in the region $r \gg L_{\text{eff}}$. For averaging we have to take for f and g the equilibrium values [Eq. (86)] on nonthrough trajectories, and perturbed values [Eqs. (90) and (91)] on the through trajectories. The number of the latter trajectories in the region under consideration is small, as a weak link is "observed" within the small solid angle $\Omega(r) \ll 4\pi$. The values of $\langle f \rangle$ and $\langle g \rangle$ (and consequently the value of Δ , too) therefore differ from the equilibrium values only by a quantity of the order of $\Omega/4\pi \ll 1$. Thus $\langle f \rangle$, $\langle g \rangle$, and Δ vary greatly only in the immediate neighborhood of a weak link ($r \sim L_{\text{eff}}$) where $\Omega \sim 2\pi$.

3. Discussion

Figure 10(b) shows the phase dependence of the supercurrent, defined by Eq. (97), while the KO-2 curve in Fig. 6 shows the dependence $V_C(T)$. The $I_S(\varphi)$ relationship is sinusoidal for $T \approx T_C$ and V_C is again defined by Eq. (14), but at lower temperatures the results are quite different from those predicted by the tunnel theory or the KO-1 theory either. For $T = 0$, the current-phase relationship is highly nonsinusoidal, and suffers jumps at $\varphi = \pi + 2\pi n$, while V_C is twice the value given by the tunnel theory

$$\max_T [eV_C] = \pi \Delta(0), \max_T [V_C] [\mu V] \approx 480 T_C [K] \quad (98)$$

Such a marked discrepancy shows an essential distinction in the physics of the processes in these three types of Josephson structures. In order to discover the causes of this discrepancy, we shall analyze the structure of the Eilenberger functions f for these three cases. For "clean" weak links, f has just been calculated and is expressed by Eqs. (86) and (90). For "dirty" links the form of $f \approx F$ is given by Eqs. (58) and (60). Therefore it remains to describe the tunnel junction in the same terms.

This is easily done by adding a small perturbation due to tunneling from the other electrode to the equilibrium

functions of each electrode [Eq. (86)]

$$f_{1,2} = (\Delta/E) \exp(i\chi_{1,2}) + T_p (\Delta\hbar\omega/E^2) \exp(i\chi_{2,1}), \quad (99)$$

where the probability T_p of electron tunneling depends on the direction of momentum $\mathbf{p}_F$. Substituting Eq. (99) into the definition of g [Eq. (80)], we obtain

$$g = (\hbar\omega/E) + iT_p (\Delta^2/E^2) \sin \varphi, \quad (100)$$

and by virtue of the relationship between T_p and R_N ,

$$R_N^{-1} = j_N S/V = (2eN(0)/V) |\langle \mathbf{v}_F T_p \text{sign}(\mathbf{v}_F)_x \rangle| S \quad (101)$$

we arrive at the result derived by Ambegaokar and Baratoff [Eqs. (9), and (10)]. Equation (99) shows that the results of the tunnel theory are only valid when a clean "mixture" (i.e., interference) of states induced by neighboring electrodes exists at all distances ($\sim \ell, \xi_0$) significant in the superconductivity theory. This will be so, of course, not only for the case of tunnel junctions.

For instance, as demonstrated by Zorin and Likharev (1978a), if two electrodes are separated by a thin opaque barrier with a large number of minute ports ($p_F L_{\text{eff}}/\hbar \ll 1$), the wave functions of such a structure in toto coincide with the functions of a tunnel junction. For this type of weak link, therefore, all the results predicted in the tunnel theory are applicable (nonetheless, the relative capacitance of these structures may, as in other types of weak links, be much less than that of tunnel junctions: $\beta \leq 1$).

For "dirty" weak links the functions f are almost isotropic (Usadel approximation), but they change considerably in space at distances L_{eff} greater than ℓ . This is precisely the cause of the difference in the results obtained within the framework of the KO-1 and tunnel theories. Finally, the functions f for a "clean" weak link change strongly both in the coordinate space (at distances $\sim \ell, \xi_0 \gg L_{\text{eff}}$) and in the momentum space (a significant difference between transparent and nontransparent trajectories). A still more essential distinction between the results of KO-2 and tunnel theories is associated with this.

Thus there is a noticeable difference in the stationary properties of weak links, even among those which are short compared to $\xi(T)$.

4. Pure one-dimensional structures

Of course, how "clean" weak links behave when their length is increased is a question of considerable interest. So far, this feature has only been studied for one-dimensional $S - N - S$ structures, i.e., in the case where a weak link material has zero critical temperature. Moreover, in these calculations, it was assumed that the condition

$$L \gg \min [\xi_0, (\xi_0 l)^{1/2}], \quad (102)$$

is satisfied, and was thus possible to disregard the proximity effect throughout a major part of the structure, and to assume that (Kulik, 1969b)

$$\Delta = 0, 0 \leq x \leq L. \quad (103)$$

In such a model the $I_S(\varphi)$ relationships for $\mathbf{T} = 0$ (Ishii, 1970, 1972) and for $T \neq 0$ (Svidzinskii et al., 1971, 1973; Bardeen and Johnson, 1972; see also the introduction to the paper by Bezuglyi et al., 1975) have been calculated for $\ell = \infty$. The influence of a finite mean free path ($\ell \sim L$) has been treated by Kulik and Mitzai (1975).²⁰

Here we shall not examine the results reported in these papers. The reason is that the equations derived by Kulik and Mitzai (1975) cannot be converted at $l \ll \xi_0, L$ into the results following

from the Usadel equations, [into Eq. (78), in particular], which hold true in this case. The author of this review feels that in the analysis of clean $S - N - S$ structures some error has probably crept into the determination of boundary conditions. Fortunately, there is a method by which we can clarify this situation: that is to make an analysis of such $S - N - S$ structures using the Eilenberger equations, where the boundary conditions are quite clear, rather than the complex formalism of the above papers.

8.2.5 E. $I_S(\varphi)$ relationship and Josephson effect

So far in the preceding pages we have assumed that the cross section of a weak link is so small that the current self-shielding (i.e., actually the Meissner effect) may be disregarded. This condition may be formulated in the form of a constraint on the maximum transverse dimension of the weak link cross section, i.e., the

$$W \ll \lambda_{\text{eff}}, \quad (104)$$

where λ_{eff} denotes some characteristic size of the selfshielding region (magnetic field penetration). As we shall see in Sec. VII, it depends substantially on both the density of the critical current in the weak link, and the geometry of the structure, and is usually much larger than L . But, even for very small widths ($W \sim \xi$), the uniform flow of the condensate, i.e., the uniform distribution of variables over the weak link cross section, may prove to be unstable with respect to the formation of vortex lines carrying one or several magnetic flux quanta (Abrikosov, 1957; see also De Gennes, 1966).

1. Abrikosov vortices

In a superconducting specimen of sizes far greater than ξ and λ , every Abrikosov vortex is an axial symmetric formation. The order parameter is zero on the vortex axis and tends to its equilibrium value at a distance of $\sim \xi$ from the axis, thereby forming a normal "core" of the vortex. A persistent ring supercurrent flows around the core at distances of about λ . The magnetic field of total flux ϕ_0 (or $n\phi_0$) directed along the vortex axis is screened by this current from the remaining portion of the superconductor. The electromagnetic region of the vortex readily suffers deformation, and this is just what happens when the vortices are of high density in a strong magnetic field or when the vortices penetrate to a specimen of small size ($< \lambda$). The vortices may penetrate into a superconducting specimen only from the side of the boundary with vacuum

(insulator),²¹ under the influence of either an external magnetic field or a current flowing through the specimen itself. These two cases differ essentially in the behavior of the vortices after penetration into the specimen. In both cases a Lorentz force of linear density

$$\mathbf{f} = \frac{\phi_0}{c} [\mathbf{n} \times \mathbf{j}] \quad (105)$$

§§

acts on the core of each vortex, where $\mathbf{n}$ is the direction of the vortex axis. A contribution to the current density $\mathbf{j}$ is made by all the currents on the vortex axis, including the circulating currents of other vortices.

²⁰ The specific influence of an external magnetic field on such structures (at $l = \infty$) has been considered by Antsygina et al. (1975). Bezuglyi et al. (1975) have examined such structures with low transparency of the $S - N$ boundary.

If the vortices are generated under the action of an external magnetic field, the force [Eq. (105)] is directed from all the boundaries towards the center of the specimen, and a steady distribution of the vortices (static mixed state) is established in the specimen. On the other hand, if the vortices are created by the current in the specimen itself, the Lorentz force pushes the vortices in a direction normal to the current, i.e., along the cross section, and tends to establish a continuous flow of additional vortices across the conductor (dynamic mixed state), the dc voltage drop in this case being nonzero.

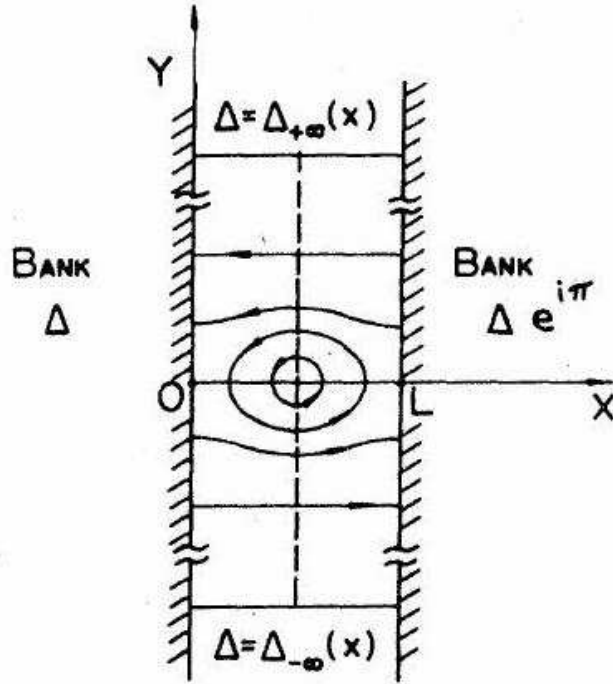
An analysis of conditions for vortex formation in samples of constant cross section, thin film strips for example (Likharev, 1971a,b), and considerable body of experimental data show that even when $W \ll \lambda_{\text{eff}}$, the critical current I_{C1} for vortex penetration may be small (much lower than the depairing current). Moreover, I_{C1} may essentially depend on small-scale ($\sim \xi$) inhomogeneities at the point where the vortex penetrates into the specimen, a fact which makes I_{C1} almost irreproducible in experiments.

It is, therefore, quite likely that the calculations of critical currents I_C in weak links may prove to be inadequate, because the vortex motion across the weak links can start at much lower currents. Fortunately, this does not occur in the most interesting case of very short weak links: the reason is that Abrikosov vortices simply cannot exist in such structures.

2. Abrikosov vortices in weak links

Following Likharev (1975b), we consider a solitary vortex which has penetrated into a weak link, within the framework of the ODSEE model in which a weak link is regarded as a span between two unperturbed bulk electrodes. Since the energy of a vortex is proportional to its length, its axis is always normal to the maximum size of the cross section, say for example, to the plane of the thin film bridge. In addition, the vortex experiences strong repulsion from the bulk superconducting banks (Likharev, 1971d). Therefore its axis is located in the middle between the electrodes (Fig. 11).

We shall carry out an analysis within the framework of the GL equations (21), and it is now necessary to assume that Δ and $\mathbf{j}_s$ may depend both on x and y (Fig. 11).



⁰²¹ Another possible process is the formation of a pair of antipolar vortices at the internal points of the specimen under the influence of current. This process is energetically less advantageous.

Representing Δ in Cartesian form as

$$\Delta = R(x, y) + I(x, y), \quad (106)$$

from Eq. (21a) we obtain the following equations for real R and I

$$\xi^2 \nabla^2 R + [\pm 1 - (R^2 + I^2) / \Delta_0^2] R = 0, \quad (107a)$$

$$\xi^2 \nabla^2 I + [\pm 1 - (R^2 + I^2) / \Delta_0^2] I = 0. \quad (107b)$$

As a rule, these equations do not easily yield to solution, but the ODSEE model offers a means of making a complete analysis, as the boundary conditions for Δ are very simple [Eq. (64)]. Since the phases in the electrodes can only change at distances $\sim \lambda_{\text{eff}}$ (see Sec. VII), under the condition (104), they are constant for each electrode. Moreover, as χ is always determined up to a constant, take $\chi_1 = 0$; hence on the first bank ($x = 0$) we have

$$R(0, y) = \Delta, \quad I(0, y) = 0. \quad (108)$$

When we pass round the vortex by angle π , the phase of the order parameter likewise should change by π , therefore the phase at the second bank is π , and

$$R(L, y) = -\Delta, \quad I(L, y) = 0. \quad (109)$$

In order to derive the boundary conditions on the lateral sides of the weak link, let us first consider the case where $W \gg L$. Thus; far from the vortex center (which we take as the origin of the transverse coordinate y) Δ should tend to a uniform solution

$$R(x, \pm\infty) = R_{\pm\infty}(x), \quad (110a)$$

$$I(x, \pm\infty) = I_{\pm\infty}(x), \quad (110b)$$

which is just the same as the one we derived earlier (Sec. IV.C). Now we shall analyze the solutions of the problem [Eqs. (107)-(110)].

3. "Explosion" of core

If the function $I_S(\varphi)$ is multivalued ($l > 1$), several (at least, three) values of I_S and consequently the same number of uniform solutions $\Delta(x)$ (Fig. 4) correspond to the value $\varphi = \pi$, which we now require. The current on the two different sides of the vortex flows in opposite directions and therefore the solutions giving different signs to $I_S(\varphi)$ have to be taken as $y \rightarrow \pm\infty$. As is clear from Fig. 3, a nonzero value of I corresponds to a nonzero current:

$$I_{+\infty} = -I_{-\infty} \neq 0, \text{ at } l > 1. \quad (111)$$

In this case the solution of the system of equations (107) describes the core of an Abrikosov vortex of a shape close to that of an ordinary axial-symmetric vortex, the only difference being that the current distribution around the core is distorted by the banks (Fig. 11). These distortions may be treated as the positive mirror images of the vortex at the banks (Likharev, 1971d). The core suffers slight distortion on decreasing L , becoming somewhat flattened along the x direction as it is stretched out in the y direction.

The situation is radically different when L is less than L_C ($l < 1$). The one-dimensional problem for this case has only one solution with $\varphi = \pi$, which, according to Fig. 4(c), has $I_S = 0$; consequently $\chi = \text{const}$ and

$$I_{+\infty} = I_{-\infty} = 0, \text{ at } l < 1. \quad (112)$$

Thus the differential equation for I (107 b) has zero boundary conditions [Eqs. (108)-(110), (112)] at all boundaries and only a trivial solution in the whole domain

$$I(x, y) = 0, \quad \Delta(x, y) = \Delta_{\pm\infty}(x) = R_{\pm\infty}(x), \quad (113)$$

which gives rise to a zero current density in the whole bulk of the weak link

$$\mathbf{j}_S(x, y) = 0 \quad (114)$$

Thus, when $l < 1$, the vortex core ceases to exist. The numerical solution of the system (107) shows that noticeable deformation of the core (expansion along the y axis) begins only for values of L several percent higher than L_c ; consequently, the vortex core literally suffers an "explosion" when the link length becomes equal to the critical length.

Where does the vortex disappear to at the time of explosion? To answer this question we have to consider a weak link of sufficient width ($W \geq \lambda_{\text{eff}}$) so that an electromagnetic region of the vortex may be embedded inside the link. If an Abrikosov vortex exists in a weak link, the circulating current [say, $(j_S)_x(L/2, y)$] builds up when $0 \leq y \leq L$, and then gradually decays in the region $L \leq y \leq S\lambda_{\text{eff}}$. After the explosion of the core, the current will grow at distances $0 \leq y \leq \lambda_{\text{eff}}$, and with a further increase in y will diminish at distances of the same order. This structure is the well known Josephson vortex with only one characteristic dimension $\lambda_{\text{eff}} \gg L$ (see, for example, Kulik and Yanson, 1970). Thus, after core explosion, the vortex remains where it was, but its nature suffers transformation, i.e., an Abrikosov vortex is transformed into a Josephson vortex.

In the course of penetration and motion of the Josephson vortices, the currents in the weak link vary only at great distances ($\sim \lambda_{\text{eff}}$); hence they do not disrupt the uniformity of current in weak links of small cross section ($W \sim \xi$). Thus all the calculations made previously are applicable to these narrow structures as well.

The fact that there are no solutions of Eq. (107) describing Abrikosov vortices is in no way connected with the particular form of these equations. Therefore, even if the equations differ from the GL ones (for example, Usadel equations are valid), the conclusions remain the same: if in a weak link the dependence of the supercurrent density on the phase difference is single-valued, two-dimensional structures of the Abrikosov vortex type may not exist in them.

For example, in $S - N - S$ -type

weak links, as we have already seen, $I_s(\varphi)$ is always a single-valued function, and consequently Abrikosov vortices cannot under any circumstance exist in such a weak link for any length of the structure.

4. Critical length versus link width

In the preceding section we analyzed a case where the link width W , i.e., the maximum size of the weak link cross section, is much greater than $L \sim \xi(\mathbf{T})$. It is worthwhile examining the way in which the critical length (below which Abrikosov vortices are forbidden) of a weak link changes with decreasing width. Such calculations have been carried out for a rectangular weak link ($0 \leq x \leq L, -W/2 \leq y \leq W/2$) by Kupriyanov et al. (1975).

In these calculations the last of the boundary conditions (110) has to be replaced by more general ones

$$\frac{\partial R}{\partial y} \left(x, \frac{\pm W}{2} \right) = 0, \quad \frac{\partial I}{\partial y} \left(x; \frac{\pm W}{2} \right) = 0 \quad (115)$$

which follow from the condition of absence of current flow through the lateral edges of the weak link. Here it is impossible to find an analytical expression for the critical length, and Eqs. (107) were solved numerically. The results are shown in Fig. 12 by a solid line for the case where $A = 1$, i.e., the weak link and the electrodes are made of the same material $T'_C = T_C$. However, the vertical asymptote

$$W \rightarrow W_C \approx 4.41\xi(T), \text{ as } L/\xi \rightarrow \infty, \quad (116)$$

does not depend on T'_C . The effective critical length increases with the decreasing width W , and reaches the asymptote (116), so that Abrikosov vortices cannot exist in the center of a strip less than W_C in width at any length.

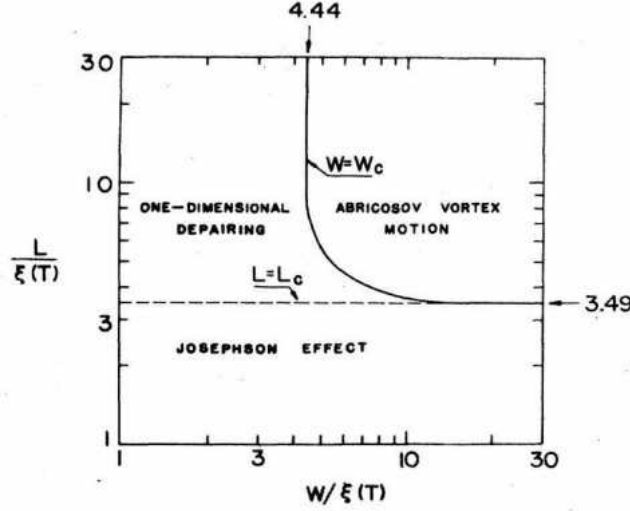


FIG. 12. Variety of possible sizes of dirty weak links (is correct qualitatively at $T \approx T_C = T'_C$). Solid line separates the region of large sizes, where Abrikosov vortex appearance is possible, from the region of small lengths and/or small widths, where de state is destroyed only due to one-dimensional instabilities, with all variables changing only along weak link length (x-axis). Dashed line corresponds to $L = L_C$ and separates the range of small length, where the Josephson effect with singlevalued $I_S(\varphi)$ relationship takes place. After Likharev, 1971d; Kupriyanov et al., 1975.

It should be emphasized that the results calculated for finite widths are somewhat uncertain. In the calculations a vortex was assumed to exist in a static position at the center of the weak link. Such a position, however, on account of the attraction toward the lateral edges, is unstable, and a vortex may be sustained in the center only with the help of an external magnetic field applied parallel to the vortex field. If $W \gg L$, the fields necessary for this purpose prove to be rather weak ($\ll H_{C2}$), and there is no need to introduce any corrections into the calculations. For $W \sim L \sim \xi$, the field should be quite strong, so that the asymptote (116) does not define a strict lower limit to W_C for any condition. When $B \approx H_{C2}$, in particular, the lower boundary shifts leftwards to values $\approx 3\xi(T)$.

In contrast, when vortex motion occurs under the action of the current passing through the weak link alone, the boundary may move from the position shown to the right only. In fact, as follows from our previous analysis, if $W < W_c$ the vortex core spreads out along the y axis normal to the current direction, thereby forming a one-dimensional structure $\Delta = \Delta(x)$. When $W \gtrsim W_C$, Abrikosov vortices may penetrate only under very large currents which move at high speeds. The finite relaxation time of the order parameters (see Secs. IV and V) likewise enhances the tendency for the vortices to "spread out" in the direction of the flux flow, i.e., along the y axis. Therefore, in the absence of a strong external magnetic field, disruption of the stationary state at $W < W_C$ may be thought of as taking place in a one-dimensional way, and consequently, the theory developed above for $I_S(\varphi)$ under this assumption is valid.

5. Limits of Josephson effect

We now have all the information needed to answer the question, to what length may the Josephson effect be supposed to take place in a superconducting weak link? Conflicting opinions have been expressed regarding this question, partly due to vagueness in defining the Josephson effect.

In particular, the current step pattern on $I - V$ curves of a weak link under microwave irradiation has often been taken as evidence of this effect. These steps appear at voltages $\bar{V} = V_{m,n}$ related to the frequency ω of microwaves by an expression

$$V_{m,n} = \frac{m}{n} \frac{\hbar\omega}{2e} (m, n \text{ are integers}) \quad (117)$$

which is a generalization of the Josephson law [Eq. (6)]. These steps are the consequence of the expression for the "classical" Josephson effect (for details, see Sec. VI). However, they may also appear in weak links of very large dimensions ($L, W \gg \xi, \lambda$). Indeed, suppose that a large number of Abrikosov vortices move simultaneously in a large structure. In the absence of inhomogeneities, they will tend to form an ordered spatial structure; say, move in m lines with identical frequency in each line. Upon application of an external microwave field, penetration of the vortices into the specimen may be locked in by the n th harmonic of frequency ω . Thus $n\omega/2\pi$ vortices per second will pass along each line. Since the intersection of the specimen by one flux quantum, according to the induction law, generates a voltage pulse across the electrodes

$$c \int V dt = \phi_0 = \frac{ch}{2e}, \quad (118)$$

and the voltage in lines is summable, we find that the mean voltage at a weak link is indeed given by the expression (117):

$$\bar{V} = \sum_m \frac{1}{T} \int V dt = m \times \frac{2\pi}{n\omega} \times \frac{h}{2e} = V_{m,n}. \quad (119)$$

Coherent motion of the vortices and current steps at voltages (117) have actually been observed in superconducting films as long as a few millimeters (Fiory, 1971) and in variable-thickness bridges of dimensions much greater than ξ (Gubankov et al., 1973). Moreover, in such large structures it is also possible to record ac voltage at frequencies given by Eq. (117) (Martinoli et al., 1976). These experiments clearly demonstrate that coherence effects in superconductivity are of a common nature.

Nonetheless, it should be stressed that even small inhomogeneities of the sample will lead to the pinning of Abrikosov vortices (see, for example, Campbell and Evetts, 1972; Shmidt and Mkrtchyan, 1974) and to differences in the velocities of the vortices in different lines, and hence to the vanishing of coherent effects. In addition, such large structures do not possess the single-valued $I_S(\varphi)$ function generally used in applications of the Josephson effect. An even more significant argument from an experimental viewpoint is that all structures admitting the appearance of small instabilities ($\sim \xi$) are almost irreproducible because unavoidable inhomogeneities determine substantially the formation of these instabilities.

It is therefore reasonable to believe that the "ideal" Josephson effect takes place only when the function $I_S(\varphi)$ is single-valued, and when, consequently, the development of instabilities is impossible. For weak links made of superconducting materials, this condition is satisfied at lengths L less than the critical length L_C found above, i.e., below the dotted line in Fig. 12, which shows the different regions for different behaviors of weak links. In the region characterized by large lengths and short widths the stationary state is disrupted by the depairing effect and by the formation of a "phase slip center" or centers at the weakest point(s) (see, for

example, Skocpol et al., 1974b, c). In the region characterized by large lengths and large widths the stationary state is disrupted due to the penetration of Abrikosov vortices; their motion has been treated by Likharev (1971d), Schmid and Hauger (1973), Larkin and Ovchinnikov (1973), and Aslamazov and Larkin (1975).

Despite interest in these nonstationary processes, we shall mainly consider the nonstationary processes in relatively short links for which $I_S(\varphi)$ is a single-valued function, i.e., in which almost the "ideal" Josephson effect takes place. Before dwelling on this topic, however, we have to outline the experimental situation in the area of stationary effects. It requires an analysis of various types of weak links and their agreement with the theoretical models used above.

8.3 IV. TYPES OF WEAK LINKS

In this section we shall briefly outline the basic types of weak links and evaluate their suitability for pure physical purposes (comparison with theory) and for the main applications of the Josephson effect. Most of the theoretical results for weak links have been derived within the framework of the ODSEE model (see Sec. III.C), in which simple boundary conditions [Eq. (64)] are assumed. In the first instance, we shall, therefore, examine the processes in the electrodes that might lead to deviations from these conditions in real weak links.

8.3.1 A. Nonlinear processes in the electrodes

We shall discuss only the processes which give rise to a deviation of $|\Delta|$ at the weak link-electrode interface from the equilibrium value, i.e., the nonlinear processes. Such linear processes as linear changes of the phase χ in the electrodes under the action of current through a weak link ($\mathbf{j}_S \propto \nabla\chi$) are consistent with the ODSEE model provided the condition (104) is valid, because noticeable changes in χ along the weak link-bank interface may take place only over λ_{eff} .

1. Proximity effect

Even in the absence of any current through a weak link, the proximity effect is observed at its interface with the electrodes: the values of $|\Delta|$ on either side of the interface tend to each other, i.e., deviate from their equilibrium values. These variations extend into the depths of the weak link and the electrodes to distances of the order of the coherence lengths of these materials. When a supercurrent I_S comparable to I_C flows through a weak link, it changes $|\Delta|$ in the weak link appreciably. Thus, if the proximity effect is significant, then $|\Delta|$ at the interface will suffer significant changes, leading to considerable deviations from the ODSEE model. Consequently, this effect cannot be avoided even if the weak link and the electrodes are made of the same material.

2. Suppression of superconductivity by a current

The critical current of short weak links, in which the Josephson effect takes place, may be rather large. For example, it is obvious from Fig. 7, that with diminishing link length the current can notably exceed the depairing current of the weak link material. This current, on entering the electrodes, can likewise produce the depairing effect accompanied by suppression of the order parameter. Therefore, for the ODSEE model to be valid [condition (64)] it is necessary that the current in the electrodes be, for some reason (say, change of material or cross section), much less than the depairing current.

3. Consequences of nonlinear effects in the banks

If at least one of the aforementioned effects occurs in the banks, then the ODSEE model is no longer applicable, and the situation, in principle, has to be reexamined. It is, however, obvious that if nonlinear effects extend into the electrodes a distance δ , then the weak link will behave qualitatively in the same manner as in the ODSEE model, but with a different effective length.

$$L_{\text{eff}} \approx L + 2\delta \quad (120)$$

We shall now consider some particular types of weakly linked structures, using the high or low current concentration in the weak link as the criterion in classifying them.

8.3.2 B. Structures without current concentration

The largest deviations from the ODSEE model may be observed in structures with no current concentration, where the electrodes have the same cross section as the weak link [Fig.1(a)-(d)].

1. Specific features of structures without current concentration

Both the above-mentioned nonlinear effects are strongly exhibited in these structures. Consider, for example, the case of a dirty structure ($\ell < \xi_0$) of length $\sim \xi'$ (primes show weak link parameters) at a temperature not very close to the critical temperature. As follows from Eq. (53), the density of the depairing current of the electrodes ($F \sim 1, \nabla \sim \xi^{-1}$) is of the order

$$j_{dp} \approx \frac{\sigma_N \Delta}{e \xi} \quad (121)$$

and the Josephson current of a weak link is of the order ($\nabla \sim L^{-1}$)

$$j_c \approx \frac{\sigma'_N \Delta}{e L} \approx \frac{\sigma'_N \Delta}{e \xi'} \quad (122)$$

Hence, in order that there may be no depairing effect at the banks, the following condition

$$\frac{\sigma'_N}{\xi'} \ll \frac{\sigma_N}{\xi} \quad (123)$$

should be satisfied. As for the proximity effect, we shall take into account the fact that the functions F and the quantities $\sigma_N \nabla F$ should be continuous on the plane of the boundary of two materials. This guarantees the continuity of the current and is in agreement with the boundary conditions of Zaitsev (1966). The deviation of F in the electrodes from the equilibrium value is of the order of $\xi \nabla F$, and is, therefore, much less than that in the weak link, if

$$\xi' |\nabla F'| \gg \xi |\nabla F|, \quad (124)$$

which again results in the condition (123) for the ratio of the parameters of the materials. Since

$$\sigma_N / \xi \propto N(0) (v_F \ell)^{1/2} \quad (125)$$

we find that condition (64) of the ODSEE model holds true in "one-dimensional" weak links (with no current concentration) only for the interlayer made either of dirty material (small ℓ) or of low concentration of conduction electrons n [in the free electron (gas) model $N(0)v_F^{1/2} \propto n^{1/2}m_e$].

If, however, σ_N/ξ is of the same order in the electrodes as in the weak link, then the effective length of the weak link (120) is of the order of $(L + 2\xi)$ i.e., it cannot be made less than ξ by diminishing L . From Figs. 7 and 8, it is evident that in such weak links it is impossible to provide the "ideal" Josephson effect with almost sinusoidal $I_S(\varphi)$ relationship and large V_C . This is confirmed by the calculations carried out for the processes taking place in such structures (De Gennes, 1966; Galaiko et al., 1969; Baratoff et al., 1970; Blackburn et al., 1972, 1975; Volkov, 1974).

Here we should mention yet another situation complicating the use of structures without current concentration for comparison with theory. We know that the basic dc characteristics of a weak link are the temperature dependence of the critical current $I_C(T)$ and the phase dependence of the supercurrent $I_S(\varphi)$, the latter carrying more information. One difficulty in the measurement of the $I_S(\varphi)$ relationship in one dimensional structures is that even in the absence of nonlinear effects the phase χ grows linearly into the electrode depth ($j_x = \text{const}$, $\partial\chi/\partial x = \text{const}$). If, in addition, the current is close to the critical depairing current, then a 2π increase in the phase takes place even at small distances $\sim \xi$ ($\sim 10^{-5}$ cm). Therefore the "three-dimensional" electrodes, across which the phase difference $\varphi = \chi_2 - \chi_1$ is measured by one method or another (Fulton and Dynes, 1970; Jackel et al., 1974, 1976a; Waldram and Lumley, 1975; Meservey and Tedrow, 1975; Rifkin and Deaver, 1976), have to be connected to the "one-dimensional" electrodes at a distance of $\sim \xi$, and this does, strictly speaking, have some influence on the processes occurring in a weak link.

We shall now take up some specific one-dimensional structures.

2. Sandwiches

Sandwiches are, as a rule, fabricated by sequential evaporation of the various layers, on a substrate. The interlayer thickness, depending on the materials used, varies from 10^{-5} to 10^{-3} cm. Experiments have been carried out mainly with sandwiches containing a relatively dirty normal metal interlayer (Clarke, 1969, 1971; Bolton and Douglas, 1971; Sheperd, 1972); similar structures were investigated by Bondarenko et al. (1970a). The value of σ_N/ξ in these studies was of the same order in a normal metal and in the electrodes, hence the boundary conditions were noticeably different from those of the ODSEE model. In experiments with interlayers much thicker than ξ' , it may, however, be possible to measure ξ' and its temperature dependence, as in this case the proportionality between I_C and $\exp(-L/\xi')$ [see Eqs. (73) and (77)] does not depend on the boundary conditions. It would, moreover, be worthwhile to study the function $I_C(T)$ for relatively clean sandwiches ($\ell \sim L, \xi_0$) and to compare them with the theoretical calculations (see Sec. III.D).

From the point of view of device applications it should be noted that sandwiches are rather unsatisfactory, as they have very low V_C ($\sim 10^{-7}V$ or even less) and R_N . For instance, in small-size sandwiches with an area of 10×10 microns and a normal metal interlayer thickness of $0.1\mu\text{m}$, the normal resistance is of the order of $10^{-5}\Omega$ (for $\sigma_N^{-1} \approx 10^{-6}\Omega\text{cm}$). Such values of V_C and R_N are too low for any application of the Josephson effect (see Likharev and Ulrich, 1978).

Thus sandwiches with an "ordinary" metal interlayer are of rather limited interest. In contrast, the "ideal" Josephson effect can be realized by using a poorly conducting interlayer (low σ_N'/ξ'). This is just the situation that probably takes place in $S - SE - S$ type junctions with a doped silicon interlayer (Huang and Van Duzer, 1974, 1975; Schyfter et al., 1977; Hu et al., 1978). In using semiconductors, the situation does, however, become complicated due to the Schottky barriers appearing at the interlayer-electrode interfaces. Under these conditions the tunneling process at the interface interferes with the direct conductivity inside the semiconductor. An important problem today in weak superconductivity is that of working out

further experiments and developing a comprehensive theory, at least, for the dc processes in these structures.

From the practical viewpoint, $S-SE-S$ type sandwiches not only provide a means of raising V_C to the maximum possible values, but also of increasing considerably the normal resistance of structures, while keeping the capacitance negligibly small. Sandwiches with $V_C \geq 1\text{mV}$ and R_N of about 0.5Ω at an area $\sim 50\mu\text{m}^2$ have already been fabricated (Schyster et al., 1977). Such structures are quite suitable for most of the microwave devices, especially those which use arrays of such junctions (see Sec. VII). Moreover, these sandwiches are quite promising for utilization in Josephson cryotrons.

3. Proximity effect bridges

Proximity effect microbridges do not, in their physical processes, differ from $S-S'-S$ or $S-N'-S$ -type sandwiches (depending on the extent to which the critical temperature of the span is suppressed by the proximity effect). Their advantage over sandwiches is their small cross section and, consequently, their relatively high resistance R_N (which may reach 10^{-2} to $10^{-1}\Omega$), which simplify both the measurements and the practical applications of these microbridges. A definite drawback is the need to provide the smallest possible dimension ($L \sim \xi$) not in thickness, but in plane of the structure. However, even when this dimension is relatively large (of the order of one micron), the condition $L \leq \xi'(T)$ can be satisfied in a narrow range near the critical temperature of the link material T'_C .

These structures may have a single-valued $I_S(\varphi)$, but their V_C , as it follows, say, from Fig. 7, is low

$$V_C \sim (V_C)_{\max} \exp[-L/\xi'(T_C)] \ll (V_C)_{\max} \quad (126)$$

In the early realizations of proximity effect microbridges, where their structures were almost one-dimensional [Fig 1 (c)], values of V_C were in the range of a few nanovolts (Notarys and Mercereau, 1971; Kirschman, 1971), i.e., about 10^6 times less than the maximum possible values which are of the order of a few millivolts [Eq. (62)]. Later, these bridges had to be modified: the entire film was placed over a normal metal film, the span superconductivity being weakened by the decreasing thickness, which stimulated the suppression of $|\Delta|$ due to the proximity effect (Mercereau, 1974; Kirschman et al., 1975). Current concentration occurs in such structures, the suppression of $|\Delta|$ at the banks is reduced (see below), and V_C increases. However, due to lack of a very great difference between the thicknesses of the span and the banks, the characteristic values of V_C even in such modified bridges do not exceed tens of microvolts.

The smallness of V_C and R_N restricts the applications of proximity effect microbridges to ac SQUIDs (Kirschman et al., 1975; Falco and Parker, 1975); dc SQUIDs based on these bridges feature rather poor characteristics (Decker and Mercereau, 1973).

4. Ion-implanted microbridges

Recently several papers have been published dealing with a new method of preparation of one-dimensional weak links by implantation of ions into superconducting films. Implantation may be carried out in the span (Arrington and Deaver, 1975; Kirschman et al., 1977; Rachford et al., 1977; Boone et al., 1977) if it diminishes the film critical parameters, or in the banks (E. Harris, 1975; Harris and Laibowitz, 1977) in the opposite case. The physical properties of these microbridges [Fig. 1(d)] are close to the one-dimensional structures already discussed and they can have a rather limited application in physics or engineering.

One work (Harris and Laibowitz, 1977) is worthy of special mention: the banks were subjected to irradiation and thus their critical temperature was increased. As ℓ and $(\xi \sim \xi_0)^{1/2}$ diminished at banks in this process, the effective length L_{eff} was close to the physical value.

Therefore, for moderate current densities, the boundary conditions for Δ were close to those of the ODSEE model, allowing verification of the theory discussed in Sec. III on the effect of length on the critical current of weak links at $T \approx T_C \approx T_C$.

8.3.3 C. Dayem bridges

The only example of structures with weak current concentration is the constant-thickness bridge [the Dayem bridge, Fig. 1(e)] in which the concentration occurs in only one direction, i.e., along the film width.

1. Specific features of the processes in Dayem bridges

The concentration of current even in one direction improves the correspondence of the weak link to the ODSEE model. However, in Dayem bridges the weak link and the banks are made of the same material ($\sigma_N/\xi = \text{const}$). Hence the banks are easily involved in nonlinear processes. Indeed, current-induced suppression of $|\Delta|$ has the greatest influence in that region where the current density J is comparable with the maximum one (J_{max}). The length of this region, as is evident from Fig. 13

$$L_{\text{eff}} \approx \max(L, W), \quad (127)$$

is always greater than the bridge geometrical length L . Moreover, even if L_{eff} is less than ξ , the simple AL and KO-1 theories are not applicable to Dayem bridges because, as can easily be verified, the function f diverges (rather weakly, as $\ln r$) on receding from the bridge into banks. Therefore any theory of the AL type

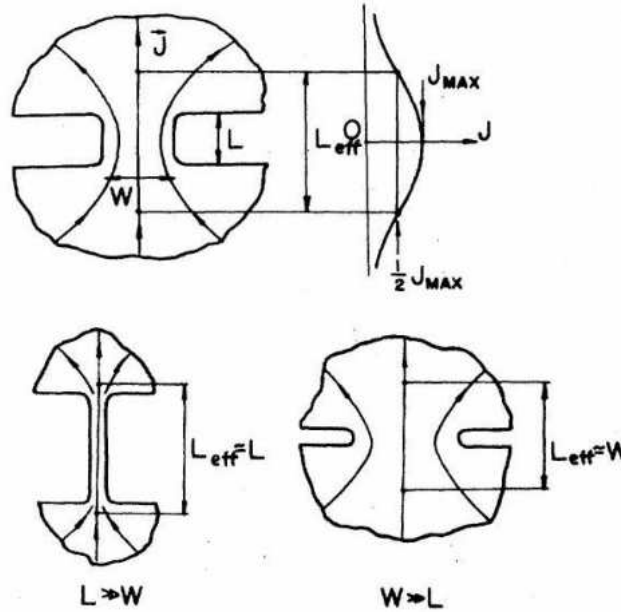


FIG. 13. Current concentration in Dayem bridges at various relationships of their length L and width W . Figure shows that current density falls off at the lengths $L_{\text{eff}} \approx \max[L, W]$. has to be reconstructed to be valid for Dayem bridges. This work has been carried out by Volkov (1973), who assumed the bridge to be bounded within hyperbolas with centers separated by a distance $2a \ll \xi$. The same result has been shown to hold valid for any shape (Likharev and Yakobson, 1975a), if the bridge boundaries at small distances ($r \ll \xi$) tend to lines emerging from one point.

The Volkov theory gives the following expression for the critical current

$$I_C = V_C/R_N(\xi) \quad (128)$$

where V_C , as in the AL theory, is expressed by Eq. (14), but $R_N(\xi)$ is the normal resistance of the bridge part inside a circle of radius ξ

$$R_N(\xi) \approx 2R_\square \ln(\xi/L_{\text{eff}}), R_\square = (\sigma_N d)^{-1}, \quad (129)$$

where R_0 is the normal (residual) resistance of the film per square. Even in the vicinity of T_C , the function $I_C(T)$ is evidently different from linear (see also Klapwijk et al., 1977b).

For Dayem bridges the $I_S(\varphi)$ relationship, just as in the case of one-dimensional structures, is somewhat uncertain, as it is not clear between which points the phase difference φ has to be taken. Since even at distances of $r \gg \xi$ inside the banks the current decreases slowly ($j_S \propto r^{-1}$), the phase continues to grow ($\propto \ln r$). Suppose that we are measuring the phase difference between bulk superconducting electrodes connected to the banks at $r \gg \xi$ (in this case they do not affect the processes occurring in the bridge). Then, as follows from Eqs. (21b) and (24), for $d \leq 2\lambda$, $dL_{\text{eff}} \leq \lambda^2$

$$\varphi_r = I_S \frac{16e\lambda^2}{c^2 \hbar d} \ln \frac{r}{L_{\text{eff}}} + \text{const}(r) \quad (130)$$

On comparing Eq. (130) with Eq. (48), we see that the electrodes make a contribution l_R to the effective kinetic inductance proportional to $\ln r$. In the dirty limit, using Eq. (24) and Eqs. (128) and (129), we obtain

$$l_R \approx \frac{1}{\pi} \ln(r/L_{\text{eff}}) \gg 1 \quad (131)$$

Even the minimum value of this contribution (when $r \sim \xi$) is of the order of unity.

Thus the dependences $I_S(\varphi_R)$ measured for Dayem bridges should be multivalued even when $L_{\text{eff}} \ll \xi$, as was observed in the experiments (Jackel et al., 1975, 1976a).

2. Large and small bridges

The aforementioned results apply, in fact, to smallsize bridges. Historically, starting with the pioneering work of Anderson and Dayem (1964),²² for a few years research was concentrated on bridges of relatively large size²³ ($L, W \approx 1 - 2\mu\text{ m}$) which, for the material used (Sn, In), satisfied the relation $L_{\text{eff}} \leq \xi$ only in a narrow temperature range close to T_C . Since this range was of the order of the phase transition width, the superconducting parameters of the film in this range were essentially inhomogeneous, and a comparison with the theory was impossible. At other temperatures, the bridge dimensions were noticeably greater than ξ , and the uniform flow of the condensate was disrupted by penetration of Abrikosov vortices (see Sec. III.E, and Fig. 12).

It is doubtful whether such large Dayem bridges could ever prove useful in any physical research or application, maybe with the exception of SQUIDS with relatively low sensitivity (see below). However, it is still more unfortunate that in the last few years considerable effort has been expended in fabricating constant-thickness bridges from such hard superconductors as Nb, Nb₃Sn, NbN, and others (Janoko et al., 1971, 1973, 1975; Laibowitz et al., 1975; Fujita et al., 1975; Holdeman and Peters, 1975, 1976; Golovashkin et al., 1976, 1978; Wu and Falco, 1977; Balla et al., 1977; Rachford et al., 1977). These films have extremely low coherence length (a few hundred angstroms or even less), and therefore remain "large" even if the bridge dimensions are as small as $0.1 - 0.2\mu\text{ m}$.

The main hurdle in microbridge fabrication in the early seventies was the formation of their sizes, usually by superposition of masks during film evaporation²⁴ Therefore, quite a significant advancement at that time was the "double scratch" technique (Gregers-Hansen et al., 1971, 1972) by which microbridges from soft materials (Sn, In, Al) of sizes of about 0.2 to $0.5\mu\text{ m}$

could be prepared without complex special equipment. Such sizes are quite sufficient for L_{eff} to be less than ξ in a noticeable temperature range close to T_C , and consequently most of the phenomena associated with the Josephson effect can be observed (Gregers-Hansen et al., 1971, 1972a, b; Gregers-Hansen and Levinsen, 1971; Jahn and Kao, 1973; Chiao et al., 1974; Pedersen et al., 1977).

Further progress in microbridge fabrication technology has been made in utilizing such methods as electron

lithography (Laibowitz, 1973; Laibowitz et al., 1974, 1975; Auracher and Van Duzer, 1974; McCarthy and Stanko, 1974; Jillie et al., 1975, 1976, 1977). These methods make possible high reproducibility in fabricating film structures with microbridges of submicron dimension.

3. Application of Dayem bridges

The main shortcoming of Dayem bridges from the application viewpoint is their low resistance R_N which is of the same order as the resistance per square $R_{\square}$ [Eq. (129)] and does not usually exceed a few tenths of an ohm.²⁵ It is not possible to increase R_N by decreasing the film thickness d or by raising R with impurities, because ℓ would decrease and the coherence length $\xi \propto l^{1/2}$ would become less than L_{eff} . Low resistance also causes strong heating effects (see Sec. V) and this further restricts the operating range of Dayem bridges close to T_C .

Moreover, the large value of l deteriorates parameters of the Dayem bridge as elements of those Josephson effect devices where the weak link is connected to the superconducting ring. For such devices, it is generally necessary that their main parameter l_{Σ} be

$$l_{\Sigma} = l_R + l, \quad l_R = \frac{2e}{\hbar c^2} \mathcal{L}_R I_C \quad (132)$$

where $\mathcal{L}_R$ is the ring inductance, to be less or slightly more than unity (for references see Footnote 2). A value of $l_{\Sigma} \gg 1$ leads to a loss of sensitivity proportional to l_{Σ} in ac SQUIDS, and proportional to l_{Σ}^{-1} in dc SQUIDD. If $l > 1$, then l_{Σ} can never be made less than unity for any inductance $\mathcal{L}_R$. This is why ac SQUIDS (Goodkind and Stolf, 1970; Murata et al., 1976) and dc SQUIDDs (Richter and Albrecht, 1973; Albrecht, 1976; Carelli and Modena, 1976; Kravchenko et al., 1976; Lobanov et al., 1976) utilizing Dayem bridges did not have very high sensitivity and had a narrow operating temperature range.

Likewise, as $l \gg 1$, the Dayem bridges are not suitable for use as quantron-type computer elements (Dynes et al., 1973; Anacker and Zappe, 1973; Likharev, 1974a, 1977a, b; Likharev et al., 1976; Hurrell, 1977). Finally, diffraction-type cryotrons cannot be based on these bridges either (see Sec. VII), as their width cannot be much greater than the effective length [Eq. (127)]. In all probability, Dayem bridges can prove useful only in several microwave devices if their microwave impedance is increased by connecting several bridges in series (see Sec. VI).

8.3.4 D. Structures with strong current concentration

In these structures both the weak link cross-section dimensions are much less than those of the electrodes, and therefore the current concentration now takes place in two dimensions. For example, in variable-thickness

⁰²² Almost at the same time these bridges began to be studied by other groups (Parks and Mochel, 1964; Lambe et al., 1964).

²³ A review of early work on Dayem bridges may be found in an article by Likharev (1971d).

²⁴ Only a recent communication (Dmitriev et al., 1976) reports the preparation of submicron-size bridges by this method.

⁰²⁵ The only exception is a work (Froome and Beck, 1976) in which Dayem bridges were reported to have R_N of the order of a few ohms. It is not clear whether the condition $L_{\text{eff}} \lesssim \xi$ was satisfied in these experiments.

bridges [Fig. 1(f)] the current is concentrated along both the thickness and the width of the film. In addition to these bridges, random microshorts in point contacts [Fig. 1(g)], blob-type junctions [Fig. 1(h)], and similar other structures belong to the class of high current concentration structures.

1. Special features of strong current concentration structures

The ODSEE model can easily be realized in these structures (Likharev, 1971d). Indeed, consider the proximity effect near the weak link-bank interface. The volume segment V' of the link contained in a distance $\sim \xi'$ from this interface is considerably less than the corresponding volume segment V of the bank at a distance $\sim \xi$ from the interface. Therefore a deviation in $|\Delta|$ from the equilibrium value at the bank would cause a much greater increase in the free energy $\Delta F \propto V$ than that created by the same deviation of $|\Delta|$ in a weak link: $\Delta F' \propto V'$. Although the order parameter in these structures is suppressed in the banks to depths $\sim \xi$ from the weak link, the amplitude of these changes is negligibly small. Hence the value of $|\Delta|$ at the boundaries remains close to the equilibrium values in the electrodes, and this happens irrespective of the materials of the links and banks.²⁶

Similarly, the second possible nonlinear effect also decreases, namely, suppression of $|\Delta|$ in the electrodes due to the current flow through the weak link. For example, in a variable-thickness bridge the current, on entering the banks from the span, rapidly diverges throughout the thickness and width. The current density is, therefore, high (of the same order as the depairing current) only at distances of the order of bridge film thickness d' from the interface, and the bridge effective length L_{eff} is almost the same as its geometrical length L . A detailed analysis (Likharev and Yakobson, 1975c) shows that the main condition [Eq. (64)] of the ODSEE model can be satisfied with any materials by decreasing the weak link cross section.

Structures with a current concentration in two dimensions have one more advantage in that the phase difference at the weak link can easily be measured. In fact, as the current in the banks diverges in two directions, its density decreases as r^{-2} with increasing distance from the weak link, and consequently the phase drop at the banks is finite

$$\varphi_r \propto \int^r j_s dr = \text{const}(r) + Cr^{-1} \quad (133)$$

Moreover, it is easy to show that with the decreasing weak link cross section this phase drop decreases (for instance, in variable-thickness bridges it is of the order of d'/L) and is usually very small. Therefore the phase difference between two remote points on the electrodes is almost exactly equal to the phase difference φ between the boundaries of the weak link and electrodes, which is the subject of calculation in the ODSEE model.

2. Point contacts and similar structures

Point contacts [see the review by Zimmerman (1972), and also paper by Gubankov et al. (1972)] are usually fabricated by lightly pressing a sharp needle point (the radius of the tip being no greater than a few tenths of a micron) onto the planar surface of another electrode; they can, however, be formed by several other methods (see, for example, Bondarenko et al., 1970a; Yanson, 1975). Numerous conducting microshorts are, as a rule, developed between electrodes in the oxide layer at the contact surface. The shape and exact dimensions of these shorts do not readily yield to investigation. By studying the magnetic field dependence of the critical current, we can only infer that the cross-section areas of these microshorts are much less than the area of mechanical contact between the electrodes.

²⁶ With the exception, of course, of the unlikely case where the normal conductivity of the bank material is much less than that of the weak link material.

Microshorts are responsible for the mechanical instability and total irreproducibility of the point contacts. While the first drawback can, to some extent, be eliminated using special mechanical adjusting systems (see, for example, Contaldo, 1967; Zimmerman et al., 1970; Nad', 1971; Buhrman et al., 1971; King, 1971; Edrich et al., 1974; Kulikov et al., 1974; Soerensen, 1974; Antyukh and Nad', 1975; Macfarlane, 1976; Neumaier, 1976; Cross and Blaney, 1977), the second shortcoming is insuperable, hindering both theoretical investigations and practical applications of point contacts.

Therefore, from the viewpoint of physical research, point contacts are only useful in verifying those theories where the geometrical structure of the weak link is unimportant. It would be very worthwhile to verify experimentally the $KO - 1$ and $KO - 2$ theories in particular, and the applicability of the tunnel theory to systems of very small microshorts (see Sec. II.C). So far such investigations have not been carried out, and only some evidence is available that $I_S(\varphi)$ functions can be single valued in point contacts (Jackel et al., 1974, 1976a; Rifkin and Deaver, 1976).

As regards the device applications, we may note that point contacts with the desired electrical characteristics can usually be prepared using repeated formation with mechanical adjusting or weak electrical discharges. The characteristic voltage V_C , in particular, can be made to attain the maximum possible theoretical value (i.e., $\sim 3\text{mV}$ at $T \sim 10\text{ K}$ and $T \leq T_C/2$), while the normal resistance can be varied from $\sim 10^2$ to $10^{-2}\Omega$, the function $I_S(\varphi)$ being almost sinusoidal. As a result of these convenient properties and ease of fabrication, point contacts find extensive application in all types of Josephson effect devices, with the exception of computers, where reproducibility is essential.

Recent advances in thin-film weak link fabrication technology, however, leave no doubt that point contacts will gradually be superseded in all fields of device applications by reproducible types of weak links. This situation is equally true of similar structures like blobtype junctions (Clarke, 1966; Manikopoulos and Hannam, 1973), microshorts in oxide films (Bondarenko et al., 1970a, 1976; Yanson, 1975; Lum and Van Duzer, 1975), and similar junctions, their only distinction from point contacts being still greater intrinsic capacitance, which complicates their applications.

3. Variable-thickness bridges

These bridges (Likharev, 1971d) are practically the only reproducible structures in which the ODSEE model is valid, i.e., the only means of avoiding nonlinear effects at the banks, for any length of the structure and for any (including identical) materials of weak link and electrodes. Even the first experiments with variablethickness bridges (Gubankov et al., 1973) confirmed the main conclusion of the theory that nonlinear effects are localized at the span film.²⁷ The relatively small-size tin bridges prepared in the same group (Gubankov et al., 1975, 1976, 1977) exhibited satisfactory Josephson properties at T close to T_C .

Similar bridges were fabricated almost at the same time by another research group (Klapwijk and Veenstra, 1974; Klapwijk and Mooij, 1975; Klapwijk et al., 1977) with a view to reducing energy dissipation (and, consequently, self-heating) by diminishing the span film thickness d' . Indeed, for fixed $\bar{V}$ and planar sizes, self-heating decreases proportionally to d' . As a result, even at temperatures far from the critical temperature, multivaluedness in the $I - V$ curves can be avoided.

In Sec. III we have already pointed out that normal metals can function as weak links just as well as superconducting metals if they have similar values of v_F . On the other hand, we have a wider choice of normal metals (Au, Ag, Cu, Al, and others) having high values of v_F ($\sim 10^8\text{ cm/s}$) than the choice of such superconductors (Sn, In). For both these classes of metals, ξ at temperatures far from the critical one is $\sim 0.1 - 0.2\mu\text{ m}$ for the real mean free paths ($\ell \sim d' \sim 10^{-5}\text{ cm}$), so that bridges with an L measuring a few tenths of a micron exhibit

the "ideal" Josephson effect. Besides, the normal metals listed above have sufficiently high melting points so as to safeguard against accidental burnouts. These arguments (Likharev, 1976) stimulated others (Komarovskikh et al., 1975) to fabricate $S - N - S$ -type bridges with niobium banks and a gold span. The resistance of such bridges was relatively high ($\sim 0.4\Omega$ for a width of $\sim 10 \mu\text{m}$) and self-heating was low even at temperatures far below T_C , where V_C was quite high ($\sim 400\mu \text{ V}$).

Some other groups, as well, have recently reported success in preparing "genuine" variable-thickness bridges. Octavio et al. (1977a, b) succeeded in fabricating tin microbridges in which self-heating noticeably suppressed the Josephson effect only at voltages as large as $\sim 3\text{mV}$. Boone et al. (1977) and Wang et al. (1977) have prepared microbridges with a tin and niobium span, while Yeh and Buhrman (1977) used lead. Sandell et al. (1977) used electron lithography to make indium variable-thickness bridges.

Some other structures closely resemble variable-

thickness bridges in their properties. For instance, the properties of proximity bridges after the above-mentioned modifications (Mercereau, 1974), which somewhat weaken the nonlinear effects at the banks, become more similar to those of the ODSEE model. The $S - N - S$ -type structures studied by Romanagan et al. (1975) and Gilabert et al. (1975) are also similar to variable-thickness bridges. However, owing to their large L , the Josephson effect was observed in them only at temperatures very close to T_C , so that the V_C 's of these bridges were rather low.

$S - N - S$ -type microbridges with Bi spans are described by Ohta et al. (1974). As a result of the low conductivity of bismuth and its moderate ξ , the ODSEE model should be valid for such structures, despite the relatively thick bridge span. Their merit from the point of view of device application is their high resistance R_N .

Furthermore, in bridges prepared by the "double scratch" method (see Sec. IV.C), if the first scratch (on the substrate) is quite shallow, the span thickness too becomes less than the bank thickness. Thus the properties of such bridges are close (though not identical) to those calculated in the ODSEE model. In particular, measurements have shown that for these bridges the $I_S(\varphi)$ relation is quite often single-valued (Jackel et al., 1976a).

Finally, in an effort to reduce the cross section of structures with a semiconductor interlayer, Schyfter et al. (1977) and also Kandyba et al. (1978) fabricated "coplanar" structures, which are, in reality, $S - N - S$ bridges in which a doped surface layer of the semiconductor substrate serves as the span. These structures have one serious drawback in that one of the planar dimensions (L) has to be kept rather small ($\sim 0.1\mu \text{ m}$), because the coherence length of the doped semiconductor is several times less than that of normal metals listed above. Their indisputable merit is relatively high normal resistance.

Naturally, variable-thickness bridges and similar structures are quite convenient in physical research into the processes occurring in weak links, especially the little studied nonstationary (ac) processes (see below). Such investigations have already been started; for instance, the basic principles of the theory of dc processes in weak links have been experimentally verified (Gubankov et al. 1976), and the first results with regard to ac processes have been published (Gubankov et al., 1977; Octavio et al., 1977b).

As regards applications, it is generally felt that variable-thickness bridges, given a proper choice of materials for the span and the banks, may meet almost all the practical requirements. Therefore, with advances in production technology (variable-thickness bridges are more complicated to produce than any other type of bridges) these weak links could certainly be incorporated into many of the Josephson effect devices. A serious competitor to them is, in all probability, the class of small-area sandwiches with an interlayer made of doped semiconductor

⁰²⁷ In these experiments the bridges were of large dimensions ($L, W \gg \xi$) and their resistive state was associated with Abrikosov vortex motion.

or any other normal material of relatively high normal resistance.

8.3.5 E. Controllable weak links

Once fabricated, the weak links considered in the previous pages should last indefinitely. Other types of weak links have been fabricated (or only conceived) which may exist only under continuous external influence.

Volkov (1971) theoretically examined a weak link between two sections of superconducting film separated by a region whose superconductivity is suppressed by incident electromagnetic waves, say, light. It was believed that suppression is effected by the nonzero mean square of the electrodynamic field.

After it was found (Testardi, 1971) that light pulses have a strong effect on superconductivity, it became evident that this effect takes place primarily due to photon breaking of the Cooper pairs, and the consequent establishment of nonequilibrium (higher) concentration of normal electrons and phonons [see reviews by Langenberg (1975) and by Chang and Scalapino (1977)]. Therefore another approach for weak links created by local irradiation of light was proposed (Wong et al., 1976, 1977). The authors believe that these structures are a particular kind of controllable weak link (CLINK) in which $|\Delta|$ is suppressed owing to the increased concentration of quasiparticles: normal electrons and phonons. In the same paper other methods of fabricating CLINKs were examined, in particular, by injection of quasiparticles through an additional tunnel junction. The first experiments (see also Gilabert et al., 1977) have shown that the processes in the system are far more complicated than what follows from an analysis based on simple phenomenological equations for the quasiparticle distribution proposed by Rothwarf and Taylor (1967).

Dolan and Lukens (1977) reported production of weak links by the action of a magnetic field on a short section of thin film strip. Outside parts were shielded from the field action with the help of another superconducting film.

Bevza et al. (1976) produced a weak link in a narrow strip, using suppression of $|\Delta|$ by a current flowing in a transversal strip. The whole structure was in the form of a cross. Suppression was probably effected by quasiparticles generated in the "control" strip, which is the first to attain its normal state.

It must be admitted that the structures created by means of external actions can be of use in studying the physics of these actions, say, of excitation and relaxation of the quasiparticles (see Secs. V, VI). Nonetheless, according to our classification, all these structures fall into the class of one-dimensional ones, i.e., do not have any noticeable current concentration at the weak link. Therefore, the banks are extensively involved in nonlinear processes, which gives rise to difficulties in their theoretical description.

8.4 V. JOSEPHSON OSCILLATIONS AND I - V CURVE

The ac processes in weak links are not as well understood as the dc processes, mainly due to the lack of appropriate microscopic theories. Thus we are compelled to describe these processes on the basis of equations derived strictly for some particular cases, or on a phenomenological basis. We shall begin our discussion with those processes that take place in the following simple experiment: a weak link is connected in series to a circuit with dc current $I = \bar{I}$.

So long as the current through the weak link is less than the critical one, it is transported by the superconducting condensate (I_S), the phase φ is constant in time, and the voltage between the electrodes is zero. If the current exceeds the critical one, it cannot be transported by only the condensate; consequently, at least a part

should be transported by normal electrons. But this "normal" current gives rise to a nonzero potential difference V with $\bar{V} \neq 0$. Hence the phase difference φ begins to change with the frequency ω_V [Eq. (6)].

In such an ac process, in the majority of weak links, the supercurrent I_s is a periodic function of the phase; thus the supercurrent oscillations occur in the element with this Josephson frequency, and probably with its harmonics, too. The manner in which this process (Josephson oscillations) occurs, and the consequent appearance of the current-voltage curves ($I - V$ curves), depends not only on the physics of conduction through the weak link, but also on the capacitance of the structure.

8.4.1 A. Influence of intrinsic capacitance

1. Impedance relationship

Josephson oscillations in real experiments have extremely high frequencies. Indeed, the typical voltage across a weak link is given by a factor $V_C = I_C R_N$ which, as we have seen in Sec. III, may be of the order of $k_B T / e$. In ordinary cases where T_C is of the order of several kelvin, we have $(V_C)_{\max} \sim 10^{-3}$ V and the frequency ω_C determined from Eq. (6) is about 10^{12} s $^{-1}$. Of course, $\bar{V}$ and ω_V may be far less; nevertheless, the arguments given below are equally applicable even to $\omega_V \sim 10^8 - 10^9$ s $^{-1}$.

At such high frequencies, the external electrodynamic system has a very large impedance Z_e as compared to that of the weak link. Indeed, $|Z_e|$ is usually on the order of the wave resistance of free space $4\pi/c \approx 377\Omega$. This is significantly greater than the intrinsic impedance of the weak link $|Z| \leq R_N$. Therefore the electric charge carried by the oscillating supercurrent cannot pass through an external circuit, and consequently at each moment the supercurrent should be compensated either by the normal current I_N or by the displacement current $I_D = C dV/dt$, where C is the capacitance between the electrodes.

The relation $|Z_e| \gg |Z|$, which holds true for the vast majority of weak links not switched into special lowohmic electrodynamic structures, precisely explains why an experiment with a constant current rather than constant voltage across the weak link is the simplest. If we attempt to set a constant voltage with the help of a source having low dc internal resistance ($R_e \ll |Z|$), the impedance relationship at the Josephson frequencies will, nevertheless, be the other way round, and it is again the current, but not the voltage, that will be constant in time. The dc resistance R_e only determines the slope of the "load line" $\bar{V} = E - R_e \bar{I}$ in the $[\bar{I}, \bar{V}]$ plane, but not the shape of the $I - V$ curves.

2. Capacitance, Josephson oscillations and $I - V$ curve

We now consider the conditions under which not only the current, but also the voltage V across the weak link may be constant in time. The effective impedance of the normal current I_N is on the order of normal resistance R_N , while the impedance due to the displacement current is $(i\omega C)^{-1}$. Therefore, if the condition

$$\omega C R_N \gg 1 \quad (134)$$

is satisfied at the frequency of the ac process, the ac component of the voltage is totally shunted by the capacitance of the structure, hence $V(t) \approx \bar{V}$. In other words, under the condition (134) the oscillating supercurrent is fully compensated by the displacement current. Therefore the normal current I_N and the voltage are constants. Substituting the characteristic frequency of the Josephson oscillations [Eq. (18)] into Eq. (134), we obtain the condition $\beta \gg 1$ [Eq. (17)] (Stewart, 1968; McCumber, 1968). If it is satisfied, then for any voltage across the

contacts, the voltage may be regarded constant in time, and phase φ varies according to the simple linear relationship

$$\varphi = \omega_V t + \text{const} , \quad \omega_V = (2e/\hbar)\bar{V}. \quad (135)$$

In contrast, if the capacitance of the structure is small ($\beta \ll 1$), then a significant ac voltage at the Josephson oscillation frequency and its harmonics is present across the junction. In this case the displacement current may be neglected in the analysis.

The influence of weak link capacitance on the ac electrodynamics is reflected in the shape of weak link $I - V$ curves. The effect of capacitance is illustrated in the following example. Let the relation $I_S = I_C \sin \varphi$ be true for a transient process. If the capacitance is large, the phase will change by the linear rule (135), and therefore there will be no constant component in the supercurrent

$$\bar{I}_S = I_C \overline{\sin}(\omega_V t + \text{const}) = 0, \quad (136)$$

i.e., the supercurrent will not contribute to $\bar{I}$ when $\bar{V} \neq 0$. If, however, $\beta \ll 1$, the phase will change at $\bar{I}_S I_C$ in a sharply nonlinear manner, the greater part of the period being close to $\pi/2 + 2\pi n$. Therefore the contribution of supercurrent in the $I - V$ curve is highly significant

$$T_S \approx I_C \sin(\pi/2) = I_C. \quad (137)$$

3. Capacitance of weakly linked superconductors

While for the tunnel junctions the capacitance is usually quite high ($\beta \gg 1$), for the majority of weak links it is negligibly small

$$\beta \ll 1. \quad (138)$$

The displacement current in metals is comparable with the conduction current only at very high frequencies, namely, of the order of the plasma frequency ω_p or the inverse scattering time v_F/l , both being $\sim 10^{14} - 10^{15} \text{ s}^{-1}$ (see, for example, Ziman, 1972). Therefore in structures where the entire space between superconducting electrodes is completely filled with metal (say, in $S - N - S$ or $S - S' - S$ -type sandwiches), the value of β is $\sim \omega_c/\omega_p \ll 1$.

In strong current concentration structures, R_N is proportional to the area of the weak link alone, while C

is proportional to a much greater area of the electrodes; therefore β becomes somewhat greater. In most cases, however, β remains much less than unity, the exception being the blob-type junctions [Fig. 1(h)] and some specially fabricated point contacts [Fig. 1(g)], for which the ratio of the electrode area to the area of conducting regions is as large as $10^4 - 10^6$.

Thus, for the majority of weak links, the displacement current is negligibly small. This is the reason for the great practical interest in weak links, since capacitance has a parasitic action on the useful characteristics of Josephson junctions. But, on the other hand, this creates some difficulties in studying the processes in weak links, since even in simple experiments the voltage cannot be assumed to be constant in time, and the phase varies according to a more complicated relation than the linear dependence of Eq. (135).

8.4.2 B. General pattern of $I - V$ curve

Although some details of the experimentally observed $I - V$ curves are as yet obscure, still their general pattern is more or less clear. We shall illustrate this with special reference to a case in which weak links may be described by tunnel theory [Eqs. (7) and (8)].

1. I-V curves in tunnel theory

In order to determine the $I - V$ curve of a structure with negligibly small capacitance within the framework of the tunnel theory, we have to find the frequency ω_V and coefficients A_n for which the current I [Eq. (8)] is constant and equal to a given value. Here in Eqs. (7) and (8) we may take $\omega = \omega_V$ because the voltage and phase oscillate only with the Josephson frequency and its harmonics.

Such a procedure has been carried out by McDonald et al. (1976) for the case $T = 0$, and by Zorin and Likharev (1977) for an arbitrary temperature. Their calculated results are shown in Fig. 14 by solid lines. For the sake of comparison, the dashed lines in the same figure show the $\text{Im } I_q(2e\bar{V}/\hbar)$ curves, which according to Eq. (15) represent the $I - V$ curve of a tunnel junction with high capacitance. We shall discuss the origin of the $I - V$ curves of weak links ($\beta \ll 1$, solid lines).

In the low-voltage range ($\bar{V} \ll V_C$) the mean current almost equals the critical level, due, as already mentioned, to the specific nature of the phase motion in this region. During the greater part of the Josephson oscillation period, the phase is close to $\pi/2$. Therefore the mean supercurrent is close to I_C (137). Once in each period the phase jumps by 2π , during which a voltage pulse of area ϕ_0/c [Eq. (118)] is generated at the weak link. This quick process, during which the current is transported primarily by normal electrons, lasts for a time of the order of

$$\tau_j = (\omega_C)^{-1} = \hbar/2eV_C, \quad (139)$$

and therefore the amplitude of the voltage pulse is

$$V_{\max} \approx \phi_0/c\tau_j = V_C. \quad (140)$$

As a consequence of these pulses, the weak link voltage spectrum is quite rich and contains about $V_C/\bar{V} \gg 1$ harmonics of the Josephson frequency.

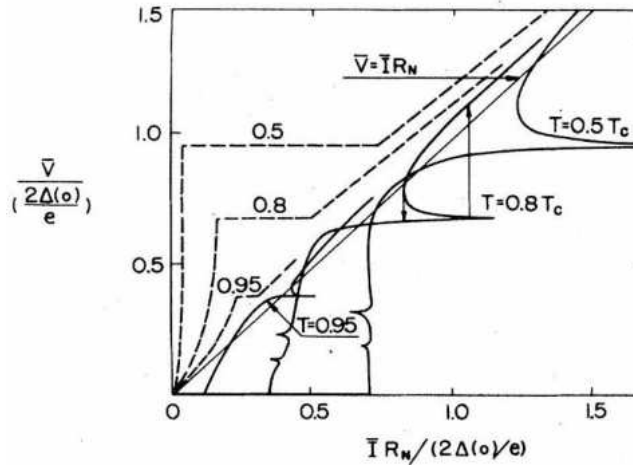


FIG. 14. $I - V$ curves of Josephson junctions within the framework of tunnel theory for negligibly small capacitance ($\beta \ll 1$, solid lines) and large capacitance ($\beta \gg 1$, dashed lines) at various temperatures. $I - V$ curves of weak links ($\beta < 1$) show singularities at voltages $\bar{V} = 2\Delta/(2n+1)e$. Arrows show the approximate hysteresis occurring at the $I - V$ curve for $T/T_c = 0.8$ when Riedel singularity is slightly smoothed.

In the high-voltage range, in contrast, the current is mainly transported by normal electrons, i.e., by the phase-independent component $\text{Im } I_q$. Hence the voltage is almost constant in time

$$\bar{V} \approx \bar{I} R_N \quad (141)$$

The phase changes almost by the linear relationship (135), supercurrent oscillates according to the same rule (15), and therefore its contribution to $\bar{I}$ is small. Thus, as $\bar{I} \rightarrow \infty$, the $I - V$

curve tends to the line (141) corresponding to normal resistance. This behavior, common to any type of weak link, may be regarded as "destruction of the superconductivity" only in a very narrow sense of the term. For example, in the tunnel theory, the supercurrent does have quite a finite amplitude despite the fact that $\bar{I}_S = 0$, and can have a noticeable influence on other effects, for instance, the emission of microwave radiation at frequency ω_V . Therefore those parts of the $I - V$ curve where $\bar{V} \neq 0$ are preferably called "resistive" rather than "normal."

In the intermediate-voltage range ($\bar{V} \sim V_C$) the current transport mechanism gradually changes from that of a supercurrent to that of a normal current. Figure 14 shows that in the case of the tunnel theory, certain peaks at voltages

$$V_n = 2\Delta(T)/(2n + 1)e, \quad (142)$$

the so-called "odd subharmonics of the gap," are superimposed on this gradual transition. They appear as a result of the fact that for each such voltage, the argument of one of the terms in Eq. (8) is of the value $4\Delta(T)/\hbar$, at which the real parts of this component have a singularity (Fig. 2). Physically, this means that electron tunneling with absorption of the quantum of the n th oscillation harmonic is possible at the voltage of Eq. (142).

2. Current relaxation time

Using the tunnel theory as an illustration, we have demonstrated that the current is periodically redistributed

between the normal and the superfluid components in the process of Josephson oscillations in weak links. In this theory redistribution occurs in a time of the order of τ_j [Eq. (139)], which, according to the expression for V_C [Eqs. (10)-(14)], is $\sim \hbar/\Delta$ when $T \rightarrow 0$, and $\sim \hbar k_B T/\Delta^2$ when $T \rightarrow T_C$. This is why we may regard the constant $\omega_C = \tau_j^{-1}$ as the characteristic frequency of the Josephson junction.

As follows from the microscopic theory of superconductivity (Mattis and Bardeen, 1958), τ_j is the characteristic time of linear processes in any superconducting specimen. It is reasonable to call τ_j the current relaxation time because it describes the result of the following simple experiment. Suppose that, starting from the moment $t = 0$, a constant current is made to flow in a long ($L \gg \xi$), thin ($S \ll \lambda^2$) superconductor. At the initial moment this current will be transported by normal electrons. Then in a time τ_j the superconducting condensate will be accelerated to an equilibrium velocity at which the whole current will be carried solely by the condensate and, consequently, the electrical field will vanish.

From Eqs. (7) and (8) it follows that τ_j is the only characteristic time in tunnel theory. Hence even in such a nonlinear effect as the Josephson effect, it is possible to have only one relaxation process, namely, relaxation of the current. Therefore, order parameter relaxation times or nonequilibrium excitation relaxation times, which have a significant bearing on the dynamics of other nonstationary effects in superconductors, do not appear in tunnel theory. This is because the electrodes are constantly in a state of equilibrium.

As a result, the Josephson effect in tunnel junctions is at present the only one of the nonlinear effects in superconductors for which a complete microscopic theory of nonstationary processes exists. Unfortunately, this theory is not adequate to describe weak links of finite dimensions ($Lp_F/\hbar \gtrsim 1$).

3. Resistively shunted junction (RSJ) model

When the temperature is close to the critical temperature, the characteristic voltage [Eq. (14)] in the tunnel theory becomes much less than the gap voltage $2\Delta/e$. Thus, in the low-voltage range

$$\bar{V} \leq 2\Delta/e, \quad (143)$$

the equations of the tunnel theory can be highly simplified. Indeed, as is obvious from Fig. 2, in this range we may take

$$\operatorname{Re} I_p \approx I_C, \quad \operatorname{Im} I_q \approx V/R_N, \quad |\operatorname{Re} I_p|, |\operatorname{Im} I_p| \ll I_C \quad (144)$$

and, retaining only two main terms in Eq. (8), we get

$$I = I_C \sin \varphi + V/R_N, \quad d\varphi/dt = (2e/\hbar)V. \quad (145)$$

This equation is known as the Resistively Shunted Junction (RSJ) model. In this model the time dependence of the phase satisfies a simple first-order differential equation.

$$\omega_c^{-1} d\varphi/dt + \sin \varphi = I(t)/I_C \quad (146)$$

Consequently it is relatively easy to analyze qualitatively many processes in the Josephson junction and in circuits with these junctions. Such an analysis can be found, for example, in the review of Vystavkin et al. (1974), and in the monograph of Likharev and Ulrich (1977).

For the simple process of Josephson oscillations ($I = \bar{I} = \text{const}$) we easily obtain the following expressions (Aslamazov et al., 1968; McCumber, 1968; Stewart, 1968; Aslamazov and Larkin, 1969)

$$\bar{V} = V_C \left[(\bar{I}/I_C)^2 - 1 \right]^{1/2} \geq 0, \quad \text{at } \bar{I} \geq I_C, \quad (147)$$

$$V(t) = V_C \frac{(\bar{V}/V_C)^2}{(\bar{I}/I_C) - \cos \omega_V t}. \quad (148)$$

Equation (147) shows that the $I - V$ curve in the RSJ model has hyperbolic resistive branches, while Eq. (148) gives a quantitative description of those features of the Josephson oscillations which we discussed qualitatively in the previous pages.

The RSJ model is really significant, in that it gives a correct description of not only a very restricted class of weak links, which obey the tunnel theory, but also of almost any weak link at temperatures close enough to the critical one.

4. RSJ model in weak links

In the same paper in which they advance a theory to explain the de Josephson effect in small weak links, Aslamazov and Larkin (1969) suggest that the ac processes occurring in weak links might be described by the RSJ model [Eq. (145)]. The logic of their argument runs approximately as follows:

The equation for the order parameter (21a) in a nonstationary case should contain some terms which depend on the time evolution of the order parameter. But if the weak link length tends to zero for a constant speed ($\sim \omega_C$) of Δ change, the magnitude of time-dependent terms should remain the same, while the gradient term grows as L^{-2} . Therefore, when $L \rightarrow 0$, the time-dependent terms may be neglected, and thus the Laplace equation (31), which leads to the solution (34), may be used as before.

In the equation for the current (21b), we have to take the normal current $\mathbf{j}_N$ into account when $V \neq 0$. An analysis of the linear nonstationary processes in superconductors (Mattis and Bardeen, 1958) shows that, for moderate frequencies ($\hbar\omega < 2\Delta$), we can use the same

expression for the supercurrent as in the stationary case, but with the addition of the normal current

$$\mathbf{j}_N(t) = \sum_{\omega} \sigma(\omega) \mathbf{E}_{\omega} e^{i\omega t}, \quad \text{when } \mathbf{E}(t) = \sum_{\omega} \mathbf{E}_{\omega} e^{i\omega t} \quad (149)$$

If $T \rightarrow T_C$, for all frequencies $\sigma(\omega)$ tends to σ_N , the conductivity of a material in its normal state. Hence we have

$$\mathbf{j}_N = \sigma_N \mathbf{E}, \quad \text{when } T \rightarrow T_C. \quad (150)$$

It is therefore natural to assume that for such nonlinear processes as the Josephson effect, we may also use Eq. (150), adding the current $\mathbf{j}_N$ to the supercurrent $\mathbf{j}_S$ [Eq. (21b)]. Thus, after repeating all the calculations of the AL theory, we arrive at Eq. (145), i.e., at the RSJ model. Nonetheless, it may be noted that the exact conditions for the applicability of this model are far from being clear.

5. Main deviations from the RSJ model

Experiments with a variety of weak links of sufficiently small dimensions ($L, W \lesssim \xi$) have shown that near the critical temperature the $I - V$ curves are indeed close to the pattern predicted by the RSJ model, provided account is taken of the influence of intrinsic and external fluctuations which are significant when $T \rightarrow T_C$. This influence is large if the critical current of a weak link is less than or comparable to the characteristic amplitudes of external (usually relatively low-frequency) interferences I_{fl} and of the intrinsic thermal noise of the junction I_{fh} (see, for example, Likharev and Ulrich, 1977). I_{fh} depends on temperature only

$$I_{fh} = 2ek_B T / \hbar, \quad I_{fh} [\mu \text{ A}] \approx 0.042 T [\text{ K}], \quad (151)$$

while I_{fl} depends on the experimental setup, and may be as high as $10 - 30 \mu \text{ A}$. By appropriate filtering, it can, however, be kept below I_{fh} .

Fluctuations smooth out the $I - V$ curves as shown in Fig. 15. Therefore, as $I_C(T)$ increases, at first an "embryo" with a characteristic size $I_f = \max [I_{fh}, I_{fl}]$ is formed on the $I - V$ straight line of a normal junction, its amplitude increasing as I_C^2 . With a further increase in the current ($I_C \gtrsim I_f$), the $I - V$ curve becomes nearly hyperbolic [Eq. (147)], with only the angles of transition from the superconducting region to the resistive branches being smoothed out. In well designed setups, where $I_{fl} \lesssim I_{fh}$, the influence of fluctuations is hardly noticeable

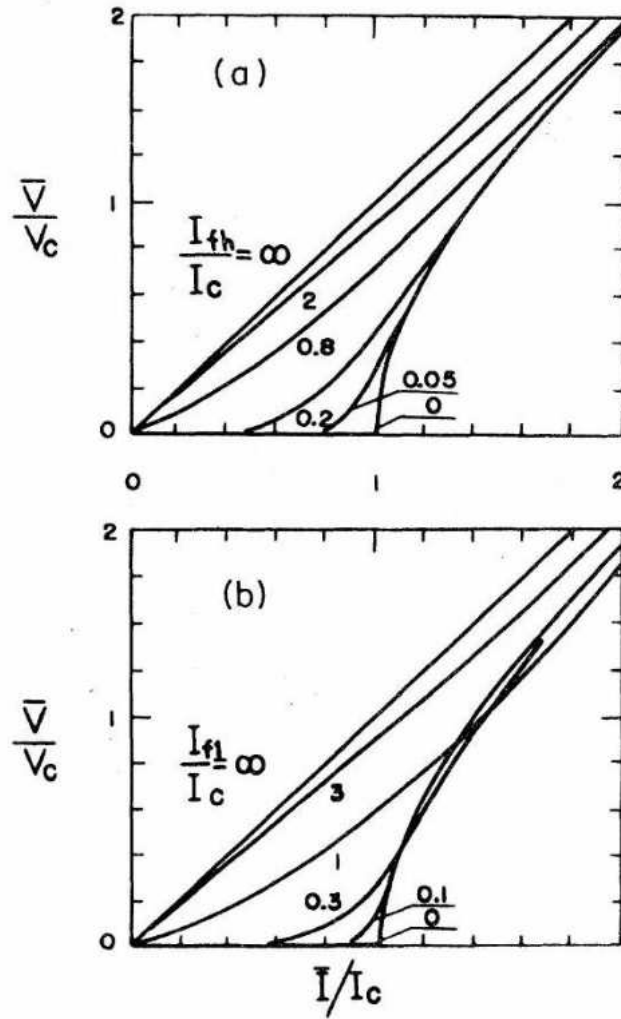


FIG. 15. Smoothing of $I - V$ curve (a) by the intrinsic thermal noise and (b) by low-frequency external interferences. $I - V$ curves are taken according the RSJ model, intrinsic noise is considered as white (δ -correlated), and external fluctuations are expected to have normal (Gauss) distribution (after Stratonovich, 1967; Ambegaokar and Halperin, 1969; Kanter and Vernon, 1970).

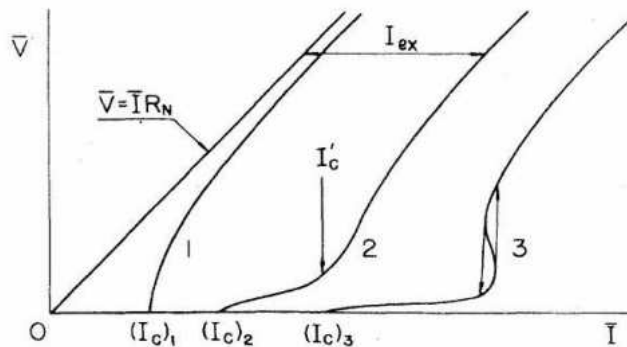


FIG. 16. $I - V$ curve of a short weak link according to the RSJ model (1) and experimental $I - V$ curves showing (2, 3) the excess current effect; (2, 3) 'bump' and (3) hysteresis at lower temperatures (3) (schematically).

at critical currents of more than tens of μA .

Nevertheless, the $I - V$ curves of small weak links²⁸ are more or less close to the hyperbolic form only in a narrow region near T_C (Fig. 16). At lower temperatures, at least three peculiarities not consistent with the RSJ model appear on the $I - V$ curves (see, for example, Clarke and Lindelof, 1976; Gubankov et al., 1977; Klapwijk et al., 1977b):

(i) As the current increases, the voltage tends to a straight line

$$\bar{V} = R_N (\bar{I} - I_{\text{ex}}) = \bar{I} R_N - V_{\text{def}}, \quad (152)$$

which is shifted up the I axis by an "excess current" I_{ex} relative to the normal $I - V$ curve, or in other words, by an amount V_{def} down the V axis. The excess current is usually of the order of I_C (Pankove, 1966).

(ii) As soon as the current I exceeds I_C , the voltage grows relatively slowly in the first stages with a slope

$$R_d = d\bar{V}/d\bar{I} \ll R_N, \quad (153)$$

and thereafter, when the current attains a specific value I'_C the voltage begins to increase rapidly, so that the differential resistance R_d sharply increases and then proceeds almost as in the RSJ model. This behavior is usually referred to as a "bump" on the $I - V$ curve.

(iii) When the temperature falls to a certain value, the differential resistance at points above the bump becomes infinite, and a further reduction in T gives rise to a negative slope region on the $I - V$ curves. If the internal resistance of the current source is high ($R_e \gg R_N$), it results in the hysteresis observed on the $\bar{V}(\bar{I})$ dependence (Fig. 16).

At present, at least one theoretical explanation has been found for each of these phenomena within the framework of the models, which are all generalizations of the RSJ model. Let us start discussion of the models. ²⁹

8.4.3 C. Description by time-dependent Ginzburg-Landau (TDGL) equations

1. TDGL approximation

The main difficulty in the microscopic theory of nonlinear nonstationary effects in superconductors (Gor'kov and Eliashberg, 1968) lies in the fact that the gap in the electron energy spectrum changes with the variations in $|\Delta|$, together with the position of square-root singularity in the density of the electron states (Bardeen et al., 1957). But in some cases this singularity does not exist, although the order parameter is finite in magnitude. This happens, for instance, in strong magnetic fields or in superconductors with a high concentration of paramagnetic impurities (see, for example, De Gennes, 1966).

Gor'kov and Eliashberg (1968b) have shown that the microscopic theory for this case allows for a simple way of generalizing GL equations [Eqs. (21)] in a nonstationary case. ³⁰ These Time-Dependent Ginzburg-Landau (TDGL) equations differ from the stationary equations (21) in that their right-hand sides contain new terms

$$\tau_\Delta \left(\frac{\partial}{\partial t} + i \frac{2e}{\hbar} \mu \right) \Delta, \quad (154a)$$

$$\sigma_N \left(-\nabla \mu - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} \right). \quad (154b)$$

⁰²⁸ Here and in the succeeding pages we shall discuss only the best investigated case of dirty weak links ($l \ll L$), of which all the bridges are a particular case. In relatively pure structures, in addition to the peculiarities listed below, some small singularities are observed on $I - V$ curves at voltages $V_n = 2\Delta / ne$; see, for example, Puma and Deaver, 1971; Gregers-Hansen et al., 1973. They were also discussed by Werthamer (1966), and by Hasselberg et al. (1974).

²⁹ The $I - V$ curves of wide weak links ($W \gtrsim \lambda_{\text{eff}}$) have peculiarities of their own; see Sec. VII.

Here μ and $\mathbf{A}$ are the scalar and vector potentials of the electromagnetic field; hence Eq. (154b) gives the normal current $\sigma_N \mathbf{E}$.

Let a uniform specimen ($\nabla = 0$), free of current ($\mu = \mathbf{A} = 0$), exist in a state with $\Delta = 0$ at $T < T_C$. From Eq. (21a), after adding (154 a), it follows that the initial stage of the Δ variation will be described by the equation

$$\tau_\Delta (\partial \Delta / \partial t) = \Delta. \quad (155)$$

Hence it is obvious that the constant τ_Δ has the meaning of a characteristic time of the order parameter modulus relaxation. For the case examined by Gor'kov and Eliashberg (1968b) the time constants τ_Δ and τ_j are related by the equation

$$\tau_\Delta / \tau_j = 12 \quad (156)$$

However, the TDGL equations have repeatedly been examined for their feasibility for the usual ("gap") superconductors (Maki, 1963; Schmid, 1966; Abrahams and Tsuneto, 1966). In such a phenomenological generalization of GL equations, the ratio between τ_Δ and τ_j remains close to Eq. (156). Indeed, substituting the expression for ξ in the dirty limit into (21a), and rewriting it in the form of a stationary diffusion equation (for the sake of simplicity, we shall take $\mathbf{A} = 0$)

$$D \nabla^2 \Delta + \frac{8k_B (T_C - T)}{\pi \hbar} \left[\pm 1 - \frac{|\Delta|^2}{\Delta_0^2} \right] \Delta = 0 \quad (157)$$

If we assume that Eq. (157) can be generalized as an ordinary diffusion equation by the mere addition of $\partial \Delta / \partial t$,

then we obtain the first TDGL equation with

\$\$

$$\tau_\Delta = \pi \hbar / 8k_B (T_C - T) \quad (158)$$

\$\$

Now using Eqs. (14) and (18) for $\omega_C = \tau_j^{-1}$ at $T \approx T_C$, we find

$$\tau_\Delta = (\pi^4 / 14 \zeta(3)) \tau_j \approx 5.79 \tau_j, \quad (159)$$

which differs from Eq. (156) only by a factor of about 2. Of course, extreme caution should be taken in using this value, as the TDGL generalization is not justified in the case of gap superconductors.³¹

2. Boundary effects and TDGL equations

The TDGL equations not only provide a simple means of describing the relaxation of $|\Delta|$ in time, but also give a simple description of a radically different, but equally fundamental process, that is, of the flow of current normal to the interface between normal and superconducting phases (Reiger et al., 1971; Yu and Mercereau, 1972; Volkov, 1974). Similar effects occur during the motion of Abrikosov vortices (Kupriyanov and Likharev, 1972).

We shall rewrite these equations for a one-dimensional geometry $\Delta = \Delta(x)$, again considering the case of small cross sections and low fields, where we may take $\mathbf{A} = 0$.

³⁰ Recently the validity of this generalization has been demonstrated experimentally (Amato and McLean, 1976).

$$\tau_{\Delta} \left(\frac{\partial \Delta}{\partial t} + i \frac{2e}{\hbar} \mu \right) \Delta = \xi^2 \frac{\partial \Delta^2}{\partial x^2} + \Delta \left(\pm 1 - \frac{|\Delta|^2}{\Delta_0^2} \right), \quad (160a)$$

$$j = C_j \operatorname{Im} \left(\Delta^* \frac{\partial}{\partial x} \Delta \right) - \sigma_N \frac{\partial \mu}{\partial x} \quad (160b)$$

After expressing Δ in the form $|\Delta| \exp(i\chi)$, and separating the real and imaginary parts in (160a), we find that the latter give only the natural equality $\partial j / \partial x = 0$. After differentiating (160b) with respect to x , we obtain a system of equations equivalent to (160)

$$\tau_{\Delta} \frac{\partial |\Delta|}{\partial t} = \xi^2 \frac{\partial^2 |\Delta|}{\partial x^2} + \left[\pm 1 - \frac{|\Delta|^2}{\Delta_0^2} - \left(\xi \frac{\partial \chi}{\partial x} \right)^2 \right] |\Delta| \quad (161a)$$

$$\tau_{\Delta} \left(\frac{\hbar}{2e} \frac{\partial \chi}{\partial t} + \mu \right) (|\Delta|^2 / \Delta_0^2) = \xi^2 \tau_j \frac{\partial^2 \mu}{\partial x^2} \quad (161b)$$

The first equation describes relaxation of $|\Delta|$ to the stationary value $|\Delta|(x)$ in time, while the second gives the time and space variations of μ , i.e., of the electrical field.

Let, for instance, the distribution $\Delta(x)$ be established at the interface between a normal and a superconducting phase in the absence of current. The order parameter changes, according to Eq. (161a), at a distance $\sim \xi$ from 0 in the N phase to Δ_0 in the S phase (Fig. 17). Now if a weak current is induced to flow through the $S - N$ interface [see the experiments of Pippard et al. (1971), Yu and Mercereau (1972, 1975), Hardin et al. (1974)], $\xi \partial \chi / \partial x$ in (161a) will be small, the distribution $\Delta(x)$ will not change, and Eq. (161b) will have a solution

$$\partial \chi / \partial t = -(2e/\hbar) \mu_p = \text{const}, \quad \mu = \mu(x)$$

This shows that the superconducting condensate has

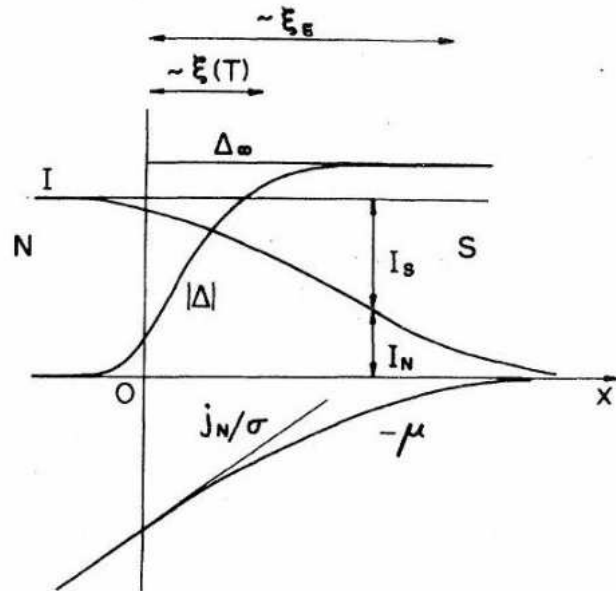


FIG. 17. Modulus $|\Delta|$ of order parameter, electric field potential μ , normal current and supercurrent vs distance from normal-superconductor interface (schematically).

the same potential μ_p , related to the order parameter phase by the Josephson equation (4), over the entire region with $\Delta \neq 0$ (even in those places where the electrical field $E = -\nabla \mu$

⁰³¹ The TDGL type of equation may possibly hold true in $S - N - S$ structures, as there is no gap in the excitation spectrum of a material with $\Delta = 0$.

exists). As regards μ , it defines the potential of normal electrons, and its evolution in space is given by Eq. (161b)

$$\xi_\mu^2 \frac{d^2 \mu}{dx^2} = (\mu - \mu_p) (\Delta^2 / \Delta_0^2), \quad \xi_\mu = \xi (\tau_j / \tau_\Delta)^{1/2} \quad (162)$$

This equation describes the change of μ from a constant value ($\mu = \mu_p$) inside the region S to a linear increase ($\partial\mu/\partial x = -j/\sigma_N$) inside the region N (Fig. 17). This change, and consequently the conversion of supercurrent into a normal current, takes place over a length ξ_μ .

Thus the TDGL equations contain two characteristic length constants (ξ and ξ_μ) which describe the scale of space variations in the modulus of the order parameter Δ and normal electrical current j_N , respectively. When Eqs. (156) or (159) are valid, ξ_μ is somewhat less than ξ .

3. Applications of TDGL equations

It is relatively easy to analyze the nonlinear ac processes occurring in superconductors with the help of TDGL equations. They have, therefore, been applied to many specific processes, such as the penetration of a strong microwave field into a bulk superconductor (Gor'kov and Eliashberg, 1968c), the action of a microwave field on thin films of infinite (Kulik, 1969a) and of finite (Likharev, 1971c) dimensions, the motion of Abrikosov vortices under the action of dc current (Schmid, 1966; Gor'kov and Kopnin, 1971; Kupriyanov and Likharev, 1972; Danilov et al., 1974) and microwave fields (Kupriyanov and Likharev, 1975), fluctuations at $T > T_C$ (Schmid, 1969) and $T < T_C$ (Kulik, 1970), and processes in superconducting thinfilaments when $\bar{I} > I_C$ (Fink, 1973; Fink and Poulsen, 1974, 1975; Likharev, 1974; Kramer and Baratoff, 1977). These equations have also been used in analyzing the processes in weak links (Reiger et al., 1972; Volkov, 1974; Likharev and Yakobson, 1974, 1975b, c; Volkov and Kasatkin, 1974; Jensen and Lindelof, 1976; Kramer and Baratoff, 1977).

Setting aside the last group of papers for the time being, we find that a comparison of the results derived from the TDGL equations with the experimental data on gap superconductors shows (Kupriyanov and Likharev, 1976) that this approximation does not hold true for all the above processes. This is quite evident, especially, in the experiments on the measurement of the uniform relaxation time of the order parameter. Thus, for thin film strips of tin (Peters and Meissner, 1973; Andratskii et al., 1976), it was found that τ_Δ is $\sim 10^{-10}$ s for $T \approx 0.9T_C$, and the temperature dependence of τ_Δ is close to $(T_C - T)^{1/2}$. This is not consistent with Eqs. (156) and (159), which give $\tau_\Delta \sim 10^{-11} - 10^{-12}$ s and $\tau_\Delta \propto (T_C - T)^{-1}$.

A noticeable discrepancy is also observed for the processes in long filaments when $\bar{I} > I_C$, for which the experiments (Scopol et al., 1974b; Meyer and Tidecks, 1976; Klapwijk et al., 1977) give somewhat greater length for the penetration of normal current into superconductors than the length derived from Eq. (162). Essentially, a value of τ_Δ cannot be chosen so as to obtain agreement with the two groups of experiments, because in the first group τ_Δ has to be greater than τ_j , whereas in the second group it is vice versa. (The discrepancy can be explained only by taking into account the nonequilibrium distributions of quasiparticles—see Sec. V.D.)

In a large group of experiments (mainly on the motion of Abrikosov vortices) however, the TDGL equations give results in satisfactory agreement with experiment, provided $\tau_\Delta/\tau_j \sim 10$. In these experiments Δ changes at distances $\sim \xi(T)$, i.e., approximately as in weak links of length $L \sim \xi$. This gives grounds for believing that the TDGL equations may, at least qualitatively, give a correct description of some of the effects in weak links if we take $\tau_\Delta \sim 10\tau_j$.

4. Processes in weak links

Several essential errors were committed in the paper (Reiger et al., 1972) in which the TDGL approximation was first applied to an analysis of ac processes in weak links.³² They were subsequently detected and eliminated by Likharev and Yakobson (1975b, c).

For weak links of sufficiently small length [$L \leq \xi(T)$], according to the results derived in Sec. III, we may neglect the term $\Delta(\pm 1 - |\Delta|^2/\Delta_0^2)$ in Eq. (160a) as compared to the gradient term $\sim (\xi/L)^2 \Delta$ and to the relaxation term $\sim (\tau_\Delta \partial \Delta / \partial t)$. The characteristic value of $\partial / \partial t$ is $\sim \tau_j^{-1}$, where τ_j is a parameter of the electrode material, or $\sim \tau_j^{-1} A^2$, if we mean τ_j of the weak link material. From

Eq. (160a) it then follows that the TDGL equations for short links have a dimensionless parameter

$$\gamma = \frac{\tau_\Delta}{\tau_j} \left(\frac{L}{\xi} \right)^2 A^2, \quad (163)$$

which characterizes the influence of the finite relaxation time of the order parameter. According to the estimate of τ_Δ / τ_j made above, γ can be taken as small for short structures, i.e., $\gamma \rightarrow 0$ if $L/\xi \rightarrow 0$. In this case the system of equations (160) under the boundary conditions of the ODSEE model [Eq. (64)] makes possible a complete analytical solution.

First, let us consider a region near the critical current: $\bar{I} \gtrsim I_C$. Since $\gamma \rightarrow 0$, the relaxation terms make only a small correction to the AL theory. According to these equations, Δ is given by Eq. (71), and μ is the same as in the normal state

$$\mu = \mu_1 - \frac{x}{L} \mu_2, \quad V = \mu_1 - \mu_2. \quad (164)$$

After substituting the solution into the left-hand side of (160a), we obtain the correction $\tilde{\Delta}$ to this solution. Now by substituting the total solution $(\Delta_{A1} + \tilde{\Delta})$ in (160 b), and then integrating it with respect to length ($0 \leq x \leq L$), we obtain an expression for the current $I = I_C \sin \varphi + \frac{V}{R_N} [1 + \frac{\gamma}{15}(1 - \cos \varphi)]$, if $\gamma/15 \ll 1$, where I_C has the same value as in the dc effect.

Now consider the region $\bar{I} \gg I_C$, in which the normal current is much greater than the supercurrent, allowing us to use Eq. (164) as a first approximation for μ . In this case the solution of the TDGL equation for Δ may be written as

$$\Delta(x, t) = \Delta \left[\Psi \left(\frac{x}{x_0} \right) \exp(i\chi_1) + \Psi \left(\frac{L-x}{x_0} \right) \exp(i\chi_2) \right] \quad (166)$$

where the function Ψ describes the decay of the electrode wave function inside the weak link and satisfies the equation

$$\frac{d^2 \Psi(u)}{du^2} + iu \Psi(u) = 0, \quad \Psi(0) = 1, \quad \Psi(L/x_0) = 0. \quad (167)$$

Here x_0 is the characteristic length which depends on $\bar{V}$, i.e., on the position of the bias point on the $I - V$ curve

³² The fundamental error is that τ_Δ has been taken as zero in all numerical calculations. It can easily be shown that in this case the system (160) is reduced to a phase equation similar to that given by the RSJ model.

$$I = I_S(\varphi) + V/R_N$$

where $I_S(\varphi)$ is the dc current-phase relation. It follows, in particular, from this equation that an excess current only exists in weak links for which $I_S(\varphi)$ is multivalent, which is in obvious contradiction to experiment. The validity of this model for not very large currents inferred by Jackel et al. (1976a) is, in all probability, a consequence of the rather limited amount of experimental data used in the analysis.

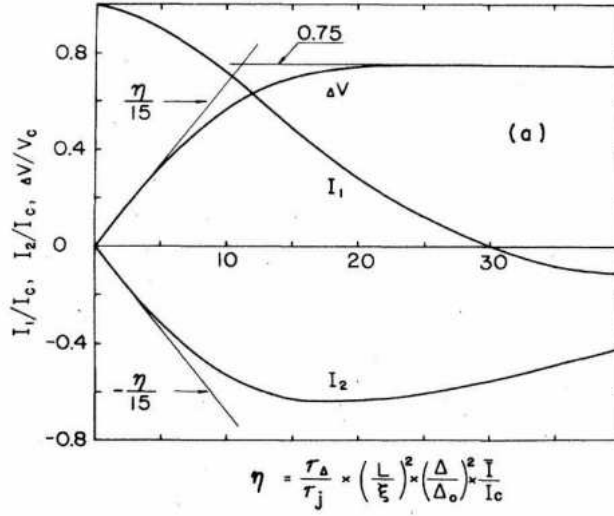
$$x_0/L = \eta^{-1/3}, \quad \eta = \gamma \frac{\bar{I}}{I_C} = \left(\frac{L\Delta}{\xi\Delta_0} \right)^2 \omega_V \tau_\Delta. \quad (168)$$

After finding the solution of Eq. (167) in terms of Airy functions (see, for example, Abramowitz and Stegun, 1965), then substituting (166) in Eq. (160b), and integrating with respect to x , we obtain an expression for the current, which may be written as follows (Jensen and Lindelof, 1976)

$$I = I_1(\eta) \sin \varphi + I_2(\eta) \cos \varphi + (V - \Delta V(\eta))/R_N \quad (169)$$

The dependences of I_1 , I_2 , and ΔV on η , i.e., on the position of the bias point, are shown in Fig. 18.

We note here that Eqs. (165) and (169) agree in the intermediate range $1 \ll \bar{I}/I_c \ll \gamma^{-1}$; therefore together they give a complete solution which is valid for any current through the weak link.



(b)

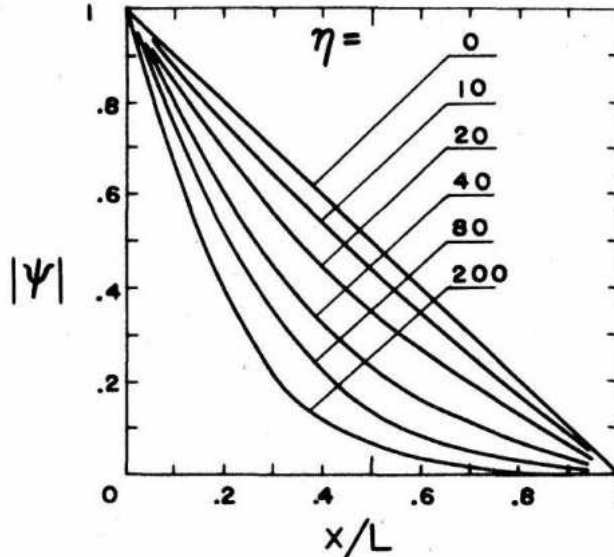


FIG. 18. (a) Frequency dependence of I_1 , I_2 , and ΔV from Eq. (169), and (b) $|\psi|$ distribution over the length of a weak link at different currents in TDGL approximation. Quantities $I_{1,2}$ give the amplitudes of Josephson supercurrent and fall as $\eta^{1/4} \exp\left(-\frac{\sqrt{2}}{3}\eta^{1/2}\right)$ at $\eta \rightarrow \infty$. Deviation of V from the normal value (ΔV) is close to $\bar{V}(\gamma/15) = V_C(\eta/15)$ when $\eta \lesssim 5$, and is almost constant ($\Delta V \approx V_{\text{def}}$) at $\eta \gtrsim 20$, describing the "excess current" effect: $I_{\text{ex}} = \Delta V/R_N \approx 0.75I_C$.

Function ψ enters in to Eq. (166) and shows the decrease of the electrode wave function in the weak link. At $\eta = 0$, $\psi = 1 - x/L$, and ψ tends to the universal function of x/x_0 (Fig. 19) when $\eta \rightarrow \infty$. After Likharev and Yacobson, 1975b, c; Jensen and Lindelof, 1976.

5. Discussion

Equation (165) shows that the $I - V$ curve of a short weak link is nearly hyperbolic (as predicted in the RSJ model). Its asymptote, however, differs from the normal $I - V$ curve by ΔV (Fig. 18). For small currents this gives only a slight decrease in the effective resistance of the weak link

$$\frac{R_N - R_{\text{eff}}}{R_N} = \frac{\gamma}{15} = \frac{1}{15} \frac{\tau_\Delta}{\tau_j} \left(\frac{L\Delta}{\xi\Delta_o} \right)^2 \quad (170)$$

For large current ΔV tends to the constant

$$\Delta V \rightarrow V_{\text{def}} = 2\epsilon V_C / R_N, \quad \epsilon \approx 0.375, \quad V_{\text{def}} [\mu \text{ V}] \approx 475 (T_C - T) [\text{K}]. \quad (171)$$

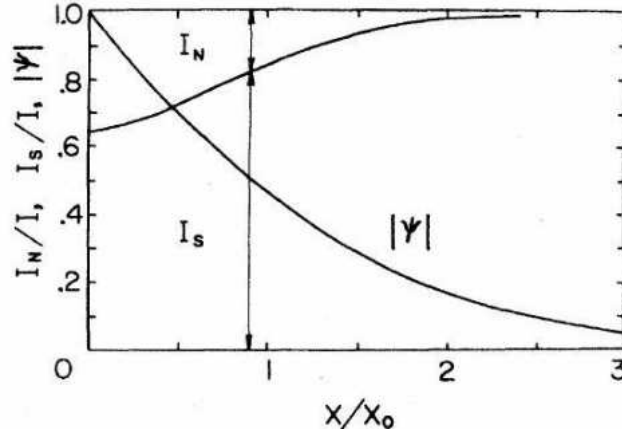


FIG. 19. Space dependences of $\psi \propto \Delta$, I_N , and I_S near the weak link-bank boundary at the large currents ($x_0 \ll L$; $\eta \gg 1$) according to the ODSEE-TDGL model. Normal current I_N is equal to zero in the bank, so I_N and I_S have jumps at the boundary. Variables Δ and μ are continuous. After Likharev and Yacobson, 1975b, c.

Consequently the $I - V$ curve tends to the straight line (152) with an excess current $I_{\text{ex}} = V_{\text{def}} / R_N \approx 0.75 I_C$ which does not depend on τ_Δ and other parameters of the weak link material (T'_C, ξ), and on L as well.

The reason for the nonzero values of I_{ex} within the framework of the TDGL equations is that with increasing current, the distribution of $|\Delta|$ over the link length begins to deviate from that derived in the AL theory [Eq. (71)], where $\Psi = 1 - x/L$ (see Fig. 18b). When the length x_0 , which characterizes the level of current-caused suppression of the order parameter, becomes less than L , the function Ψ begins to decay from 1 at the banks to zero just over this range. Thus the electrode wave functions overlap range decreases and the Josephson effect, characterized by the supercurrent amplitude $I_A = (I_1^2 + I_2^2)^{1/2}$, is suppressed.

For large currents ($\eta \gg 1$), the wave functions decay so rapidly that they are hardly overlapped at all. Therefore the weak link is wholly in a normal state, except that $|\Delta|$ is different from zero in regions of length x_0 near the banks, where the superconductivity is supported by the proximity effect between the weak link and the banks (Fig. 19). In these regions, part of the current is transported by the superfluid component, hence the total voltage drop V across the weak link is less than the normal one by V_{def} . Thus the "excess current" effect³³ is a natural phenomenon in the TDGL model, and it is described by Eq. (171), which

does not depend on τ_Δ . The latter fact suggests that the result is not very critical to changes in the model.

Other effects, such as bumps and hysteresis on the $I - V$ curves, do not appear for short weak links in this model. The following procedure was adopted by Jensen and Lindelof (1975) and Clark and Lindelof (1976): the equation for the phase $\varphi(t)$ derived from Eq. (169) was solved for all currents. A bump resembling the experimentally observed one appears on the $I - V$ curves if $\gamma \gg 1$.

It should, however, be borne in mind that Eq. (169) is only valid when either $\gamma \ll 1$, or $\gamma \sim 1$ and $\bar{I} \gg I_C$. If $\gamma \gtrsim 1$,

³³ Our analysis has shown that this should be called a "voltage deficiency" effect.

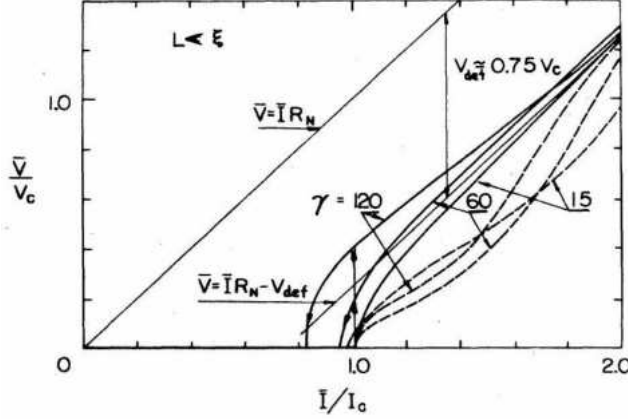


FIG. 20. $I - V$ curve of short ($L \lesssim \xi$) weak link, calculated numerically from Eqs. (160) for various γ [Eq. (163)]. Dashed lines show results of Jensen and Lindelof (1976) obtained from Eq. (169), which is really applicable only if $\gamma \ll 1$ and/or $\bar{I} \gg I_C$. Figure shows that TDGL equations in fact do not describe the "bumps" on $I - V$ curves. Small hysteresis appears on $I - V$ curves at very large γ .

and $\bar{I} \sim I_C$, the system of equations (160) has to be solved numerically (Likharev and Yakobson, 1975b, c; Kramer and Baratoff, 1977). Such numerical results are shown in Fig. 20 by solid lines. It is obvious that there is no bump on the $I - V$ curves, and the only new situation is the appearance of slight hysteresis on the $I - V$ curve near the critical current. Such a "relaxation" mechanism (Likharev, 1974b; Likharev and Yakobson, 1975b; Song, 1976) can hardly explain the experimentally observed hysteresis, since a very large τ_Δ/τ_j has to be introduced. It is also easy to demonstrate that potential variations inside the weak links (Yu and Mercereau, 1976) can be explained within the framework of the TDGL equations.

8.4.4 D. Nonequilibrium quasiparticles

1. Nonequilibrium excitations

In the TDGL approximation no account is taken of possible deviations from the equilibrium concentration and the energy distribution of the two most important types of quasiparticle excitations in superconductors, namely, normal electrons and phonons. ³⁴ These deviations, however, can be quite high in weak links, as nonequilibrium electrons are intensively generated when the order parameter suffers periodic variations. On relaxing in energy, these electrons transfer this energy in the form of phonons to the crystal lattice. Therefore the concentration and distribution of quasiparticles may differ markedly from their equilibrium values.

On the other hand, the quasiparticle distribution has considerable influence on the order parameter. For example, the suppression of Δ as $T \rightarrow T_C$ is due to the in-

³⁴ The influence of photons is only significant at the external irradiation, because the probability of photon generation in the bulk of the superconductor itself is extremely low (Fife and Gyax, 1972).

creasing number of quasiparticles with increasing temperature. Therefore even a small deviation in this number from the equilibrium value will have a significant influence on $|\Delta|$, especially when $T \approx T_C$.

Today, we know how the excitation and relaxation of nonequilibrium excitations occur in spatially uniform specimens carrying no current: see, for example, comprehensive reviews by Langenberg (1975) and by Chang and Scalapino (1977). The generation and relaxation of quasiparticles under strong gradients of Δ have, however, as yet been little studied, because the condensate and quasiparticles form here a rather complicated coupled system. At present, therefore, we can only give a qualitative description of these processes.

We shall begin with the important case in which the deviations of the phonon subsystem from equilibrium may be disregarded, and consequently only the distribution of normal electrons is out of equilibrium.

2. Nonequilibrium electrons

A general picture of the relaxation of a superconductor to its stationary state from an initial nonequilibrium state has been examined in several papers (Galaiko, 1971; Eliashberg, 1971; Larkin and Ovchinnikov, 1973; Aronov and Gurevich, 1974; Kulik, 1976). Qualitatively, it can be explained as follows: an ordered condensate of the Cooper pairs and a particular density of states of normal electrons is formed in a time interval $\hbar/|\Delta| \sim 10^{-12}$ s. As is known (Bardeen et al., 1957), this density is

$$N(\epsilon) = \begin{cases} N(0)\epsilon/(\epsilon^2 - |\Delta|^2)^{1/2}, & \text{at } \epsilon > |\Delta|, \\ 0, & \text{at } \epsilon < |\Delta|, \end{cases} \quad (172)$$

where ϵ is the energy measured from the Fermi level.

These states are populated by electrons with the equilibrium density

$$n(\epsilon) = [\exp(\epsilon/k_B T) + 1]^{-1} \quad (173)$$

in a much larger time interval, which is of the order of the energetic relaxation time in the normal state τ_ϵ . The frequent (in times of $\sim l/v_F \sim 10^{-13} - 10^{-14}$ s) collisions of electrons with impurities do not contribute to this relaxation, as they are elastic and do not change the energy of electrons. Therefore τ_ϵ is of the order of the electron-phonon collisions time. The density of phonons at temperatures far below the Debye temperature $T_D \sim 10^2$ K is rather low ($\propto T^3$)

$$\frac{\tau_\epsilon}{\hbar} \approx \frac{1}{k_B T} \left(\frac{T_D}{T} \right)^2 \gg \frac{1}{k_B T_C}. \quad (174)$$

Calculations (Kaplan et al., 1976) give τ_ϵ of the order of 10^{-9} to 10^{-10} s for a majority of superconductors, and $\tau_\epsilon \sim 10^{-6}$ s for aluminum ($T_C \approx 1.19$ K) and zinc ($T_C \approx 0.87$ K). On the other hand, this equilibrium value of $|\Delta|$ itself depends on the distribution of the normal electrons $n(\epsilon)$ by virtue of the self-consistency equation

$$|\Delta| = \lambda N(0) \int_{|\Delta|}^{k_B T_D} \frac{|\Delta| f(\epsilon) d\epsilon}{(\epsilon^2 - |\Delta|^2)^{1/2}}, \quad f(\epsilon) = 1 - 2n(\epsilon), \quad (175)$$

where λ is an electron-phonon coupling parameter.

Therefore the equilibrium values of $f(\epsilon)$ and $|\Delta|$ are established simultaneously in the time $\sim \tau_\epsilon$ [near T_C in a time $\sim \tau_\epsilon (k_B T_C / \Delta)$], i.e., in $\sim 10^{-9}$ s for a typical case.

If for some reason $|\Delta|$ varies with a frequency greater than τ_ϵ^{-1} , then some stationary electron distribution is established in the specimen, which in general differs from the equilibrium distribution. Its influence on the order parameter may be accounted for as follows.

Let the temperature be close to the critical value, i.e., we are dealing with a region where the GL equations (21) are applicable. The term in square brackets appears in (21a) from Eq. (175) after substitution of the stationary distribution, $f(\epsilon) \approx \epsilon/2k_B T$, from Eq. (173), and taking into account the smallness of $|\Delta|$. Making the same calculations for an arbitrary $f(\epsilon)$ and taking as before $|\Delta| \ll k_B T$, we find that Eq. (21a) is satisfied if T_C in (12) is replaced by a new effective critical temperature

$$k_B T'_C = k_B T_C \left\{ 1 + \int_{1\Delta_1}^{\infty} \left[f(\epsilon) - \frac{\epsilon}{2k_B T} \right] \frac{d\epsilon}{(\epsilon^2 - |\Delta|^2)^{1/2}} \right\}. \quad (176)$$

Equation (176) shows that if $f(\epsilon)$ is greater than its value in equilibrium (i.e., the electron density $n(\epsilon)$ is less than its equilibrium value), T'_C may increase, i.e., enhancement of superconductivity occurs. The value of T'_C is especially sensitive to changes of $f(\epsilon)$ in the region $\epsilon \approx |\Delta|$, and therefore even a simple redistribution of electrons towards higher energies, with the total concentration conserved, can raise T'_C .

3. Current enhancement of superconductivity

Aslamazov and Larkin (1976) advanced a theory which provides an explanation of the bumps on the $I - V$ curves of weak links. They considered a one-dimensional geometry identical with the ODSEE model for the case of small lengths ($L \ll \xi$). The critical current in this case is given by the AL theory [Eq. (41)]. If the current exceeds the critical value, the frequency of the Josephson oscillations rapidly exceeds τ_ϵ^{-1} , and a stationary nonequilibrium distribution $f(\epsilon)$ is established in the specimen. After calculating this distribution with the help of equations derived earlier (Larkin and Ovchinnikov, 1975), Aslamazov and Larkin obtained the following expression

$$f(\epsilon) = \begin{cases} (\Delta/2k_B T) \times [1 - (1 - \epsilon/\Delta)^{5/4}], & \text{at } \epsilon < \Delta \\ \epsilon/2k_B T, & \text{at } \Delta < \epsilon \ll k_B T \end{cases} \quad (177)$$

where Δ is the equilibrium value of $|\Delta|$ at the banks [Eq. (12)]. The deviation of $f(\epsilon)$ from its equilibrium value occurs only for $\epsilon < \Delta$, because high-energy electrons may easily diffuse from weak links into the banks in a time $\sim L^2/D \ll \tau_\epsilon$. For low-energy electrons there are no free states in the banks [Eq. (172)], and they are reflected back into the weak link from those points where $|\Delta|$ is equal to their energy ϵ (Andreev, 1964).

Since $f(\epsilon)$ appears to be greater than the equilibrium value, superconductivity is enhanced, i.e., the amplitude of the supercurrent in ac Josephson effects becomes greater than the critical current I_C . Therefore the Josephson oscillations take place just as in the RSJ model with increased critical current I'_C and the $I - V$ curves are close to hyperbolic everywhere, except in a small region when $\bar{V} \rightarrow 0$, where the average current $\bar{I}$ sharply falls from I'_C to I_C . This corresponds to the usual "bump" on the $I - V$ curve.

In order to find I'_C , we have to substitute Eq. (177) into Eq. (176), and thus find the new value of the critical current from Eq. (21) for the new effective value of T'_C . For T very close to T_C , there appears to be a region of lengths $\xi \times (T_C - T/T_C)^{1/4} \ll L \ll \xi$ where I'_C is independent of the length, and is given by the expression

$$I'_C \approx 0.12 C_j S (8/\pi \hbar D)^{1/2} \Delta^{5/2} \approx 0.2 I_C [L/\xi(T)] \times [T/(T_C - T)]^{1/4} \gg I_C. \quad (178)$$

In this region $|\Delta|$ is almost constant over the whole length of the weak link, and tends to Δ only at the boundaries with the banks.

Another explanation of the bump on the $I - V$ curve was given by Deaver et al., (1976a, b). They assumed that an increase in the number of quasiparticles with a decrease in Δ at the phase shifts $\varphi \approx \pi$ is responsible for the bump (see the discussion in Sec. VI.A). One more approach to nonequilibrium in weak links was discussed by Golub (1976).

4. Balance of holes and electrons

Quasiparticle excitations, which we have so far called electrons, change their properties depending on whether their momentum p is more or less than the Fermi momentum p_F . When $p > p_F$, a quasiparticle resembles an ordinary conduction electron: it has a negative charge ($-e$) and differs from an electron only in that its dispersion at $p \gtrsim p_F$ is different

$$\epsilon = [|\Delta|^2 + v_F^2 (p - p_F)^2]^{1/2}, \quad (179)$$

which is just the cause of the anomalous density of states [Eq. (172)]. If $p < p_F$, a quasiparticle behaves like a hole, i.e., like a particle with a positive charge ($+e$) and with the same dispersion law [Eq. (179)].

As a result of the different signs of the charge, the elastic conversion of an electron into a hole (or vice versa) is impossible, because in this process an additional Cooper pair with an energy corresponding to the Fermi level should appear (or disappear). If, therefore, an imbalance exists between the electrons and holes in a superconductor, it relaxes to zero in a time of the order of $\tau_Q \sim \tau_e$ [more exactly, $\tau_Q \approx \tau_e (k_B T / |\Delta|)$ when $T \rightarrow T_C$ (Kaplan et al., 1976)].

The nonzero space charge of a superconductor has been shown to be directly connected with imbalance in the number of the electrons and holes (Tinkham and Clarke, 1972; Clarke, 1972; Tinkham, 1972). But, by virtue of the Maxwell equation, a space charge must appear on any interface between superconducting and normal phases crossed by a current. Since $\mathbf{E} = 0$ in the superconducting phase and $\mathbf{E} = \mathbf{j} / \sigma_N$ in the normal phase, a space charge appears at the interface with a total density

$$\int \rho dx = j / 4\pi\sigma_N \quad (180)$$

Thus, an imbalance between the electron and the hole components always arises at the $S - N$ interface when

the current flows through this boundary. Since, during the relaxation time τ_Q , electrons and holes diffuse from the boundary to a distance

$$\xi_\epsilon \approx (D\tau_Q)^{1/2} = (v_F l \tau_Q / 3)^{1/2} \propto (T_C - T)^{-1/4}, \quad \text{at } T \rightarrow T_C, \quad (181)$$

the electrical field and the normal current should penetrate into the superconductor to a depth of the same order. The magnitude of ξ_ϵ derived from Eq. (181) is much greater (several units or tens of microns) than that predicted by the TDGL theory [Eq. (162)]. The results obtained in experiments with long specimens (Skocpol et al., 1974b, c; Meyer and Tidecks, 1976; Yu and Mercereau, 1975; Klapwijk et al., 1976b) are generally in satisfactory agreement with Eq. (181), although it is not easy to detect the weak temperature dependence of T at $T \rightarrow T_C$.

There can be no doubt that slow relaxation of the normal current in time and space should have some kind of influence on the ac processes in weak links, on the "excess current" effect in particular, thereby giving rise to deviations from the TDGL result [Eq. (171)]. But the excess currents recorded in the experiments are quite well described by Eq. (171) [see, for example, Gubankov et al. (1977)]. The Andreev reflection of electrons at the $S - N$ interface (Andreev, 1964) seems to have a rather strong influence on this effect, resulting in partial changes in the

electrical field at distances $\sim \xi \ll \xi_\epsilon$. In other words, just as in the case of the TDGL model, part of the electrical charge [Eq. (180)] is localized in a layer of thickness $\sim \xi(T)$. (See also the works Artemenko and Volkov, 1976, 1977; Schmid and Schön, 1975; Artemenko et al., 1978.)

5. Self-heating

Scattering of the nonequilibrium electrons by the phonons leads to an increase in phonon concentration. Although the nonequilibrium spectrum of these phonons can differ markedly from the equilibrium one, qualitatively these phonons have the same effect as would be produced by a simple increase in their concentration, i.e., as the heating of the crystal lattice. Without giving an exact description of the thermal effects, which differ depending upon the weak link form and substrate properties, etc., we shall demonstrate that even a simple account of these effects (Gubankov et al., 1972) gives rise to a negative slope region on the $I - V$ curve of the weak link.

Let us suppose that dissipation of energy P in a weak link leads to a proportional increase in the temperature of the system ³⁵

$$\Delta T = T' - T = \kappa P = \kappa \bar{I} \bar{V} \quad (182)$$

where T is the temperature of the cryostat (surrounding medium). The temperature increase leads to a drop in the critical current. Let T be close to T_C ; then Eq. (14) holds valid; hence

$$V_C(T') = V_C(T) - \alpha(T' - T) = V_C(T) - \bar{I} \bar{V} / I_T, \quad (183)$$

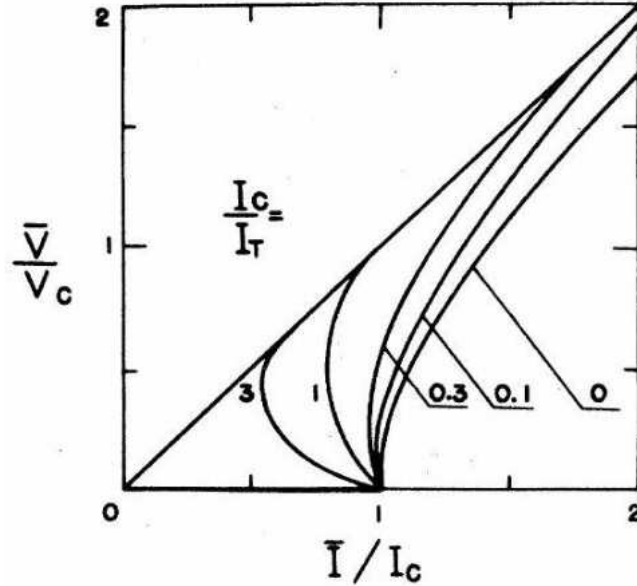


FIG. 21. $I - V$ curves according to the RSJ model with finite self-heating [Eq. (184)]. Parameter is the relation of the critical current I_C to the thermal current $I_T = |dV_C/dT|^{-1} (dT/dP)^{-1}$, which is larger, thermoconductivity is better. After Gubankov et al., 1972. where the larger the constant $I_T = (\alpha\kappa)^{-1}$, the better the heat transfer from the weak link. Substituting Eq. (183) into Eq. (147) derived in the RSJ model, we obtain a relationship for the resistive part on the $I - V$ curve for a nonisothermic case

$$\bar{I}^2 R_N^2 - \bar{V}^2 = \begin{cases} (V_C - \bar{I} \bar{V} / I_T)^2, & \text{at } T' < T_C, \\ 0, & \text{at } T' > T_C, \end{cases} \quad (184)$$

which is shown in Fig. 21.

This figure shows that any nonzero value of κ gives rise to a region of negative slope on the

³⁵ For $L \lesssim \xi$, only the temperature of those bank regions which border on the weak link is essential.

$I - V$ curve due to the diminished critical current of the junction owing to heating in resistive state ($\bar{V} \neq 0$). We know that the initial portion of the resistive part can really be smoothed out as a result of the dc current enhancement of superconductivity. Consequently, a negative slope region appears only for finite κ (Fig. 16).

From Fig. 21 it is also evident that the thermal effects are enhanced with the increasing I_c , i.e., with decreasing temperature of the cryostat. This is in complete agreement with experiment. The dependence of the self-heating effect on geometry is not so pronounced. Although the dependence of $\alpha = |dV_c/dT|$ on the link dimensions, as a rule, is not very great, nonetheless it may be quite significant in some cases; therefore the specific calculation varies for different cases (Iwanishin and Smith, 1972; Nad and Polyanskii, 1973; Fulton and Dunkleberger, 1974; Skocpol et al., 1974a, b; Kaminaga, 1975; Desmons, Martin, and Thomas, 1975; Jahn and Kao, 1976; Tinkham et al., 1977; Decker and Palmer, 1977).

Outwardly, the changes in the $I - V$ curves caused by the self-heating effect resemble those induced under the influence of finite capacitance of the structure (McCumber, 1968; Stewart, 1968): for β [Eq. (17)] greater than unity, a negative slope region appears on $I - V$ curves. Its depth increases with the growing β , i.e., with the growth of the critical current (for fixed R_N and C). Such a close similarity between the two phenomena has resulted in the fallacious opinion that the negative slope regions on $I - V$ curves, usually observed as hysteresis if the current source ($R_e \gg R_N$) is used, originate from the capacitance of the structure. Although even the simplest estimates give $\beta \ll 1$ for weak links, this opinion has persisted long after the publication of the first paper in which the thermal nature of hysteresis was proved (Gubankov et al., 1972). In all probability, the unending stream of publications in which the influence of heat on $I - V$ curves of weak links is rediscovered, is an illustration of poor communication even in this narrow branch of physics.

In structures without current concentration or with weak current concentration a specific type of heating may take place in the electrodes, i.e., field-induced heating of electrons even without lattice heating (Aslamazov and Larkin, 1976). This effect is also observed in long superconducting films (Shklovskii, 1975; Volotskaya et al., 1976). Qualitatively it leads to the same results as the "true" heating described above.

8.5 VI. OTHER NONSTATIONARY PROCESSES

From the preceding section it is evident that we are still in need of a complete theory of the nonstationary processes occurring in weak links, that would adequately explain the phenomena observed in real structures. The difficulty encountered in verifying the theoretical hypotheses experimentally is that the $I - V$ curves of weak links are not the best touchstone for them.

On the one hand, certain special features of $I - V$ curves (say, bumps) may be described by various theories. On the other hand, in the total voltage range from fractions of a microvolt (typical of the fluctuation bend on $I - V$ curves) to several tens of millivolts (where the Josephson effect is still frequently observed), the frequency of the processes (ω_V) changes by a factor of $10^4 - 10^5$. Therefore it is hard to believe that a theory that describes the processes satisfactorily for a fixed $\bar{V}$, could give an exact description of the $I - V$ curve as a whole.

Use of other nonstationary phenomena, even if more complicated in experimentation, seems promising in comparing theories with experiments. In this section we shall discuss some of the possible experiments.

8.5.1 A. High-frequency impedance

1. Impedance in a superconducting state

Let a bias point be located on the superconducting part of the $I - V$ curve, i.e., let the constant current through the weak link be less than the critical current: $\bar{I} < I_C$, and the ac current of frequency ω and amplitude I_ω be superimposed on it. The current I_ω will cause the phase to deviate from its constant value $\bar{\varphi}$, corresponding to the current $\bar{I}$, the amplitude of these deviations being small for small I_ω . Finite-frequency oscillations in φ , by virtue of the Josephson relation (4), will give rise to voltage oscillations with an amplitude

$$V_\omega = i(\hbar\omega/2e)\varphi_\omega \quad (185)$$

proportional to I_ω when $I_\omega \rightarrow 0$

$$V_\omega = Z(\omega)I_\omega, \quad Z(\omega) = i(\hbar\omega/2e)\varphi_\omega/I_\omega \quad (186)$$

The complex quantity Z is the total ac resistance (impedance) of a weak link. Frequently it is more convenient to use the inverse of impedance, i.e., the total conductance (admittance)

$$Y(\omega) = Z^{-1}(\omega) = -i(2e/\hbar\omega)I_\omega/\varphi_\omega. \quad (187)$$

In order to estimate impedance, we shall first determine it at low and high frequencies. If $\omega \ll \omega_c$, then I_s and φ are related by the dc $I_S(\varphi)$ relationship, hence

$$I_\omega/\varphi_\omega = dI_S/d\varphi|_{\varphi=\bar{\varphi}}, \quad Z(\omega) = i(\omega/c^2)\mathcal{L}(\bar{\varphi}), \quad \mathcal{L}^{-1} = \frac{2e}{\hbar c^2} \frac{dI_S}{d\varphi}. \quad (188)$$

We find that the impedance is inductive. For example, for short weak links with a sinusoidal $I_S(\varphi)$, we have

$$\mathcal{L}^{-1} = \mathcal{L}_C^{-1} \cos \bar{\varphi}, \quad \mathcal{L}_C = c^2 R_N / \omega_C, \quad (189)$$

therefore the impedance is much less than R_N

$$Z = iR_N (\omega/\omega_C) \cos^{-1} \bar{\varphi}. \quad (190)$$

On the other hand, the main current component at very high frequencies is the normal current, and

$$Z = R_N, \text{ at } \omega \gg \omega_C, \Delta/\hbar. \quad (191)$$

Thus the impedance grows with increasing frequency, and it is therefore easiest to measure it at microwave frequencies, where ω is of the order of ω_C . A typical experiment here is measurement of the reflection coefficient from the resonator or stub containing a weak link in a microwave guide (see, for example, Vystavkin et al., 1973).

The impedance can also be measured at relatively low frequencies (units or tens of megahertz) by inserting a weak link into a superconducting ring of low inductance ($l_R < 1$), i.e., with the help of a quantum interferometer (SQUID) operating in a nonhysteretic mode (see, for example, Hansma, 1975; Danilov and Likharev, 1975). One merit here is that in this method any value of $\bar{\varphi}$ can steadily be set up if $l + l_R < 1$, not only values in the range $|\varphi| \leq \pi/2$ as in the absence of the ring.

2. $G_1 \cos \varphi$ term

In the RSJ model we have

$$I_\omega = I_C \cos \bar{\varphi} \varphi_\omega + V_\omega / R_N, \quad (192)$$

hence, with the help of (185), we get

$$\text{Re } Y(\omega) = R_N^{-1}, \quad \text{Im } Y(\omega) = -(2eI_C / \hbar \omega) \cos \bar{\varphi}. \quad (193)$$

This expression shows that with increasing ω , the transition from the asymptote of Eq. (190) to the asymptote of Eq. (191) occurs gradually. We may note that in the RSJ model the losses in the weak link ($\text{Re } Y$) do not depend on $\bar{\varphi}$. In the experiments, however, $\text{Re } Y$ always increases as $\bar{\varphi}$ approaches π .

The dependence of $\text{Re } Y$ on φ appears in the tunnel theory [Eqs. (7) and (8)]. Indeed, taking φ as

$$\varphi = \bar{\varphi} + \varphi_\omega \exp(i\omega t) + \text{c.c.}$$

we express $\exp(i\varphi/2)$ contained in Eq. (7) in the first approximation of φ_ω as

$$\exp(i\varphi/2) = \exp(i\bar{\varphi}/2) [1 + i(\varphi_\omega/2) \exp(i\omega t) + \text{c.c.}] \quad (194)$$

Hence we obtain an expression for the coefficients A_n

$$A_0 = 1, \quad A_{\pm 1} = \pm \varphi_\omega / 2; \quad A_n = 0, \quad \text{at } |n| > 1. \quad (195)$$

Substituting these coefficients into Eq. (8), and retaining as before only the first terms with respect to $|A_{\pm 1}| \ll A_0$, we get

$$I = I_C \sin \bar{\varphi} + Y(\omega) V_\omega \exp(i\omega t) + \text{c.c.} \quad (196)$$

where the admittance $Y(\omega)$ has the following components (R. Harris, 1975)

$$\begin{aligned} \text{Re } Y &= \frac{e}{\hbar \omega} \{ \text{Im } I_q(\omega) + \cos \bar{\varphi} \text{Im } I_p(\omega) \} \\ \text{Im } Y &= \frac{e}{\hbar \omega} \{ \text{Re } I_q(\omega) - \cos \bar{\varphi} [\text{Re } I_p(\omega) + \text{Re } I_p(0)] \} \end{aligned} \quad (197)$$

Since $\text{Re } I_p(0)$ is simply the critical current, from Eq. (187) we get for low frequencies ($\omega \ll \Delta(T)/e$)

$$\begin{aligned} \text{Re } Y &= G_0 + G_1 \cos \bar{\varphi}, & G_0 &= d [\text{Im } I_q(2e\bar{V}/\hbar)] / d\bar{V}, \\ \text{Im } Y &= -(2eI_C / \hbar \omega) \cos \bar{\varphi}, & G_1 &= d [\text{Im } I_p(2e\bar{V}/\hbar)] / d\bar{V}. \end{aligned} \quad (198)$$

We can see that the basic distinction from the RSJ model is that $\text{Re } Y$ becomes a function of $\cos \bar{\varphi}$. Equation (198) and Fig. 2, however, show that $G_1 > 0$ and the losses ($\text{Re } Y$) diminish as $\varphi \rightarrow \pi$ in the tunnel theory, a fact which is in contradiction to the experiments both with tunnel junctions (Pedersen et al., 1972) and with weak links (Vincent and Deaver, 1974; Nisenoff and Wolf, 1975; Callegari et al., 1976; Rifkin and Deaver, 1976; Rifkin et al., 1976). The discrepancy between theory and experiment was discussed by Poulsen (1973), Langenberg (1974), and Harris (1974). Recently it was found (Zorin and Likharev, 1978b) that the observed negative sign of G_1 can be explained if the finite width of the Riedel peak is taken into account in tunnel theory.

The sign of the fourth term in Eq. (197) also seems unsatisfactory. Indeed, we see from Fig. 2, that at high frequencies

$$\text{Re } I_q(\omega) \approx \text{Re } I_p(\omega) + \text{Re } I_p(0) \approx I_C, \text{ at } \omega \gg \Delta/\hbar, \quad (199)$$

hence $\text{Im } Y$ is positive for any $\bar{\varphi}$. This means that the equivalent circuit of the Josephson junction at high frequencies consists of a normal resistance R_N in parallel with a negative inductance

$$L = -L_C(1 - \cos \bar{\varphi})/2. \quad (200)$$

Consequently, the stationary state of the junction for any (i.e., for any $\bar{I}$) is unstable to high-frequency perturbations, a result which at present seems unphysical. However, because of the complex type of $\text{Re } I_q$ frequency dependence, the problem needs some further analysis. In conclusion, we may note that EqS. (15) and (198) tempt one to modify the RSJ model as follows:

$$I = I_C \sin \varphi + (G_0 + G_1 \cos \varphi) V/R_N \quad (201)$$

Expressions with a structure similar to Eqs. (15) and (198) really follow from Eq. (201). But even within the framework of these particular effects, Eq. (201) is contradictory in the sense that $G_{0,1}$ has different values in two effects. It is, moreover, meaningless to assert that the approximation (201) is valid in more complicated cases, say for instance during strong oscillations of $V(t)$, which take place on the resistive part of the $I - V$ curve.

3. Other theories

The expression for current [Eq. (165)] which directly follows from the TDGL equations may also be used for finding the impedance. For low frequencies it gives Eq. (198), where we now have to take

$$G_0 = R_N^{-1}, \quad G_1 = -(\gamma/15)R_N^{-1} \ll R_N^{-1}. \quad (202)$$

Thus, in this theory, in agreement with the experiments, we have $G_1 < 0$ and the ratio G_1/G_0 just equal to the correction to normal resistance [Eq. (170)]. In particular, as $T \rightarrow T_C$ and $\Delta = \Delta_0$ (i.e., $A = 1$), G_1 should vary as $(T_C - T)^{-1}$.

Volkov and Kasatkin (1974) obtained the same sign as for the $V \cos \varphi$ term in analyzing, with the help of the same equations, the proximity effect bridges where the banks are to a considerable extent involved in the nonlinear processes.

Proceeding from the microscopic theory, Mitsai (1976) calculated the admittance of a short weak link which had the same geometry as the ODSEE model. For the most interesting case $\hbar\omega \ll \Delta \ll k_B T$ he derived the following expressions

$$\begin{aligned} \text{Re } Y &= R_N^{-1} \left[\left(1 - \frac{7\Delta}{8k_B T} \right) + \frac{5\Delta}{8k_B T} \cos \bar{\varphi} \right], \\ \text{Im } Y &= -(2eI_C/\hbar\omega) \left[\frac{1}{2}(\bar{\varphi} - 2 \sin \bar{\varphi}) \cot \frac{\bar{\varphi}}{2} + \cos \bar{\varphi} \right], \\ &\text{at } |\bar{\varphi}| \leq \pi. \end{aligned}$$

Note that the function in square brackets in the expression for $\text{Im } Y$ is less than or equal to zero for any $\bar{\varphi}$. As a result the stationary state $\varphi = \bar{\varphi}$ is unstable to highfrequency perturbations for the same reason which we have discussed above. The Mitsai theory, in contradiction to experiment, also gives a positive sign to the $\cos \bar{\varphi}$ term in $\text{Re } Y$.

Remarkably, the Mitsai theory of nonlinear nonstationary processes has been derived on the basis of the Green's function formulation of the superconductivity theory. None of the other

results deduced in this way has as yet been confirmed by experiment.³⁶ Probably the wrong signs of two current amplitudes are due to the difficulties involved in analytical continuation procedure in a nonstationary case (Chang and Scalapino, 1977). Therefore this fact deserves the utmost attention.

Deaver et al. (1974, 1976a, b) have advanced a phenomenological model in which increased losses when $\bar{\varphi} \approx \pi$ are attributed to decrease of $|\Delta|$ under these phase values in a noticeable part of the bulk of the weak link. They believe that this situation should lead to the increased density n of the normal electrons, and consequently, to increased active conductivity $\text{Re } Y$. Nonetheless, we may note that when $T \approx T_C$, this effect should hardly be noticeable, as the variation of $|\Delta|$ does not

change the value of n

\$\$

$$n = \int_{|\Delta|}^{\infty} n(\epsilon) N(\epsilon) d\epsilon = N(0) \int_{|\Delta|}^{\infty} \times \frac{1}{2} \frac{\epsilon d\epsilon}{(\epsilon^2 - |\Delta|^2)^{1/2}} = \text{const}(|\Delta|) \quad (204)$$

\$\$

Recently (Soerensen et al., 1977) the first attempt has been made to measure Y as a function of the temperature near T_C . Such experiments with weak links of a geometry corresponding to the ODSEE model (for instance, with variable-thickness bridges) could give a better idea of the adequacy of the various existing theories.

4. Impedance in resistive state

It has been demonstrated (Vystavkin et al., 1972; see also Vystavkin et al., 1974; Likharev and Ulrich, 1978) that if the bias point is located on the resistive part of the $I - V$ curve ($\bar{V} \neq 0$), for a low-capacitance Josephson junction, we obtain quite an unusual dependence of the impedance on the bias voltage $\bar{V}$, i.e., on the frequency of the Josephson oscillations ω_V . In the RSJ model without fluctuations, in particular, it was found that impedance

$$\begin{aligned} \text{Re } Z(\omega) &= R_N [F(\omega_V/\omega_c) + 1] \\ &\times [\omega_c/(\omega + \omega_V) + \omega_c/(\omega - \omega_V)], \omega_V \neq \omega, \\ \text{Im } Z(\omega) &= -R_N \pi F(\omega_V/\omega_c) \omega_C \delta(\omega - \omega_V), \\ F(x) &= \left[(1 + x^2)^{1/2} - x \right] / 2, \end{aligned} \quad (205)$$

has a resonance-type singularity at $\omega \approx \omega_V$. The real part of Z is negative in the range of frequencies ω slightly greater than ω_V (Fig. 22)

$$\text{Re } Z < 0, \text{ at } \omega_V^2 < \omega^2 < \omega_V (\omega_C^2 + \omega_V^2)^{1/2}. \quad (206)$$

If account is taken of even small fluctuations, then the singularity in Z at $\omega = \omega_V$ vanishes (Likharev and Semenov, 1973; Likharev and Kuzmin, 1977)

$$(\omega - \omega_V) \rightarrow (\omega - \omega_V) + i\gamma, \quad (207)$$

³⁶ For instance, the results obtained for the viscous motion of Abrikosov vortices (Gor'kov and Kopnin, 1973) are in apparent contradiction to experiment (see Danilov et al., 1975).

where γ is the half-width of the Josephson oscillation line. This gives only a small reduction in the frequency range where $\text{Re } Z < 0$ (Fig. 22).

This "anomalous" impedance results from the complicated nature of the interaction between the signal and the Josephson oscillations (see Likharev and Ulrich, 1978, for details). This effect was first observed in 1972 in point contacts (Vystavkin et al., 1973) and since then has been a topic for repeated investigations in point contacts as well as in weak links of well-defined geometry, such as variable-thickness bridges (Gubankov et al., 1977), Dayem bridges (Pedersen et al., 1977), and proximity effect bridges (Franson and Mercereau, 1976). The results of the experiments are in qualitative agreement with Eq. (205) of the RSJ model. With decreasing temperature, however, the resonance at $\omega = \omega_V$ is noticeably less than the theoretical value. Unfortunately, so far the impedance at $\nabla \neq 0$ has not been calculated by means of any other theory.

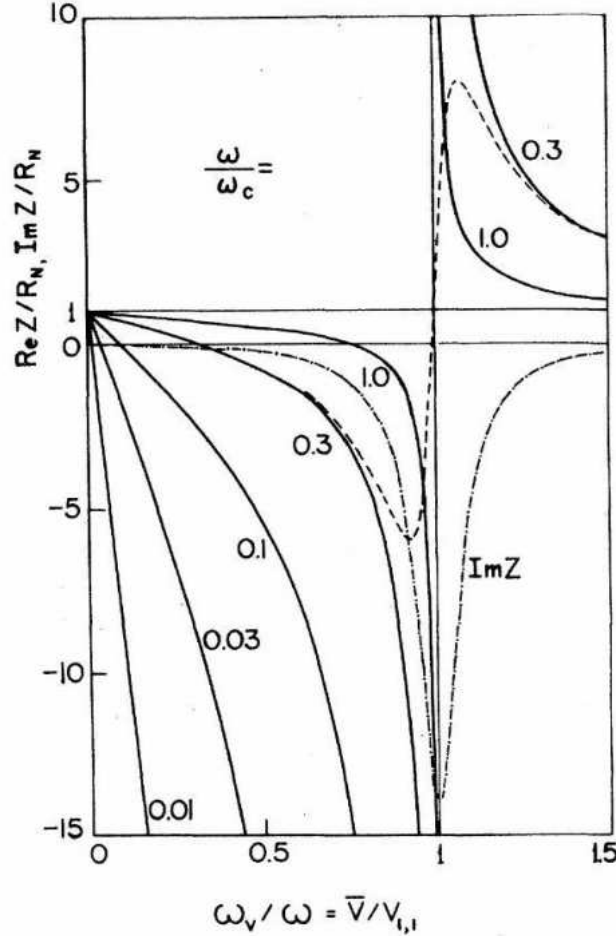


FIG. 22. Weak link impedance at the resistive branch of the $I - V$ curve as a function of the Josephson frequency ω_V for different frequencies ω of signal, as calculated from the RSJ model. Singularity in impedance occurs at $\omega = \omega_V$; dashed lines show smoothing of the singularity by small fluctuations. After Vystavkin et al., 1972; Likharev and Semenov, 1973.

8.5.2 B. $I_A(\omega)$ function and current steps on $I - V$ curves

The shape of nonstationary equations should, of course, vary for the different types of weak links. But for large voltages they should all give an expression of a form similar to Eq. (15)

$$I = \bar{I}(\bar{V}) + I_A \sin(\omega_V t + \vartheta), \quad \omega_V = (2e/\hbar)\bar{V} \quad (208)$$

describing the Josephson oscillations with an amplitude $I_A(\omega_V)$. Clearly, great interest lies in the measurement of the $I_A(\omega)$ dependence. This, in principle, may be carried out by

measuring the impedance at frequencies $\sim \omega_V$ or by direct measurement of the Josephson oscillations at the frequency ω_V . But technically all these measurements involve difficulties, especially at frequencies ω_C , which may be as high as $10^{11} - 10^{12} \text{ s}^{-1}$ or even greater.

Therefore, of considerable interest for $I_A(\omega)$ determination are the straight parts ("current steps") on $I - V$ curves, first predicted by Josephson in his pioneering paper (1962). These current steps appear at the voltages $V_{m,n}$ [Eq. (117)] on the junction when it is irradiated by external microwaves of frequency ω . The magnitude of these steps varies with signal power, i.e., with the amplitude A of the microwave current through the weak link. At the same time, the superconducting region changes as well, and it is convenient to regard it as the "zero current step" ($m = 0$). The step heights can easily be measured in experiments, and may serve as a way of measuring the $I_A(\omega)$ dependence.

1. High-frequency approximation

If the voltage $V_{m,n}$ is sufficiently high ($\gg V_c$), the main contribution to the current is made by the normal component

$$V = IR_N = (I + A \cos \omega t) R_N \quad (209)$$

hence we obtain a simple formula for the phase

$$\begin{aligned} \varphi &= (2e/\hbar) \int V dt = \omega_V t + a \sin \omega t + \vartheta, \quad \vartheta = \text{const} \\ a &= 2eAR_N/\hbar\omega \end{aligned} \quad (210)$$

which now has to be substituted in the expression for the supercurrent. For example, if the supercurrent amplitude I_A does not depend on the frequency in the range, which is the case, say, in the RSJ model, then we directly obtain:

$$\bar{I}_S = I_A \overline{\sin \varphi} = I_A \sum_m J_m(a) \overline{\sin(\omega_V t + \vartheta + m\omega t)}, \quad (211)$$

where J_m is the Bessel function of the first kind. Equation (211) shows that I_S is only nonzero when $\omega_V = m\omega$, i.e., there are no subharmonic steps ($n \neq 1$) in this approximation, while the harmonic steps oscillate with A according to the "Bessel law" ³⁷:

$$\Delta I_m = 2I_A |J_m(a)|, \quad V_m = m(\hbar\omega/2e) \quad (212)$$

It is known that the maximum value of J_m decays very slowly with increasing m , for $m \gg 1$ only as $\sim 0.67 m^{-1/3}$ (see, for example, Abramowitz and Stegun, 1965). Therefore even steps with very high numbers up to $\sim 10^3$ can readily be observed in experiments (McDonald et al., 1971; Ainitdinov et al., 1976; Octavio et al., 1977a).

It is now important to notice the following. The fact that noticeable steps are observed at voltages V_m does not necessarily mean that the Josephson effect has to take place in the junction at this voltage, i.e., that $I_A(m\omega)$ should be finite. To prove this, consider the time dependence of the phase [Eq. (210)], taking $\omega_V = m\omega$ and the value $a \approx m$, at which the size of the m th step is maximum (Fig. 23). During a part of the period $T = 2\pi/\omega$ the phase grows rapidly, hence the voltage across the weak link is rather high

$$d\varphi/dt \approx 2m\omega, \quad V \approx 2V_m \quad (213)$$

and during the other part φ is almost constant, hence the voltage is almost zero. The contribution to the average supercurrent is made in just these slow phase variation periods, and

this contribution depends on the phase shift ϑ , which shows the position of the bias point on the current step. During the fast phase change, the supercurrent oscillates almost sinusoidally, and there is, therefore, no contribution to $\bar{I}_S$.

Even if $I_A(m\omega)$ is zero, i.e., there is no supercurrent flow during rapid phase variations, nevertheless, from Fig. 23, it is evident that the contribution to the mean current is nonzero providing $I_A \neq 0$ at frequencies of the order of ω (i.e., the rate at which the phase changes on

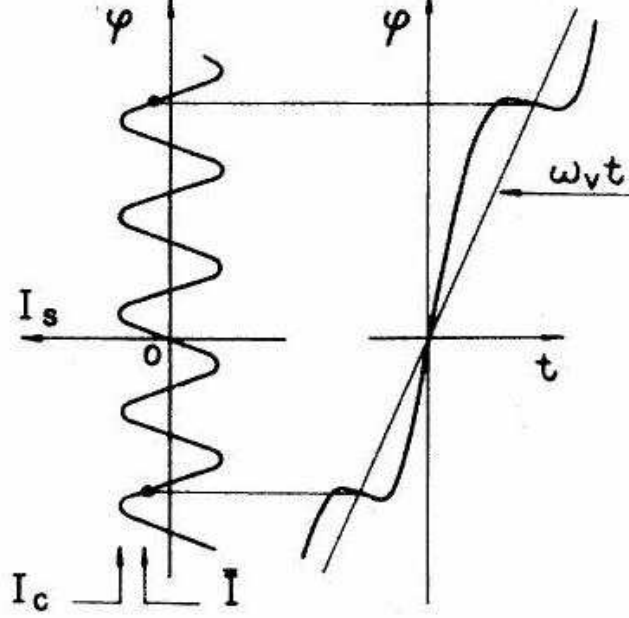


FIG. 23. Time dependence of phase difference φ when the bias point is located at the m th current step [Eq. (210)]. The figure shows that the main contribution to average supercurrent I_S occurs during the slow change of φ , when only frequencies $\sim \omega$ (but not $m\omega > \omega$) are present in its spectrum.

the flat portion of its time evolution). Thus, for $m \gg 1$, we have $\Delta I_m \propto I_A(\omega')$, $\omega' \sim \omega \ll m\omega$.

This fact, first noticed by Borovitskii and Malinovskii (1975) for the case of tunnel theory, shows that the high steps (for instance, of the order of units or tens of millivolts) observed with a relatively low-frequency source (say, X-band oscillator) show nonzero amplitude of the supercurrent only at frequencies $\sim 10^{10}$ Hz, but not at $\sim 10^{12}$ Hz. On the other hand, the first step ($m = 1$) at the same voltage induced by a very-high-frequency source is proportional to the amplitude of the Josephson current at this frequency: $I_1 \propto I_A(\omega)$. Therefore, measurement of the first step may serve as a good way of measuring the $I_A(\omega)$ dependence, say, to observe the smoothed Riedel singularity in relatively clean weak links at $\omega_V \approx 4\Delta/\hbar$ (Thome and Couder, 1975; Kofoed and Saermark, 1975; see also Vernet and Adde, 1976). In contrast, the observation of high-order steps from relatively low-frequency sources is merely an indication of the absence of heating in weak links (Octavio et al., 1977a). Such heating, being an inertial effect with a time constant $\tau \gg \omega^{-1}$, totally suppresses superconducting phenomena, including current steps.

2. Theoretical $I_A(\omega)$ dependences

As yet the frequency dependence of the supercurrent amplitude has only been calculated within the framework of two theories. In the tunnel theory it is expressed by Eq. (15) and formulas for the coefficients I_q, I_p (see, for example, Poulsen, 1973)

⁰³⁷ Strictly speaking, $\bar{I}_S \neq 0$ also in a voltage range of width $\sim \Delta I_m R_N$ near each step, where the $I - V$ curve becomes hyperbolic in shape (Volkov and Nad', 1970; Likharev and Kuzmin, (1977).

$$I_A = |I_p(\omega_V)| = \begin{cases} I_C, & \text{at } \omega_V \ll 2\Delta/\hbar, \\ \frac{2\Delta}{eR_N} \frac{\Delta}{\hbar\omega_V} \ln \frac{\hbar\omega_V}{\Delta}, & \text{at } \omega_V \gg 2\Delta/\hbar. \end{cases}$$

For $\omega_V = 4\Delta/e$, the amplitude I_A has a weak "gap" singularity (Fig. 2). At frequencies behind the gap the amplitude slowly decreases with increasing ω_V (slightly slower than ω_V^{-1}). Any other theory probably will cause a faster drop in I_A .

In the TDGL model, as follows from Eq. (169), the amplitude $I_A = (I_1^2 + I_2^2)^{1/2}$, as Fig. 18 shows, decreases exponentially for η greater than 10 – 20

$$I_A \approx 3I_C\eta^{1/4} \exp \left[-(\sqrt{2}/3)\eta^{1/2} \right] \quad (215)$$

Within the framework of this theory, the decrease in I_A starts approximately where the $I - V$ curve becomes a straight line of the excess current. This result still has to be verified experimentally.

3. Low frequencies

Measurements of current step heights as functions of microwave power not only provide a means of determining the dispersion of supercurrent amplitude, but also give an indirect method of verifying the sinusoidal shape of the dc $I_S(\varphi)$ relationship. Experimentally, this may be a much easier way than direct measurement of this relationship.

Indeed, for not-too-high frequencies ($\omega \lesssim (\omega_V)_{\max}$) the dispersion of I_S may be disregarded. Therefore, if $I_S = I_C \sin \varphi$ we may use the RSJ model [Eq. (145)] to determine the height of the steps. Such calculations have been made by several workers and their results are well known (see, for example, Fack and Kose, 1971; Likharev and Semenov, 1971; Russer, 1972; Vystavkin et al., 1974). In this model only the harmonic steps ($n = 1$) are nonzero; their sizes and positions on the current axis depend on two dimensionless variables: current amplitude A and reduced frequency ω/ω_C . These dependences are not very suitable for comparison with experiment, and therefore the curves shown in Fig. 24 (Likharev and Semenov, 1971) are used frequently.

The solid lines show the functions $\kappa_m(\omega/\omega_C)$ defined by the equation

$$\kappa_m = (P_m^{(2)}/P_m^{(1)})^{1/2} - 1 \quad (216)$$

where $P_m^{(k)}$ is the value of power P of the microwave signal ($\propto A^2$) ^{m} , at which the height of the m th current step has its k th zero. The experimental values of κ_m are to be plotted along the Y axis (Fig. 24) and then the corresponding values of ω/ω_C are to be found. If, within the limits of experimental error, all the values of κ_m give one value of reduced frequency, then the experimental weak link is consistent with the RSJ model. Such a " κ criterion" is convenient in that, firstly, there is no need for the use of absolute values of the power [Eq. (216)], and secondly the procedure is stable to slight smoothing of

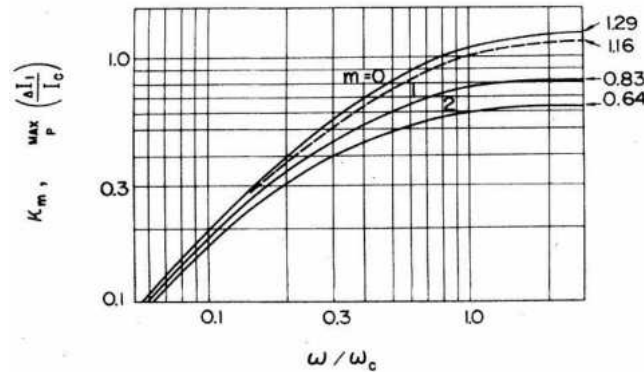


FIG. 24. Magnitudes κ_m [Eq. (216)] (solid lines) and maximum size of the first current step (dashed line) vs frequency ω of external microwave signal, reduced to the characteristic

frequency ω_c of the weak link [Eq. (18)]. Results are based on the RSJ model (Likharev and Semenov, 1971; Fack and Kose, 1971).

current step edges by the fluctuations. Sometimes it is also convenient to use the maximum size of the first current step height which is shown by a dashed line in Fig. 24 (Fack and Kose, 1971).

It should be noted that even a slight deviation of $I_S(\varphi)$ from its sinusoidal shape gives rise to noticeable changes in the magnitude of the steps; in particular, the step heights do not disappear periodically when P changes (Vystavkin et al., 1974), and subharmonic steps appear (Lübbig and Luther, 1974).

4. Other methods for I_A measurements

Several other methods are also available for measuring the frequency dependence of the supercurrent amplitude, among which the self-detection methods deserve special mention. For example, in the RSJ model [Eq. (147)] even for relatively high $\bar{V} \gg V_C$ the mean current is not equal to the normal current, and the difference is

$$\Delta \bar{I} = \bar{I} - \bar{V}/R_N \approx I_C^2/2\bar{V}R_N \quad (217)$$

This results from the fact that even when $\bar{V} \gg V_C$, the law of change of phase $\varphi(t)$ is slightly different from the linear law [Eq. (135)] for the reasons mentioned in Sec. V.A. We can say that $\bar{I}$ is a consequence of self-detection of the Josephson oscillations by the weak link itself (see, for example, Likharev and Ulrich, 1978, for details).

On connecting a weak link to an external circuit with an impedance Z_e (reduced to the junction), comparable to R_N , the self-detection effect is weakened and thus $\bar{I}$ is decreased. Extending the results obtained in the RSJ model (Likharev and Semenov, 1972; Volkov, 1972) for an arbitrary $I_A(\omega)$ dependence, we find

$$\Delta I = -\frac{I_A^2(\omega_V)}{2\bar{V}} \operatorname{Re} Y_{\parallel}, \quad Y_{\parallel} = R_N^{-1} + Z_e^{-1}, \quad \text{at } \bar{V} \gg V_C \quad (218)$$

Thus I_A can be measured by varying Z_e in a controlled manner. Such experiments have been described by Ganz and Mercereau (1975).

As $Z_e/R_N \rightarrow 0$, we have $\bar{I}_S \rightarrow 0$; this result is valid for any arbitrary $\bar{V}$. Consequently, by shunting the weak link heavily at all frequencies, the $I - V$ curve can be made to coincide with the "quasiparticle current" curve $I_N(\bar{V})$ for $\bar{V} \neq 0$. Such an experiment was recently performed by Chiao and Levinsen (1977). However, they did not succeed in obtaining the relation $|Z_e| \ll R_N$, due, probably, to the noticeable contribution of kinetic inductance of bridge banks to Z_e . More recently, Yeh and Buhrman (1977) obtained somewhat lower values of $|Z_e|_N$.

The function $I_A(\omega)$ may also be determined from the mixing between the frequencies of two external signals at the Josephson junction (Grimes and Shapiro, 1968; McDonald et al., 1974), or between an external frequency and the Josephson oscillations' frequency (Zimmerman, 1970; Vernet and Adde, 1976). The more convenient of these two experiments-mixing between two close external frequencies ω_1 and ω_2 with measuring of a low frequency difference component $\omega_- = \omega_1 - \omega_2$ - can unfortunately again give information only about $I_A(\omega_-)$, for the same reasons as mentioned above for the high current steps. The same can also be said of mixing with fre-

quency multiplication (see, for example, McDonald et al., 1972): $\omega_- = m\omega_1 - \omega_2$, where the output signal is also proportional to $I_A(\omega_-)$, but not to $I_A(\omega_2)$.

8.5.3 C. Microwave enhancement of superconductivity

According to the RSJ model, the critical current is always less in the presence of an external microwave irradiation than without it. It was found (Wyatt et al., 1966; Dayem and Wiegand, 1967), however, that in some weak links at temperatures close to T_c the critical current increases somewhat at first, with the increasing power, and begins to decrease thereafter. Later this "enhancement of the superconductivity by microwaves" was repeatedly verified experimentally. Owing to the recent publication of a review (Klapwijk et al., 1977a), we shall outline this topic only briefly.

1. Main experimental results

Critical current shows an increase only in a specific frequency range, i.e., the effect has lower (Dayem and Wiegand, 1967) and upper (Latyshev and Nad', 1974) frequency limits. These limits shift upward as the temperature decreases, although the behavior far from T_C has not been investigated in detail.

Essentially, critical temperature does not rise as the critical current increases.³⁸ The insignificant increase in T_C , observed in some experiments, does not usually exceed the width of the $S - N$ phase transition, and consequently may easily be attributed to the proximity effect between regions with different T_C .

Microwave-induced enhancement shows a strong dependence on the size of the specimen. This effect is rarely observed in point contacts (Dmitriev et al., 1973; Fjordbøge et al., 1976), i.e., in structures containing microshorts, or in small bridges (submicron dimensions) where the "ideal" Josephson effect exists. On the contrary, in bridges of the dimensions of one or several microns, where the Josephson effect is weakly exhibited, enhancement of superconductivity is distinctly pronounced. A further increase in the specimen dimensions, say, the length of the film strip, almost inhibits the effect. In those rare instances where it has been observed on long specimens (Shepard, 1971; Latyshev and Nad', 1976) this effect probably takes place at minute inhomogeneities of either a geometrical or a physical nature.

This effect seems not to be dependent upon whether the current is concentrated in one direction or in two. In any case, it was observed both in Dayem bridges [current concentration only along the width, Fig. 1(e)], and in variable-thickness bridges [current concentration along the width and thickness, Fig. 1(f)].

2. Theoretical interpretations

Microwave enhancement of superconductivity has been treated differently in phenomenological terms (see, for example, Hunt and Mercereau, 1967; Shepard, 1971;

Notarys et al., 1973; Lindelof, 1976). None of these interpretations, however, is based on modern concepts of the physics of superconductors and is able to give a proper explanation to the whole body of experimental data.

At present, most of the workers in this field are inclined towards the hypothesis proposed by Eliashberg (1970), who attributed this effect to the fact that microwave oscillations cause a redistribution of normal electrons in the energy scale, the maximum of this distribution shifting from $|\Delta|$ to higher ϵ . By virtue of Eq. (175), such a redistribution should lead to a weakening of $|\Delta|$ suppression by normal electrons, i.e., to an increase in $|\Delta|$ at $T \approx T_C$. Accordingly, all critical parameters of the superconducting state, including the critical current, should increase. Although several works have been published with regard to Eliashberg's idea (Ivlev, 1970; Ivlev and Eliashberg, 1971; Ivlev et al., 1973; Schmid, 1977), this theory still cannot explain all the

³⁸ The only exception is the recent experiment (Klapwijk and Mooij, 1976) with Al bridges. So far these results have not been confirmed by others.

experimentally observed facts, especially the dependence of microwave enhancement on the dimensions of the weak link. This can, probably, be done with the help of the ideas developed by Aslamazov and Larkin (1976) for the dc current enhancement of superconductivity. Indeed, an ac current of frequency $\omega \gtrsim \tau_c^{-1}$ flowing through a weak link should affect the distribution of normal excitations approximately as the Josephson oscillations do. Consequently, the critical current should raise approximately to I'_c , the height of the bump on the $I - V$ curve. Such an equality of superconductivity enhancement by dc and ac (microwave) currents has actually been observed in experiment (Gubankov et al., 1977).

Recently Aslamazov and Larkin (1978) advanced a theory where the discussion runs along these lines.

8.5.4 D. Fluctuations

The nonstationary properties of weak links also have an influence upon the fluctuations taking place in them. In analyzing fluctuations, a clear distinction should be made between real (measurable) fluctuations of a particular quantity (say, voltage) and the intensity of the fluctuation sources, which may conveniently be represented as a "fluctuating current" I_f source connected in parallel to the weak link. In fact, the current fluctuations I_f interact with the supercurrent in a rather complex way, especially in the presence of the ac Josephson effect. Therefore the observed dependences of the real fluctuations on the parameters may be quite complicated. Conversely, I_f is directly associated with the physics of conductivity of the structure.

1. The nature of sources

The most important result of all the theories of fluctuations in the Josephson effect is that the only intrinsic source of fluctuations is the normal current rather than the supercurrent of the junction.³⁹ The reason is that the

superconducting condensate is an ordered set of electron pairs, which cannot fluctuate independently. The collective fluctuations of the condensate as a whole under the usual conditions are negligibly small. Indeed, the order parameter can noticeably vary only at distances of the order of the coherence length. Therefore, in a bulk specimen, for example, the probability of a large fluctuation is of the order

and since $|\Delta| \sim k_B T$ and $n\xi^3 \gg 1$, is negligibly small.

The intensity of fluctuation sources in a superconducting weak link differs from the intensity in the normal state only to the extent that the normal (but not the total) conductivity of the structure differs. When this difference is not essential (say, within the framework of the RSJ model), the intensity of the sources is simply the same as in the normal state.

In the literature we sometimes come across wrong statements that the Stephen theory (1969) shows that the supercurrent also contributes to the source of fluctuations. Actually, the Stephen equations, valid only for a particular case, are obtained when only the normal conductivity of the junction and the equilibrium fluctuations of a resonator connected to the junction are taken as the fluctuation sources (see Likharev and Kuzmin, 1977, for details).

2. Equilibrium fluctuations

The fluctuation-dissipation theorem (see, for example, Stratonovich, 1962) holds true as long as the process occurring in the weak link is an equilibrium one. This theorem gives a

³⁹ The only exception is the recent publication by Landa and Tarankova (1977). The author of this review is not convinced of the correctness of the results reported therein.

simple relationship between the active conductivity $\text{Re } Y$ of the two-terminal circuit with the spectral density $S_I(\omega)$ of the fluctuating current I_f source

$$S_I(\omega) = (2/\pi) \text{Re } Y(\omega) (\hbar\omega/2) \coth(\hbar\omega/2k_B T)$$

In particular, at low frequencies ($\hbar\omega \leq k_B T$) we obtain the Nyquist formula

$$S_I(\omega) = \text{Re } Y(\omega) 2k_B T / \pi \quad (219)$$

For example, in the RSJ model, the value of Y is the same as in a normal structure and

$$S_I(\omega) = 2k_B T / \pi R_N \quad (220)$$

3. Shot noise

As was shown by Larkin and Ovchinnikov (1967), when the voltage across a tunnel junction is greater than $\sim k_B T/e$ (about $360\mu\text{ V}$ at $T = 4.2\text{ K}$), the noise level is greater than that predicted by the fluctuation-dissipation theorem. For example, if the voltage across the junction is constant, Eq. (220) takes the form (for details see also Dahm et al., 1969).

$$S_I(0) = (e\bar{V}/\pi R_N) \coth(e\bar{V}/2k_B T) \quad (221)$$

On increasing the factor $e\bar{V}/k_B T$ to a value slightly greater than unity, $\coth$ may be taken to be unity, and thus the noise is not dependent on temperature. Therefore we obtain the well known formula

$$S_I(0) = e\bar{I}/\pi, \quad (222)$$

for the shot noise of a diode.

The physical meaning of the noise change at $e\bar{V} \sim k_B T$ lies in the following: normal electrons, responsible for the noise and usually having random energy of thermal motion $k_B T$, begin to be "overheated" in passage from one electrode to another due to the absorption of electrostatic energy $e\bar{V}$. Such a change should take place not only in tunnel contacts, but also in any case where the length ξ_e , over which an electron transfers the energy $e\bar{V}$ gained in a field to lattice phonons, is much greater than the length of the weak link L . According to the facts mentioned in the previous section, we obtain the condition

$$L \lesssim (v_F \ell \tau_e / 3)^{1/2},$$

which is generally satisfied in weak links of small or moderate lengths. So the transition from thermal noise to shot noise should take place in such weak links under voltage increase.

4. $1/f$ noise

Besides the noise sources described above, which have a wide frequency spectrum, low-frequency noise of the $1/f$ type (flicker noise) also arises in weak links. This noise can be of a diverse nature, and has a spectral density proportional to approximately $\omega^{-\alpha}$, $\alpha \sim 1$. The noise may be characterized by the cutoff frequency ω_{cf} at which the spectral density becomes equal to that of thermal noise.

Flicker noise is especially strong in point contacts (Kurdyumov, 1976), in which the cutoff frequency may be as high as hundreds of kilohertz. In all other types of weak links this noise is much less; for example, the cutoff frequency in the proximity effect bridges was found to be less than 20 Hz (Decker and Mercereau, 1975).

Recently it was found that some fluctuations of the $1/f$ type might appear even in an equilibrium case, where they are originated by the fundamental temperature fluctuation T (Clarke and Voss, 1974). Thus, for "zerodimensional" objects (particles) with a total heat capacity C_V and connected with the thermal bath via the total heat conductivity G , the general equations of statistical physics show that

$$\langle \tilde{T}^2 \rangle = k_B T^2 / C_V \quad (223)$$

The spectral density of these fluctuations is constant up to a frequency of the order of the inverse thermal time constant $\tau = C_V/G$, and then falls as $\omega^{-3/2}$. For larger specimens the frequency dependence is more complicated; in particular, it may be proportional to ω^{-1} over a wide frequency range (Clarke and Voss, 1974; Voss and Clarke, 1976).

5. Fluctuation measurement

It is reasonable to measure the intensity of fluctuation sources in a weak link only in those experiments which provide the best controlled conditions for the transformation of S_I to the measured variable. Some examples of these conditions are the following (for details see Chap.

6 in Likharev and Ulrich, 1978).

(i) If the bias point is located far along the resistive branch of the $I - V$ curve ($\bar{V} \gg V_C$), then $V = IR_N$ and the spectral density of low-frequency voltage fluctuations is simply connected with the density of I_f .

$$S_V(0) = S_I(0) R_N^2 \quad (224)$$

Essentially, no such relationship exists for an arbitrary $\bar{V}$, even if the differential resistance R_d at the bias point is taken instead of R_N (see, for example, Likharev and Semenov, 1972).

(ii) If the fluctuation intensity is not very high, instead of S_V , we can measure the half-width of the Josephson oscillations line, which is simply connected with S_V (Larkin and Ovchinnikov, 1967)

$$\Delta\omega = (\pi/2)(2e/\hbar)^2 S_V(0) \quad (225)$$

The half-width $\Delta\omega$ may be measured either directly (by microwave emission of the weak link at the frequency ω_V) or indirectly by means of the width of the current step "embryo" formed on the $I - V$ curve by the action of a weak, external, monochromatic signal of frequency ω . This embryo is of the following form (see, for example, Likharev and Ulrich, 1978)

$$\Delta\bar{I} \propto (\bar{V} - V_\omega) / \left[(\bar{V} - V_\omega)^2 + (\Delta V)^2 \right], \quad \Delta V = (\hbar/2e)\Delta\omega, \\ V_\omega = (\hbar/2e)\omega. \quad (226)$$

Measurements of $\Delta\omega$ (or ΔV) suffer from one drawback: all the fluctuations (including the low-frequency external interferences and flicker noise) whose spectra are located on the frequency axis between zero and $\sim \Delta\omega$, make a contribution to these quantities. Therefore the influence of undesirable fluctuations has to be effectively controlled in the experiments.

8.6 VII. MORE COMPLICATED WEAKLY LINKED STRUCTURES

In terms of circuit theory all the types of weak links considered in Sec. IV are simple two-terminal devices, i.e., they have two electrodes, which serve as terminals. In the last few

years more complicated weakly linked structures have been acquiring increasing importance. We shall deal with these structures in this section. They can be formed either as single weak links of a complicated geometry, or by several ordinary weak links (two-terminal structures) connected in a complicated manner.

8.6.1 A. Multiterminal weak links

The multiterminal structure considered by Likharev (1975) is an example of the first type of structure. It contains a few superconducting electrodes connected by a common weak link. The possible realization of a threeelectrode structure ("Josephson triode") is shown in Fig. 25(a), where a spot of thin film connects three superconducting electrodes made of a bulk superconducting film.

In order to analyze the processes occurring in a multiterminal structure we shall assume that the conditions of the AL theory are satisfied. In particular, all the di-

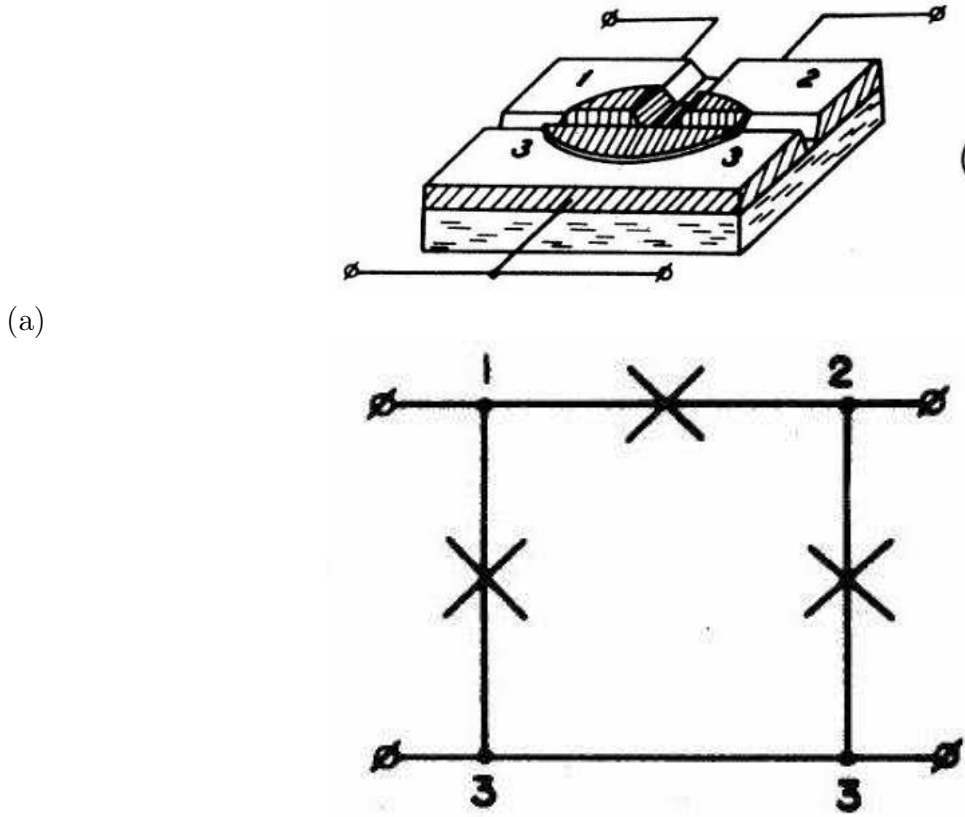


FIG. 25. Possible realization of Josephson triode, i.e., a weak link connecting three superconducting electrodes (a), and its equivalent circuit (b). Crosses denote usual Josephson junctions, which are really superimposed in the space. These imaginary junctions can have different values of I_C and R_N , but their characteristic voltages $V_C = I_C R_N$ are equal if the material of all electrodes is the same.

mensions of the structure are supposed to be less than ξ [if the weak link dimensions are great, then the currents from different terminals are simply added (Prans and Meissner, 1974)]. The solution to the Laplace equations for the order parameter $\Delta(\mathbf{r}, t)$ [Eq. (29)] and the electrical field potential $\mu(\mathbf{r}, t)$ [Eq. (35)] may now be written in the form

$$\Delta = \sum_k \Delta_k f_k \exp(i\chi_k), \quad \mu = \sum_k f_k \mu_k, \quad f_k = f_k(\mathbf{r}), \quad (227)$$

where f_k is a particular solution to the Laplace equation for a given geometry of the structure, which is equal to unity inside the k th electrode and to zero in other electrodes. Variables Δ_k , $\chi_k(t)$, and $\mu_k(t)$ are values of the modulus and phase of the order parameter and scalar

potential inside the k th electrode.

Substituting Eq. (227) into the GL equation for current (21b), we easily find the current in the n th electrode in the general case

$$I_n = \sum_k \left[(I_C)_{k,n} \sin \varphi_{k0n} + V_{k,n} / (R_N)_{k,n} \right], \quad (228)$$

$$\varphi_{k,n} = \chi_n - \chi_k, \quad V_{k,n} = \mu_k - \mu_n$$

where the constants I_C and R_N are interrelated by the condition

$$(I_C)_{k,n} (R_N)_{k,n} = V_C \quad (229)$$

Equation (228) is a generalization of the RSJ model [Eq. (145)] for the case of several electrodes. It shows that the equivalent circuit of a multiterminal structure consists of "ordinary" (two-terminal) Josephson junctions interconnecting each pair of electrodes. For example, three such imaginary "junctions" represent the triode [Fig. 25(b)]. The critical currents and normal resistances of these junctions depend on the geometry of the structure, but products V_C [Eq. (229)] are expressed

by the usual formula (40), and therefore depend on the product $\Delta_k \Delta_n$ only. If all the electrodes are made from the same material, the quantities Δ_k and V_C are the same for all two-terminal junctions.

From Fig. 25(b) it is evident that the Josephson triode properties are close to the properties of three separate junctions inserted into a superconducting ring. The electrodynamic properties of the triode have been analyzed by Likharev (1975a), and it has been proven that, given a proper choice of parameters, a triode can be utilized particularly for microwave mixing, using the Josephson oscillations from one of the two-terminal junctions as the local oscillator signal.

8.6.2 B. Connections of weak links

Some interesting new processes may also take place in those structures where the usual (two-terminal) weak links are interconnected in a system by means of superconducting electrodes. The properties of the two simplest types of connections, series and parallel ones (Fig. 26), appear to be radically different.

1. Series connection

In series connections (Palmer and Mercereau, 1974, 1975, 1977; Jillie et al., 1975; Feldman et al., 1975; Dupart and Baixeras, 1977) the processes occurring in weak links are independent in the first approximation (Likharev, 1973). In fact, in Sec. V.A we have seen that the processes in the weak link only affect the phase shift and voltage drop on it, but have hardly any influence on the time dependence of the current through it. Since current is the only common variable for the elements connected in series, the contacts hardly interact at all. Of course, in real situations, there is always some weak interaction due to one mechanism or another, discussed below.

First, owing to the finite magnitude of impedance Z_e of the external system, Josephson oscillations give rise to an ac current, although a very weak one, at the Josephson frequency ω_V . This current will have a synchronizing action on the other weak links in the system.

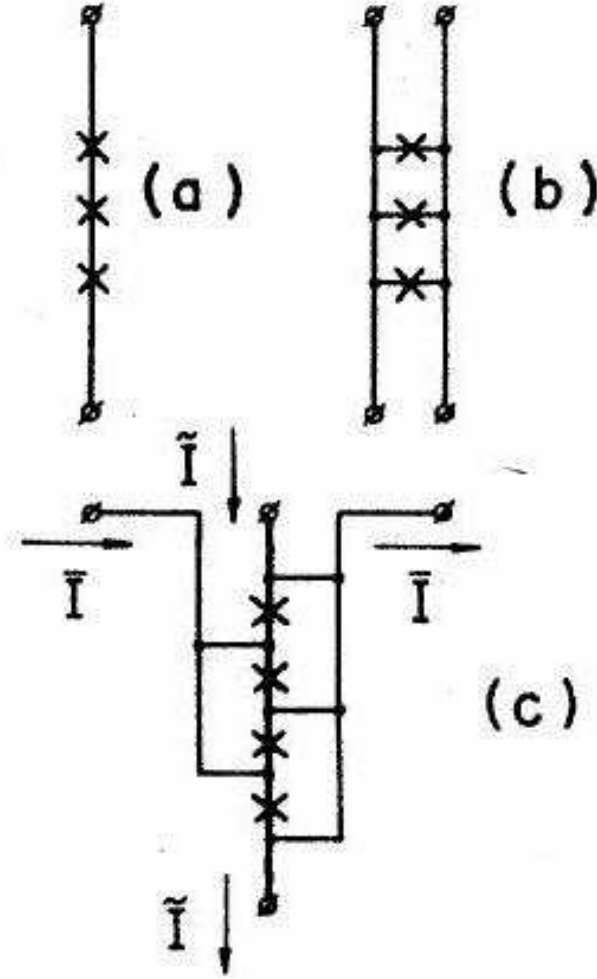


FIG. 26. Series (a), parallel (b), and one of the possible complex connections (c) of weak links. In the last case (Lukens, 1976) weak links are connected in series for the microwave signal (current $\tilde{I}$), but in parallel for dc bias (current $\bar{I}$), so that dc voltages $|\bar{V}|$ are equal at all junctions. Cross denotes a single element (weak link).

Therefore, mutual locking of the oscillations may take place if the link parameters are close in value, i.e., if the frequencies ω_V are almost the same for the same dc bias current.

Second, if the distance between the weak links is comparable to the coherence length of the electrodes ξ , there may be a finite interaction through the variation in the modulus of the order parameter at the banks. We have already seen that the greater the deviation of the weak links from the ODSEE model, the greater these variations will be. In other words, the variations are least in the variable-thickness bridges and greatest in onedimensional structures with weak links and electrodes made from almost identical materials. This coupling mechanism has been theoretically analyzed in the literature (Blackburn et al., 1972, 1975; Howard and Kao, 1975; Kovalenko, 1976; Way et al., 1978).

Third, the generation of excess electron-like quasiparticles with an intensity varying with the Josephson frequency ω_V takes place in weak links during the ac Josephson effect. These normal electrons, on penetrating into the banks, may diffuse to distances $\sim \xi_e$ [Eq. (181)] which may be of the order of several microns (in aluminum tens or even hundreds of microns). If another weak link is located within this distance, the normal electrons will act upon it at a frequency ω_V . This mechanism has indeed been detected experimentally by Jillie et al. (1977a, b), but no detailed theoretical description is yet developed.

Finally, for very low capacitances C between the connected electrodes: $e/C \gtrsim V_C$, one more specific coupling mechanism is also possible. In this case the passage of even one electron through one of the weak links changes the potential of the "middle" electrode (connected be-

tween two weak links) by a value e/C comparable to V_C . Therefore the conventional description of weak links (at least for those with a relatively large mean free path ℓ) is not valid, and the discrete nature of the current carriers has to be taken into consideration in the theory. Intuitively, it is evident that this will give a specific "electrostatic" mechanism of interaction between neighboring weak links. Although there are some communications describing such effects (Zeller and Giaever, 1969; Kulik and Shekhter, 1975, 1976), neither the exact boundaries, nor the main consequences of such interaction are as yet clear.

It should be pointed out that it may be rather difficult to separate the various synchronization mechanisms in the experiment.

2. Parallel connection

Unlike the case of series connection, weak links connected in parallel (two superconducting electrodes connected by several junctions) always interact between each other. Indeed, there can be no voltage drop across the sections of superconducting electrodes separating the junctions. Therefore the mean voltage $\bar{V}$, and consequently the Josephson oscillation frequency ω_V [Eq. (6)], is the same at all the weak links connected in parallel.

A similar strong interaction is observed in the stationary state too; it is responsible for a number of specific properties described in detail in the existing literature on the Josephson effect. Use of weak links in this case does not give rise to any new effects, and therefore we recommend that the reader consult some of the available publications (see Footnote 1).

3. Use of parallel and series connections

Parallel connections of Josephson junctions are used in dc SQUIDS (Clarke, 1977), and may also be used in cryotron-type (Zappe, 1975) and quantron-type (Fulton et al., 1973; Likharev, 1977b) digital circuits. Moreover, distributed (wide) weak links (see below) can also be represented as a parallel connection of a large number of junctions.

Series connections of weak links into arrays seem to be highly promising for increasing the resistivity in microwave applications of the Josephson effect. The first experiments with such circuits (Feldman et al., 1975; Chiao and Parrish, 1976), however, have shown that the inevitable variation of the parameters of the weak links in the array leads to noticeable differences in the values of dc voltage across them, and consequently to differences in the Josephson oscillation frequencies. It is, at the same time, highly desirable to fix a definite bias point $\bar{V}$ in most microwave devices. In this situation combined parallel-and-series connections can be used. An example of such a connection suggested by Lukens (1976) is shown in Fig. 26(c). In this circuit the ac current flows along a wide central line through all the elements in series. At the same time, thin lines of high inductance (to avoid shunting of the ac current) connect all the elements in parallel with respect to the dc bias; therefore $\bar{V}$ across them is the same even in the case of wide scattering of the parameters.

Some other weak link systems have also been studied (Mooij et al., 1974; Tsang and Wang, 1974; Lindelof et al., 1977), their utility for device applications being more doubtful. Moreover, a considerable number of papers have been published dealing with granular superconductors which may be looked upon as a complicated connection between chaotically located weak links with random parameters. Perhaps a separate paper is needed to review all these works. Today, regular weak link systems seem to be more promising both for physical research and for device applications.

8.6.3 C. Distributed structures

In the preceding pages we considered relatively narrow ($W \sim \xi \ll \lambda_{\text{eff}}$) weak links in which the current is uniformly distributed throughout the cross section. If the link width W becomes large, the phase difference φ and the current density J may depend on the transverse coordinate y . This dependence can be caused by an external magnetic field, and if the width W is greater than some characteristic size λ_{eff} , it can also be caused by the magnetic field of the current flowing through the weak link. In the case where the relation between J_S and φ is sinusoidal, a special term, the Josephson penetration depth λ_J , is used for λ_{eff} .

Such dependences $[J(y)]$ are well understood for tunnel junctions (see, for example, Ferrell and Prange, 1963; Owen and Scalapino, 1967; Kulik and Yanson, 1970), and our aim here is only to show that they can have some specific features in weak links.

1. Local and nonlocal self-limitation

Self-limitation, that is variation of φ along the width of the structure under the influence of the current flowing through it, is conveniently described in the following way (Lapir et al., 1975). An elementary current $J(\mathbf{r}') d\mathbf{r}'$ flowing through a particular region of a weak link penetrates into the banks, and thus creates a gradient $d(\nabla\chi)$ of the order parameter phase inside the bank. If the nonlinear processes are localized in the weak link itself, i.e., condition (64) of the ODSEE model is applicable, the electrodynamics of the banks is linear, and therefore $d(\nabla\chi(\mathbf{r})) \propto J(\mathbf{r}') d\mathbf{r}'$. Summing up the contributions of all regions of the weak link to the phase gradient, we obtain

$$\chi(\mathbf{r}) = \chi_0(\mathbf{r}) + \int K(\mathbf{r}, \mathbf{r}') J(\mathbf{r}') d\mathbf{r}', \quad (230)$$

where the integral is taken along the weak link width, and is the phase distribution in the absence of the current through the link, for instance, under the effect of external magnetic fields or currents along the banks. $K(\mathbf{r}, \mathbf{r}')$ is a kernel, which depends on the geometry and the material of the banks.

After subtracting Eq. (230) written for the points on opposite sides of the weak link, and taking into consideration that $\varphi(y) = \chi_2 - \chi_1$, we obtain the self-consistency equation:

$$\varphi(y) = \varphi_0(y) + \int_0^W K(y, y') J(y') dy', \quad (231)$$

where the kernel K is simply the sum of the kernels $K_{1,2}$ for the banks taken at $x = x' = 0$. In sandwich-type structures [Fig. 1(a), (b)]⁴⁰ the current penetrates into the banks only at distances of the order of penetration depth λ , which is usually much less than λ_J . Therefore at distances λ_J from the source the current flows parallel to the interlayer, and thus, by virtue of Eqs. (21b) and (24), creates a constant phase gradient:

$$\nabla\varphi = (2\pi/c\phi_0) \mathcal{L}_{\square} J, \quad \phi_0 = \pi\hbar/2e, \quad (232)$$

along the y coordinate. Here $\mathcal{L}_{\square}$ stands for the inductance per square of the system consisting of two superconducting films, in which the currents flow in opposite directions along adjacent surfaces

$$\mathcal{L}_{\square} = 4\pi t_{\text{eff}}, \quad (233)$$

where, for the case under consideration, $t_{\text{eff}} = L + 2\lambda$, and L is the thickness of the interlayer. From Eqs. (232) and (233) it follows that in sandwichtype structures we may take that

$$K(y, y') = (4\pi^2 t_{\text{eff}} / c\phi_0) |y - y'|. \quad (234)$$

Substituting this kernel in the self-consistency equation

(231), and then differentiating twice with respect to y , we arrive at the usual result that the distribution of the phase over the width of the junction obeys the differential equation
 $\S\S$

$$\frac{\partial^2 \varphi}{\partial y^2} = \frac{\partial^2 \varphi_0}{\partial y^2} + \lambda_J^{-2} J(y) / J_C \quad (235a)$$

$$\lambda_J = (c\phi_0 / 8\pi^2 t_{\text{eff}} j_C)^{1/2} \quad (235b)$$

$\S\S$

Here we can say that self-limitation is a local effect in which the current at a given point affects the phase changes only at the same point. However, Eq. (231) shows that this is not so in the general case, where self-limitation is described by an integral equation, i.e., is a nonlocal effect.

2. Variable-thickness bridge over ground plane

Consider the system shown in Fig. 27 (Lapir et al., 1975) as an example of a structure with nonlocal selflimitation. In this system a variable-thickness bridge connecting two banks made of relatively thick ($d \geq 2\lambda$) film is located above a superconducting ground plane, say, above a thick film ($d_{\text{gp}} \geq 2\lambda_{\text{gp}}$). The spacing t between banks and ground plane, just like λ and d , is small compared to the planar dimensions of the systems, say the bridge width W . These systems are interesting not only for their unusual electrodynamics, but also for their application as Josephson cryotrons.

In this case, to find the kernel K in Eq. (231), note that Eq. (23) is also valid for the bank-ground plane gap, if $t_{\text{eff}} = t + \lambda + \lambda_{\text{gp}}$ is the effective spacing between the bank and the ground plane. In the dc state the current in the banks is conserved ($\text{div}_{x,y} J = 0$); therefore the phase χ satisfies the Laplace equation

$$\nabla_{x,y}^2 \chi = 0 \quad (236a)$$

which has to be solved under the boundary condition following from (232)

$$\left. \frac{\partial \chi}{\partial n} \right|_C = \frac{8\pi^2 t_{\text{eff}}}{c\phi_0} J_n, \quad (236b)$$

where C is the contour of the bank boundaries, and n is

⁰⁴⁰ For the sake of simplicity, we consider a long sandwich with one of its dimensions (W') much less than the other (W), so that it can be regarded as a structure distributed in one dimension (along W) with a linear current density $J = jW'$.

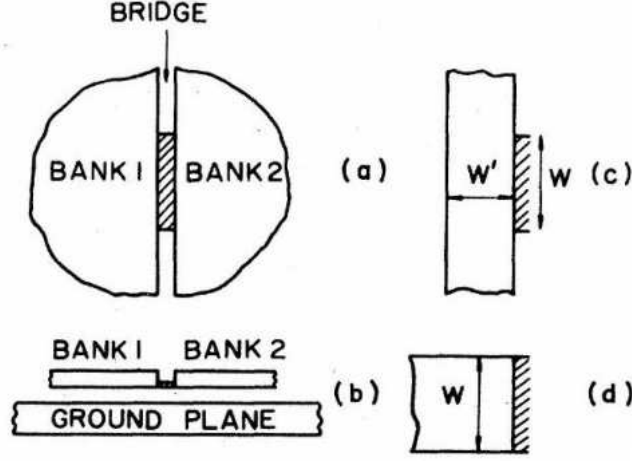


FIG. 27. Variable-thickness bridge over the superconducting ground plane: (a) view from above and (b) from the side, and also particular shapes of one of the banks (c), (d). Views from above (a, c, d) do not show the ground plane, sizes of which are much greater than the sizes of the banks.

a normal to this contour. Solving Eq. (236) by the usual potential method, we arrive at Eq. (231) in which the kernels $K_{1,2}$ satisfy the equations

$$\nabla_{x,y}^2 K_{1,2} = 0, \quad K_{1,2} - (2\pi\lambda_0 J_C)^{-1} \ln |\mathbf{r} - \mathbf{r}'|, \quad \text{at } \mathbf{r} \rightarrow \mathbf{r}' \quad (237)$$

Here λ_0 is a constant

$$\lambda_0 = c\phi_0/16\pi^2 t_{\text{eff}} J_C, \quad \lambda_0 [\mu\text{ m}] \approx 1.31/J_C [A/\text{cm}] t_{\text{eff}} [\mu\text{ m}], \quad (238)$$

which determines the characteristic scale of the selflimitation effect.⁴¹ For the real values of the parameters ($t_{\text{eff}} \approx 0.3\mu\text{ m}$, $J_C \sim 1 - 10\text{ A/cm}$) λ_0 is in the range of units or tens of microns, i.e., much greater than the typical values of W and L (tenths of a micron).

The kernels $K_{1,2}$ essentially depend on the shape of the banks. If the bank planar dimensions are much greater than λ_0 , the kernels are simple

$$K_{1,2} = (2\pi\lambda_0 J_C)^{-1} \ln |\mathbf{r} - \mathbf{r}'| \quad (239)$$

In this case the current from the bridge penetrates the bank-ground plane gap at large distances of the order of λ_0 and the self-limitation effect is essentially nonlocal, i.e., Eq. (231) cannot be reduced to a differential equation of the type (235a). In this case various electro-dynamical characteristics of wide ($W \gg \lambda_0$) bridges (for example, field distribution across the width of structures, dependence of critical current on external magnetic field, shape and characteristics of Josephson vortices) are somewhat different from the characteristics of Josephson sandwich-type structures. A detailed analysis of these characteristics is given in papers by Lapid et al. (1975) and Kupriyanov et al. (1976a).⁴²

But if the penetration of the current into the depths of the bank-ground plane gap is limited by the shape of the bank, the effective size becomes less than λ_0 . For example, for the bank shape shown in Fig. 27(c), applying conformal transformation of a semi-infinite strip to half-plane, from Eq. (237) we find that

$$K_1 = (2\pi\lambda_0 J_C)^{-1} \ln \left| \sinh \frac{\pi(y - y')}{2W'} \right|. \quad (240)$$

If the bank "depth" W' decreases and becomes less than λ_0 , the kernel K_1 [Eq. (240)] assumes the shape of Eq. (234) at the distances λ_J , which we are interested in. Therefore, here Eq. (231) is again reduced to the differential equation (235a) where we now have

$$\lambda_J = (2\lambda_0 W')^{1/2} = (c\phi_0 W' / 8\pi^2 t_{\text{eff}} J_C)^{1/2} \ll \lambda_0. \quad (241)$$

Thus, if at least one of the banks is sufficiently narrow, the self-limitation effect once again becomes local and the expression for the characteristic size λ_J merely coincides with the expression for sandwich-type junctions (235b). The "depth" W' of the narrow bank now serves as the second planar dimension of the tunnel junction, and the effective gap between the bank and the ground plane functions as the effective thickness of the sandwich t_{eff} .

3. Applications of distributed weak links

It has been seen that the electrodynamical properties of variable-thickness bridges over the ground plane with at least one narrow bank are almost analogous to the properties of wide Josephson sandwich-type structures (for example, tunnel junctions), but have a negligibly small intrinsic capacitance and, therefore, singlevalued $I - V$ characteristics. Hence variable-thickness bridges can easily be used as Josephson cryotrons (or "Josephson gates"), particularly diffraction-type cryotrons.

A cryotron (see, for instance, Bremer, 1962) is a four-terminal superconducting device, through which two currents can pass, usually without direct (galvanic) contact with each other: gate current I_g and control current I_c . Switching on the control current sharply decreases the critical value $(I_g)_{\text{max}}$ of the gate current, resulting in decrease of gate current I_g . Figure 28 shows the simplest ("in-line") Josephson cryotron (Matisoo, 1967) and the distributions of the gate current (I_g) and the control current (I_c) density in it. These figures show that the equivalent circuit of such a cryotron is a distributed Josephson junction fed by the currents I_g and I_c as shown in Fig. 28(d). To find the main characteristics of the cryotron (control characteristics), i.e., the dependence of the critical value of the current I_g on the control current

$$(I_g)_{\text{max}} = f(I_c), \quad (242)$$

we have to solve Eq. (235a) with $\partial^2 \varphi_0 / \partial y^2 = 0$ under the following boundary conditions (see, for example, Kulik and Yanson, 1970)

$$\lambda_J \frac{d\varphi}{dy} \Big|_{y=0} = 2 \frac{I(0)}{I_J}, \quad \lambda_J \frac{d\varphi}{dy} \Big|_{y=W} = 2 \frac{I(W)}{I_J}, \quad I_J = 2\lambda_J J_C. \quad (243)$$

Here $I(0)$ and $I(L)$ are the currents fed to the junction edges. From Fig. 28(d), it is evident that for an in-line cryotron

$$I(0) = I_c, \quad I(L) = I_c + I_g \quad (244)$$

This problem has been analyzed both numerically (see, for example, Basavaiah and Broom, 1975), and analytically within the limits $W \ll \lambda_J$ (Jaklevic et al., 1964) and $W \gg \lambda_J$ (Kupriyanov et al., 1976b). Figure 28(e) shows the shape of the control characteristics for the case $W \gg \lambda_J$, where it is described by the equalities

In actual wide junctions ($W \approx 4 - 10\lambda_J$), the curve lies slightly lower, and small oscillations exist at its edges, the curve (245) being the envelope of the oscillations.

⁰⁴¹ For variable-thickness bridges without a ground plane, simple estimates give $\lambda_J = (c\phi_0 / 16\pi^2 J_C)^{1/2}$, i.e., about one micron at $J_C = 1$ A/cm.

⁰⁴² Nonstationary effects under nonlocal self-limitation, in particular, the shape of the $I - V$ curves of wide weak links (Kupriyanov et al., 1976a) likewise differ from those given by the local equation (235) (Clarke, 1971).

From the above it follows that a structure based on

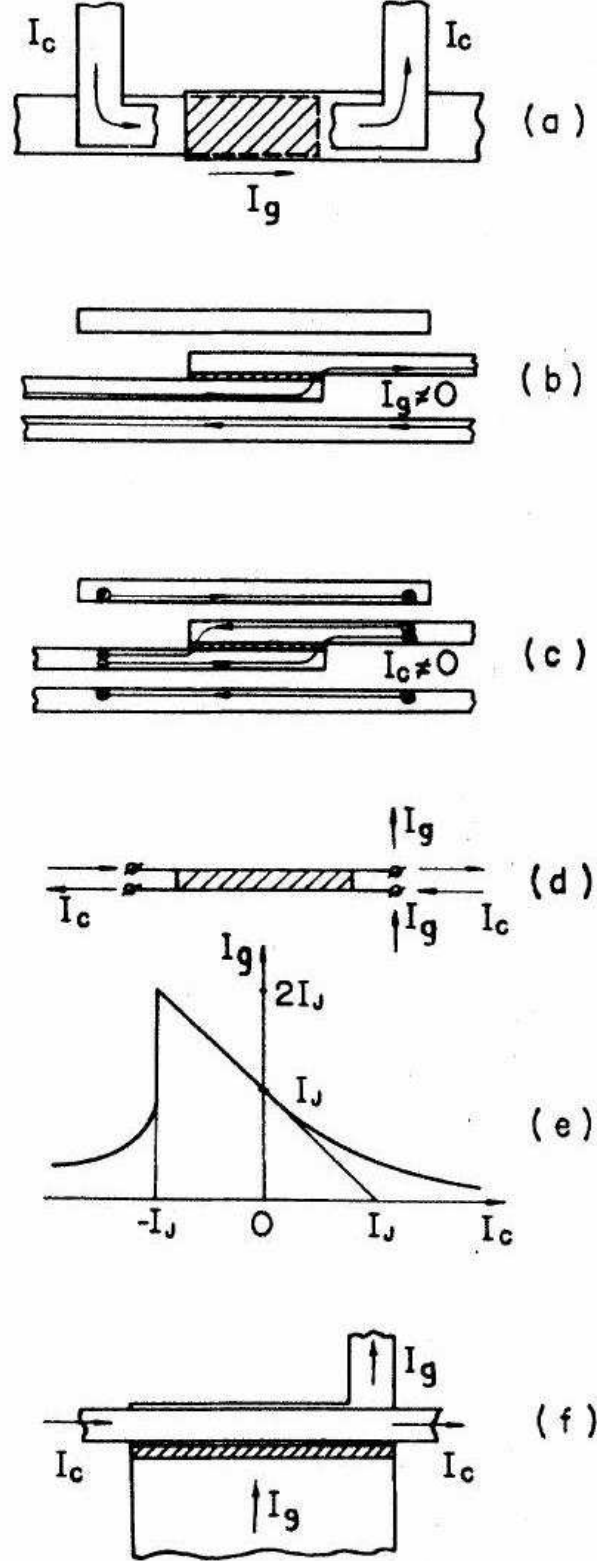


FIG. 28. In-line Josephson cryotron with sandwich-type junction: (a) view from above; (b) distributions of gate current; (c) distribution of control current; (d) equivalent circuit; (e) control characteristics for the case $W \gg \lambda_j$; and (f) possible realization of the cryotron with variable-thickness bridge.

a variable-thickness bridge will have the same control characteristics if the currents are fed into the bridge as shown in Fig. 28(f) (superconducting ground plane not shown). It is clear that the gate current I_g is fed precisely as shown in Fig. 28(d). In order to verify the validity of the

circuit shown in Fig. 28(d) for the control current, we have to take account of the fact that I_c excites an equal countercurrent along the upper surface of the bank, which should return to the lower surface of the wide bank through the bridge.

As is clear from Fig. 28(e), the effective gain of an inline cryotron

$$G = |\Delta(I_g)_{\max} / \Delta I_c| \quad (246)$$

for relatively large logical signals $|\Delta I| \sim I_J$ is small: $G \lesssim 1$. Kupriyanov et al. (1976) have found that the gain G can be significantly increased by feeding the current in the middle of the Josephson junction. For example, for structures with an equivalent circuit shown in Fig. 29(a), the control characteristics are of the form [Fig. 29(b)]

$$g = \begin{cases} 4, & \text{at } |h| < 1, \\ 2 \left[|h| + (h^2 - 1)^{1/2} \right]^{-1}, & \text{at } |h| > 1 \end{cases} \quad (247)$$

The gain G of such a cryotron may be as high as 3 or 4,

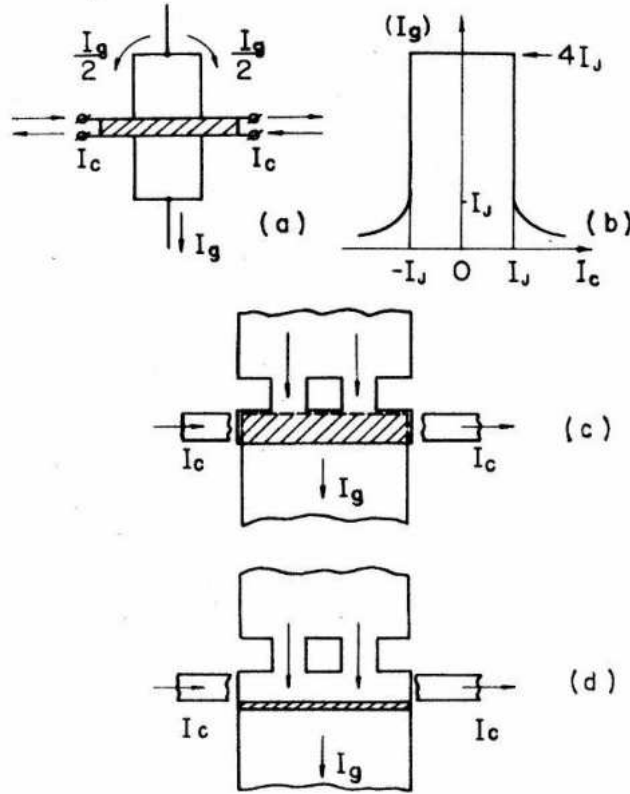


FIG. 29. Cryotron with gate current fed into two internal points: (a) equivalent circuit; (b) control characteristics; (c) realization of the cryotron with sandwich-type junction and (d) variable-thickness bridge (Kupriyanov et al., 1976b; Likharev and Ulrich, 1978).

and this level is quite sufficient for most of the simple cryotron-based circuits (Bremer, 1962). Current fed into the internal points can easily be realized both with sandwich-type structures [Fig. 29(c)] and with variable-thickness bridges [Fig. 29(d)].

Interference cryotrons (Zappe, 1975, 1977) can easily be built up with microbridges too. These circuits do not have such good control characteristic as the cryotrons shown in Fig. 29, but may be somewhat simpler in fabrication. In particular, structures with $W \sim L$, for instance, Dayem bridges, can be used in these cryotrons.

8.7 VIII. CONCLUSIONS

Superconducting weak links-weak conducting connections between superconducting electrodes—are unique in their variety of electrodynamic phenomena. Intensive research in the past few years has explained many of these phenomena, but also brought to the fore many new questions. Below we attempt to list those phenomena that are clearly understood now, and those unresolved problems which are, in the author's opinion, key questions today.

8.7.1 A. Basic results

1. Stationary processes

(1) The Josephson effect occurs in all weak links of sufficiently small length, namely, in those having an ef-

fective length L_{eff} less than the coherence length of the weak link material (but not of the electrode material!). By the Josephson effect we mean that the dependence of supercurrent I_S flowing through a weak link on the phase difference φ of the electrodes is single-valued and 2π -periodic (Fig. 4). Thus the Josephson effect is not directly associated with superconducting tunneling: tunnel junctions are just one of the systems that exhibit the Josephson effect (Fig. 1).

(2) Only in some rare and very special cases does the Josephson effect take place in weak links in the same manner as in tunnel junctions. In most real weak links, even in a dc state, the function $I_S(\varphi)$ differs both in shape and in amplitude (the critical current) from the function in tunnel junctions. The function $I_S(\varphi)$ is substantially dependent, also, on the ratio of the effective length L_{eff} to the mean free path ℓ of electrons in a weak link (Fig. 6).

(3) For small weak links ($L_{\text{eff}} \lesssim \xi$) it is unimportant whether the link material at a given temperature is superconducting or normal, and the main characterizing parameter of the weak link $V_C = I_C R_N$ does not depend on the link material (but only on the electrode material). However, if the effective length of the weak link L_{eff} is increased to several coherence lengths of the link material, the function $I_S(\varphi)$ begins to be dependent of the parameters of the link material. If the weak link is made of a normal material, the Josephson effect is limited by an exponential drop in the supercurrent as the length increases (Fig. 7). If the link material is superconducting, then at approximately the same lengths, the dependence $I_S(\varphi)$ becomes multivalued (Figs. 7 and 8).

(4) The uniform flow of the superconducting condensate in weak links with a multivalent function $I_S(\varphi)$ [Fig. 4(b), 8(a)] may be violated even if their dimensions are relatively small ($L, W \sim \xi$), due to the onset of Abrikosov vortices (if $W > W_C$) or the formation of phase-slip centers (if $W < W_C$) (see Fig. 12). Both these effects are highly sensitive to the small space inhomogeneities which are responsible for the strong irreproducibility of the phenomena. They are impossible if the $I_S(\varphi)$ relationship is single-valued, and thus the structure is stable with respect to small space inhomogeneities. Therefore the situation where the function $I_S(\varphi)$ ceases to be single-valued may reasonably be taken to be the limit of the "ideal" Josephson effect. For dirty weak links ($\ell \ll L_{\text{eff}}$) made from superconducting material, this limit lies within several coherence lengths of the link material.

(5) Different types of weak links (Fig. 1) mainly vary in the degree of involvement of the electrodes in the nonlinear processes occurring in the weak link. In the limit when this involvement is negligible, the simple model [One-Dimensional Structure with Electrodes in Equilibrium

⁰⁴³ Of course, we can also use other types of wide ($W \gg L_{\text{eff}}$) bridges as well, say, one-dimensional structures where the nonlinear effects do not extend to the banks [Fig. 1(C), (D)], provided the critical current of such structures is small. But in this case the parameter V_C that determines the amplitude of the cryotron output signal is small.

(ODSEE)] is applicable in which the values of the order parameter modulus at the weak link-electrode interfaces are taken to be equal to the equilibrium values inside the electrodes. In those types of weak links where geometric current concentration does not occur [constant cross-section, Fig. 1(d)], the ODSEE model can be made applicable if the parameter σ_N/ξ of the link material is much less than that of the bank material. For structures with strong current concentration the ODSEE

model can be made valid practically for any material of the weak link and electrodes. Such behavior is only exhibited by variable-thickness bridges [Fig. 1(f)] among the structures with controllable geometry.

(6) The higher the V_C and R_N , the better the application properties of the weak links for utilization in most Josephson effect devices. The values of V_C are maximal if $L_{\text{eff}} \lesssim \xi$, a condition which is easier to obtain in structures approximately obeying the ODSEE model. At present, the most promising seem to be the sandwich-type structures with an interlayer made of low-conduction material, and variable-thickness bridges, which both are superior to tunnel junctions because of low intrinsic capacitance.

(7) When the cross section (i.e., width) is increased, the bridge-type weak links may exhibit self-limitation effects which differ from those in sandwich-type structures, for example, in tunnel junctions. In particular, the effect of the current on the phase changes along the structure width may be essentially nonlocal. Nonetheless, variable-thickness bridges can be utilized in the design of magnetic-field-controlled structures (Josephson cryotrons), just as can tunnel junctions.

2. Nonstationary processes

(1) In analyzing the nonstationary (ac) processes, one should bear in mind that the intrinsic capacitance of weak links is relatively small, since the capacitance of Josephson structures may have somewhat greater influence on their observed properties than the physics of their conductivity.

(2) The basic ac properties of short weak links [with single-valued $I_S(\varphi)$] may be interpreted even on the basis of very simple models, for example, the RSJ model in which the supercurrent and normal current are supposed to be superimposed in a simple linear manner. Among the main effects which this model fails to explain are enhancement of superconductivity by ac and dc currents, and the existence of "excess" current and of the upper frequency limit for the Josephson effect. The last two of these effects can easily be interpreted within the framework of simple TDGL equations, which are strictly suitable only for gapless superconductors. Enhancement of superconductivity is most likely to be associated with the decrease in the suppression of the order parameter by electron-like excitations of energies $\epsilon \sim |\Delta|$.

(3) In general, the nonstationary effects are highly complicated as a result of nonequilibrium distributions of quasiparticles (normal electrons and phonons) and of their consequent back response to the ac processes.

(4) For comparing theories of nonstationary processes with reality, it is frequently not simple experiments (say, observation of $I-V$ curve of an autonomous weak link) but more complicated ones (usually with microwave irradiation from an external source) that are most convenient. For instance, a study of the real and imaginary parts of the high-frequency conductivity of a weak link as functions of the dc bias point in the superconducting region of the $I-V$ curve and of temperature seems to be quite useful because the corresponding theoretical problems admit linearization with respect to variables. In contrast, such well known methods as observation of current steps on $I-V$ curves with high serial numbers under microwave radiation can give only a limited amount of information.

(5) The weak links which satisfy the ODSEE model are remarkably useful for the comparison mentioned above. Since it is desirable to use structures with an arbitrary relation between the

parameters of the weak link and the electrodes, variable-thickness bridges seem to be particularly promising.

8.7.2 B. Unsolved problems

1. dc processes

- (1) Is it possible to derive the familiar theoretical results for clean ($\ell \gg \xi_0$), thick ($L \gg \xi_0$), $S - N - S$ -type sandwiches from the Eilenberger equations? How does the transition to the "dirty" limit occur in these structures with decreasing mean free path?
- (2) How does the finite critical temperature of the link material affect the critical current I_C and the $I_S(\varphi)$ relationship at arbitrary temperatures (dirty limit, ODSEE model)?
- (3) Is it experimentally possible to prepare weak links which completely obey the tunnel theory? Are the microshort dimensions $L \lesssim \hbar/p_F$ unrealizable? What are the superconducting properties of one such microshort?
- (4) What are the optimum parameters of the material of the interlayer and its thickness in a sandwich-type weak link, so that V_C would be maximum while $I_S(\varphi)$ is single-valued? How does the presence of the Schottky barriers in semiconductors affect these results?

2. ac processes

- (1) Is the discrepancy in the signs of the same two terms between experiment and the Mitsai theory indicative of certain fundamental inconsistencies in the Green's function formulation of the theory of nonstationary superconductivity?
- (2) What is the physical pattern of the distribution of normal electrons in a weak link during Josephson oscillations? What are the corresponding changes in the density of excitations? Can electrons with energy $\epsilon < |\Delta|$ exist in a weak link when the usual relation $\omega_V \tau_\epsilon \gg 1$ is satisfied?
- (3) What is the distribution (at arbitrary temperatures) of an electric field when a weak current flows through the interface between the normal and superconducting phase? What is the effect of the current density on this distribution? How does this pattern affect the simple expression for the excess current of weak links obtained in the TDGL approximation?
- (4) Why does the weak link length have such a considerable influence on superconductivity enhancement by microwaves? What is the relationship between this effect and the dc-induced enhancement?
- (5) What is the high-frequency limit for the Josephson effect, i.e., the characteristic frequency from which the function $I_A(\omega_V)$ falls off, if one takes into account nonequilibrium excitations? How does the weak link length affect this limit?
- (6) All the listed questions relate to the usual model of weak links (ODSEE model, for $l \ll L$). But how do the nonstationary processes occur in clean weak links, if only of small dimensions ($L_{\text{eff}} \ll l, \xi_0$)?

8.7.3 C. Concluding remarks

One can easily notice that while the unsolved problems in the dc field are rather particular in nature [although question (4) is quite essential for device applications], in the field of ac effects there are more questions than answers. This is due to the lack of an adequate theory to explain the nonstationary processes in superconductors and due to the complexity of the processes themselves, which depend to a greater extent on the various specific properties of weak links than do the stationary ones. In the first place, this results from the fact that excitation of the nonequilibrium electrons and phonons plays a decisive role in nonstationary processes; their subsequent diffusion, energy relaxation, and interaction are greatly affected by the properties

of materials and the shape of both the weak link itself and the banks, and frequently by the properties of surrounding objects, e.g., of the substrate and cryogenic gas or liquid (helium) in contact with it.

It should be pointed out that the questions about the nonstationary behavior of weak links which still remain obscure have no direct immediate practical importance. Without even answering them, we are able to say to what extent a specific type of weak link is suitable for a particular device based on the Josephson effect. However, elucidating these problems may be very important for the development of the theory of nonlinear nonstationary processes in superconductors and, consequently, for their possible applications in future. The geometry of weak links (especially those obeying the ODSEE model) is remarkably useful for developing this theory. The recent advent of an appreciable number of theoretical studies and the development of such convenient structures for physical investigations as variable-thickness bridges gives us every hope that progress in this field is forthcoming.

9 Current-phase relation in Josephson junctions by Golubov, Kupriyanov, Il'ichev

9.1 IX. SUMMARY AND OUTLOOK

We have discussed various possible forms of the CPR in different types of Josephson junctions. Due to space limitations, only a small fraction of the research work in this area was reviewed. We did not consider the possibility to control the CPR by high frequency irradiation or dc current injection into a weak link, as well as Josephson transistors and multiterminal Josephson devices. Nevertheless, even under these limitations we have demonstrated the existence of a variety of the CPR as a result of complex fundamental processes in superconductors.

In tunnel SIS structures it is the resonant tunnelling that may transform the classical tunnel junction into a set of parallel constrictions or double barrier devices. Ballistic electron emission microscopy done by Rippard et al. (2002) indicates that when the AlO_x oxide layer just begins to form, the hybridization of low energy states in oxide with the electrodes provides direct conduction channels through the barrier. The density of these LS is sufficiently large and according to Naveh et al. (2000) may lead to high critical current density. We have argued in this review that the supercurrent across such tunnel junctions may be represented by an equivalent double barrier structure in the coherent regime.

The type of CPR in SNS junctions makes it possible to judge about the degree of pair-breaking in the S electrodes. This may provide information important for practical applications, since the pair-breaking by current or proximity effect may lead to the reduction of high frequency cut-off by orders of magnitude compared to a classical SNS junction.

The CPR in ballistic SINIS structures is controlled by an interplay between ABS and transmission resonances. Götz et al. (2002a,b) provided the procedure which allows to extract all the set of parameters of SINIS junctions from the CPR measurements. Therefore, the developed method opens the way for calibration of the second oxidation process in SINIS junction technology. Further, the CPR measurements in a SIS'IS junction may provide an information about the sign of the effective electron-electron interaction in thin metallic films.

In Josephson junctions with a magnetoactive interlayer material it is possible to invert the sign of classical CPR and realize the so-called π junction. Moreover, the CPR modifications are generally not reduced to the $0 - \pi$ transition. The energy-phase relation in such junctions may have two minima: at $\varphi = 0$ and $\varphi = \pi$. The effect can be observable in a variety of structures: SFS sandwiches or SFcFS constrictions using metallic ferromagnets, tunnel junctions with a ferromagnetic insulator or junctions with magnetically active interfaces. These phenomena may

be used for the engineering cryoelectronic devices manipulating spin-polarized electrons and in qubit structures.

For a long time the real shape of the CPR was not considered as an important factor. Practical Josephson based devices have been implemented on tunnel junctions with $I_S(\varphi) \propto \sin \varphi$. The I-V characteristics of SIS junctions have strong hysteresis at low voltages, which is usually avoided by shunting a junction by a normal resistor. This leads to the reduction of high frequency cut-off, additional shunt noise and limits the integration scale in LSI circuits. Recent developments in design of superconducting circuits require the transition to new technology not limited by externally shunted tunnel junctions. These junctions must not require the shunting and hence they could be SNS, SINIS or high- J_C tunnel structures. As we have seen above, in all these cases the CPR is no more sinusoidal. This should be taken into account both at the level of data interpretation and systems design.

Study of the CPR is also important for understanding the fundamental properties in superconducting materials, such as the symmetry of superconducting correlation, peculiarities of spin transport in multilayered systems based on superconducting and ferromagnetic materials. As we just saw from the examples described in Sec. VIII, the monitoring of CPR may provide the information which sometimes is either impossible to get by any other existing tools, or application of these tools requires drastically increased cooling efforts and careful electromagnetic filtering. The CPR measurements are of fundamental importance also as a possible direct tool to detect the time reversal symmetry breaking at the interfaces of unconventional superconductors.

The modern technology is focused on fabrication of the sub-micron size Josephson junctions and will require the developments in precise measuring technique. And as shown in this review, the measurement of the CPR permits to determine experimentally the critical current even in the case when the Josephson energy is smaller than the energy of thermal fluctuations. Measurements of CPR may also provide the information on the transparency of interfaces, important for the junction fabrication technology, and may be used as a routine method for control some technological steps or processes.

Finally the peculiarities of the φ dependence of Josephson junction energy, which could be directly reconstructed from experimental CPR of a junction of interest may be important for designing of superconducting quantum bits. Recently Il'ichev et al. (2002) have applied the method, developed for the CPR measurements, for the characterization of the superconducting qubit circuits.

We believe that the CPR spectroscopy may become the useful tool for novel fundamental and technological developments.

9.2 I. INTRODUCTION

The Josephson effect was discovered by Brian Josephson in 1962 (see also Josephson, 1964, 1965) and first observed experimentally by Anderson and Rowell (1963) and Rowell (1963). Since that time, there was a continuously growing interest to the fundamental physics and application of this effect. The achievements in Josephson junction technology made it possible to develop a variety of sensors for detecting ultra-low magnetic fields and weak electromagnetic irradiation, to fabricate and to test ultrafast digital RSFQ circuits and to pose a problem of designing large scale integrated circuits for signal processing (Likharev, 1996, 2000; SEMATECH, 2001; SCENET). In the present literature, there is a lack of overview of this area of physics available to both experts and people entering the field.

Classical developments were summarized in the review by Likharev (1979). Several recent review articles by Kupriyanov and Likharev, 1990; Braginski, 1992; Sigrist and Rice, 1995; van Harlingen, 1995; Delin and Kleinsasser, 1996; Tsuei and Kirtley, 2000; Kashiwaya and Tanaka, 2000; Lofwander, et al., 2001; Makhlin et al., 2001; Hilgenkamp and Mannhart, 2002

are devoted to a number of applications of the Josephson effect, e.g. in high T_C superconductors (HTS) and Josephson quantum bits. At the same time, recent progress in the physics of the Josephson effect justifies an overview of the fundamentals of the Josephson effect on a more general level.

The purpose of the present review is to provide a theoretical basis and to discuss various possible

38 forms of the dependence of the supercurrent I_S on the phase difference φ in Josephson junctions of different type: superconductor-normal-superconductor (SNS), superconductor-insulator-superconductor (SIS), double-barrier (SINIS), superconductor-ferromagnet-superconductor (SFS) and superconductor-two dimensional gas-superconductor (S-2DEG-S) junctions. Manifestation of unconventional symmetry of the order parameter in HTS in CPR will also be discussed.

Recently, a new experimental tool has been developed (Rifkin and Deaver, 1976; Il'ichev, 1998b, 2000a). It was applied to study the CPR for a variety of Josephson junctions. We will describe this method and its practical applications. It is important that the method has a resolution permitting to study weak links with Josephson energies smaller than the intensity of thermal fluctuations. We shall illustrate by a number of examples the importance of these experimental studies for fundamental physics and applications. The examples are related to measurements of small critical currents in Josephson junctions, a calibration of second oxidation in SINIS junctions, study of hybrid S-2DEG-S Josephson devices, CPR anomalies in HTS junctions, physics of π states in SFS junctions and qubit structures.

The main emphasis in the review is on the general physical nature of the phenomenon rather than on a mere enumeration of experiments and details regarding the technology of fabrication of various weak links. Nevertheless the references to original experimental works are given in the appropriate places.

9.3 II. DC JOSEPHSON EFFECT

Josephson (1962) predicted that a supercurrent I_S can exist between two superconductors separated by a thin isolating layer and that its value would be proportional to the sine of the difference $\varphi = \chi_1 - \chi_2$ of the phases of the superconductor order parameters $\Delta_1 \exp \{i\chi_1\}$ and $\Delta_2 \exp \{i\chi_2\}$

$$I_S(\varphi) = I_C \sin \varphi \quad (1)$$

(so-called *dc* Josephson effect). The maximum current I_C in that CPR is the critical current. Further studies have shown (for a review see e.g. Likharev, 1979) that the effect extends beyond Josephson's predictions and can exist if superconductors are connected by a "weak link" of any physical nature (normal metal, semiconductor, superconductor with smaller critical temperature, geometrical constriction etc.)

The CPR is an important characteristic of a Josephson junction. Only in a few cases it reduces to the wellknown simple form of Eq.(1), which is ordinarily used to study the dynamics and ultimate performance of analogous and digital devices based on Josephson junctions, see Van Duzer and Turner (1981), Barone and Paterno (1982), Likharev (1986), Gallop (1991), Orlando and

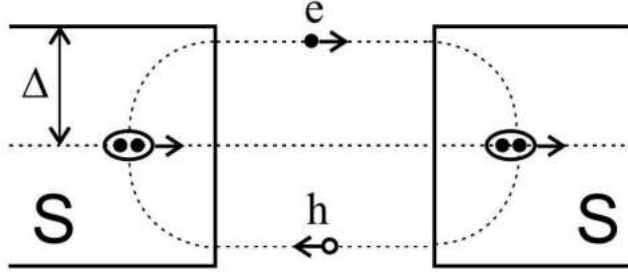


FIG. 1: Formation of Andreev levels in a Josephson junction.

Delin (1991), Tinkham (1996), Schmidt (1997), Kadin (1999).

The physics of the dc Josephson effect can be understood if we take into account that a quasiparticle located in the weak link can not penetrate directly into a superconductor if its energy is smaller than the superconducting energy gap. However, another form of charge transport, the so-called Andreev reflection (Andreev, 1964) can occur. An electron with momentum k impinging on one of the interfaces is converted into a hole moving in the opposite direction, thus generating a Cooper pair in a superconductor. This hole is consequently Andreevreflected at the second interface and is converted back to an electron, leading to a destruction of a Cooper pair (see Fig. 1). As a result of this cycle, a pair of correlated electrons is transferred from one superconductor to another creating a supercurrent flow across a junction. Since Andreev reflection amplitudes depend on corresponding phases $\chi_{1,2}$, the resulting current depends on the phase difference φ thus leading to the dc Josephson effect.

Due to the electron-hole interference in the quantum well formed by the pairing potentials of the superconducting electrodes, standing waves with quantized energy, E_{AB} , appear in weak link region. The corresponding quantum states are referred to as Andreev bound states (ABS). The physics of ABS in Josephson junctions was extensively studied starting from pioneering work by Kulik (1969).

It follows from the microscopic theory of superconductivity (see review by Lambert and Raimondi, 1998; Belzig et al., 1999; Kopnin, 2002), that in stationary situations the supercurrent across Josephson junctions

$$I_S(\varphi) \propto \int_{-\infty}^{\infty} dE [1 - 2f(E)] \text{Im} \{I_E(\varphi)\} \quad (2)$$

depends on the electron energy distribution function $f(E)$ and the spectral current $\text{Im} \{I_E\}$. Spectral current incorporates the information on the energy distribution of the ABS in a junction. $\text{Im} \{I_E\}$ depends on the distance between the superconductors d , transport parameters of the junction's materials (resistivities $\rho_{1,2}$, Fermi velocities $v_{F1,2}$ and parameters of interfaces).

In the structure where the momentum of an electron in the weak link region is a good quantum number (so-called "clean" Josephson junctions), ABS form a regular structure in energy and $\text{Im} \{I_E(\varphi)\}$ is peaked at the corresponding energies (Kulik, 1969; Ishii, 1970; Bardin and Johnson, 1972; Bagwell, 1992; Schüssler and Kummel, 1993; Tang et al., 1996).

An increase of the degree of disorder in the weak link leads to broadening and decreasing of the amplitudes of the peaks. The disorder generates a distribution of the lengths of the electron trajectories in the weak link. Therefore $\text{Im} \{I_E(\varphi)\}$ for a disordered junction is a weighted average of the ballistic result over the corresponding distribution. This makes the spectral current $\text{Im} \{I_E(\varphi)\}$ a continuous function of energy.

The electron energy distribution function in Eq. (2) defines the population of ABS at a given temperature. Thus the expression (2) shows that the whole supercurrent $I_S(\varphi)$ is the sum of the partial currents transported via ABS. Therefore one can modify the shape of $I_S(\varphi)$ in two ways: (a) by modifying the spectral current $\text{Im} \{I_E(\varphi)\}$, changing the material parameters

or geometry of a junction; (b) by manipulating the occupation numbers of ABS , i.e. creating a nonequilibrium distribution function $f(E)$ in a weak link.

The nonequilibrium effects in weak links were intensively studied by many authors (Zaitsev 1976, 1993, 1995; Artemenko et al., 1979; Zaikin and Zharkov, 1981; Aslamazov and Lempitskii, 1982; Lempitskii, 1983; Zaikin 1983, 1988; Aslamazov and Volkov, 1986; van Wees et al. 1991; Shumeiko et al. 1993; Gorelik et al. 1995; Volkov 1995a; Volkov and Takayanagi, 1997; Chang and Bagwell, 1997; Samuelsson et al. 1997, 2000, 2001; Wilhelm et al. 1998, 2000; Yip 1998; Morpurgo et al. 1998; Tolga Ilhan et al. 1998; Neurohr et al. 1999; Kutchinsky et al. 1999, 2000; Seviour and Volkov, 2000; Sun et al. 2000; Baselmans et al. 1999, 2001, 2002a,b; Heikkilä et al. 2000, 2002; Zhu et al. 2002; Huang, et al. 2002). These effects occur either in the nonstationary regime, when $I > I_C$ and a voltage is generated across the junction (stimulation by current), under microwave irradiation or in multiterminal structures, when the junction is in a stationary regime, but voltage is applied via additional terminals.

In this review we will restrict ourselves to the stationary Josephson effect in two-terminal weak links when nonequilibrium effects do not play a role. This corresponds to the above case (a), when the CPR is determined by the spectral current $\text{Im}\{I_E(\varphi)\}$, depending on material parameters and junction geometry.

9.3.1 A. General properties of the Josephson CPR

There are several general properties of the CPR which are rather general and depend neither on the junctions materials and geometry nor on the theoretical model used to describe the processes in the junction.

1. A change of phase of the order parameter of 2π in any of the electrodes is not accompanied by a change in their physical state. Consequently this change must not influence the supercurrent across a junction and $I_S(\varphi)$ should be a 2π periodic function

$$I_S(\varphi) = I_S(\varphi + 2\pi) \quad (3)$$

A recent analysis made by Ovchinnikov and Larkin (2002, 2003) provided some more arguments supporting this statement.

2. Changing the direction of a supercurrent flow across the junction must cause a change of the sign of the phase difference, therefore

$$I_S(\varphi) = -I_S(-\varphi) \quad (4)$$

Note that Eq. (4) is violated in superconductors with broken time-reversal symmetry (Geshkenbein et al., 1986, 1987; Yip, 1995; Tanaka and Kashiwaya, 1997b; Fogelstrom et al., 1997).

3. A dc supercurrent can flow only if there is a gradient of the order parameter phase. Hence in the absence of the difference $\varphi = 0$ there should be zero supercurrent

$$I_S(2\pi n) = 0, \quad n = 0, \pm 1, \pm 2 \dots \quad (5)$$

4. It follows from (1) and (2) that supercurrent should also be zero at $\varphi = \pi n$

$$I_S(\pi n) = 0, \quad n = 0, \pm 1, \pm 2 \dots \quad (6)$$

Therefore it is sufficient to consider the $I_S(\varphi)$ only in the interval $0 < \varphi < \pi$.

As follows from (1)-(4), $I_S(\varphi)$ can in general be decomposed into the series of different harmonics (see e.g. Tanaka and Kashiwaya, 1997b)

$$I_S(\varphi) = \sum_{n \geq 1} \{I_n \sin(n\varphi) + J_n \cos(n\varphi)\} \quad (7)$$

and the coefficients J_n vanish if the time-reversal symmetry is not broken.

The free energy E_J of a Josephson junction is generally given by an integral $(\hbar/2e) \int_0^\varphi I_S(\chi) d\chi$. If the CPR is sinusoidal, $I_S(\varphi) = I_C \sin \varphi$ (Fig. 2, the curve *a*), the dependence of E_J on φ has the standard form

$$E_J(\varphi) = \frac{\hbar I_C}{2e} (1 - \cos \varphi) \quad (8)$$

also well-known as the 2π -periodic 'washboard potential'. The standard junction has an energy minimum at $\varphi = 0$ when there is no current flowing across the junction. For junctions with small capacitance charging energy becomes important and should be added to $E_J(\varphi)$ (Likharev, 1986; Tinkham, 1996).

Possible types of CPR in Josephson junctions are shown in Fig. 2. The curve *a* is the standard sinusoidal CPR. The critical current $I_C = \max(I_S(\varphi))$ can

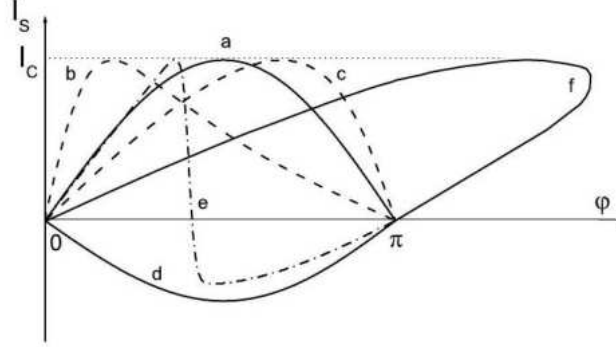


FIG. 2: Various types of CPR.

be achieved both at $\varphi_{\max} \leq \pi/2$ and $\varphi_{\max} \geq \pi/2$ (the curves *b* and *c* respectively).

An interesting special case is the junction with $I_C < 0$ (Fig. 2, the curve *d*), so-called π -junction (Bulaevskii et al. 1977). According to Eq. (8), such a junction has an energy minimum at $\varphi = \pi$, i.e. it provides a phase shift of π in the ground state.

A π -junction may be used as the phase inverter in superconducting digital circuits, the so-called complementary Josephson digital devices (e.g. π -SQUID) were discussed by Terzioglu and Beasley (1998). Ioffe et al. (1999) and Blais and Zagoskin (2000) proposed the π -junctions as candidates for engineering a quantum two-level system - a qubit, which is the basic element of a quantum computer. Blatter et al. (2001) suggested structures including arrays of conventional 0 -and π -junctions for the realization of a "quiet" phase qubit. A topologically stable qubit, based on a triangular or more complicated π -junction arrays was discussed by Blatter et al. (2001) as well.

In some special cases $I_S(\varphi)$ may cross the horizontal axis at a position in between $\varphi = 0$ and $\varphi = \pi$, as shown in Fig. 2, the curve *e*. The energy-phase relation in this case has two minima at $\varphi = 0$ and $\varphi = \pi$.

One might consider the possibility of a multivalued CPR with $I_S(\pi n) \neq 0$, as shown in Fig. 2, the curve *f*. However, structures with multivalued CPR can not be regarded as real Josephson junctions. Such CPR may be realized either due to the Abrikosov vortices (see Kupriyanov et al., (1975); Likharev, 1979) or phase slip centers inside the junction (Martin-Rodero et al., 1994; Sols and Ferrer, 1994; Ivlev and Kopnin, 1984). The CPR in this case is controlled by completely different physical processes than in the standard Josephson junctions.

The $I_S(\varphi)$ dependencies of the type shown in Fig. 2 by the curves b and c may be roughly described by the expression (see Likharev, 1976, 1979; Zubkov et al., 1981; Schüssler and Kümmel, 1993)

$$I_{\mathcal{L}}(\varphi) = I_C \sin(\varphi - \mathcal{L} I_{\mathcal{L}}(\varphi)) \quad (9)$$

by which a weak link is represented as an 'ideal' Josephson junction with $I_S(\varphi) \propto \sin \varphi$ connected in series with a nonlinear inductance $\mathcal{L}$. This inductance can be either positive (Fig. 2, the curve b) or negative (Fig. 2, the curve c) and is a consequence of specific properties of a superconducting condensate. The presentation of CPR in the form Eq. (9) may be useful for the analysis of a mode of operation of systems containing several Josephson junctions with a nonsinusoidal CPR.

9.3.2 B. Basic Josephson structures.

The Josephson effect may be observed in a variety of structures. To realize this structures it is enough to fabricate a 'weak' place interrupting the supercurrent flow in a superconductor or suppress the ability of a superconductor to carry a current, e.g. by deposition on its top a normal metal, by implantation of impurities in its restricted volume or by changing the geometry of a sample. All these possibilities were extensively discussed in reviews and textbooks (see Likharev, 1979; Van Duzer and Turner, 1981; Barone and Paterno 1982; Likharev, 1986; Gallop, 1991; Orlando and Delin, 1991; Tinkham, 1996; Schmidt, 1997; Kadin, 1999). Among them only several configurations have an importance for practical applications. They are point contacts, tunnel junctions, sandwiches and variable thickness bridges (having a normal metal, a semiconductor or a weak ferromagnet as a weak link material), double barrier structures.

We will start our analysis from considering the dc Josephson effect in point contacts in Sec. III. The point contact is the structure with strong supercurrent concentration and provides the simplest system in which the Josephson effect may be observed. The point contacts may be fabricated by placing a superconducting tip on top of a bulk superconductor or by depositing a superconductor on the top superconductor-dielectric bilayer with a submicron hole in the isolator which defines a small contact area. The contacts using a two-dimensional electron gas are also close to this type of structures. Point contacts are well-defined systems where the fundamental physics of the Josephson effect can be most easily studied theoretically and verified experimentally. The reason is that all nonlinear, nonstationary and nonequilibrium processes are localized in the weak link, while the electrodes may be considered as underdisturbed and equilibrium.

In tunnel junctions discussed in Sec. IV the weak place is formed by a dielectric layer separating two superconducting electrodes. Ideal tunnel junction is characterized by a sinusoidal CPR. In real tunnel junctions, based on NbAl/AlO_x/Nb technology (Gurvitch et al., 1983), the base electrode has the form of a NbAl bilayer. The proximity effect may strongly influence the supercurrent in such junctions, while CPR remains sinusoidal (see Golubov and Kupriyanov 1988, 1989; Golubov et al., 1984, 1993, 1995). AlO_x forms a thin dielectric layer with a high potential barrier. In order to increase the current density, this barrier can be made extremely thin, of the order of few atomic layers, in which case electronic transport is dominated by resonant tunneling (Naveh et al., 2000). There exist another type of tunnel junctions consisting from homogeneous electrodes divided by a thick and relatively low potential barrier (like in high T_c junctions with PrBaCuO barrier, see e.g. Gao et al., 1990). The situation in structures with extended barriers may be more complex due to the appearance of additional ABS (see Wendin and Shumeiko, 1996) or localized states inside the barrier (see Gubankov et al., 1985; Amatuni et al., 1987; Xu et al., 1990; Golubov et al., 1994; Vengrus

et al., 1994; Kupriyanov and Tsai, 1995; Verhoeven et al., 1996; Verbist et al., 1997; Satoh et al., 1995; Dömmel et al., 1995; Yoshida et al., 1997; Wen et al., 1995; Gerritsma et al., 1997; Horstmann et al., 1997, 1998; Satoh et al., 1998; Yoshida, 2000; Sun and Gao, 2000). These effects as well as the influence of depairing processes in the electrodes on the shape of CPR will be discussed in Sec. IV.

In Sec. V the SNS sandwiches and variable thickness bridges are discussed. In these junctions the weak link is formed by a normal metal separating two superconducting electrodes. In the last few years, interest in SNS weak links has continuously grown (see Imamura and Hasio, 1989; Benz, 1995, 1997; Benz and Burroughs, 1997; Sachse et al., 1997; Fritzsche et al., 1998, 1999; Whiteley et al., 1998; Pöpel et al., 2000; Niemeyer, 2000; Lacquaniti et al., 1999, 2000, 2001; Schubert et al., 2001; Ohta et al., 2001; Hadfield et al., 2001; Kaul and Van Duzer, 2001; Hagedorn et al., 2001, 2002; Van Duzer et al., 2002) since they have been considered to provide the necessary critical current density for the realization of a programmable voltage standard (see Benz and Burroughs, 1997; Hamilton et al., 1997; Jeanneret et al., 2001, Hagedorn et al., 2001, 2002) and RSFQ circuits in submicron technology (see Van-Duzer et al., 2002). In these junctions the shape of $I_S(\varphi)$ not only depends on temperature and the distance between electrodes, but also on the critical temperature of the interlayer, transport parameters of both metals and the interfaces. As we will see in Sec. V below, one may expect in these junctions the realizations of CPR presented in Fig. 2.

Sec. VI deals with SINIS double barrier junctions. Here the weak link is formed by a N metal separated by tunnel barriers from each of the S electrodes (N could also be a weak superconductor, S' , e.g. Al). The barriers may be naturally formed at the NS interfaces during the fabrication or made artificially. Double-barrier junctions are unique devices, which combine the main advantages of both weak links and tunnel junctions (see Kupriyanov and Lukichev, 1988). It is important to note that these structures can be fabricated, making use of a modification of the standard SIS junctions fabrication process ((Gurvitch et al., 1983). Their parameters are intrinsically homogeneous due to additional averaging of the irregularities of the interface transparency by the intermediate Al layer, which is characterized by a large coherence length. They are reproducible on the on-chip level. This fact was successfully demonstrated by several tech-

nological groups (see Amin et al., 1993; Nevirkovets 1995, 1997; Capogna and Blamire, 1996; Capogna et al., 1997; Maezawa and Shiji 1997; Sugiyama et al., 1997, 1999; Balashov et al., 1998, 1999, 2000, 2001; Nevirkovets et al., 1996, 1999, 2000, 2001a,b; Schulze et al., 1998, 1999, 2000; Brinkman et al., 1999, 2001; Kupriyanov et al., 1999; Behr et al., 1999; Khabipov et al., 1999, 2002a,b; Cassel et al., 2001; Shaternicet et al., 2001; Buchholz et al., 1999, 2001; Kohlmann et al., 2001, 2002; Kieler et al., 2002; Nevirkovets and Ketterson, 2002; Yamamori et al., 2002; Tolpygo et al., 2003). SINIS junctions are considered now as perspective elements for superconducting large scale integrated circuits. We will show below in Sec. VI that the shape of $I_S(\varphi)$ in these junctions strongly depends on the critical temperature of the S' material.

In SFS Josephson junctions, discussed in Sec. VII, the weak link is formed by a ferromagnetic metal placed between the S electrodes. Up to now, only a planar geometry was realized in experiments by Ryazanov et al. (2000, 2001, 2002), Kontos et al. (2002), Blum et al. (2002) and Surgers et al. (2002). In these experiments the transition from the '0' state (the curve *a* Fig. 2) to the ' π ' state (the curve *d* Fig. 2) was observed experimentally. Theory predicts that if the weak link in a Josephson junction is formed by a magnetic material, either metal or dielectric, the shape of CPR may have the variety of forms shown in Fig. 2.

In Sec. VIII we discuss the experimental method for measuring the CPR and its applications. The range of validity of the method, measuring technique and calibration are discussed. The applications include measurements of small critical current, CPR in SINIS, S/2DEG/S and HTS junctions. Of particular interest are the CPR anomalies observed in HTS junctions. HTS

superconductors are characterized by a unconventional d-wave symmetry of the order parameter as was proved by phase sensitive tests, see review papers by van Harlingen (1995), and by Tsuei and Kirtley (2000). Since theory of the Josephson effect between d-wave superconductors was extensively discussed in review papers by Kashiwaya and Tanaka (2000) and Löfwander et al. (2001), we will introduce the relevant theoretical concepts only in relation with the measurements in Sec.VIII.

9.4 III. JOSEPHSON EFFECT IN POINT CONTACTS

Point contacts are structures with strong supercurrent concentration. If the constriction size is smaller than the electronic mean free path, it is usually called a Sharvin point contact (Sharvin, 1965). Further reducing the transverse size and making it comparable with the Fermi wave length, one enters the quantum regime when conductance is quantized. This regime was realized in normally conducting structures with two-dimensional electron gas by van Wees et al. 1988 and in superconducting structures by Takayanagi and Kawakami (1985),

Takayanagi et al. (1995a, 1995b) (see recent review by Schäpers, 2001). Another regime is the diffusive point contact, when the its size is larger than the electronic mean free path (the so-called Maxwell contact).

Superconducting constrictions serve as a model systems to understand the fundamental physics of the Josephson effect. In this Section we discuss the classical models of Aslamazov-Larkin and Kulik-Omelyanchuk, their extensions and physics of the ABS.

9.4.1 A. Aslamazov-Larkin model

The model suggested by Aslamazov and Larkin (1969) illustrates the basic principle of Josephson coupling: the coupling occurs due to an overlap of superconducting order parameters from both electrodes in the weak link region. The model is applicable near the T_C of the electrodes if the characteristic size of a weak link is sufficiently small, $L_{eff} \ll \xi(T)$. Here

$$\xi(T) = (\pi/2)\xi (1 - T/T_C)^{-1/2} \quad (10)$$

is the Ginzburg-Landau coherence length introduced in Eq. (10). The details of the derivation for the supercurrent can be found in the review by Likharev (1979). The result is (Aslamazov and Larkin, 1969):

$$I_S(\varphi) = \frac{\pi\Delta_1\Delta_2}{4eR_N T} \sin \varphi \quad (11)$$

where $\Delta_{1,2}$ are the magnitudes of the pair potentials in the superconducting electrodes and R_N is the normal state resistance of the weak link. To simplify notations, we put the Boltzman constant $k_B = 1$ in Eq. (11) and all other expressions in this article if not stated otherwise.

The expression (11) is very general, it does not depend on the electronic mean free path in the weak link and is applicable to all types of weak links near T_C . According to Eq. (11), near T_C the CPR is always sinusoidal, independently on the material of the weak link, which is characterized only by its resistance R_N .

The Aslamazov-Larkin model can be generalized to arbitrary temperatures both in the clean and dirty limit. At low temperature the CPR is not universal any more and depends on the electronic mean free path in the weak link.

9.4.2 B. Kulik-Omelyanchuk model, dirty limit (KO-1)

Kulik and Omelyanchuk (1975a) considered a point contact as a diffusive quasi-one-dimensional wire with length $L \ll \xi_0 = \hbar v_F / 2\pi T_C$ and transverse size $a \ll d$,

connecting two superconductors. In this case supercurrent can be calculated from the Usadel equations (178)(181) in which non-gradient terms are neglected. For a

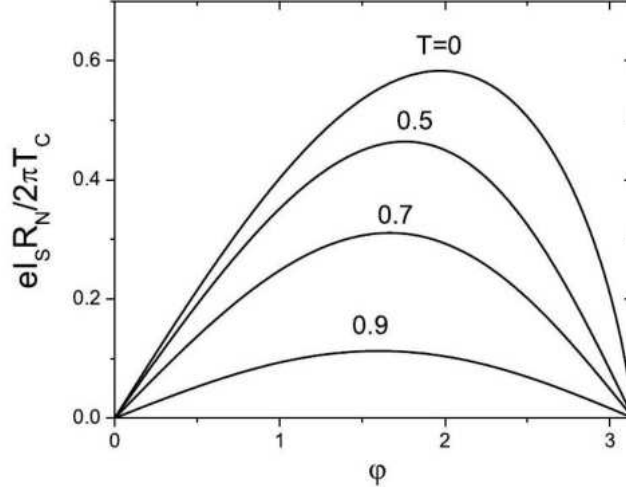


FIG. 3: Current-phase relations for a symmetric diffusive point contact (KO-1 model) at various temperatures.

symmetric junction formed by identical superconductors the result is

$$I_S(\varphi) = \frac{4\pi T}{eR_N} \sum_{\omega>0} \frac{\Delta \cos(\varphi/2)}{\delta} \arctan \frac{\Delta \sin(\varphi/2)}{\delta} \quad (12)$$

where $\delta^2 = \Delta^2 \cos^2(\varphi/2) + \omega^2$ and Δ is the magnitude of the pair potential in the electrodes. The CPR at arbitrary temperature can be calculated numerically from Eq. (12) and are shown in Fig. 3. The curves $I_S(\varphi)$ are non-sinusoidal at low temperatures and reduce to the Aslamazov-Larkin result for $T \rightarrow T_C$.

Eq. (12) was generalized to the asymmetric case ($\Delta_1 \neq \Delta_2$) by Zubkov et al. (1981). With the increase of the ratio Δ_2/Δ_1 the CPR becomes closer to sinusoidal. For $\Delta_2/\Delta_1 \gg 1$ the result is

$$I_S(\varphi) = \frac{\Delta_1}{eR_N} \left[\ln \frac{2\Delta_2}{\Delta_1(1 + \cos \varphi)} \right] \sin \varphi \quad (13)$$

i.e. there is slow logarithmic crossover to the $\sin \varphi$ dependence.

9.4.3 C. Kulik-Omelyanchuk model, clean limit (KO-2)

A clean point contact provides a model system to study fundamental aspects of Josephson CPR. As we will see below, supercurrent per single channel is a building block for a more complex structures. A clean point contact is a constriction with the size in both directions (transverse and along the current) smaller than the electronic mean free path. It may contain a single scattering center, e.g. a tunnel barrier of finite transparency, characterized by the transmission coefficient D , where $0 \leq D \leq 1$. Highly idealized model for a clean constriction is the aperture in a thin impenetrable screen.

Kulik and Omelyanchuk (1977) advanced a theory to describe the Josephson effect in fully ballistic case, $D = 1$. Supercurrent was calculated in the framework of the Eilenberger equations Eq. (173)-(175). For $D = 1$, the boundary conditions (176),(177) provide continuous matching of the Greens' functions along the trajectories crossing the constriction. The Eilenberger equations can be then straightforwardly solved (see also discussion by Likharev, 1979) and the supercurrent is given by simple expression

$$I_S(\varphi) = \frac{\pi\Delta}{eR_N} \sin(\varphi/2) \tanh \frac{\Delta \cos(\varphi/2)}{2T} \quad (14)$$

The only characteristic of the weak link material in Eq. (14) is the Sharvin resistance $R_N^{-1} = e^2 k_F^2 S / (4\pi^2 \hbar)$, where k_F is the Fermi wave-vector, S is the constriction area.

The conductance of a ballistic point contact is quantized in units $2e^2/h$ (see review by Beenakker and van Houten, 1991a). Generalization of Eq. (14) to the quantum regime was provided by Beenakker and van Houten (1991b) and by Furusaki and Tsukada (1991a,b). It was shown theoretically that the critical current of a smooth and impurity-free superconducting constriction increases stepwise as a function of its width, with the step height at zero temperature equal to $e\Delta/\hbar$. This result is a natural extension of Eq. (14) with the quantum resistance $R_N^{-1} = e^2/\pi\hbar$. If N channels are open, the general expression for the supercurrent is

$$I_S(\varphi) = N \frac{e\Delta}{\hbar} \sin(\varphi/2) \tanh \frac{\Delta \cos(\varphi/2)}{2T} \quad (15)$$

Supercurrent quantization with variation of the junction width was observed Takayanagi et al. (1995b) in ballistic S/2DEG/S structures.

9.4.4 D. Point contact, the general case

Haberkorn et al. (1978) generalized the KulikOmelyanchuk result to the case of arbitrary transparency of a tunnel barrier inserted in the constriction by solving directly the Gor'kov equations (168). The result is

$$I_S(\varphi) = \frac{\pi\Delta}{2eR_N} \frac{\sin \varphi}{\sqrt{1-\bar{D} \sin^2 \frac{\varphi}{2}}} \tanh \left[\frac{\Delta}{2T} \sqrt{1-\bar{D} \sin^2 \frac{\varphi}{2}} \right],$$

where $\bar{D}$ is the angle-averaged transmission probability and the contact resistance R_N is

$$R_N^{-1} = \frac{e^2 k_F^2 S}{4\pi^2 \hbar} \bar{D} \quad (17)$$

Equation (16) interpolates nicely between the clean ballistic constriction and tunnel junction. Its expression reduces to the Aslamazov-Larkin result near T_C and to

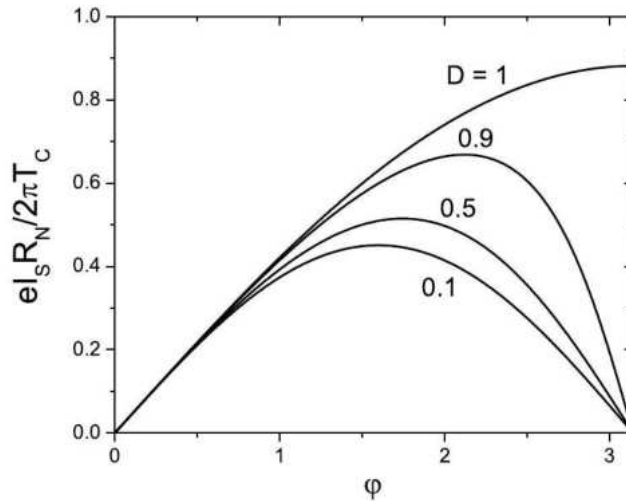


FIG. 4: Current-phase relations for a clean point contact at zero temperature and various $\bar{D}$.

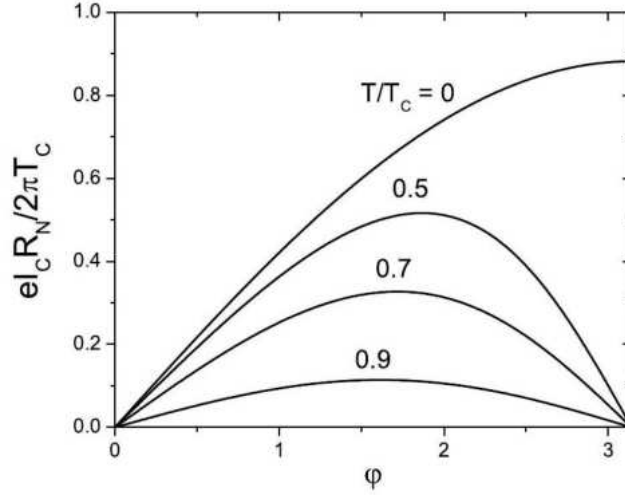


FIG. 5: Current-phase relations for a clean point contact for $D = 1$ and various temperatures.

the Ambegaokar-Baratoff result for a tunnel junction for $\bar{D} \ll 1$. It was later derived by several authors using different approaches (see Zaitsev, 1984; Arnold, 1985; Furusaki and Tsukada, 1990; Beenakker and van Houten, 1991a; Bagwell, 1992).

Figures 4 and Fig. 5 show the modifications of the CPR with variation of temperature and average transparency according to Eq. (16). Strongest deviations from the sinusoidal CPR occur for the ballistic contact with $\bar{D} = 1$ at low T , when $I_S(\varphi) \propto \sin(\varphi/2)$. It crosses over to the harmonic dependence with increasing temperature or decreasing $\bar{D}$. According to the above discussion, CPR becomes sinusoidal for $\bar{D} \ll 1$ in the whole temperature range or near T_C for arbitrary $\bar{D}$.

Multichannel generalization of Eqs.(16) and (15) was performed by Beenakker (1991), (see also Böttcher and Kopp (1997)):

$$I_S(\varphi) = \frac{e\Delta^2}{2\hbar} \sin \varphi \sum_{n=1}^N \frac{D_n}{E_n} \tanh \left[\frac{E_n}{2T} \right] \quad (18)$$

where D_n are the eigenvalues of an arbitrary transmission matrix, which describes a disordered junction, n is the number of channel and $E_n = \Delta \sqrt{1 - D_n \sin^2(\varphi/2)}$ is the ABS energy. Eqs.(18) is valid for an arbitrary disorder. If the number of channels N is large, one can go from summation over n to integration over transmissions D according to $\sum_{n=1}^N \dots = \int_0^1 \rho(D) dD \dots$, where $\rho(D)$ is the transmission eigenvalue density. In the diffusive point contact, when $l \ll \xi_0$, the eigenvalue density function is universal (Dorokhov, 1984; Nazarov, 1994a)

$$\rho(D) = (\pi\hbar/2e^2 R_N) D^{-1/2} (1 - D)^{-1/2} \quad (19)$$

Integrating Eq. (16) with this function provides exactly the Kulik-Omelyanchuk expression for the supercurrent in the dirty limit, Eq. (12). Below we will discuss the applicability of this result to another types of short weak links. For the derivation based on Eilenberger equations and further applications to nonstationary problems see Zaitsev and Averin (1998); Bardas and Averin (1997).

The most general formula for a *dc* supercurrent through a classical point contact (large number of channels N) was derived by Zaitsev (1984). Theory of Zaitsev takes into account asymmetry of the electrodes as well as possible non-BCS structure of the densities of states in superconducting electrodes (e.g. due to the proximity effect or a magnetic field), see also Aminov et al. (1996) and Zaitsev and Averin (1998). As a model for the constriction, an aperture of a small radius was considered in a thin impenetrable screen dividing two different superconductors. The angle-dependent transmission coefficient $D(\alpha)$ was introduced within

the aperture, where α is the angle between the electronic trajectory and normal to the junction plane. Solving the Eilenberger equations (173)-(175) with the boundary conditions (176), (177), Zaitsev (1984) derived the following general expression for the supercurrent

$$I_S(\varphi) = \frac{8\pi T}{eR_N} \sum_{\omega>0} \left\langle \frac{f_1 f_2 \sin \varphi}{2 + D(g_1 g_2 + f_1 f_2 \cos \varphi - 1)} \right\rangle \quad (20)$$

where $\omega = \pi T(2n + 1)$ is the Matsubara frequency and the angular brackets denote angle-averaging $\langle \dots \rangle = \int_0^1 (\dots) x dx$. For a symmetric junction with bulk values for the Green's functions in both electrodes, $g_{1,2} = \omega/\sqrt{\omega^2 + \Delta^2}$ and $f_{1,2} = \Delta/\sqrt{\omega^2 + \Delta^2}$, the summation in Eq. (20) can be performed analytically and it reduces to the result of Haberkorn et al. (1978).

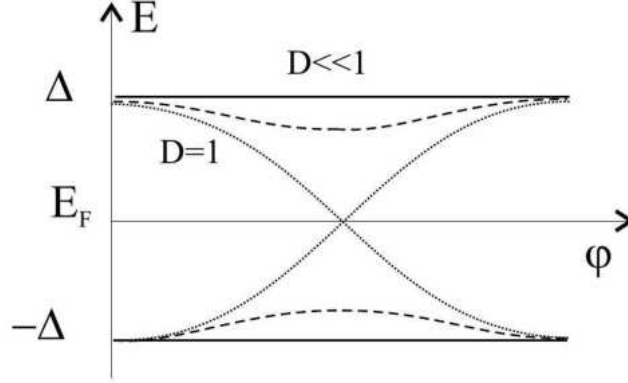


FIG. 6: Energy-phase relation for ABS in a short weak link.

Eq. (20) can be rewritten in the real energy representation of Eq. (2) with equilibrium distribution function $f_L = \tanh(E/2T)$ and spectral supercurrent $\text{Im } I_S(E)$ given by the analytical continuation of the expression in the angular brackets in Eq. (20) from Matsubara frequencies $i\omega = E + i0$. For fixed D , the resulting expression has poles at energies

$$E(\varphi) = \pm \Delta \sqrt{1 - D \sin^2(\varphi/2)} \quad (21)$$

corresponding to the ABS (see also Beenakker (1991), Bagwell (1992) where this result was derived from the Bogoljubov - de Gennes equations).

Two bound states at $E_{\pm} = \pm \Delta \sqrt{1 - D \sin^2(\varphi/2)}$ describe two possible processes (one of them is shown in Fig. 6), which differ by the directions of propagation of electron and a hole and correspond to currents flowing in opposite directions.

Each bound state carries a current

$$I(\varphi) = 2ef_0[E(\varphi)] \frac{d}{d\varphi} E(\varphi) = ef_0[E(\varphi)] D \frac{\Delta^2}{2E(\varphi)} \sin(\varphi) \quad (22)$$

where $f_0[E(\varphi)]$ is the Fermi occupation factor of the bound state. Therefore factor $1 - 2f = \tanh(E/2T)$ in Eq. (2) has simple physical interpretation (Bagwell, 1992): it arises due to difference of populations of two ABS, $\tanh(E/2T) = f_0(E_+) - f_0(E_-)$, where $E = E_+ = -E_-$. This can be viewed as thermal average of the contributions of two bound states (i.e. left and right-moving electrons) to the supercurrent.

Figure 6 shows the ABS as a function of the phase difference φ across the single-channel junction characterized by a single transmission coefficient D . In the ballistic regime $D = 1$ two levels, corresponding to a left-moving (E_+) and right-moving (E_-) electrons cross at $\phi = \pi$, while the gap $2\Delta\sqrt{1-D}$ opens for $D < 1$.

As was mentioned above, Eq. (20) is quite general and describes not only an ideal ScS junction, but also the junction with spatially-inhomogeneous electrodes.

Aminov et al. (1996) considered practically interesting case of $SNcNS$ point contact, when two NS bilayers are coupled by a clean constriction with an arbitrary transparency D . If further the NS bilayers are in the dirty limit, the Green's functions $f_{1,2}$ and $g_{1,2}$ in Eq. (20) suitable for this problem should be found from the Usadel equations (178)-(181). For the symmetric $SNcNS$ contact with thin N -layers, $d_N \ll \xi_2$, supercurrent is given by rather simple expression (Aminov et al., 1996)

$$I_S(\varphi) = \frac{2\pi T}{eR_N} \sum_{\omega>0} \left\langle \frac{\Phi^2(\omega) \sin \varphi}{\omega^2 + [1 - D \sin^2(\varphi/2)] \Phi^2(\omega)} \right\rangle \quad (23)$$

Here the function $\Phi(\omega) = \Delta / [1 + (\Delta^2 + \omega^2)^{1/2} \gamma_{BM} / \pi T_C]$, where $\gamma_{BM} = \gamma_B d / \xi_2$ with γ_B given by Eq. (186). Such a constriction model for $SNcNS$ contacts was applied to interpret experimental data for some HTS-based junctions and junctions and $S/2DEG/S$ Josephson junctions by Aminov et al. (1996), Golubov et al. (1997, 1998b); Chrestin et al. (1997); Schäpers (2001); Grajcar et al. (2002a).

At nonzero bias voltage the mechanism of conduction in point contacts is the process of multiple Andreev reflections (MAR) (Klapwijk et al., 1982; Zaitsev, 1980). Discussion of MAR theory in short weak links is outside the scope of the present review. This theory was advanced by Arnold (1985), Günsenheimer and Zaikin (1994), Averin and Bardas (1995), Bratus' et al. (1995), Martin-Rodero et al. (1996). MAR was observed in various types of the nanoscale contacts by Kleinsasser et al. (1994), van der Post et al. (1994), Vleeming et al. (1994), Takayanagi et al. (1995a, 1995b), Mur et al. (1996), Scheer et al. (1997), Naveh et al. (2000).

9.5 IV. JOSEPHSON JUNCTIONS WITH TUNNEL TYPE OF CONDUCTIVITY

The Josephson effect in a tunnel junction deals with coherent transmission of Cooper pairs through a tunnel barrier which separates superconducting electrodes. The simplest case of an 'ideal tunnel junction' with the barrier modeled by a delta functional potential was originally considered by Josephson (1962). Further developments of theory include more complex situations, like extended barriers, resonant tunneling through localized states and depairing by current.

9.5.1 A. Ideal tunnel junctions

Ambegaokar and Baratoff (1963) used the Green's function method to generalize the calculation of Josephson (1962) to finite temperatures. Ambegaokar and Baratoff (1963) also explicitly pointed out that the Josephson effect occurs because the Cooper pair amplitudes (the Gor'kov F -functions, see Eq. 168), which are nonlocally dependent on superconducting pair potential Δ , may overlap in the oxide layer even though $\Delta = 0$ there.

The CPR in SIS tunnel junction is sinusoidal and is given by the following expression (Ambegaokar and Baratoff, 1963)

$$\frac{eI_S(\varphi)R_N}{2\pi T_C} = \frac{T}{T_C} \sum_{\omega>0} \frac{|\Delta_1| |\Delta_2|}{\sqrt{(\omega^2 + |\Delta_1|^2)(\omega^2 + |\Delta_2|^2)}} \sin \varphi. \quad (24)$$

where $\Delta_{1,2}$ are pair potentials in the superconducting electrodes. This result can be also obtained directly from Eq. (20) in the limit $D \ll 1$.

The Ambegaokar-Baratoff formula (24) (further referred as AB) is a particular case of the general theory formulated by Werthamer (1966) and Larkin and Ovchinnikov (1966) for the SIS tunnel junction at arbitrary constant bias voltage V

$$I_S = \frac{\sin \varphi}{2eR_N} \int_{-\infty}^{\infty} \{ \text{Re } F_1(E) \text{Im } F_2(E + eV) \quad (25)$$

$$+ \text{Im } F_1(E) \text{Re } F_2(E + eV) \} \tanh(E) dE \quad (26)$$

where time-dependent phase difference $\varphi = 2eVt + \varphi_0$ and $F_{1,2}$ are the quasiclassical Green's functions in superconductors, $F_{1,2} = |\Delta_1| / \sqrt{|\Delta_{1,2}|^2 - E^2}$ (compare with Eq. (182) for $\omega = -iE$). In the limit $V \rightarrow 0$ Eq. (25) is reduced to Eq. (24).

The physical mechanism of a supercurrent transfer across the tunnel barrier can be understood in terms of ABS. Comparing Eq. (25) with Eq. (2), one can conclude that in an asymmetric tunnel junction with $|\Delta_1| \leq |\Delta_2|$ all states with energies in the interval $|\Delta_1| \leq |E| \leq |\Delta_2|$ contribute to the spectral current. That means that the ABS are distributed in the whole energy range between $|\Delta_1|$ and $|\Delta_2|$. In a symmetric tunnel junction ($|\Delta_1| = |\Delta_2| = \Delta$) the bound state energies are $E(\varphi) = \pm \Delta$ (see Fig. 7) and in accordance with Eq. (24) the supercurrent

$$I_S(\varphi) = \frac{e\Delta}{2} D \tanh \frac{\Delta}{2T} \sin(\varphi) \quad (27)$$

is proportional to the normal electron transparency of the junction.

In real tunnel junctions the barrier thickness $d > \alpha$ and the supercurrent transport is more complex due to existence of a resonant coupling between superconducting surface states situated at the two SI interfaces (Furusaki and Tsukada, 1991, Wendin and Shumeiko, 1996 of the SIS junction and appearance of localized states inside the barrier (Knauer, 1977; Itskovich and Shekhter, 1981; Aslamazov and Fistul', 1982, Larkin and Matveev, 1987; Glazman and Matveev, 1988, 1989; Tartakovsky and Fistul', 1988; Beenakker and van Houten, 1991, Devyatov and Kupriyanov, 1993, 1994a,b, 1997a,b,c, 1998; Devyatov et al., 1995, 1997 2001a,b; Aleiner, 1996; Naveh et al. 2000; Halbritter, 2002).

9.5.2 B. Tunnel junctions with extended barriers

In the structures with thick tunnel barrier with small transparency $D \ll 1$ the structure of the bound state spectrum is determined by the coupling of superconducting surface states situated at the two SI interfaces of the SIS junction (Wendin and Shumeiko, 1996):

$$E_{\pm}^2(\varphi) = \Delta^2 [1 - \Omega^2], \quad \Omega = \left(\frac{|\beta|}{2} \pm \sqrt{D} \left| \sin \frac{\varphi}{2} \right| > 0 \right) \quad (28)$$

$$\varphi \gg d\delta k, \quad \delta k = (\sqrt{2m(W - (\mu + E))} - \sqrt{2m(W - (\mu - E))})$$

Here m is effective mass, W is the height of the barrier, μ is the Fermi energy, E is the quasiparticle energy.

Each bound state carries large current

$$I_S(E_{\pm}(\varphi)) = -\text{sign}(E(\varphi)) n_F(E(\varphi)) \frac{e\Delta D}{2} \sin(\varphi) \times \quad (29)$$

$$\left(1 \pm \frac{|\beta|}{2\sqrt{D}} \frac{1}{\sqrt{\sin^2(\varphi/2) + (\delta kd/2)^2}} \right)$$

The magnitude of $I(E_{\pm}(\varphi))$ is proportional to the amplitude rather than the probability of normal electron tunneling. Nevertheless the currents are distributed among the bound states in such a way that they almost cancel each other in equilibrium, giving rise to a comparatively small residual current, including the contribution from the continuum. This current coincides with the AB expression (27).

The large current of the single bound state as well as the nonsinusoidal CPR can be revealed under nonequilibrium conditions when the bound level population is imbalanced by means of microwave pumping or tunnel injection.

9.5.3 C. Tunneling via localized states (LS)

The problem of supercurrent transfer across Josephson tunnel junction containing resonant impurity levels in the tunnel barrier was first considered by Aslamazov and Fistul' (1982). They have shown that resonant propagation of coherent electrons along trajectories made up of periodically arranged impurity atoms results in slower decrease of the critical current with the barrier thickness compared to that for the direct tunneling. The CPR remains sinusoidal in this case. The last fact is understandable since in this model the LS form long metallic channels so that the junction is similar to a long SNS junction which has sinusoidal CPR.

The deviation from the $\sin \varphi$ law may occur if the charge transport goes via single localized state (Larkin and Matveev, 1987; Glazman and Matveev, 1989; Beenakker and van Houten, 1991, Devyatov and

Kupriyanov, 1997b, 1998; Devyatov et al., 1997; Naveh et al. 2000).

Devyatov and Kupriyanov, 1997b have shown that properties of the structure depends on the relation of the barrier thickness d and the decay lengths ξ_i and ξ_b

$$\xi_i = \frac{\sqrt{W - \mu}}{2mE}, \quad \xi_b = \frac{m}{\sqrt{W - \mu}} \ln \frac{W - \mu}{E} \quad (30)$$

If the thickness of the tunnel barrier $d \ll \xi_i, \xi_b$, then the ABS spectrum is determined by the equation (Devyatov and Kupriyanov, 1997b)

$$\begin{aligned} \sqrt{\Delta^2 - E^2} \left(E^2 - E_R^2 - \frac{\Gamma^2}{4} \right) + \frac{\Delta^2 \Gamma_1 \Gamma_2}{\sqrt{\Delta^2 - E^2}} \sin^2 \left(\frac{\varphi}{2} \right) + \Gamma E^2 &= 0 \\ \Gamma &= \Gamma_1 + \Gamma_2, \quad \Gamma_1 = \Gamma_0 \exp \left(-\frac{x_0}{\alpha} \right), \quad \Gamma_2 = \Gamma_0 \exp \left(\frac{x_0}{\alpha} \right) \end{aligned} \quad (31)$$

where $\Gamma_0 = 2(W - \mu)\sqrt{D}$ is the width of the LS resonant level, E_R is the renormalized energy of the localized state and x_0 is the position of LS inside the barrier, $\alpha = (2m(W - \mu))^{-1/2}$. Equation (31) was obtained previously by Beenakker and van Houten (1991) for the problem of tunnelling via quantum dot connecting two superconductors.

In the limit of a broad resonance $\Gamma_0 \gg \Delta$, i.e. in the range of barrier thickness

$$\alpha \ll d \ll \alpha \ln \left[8 \frac{W - \mu}{\Delta} \sqrt{\frac{W - \mu}{\mu}} \right] \quad (32)$$

the dispersion relation (31) yields an expression for the energy of the ABS similar to Eq. (21) in constrictions

$$E(\varphi) = \pm \Delta \sqrt{1 - D(E_R) \sin^2 \left(\frac{\varphi}{2} \right)} \quad (33)$$

with the coefficient D replaced by $D(E_R)$

$$D(E_R) = \frac{\Gamma_0^2}{(E - E_R)^2 + \Gamma_0^2 \cosh^2(x_0/\alpha)} \quad (34)$$

describing the Breit-Wigner resonance for the LS with a energy E_R .

Expressions (22), (33) immediately provide the Josephson current through the structure (Glazman and Matveev, 1989):

$$I_S(\varphi) = \frac{e\Delta^2}{2} D(E_R) \frac{\sin(\varphi)}{|E(\varphi)|} \tanh \frac{|E(\varphi)|}{2T} \quad (35)$$

which is formally the same as the CPR for the single mode S-c-S constriction (Haberkorn et al., 1978).

In the opposite case of narrow resonance $\Gamma_0 \ll \Delta$ both $E(\varphi)$ and $I_S(\varphi)$ have the same dependence on φ , as in (33) and (35) respectively, but with strongly suppressed prefactor

$$E(\varphi) = \pm \frac{\Delta \Gamma_0 \sqrt{1 - D(E_R) \sin^2(\varphi/2)}}{\sqrt{D(E_R) (E_R^2 + (\Delta + \Gamma_0 \cosh(x_0/\alpha))^2)}} \quad (36)$$

$$I_S(\varphi) = \frac{e\Delta^2 \Gamma_0 \sqrt{D(E_R)} \sin(\varphi) \tanh[|E(\varphi)|/2T]}{2|E(\varphi)| \sqrt{(E_R^2 + (\Delta + \Gamma_0 \cosh(x_0/\alpha))^2)}} \quad (37)$$

Equations (35) and (37) imply that in both limiting cases $I_S(\varphi)$ differs from sinusoidal for LS energies $E_R < \Gamma_0$. In particular, for $E_R = 0$ and $x_0 = 0$ the structure is fully transparent and $I_S(\varphi) \propto \sin(\varphi/2)$ at $T = 0$ in both cases of broad and narrow resonance.

To calculate the supercurrent one has to average the Eq. (35) over the coordinates and energies of LS

$$\langle I_S(\varphi) \rangle = \int I_S(\varphi) \rho(E_R, x_0) dx_0 dE_R \quad (38)$$

where $\rho(E_R, x_0)$ is LS distribution function.

Naveh et al. (2000) have shown that if Γ_0 in Eq. (34) is so large that the first term in the denominator is unimportant, then $D(E_R)$ depends only on position of LS inside the barrier x_0 and to calculate a current it is enough to average the Eq. (35) only on x_0 . Then $D(E_R, x_0) \approx \cosh(x_0/\alpha)^{-2}$ and for a system with a uniform spatial distribution of LS $\rho(E_R, x_0) = \rho(x_0) = \text{const}$. The averaging over positions of LS

$$dx_0 = \left(\frac{dD}{dx_0} \right)^{-1} dD \equiv \rho(D) dD$$

is then equivalent to the averaging of the supercurrent (35) with the Dorokhov's distribution function $\rho(D)$ given by Eq. (19). This provides exactly the result of KO-1 theory Eq. (12).

On the other hand, if LS are distributed uniformly in energy and are located in the middle of the barrier so that $x_0 \approx 0$ and $D(E_R, x_0) \approx (1 + ((E - E_R)^2 / \Gamma_0^2))^{-1}$, then

$$dE_R = \left(\frac{dD}{dE_R} \right)^{-1} dD \equiv \rho(D) dD$$

with the distribution function

$$\rho(D) = (\hbar/e^2 R_N) D^{-3/2} (1 - D)^{-1/2} \quad (39)$$

derived by Schep and Bauer (1997) for disordered interface. The result of this averaging provides the temperature dependence of the supercurrent, which fits well the data by Naveh

et al. (2000), where high J_C SIS junctions were fabricated and studied experimentally. The same distribution is also valid for double-barrier junctions (Beenakker, 1997). As a result, the averaging gives exactly the result of KL theory in the coherent regime Eq. (117) (Kupriyanov and Lukichev, 1988) for SINIS junctions (see Section VI below). Therefore the supercurrents in high J_C SIS junctions and in SINIS junctions in the coherent regime are identical.

If $\rho(E_R, x_0)$ is a function of both arguments, then averaging of supercurrent Eq. (35) over coordinates and energies of LS results in $I_S(\varphi) \propto \sin(\varphi)$ over almost the entire temperature range. An averaging procedure carried

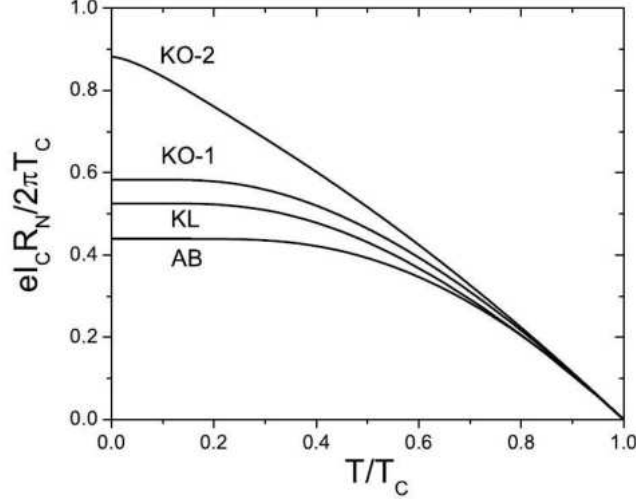


FIG. 7: Temperature dependencies of the critical current for various types of short weak links.

out in three dimensional case using the Green's function formalism yields the same result. (Glazman and Matveev, 1989; Devyatov and Kupriyanov, 1994a).

Comparison of temperature dependencies of critical current for different types of short weak links and tunnel junctions is shown in Fig. 7. At $T = 0$ $I_C R_N$ product for a ballistic ScS point contact (KO-2) is twice as large as for the tunnel junctions (AB). The $I_C R_N$ product for the diffusive point contact (KO-1) and double barrier junctions in coherent regime have intermediate values between KO1 and AB.

It can be shown that further increase of the barrier thickness ($\xi_b \ll d \ll \xi_i$) leads to the appearance of two additional Andreev levels located in the vicinity of $E = \Delta$ (Devyatov and Kupriyanov, 1997b). These two new solutions are direct analogs of the 'split' Andreev levels examined by Wendin and Shumeiko (1996) for the case of a long SIS junction and discussed in the previous paragraph. The expressions for $E(\varphi)$ and $I_S(\varphi)$ obtained in this limit have the same form as Eqs.(28) and (29) with the factors D and $\sin(\varphi/2)$ replaced by the resonance transparency Eq. (34) and $\cos(\varphi/2)$ respectively. Similarly, the large currents carried by individual levels flow in the opposite directions and effectively cancel each other. The resulting current appears to be exponentially small compared to the contribution from the lower energy levels (37) and does not effect the resonant transport.

A further increase in the junction thickness does not affect the expression for the Andreev level (36) near the chemical potential ($E_R \approx \Gamma_0$). Numerical analysis (Devyatov and Kupriyanov, 1997b) has shown that, as before, two split roots exist near $E = \Delta$, but they become flatter and as before, their presence does not contribute to the full equilibrium resonance Josephson current determined by (36).

The CPR associated with the tunneling via individual

LS has the same form as for a geometrical constriction providing the consideration of LS as a 'constriction in the energy space'. The contribution of an individual LS as a function of Γ_0 saturates at large $\Gamma_0 \gg \Delta$, while at small Γ_0 it scales as Γ_0/Δ .

Experimental observation of deviations of $I_S(\varphi)$ from $\sin \varphi$ law in tunnel structures is diffi-

cult since in real SIS junctions the resonant contributions to the full supercurrent is effectively averaged over positions and energies of LS inside the barrier. Together with the contribution from the direct tunneling channel, this provides $\sin(\varphi)$ law with the critical current larger compared to the AB theory prediction Eq. (24).

Nevertheless nonsinusoidal $I_S(\varphi)$ can be found in high J_C tunnel junctions (Naveh et al., 2000) or as a result of depairing in the electrodes by the supercurrent, which may occur with increase of the barrier transparency in 'in-line' Josephson junctions with very narrow width of its electrodes.

9.5.4 D. SIS structures with high transparency: depairing by current

The supercurrent across the SIS structure with arbitrary transparency D can be found in the frame of Usadel equations (190) with the boundary conditions Eqs.(192), (193) at the barrier and the condition of realization of spatially homogeneous current carrying state far from the SIS interface (Kupriyanov 1992). The problem is simplified by the fact that in symmetric junctions boundary conditions Eqs. (192), (193) do not contain Matsubara frequencies and as in the depairing problem (Kupriyanov and Lukichev, 1980) the phases of all Usadel functions can be chosen equal to the phase of Δ in the electrodes.

With this simplification it can be shown that at $T \lesssim T_C$ the CPR can be found from the first integral of the GL equation

$$\left(\frac{2\alpha \sin^2(\varphi/2)}{\Gamma_B} \right)^2 + (\alpha - 1) \times \times \left[1 - \frac{I^2 \Delta_0^4}{\alpha I_0^2 \Delta^4(\infty)} - \frac{\Delta^2(\infty)(1 + \alpha)}{2\Delta_0^2} \right] = 0 \quad (40)$$

The supercurrent is given by

$$I = I_0 v_S (1 - v_S^2), \quad I_0 = \frac{\pi \Delta_0^2}{4e\rho T_C \xi(T)} \quad (41)$$

where $\Delta(\infty)$ is the magnitude of the pair potential far from the interface

$$\Delta^2(\infty) = \Delta_0^2 (1 - v_S^2), \quad v_S = \alpha \frac{\sin(\varphi)}{\Gamma_B}, \quad \alpha = \frac{\Delta(0)}{\Delta(\infty)} \quad (42)$$

Here $\Delta_0^2 = (8\pi^2/7\zeta(3)) T_C (T_C - T)$, $\Gamma_B = \gamma_B \xi(T)/\xi$ is the suppression parameter and $\Delta(0)$ is the magnitude of the pair potential at the interface.

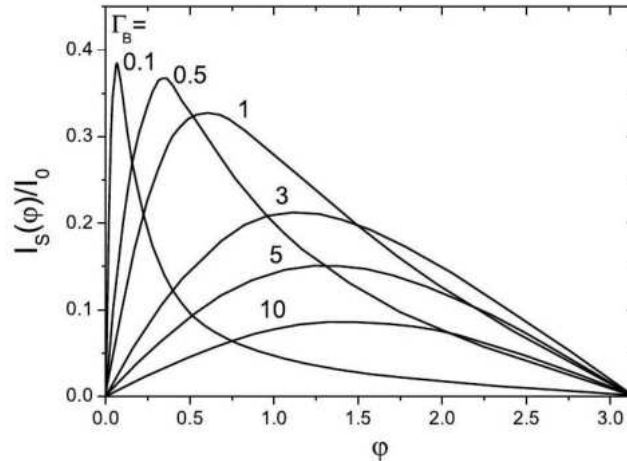


FIG. 8: The CPR for SIS junction for various values of the barrier transparency parameter Γ_B . Adapted from Kupriyanov (1992).

In the limit of small transparency ($\Gamma_B \gg 1$) from (40) in first approximation on Γ_B^{-1} , parameter $\alpha = 1$ and the result of AB theory (27) is reproduced. The correction to this result are caused by suppression of $\Delta(\infty)$ and $\Delta(0)$ by supercurrent. The last of these factors is more essential at $\Gamma_B \gg 1$ and leads to corrections to the critical current

$$I_C = I_0 \Gamma_B^{-1} \left(1 - \sqrt{2} \Gamma_B^{-1} \right) \quad (43)$$

and small deformations of $I_S(\varphi)$ having the maximum at $\varphi_{\max} = \pi/2 - \sqrt{2} \Gamma_B^{-1}$ smaller than $\pi/2$.

In the opposite limit ($\Gamma_B \ll 1$) the $I_S(\varphi)$ is mainly controlled by the depairing in electrodes and $I_S(\varphi)$ is given by

$$I_S(\varphi) = \begin{cases} \beta (1 - \beta^2), & 0 \leq \varphi \leq \varphi_{\max} \\ q (1 - q^2), & \varphi_{\max} \leq \varphi \ll 1 \\ \Gamma_B / 4 \cot(\varphi/2), & \varphi_{\max} \ll \varphi \leq \pi \end{cases} \quad (44)$$

The position of maximum of $I_S(\varphi)$ shifts to $\varphi_{\max} = \Gamma_B / \sqrt{3} \ll 1$ and the critical current equals to the depairing current $I_C = (2/3\sqrt{3})I_0$. Here $\beta = \varphi/\Gamma_B$, and $q^{-1} = \beta + \sqrt{1 + \beta^2}$

The $I_S(\varphi)$ dependencies for arbitrary Γ_B has been obtained numerically by Kupriyanov (1992) and are shown in Fig. 8. In the full agreement with the above analysis decrease of Γ_B leads to transition from $\sin(\varphi)$ law to $I_S(\varphi)$ defined by (44). At $\Gamma_B \approx 10$ (corresponding to the specific barrier resistance $R_B \approx 10\rho\xi$), the $\varphi_{\max}$ shifts more than 10% from $\pi/2$.

For typical superconducting LTS materials this value $R_B \approx 10\rho\xi \sim 10^{-11}\Omega \text{ cm}^2$. Such values have been found experimentally for Nb/Cu (Baxter et al., 1999; Park et al., 2000), Nb/Al (Dmitriev 1999; Zehnder et al., 1999; Brammertz et al., 2001; den Hartog et al., 2001) Pd/Cu and Pt/Cu interfaces (Kurt et al., 2002).

9.6 V. SNS JUNCTIONS.

SNS Josephson junctions are the structures composed from two superconducting electrodes connected by a normal metal or superconductor with the smaller critical temperature T'_C . These structures further classified on "clean" ($\ell \gg \xi_0, d$) and "dirty" ($\ell \ll \xi_0, d$) depending on relation between electron mean free path ℓ , decay length in the interlayer material ξ_0 and the distance d between electrodes.

The shape of $I_S(\varphi)$ dependence in SNS junctions depends on a variety of parameters. They are the ratio between decay length of interlayer material and distance between electrodes, the ratio between critical temperatures T_C and T'_C and the transport properties of interfaces. We will see that there are several practically important limits in which $I_S(\varphi)$ has the $\sin(\varphi)$ form. The deviations resulting in the shift of $\varphi_{\max}$ to the region $\varphi_{\max} \geq \pi/2$ are mainly controlled by the processes in the interlayer material. Contrary to that, $\varphi_{\max} \leq \pi/2$ when the processes in the electrodes are more important.

To demonstrate this we will start with the analysis of the most simple situation, when the electrodes have equal critical temperatures, $T'_C = 0, T \lesssim T_C$, the dirty limit conditions are valid for both in N and S metals and the width of the structure is small compared to Josephson penetration depth λ_J .

9.6.1 A. The limit of high temperature $T \approx T_C$

The analysis of the shape of CPR at temperatures close to T_C has been done by Ivanov et al. (1978, 1981). It has been shown that in the practically interesting limit

$$\gamma \ll \gamma_B + \min \{1, d/\xi_2\} \quad (45)$$

the shape of $I_S(\varphi)$ may be found from the solution of two algebraic equations (see the definitions of γ, γ_B in Eqs. (188),(186)). One of them follows from the first integral of GL equations

$$\Gamma^2 \alpha^2 [p \cos^2(\varphi/2) + q \sin^2(\varphi/2)]^2 + (\alpha - 1) \left[1 - \left(\frac{I_S^2(\varphi) \Delta_0^4}{\alpha I_0^2 \Delta^4(\infty)} \right) - \frac{\Delta^2(\infty)(1 + \alpha)}{2\Delta_0^2} \right] = 0 \quad (46)$$

Another equation follows from the expression for the supercurrent

$$I_S(\varphi) = \frac{\alpha}{2} I_0 \Gamma (q - p) \frac{\Delta^2(\infty)}{\Delta_0^2} \sin(\varphi) \quad (47)$$

Here $\alpha = \Delta(d/2)/\Delta(\infty)$ and $\Delta(d/2), \Delta(\infty)$ are the magnitudes of the pair potentials at the NS interface and in the bulk, $\Gamma = \gamma \xi(T)/\xi_2$ is the suppression parameter, and the functions $p(\gamma_B, d)$ and $q(\gamma_B, d)$ are defined by equations

$$p \sum_{\omega} \omega^{-2} = \sum_{\omega} \frac{\omega^{-2} \beta}{\beta \gamma_B + \coth(\beta d/2\xi_2)} \quad (48)$$

$$q \sum_{\omega} \omega^{-2} = \sum_{\omega} \frac{\omega^{-2} \beta}{\beta \gamma_B + \tanh(\beta d/2\xi_2)}. \quad (49)$$

9.6.2 1. Weak depairing in the electrodes (rigid boundary conditions)

As follows from Eq. (46), the critical current is smaller than I_0 and in the first approximation $\Delta(d/2) = \Delta(\infty) = \Delta_0$ under the conditions $\Gamma q \ll 1$ and $\Gamma p \ll 1$, i.e. when

$$\Gamma \ll \gamma_B + \min \{1, d/\xi_2\} \quad (50)$$

Hence, in the considered limit the CPR is sinusoidal (see Kupriyanov and Lukichev, 1988; Heslinga and Klapwijk, 1993) with I_C given by

$$I_C = I_0 \frac{8\Gamma}{\pi^2} \sum_{\omega} \frac{\beta^{-3} \sinh^{-1}(\beta d/\xi_2)}{[1 + \beta^2 \gamma_B^2 + 2\beta \gamma_B \coth(\beta d/\xi_2)]} \quad (51)$$

In the limit of small γ_B Eq. (51) coincides with the result of Likharev (1976) for variable thickness SNS bridges.

The normal junction resistance is the sum of the resistance of the weak link material, $\tilde{R}_N = \rho_2 d/S$, and the boundary resistances

$$R_N = \frac{2R_B + \rho_2 d}{S} = \tilde{R}_N (1 + 2\Gamma_B), \tilde{\Gamma}_B = \gamma_B \frac{\xi_2}{d} \quad (52)$$

where S is the cross-section area of the interlayer, which is supposed to be equal to the area of the interfaces.

The $I_C R_N$ product decreases with the increase of γ_B . In particular, for $d \ll \xi_2$ it follows from (51), (52)

$$I_C R_N = V_0 \left[1 - \frac{4}{\alpha \pi^2} \left\{ \psi \left(\frac{1 + \vartheta}{2} \right) - \psi \left(\frac{1}{2} \right) \right\} \right] \quad (53)$$

$$V_0 = \frac{\pi \Delta_0^2}{4eT_C}, \quad \vartheta = \frac{1 + \tilde{\Gamma}_B}{\gamma_B^2}$$

where $\psi(x)$ is the digamma function. It follows from Eq. (53) that the decrease of the transparency leads to the decrease of the slope of $I_C R_N(T)$ dependence at $T \approx T_C$

$$R_N \frac{dI_C}{dT} = \begin{cases} 635 \left(1 - \frac{4}{\vartheta \pi^2} \ln \frac{\vartheta}{2} \right) \left[\frac{\mu V}{K} \right], & \gamma_B^2 \ll 1 + \frac{\gamma_B \xi_2}{d} \\ 1082 \vartheta, & \gamma_B \gg \frac{\xi_2}{d} \end{cases} \quad (54)$$

Eqs.(53), (54) allow to estimate the interface resistance from the $I_C(T)$ data..

In the limit of large thickness $d \gg \xi_2$ both the critical current and $I_C R_N$ product are exponentially small

$$I_C R_N = V_0 \frac{16}{\pi^2} \frac{1 + 2\tilde{\Gamma}_B}{(1 + \gamma_B)^2} \frac{d}{\xi_2} \exp \left\{ -\frac{d}{\xi_2} \right\}, \quad d \gg \xi_2 \quad (55)$$

As follows from Eqs.(54), (55) and (52), the $I_C R_N$ product is independent on weak link transport parameters only if γ_B is relatively small

$$\gamma_B \ll \max \{1, \xi_2/d\} \quad (56)$$

In the opposite case $I_C R_N \propto R_N^{-1}$.

In the parameter range opposite to (56), $I_C \propto \gamma_B^{-2}$ and $R_N \propto \gamma_B$ for large γ_B , thus leading to $I_C R_N \propto R_N^{-1}$.

9.6.3 2. Depairing in the S electrodes by the proximity effect

As follows from Eqs. (48), (49), Γp and $\Gamma q \gg 1$ under the condition

$$\Gamma \gg \gamma_B + \min \{1, d/\xi_2\}$$

In this regime one can neglect the magnitude of $\Delta(d/2) \propto \Delta_0 / \max(\Gamma p, \Gamma q)$ in comparison with $\Delta(\infty)$ in Eq. ((47). Further, $\Delta(\infty) \simeq \Delta_0$ due to the smallness of the current through the superconductor $I \propto I_0 / (\Gamma p, \Gamma q)$. With the simplifications made we obtain directly from Eqs. (46), (47)

$$I_S(\varphi) = I_C \frac{qp \sin(\varphi)}{p \cos^2(\varphi/2) + q \sin^2(\varphi/2)}, \quad \frac{I_C}{I_0} = \frac{q - p}{4\Gamma qp} \quad (57)$$

The critical current $I_C \propto (T - T_C)^2$ and is achieved at $\varphi = \arctan(q/p)$. Its asymptotic behavior at small and large values of d is given by the expressions

$$\frac{I_C}{I_0} = \frac{\pi^2}{8\Gamma(1 + \gamma_B)^2} \left[\sum_{\omega} \frac{1}{\beta^3(1 + \beta\gamma_B)} \right]^{-2}, \quad d \gg \xi_2, \quad (58)$$

$$\frac{I_C}{I_0} = \frac{\pi^2 \xi_2}{16\Gamma d} \frac{1 - (4/\vartheta \pi^2) \Psi((1 + \vartheta)/2) - \Psi(1/2)}{\Psi((1/2 + \Gamma/\gamma_B^2)) - \Psi(1/2)}, \quad d \ll \xi_2. \quad (59)$$

Note that at fixed $\Gamma \gg 1$ the reduction of the weak link thickness deforms the CPR shifting its maximum into the region $\varphi_{\max} < \pi/2$.

9.6.4 3. Depairing in the S electrodes by the supercurrent

As follows from Eqs. (48), (49), if the N-layer thickness is small enough compared to ξ_2 , the parameter range exists

$$\gamma_B + \min\{1, d/\xi_2\} \ll \Gamma \ll \gamma_B + \max\{1, \xi_2/d\} \quad (60)$$

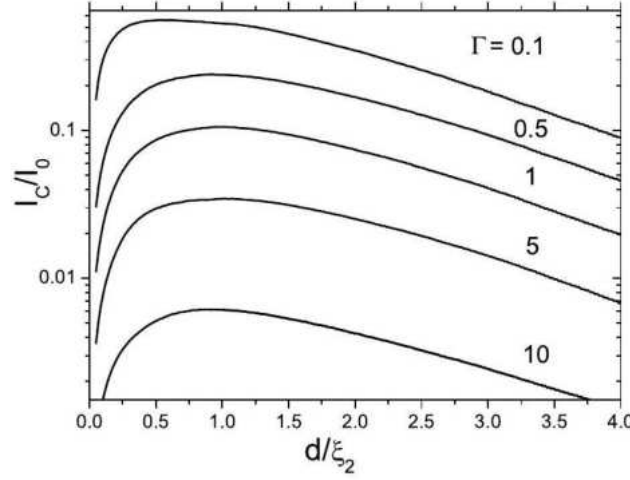


FIG. 9: Thickness dependence of the critical current in SNS junction. Adapted from Ivanov et al. (1981).

when $\Gamma q \gg 1$ and $\Gamma p \ll 1$. Neglecting the term proportional Γp in Eqs. (46), (47), one arrives at

$$\frac{I_S(\varphi)}{I_0} = \begin{cases} I_0(2\Gamma q)^{-1} \cot(\varphi/2), & \varphi \gg \frac{1}{\sqrt{\Gamma q}} \\ \Gamma q \varphi (1 - (\Gamma q \varphi/2)^2), & 0 \leq \varphi \ll 1 \leq \frac{2}{\sqrt{3\Gamma q}} \\ 4\Gamma q \varphi / \left(\Gamma q \varphi + \sqrt{4 + (\Gamma q \varphi)^2} \right), & \frac{2}{\sqrt{3\Gamma q}} \leq \varphi \ll 1 \end{cases} \quad (61)$$

The last two expressions in Eq. (61) merge asymptotically in the interval $(\Gamma q)^{-1/2} \ll \varphi \ll 1$. The supercurrent reaches its critical value at $\varphi = 2(\sqrt{3\Gamma q})^{-1}$ and is equal to the pair breaking current of the superconducting electrodes. In this regime $I_C \propto (T_C - T)^{3/2}$.

Figure 9 shows the thickness dependencies of $I_C R_N$ numerically determined by Ivanov et al. (1978, 1981) for various values of parameter Γ . From the shape of this dependences one may conclude that under reasonably small Γ there is a whole range of parameters

$$\Gamma \ll \gamma_B + d/\xi_2, \quad \gamma_B \ll \gamma_B^{-1} + \xi_2/d, \quad d \lesssim \xi_2 \quad (62)$$

when the $I_C R_N$ is close to its maximum value V_0 . Violation of the first inequality in Eq. (62) leads to the transition into depairing current regime and linear decrease of $I_C R_N$ with decreasing of d . The violation of the last two inequalities in Eq. (62) results in suppression of the critical current either in the form of $I_C R_N \propto V_0/\gamma_B$ or exponential with thickness $I_C R_N \propto V_0 \exp\{-d/\xi_2\}$ laws. Since $\Gamma = \gamma/\sqrt{1 - T/T_C}$, the smaller the γ , i.e. the higher the resistivity of the normal interlayer, the larger is the interval of d when $I_C R_N$ product is large.

Note that the increase of T is accompanied by increase of Γ under fixed values of the parameters $\gamma_B, \gamma, d/\xi_2$. Therefore, the crossover takes place from the linear temperature dependence $I_C R_N \propto (T_C - T)$ predicted by the AL theory to the quadratic one $I_C R_N \propto (T_C - T)^2$ first

suggested by de Gennes (1964).

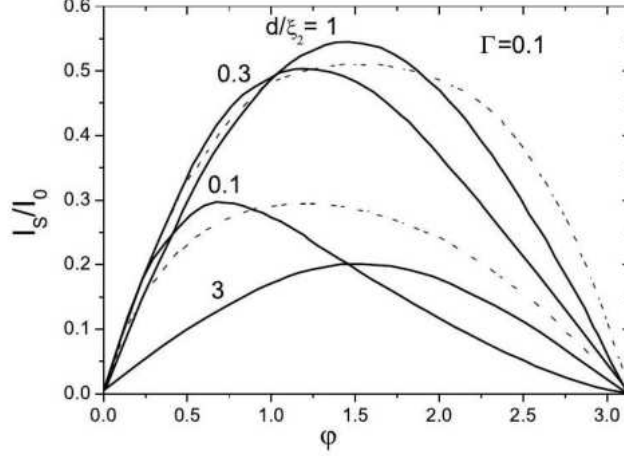


FIG. 10: The CPR in SNS junction in the depairing regime. Adapted from Ivanov et al. (1981).

9.6.5 4. Nonlinear phase shift in the electrodes

Figure 10 shows a set of $I_S(\varphi)$ curves calculated in Ivanov et al. (1978, 1981) for $\gamma_B = 0$, $\Gamma = 0.1$ and various ratios of d/ξ_2 . It is clearly seen that the reduction of d/ξ_2 deforms the CPR in such a way that their maximum shifts into the region $\varphi < \pi/2$.

The CPR shown in Fig. 10 by solid lines correspond to experimentally observed relations only at relatively low critical currents, when the kinetic inductance of the superconducting electrodes L_K can be neglected in comparison with the Josephson junction inductance $L_J = \Phi_0/2\pi I_C$. However at sufficiently small d , namely, in the limit of large supercurrents, $L_K \gg L_J$, and the phase difference increases linearly with the distance from weak link region.

$$\varphi(x) = \varphi(\infty) + 2 \frac{I_S(\varphi)}{I_0} \frac{\Delta_0^2}{|\Delta|^2(\infty)} \frac{x}{\xi(T)} \quad (63)$$

$$\varphi(\infty) = \varphi + \int_{d/2}^{\infty} \frac{I_S(\varphi)}{I_0} \left[\frac{\Delta_0^2}{|\Delta|^2(d/2)} - \frac{\Delta_0^2}{|\Delta|^2(\infty)} \right] dy \quad (64)$$

The main linear part of L_K can be subtracted by using compensating measuring scheme, thus the measured phase difference is then given by $\varphi(\infty)$, which does not contain a term proportional to the length of superconducting electrodes. The curves $I_S(\varphi(\infty))$ (dashed lines in Fig. 10) also have maxima at $\varphi_{\max} \leq \pi/2$. The smaller d/ξ_2 , the closer the system to the depairing limit and the larger is the deviation of $\varphi_{\max}$ from $\pi/2$.

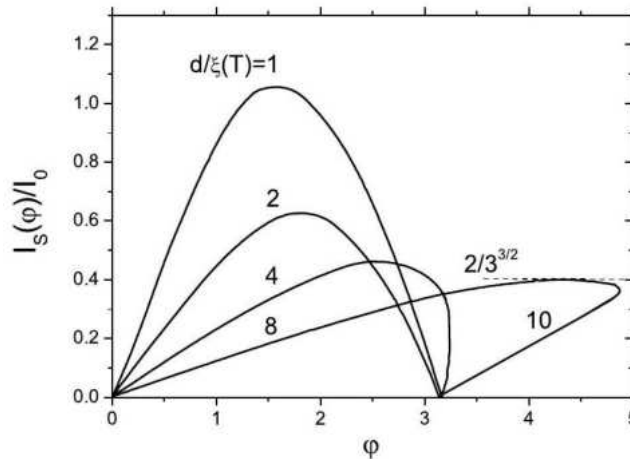


FIG. 11: Variation of the CPR with the length of a weak link at $T < T'_C$. Adapted from Likharev and Yakobson (1975).

9.6.6 5. Influence of the critical temperature of weak link material on $I_C R_N$.

The influence of the critical temperature of weak link material on $I_C R_N$ at $T \approx T_C$ has been discussed by Likharev and Yakobson (1975) and by Barone and Ovchinnikov (1979).

Likharev and Yakobson (1975) considered the situation when both critical temperatures of the weak link material T'_C and of the electrode material T_C are close to the physical temperature and, consequently, $T'_C \approx T_C$. Assuming further that the pair potential at the interfaces equals to its bulk value (the so-called 'rigid boundary conditions') Likharev and Yakobson (1975) reduced the problem of CPR calculation to the solution of GL equations in the interlayer. They have found that is the junction is sufficiently short

$$d \ll \min\{\xi(T), \xi(T)/A\}, \quad A^2 = \frac{T_C - T}{|T'_C - T|}, \quad (65)$$

then in the first approximation on d the CPR is given by

$$\frac{I_S(\varphi)}{I_0} = \left[\frac{\xi(T)}{d} + \left(\frac{1}{6} - \frac{A^2}{10} \right) \frac{d}{\xi(T)} \right] \sin \varphi - \frac{1}{30} \frac{d}{\xi(T)} \sin 2\varphi \quad (66)$$

According to (66), the corrections to sinusoidal form of CPR are small even for small d of the order of $\xi(T)$ or $\xi(T)/A$. A further increase of d results in effects that are different for $T > T'_C$ and $T < T'_C$.

For $T > T'_C$ weak link material is in the normal state, superconducting correlations induced from the electrodes decays exponentially with increase of d and the function $I_S(\varphi)$ remains almost sinusoidal. Maximum deviations from $\sin \varphi$ occur at $d \approx \xi(T)$, and according to (66) they are of the order of few percents.

For $T < T'_C$ weak link material is in the superconducting state and with increase of d the transition from

Josephson effect to depairing effect takes place. As a result, the critical current first decreases with d (see Fig. 11) and then saturates at a constant value when d exceeds approximately $10\xi(T)$. Simultaneously, the shape of $I_S(\varphi)$ function is radically changes. At $d \gtrsim 3.6\xi(T)$ the CPR becomes multivalued with shape close to a cubic parabola

$$I_S(\varphi) = I_C \frac{\varphi \xi(T)}{d} \left[1 - \left(\frac{\varphi \xi(T)}{d} \right)^2 \right], \quad \frac{I_C}{I_0} = \frac{2}{3\sqrt{3}A^2} \quad (67)$$

The unstable reverse branch of the $I_S(\varphi)$ function corresponds to the sharp drop of $|\Delta|$ in the middle of the junction, i.e. to the nucleation of the phase-slip centers (see Ivlev, B. I., and N. B. Kopnin, 1984).

Summarizing the discussion above we can conclude that even at high temperatures there exist a variety of shapes of the CPR. The deviations from $\sin \varphi$ depend on the relation between the parameters characterizing either the properties of the weak link material like $d/\xi_2, T_C, T'_C$, or the influence of the processes in the electrodes on the junctions characteristics (suppression parameters $\gamma, \gamma_B, T/T_C, d/\xi(T)$).

To estimate the range of validity of different regimes discussed above at arbitrary T , we will subdivide the general problem of CPR on two parts: (1) influence of the processes in the weak link, (2) influence of the processes in the electrodes.

9.6.7 B. Dirty limit, arbitrary temperature.

To calculate the CPR for dirty SNS junctions at arbitrary temperature the general approach is to solve the Usadel equations (190), (191) in S and S' materials and match them by the boundary conditions Eq. (192), (193) at the interfaces. Far from the interfaces the solution should tend to a uniform current-carrying superconducting state

$$\operatorname{Re} \Phi_1(\mp\infty) = \quad (68)$$

$$\Phi_\infty \cos \varphi, \operatorname{Im} \Phi_1(\mp\infty) = \mp \Phi_\infty \sin \varphi \quad (69)$$

$$\Phi_\infty = \frac{|\Delta_1(\infty)|}{1 + u^2 \pi T_C / \sqrt{\omega^2 + |\Phi_S|^2}} \quad (70)$$

$$\varphi = \varphi(\infty) - 2ux/\xi_1$$

where $\varphi(\infty)$ is the asymptotic phase difference across the junction, $u = 2mv_s\xi_1$, m is the electron's mass, v_s is the superfluid velocity.

Note that since the boundary conditions Eqs.(192), (193) include the Matsubara frequency ω , the phases of Φ_1 functions depend on ω and in general differ from the phase of the pair potential Δ at the SN interfaces, $\varphi(\pm d/2)$. Therefore it is the value $\varphi(\infty)$ rather than $\varphi(d/2)$, that can be measured experimentally by using a scheme compensating the linear in x part in Eq. (70) (see also the related discussion in Sec. V.A).

Several important limiting cases of the solution of the general problem can be considered.

9.6.8 1. Rigid boundary conditions

In this approximation it is assumed that the suppression of superconductivity in S layer is weak and the solution of Usadel equations in the superconductor has the form

$$\Phi_1\left(\pm\frac{d}{2}\right) = \Delta_1(T) \exp\left\{\pm\frac{\varphi}{2}\right\}, G_S = \frac{\omega}{\sqrt{\omega^2 + \Delta_1^2(T)}} \quad (71)$$

where $\Delta_1(T)$ is the modulus of the electrode order parameter. As was discussed in Section III, these conditions are valid for the constriction weak link geometry. The quantitative criteria for the validity of this approximation for planar SS'S sandwiches and variable thickness bridges will be discussed below for different parameter ranges.

In this approximation the CPR depends on reduced temperature T/T_C , reduced weak link length d/ξ_2 and the ratio T'_C/T_C (see Kulik and Omelyanchuk, 1975a; Likharev, 1976; Kupriyanov and Lukichev, 1981; Kupriyanov et al., 1981, 1982; Zaikin and Zharkov, 1981; Dubos et al, 2001).

In the limit of small interlayer thickness $d \ll \xi_2$ the KO-1 theory is valid and, as was discussed in Section IIIC, the CPR is independent on T'_C/T_C , the masimum position $\varphi_{\max} \geq \pi/2$ and tends to 1.86 at low T . Increase of d leads to different effects depending on relation between T, T_C and T'_C .

a. Structures made from one material ($T_C = T'_C$). Supercurrent across the junction consisting of bulk electrodes connected by a one-dimensional filament from the same material ($T_C = T'_C$) was analyzed by Kupriyanov and Lukichev (1981). The main effect predicted here is the transformation from Josephson to depairing effect with the temperature decrease under fixed value of the filament length. This is clearly demonstrated in Figs. 1214

Figure 12 shows the CPR calculated numerically for the structures with $d = 10\xi_2$ and $T/T_C = 0, 28; 0.53$ and 0.8 . The deformation of the CPR from the $\sin \varphi$ form to multivalued function of φ at low T is clearly seen.

The physical reason of this transformation becomes clear if we take into account that the characteristic length of variation of the order parameter and Usadel functions is temperature dependent. At $T \lesssim T_C$ the properties of the junction depends on relation between d and GL coherence length $\xi(T)$ given by Eq. (10). In the vicinity of critical temperature $\xi(T) \gg d$ and CPR is sinusoidal, according to the AL theory (Aslamazov and Larkin, 1969). With the temperature decrease the coherence length $\xi(T)$ decrease also. It is equivalent to increase of the effective

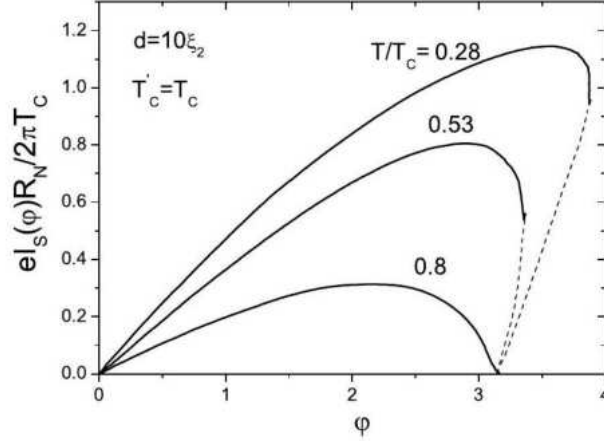


FIG. 12: Transformation from Josephson to depairing effect in a long SS'S junction with $T'_c = T_c$. Adapted from Kupriyanov and Lukichev (1981).

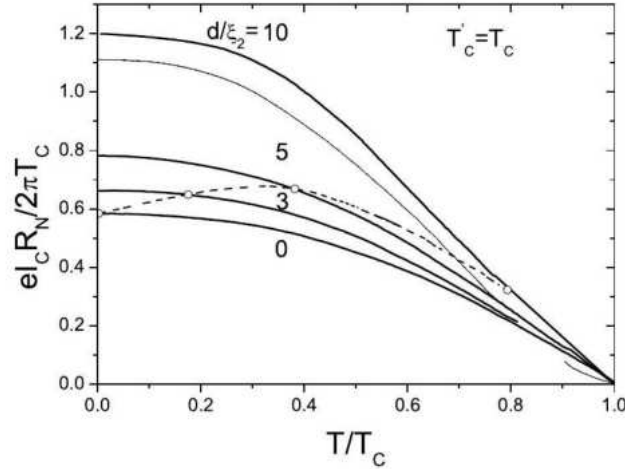


FIG. 13: Temperature dependence of the critical current in SS'S junction with $T'_c = T_c$. Adapted from Kupriyanov and Lukichev (1981).

length of the structure leading to the transition to the depairing regime, most clearly visible at $T \leq 0.5T_C$. In this temperature interval the characteristic length of variation of the function Φ_2 is close to ξ_2 and the effective length of the junction is large in this scale. As a result, for $T = 0.28T_C$ the Josephson critical current becomes rather close to the depairing current calculated for the same temperature by Kupriyanov and Lukichev (1980).

Figure 13 shows the temperature dependencies of $I_C R_N$ product calculated numerically for various d/ξ_2 ratios. The thin line in Fig. 13 is the $I_C(T)R_N$ dependence, where $I_C(T)$ is the density of the depairing current (Kupriyanov and Lukichev, 1980) and R_N is the normal resistance calculated for $d = 10\xi_2$. It gives smaller $I_C(T)R_N$ than $I_C R_N$ product for the junction of

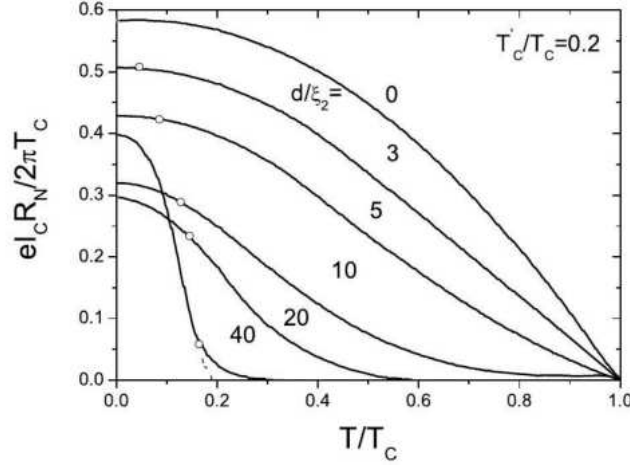


FIG. 14: Temperature dependence of the critical current in SS'S junction with $T'_C < T_C$. Adapted from Kupriyanov and Lukichev (1981).

the same length ($d = 10\xi_2$). The Josephson I_C exceeds the depairing current because the electrodes additionally support superconductivity in the line.

At T close to T_C all the curves have similar temperature dependence $I_C R_N \propto (T_C - T)$, predicted by the AL theory, except for the thin line, ($I_C(T) R_N \propto (T_C - T)^{3/2}$). The deviations from this linear dependence take place depend on the length of the structure. The open circles mark the temperatures when the CPR starts to be multivalued. The locus of this positions determines the boundary between Josephson and depairing effects and is shown in Fig. 13 by dashed dotted line. It is interesting to note that this curve has the maximum at $T \approx 0.5T_C$, which corresponds to $d \approx (4\nabla \cdot 6)\xi_2$, and that the maximum value of $I_C R_N$ in this region is even larger than $I_C R_N$ in a dirty point contact at $T = 0$ provided by the KO-1 theory.

b. $SS'S$ junctions ($T_C \neq T'_C$). The Josephson effect in SS'S junctions at arbitrary temperatures was analyzed by Kupriyanov et al. (1981, 1982) in the framework of the Usadel equations.

The results of this study are summarized in Fig. 14 which demonstrates the $I_C R_N$ products at $T'_C/T_C = 0.2$ for various ratios of d/ξ_2 . Similar to Fig. 13, the open circles mark the temperatures at which the CPR starts to be multivalued at low T when the depairing regime occurs. The smaller are T'_C/T_C and d/ξ_2 , the faster is the transformation from depairing to the Josephson effect with increasing T .

In the limit of large d and $T \geq T'_C$ the solution of the Usadel equations in the interlayer is a linear superposition of the functions $\tilde{\Phi}_2(x)$ describing the penetration of superconductivity from the S electrode into semi-infinite S'

$$\Phi_2(x) = \tilde{\Phi}_2(x - d/2) \exp\{i\varphi/2\} + \tilde{\Phi}_2(x + d/2) \exp\{-i\varphi/2\}.$$

Substitution of Eq. (72) into expression for the supercurrent Eq. (194) leads to the sinusoidal CPR and exponential thickness dependence of the $I_C R_N$ product

$$\frac{eI_C R_N}{2\pi T_C} = V^* \frac{d}{\xi^*} \exp\left\{-\frac{d}{\xi^*}\right\}, \quad d \gg \xi^* \quad (73)$$

Exponential decrease of $I_C R_N$ as $d/\xi^* \rightarrow \infty$ is typical for weak links (see de Gennes, 1964; Clarke, 1969; Aslamazov and Larkin, 1969), but the exact values of prefactor V^* and the effective decay length ξ^* in Eq. (73) depend on T'_C/T_C and T/T_C . Analytical expressions for these functions at $T'_C = 0$ were found by Likharev (1976) and by Zaikin and Zharkov (1981)

$$\xi^* = \xi_2 \sqrt{\frac{T_C}{T}}, \quad V^* = \frac{32T\Delta^2(T)}{T_C \left[\pi T + \Delta^{*2} + \sqrt{2\Delta^* (\pi T + \Delta^{*2})} \right]^2} \quad (74)$$

At arbitrary T'_C the determination of V^* and ξ^* is simplified by the existence of two different

decay lengths for functions Φ_2 and pair potential Δ_2 in the S' material at low T . In the vicinity of the interfaces this decay is described by the nonlinear Usadel equations and occurs at distances of the order of ξ_2 . Further decay of both Φ_2 and Δ_2 takes place with on the scale $\xi^* > \xi_2$ which with the accuracy better then 3% is given by the expression (Kupriyanov et al., 1982)

$$\xi^* = \xi_2 \sqrt{\frac{T_C}{T} \left[1 + \frac{\pi^2}{4} \ln^{-1} \frac{T}{T'_C} \right]}, \quad T \geq T'_C \quad (75)$$

The $V^*(T)$ dependencies calculated numerically by Kupriyanov et al. (1982) are shown in Fig. 15. The curve for $T'_C = 0$ is close to envelope of the curves corresponding to different values of T'_C . It means that SS'S junctions with $T'_C \neq 0$ can be described by Eq. (73) with parameters V^* and ξ^* defined by Eqs.(74) and (75) in a sufficiently broad temperature interval $T \gtrsim T'_C$.

From (75) it follows that at fixed d there is the temperature interval at $T \approx T'_C$ when $\xi^* \geq d$ and Eq. (73) is violated. Instead in the limit of $d \gg \xi_2$ the $I_C R_N$ product falls as $I_C R_N \propto (\xi_2/d)^2$. The CPR in this limit is close to that in the KO-1 theory.

For the SNS junctions with $T'_C = 0$ this fact was first analyzed by Likharev (1976). More extensive calculations in this limit has been performed by Dubos et al. (2001) who demonstrated that at $T = 0$

$$\frac{e I_C R_N}{2\pi T_C} \approx 10.82 \left[\frac{\xi_2}{d} \right]^2, \quad d \gtrsim 6\xi_2 \quad (76)$$

The CPR in this case has $\varphi_{\max} \approx 1.27\pi/2$ (for comparison, the KO-1 theory valid for $d \lesssim \xi_2$ provides $\varphi_{\max} \approx 1.25\pi/2$).

The detailed study of the temperature dependences of $I_C R_N$ product for SNS junctions with $T'_C = 0$ are summarized in the review by Delin and Kleinsasser (1996).

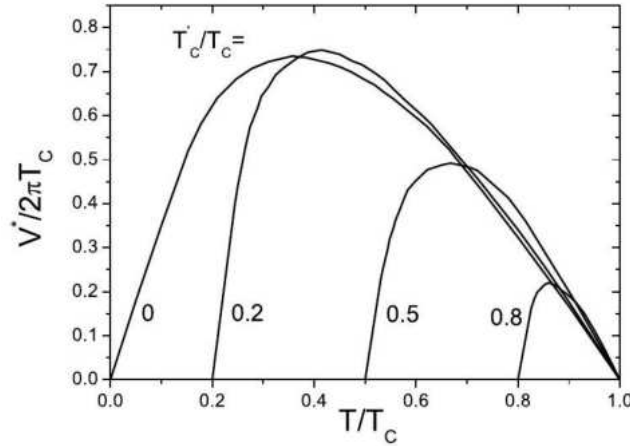


FIG. 15: Temperature dependence of the characteristic voltage for SS'S junction for various T'_C . Adapted from Kupriyanov et al. (1982).

The Josephson effect can take place in SNS weak links even if the effective electron-phonon interaction in the N -layer is repulsive ($\lambda_N < 0$). The repulsion suppresses the ability of the weak link material to carry the supercurrent and leads to decrease of $I_C R_N$. This effect was analysed by Kupriyanov et al. (1982) who have shown that the main influence of negative λ_N is the decrease of the decay length from $\xi^* = \xi_2$ for $\lambda_N = 0$ to $\xi^* \approx 0.65\xi_2$ in the limit of strong repulsion. Even for $|\lambda_N| = 0.9\lambda_S$ (where λ_S is the BCS coupling constant in superconducting electrodes) the temperature dependences of $I_C R_N$ for $d/\xi_2 \leq 5$ only slightly differ from those for $\lambda_N = 0$.

Except negative λ_N there are several other physical mechanisms leading to suppression of $I_C R_N$ product in weak links and deformation the shape of the CPR. They are

- suppression of the superconductivity in the electrodes due to proximity effect at $S'S$ interfaces,
- the depairing of electrodes by the current across the junction,
- the potential barriers at the $S'S$ interfaces.

Fortunately, the first mechanism is the most effective at $d > \xi_2$, while the second is important in the opposite limit $d < \xi_2$. This provides the opportunity for independent analysis of the influence of these factors on weak link properties. The influence of transparency of $S'S$ interfaces will be discussed in [Sec. VI](#) on SINIS junctions.

9.6.9 2. SNS junction at arbitrary T. Depairing by the proximity effect

The proximity effect between N and S materials results not only in penetration of superconducting correlation into N metal, as was studied above in the frame of rigid boundary conditions. The quasiparticle excitations, in turn, can diffuse into the superconductor resulting in

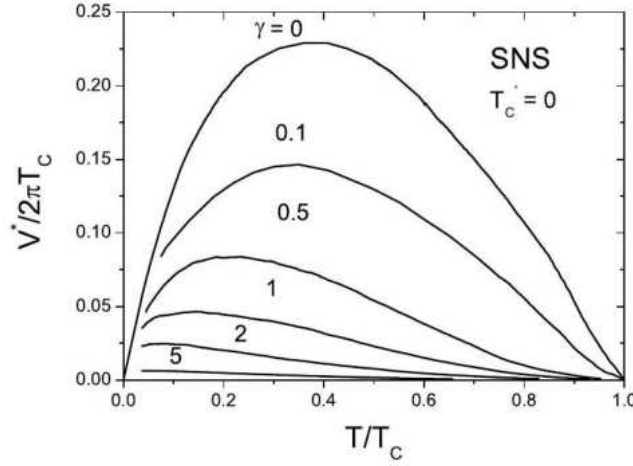


FIG. 16: Temperature dependence of the critical voltage for SS'S junction for various values of the proximity effect parameter γ . Adapted from Kupriyanov and Lukichev (1982).

suppression of superconductivity near the interface. The influence of this effect on a long one-dimensional planar type SNS junction ($d > \xi_2$) was analyzed by Kupriyanov and Lukichev (1982). It was supposed that both interfaces are transparent and $T'_C = 0$. The calculations lead to sinusoidal CPR and exponential dependence of $I_C R_N$ product (73) with $\xi^* = \xi_2$ and temperature-dependent V^* presented in Fig. 16 for various values of γ defined by Eq. (188).

It is clearly seen that for $\gamma \lesssim 0.1$ the values of $I_C R_N$ product are close to that calculated within the rigid boundary conditions (74) in a broad temperature range. Increase of γ leads to suppression of $I_C R_N$. If the normal metals like Au, Ag, Cu are used, then $\gamma \geq 1$ and the magnitude of the suppression exceeds two orders of magnitude at $T \approx 0.5T_C$.

Thus to get high values of $I_C R_N$ it is necessary to use high-ohmic weak link material.

In variable thickness bridges composed from the two S-banks placed on the top of thin normal film of thickness d_n , the situation is close to the described above (see Golubov et al., 1983; Kupriyanov, 1989) and the $I_C R_N$ product is weakly suppressed if $\gamma \leq 0.3 \max \{1, \xi_2/d_n\}$.

9.6.10 3. $SS'S$ junction at arbitrary T . Depairing by the supercurrent

The suppression of the superconductivity in electrodes by a supercurrent across the $SS'S$ junction may become essential if

$$d \ll \xi_2. \quad (77)$$

and the transparency of $S'S$ interfaces is high. This situation has been studied by Zubkov and Kupriyanov (1983) in the frame of the Usadel equation. Based on Eq. (77) they reduced the problem to the solution of the Eq. (190), (191) with the boundary conditions at the $S'S$ interface

$$\xi_1 G_1 \partial_x \text{Im } \Phi_1 = 2\omega \gamma_J \arctan \left[\frac{G_1 \text{Im } \Phi_1}{\omega} \right] \quad (78)$$

$$G_1 = \frac{\omega}{\sqrt{\omega^2 + \text{Re } \Phi_1^2}}, \quad \gamma_J = \gamma \frac{\xi_2}{d} \quad (79)$$

and the with conditions given by Eq. (68)-(70) in the bulk.

Figure 17 shows the CPR calculated for $T = 0.1T_C$ by Zubkov and Kupriyanov (1983). Here $\varphi(\infty)$ is the asymptotic phase difference defined by Eq. (70). At $\gamma_J = 0$ the KO-1 theory is valid and $\varphi_{\max} \approx 1.25\pi/2$ as it is shown in Fig.17. Increase of γ_J makes $I_S(\varphi(\infty))$ close to $\sin(\varphi)$ form at $\gamma_J \approx 2.5$ and shifts $\varphi_{\max}$ to the value smaller than $\pi/2$ for larger γ_J . At high temperatures in the absence of depairing effect ($\gamma_J = 0$, rigid boundary conditions) the shape of $I_S(\varphi)$ is close to the sinusoidal form. In complete agreement with the situation discussed above in Sec. V, increase of γ_J leads to the deformation of the shape of $I_S(\varphi(\infty))$ so that the position of its maximum shifts to $\varphi_{\max} < \pi/2$.

Numerical calculations made by Zubkov and Kupriyanov (1983) have shown that at

$$\gamma_J \lesssim 0.1 \quad (80)$$

there is the large temperature interval where the suppression of superconductivity in S-electrodes is small. At larger γ_J and $T \lesssim 0.4T_C$ the ratio decay exponentially with γ_J

$$\frac{I_C R_N(T, \gamma_J)}{I_C R_N(T, 0)} \approx 0.8 \exp \left\{ -\frac{\gamma_J}{1.45} \right\}, \quad \gamma_J \gtrsim 0.2 \quad (81)$$

At $T \approx T_C$ additional temperature dependent factor should be taken into account and the parameter $\Gamma_J = \gamma_J / \sqrt{1 - T/T_C}$ should be used instead of γ_J . It means that under fixed γ_J there will be the transition to the depairing regime similar to the discussed in Sec. 5.

Summarizing the above results, we can conclude that the processes in the electrodes are not essential if

$$\gamma \leq 0.1 (\gamma_B + \min \{1, d/\xi_2\}) \sqrt{1 - T/T_C} \quad (82)$$

The violation of this inequality leads to suppression of $I_C R_N$ product and the deformation of the shape of $I_S(\varphi)$ calculated on the base of rigid boundary conditions. The deviations from Eqs.(82) make the dependence $I_S(\varphi(\infty))$ closer to the sinusoidal. In some cases $\varphi_{\max} < \pi/2$ may be expected. It is important that due to nonlinear effects in the electrodes the phase difference should be considered in the sense of Eqs.(70). Strictly speaking, $\varphi(\infty) \approx \varphi$ only if the conditions

$$\gamma \leq 0.1 (\gamma_B + d/\xi_2) \sqrt{1 - T/T_C} \quad (83)$$

is not violated.

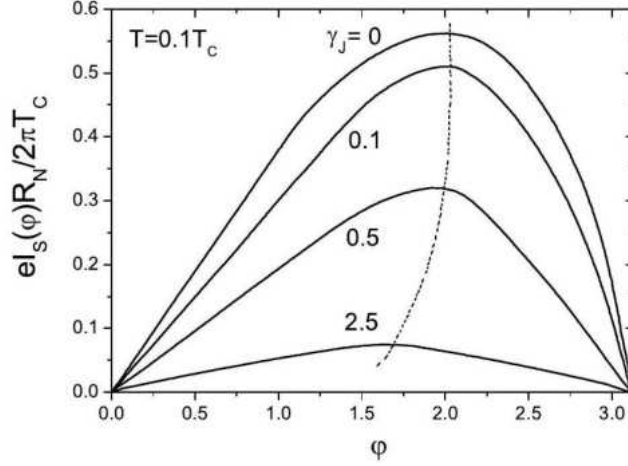


FIG. 17: The CPR in SS'S junction for various values of the depairing current parameter γ_J . Adapted from Zubkov and Kupriyanov (1983).

9.6.11 C. Clean SNS junctions

Now we shall briefly examine the properties of "clean" weak links in which the mean free path of electron ℓ_2 is larger than the distance between the superconducting electrodes d and weak link coherence length $\xi_{02} = v_{F2}/2\pi T_C$ and the condition of clean limit $\ell_1 \gg \xi_{01} = v_{F1}/2\pi T_C$ is also fulfilled in superconducting electrodes.

The study Josephson coupling and the shape of the CPR in these structures has been done by Kulik (1969), Ishii (1970), Svidzinskii et al. (1971), Bardeen and Johnson (1972). In these models it was supposed that the superconductor and normal metal have the same transport parameters, $T'_C = 0$ in the N-material, the distance between S-electrodes is large ($d \gg \xi_{02}$) and $T \ll T_C$. The suppression of the superconductivity in the electrodes was neglected and rigid boundary conditions has been used for the calculation of $I_S(\varphi)$.

Zaikin and Zharkov (1983) have found that under the above conditions the pair potential is suppressed at the interface and its magnitude at $T \ll T_C$ is twice smaller than in the bulk. Thus the rigid boundary conditions are not strictly valid.

On the other hand, it was shown by Kupriyanov (1981) that the rigid boundary conditions are still applicable if there is large difference between Fermi velocities of the metals, namely

$$v_{F2} \ll v_{F1} \quad (84)$$

Under this condition, only small amount of electrons from S metals moving almost perpendicular to the SN interface can penetrate from S into N and finally from N into the second electrode thus providing the Josephson coupling in the structure (see Fig. 18). The smooth transparent interfaces are assumed in this model. The electrons in S with the parallel to the interface momentum component

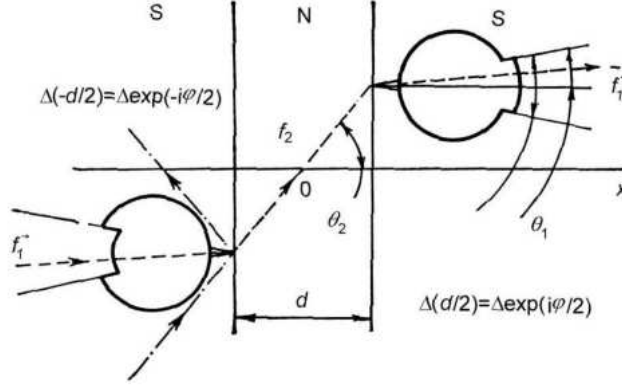


FIG. 18: Processes of current transport in a clean SNS junction larger than v_{F2} are reflected back into S. It means, that exactly as in the KO-2 model (Kulik and Omelyanchuk, 1977, 1978), the electron distribution function in the momentum space in superconductors deviates from its bulk value only in the small angle range

$$|\theta_1| \leq v_{F2}/v_{F1} \quad (85)$$

where θ_1 is the angle between the x -axis and direction of $\mathbf{v}_{F1}$. Therefore, to solve Eilenberger equations (173), (174), one may assume that the pair potential Δ and the isotropic parts of functions $\langle f_1 \rangle$, $\langle f_1^+ \rangle$ and $\langle g_1 \rangle$ in the electrodes are spatially homogenous and equal to their bulk values, which in the symmetrical case are given by

$$\langle f_1 \rangle = \frac{\Delta}{E_0}, \quad \langle f_1^+ \rangle = \frac{\Delta^*}{E_0}, \quad \langle g_1 \rangle = \frac{\omega}{E_0} \quad (86)$$

$$E_0 = \sqrt{\omega^2 + \Delta^2}, \quad \Delta\left(\pm \frac{d}{2}\right) = \Delta \exp\left\{\pm i \frac{\varphi}{2}\right\}. \quad (87)$$

As a result, Eq. (173) is reduced to a linear differential equation similar to that solved in the KO-2 model. The solutions for the 'outgoing' ($\rightarrow$) and the 'incoming' ($\leftarrow$) trajectories have the form

$$f_1^{\leftarrow} = \frac{\Delta}{E_0} + f^{\leftarrow} \exp\left\{-\frac{|d/2 - x|}{\xi^* \cos \theta_1}\right\} \quad (88)$$

$$g_1^{\leftarrow} = \frac{\omega}{\xi} + g^{\leftarrow} \exp\left\{-\frac{|d/2 - x|}{\xi^* \cos \theta_1}\right\} \quad (89)$$

where

$$f_1 \rightarrow g^{\rightarrow} \quad (90)$$

$$\omega - E_0 \Delta$$

$$\omega f_1^{\leftarrow} = g^{\leftarrow} \frac{\Delta}{\omega + E_0}$$

$$\xi^* = \xi_{01} \left[\frac{E_0}{\pi T_C} + \frac{\xi_{01}}{\ell_1} \right]^{-1}$$

integral comes from small u and the result is (Ishii, 1970)

$$I_S(\varphi) = \frac{1}{2} e N_2 v_2 \Delta \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin k\varphi}{k} \quad (95)$$

The Fourier series Eq. (95) determines the periodic $I_S(\varphi)$ function in the form

$$I_S(\varphi) = eN_2v_2\Delta \left\{ \frac{\varphi}{2} - \pi \left[\frac{\varphi}{2\pi} + \frac{1}{2} \right] \right\} \quad (96)$$

where $[x]$ is integer part of x . The details of calculations can be found in the textbook Svidzinskii (1982).

Thus in clean SNS junctions the CPR transforms from the sinusoidal form at $T \approx T_C$ to the saw-toothed curve (96) at low T if the suppression of superconductivity by current or proximity effect is negligibly small and the rigid boundary conditions are valid.

The analysis of the validity range of the rigid boundary conditions at the clean NS interface was done by Kupriyanov and Lukichev (1988b). They considered thick ($d \gg \xi_{0T}$) SNS junctions with fully transparent interfaces under arbitrary ratio between Fermi velocities of N and S materials. Starting with the analytical solution of Eilenberger equations in the N metal (91) they derived the general expression for the supercurrent

$$I_S(\varphi) = \frac{8\pi T e N_2 v_{F2} \Delta \sin(\varphi)}{\gamma_C^2 \left[\pi T + \sqrt{(\pi T)^2 + \Delta^2} \right]} \times \\ \times \int_0^{\theta_p} \Phi_+(\theta) \exp \left\{ -\frac{d}{\xi_{0T} \sqrt{1 - \gamma_C^{-2} \sin^2(\theta)}} \right\} \sin(\theta) \cos(\theta) d\theta \quad (97)$$

$$\gamma_C = \frac{v_{F2}}{v_{F1}}, \quad \Phi_+(\theta) = \frac{1}{2} \left(f_1 \left(\frac{d}{2}, \theta \right) + f_1^+ \left(\frac{d}{2}, \theta \right) \right)$$

Here $\theta_p = \pi/2$, if $\gamma_C \geq 1$, and $\theta_p = \arcsin \gamma_C$, if $\gamma_C \leq 1$ and $f_1(\frac{d}{2}, \theta)$ is the Eilenberger function on the S side of the NS interface. Thus the problem is reduced to calculation of function $\Phi_+(\theta)$ by numerical solution of the Eilenberger equations in the S electrodes.

In the limit of small $\gamma_C \ll 1$ the integration in Eq. (97) takes place only in the very small cone of angles and, as discussed above, the supercurrent is defined by the function $\Phi_+(\theta) = \Delta / \left(\pi T + \sqrt{(\pi T)^2 + \Delta^2} \right)$ on the trajectory perpendicular to the interface and Eq. (97) is reduced to Eq. (94).

In the opposite case $\gamma_C \gg 1$ the supercurrent falls as γ_C^{-2} and has slightly different dependence on d . Note that function $\Phi_+(\theta)$ depends on γ_C only for $\gamma_C \leq 1$. Moreover, the suppression $\Phi_+(\theta)$ with increase of γ_C is essentially weaker compared to the dirty limit when for $\gamma \approx 1$ the $I_C R_N$ is reduced by two orders of magnitude (see Sec.V.B2).

As a result, for $\gamma_C \lesssim 1$ and in a wide temperature range $T \lesssim 0.8T_C$ the suppression of superconductivity in electrodes does not essentially modify the results obtained in the framework of the rigid boundary conditions.

Consider briefly the opposite limiting case $T'_C = T_C$, when two superconductors are connected by a narrow wire of the length d made from the same material. In a short contact, $d \ll \xi_{0T}$, $I_S(\varphi)$ is described by KO-2 model. However, strong modifications take place as one moves to the opposite case $d \gg \xi_{0T}$. Starting from a critical value $d_C \sim \xi_{0T}$, the function $I_S(\varphi)$ becomes multivalued (Likharev, 1979; Kulik and Omelyanchuk, 1975b; Ferrer adn Sols, 1994; Martin-Rodero et al., 1994). This situation corresponds to the appearance of the solution with a phase slip center inside the contact and is similar to that discussed above in Sec.V.B. 3 for the dirty limit.

Interesting peculiarities in the CPR of the ballistic and symmetrically stacked double SNSNS junctions were predicted by Chang and Chu (1997).

9.7 VI. DOUBLE BARRIER SINIS JUNCTIONS

Double-barrier junctions recently attracted considerable interest both in experiments and in theory. A supercurrent in a tunnel SIS contact depends linearly on the barrier transparency since Cooper pairs tunnel coherently (Josephson, 1962). Supercurrent transport in the SINIS structure having two barriers has more complicated nature. It is well known from the BTK theory (Blonder et al., 1982) that the subgap resistance of a ballistic SIN junction has a quadratic dependence on the transparency of the interface, since Andreev reflection is a two-particle process. The disorder in a normal region enhances the Andreev current due to opening of some fraction of tunneling channels, and the resistance has a linear dependence on the transparency. This effect is known as reflectionless tunneling in SIN junctions (see Volkov et al. (1993), Nazarov (1994a) and further references in the review article by Beenakker (1997)). Interestingly, the opening of tunnel channels may be realized in a ballistic NI_1NI_2S junction as well by placing a second tunnel barrier Melsen et al. (1996). Therefore it is important to understand how the barrier transparency and disorder influence on the coherence in SINIS junctions and what are the corresponding modifications of CPR.

Coherent effects in SINIS junctions are of significant practical importance. Recent experiments demonstrated the possibility of engineering Josephson junctions with desired properties using existing multilayered techniques. As already mentioned in Sec. IIB, present interest to SINIS junctions is related to their possible applications in superconducting electronics and the references to experiments are given in Sec. IIB.

An overview of early experimental work on supercurrent flow in superconductor-semiconductorsuperconductor junctions was given by Huffelen et al. (1993). Carrier transport through these structures was found to be dominated by interfaces, therefore they are of SINIS-type. Several types of weak links using various semiconducting materials that carry large supercurrents, were reported by Kleinsasser et al. (1989) and Klapwijk et al. (1989). The operation of a superconducting field-effect transistor, with a gate electrode controlling the

supercurrent has been demonstrated. More recently, the structures using two-dimensional electron gas (2DEG) as a weak link materials were successfully fabricated and investigated by Marsh et al. (1994), Takayanagi et al. (1995a), Chrestin et al. (1997, 1999), Mur et al. (1996), Lehnert et al. (1999), Neurohr et al. (1999), Heida et al. (1999), Grajcar et al. (2002a) and Schäpers et al. (2003). A qubit device based on manipulations of ABS in SINIS junctions was recently proposed by Shafranjuk et al. (2002).

Theory of long SINIS junctions with low barrier transparency was developed by Aslamazov et al. (1968) (see also discussion in the previous Section). Nikolić et al. (2002) recently reported the results of theoretical study of double-screened dipole barrier SINIS junctions, which are of relevance for HTS grain boundary junctions.

In this Section, recent developments in theory of SINIS structures with low transparent barriers is discussed. We will concentrate on SINIS junctions with thin metallic interlayers, which are of the most practical importance.

9.7.1 A. SINIS junctions, clean limit

9.7.2 1. The general case

Supercurrent in ballistic SINIS structures is controlled by an interplay between ABS and transmission (BreitWigner) resonances. Much theoretical work was concentrated on studying resonance supercurrents in lowdimensional structures, in particular in S-2DEG-S (where 2DEG is two-dimensional electron gas).

Supercurrents in such structures were calculated using various theoretical approaches by

Kresin (1986), Gudkov et al. (1988), Furusaki et al. (1992), Schüssler and Kümmel (1993), Chrestin et al. (1994), Rittenhouse and Graybeal (1994), Wendin and Shumeiko (1996b), Riedel et al. (1996), Tang et al. (1996a,b), Buttcher and Kopp (1997), Johansson et al. (1999), Gogadze and Kosevich (1998), Nevirkovets and S.E. Shafranjuk (1999), Samuelsson (2000), Shchelkachev (2000), Kuhn et al. (2001), Otadov and Lodder (2002), Lodder and Nazarov (2002). Much of this work was devoted to the physics of supercurrent flow through Breit-Wigner resonances in ballistic structures. Common conclusion from these studies is that the physics of supercurrent via a single Breit-Wigner resonance is similar to that of resonant tunneling via a localized state in SIS tunnel junctions, discussed above in Section III. The CPR depends on the width of the resonance and its position with respect to the Fermi level. Important consequence is that supercurrent is influenced significantly by phase-coherent normal reflections, which may lead to oscillation of supercurrent as a function of interlayer thickness or electronic momentum. These effects are most strongest in low dimensions. Supercurrent oscillations as a function of a gate voltage in S-2DEG-S junction were observed by Takayanagi et al. (1995a).

For most of the practical applications the universal features of supercurrent a three-dimensional SINIS junction are of interest, where proper averaging over the resonances is important. Theory of stationary properties in such structures was developed by Brinkman and Golubov (2000), Galaktionov and Zaikin (2002), Osana et al. (2002). For thin interlayers, the existence of the coherent regime was demonstrated, when the supercurrent averaged over the transmission resonances is proportional to the barrier transparency D . With increasing thickness supercurrent becomes incoherent, of the order of D^2 , as expected for two uncorrelated sequential tunneling processes. Quantitative calculations were done by Brinkman and Golubov (2000), Galaktionov and Zaikin (2002) to study the crossover between these two regimes. The results are briefly summarized below.

In a clean SINIS junction Eilenberger equations are in general not applicable because of interference of two interfaces. Starting from a set of Gor'kov equations, an expression for the temperature Green's functions in a clean three-dimensional SINIS structure was found by Brinkman and Golubov (2000). Here N is either a normal metal or a weak superconductor (S') with $T_C' < T_C$. In the case of symmetric low-transparent barriers the supercurrent is given by

$$I_s = \frac{e}{\hbar} \int \frac{d^2 k_{\parallel}}{(2\pi)^2} T \sum_{\omega \geq 0} \times \frac{\Delta_1^2 \sin \varphi + \Delta_1 \Delta_2 \sqrt{E_1/E_2} d/\xi_x W^2 \sin \frac{\varphi}{2}}{2W^4 E_1^2 (\cosh d/\xi_x - \cos 2k_x d) + E_3^2}. \quad (98)$$

Here $k_{\parallel}$ is the wave-vector component parallel to the interfaces, $E_1 = \sqrt{\omega^2 + \Delta_1^2}$, $E_2 = \sqrt{\omega^2 + \Delta_2^2}$, $E_3 = \sqrt{\omega^2 + \Delta_1^2 \cos^2 \varphi/2}$, $W = W_{1,2}/\hbar v_x \gg 1$, where $W_{1,2}$ is the strength of the interface potential $V(x) = W_1 \delta(x) + W_2 \delta(x-d)$, and Δ_1, Δ_2 are the pair potentials in S and N -layers, respectively. Further, $v_x = v_F \cos \theta$, $k_x = k_F \cos \theta$, $\xi_x = \hbar v_F \cos \theta/2\omega$, where v_F and k_F are the Fermi velocity and Fermi momentum in the interlayer. For the case of normal interlayer, $\Delta_2 = 0$, more general expression for the supercurrent in a clean SINIS junction for arbitrary barrier transparency was obtained by Galaktionov and Zaikin (2002) and Osana et al. (2002), which agree with Eq. (98) in the relevant limits.

Eq. (98) describes the interplay between quasiparticle Breit-Wigner and Andreev resonances for arbitrary values of the parameter $k_F d$. It is instructive to discuss this crossover in terms of the width of transmission resonances in a double-barrier junction, which in the symmetrical case is given by $\Gamma = \hbar v_F \langle x D(x) \rangle / 2d = \hbar v_F / 8d \bar{W}^2$, where $\langle x D(x) \rangle$ is the angle-averaged transparency of a single barrier ($x = \cos \theta$).

It was proved by Galaktionov and Zaikin (2002) and Osana et al. (2002) that the supercurrent given by Eq. (98) for large $k_F d$ is equivalent to that obtained directly from the

quasiclassical Eilenberger equations.

The supercurrent of SINIS junction is controlled by a single parameter γ_{eff}

$$\gamma_{eff} = \frac{\pi T_C}{\Gamma} = \frac{2\pi T_C d}{\hbar v_F \langle xD(x) \rangle} \quad (99)$$

which is equal to the ratio of the widths of Andreev and Breit-Wigner resonances. Since the ratio $2\pi T_C / \hbar v_F \sim k_e - k_h$, where $k_{e,h}$ are the wave-vectors of the electrons and holes forming the ABS, and since $d / \langle xD(x) \rangle$ is average path, L , of the particles inside the N -layer, the parameter γ_{eff} determines the dephasing of the ABSs. We will call the regime of small γ_{eff} , when there is no dephasing, as the coherent regime.

9.7.3 2. Coherent regime (broad resonances)

As follows from Eq. (98), the coherent regime takes place for a thin interlayer when the transmission resonances are broad $\Gamma \gg \pi T_C$. The supercurrent is then given by

$$I_S(\varphi) = \frac{e}{\hbar} \int \frac{d^2 k_{\parallel}}{(2\pi)^2} T \sum_{\omega} \frac{\Delta_1^2 \sin \varphi}{E_1^2 D_2^{-1} - \Delta_1^2 \sin^2 \varphi / 2} \quad (100)$$

where D_2 is the transparency of a double barrier NININ contact, which has a resonant structure, $D_2^{-1} = 1 + (2W \cos k_x d + 2W^2 \sin k_x d)^2$. Integration over the directions of $k_{\parallel}$ (over the resonances) yields the supercurrent

$$eI_S(\varphi)R_N = 2\pi T \sum_{\omega} \frac{\Delta_1^2 \sin \varphi}{\sqrt{\omega^2 + \Delta_1^2} \sqrt{\omega^2 + \Delta_1^2 \cos^2 \varphi / 2}} \quad (101)$$

which does not depend on the properties of the interlayer and coincides with the dirty-limit result (Kupriyanov and Lukichev, 1988). Here $R_N^{-1} = e^2 k_F^2 \gamma / 4\pi^2 \hbar$ is the normal state contact resistance per square, where $\gamma = \langle xD(x) \rangle = 1/4W^2$. This expression can be generalized to the asymmetric case: $eI_S(\varphi)R_N = 2\pi T \sum_{\omega} |\Delta_1|^2 \sin \varphi / E_1 E'_3$, with $E'_3 = \sqrt{\omega^2 + |\Delta_1|^2 (\cos^2 \varphi / 2 + \gamma_-^2 \sin^2 \varphi / 2)}$, $\gamma_- = (\gamma_1 - \gamma_2) / (\gamma_1 + \gamma_2)$, $\gamma_{1,2} = \langle xD_{1,2}(x) \rangle$, where $D_{1,2}$ are the individual barrier transparencies. In this case, $R_N^{-1} = e^2 k_F^2 \gamma_c / 2\pi^2 \hbar$, where $\gamma_c = \gamma_1 \gamma_2 / (\gamma_1 + \gamma_2)$.

For $T = 0$ the maximum value of $eI_S(\varphi)R_N$ is achieved at $\varphi \approx 1.86$ and exceeds the $eI_C R_N$ value of $(\pi/2) |\Delta_1|$ for a tunnel SIS contact (see Fig. 7).

Expression (101) is identical to that found by Kupriyanov and Lukichev (1988) for the SINIS junction with diffusive interlayer in the same limit of $\gamma_{eff} \ll 1$ (see Eq. (117) below). This means that Eq. (117) is more general than the model used to derive it, since it holds even when the dirty limit condition is violated in the interlayer.

Note that in the coherent regime $\gamma_{eff} \ll 1$ the supercurrent in SINIS junction scales with $\langle xD(x) \rangle$ (similar to the SIS case) and not with $\langle xD(x) \rangle^2$. The reason is that the transmission resonances than Δ for $\gamma_{eff} \ll 1$.

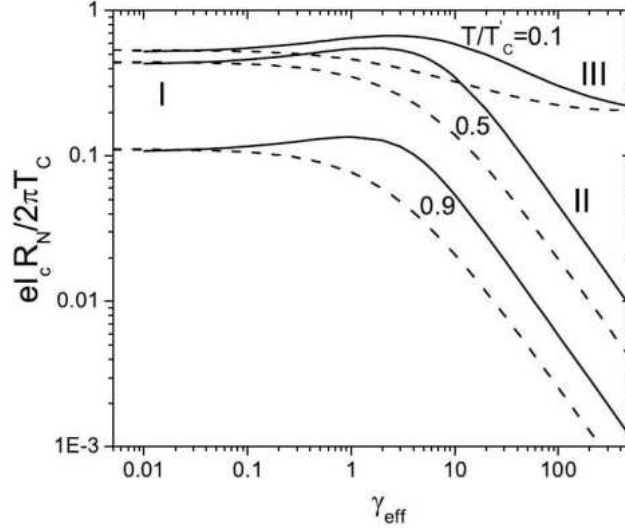


FIG. 19: $I_C R_N$ product in a SINIS junction in the clean limit (solid lines) and the dirty limit (dashed lines) for the ratio $T_C/T'_C = 7.4$ corresponding to the Nb/Al combination. Adapted from Brinkman and Golubov (2000).

Brinkman and Golubov (2000) have shown that Eq. (101) can be derived by integrating the supercurrent per single channel over the transmission eigenvalue density for a normal double-barrier contact. This is proven by integrating $\int_0^1 I_C(D) \rho(D) dD$ with $I_C(D)$ from Eq. (16) and with the density function $\rho(D) = (\hbar/e^2 R_N) D^{-3/2} (1-D)^{-1/2}$ for a double-barrier contact (Beenakker, 1997). Note that the same function $\rho(D)$ describes a disordered barrier, see Eq. (39)), which was applied by Naveh et al. (2000) to high J_c SIS junctions.

Analytical continuation in Eq. (101) provides the spectral supercurrent in the coherent regime (Brinkman and Golubov, 2000).

$$\text{Im } I_E = \text{Im} \frac{\Delta_1^2 \sin \varphi}{\sqrt{\Delta_1^2 - E^2} \sqrt{\Delta_1^2 \cos^2(\varphi/2) - E^2}} \quad (102)$$

It follows from Eq. (102), supercurrent is carried by ABS in the energy range $\Delta_1 \cos \varphi/2 < E < \Delta_1$. The reason for this broad energy distribution is that all channels with transmission coefficients $0 \leq D \leq 1$ contribute to the supercurrent.

9.7.4 3. Incoherent regime (narrow resonances)

In this case $\Gamma \ll \pi T_C$ or, equivalently $\gamma_{\text{eff}} \gg 1$. With the increase of the interlayer thickness the coherent regime breaks down due to the dephasing of the transmission resonances. After performing the angle averaging in Eq. (98) the general expression for a double-barrier junction becomes

$$eI_S(\varphi)R_N = 2\pi T \sum_{\omega} \frac{\Delta_1^2 \sin \varphi}{E_1^2} \int_0^1 \frac{4x^5 dx}{\widetilde{W}^2 \sqrt{a^2 - 1}} + \frac{\Delta_1 \Delta_2 \sin \frac{\varphi}{2}}{E_1 E_2} \frac{d}{\xi_{s'}} \int_0^1 \frac{8x^2 dx}{\sqrt{a^2 - 1}} \quad (103)$$

where $a = \cosh(d/\xi_x) + (1/2W^4) E_3/E_1$. The pair potential in S' is determined self-consistently with

$$-\Delta_2 \ln \frac{T}{T'_C} = 2\pi T \sum_{\omega} \left(\frac{\Delta_2}{\omega} - \langle F_2 \rangle \right) \quad (104)$$

where $\langle F_2 \rangle$ is the angle-averaged anomalous Green's function in the interlayer

$$\langle F_2 \rangle = 4 \int_0^1 \left(\frac{\Delta_2}{E_2} \sinh \left(\frac{d}{\xi_x} \right) + \frac{2\Delta_1}{E_1} x^4 \cos \frac{\varphi}{2} \right) \frac{xdx}{\sqrt{a^2 - 1}} \quad (105)$$

It follows from Eq. (103), that at low temperatures the CPR has two components, $\sin \varphi$ and $\sin \varphi/2$. The sign of the $\sin \varphi/2$ component is determined by the sign of the interlayer pair potential Δ_2 , which is determined selfconsistently from Eq. (104) and depends on the sign of the effective electron-electron interaction in an S' material.

Due to proximity effect, the interlayer pair potential Δ_2 does not vanish at temperatures $T > T_C$. If the effective electron-electron interaction is attractive, Δ_2 has a positive sign, while for a repulsive interaction Δ_2 is negative at $T > T'_C$ (where T'_C is defined using the absolute value of the coupling constant). Thus the measurements of a CPR in a SIS'IS junction can in principle be used for measuring the sign of the electron-electron interaction in metallic films (Brinkman and Golubov, 2000).

As follows from Eq. (103), for $T > T'_C$ and $\gamma_{eff} \gg 1$ the supercurrent is the second order in $\langle xD(x) \rangle$

$$eI_S(\varphi)R_N = \frac{32\pi T}{7\gamma_{eff}} \sum_{\omega} \frac{\Delta_1^2 \sin \varphi}{E_1^2} \quad (106)$$

Therefore, with increasing γ_{eff} the crossover from the direct tunnelling with nonsinusoidal CPR and large $I_C R_N$ to the sequential tunnelling with sinusoidal CPR and $I_C R_N 1/\gamma_{eff}$ takes place.

9.7.5 B. SINIS junctions, dirty limit

9.7.6 1. The general case

Now we consider the situation when the interlayer is in the dirty limit $\ell \ll (d, \xi_2)$ with $\xi_2 = \sqrt{D_N/2\pi T_C}$. In this case the stationary Josephson effect in SINIS structure was analyzed in the framework of the Usadel equations by Kupriyanov and Lukichev (1988), Zaitsev (1991), Golubov and Kupriyanov (1994), Kupriyanov et al. (1999), using various approximations.

Kupriyanov et al. performed the detailed analysis in the practically interesting regime of small interlayer thickness $d \ll \xi_2$, and the following general expression for the CPR was obtained

$$\frac{I_S(\varphi)R_N}{2\pi T_C} = \frac{\pi T}{\gamma_{eff}} \sum_{\omega \geq 0} \frac{\Delta_1^2 g^2 \sin(\varphi)}{\omega^2 \tilde{\Omega}} + \frac{T}{T_C} \sum_{\omega \geq 0} \frac{g \Delta_1 \Delta_2 \sin(\varphi)}{\tilde{\Omega} \omega \eta(\varphi)},$$

where

$$\gamma_{eff} = \frac{\gamma_{B1}\gamma_{B2}}{\gamma_{B2} + \gamma_{B1}} d, \quad \gamma_- = \frac{\gamma_{B1} - \gamma_{B2}}{\gamma_{B2} + \gamma_{B1}} \quad (108)$$

$$q = g \frac{\pi T_C}{\gamma_{eff} \omega}, \quad \eta(\varphi) = \sqrt{\cos^2 \left(\frac{\varphi}{2} \right) + \gamma_-^2 \sin^2 \left(\frac{\varphi}{2} \right)}$$

$$\text{and } g = \omega / \sqrt{\omega^2 + \Delta_1^2}, \quad \tilde{\Omega} = \sqrt{\omega^2 (1 + q)^2 + [q \Delta_1 \eta(\varphi) + \Delta_2]^2}.$$

The magnitude of the pair potential in the interlayer Δ_2 is determined by the selfconsistency equation

$$\Delta_2 \left\{ \ln \frac{T}{T'_C} + 2\pi T \sum_{\omega \geq 0} \left[\frac{1}{\omega} - \frac{1}{\tilde{\Omega}} \right] \right\} = 2\pi T \sum_{\omega \geq 0} \left[\frac{q \Delta_1 \eta(\varphi)}{\tilde{\Omega}} \right] \quad (109)$$

The first term in (107) describes the direct coupling of the superconducting electrodes, while the second term describes the contribution to the supercurrent due to sequential tunnelling in two SIS' junctions connected in series. The interplay between these two channels depends on the barrier parameters $\gamma_{B1,2}$, temperature and the ratio between T'_C and T_C .

Figure 19 shows the results of calculation of the critical current in a SINIS junction with selfconsistently determined pair potential both in the clean and dirty limits. The coherent (I) and incoherent (II) regimes are indicated. III is the regime of two SIS' tunnel junctions connected in series for $T < T'_C$. In the coherent regime I the supercurrent does not depend on whether the interlayer is in the clean or dirty limit.

Below we will consider several limiting cases when the CPR can be calculated analytically.

9.7.7 2. Limit of high temperatures

If the temperature is high enough to guarantee the inequality

$$\pi T \gg \frac{\gamma_{eff}\Delta_2}{1 + \gamma_{eff}} + \frac{\Delta_1}{1 + \gamma_{eff}} \quad (110)$$

the magnitude of the interlayer order parameter $\Delta \propto \eta(\varphi)$ and the CPR is sinusoidal. For T close to T_C the selfconsistency equation Δ_2 can be solved analytically in two limits of small and large values of γ_{eff} . For $\gamma_{eff} \ll 1$ the supercurrent is given by (Kupriyanov et al., 1999)

$$I_S(\varphi)R_N = \frac{\Delta_B^2}{\pi T_C} \left\{ \frac{\pi^2}{4} + \frac{\gamma_{eff} \ln^2(1/2\gamma_{eff})}{\ln(T_C/2\gamma_{eff}T'_C)} \right\} \sin(\varphi) \quad (111)$$

while for $\gamma_{eff} \gg 1$

$$I_S(\varphi)R_N = \frac{\Delta_B^2 \sin(\varphi)}{4\pi T_C \gamma_{eff}} \left\{ 7\zeta(3) + \frac{\gamma_{eff}\pi^4/4}{\pi^2/4 + \gamma_{eff} \ln(T_C/T'_C)} \right\} \quad (112)$$

For small γ_{eff} expression (111) coincides with the Aslamazov-Larkin result Eq. (11). For large γ_{eff} it follows from (112) that $I_C R_N$ product is enhanced with the increase of the T'_C .

9.7.8 3. Limit of low temperatures

If $T \ll T'_C$ and $\gamma_{eff} \gg T_C/T'_C$, the maximum value of the supercurrent flowing through the boundaries $I \sim \sigma_{s'}\Delta_1\Delta_2/\xi_{s'}\gamma_B$ is much smaller than the depairing current in the interlayer material. In this regime the structure is equivalent to two junctions in series and the CPR is given by

$$I_S(\varphi)R_N = 2\pi T \sum_{\omega \geq 0} \frac{\Delta_1\Delta_2}{\sqrt{(\omega^2 + \Delta_1^2)(\omega^2 + \Delta_2^2(T))}} \frac{\sin(\varphi)}{\eta(\varphi)} \quad (113)$$

The critical current is achieved at

$$\cos(\varphi) = \min \left\{ \frac{\gamma_{B1}}{\gamma_{B2}}, \frac{\gamma_{B2}}{\gamma_{B1}} \right\} \quad (114)$$

and equals to

$$I_C R_N = 2\pi T \sum_{\omega \geq 0} \frac{\Delta_1\Delta_2}{\sqrt{(\omega^2 + \Delta_1^2)(\omega^2 + \Delta_2^2(T))}} \frac{\gamma_{B1} + \gamma_{B2}}{\max\{\gamma_{B1}, \gamma_{B2}\}} \quad (115)$$

At $T = 0$ Eq. (113) yields

$$I_S(\varphi) = \frac{I_{C1}I_{C2}\sin(\varphi)}{\sqrt{I_{C1}^2 + I_{C2}^2 + 2I_{C1}I_{C2}\cos(\varphi)}} \quad (116)$$

where I_{C1}, I_{C2} are the critical currents of each of two consecutive Josephson junctions. For symmetric structure Eq. (116) yields the CPR $\sin(\varphi/2)$.

9.7.9 4. Limit of small γ_{eff}

As follows from Eqs. (107), (109), the direct coupling is dominating for $\gamma_{eff} \ll 1$ and the CPR is determined by

$$I_S(\varphi)R_N = 2\pi \sum_{\omega \geq 0} \frac{\Delta_1^2 \sin(\varphi)}{\sqrt{\omega^2 + \Delta_1^2} \sqrt{\omega^2 + [\Delta_1 \eta(\varphi)]^2}} \quad (117)$$

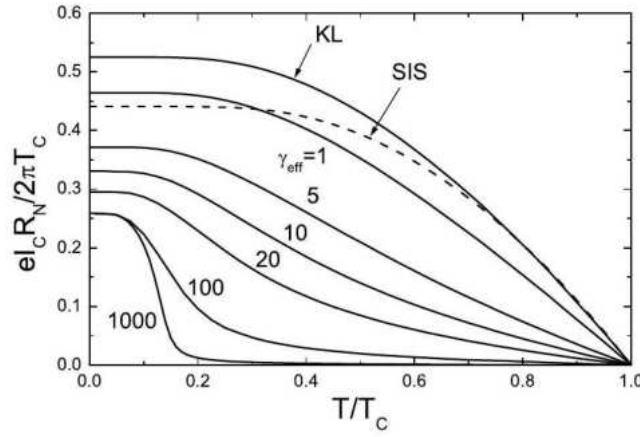


FIG. 20: Temperature dependence of critical current in the symmetric SINIS junction in the dirty limit. The KL curve is described by Eq. (101), the AB curve corresponds to the SIS junction. Adapted from Kupriyanov et al. (1999).

As soon as $\eta(\varphi) \leq 1$ at any φ , the $I_C R_N$ product of SINIS junction in this limit is larger than that for the symmetric SIS junction with the gap Δ_1 in both electrodes. At $T=0$ Eq. (117) yields

$$I_S(\varphi)R_N = \Delta(0)K\left(\sqrt{1 - \gamma_-^2} \sin \frac{\varphi}{2}\right) \sin \frac{\varphi}{2}$$

where $K(x)$ is the elliptic integral of the first kind. At $\gamma_- = 0$, critical current is achieved at $\varphi = 1.86$ with a $I_C R_N$ product 22% larger than in SIS junctions. In the symmetric case Eq. (117) coincides with Eq. (101) for the clean SINIS junction.

9.7.10 5. Limit of large γ_{eff}

In the limit of the large γ_{eff} there are three characteristic temperature intervals in $I_C(T)$ dependence.

At temperatures larger than the critical temperature of the interlayer material

$$T/T_C' \gtrsim 1 + \left[\sqrt{T_C(T_C - T_C') / (\gamma_{eff} T_C')} \right]^{2/3}$$

the CPR is sinusoidal. The general expression for the CPR was derived by Kupriyanov et al. (1999). The $I_C R_N$ product in this case is proportional to γ_{eff}^{-1} .

In the rather narrow temperature interval,
 $|T/T'_C - 1| \lesssim T/T'_C \lesssim 1 + \left[\sqrt{T_C(T_C - T'_C)} / (\gamma_{eff} T'_C) \right]^{2/3}$,
 $I_S(\varphi)$ is modified

$$\frac{I_S(\varphi) R_N}{2\pi T_C} \propto \frac{\sin(\varphi)}{(\eta(\varphi))^{2/3}} \quad (118)$$

Finally, for $T \ll T'_C$ CPR corresponds to two tunnel junctions in series and is given by Eq. (113).

Figure 20 summarizes the temperature dependencies of the $I_C R_N$ in symmetric SINIS junction in the dirty limit for various values of γ_{eff} and T'_C . The sinusoidal CPR is realized at large γ_{eff} and high temperature. The CPR for small γ_{eff} is described by Eq. (117)), at low temperatures by Eq. (113) and near T'_C by Eq. (118).

9.8 VII. SFS JOSEPHSON JUNCTIONS

There is a continuously growing interest to charge and spin transport in contacts between superconductors and ferromagnets (F). During long time, the observation of Josephson coupling in SFS junctions was a serious challenge. The solution of the problem of measuring supercurrents in SFS Josephson junctions was found by employing a dilute ferromagnetic alloys. The first experimental observation of supercurrents in SFS junctions and the crossover from 0 - to π -state in Nb/Cu_xNi_{1-x}/Nb Josephson junctions was reported by Ryazanov et al. (2001a,b) and further phase-sensitive measurements were reported by Ryazanov et al. (2002). According to Eq. (8), the π state is characterized by the phase shift of π in the ground state of junction and is formally described by the negative critical current I_C in the CPR $I_S(\varphi) = I_C \sin(\varphi)$.

The possibility of π state was first predicted by Bulaevskii et al. (1977) in a Josephson tunnel junction with magnetic impurities localized in the barrier. Bulaevskii et al. (1977) also predicted that a superconducting ring containing π -junction can generate a spontaneous current and magnetic flux.

The discovery of a π -junction by Ryazanov et al. (2001a,b) stimulated further experimental activity in this field and led to observation of new phenomena in Josephson junctions with interlayers made from various ferromagnetic alloys (Kontos et al., 2001,2002; Blum et al. 2002; Surgers et al., 2002).

At the same time, theory of SFS π junctions also attracted broad interest. Physics of 0 - to π crossover in SFS junctions was studied theoretically using different approaches by Buzdin et al. (1982,1992,2002), Buzdin and Kupriyanov (1991), Radović et al. (1991), Tanaka and Kashiwaya (1997a), Pročić et al. (1999), Dobrosavljević-Grujić et al. (2000), Fogelström (2000), Barash et al. (2002a,b), Kulić and Kulić (2001), Krivoruchko and Koshina (2001a, 2001b), Chtchelkatchev et al. (2001, 2002), Bergeret et al. (2001a,b), Radović et al. (2001), Golubov et al (2002a,b), Buzdin and Baladie (2002). In addition to π -transitions, new intriguing predictions have been made of complex CPR.

Recently, the generation of the long-range triplet order parameter was predicted in structures with inhomogeneous magnetization or with non-collinear orientations of magnetization in different F-layers (Bergeret et al., 2001c; Kadigrobov et al., 2001; Volkov et al., 2003).

Josephson effect in the junctions between unconventional superconductors across different types of magnetic barriers was studied theoretically by Tanaka and Kashiwaya (1999, 2000).

Variety of systems exhibiting π -states includes planar SFS proximity effect structures, tunnel junctions with magnetic insulator or magnetically active interfaces and structures with the barriers containing more than one magnetic layer. In this Section we will concentrate on the basic features of the Josephson effect in the junctions containing weak homogeneous metallic ferromagnets with collinear magnetization directions.

9.8.1 A. Proximity effect in SF bilayer

9.8.2 1. The formalism

Consider a bilayer consisting of semi-infinite superconductor and a ferromagnet, separated by an interface with arbitrary transparency. Superconducting correlations induced into a ferromagnet differ qualitatively from those in SN proximity systems. Generally, the proximity effect can be understood as a penetration of a Cooper pair amplitude into a non-superconducting material. An electron and a hole having opposite spins and momenta are correlated via Andreev reflection (see Fig.1) thus providing an extension of superconductivity into a nonsuperconducting region. In the NS bilayer, at nonzero temperature these correlations decay exponentially with the distance from interface to the N-metal due to dephasing between the wave functions of electrons and holes. In the SF bilayer the correlated electrons and holes, having opposite spin directions, are under the exchange field of a ferromagnet. This results in an energy shift between these quasiparticles and creation of a non zero momentum Q of Cooper pairs (Demler et al., 1997, 1998).

As a result, the amplitude of superconducting correlations oscillates spatially in the F-metal as $\cos Qx$. Sign change of this amplitude is equivalent to periodic $0 - \pi$ phase jumps at certain points in the F-metal. Such oscillating Cooper pair amplitude is an analog of the so-called Larkin-Ovchinnikov-Fulde-Ferrel (LOFF) state in magnetic superconductors (Larkin and Ovchinnikov, 1964; Fulde and Ferrel, 1964).

The oscillations in F decay with the distance from the SF interface. There is quantitative difference between clean and diffusive ferromagnets regarding the decay length. At $T = 0$, the decay length in the dirty limit exactly coincides with the oscillation period (Radović et al., 1991, Buzdin and Kupriyanov, 1991). In the clean limit the decay length is infinite $T = 0$ (Buzdin et al., 1982) and is limited only by elastic impurity scattering (Bergeret et al., 2002) or spin-orbit scattering (Demler et al., 1997) and typically exceeds the oscillation period. Therefore, the spatial oscillations are easier to observe in cleaner systems, but there is no qualitative difference between these two regimes. Having in mind that the most of the practical thin-film junctions are in the dirty limit, we will concentrate on this regime below.

Consider the SF bilayer where the dirty limit conditions are fulfilled in the S and F metals, the interface is not magnetically active and can be described by spin independent parameters γ and γ_B

$$\gamma_B = R_B / \rho_F \xi_F, \quad \gamma = \rho_S \xi_S / \rho_F \xi_F \quad (119)$$

where R_B is the specific resistance of the SF interface, $\rho_{S(F)}$ is the resistivity of the S(F) layer, while the coherence lengths are related to the diffusion constants $D_{S(F)}$ as $\xi_{S(F)} = \sqrt{D_{S(F)} / 2\pi T_C}$. This approach is valid for relatively weak ferromagnetic materials - the case we are mostly interested in, when the exchange integral is smaller than 0.1 eV and spin dependent corrections to resistivity can be neglected.

The problem is described by the Usadel equations, Eq. (178), see Appendix, where the physical interpretation of γ, γ_B is also discussed. In a ferromagnet, the Matsubara frequency ω is replaced by the complex frequency $\tilde{\omega} = \omega + iH$, where H is the exchange energy. Thus the following parametrization can be used in a ferromagnet

$$G_\omega = \frac{\tilde{\omega}}{\sqrt{\tilde{\omega}^2 + \Phi_\omega \Phi_{-\omega}^*}}, \quad F_\omega = \frac{\Phi_\omega}{\sqrt{\tilde{\omega}^2 + \Phi_\omega \Phi_{-\omega}^*}}. \quad (120)$$

The Usadel equation in F has the form

$$\xi_F^2 \frac{\pi T_C}{\tilde{\omega} G_F} \frac{\partial}{\partial x} \left[G_F^2 \frac{\partial}{\partial x} \Phi_F \right] - \Phi_F = 0 \quad (121)$$

while in the S layer $\tilde{\omega}_k = \omega$. The boundary conditions at the SF interface ($x = d_F$) are

$$\begin{aligned} \frac{\xi_S G_S^2}{\omega} \frac{\partial}{\partial x} \Phi_S &= \gamma \frac{\xi_F G_F^2}{\tilde{\omega}} \frac{\partial}{\partial x} \Phi_F, \\ \gamma_B \frac{\xi_F G_F}{\tilde{\omega}} \frac{\partial}{\partial x} \Phi_F &= G_S \left(\frac{\Phi_S}{\omega} - \frac{\Phi_F}{\tilde{\omega}} \right). \end{aligned} \quad (123)$$

One can directly see from Eq. (121) that the effective coherence length in a ferromagnet is complex. This point will be discussed in detail in the following subsection. Another interesting fact is that Φ_S near the SF interface is a complex function. This is consequence of the nonlocal nature of the proximity effect: the influence of exchange field on Cooper pairs penetrates into a superconductor at the distance ξ_S (Buzdin and Kupriyanov (1991); Krivoruchko and Koshina, 2002).

9.8.3 2. Complex coherence length

Far from the SF interface in a ferromagnet Usadel equations can be linearized ($G_F = 1$). Then it follows from Eq. (121) that the solution in a ferromagnet has a form

$$\Phi(x) \propto e^{-x/\tilde{\xi}_F} \propto e^{-x/\xi_{F1}} e^{-ix/\xi_{F2}} \quad (124)$$

with the complex coherence length $\tilde{\xi}_F$ given by

$$\tilde{\xi}_F = \sqrt{\frac{\hbar D_F}{2(\omega + iH)}} \quad (125)$$

The main consequence of the complex coherence length $\tilde{\xi}_F$ is that that $\Phi(x)$ oscillates with a period defined by ξ_{F2} and the oscillations decay on the scale of ξ_{F1} . Further, in a strong ferromagnet, when $T \ll H$, one can neglect ω in the denominator of Eq. 125 and get $\xi_{F1} = \xi_{F2} = \sqrt{\hbar D_F / H}$, i.e. in a strong diffusive ferromagnet both scales exactly coincide. On the other hand, for a weak ferromagnet the exchange field and the temperature are equally important, which yields for ξ_{F1} and ξ_{F2} (Ryazanov et al., 2001a,b):

$$\xi_{F1,2} = \sqrt{\frac{\hbar D_F}{(H^2 + (\pi T)^2)^{1/2} \pm \pi T}} \quad (126)$$

Here for simplicity the Matsubara frequency $\omega = \pi T$ for $n = 0$ was taken (exact treatment will be presented below).

For $H \gg T$ Eq. (126) yields $\xi_{F1} = \xi_{F2} = \sqrt{\hbar D_F / H}$ as encountered with classical ferromagnets (Fe, Co, Ni) where $H \sim 1\text{eV}$ is much higher than superconducting T_C .

On the other hand, in a weak ferromagnet $T \approx H$. Then the decay length ξ_{F1} increases with decreasing temperature whereas ξ_{F2} decreases. As a result, temperature variation changes the oscillation period and therefore provides the possibility to change the phase of the superconducting order parameter at a given point in F from a 0 to a π value (Heikkilä et al., 2000; Ryazanov et al., 2001a,b). This fact, together with large decay length ξ_{F1} for a small value for H (long range proximity effect), makes weak ferromagnets very interesting for experimental investigation.

Let's compare these results with those for a clean ferromagnet. In this case, as follows directly from Eilenberger equations (Buzdin et al 1982), the coherence length is also complex and $\tilde{\xi}_F = \hbar v_F / 2(\omega + iH)$ where v_F is the Fermi velocity in a ferromagnet. In a strong ferromagnet, $T \ll H$, $\tilde{\xi}_F \approx \hbar v_F / 2iH$ is purely imaginary, which corresponds to the oscillating

order parameter without decay. Detailed treatment (Volkov et al, 2002) has revealed that the order parameter decays at the scale of the mean free path $l \gg \xi_0 = \hbar v_F / 2\pi T_C$ where ξ_0 is the standard clean limit coherence length.

9.8.4 3. Phase variation at the SF interface

A jump of superconducting phase is possible at the SF interface, if relatively low-transparent barrier is present (Koshina and Krivoruchko, 2001; Golubov et al., 2002a). In the limit of small F layer thickness

$$d_F \ll \min \left(\xi_F, \sqrt{\frac{D_F}{2H}} \right) \quad (127)$$

the gradients in (121) are small and in the first approximation on d_F/ξ_F the solution of (121) has the form

$$\Phi_F = \frac{\tilde{\omega} \Phi_S G_S}{\omega (G_S + \tilde{\omega} \gamma_{BM} / \pi T_C)}, \quad \gamma_{BM} = \gamma_B \frac{d_F}{\xi_F} \quad (128)$$

Further, if the rigid boundary conditions hold ($\gamma/(1 + \gamma_{BM}) \ll 1$), $G_S = \omega / \sqrt{\omega^2 + \Delta^2}$ and one can calculate the anomalous Green's function $F_{F,\omega} = \Phi_{F,\omega} / \sqrt{\tilde{\omega}^2 + \Phi_{F\omega} \Phi_{F,-\omega}^*}$, which defines the superconducting order parameter. Function F_F is complex and the phase shift χ between F_F and the pair potential Δ in a superconductor equals to

$$\chi = \frac{1}{2} \arctan \frac{q}{p} + \frac{\pi}{4} (1 - \text{sgn } p) \text{sgn } H \quad (129)$$

where

$$p = 1 + \frac{\omega^2 - H^2}{(\pi T_C)^2} \gamma_{BM}^2 + 2 \frac{\omega^2 \gamma_{BM}}{\pi T_C \sqrt{\omega^2 + \Delta_0^2}} \quad (130)$$

$$q = 2 \gamma_{BM} \frac{H \omega}{\pi T_C} \left(\frac{\gamma_{BM}}{\pi T_C} + \frac{1}{\sqrt{\omega^2 + \Delta_0^2}} \right) \quad (131)$$

At low ω the phase shift χ monotonously increases with H achieving maximum value

$$\chi_{\max} = \frac{\pi}{2} \quad (132)$$

at $H^* \sim \pi T_C / \gamma_{BM}$. The characteristic exchange field H^* increases with increasing the interface transparency or decreasing thickness d_F . Thus the 'phase rotation' occurs at the interface between a superconductor and a ferromagnet.

9.8.5 B. CPR in SFS junctions: simple geometry

The most prominent feature of a CPR of the SFS junction is the possibility to realize the π state. Below we will discuss several mechanisms of $0 - \pi$ transitions and their quantitative criteria in terms of S, F material parameters and the SF interface resistance. Further, under certain conditions new types of nontrivial CPR can be realized, having two energy minima for $\varphi = 0$ and $\varphi = \pi$. We will discuss corresponding physical mechanisms in terms of spin splitting of ABS.

9.8.6 1. $0 - \pi$ transitions due to oscillating order parameter

We start from the simplest model of SFS junction with metallic ferromagnet: one-dimensional geometry and resistivity of a ferromagnet high enough so that the rigid boundary conditions are fulfilled at the SF interface. This model was studied by Buzdin et al. (1982), where the π states were predicted for the first time. Buzdin et al. (1982) have assumed that clean limit conditions are fulfilled in a ferromagnet. Later Buzdin et al. (1991, 1992) considered the experimentally more relevant case when both S and F metals are in the dirty limit. Strong ferromagnetism was assumed such that $H \gg \pi T_C$. At temperatures near T_C the GL equations were solved for this structure leading to sinusoidal CPR with the critical current (Buzdin et al., 1991, 1992)

$$I_C R_N = \frac{\pi \Delta^2}{4eT_C} y \frac{\sinh y \cos y + \cosh y \sin y}{\sinh^2 y \cos^2 y + \cosh^2 y \sin^2 y} \quad (133)$$

$$y = \frac{d_F}{\xi_F} \sqrt{\frac{H}{2\pi T_C}}.$$

Here d_F is the thickness of the F -layer, R_N is the resistance of the junction and Δ is the value of the pair potential in a superconductor near SF interface. Equation (133) describes damped oscillations of the critical current as a function of d_F , where the negative values of I_C correspond to a π junction. At large thicknesses critical current decays as $I_C \propto \exp(-d_F/\xi_F)$, while the oscillation period is given by $2\pi\xi_{F2}$, with $\xi_{F1,2} = \sqrt{\hbar D_F/H}$ in accordance with Eq. (126). The critical current vanishes at $y \approx 3\pi/4 + \pi n$.

At arbitrary T analytical result can be obtained in the limit of large thicknesses $y \gg 1$ and under rigid boundary conditions. CPR is sinusoidal with critical current given by (Buzdin et al., 1991, 1992)

$$I_C R_N = 32\sqrt{2} \frac{\Delta}{e} \mathcal{F}(\Delta/T) y \exp(-y) \sin(y + \pi/4) \quad (134)$$

where $\mathcal{F}(\Delta/T)$ is monotonous function of temperature, $\mathcal{F}(\Delta/T) = (\pi/128)\Delta/T_C$ at $T \approx T_C$ and $\mathcal{F}(\Delta/T) \approx 0.071$ at $T \ll T_C$. Near transition temperature the expression (134) coincides with Eq. (133). Thus, the damped oscillatory behavior of I_C vs d_F holds in whole temperature range.

Let us now consider the case of a weak ferromagnet, when thermal energy T is not negligible compared to exchange energy H and the complex coherence length ξ_F is given by Eq. (126).

If interface transparency is low ($\gamma_B \gg 1$), the Usadel equations in the F layer can be linearized and the supercurrent is given by (Ryazanov et al., 2001b)

$$I_S = \frac{4\pi T}{eR_N} \frac{d_F}{\gamma_B \xi_F} \operatorname{Re} \sum_{\omega > 0} \frac{\Delta^2 \sin(\varphi)}{(\omega^2 + \Delta^2) \tilde{d}_F \sinh \tilde{d}_F} \quad (135)$$

with $\tilde{d}_F = d_F/\tilde{\xi}_F$. Eq. (135) is valid for sufficiently high temperatures when $T/T_C \gg d_F/(\xi_F \gamma_B)$. In this regime the temperature dependence of I_C is not sensitive to the value of γ_B , while the magnitude of $I_C R_N$ product is suppressed as this parameter rises. It also follows from Eq. (135) that I_C oscillates vs d_F with the temperature-dependent period ξ_{F2} given by Eq. (126). This provides the possibility of $0 - \pi$ crossover as temperature decreases, if $H \sim \pi T_C$ and $d_F \sim \xi_{F2}$.

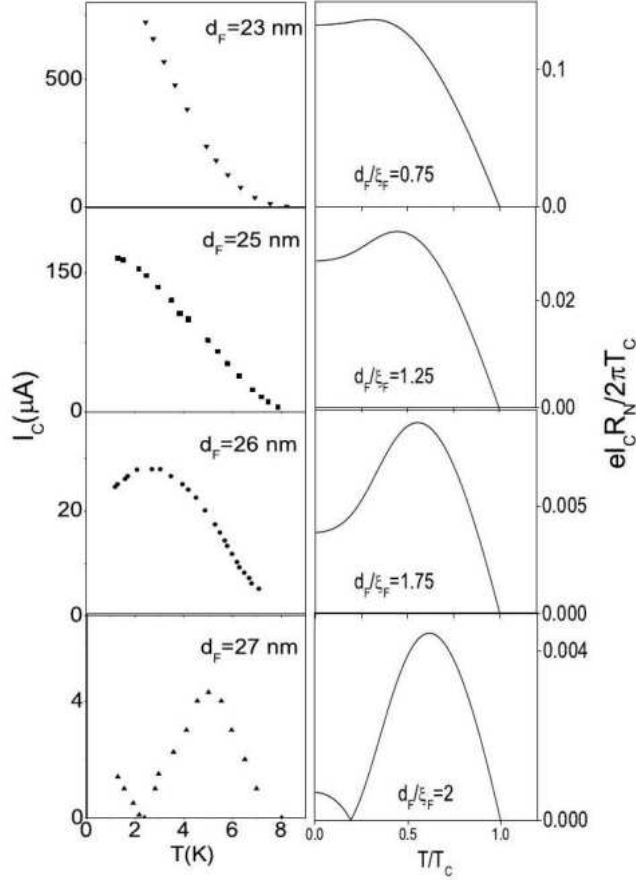


FIG. 21: Left panel: critical current I_C as a function of temperature for Nb—Cu_{0.48}Ni_{0.52}—Nb junctions with different Flayer thicknesses between 23 and 27 nm as indicated. Right panel: model calculations of the temperature dependence of I_C in a SFS junction from Eq. (135) for exchange energy $H = 0.8\pi T_C$, $\gamma_B = 10$ and different values of d_F/ξ_F ratio. Adapted from Ryazanov et al. (2001a).

Such crossover was observed by Ryazanov et al. (2001a,b) in Nb/Cu_{1-x}Ni_x/Nb junctions Josephson junctions and explained using Eq. (135). Figure 21 shows the experimental data for Nb/Cu_{1-x}Ni_x/Nb junctions for $x = 0.52$ (left panel). This crossover is reasonable well reproduced by simple calculation using Eq. (135) (right panel).

Eq. (135) is not applicable at low temperatures and nonlinear Usadel equations must be solved. The results of selfconsistent numerical calculations of critical current of SFS junction are presented in Fig. 22. Here the more general case is considered when the F-layer is splitted into two parts with parallel or antiparallel magnetization directions. I_C monotonously decreases with H for antiparallel magnetizations of the F layers and the 0π crossover takes place for the parallel case. Thus the switching of supercurrent between 0 to π states may be achieved by changing the magnetization directions of the layers. Thermally induced $0-\pi$ crossover is shown in the inset to Fig. 22 using parameters relevant for the junc-

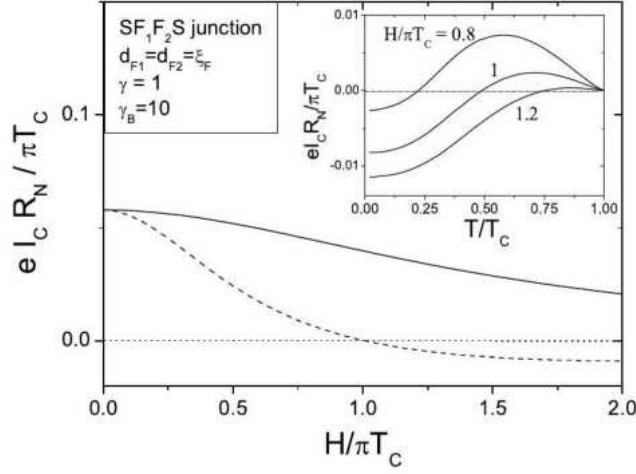


FIG. 22: Critical current in the SFS junctions calculated selfconsistently at $T/T_c = 0.5$. The solid and dashed lines correspond to the antiparallel and parallel orientations of magnetizations, respectively. Inset: thermally induced $0 - \pi$ crossover in the parallel case. Adapted from Golubov et al. (2002a).

tions studied by Ryazanov et al. (2001a,b).

Therefore, the theory based on the Usadel equations reasonably well accounts for the observed thermally induced crossover to the π state in SFS junctions. Phasesensitive proof of the π -shift below the crossover temperature in Nb/Cu_{1-x}Ni_x/Nb junctions was reported by Ryazanov et al. (2002).

9.8.7 2. $0 - \pi$ transitions due to phase jumps at the SF interfaces

As was shown above, the phase of the order parameter χ is discontinuous at the SF interface if a potential barrier exists between the two metals, see Eq. 129. Such phase jumps may lead to a π transition even if F -layer thickness is smaller than ξ_{F2} so that the order parameter in F does not oscillate (Koshina and Krivoruchko, 2001; Golubov et al., 2002a).

To illustrate this, consider tunnel junction formed by two SF bilayers (SFIFS structure) with thin diffusive F layers, $d_F \ll \xi_F$. We also assume the rigid boundary conditions hold at the both SF interfaces. In this case, the general expression for the supercurrent has the form

$$I = \frac{\pi T}{e R_N} \sum_{\omega} \text{Im} [F_{F,L}^*(-H_L) F_{F,R}(H_R)] \quad (136)$$

where $F_{F,L(R)}$ are the solutions of the Usadel equation in the left (right) F -layers (128) and R_N is the barrier resistance.

Let us consider for simplicity symmetric structure in which the SF bilayers may differ only by the magnetization directions in the F -layers and $|H_L| = |H_R|$. In this case CPR is sinusoidal, $I_S(\varphi) = I_C \sin(\varphi)$, with the critical current given by

$$I_C = \frac{\pi T}{e R_{B,I}} \sum_{\omega} \frac{|\Phi_S|^2}{\omega^2 + |\Phi_S|^2} \frac{\cos \Psi}{(p^2 + q^2)^{1/2}}$$

with $\Psi = \frac{1}{2}(\chi_R(H_R) - \chi_L(-H_L))$, where constants p, q are given by Eqs.(130),(131) and the phases $\chi_{L,R}$ - by Eq. (129).

For the antiparallel orientation of magnetizations, $H_R = -H_L$, the phase shift across the junction $\Psi_a = 0$, since the phase drops at two SF interfaces are subtracted from each other. For the parallel orientation, $H_R = H_L$, we have

$$\Psi_p = \arctan \frac{q}{p} + \frac{\pi}{2}(1 - \text{sgn } p) \quad (137)$$

In this case the phase shifts at two SF interfaces are added and provide the possibility of π -state. Such crossover happens for sufficiently large exchange fields. For $|H_{L,R}| < \pi T_C/\gamma_{BM}$, the parameter p is positive, so that $\Psi_p < \pi/2$ and the junction is always in 0-state ($I_S(\varphi) = I_C \sin(\varphi)$). However, with the increase of exchange field above the threshold value $\pi T_C/\gamma_{BM}$, the parameter p changes sign at small ω . As a result, sharp transition to a π -state occurs at $H = \pi T_C/\gamma_{BM}$ and low T .

We emphasize that F-layers are thin so that there are no spatial oscillations of the order parameter in these layers. There is no phase variation in S-layers either. Therefore, the considered scenario of $0 - \pi$ transition layer is totally related to the phase jumps at the SF interfaces. Indeed, according to Eq. (129), the phase drop χ between Usadel functions $F_S \propto \Delta_0$ and $F_F \propto \Delta_0 \exp\{i\chi\}$ at each SF interface is equal $\pi/2$ in the limit of large H , providing the total π -shift between $F_{F,L}(-H_L)$ and $F_{F,R}(H_R)$ [it is the phase difference between these two functions that determines the supercurrent according to Eq. (136)].

At $H = \pi T_C/\gamma_{BM}$ and low temperature the critical current is enhanced logarithmically $I_c^{(a)} \propto \ln(T_C/T)$ for antiparallel directions of the magnetizations. Such an enhancement of I_C was demonstrated by Bergeret et al. (2001b) in a simple model. More detailed calculations were performed by Krivoruchko and Koshina (2001), Golubov et al. (2002a), Chtchelkatchev et al. (2002) and Barash et al. (2002b). It was shown that the origin of this effect is the exchange splitting of the proximityinduced density of states and of the ABS in the F -layers, which becomes equal to Δ at $H = \pi T_C/\gamma_{BM}$.

The above results are illustrated in Fig. 23, where critical current of SFIFS is shown for parallel and antiparallel configurations. For arbitrary F-layer thicknesses and interface parameters the boundary problem (120)-(121) can be solved by using fully self-consistent numerical procedure (Golubov et al., 2002a), which includes the self-consistency over the superfluid velocity v_s , essential (contrary to the constriction case) in the quasi-one-dimensional geometry.

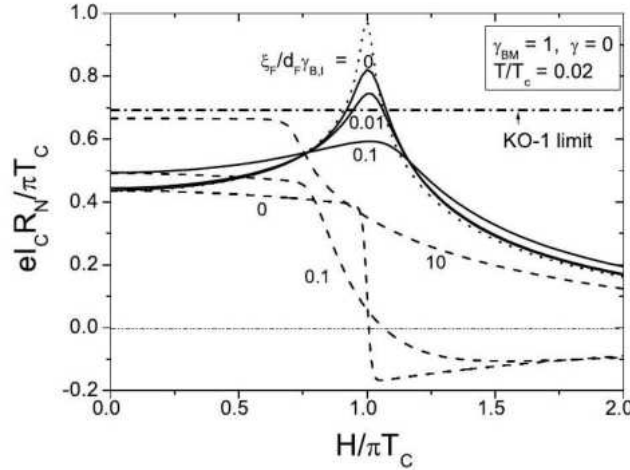


FIG. 23: Enhancement of the critical current (antiparallel magnetizations, solid lines) and the $0 - \pi$ transition at which I_c changes its sign (parallel magnetizations, dashed lines) in the SFIFS junction: influence of temperature and barrier transparency. The dotted line corresponds to $T/T_c = 0.01$ and $\xi_F/d_F\gamma_{B,I} = 0$. Adapted from Golubov et al. (2002a).

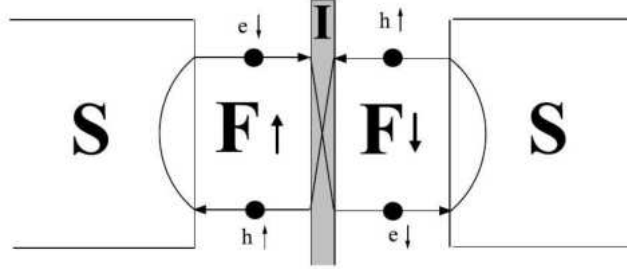


FIG. 24: Processes leading to critical current enhancement in SFIFS junction at antiparallel orientations of magnetizations at $H = \pi T_c / \gamma_{BM}$.

Fig. 23 demonstrates sharp $0 - \pi$ transition in a SFIFS junctions in the parallel case at low temperature and small barrier transparency (described by the parameter $\gamma_{B,I} \gg 1$ which characterizes the barrier I). The transition broadens with decreasing $\gamma_{B,I}$ and shifts to large values of H . The same holds for the magnitude of I_C enhancement in the antiparallel case: the peak magnitude drops with the decrease of the barrier strength, when the ratio $d_F \gamma_{B,I} / \xi_F$ becomes comparable to T / T_C .

Figure 24 illustrates the supercurrent flow in a SFIFS tunnel junction with two antiparallel orientated ferromagnetic layers. Electrons and holes shift their energies upon crossing the barrier. Due to the spin flip under Andreev reflection, the reflected hole gains exactly the same energy as lost by the electron. The corresponding resonance transmission leads to I_C enhancement in the antiparallel case (Golubov et al., 2002a; Chtchelkatchev et al., 2002).

9.8.8 C. Asymmetric case: SIFS tunnel junction

Practically interesting geometry is one-dimensional SIFS tunnel junction, where superconducting phase drops at the tunnel barrier between S and F materials. CPR is sinusoidal in this case, but supercurrent oscillates as a function of the F-layer thickness d_F . Such junction was realized recently by Kontos et al (2002), where oscillating dependence of I_C on d_F was reported.

Critical current is given by the following expression

$$I_C = \frac{2\pi T}{eR_N} \sum_{\omega} \frac{\Delta^2}{\omega^2 + \Delta^2} \text{Re} \left\{ \frac{1}{\tilde{\gamma}_B \sinh \tilde{d}_F + G_S \cosh \tilde{d}_F} \right\},$$

where $\tilde{\gamma}_B = \gamma_B \sqrt{\tilde{\omega}} / \pi T_C$, $\tilde{d}_F = d_F \sqrt{\tilde{\omega}} / \pi T_C$. This expression is valid for arbitrary thickness d_F and exchange field H assuming rigid boundary conditions at the FS interface. For large d_F this expression describes the damped oscillation behavior of I_C vs d .

In the case of a strong ferromagnet, $H \gg \pi T_C$, simple analytical expression for the critical current can be obtained. In the vicinity of the transition temperature and for highly transparent SF interface Buzdin and Baladie (2002) have obtained the critical current

$$I_C R_{B,I} = \frac{\pi \Delta^2}{2T_C} \frac{\cos y \cosh y}{\cos 2y + \cosh 2y}. \quad (139)$$

where $y = (d_F / \xi_F) \sqrt{H / 2\pi T_C}$.

For arbitrary T and low transparent SF interface, when $\gamma_B \sqrt{\frac{H}{2\pi T_C}} \gg \coth \left(\frac{d_F}{\xi_F} \sqrt{\frac{H}{2\pi T_C}} \right)$, one arrives the expression

$$I_C R_{B,I} = \frac{\pi \Delta}{4e} \frac{(\sinh y \cos y - \cosh y \sin y)}{z (\sinh^2 y + \sin^2 y)} \tanh \left(\frac{\Delta}{2T} \right) \quad (140)$$

where $z = \left(\gamma_B / \sqrt{H / 2\pi T_C} \right)$. Similar expression was used at low T by Kontos et al. (2002) to fit their experimental data on $I_C(d_F)$ in SIFS junctions and excellent agreement with theory was reported.

9.8.9 D. Clean SFcFS point contact

Consider a clean point contact connecting two superconducting SF bilayers, SFcFS constriction, may exhibit rather peculiar $I_S(\varphi)$ dependencies (Golubov et al., 2002b).

Assume that the constriction has transparency D and its size is much smaller than the mean free path, $a \ll l$. For illustration we consider the generic system when both the F -layers are thin, $d_F \ll \xi_F$, and their magnetization directions are parallel.

The supercurrent can be calculated by the general expression Eq. (20) with functions $f_{1,2}$ and $g_{1,2}$ in the F layers. Importantly, since $f_{1,2}(-\omega) \neq f_{1,2}(\omega)$, the summation over ω should be extended from $-\infty$ to ∞ .

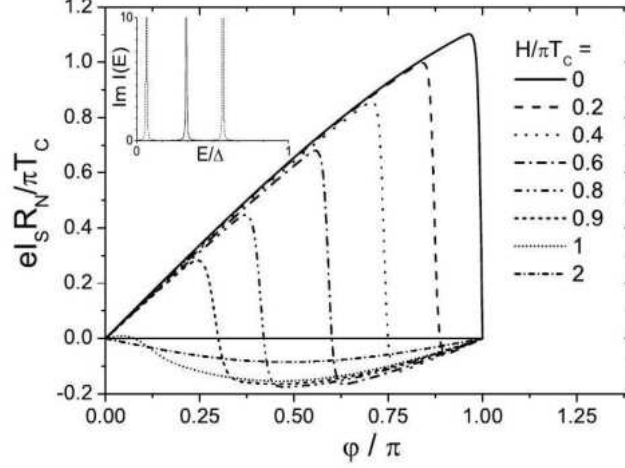


FIG. 25: Current-phase relation in clean SFcFS junction with ideally transparent constriction ($D = 1$) at $T/T_C = 0.01$, $\gamma_{BM} = 1$ for different values of the normalized exchange field H . Inset: spectral supercurrent density at $\varphi = 2\pi/3$ for $H = 0$ (solid line) and $H = 0.4$ (dashed lines). Adapted from Golubov et al. (2002b).

In the symmetric junction, $I_S(\varphi)$ is determined by the expression

$$I_S(\varphi) = \frac{4\pi T}{eR_N} \sum_{\omega > 0} \text{Re} \frac{\Delta^2 \sin \varphi}{W^2 + \Delta^2 (1 - D \sin^2(\varphi/2))} \quad (141)$$

$$W = \omega + \gamma_{BM} \tilde{\omega} \sqrt{\omega^2 + \Delta^2 / \pi T_C} \quad (142)$$

At small ω the function $I_S(\varphi)$ changes its sign at finite phase difference $\varphi_c = 2 \arcsin \sqrt{[1 - (\gamma_{BM} H / \pi T_C)^2] / D}$ if the exchange field is in the range $1 - D < (\gamma_{BM} H / \pi T_C)^2 < 1$. The resulting CPR for is shown in Fig. 25.

In order to understand this result, we note that Eq. (141) has the form similar to the expression for supercurrent in a clean ScS point contact, with renormalized frequency $\omega \rightarrow W$. Going to the spectral current $\text{Im } I(\varepsilon)$ by the analytical continuation $\omega \rightarrow -i(\varepsilon + i0)$, we see that $\text{Im } J(E) \propto \sum_i \delta(E - E_{Bi})$. The ABS energies E_{Bi} are the roots of the equation

$$E_B + \gamma_{BM} (E_B - H) \frac{\sqrt{\Delta^2 - E_B^2}}{\pi T_C} = \pm \Delta_0 \sqrt{1 - D \sin^2 \frac{\varphi}{2}}.$$

For $\gamma_{BM} = 0$ the standard result $E_B = \pm \Delta \sqrt{1 - D \sin^2(\varphi/2)}$ for the *ScS* contact is recovered, while in the presence of exchange field this bound state splits into two (see inset in Fig. 25). At $\varphi = \varphi_c$ one of these split (positive) peaks crosses zero leaving the domain $E > 0$, and simultaneously a negative peak

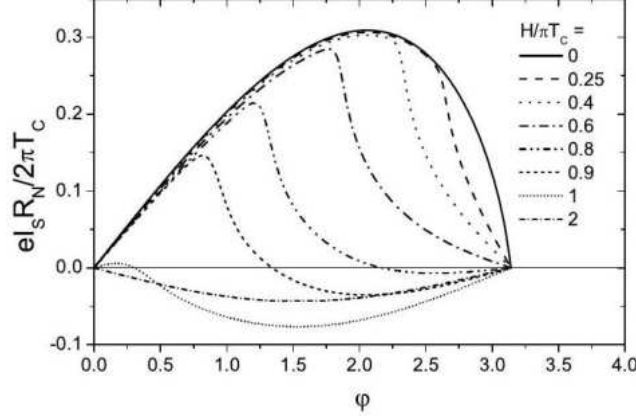


FIG. 26: The CPR in diffusive SFcFS point contact at $T/T_c = 0.01$, $\gamma_{BM} = 1$ for different values of the exchange field h . Adapted from Golubov et al. (2002b).

moves from the region $E < 0$ into $E > 0$, thus reversing the sign of the supercurrent.

The splitting of ABS in SFS junctions was first predicted by Kuplevakhskii and Fal'ko (1990, 1993) for the SFS junction either with a clean metallic ferromagnet with fully transparent interfaces or with a paramagnetic tunnel barrier. The splitting of ABS was later demonstrated in a number of various types of weak links involving ferromagnets by Tanaka and Kashiwaya (1997a), Dobrosavljević-Grujić et al. (2000), Chtchelkatchev et al. (2000), Barash and Bobkova (2000), Fogelström (2000), Radović et al. (2001), Golubov et al. (2002b).

The sign-reversal of the supercurrent in SFS junctions can also be achieved at fixed H due to nonequilibrium population of levels as discussed by Heikkilä et al. (2000) and Yip (2000).

For antiferromagnetic orientation of F layers $0 - \pi$ transition does not occur due to compensation of the phase shifts in both electrodes.

9.8.10 E. Diffusive SFcFS point contact

Now we consider the case of a diffusive point contact connecting two SF bilayers. In the symmetric structure with rigid boundary conditions at SF interfaces and thin F layers supercurrent can be derived from the Usadel equations (Golubov et al., 2002)

$$I_S(\varphi) = \frac{4\pi T}{eR_N} \operatorname{Re} \sum_{\omega > 0} \frac{\Delta \cos(\varphi/2)}{\sqrt{W^2 + \Delta^2 \cos^2(\varphi/2)}} \quad (144)$$

$$\times \arctan \left(\frac{\Delta \sin(\varphi/2)}{\sqrt{W^2 + \Delta^2 \cos^2(\varphi/2)}} \right) \quad (145)$$

where W is given by Eq. (142).

At $\gamma_{BM} = 0$, the above expression reproduces the KO1 formula for the diffusive ScS constriction, in which case

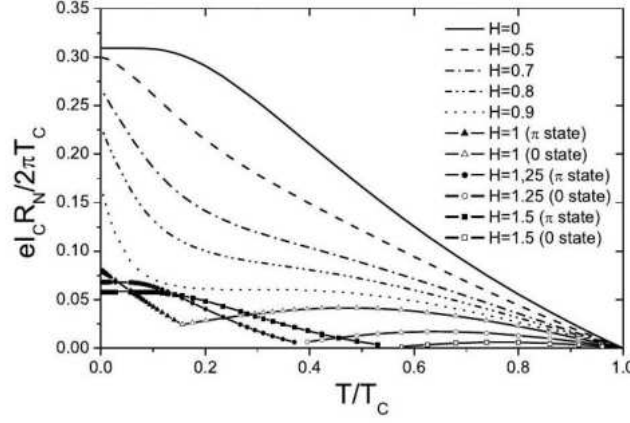


FIG. 27: Temperature dependence of the critical current in diffusive SFcFS point contact at $\gamma_{BM} = 1$ for different values of the exchange field H .

the supercurrent does not depend on the properties of the interlayer. Thus the exchange field does not influence $I_S(\varphi)$ if the SF interfaces are transparent, $\gamma_{BM} = 0$. Still, for finite γ_{BM} strong modifications of the CPR take place in the SFcFS compared to the *ScS* contact. Calculation of $I_S(\varphi)$ using the above expression yields results similar to those for the clean point contact, with less sharp transition from 0 - to π -state (see Fig. 26).

Temperature dependence of the critical current in this case shows thermally-induced 0 - π crossover with nonzero critical current at the transition point, in agreement with results of Chtchelkatchev et al. (2001). The curves $I_C(T)$ are presented Fig. 27. The nonvanishing supercurrent at the 0 - π transition point is generally realized when higher harmonics are present in $I_S(\varphi)$ (similar situation takes place in *d*-wave junctions, see Section X). Zero value of I_C at the 0 - π transition by Ryzanov et al. (2001) (see Fig. 21) is due to the fact that the low transparency regime (and hence the sinusoidal CPR) was realized in that experiment.

Note that Eq. (144) can be also derived by integration over transmission distribution $\int_0^1 \rho(D)I(D)dD$, where $I(D)$ is given by Eq. (141) and $\rho(D)$ is the Dorokhov's distribution function Eq. (19).

9.8.11 F. Double barrier SIFIS junction

Double-barrier SIFIS junction with diffusive *F* interlayer is easier for experimental implementation than a SFcFS junction, therefore we consider this case separately.

We assume that the condition $\gamma \ll \max(1, \gamma_B)$ is satisfied, then the rigid boundary conditions can be used. For the low transparent barriers, $\gamma_B \gg d_F/\xi_F$, the expression for the supercurrent is given by (Golubov et al.,

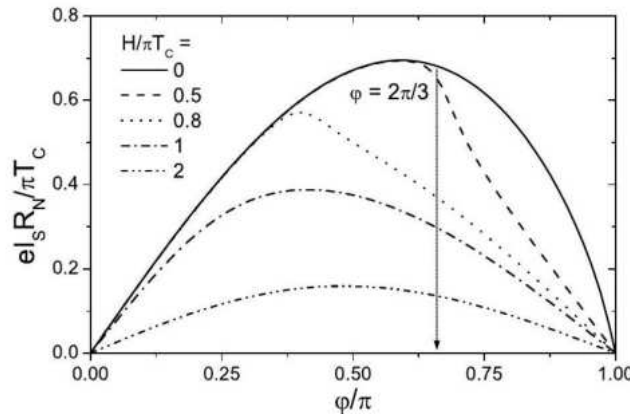


FIG. 28: The CPR in double-barrier SIFIS junction at $T/T_C = 0.02$, $\gamma_{BM} = 1$ for different values of the exchange field H . Adapted from Golubov et al. (2002b).

$$I_S(\varphi) = \frac{2\pi T}{eR_N} \operatorname{Re} \sum_{\omega>0} \frac{\Delta^2 \sin \varphi}{\sqrt{\omega^2 + \Delta^2} \sqrt{W^2 + \Delta^2 \cos^2(\varphi/2)}} \quad (146)$$

where W is given by Eq. (142). $I_S(\varphi)$ is strongly modified by finite H (see Fig. 28), especially at low temperatures. An increase of H results in the shift of the $I_S(\varphi)$ maximum from $\varphi_{\max} \approx 1.86$ at $H = 0$ to the values smaller than $\pi/2$. In the limit of large exchange fields, $H/\pi T_C \gg \gamma_{BM}^{-1}$, $I_S(\varphi)$ returns to the sinusoidal form.

The physical origin of these results can be clarified in the real energy representation. Making analytical continuation in Eq. (146), we obtain the spectral supercurrent (compare with Eq. (102) for SINIS junction)

$$\begin{aligned} \operatorname{Im} I_E &= \operatorname{Im} \frac{\Delta_0^2 \sin \varphi}{\sqrt{\Delta^2 - E^2} \sqrt{\Delta^2 \cos^2(\varphi/2) - \tilde{E}^2}} \\ \tilde{E} &= E + \gamma_{BM}(E - H)\sqrt{\Delta^2 - E^2}/\pi T_C \end{aligned} \quad (147)$$

Equation (147) implies that at $\varphi_c = 2 \arccos(\gamma_{BM}H/\pi T_C)$ singularities in $\operatorname{Im} I(E)$ are shifted to the Fermi level. At $\varphi > \varphi_c$ the negative singularity in $\operatorname{Im} I(E)$ for one spin projection crosses the Fermi level and appears in the positive energy domain, whereas the positive peak for the other projection leaves the domain $E > 0$ (this process is illustrated in Fig. 29). As a result, the supercurrent starts to decrease at $\varphi > \varphi_c$ (see Fig. 28).

Figure 30 shows the temperature dependencies of $I_C R_N$ product for various H and $\gamma_{BM} = 1$. For $H = 0$ the KL curve is reproduced with maximum $I_S(\varphi)$ at $\varphi_m \approx 1.86$. For larger H the thermal effects suppress I_C at low T due to the presence of low energy peaks of $\operatorname{Im} I(E)$. Strongest modifications of $I_C(T)$ take place for $H/\pi T_C \sim 1$. In contrast to the SFIFS case considered

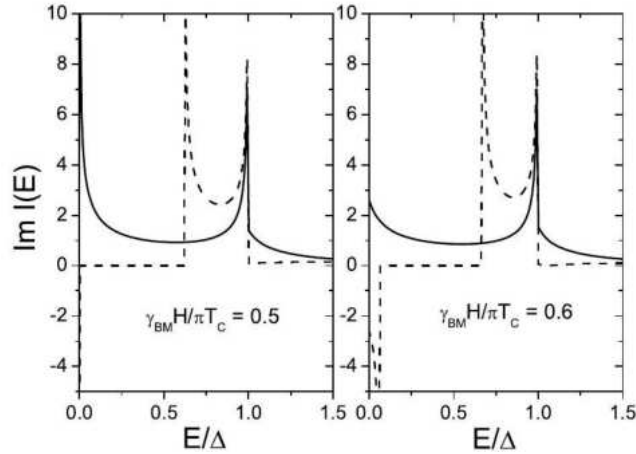


FIG. 29: Spectral supercurrent in diffusive double-barrier junction with thin ferromagnetic interlayer at $\gamma_{BM} = 1$, $\varphi = 2\pi/3$ for two different values of the exchange field H . The chosen value of φ corresponds to φ_c at $\gamma_{BM}H/\pi T_C = 0.5$. Adapted from Golubov et al. (2002b).

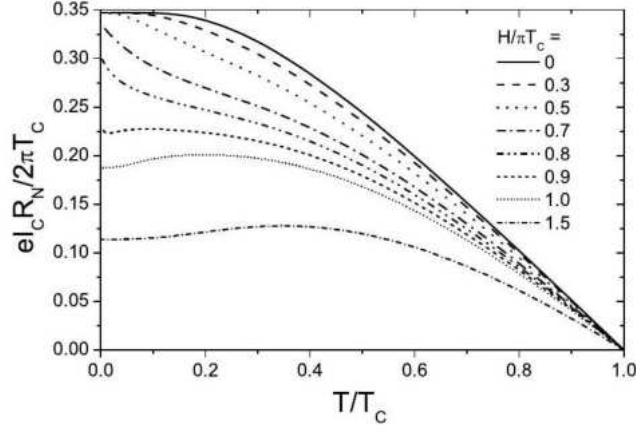


FIG. 30: The non-monotonous temperature dependence of the critical current in diffusive double-barrier junction with thin ferromagnetic interlayer at $\gamma_{BM} = 1$ and various values of the exchange field H .

above, there is no transition to the π -state in the doublebarrier SIFIS structure.

9.9 VIII. EXPERIMENTAL METHOD OF MEASURING $I_S(\varphi)$ AND ITS APPLICATION

From the above consideration we were able to find out that study of the CPR may provide the important information about the junction parameters. Below we will describe the method for the CPR measurement and some of its practical applications.

9.9.1 A. Description of the experimental method

For the experimental investigation of the CPR, most commonly, the weak link of interest is incorporated in a superconducting ring with a sufficiently small inductance L . This circuit is usually called the single junction interferometer. When an external flux Φ_e is applied to a superconducting ring, the circulating supercurrent is induced and the flux Φ_i through the ring is given by

$$\Phi_i = \Phi_e - IL \quad (148)$$

Here I is the full current in the loop. In this case the phase difference φ across the weak link is

$$\varphi = 2\pi \frac{\Phi_i}{\Phi_0} + 2\pi n \quad (149)$$

where Φ_0 is the flux quantum and n is an integer number. In general, the total current may consist of several components. We will consider below the situations when its superconducting part $I_S(\varphi) = I_C f(\varphi)$ essentially exceeds all other components, namely

$$\omega L/R_N \equiv \omega\tau \ll 1, LC\omega^2 \ll 1 \quad (150)$$

where R_N and C are the normal junction resistance and the capacitance, respectively. Under this limitation we get from Eqs.(148) and (149) that

$$\varphi = \varphi_e - \beta f(\varphi) \quad (151)$$

where $\varphi_e = 2\pi\Phi_e/\Phi_0$ is the normalized external flux and the normalized critical current β is defined as

$$\beta = 2\pi LI_c/\Phi_0 \quad (152)$$

Jackel et al. (1974, 1976) demonstrated that in this case the CPR can be obtained by monitoring φ with a flux detector as a function of φ_e .

However, this method is not suited for the investigation of samples made in thin film technology because, for instance, inhomogeneities of the magnetic field distribution inside the loop crucially affect the measurement results.

The difficulties can be resolved by the radio-frequency (*rf*) method proposed by Silver and Zimmerman (1967). The method was further developed by Rifkin and Deaver (1976) who have shown that the CPR and the phasedependent conductance can be extracted from experimental data without any fitting parameters.

In the *rf* method the interferometer and the weak link of interest are coupled through a mutual inductance M

$$M = k\sqrt{LL_T} \quad (153)$$

to a tank circuit with known inductance L_T , capacitance C and quality factor Q . Here k is the coupling coefficient between the tank circuit and the interferometer. The tank circuit is driven by a *dc* bias current I_{dc} and

an *rf* current I_{rf} of a frequency ω close to the resonant frequency $\omega_0 = 1/\sqrt{L_T C}$ of the tank circuit. Usually in the experiment $Q \gg 1$ and $k \ll 1$ are chosen and the *rf* current in the tank circuit is sinusoidal. The total magnetic flux applied to the interferometer is $\Phi_e = \Phi_{dc} + \Phi_{rf}$. If the amplitude of Φ_{rf} is small, $\Phi_{rf} \ll \Phi_0$, then $\varphi_e \cong \varphi_{dc}$ and the CPR can be determined by monitoring the effective impedance of the tank circuit-interferometer combination as a function of the externally applied magnetic flux.

Rifkin and Deaver (1976) and Il'ichev et al. (2001a,c) have shown that if in addition to restrictions given by Eq. (150) the condition of small signal limit $\varphi_{rf} \ll 1$ is valid, then back influence of the superconducting loop on the tank circuit can be expressed via the effective tank circuit inductance L_{eff} and the resistance R_{eff}

$$L_{eff} = L_T \left(1 - \frac{k^2 \beta f'(\varphi)}{1 + \beta f'(\varphi)} \right) \quad (154)$$

$$R_{eff} = R_T \left(1 + \frac{k^2 Q \omega \tau}{(1 + \beta f'(\varphi))^2} \frac{\omega}{\omega_0} \right). \quad (155)$$

Here, R_T determines the damping of the *rf* oscillations when the interferometer is decoupled. The phase angle α between the drive current I_{rf} and the tank voltage U is given by

$$\tan \alpha = \frac{1}{R_{eff}(\varphi_e)} \cdot \left(\frac{1}{\omega C_T} - \omega L_{eff}(\varphi_e) \right) \quad (156)$$

It follows from Eqs. (154), (155) and (156) that at $\omega = \omega_0$ and $\omega_0 \tau \ll 1$ the phase angle is

$$\tan \alpha = \frac{k^2 Q \beta f'(\varphi)}{1 + \beta f'(\varphi)} \quad (157)$$

Futhermore, in the limits $\varphi_{rf} \ll 1$ and $\omega \tau \ll 1$ Eq. (151) can be rewritten as

$$d\varphi = \frac{d\varphi_{dc}}{1 + \beta f'(\varphi)} \quad (158)$$

Combining Eq. (157) and Eq. (158) one can easily obtain:

$$I_S(\varphi_{dc}) = \frac{I_c}{k^2 Q \beta} \int_0^{\varphi_{dc}} \tan \alpha(\varphi_{dc}) d\varphi_{dc} \quad (159)$$

The numerical factor in Eq. (159) can be obtained from the experiment. Indeed, the mutual inductance M is extracted from the period ΔI_{dc} of the $\alpha - I_{dc}$ dependence according to $M\Delta I_{dc} = \Phi_0$. Combining Eq. (152) and Eq. (153) this factor can be rewritten as

$$\frac{I_c}{k^2 Q \beta} = \frac{L_T (\Delta I_{dc})^2}{2\pi Q \Phi_0} \quad (160)$$

The inductance L_T can be measured in a conventional setup. The quality factor Q can be determined from the

resonant response curve of the tank circuit. Thus all the parameters in Eq. (159) are known. Note that for the analysis of the experimental data no fitting parameter is needed.

If the parameter $\beta f'(\varphi)$ in Eq. (158) is small, then the difference between φ and φ_{dc} is also negligible, $\varphi \simeq \varphi_e$, and Eq. (156) is simplified to

$$\tan \alpha = k^2 Q \beta f'(\varphi), \beta f'(\varphi) \ll 1 \quad (161)$$

Moreover, since in experiments the typical value of $k^2 Q \lesssim 1$, then $\tan \alpha \simeq \alpha$ and the measured dependence $\alpha(\varphi_{dc})$ is proportional to the derivative of the CPR $f'(\varphi)$ with respect to the phase difference across the junction of interest.

9.9.2 B. The range of validity of the method

Let us now briefly discuss physical limitations on the key parameters. First of all it is important to note that in order to reconstruct the CPR in the complete phase range $-\pi < \varphi < \pi$, the inequality

$$|\beta f'(\varphi)| < 1 \quad (162)$$

should be fulfilled. Otherwise the CPR becomes multivalued in the vicinity of $\varphi_e = \pi + 2\pi n$ (see Eq. (151)) and certain values of φ could not be fixed by applying the external magnetic flux.

For a given critical current of the Josephson element of interest, an upper limit for the loop inductance L is set by inequality (162). For moderate critical currents of the order of 20μ A, the inductance $L \simeq 10$ pH, which is close to practical geometrical limitations on L .

On the other hand, if the critical current is much smaller, say of the order of 10 nA, inequality (162) does not limit the loop inductance. Instead the fluctuation inductance L_f restricts the value of L . That is why measurements of very small supercurrents, which become accessible by the method described here, are typically realized within the limit $\beta \ll 1$.

The achievable current resolution can be estimated using the model developed by Kurkijärvi (1973) for $r f$ SQUIDs which predicts

$$\langle \delta \Phi_N^2 \rangle^{1/2} = \frac{1}{2\beta} \sqrt{\frac{T_e L}{k^2 Q \omega_0}} \Delta \omega \quad (163)$$

Here $\langle \delta \Phi_N^2 \rangle^{1/2}$ is the equivalent flux noise in the interferometer loop as found in the bandwidth $\Delta \omega$ at a given effective temperature T_e characterizing the thermal fluctuations in the tank circuit. The relevant bandwidth depends on the resonant response curve of the tank circuit and on the bandwidth of the phase detector used in experiment. If $\langle \delta \Phi_N^2 \rangle^{1/2} \approx \Phi_0/2$ the output signal is masked by the noise.

Il'ichev et al. (2001a, c) have also shown that the condition $\beta \ll 1$ can be achieved by the reducing the critical current even to the limit when the Josephson coupling energy $E_J = I_C \Phi_0 / 2\pi$ becomes smaller than the thermal energy $k_B T$. According to Khlus and Kulik (1975), an exponential reduction of the observable critical current with temperature occurs in this limit

$$I_C \rightarrow I_C \exp(-L/2L_f) \quad (164)$$

where

$$L_f = (\Phi_0 / 2\pi)^2 / k_B T \quad (165)$$

is the so-called fluctuation-threshold inductance. At $T = 4.2$ K the value of $L_f \approx 1.7$ nH exceeds the geometrical inductance limit (10 pH) by several orders of magnitude. Therefore required interferometer inductance $L \ll L_f$, permitting to neglect the reduction of I_C can be easily realized experimentally.

If $L \sim L_f$, thermal noise modifies the internal flux in the ring and therefore the measured (apparent) CPR is also modified compared to the real CPR (see Jackel, et al., 1974; de Bruyn Ouboter and Omelyanchouk, 1995; Il'ichev et al., 1998b, 2000a). For instance, in a junction with the sinusoidal CPR, the apparent CPR is deformed in such way that its maximum shifts to phase $\varphi_{\max} < \pi/2$, and the slope of the CPR curve at $\varphi = 0$ becomes larger than unity (see de Bruyn Ouboter and Omelyanchouk, 1995; Il'ichev et al., 1998b, 2000a). Nevertheless, Il'ichev et al. (2000a) have shown that the real CPR can be reconstructed from the data, thus the method is still applicable even for the case $L \sim L_f$.

Therefore, if the inequality (162) is fulfilled, then quite generally the CPR can be extracted from the data in the complete phase range by using Eqs. (151), (159) without any fitting parameters. This method, being differential with respect to φ , provides a high sensitivity of the CPR measurement. In practice, however, this procedure works perfectly well when $\beta f'(\varphi)$ is not close to unity. Indeed, even small thermal fluctuations and the applied rf flux φ_{rf} can distort the $\varphi(\varphi_e)$ dependence and hence the apparent CPR, because for $\beta f'(\varphi) = 1$ the derivation $d\varphi/d\varphi_e$ diverges (see Eq. (158)).

9.9.3 C. Measuring technique and calibration

Several parameters of the measurement setup have been tested and applied for the CPR study. The inductance of tank circuits was in the range of $10^{-2} - 1 \mu\text{H}$ providing the resonant frequencies of 5–40 MHz. The unloaded quality factor measured for all tank circuits at different temperatures was in the range of $10^2 < Q < 10^3$. Usually the inductive coupling between an interferometer and a tank circuit was provided by the flip-chip configuration. In some particular cases the interferometer of interest was integrated with the coil on the same chip.

The interferometer inductances were in the range of 10 – 100 pH providing $L \ll L_f$ for liquid helium temperatures, where the deviation of the CPR from conventional sinusoidal form is expected. At earlier studies the

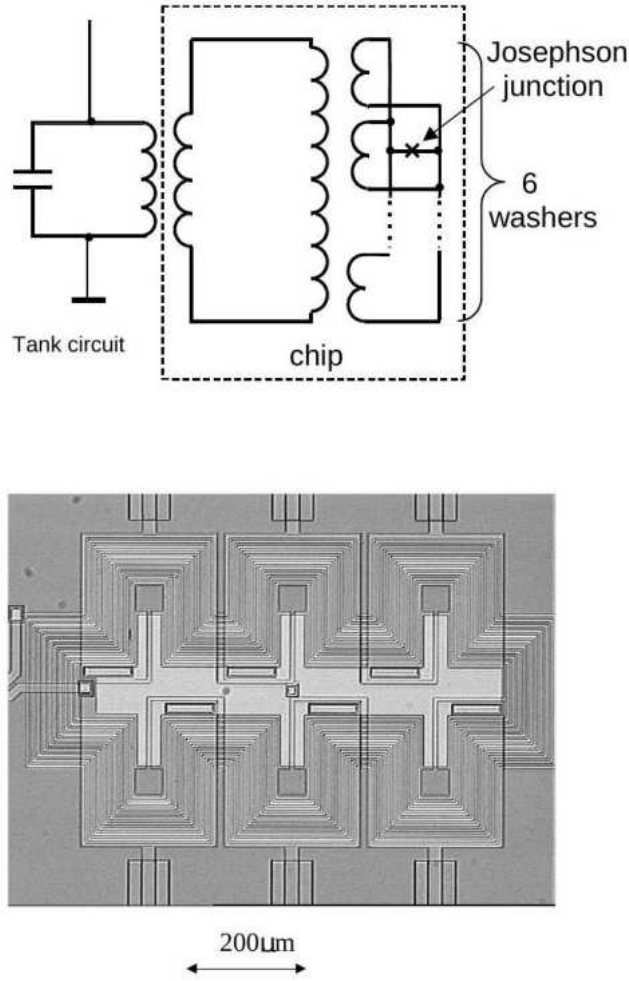


FIG. 31: Equivalent circuit and coupling scheme for the sample with Nb tunnel junction (top), and foto of the six-washer structure (bottom). After Il'ichev et al. (2001c).

tank voltage was detected by a room-temperature amplifier with high input impedance. In order to increase the sensitivity of the setup Oukhanski et al. (2003) recently developed the cold amplifier with low noise temperature, which operates down to 300 mK. To satisfy the small signal limit, the measurements were carried out with amplitude of I_{rf} providing the corresponding flux of $(10^{-3} - 10^{-1}) \Phi_0$ in the interferometer loop.

At present, several setups for the CPR measurements are available which work in the temperature range from T_C down to 10 mK. External magnetic field is shielded by a magnetic shield or by a combination of magnetic and superconducting shields.

From the periodicity of $I_S(\varphi)$ it follows that $\int_0^\pi \tan \alpha(\varphi_{dc}) d\varphi_{dc} = 0$. This condition enables a subtraction of a constant phase shift coming from electronics and cables. Since the only sample characteristic entering Eq. (159) is $I_S(\varphi)$, the measurement system can be calibrated using samples with known critical current and $I_S(\varphi)$. For this purpose Nb interferometers with conventional Nb/ AlO_x /Nb tunnel junctions are used for the calibration.

As follows from the inequality (162), in order to obtain the CPR in the complete phase range for the $f(\varphi) = \sin \varphi$ the regime $\beta < 1$ is required. For a junction with critical current of the order of $20 \mu\text{A}$ this leads to the restriction on the interferometer inductance of $L < 16 \text{pH}$. Such

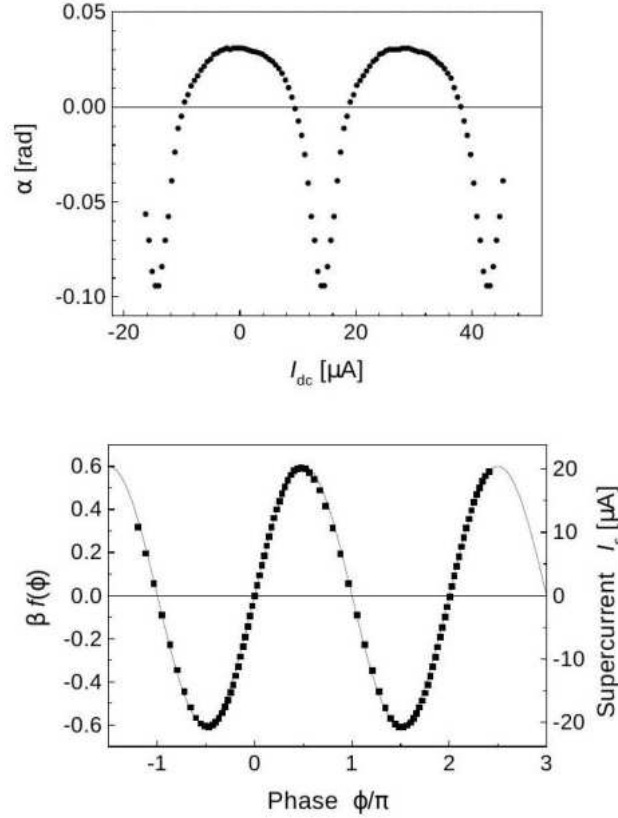


FIG. 32: Phase angle between the driving current and the output voltage as a function dc current I_{dc} (top), and the current-phase relationship, determined from the upper curve (bottom). After Il'ichev et al. (2001c).

small value of L limits the size of the interferometer loop to be below $10\mu\text{ m} \times 10\mu\text{ m}$. For conventional tank coils with diameter of about 1 mm it is difficult to get reasonable M values between a tank circuit and an interferometer for flip-chip arrangement. Moreover, the external magnetic field, producing one flux quanta in the interferometer loop, simultaneously produces the finite value of the magnetic flux in the junction (the size of the tunnel junction is $3.5\mu\text{ m} \times 3.5\mu\text{ m}$), suppressing the critical current value due to $I_C(H)$ dependence (E. Il'ichev et al., 2000b). In order to avoid these difficulties, the value of L was decreased by using the series of samples with the interferometer loop consisting of 6 washers, each having an inductance of 55 pH, and all connected in parallel. The 10 -turns coupling coils were placed on each washer and connected in series (see Fig. 31). The coupling coil was connected to a large 5 mm $\times$ 5 mm loop integrated on the chip. In this configuration the total inductance of the interferometer is evaluated to be 9 pH.

An example of the experimental data obtained for the structure presented in Fig. 31, is shown in Fig. 32. From the measured $\alpha(I_{dc})$ dependence (see Fig. 32, upper curve), the corresponding CPR was reconstructed (see Fig. 32, bottom curve). As expected, the sinusoidal CPR was obtained and the critical current was in a good agreement with I_C of a similar shunted tunnel junction from the same chip obtained by standard transport measurements.

9.9.4 D. Measurements of small critical currents

The properties of supercurrent in submicron-size Josephson junctions are of considerable fundamental and practical interest. Josephson junction circuits have been recently proposed as qubits (the elementary units in quantum information processing) based either on the flux (Bosco et al., 1997; Mooij et al., 1999; Ioffe et al., 1999) or charge (Averin, 1998; Makhlin et al., 1999) degrees of freedom. Such circuits have been recently successfully tested by Nakamura et al. (1999), Friedman et al. (2000), van de Wal et al. (2000), Vion et al. (2002), Yu et al.

(2000) and Wallraff et al. (2003). Further practical realization of quantum networks requires comprehensive investigation of quantum coherence in small Josephson junctions.

In SIS tunnel junctions with sub-micron dimensions the capacitance is reduced to less than 100fF, I_C is typically below $1\mu\text{ A}$, while the normal state resistance R_N is of the order of a few $k\Omega$. Under these parameters the Josephson coupling energy is smaller than or comparable to the thermal energy $k_B T$, and it becomes difficult to distinguish the critical current from the switching current on the $I-V$ curve. The routine solution requires drastically increased cooling efforts as well as careful electromagnetic filtering (Vion et al., 1995).

Il'ichev et al. (2001a,c) demonstrated that the CPR measurements are much less sensitive to thermal fluctuations. In this work, in particular, double-junction samples consisting of the two SIS contacts in series were investigated. The values of critical currents were chosen in the range $E_J/k_B T = 0.3 \cdots 20$. In this case the thermal noise substantially affects the switching current measured by standard $I-V$ method.

For these samples, the experimentally measured CPR have the shape given by Eq. (116). For highly asymmetric double junction the shape was close to $\sin\varphi$ as expected. The most remarkable result of this experiment is the demonstration of the ability to resolve the CPR even though the thermal energy exceeded the Josephson coupling energy by a factor of 3.3. Further, in accordance with Eq. (116), the CPR in the symmetric case was far from $\sin\varphi$ and was fitted well by $\kappa \approx 0.4$. From this fitting parameter and the overall critical current of the sample, the individual critical currents $1.1\mu\text{ A}$ and $2.8\mu\text{ A}$ were found.

Besides the high resolution, another promising feature of the rf -readout technique was demonstrated, namely large range of supercurrents that can be investigated. At the present level, it is possible to reconstruct the CPR from samples characterized by $0.005 < \beta < 1$, i.e. for fixed loop inductance the CPR can be studied over more than two orders of magnitude range of critical currents.

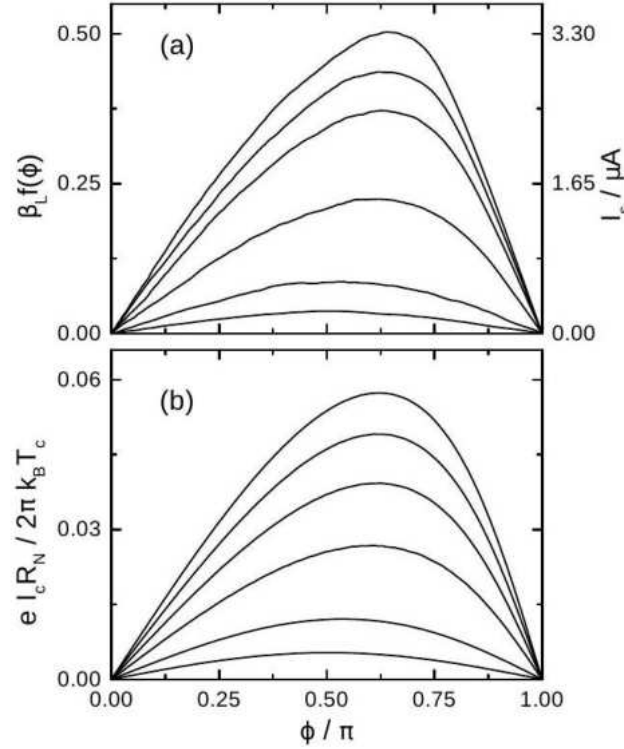


FIG. 33: The CPR of the SINIS junction in the temperature range 1.90 K – 1.95 K (from top to bottom increasing in steps of 0.01 K). The experimental results are presented in (a), the calculated curves are shown in (b). After Götze et al. (2000).

9.9.5 E. Measurement of asymmetry of barrier transparency in SINIS junctions

Cassel et al. (2001) and Tolpygo et al. (2003) have recently shown that one of the hidden difficulties for implementation of SINIS double barrier junction technology is the asymmetry in the barrier transparency of interfaces.

For the junctions with relatively small critical current density, e.g. for Josephson voltage standards, intrinsically shunted junctions with low asymmetry were fabricated (see Maezawa and Shiji 1997; Sugiyama et al., 1997, 1999; Balashov et al., 1998, 1999, 2000, 2001; Schulze et al., 1998, 1999, 2000; Behr et al., 1999; Khabipov et al., 1999, 2002a,b; Cassel et al., 2001; Buchholz et al., 1999, 2001; Kohlmann et al., 2001, 2002; Kieler et al., 2002).

However, for digital applications the junctions with critical current density in the range $1 \nabla \cdot 10 \text{ kA/cm}^2$ are required. This may be achieved by shorter oxidation of both Al layers in a NbAl/ AlO_x /Al/ AlO_x /Nb structure. The inherently different deposition conditions for the first and second Al films in the multilayer fabrication result in different morphology of the two Al films. Therefore oxidation performed under the same oxidation conditions should lead to the formation of the AlO_x barriers having different transparency (see Cassel et al., 2001; Tolpygo et al., 2003). The barrier asymmetry reduces the suppressed $I_C R_N$ product of SINIS junctions and strongly limits their applicability in digital circuits.

Cassel et al. (2001) have shown that among different methods for monitoring the asymmetry, the CPR measurement should provide the most accurate results. In accordance with Eqs. (107), (109) the CPR in SINIS

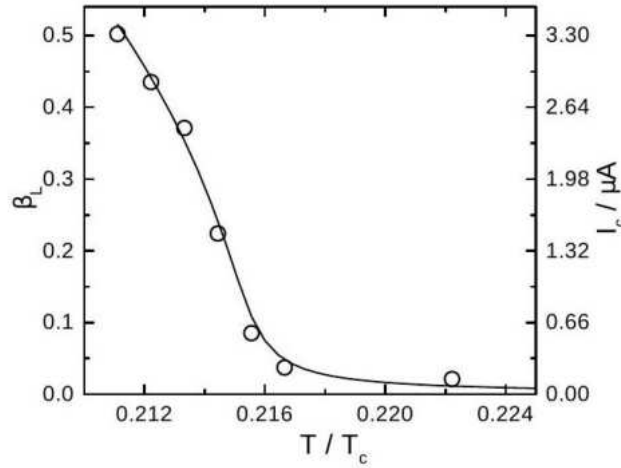


FIG. 34: The fit of the temperature dependence of the critical current. The solid line represents the calculation and symbols are the experimental data. After Götz et al. (2000). junctions depends on asymmetry parameter γ_- and the ratio T'_C/T , where T'_C is the critical temperature of Al interlayer. At $T > T'_C$ Al is in the normal state and $I_S(\varphi)$ is close to $\sin \varphi$ if γ_{eff} is large. At $T < T'_C$ the Al is in superconducting state, the sequential tunnelling dominates the supercurrent flow and the CPR transforms to that given by Eq. (116).

Götz et al. (2000) confirmed the above considerations by measuring the CPR in SINIS junctions, fitting the data by curves calculated from Eqs. (107), (109). The set of measured CPR is presented in Fig. 33(a). The corresponding theoretical curves are shown in Fig. 33(b). As it was reported in detail by Götz et al. (2000) the CPR shown in Fig. 33(a) as well as $I_C(T)$ dependence shown in Fig. 34 can be fitted with the single set of parameters. This procedure allows one to extract all the the junction characteristics for this series of samples, namely $T'_C \approx 1.935$, $\gamma_{eff} \approx 10^5$, $R_N \approx 83 \Omega$ and $\gamma_- \approx 0.45$.

Therefore, the developed method opens the way for calibration of the second oxidation process in SINIS junction technology based on accurate measurement of the barrier asymmetry parameter (γ_-).

9.9.6 F. CPR in hybrid S-2DEG-S junctions

For low temperature electronics it is quite natural to combine modern superconducting technology with semiconducting one. One of the practical realization of such contacts is to couple two superconducting electrodes by the 2DEG (Marsh et al., 1994; Takayanagi et al., 1995a; Chrestin et al., 1997, 1999; Lehnert et al., 1999; Neurohr et al., 1999; Heida et al., 1999; Grajcar et al., 2002a and Schäpers et al., 2003).

For practical applications it is important to estimate the degree of the critical current limitation caused by the S-2DEG interfaces. Grajcar et al. (2002a) have demonstrated that the CPR measurements may provide this information.

Grajcar et al. (2002) studied the samples consisting of

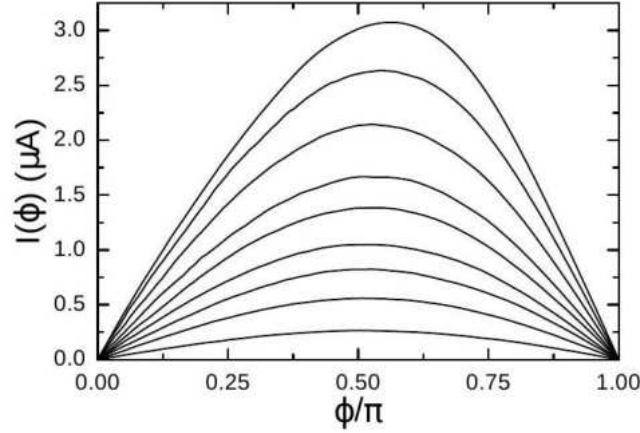


FIG. 35: The current-phase relation a Nb/ InAs(2DEG)/Nb Josephson junction. The curves from top to bottom correspond to $T = 2.75, 3, 3.25, 3.5, 3.75, 4, 4.2, 4.5, 5$ K. After Grajcar et al. (2002).

two Nb electrodes coupled by the 2DEG formed in the native inversion layer at the surface of p-type bulk InAs (see Chrestin et al., 1997 for details). The electron density and the Fermi velocity of the 2DEG in the channel can be adjusted via field-effect using a gate electrode. At a typical electron density of $n_s = 1.2 \cdot 10^{12} \text{ cm}^{-2}$ and mobility $\mu \approx 10000 \text{ cm}^2/\text{Vs}$, the electron mean free path is $\ell \approx 240 \text{ nm}$ is comparable to the coherence length $\xi_2 \approx 145 \text{ nm}$ at 1.8 K and the channel length $d \approx 120 \text{ nm}$. Thus the studied quasi-ballistic junction was in the intermediate regime between a short and a long weak link regimes. In addition, thin InAs layer underneath the Nb electrodes was proximized, resulting in formation of a SIN-2DEG-NIS junction.

To describe theoretically the Josephson effect in such a structure, more general expression compared to that discussed in Sections III and IV is needed. It should take into account both scattering of electrons inside 2DEG and proximity effect in the NS electrodes. The expression for the supercurrent was derived by Grajcar et al. (2002)

$$\frac{eIR_0}{2\pi T_c} = \frac{8T}{T_c} \sum_{n=0}^{N-1} \sum_{\omega} \frac{-a^2 |S_{12}|^2 \sin(\varphi)}{1 - 2a^2 (|S_{12}|^2 \cos(\varphi) + |S_{11}|^2) + a^4 |S_{11}^2 + S_{12}^2|^{(166)}}$$

and combines the result of Brouwer and Beenakker (1997) for $I_S(\varphi)$ in chaotic Josephson junction with the constriction model developed by Aminov et al. (1996) and Golubov et al. (1997a,b). Here $R_0 = h/e^2$ is the resistance quantum, a is the amplitude of the Andreev reflection at the InAs(2DEG)/ InAsNb NIS interface and S_{ij} are the elements of the scattering matrix for electron propagation across the structure and N is the number of open transmission channels.

Experimental CPR are shown in Fig. 35. A gradual deviations from sinusoidal behavior with decreasing temperature are clearly seen. Reasonable agreement between experimental and theoretical data was obtained (for details see Grajcar et al. (2002)).

We should point out that a relatively large inductance (80pH) of interferometers was used in this set of experiments. In this case the inequality (162) requires that the current should be smaller than 4μ A. Usually the critical current of the Nb/InAs(2DEG)/Nb Josephson junctions exceeds this value considerably. The junctions measured by Crajcar et al. (2002) had reduced critical current and lower interface transparency. Still, the shift of the maximum of the CPR towards π was observed. The investigation of the CPR of junctions with higher critical current requires a more sophisticated technique.

9.9.7 G. CPR anomalies in HTS junctions

9.9.8 1. d-wave effects in HTS

The most important phenomenological difference between the HTS cuprates and conventional superconductors regards the orbital symmetry of the superconducting order parameter. In HTS the pair potential changes sign depending on the direction in momentum space according to $\Delta(\theta) = \Delta_0 \cos(\theta - \alpha)$, where θ and α are the angles between the x -axis and the wave vector and the Cu – Cu bond direction of the superconductor, respectively. As a result, the π - shift of the superconducting phase occurs between the a and b crystallographic directions. The unconventional symmetry in cuprates was experimentally confirmed by phase-sensitive experiments (see van Harlingen, 1995; Tsuei and Kirtley, 2000).

Several comprehensive review articles exist on the Josephson effect in HTS junctions, both on fundamental aspects and applications, see Sigrist and Rice (1995), van Harlingen (1995), Delin and Kleinsasser (1996), Tsuei and Kirtley (2000), Kashiwaya and Tanaka (2000), Hilgenkamp and Mannhart (2002). Theory of the effects of d-wave symmetry on CPR was discussed in detail in review papers by Kashiwaya and Tanaka (2000) and Löfwander et al. (2001).

There are several important consequences of an unconventional order parameter symmetry in HTS junctions. First, in HTS electrodes the superconducting state in the vicinity of interfaces differs essentially from that in the bulk. Due to the angular dependence of the pair potential in HTS incoming and reflected quasiparticles at certain trajectories through the junction probe a sign change of the pair potential upon reflection. This sign change has fundamental consequences - it leads to the appearance of robust mid-gap states (MGS) at zero energy (Hu, 1994). The MGS do not occur at the sign-conserving trajectories, still in both cases the interface is pair-breaking and the self-consistently determined magnitude of the pair potential near the interface is reduced in comparison with its bulk value. The nucleation of an order parameter with different symmetry is possible as well.

Different types of HTS Josephson junctions were fabricated, including ramp-type junctions (Gao et al., 1990), grain-boundary junctions (see reviews by Tsuei and Kirtley, 2000; Hilgenkamp and Mannhart, 2002), intrinsic junctions (Kleiner and Müller, 1994; Krasnov et al., 2000, 2001; Latyshev et al., 2001), HTS/LTS junctions (van Harlingen, 1995; Smilde et al., 2002) and SNS-type junctions (Delin and Kleinsasser, 1996).

Physics of Josephson effect in HTS junctions is strongly influenced by a number of physical processes. They are (a) robust mid-gap states at zero energy, which may contribute to the supercurrent (see Hu, 1994; Tanaka and Kashiwaya, 1995, 1996; Barash et al., 1996; Fogelström et al., 1997; Riedel and Bagwell, 1998; Barash 2000); (b) bound states with finite energy due to suppression of the pair potential at the interface (see Barash et al., 1997); (c) the subdominant order parameter with different symmetry compared to that in the bulk (see Fogelström et

al., 1997); (d) gapless singlet superconductivity nucleated at rough interfaces (see Golubov and Kupriyanov, 1998a, 1999a,b); (f) spontaneous half-flux vortices (Mannhart et al., 1996; Tsuei and Kirtley, 2000; Hilgenkamp et al., 2003), which may have peculiar static and dynamic properties (Goldobin et al., 2002).

The possibility of broken time-reversal symmetry in cuprates, when the order parameter in the bulk has an imaginary subdominant component, was suggested by Sigrist and Rice (1995). The consequences for HTS junctions were discussed and some experimental findings have been interpreted as realizations of this state (for references see Tsuei and Kirtley, 2000; Tanaka and Kashiwaya, 2000; Löfwander et al., 2001). Further, it was predicted by Löfwander et al. (2000) and Amin et al. (2001) that spontaneous time-reversal symmetry breaking at HTS junctions may occur even in the absence of subdominant components in the bulk. Note that according to Geshkenbein et al. (1986, 1987), the symmetry property of the CPR, Eq. (4), should be violated when the time-reversal symmetry is broken.

So far the experimental data, including the CPR measurements discussed below, could not unambiguously resolve the issue of time-reversal symmetry breaking in HTS. Further study is needed to realize the full potential of the CPR measuring technique in this respect.

Here we will concentrate on the experimental measurements of the CPR in HTS junctions, where the anomalies due to pure d -wave symmetry were revealed by CPR measurements: (1) detection of the strong second harmonic in the CPR, (2) manifestation of the MGS and (3) CPR anomalies in c -axis tunneling.

9.9.9 2. Detection of the second CPR harmonic

Tanaka (1994), Yip (1995) pointed out that the Josephson coupling energy may have minima at φ of neither 0 nor π even in a junction between pure d -wave superconductors, even in the absence of the time-reversal symmetry breaking. This may occur due to strong reduction of the first harmonic I_1 in Eq. (7) so that the second harmonic I_2 may become comparable or larger than I_1 .

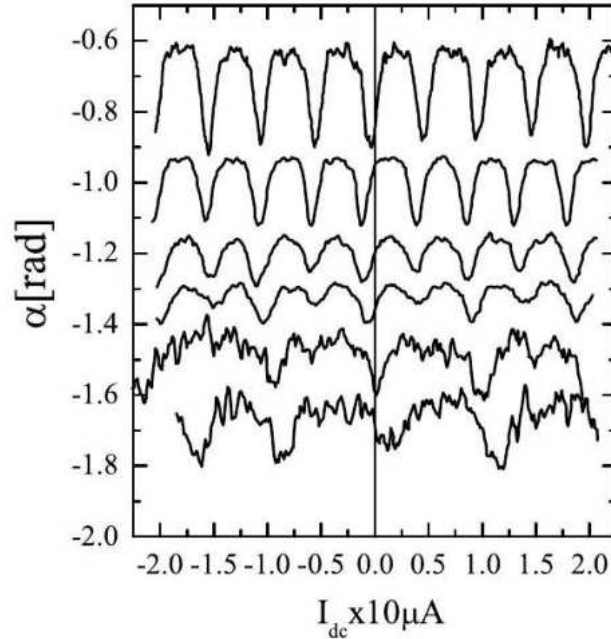


FIG. 36: Phase angle α as a function of the dc current flowing through the tank coil for the interferometer with asymmetric 45°GB junction at various temperatures. After Il'ichev et al. (1999a).

Walker and Lüttmer-Strathmann (1996) put these arguments further. Since the order parameter is bound to the crystal lattice, the supercurrent of a junction between two d -wave super-

conductors depends on the orientation of the d -wave electrodes with respect to their boundary. Let θ_1 (θ_2) denote the angle between the normal to the junction interface and the a axis in the electrode 1 (2). The symmetry of the problem dictates that (Walker and Lüttmer-Strathmann, 1996)

$$I_1 = I_S \cos 2\theta_1 \cos 2\theta_2 + I_S \sin 2\theta_1 \sin 2\theta_2 \quad (167)$$

The coefficients I_C, I_S are functions of the barrier strength and temperature. As follows from Eq. (167), the criterion for the observation of an anomalous $I_S(\varphi)$, $I_1 = 0$, is realized for an asymmetric 45° junction, i.e. a junction with $\theta_1 = 45^\circ$ and $\theta_2 = 0$. In this case one expects double-periodic $I_S(\varphi)$ dependence. Note that such type of the CPR was discussed by Ioffe et al. (1999) in their proposal for the d -wave qubit.

Before discussing the detection of the second CPR harmonic, let us mention that in practice the properties of grain-boundaries (GB) are strongly influenced by the effect of meandering (faceting) of the junction boundary (Copetti et al., 1995; Hilgenkamp et al., 1996). Within the d -wave scenario, the faceting has a more significant influence on the electronic properties of boundaries with a misorientation close to 45° than on boundaries with considerably smaller misorientation angles (Hilgenkamp et al., 1996). The GB junctions, fabricated on bicrystal substrates with the misorientation angle 45° contain a higher density of facets which by themselves show anomalous behavior. Due to the sign difference of the adjacent lobes of the d -wave order parameter, many facets are biased with an additional π -phase shift (π -facets). Therefore, the supercurrent density J_S is a random alternating

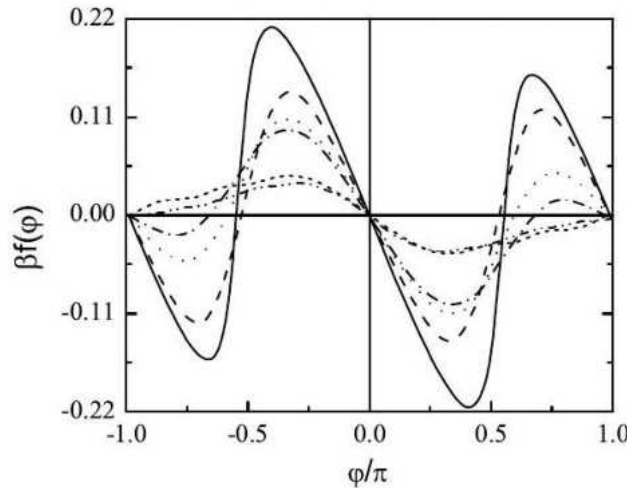


FIG. 37: Josephson current through the junction as a function of the phase difference φ , determined from the data in Fig. 36. After Il'ichev et al. (1999a).
function in the plane of the contact. This effect must be taken into account in the interpretation of the data.

Il'ichev et al. (1998, 1999 b) measured CPR in 45° GB junctions with $\theta_{1,2} = 22.5^\circ$ as well as $\theta_1 = 19^\circ$ and $\theta_2 = 26^\circ$. The junction size was between $1 - 3\mu\text{m}$. These junctions exhibit deviations of the CPR from the conventional sinusoidal form. The relation between first and second harmonics of the CPR was not too large, $I_2/I_1 \sim 0.1 - 0.2$. But surprisingly the second harmonic had positive sign, $I_2 > 0$. This means, that the supercurrent reaches its maximum value at $\varphi_{\text{max}} < \pi/2$. As discussed in Sec.IV.D, such type of CPR may be realized in s -wave SIS junction with high barrier transparency due to depairing by current (Kupriyanov, 1992). However, high barrier transparency is hardly the case in 45° GB junctions. To our knowledge, the observed CPR with $\varphi_{\text{max}} < \pi/2$ cannot be explained in the frame of both s wave or d -wave scenario in the assumption that junction has flat interface.

Il'ichev et al. (1999b) applied the model developed by Mints (1998) for the faceted junction to derive the CPR of a Josephson junction in the presence of a randomly distributed alternating

current density, assuming that the local CPR is sinusoidal. It was shown that the model is in good agreement with experimental results.

In the other set of measurements by Il'ichev et al. (1999a) the CPR of GB junction ($\theta_1 = 0^\circ$ and $\theta_2 = 45^\circ$, the so-called asymmetric GB junction) with the size $12\mu\text{ m}$ were measured. It was found that $|I_2/I_1| > 1$ with negative second harmonic. The results of measurements at different temperatures are shown in Fig. 36. At $T = 15\text{ K}$ and 20 K the curves are 2π -periodic, but the local minima show the presence of the second harmonic of the CPR. At $T < 10\text{ K}$ the amplitude of the second harmonic becomes larger than the first one.

The CPR determined from the data in Fig. 36 is shown in Fig. 37. For all curves a minimal necessary shift consistent with $I_S(\varphi = 0) = 0$ was performed. Thus, it is assumed that at $\varphi_{dc} = 0$ a minimum of $\alpha(\varphi_{dc})$ is real-

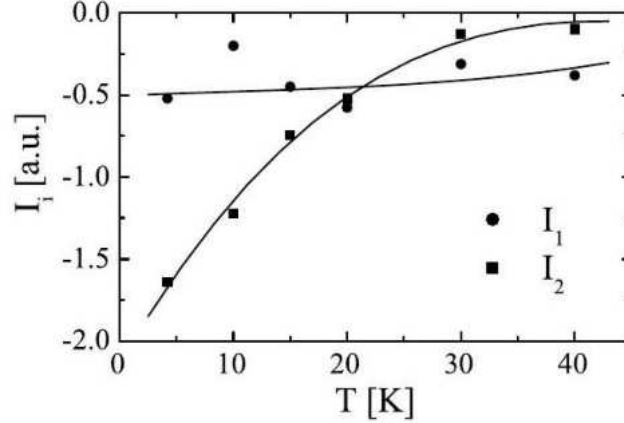


FIG. 38: Temperature dependence of the Fourier expansion the CPR, determined from the data in Fig. 36. After Il'icel et al. (1999a).

ized. For an interferometer with a conventional s -wave weak link, at $\varphi_{dc} = 0$ a maximum of $\alpha(\varphi_{dc})$ is realized. Note that the maximum (minimum) of $\alpha(\varphi_{dc})$ at $\varphi_{dc} = 0$ implies a diamagnetic (paramagnetic) response of the interferometer in the limit of small applied fields. Fig. 38 shows the coefficients I_1 and I_2 determined by Fourier analysis of the CPR at various temperatures. With decreasing T , $|I_2|$ grows monotonically down to $T = 4.2\text{ K}$, while the I_1 component exhibits only a weak temperature dependence.

In order to explain the fact that $|I_2/I_1| > 1$ within the faceted scenario, the alternating current density should average to zero (Hlubina et al., 2003). This is possible only for sufficiently long junctions, at least much longer than Josephson penetration depth, whereas in the experiment by Il'ichev et al. (1999a) much shorter junctions with $L \sim 1\mu\text{ m}$ were studied. Therefore it is impossible to explain the large measured I_2/I_1 ratio within the faceted scenario. Possible explanation is that for some reasons only few facets carry the supercurrent, therefore averaging does not occur. In this case the models of Tanaka (1994) and Walker and Lüttmer-Strathmann (1996) for the flat interface are applicable and can explain the results.

In order to avoid the influence of the faceting, submicron junctions are required. These results will be discussed in the next section.

9.9.10 3. Manifestation of MGS in the CPR

The presence of MGS is expected to influence in a spectacular way the Josephson junctions between d -wave superconductors with different crystallographic orientations. As shown by Tanaka and Kashiwaya (1995,1996), Barash et al. (1996), Fogelström et al. (1997), Riedel and Bagwell (1998) and Barash (2000), the MGS in symmetric d -wave junctions provide the anomalous contribution to the supercurrent with the sign opposite to the contribution of the regular ABS. As a result, the sign change of the first CPR harmonic, I_1 , i.e. the crossover to π

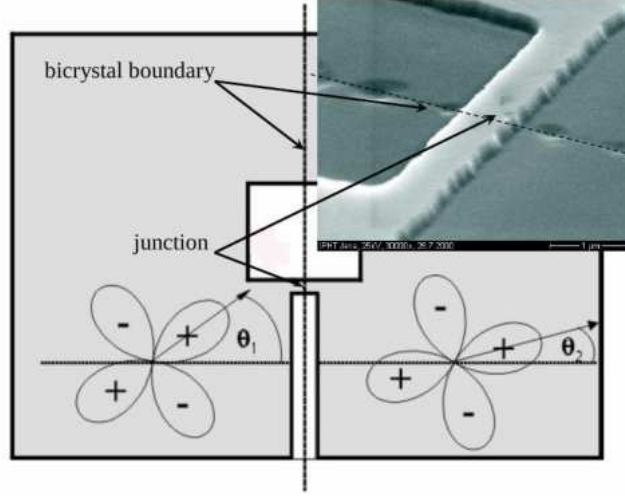


FIG. 39: Schematic picture of the interferometer. The YBCO thin film occupies the gray area. The inset shows an electron microscope image of the grain boundary Josephson junction. After Il'ichev et al. (2001b).

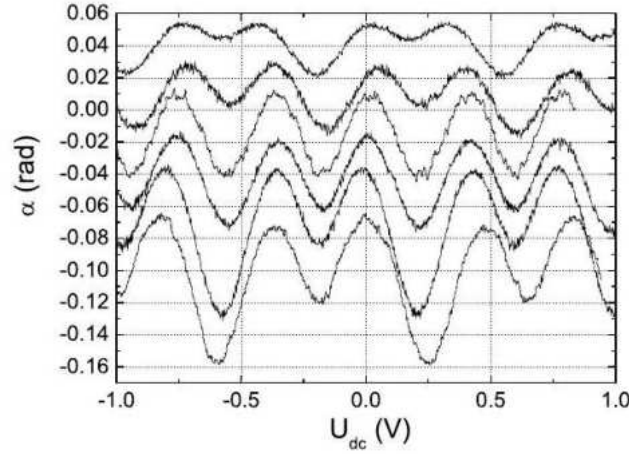


FIG. 40: The phase angle α as a function of the dc current flowing through the tank coil for the interferometer with submicron symmetric 45° GB JJ at various temperatures. From top to the bottom the data correspond to $T = 20, 15, 11, 10, 5$, and 1.6 K. After Il'ichev et al. (2001b). junction should take place at low T .

Motivated by search for the MGS, Il'ichev et al. (2001b) have studied symmetric GB junctions ($\theta_1 = \theta_2 = 22.5^\circ$) of small sub-micron size. It is important to note that in this case standard transport measurements of I_C are only possible at low temperatures, when the thermal energy $k_B T$ is smaller than the Josephson coupling energy. As was describe above, the radio-frequency method offers the possibility to investigate $I_S(\varphi)$ at temperatures T much higher than the Josephson coupling energy.

The temperature-controlled crossover to a π -junction should manifest itself as a π shift of the phase $\alpha - v s - I_{dc}$ oscillations. For a junction with nominal width of about $0.5 \mu\text{m}$ as shown in Fig. 39, such behavior was observed experimentally (see Fig. 40). At high temperatures the curves are 2π -periodic, but the local minima show the presence of the second harmonic of the CPR. At $T \approx 11$ K

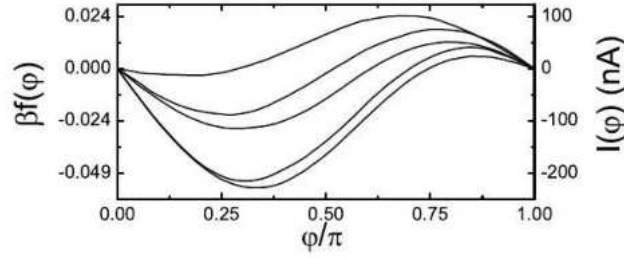


FIG. 41: The CPR for submicron symmetric 45° GB JJ, calculated from the data in Fig. 40. From top to the bottom the data correspond to $T = 15, 11, 10, 5$, and 1.6 K . After Il'ichev et al. (2001b).

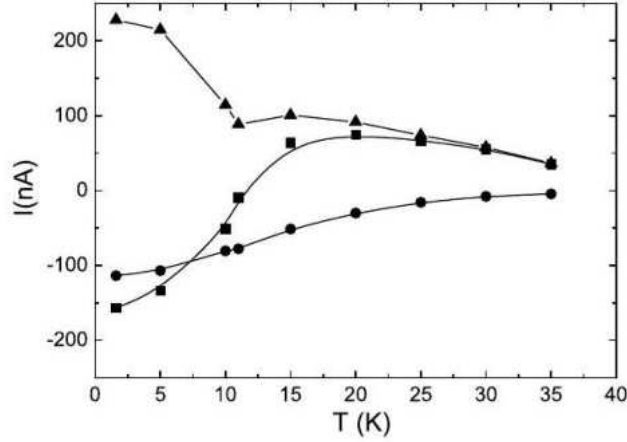


FIG. 42: The critical current (triangles) and the harmonic components I_1 (squares) and I_2 (circles) as a function of temperature for sub-micron symmetric 45° GBJJ, calculated from the data in Fig. 40. After Il'ichev et al. (2001b).

the $\alpha - vs - I_{dc}$ oscillations are π -periodic, i.e. the amplitude of the first harmonic is equal zero. At lower temperatures the local minima transform to the local maxima, hence the amplitude of the first harmonic changes sign. The corresponding CPR and $I_C(T)$ dependence are shown in Fig. 41 and Fig. 42 respectively.

Note also the remarkably large magnitude of the second harmonic I_2 . As was argued by Il'ichev et al. (2001b), this enhancement may be also explained by the contribution of the MGS. Alternative model which explains large second harmonic by quantum fluctuations of the order parameter was proposed by Hlubina (2002, 2003).

We conclude that the data for the sub-micron GB junctions are in a qualitative agreement with theoretical predictions for the CPR for HTS junctions with ideally flat interfaces. Thus the experiment by Il'ichev et al. (2001b) has revealed the contribution of the MGS to the Josephson effect in HTS junctions.

9.9.11 4. c-axis HTS junctions

Recently, it became clear that a finite admixture of the order parameter component with s -wave symmetry should exist in addition to the dominant d -wave order

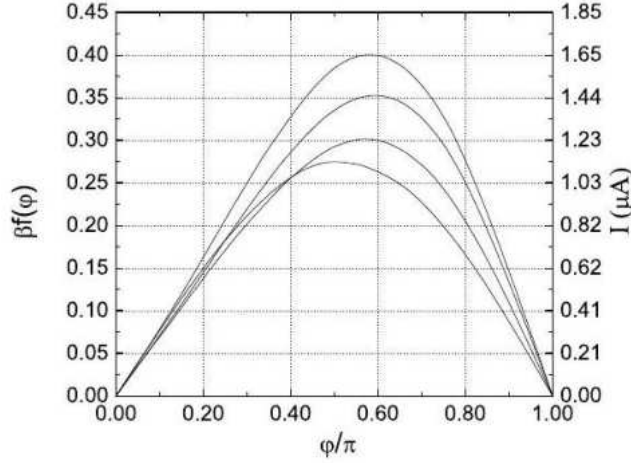


FIG. 43: The CPR for Nb/Au/(001)YBCO heterojunctions (c-contact). From top to the bottom the data correspond to $T = 1.66, 2.5, 4.2$, and 6 K . parameter in YBCO. An elegant c -axis tunnelling experiment by Kouznetsov et al. (1997) has directly shown that the s -wave component changes sign upon crossing the twin boundary in YBCO.

The above picture of the YBCO pairing state is challenged by the experimental observation of a finite c -axis Josephson current between heavily twinned YBCO and a Pb counterelectrode (Sun et al., 1996). Namely, the contribution of the s -wave part of the YBCO order parameter to the Josephson coupling between Pb and YBCO should average to zero for equal abundances of the two types of twins. In other words, the 'macroscopic pairing symmetry' of twinned YBCO samples should be a pure d wave (Walker and Lüttmer-Strathmann, 1996). Tanaka (1994) has shown that a finite value of the Josephson current occurs for second-order tunnelling between YBCO and Pb . However, measurements of microwave induced steps at multiples of $hf/2e$ on the $I - V$ curves of Pb/Ag/YBCO tunnel junctions imply dominant first order tunnelling (Kleiner et al., 1996). Therefore the finite c -axis Josephson current has to result from a nonvanishing s -wave component of the 'macroscopic pairing state' (Walker and Lüttmer-Strathmann, 1996). Two alternatives of how this can take place have been discussed in the literature:

- (i) Sigrist et al. (1996) have suggested that the phase of the s -wave component in YBCO does not simply jump from 0 to π upon crossing the twin boundary, but rather changes in a smooth way, attaining the value of $\pi/2$ right at the twin boundary. The twinned YBCO sample is thus assumed to exhibit a macroscopic $d+is$ pairing symmetry. A related picture has been proposed by Haslinger and Joynt (2000), who propose a $d + is$ surface state of YBCO.
- (ii) Donovan et al. (1997) argued that a difference in the abundances of the two types of twins implies a $d + s$ symmetry of the macroscopic pairing state. Let us point out that also structural peculiarities of other type (such as a lamellar structure in a preferred direction) may lead to the $d + s$ macroscopic pairing symmetry.

Kommissinski et al. (2002) have measured the CPR in c -axis Nb/Au/(001)YBCO heterojunctions and the deviation from the sinusoidal shape was found, as shown in Fig. 43. The obtained results are consistent with the statement that the macroscopic pairing symmetry in YBCO films is of the $d + s$ type, most probably due to a difference in the abundances of the two types of twins. Kommissinski et al. (2002) used a single set of parameters to explain the measured values of the normal-state resistance R_N together with I_1 and I_2 within the standard microscopic $d + s$ picture of the YBCO pairing state, under additional assumption that the barrier fluctuates along the junction.

9.9.12 5. General remarks

Due to the relatively complicated structure of high T_c materials there is no generally accepted picture of the nature of barriers in the GB junctions, where many mechanisms have been suggested (see Hilgenkamp and Mannhart (2002) and references therein). The CPR measurements by Il'ichev et al. (1999c) and by Grajcar et al. (2002b) for the 24° and 36°GB junctions are consistent with the tunnel limit prediction that $I_S(\varphi) \propto \sin \varphi$. According to the data, transition from low-angle boundaries to large-angle ones increases the GB resistance (detailed theoretical analysis was done by Gurevich and Pashitskii, 1998). The average transparency of 45° is smaller than for low-angles grain boundary junctions. Therefore it is natural to try to explain the large amplitude of the second harmonic of the CPR in the tunnel limit. Recently Hlubina (2002, 2003) proposed to take into account quantum fluctuations of the order parameter in order to explain the experimental data on CPR in GB junctions.

The barrier in GB junctions may also be represented as a disordered normal metal, such that the structure is of the SNS type. This may occur e.g. due to short coherence length in HTS (Deutscher and Müller, 1987), deviations from stoichiometry (Gross and Mayer, 1991; Halbritter, 1993) or the band-bending (Hilgenkamp and Mannhart, 1998; Nikolić et al., 2002). The generation of second harmonic in SNS junctions was discussed by Zagoskin (1997) and Huck et al. (1997). Asano (2001) has studied theoretically the CPR in SNS junctions of unconventional superconductors, including the cases of spin-singlet, spintriplet unitary and spin-triplet non-unitary superconductors. This study generalized previous theories by Sigrist and Ueda (1991) and Millis et al. (1988) to the case when MGS and disorder are present in the junction. The challenging problem for future CPR measurements remains to verify the predictions of various models in order to shed light on the conductance mechanisms across the GB in HTS.

9.10 XI. APPENDIX: MICROSCOPIC THEORY OF SUPERCONDUCTIVITY

We describe below the formalism which forms the basis for description of the stationary Josephson effect in various types of Josephson junctions.

9.10.1 A. Green's functions

Quantum field theoretical methods in terms of Green's functions are a powerful tool in all many-body problems (see e.g. Abrikosov et al., 1963; Rammer and Smith, 1986; Zagoskin, 1998).

The starting point for all problems in superconductivity is the Green's function in Nambu space, which combines the particle and hole space (Nambu, 1960). In a superconductor the Green's function has an 'anomalous' components $F(x, x')$, $F^\dagger(x, x')$ and normal component $G(x, x')$ which obey a set of coupled Gor'kov equations (Gor'kov 1958, 1959a,b):

$$\left\{ i\omega\tau_3 + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial \mathbf{r}_1^2} + \mu + U(\mathbf{r}_1) + \hat{\Delta}(\mathbf{r}_1) \right\} \hat{G}(\mathbf{r}_1, \mathbf{r}_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2) \quad (168)$$

$$\hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ -\Delta^* & 0 \end{pmatrix}, C = \begin{pmatrix} G & F \\ F^\dagger & -G \end{pmatrix}$$

The pair potential Δ is determined by the selfconsistency equation

$$\Delta(\mathbf{r}) = -\frac{\lambda}{2} T \sum_{\omega} \text{Tr}(\tau_1 + i\tau_2) \hat{G}(\mathbf{r}, \mathbf{r}) \quad (169)$$

Here $\omega = \pi T(2n + 1)$ is the Matsubara frequency, τ_i are Pauli matrices, $U(\mathbf{r}_1)$ is electric potential, λ is the strength of the attractive interaction. For a normal metal we have $\lambda = 0$ and hence $\Delta = 0$, but the pair amplitude F can still be nonzero in various situations (e.g. due to proximity effect in NS structures, as will be considered in detail below).

The solutions of the Gor'kov equations oscillate as a function of the relative coordinate $\mathbf{r}_1 - \mathbf{r}_2$ on a scale of the Fermi wavelength λ_F . However, characteristic length scales in the typical problems in superconductivity $\xi_0 = \hbar v_F / \Delta$ and $\xi_T = \hbar v_F / 2\pi T$ are much larger than λ_F . If additionally the characteristic spatial scale of a system exceeds λ_F , it is sufficient to know only the dependence of the Green's functions on the center of mass coordinate $(\mathbf{r}_1 + \mathbf{r}_2)/2$ and integrate out the dependence on the relative coordinate. This procedure leads to the quasiclassical equations of motion for the Green's functions, formulated by Eilenberger (1968) and by Larkin and Ovchinnikov (1968).

9.10.2 B. The quasiclassical approximation

1. Eilenberger equations

We will now discuss the quasiclassical approximation in equilibrium and stationary situations. The formulation for nonequilibrium and time-dependent problems has been done by Eliashberg (1971) and Larkin and Ovchinnikov (1975, 1977) (see also review papers by Schmid, 1981; Larkin and Ovchinnikov 1986; Schön, 1984; Rammer and Smith, 1986; Lambert and Raimondi, 1998; and Belzig et al., 1999).

The derivation of quasiclassical equations is based on gradient expansion of Gor'kov equations using small parameter λ_F / ξ_0 . These equations are written for the functions $f(\mathbf{r}, \mathbf{n})$, $g(\mathbf{r}, \mathbf{n})$ which are the energy-integrated Gor'kov functions:

$$f(\mathbf{r}, \vec{n}) = \frac{i}{\pi} \iint F_p d\xi, g(\mathbf{r}, \mathbf{n}) = \frac{i}{\pi} \iint G_p d\xi \quad (170)$$

Here $\xi = p^2 / 2m - \mu$, $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2) / 2$ and $\vec{n}$ is the unit vector normal to the Fermi surface. Functions F_p, G_p are defined by the following relations:

$$F_p = \frac{1}{(2\pi)^3} \iint F(\mathbf{r}_1, \mathbf{r}_2) \exp \{i\mathbf{p} \cdot ((\mathbf{r}_1 - \mathbf{r}_2) / \hbar)\} d(\mathbf{r}_1 - \mathbf{r}_2),$$

$$G_p = \frac{1}{(2\pi)^3} \iint G(\mathbf{r}_1, \mathbf{r}_2) \exp \{i\mathbf{p} \cdot ((\mathbf{r}_1 - \mathbf{r}_2) / \hbar)\} d(\mathbf{r}_1 - \mathbf{r}_2).$$

Functions f, g satisfy the following normalization and symmetry relations

$$g^2(\mathbf{r}, \mathbf{n}) + f^\dagger(\mathbf{r}, \mathbf{n})f(\mathbf{r}, \mathbf{n}) = 1 \quad (171)$$

$$f^\dagger(\mathbf{r}, \mathbf{n}) = f(\mathbf{r}, -\mathbf{n}) \quad (172)$$

and obey the Eilenberger equations

$$\mathbf{v}_F \partial_r \hat{g} + \left[\omega \hat{\tau}_3 + \hat{\Delta} - \frac{1}{\tau} \langle \hat{g} \rangle, \hat{g} \right] = 0 \quad (173)$$

$$\Delta \ln \frac{T}{T_C} - 2\pi T \sum_{\omega} \left(\frac{\Delta}{\omega} - \langle f \rangle \right) = 0 \quad (174)$$

Here $\partial_r = \nabla - i \frac{2e}{\hbar c} \mathbf{A}$, where $\mathbf{A}$ is vector potential, $\mathbf{v}_F$ is the Fermi velocity, τ is the impurity scattering time, square brackets denote commutator, angular brackets denote angle averaging. The matrices $\hat{\Delta}, \hat{g}$ are defined as

$$\hat{\Delta} = \begin{pmatrix} 0, & \Delta \\ \Delta^*, & 0 \end{pmatrix}, \hat{g} = \begin{pmatrix} g, & f \\ f^\dagger, & -g \end{pmatrix}$$

Electrical current density is expressed through g in the following way:

$$J = 4eN(0)\pi T \sum_{\omega} \langle \mathbf{v}_F \text{Im } g \rangle \quad (175)$$

where $N(0)$ is the density of states per spin at the Fermi level.

Quasiclassical approximation is not valid in the vicinity of atomically sharp interfaces and potential barriers. Therefore Eq. (173) should be completed by effective boundary conditions. A derivation of such conditions valid at arbitrary transmission of the interface has been given by Zaitsev (1984) on the basis of the full theory by matching the solutions of Gor'kov and Eilenberger equations along the classically transmitted and reflected trajectories. For the trajectories connecting two metals the boundary conditions have the form

$$\hat{g}_1^a = \hat{g}_2^a \equiv \hat{g}^a, \quad (176)$$

$$\hat{g}^a \left\{ R - R(\hat{g}^a)^2 + D(\hat{g}_-^c)^2 \right\} = D\hat{g}_-^c \hat{g}_+^c, \quad (177)$$

where D is the transmission probability across the interface, $R = 1 - D$, and functions $\hat{g}^{c(a)}$ are defined by the relations

$$\hat{g}_\pm^c = \frac{1}{2}(\hat{g}_1^c \pm \hat{g}_2^c), \hat{g}_{1,2}^{c(a)} = \frac{1}{2}\{\hat{g}_{1,2}(n_z) \pm \hat{g}_{1,2}(-n_z)\}$$

Here n_z is unit vector along the normal to the interface.

For the fully reflected trajectories, when the parallel to the interface momentum component $p_\parallel$ satisfies the condition $p_{F1} < p_\parallel < p_{F2}$, the following condition is satisfied

$$\hat{g}_1^c = \hat{g}_2^c$$

Different form of boundary conditions, using the so-called Ricatti parametrization (Schopohl and Maki, 1995) was derived by Eschrig (2000), Fogelström (2000) and Yip (1997). This method is more convenient for stable numerical solution of Eilenberger equations.

2. The dirty limit: Usadel equations

Important practical case is the superconducting material with strong impurity scattering. If the mean free path is much smaller than the coherence length (quantitative criteria will be given below), in the so-called 'dirty limit', Eilenberger equations can be further simplified. In this limit the electron motion is diffusive and the Green's functions are nearly isotropic, i.e. in the main approximation they do not depend on the direction of the vector $\vec{n}$. In this case, expanding the Green's functions in spherical harmonics, it is possible to obtain the equations (see Usadel, 1970) for the isotropic parts $\hat{G}$ of the Green's functions:

$$i\hbar D \partial_r \left(\hat{G}(r) \partial_r \hat{G}(r) \right) = \left[\omega \hat{\tau}_3 + \hat{\Delta}(r), \hat{G}(r) \right] \quad (178)$$

where

$$\hat{\Delta} = \begin{pmatrix} 0, & \Delta \\ \Delta^*, & 0 \end{pmatrix}, \hat{G} = \begin{pmatrix} G, & F \\ F^*, & -G \end{pmatrix}$$

and $D = v_F l / 3$ is the diffusion coefficient. The matrix Green's function $\hat{G}$ satisfies the normalization condition

$$\hat{G}^2 = 1, G^2 + FF^* = 1, \quad (179)$$

and the pair potential Δ is determined by the selfconsistency equation

$$\Delta \ln \frac{T_C}{T} = \pi T \sum_{\omega} \left(\frac{\Delta}{\omega} - F \right) \quad (180)$$

Eqs.(178)-(180) are called in literature the Usadel equations.

The current density in the dirty limit is given by

$$J(r) = -\frac{i\pi\sigma}{2e} T \sum_{\omega} \text{Tr} \left(\hat{\tau}_3 \hat{G}(r) \partial_r \hat{G}(r) \right) \quad (181)$$

where $\sigma = 2e^2 N(0) D$ is the normal-state conductivity.

In a homogeneous bulk s-wave superconductor under equilibrium conditions the solutions for G, F are

$$G = \frac{\omega}{\sqrt{\Delta^2 + \omega^2}}, F = \frac{\Delta \exp(i\chi)}{\sqrt{\Delta^2 + \omega^2}} \quad (182)$$

where Δ and χ are the magnitude and phase of the pair potential at a given temperature.

The boundary conditions for the Usadel equations at the interface between two superconductors $S_{1,2}$ were derived by Kupriyanov and Lukichev, 1988

$$p_{F1}^2 l_1 \hat{G}_1 \frac{d}{dx} \hat{G}_1 = p_{F2}^2 l_2 \hat{G}_2 \frac{d}{dx} \hat{G}_2 \quad (183)$$

$$\gamma_B \xi_1 \hat{G}_1 \frac{d}{dx} \hat{G}_1 = [\hat{G}_1, \hat{G}_2] \quad (184)$$

Here x is the coordinate along the normal direction to the interface, $p_{F1,2}$ and $l_{1,2}$ are the Fermi momenta and ξ_1 is the dirty limit coherence length in S_1

$$\xi_1 = \sqrt{\frac{\hbar D_1}{2\pi T_C}} \quad (185)$$

where $D_1 = v_{F1} l_1 / 3$ and T_C is the transition temperature of S_1 .

The dimensionless parameter γ_B describes the effect of the interface barrier (Kupriyanov and Lukichev, 1988)

$$\gamma_B = \frac{2l_1}{3\xi_1} \left\langle \frac{1-D}{xD} \right\rangle = \frac{R_B}{\rho_1 \xi_1} \quad (186)$$

where angular brackets denote averaging over angles of trajectories, $x = p_{F1x} / p_{F1}$, D is the barrier transmission coefficient and ρ_1 is the normal state resistivity of a superconductor S_1 .

The boundary conditions (183), (184) and have simple physical interpretation: Eq. (183) manifests current conservation on both sides of the interface while Eq. (184) the current continuity across the interface.

According to the definition (186), $\gamma_B = 0$ for fully transparent interface. It follows from Eq. (184) that Green's functions $\hat{G}_{1,2}$ are continuous at the interface in this case. In the regime of low barrier transparency (tunnel junction) $\gamma_B \gg 1$ and $\hat{G}_{1,2}$ have discontinuities at the interface. Lambert et al. (1997) have shown that the condition (184) is exact in two limits

of high and low barrier transparency, $\gamma_B = 0$ and $\gamma_B \gg 1$, and found the corrections at the intermediate values of $\gamma_B \sim 1$ which however do not exceed 10%.

It is useful to introduce another dimensionless parameter describing the proximity effect between $S_{1,2}$, which will be used extensively below. For this purpose we rewrite Eq. (183) in the form

$$\gamma \xi_1 \hat{G}_1 \partial_x \hat{G}_1 = \xi_2 \hat{G}_2 \partial_x \hat{G}_2 \quad (187)$$

where

$$\gamma = \frac{\rho_2 \xi_2}{\rho_1 \xi_1} \quad (188)$$

Parameter γ determines the magnitude of $\hat{G}_2$ gradients near the interface: the higher is conductivity in S_1 compared to that in S_2 , the larger is the gradient of $\hat{G}_2$ and thus the stronger is deviation of surface $\hat{G}_2$ value compared to the bulk one. In particular, if S_1 is a weak superconductor (or a normal metal) and S_2 is a strong superconductor, γ determines the strength of suppression of superconductivity in S_2 near the interface compared to the bulk: no suppression occurs for $\gamma = 0$ while strong suppression of F_2 and Δ_2 takes place for $\gamma \gg 1$.

3. Φ -parametrization

New function Φ can be introduced in the following way, which fulfills automatically the normalization condition

$$G = \frac{\omega}{\sqrt{\omega^2 + \Phi \Phi^*}}, F = \frac{\Phi}{\sqrt{\omega^2 + \Phi \Phi^*}} = \frac{G \Phi}{\omega} \quad (189)$$

The Usadel equation, the selfconsistency equation and the boundary conditions now take the form

$$\Phi = \Delta + \frac{D}{2\omega} \partial_x (G^2 \partial_x \Phi) \quad (190)$$

$$\Delta \ln \frac{T_C}{T} = \pi T \sum_{\omega} \frac{\Delta - \Phi G}{\omega} \quad (191)$$

$$\xi_2 G_2^2 \partial_x \Phi_2 = \gamma \xi_1 G_1^2 \partial_x \Phi_1 \quad (192)$$

$$\xi_2 \gamma_B G_2 \partial_x \Phi_2 = G_1 (\Phi_1 - \Phi_2) \quad (193)$$

The current density is given by

$$J(r) = \frac{2\pi\sigma}{e} T \sum_{\omega} \text{Im} (G^2 \Phi^* \partial_x \Phi) \quad (194)$$

Φ -parameterization simplifies analytical treatment in the cases of small gradients (when $\Phi \simeq \Delta$) and large gradients (when nongradient terms in Eq. (190) may be neglected). Moreover, equation (190) becomes linear when Φ is small compared to ω .

9.10.3 C. Ginzburg-Landau equations

At temperature near the critical temperature T_C a superconductor is described by the Ginzburg-Landau (GL) equations

$$\xi^2(T) \partial_x^2 \Delta + \Delta (1 - |\Delta|^2 / \Delta_0^2) = 0 \quad (195)$$

Here $\Delta_0^2 = (8\pi^2/7\zeta(3)) T_C (T_C - T)$, $\xi(T)$ is the GL coherence length which in the dirty limit is given by

$$\xi(T) = (\pi/2)\xi(1 - T/T_C)^{-1/2} \quad (196)$$

where ξ is given by Eq. 185. Equation (195) can be derived directly from the Usadel equations (190),(191) using expansion of Φ over small gradients in (190) and substituting it into selfconsistency equation (191).

Supercurrent density in the GL theory is given by

$$J = \frac{\pi}{4e\rho T_C \xi(T)} \text{Im}(\Delta^* \partial_x \Delta) \quad (197)$$

Despite the GL theory is limited to temperatures close to T_C , it was proved very useful in describing currentcarrying states in various types of Josephson junctions, due to computational simplicity compared to the quasiclassical methods. Moreover, the GL method is physically more transparent as well, since a superconductor is described by a single function Δ which has a meaning of the superconducting order parameter. It is also commonly referred to as a condensate wave function.

The GL equations describe the spatial evolution of Δ at a length scale of the order of $\xi(T)$, at the interfaces they should be completed by corresponding boundary conditions. Their knowledge is crucial for application of the GL theory.

As was first shown by de Gennes (1964, 1966), the general form of the boundary condition is

$$\mathbf{n} \partial_r \Delta = \frac{1}{b} \Delta \quad (198)$$

where $\mathbf{n}$ is the unit vector along the direction normal to the interface and b is the so-called 'extrapolation length'. The quantity $\xi(T)/b$ determines the slope of the order parameter a superconductor S near the interface, i.e. the spatial scale of a penetration of the order parameter from S into another weaker superconductor to a normal metal brought to a contact with S . At the contact with a dielectric or at a free surface of a superconductor $b = \infty$. For the contact with a normal metal, parameter b was calculated by de Gennes with accuracy up to numerical prefactor

$$b = \xi(T)/\gamma \quad (199)$$

where γ is defined in Eq. (188). The dependence of b on parameters of contacting S and N metals was later studied quantitatively in Ivanov et al. (1978), Barone and Ovchinnikov (1979) and Ivanov et al. (1981).

10 Josephson and Andreev transport through quantum dots by Martín-Rodero and A. Levy Yeyati

Abstract

In this article we review the state of the art on the transport properties of quantum dot systems connected to superconducting and normal electrodes. The review is mainly focused on the theoretical achievements although a summary of the most relevant experimental results is also given. A large part of the discussion is devoted to the single level Anderson type models generalized to include superconductivity in the leads, which already contains most of the interesting physical phenomena. Particular attention is paid to the

competition between pairing and Kondo correlations, the emergence of π -junction behavior, the interplay of Andreev and resonant tunneling, and the important role of Andreev bound states which characterized the spectral properties of most of these systems. We give technical details on the several different analytical and numerical methods which have been developed for describing these properties. We further discuss the recent theoretical efforts devoted to extend this analysis to more complex situations like multidot, multilevel or multiterminal configurations in which novel phenomena is expected to emerge. These include control of the localized spin states by a Josephson current and also the possibility of creating entangled electron pairs by means of non-local Andreev processes.

10.1 Introduction

The field of electronic transport in nanoscale devices is experiencing a fast evolution driven both by advances in fabrication techniques and by the interest in potential applications like spintronics or quantum information processing. Within this context quantum dot (QD) systems are playing a central role. These devices have several different physical realizations including semiconducting heterostructures, small metallic particles, carbon nanotubes or other molecules connected to metallic electrodes. In spite of this variety a very attractive feature of these devices is that they can usually be described theoretically by simple "universal-like" models characterized by a few parameters. In addition to their potential applications, these systems provide a unique test-bed for analyzing the interplay of electronic correlations and transport properties in nonequilibrium conditions.

Electron transport in semiconducting QDs has been studied since the early 90's when phenomena like Coulomb blockade (CB) was first observed [65]. It soon became clear that QDs could allow to study the effect in transport properties of basic electronic correlations phenomena like the Kondo effect as suggested in early predictions [45, 76]. These predictions were first tested in metallic nanoscale junctions containing magnetic impurities [92]. However, a definitive breakthrough in the field came with the observation of this effect in semiconducting QDs by Goldhaber et al. [46] and Cronenwett et al. [28]. A great advantage of these devices is to offer the possibility of controlling the relevant parameters, thus allowing a more direct comparison with the theoretical predictions. Since then the effect of Kondo correlations in electronic transport has been observed in several physical realizations of QDs based on carbon nanotubes (CNTs) [86] and big molecules like fullerenes [94].

In parallel to these advances the study of superconducting (SC) transport in nanoscale devices has also experienced a great development. From a theoretical point of view, with the advent of mesoscopic physics, a more detailed understanding of superconducting transport was developed around the central concept of coherent Andreev reflection (AR) [3, 9]. This concept has allowed to unify the description of superconducting transport in different types of structures like normal metal-superconductor (N-S), S-N-S junctions and superconducting quantum point contacts (SQPC). Due to the multiple AR (MAR) mechanism the spectral density of systems like S-N-S or SQPCs is characterized by the presence of the so-called Andreev bound states (ABS) inside the superconducting gap. These states are sensitive to the superconducting phase difference and are thus current-carrying states which usually give the dominating contribution to the Josephson effect.

In recent years it has become feasible to produce hybrid systems combining different physical realizations of QDs well contacted to superconducting electrodes (for a review see [42]). Superconducting transport through QDs provides the interesting possibility to explore the interplay of the AR mechanism and typical QD phenomena like CB and Kondo effect. The central aim of this review article is to discuss the main advances which have taken place on this issue during the last years.

A usual assumption in these studies is that a basic description of the main properties of

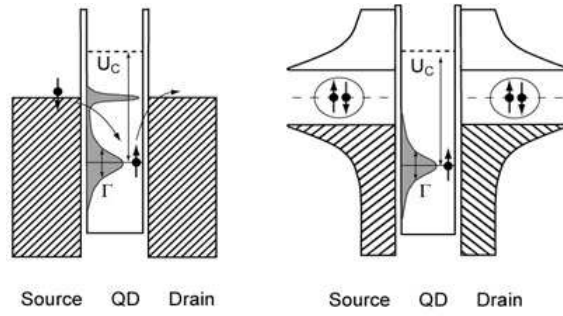


Figure 10.1: Schematic representation of a single spin-degenerate level QD connected to normal (left panel) and superconducting (right panel) leads with a large charging energy U_c . In the normal case the local density of states (LDOS) in the dot exhibits the typical form corresponding to the Kondo regime with a narrow resonance at the Fermi energy and a broad resonance (of width Γ) below it. In the superconducting case the Kondo resonance (assumed to be narrower than the superconducting gap) disappears due to the competition with the pairing correlations in the leads. Courtesy of C. Schönenberger.

these hybrid systems can be provided by the Anderson model and its generalizations to include SC leads, orbital degeneracy, etc. The single level model applies when the dot level spacing $\delta\epsilon$ is larger than all other relevant energy scales. In the normal state the model allows to describe in a unified way CB and the Kondo effect both in and out of equilibrium conditions [71]. With two superconducting leads interesting new physics already appear in the equilibrium case. Due to the Josephson effect, electron transport is possible without an applied bias voltage and the model describes the competition between Kondo effect and induced pairing within the dot. Figure 10.1 illustrates this competition: depending on the ratio between the Kondo temperature, T_K , and the superconducting order parameter, Δ , there is a phase transition between a Kondo dominated spin-singlet ground state to a degenerate magnetic ground state. This transition is accompanied by a reversal of the sign of the Josephson current. Thus in the magnetic case the S-QD-S system constitutes a realization of the so-called π -junction [44, 110]. A more detailed understanding of this transition describing the appearance of intermediate phases with metastable states was achieved more recently [97, 122]. The realization of a π -junction in QD systems should be distinguished from the similar phenomena in SFS junctions, where F denotes a ferromagnetic material [48].

Another basic situation which has been extensively explored (both theoretically and experimentally) is the N-QD-S case. This situation has been mainly analyzed in the linear transport regime in which it exhibits an interesting interplay between Kondo behavior and resonant Andreev reflection. In contrast to the S-QD-S case this system does not exhibit a quantum phase transition but there is instead a crossover from a Kondo dominated regime for large T_K/Δ to a singlet superconducting regime in the opposite limit.

A third paradigmatic situation which has been studied is the voltage biased S-QD-S system. This situation constitutes a much more demanding task for the theory due to the need of describing properly the strong out-of-equilibrium distribution which is generated by the infinite series of multiple Andreev reflection (MAR) processes together with the effects of Coulomb interactions. The problem becomes simpler in two limiting cases: 1) when Coulomb interactions are small and treated in a mean field approximation thus allowing to analyze the interplay of MAR and resonant tunneling and 2) when the Coulomb energy is the larger energy scale in the problem and the contribution of MAR processes are largely suppressed.

More recent developments include the study of several QDs (connected either in series or in parallel) coupled to one or more SC electrodes. In these situations there is a competition

between not only the Kondo and the SC correlations but also the possible magnetic coupling of the spins localized within the dots. This could open the possibility to control the spin state of the dots system by means of the Josephson current.

In addition there is a growing interest in analyzing transport in these hybrid structures in a multiterminal configuration. One of the aims of these studies is the detection and control of non-local Andreev processes, which offer the possibility of producing entangled electron pairs [93].

This review article is organized as follows: in Section 10.2 we introduce the basic theoretical models used to describe the hybrid QD systems, which are largely based on the Anderson model and its generalizations. In this section we also give a brief summary of the application of nonequilibrium Green functions techniques for the calculation of the electronic properties within these type of models. Section 10.3 is devoted to review the main results for the S-QD-S systems in equilibrium, i.e. in the dc Josephson regime. We first discuss the case of a non-interacting resonant level which is useful to illustrate the emergence of ABSs and its contribution to the Josephson current. In the subsequent subsections we give account of the different theoretical methods which have been used to analyze the effect of interactions in the dc Josephson regime. A main issue which is discussed in this section are the phase diagrams describing the transition to the π -state as a function of the model parameters. We also give a brief account of the existing experimental results for S-QD-S devices in this regime. The case of a QD coupled to both a normal and a superconducting leads (N-QD-S) is addressed in Section 10.4. Most of the results obtained for this systems correspond to the linear regime with different levels of approximation to include the Coulomb interactions. We also briefly mention existing results for the non-linear regime and the few experiments which have been reported of this case up to date. Section 10.5 is devoted to the voltage biased S-QD-S system. We first give some technical details on the calculations for the non-interacting case in order to illustrate how to deal with the out of equilibrium MAR mechanism. We also comment in this section the few existing results including interactions in this regime and give an account of the related experiments. Finally, in Section 10.6 we discuss several different situations which go beyond the single-level two-terminal case discussed in the previous sections. These include: multidot systems connected either in parallel or in series, the multilevel situation and setups in a multiterminal configuration. We conclude this article with a brief discussion of related issues not included in the present review and of topics which, in our view, deserve to be further analyzed in the near future.

10.2 Basic models and formalism

The minimal model for a QD coupled to metallic electrodes in the regime where $\delta\epsilon$ is sufficiently large to restrict the analysis to a single spin-degenerate level is provided by the single level Anderson model [2], with the Hamiltonian $H = H_L + H_R + H_T + H_{QD}$ where H_{QD} corresponds to the uncoupled dot given by

$$H_{QD} = \sum_{\sigma} \epsilon_0 c_{0\sigma}^{\dagger} c_{0\sigma} + U n_{0\uparrow} n_{0\downarrow}, \quad (10.1)$$

where $c_{0\sigma}^{\dagger}$ creates an electron with spin σ on the dot level located at ϵ_0 and U is the local Coulomb interaction for two electrons with opposite spin within the dot ($n_{0\sigma} = c_{0\sigma}^{\dagger} c_{0\sigma}$). On the other hand, $H_{L,R}$ describe the uncoupled left and right leads which can be either normal or superconducting. In this last more general case, they are usually represented by a BCS Hamiltonian of the type

$$H_{\nu} = \sum_{k\sigma} \xi_{k,\nu} c_{k\sigma,\nu}^{\dagger} c_{k\sigma,\nu} + \sum_k \left(\Delta_{\nu} c_{k\uparrow,\nu}^{\dagger} c_{-k\downarrow,\nu}^{\dagger} + \text{h.c.} \right), \quad (10.2)$$

where $c_{k\sigma,\nu}^\dagger$ creates an electron with spin σ at the single-particle energy level $\xi_{k,\nu}$ of the lead $\nu = L, R$ (usually referred to the lead chemical potential, i.e. $\xi_{k,\nu} = \epsilon_{k,\nu} - \mu_\nu$) and $\Delta_\nu = |\Delta_\nu| \exp\{i\phi_\nu\}$ is the (complex) superconducting order parameter on lead ν . Finally, H_T describes the coupling between the QD level to the leads and has the form

$$H_T = \sum_{k\sigma,\nu} \left(V_{k,\nu} c_{k\sigma,\nu}^\dagger c_{0\sigma} + \text{h.c.} \right). \quad (10.3)$$

In order to reduce the number of parameters it is usually assumed that the normal density of states of the leads $\rho_\nu(\omega)$ is a constant in the range of energies around the Fermi level of the order of the superconducting gap and that the k dependence of the hopping elements $V_{k\nu} \simeq V_\nu$ can be neglected within this range. The coupling to the leads is then characterized by a single parameter $\Gamma_\nu = \pi \rho_\nu |V_\nu|^2$, which can be interpreted as the normal *tunneling rate* from the dot to the leads.

Within the above model $\Delta_\nu = 0$ would correspond to the normal state. For vanishing Γ_ν the model is in the so-called atomic limit which is characterized by sharp peaks in the spectral density at ϵ_0 and $\epsilon_0 + U$. This limit corresponds to the Coulomb blockade regime in an actual QD where the conductance is strongly suppressed except at the charge degeneracy points. When the couplings to the leads increase (Γ_ν become larger than temperature) virtual processes allow the charge and spin in the dot to fluctuate and a resonance at around the Fermi energy appears close to half-filling due to the Kondo effect. This simple model thus already captures the most relevant Physics of ultrasmall QDs with well separated energy levels, like the crossover from the Coulomb blockade to the Kondo regime as the temperature is lowered.

The simplicity of this model has allowed to obtain exact results in the equilibrium case by means of the Bethe ansatz [124]. The most basic of these results is the expression for the Kondo temperature [53]

$$T_K = \sqrt{\frac{U\Gamma}{2}} \exp \left\{ \left(-\frac{\pi |\epsilon_0(\epsilon_0 + U)|}{2U\Gamma} \right) \right\}, \quad (10.4)$$

where $\Gamma = \Gamma_L + \Gamma_R$.

This temperature characterizes the crossover from the so-called local moment regime for $T \gg T_K$ to the regime where Kondo correlations between the localized spin within the QD and the spin of the electrons in the leads sets in. Although this physics is basically well understood since the 70's for the case of magnetic impurities in metals, its consequences for transport in artificial nanostructures has started to be developed much more recently specially driven by the advances in fabrication techniques. In this respect while the linear transport properties are well understood still open questions remain regarding the non-equilibrium regime.

In the superconducting case another energy scale, associated with the superconducting gap, appears bringing additional complexity to the problem, whose description is in fact the scope of this review. Even in the equilibrium situation the Anderson model with superconducting leads contains the non-trivial Physics associated to the Josephson effect. A relevant parameter is then provided by the superconducting phase difference $\phi = \phi_L - \phi_R$.

In order to analyze the electronic and transport properties of a general superconducting system in the presence of interactions and in a non-equilibrium situation it is convenient to use Green function techniques. The Keldysh formalism provides the basic tools for this purpose.

Due to the presence of superconducting correlations it is convenient to introduce the Nambu spinor field operators $\Psi_j, \Psi_j^\dagger$, with $\Psi_j^\dagger = \begin{pmatrix} c_{j\uparrow}^\dagger, c_{j\downarrow} \end{pmatrix}$ where $j = k\nu, 0$ denotes the $\nu = L, R$ electrodes and the dot level respectively. The different terms in the model Hamiltonian of Eqs.

(10.1), (10.2) and (10.3) can then be written as

$$\begin{aligned} H_{QD} &= \Psi_0^\dagger \hat{h}_0 \Psi_0 + U n_{0\uparrow} n_{0\downarrow} \\ H_\nu &= \sum_k \Psi_{k\nu}^\dagger \hat{h}_{k\nu} \Psi_{k\nu} \\ H_T &= \sum_{k,\nu} \left(\Psi_{k\nu}^\dagger \hat{V}_{k\nu} \Psi_0 + \text{h.c.} \right) , \end{aligned} \quad (10.5)$$

where $\hat{h}_0 = \epsilon_0 \tau_3$, $n_{0\sigma} = \frac{1}{2} \Psi_0^\dagger [\tau_0 + \text{sign}(\sigma) \tau_3] \Psi_0$, $\hat{h}_{k\nu} = \xi_{k\nu} \tau_3 + \text{Re} \Delta_\nu \tau_1 + \text{Im} \Delta_\nu \tau_2$, $\tau_{i=0,1,2,3}$ being the Pauli matrices defined in Nambu space.

Starting from these spinor field operators, generalized single-particle propagators can be defined along the Keldysh closed time loop as

$$\hat{G}_{j,j'}^{\alpha\beta}(t, t') = -i \langle T_c \left[\Psi_j(t_\alpha) \Psi_{j'}^\dagger(t'_\beta) \right] \rangle , \quad (10.6)$$

where $\alpha, \beta \equiv \pm$ denote the two branches in the Keldysh contour. These propagators allow to calculate in a straightforward way most of the relevant quantities like the mean charge and the superconducting order parameter within the dot as well as the mean current through it, which are given by

$$\begin{aligned} n_0(t) &= i \text{Tr} \left(\tau_3 \hat{G}_{00}^{+-}(t, t) \right) - 1 \\ \text{Re} \Delta_0(t) &= U \text{Tr} \left(\tau_1 \hat{G}_{00}^{+-}(t, t) \right) \\ I_\nu(t) &= \frac{e}{\hbar} \sum_k \text{Tr} \left(\tau_3 \left[\hat{V}_{k\nu} \hat{G}_{k\nu,0}^{+-}(t, t) - \hat{V}_{\nu k} \hat{G}_{\nu k,0}^{+-}(t, t) \right] \right) . \end{aligned} \quad (10.7)$$

These expressions are formally exact but of little use unless the single-particle propagators are known. Fully analytical and exact results can only be obtained in the non-interacting case. In the presence of interactions numerical methods allow to obtain exact results in the equilibrium case. In a more general case one is bound to find reasonable approximations for these propagators valid for a restricted range of parameters. It is usually convenient to express these approximations in terms of a self-energy Σ which is related to the propagators by the usual Dyson equation

$$\check{\check{G}}_{00} = \check{\check{G}}_{00}^{(0)} + \check{\check{G}}_{00}^{(0)} \check{\check{\Sigma}}_{00} \check{\check{G}}_{00} , \quad (10.8)$$

where $\check{\check{G}}^{(0)}$ denotes the unperturbed propagators corresponding to an appropriately defined non-interacting Hamiltonian H_0 (the $\check{\check{}}$ symbol indicates matrix structure in Keldysh space) and where integration over internal times is implicitly assumed. Different approximations for the self-energy associated with electron-electron interactions are discussed in the forthcoming sections. The analysis of the problem is greatly simplified in the stationary case where Fourier methods can be applied both in the equilibrium and in the non-equilibrium situation. We shall start discussing in the next section the simplest possible case of an equilibrium situation. In this case further simplification arises from the possibility of expressing all Keldysh propagators in terms of the retarded-advanced propagators and the Fermi equilibrium distribution function, $n_F(\omega) = 1/[1 + \exp\{\beta(\omega - \mu)\}]$, where μ is the chemical potential and $\beta = 1/k_B T$ as

$$\hat{G}^{+-}(\omega) = n_F(\omega) [G^a(\omega) - G^r(\omega)] . \quad (10.9)$$

Thus, for instance, the mean current can be written as

$$I_\nu = \frac{e}{\hbar} \sum_k \int d\omega n_F(\omega) \text{Tr} \left[V_{k\nu} \text{Re} (G_{k\nu,0}^a - G_{k\nu,0}^r) \right] . \quad (10.10)$$

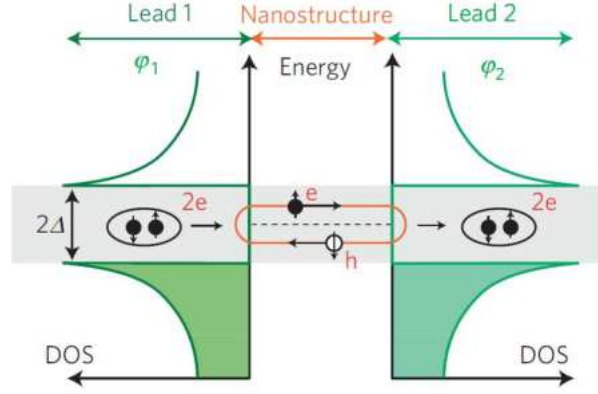


Figure 10.2: Schematic representation of the physical mechanism responsible for the formation of ABS in a generic nanostructure coupled to superconducting leads. Reprinted by permission from Macmillan Publishers Ltd: Nature Physics [90], copyright (2010).

10.3 Equilibrium properties of quantum dots with superconducting leads

For a nanoscale system coupled to superconducting electrodes a finite current can flow even in the absence of an applied bias voltage due to the Josephson effect. We shall illustrate this effect starting from the non-interacting situation ($U = 0$) within the single level Anderson model introduced above. In this case the advanced-retarded Green function of the coupled dot can be expressed as

$$G_{00}^{a,r}(\omega) = \begin{pmatrix} \omega - \epsilon_0 - \Gamma_L g_L^{a,r} - \Gamma_R g_R^{a,r} & \Gamma_L e^{i\phi_L} f_L^{a,r} + \Gamma_R e^{i\phi_R} f_R^{a,r} \\ \Gamma_L e^{-i\phi_L} f_L^{a,r} + \Gamma_R e^{-i\phi_R} f_R^{a,r} & \omega + \epsilon_0 - \Gamma_L g_L^{a,r} - \Gamma_R g_R^{a,r} \end{pmatrix}^{-1}, \quad (10.11)$$

where $f_{L,R}^{a,r} = |\Delta_{L,R}| / \sqrt{|\Delta_{L,R}|^2 - (\omega \pm i0^+)^2}$ and $g_{L,R}^{a,r} = -(\omega \pm i0^+) f_{L,R}^{a,r} / |\Delta_{L,R}|$ are the dimensionless BCS green functions of the uncoupled leads. For simplicity we focus below on the case where both leads are of the same material for which $|\Delta_L| = |\Delta_R| = \Delta$.

The spectral density associated with this model exhibits bound states within the superconducting gap (i.e. $|\omega| \leq \Delta$). Physically, the ABSs arise from virtual multiple Andreev reflection processes at the interface between the dot and each of the leads. In such processes and for energies inside the gap, the electrons (holes) incident towards the leads are reflected back as holes (electrons), as illustrated in Fig. 10.2. The condition for the appearance of ABSs is that the accumulated phase in the closed trajectory be a multiple of 2π , which is equivalent to satisfying the equation

$$D(\omega) \equiv [\omega - \epsilon_0 - \Gamma g(\omega)] [\omega + \epsilon_0 - \Gamma g(\omega)] - |\Gamma_L e^{i\phi_L} + \Gamma_R e^{i\phi_R}|^2 f(\omega)^2 = 0, \quad (10.12)$$

where we have used that the dimensionless BCS Green functions g, f become real for energies inside the superconducting gap. It can be shown [95] that this equation has two real roots inside the gap $\pm\omega_s$, i.e. symmetrically located with respect to the Fermi energy.

An essential property of the ABSs is that they correspond to current-carrying states. In fact, due to its dependence on the superconducting phase difference they have associated a Josephson current $i_s(\phi) = 2e/\hbar (\partial\omega_s/\partial\phi)$. The total Josephson current is obtained by adding the contribution of all states with finite occupation. Thus, at zero temperature only the lower

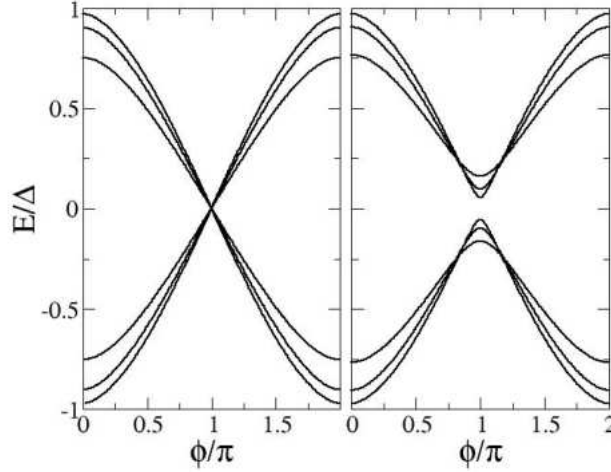


Figure 10.3: Andreev bound states for a non-interacting S-QD-S with different Γ/Δ values: 1.0, 2.0 and 4.0. Left panel corresponds to the e-h symmetric case ($\epsilon_0 = 0$) and the right panel to a case with $\epsilon_0 = 0.5\Delta$.

ABS contributes and there is an additional contribution from the continuous spectrum $\omega < -\Delta$ which we discuss below.

The ABS equation (10.12) becomes particularly simple for an electron-hole and left-right symmetric case (i.e. $\epsilon_0 = 0$ and $\Gamma_L = \Gamma_R$) when it can be reduced to

$$\omega \pm \Delta \cos \phi/2 + \frac{\omega \sqrt{\Delta^2 - \omega^2}}{\Gamma} = 0, \quad (10.13)$$

which for $\Gamma \gg \Delta$ has the simple solutions $\omega_s \simeq \pm \tilde{\Delta} \cos \phi/2$, where $\tilde{\Delta}$ is a reduced gap parameter which for $|\phi| \ll 1$ is given by $\tilde{\Delta} = \Delta [1 - 2(\Delta/\Gamma)^2]$. The ABSs for this case tend to the ones of a perfectly transmitting one channel superconducting contact $\pm \Delta \cos \phi/2$, the main qualitative difference being the reduced amplitude of their dispersion detaching them from the gap edges at $\phi = 2n\pi$. This is illustrated in Fig. 10.3(a).

In the absence of electron-hole symmetry (i.e. $\epsilon_0 \neq 0$) a finite internal gap between the upper and lower ABSs appears as in the case of a non-perfect transmitting one channel contact. When $\Delta/\Gamma \rightarrow 0$ the ABSs for this case are given by $\omega_s = \pm \Delta \sqrt{1 - \tau \sin^2(\phi/2)}$, where $\tau = 1/(1 + (\epsilon_0/\Gamma)^2)$ is the normal transmission at the Fermi energy. Outside this limiting case the ABSs exhibit both the internal gap and the detachment from the continuum states at $\phi = 2n\pi$, as it is illustrated in Fig. 10.3(b).

An interesting issue to comment is that the contribution from the states in the continuous spectrum becomes negligible when $\Delta/\Gamma \rightarrow 0$. In this limit the zero temperature current-phase relation (CPR) is simply given by

$$i_s(\phi) = \frac{e\Delta}{2\hbar} \frac{\tau \sin \phi}{\sqrt{1 - \tau \sin^2(\phi/2)}}, \quad (10.14)$$

which is the CPR of a one channel contact with transmission τ . For finite Δ/Γ there is a contribution from the continuum states. The expression for the total Josephson current in the non-interacting case can be derived from Eq. (10.10), which yields the following compact form

$$I = \frac{8e}{\hbar} \Gamma_L \Gamma_R \sin \phi \int d\omega n_F(\omega) \text{Im} \left(\frac{f_L^r f_R^r}{D^r} \right). \quad (10.15)$$

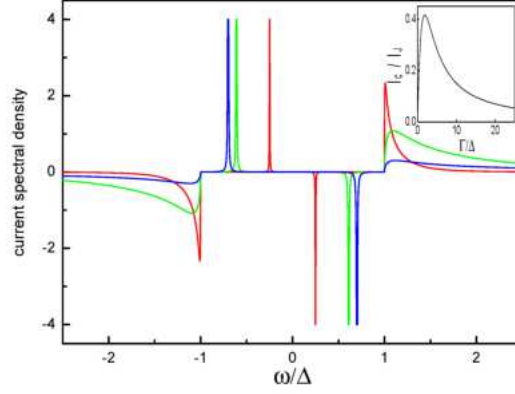


Figure 10.4: Current spectral density for the non-interacting S-QD-S system with ϵ_0 and $\phi = 1.5$ for increasing values of $\Gamma/\Delta = 0.5$ (red), 4 (green) and 16 (blue). A small broadening has been introduced to help visualizing the ABS contribution, which has been truncated for the sake of clarity. The inset shows the relative contribution of the continuous spectrum I_c compared to the total value I_J .

The current density in Eq. (10.15) contains both the contribution from the ABSs (region $|\omega| < \Delta$) and from the continuous spectrum $|\omega| > \Delta$. The behavior of the current density as a function of Δ/Γ is depicted in Fig. 10.4. As can be observed the contribution from the continuous spectrum has the opposite sign compared to the one arising from the ABS. The inset shows that this contribution becomes negligible in the limit $\Delta/\Gamma \rightarrow 0$ and reaches a maximum for $\Delta/\Gamma \sim 2$.

In the rest of this section we shall discuss the different theoretical approaches to include the effect of interactions in the dc Josephson effect through single level QD models. We also include a subsection on experimental results.

10.3.1 Cotunneling approach

From a theoretical point of view the simplest approach to account for the effect of interactions in the Josephson current is to perform a perturbative expansion to the lowest non-zero order in the tunnel Hamiltonian. This so-called cotunneling approach was first used by Glazman and Matveev [44], who predicted the onset of the π -junction behavior by this method. More precisely they obtained for the $U \rightarrow \infty$ limit

$$I(\phi) = \lambda \frac{e}{\hbar} \frac{\Gamma_L \Gamma_R}{\Delta} F\left(\frac{|\epsilon_0|}{\Delta}\right) \sin \phi, \quad (10.16)$$

where λ changes its value from 2 ($\epsilon_0 > 0$) to -1 ($\epsilon_0 < 0$), thus describing the transition to the π -phase, the function $F(x)$ having the form

$$F(x) = \frac{1}{\pi^2} \int \frac{dt_1 dt_2}{(\cosh t_1 + \cosh t_2)(x + \cosh t_1)(x + \cosh t_2)}.$$

This approximation is clearly not valid for describing the Kondo regime ($T_K \gg \Delta$) which requires non-perturbative approaches like the ones discussed in following subsections.

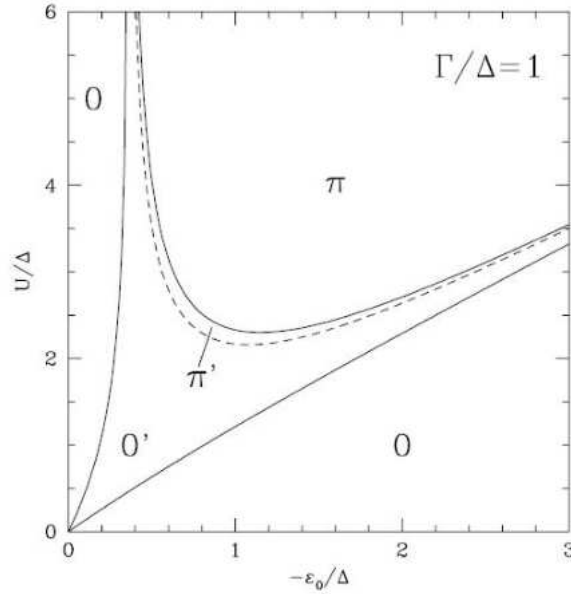


Figure 10.5: Phase diagram in the $U, -\epsilon_0$ plane for $\Gamma/\Delta = 1$ obtained using the HFA [97]). The phases are classified into 0, 0', π' and π as explained in the text. Reprinted figure with permission from A.V. Rozhkov and D. Arovas, Physical Review Letters **82**, 2788, 1999 [97]. Copyright (1999) by the American Physical Society.

10.3.2 Mean field and variational methods

Another simple approximate methods to deal with interactions are those of a mean field type like the Hartree-Fock approximation (HFA) or the slave-boson mean field (SBMF). In spite of their simplicity these approximations are able to capture important qualitative features due to interactions in certain limits.

We start by analyzing the HFA. In the context of magnetic impurities in superconductors this method was first applied by Shiba [104], while for the analysis of the Josephson effect it was first considered in Ref. [97] and further analyzed in [122]. Within this approximation electrons with a given spin “feel” a static potential due to the average occupation of the opposite spin, which corresponds to a simple constant self-energy $(\Sigma_{00})_{11} = U < n_{0\downarrow} >$ and $(\Sigma_{00})_{22} = -U < n_{0\uparrow} >$. In principle within the same level of approximation there appears a non-diagonal self-energy taking into account the induced pairing within the dot due to proximity effect, which can be written as $(\Sigma_{00})_{12} = U < c_{0\uparrow}^\dagger c_{0\downarrow}^\dagger >$. The effect of this non-diagonal contribution, which was not included in Ref. [97], was analyzed in Ref. [122]. The determination of the self-energy in the HFA requires a self-consistent calculation by using Eqs. (10.8) which cannot be performed analytically in general.

The most significant result within the HFA is the appearance of a broken symmetry state in which the dot acquires a finite magnetic moment (i.e. $< n_{0\uparrow} > \neq < n_{0\downarrow} >$) for certain ranges of parameters. In this respect one should be cautious in principle as the HFA is known to predict also broken symmetry states for the same model with normal leads [2], which are known to be spurious. However, for the superconducting case ground states with a finite magnetization do exist for certain parameter range as commented in the introduction. As it is shown below the HFA gives a rather good estimate of the magnetic ground state energy in the regions where it is the most stable phase.

The general properties of the ground state within the HFA are most conveniently displayed by a phase-diagram like the one in Fig. 10.5. In this diagram the notation “0”, “0’”, “ π' ” and “ π ” corresponds to the different ground state symmetries. Thus, “0” corresponds to the non-

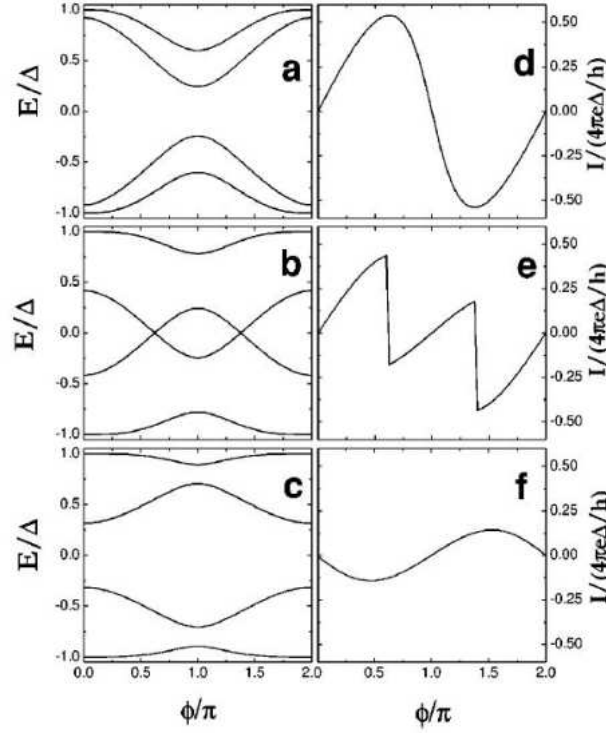


Figure 10.6: Evolution of the ABSs and Josephson current in the “toy” model of Ref. [122] for $\epsilon_0/\Gamma = -0.5$ and increasing E_{ex}/Γ parameter: 0.25 (upper panels), 0.75 (middle panels) and 1.5 (lower panels). Reprinted figure with permission from E. Vecino *et al.*, Physical Review B **68**, 035105, 2003 [122]. Copyright (2003) by the American Physical Society.

magnetic case for all values of ϕ (the absolute energy minimum being located at $\phi = 0$), while the “ π ” denotes that the magnetic solution is the most stable for all ϕ values (with the absolute minimum at $\phi = \pi$). On the other hand, “0’” and “ π ’” refer to intermediate situations with mixed magnetic and non-magnetic solutions as a function of ϕ , the name indicating whether the absolute energy minimum corresponds to a non-magnetic or a magnetic solution. From Fig. 10.5 the broken symmetry ground states are predicted to appear around the $\epsilon_0 = -U/2$ line, which corresponds to the half-filled case for sufficiently large U , i.e. $U > \Gamma, \Delta$. It is worth noticing that for normal leads this region corresponds to the deep Kondo regime, which anticipates an interesting interplay between both effects in the superconducting case beyond the HFA.

Further insight into the HFA solution can be provided by a “toy” model introduced in Ref. [122] (A similar model was analyzed in Ref. [10]). In this simplified model the finite magnetization which appears in the HFA is simulated by means of an exchange field parameter, E_{ex} , corresponding to the splitting of the diagonal dot levels for each spin, i.e. $\epsilon_{0\sigma} = \epsilon_0 + \sigma E_{ex}$. The analysis of the Andreev states within this toy model is similar to the one given at the beginning of this section for the non-interacting case, and becomes particularly simple in the limit $\Delta \ll \Gamma$ in which they adopt the analytical expression

$$\left(\frac{\omega_{\pm}}{\Delta}\right)^2 = \frac{\cos^2 \phi/2 + 2E^2 + Z^2(Z^2 + \sin^2 \phi/2) \pm 2XS(\phi)}{Z^4 + 2(X^2 + E^2) + 1}, \quad (10.17)$$

where $E = \epsilon_0/2\Gamma$, $X = E_{ex}/2\Gamma$ and $Z^2 = X^2 - E^2$ and $S(\phi)$ is given by

$$S(\phi) = \sqrt{Z^2 \cos^2 \phi/2 + E^2 + \sin^2 \phi/4}.$$

This expression clearly shows that the effect of the exchange field is to break the spin degeneracy producing an splitting of the ABSs. Consequently for $E_{ex} \neq 0$ one could in principle observe up to four ABSs in the spectral density. The evolution of these states with increasing E_{ex} is shown in Fig. 10.6 together with the corresponding Josephson current. While for $E_{ex} < \Gamma$ the splitting is small and states corresponding to different spin orientation do not cross, for increasing E_{ex} the upper and lower states closer to the Fermi energy begin to cross yielding a current-phase relation of $0'$ or π' character. Eventually for sufficiently large E_{ex} these two states completely interchange position with a complete reversal of the sign of the current (π -phase). It should be mentioned that although in this toy model the spin degeneracy is artificially broken, it nevertheless qualitatively simulates the behavior of the exact solution in the magnetic phase.

Another simple approach of a mean field type is provided by the slave boson mean field approximation (SBMFA). This method was introduced by Coleman for the normal Anderson model [27, 53]. It is based in the introduction of auxiliary boson fields $b_0^\dagger, b_0$ which act as projectors onto the empty impurity state. At the same time fermion creation and annihilation operators $f_{0\sigma}^\dagger, f_{0\sigma}$ are introduced for describing the singly occupied states. In order to get rid of the doubly occupied states in the $U \rightarrow \infty$ these operators should satisfy the completeness relation

$$b_0^\dagger b_0 + \sum_{\sigma} f_{0\sigma}^\dagger f_{0\sigma} = 1 \quad (10.18)$$

In terms of these operators the terms H_{QD} and H_T become

$$\begin{aligned} H_{QD} &= \sum_{\sigma} \epsilon_0 f_{0\sigma}^\dagger f_{0\sigma} \\ H_T &= \sum_{k\sigma, \nu} \left(V_{k\nu} c_{k\nu, \sigma}^\dagger b_0^\dagger f_{0\sigma} + \text{h.c.} \right). \end{aligned} \quad (10.19)$$

So far this transformation is exact in the $U \rightarrow \infty$ limit. Specific diagrammatic methods to obtain the impurity self-energy in this slave boson formulation have been developed [13]. Within this formulation the simplest solution is provided by the mean field approximation in which the boson operator is treated as a c-number. In fact the dot Hamiltonian reduces in this case to

$$H_{QD}^{MF} = \sum_{\sigma} \epsilon_0 f_{0\sigma}^\dagger f_{0\sigma} + \lambda \left(|b_0|^2 + \sum_{\sigma} f_{0\sigma}^\dagger f_{0\sigma} - 1 \right), \quad (10.20)$$

where λ is a Lagrange multiplier associated to the constraint (10.18). The problem becomes equivalent to a non-interacting impurity model with renormalized parameters $\tilde{\epsilon}_0 = \epsilon_0 + \lambda$ and $\tilde{\Gamma}_\nu = |b_0|^2 \Gamma_\nu$. Self-consistency is achieved by minimizing the system free energy.

Strictly speaking the mean field approximation is only valid in the $N \rightarrow \infty$ limit where N is the level degeneracy of the Anderson model (for the single level case $N=2$). However, the mean field approximation yields a reasonably good description of quantities like the Kondo temperature in the normal case [53]. When applied to the Anderson model with superconducting electrodes the SBMFA is only valid in the regime $T_K \gg \Delta$ as it is not able to describe the transition to the π -phase when $T_K \sim \Delta$. In the regime $T_K \gg \Delta$ the ABSs as described by the SBMFA corresponds to the non-interacting case with renormalized parameters $\tilde{\epsilon}_0$ and $\tilde{\Gamma}$. In this way the ABSs within the SBMFA in this regime would be given by $\omega_s(\phi) = \pm \sqrt{1 - \tilde{\tau} \sin^2 \phi/2}$

with $\tilde{\tau} = 4T_K^2/(\tilde{\epsilon}_0^2 + 4T_K^2)$. In principle, the self-consistent effective parameters in the superconducting state can differ from those in the normal state. However, in the limit $T_K \gg \Delta$ in which the approximation is supposed to work this difference can be neglected. The SBMFA in the $U \rightarrow \infty$ limit has only been applied for the case of superconducting leads in a few references: Avishai et al. [6] for analyzing the dc current with an applied bias voltage, and in Ref. [133] for studying the dynamics of Andreev states in the Kondo regime. Both works correspond to the non-equilibrium situation which will be discussed in Section 10.5.

For a proper description of the phase-diagram within a mean-field slave boson approach a finite- U version of the method, like the one introduced by Kotliar and Ruckenstein [70], is necessary. Within this method the number of auxiliary boson fields is extended up to four, denoted by e, p_σ and d , which project into the empty, singly occupied (with either spin orientation) and doubly occupied dot states respectively. These operators must satisfy the constraints

$$\sum_{\sigma} p_{\sigma}^{\dagger} p_{\sigma} + e^{\dagger} e + d^{\dagger} d = 1$$

$$c_{0\sigma}^{\dagger} c_{0\sigma} = p_{\sigma}^{\dagger} p_{\sigma} + d^{\dagger} d. \quad (10.21)$$

For recovering the non-interacting limit it is necessary to introduce also an auxiliary operator $z_{\sigma} = (1 - d^2 - p_{\sigma}^2)^{-1/2}(e p_{\sigma} + p_{\bar{\sigma}} d)(1 - e^2 - p_{\bar{\sigma}}^2)^{-1/2}$, in terms of which the model Hamiltonian becomes

$$H = H_L + H_R + \sum_{\sigma} \epsilon_0 \hat{f}_{0\sigma}^{\dagger} \hat{f}_{0\sigma} + U d^{\dagger} d + \sum_{k\nu, \sigma} \left(V_{k\nu} z_{\sigma}^{\dagger} \hat{f}_{0\sigma}^{\dagger} c_{k\nu\sigma} + h.c. \right)$$

$$- \lambda \left(e^{\dagger} e + d^{\dagger} d + \sum_{\sigma} p_{\sigma}^{\dagger} p_{\sigma} - 1 \right) - \sum_{\sigma} \lambda_{\sigma} \left(f_{0\sigma}^{\dagger} f_{0\sigma} - p_{\sigma}^{\dagger} p_{\sigma} - d^{\dagger} d \right), \quad (10.22)$$

where λ and λ_{σ} are the Lagrange multipliers associated with the constraints (10.21). Again, in a mean field approximation, the auxiliary fields are treated as c-numbers to be determined self-consistently.

The type of phase-diagram that is obtained within the finite- U SBMFA will be analyzed in subsection 10.3.4, for the zero band-width limit which allows a comparison with exact diagonalizations. As it is shown in that subsection the finite- U SBMFA tends to underestimate the stability of the π -phase in contrast with the HFA, which typically overestimates it.

Another relatively simple approach is provided by the use a variational wave-function. This approach was used by Rozhkov and Arovas [95] extending previous works [120] in which variational wave-functions were proposed for analyzing the normal Kondo problem. In their work Rozhkov and Arovas propose different many-body variational states in the $U \rightarrow \infty$ limit for the singlet and the doublet states, looking for the their relative stability. They find a transition between both ground states for $\Delta/T_K \sim 2$ and also predict the appearance of the intermediate phases $0'$ and π' in addition to the pure 0 and π ones.

10.3.3 Diagrammatic approaches

Perturbation theory in the Coulomb interaction

A natural extension over the HFA is provided by applying diagrammatic perturbation theory in the Coulomb parameter U . Already at the level of second order one can obtain an approximation for the self-energy which is able to capture part of the interplay between Kondo effect and pairing interactions. This approximation has been applied both for the S-QD-S case in equilibrium [84, 122], as well as for the N-QD-S case [30, 127].

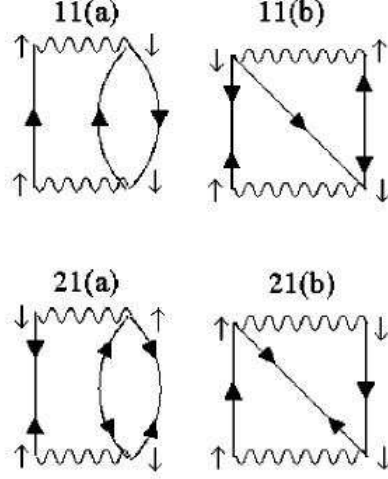


Figure 10.7: Feynmann diagrams for the second-order self-energy in the Anderson model with superconducting electrodes. Reprinted figure with permission from E. Vecino *et al.*, Physical Review B **68**, 035105, 2003 [122]. Copyright (2003) by the American Physical Society.

The diagrams contributing to the second-order self-energy in the superconducting Anderson model are depicted in Fig. 10.7. The first diagrams (denoted as 11(a)) is equivalent to the one appearing in the normal case, describing interaction of an electron with an electron-hole pair with opposite spin. The other diagrams include anomalous superconducting propagators and are therefore characteristic of the superconducting state. The presence of these propagators gives several effects: the appearance of non-diagonal elements of the self-energy in Nambu space, and the presence of diagrams like 11(b) and 21(b) in Fig. 10.7 which corresponds to a double-exchange process. Finally, diagram 21(a) describes the interaction of a Cooper pair with fluctuations in the pairing amplitude within the dot.

Formally, these diagrams can be computed from the full Green functions of the non-interacting case by means of the expressions [122]

$$\Sigma_{11,a}^{r(2)}(\omega) = \frac{U^2}{(2\pi i)^3} \int d\epsilon_1 \int d\epsilon_2 \int d\epsilon_3 \frac{G_{11}^{(0)+-}(\epsilon_1)G_{22}^{(0)+-}(\epsilon_2)G_{22}^{(0)-+}(\epsilon_3) + G_{11}^{(0)-+}(\epsilon_1)G_{22}^{(0)-+}(\epsilon_2)G_{22}^{(0)+-}(\epsilon_3)}{\omega - \epsilon_1 - \epsilon_2 + \epsilon_3 + i0^+}$$

$$\Sigma_{11,b}^{r(2)}(\omega) = \frac{U^2}{(2\pi i)^3} \int d\epsilon_1 \int d\epsilon_2 \int d\epsilon_3 \frac{G_{12}^{(0)+-}(\epsilon_1)G_{21}^{(0)+-}(\epsilon_2)G_{22}^{(0)-+}(\epsilon_3) + G_{12}^{(0)-+}(\epsilon_1)G_{21}^{(0)-+}(\epsilon_2)G_{22}^{(0)+-}(\epsilon_3)}{\omega - \epsilon_1 - \epsilon_2 + \epsilon_3 + i0^+}$$

$$\Sigma_{21,a}^{r(2)}(\omega) = -\frac{U^2}{(2\pi i)^3} \int d\epsilon_1 \int d\epsilon_2 \int d\epsilon_3 \frac{G_{21}^{(0)+-}(\epsilon_1)G_{12}^{(0)+-}(\epsilon_2)G_{21}^{(0)-+}(\epsilon_3) + G_{21}^{(0)-+}(\epsilon_1)G_{12}^{(0)-+}(\epsilon_2)G_{21}^{(0)+-}(\epsilon_3)}{\omega - \epsilon_1 - \epsilon_2 + \epsilon_3 + i0^+}$$

$$\Sigma_{21,b}^{r(2)}(\omega) = \frac{U^2}{(2\pi i)^3} \int d\epsilon_1 \int d\epsilon_2 \int d\epsilon_3 \frac{G_{22}^{(0)+-}(\epsilon_1)G_{11}^{(0)+-}(\epsilon_2)G_{21}^{(0)-+}(\epsilon_3) + G_{22}^{(0)-+}(\epsilon_1)G_{11}^{(0)-+}(\epsilon_2)G_{21}^{(0)+-}(\epsilon_3)}{\omega - \epsilon_1 - \epsilon_2 + \epsilon_3 + i0^+}$$

The evaluation of these expressions for a general range of parameters requires a significant numerical effort. An efficient algorithm can be implemented to evaluate these expressions based on Fast Fourier transformations, as discussed in [102].

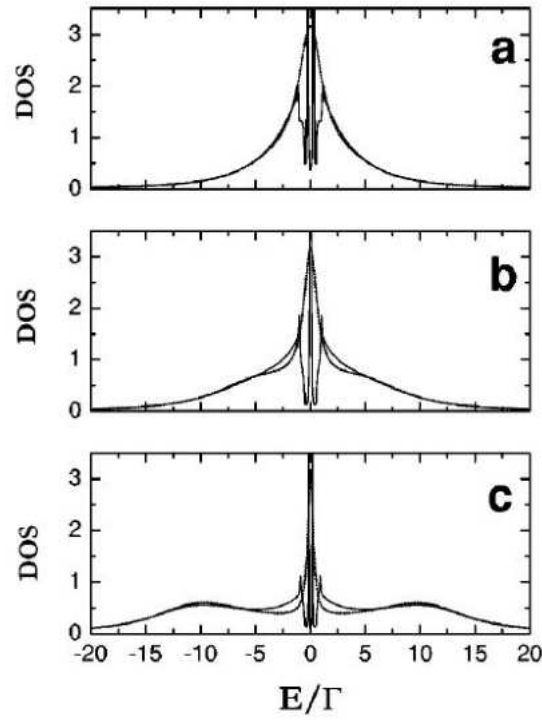


Figure 10.8: Evolution of the DOS in the S-QD-S system in equilibrium within the second-order self-energy approximation for an electron-hole symmetric case with $\Delta/\Gamma = 0.1$. The U/Γ parameter takes the values 2.5 (a), 5 (b) and 10 (c). Reprinted figure with permission from E. Vecino *et al.*, Physical Review B **68**, 035105, 2003 [122]. Copyright (2003) by the American Physical Society.

In the limit $\Delta/\Gamma \ll 1$ Kondo correlations dominate over pairing ones. The results of the second-order self-energy approach for the half-filled case capture the main features of the onset of Kondo correlations in the spectral density when $U > \Gamma$. This is illustrated in Fig. 10.8 taken from Ref. [122], which shows its evolution for increasing U/Γ . The spectral density is similar to the one found in the normal state except for the superimposed features inside the superconducting gap. The overall shape evolves from the single Lorentzian broad resonance for $U < \Gamma$ to the three peaked structure characteristic of the Kondo regime when $U > \Gamma$. In this regime the width of the central Kondo peak is set by the scale T_K , which in the present approximation is given by $T_K \sim \Gamma/(1 - \alpha_0)$, where

$$\alpha_0 = \frac{\partial \Sigma_{11}}{\partial \omega}(0) \simeq - \left(\frac{U}{2\pi\Gamma} \right)^2 \left(3 - \frac{\pi^2}{4} \right), \quad (10.23)$$

thus coinciding with the perturbative result in the normal state (see Ref. [126]). Although this perturbative approach fails to yield the exponential behavior of T_K for large U/Γ , it provides a reliable description of the spectral density for moderate values of this parameter [41].

The second-order self-energy allows also to analyze the renormalization of the ABSs due to the presence of Coulomb interactions. For values of $U/\Gamma < 10$ the renormalized ABSs maintain approximately the $\sim \cos \phi/2$ behavior of the non-interacting case but with a narrower dispersion set by $\omega_s(0) \simeq \Delta [1 - (U/U_0)^2]$, where $(U_0/\Gamma)^2 = (\Gamma/\Delta)\pi^2/(2\pi + 2)$.

On the other hand, when $T_K \sim \Delta$ a transition to the π -phase is expected. Within the second-order self-energy approach the transition can be identified by allowing for a breaking of the spin-symmetry in the initial non-interacting problem and searching for self-consistency. Rather than imposing the consistency condition of the HFA, i.e. $\tilde{\epsilon}_{0\sigma} = \epsilon_0 + U < n_{0\bar{\sigma}} >$ in Ref. [122] it was imposed that the effective dot level for each spin-orientation be determined by the charge-consistency condition, i.e. $< n_{0\sigma} > = < n_{0\sigma}^0 >$, where $< n_{0\sigma}^0 >$ is the dot charge corresponding to the broken-symmetry non-interacting Hamiltonian. Such a procedure was shown to eliminate the unstable behavior of perturbation theory when developed from the HFA [132].

NCA approximation

Within the diagrammatic approximations one can include the so-called non-crossing approximation (NCA). In this case an infinite order resummation of the perturbation theory is performed starting from the $U \rightarrow \infty$ slave boson representation of the Anderson Hamiltonian [13]. To the lowest order in $1/N$ (where $N = 2$ is the spin degeneracy) the family of diagrams in this resummation is represented in Fig. 10.9. The dashed lines correspond to the fermion propagators and the wavy lines to the slave bosons. In the normal case the NCA include only the first two diagrams in the Dyson equation for the fermion and boson propagators. The extension to the superconducting case was proposed in Ref. [25] and corresponds to including the anomalous propagators for describing multiple Andreev reflection processes (last two diagrams in the fermion and boson self-energies represented in Fig. 10.9).

In the normal case the NCA theory has been shown to yield reliable results for temperatures down below T_K [13, 53] in spite of certain pathologies like its failure to fulfill the Friedel sum-rule. The self-consistent extension for the superconducting case by Clerk et. al. [24] is also formally exact to order $1/N$ and it is thus expected to yield reasonable results even in the presence of MAR processes.

Further analysis of the NCA applied to the S-QD-S system was provided in Ref. [103]. Their results for the Josephson current and LDOS in the superconducting gap region are summarized in Fig. 10.10. The fact that the calculations are performed for temperatures which are a quite large fraction of Δ yields very broad resonances for the subgap states. As can be observed in

$$\begin{aligned}\Sigma(i\omega) &= \text{diagram 1} + \text{diagram 2} \\ \Pi(iv) &= \text{diagram 3} + \text{diagram 4}\end{aligned}$$

Figure 10.9: Fermion (top) and boson (bottom) self-energy diagrams in the NCA approximation extended to the superconducting case. Reprinted figure with permission from G. Sellier *et al.*, Physical Review B **72**, 174502, 2005 [103]. Copyright (2005) by the American Physical Society.

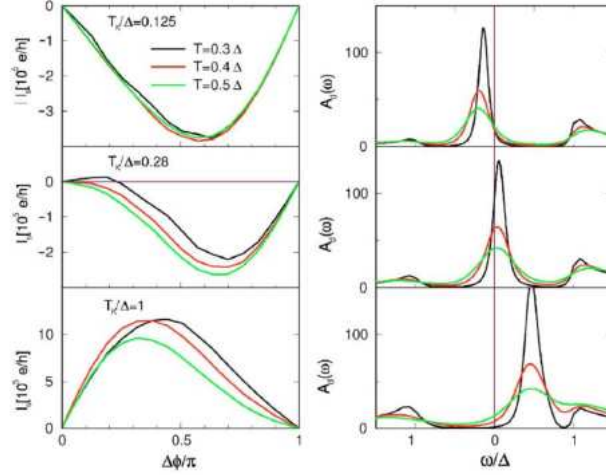


Figure 10.10: Josephson current and subgap LDOS for the equilibrium S-QD-S model within the NCA for three values of T_K/Δ and three different temperatures. Reprinted figure with permission from G. Sellier *et al.*, Physical Review B **72**, 174502, 2005 [103]. Copyright (2005) by the American Physical Society.

Fig. 10.10, only one broad resonance can be clearly resolved within the gap. These results are in contrast to what is obtained using exact numerical methods as will be discussed in Sect. 10.3.4. In this approximation the transition to the π -phase appears as a smooth crossover which can be associated to the crossing of this resonance through the Fermi energy.

Real time diagrammatic approach

Another technique which has been applied to the study of quantum dots coupled to superconducting leads is the real time diagrammatic approach first introduced by König *et al.* [69] for the normal Anderson model. The main idea of this technique is to integrate out the fermionic degrees of freedom of the electrodes leading to a reduced description of the density matrix projected on the Hilbert space of the isolated dot states. In the superconducting case this reduced density matrix also depends on the number of Cooper pairs in the leads relative to some chosen reference. The aim of the technique is to determine the time evolution of this reduced density matrix in the Keldysh contour thus allowing to consider both equilibrium and non-equilibrium situations. In Ref. [49] the method has been applied to the equilibrium S-QD-S case obtaining results in agreement with those of Ref. [44] in the cotunneling limit. The method has been mainly applied to analyze the properties of quantum dots connected to both normal and superconducting leads in multiterminal configurations out of equilibrium, an issue which will be commented in Sect. 10.6.

10.3.4 Diagonalization by numerical methods

Within this subsection we will review methods which attempt a direct diagonalization of the superconducting Anderson model, either by truncating the initial Hilbert space using physical arguments valid for certain parameter region or by using the Numerical Renormalization group (NRG) method.

Exact diagonalization for the large Δ limit

An exact diagonalization of the model is possible in the limit $\Delta \rightarrow \infty$. In this limiting case the Hilbert space of the problem is automatically reduced to states spanned by the different electronic configuration of the dot levels. The effect of the superconducting leads appears as a pairing term between the electrons within the dot. The effective Hamiltonian for the truncated Hilbert space becomes [122, 118, 114]

$$H^{eff} = 2\Gamma \cos \phi/2 \left(c_{0\uparrow} c_{0\downarrow} + c_{0\downarrow}^\dagger c_{0\uparrow}^\dagger \right) + \epsilon_0 \sum_{\sigma} n_{0\sigma} + U n_{0\uparrow} n_{0\downarrow}. \quad (10.24)$$

The eigenvalues of this reduced Hamiltonian can be determined straightforwardly by noting the decoupling of subspaces with even and odd number of electrons. The ground state for the even case (corresponding to total spin $S = 0$) is a linear combination of the empty and doubly occupied dot state with an energy

$$E_{0(S=0)}(\phi) = \epsilon_0 + U/2 - \sqrt{(\epsilon_0 + U/2)^2 + 4\Gamma^2 \cos^2 \phi/2}. \quad (10.25)$$

On the other hand, the odd number subspace simply corresponds to a single uncoupled spin with energy $E_{0(S=1/2)} = \epsilon_0$. The transition to the magnetic ground state thus occurs for $E_{0(S=1/2)} = E_{0(S=0)}(\phi)$. In the simpler electron-hole symmetric case ($\epsilon_0 = -U/2$) this condition reduces to $2\Gamma \cos \phi/2 = U/2$ and thus the full π state appears for $\Gamma < U/4$. This simple model already gives a rough qualitative account of the $0 - \pi$ quantum phase transition.

A further step in the idea truncating the Hilbert space is performed in the so called zero band-width limit [1, 122, 12]. In this approximation the superconducting leads are represented by a single localized level (which formally corresponds to the limit of vanishing width of the leads spectral density). This approximation is justified when the superconducting gap is large compared to the other energy scales in the problem, and thus can be considered as a refinement with respect to the previous approach. The corresponding Hamiltonian can be written as $H = H_d + H_T + H_L + H_R$, with

$$H_\nu = \sum_{\sigma} \epsilon_\nu c_{\nu\sigma}^\dagger c_{\nu\sigma} + (\Delta_\nu c_{\nu\uparrow} c_{\nu\downarrow} + \text{h.c.})$$

$$H_T = \sum_{\nu,\sigma} (V_\nu c_{\nu\sigma}^\dagger c_{0\sigma} + \text{h.c.}), \quad (10.26)$$

where $\nu = L, R$ denotes the left-right sites describing the leads in this approximation.

Although the total number of particles is not a good quantum number, their parity is conserved as in the previous case. This allows to reduce the initial 64 states in the Hilbert space to a subspace of 20 states for even parity with a total spin z -component $S_z = 0$ and 15 states for odd parity with $S_z = \pm 1/2$. These values of S_z are the ones corresponding to the ground state in each subspace with total spin $S = 0$ and $S = 1/2$. In addition to providing a

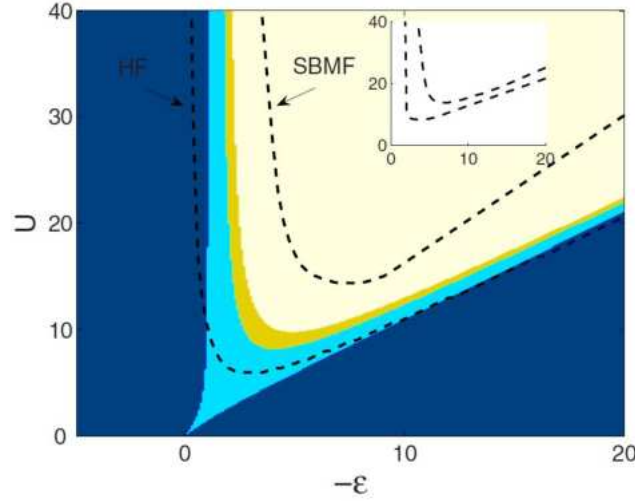


Figure 10.11: Phase diagram of the S-QD-S system in the ZBW model for the leads obtained by exact diagonalization for $\Delta = V_L = V_R$, taken as unit of energy. The dashed lines correspond to the boundary between the 0 and π regions within the HFA (lower line) and the finite-U SBMF (upper line). The inset show the corresponding results for these boundaries within the full model. Reprinted figure with permission from F.S. Bergeret *et al.*, Physical Review B **76**, 174510, 2007 [12]. Copyright (2007) by the American Physical Society.

qualitative description of the phase diagram of the full model, this simplified calculation can furthermore be useful as a test for comparing different approximation methods.

The phase diagram obtained within this approximation was discussed in Refs. [122, 12] and is shown in Fig. 10.11 for $\Gamma \equiv V = \Delta$, with $V_L = V_R = V$. As can be observed, the overall diagram is very similar to the one shown before for the HFA (Fig. 10.5) exhibiting the four phases 0, 0', π' and π in the same sequence. For a more direct comparison it is necessary to perform the HFA of the ZBW model, which as an exactly solvable model also provides a stringent test of the approximation. The lower broken line in Fig. 10.11 indicates the boundary of the π -phase within the HFA. It can be noticed that the HFA overestimates the stability of this magnetic phase. On the other hand, it is also possible to test the finite-U SBMF approximation in this ZBW model. The corresponding boundary for the π -phase is indicated by the upper broken line in Fig. 10.11. In opposition to the HFA this approximation overestimates the stability of the 0 phase, the exact boundary therefore lying in between the two different mean-field approximations. It is interesting to point out that a similar difference between both approximations is also found for the full model (see inset in Fig. 10.11). One would then expect that the exact boundary for the full model lays in between these two.

The ZBW model can also be used to test approximate methods beyond the mean field ones. In Ref. [122] this was done for the second-order self-energy approximation. As it is illustrated in Fig. 10.12 for the symmetric case ($\epsilon_0 = -U/2$) the second-order self-energy approach matches quite well the exact ground state energy for U/Δ values up to $\simeq 2.5$. It is interesting to note that for larger U values even when the HFA already yields a full π -state, the second-order approximation predicts a mixed ground state in agreement with the exact solution.

Numerical Renormalization Group (NRG)

The NRG method is based on the ideas of Wilson on logarithmic discretization for magnetic impurity problems [125] and was first applied to the Anderson model by Krishna-murthy et al. [73]. The idea behind the method is to discretize the energy levels in the leads on a logarithmic grid of energies Λ^{-n} (with $\Lambda > 1$ and $1 \leq n \leq N \rightarrow \infty$) with exponentially high resolution on

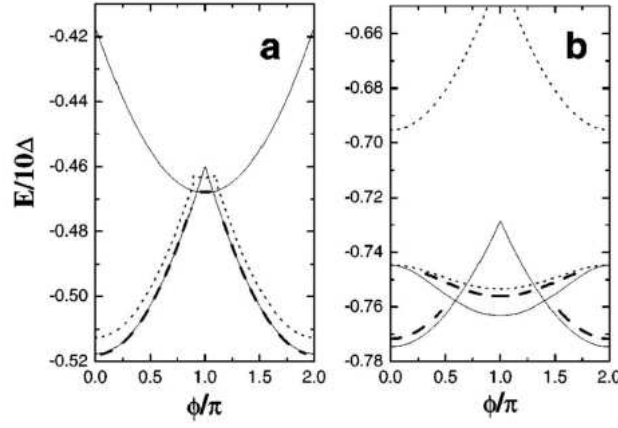


Figure 10.12: Ground state energy for the $S = 0$ and $S = 1/2$ states of the S-QD-S system in the ZBW model for the leads. Full lines correspond to the exact results, dotted lines to the HFA and dashed lines to the second-order self-energy approximation. Reprinted figure with permission from E. Vecino *et al.*, Physical Review B **68**, 035105, 2003 [122]. Copyright (2003) by the American Physical Society.

the low-energy excitations. This discretization allows then to map the impurity model into a linear "tight-binding" chain with hopping matrix elements decaying as $\Lambda^{-n/2}$ with increasing site index n . The sequence of Hamiltonians which is constructed by adding a new site in the chain is then diagonalized iteratively. As the number of states grows exponentially an adequate truncation scheme is required.

The NRG scheme has been first generalized to the case of an Anderson impurity in a superconducting host by Yoshioka and Ohashi [134] and implemented by several authors to analyze the S-QD-S model with a finite phase difference [20, 88, 118, 78]. For the left-right symmetric case (i.e. $\Delta_L = \Delta_R$ and $\Gamma_L = \Gamma_R$) the sequence of Hamiltonians can be written as [20]

$$\begin{aligned} \tilde{H}_{N+1} = & \sqrt{\Lambda} \tilde{H}_N + \xi_N \sum_{\mu, \sigma} \left(f_{\mu, N+1, \sigma}^\dagger f_{\mu, N+1, \sigma} + \text{h. c.} \right) \\ & - \Lambda^{N/2} \sum_{\mu} \tilde{\Delta}_{\mu} \left(f_{\mu, N+1, \uparrow}^\dagger f_{\mu, N+1, \downarrow} + \text{h. c.} \right), \end{aligned} \quad (10.27)$$

where the initial Hamiltonian is given by

$$\begin{aligned} \tilde{H}_0 = & \frac{1}{\sqrt{\Lambda}} \left[\tilde{H}_{QD} + \sum_{\mu=e,o} \sum_{\sigma} \tilde{V}_{\mu} \left(c_{0\sigma}^\dagger f_{\mu,0,\sigma} + \text{h. c.} \right) \right. \\ & \left. - \sum_{\mu} \tilde{\Delta}_{\mu} \left(f_{\mu,0,\sigma}^\dagger f_{\mu,0,\sigma} + \text{h. c.} \right) \right]. \end{aligned} \quad (10.28)$$

The fermion operators $f_{\mu, N, \sigma}$ correspond to an effective tight-binding chain resulting from the logarithmic discretization and the canonical transformation into the even-odd linear combination of original left-right states in the leads and

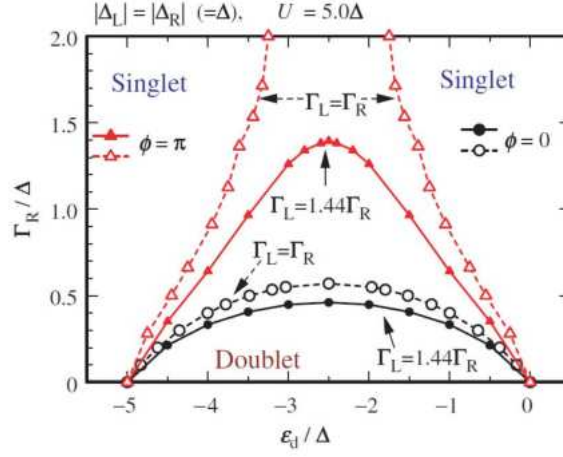


Figure 10.13: Phase diagram of the S-QD-S system in the Γ_R, ϵ_0 plane for fixed U and different values of Γ_L/Γ_R obtained using the NRG method. Reprinted figure with permission from Y. Tanaka *et al.*, New Journal of Physics **9**, 115, 2007 [118]. Copyright (2007) by IOP Publishing Ltd.

$$\begin{aligned}\tilde{H}_{QD} &\equiv \chi \frac{H_{QD}}{D} \quad , \quad \tilde{\Delta}_\mu \equiv \chi \frac{\Delta_\mu}{D} \\ \tilde{V}_e &= \chi \sqrt{\frac{2\Gamma}{\pi D}} \cos \phi/4 \quad , \quad \tilde{V}_o = -\chi \sqrt{\frac{2\Gamma}{\pi D}} \sin \phi/4,\end{aligned}\tag{10.29}$$

with $\chi = 2/(1 + 1/\Lambda)$ and D being an energy cut-off in the leads spectral density. The original Hamiltonian is recovered in the limit $H/D = \lim_{\Lambda \rightarrow \infty} \tilde{H}_N/(\chi \Lambda^{(N-1)/2})$.

The NRG method was applied to analyzing the Josephson current in a S-QD-S system in Ref. [20], this work confirming the predicted $0 - \pi$ quantum phase transition at $\Delta \sim T_K$ for the electron-hole symmetric case. It should be mentioned that more recent calculations [64] using NRG obtain Josephson currents which are approximately a factor 2 larger than the ones of [20]. On the other hand, Oguri *et al.* [88] used NRG to analyze this model in the case of $|\Delta_L| \gg |\Delta_R|$. In this case the model can be exactly mapped into a single channel model consisting on the right lead coupled to the Anderson impurity with a local pairing $\Delta_d \equiv \Gamma_L e^{i\phi_L}$, thus allowing a simpler implementation of the NRG algorithm. Further work for the $\Gamma_L \neq \Gamma_R$ case although with $\Delta_L = \Delta_R$ by Tanaka *et al.* [118] confirmed the presence of intermediate $0' - \pi'$ phases even in the left-right asymmetric case. A characteristic phase diagram obtained for this case is shown in Fig. 10.13.

In addition to the ground state properties, NRG methods have been applied in an attempt to clarify the structure of the subgap ABSs. In Ref. [78] the spectral density inside the gap obtained from the NRG algorithm was analyzed, showing that a pair of ABSs located symmetrically respect to the Fermi energy is present in the $U \rightarrow \infty$ case. This is in contrast to the NCA results discussed previously (shown in Fig. 10.10) where a single broad resonance appears. Similar conclusions are obtained in Ref. [7] although for the single lead case and for finite U . A word of caution should be said regarding the analysis of the ABSs in this last work in which the relation $\Sigma_{22}(\omega) = -\Sigma_{11}(-\omega)$ is assumed in their Eq. (8) for the states inside the gap. This relation would not be strictly valid for the doublet ground state when choosing a given spin orientation. In this case the quasi-particle excitation energies would become spin dependent and the electron-hole symmetry would be broken. This would allow in principle to have up to 4 ABSs inside the gap as predicted both by the Hartree-Fock approximation and

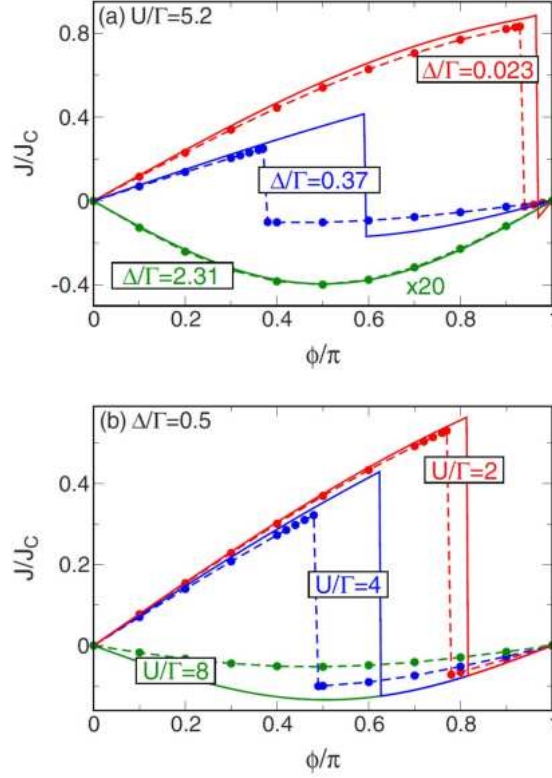


Figure 10.14: Current phase-relations for the S-QD-S system obtained using the fRG approach truncated at the HF level for different values of U/Γ and Δ/Γ . For comparison the results obtained using the NRG method are also plotted (indicated by the filled dots). Reprinted figure with permission from C. Karrash *et al.*, Physical Review B **77**, 024517, 2008 [64]. Copyright (2008) by the American Physical Society.

in the exact $\Delta \rightarrow \infty$ limit. Of course, in the π phase the spin is not frozen but is fluctuating. In this sense the above relation between the self-energy components would be recovered when averaging over the $S_z = 1/2$ and $S_z = -1/2$ states. We believe in any case that a more detailed analysis of the ABSs using the NRG method is still lacking.

10.3.5 Functional renormalization group

The functional renormalization group (fRG) method is based on the application of an RG analysis to the diagrammatic expansions in terms of electronic Green functions. This is an approximate method whose accuracy depends on the initial diagrams used in the evaluation of the electron self-energies. The starting point is the introduction of an energy cut-off Λ into the Matsubara non-interacting Green-functions

$$G^{0,\Lambda}(i\omega) = \Theta(|\omega| - \Lambda)G^0(i\omega)$$

Using these propagators the n -particle vertex functions acquires a Λ dependence. The flow equations are determined differentiating these vertex functions with respect to Λ which are then solved iteratively for increasing Λ . In Ref. [64] the method has been applied to the S-QD-S system employing a truncation scheme which keeps only diagrams corresponding to the static Hartree-Fock approximation. The Λ -dependent Green function used in Ref. [64] was of the form,

$$G^\Lambda(i\omega) = \begin{pmatrix} i\tilde{\omega} - \epsilon_0 - \Sigma^\Lambda & \tilde{\Delta}(i\omega) - \Sigma_\Delta^\Lambda \\ \tilde{\Delta}(i\omega) - \Sigma_\Delta^\Lambda & i\tilde{\omega} + \epsilon_0 + \Sigma^\Lambda \end{pmatrix}^{-1}, \quad (10.30)$$

where $i\tilde{\omega} = i\omega(1 + \sum_\mu \Gamma_\mu g(i\omega))$ and $\tilde{\Delta}(i\omega) = \sum_\mu \Gamma_\mu f(i\omega)e^{i\phi_\mu}$, g and f being the dimensionless BCS Matsubara Green functions of the uncoupled leads. Within this approximation the flow equations lead to energy-independent self-energies, corresponding to an effective non-interacting model with renormalized parameters. It is important to notice that this approximation exhibits also the limitation already pointed out in the previous section as it imposes electron-hole symmetry which is not satisfied in the magnetic phase. Nevertheless the approximation allows to identify a transition to a phase with inversion of the Josephson current which is driven by an "overscreening" of the induced pairing determined by Σ_Δ . Fig. 10.14 shows the comparison of fRG results with those obtained with the NRG method. The agreement is rather good for large Δ/Γ but it becomes poorer in the π -phase. We believe that the agreement could be improved allowing for a broken symmetry state within the same fRG approach.

10.3.6 Quantum Monte-Carlo

The Josephson current in the S-QD-S system has also been analyzed using Quantum Monte Carlo (QMC) simulations by Siano and Egger [106]. The method used was the Hirsh-Fisher algorithm adapted to this particular problem. They consider the deep Kondo regime $U/\Gamma \gg 1$ and $\epsilon_0/\Gamma \ll -1$ and show that the results for the Josephson current exhibit a universal dependence with T_K provided that $U/\Gamma > 5$. They identify the transition between the different phases at $\Delta/T_K \simeq 0.51, 0.875$ and 1.105 for $0 - 0'$, $0' - \pi'$ and $\pi' - \pi$ respectively [107]. Being a finite temperature calculation the resulting current-phase relations do not exhibit sharp discontinuities in the intermediate phases. This smooth behavior was criticized in Ref. [21] pointing out that the QMC results did not match the NRG ones of Ref. [20] at finite temperatures, which was attributed by Siano and Egger in their reply [108] to a possible limited accuracy of the NRG calculation of Ref. [20]. More recent NRG calculations of Ref. [64] give a good agreement with QMC results at finite temperature for ϕ values between $\pi/2$ and π , whereas QMC underestimates the current in the range $0 - \pi/2$. It is claimed in Ref. [64] that the origin of the discrepancy lies in the fact that the first excited state in this phase range is smaller or of the order of the temperature values used in the calculations of [106].

The QMC method has more recently been applied to analyze the spectral properties of this model in Ref. [81]. The authors employ the so-called weak-coupling continuous-time version of the method which is based on a perturbative expansion around the $U = 0$ limit. They show that the results for the spectral densities are in qualitative good agreement with the ones obtained in the zero band-width approximation introduced in Ref. [122], which was discussed before in this section.

10.3.7 Experimental results

Several physical realizations of the S-QD-S system have been obtained in the last few years by means of contacting carbon nanotubes (CNT), C60 molecules or semiconducting nanowires with superconducting electrodes (for a recent review see Ref. [42]). In view of the existence of this review on the experiments in this section we give only a brief summary of the main findings and its relation to the theoretical work.

CNTs have provided so far the most promising setups for a direct test of the theoretical predictions concerning the Josephson effect through a QD. The first experiments detecting a supercurrent through a CNT-QD strongly coupled to the leads (i.e. $\Gamma \gg \Delta, U$) were performed

by Jarillo-Herrero et al [58]. These experiments were basically performed in the resonant-tunneling regime with a single-level spacing $\delta\epsilon \gg \Gamma$. The results indicated a strong correlation between the critical current I_c and the normal conductance G_N . However, the product $I_c R_N$ deviated from a constant value due to the effect of the electromagnetic environment suppressing the critical current more strongly in off-resonance conditions due to phase fluctuations.

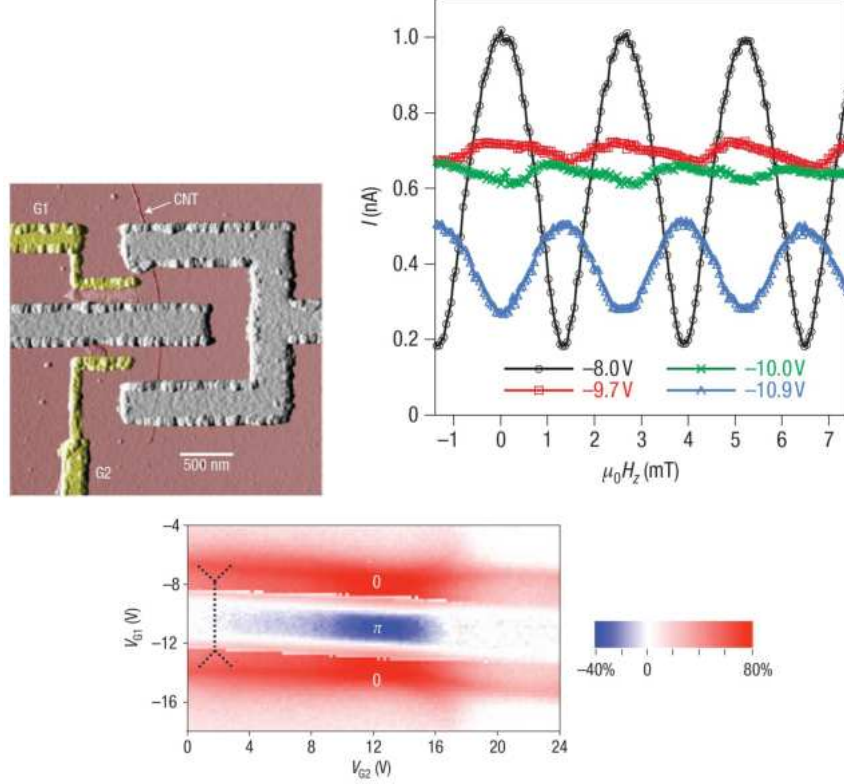


Figure 10.15: Experimental setup used in Ref. [26] for analyzing the $0 - \pi$ transition in CNT QDs (upper panel). The middle panel shows the reversal of the oscillatory pattern of the current as a function of the magnetic flux through the SQUID across the transition. The lower panel shows the correlation of the π -phase region with the Kondo ridges in the normal state (indicated by the dashed line). Reprinted by permission from Macmillan Publishers Ltd: Nature Physics [26], copyright (2006).

In a subsequent work by the same group the first experimental evidence of π -junction behavior in a S-QD-S system was obtained using semiconducting InAs nanowires with Al leads in a SQUID configuration [119]. However, the observed features in this experiment could not be explained completely on the basis of a single-level model but rather a multilevel description was necessary. In particular the authors showed that in this case the π -junction behavior is not necessarily linked to the parity of the number of the electrons in the dot (this will be further discussed in Sect. 10.6). On the other hand, π -junction behavior have also been demonstrated in Ref. [26] using CNT-QDs in a SQUID geometry. The corresponding experimental setup is depicted in Fig. 10.15. In this SQUID configuration the transition to the π -phase is directly demonstrated in the measured current-phase relation as a function of one of the applied gate voltages (see Fig. 10.15). Remarkably, this experiment allows to correlate the appearance of the π -junction behavior with the presence of Kondo correlations in the normal state. The dotted lines in the lower panel of Fig. 10.15 indicate the Kondo ridge appearing in the normal state (which is recovered by applying a magnetic field).

Evidence of a $0 - \pi$ transition has also been found in CNT-QD systems by analyzing the

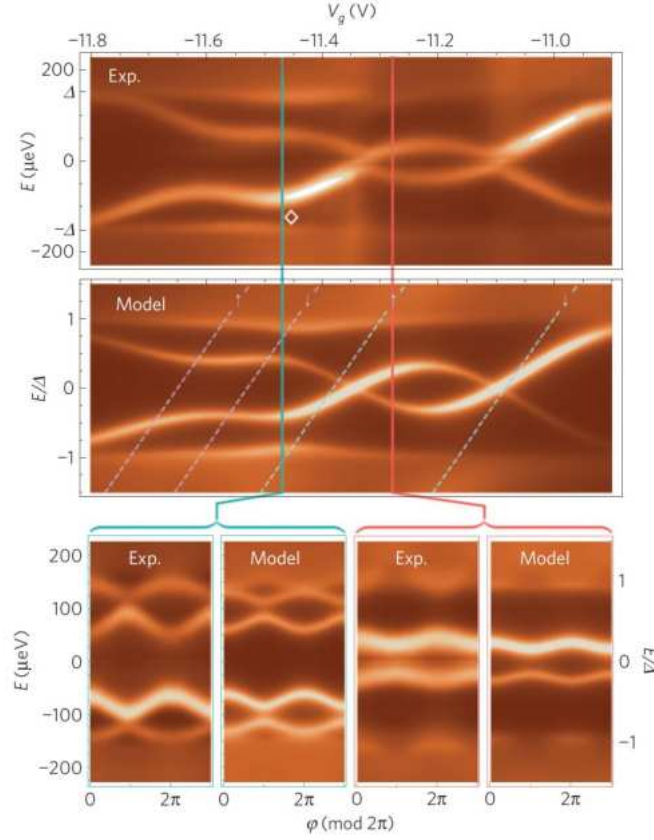


Figure 10.16: Color scale plots of the local density of states in a CNT QD coupled to SC leads as reported in Ref. [90] as a function of gate voltage and phase difference. The figure also shows the results obtained from a phenomenological model, as discussed in the text. Reprinted by permission from Macmillan Publishers Ltd: Nature Physics [90], copyright (2010).

current-voltage characteristic [62]. In this work it was shown that the evolution of the zero-bias conductance can be correlated to the behavior of the critical current when transversing the $0 - \pi$ boundary. As it corresponds to a non-equilibrium situation this analysis will be discussed later in Sect. 10.5. A similar observation holds for other experimental works [121, 51, 36] which will be discussed in that section.

Finally, it is worth pointing out recent experimental developments which have allowed to directly measure the Andreev bound states spectrum of a CNT-QD coupled to superconducting leads in a SQUID configuration [90]. In this experiment a weakly coupled lead was deposited at the center of the CNT to allow for tunneling spectroscopy measurements. In this way both the phase and gate voltage variation of the Andreev bound states were determined. The results could be fitted satisfactorily using a simplified phenomenological model corresponding to the superconducting Anderson model within a mean field approximation, in which an exchange field is included to represent the magnetic ground state. An example of the comparison between theory and experiment is given in Fig. 10.16. This analysis showed that a double dot model was in general necessary to fit the experimental results. Recent experimental work on graphene QDs coupled to SC leads [34] provided also evidence on the crossing of ABs as a function of the gate potential which is consistent with a magnetic ground state.

10.4 Quantum dots with normal and superconducting leads

A single-level quantum dot coupled to a normal and a superconducting lead provides a basic model system to study electron transport in the presence of Coulomb and pairing interactions. Compared to the S-QD-S situation, this case has a simpler response in non-equilibrium conditions due to the absence of the ac Josephson effect. For this reason this system has been widely analyzed theoretically.

As in any N-S junction the low bias transport properties are dominated by Andreev processes. This mechanism is in general highly modified by resonant tunneling through the localized levels in the dot. In addition, charging effects can strongly suppress the Andreev reflection in certain parameter ranges. Furthermore there is also an interesting interplay between Kondo and pairing correlations as in the case of the S-QD-S system discussed above.

As an illustration of the general formalism we derive here the linear transport properties of the non-interacting model. One can straightforwardly write the dot retarded Green function for this case from expression (10.11) by setting $\Delta_L = 0$. Then, from the expression of the current in terms of Keldysh Green functions and using the corresponding Dyson equation one can write

$$I_L = \frac{e}{h} \sum_k \int d\omega \text{Tr} \left[\tau_3 \left(V_{kL} \hat{g}_{kL}^{+-} V_{Lk} \hat{G}_{00}^{-+} - V_{kL} \hat{g}_{kL}^{-+} V_{Lk} \hat{G}_{00}^{+-} \right) \right], \quad (10.31)$$

where $\hat{g}_{kL}^{+-, -+}$ are the Keldysh Green functions of the uncoupled normal lead. By further using $\hat{G}_{00}^{+-, -+} = \sum_{\mu,k} G_{00}^r V_{\mu k} \hat{g}_{k\mu}^{+-, -+} V_{\mu k}^* \hat{G}_{00}^a$ and taking the wide band approximation for the uncoupled leads one can obtain the following expression for the temperature dependence linear conductance [101, 30]

$$G = \frac{16e^2}{h} \Gamma_N \Gamma_S \int d\omega \text{Im} (G_{12}^r G_{11}^a) \left(-\frac{\partial n_F}{\partial \omega} \right) \quad (10.32)$$

which, at zero temperature reduces to the simple expression arising from the contribution of pure Andreev reflection processes

$$G = \frac{4e^2}{h} \frac{4\Gamma_N^2 \Gamma_S^2}{(\epsilon_0^2 + \Gamma_N^2 + \Gamma_S^2)^2} \quad (10.33)$$

As shown in Ref. [8] this expression for the non-interacting case is equivalent to the formula $G = (4e^2/h) \tau^2 / (2 - \tau)^2$, with $\tau = 4\Gamma_N \Gamma_S / (\epsilon_0^2 + (\Gamma_N + \Gamma_S)^2)$ being the normal transmission through the dot at the Fermi energy. In contrast to the case of a NS quantum point contact with essentially energy independent transmission, in the dot case the Andreev processes become resonant at $\epsilon_0 = 0$ reaching the maximum value $G = 4e^2/h$.

10.4.1 Effect of interactions (linear regime)

One of the first attempts to describe the effect of Coulomb interactions in the NDQS system was presented by Fazio and Raimondi [39] using the equation of motion technique truncated to the second order in the tunneling to the leads. They derived expressions for the mean current using the Keldysh formalism and extending the so-called Ng ansatz [85] to the superconducting case. The claim of an extended temperature range for the zero bias anomaly due to the Kondo resonance was subsequently corrected in [40].

The problem was addressed in Ref. [63] by assuming that the relation $G = (4e^2/h)\tau^2/(2-\tau)^2$ of the non-interacting case still holds by substituting τ by the normal transmission of the interacting case. Within this assumption that work suggested an increase of the conductance in the Kondo regime by a factor of two with respect to the normal case. As shown by subsequent works which we discuss below, this enhancement is not always possible, the general case is rather the opposite.

The conductance of the interacting N-QD-S system was also analyzed in Ref. [101] within the infinite-U slave-boson mean field approximation. This approximation reduces the problem into an effective Fermi liquid description with renormalized parameters $\tilde{\Gamma}_{N,S}$ and $\tilde{\epsilon}_0$. Within this approximation both $\Gamma_{N,S}$ are renormalized equally and therefore the condition for the maximum conductance is reached for the symmetric case as in the normal state. As the authors acknowledge this result is valid only in the deep Kondo regime $\Delta \ll T_K$, otherwise residual interactions would renormalize the left and right tunneling rates differently, in particular Γ_S coupling the dot with the superconductor would be significantly suppressed by interactions.

The problem was subsequently addressed by Clerk et al. [25] using the extension of the NCA to the superconducting case already mentioned in Sect. 10.3. They analyze three different models: the N-QD-S Anderson model and a single channel magnetic and a two-channel non-magnetic contact between normal and superconducting electrodes. We comment here only the results for the first model. They find an overall decrease of the quasiparticle spectral density at the Fermi energy together with the appearance of additional Kondo peaks at $\pm\Delta$. As a consequence, their conclusion was that there is no enhancement of the linear conductance due to Andreev processes in contrast to the claim of previous works.

Diagrammatic techniques for the finite-U N-QD-S Anderson model were applied in Ref. [30]. Within this approach the linear conductance can be expressed as the one corresponding to the non-interacting model with asymmetrically renormalized parameters

$$G = \frac{4e^2}{h} \frac{4\Gamma_N^2 \tilde{\Gamma}_S^2}{(\tilde{\epsilon}_0^2 + \Gamma_N^2 + \tilde{\Gamma}_S^2)^2} \quad (10.34)$$

where $\tilde{\Gamma}_S = \Gamma_S - \Sigma_{12}(0)$ and $\tilde{\epsilon}_0 = \epsilon_0 - \Sigma_{11}(0)$, $\Sigma_{\mu,\nu}$ being the dot self-energy elements in Nambu space. Although this result is valid in general within a diagrammatic analysis, concrete results were obtained in this work by means of an interpolated second-order approach. From Eq. (10.34) it was found that even when the starting bare parameters correspond to the symmetric case $\Gamma_N = \Gamma_S$, interactions would tend to reduce the conductance by inducing an asymmetry, i.e. leading to $\tilde{\Gamma}_S \neq \Gamma_N$, as was suggested in Ref. [101]. However, this equation also predicts the possibility that an adequate tuning of the bare coupling parameters could yield an enhancement of the conductance up to the unitary limit ($4e^2/h$ in the NS case) which would not correspond to the maximum conductance in the normal case.

The approximation of Ref. [30] is based on the evaluation of the second order diagrams, which due to the proximity induced pairing in the dot are formally the same as those of Fig. 10.7 which were discussed for the S-QD-S case. For the non-symmetric case the authors used an interpolative ansatz which recovers the correct behavior in the $\Gamma/U \rightarrow 0$ (atomic) limit. The obtained behavior of the conductance as a function of U/Γ for the symmetric and non-symmetric cases is illustrated Fig. 10.17.

As can be observed, in the symmetric case the conductance drops steadily from the unitary limit as U/Γ increases, the scale of this decay being set by the parameter Γ/Δ . On the other hand the right panel of Fig. 10.17 illustrates that the unitary limit can be restored by an adequate tuning of the ratio Γ_N/Γ_S .

The behavior of the linear conductance in the N-QD-S was also analyzed in [72, 35] using the EOM technique with different decoupling schemes. While in Ref. [72] it was obtained that

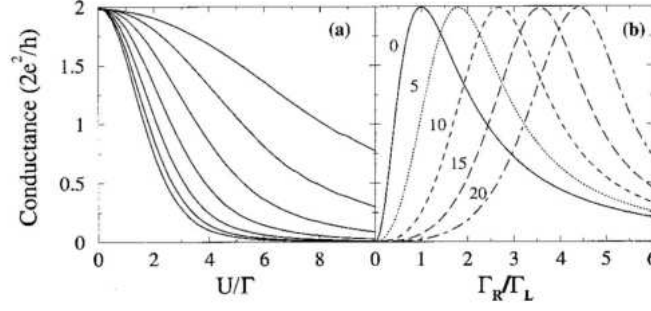


Figure 10.17: Linear conductance for the N-QD-S system in the second-order self-energy approximation of Ref. [30]. (a) symmetric case as a function of U/Γ for different values of $\Gamma/\Delta = 0.125, 0.25, 0.5, 1.0, 2.0, 4.0$ and 8.0 , from top to bottom. (b) Same as (a) for asymmetric coupling to the leads $\Gamma_L \neq \Gamma_R$ and different values of U/Γ_L (within the text we have set $\Gamma_L \equiv \Gamma_N$ and $\Gamma_R \equiv \Gamma_S$). Reprinted figure with permission from J.C. Cuevas *et al.*, Physical Review B **63**, 094515, 2001 [30]. Copyright (2001) by the American Physical Society.

the conductance due to Andreev processes was completely suppressed in the Kondo regime, an improved approximation for the EOM decoupling procedure in Ref. [35] showed that there is in fact a finite zero bias anomaly in the Andreev conductance although it is in general much smaller than the one in the normal case, its value depending on the ratio Γ_N/Γ_S . This behavior is in qualitative agreement with the results of the diagrammatic approach discussed before. However, the unitary limit is not reached within this approach.

More recently the linear conductance of the N-QD-S model has been studied using the NRG method [115]. In this work the limit $\Delta \rightarrow \infty$ was taken from the start, which allows to map the problem into the case of a QD with a local pairing amplitude $\Delta_d = \Gamma_S$ coupled to a single normal electrode (the $\Delta \rightarrow \infty$ limit for the S-QD-S case was discussed in Sect. 10.3). The NRG algorithm can be considerably simplified by this assumption because a simple Bogoliubov transformation allows to get rid of the local pairing term leading to a problem which is formally equivalent to a normal Anderson model, which implies Fermi liquid behavior. As in the diagrammatic approach discussed before, the main effect of the interactions is to renormalize the couplings $\Gamma_{N,S}$ and the dot level position. Figure 10.18 illustrates the evolution of the renormalized parameters $\tilde{\Gamma}_{N,S}$ as a function of U for the half-filled case $\epsilon_0 = -U/2$ with initial parameters $\Gamma_S = 5\Gamma_N$. As can be observed in the upper panel of Fig. 10.18, the main effect of increasing the interaction U is to reduce the effective coupling to the superconductor, $\tilde{\Gamma}_S$, while the effective coupling to the normal lead $\tilde{\Gamma}_N$ remains almost constant up to the region $U \sim 10\Gamma_N$ when the Kondo effect is significant. Eventually the system reaches the condition $\tilde{\Gamma}_N \simeq \tilde{\Gamma}_S$ and the conductance increases up to the unitary limit, as shown in the lower inset of Fig. 10.18. This behavior is in good agreement with the prediction of the diagrammatic theory of Ref. [30]. On the other hand, the induced pairing amplitude (middle panel in Fig. 10.18), exhibits the expected monotonous decrease for increasing intradot repulsion.

The behavior of the linear conductance outside the half-filled case obtained from these NRG calculations is illustrated in Fig. 10.19. In this color-scale map it can be clearly observed that the unitary limit is reached mainly along the curve $((\epsilon_0/U + 1/2)^2 + (\Gamma_S/U)^2)^{1/2} = 1/2$, which corresponds to the single-doublet transition in the $\Gamma_N \rightarrow 0$ limit. When $\Gamma_S/U < 0.5$ (corresponding to the doublet state in the $\Gamma_N \rightarrow 0$ limit within the dashed curve in Fig. 10.19), the conductance as a function of ϵ_0 exhibits a double peaked structure, whereas for $\Gamma_S/U > 0.5$ only a single peak is found which can be correlated to the superconducting singlet ground state of the system in the $\Gamma_N \rightarrow 0$ limit.

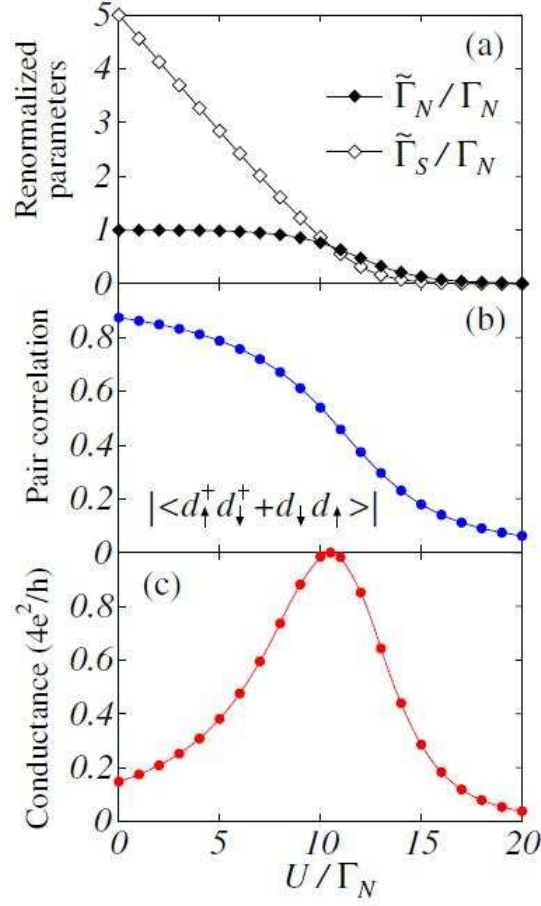


Figure 10.18: Renormalized couplings to the leads $\tilde{\Gamma}_{N,S}$ (upper panel); Induced pairing amplitude (middle panel) and linear conductance (lower panel) as a function of U/Γ_N for the N-QD-S system obtained using NRG method in the $\Delta \rightarrow \infty$ limit. Reprinted figure with permission from Y. Tanaka *et al.*, Journal of the Physical Society of Japan **76**, 074701, 2007 [115]. Copyright (2007) by the Physical Society of Japan.

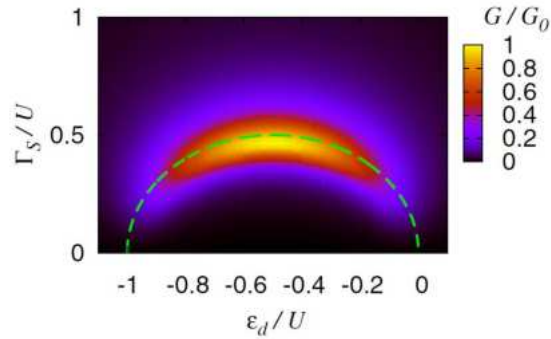


Figure 10.19: Color-scale map of the linear conductance in the N-QD-S system in the Γ_S/U - ϵ_0/U plane obtained in Ref. [115] using the NRG method as in Fig. 10.18. Reprinted figure with permission from Y. Tanaka *et al.*, Journal of the Physical Society of Japan **76**, 074701, 2007 [115]. Copyright (2007) by the Physical Society of Japan.

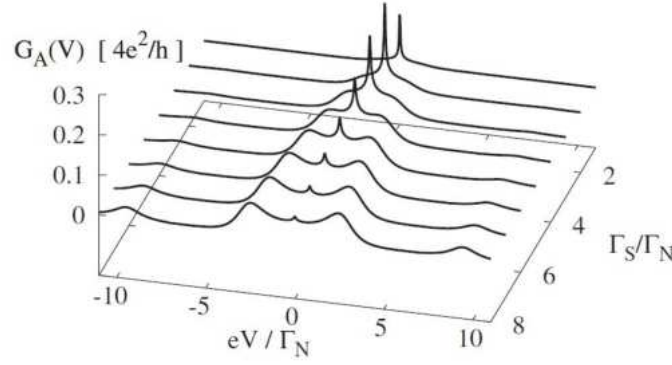


Figure 10.20: Non-linear conductance for the N-QD-S system obtained using the EOM method in the decoupling scheme of Ref. [35] in the $\Delta \rightarrow \infty$ limit as a function of eV/Γ_N and Γ_S/Γ_N . Reprinted figure with permission from T. Domanski and A. Donabidowicz, Physical Review B **78**, 073105, 2008 [35]. Copyright (2008) by the American Physical Society.

10.4.2 Non-linear regime

The non-linear regime in the N-QD-S system has received so far much less attention and there are still aspects, specially those related to the Kondo effect which are not sufficiently understood. This regime has been analyzed using the EOM technique employing different decoupling schemes in Refs. [39, 112, 72] and [35]. The approximation used in Ref. [39] has been already described in the context of the linear regime. On the other hand, the approximation used in Ref. [112] consisted in truncating the EOM equations at the level of the two particle Green functions by substituting the leads fermionic operators by their average values. Within this decoupling the authors find the appearance of Kondo features in the dot spectral density. In addition to the usual features at $\omega = \pm eV$ for the regime $\Delta > \Gamma_S > \Gamma_N$, they also find excess Kondo like features at $\omega = \pm(2\epsilon_0 + U - eV)$. These features have been explained as arising from co-tunneling processes involving Andreev tunneling from the QD-S interface and normal tunneling from the N-QD interface.

In Refs. [72, 35] already commented for the linear regime, the finite voltage case was also analyzed. The non-linear conductance obtained for the $\Delta \rightarrow \infty$ limit in Ref. [35] both as a function of the bias voltage and the asymmetry in the coupling parameters Γ_S/Γ_N is shown in Fig. 10.20. The parameters of this case correspond to the Kondo regime of the normal state $\epsilon_0 = -1.5\Gamma_N$ and $U = 10\Gamma_N$. This figure exhibits the expected features like the qualitative evolution of the zero bias anomaly already discussed in the previous section and the splitting of the dot resonances due to the proximity effect. Additional peaks at $eV \sim \pm U$ can be observed corresponding to the population of the higher charge states.

The non-linear case has been analyzed more recently in Ref. [127] by extending the interpolative self-energy approach of Ref. [30] to the non-equilibrium case. The authors consider the case $\Delta = \Gamma_N$ and $\epsilon_0 = -U/2$ for different values of Γ_S and U . Their main findings are illustrated in Fig. 10.21 corresponding to the non-linear conductance for $U = 20\Gamma_N$. They observe a Kondo peak which is displaced from zero bias, whose height increases with increasing Γ_S while its position is only weakly modified. When reducing Γ_S a second peak develops which shift progressively towards the gap edge $eV/\Gamma_N = 1$.

We should also mention the work of Ref. [68] in which the non-equilibrium transport through a N-QD-S system is studied within an effective cotunneling model. Within this approach the self-energy is calculated to leading order in the cotunneling amplitude from which the nonlinear cotunneling conductance can be obtained. By neglecting charge fluctuations in the dot two

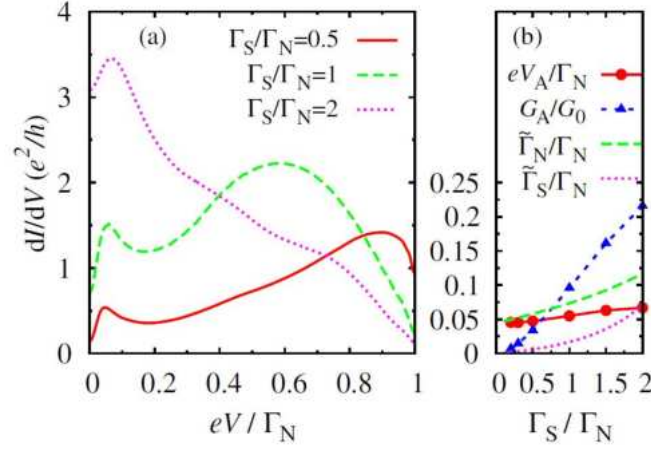


Figure 10.21: Non-linear conductance for the N-QD-S system using the interpolative self-energy approach of Ref. [127]. The results correspond to the symmetric case with $\Gamma_N = \Delta$ and different values of Γ_S and U . Reprinted figure with permission from Y. Yamada *et al.*, Journal of the Physical Society of Japan **79**, 043705, 2010 [127]. Copyright (2010) by the Physical Society of Japan.

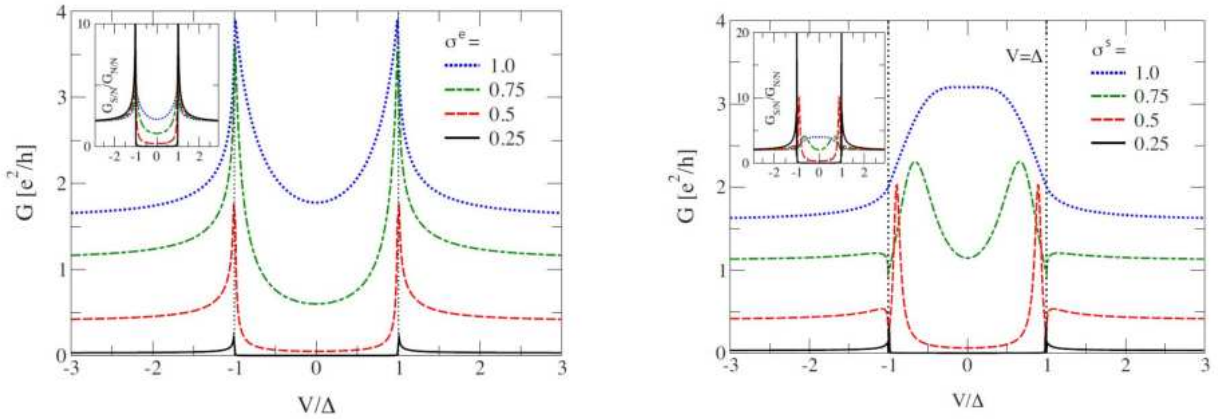


Figure 10.22: Non-linear conductance for the N-QD-S system in the effective cotunneling model of Ref. [68] for even (upper panel) and odd (lower panel) cases. The parameters σ^e and σ^s control the normal transmission through the dot in the even and odd cases respectively. The inset shows the differential conductance normalized to the one in the normal case. Reprinted figure with permission from V. Koerting *et al.*, Physical Review B **82**, 245108, 2010 [68]. Copyright (2010) by the American Physical Society.

different regimes are found corresponding to the case of even and odd number of electrons. For the even case the system becomes equivalent to an effective S/N junction with the subgap transport due to Andreev reflection processes. On the other hand, for the odd case they find that the net spin within the dot leads to the appearance of subgap resonances giving rise to a peak-dip structure in the differential conductance. The typical conductance curves that are found for both cases are shown in Fig. 10.22. As can be observed, in the even case (upper panel in Fig. 10.22) the behavior of the conductance is similar to a conventional NS junction with an effective transmission set by the second-order cotunneling amplitude. In the odd case (lower panel) the double peak structure within the subgap region evolves into a single zero bias peak as the cotunneling amplitude increases.

To conclude this section it appears that our present knowledge of the non-equilibrium N-QD-S system is still limited and further research would be desirable, particularly to understand the behavior of Kondo features at finite applied voltages and the crossover between the different parameter regimes so far analyzed. In this respect we refer the interested reader to a recent work [128] that has been published after submitting this review.

10.4.3 Experimental results

Unlike the case of the S-QD-S system, only a few works have addressed the issue of the transport properties of N-QD-S systems experimentally. This is probably due to the technical difficulties associated to the fabrication of such hybrid systems. The first experimental realization of this configuration was presented by Gräber et al. [50] using a multiwall CNT as a QD coupled to Au (normal) and Al/Au (superconducting) leads at each side. They first analyzed the normal case by applying a magnetic field of $25mT$, clearly observing Kondo features in the linear conductance, as shown in the upper panel of Fig. 10.23. At the lowest temperature of $90mK$ the normal conductance reached values $\sim 1.5e^2/h$, indicating good and rather symmetric coupling to the leads. When one of the leads become superconducting it was observed that the temperature dependence characteristic of the Kondo regime was very much suppressed, as shown in the lower panel of Fig. 10.23. This behavior is in qualitative agreement with the theoretical results of Refs. [30, 115] for the case of a nearly symmetrically coupled dot.

A different experimental realization of the N-QD-S system was presented in Refs. [32, 31]. These authors used a self-assembled InAs QD with diameters of the order of $\sim 100nm$ coupled to a Ti/Au (N lead) and a Ti/Al (S lead). In the first of these works the authors focused on devices with large coupling asymmetry $\Gamma_S \gg \Gamma_N$ in which the Kondo effect is suppressed by the strong proximity effect. In this limiting situation the normal lead is basically providing a means to probe spectroscopically the Andreev spectra of the QD-S system. The experiment provided evidence of the transition between the singlet and the doublet ground state for the QD-S system when the number of electrons changed from even to odd. As shown in Fig. 10.24 this was reflected in the behavior of the Andreev states within the gap exhibiting a crossing point together with a large drop in the conductance. These experimental results were in good qualitative agreement with NRG calculations for the QD-S system.

In a subsequent work by this group [31] the same experimental realization but with varying coupling asymmetry was analyzed. The main results of this work are shown in Fig. 10.25 corresponding to asymmetries $\Gamma_S/\Gamma_N = 0.045$ (upper panel) and 8.0 (two lower panels). For the first case with sufficiently large Γ_N one would expect the formation of a Kondo resonance due to the good coupling of the QD with the normal electrode. However, the conductance which is mediated by Andreev processes is suppressed inside the gap due to the very small coupling to the superconductor. The cases with asymmetries of the order of 8.0 were not in the extreme situation of the previous work (with $\Gamma_S/\Gamma_N \sim 50$) and thus did exhibit Kondo features as can be observed in the lower panels of Figs. 10.25.

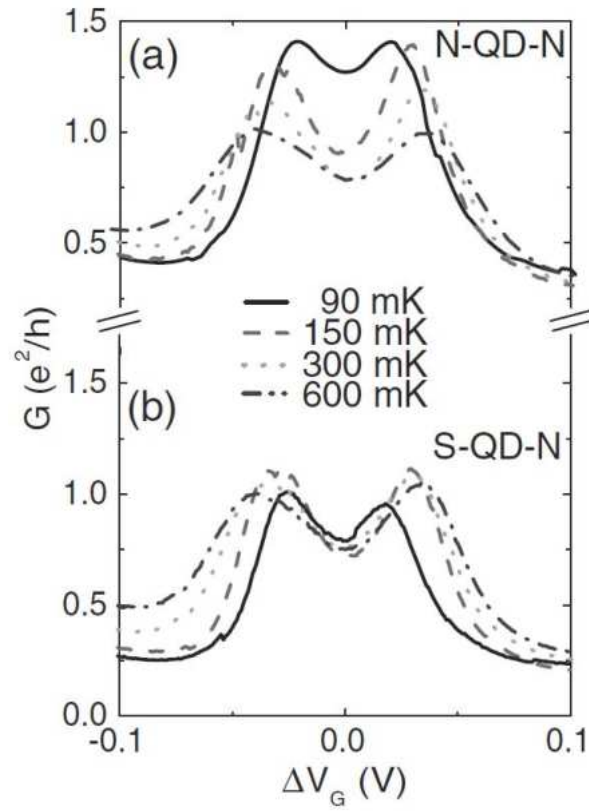


Figure 10.23: Comparison between the linear conductance in the NQDN and SQDN systems as a function of the gate voltage for different temperatures in the experimental realization of Ref. [50]. Reprinted figure with permission from M.R. Graber *et al.*, Nanotechnology **15**, S479, 2004 [50]. Copyright (2004) by IOP Publishing Ltd.

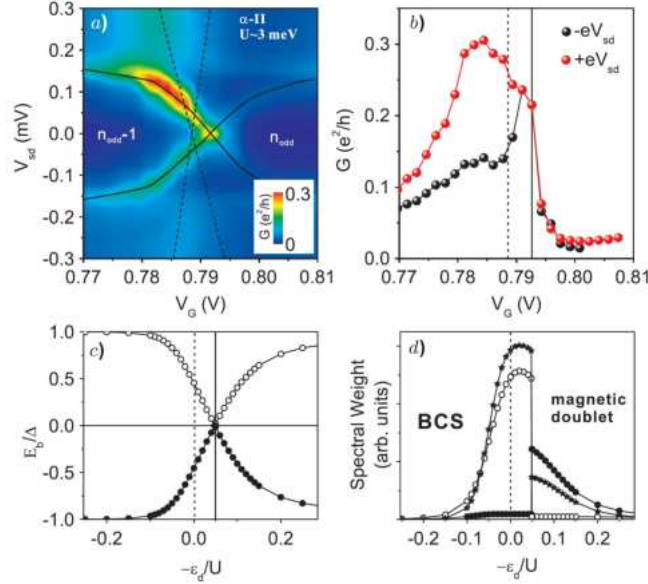


Figure 10.24: (a) Color scale map of the differential conductance as a function of the source-drain and gate voltages in the experimental realization of the N-QD-S system of Ref. [32]. (b) plot of the peak conductance following the subgap resonances indicated by the full lines in (a). Panels (c) and (d) show the Andreev levels position and weight respectively obtained using the NRG method for $U/\Delta = 20$ and $\Gamma_S/\Delta = 0.7$. Reprinted figure with permission from R.S. Deacon *et al.*, Physical Review Letters **104**, 076805, 2010 [32]. Copyright (2010) by the American Physical Society.

10.5 Voltage biased S-QD-S systems

Including a finite bias voltage between the superconducting electrodes in the S-QD-S system poses an additional difficulty in the theory due to the intrinsic time-dependence of the ac Josephson effect. Even in the non-interacting case the inclusion of MAR processes up to infinite order constitutes a challenging problem which in general requires a numerical analysis.

Although the mechanism of MAR processes for explaining the subgap structure in superconducting junctions was introduced in the early '80s [67, 87], it was not until the mid '90s that a full quantitative theory of the I-V characteristics in a superconducting contact of arbitrary transparency was developed using either a scattering approach [15, 5] or a non-equilibrium Green function approach [29]. This theoretical progress allowed a very accurate description of experimental results for atomic size contacts [100, 99]. For the discussion in the present section it could be useful to remind the main features of the I-V characteristics of a one channel contact. Fig. 10.26 shows the evolution of the dc current as a function of the contact transmission obtained using the theory of Ref. [29]. As can be observed, at sufficiently low transmission the current exhibits a subgap structure with jumps at $V = 2\Delta/n$, corresponding to the threshold voltage for an n -order MAR process. As the transmission is increased the subgap structure is progressively smeared out and eventually at $\tau = 1$ the behavior of the I-V curve is almost linear except in the limit $V \rightarrow 0$ where it saturates to a finite value $2e\Delta/h$ [5, 29].

The case of a non-interacting resonant level coupled to superconducting electrodes was first analyzed in Ref. [130]. We discuss briefly here the Green function formalism for this case which has the advantage of allowing to include the effect of interactions in a second step. The main point in this formalism is to realize that even when the Green functions $\tilde{G}(t, t')$ depends on the two time arguments, in the case of a constant voltage bias the dependence on the mean time $(t + t')/2$ can only correspond to the harmonics of the fundamental frequency $eV/\hbar$ [29]. This

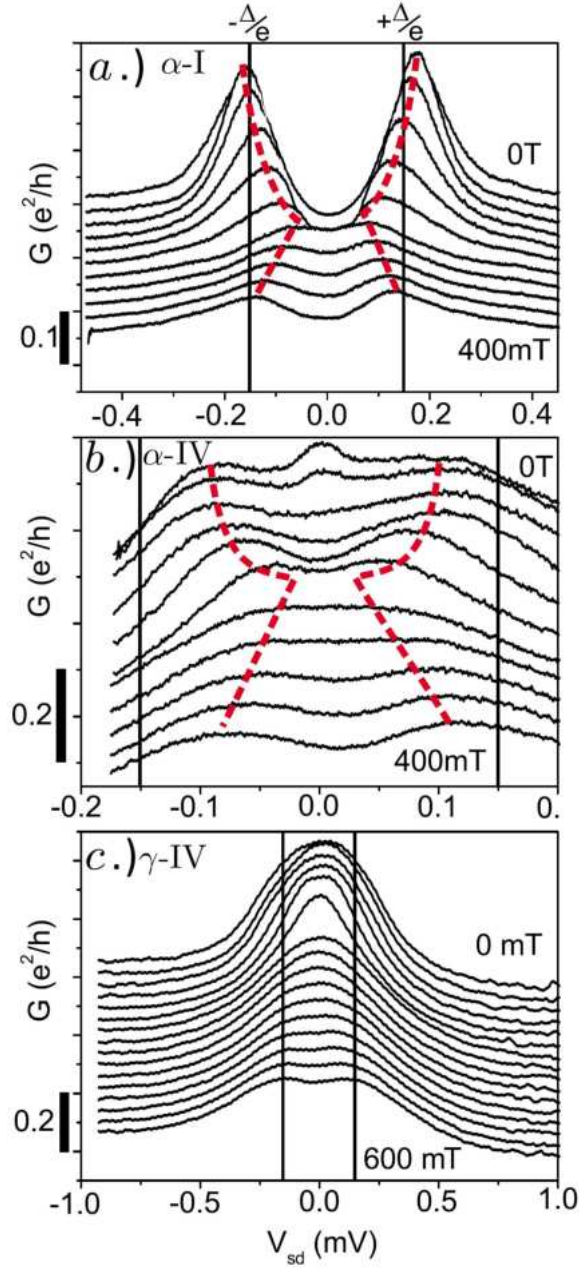


Figure 10.25: Differential conductance at the center of odd occupation regions for three different samples of the experimental realization of the N-QD-S system of Ref. [31] as a function of the applied magnetic field. The three cases correspond to different values of the Γ_S/Γ_N parameter: 0.045 (upper panel) and 8.0 (middle and lower panels). Reprinted figure with permission from R.S. Deacon *et al.*, Physical Review B **81**, 121308, 2010 [31]. Copyright (2010) by the American Physical Society.

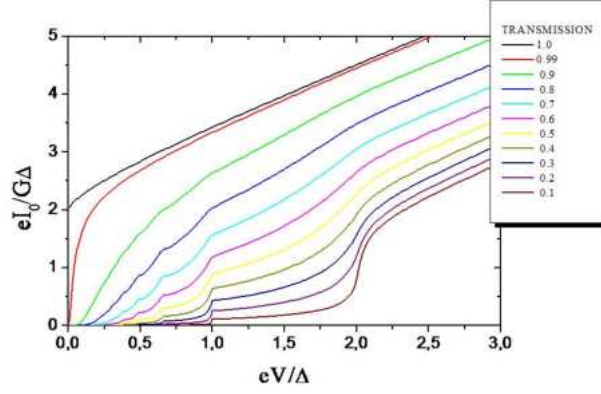


Figure 10.26: dc I-V characteristic for a superconducting single channel contact for different values of the normal transmission.

allows to express all quantities in terms of the components $\check{G}_{nm}(\omega)$ corresponding to a double Fourier transformation [4] of $\check{G}(t, t')$ defined as [82]

$$\check{G}_{nm}(\omega) = \int dt \int dt' e^{-iV(nt-mt')} e^{i\omega(t-t')} \check{G}(t, t') \quad (10.35)$$

The Fourier components $\check{G}_{nm}$ obey an algebraic Dyson equation in the discrete space defined by the harmonic indices which can be solved using a standard recursive algorithm. A compact expression of these equations for the dot case, given in Ref. [33], is

$$(\check{G}_{00})_{nm}^{-1} = (\omega_n - \epsilon_0 \sigma_z) \tau_z - \sum_{\mu=L,R} \Gamma_\mu \sigma_z \tau_z \check{g}_{nm}(\omega) \sigma_z \tau_z \quad (10.36)$$

where $\omega_n = \omega + nV$, while σ_i and τ_i correspond to Pauli matrices in the Nambu and Keldysh space respectively and

$$\check{g}_{nm} = \begin{pmatrix} \delta_{nm} \check{X}(\omega_n \mp V/2) & \delta_{n,m\mp 1} \check{Y}(\omega_n \mp V/2) \\ \delta_{n,m\pm 1} \check{Y}(\omega_n \pm V/2) & \delta_{nm} \check{X}(\omega_n \pm V/2) \end{pmatrix} \quad (10.37)$$

where the matrix $\check{X}(\omega) = -\omega \check{Y}/\Delta$ in Keldysh space are given by

$$\check{X}(\omega) = \begin{cases} -\frac{\omega}{\sqrt{\omega^2 - \Delta^2}} \tau_z & |\omega| > \Delta \\ \frac{i|\omega|}{\sqrt{\Delta^2 - \omega^2}} \begin{pmatrix} 2n_F(\omega) - 1 & 2n_F(\omega) \\ 2n_F(-\omega) & 2n_F(\omega) - 1 \end{pmatrix} & |\omega| < \Delta \end{cases} \quad (10.38)$$

The presence of a discrete resonant level between the superconducting leads can strongly modify the I-V characteristics with respect to the quantum point contact case. This is illustrated in Fig. 10.27 which corresponds to a resonant level located at zero energy with decreasing tunneling rates to the leads. As the figure shows, in the limit of large Γ the I-V curves tend to that of a perfect transmitting contact. In the opposite limit $\Gamma \ll \Delta$ there appears a pronounced subgap structure. In contrast to the contact case, the current jumps associated to the threshold of MAR processes appear only for the condition $V = 2\Delta/n$ with n being an odd integer, while the features at $2e\Delta/n$ with even n are suppressed. This can be understood qualitatively from the schematic pictures of Fig. 10.28. They represent the $n = 2$ and $n = 3$ MAR processes with arrows indicating propagation of electrons (full lines) or holes (broken lines). In the $n = 2$ case

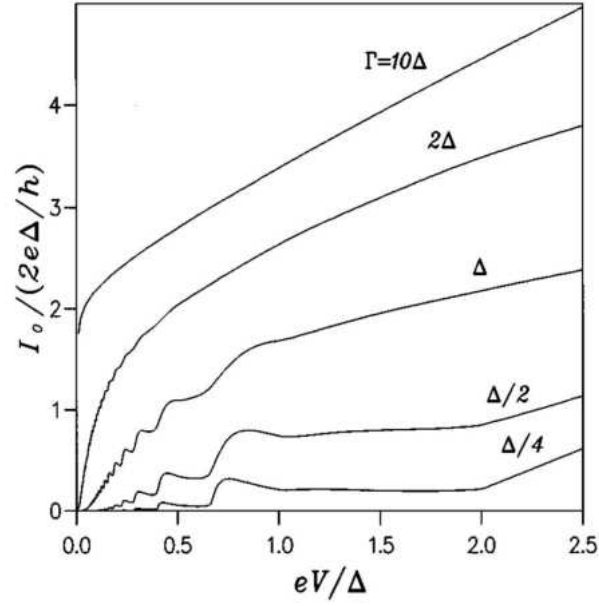


Figure 10.27: dc I-V characteristic for the non-interacting S-QD-S system with $\epsilon_0 = 0$ and different values of $\Gamma = \Gamma_L = \Gamma_R$. Reprinted figure with permission from A. Levy Yeyati *et al.*, Physical Review B **55**, 6137, 1997 [130]. Copyright (1997) by the American Physical Society.

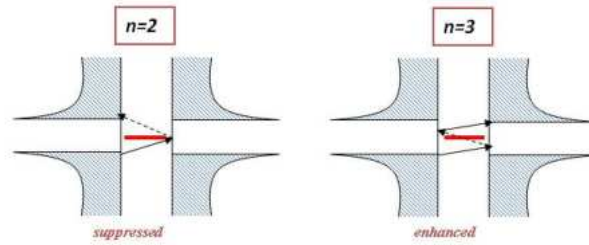


Figure 10.28: Schematic representation of MAR processes of order $n = 2$ and $n = 3$. The central horizontal line represents the level position, assumed to be located at the Fermi level.

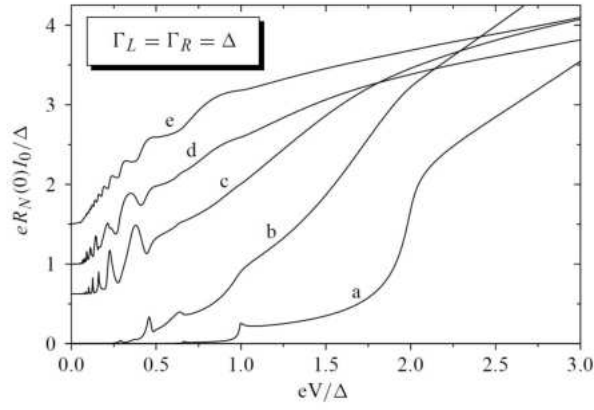


Figure 10.29: dc I-V characteristic for the non-interacting S-QD-S system with $\Gamma_L = \Gamma_R = \Delta$ and different dot level positions: 5 (a), 2 (b), 1 (c), 0.5 (d) and 0 (e) in units of Δ . Reprinted figure with permission from A. Martín-Rodero *et al.*, Superlattices and Microstructures **25**, 925, 1999 [83]. Copyright (1999) by Elsevier.

the MAR “trajectory” in energy space does not cross the resonant level while in the $n = 3$ case the resonant condition is fulfilled.

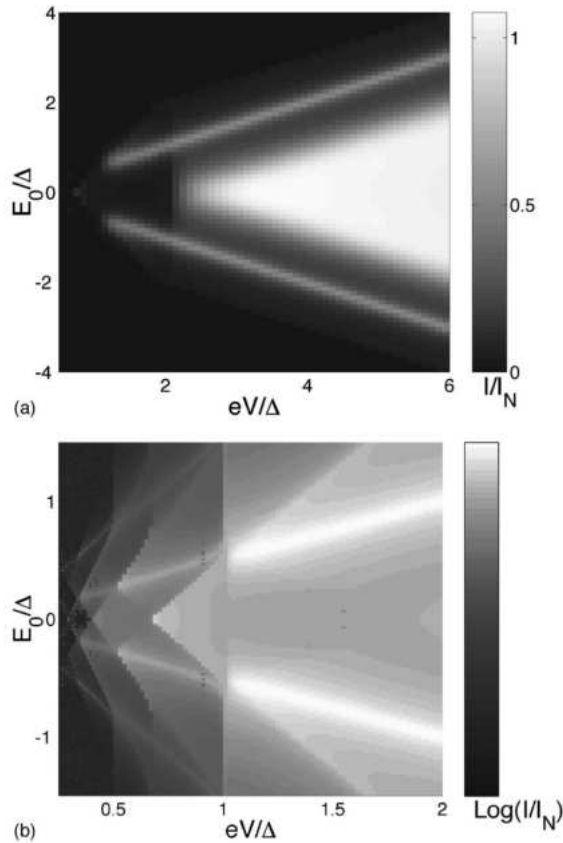


Figure 10.30: Color scale map of the the dc current in the ϵ_0 - eV plane for the non-interacting S-QD-S system. The upper panel corresponds to $eV > \Delta$ and $\Gamma = 0.2\Delta$ while the lower panel corresponds to $eV < 2\Delta$ and $\Gamma = 0.05\Delta$. Reprinted figure with permission from G. Johansson *et al.*, Physical Review B **60**, 1382, 1999 [59]. Copyright (1999) by the American Physical Society.

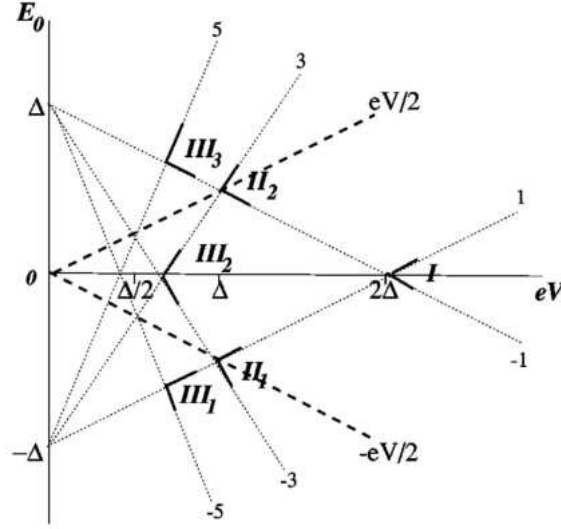


Figure 10.31: Lines indicating the boundaries for the resonant regions of the single, double and triple quasiparticle currents in the ϵ_0 - eV plane for the non-interacting S-QD-S system. Reprinted figure with permission from G. Johansson *et al.*, Physical Review B **60**, 1382, 1999 [59]. Copyright (1999) by the American Physical Society.

The subgap features are quite sensitive to the level position. Fig. 10.29 illustrates the evolution of the I-V characteristics as a function of the level position ϵ_0 for the case $\Gamma_L = \Gamma_R = \Delta$. As can be observed, when the level is far from the gap region the behavior of a weakly transmitting contact is recovered, while in the case where the level approaches the gap, the subgap features become more pronounced and correspond to resonant conditions which depend both on Δ and ϵ_0 . In this complex situation a more clear picture of the overall behavior was presented in Ref. [59]. Figs. 10.30 show the intensity plot of the current in the $\epsilon_0 - V$ plane. The upper panel illustrates the behavior of the current for $eV > \Delta$ for $\Gamma = 0.2\Delta$, showing clearly the onset of single quasiparticle current for $eV > 2\Delta$ at $\epsilon_0 = 0$. For $\epsilon_0 \neq 0$ this current is only significant in a wedge-like zone limited by $\epsilon_0 = \pm(V - 2\Delta)/2$. It can also be noticed in addition the presence of resonant peaks at $V/2 = \pm\epsilon_0$ which are reminiscent of the resonant condition for the normal case. The lower panel shows the intensity map in the region $eV < 2\Delta$ for $\Gamma = 0.05\Delta$. This illustrates the onset of higher order MAR processes, which also appear to be limited into wedge-like regions bounded by the condition $\epsilon_0 = \pm(\Delta - neV/2)$ with odd n . The schematic figure 10.31 indicates the different resonant regions for the single, double and triple quasi-particle currents.

In subsequent works further analysis of the non-interacting S-QD-S case out of equilibrium was presented [113, 60]. While in Ref. [113] the Hamiltonian approach was used to analyze the out of equilibrium dot spectral density and the ac components of the current, in Ref. [60] the effect of dephasing simulated by a third normal reservoir coupled to the dot has been studied. This work will be further commented in Sect. 10.6.

10.5.1 Effect of Coulomb interactions

The inclusion of intradot interactions in the out of equilibrium S-QD-S system introduces an additional difficulty in an already challenging theoretical problem, as shown in the previous section. So far the attempts have been restricted to some limiting cases which have been treated using approximate methods. One of these special limiting situations which was first analyzed was the case of a quantum dot in the strong Coulomb blockade regime [123, 130].

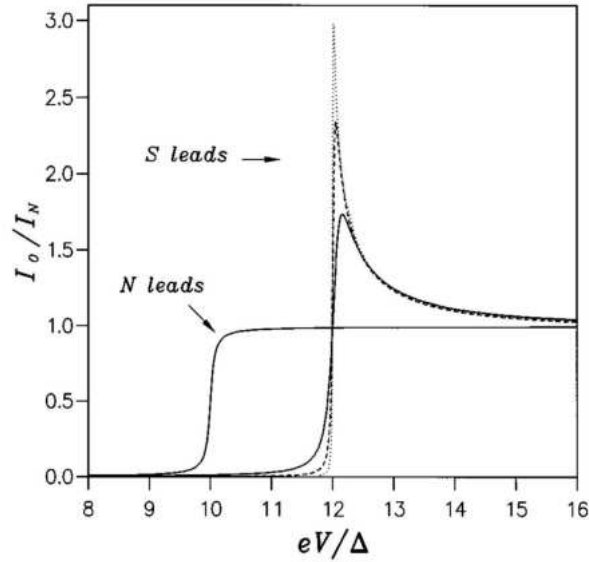


Figure 10.32: dc current-voltage characteristic for a S-QD-S system in the strong Coulomb blockade regime for different values of $\Gamma_L = 5 \times 10^{-3}\Delta$ (full line), $10^{-3}\Delta$ (dashed line) and $2 \times 10^{-4}\Delta$ (dotted line), with $\epsilon = 5\Delta$. Reprinted figure with permission from A. Levy Yeyati *et al.*, Physical Review B **55**, 6137, 1997 [130]. Copyright (1997) by the American Physical Society.

These works were motivated by the experimental results of Ref. [91] for transport through small metallic nanoparticles coupled to superconducting leads. In this strong blockade regime multiple quasiparticle processes are suppressed and the current is basically due to single quasiparticle tunneling.

In Ref. [123] the current was calculated in this regime by means of a master equation approach assuming a sequential tunneling regime. The single-particle tunneling rates were calculated using the Fermi golden-rule. A slightly different method was used in Ref. [130] where resonant tunneling through an effective one-electron level describing the dot in the limit $U \rightarrow \infty$ was considered. The corresponding expression for the tunneling current was given by

$$I(V) = \frac{4e}{h} \int d\omega \frac{\Gamma_L(\omega)\Gamma_R(\omega)}{(\omega - \epsilon)^2 + [\Gamma_L(\omega) + \Gamma_R(\omega)]^2} \times [n_F(\omega - eV/2) - n_F(\omega + eV/2)], \quad (10.39)$$

where ϵ denotes the effective level and

$$\Gamma_{L,R}(\omega) = (\Gamma/2) \text{Re} \left[|\omega \pm V/2| / \sqrt{(\omega \pm V/2)^2 - \Delta^2} \right].$$

The corresponding result for different values of Γ are shown in Fig. 10.32. This result differs from the simple sequential tunneling picture, which would predict $I(V) \sim \Gamma_L(\epsilon)\Gamma_R(\epsilon)/(\Gamma_L(\epsilon) + \Gamma_R(\epsilon))$, exhibiting an intrinsic broadening of the BCS-like feature, in agreement with the experimental observation [91]. A similar expression was obtained in Ref. [63] using the equation of motion approach in the atomic limit which produces a correction factor in the current, $\sim \sum_{\sigma}(1 - \langle n_{0\sigma} \rangle)$ due to the strong Coulomb interaction.

In order to analyze the interplay of MAR and Kondo correlations it is necessary to rely on other approaches. In Ref. [6] the current-voltage of this system was obtained using the slave boson mean field approximation already discussed in Sect. 10.3. In the infinite- U version of the

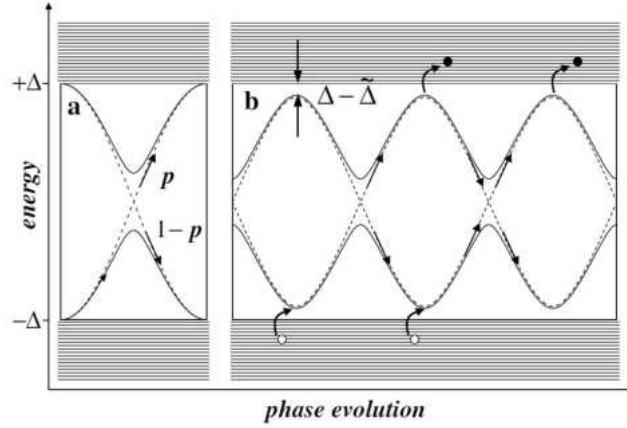


Figure 10.33: Schematic representation of the ABS dynamics explaining the low bias dc current in the case of a single channel contact (panel a) and for the S-QD-S system in the Kondo regime (panel b). Reprinted figure with permission from A. Levy Yeyati *et al.*, Physical Review Letters **91**, 266802, 2003 [133]. Copyright (2003) by the American Physical Society.

method used in Ref. [6] the problem becomes equivalent to an effective non-interacting model with renormalized parameters $\tilde{\Gamma}$ and $\tilde{\epsilon}_0$, as indicated in subsection 10.3.2. The authors analyze the evolution of the I-V curves and the shot-noise as a function of the parameter T_K/Δ . As is physically expected, the behavior is similar to that of a perfectly transparent contact when $T_K/\Delta \gg 1$ developing a clear subgap structure in the opposite limit.

A similar approach was used in Ref. [133] in order to analyze the low bias transport properties of a S-QD-S in the Kondo regime $T_K \gg \Delta$. It was shown that these properties can be understood in terms of the dynamics of the subgap Andreev states. In this limit the ABSs satisfy the equation corresponding to the non-interacting case, ie. Eq. (10.12), with renormalized parameters $\tilde{\Gamma}_{L,R}$ and $\tilde{\epsilon}_0$ instead of $\Gamma_{L,R}$ and ϵ_0 . The low bias quasiparticle current through the system arises from transitions between the continuum occupied and empty states below and above the superconducting gap which occur by means of Landau-Zener like processes involving the ABSs. This is illustrated in Fig. 10.33 first for the case of a quantum point contact (left panel) and then for the S-QD-S case (right panel). In the last case it is necessary to have a transition between the ABSs and the continuum in addition to the Landau-Zener transition between the lower and upper ABSs. The results for the low bias dc current which are obtain from this analysis, shown in Fig. 10.34, are in good agreement with the results of a full numerical calculation including MAR processes up to infinite order. The analysis based on the dynamics of the ABSs was also used in Ref. [121] for a comparison of experimental results in the low bias regime. In contrast to Ref. [133], in this work a phenomenological damping rate η was introduced in order to fit the experimental data obtained for multiwall CNTs connected to Au/Al leads.

A step beyond the infinite- U SBMF approach was taken in Ref. [37] where the finite- U SBMF method was used to determine the I-V characteristics of a S-QD-S system. This approach allowed to describe the observed differences in the subgap structure between situations with even and odd number of electrons in the dot for a SWCNT QD coupled to Al/Ti electrodes. In Ref. [37] the effective parameters of the finite- U SBMF approach were obtained for the leads in the normal state and assumed to remain unmodified in the superconducting case. The results of this work for the subgap structure are shown in Fig. 10.35 where the comparison with the experimental data is given. The figure corresponds to an *odd* valley exhibiting clear Kondo features in the normal state. The more intriguing feature of the differential conductance

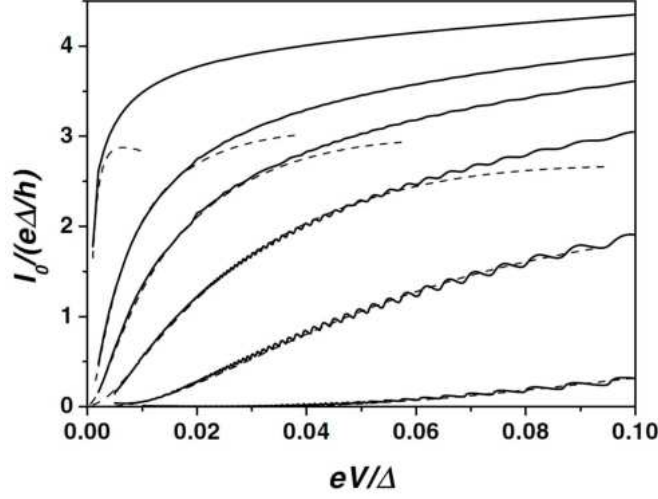


Figure 10.34: Comparison of the low bias dc current in the S-QD-S system in the Kondo regime obtained from the ABS dynamics (dashed lines) and by the full numerical calculation (full line) for different values of the effective coupling to the leads $\tilde{\Gamma}_L = \tilde{\Gamma}_R = 1, 2, 3, 4, 5$ and 10 in units of Δ , from bottom to top. Reprinted figure with permission from A. Levy Yeyati *et al.*, Physical Review Letters **91**, 266802, 2003 [133]. Copyright (2003) by the American Physical Society.

in the superconducting case was the presence of a pronounced structure for $eV \sim \Delta$, which cannot be explained by the non-interacting theory. This feature was attributed to the large asymmetry $\Gamma_L/\Gamma_R \sim 30$ which produces a Kondo resonance pinned at the chemical potential of the left lead. As is schematically depicted in Fig. 10.35(e) this resonance would produce an enhancement of the current for $eV \sim \Delta$.

The effect of interactions in the subgap structure has also been considered in Ref. [33] by means of a perturbative approach in which the dot self-energy was calculated up to second order in U . The method thus includes the diagrams already discussed in Sect. 10.3 but extended to the non-equilibrium situation. To avoid heavily time-consuming computation of the multiple frequency integrals, the authors calculate the diagrams in time representation and then Fourier transform the final result. They consider the weak interaction regime $U/\Gamma < 1$ thus avoiding the regime of π -junction behavior. The most remarkable result is the observation of an enhancement of the current due to the interactions, which is more pronounced for voltages approaching the odd MAR onset conditions $2\Delta/(2r + 1)$. This is illustrated in Fig. 10.36 for the case $\epsilon_0 = 0$ where the difference $I(U) - I(0)$ as a function of $2\Delta/V$ is represented. The enhancement is observed both in the self-consistent first order approximation and when including the second-order diagrams. This current enhancement is reminiscent yet different from the “antiblockade” behavior due to dynamical Coulomb blockade effects on MAR transport, as discussed in Ref. [129]. It is worth mentioning that MAR transport through a resonant level coupled to a localized phonon mode was studied in Ref. [135] using second order perturbation theory in the electron-phonon coupling.

10.5.2 Summary of Experimental results

The already commented work by Ralph *et al.* [91] can be considered one of the first realizations of a S-QD-S system in which the current-voltage characteristic was measured. This case corresponded, however, to the strong blockade regime in which the subgap structure is absent. It was not until 2002 that experiments on CNTs coupled to Al leads [17, 16] allowed a clear observation of the subgap features. Ref. [17] mainly focused in the linear conductance which

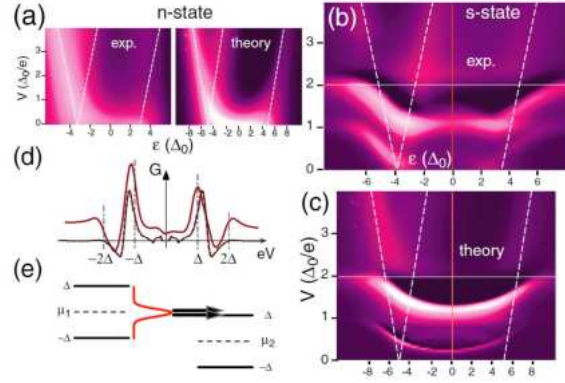


Figure 10.35: Experimental and theoretical results for the differential conductance of the S-QD-S system in an odd valley. Panel (a) corresponds to the normal state (experimental and theoretical results), while (b) and (c) show the corresponding results in the superconducting state. Panel (d) shows the conductance vs voltage bias along the line indicated in red in panels (b) and (c). The theoretical curves corresponds to $\Gamma_L = \Delta$, $\Gamma_R = 0.03\Delta$ and $U = 10\Delta$. Reprinted figure with permission from A. Eichler *et al.*, Physical Review Letters **99**, 126602, 2007 [37]. Copyright (2007) by the American Physical Society.

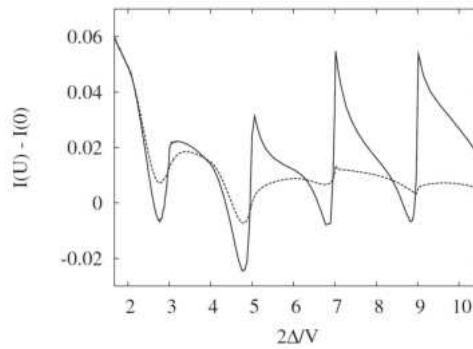


Figure 10.36: Variation of the dc current in the symmetric S-QD-S system with respect to the non-interacting case obtained in Ref. [33] using the second-order self-energy approach for $U/\Gamma = \Gamma/\Delta = 0.5$. The dashed curve gives the self-consistent first order result. Reprinted figure with permission from L. Dell'Anna *et al.*, Physical Review B **77**, 104525, 2008 [33]. Copyright (2008) by the American Physical Society.

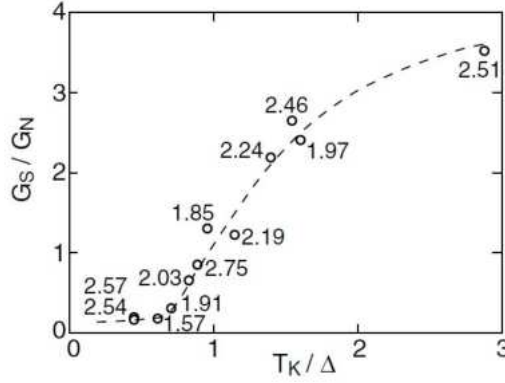


Figure 10.37: Linear conductance in the Kondo regime for the experimental realization of the S-QD-S system of Ref. [17] normalized to its value in normal state as a function of T_K/Δ . Reprinted figure with permission from M.R. Buitelaar *et al.*, Physical Review Letters **89**, 256801, 2002 [17]. Copyright (2002) by the American Physical Society.

can exhibit either an enhancement or a suppression with respect to the normal case depending on the ratio T_K/Δ . The results of this work are summarized in Fig. 10.37. On the other hand in Ref. [16] the authors analyzed the MAR induced subgap structure for the same type of systems in more detail. As shown in Fig. 10.38 clear peaks in the differential conductance are observed at the positions $eV \sim 2\Delta$, Δ and $\Delta/2$. The intensity of these peaks evolves as a function of the dot level position (controlled by the gate voltage V_g). Contrary to the theoretical expectations the peak at Δ is still visible at resonance whereas the expected feature from the non-interacting model at $2\Delta/3$ is not observed. The authors of Ref. [16] suggest that the discrepancy can be attributed to the effect of interactions not included in their theoretical analysis. These results could be analyzed in the light of the already commented arguments of Ref. [37] which attributed the pronounced Δ peak to the combined effect of coupling asymmetry and Kondo effect.

The competition between Kondo effect and superconductivity was also observed in Ref. [18] in which self-assembled InAs quantum dots contacted with Al leads were analyzed. A magnetic field was used to control the size of the superconducting gap parameter and the linear conductance as a function of $\Delta/k_B T_K$ was measured. The results exhibited a rather universal behavior as a function of this parameter. However, in contrast to Ref. [17] the ratio G_S/G_N did not exceed unity for $\Delta/k_B T_K < 1$ while decreasing as expected for $\Delta/k_B T_K > 1$. Although the authors attributed this difference to a stronger Coulomb repulsion which would in their case heavily damp the MAR processes, one would expect that this effect would be already included when scaling Δ in units of T_K . The absence of conductance enhancement for large T_K could be also pointing out to an ingredient in this system not included in the simplest Anderson model like spin-orbit interactions.

10.6 Beyond the single level model: multidot, multilevel and multi-terminal systems

In recent years there has been an increasing interest in more complex situations which cannot be described by the simplest single level Anderson model. These include situations where transport occurs through more than a single dot or where several quantum channels in a single dot are involved. In addition, there is also great interest in analyzing the transport properties of hybrid quantum dot systems coupled to several superconducting and/or normal electrodes in

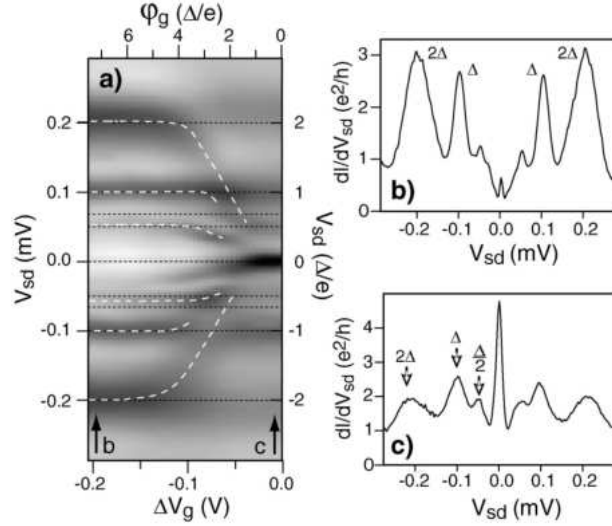


Figure 10.38: Results for the differential conductance in the experimental realization of the S-QD-S system of Ref. [16]. Panel (a) color map for this quantity in the bias voltage-gate voltage plane. The dashed lines indicate the evolution of the MAR resonances. Panels (b) and (c) show plot of the conductance as a function of bias for the lines indicated by the arrows in panel (a). Reprinted figure with permission from M.R. Buitelaar *et al.*, Physical Review Letters **91**, 057005, 2003 [17]. Copyright (2003) by the American Physical Society.

a multiterminal configuration. These configurations could allow to explore non-local electronic transport, in particular the possibility of creating entangled electron pairs by means of crossed or non-local Andreev processes. To describe these developments we organize this section as follows: in the first subsection we discuss the case of Josephson transport through double and multiple dot systems, in the second one we consider this effect for a multilevel dot, and finally we consider multiple dot systems including both normal and superconducting electrodes as well as in a multiterminal configuration.

10.6.1 Josephson effect through multidot systems

Transport through double quantum dots connected either in parallel or in series to superconducting electrodes has been extensively analyzed in the literature. Most of the theoretical works describe this situation by using a single-level Anderson model to represent each dot and introducing extra terms describing the coupling to the leads. Choi et al. considered in Ref. [22] the case of two dots connected in parallel to superconducting leads as depicted in Fig. 10.39. By analyzing the problem to the fourth order in the tunneling to the leads they derived an effective Hamiltonian coupling the localized spins in both dots. In the regime $0 < -\epsilon_0 < \Delta < U$ this Hamiltonian adopts the form

$$H_{eff} \simeq J(1 + \cos \varphi) \left[\mathbf{S}_a \cdot \mathbf{S}_b - \frac{1}{4} \right], \quad (10.40)$$

where J is an exchange coupling between the localized spins in dots a and b ; and φ is defined as

$$\varphi = \phi_L - \phi_R - \frac{\pi}{\Phi_0} \int (dl_a + dl_b) \cdot \mathbf{A}, \quad (10.41)$$

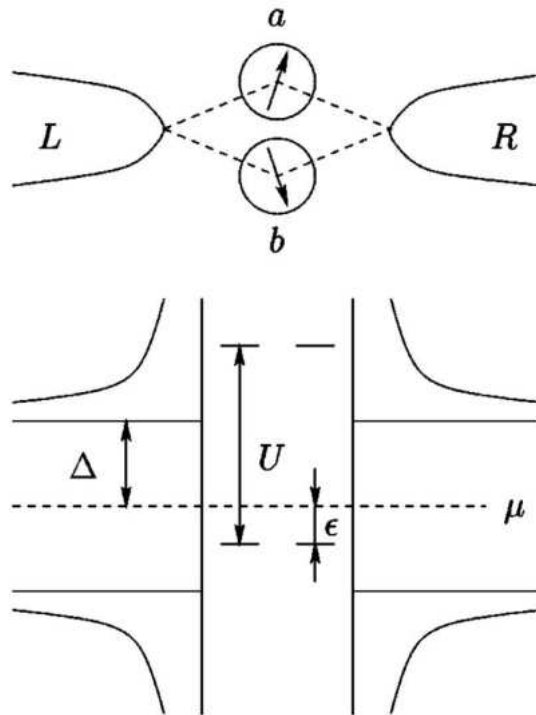


Figure 10.39: Schematic representation of a double dot system coupled in parallel to two superconducting electrodes (upper panel) considered in Ref. [22]. The lower panel describe the leads spectral density and the parameters of the double dot Anderson model used in this reference. Reprinted figure with permission from M.S. Choi *et al.*, Physical Review B **62**, 13569, 2000 [22]. Copyright (2000) by the American Physical Society.

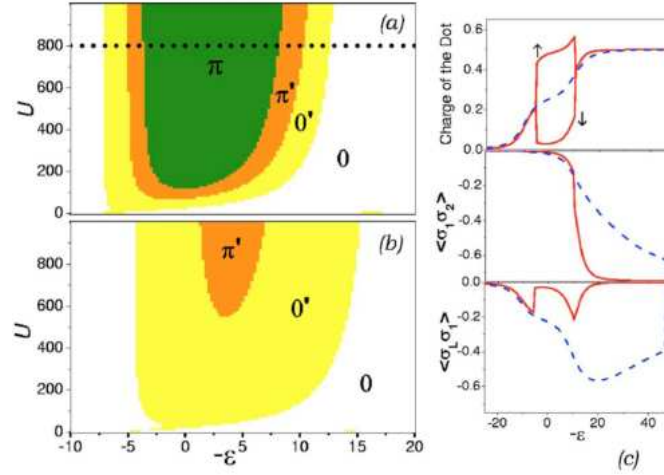


Figure 10.40: $U - \epsilon$ phase diagrams, dot population and spin correlation functions for the series double dot model considered in Ref. [11] obtained using the zero band-width approximation to describe the superconducting leads. The parameters are $t_{12} = 10\Delta$, $t_L = t_R = 2\Delta$ (a) and $t_L = t_R = 2.5\Delta$ (b). The plots in panel (c) are taken along the dotted line corresponding to $U = 800\Delta$ shown in panel (a). Reprinted figure with permission from F.S. Bergeret *et al.*, Physical Review B **74**, 132505, 2006 [11]. Copyright (2006) by the American Physical Society.

where the last term corresponds to the phase accumulated on each path of the loop due to the magnetic field. These results indicate that the Josephson current through such a device would be sensitive to the total spin of the double dot. In order to probe the spin state of the double dot system the authors propose to incorporate it into a SQUID geometry in which an additional tunnel junction is included in one of its arms. Further elaboration on similar ideas were presented in Ref. [55] for a triple dot between superconducting leads. This system is shown to behave under certain conditions as a *mesoscopic pendulum* where the singlets injected through a pair of dots oscillate between two different configurations like in the resonating valence bond model.

The case of two QDs in series connected to superconducting leads was first analyzed in Ref. [139]. The authors start by diagonalizing exactly the isolated double dot Hamiltonian including a term describing the interdot repulsion V and then introducing the external leads by a Dyson-type equation. This approach should be valid in the limit of vanishing coupling to the leads and is equivalent to the one commented in Sect. 10.3 for the single dot case. For the case $U > V > t$, where t is the interdot hopping parameter, and for $k_B T \gg \Gamma$ they find that the system exhibits a 0 type current-phase relation except when a finite Zeeman splitting is included. The absence of a π -junction behavior for any dot filling is probably due to the extremely small value of Γ compared to $k_B T$. This will be further discussed below.

A similar situation was later considered in Ref. [11], where a double dot model was analyzed using both small cluster numerical diagonalizations (discussed before in Subsect. 10.3.4 for the single dot case) together with the finite- U SBMF technique. In addition to the transition to the π -phase this work aimed to investigate the interplay between different possible magnetic correlations including Kondo and anti-ferromagnetic coupling between the localized spins within each dot. Part of the results are shown in Fig. 10.40 where the phase-diagram in the U vs $-\epsilon$ plane is given for the case $t_{12} \gg \Delta$, t_{12} being the interdot hopping parameter. In the range of parameters of Fig. 10.40 the system exhibits a π -phase region associated to the transition between the empty and singly occupied double dot. When increasing the dot population to the level of one electron per dot antiferromagnetic correlations dominate and the π -phase is

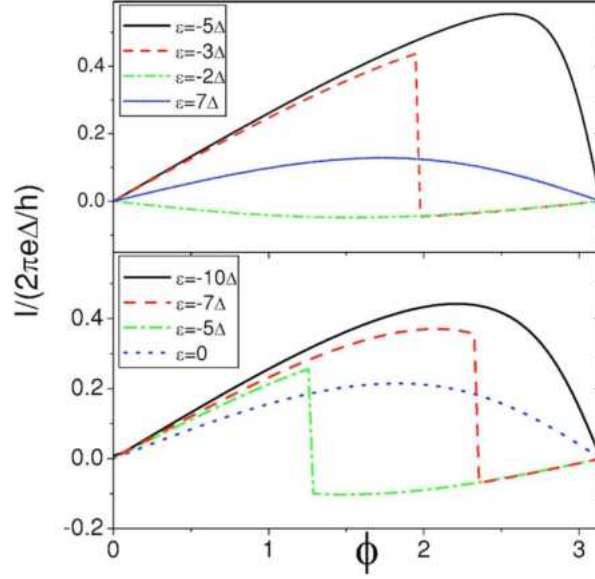


Figure 10.41: Current-phase relation for the model considered in Ref. [11] obtained using the finite-U SBMF method with parameters $U = 800\Delta$, $t_{12} = 10\Delta$, $\Gamma_L = \Gamma_R = 2.25\Delta$ (upper panel) and $\Gamma_L = \Gamma_R = 4\Delta$ (lower panel). Reprinted figure with permission from F.S. Bergeret *et al.*, Physical Review B **74**, 132505, 2006 [11]. Copyright (2006) by the American Physical Society.

no longer stable. This is further illustrated in panel (c) of Fig. 10.40 where different spin-spin correlation functions are shown both for the normal and the superconducting case. The results obtained using the small cluster diagonalizations were confirmed by the finite-U SBMF calculations. In particular these last calculations also show the gradual disappearance of the full π -phase when increasing the coupling to the leads, as illustrated in Fig. 10.41. This work also provided a possible scenario for explaining the experimental results of Ref. [66] for fullerene dimers containing Gd magnetic atoms and suggested the possibility to control the magnetic configuration of these atoms by means of the Josephson current.

The SBMF approach in the infinite-U version was applied to series and parallel double quantum dots in Ref. [80]. As in the case of a single QD this method cannot account for the appearance of a π -junction phase. The results are nevertheless relevant for the regime $T_K \gg \Delta$ where Kondo correlations dominate over pairing. While in the parallel case it is found that the Josephson critical current, I_c , decreases monotonically with the interdot hopping parameter t , in the series case a non-monotonous behavior is found. This is illustrated in Fig. 10.42 where I_c exhibits a maximum at $t/\Gamma \sim 1$. The authors interpret the change in behavior of I_c as a transition from a regime characterized by two independent Kondo singlets involving each dot and the corresponding lead to the formation of bonding and antibonding Kondo resonances.

Quite recently, the series double dot system coupled to SC leads has been analyzed using the NRG method [140]. The authors considered the regime $U \rightarrow \infty$ and $-\epsilon \gg \Gamma$, which would correspond to the deep Kondo regime in a normal single QD. In this range of parameters they find a rich phase diagram as a function of the interdot coupling t and the ratio T_K/Δ . Some of their results are illustrated in Fig. 10.43 showing the regions corresponding to 0, π and π' phases in the t/Γ vs ϕ plane for different values of T_K/Δ . One can notice the abrupt transition between 0 and π phases for $t \sim 10\Gamma$ which can be associated to a change in the DQD population from an even to an odd number of electrons. An additional remarkable feature is the appearance of a π' "island" close to $\phi = \pi$ and for $t \sim \Gamma$ in the intermediate coupling

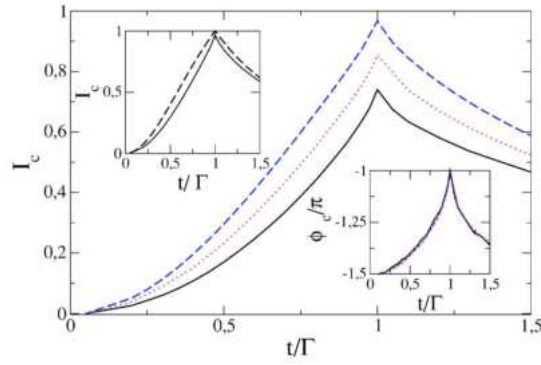


Figure 10.42: Josephson critical current as a function of t/Γ for the series double dot model analyzed in Ref. [80] using the infinite- U SBMF method for $\Delta/T_K = 0.1, 0.25$ and 0.5 (from bottom to top). The upper inset shows the comparison of the numerical result for $\Delta/T_K = 0.1$ (full line) with the prediction from a non-interacting model with an effective transmission (dashed line). The lower inset shows the critical phase at which the maximum current occurs. Reprinted figure with permission from R. López *et al.*, Physical Review B **75**, 045132, 2007 [80]. Copyright (2007) by the American Physical Society.

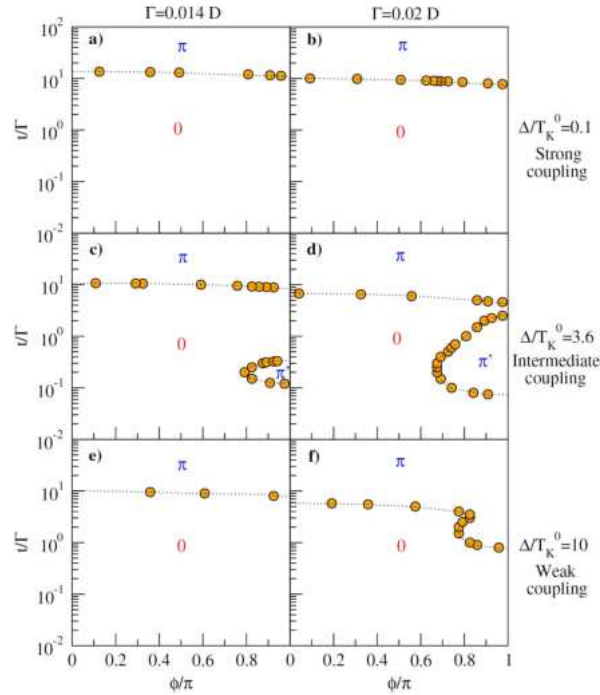


Figure 10.43: Phase boundaries between the 0 and π states in the series double dot model of Ref. [140] obtained using the NRG method for $\Gamma = 0.014D$ and $\Gamma = 0.02D$, where D is the leads band-width. Reprinted figure with permission from R. Zitko *et al.*, Physical Review Letters **105**, 116803, 2010 [140]. Copyright (2010) by the American Physical Society.

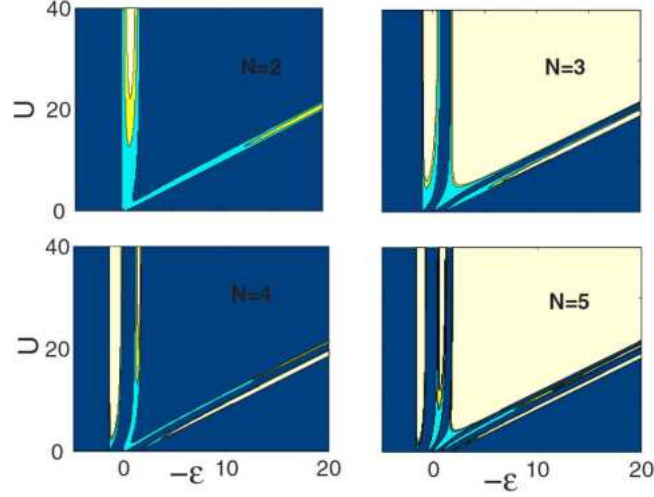


Figure 10.44: $\epsilon - U$ phase diagrams for the quantum dot array coupled to superconductors considered in Ref. [12] with increasing number of dots $N = 2, 3, 4$ and 5 . The results were obtained using the zero band-width approximation described in Subsect. 10.3.4 with parameters $t_L = t_R = t = \Delta$. Reprinted figure with permission from F.S. Bergeret *et al.*, Physical Review B **76**, 174510, 2007 [12]. Copyright (2007) by the American Physical Society.

regime $T_K \sim \Delta$.

The evolution of the Josephson effect as a function of the number of dots connected in series was studied in Ref. [12]. The model considered in this work with all dot levels fixed at a same value ϵ , with the same local Coulomb repulsion U and with dots connected by a fixed hopping parameter t , is equivalent to a finite Hubbard chain connected to two superconducting leads. The ground state properties of this model were obtained using the zero band-width limit description of the leads discussed in Subsect. 10.3.4 and employing the Lanczos algorithm. Fig. 10.44 illustrates the evolution of the phase diagram as N , the number of dots in the chain, is increased. One can clearly distinguish the case of even and odd N . In the last case the diagram is similar to the single dot case with a central π -phase region corresponding to the half-filled case. On the contrary, for even N the π -phase is absent around half-filling due to the dominance of antiferromagnetic correlations between spins in neighboring dots. One can also notice the appearance of additional narrower regions of π -phase character corresponding to fillings with odd number of electrons in the dots region. The authors also analyzed the current-phase relation as a function of N for the half-filled case. Fig. 10.45 shows that the critical current scales as $e^{-\alpha N}$ with a different sign depending on the parity of N . This behavior is consistent with the prediction of field theoretical calculations for a 1D Luttinger liquid with repulsive interactions where the fixed point corresponds to the absence of Josephson coupling in the limit of an infinite long chain [1].

10.6.2 Multilevel quantum dots

So far only the single-level Anderson model has been considered for describing an individual quantum dot. A proper description of actual physical realizations of quantum dots could require to consider a multilevel generalization of this model. This has been already pointed out in connection with the experiments of Ref. [119] on InAs nanowires, whose results were qualitatively accounted for using a multilevel model in which two orbitals with different parity were involved.

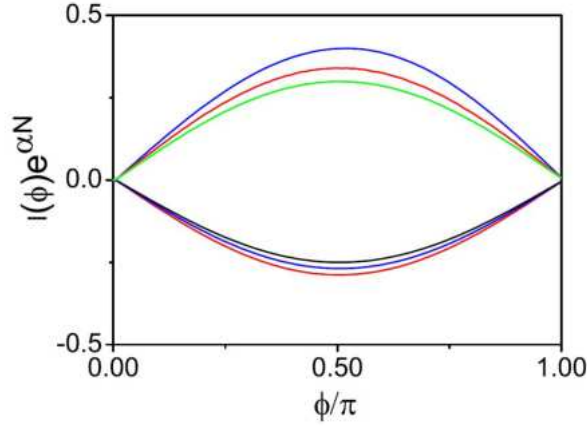


Figure 10.45: Josephson current-phase relation for the quantum dot array model analyzed in Ref. [12] with $U = 10\Delta$ at half-filling. The curves in the positive half plane correspond to $N = 2, 4$ and 6 (from top to bottom) while those taking negative values correspond to $N = 1, 3$ and 5 (from bottom to top). The current is scaled by an exponential factor $\exp\{\alpha N\}$ with $\alpha \simeq 1.8$ in units of $e\Delta/h$. Reprinted figure with permission from F.S. Bergeret *et al.*, Physical Review B **76**, 174510, 2007 [12]. Copyright (2007) by the American Physical Society.

Multilevel effects in the Josephson current through a QD were first addressed in Ref. [105] by means of the Hartree-Fock approximation. The authors showed that when non-diagonal processes involving different dot levels are relevant the system can behave as a π -junction even in the absence of a magnetic ground state. This is illustrated in Fig. 10.46 where the Josephson current at $\phi = \pi/2$ is plotted as a function of the dot levels position, ϵ_0 , for the cases with 1, 2 and 3 levels and including diagonal and non-diagonal couplings to the leads with the same value Γ . One can clearly notice that whereas in the lower panel (single level) the π -phase is only present for odd number of electrons ($n_d = 1$) in the 2 and 3 level cases (panels b and c) the π behavior is also present for even occupancy plateaus.

Similar ideas were discussed in Ref. [96] for a multilevel situation with nearly degenerate levels ($\delta\epsilon \ll \Delta$) connected at the same two points to the leads and using the cotunneling approach. The authors pointed out that the π -junction behavior is linked to two-particle processes in which one of the electrons proceeds through an occupied state and the other through an empty one.

Further analysis of this multilevel case was presented in Ref. [75] using both perturbation theory (cotunneling approach) and NRG calculations. They considered a two-level situation coupled in parallel to single channel leads and including an exchange term J between the electron spins in each dot level. In the normal case this exchange term for $J < 0$ would drive a singlet-triplet transition. Two different situations are distinguished: a case in which the two orbitals have the same "parity" (i.e. $\text{sign}(t_{1L} * t_{1R}) = \text{sign}(t_{2L} * t_{2R})$, where $t_{j\alpha}$ are the hopping from the j level to the lead α) and a case in which the parities are different. In the first case and for $t_{1L}/t_{1R} = t_{2L}/t_{2R}$ the dot levels are only coupled to the symmetric combination of the two leads, yielding an effective one channel problem. In the second case the problem is equivalent to a two channel two impurity model with exchange coupling.

The system exhibits a very rich phase diagram depending on the several model parameters. For illustration we show in Fig. 10.47 and 10.48 the obtained phase diagrams in the Δ, J plane for the single channel and the two channel cases respectively for symmetric coupling to the leads. The different phases are denoted by a capital letter indicating the spin of the dominant ground state, which can be either S , D , or T for spin 0, 1/2 or 1. In the case of mixed phases

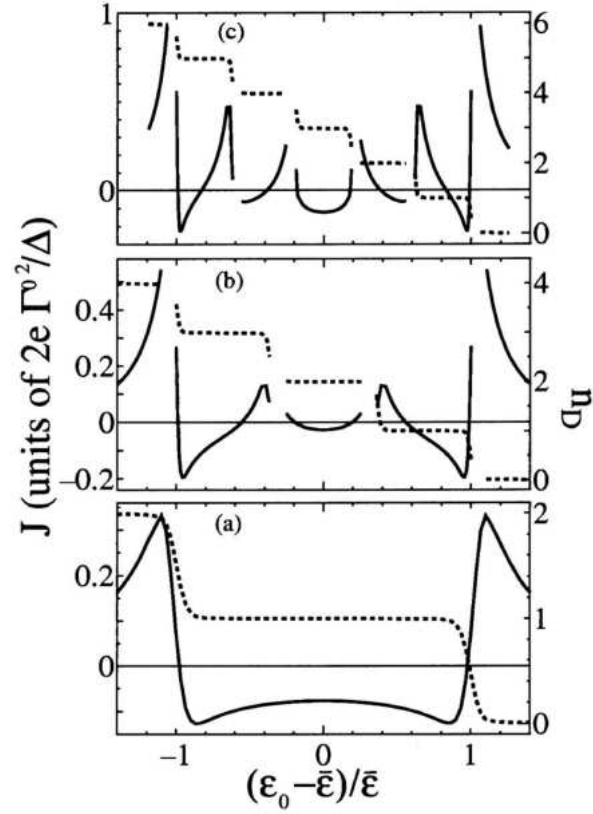


Figure 10.46: Josephson current at $\phi = \pi/2$ in the multilevel S-QD-S model of Ref. [105] as a function of the level position for the cases $n = 1$ (a), $n = 2$ (b) and $n = 3$ (c), where n denotes the level degeneracy. The dotted lines indicate the evolution of the dot total charge n_d . Reprinted figure with permission from Y. Shimizu *et al.*, Journal of the Physical Society of Japan **67**, 1525, 1998 [105]. Copyright (1998) by the Physical Society of Japan.

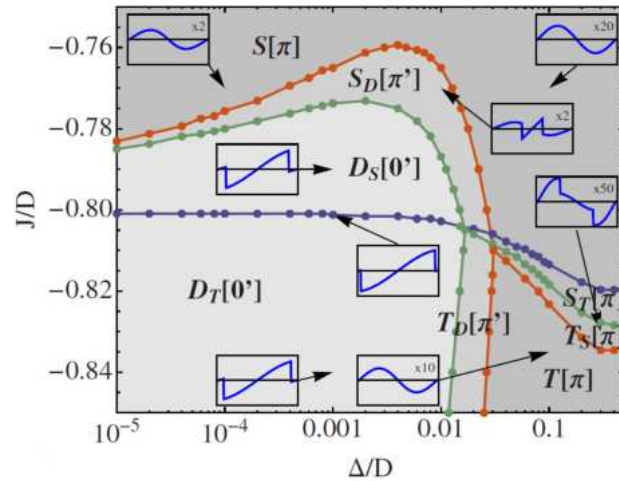


Figure 10.47: Phase diagram in the $J - \Delta$ plane for the two level dot model of Ref. [75] corresponding to an effective one channel situation with symmetric coupling to the leads. The notation for the different phases is indicated in the main text. Reprinted figure with permission from M. Lee *et al.*, Physical Review B **81**, 155114, 2010 [75]. Copyright (2010) by the American Physical Society.

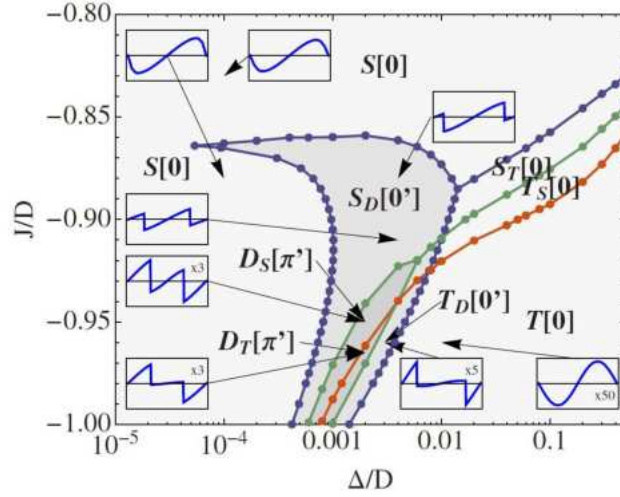


Figure 10.48: Same as Fig. 10.47 for the case corresponding to an effective two-channel two-impurity model. Reprinted figure with permission from M. Lee *et al.*, Physical Review B **81**, 155114, 2010 [75]. Copyright (2010) by the American Physical Society.

where the ground state changes with ϕ a subindex is included indicating the character of the metastable state. There is finally a label which can be either 0 or π indicating the character of the current-phase relation in the dominant phase. The different types of current-phase relations are shown as insets in Figs. 10.47 and 10.48.

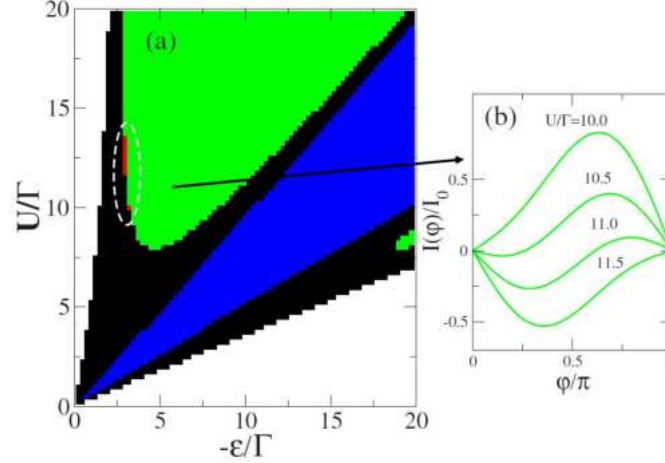


Figure 10.49: Phase diagram in the $U - \epsilon$ plane for the SU(4) Anderson model with superconducting leads analyzed in Ref. [136] obtained in the ZBW approximation for $\Delta = 10\Gamma$. White regions correspond to the $(S, T) = (0, 0)$ and green to $(S, T) = (1/2, 1/2)$. The black and blue regions correspond to mixed phases with $(S, T) = (0, 0)$ at $\phi = 0$ and $(S, T) = (1/2, 1/2$ or $(S, T) = (1, 0) - (0, 1)$ at $\phi = \pi$ respectively. The right panel shows the evolution of the current-phase relation for $\epsilon/\Gamma = -5$ and several values of U/Γ . The small region in red inside the dashed white line corresponds to a π' mixed phase. Reprinted figure with permission from A. Zazunov *et al.*, Physical Review B **81**, 012502, 2010 [136]. Copyright (2010) by the American Physical Society.

Another multilevel situation which has been recently analyzed is the case of a four-fold de-

generate level coupled to multichannel leads [136]. This situation has been found in experiments on high quality CNT quantum dots with normal leads showing clear signatures of four-fold degeneracy both in the Coulomb blockade and in the Kondo regimes [56, 57]. The SU(4) Kondo effect has been analyzed in the normal case by several authors like [131], [23] and [79]. In Ref. [136] Zazunov et al. have studied the Josephson effect in this case by considering a generalized SU(4) Anderson model with superconducting leads. They obtained analytical results in two opposite regimes corresponding to the deep Kondo limit $T_K \gg \Delta$ and the cotunneling limit. In the first case they obtained a current-phase relation which corresponds to the superposition of effective non-interacting channels with renormalized transmission $\tau = 1/2$, therefore deviating from the SU(2) case where the unitary limit can be reached. In the cotunneling case with $U \rightarrow \infty$ a $0-\pi$ transition at $\epsilon_0 = 0$ was obtained, as in the SU(2) case (see Ref. [44]) but with a different ratio of the critical currents $I_c(-|\epsilon_0|)/I_c(|\epsilon_0|) = -1/4$. The reduction of this ratio by a factor 2 with respect to the SU(2) case can be readily understood by considering the number of processes leading to Cooper pair transfer through the dot.

The authors also analyzed numerically the phase diagram of the model in the regime $\Delta \gg \Gamma$. A first insight is obtained by taking the limit $\Delta \rightarrow \infty$ in which case the relevant Hilbert space is reduced to the 2^4 dot states. Conservation of the total spin, S , and orbital pseudo-spin, T , allows to further decouple this Hilbert space into three different sectors: $(S, T) = (0, 0)$, $(S, T) = (1/2, 1/2)$ and $(S, T) = (1, 0)$ or $(0, 1)$ (these last two are degenerate in the SU(4) case). The main features of the phase diagram obtained in this limit were shown to be preserved in the finite $\Delta \gg \Gamma$ regime, which was analyzed by means of the zero band-width model for the leads. The phase diagram, illustrated in Fig. 10.49, is essentially the same as in the $\Delta \rightarrow \infty$ limit except for two properties: first, the appearance of tiny π' type mixed phase (indicated by the red regions in Fig. 10.49) and second by the change in the character of the current-phase relation of the $(S, T) = (1/2, 1/2)$ with increasing U . The panel on the right shows that this relation evolve with U/Δ : while for $\Delta \gg U$ it is typically of 0 type, for $U > \Delta$ it becomes of π type.

In closing this subsection we mention the recent appearance of a work by Lim et al. analyzing the effect of including spin-orbit interactions within this model [77], which can be relevant for small radius CTNs [74].

10.6.3 Multidot-multiterminal systems with normal and superconducting leads

Josephson effect through a quantum dot in a three terminal configuration

The Josephson effect through a single dot coupled to two superconductors and to a third normal lead has been analyzed by several authors. This configuration is schematically depicted in the upper panel of Fig. 10.50. An interesting non-equilibrium enhancement of the Josephson effect was predicted in Ref. [89]. The authors applied the real time diagrammatic approach commented in Subsect. 10.3.3 with the tunneling rates to the leads calculated to the first order in $\Gamma_{S,N}$. They found that a significant Josephson current can be induced by the voltage bias applied to the normal leads even in the situation $\Gamma_S < k_B T$ where the equilibrium Josephson current would be negligible.

Fig. 10.50 shows a color map of the Josephson critical current as a function of both the level position and the chemical potential on the normal lead, μ_N . One can notice the presence of a white region for $|\mu_N| \sim < U/2$ where the Josephson current is negligible. Outside this region it becomes of the order of $\sim e\Gamma_S/\hbar$ and exhibits a transition from 0 to π behavior depending on the level position. The origin of this peculiar behavior can be traced to the enhancement of the proximity effect pairing amplitude on the dot due to the non-equilibrium population which

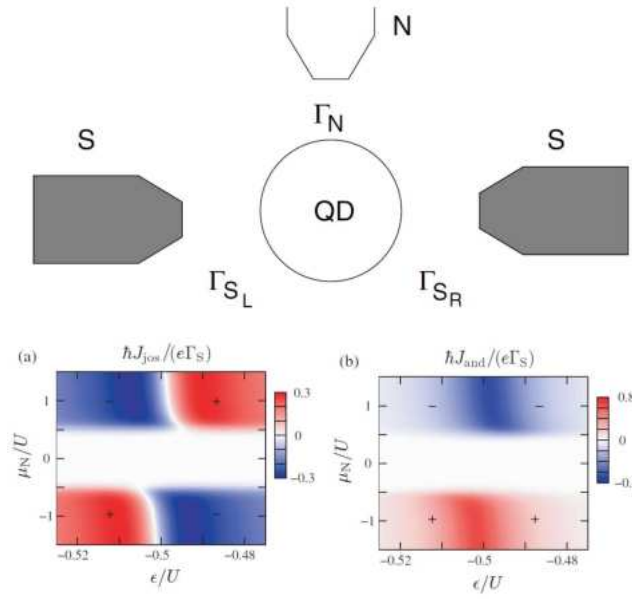


Figure 10.50: Color map of the Josephson critical current (lower left panel) and Andreev current (lower right panel) for the QD dot coupled to two superconducting leads and an additional voltage biased normal one in the $\mu_N/U - \epsilon/U$ plane. The upper panel gives a schematic representation of the setup considered in this work. Reprinted figure with permission from M. Pala *et al.*, New Journal of Physics **9**, 278, 2007 [89]. Copyright (2007) by IOP Publishing Ltd.

increases the double population probability that is strongly suppressed at $\mu_N = 0$ due to the charging energy. In a subsequent publication by the same group [49] the authors considered the same effect in the limit $\Delta \rightarrow \infty$ which allows to account for the Josephson effect to all orders in Γ_S . The results obtained are qualitatively similar and allow to identify the lines separating different regions with the Andreev bound states of the S-QD-S system.

It should be also mentioned within this context the work by Jonckheere *et al.* [61] in which the effect of a third normal lead on the ac Josephson effect in a non-interacting voltage biased S-QD-S system was analyzed. The main idea of this work was to study the transition from the coherent MAR regime to the incoherent limit controlled by the coupling Γ_N to the normal lead. They show that while the dc Josephson current exhibits a monotonous decrease with increasing Γ_N the behavior of the dc quasiparticle current and its first ac harmonics have a much more involved evolution, which is illustrated in Fig. 10.51.

Andreev transport through double quantum dots

Andreev transport in double quantum dot systems connected to normal and superconducting leads has been studied so far in a few works. Tanaka *et al.* [116] considered the case of a T-shape geometry where a central dot is coupled to both electrodes and a second dot is only side-coupled to the central one, as shown in the upper panel of Fig. 10.52. The authors used the NRG method in the $\Delta \rightarrow \infty$ limit where, as discussed in Subsect. 10.3.4, the superconducting lead acts as a simple boundary condition for Andreev reflection. They focused in the case where interactions are neglected in the central QD analyzing the effect of increasing U in the lateral dot on the conductance through the system. The results of Fig. 10.52 show that the Andreev conductance gradually approaches the unitary limit as the side dot U increases for the case $\Gamma_N = t$, where t is the hopping between the two dots. On the other hand, for smaller values of Γ_N the evolution of the conductance with U is non-monotonous exhibiting for the symmetric case first an increase followed by a decrease. In this work it was also shown that the

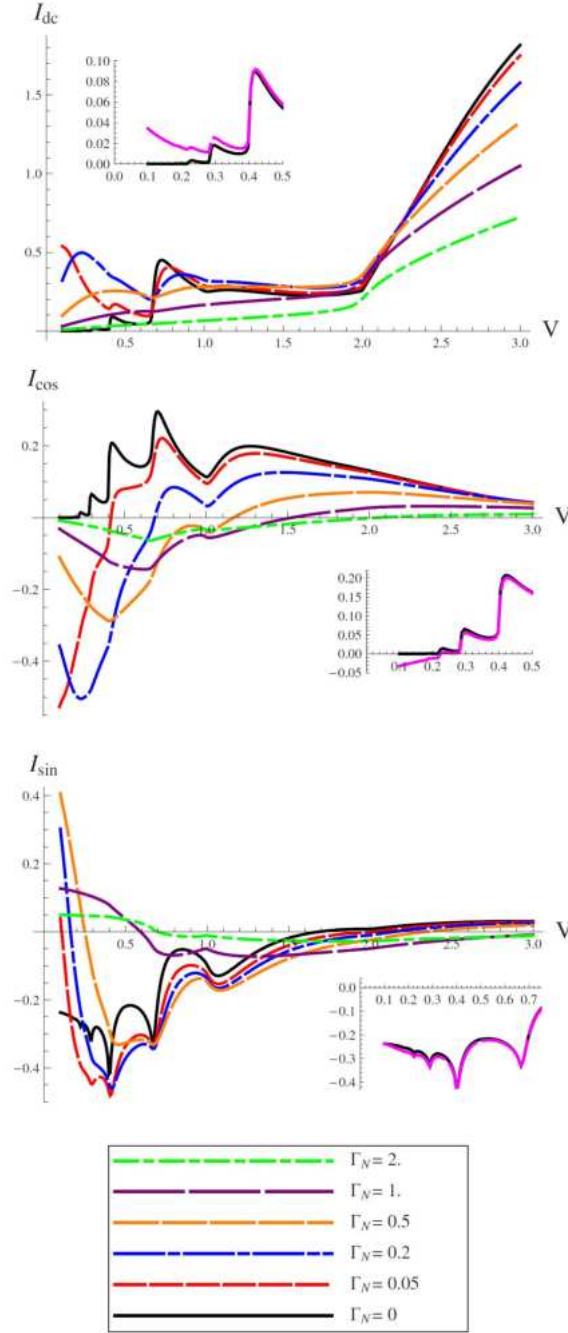


Figure 10.51: dc and the two first harmonics of the current-voltage characteristic of a non-interacting single level S-QD-S system with an additional normal lead connected to the dot with increasing coupling Γ_N analyzed in Ref [61]. The other parameters are ϵ_0 and $\Gamma_S = 0.2\Delta$. The insets show the comparison of the dc current and the corresponding harmonics for $\Gamma_N = 0$ (black line) and $\Gamma_N = 0.02\Delta$ (red line). Reprinted figure with permission from T. Jonckheere *et al.*, Physical Review B **80**, 184510, 2009 [61]. Copyright (2009) by the American Physical Society.

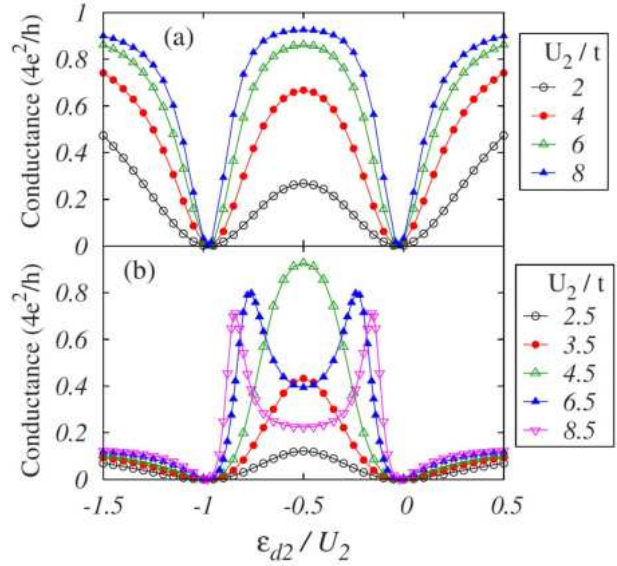
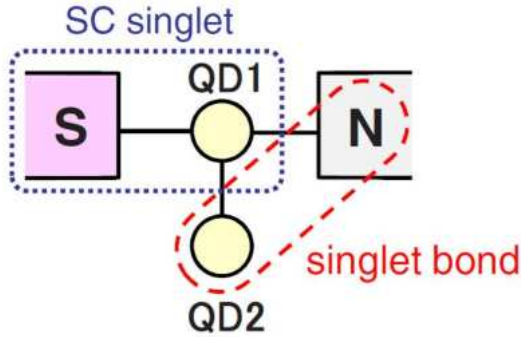


Figure 10.52: Linear conductance for the side coupled dot between a normal and a superconducting lead considered in Ref. [116] as a function of the lateral dot level ϵ_2 and for different values of the corresponding Coulomb interaction parameter U_2 . The upper panel shows a schematic representation of the setup considered in this work. Reprinted figure with permission from Y. Tanaka *et al.*, Physical Review B **78**, 035444, 2008 [116]. Copyright (2008) by the American Physical Society.

conductance in the electron-hole symmetric case can be tuned to the unitary limit for fixed U by varying the coupling to the superconducting lead Γ_S . This possibility is analogous to the one already discussed for the single SQDN system in Sect. 10.4.

In a subsequent paper [117], the same authors considered the case of a double QD in series connected to a normal and a superconducting lead (see upper panel of Fig. 10.53) using the same theoretical approach. They first showed that for the case where the dot coupled to the normal lead is in the electron-hole symmetric condition the problem can be mapped into an effective normal two impurity Anderson model in terms of the Bogoliubov operators. Furthermore, this transformation allows to calculate the conductance using the Friedel sum rule in terms of a phase-shift given by

$$G = \frac{4e^2}{h} \left(\frac{\Gamma_S}{E} \right) \sin(\pi Q), \quad (10.42)$$

where $E = \sqrt{\epsilon_2^2 + \Gamma_S^2}$, ϵ_2 being the level for the dot coupled to the superconductor and $Q = \sum_{\sigma} \left(\langle \gamma_{1\sigma}^\dagger \gamma_{1\sigma} \rangle + \langle \gamma_{2\sigma}^\dagger \gamma_{2\sigma} \rangle \right)$, $\gamma_{i\sigma}$ indicating the Bogoliubov operators. The authors identified three different regimes. For small t and $U < 2\Gamma_s$ a regime corresponding to the a local superconducting singlet is found while for $U > 2\Gamma_s$ the Kondo singlet state is formed. On the other hand, for large t the antiferromagnetic coupling between the dots dominates. The behavior of the conductance in these three different regimes is summarized in Fig. 10.53. As can be observed, for sufficiently small U/Γ_s the conductance can reach the unitary limit, the maximum displacing towards smaller t values as U increases and eventually for $U/\Gamma_s \sim > 2$ the unitary limit cannot be reached. A surprising feature appears for $U/\Gamma_s \sim 2$ where the conductance reaches the unitary limit for two different t values, indicated by the dashed rectangle in Fig. 10.53.

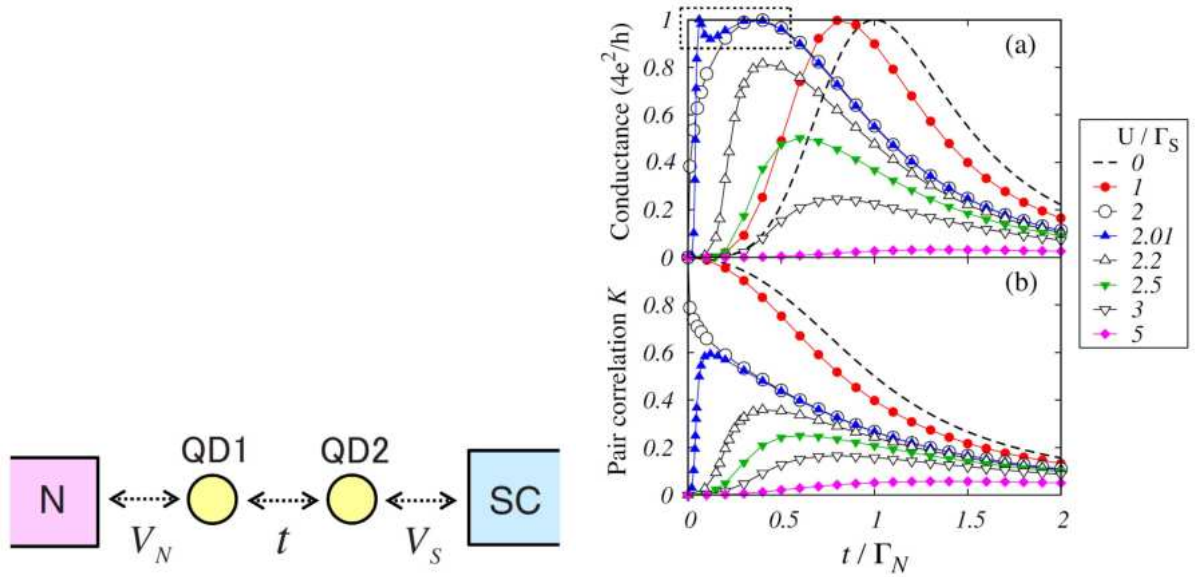


Figure 10.53: (a) Linear conductance for the series double quantum dot between a normal and a superconducting lead discussed in Ref. [117] as a function of the interdot tunneling t and for different values of U/Γ_s . Panel (b) shows the corresponding induced pairing correlation in the dots region. The results were obtained using the NRG method in the $\Delta \rightarrow \infty$ limit. Upper panel: setup considered in this work. Reprinted figure with permission from Y. Tanaka *et al.*, Physical Review B **81**, 075404, 2010 [117]. Copyright (2010) by the American Physical Society.

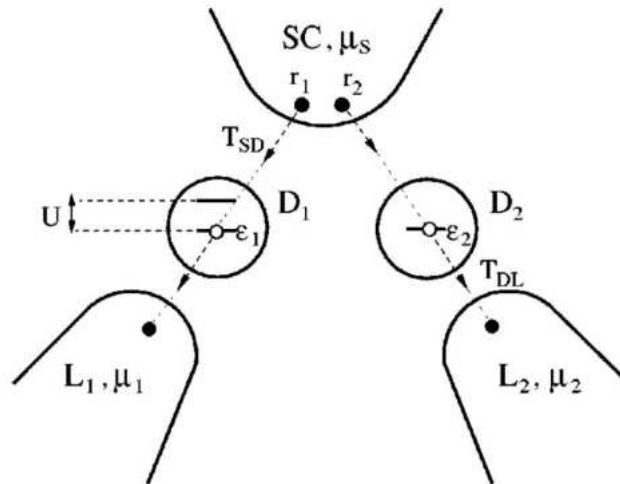


Figure 10.54: Schematic representation of a double quantum dot coupled to a central superconducting and two lateral normal leads in the proposal of Ref. [93] for producing entangled electron pairs by means of non-local Andreev processes. Reprinted figure with permission from P. Recher *et al.*, Physical Review B **63**, 165314, 2001 [93]. Copyright (2001) by the American Physical Society.

Non-local Andreev transport through single or double quantum dots

Double dots coupled to several normal and superconducting leads are receiving recently considerable attention in connection to the possibility of producing non-local entangled electrons from the splitting of Cooper pairs [93, 54, 52]. The basic ideas were first put forward in Ref.

[93] where the multiterminal geometry of Fig. 10.54 was considered. In this configuration, when a Cooper pair is injected from the SC lead it can either be transmitted as a whole to one of the normal leads by a *local* Andreev process or split with each of the electrons in the pair transmitted to a different lead (corresponding to a *non-local* or crossed Andreev reflection process). The advantage of the DQD set up is twofold: on the one side it allows to tune independently the two dot levels and on the other hand Coulomb interactions could be used to favor the splitting processes compared to the local ones. While this issue is of a great current interest it goes beyond the scope of the present review. We would just mention several works addressing the non-local Andreev transport involving quantum dots and multiterminal configurations in Refs. [43, 38].

10.7 Concluding remarks

In this review article we have summarized the most relevant published work related to superconducting transport in quantum dots systems. The large variety of topics that we have covered give an indication of the great activity which this field has shown in recent years. Due to the limited space it has become necessary to restrict somehow its content. For this purpose we have chosen to give priority to the more basic topics like Josephson effect and Andreev transport through single level quantum dots, and had left aside some interesting but more specialized situations like those involving ferromagnetic materials or unconventional superconductors. In the same way, we have not covered in this review the response of these systems to external ac fields, like photon assisted transport in S-QD-S [19, 139] or N-QD-S systems [137, 138] and adiabatic pumping in NDQS systems [111].

Even within the basic topics discussed in this review there remain several issues which are not completely understood and deserve further analysis. Among them we may point out: 1) the conflicting description of ABs within the different approximation schemes for S-QD-S systems, as discussed in Sect. 10.3; 2) a more complete analysis of the phase diagrams of double QDs which so far has been restricted to certain parameter ranges, as commented in Sect. 10.6; 3) clarifying the interplay of Kondo and Andreev transport in N-QD-S beyond the linear regime (discussed in Sect. 10.4, and 4) extending the analysis of the MAR regime in S-QD-S beyond the limit of weak interactions analyzed in Sect. 10.5.

It could be expected that the intense experimental and theoretical activity within this field will continue to grow in the next years. In addition to the already commented open issues there are several directions in which the research can be oriented. There are, on the one hand, other transport properties to be explored in the systems considered in this review, specially those related to current fluctuations. Some recent work address in fact the full-counting statistics in a non-interacting N-QD-S system [109], but certainly there is a lot of open issues regarding the effect of interactions and the non-local current correlations in multiterminal configurations. In fact the analysis of these correlations can provide insight on the issue of non-local entanglement produced by the splitting of Cooper pairs, as has been shown in the case of diffusive samples [14] and ballistic conductors [98].

On the other hand one could expect a renewed interest in these systems arising from the inclusion of recently discovered materials, like graphene and topological insulators. While graphene quantum dots have been already successfully produced experimentally and combined with superconductors like in Ref. [34], proposals of combining these systems with topological insulators are still on an speculative level [47]. We expect nevertheless that these issues would exhibit a great development in the near future.

11 Josephson Current through a Semiconductor Nanowire: effect of strong spin-orbit coupling and Zeeman splitting Cheng, Lutchyn

Abstract

We study coherent transport through a semiconductor nanowire in the presence of spin-orbit coupling and Zeeman splitting due to an applied magnetic field. By employing analytical and numerical techniques we develop a theory for the Josephson effect in the superconductor-semiconductor nanowire-superconductor structure. We show that Josephson current through the clean semiconductor nanowire exhibits a number of interesting features due to the interplay between the Zeeman splitting and spin-orbit coupling. We also study effect how disorder in the nanowire affects Andreev bound-state energy spectrum and calculate local density of states at the junction.

11.1 Introduction

Josephson effect is an example of macroscopic quantum phenomena and is of fundamental importance in condensed matter physics. The applications of the Josephson effect are both numerous and profound [1, 2, 3]. Recently, there has been a growing interest in understanding coherent transport phenomena in hybrid heterostructures involving s-wave superconductors and non-superconducting nanostructures [4, 5, 6, 7, 8, 9, 10, 11]. The possibility to control various parameters of the nanostructures allows one to explore different aspects of the Josephson effect in these devices.

In this paper, we study Josephson effect in superconductor-semiconductor nanowire-superconductor heterostructure (SC-SM-SC) shown in Fig 11.1a. The system consists of a semiconductor nanowire with strong Rashba spin-orbit coupling(SOC) subject to a large in-plane magnetic field. The wire is connected to the superconducting leads as shown in Fig. 11.1a. When chemical potential is in the gap, see Fig. 11.1b, there is only one Fermi surface [12]. Coherent transport through the nanowire in the helical phase, which originates from the non-trivial interplay between large Zeeman splitting and strong Rashba SOC, have not been previously explored [13]. Previous studies of the Josephson effect [15, 15, 16, 17] considered the case of either small Zeeman field or small SOC whereas we are interested in a non-perturbative limit where *both* Zeeman and SO terms are large leading to a significant change of the nanowire bandstructure. Such heterostructures involving semiconductor nanowires and s-wave superconductors exhibit a number of interesting effects and, in particular, are very promising for realizing topological superconducting phase hosting Majorana zero-energy modes [20, 21]. We note that the present setup is different from the one in Ref. [[20]] leading to 4π ac Josephson effect because the SC parts are topologically trivial and there are no Majorana bound states.

In this paper, we develop a quantitative theory for the coherent transport through the nanowire in the clean limit by employing a combination of analytical and numerical techniques. The supercurrent through the nanowire in the short-constriction limit [24, 25] is mainly transmitted by the discrete electron-hole states (Andreev bound states) confined in the nanowire. These bound states appear because an electron incident on the semiconductor/superconductor interface is Andreev reflected as a hole with the opposite spin and momentum. To understand how Zeeman splitting and spin-orbit coupling affect Andreev bound states, it is instructive to consider two limiting cases: (a) magnetic-field-dominated regime where Zeeman splitting V_z is much larger than the spin-orbit coupling E_{so} . In this regime, propagating electron and hole modes are spin-polarized, and, thus, Andreev reflection is suppressed. (b) spin-orbit dominated

regime $E_{\text{so}} \gg V_z$. In this case, the propagating electron and hole have helical character and Andreev bound states can be formed by the helical electrons. The intuition developed for these two limits allows one to understand the general case of both strong Zeeman and SO couplings. We first consider short-junction limit and show that the helical phase in the semiconductor nanowires can be identified by a sharp decrease of the supercurrent as one increases an in-plane magnetic field. We also consider here long-junction limit and show that our findings persist in this case as well.

To have a better understanding of the Josephson transport in realistic situations, we consider the Andreev spectrum in disordered nanowires and calculate local density of states (LDoS) which can be probed in tunneling experiments. It is generally found that disorder smears out the peaks in the local density of states corresponding to the Andreev bound states and reduces transmission probability. However, we also show that magnetic field induced zero-energy crossings generically appearing in Andreev spectrum are robust against moderate disorder. This phenomenon appears when Andreev energy levels are non-degenerate due to the combination of spin-orbit and Zeeman couplings and is a consequence of the particle-hole symmetry which constraints certain matrix elements between a pair of zero-energy states at an isolated crossing. Thus, we predict that in this regime there will be generically zero-bias peaks in tunneling conductance as a function of the superconducting phase difference across the junction.

The paper is organized as follows: In Sec. I we present the theoretical model of the superconductor-nanowire-superconductor junction and the scattering matrix formalism for calculating the Josephson current. In Sec. II we examine in great detail the Andreev spectrum and Josephson current in short-junction limit. In Sec. III we study Josephson transport by numerically diagonalizing the Hamiltonian in a finite-size system which goes beyond the short-junction limit. In Sec. IV we consider the disorder effect.

11.2 Theoretical Model

We consider SC-SM-SC junction shown in Fig. 11.1 where the single-channel semiconductor nanowire is modeled using the following Hamiltonian:

$$\mathcal{H}_0 = \psi^\dagger(x) \left(-\frac{\partial_x^2}{2m^*} - \mu + V_z \sigma_x - i\alpha \sigma_y \partial_x \right) \psi(x), \quad (11.1)$$

where m^*, μ, α are the effective mass, chemical potential and strength of Rashba spin-orbit coupling, respectively; ψ is the spinor $(\psi_\uparrow, \psi_\downarrow)^T$ and we set $\hbar = 1$. The magnetic field B applied along the nanowire leads to the Zeeman energy $V_z = g_{\text{SM}} \mu_B B/2$ where g_{SM} is the g -factor in the semiconductor which is assumed to be much larger than in the superconductor. One notices that the Hamiltonian is invariant under the combined operation $U = i\sigma_x P$ of spatial inversion $P : Pf(x) \rightarrow f(-x)$ and spin π -rotation $i\sigma_x$. The eigenvalues of H_{SM} in momentum space are

$$\varepsilon_\pm(k) = \frac{k^2}{2m^*} - \mu \pm \sqrt{\alpha^2 k^2 + V_z^2}. \quad (11.2)$$

Taking $\mu = 0$ to be mid-way in the gap as a reference point, one can see that depending on whether $|\mu|$ exceeds $|V_z|$ or not, the semiconductor can have one or two Fermi surfaces. The phase with only one Fermi surface is called “helical” because the motion of an electron and its spin are correlated. Conversely, if $|V_z| < \mu$, there are two Fermi surfaces and we refer to this phase as non-helical (i.e. normal), see Fig. 1(b).

Two superconducting leads on both sides of the nanowire are described by s-wave BCS Hamiltonian with $\Delta_{R/L} = \Delta_0 e^{i\phi_{R/L}}$ being the corresponding order parameters, see Fig 11.1(a). We assume that one can neglect Zeeman splitting in the superconductor due to the large difference in the Lande g -factors, *e.g.* the g -factor in InSb is $|g_{\text{SM}}| \sim 50$ whereas in the superconductor

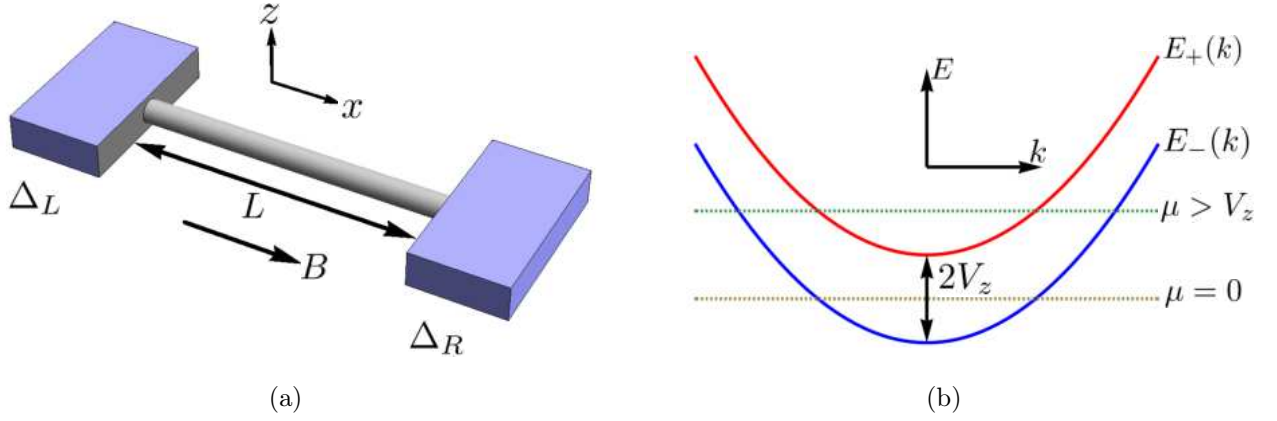


Figure 11.1: (a) Schematic view of the SC-SM-SC junction. Semiconducting nanowire is coupled to s-wave superconducting electrodes. The chemical potential in the nanowire can be tuned by the gate (not shown). (b) Energy spectrum of the semiconductor with strong Rashba spin-orbit coupling and Zeeman splitting. The two values of the chemical potential at $\mu = 0$ and $\mu > V_z$ correspond to helical and non-helical states of the nanowire, respectively.

$g_{\text{SC}} \approx 2$. Thus, one can tune the field to the regime where magnetic field induces large Zeeman splitting in the semiconductor without substantially suppressing superconductivity in the leads.

The quasiparticle spectrum in this system can be found by solving the Bogoliubov-de Gennes(BdG) equation:

$$\begin{pmatrix} \mathcal{H}_0 & \Delta(x) \\ \Delta^*(x) & -\sigma_y \mathcal{H}_0^* \sigma_y \end{pmatrix} \Psi(x) = E \Psi(x). \quad (11.3)$$

Here $\Psi = (u_\uparrow, u_\downarrow, v_\downarrow, -v_\uparrow)^T$ is the wave function in the Nambu spinor space.

The dc Josephson current in the junction, as a function of the superconducting phase difference φ , is given by $I(\varphi) = -\frac{2e}{\hbar} \sum_{\varepsilon_n > 0} \tanh \frac{\varepsilon_n(\varphi)}{2T} \frac{d\varepsilon_n(\varphi)}{d\varphi}$ with ε_n being positive energy eigenvalues of Andreev states. Interestingly, the symmetry U defined above implies that Andreev spectrum should be symmetric under $\varphi \rightarrow -\varphi$, provided the two interfaces are mirror symmetric, which is assumed in our calculations. If magnetic field and the direction of the spin-orbit coupling are not perpendicular to each other (i.e., there is a nonzero parallel component), the inversion symmetry with respect to left and right leads is completely broken and the spectrum can be asymmetric, leading to a spontaneous formation of supercurrent at the junction, see Ref.[23] for details.

We now consider the short-junction geometry corresponding to the length of the nanowire L such that the dwell time of electrons τ_{dw} in the nanowire $\tau_{\text{dw}} \sim L/v_F \ll \Delta_0^{-1}$, which means that $L \ll \xi = \frac{v_F}{\Delta_0}$. In this case, the main contribution to the Josephson current comes from the Andreev bound states formed in the junction [24, 25]. The spectrum of the Andreev bound states in the nanowire can be obtained using scattering matrix formalism [26]. For technical reasons, we insert thin ballistic normal-state leads (N) to the left and the right of the semiconducting nanowire allowing for a well-defined scattering problem [26]. The spatial separation of Andreev and normal scattering simplifies the problem substantially and allows one to relate the Josephson current to normal-state scattering matrix for N-SM-N system.

The spectrum of Andreev states is determined by the following condition [26]:

$$\det[1 - r_{\text{he}} S_e(\varepsilon) r_{\text{eh}} S_h(\varepsilon)] = 0, \quad (11.4)$$

where

$$r_A = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \quad (11.5)$$

$$r_{he} = \gamma r_A, \quad r_{eh} = \gamma r_A^*, \quad \gamma = e^{-i \arccos \frac{\varepsilon}{\Delta_0}},$$

and $S_{e(h)}$ is the normal-state scattering matrix for electrons (holes). The particle-hole symmetry relates S_h to S_e as $S_h(\varepsilon) = \Theta S_e(-\varepsilon) \Theta^{-1}$, where $\Theta = i\sigma_y K$ is the particle-hole operator for spin-1/2 electrons and K is the operator of complex conjugation [15]. We further neglect the energy dependence of scattering matrices $S_e(\varepsilon) \approx S_e(0)$, which is justified in the short-junction limit. We will examine the validity of the short-junction approximation later. We assume perfect transparency at the superconductor-nanowire interface unless stated otherwise.

11.3 Short-Junction Results

11.3.1 Zeeman-field dominated regime

We consider first the case of a large Zeeman splitting $V_z \gg E_{so} \equiv m^* \alpha^2$, and show that the Josephson current exhibits two remarkable features depending on the value of the chemical potential: in the non-helical regime $V_z < \mu$, supercurrent exhibits oscillations with the applied magnetic field whereas in the helical phase $V_z > \mu$ the supercurrent is significantly suppressed.

The normal electron scattering matrix can be found exactly in the $\alpha = 0$ limit.

$$S_e = \begin{pmatrix} -iB_\uparrow/A_\uparrow^* & 0 & 1/A_\uparrow^* & 0 \\ 0 & -iB_\downarrow/A_\downarrow^* & 0 & 1/A_\downarrow^* \\ 1/A_\uparrow^* & 0 & -iB_\uparrow/A_\uparrow^* & 0 \\ 0 & 1/A_\downarrow^* & 0 & -iB_\downarrow/A_\downarrow^* \end{pmatrix}, \quad (11.6)$$

where the coefficients A_σ, B_σ are defined as:

$$A_\sigma = \cos k_\sigma L + \frac{i}{2} \left(\frac{k_F}{k_\sigma} + \frac{k_\sigma}{k_F} \right) \sin k_\sigma L$$

$$B_\sigma = \frac{1}{2} \left(\frac{k_F}{k_\sigma} - \frac{k_\sigma}{k_F} \right) \sin k_\sigma L. \quad (11.7)$$

Here k_F and $k_\sigma = \sqrt{2m(\mu - \sigma V_z)}$ are the Fermi momentum in the normal leads and electron momentum in the nanowire, respectively.

Having particle and hole scattering matrices, the spectrum of Andreev levels can be obtained using Eq.(5) of the main text. In this case of $V_z \neq 0, \alpha = 0$, the spectrum is determined by the following equation:

$$2 \arccos \frac{\varepsilon}{\Delta_0} = \arccos \frac{B_\uparrow B_\downarrow + \cos \varphi}{|A_\uparrow A_\downarrow|} \pm \arg \frac{A_\uparrow}{A_\downarrow} + 2n\pi. \quad (11.8)$$

Here $n \in \mathbb{Z}$.

Let us now consider the limit $V_z \ll \mu$. In this case, the spectrum of Andreev levels to the leading order in V_z/μ becomes $\varepsilon(\varphi) \approx \pm \Delta_0 \cos \left(\frac{\varphi}{2} \pm \varphi_0 \right)$ with $\varphi_0 = V_z L / v_F$, and the Josephson current is given by

$$\frac{I(\varphi)}{2e\Delta_0} = \begin{cases} \cos \varphi_0 \sin \frac{\varphi}{2} & 0 < \varphi < \pi - 2\varphi_0 \\ -\sin \varphi_0 \cos \frac{\varphi}{2} & \pi - 2\varphi_0 < \varphi < \pi + 2\varphi_0, \\ -\cos \varphi_0 \sin \frac{\varphi}{2} & \pi + 2\varphi_0 < \varphi < 2\pi \end{cases} \quad (11.9)$$

The current $I(\varphi)$ exhibits a jump at $\varphi = \pi \pm 2\varphi_0$, corresponding to the zero-energy crossings in the Andreev spectrum. The energy spectrum and the position of the crossings can be understood from the doubly-degenerate Andreev spectrum without any Zeeman field $\varepsilon(\varphi) = \pm\Delta_0 \cos(\varphi/2)$, with zero-energy crossing at $\varphi = \pi$. The Zeeman field splits the degeneracy and the zero-energy crossings move away from π by an amount φ_0 . Note that zero-energy crossings are not robust and appear due to our assumption of perfect interface transparency $\mathcal{T} = 1$. In realistic conditions $\mathcal{T} \neq 1$, and zero-energy crossings will become avoided crossings corresponding to the local minima of the energy spectrum and therefore vanishing of the Josephson current, which are accompanied with the sign reversal of the supercurrent [15], see Fig.11.2.

The physical origin of the phase shift φ_0 can be understood quasi-classically from the Bohr-Sommerfeld quantization condition. Indeed, Andreev bound states can only form when the total phase acquired by the electron undergoing two Andreev reflections is quantized: $\oint p dx + \Omega = 2\pi m, m \in \mathbb{Z}$. Here $\Omega = \pm\varphi - 2 \arccos \frac{\varepsilon}{\Delta}$ is the total scattering phase acquired by the electron (hole) at the interfaces and $\oint p dx = \pm(k_\uparrow - k_\downarrow)L \approx \pm 2V_z L/v_F$. The latter agrees perfectly with the scattering matrix calculation. As follows from this analysis, the Josephson current in the limit of $V_z \ll \mu$ should oscillate with the Zeeman field V_z with period which is proportional to $1/L$. Similar oscillations of the critical current have been found in SC-ferromagnet-SC junction [16, 17].

In the polarized limit ($V_z \gg \mu$), the propagation of the spin-up electrons in the nanowire is exponentially suppressed. The calculation of the Josephson current in this limit yields the following result:

$$I(\varphi) \approx 2e\Delta_0 e^{-\kappa L} \left(1 - \frac{\mu}{8V_z} \sin^2 k_\downarrow L\right) \sin \varphi, \quad (11.10)$$

where $\kappa = \sqrt{2m^*(V_z - \mu)}$ and $k_\downarrow = \sqrt{2m^*(V_z + \mu)}$. Thus, if the length of the wire $\xi \gg L \gg \kappa^{-1}$, the supercurrent through the nanowire will be significantly suppressed. This result is strictly speaking only valid when $\alpha = 0$ as the presence of spin-orbit coupling leads to spin flips. However, the probability for the spin flips to occur within the distance κ^{-1} is small because $E_{\text{so}} \ll V_z$. Thus, the suppression of the supercurrent in the helical phase should be observable even in the presence of the small SOC as we explicitly demonstrate below.

11.3.2 SOC dominated regime

We now discuss Josephson effect in the case of small Zeeman splitting $V_z \ll E_{\text{so}}$. The effect of magnetic field is perturbative here and we begin with $V_z = 0$ limit. Without any loss of generality, we can choose the direction of the spin-orbit coupling to be along $\hat{z}$ -axis. The s-wave superconductor is spin-SU(2) invariant. Thus, the Hamiltonian for the nanowire reads

$$H = \frac{p^2}{2m^*} - \mu - \alpha p \sigma_z. \quad (11.11)$$

The energy spectrum for spin-up(-down) electrons is shifted to the right (left). Assuming that the chemical potential $\mu > 0$, we can write down the wavefunction in the nanowire as

$$\psi(x) = \sum_{\sigma=\uparrow,\downarrow} (b_\sigma^+ e^{ik_\sigma x} + b_\sigma^- e^{-ik_\sigma x}) |\sigma\rangle. \quad (11.12)$$

Here k_σ are defined as

$$k_\sigma = k_0 + m^* \alpha \sigma, k_0 = \sqrt{2m^* \mu + (m^* \alpha)^2}. \quad (11.13)$$

Matching the boundary conditions at $x = \pm L/2$, we find the normal state scattering matrix

to be

$$S_e = \frac{1}{A^*} \begin{pmatrix} iB & 0 & e^{-im^*\alpha L} & 0 \\ 0 & iB & 0 & e^{im^*\alpha L} \\ e^{im^*\alpha L} & 0 & iB & 0 \\ 0 & e^{-im^*\alpha L} & 0 & iB \end{pmatrix}. \quad (11.14)$$

Here

$$\begin{aligned} A &= \cos k_0 L + \frac{i}{2} \left(\frac{k_F}{k_0} + \frac{k_0}{k_F} \right) \sin k_0 L \\ B &= \frac{1}{2} \left(\frac{k_0}{k_F} - \frac{k_F}{k_0} \right) \sin k_0 L. \end{aligned} \quad (11.15)$$

and k_F is the Fermi momentum in the normal leads.

The spectrum of Andreev levels follow directly:

$$\varepsilon(\varphi) = \pm \Delta_0 \sqrt{1 - \frac{\sin^2(\varphi/2)}{1 + \lambda^2 \sin^2 k_0 L}}. \quad (11.16)$$

Here $\lambda = \frac{1}{2}(\frac{k_F}{k_0} - \frac{k_0}{k_F})$ and $k_0 = \sqrt{2m^*\mu + m^{*2}\alpha^2}$, k_F is the Fermi momentum in the superconductor. Andreev levels are doubly degenerate due to the presence of the time-reversal symmetry as well as the approximation $S_e(\varepsilon) \approx S_e(0)$ used here. The small corrections to the spectrum due to the presence of the spin-orbit coupling were considered in Refs.[[18, 19]]. The main effect of spin-orbit coupling here is actually the modification of the Fermi momenta which affects normal reflection and thus the Josephson current due to the mismatch of the Fermi momenta in the nanowire and superconductor. For special values of L satisfying the resonance condition $k_0 L = n\pi$, $n \in \mathbb{Z}$, transmission of electrons through the spin-orbit coupled wire becomes perfect and the supercurrent is the same as in the absence of SOC.

11.3.3 General case

We now turn to the general case when both SO and Zeeman terms are large. We can develop some intuition by considering Andreev reflection in terms of the eigenstates $|\pm\rangle$ of the Hamiltonian (11.1). From the discussion of the Zeeman-dominated regime, we recall that there are two major consequences caused by Zeeman polarization: (a) the oscillation of the critical current with V_z and L . (b) Significant suppression of the critical current when $V_z > \mu$. However, the spin-orbit interaction causes precession of the spin of electrons/holes within the characteristic length scale l_{so} , which tends to wash out the spin polarization of electrons and holes. To this end, it is instructive to rewrite the s-wave pairing in the chiral basis:

$$\Delta \psi_{\uparrow}^{\dagger}(k) \psi_{\downarrow}^{\dagger}(-k) = \Delta \sum_{\lambda, \lambda'=\pm} \chi_{\lambda\lambda'}(k) \psi_{\lambda}^{\dagger}(k) \psi_{\lambda'}^{\dagger}(-k), \quad (11.17)$$

where $\chi_{\lambda\lambda}(k) = \alpha k / \sqrt{V_z^2 + \alpha^2 k^2}$ and $\chi_{+-}(k) = V_z / \sqrt{V_z^2 + \alpha^2 k^2}$. Thus, both inter-band and intra-band pairing are generated in the chiral basis. Therefore, one expects that Andreev scattering at the N-S boundary has qualitatively similar behavior, i.e. there is both inter- and intra-band scattering. We can understand our analytical results in the two limiting cases as follows: first in the Zeeman-dominated regime inter-band scattering dominates over intra-band scattering. Once the $|+\rangle$ band is gapped, quasiparticles in this band cannot propagate in the junction whose length is much larger than the length scale associated with the Zeeman gap. Therefore, the supercurrent will be suppressed, similar to the spin-polarized case discussed above. In the opposite regime $E_{so} \gg V_z$, the Andreev reflection is dominated by the intra-band scattering and the situation is similar to the SOC-dominated regime where there is no abrupt change in the Josephson current across the boundary $\mu = V_z$.

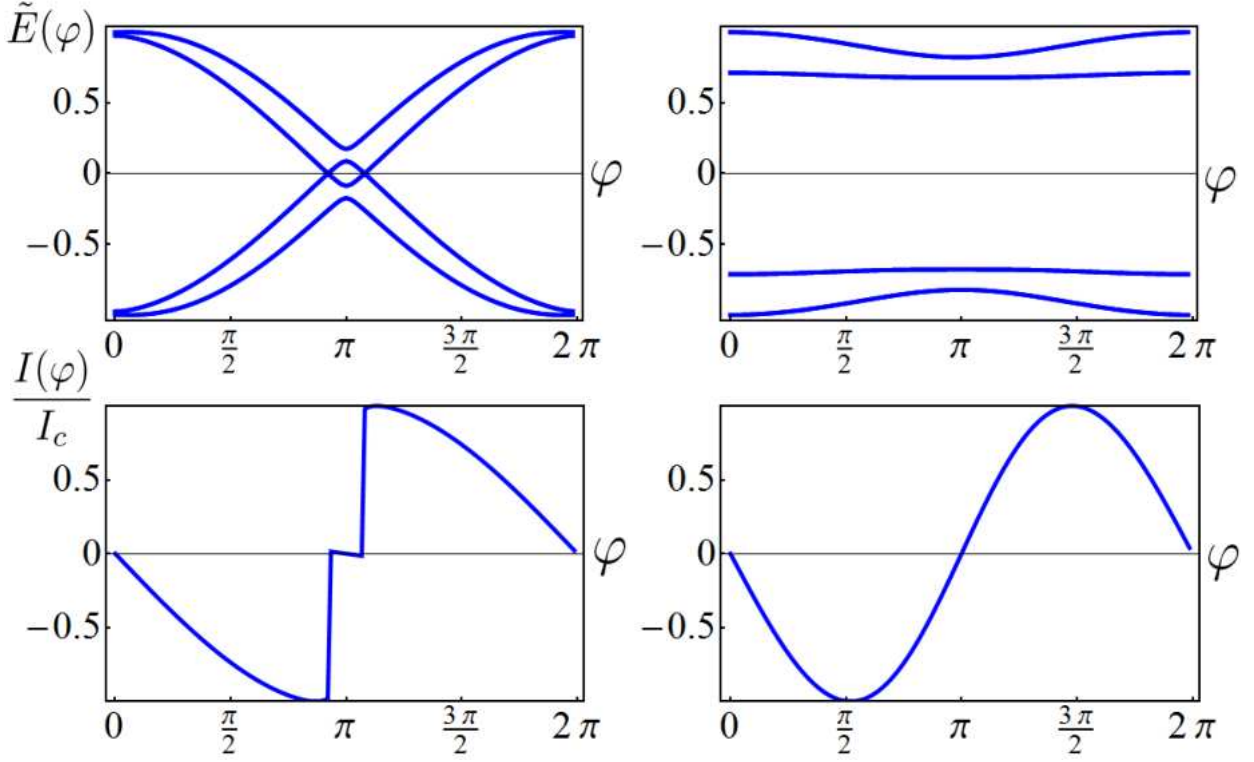


Figure 11.2: $I(\varphi)/I_c$ and $E(\varphi)$ in the non-helical phase (left panel) and the helical phase (right panel). The parameters are $\tilde{V}_z = 0.1$, $\tilde{\mu} = 0.5$ in the non-helical phase and $\tilde{\mu} = 0.01$ in the helical phase.

The discussion above provides some intuition about Andreev reflection. However, in order to understand the coherent transport quantitatively, one needs to consider also normal reflection at the SC-SM interface. This can be done by appropriately matching the boundary conditions and finding the corresponding scattering matrix $S(\varepsilon)$ as described in the appendix. The expression of Josephson current in the general case is not particularly enlightening and we present here numerical results instead. We proceed by first re-scaling all the lengths and energies in units of $l_{\text{so}} = \hbar^2/m^*\alpha$ and $E_{\text{so}} = m^*\alpha^2/\hbar^2$, respectively. Using typical parameters for InAs $m^* \approx 0.04m_e$, $\alpha \approx 0.1 \text{ eV}\cdot\text{\AA}$, we estimate $l_{\text{so}} \sim 100 \text{ nm}$, $E_{\text{so}} \sim 1 \text{ K}$. The Josephson current is measured in units of $2e\Delta_0/\hbar$. Henceforth, all the re-scaled physical quantities are denoted by tilde.

The Andreev spectra and current-phase relation for the SC-SM-SC junction are shown in Fig. 11.2. In the non-helical phase, Zeeman field splits the degenerate Andreev levels and results in zero-energy crossings. As discussed above, finite transparency of the SM-SC interface will lead to avoided level crossings at these points. One can see that as we change chemical potential from non-helical to helical phase, the dispersion of Andreev levels with φ decreases resulting in the suppression of the Josephson current.

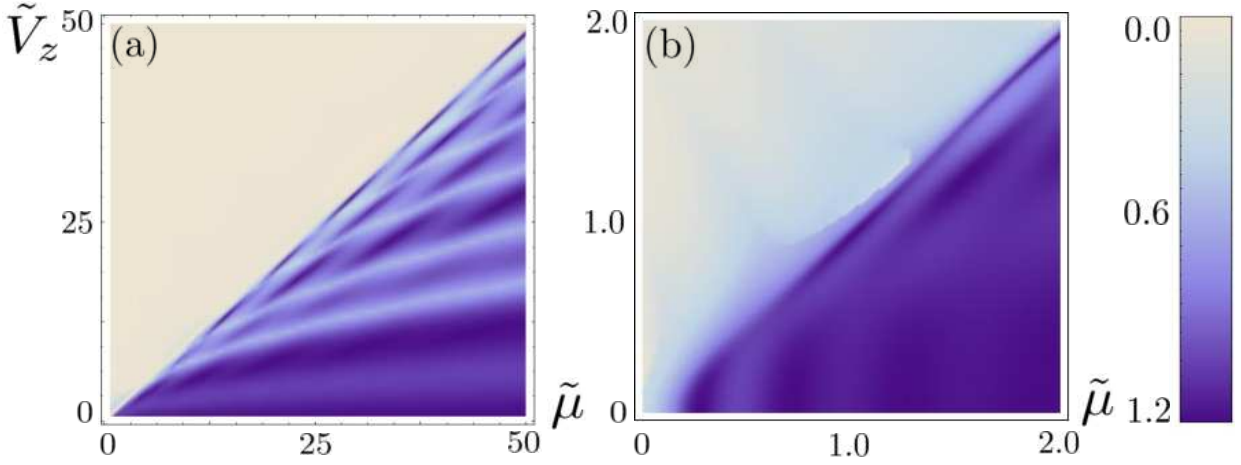


Figure 11.3: Density plot of the current $|I(\pi/2)|$ as a function of $\tilde{V}_z$ and $\tilde{\mu}$. In order to understand the magnitude of the critical current, we need to decouple the phase dependence of the Josephson current which is non-trivial, especially around $\phi = \pi$. We find that it is more useful to plot $I(\varphi = \pi/2)$ instead of I_c . In Fig. (a) we show $|I(\pi/2)|$ for junction length $L = 10$. Fig. (b) is a zoom-in of Fig. (a) for small $\tilde{V}_z$ and $\tilde{\mu}$ (or equivalently, large E_{so}).

The dependence of the Josephson current on chemical potential and Zeeman splitting is summarized in Fig. 11.3 in which we plot $|I(\pi/2)|$ as a measure of the size of the critical current. For $V_z \gg E_{so}$, the suppression of the Josephson current between helical and non-helical phases manifests as a sharp borderline $\tilde{V}_z = \tilde{\mu}$ in Fig. 11.3(a), which actually persists to $V_z \sim E_{so}$. As one further increases SOC to $E_{so} \gg V_z$, such suppression is smeared out as shown in Fig. 11.3(b). The critical current oscillations with magnetic field are also reflected in the “stripes” in the region $\tilde{V}_z < \tilde{\mu}$. One can notice that as μ increases the onset of the oscillations requires a larger V_z . Thus, the main features discussed qualitatively above within the simple models are also present in the general case.

11.4 Long-junction results

So far we have discussed Josephson effect for $L \ll \xi$. We now present results for the long-junction obtained using finite-size lattice calculation which involves full diagonalization of the BdG equations. This calculation is suitable for studying the Josephson effect beyond the short-junction limit allowing one to take into account the contribution to the current from the continuum spectrum. The tight-binding Hamiltonian reads

$$\begin{aligned}
H = & \sum_{\langle ij \rangle, \sigma, \sigma'} [\delta_{\sigma, \sigma'} t_{ij} c_{j\sigma}^\dagger c_{i\sigma'} + \alpha_i c_{j, \sigma}^\dagger (\tau_x)_{\sigma\sigma'} c_{i\sigma'} + \text{h.c.}] \\
& + \sum_i (V_{zi} \sigma - \mu_i) c_{i\sigma}^\dagger c_{i\sigma} + \sum_i (\Delta_i c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger + \text{h.c.}).
\end{aligned} \tag{11.18}$$

Here we allow V_z, μ, α, Δ to be position-dependent to model the junction. τ are the Pauli matrices in spin space. We take hopping amplitudes to be $t_{ij} = -t(\delta_{i,j-1} + \delta_{i,j+1})$. We assume that $\alpha_i = \alpha$ and $V_{zi} = V_z$ in the wire and zero otherwise; the SC pairing potential Δ_i is chosen to be non-zero outside of the wire with the magnitude $\Delta_0 = 0.05t$ which corresponds to the coherent length $\xi \sim 10a$. In the numerical calculations, we take the length of the

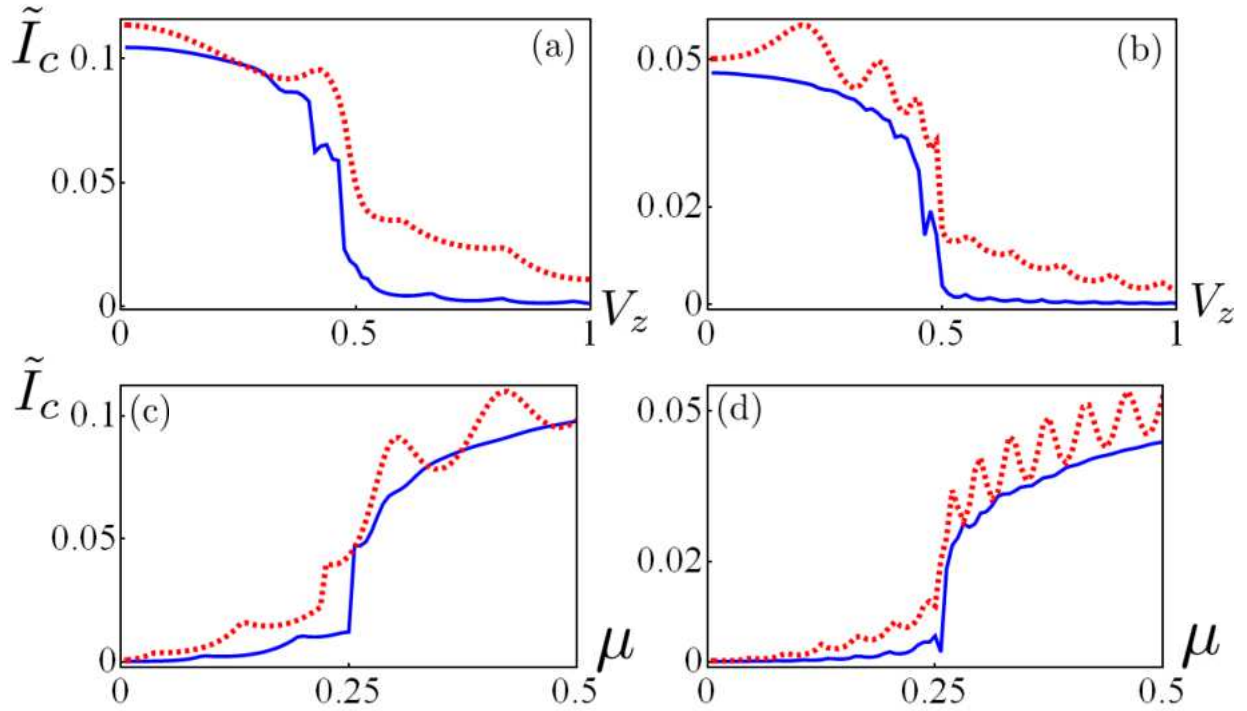


Figure 11.4: Dependence of $\tilde{I}_c = |I(\pi/2)|$ on μ and V_z from finite-size lattice diagonalization. The blue(solid) lines are results for $\alpha = 0.5$ and the red(dashed) lines are results for $\alpha = 1.5$. The parameters for each subfigure are: (a) $\mu = 0.5, L = 20$. (b) $\mu = 0.5, L = 60$. (c) $V_z = 0.25, L = 20$. (d) $V_z = 0.25, L = 60$. Here the current is normalized by the corresponding value at $V_z = 0$. Here μ is measured from the band bottom.

superconductors on both sides to be $L_{SC} = 200a$ and measure the energies and length in units of t and a , respectively.

Having the excitation spectrum of the system as a function of the phase difference φ , one can compute Josephson current through the nanowire. The dependence of the Josephson current on μ and V_z is shown in Fig. 11.4. The suppression of the Josephson current as one sweeps parameters between helical and non-helical phases ($\mu \approx V_z$) is also visible in the long-junction limit for small values of the SOC, see results for $\alpha = 0.5$ in Fig. 11.4. As SOC is increased to $\alpha = 1.5$, the sharp suppression of the current is smoothen out and the difference in the critical current between the helical and non-helical phase becomes less significant. Overall, we find that the interesting features in the Josephson effect discussed above persist in the long-junction limit.

11.5 Local Density of States at the junction

In addition to measuring the Josephson current discussed in the previous sections, one can also extract information about Andreev energy spectrum using tunneling spectroscopy experiments, effectively measuring local Density of States (LDoS). In these experiments, it is important to take into account effect of disorder since it will lead to a reduction of the transmission probability and, thus, will strongly affect the energy spectrum. For example, in the SNS system in the short junction limit, the zero-energy crossing at $\varphi = \pi$ as evident from Andreev spectrum $\varepsilon(\varphi) = \pm\Delta \cos \frac{\varphi}{2}$ is removed once we introduce impurities which lead to the reduction of the junction transparency[26]. The situation is different in semiconductor

nanowires in a magnetic field as we explain below.

Let's consider LDoS defined as

$$\rho(\mathbf{r}, \omega) = \sum_n [|u_{n\sigma}(\mathbf{r})|^2 \delta(\omega - E_n) + |v_{n\sigma}(\mathbf{r})|^2 \delta(\omega + E_n)], \quad (11.19)$$

Here u_σ, v_σ are eigenstates of the BdG equation corresponding to eigenenergy E_n . In our numerical calculations, we use Lorentzian function instead of the δ function and introduce some broadening of the energy levels. The LDoS in the clean system is shown in Fig. 11.5(a) and b). As discussed above, magnetic field generically leads to the appearance of the zero-energy crossings in Andreev spectrum. However, unlike in SNS junction, these crossings are robust against disorder which is a distinct signature of semiconductor nanowires in a magnetic field where Andreev states are non-degenerate in contrast with the conventional normal metals having spin-degenerate spectrum. These differences can be seen in LDoS measurements in a magnetic field which can be done experimentally since LDoS is directly related to the local differential tunneling conductance:

$$\frac{dI(\mathbf{r}, eV)}{dV} \propto \rho(\mathbf{r}, \omega = eV). \quad (11.20)$$

We now discuss the disorder-averaged LDoS $\rho(\mathbf{r}, \omega)$ where these features are visible. The disorder can be included in the lattice calculations by adding an on-site random chemical potential to the Hamiltonian (11.18):

$$H_{\text{disorder}} = \sum_i V_i c_i^\dagger c_i. \quad (11.21)$$

The random potential $V_i = V(\mathbf{r}_i)$ satisfies Gaussian distribution with the correlation function $\langle V(\mathbf{r})V(\mathbf{r}') \rangle_{\text{dis}} = W^2 \delta(\mathbf{r} - \mathbf{r}')$. In conventional metals, the strength of the disorder potential determines the electron mean-free path $l = v_F \tau$ with $\tau^{-1} = 2\pi W^2 \nu(\varepsilon_F)$ which is the same for both spins. Here $\nu(\varepsilon_F)$ is the normal-state density of states at Fermi level, $\nu(\varepsilon_F) = \frac{1}{\pi} \left(\frac{d\varepsilon(k)}{dk} \right)^{-1} \big|_{k=k_F}$. In the presence of spin-orbit coupling and Zeeman field, electrons in different bands have different mean-free paths due to, for example, Fermi velocity mismatch. In addition, the singularity of the DoS at the bottom of the band leads to strong energy dependence of the mean-free path. Thus, unlike in conventional metals l is not a good quantity to characterize the strength of the disorder. However, for the lack of a better quantity to characterize the disorder, we roughly estimate $l \approx 5$ for the parameters used in the numerical calculation and away from the bottom of the band. Thus, in our simulations l is much smaller than the length of the junction $L = 20$ and, thus, effect of the disorder is quite important here.

For the sake of concreteness, we now consider LDoS in the short-junction limit in the presence of spin-orbit coupling. From the scattering matrix calculation in Sec. 11.3, we already know that the spectrum of Andreev bound states has no zero-energy crossings, see Fig. 11.5(a). It is naturally expected that disorder does lead to any qualitative changes in this case apart from broadening the peaks in the LDoS and increasing the gap at $\varphi = \pi$. This is confirmed by the numerical simulations shown in Fig. 11.5(a) and (c). The Zeeman field splits the spin SU(2) degeneracy and results in a complicated spectrum as demonstrated in Fig. 11.5(b) for the clean case and Fig. 11.5(d) for the disordered case. One can notice that there are two magnetic field-induced zero-energy crossings in the spectrum of Andreev bound states on the interval $\phi = [0, 2\pi]$. From the disorder-averaged LDoS shown in Fig. 11.5(d), we see that the crossing at zero energy persists at least for moderate disorder strength. The reason of this robustness is due to particle-hole symmetry as we explain below.

Let's consider an isolated crossing in the energy spectrum for certain superconducting phase difference $\varphi = \varphi_0$. In the spirit of $k \cdot p$ perturbation theory, one can analyze effect of various

perturbations around this level crossing. The two zero-energy states Ψ_0 and Ψ_1 at the crossing are related by particle-hole symmetry, namely $\Psi_1 = \Xi\Psi_0^*$ where $\Xi = \sigma_y\tau_y$ in the Nambu spinor notation adopted here. To understand the stability of the crossing, we consider now an arbitrary local perturbation $O(x)$ (e.g. disorder potential) and calculate its matrix element between zero-energy states. The perturbation $O(x)$ has to satisfy particle-hole symmetry $\Xi^\dagger O(x)\Xi = -O^T(x)$. Using this relation, one can show that the off-diagonal matrix element $O_{10} = \langle\Psi_1|O|\Psi_0\rangle$ is zero. Indeed, one can show that

$$\begin{aligned} O_{10} &= \langle\Psi_1|O|\Psi_0\rangle = \langle\Psi_0^*|\Xi^\dagger O|\Psi_0\rangle \\ &= -\langle\Psi_0^*|O^T\Xi^\dagger|\Psi_0\rangle \\ &= -\int dx \Psi_0^T(x)O^T(x)\Xi^\dagger\Psi_0(x) \end{aligned} \quad (11.22)$$

Since $\Xi^T = \Xi$, one finds that $\delta O_{10}^T = -\delta O_{10} = 0$. Thus, any on-site perturbation preserving particle-hole symmetry does not mix Ψ_0 and Ψ_1 states. Another way of proving this fact is to show that the two zero-energy levels actually correspond to different fermion parity[28] and as such are robust as long as there are no other zero-energy crossings nearby. However, generic perturbation produces non-zero diagonal contributions $\langle\Psi_0|O|\Psi_0\rangle = -\langle\Psi_1|O|\Psi_1\rangle$ which shift the position of the crossing. Thus, in the subspace spanned by the zero-energy eigenstates, an on-site perturbation acts as μ_z .

We now apply these results to the zero-energy crossings in the Andreev bound spectrum. As discussed in Sec.11.3, for $V_z < \mu$ there might be two such crossings in the spectrum located symmetrically with respect to $\varphi = \pi$. Each crossing has a pair of zero-energy states since spin degeneracy is removed by the magnetic field. One of the relevant perturbation near the crossing point is the deviation of the superconducting phase from $\delta\varphi$. According to our general argument, we can expand in small deviation of the phase difference $\Delta\varphi \equiv \varphi - \varphi_{0i}$ to obtain:

$$\delta H_\varphi = (a\Delta\varphi + b\Delta\varphi^2 + \dots)\mu_z. \quad (11.23)$$

Here μ_z is the Pauli matrix acting in the space spanned by the two zero-energy eigenstates. Other perturbations, such as on-site disorder, also take the form

$$\delta H' = \delta O_{00}\mu_z. \quad (11.24)$$

Combining the two contributions, we find that the spectrum now reads

$$E = \pm |a\Delta\varphi + \delta O_{00}|. \quad (11.25)$$

Therefore, the perturbation shifts the crossing point $\varphi_0 \rightarrow \varphi_0 + \frac{\delta O_{00}}{a}$. If the perturbation is strong enough, then the two crossings located at different φ_i are brought close to each other and can annihilate each other since these perturbations can now mix the states corresponding to different crossings and, thus, can generically open a gap in the spectrum. However, such a perturbation needs to be larger than certain threshold which explains why zero-energy crossings survive in the presence of a moderate disorder.

11.6 Conclusions and Discussions

We have developed a theory for the coherent transport through semiconductor nanowire with strong spin-orbit coupling and Zeeman splitting. We find a number of remarkable features in the Josephson current through the nanowire originating from the interplay between two competing terms: spin-orbit coupling and Zeeman splitting. In particular, we show that the transition between helical and non-helical states of the nanowire is accompanied by a large

suppression of the critical current through the junction when Zeeman energy is large compared to the spin-orbit energy scale. We support these findings by calculating the Josephson current both in the short-junction limit using scattering matrix formalism and in the long-junction case using the numerical diagonalization. The suppression of the critical current can be used as a diagnostic of the helical state in the nanowire. We also show that the critical current exhibits oscillations with an applied magnetic field. These oscillations are generic and might be relevant for recent experiments on topological insulators/superconductor structures [27].

We also compute local density of states and study effect of disorder on the Andreev energy spectrum. Without an applied magnetic field, the disorder-averaged local density of states is qualitatively similar to that of the clean case - the disorder mainly changes the junction transparency. In the presence of a magnetic field removing spin-degeneracy of the Andreev energy spectrum, we find zero-energy crossings which are robust against moderate disorder. We explain this robustness by invoking general arguments based on particle-hole symmetry and show analytically that any on-site perturbation only shifts the position of an isolated zero-energy crossing but does not open a gap in the spectrum.

11.7 Derivation of the scattering matrix $S(\varepsilon)$

We now briefly outline the derivation of the particle component of the scattering matrix $S_e(\varepsilon)$. The system we consider here consists of the semiconductor nanowire $-L/2 < x < L/2$ coupled to the superconducting electrodes at $x = \pm L/2$. The particles propagate freely within the nanowire $-L/2 < x < L/2$ in the presence of a finite spin-orbit coupling and Zeeman field (i.e. $\alpha \neq 0, V_z \neq 0$). For technical reasons, we insert thin ballistic normal leads to the left and right of the semiconducting nanowire [26]. We then calculate particle (electron) scattering matrix by considering N-SM-N system where the scattering at the SM-N interface occurs, for example, due to the mismatch of the Fermi momenta. The scattering at the N-SC interface is assumed to be ideal (i.e. normal reflection-free) consisting of Andreev reflection at the SC-N interface. This spatial separation allows one to substantially simplify the problem and relate supercurrent to the normal-state scattering matrix.

Let us first construct a basis for the particle component of the total scattering matrix. The wave incident on the nanowire corresponds to a vector:

$$c^{\text{in}} = (c_{\uparrow}^+(L), c_{\downarrow}^+(L), c_{\uparrow}^-(R), c_{\downarrow}^-(R)). \quad (11.26)$$

The corresponding outgoing wave is given by

$$c^{\text{out}} = (c_{\uparrow}^-(L), c_{\downarrow}^-(L), c_{\uparrow}^+(R), c_{\downarrow}^+(R)) \quad (11.27)$$

where $c_{\sigma}^{\pm}(L)$ correspond to right/left-moving quasiparticles with spin σ in the normal region at the left boundary. The normal scattering matrix is defined as $c^{\text{out}} = S_e(\varepsilon)c^{\text{in}}$. The wavefunction in the N-SM-N system is given by

$$\psi(x) = \begin{cases} \sum_{\sigma=\uparrow,\downarrow} c_{\sigma}^+(L) \varphi_{\sigma} e^{ik(x+L/2)} + c_{\sigma}^-(L) \varphi_{\sigma} e^{-ik(x+L/2)} & x < -L/2 \\ \sum_{s=\pm} b_s^+ \chi_s(k_s) e^{ik_s x} + b_s^- \chi_s(-k_s) e^{-ik_s x} & -L/2 < x < L/2 \\ \sum_{\sigma=\uparrow,\downarrow} c_{\sigma}^+(R) \varphi_{\sigma} e^{ik(x-L/2)} + c_{\sigma}^-(R) \varphi_{\sigma} e^{-ik(x-L/2)} & x > L/2 \end{cases} \quad (11.28)$$

Here $k = \sqrt{2m(\varepsilon + \mu)}$ where μ is the chemical potential in the normal metal. The spinor

wavefunction φ_σ and $\chi_s(k)$ are given by

$$\begin{aligned}\varphi_\uparrow &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \varphi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\ \chi_+(k) &= \begin{pmatrix} \cos \theta_k \\ \sin \theta_k \end{pmatrix}, \chi_-(k) = \begin{pmatrix} -\sin \theta_k \\ \cos \theta_k \end{pmatrix}.\end{aligned}\tag{11.29}$$

Here $\tan 2\theta_k = \alpha k/V_z$.

At the right/left boundaries $x = \pm L/2$, the wavefunction and the probability current have to be continuous. For example, at $x = L/2$ we have the following boundary conditions:

$$\begin{aligned}\psi(L/2 + 0^+) &= \psi(L/2 - 0^+), \\ \partial_x \psi(x)|_{x=L/2+0^+} &= (\partial_x + im\alpha\sigma_x)\psi(x)|_{x=L/2-0^+}\end{aligned}\tag{11.30}$$

Matching the boundary conditions yields a system of equations for $c_\sigma^\pm, b_s^\pm$. For example, at $x = -L/2$ we find

$$\begin{aligned}c_\uparrow^+(L) + c_\uparrow^-(L) &= (b_+^+ e^{-ik_+ L/2} + b_+^- e^{ik_+ L/2}) \cos \theta_+ - (b_-^+ e^{-ik_- L/2} - b_-^- e^{ik_- L/2}) \sin \theta_- \\ c_\downarrow^+(L) + c_\downarrow^-(L) &= (b_+^+ e^{-ik_+ L/2} - b_+^- e^{ik_+ L/2}) \sin \theta_+ + (b_-^+ e^{-ik_- L/2} + b_-^- e^{ik_- L/2}) \cos \theta_- \\ k(c_\uparrow^+(L) - c_\uparrow^-(L)) &= (k_+ \cos \theta_+ + k_R \sin \theta_+)(b_+^+ e^{-ik_+ L/2} - b_+^- e^{ik_+ L/2}) + (k_R \cos \theta_- - k_- \sin \theta_-)(b_-^+ e^{-ik_- L/2} + b_-^- e^{ik_- L/2}) \\ k(c_\downarrow^+(L) - c_\downarrow^-(L)) &= (k_R \cos \theta_+ + k_+ \sin \theta_+)(b_+^+ e^{-ik_+ L/2} + b_+^- e^{ik_+ L/2}) + (k_- \cos \theta_- - k_R \sin \theta_-)(b_-^+ e^{-ik_- L/2} - b_-^- e^{ik_- L/2})\end{aligned}\tag{11.31}$$

Here $k_R = m\alpha$. The equations at $x = L/2$ are similar. Eliminating $b_s^\pm$, we obtain the particle scattering matrix $S_e(\varepsilon)$. Using the particle-hole symmetry of the BdG equations one can obtain $S_h(\varepsilon)$ and compute the spectrum of Andreev levels using Eq.(5) of the main text.

11.8 Validity of the short-junction approximation

In the calculation of the Josephson current, we have made two approximations: (1) We assumed that the Josephson current is dominated by the Andreev bound states in the junction (i.e. short-junction limit), and neglected the contribution of the continuum states. (2) In evaluating the Andreev spectrum we have neglected the energy dependence of the normal state scattering matrix: $S_e(\varepsilon) \approx S_e(0)$. In this section we examine the validity of the second approximation. We start with the case $\alpha = 0$ and show that such approximation is justified apart from the points close to $\varphi = 0, 2\pi$.

We first derive the equation determining the Andreev levels with full energy dependence of the scattering matrix $S_e(\varepsilon)$. When $\alpha = 0$, the dispersion of the Andreev levels is given by

$$\begin{aligned}2 \arccos \frac{\varepsilon}{\Delta_0} &= \pm \left[\arccos \frac{B_\uparrow(\varepsilon)B_\downarrow(-\varepsilon) + \cos \varphi}{|A_\uparrow(\varepsilon)A_\downarrow(-\varepsilon)|} + \arg \frac{A_\uparrow(\varepsilon)}{A_\downarrow(-\varepsilon)} \right], \\ 2 \arccos \frac{\varepsilon}{\Delta_0} &= \pm \left[\arccos \frac{B_\uparrow(-\varepsilon)B_\downarrow(\varepsilon) + \cos \varphi}{|A_\uparrow(-\varepsilon)A_\downarrow(\varepsilon)|} - \arg \frac{A_\uparrow(-\varepsilon)}{A_\downarrow(\varepsilon)} \right],\end{aligned}\tag{11.32}$$

where A_σ and B_σ are defined in Eq. (11.7) with $k_\sigma \equiv k_\sigma(\varepsilon) = \sqrt{2m(\mu + \varepsilon - \sigma V_z)}$.

Assuming that both V_z and Δ are much smaller than μ , one can treat V_z and ε as small parameters and perform perturbative expansion of Eq. (11.32). To the leading order in ε/μ and V_z/μ , the coefficients B_σ and A_σ are given by:

$$A_\sigma \approx e^{ik_\sigma(\varepsilon)L}, B_\sigma \approx -\frac{\varepsilon - \sigma V_z}{\mu} \sin k_\sigma L\tag{11.33}$$

Substituting into (11.32), we obtain

$$\begin{aligned} 2 \arccos \frac{\varepsilon}{\Delta_0} &= \pm \left\{ \arccos \left[-\frac{(\varepsilon - V_z)^2}{\mu^2} \sin^2 k_F L + \cos \varphi \right] - \frac{2(\varepsilon - V_z)L}{v_F} \right\} \\ 2 \arccos \frac{\varepsilon}{\Delta_0} &= \pm \left\{ \arccos \left[-\frac{(\varepsilon + V_z)^2}{\mu^2} \sin^2 k_F L + \cos \varphi \right] + \frac{2(\varepsilon + V_z)L}{v_F} \right\}. \end{aligned} \quad (11.34)$$

Compared to the short-junction limit, an extra term $\varepsilon L/v_F$ appears in the equation. As a rough estimate, one can put in $\varepsilon \sim \Delta_0$. Then, this term becomes $\Delta_0 L/v_F = L/\xi$. When $L/\xi \ll 1$, we can disregard this term and the short-junction approximation is justified.

However, in some cases that even for small L/ξ this correction can result in a qualitatively different dependence on V_z . Let us consider the spectrum of Andreev levels at $\varphi \approx 0$. Since the time-reversal symmetry of the system is broken by the Zeeman term, we expect the Andreev levels to be split. However, if we start with the approximation of $S_e(\varepsilon) \approx S_e(0)$, we find

$$\varepsilon(\varphi = 0) = \pm \Delta_0 \cos \left(\frac{V_z \sin k_F L}{\mu} \pm \frac{V_z L}{v_F} \right), \quad (11.35)$$

which gives in the limit of $V_z \rightarrow 0$ the energy splitting quadratic in V_z :

$$\delta \varepsilon \approx 2 \Delta_0 \left(\frac{V_z}{\mu} \right)^2 k_F L \sin k_F L. \quad (11.36)$$

However, this scaling is qualitatively different if we include the energy dependence. Indeed, assuming $k_F L \gg 1$, we find that

$$\delta \varepsilon \approx \left(\frac{L}{\xi} \right)^2 V_z \quad (11.37)$$

as one would expect from qualitative considerations. Thus, apart from the points $\varphi \approx 0, 2\pi$, the approximation $S_e(\varepsilon) \approx S_e(0)$ gives qualitatively correct results.

Let us now discuss the case with a finite SOC. Generally, one has to rely on numerical solution but under the assumption that both α and V_z are small with respect to μ , we obtain the following analytical equation for the Andreev levels, which agrees with the one derived in Ref. [[29]]:

$$\arccos \frac{\varepsilon}{\Delta_0} = \frac{\varepsilon L}{v_F} + \frac{\varphi}{2} \pm \arcsin \left[\frac{V_z |\sin(EL/v_F)|}{E} \right] + n\pi. \quad (11.38)$$

Here $E = \sqrt{V_z^2 + \alpha^2 k_F^2}$. However there is a subtle point here: at $\alpha = 0$ the above equation reduces to (11.34) for $\varphi \neq 0$ up to linear order in V_z . However, as explained above Eq. (11.34) around $\varphi = 0$ contains an additional term $\frac{(\varepsilon \pm V_z) \sin k_F L}{\mu}$ which was neglected in Ref. [[29]].

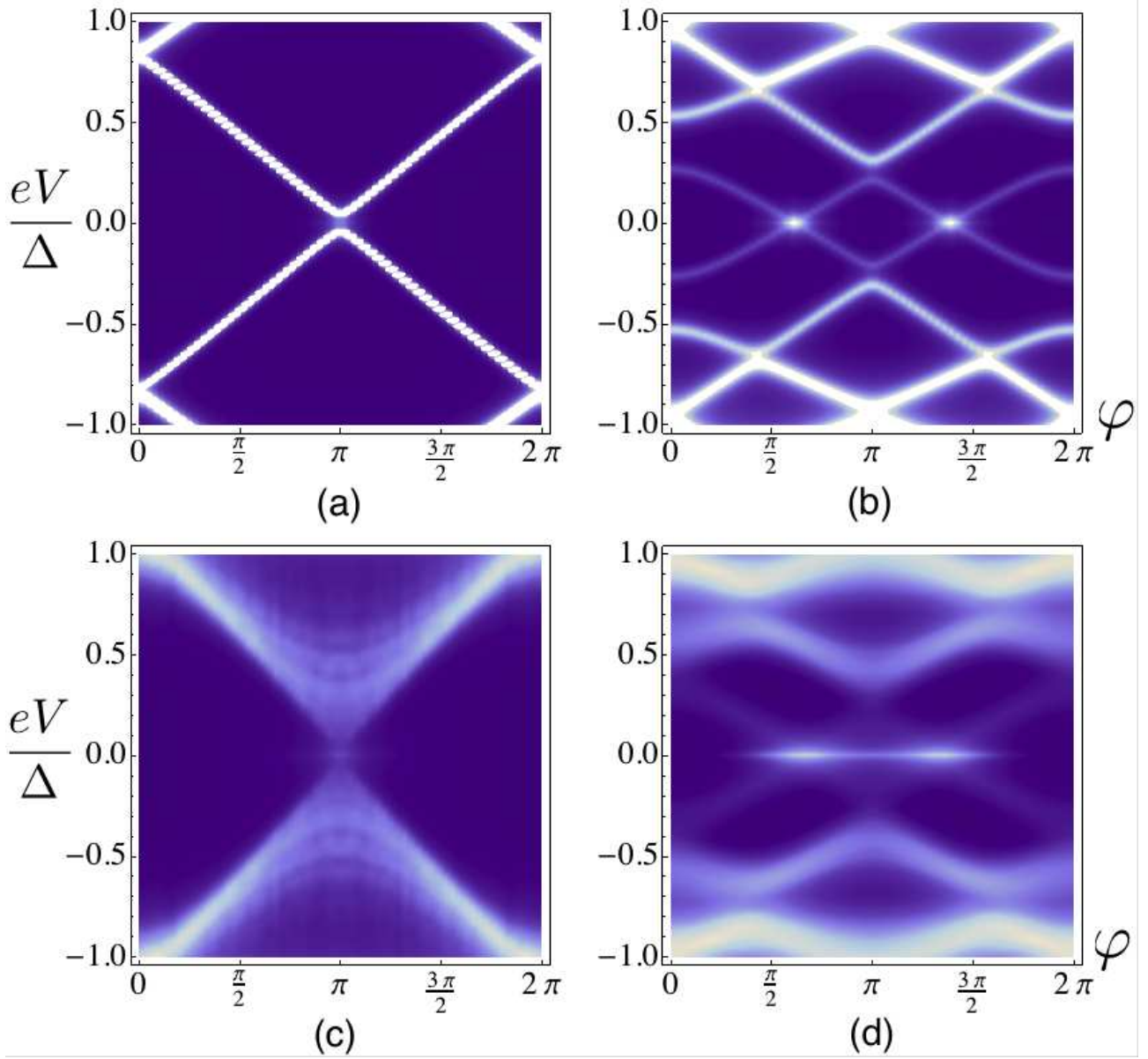


Figure 11.5: LDOS measured in the middle of the junction. In both plots, $\Delta_0 = 0.05$, $L = 20$, $\alpha = 0.5$, $\mu = 0.5$. (a) $V_z = 0, W = 0$ (b) $V_z = 0.4, W = 0$ (c) $V_z = 0, W = 0.3$. (d) $V_z = 0.4, W = 0.3$.

12 Josephson effect in junctions of conventional and topological superconductor by Zazunov, Iks, Alvarado, Levy Yeyati

Abstract

We present a theoretical analysis of the equilibrium Josephson current-phase relation in hybrid devices made of conventional s -wave spin-singlet superconductors (S) and topological superconductor (TS) wires featuring Majorana end states. Using Green's function techniques, the topological superconductor is alternatively described by the low-energy continuum limit of a Kitaev chain or by a more microscopic spinful nanowire model. We show that for the simplest S-TS tunnel junction, only the s -wave pairing correlations in a spinful TS nanowire model can generate a Josephson effect. The critical current is much smaller in the topological regime and exhibits a kink-like dependence on the Zeeman field along the wire. When a correlated quantum dot (QD) in the magnetic regime is present in the junction region, however, the Josephson current becomes finite also in the deep topological phase as shown for the cotunneling regime and by a mean-field analysis. Remarkably, we find that the S-QD-TS setup can support φ_0 -junction behavior, where a finite supercurrent flows at vanishing phase difference. Finally, we also address a multi-terminal S-TS-S geometry, where the TS wire acts as tunable parity switch on the Andreev bound states in a superconducting atomic contact.

12.1 Introduction

The physics of topological superconductors (TSs) is being vigorously explored at present. After Kitaev [8] showed that a one-dimensional (1D) spinless fermionic lattice model with nearest-neighbor p -wave pairing ('Kitaev chain') features a topologically nontrivial phase with Majorana bound states (MBSs) at open boundaries, Refs. [6, 7] have pointed out that the physics of the Kitaev chain could be realized in spin-orbit coupled nanowires with a magnetic Zeeman field and in the proximity to a nearby s -wave superconductor. The spinful nanowire model of Refs. [6, 7] indeed features p -wave pairing correlations for appropriately chosen model parameters. In addition, it also contains s -wave pairing correlations which become gradually smaller as one moves into the deep topological regime. Topologically nontrivial hybrid semiconductor nanowire devices are of considerable interest in the context of quantum information processing [42, 10, 2, 4, 5, 9, 10, 11, 12], and they may also be designed in two-dimensional layouts by means of gate lithography techniques. Over the last few years, several experiments employing such platforms have provided mounting evidence for MBSs, e.g., from zero-bias conductance peaks in N-TS junctions (where N stands for a normal-conducting lead) and via signatures of the 4π -periodic Josephson effect in TS-TS junctions [30, 15, 10, 9, 10, 11, 12, 20, 51, 19, 23, 24, 25]. Related MBS phenomena have been reported for other material platforms as well, see, e.g., Refs. [14, 15, 16, 17, 18], and most of the results reported below also apply to those settings. Available materials are often of sufficiently high quality to meet the conditions for ballistic transport, and we will therefore neglect disorder effects.

In view of the large amount of published theoretical works on the Josephson effect in such systems, let us first motivate the present study. (For a more detailed discussion and references, see below.) Our manuscript addresses the supercurrent flowing in Josephson junctions with a magnetic impurity. By considering Josephson junctions between a topological superconductor and a non-topological superconductor, we naturally extend previous works on Josephson junctions with a magnetic impurity between two conventional superconductors, as well as other works on Josephson junctions between topological and non-topological superconductors but

without a magnetic impurity. In the simplest description, Josephson junctions between topological and non-topological superconductors carry no supercurrent. Instead, a supercurrent can flow only with certain deviations from the idealized model description. The presence of a magnetic impurity in the junction is one of these deviations, and this effect allows for novel signatures for the topological transition via the so-called φ_0 behavior and/or through the kink-like dependence of the critical current on a Zeeman field driving the transition. We consider two different geometries in various regimes, e.g., the cotunneling regime where a controlled perturbation theory is possible, and a mean-field description of the stronger-coupling regime. We study both idealized Hamiltonians (allowing for analytical progress) as well as more realistic models for the superconductors.

To be more specific, we address the equilibrium current-phase relation (CPR) in different setups involving both conventional *s*-wave BCS superconductors ('S' leads) and TS wires, see Fig. 12.1 for a schematic illustration. In general, the CPR is closely related to the Andreev bound state (ABS) spectrum of the system. For S-TS junctions with the TS wire deep in the topological phase such that it can be modeled by a Kitaev chain, the supercurrent vanishes identically [35]. This supercurrent blockade can be traced back to the different (*s/p*-wave) pairing symmetries for the S/TS leads, together with the fact that MBSs have a definite spin polarization. For an early study of Josephson currents between superconductors with different (*p/d*) pairing symmetries, see also Ref. [4]. A related phenomenon concerns Multiple Andreev Reflection (MAR) features in nonequilibrium superconducting quantum transport at subgap voltages [33, 17, 47, 14]. Indeed, it has been established that MAR processes are absent in S-TS junctions (with the TS wire in the deep topological regime) such that only quasiparticle transport above the gap is possible [19, 37, 38, 20, 39, 40, 70, 44].

There are several ways to circumvent this supercurrent blockade in S-TS junctions. (i) One possibility has been described in Ref. [70]. For a trijunction formed by two TS wires and one S lead, crossed Andreev reflections allow for the nonlocal splitting of Cooper pairs in the S electrode involving both TS wires (or the reverse process). In this way, an equilibrium supercurrent will be generated unless the MBS spin polarization axes of both TS wires are precisely aligned. (ii) Even for a simple S-TS junction, a finite Josephson current is expected when the TS wire is modeled as spinful nanowire. This effect is due to the residual *s*-wave pairing character of the spinful TS model [6, 7]. Interestingly, upon changing a control parameter, e.g., the bulk Zeeman field, which drives the TS wire across the topological phase transition, we find that the critical current exhibits a kink-like feature that is mainly caused by a suppression of the Andreev state contribution in the topological phase. (iii) Yet another possibility is offered by junctions containing a magnetic impurity in a local magnetic field. We here analyze the S-QD-TS setup in Fig. 12.1(a) in some detail, where a quantum dot (QD) is present within the S-TS junction region. The QD is modeled as an Anderson impurity [14], which is equivalent to a spin-1/2 quantum impurity over a wide parameter regime. Once spin mixing is induced by the magnetic impurity and the local magnetic field, we predict that a finite Josephson current flows even in the deep topological limit. In particular, in the cotunneling regime, we find an anomalous Josephson effect with finite supercurrent at vanishing phase difference (φ_0 -junction behavior) [45, 46, 47], see also Refs. [55, 49, 50, 51]. The 2π -periodic CPR found in S-QD-TS junctions could thereby provide independent evidence for MBSs via the anomalous Josephson effect. In addition, we compute the CPR within the mean-field approximation in order to go beyond perturbation theory in the tunnel couplings connecting the QD to the superconducting leads. Our mean-field analysis shows that the φ_0 -junction behavior is a generic feature for S-QD-TS devices in the topological regime which is not limited to the cotunneling regime.

In the final part of the paper, we turn to the three-terminal S-TS-S setup shown in Fig. 12.1(b), where the S-S junction by itself (with the TS wire decoupled) represents a standard superconducting atomic contact (SAC) with variable transparency of the weak link.

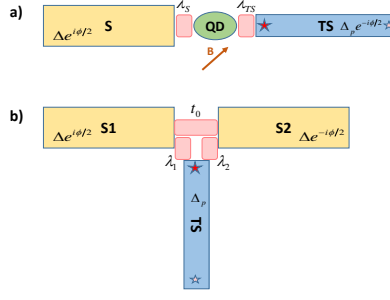


Figure 12.1: Schematic setups studied in this paper. **a)** S-QD-TS geometry: S denotes a conventional *s*-wave BCS superconductor with order parameter $\Delta e^{i\phi/2}$, and TS represents a topologically nontrivial superconducting wire with MBSs (shown as stars) and proximity-induced order parameter $\Delta_p e^{-i\phi/2}$. The interface contains a quantum dot (QD) corresponding to an Anderson impurity, connected to the S/TS leads by tunnel amplitudes $\lambda_{S/TS}$ (light red). The QD is also exposed to a local Zeeman field **B**. **b)** S-TS-S geometry: Two conventional superconductors (S1 and S2) with the same gap Δ and a TS wire with proximity gap Δ_p form a trijunction. The order parameter phase of S1 (S2), $\phi_1 = \phi/2$ ($\phi_2 = -\phi/2$), is taken relative to the phase of the TS wire, and tunnel couplings $\lambda_{1/2}$ connect S1/S2 to the TS wire. When the TS wire is decoupled ($\lambda_{1,2} = 0$), the S-S junction becomes a standard SAC with transparency $\mathcal{T}$ determined by the tunnel amplitude t_0 , see Eq. (12.42).

Recent experiments have demonstrated that the many-body ABS configurations of a SAC can be probed and manipulated to high accuracy by microwave spectroscopy [52, 53, 116]. When the TS wire is coupled to the S-S junction, see Fig. 12.1(b), the Majorana end state acts as a parity switch on the ABS system of the SAC. This effect allows for additional functionalities in Andreev spectroscopy. We note that similar ideas have also been explored for TS-N-TS systems [55].

12.2 S-QD-TS junction

12.2.1 Model

Let us start with the case of an S-QD-TS junction, where an interacting spin-degenerate single-level quantum dot (QD) is sandwiched between a conventional *s*-wave superconductor (S) and a topological superconductor (TS). This geometry is shown in Fig. 12.1(a). The corresponding topologically trivial S-QD-S problem has been studied in great detail over the past decades both theoretically [56, 57, 13, 59, 60, 61, 62, 63] and experimentally [64, 65, 66, 67, 68, 69]. A main motivation for those studies came from the fact that the QD can be driven into the magnetic regime where it represents a spin-1/2 impurity subject to Kondo screening by the leads. The Kondo effect then competes against the superconducting bulk gap and one encounters local quantum phase transitions. By now, good agreement between experiment and theory has been established. Rather than studying the fate of the Kondo effect in the S-QD-TS setting of Fig. 12.1(a), we here pursue two more modest goals. First, we shall discuss the cotunneling regime in detail, where one can employ perturbation theory in the dot-lead couplings. This regime exhibits π -junction behavior in the S-QD-S case [56]. Second, in order to go beyond the cotunneling regime, we have performed a mean-field analysis similar in spirit to earlier work for S-QD-S devices [57, 13].

The Hamiltonian for the setup in Fig. 12.1(a) is given by

$$H = H_S + H_{TS} + H_{QD} + H_{\text{tun}}, \quad (12.1)$$

where $H_{S/TS}$ and H_{QD} describe the semi-infinite S/TS leads and the isolated dot in between, respectively, and H_{tun} refers to the tunnel contacts. We often use units with $e = \hbar = k_B = 1$, and $\beta = 1/T$ denotes inverse temperature. The QD is modeled as an Anderson impurity [14], i.e., a single spin-degenerate level of energy ϵ_0 with repulsive on-site interaction energy $U > 0$,

$$H_{QD} = \sum_{\sigma=\uparrow,\downarrow} \epsilon_0 \left(n_\sigma - \frac{1}{2} \right) + U n_\uparrow n_\downarrow - \mathbf{B} \cdot \mathbf{S}, \quad (12.2)$$

where the QD occupation numbers are $n_\sigma = d_\sigma^\dagger d_\sigma = 0, 1$, with dot fermion operators d_σ and $d_\sigma^\dagger$ for spin σ . Using standard Pauli matrices $\sigma_{x,y,z}$, we define

$$\mathbf{S}_{i=x,y,z} = \sum_{\sigma,\sigma'} d_\sigma^\dagger (\sigma_i)_{\sigma\sigma'} d_{\sigma'}, \quad (12.3)$$

such that $\mathbf{S}/2$ is a spin-1/2 operator. In the setup of Fig. 12.1(a), we also take into account an external Zeeman field $\mathbf{B} = (B_x, B_y, B_z)$ acting on the QD spin, where the units in Eq. (12.2) include gyromagnetic and Bohr magneton factors. The spinful nanowire proposal for TS wires [6, 7] also requires a sufficiently strong bulk Zeeman field oriented along the wire in order to realize the topologically nontrivial phase, but for concreteness, we here imagine the field $\mathbf{B}$ as independent local field coupled only to the QD spin. One could use, e.g., a ferromagnetic grain near the QD to generate it. This field here plays a crucial role because for $\mathbf{B} = 0$, the S+QD part is spin rotation [SU(2)] invariant and the arguments of Ref. [35] then rule out a supercurrent for TS wires in the deep topological regime. We show below that unless $\mathbf{B}$ is inadvertently aligned with the MBS spin polarization axis, spin mixing will indeed generate a supercurrent.

The S/TS leads are coupled to the QD via a tunneling Hamiltonian [70],

$$H_{\text{tun}} = \lambda_S \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger d_\sigma + \lambda_{TS} e^{-i\phi/2} \psi^\dagger d_\uparrow + \text{h.c.}, \quad (12.4)$$

where ψ_σ and ψ are boundary fermion fields representing the S lead and the effectively spinless TS lead, respectively. For the S lead, we assume the usual BCS model [62], where the operator ψ_σ annihilates an electron with spin σ at the junction. The TS wire will, for the moment, be described by the low-energy Hamiltonian of a Kitaev chain in the deep topological phase with chemical potential $\mu = 0$ [8, 10]. The corresponding fermion operator ψ at the junction includes both the MBS contribution and above-gap quasiparticles [20]. Without loss of generality, we choose the unit vector $\hat{e}_z$ as the MBS spin polarization direction and take real-valued tunnel amplitudes $\lambda_{S/TS}$, see Fig. 12.1(a), using a gauge where the superconducting phase difference ϕ appears via the QD-TS tunneling term. These tunnel amplitudes contain density-of-states factors for the respective leads. The operator expression for the current flowing through the system is then given by

$$\hat{I} = \frac{2e}{\hbar} \partial_\phi H_{\text{tun}}. \quad (12.5)$$

We do not specify $H_{S/TS}$ in Eq. (12.1) explicitly since within the imaginary-time (τ) boundary Green's function (bGF) formalism [20] employed here, we only need to know the bGFs. For the S lead with gap value Δ , the bGF has the Nambu matrix form [20]

$$g(\tau) = -\langle \mathcal{T}_\tau \Psi_S(\tau) \Psi_S^\dagger(0) \rangle_0 = \beta^{-1} \sum_{\omega} e^{-i\omega\tau} g(\omega), \quad (12.6)$$

$$\Psi_S = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \quad g(\omega) = -\frac{i\omega\tau_0 + \Delta\tau_x}{\sqrt{\omega^2 + \Delta^2}},$$

where the expectation value $\langle \cdots \rangle_0$ refers to an isolated S lead, $\mathcal{T}_\tau$ denotes time ordering, ω runs over fermionic Matsubara frequencies, i.e., $\omega = 2\pi(n + 1/2)/\beta$ with integer n , and we define Pauli (unity) matrices $\tau_{x,y,z}$ (τ_0) in particle-hole space corresponding to the Nambu spinor Ψ_S . Similarly, for a TS lead with proximity-induced gap Δ_p , the low-energy limit of a Kitaev chain yields the bGF [20]

$$\begin{aligned} G(\tau) &= -\langle \mathcal{T}_\tau \Psi_{\text{TS}}(\tau) \Psi_{\text{TS}}^\dagger(0) \rangle_0, \quad \Psi_{\text{TS}} = \begin{pmatrix} \psi \\ \psi^\dagger \end{pmatrix}, \\ G(\omega) &= \frac{1}{i\omega} \left(\sqrt{\omega^2 + \Delta_p^2} \tau_0 + \Delta_p \tau_x \right). \end{aligned} \quad (12.7)$$

The matrices $\tau_{0,x}$ here act in the Nambu space defined by the spinor Ψ_{TS} . Later on we will address how our results change when the TS wire is modeled as spinful nanowire [6, 7], where the corresponding bGF has been specified in Ref. [70]. We emphasize that the bGF (12.7) captures the effects of both the MBS (via the $1/\omega$ term) and of the above-gap continuum quasiparticles (via the square root) [20, 21].

In most of the following discussion, we will assume that U is the dominant energy scale, with the single-particle level located at $\epsilon_0 \approx -U/2$. In that case, low-energy states with energy well below U are restricted to the single occupancy sector,

$$n_\uparrow + n_\downarrow = 1, \quad (12.8)$$

and the QD degrees of freedom become equivalent to the spin-1/2 operator $\mathbf{S}/2$ in Eq. (12.3). In this regime, the QD acts like a magnetic impurity embedded in the S-TS junction. Using a Schrieffer-Wolff transformation to project the full Hamiltonian to the Hilbert subspace satisfying Eq. (12.8), $H \rightarrow H_{\text{eff}}$, one arrives at the effective low-energy Hamiltonian

$$H_{\text{eff}} = H_0 + H_{\text{int}}, \quad H_0 = H_S + H_{\text{TS}} - \mathbf{B} \cdot \mathbf{S}, \quad (12.9)$$

with the interaction term

$$\begin{aligned} H_{\text{int}} &= -\frac{2}{U} \sum_{\sigma, \sigma'} \left(\eta_\sigma^\dagger d_\sigma d_{\sigma'}^\dagger \eta_{\sigma'} + \text{h.c.} \right) \\ &= \frac{2}{U} \sum_{\sigma=\uparrow/\downarrow=\pm} \left(\sigma S_z \eta_\sigma^\dagger \eta_\sigma + S_\sigma \eta_{-\sigma}^\dagger \eta_\sigma \right) + \frac{2}{U} \delta n \sum_\sigma \eta_\sigma^\dagger \eta_\sigma - \frac{2\Lambda}{U} (\delta n + 1), \end{aligned} \quad (12.10)$$

where $S_\pm = S_x \pm iS_y$ and $\delta n = \sum_\sigma n_\sigma - 1$. Moreover, $\Lambda = [\eta_\sigma, \eta_\sigma^\dagger]_+$ is the anticommutator of the composite boundary fields

$$\eta_\sigma = \lambda_S \psi_\sigma + \delta_{\sigma,\uparrow} \lambda_{\text{TS}} e^{i\phi/2} \psi. \quad (12.11)$$

We note that Λ is real-valued and does not depend on ϕ . Due to the constraint (12.8) on the dot occupation, the last two terms in Eq. (12.10) do not contribute to the system dynamics and we obtain

$$\begin{aligned} H_{\text{int}} &= \frac{4}{U} \sum_{\sigma, \sigma'} \mathcal{Q}_{\sigma\sigma'} \eta_\sigma^\dagger \eta_{\sigma'}, \\ \mathcal{Q}_{\sigma\sigma} &= \frac{\sigma}{2} S_z, \quad \mathcal{Q}_{\sigma, -\sigma} = \frac{1}{2} S_{-\sigma}. \end{aligned} \quad (12.12)$$

A formally exact expression for the partition function is then given by

$$Z = \text{Tr} \Big|_{\delta n=0} \left(e^{-\beta H_0} \mathcal{T}_\tau e^{-\int_0^\beta d\tau H_{\text{int}}(\tau)} \right), \quad (12.13)$$

where $H_{\text{int}}(\tau) = e^{\tau H_0} H_{\text{int}} e^{-\tau H_0}$ with H_0 in Eq. (12.9) and the trace extends only over the Hilbert subspace corresponding to Eq. (12.8). We can equivalently write Eq. (12.13) in the form

$$\begin{aligned} Z &= Z_0 \left\langle \mathcal{T}_\tau e^{-\beta \hat{W}} \right\rangle_0 = e^{-\beta F}, \\ \hat{W} &= \beta^{-1} \int_0^\beta d\tau H_{\text{int}}(\tau), \\ Z_0 &= \text{Tr} \Big|_{\delta n=0} e^{-\beta H_0} = e^{-\beta F_0}, \end{aligned} \quad (12.14)$$

where F is the free energy. The Josephson current then follows as $I = (2e/\hbar) \partial_\phi F$, see Eq. (12.5).

12.2.2 Cotunneling regime

We now address the CPR in the elastic cotunneling regime,

$$\lambda_S \lambda_{TS} \ll \min\{\Delta, \Delta_p, U\}, \quad (12.15)$$

where perturbation theory in H_{int} is justified. We thus wish to compute the free energy $F(\phi)$ from Eq. (12.14) to lowest nontrivial order. With $W_0 = \langle \hat{W} \rangle_0$, the standard cumulant expansion gives

$$F - F_0 = W_0 - \frac{\beta}{2} \left(\langle \hat{W}^2 \rangle_0 - W_0^2 \right) + \mathcal{O}(W^3). \quad (12.16)$$

By virtue of Wick's theorem, time-ordered correlation functions of the boundary operators (12.11) are now expressed in terms of S/TS bGF matrix elements, see Eqs. (12.6) and (12.7),

$$\langle \mathcal{T}_\tau \eta_\sigma(\tau) \eta_{\sigma'}^\dagger(0) \rangle_0 = \delta_{\sigma\sigma'} \left[\lambda_S^2 \langle \mathcal{T}_\tau \psi_\sigma(\tau) \psi_\sigma^\dagger(0) \rangle_0 + \delta_{\sigma,\uparrow} \lambda_{TS}^2 \langle \mathcal{T}_\tau \psi(\tau) \psi^\dagger(0) \rangle_0 \right] \quad (12.17)$$

and similarly

$$\langle \mathcal{T}_\tau \eta_\sigma(\tau) \eta_{\sigma'}(0) \rangle_0 = \delta_{\sigma,-\sigma'} \lambda_S^2 \langle \mathcal{T}_\tau \psi_\sigma(\tau) \psi_{-\sigma}(0) \rangle_0 + e^{i\phi} \delta_{\sigma\sigma'} \delta_{\sigma,\uparrow} \lambda_{TS}^2 \langle \mathcal{T}_\tau \psi(\tau) \psi(0) \rangle_0. \quad (12.18)$$

Next we observe that $\partial_\phi \langle H_{\text{int}} \rangle_0 = 0$. As a consequence, the ϕ -independent terms W_0 and W_0^2 in Eq. (12.16) do not contribute to the Josephson current. The leading contribution is then of second order in H_{int} ,

$$\begin{aligned} I(\phi) &= -\beta^{-1} \partial_\phi \int_0^\beta d\tau_1 d\tau_2 \langle \mathcal{T}_\tau H_{\text{int}}(\tau_1) H_{\text{int}}(\tau_2) \rangle_0 \\ &= -\frac{\kappa^2}{\beta} \int_0^\beta d\tau_1 d\tau_2 g_{12}(\tau_1 - \tau_2) G_{12}(\tau_1 - \tau_2) \\ &\quad \times i e^{i\phi} \sum_\sigma \sigma \langle \mathcal{T}_\tau \mathcal{Q}_{\sigma,\uparrow}(\tau_1) \mathcal{Q}_{-\sigma,\uparrow}(\tau_2) \rangle_0 + \text{h.c.}, \end{aligned} \quad (12.19)$$

with $\mathcal{Q}_{\sigma,\sigma'}$ in Eq. (12.12) and the small dimensionless parameter

$$\kappa = \frac{4\lambda_S \lambda_{TS}}{U} \ll 1. \quad (12.20)$$

From Eqs. (12.6) and (12.7), the bGF matrix elements needed in Eq. (12.19) follow as

$$\begin{aligned} g_{12}(\tau) &= -\frac{\Delta}{\beta} \sum_\omega \frac{\cos(\omega\tau)}{\sqrt{\omega^2 + \Delta^2}}, \\ G_{12}(\tau) &= -\frac{\Delta_p}{\beta} \sum_\omega \frac{\sin(\omega\tau)}{\omega} \simeq -\frac{\Delta_p}{2} \text{sgn}(\tau). \end{aligned} \quad (12.21)$$

Now $|g_{12}(\tau)|$ is exponentially small unless $\Delta|\tau| < 1$. In particular, $g_{12}(\tau) \rightarrow -\delta(\tau)$ for $\Delta \rightarrow \infty$. Moreover, for $B \ll \Delta$ with $B \equiv |\mathbf{B}|$, the magnetic impurity ($\mathbf{S}$) dynamics will be slow on time scales of order $1/\Delta$. We may therefore approximate the spin-spin correlators in Eq. (12.19) by their respective equal-time expressions,

$$\lim_{\tau_1 \rightarrow \tau_2} \langle \mathcal{T}_\tau \mathcal{Q}_{\sigma,\uparrow}(\tau_1) \mathcal{Q}_{-\sigma,\uparrow}(\tau_2) \rangle_0 = \frac{\sigma}{4} \text{sgn}(\tau_1 - \tau_2) \langle S_+(\tau_1) \rangle_0. \quad (12.22)$$

Inserting Eqs. (12.21) and (12.22) into the expression for the supercurrent in Eq. (12.19), the time integrations can be carried out analytically.

We obtain the CPR in the cotunneling regime as

$$\begin{aligned} I(\phi) &= I_x \sin \phi + I_y \cos \phi, \\ I_{x,y} &= \frac{e\kappa^2 \Delta_p}{2\hbar} \frac{B_{x,y}}{B} \tanh(\beta B), \end{aligned} \quad (12.23)$$

with κ in Eq. (12.20). We note that while $I(\phi)$ is formally independent of Δ , the value of Δ must be sufficiently large to justify the steps leading to Eq. (12.23). Remarkably, Eq. (12.23) predicts anomalous supercurrents for the S-QD-TS setup, i.e., a finite Josephson current for vanishing phase difference ($\phi = 0$) [45, 46, 72]. One can equivalently view this effect as a φ_0 -shift in the CPR, $I(\phi) = I_c \sin(\phi + \varphi_0)$. An observation of this φ_0 -junction behavior could then provide additional evidence for MBSs (see also Ref. [47]), where Eq. (12.23) shows that the local magnetic field is required to have a finite B_y -component with $\hat{e}_z$ defining the MBS spin polarization direction. In particular, if $\mathbf{B}$ is aligned with $\hat{e}_z$, the supercurrent in Eq. (12.23) vanishes identically since s -wave Cooper pairs cannot tunnel from the S lead into the TS wire in the absence of spin flips [35]. Otherwise, the CPR is 2π -periodic and sensitive to the MBS through the peculiar dependence on the relative orientation between the MBS spin polarization ($\hat{e}_z$) and the local Zeeman field $\mathbf{B}$ on the QD. The fact that $B_y \neq 0$ (rather than $B_x \neq 0$) is necessary to have $\varphi_0 \neq 0$ can be traced back to our choice of real-valued tunnel couplings. For tunable tunnel phases, also the field direction where one has $\varphi_0 = 0$ will vary accordingly.

Noting that the anomalous Josephson effect has recently been observed in S-QD-S devices [73], we expect that similar experimental techniques will allow to access the CPR (12.23). We mention in passing that previous work has also pointed out that experiments employing QDs between N (instead of S) leads and TS wires can probe nonlocal effects due to MBSs [9, 12, 27, 28, 29, 33, 34]. In our case, e.g., by variation of the field direction in the xy -plane, Eq. (12.23) predicts a tunable anomalous supercurrent. We conclude that in the cotunneling regime, the π -junction behavior of S-QD-S devices is replaced by the more exotic physics of φ_0 -junctions in the S-QD-TS setting.

12.2.3 Mean-field approximation

Next we present a mean-field analysis of the Hamiltonian (12.1) which allows us to go beyond the perturbative cotunneling regime. For the corresponding S-QD-S case, see Refs. [13, 79]. We note that a full solution of this interacting many-body problem requires a detailed numerical analysis using, e.g., the numerical renormalization group [60, 61] or quantum Monte Carlo simulations [59, 63], which is beyond the scope of the present work. We start by defining the GF of the QD,

$$G_d(\tau) = -\langle \mathcal{T}_\tau \Psi_d(\tau) \Psi_d^\dagger(0) \rangle, \quad \Psi_d^\dagger = (d_\uparrow^\dagger, d_\downarrow^\dagger, -d_\uparrow^\dagger)^T. \quad (12.24)$$

Note that this notation introduces double counting, which implies that only half of the levels are physically independent. Of course, the results below take this issue into account.

With the above Nambu bi-spinor basis, the mean-field Hamiltonian has the 4×4 matrix representation

$$\begin{aligned}\mathcal{H}_{\text{MF}} &= \begin{pmatrix} \epsilon_{\uparrow} & \Delta_d & \alpha_d & 0 \\ \Delta_d^* & -\epsilon_{\downarrow} & 0 & \alpha_d \\ \alpha_d^* & 0 & \epsilon_{\downarrow} & \Delta_d \\ 0 & \alpha_d^* & \Delta_d^* & -\epsilon_{\uparrow} \end{pmatrix}, \\ \epsilon_{\uparrow} &= \epsilon_0 - B_z + U\langle n_{\downarrow} \rangle, \quad \epsilon_{\downarrow} = \epsilon_0 + B_z + U\langle n_{\uparrow} \rangle, \\ \alpha_d &= B_x + iB_y - U\langle d|_{\downarrow}^{\dagger} d_{\uparrow} \rangle, \quad \Delta_d = U\langle d|_{\downarrow} d_{\uparrow} \rangle.\end{aligned}\tag{12.25}$$

The mean-field parameters appearing in Eq. (12.25) follow by solving the self-consistency equations

$$\begin{aligned}\langle n_{\uparrow} \rangle &= \frac{1}{\beta} \sum_{\omega} G_{d,11}(\omega), \quad \langle n_{\downarrow} \rangle = \frac{1}{\beta} \sum_{\omega} G_{d,33}(\omega), \\ \langle d|_{\downarrow}^{\dagger} d_{\uparrow} \rangle &= \frac{1}{\beta} \sum_{\omega} G_{d,13}(\omega), \quad \langle d|_{\downarrow} d_{\uparrow} \rangle = \frac{1}{\beta} \sum_{\omega} G_{d,21}(\omega),\end{aligned}\tag{12.26}$$

where the mean-field approximation readily yields

$$G_d(\omega) = [i\omega - \mathcal{H}_{\text{MF}} - \Sigma_S(\omega) - \Sigma_{TS}(\omega)]^{-1}.\tag{12.27}$$

The self-energies $\Sigma_{S/TS}(\omega)$ due to the coupling of the QD to the S/TS leads have the matrix representation

$$\Sigma_S = \Gamma_S \begin{pmatrix} g_{11} & -g_{12} & 0 & 0 \\ -g_{21} & g_{22} & 0 & 0 \\ 0 & 0 & g_{11} & -g_{12} \\ 0 & 0 & -g_{21} & g_{22} \end{pmatrix}\tag{12.28}$$

and

$$\Sigma_{TS} = \Gamma_{TS} \begin{pmatrix} G_{11} & 0 & 0 & -G_{12}e^{-i\phi} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -G_{21}e^{i\phi} & 0 & 0 & G_{22} \end{pmatrix}\tag{12.29}$$

with the hybridization parameters $\Gamma_{S/TS} = \lambda_{S/TS}^2$. The bGFs $g(\omega)$ and $G(\omega)$ have been defined in Eqs. (12.6) and (12.7), respectively. Once a self-consistent solution to Eq. (12.26) has been determined, which in general requires numerics, the Josephson current is obtained from Eq. (12.5) as

$$I(\phi) = -\frac{e}{\hbar\beta} \sum_{\omega} \frac{\partial_{\phi} \det [G_d^{-1}(\omega)]}{\det [G_d^{-1}(\omega)]}.\tag{12.30}$$

In what follows, we study a setup with $\Delta_p = \Delta$ and consider the zero-temperature limit.

In order to compare our self-consistent mean-field results to the noninteracting case, let us briefly summarize analytical expressions for the $U = 0$ ABS spectrum in the atomic limit defined by $\Gamma_{S,TS} \ll \Delta$. First we notice that at low energy scales, the self-energy $\Sigma = \Sigma_S + \Sigma_{TS}$, see Eqs. (12.28) and (12.29), simplifies to

$$\Sigma \simeq \begin{pmatrix} \frac{2\Delta}{i\omega} \Gamma_{TS} & -\Gamma_S & 0 & -\frac{2\Delta}{i\omega} \Gamma_{TS} e^{-i\phi} \\ -\Gamma_S & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Gamma_S \\ -\frac{2\Delta}{i\omega} \Gamma_{TS} e^{i\phi} & 0 & -\Gamma_S & \frac{2\Delta}{i\omega} \Gamma_{TS} \end{pmatrix}.\tag{12.31}$$

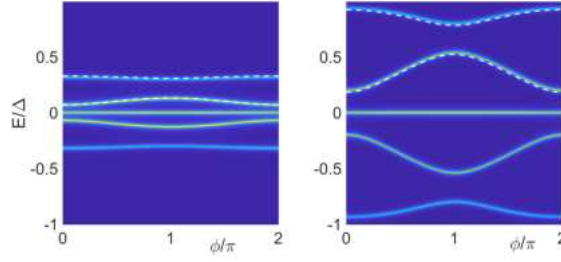


Figure 12.2: Phase dependence of the subgap spectrum of an S-QD-TS junction in the noninteracting case, $U = 0$. The TS wire is modeled from the low-energy limit of a Kitaev chain, and we use the parameters $B_y = 0$, $B_x = B_z = B/\sqrt{2}$, $\epsilon_0 = 0$, $\Delta_p = \Delta$, and $\Gamma_S = \Gamma_{TS} = \Gamma$. From blue to yellow, the color code indicates increasing values of the spectral density. The left (right) panel is for $\Gamma = 0.045\Delta$ and $B = 0.1\Delta$ ($\Gamma = B = 0.5\Delta$). Solid curves were obtained by numerical evaluation of Eq. (12.30). Dashed curves give the analytical prediction (12.32). In the right panel, the energies resulting from Eq. (12.32) have been rescaled by the factor $1 + \Gamma/\Delta$.

The ABS spectrum of the S-QD-TS junction then follows by solving a determinantal equation, $\det[G_d^{-1}(\omega)] = 0$. One finds a zero-energy pole which is related to the MBS and results from the $1/\omega$ dependence of $\Sigma_{TS}(\omega)$. In addition, we get finite-energy subgap poles for

$$i\omega \equiv E_A^{(\sigma_1=\pm, \sigma_2=\pm)} = \sigma_1 \sqrt{\frac{b_0 + \sigma_2 \sqrt{b_0^2 + 4c_0}}{2}}, \quad (12.32)$$

with the notation

$$\begin{aligned} b_0 &= \epsilon_\downarrow^2 + \epsilon_\uparrow^2 + 4\Gamma_{TS}\Delta + 2\Gamma_S^2 + 2|\alpha_d|^2, \\ c_0 &= -4\Gamma_{TS}\Delta (\epsilon_\downarrow^2 + \Gamma_S^2 + |\alpha_d|^2) - \epsilon_\uparrow^2 \epsilon_\downarrow^2 \\ &\quad - (|\alpha_d|^2 - \Gamma_S^2) (|\alpha_d|^2 - \Gamma_S^2 - 2\epsilon_\uparrow \epsilon_\downarrow) \\ &\quad + 8\Delta\Gamma_S\Gamma_{TS}\text{Re}(\alpha_d e^{i\phi}). \end{aligned} \quad (12.33)$$

In Fig. 12.2, numerically exact results for the $U = 0$ ABS spectrum are compared to the analytical prediction (12.32). We first notice that, as expected, Eq. (12.32) accurately fits the numerical results in the atomic limit, see the left panel in Fig. 12.2. Deviations can be observed for larger values of $\Gamma_{S,TS}/\Delta$. However, as shown in the right panel of Fig. 12.2, rather good agreement is again obtained by rescaling Eq. (12.32) with a constant factor of the order of $(1 + \Gamma_{S,TS}/\Delta)$. For finite B_y , we find (data not shown) that the phase-dependent ABS spectrum is shifted with respect to $\phi = 0$. In fact, since the phase dependence of the subgap states comes from the term $\text{Re}(\alpha_d e^{i\phi})$ in the atomic limit, see Eqs. (12.25) and (12.33), B_y can be fully accounted for in this limit by simply shifting $\phi \rightarrow \phi + \varphi_0$. We thereby recover the φ_0 -junction behavior discussed before for the cotunneling regime, see Eq. (12.23).

We next turn to self-consistent mean-field results for the phase-dependent ABS spectrum at finite U . Figure 12.3 shows the spectrum for the electron-hole symmetric case $\epsilon_0 = -U/2$, with other parameters as in the right panel of Fig. 12.2. For moderate interaction strength, e.g., taking $U = \Delta$ (left panel), we find that compared to the $U = 0$ case in Fig. 12.2, interactions push together pairs of Andreev bands, e.g., the pair corresponding to $E_A^{(+,\pm)}$ in Eq. (12.30). On the other hand, for stronger interactions, e.g., $U = 10\Delta$ (right panel), the outer ABSs leak into the continuum spectrum and only the inner Andreev states remain inside the superconducting gap. The ABS spectrum shown in Fig. 12.3 is similar to what is observed in mean-field calculations for S-QD-S systems with broken spin symmetry and in the magnetic

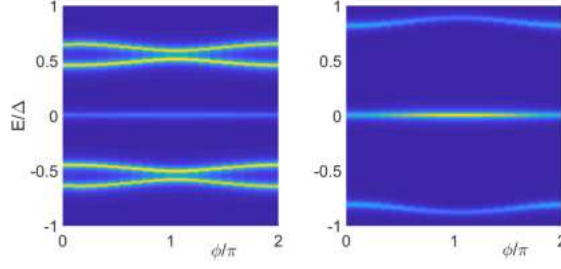


Figure 12.3: Phase-dependent ABS spectrum from mean-field theory for S-QD-TS junctions as in Fig. 12.2 but with $U > 0$ and $\epsilon_0 = -U/2$. We put $\Delta_p = \Delta$, $B_y = 0$, and $\Gamma_S = \Gamma_{TS} = \Gamma$. The color code is as in Fig. 12.2. The left panel is for $U = \Delta$, $\Gamma = 0.5\Delta$, and $B_x = B_z = B/\sqrt{2}$ with $B = 0.5\Delta$ [cf. the right panel of Fig. 12.2]. The right panel is for $U = 10\Delta$, $\Gamma = 4.5\Delta$, $B_x = 15\Delta$, and $B_z = 0$.

regime of the QD, where one finds up to four ABSs for $U < \Delta$ while the outer ABSs merge with the continuum for $U > \Delta$ [79]. Interestingly, the inner ABS contribution to the free energy for $U = 10\Delta$ is minimal for $\phi = \pi$, see right panel of Fig. 12.3, and we therefore expect π -junction behavior for $B_y = 0$ also in the regime with $U \gg \Delta$ and $B \gg \Delta$. We notice, however, that changing the sign of B_x would result in zero junction behavior. We interpret the inner ABSs for $U \gg \Delta$ as Shiba states with the phase dependence generated by the coupling to the MBS. Without the latter coupling, the Shiba state has ϕ -independent energy slightly below Δ determined by the scattering phase shift difference between both spin polarizations [80].

As illustrated in Fig. 12.4, the CPR computed numerically from Eq. (12.30) for different values of $\Gamma_{S,TS}/\Delta$, where B_x has been inverted with respect to its value in Fig. 12.3, results in zero junction behavior for $B_y = 0$. This behavior is expected from Eq. (12.23) in the cotunneling regime, and Fig. 12.4 shows that it also persists for $\Gamma_{S,TS} \gg \Delta$. In contrast to Eq. (12.23), however, the CPR for $\Gamma_{S,TS} \gg \Delta$ differs from a purely sinusoidal behavior, see Fig. 12.4. Moreover, for $B_y \neq 0$, we again encounter φ_0 -junction behavior, cf. the inset of Fig. 12.4, in accordance with the perturbative result in Eq. (12.23). Our mean-field results suggest that φ_0 -junction behavior is very robust and extends also into other parameter regimes as long as the condition $B_y \neq 0$ is met.

Next, Fig. 12.5 shows mean-field results for the critical current, $I_c = \max_\phi |I(\phi)|$, as function of the local magnetic field B_x and otherwise the same parameters as in Fig. 12.4. The main panel in Fig. 12.5 shows that I_c increases linearly with B_x for small $B_x < \Delta$, then exhibits a maximum around $B_x \approx \Gamma$, and subsequently decreases again to small values for $B_x \gg \max\{\Gamma_{S,TS}, \Delta\}$. On the other hand, for a fixed absolute value B of the magnetic field and $B_y = 0$, the critical current also exhibits a maximum as a function of the angle θ_B between $\mathbf{B}$ and the MBS spin polarization axis ($\hat{e}_z$). This effect is illustrated in the inset of Fig. 12.5. As expected, the Josephson current vanishes for $\theta_B \rightarrow 0$, where the supercurrent blockade argument of Ref. [35] implies $I_c = 0$, and reaches its maximal for $\theta_B = \pi/2$.

12.3 Spinful nanowire model for the TS

12.3.1 Model

Before turning to the S-TS-S setup in Fig. 12.1(b), we address the question of how the above results for S-QD-TS junctions change when using the spinful nanowire model of Refs. [6, 7] instead of the low-energy limit of a Kitaev chain, see Eq. (12.7). In fact, we will first describe the Josephson current for the elementary case of an S-TS junction using the spinful nanowire model. Surprisingly, to the best of our knowledge, this case has not yet been addressed in the

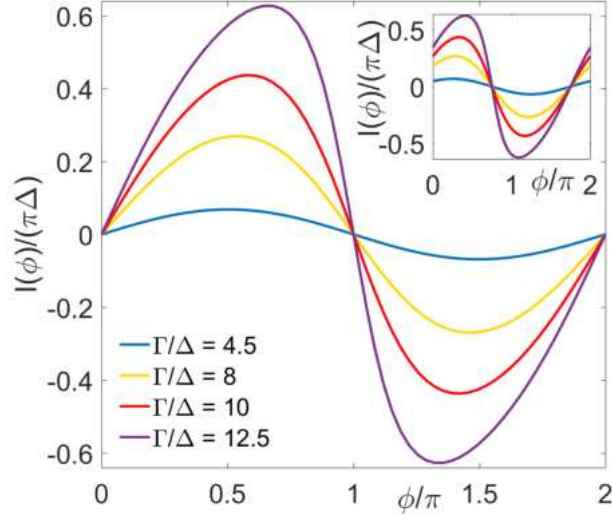


Figure 12.4: Main panel: Mean-field results for the CPR of S-QD-TS junctions with different Γ/Δ values, where we assume $\Delta_p = \Delta$, $U = 10\Delta$, $\epsilon_0 = -U/2$, $\Gamma_S = \Gamma_{TS} = \Gamma$, $B = 15\Delta$, and $B_z = 0$. Main panel: For $B_x = -B$ and $B_y = 0$, we find π -junction behavior. Inset: Same but for $B_y = -B_x = B/\sqrt{2}$, where φ_0 -junction behavior occurs.

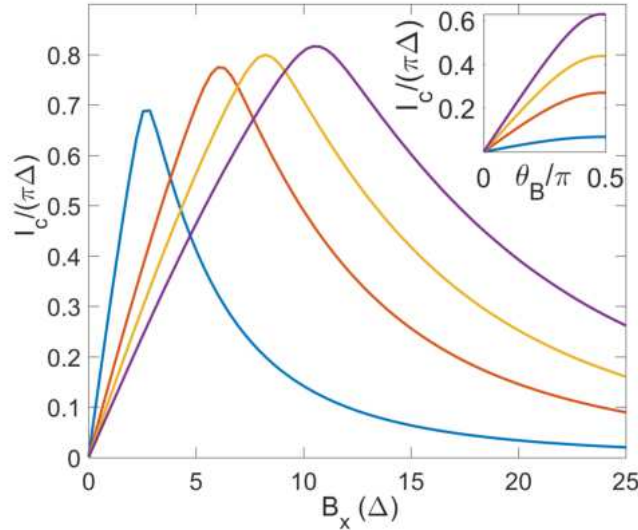


Figure 12.5: Main panel: Mean-field results for the critical current I_c vs local magnetic field scale B_x in S-QD-TS junctions. Parameters are as in the main panel of Fig. 12.4, i.e., $U = 10\Delta$, $\epsilon_0 = -U/2$, and $B_{y,z} = 0$. From left to right, different curves are for $\Gamma/\Delta = 4.5, 8, 10$ and 12.5 . Inset: I_c vs angle θ_B , where $\mathbf{B} = B(\sin \theta_B, 0, \cos \theta_B)$ with $B = 15\Delta$.

literature.

In spatially discretized form, the spinful nanowire model for TS wires reads [6, 7, 70]

$$\begin{aligned} H_{\text{TS}} &= \frac{1}{2} \sum_j \left[\psi_j^\dagger \hat{h} \psi_j + \left(\psi_j^\dagger \hat{t} \psi_{j+1} + \text{h.c.} \right) \right], \\ \hat{h} &= (2t - \mu) \tau_z \sigma_0 + V_x \tau_0 \sigma_x + \Delta_p \tau_x \sigma_0, \\ \hat{t} &= -t \tau_z \sigma_0 + i \alpha \tau_z \sigma_z, \end{aligned} \quad (12.34)$$

where the lattice fermion operators $c_{j\sigma}$ for given site j with spin polarizations $\sigma = \uparrow, \downarrow$ are combined to the four-spinor operator $\psi_j = \left(c_{j\uparrow}, c_{j\downarrow}, c_{j\downarrow}^\dagger, -c_{j\uparrow}^\dagger \right)^T$. The Pauli matrices $\tau_{x,y,z}$ (and unity τ_0) again act in Nambu space, while Pauli matrices $\sigma_{x,y,z}$ and σ_0 refer to spin. In the figures shown below, we choose the model parameters in Eq. (12.34) as discussed in Ref. [70]. The lattice spacing is set to $a = 10$ nm, which results in a nearest-neighbor hopping $t = \hbar^2/(2m^*a^2) = 20$ meV and the spin-orbit coupling strength $\alpha = 4$ meV for InAs nanowires. The proximity-induced pairing gap is again denoted by Δ_p , the chemical potential is μ , and the bulk Zeeman energy scale V_x is determined by a magnetic field applied along the wire. Under the condition

$$V_x > V_x^c = \sqrt{\mu^2 + \Delta_p^2}, \quad (12.35)$$

the topologically nontrivial phase is realized [6, 7]. As we discuss below, the physics of the S-QD-TS junction sensitively depends on both the bulk Zeeman field V_x and on the local magnetic field $\mathbf{B}$ acting on the QD, where one can either identify both magnetic fields or treat $\mathbf{B}$ as independent field. In any case, the bGF $\tilde{G}(\omega)$ for the model in Eq. (12.34), which now replaces the Kitaev chain result $G(\omega)$ in Eq. (12.7), needs to be computed numerically. The bGF $\tilde{G}$ has been described in detail in Ref. [70], where also a straightforward numerical scheme for calculating $\tilde{G}(\omega)$ has been devised. With the replacement $G \rightarrow \tilde{G}$, we can then take over the expressions for the Josephson current discussed before. Below we study these expressions in the zero-temperature limit.

12.3.2 S-TS junction

Let us first address the CPR for the S-TS junction case. The Josephson current can be computed using the bGF expression for tunnel junctions in Ref. [20], which is a simplified version of the above expressions for the S-QD-TS case. The spin-conserving tunnel coupling λ defines a transmission probability (transparency) $\mathcal{T}$ of the normal junction [20, 70]. Close to the topological transition, the transparency is well approximated by

$$\mathcal{T} = \frac{4(\lambda/t)^2}{[1 + (\lambda/t)^2]^2}, \quad (12.36)$$

where $t = 20$ meV is the hopping parameter in Eq. (12.34). We then study the CPR and the resulting critical current I_c as a function of $\mathcal{T}$ for both the topologically trivial ($V_x < V_x^c$) and the nontrivial ($V_x > V_x^c$) regime, see Eq. (12.35).

In Fig. 12.6, we show the V_x dependence of the critical current I_c for the symmetric case $\Delta = \Delta_p$. In particular, it is of interest to determine how I_c changes as one moves through the phase transition in Eq. (12.35). First, we observe that I_c is strongly suppressed in the topological phase in comparison to the topologically trivial phase. In fact, I_c slowly decreases as one moves into the deep topological phase by increasing V_x . This observation is in accordance with the expected supercurrent blockade in the deep topological limit [35]: $I_c = 0$ for the corresponding Kitaev chain case since p -wave pairing correlations on the TS side are incompatible with s -wave

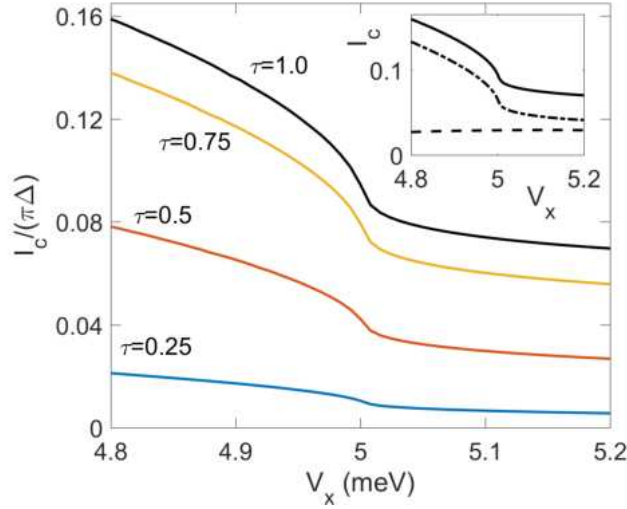


Figure 12.6: Main panel: Critical current I_c vs Zeeman energy V_x for an S-TS junction using the spinful TS nanowire model (12.34) for $\Delta_p = \Delta = 0.2$ meV, $\mu = 5$ meV, and different transparencies $\mathcal{T}$ calculated from Eq. (12.36). All other parameters are specified in the main text. Inset: Decomposition of I_c for $\mathcal{T} = 1$ into ABS (dotted-dashed) and continuum (dashed) contributions.

correlations on the S side. However, a residual finite supercurrent can be observed even for rather large values of V_x . We attribute this effect to the remaining s -wave pairing correlations contained in the spinful nanowire model (12.34). Second, Fig. 12.6 shows kink-like features in the $I_c(V_x)$ curve near the topological transition, $V_x \approx V_x^c$. The inset of Fig. 12.6 demonstrates that this feature comes from a rapid decrease of the ABS contribution while the continuum contribution remains smooth. This observation suggests that continuum contributions in this setup mainly originate from s -wave pairing correlations which are not particularly sensitive to the topological transition.

In Fig. 12.7, we show the CPR for the S-TS junction with $\mathcal{T} = 1$ in Fig. 12.6, where different curves correspond to different Zeeman couplings V_x near the critical value. We find that in many parameter regions, in particular for $\mathcal{T} < 1$, the CPR is to high accuracy given by a conventional 2π -periodic Josephson relation, $I(\phi) = I_c \sin \phi$. In the topologically trivial phase, small deviations from the sinusoidal law can be detected, but once one enters the topological phase, these deviations become extremely small.

12.3.3 S-QD-TS junction with spinful TS wire: Mean-field theory

Apart from providing a direct link to experimental control parameters, another advantage of using the spinful nanowire model of Refs. [6, 7] for modeling the TS wire is that the angle between the local Zeeman field $\mathbf{B}$ and the MBS spin polarization does not have to be introduced as phenomenological parameter but instead results from the calculation [70]. It is thus interesting to study the Josephson current in S-QD-TS junctions where the TS wire is described by the spinful nanowire model. For this purpose, we now revisit the mean-field scheme for S-QD-TS junctions using the bGF $\tilde{G}(\omega)$ for the spinful nanowire model (12.34). In particular, with the replacement $G \rightarrow \tilde{G}$, we solve the self-consistency equations (12.26) and thereby obtain the mean-field parameters in Eq. (12.25). The resulting QD GF, $G_d(\omega)$ in Eq. (12.27), then determines the Josephson current in Eq. (12.30). Below we present self-consistent mean-field results obtained from this scheme. In view of the huge parameter space of this problem, we here only discuss a few key observations. A full discussion of the phase diagram and the corresponding physics will be given elsewhere.

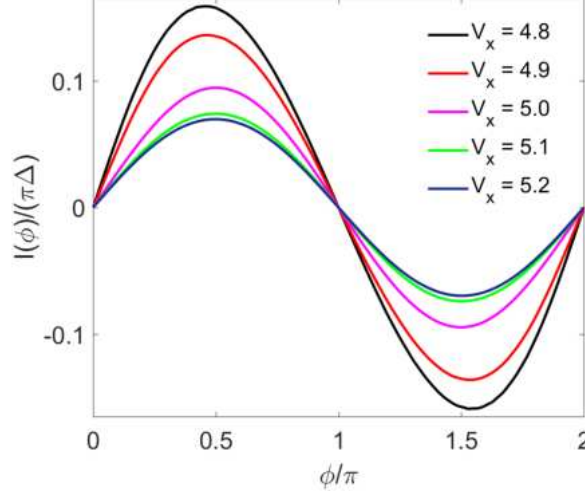


Figure 12.7: CPR for the S-TS junction with $\mathcal{T} = 1$ in Fig. 12.6, for different bulk Zeeman fields V_x (in meV) near the critical value $V_x^c = 5.004$ meV.

The main panel of Fig. 12.8 shows the critical current I_c vs the bulk Zeeman energy V_x for several values of the chemical potential μ , where the respective critical value V_x^c in Eq. (12.35) for the topological phase transition also changes with μ . The results in Fig. 12.8 assume that the local magnetic field $\mathbf{B}$ acting on the QD coincides with the bulk Zeeman field V_x in the TS wire, i.e., $\mathbf{B} = (V_x, 0, 0)$. For the rather large values of $\Gamma_{S,TS}$ taken in Fig. 12.8, the I_c vs V_x curves again exhibit a kink-like feature near the topological transition, $V_x \approx V_x^c$. This behavior is very similar to what happens in S-TS junctions with large transparency $\mathcal{T}$, cf. Fig. 12.6. As demonstrated in the inset of Fig. 12.8, the physical reason for the kink feature can be traced back to a sudden drop of the ABS contribution to I_c when entering the topological phase $V_x > V_x^c$. In the latter phase, I_c becomes strongly suppressed, in close analogy to the S-TS junction case shown in Fig. 12.6.

In Fig. 12.8, both the QD and the TS wire were subject to the same magnetic Zeeman field. If the direction and/or the size of the local magnetic field $\mathbf{B}$ applied to the QD can be varied independently from the bulk magnetic field $V_x \hat{e}_x$ applied to the TS wire, one can arrive at rather different conclusions. To illustrate this statement, Fig. 12.9 shows the I_c vs B_z dependence for $\mathbf{B} = (0, 0, B_z)$ perpendicular to the bulk field, with $V_x > V_x^c$ such that the TS wire is in the topological phase. In this case, Fig. 12.9 shows that I_c exhibits a maximum close to $B_z = 0$. This behavior is reminiscent of what we observed above in Fig. 12.5, using the low-energy limit of a Kitaev chain for the bGF of the TS wire. Remarkably, the critical current can here reach values close to the unitary limit, $I_c \sim e\Delta/\hbar$. We note that since B_z does not drive a phase transition, no kink-like features appear for the $I_c(B_z)$ curves shown in Fig. 12.9. Finally, the inset of Fig. 12.9 shows that for $\mathbf{B}$ perpendicular to $V_x \hat{e}_x$, where $V_x > V_x^c$ for the parameters chosen in Fig. 12.9, above-gap quasiparticles again provide a more significant contribution to I_c than Andreev states.

12.4 S-TS-S junctions: Switching the parity of a superconducting atomic contact

12.4.1 Model

We now proceed to the three-terminal S-TS-S setup shown in Fig. 12.1(b). The CPR found in the related TS-S-TS trijunction case has been discussed in detail in Ref. [70], see also Ref. [44]. Among other findings, a main conclusion of Ref. [70] for the TS-S-TS geometry was that the

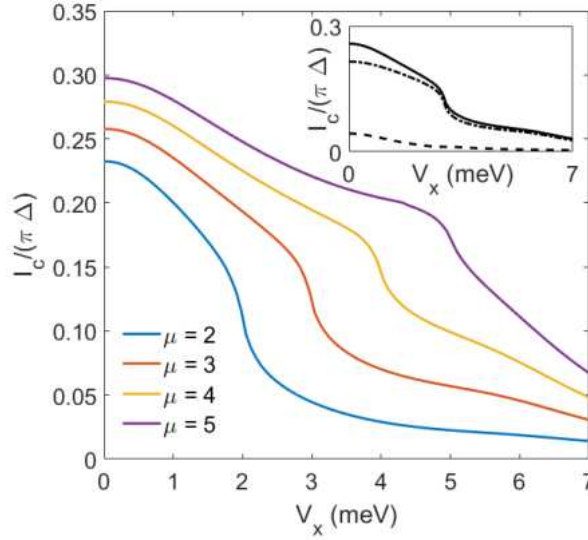


Figure 12.8: Main panel: Critical current I_c vs Zeeman energy V_x for S-QD-TS junctions from mean-field theory using the spinful TS nanowire model (12.34). Results are shown for several values of the chemical potential μ (in meV), where we assume $U = 10\Delta$, $\epsilon_0 = -U/2$, $\Delta_p = \Delta = 0.2$ meV, $\Gamma_S = 2\Gamma_{TS} = 9\Delta$, and $\mathbf{B} = (V_x, 0, 0)$. Inset: Detailed view of the transition region $V_x \approx V_x^c$ for $\mu = 4$ meV, including a decomposition of I_c into the ABS (dotted-dashed) and the continuum (dashed) contribution.

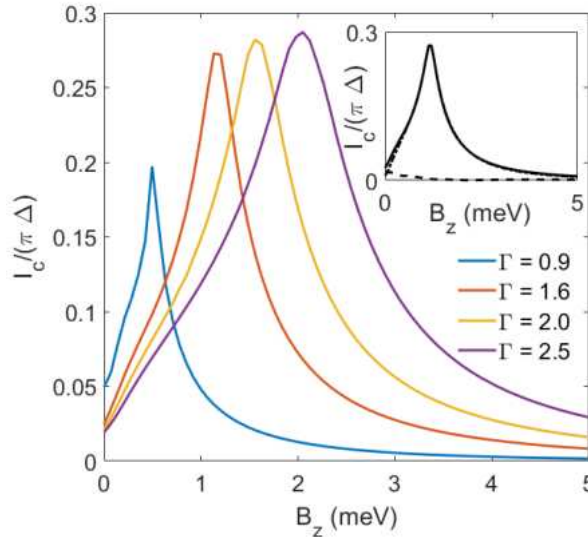


Figure 12.9: Main panel: Mean-field results for I_c vs B_z in S-QD-TS junctions for several values of $\Gamma_S = \Gamma_{TS} = \Gamma$ (in meV) and $\mu = 4$ meV. The bulk Zeeman field $V_x = 5$ meV along $\hat{e}_x$ (where $V_x > V_x^c$ for our parameters) is applied to the spinful TS wire, while the QD is subject to the local magnetic field $\mathbf{B} = B_z \hat{e}_z$. All other parameters are as in Fig. 12.8. Inset: Decomposition of I_c into ABS (dotted-dashed) and continuum (dashed) contributions for $\Gamma = 1.6$ meV.

CPR can reveal information about the spin canting angle between the MBS spin polarization axes in both TS wires. In what follows, we study the superficially similar yet rather different case of an S-TS-S junction. Throughout this section, we model the TS wire via the low-energy theory of a spinless Kitaev chain, where the bGF $G(\omega)$ in Eq. (12.7) applies.

One can view the setup in Fig. 12.1(b) as a conventional superconducting atomic contact (SAC) with a TS wire tunnel-coupled to the S-S junction. Over the past few years, impressive experimental progress [52, 53, 116] has demonstrated that the ABS level system in a SAC [13] can be accurately probed and manipulated by coherent or incoherent microwave spectroscopy techniques. We show below that an additional TS wire, cf. Fig. 12.1(b), acts as tunable parity switch on the many-body ABS levels of the SAC. As we have discussed above, the supercurrent flowing directly between a given S lead and the TS wire is expected to be strongly suppressed. However, through the hybridization with the MBS, Andreev level configurations with even and odd fermion parity are connected. This effect has profound and potentially useful consequences for Andreev spectroscopy.

An alternative view of the setup in Fig. 12.1(b) is to imagine an S-TS junction, where S1 plays the role of the S lead and the spinful TS wire is effectively composed from a spinless (Kitaev) TS wire and the S2 superconductor. The p - and s -wave pairing correlations in the spinful TS wire are thereby spatially separated. Since the s - and p -wave bands represent normal modes, they do not directly couple to each other in this scenario, i.e., we have to put $\lambda_2 = 0$. We discuss this analogy in more detail later on.

We consider a conventional single-channel SAC (gap Δ) coupled via a point contact to a TS wire (gap Δ_p), cf. Fig. 12.1(b). The superconducting phase difference across the SAC is denoted by $\phi = \phi_1 - \phi_2$, where ϕ_j is the phase difference between the respective S arm ($j = 1, 2$) and the TS wire. In practice, the SAC can be embedded into a superconducting ring for magnetic flux tuning of ϕ . To allow for analytical progress, we here assume that Δ_p is so large that continuum quasiparticle excitations in the TS wire can be neglected. In that case, only the MBS at the junction has to be kept when modeling the TS wire. However, we will also hint at how one can treat the general case.

For the two S leads, boundary fermion fields are contained in Nambu spinors as in Eq. (12.6),

$$\Psi_{S,j=1,2} = \begin{pmatrix} \psi_{j,\uparrow} \\ \psi_{j,\downarrow}^\dagger \end{pmatrix}, \quad (12.37)$$

where their bGF follows with the Nambu matrix $g(\omega)$ in Eq. (12.6) as

$$g_j^{-1}(\omega) = g^{-1}(\omega) + b_j \tau_0. \quad (12.38)$$

We again use Pauli matrices $\tau_{x,y,z}$ and unity τ_0 in Nambu space. The dimensionless parameters $b_{1,2}$ describe the Zeeman field component along the MBS spin polarization axis, see below. Since above-gap quasiparticles in the TS wire are neglected here, the TS wire is represented by the Majorana operator $\gamma = \gamma^\dagger$, with $\gamma^2 = 1/2$, which anticommutes with all other fermions. We may represent γ by an auxiliary fermion $f_\uparrow$, where the index reminds us that the MBS spin polarization points along $\hat{e}_z$,

$$\gamma = (f_\uparrow + f_\uparrow^\dagger)/\sqrt{2}. \quad (12.39)$$

The other Majorana mode $\gamma' = -i(f_\uparrow - f_\uparrow^\dagger)/\sqrt{2}$, which is localized at the opposite end of the TS wire, is assumed to have negligible hybridization with the $\Psi_{S,j}$ spinors and with γ . Writing the Euclidean action as $S = S_0 + S_{\text{tun}}$, we have an uncoupled action contribution,

$$\begin{aligned} S_0 = & \sum_{j=1,2} \int_0^\beta d\tau d\tau' \bar{\Psi}_{S,j}(\tau) g_j^{-1}(\tau - \tau') \Psi_{S,j}(\tau') + \\ & + \frac{1}{2} \int_0^\beta d\tau \gamma(\tau) \partial_\tau \gamma(\tau). \end{aligned} \quad (12.40)$$

The leads are connected by a time-local tunnel action corresponding to the tunnel Hamiltonian

$$H_{\text{tun}} = t_0 \left(\Psi_{S,1}^\dagger \tau_z e^{i\tau_z \phi/2} \Psi_{S,2} + \text{h.c.} \right) + \sum_{j=1,2} \frac{\lambda_j}{\sqrt{2}} \left(\psi_{j,\uparrow}^\dagger e^{i\phi_j/2} - \text{h.c.} \right) \gamma. \quad (12.41)$$

Without loss of generality, we assume that the tunnel amplitudes t_0 and $\lambda_{1,2}$, see Fig. 12.1(b), are real-valued and that they include density-of-state factors again. The parameter t_0 (with $0 \leq t_0 \leq 1$) determines the transparency $\mathcal{T}$ of the SAC in the normal-conducting state [14], cf. Eq. (12.36),

$$\mathcal{T} = \frac{4t_0^2}{(1+t_0^2)^2}. \quad (12.42)$$

Note that in Eq. (12.41) we have again assumed spin-conserving tunneling, where only spin- $\uparrow$ fermions in the SAC are tunnel-coupled to the Majorana fermion γ , cf. Eq. (12.4).

At this stage, it is convenient to trace out the $\Psi_{S,2}$ spinor field. As a result, the SAC is described in terms of only one spinor field, $\Psi \equiv \Psi_{S,1}$, which however is still coupled to the Majorana field γ . After some algebra, we obtain the effective action

$$\begin{aligned} S_{\text{eff}} = & \int_0^\beta d\tau d\tau' \left\{ \bar{\Psi}(\tau) K^{-1}(\tau - \tau') \Psi(\tau') \right. \\ & + \Phi^T(\tau) \left[\frac{1}{2} \delta(\tau - \tau') \partial_{\tau'} - \lambda_2^2 P_\uparrow g_2(\tau - \tau') P_\uparrow \right] \Phi(\tau') \\ & + \left[\bar{\Psi}(\tau) \left(\lambda_1 e^{i\phi_1/2} \delta(\tau - \tau') \right. \right. \\ & \left. \left. - \lambda_2 e^{i\phi_2/2} t_0 \tau_z e^{i\tau_z \phi/2} g_2(\tau - \tau') \right) P_\uparrow \Phi(\tau') + \text{h.c.} \right] \left. \right\}, \end{aligned} \quad (12.43)$$

where the operator $P_\uparrow = (\tau_0 + \tau_z)/2$ projects a Nambu spinor to its spin- $\uparrow$ component. Moreover, we have defined an effective GF in Nambu space with frequency components

$$K^{-1}(\omega) = g_1^{-1}(\omega) - t_0^2 \tau_z e^{i\tau_z \phi/2} g_2(\omega) e^{-i\tau_z \phi/2} \tau_z, \quad (12.44)$$

and the TS lead has been represented by the Majorana-Nambu spinor

$$\Phi(\tau) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \gamma(\tau) = \tau_x \Phi^*(\tau). \quad (12.45)$$

We note in passing that Eq. (12.43) could at this point be generalized to include continuum states in the TS wire. To that end, one has to (i) replace $\Phi \rightarrow (\psi, \psi^\dagger)^T$, where ψ is the boundary fermion of the effectively spinless TS wire, and (ii) replace $\delta(\tau - \tau') \partial_{\tau'} \rightarrow G^{-1}(\tau - \tau')$ with G in Eq. (12.7). Including bulk TS quasiparticles becomes necessary for small values of the proximity gap, $\Delta_p \ll \Delta$, and/or when studying nonequilibrium applications within a Keldysh version of our formalism.

In any case, after neglecting the above-gap TS continuum quasiparticles, the partition function follows with S_{eff} in Eq. (12.43) in the functional integral representation

$$Z = \int \mathcal{D}[\bar{\Psi}, \Psi, \gamma] e^{-S_{\text{eff}}} \equiv e^{-\beta F(\phi_1, \phi_2)}. \quad (12.46)$$

As before, the Josephson current through S lead no. j then follows from the free energy via $I_j = (2e/\hbar) \partial_{\phi_j} F$. The supercurrent flowing through the TS wire is then given by

$$I_{\text{TS}} = -(I_1 + I_2), \quad (12.47)$$

as dictated by current conservation.

12.4.2 Atomic limit

In order to get insight into the basic physics, we now analyze in detail the atomic limit, where Δ represents the largest energy scale of interest and hence the dynamics is confined to the subgap region. In this case, we can approximate $\sqrt{\Delta^2 + \omega^2} \approx \Delta$. After the rescaling $\Psi \rightarrow \sqrt{\Delta/(1+t_0^2)}\Psi$ in Eq. (12.43), we arrive at an effective action, $S_{\text{eff}} \rightarrow S_{\text{at}}$, valid in the atomic limit,

$$S_{\text{at}} = \int_0^\beta d\tau \left\{ \frac{1}{2} \gamma \partial_\tau \gamma + \bar{\Psi} \left[\partial_\tau + \Delta \cos(\phi/2) \tau_x + \right. \right. \\ \left. \left. + r \Delta \sin(\phi/2) \tau_y + B_z \tau_0 \right] \Psi + \right. \\ \left. + \frac{1}{\sqrt{2}} \sum_{\sigma=\uparrow,\downarrow} (\lambda_\sigma \psi_\sigma^\dagger - \text{h.c.}) \gamma \right\}, \quad (12.48)$$

where $r = \sqrt{1 - \mathcal{T}}$ is the reflection amplitude of the SAC, see Eq. (12.42). We recall that $\Psi = (\psi_\uparrow, \psi_\downarrow)^T$, see Eq. (12.37). Moreover, we define the auxiliary parameters

$$\begin{aligned} \lambda_\uparrow &= \lambda_1 \sqrt{(1+r)\Delta/2} e^{i\phi_1/2}, \\ \lambda_\downarrow &= -\lambda_2 \sqrt{(1-r)\Delta/2} e^{-i\phi_2/2}, \\ B_z &= \left(\frac{1+r}{2} b_1 + \frac{1-r}{2} b_2 \right) \Delta. \end{aligned} \quad (12.49)$$

The parameters $b_{1,2}$ in Eq. (12.38) thus effectively generate the Zeeman scale B_z in Eq. (12.49).

As a consequence of the atomic limit approximation, the action S_{at} in Eq. (12.48) is equivalently expressed in terms of the effective Hamiltonian

$$\begin{aligned} H_{\text{at}} &= \sum_{\sigma=\uparrow,\downarrow=\pm} \sigma B_z \psi_\sigma^\dagger \psi_\sigma + \left(\delta_A \psi_\uparrow^\dagger \psi_\downarrow^\dagger + \text{h.c.} \right) \\ &+ \frac{1}{\sqrt{2}} \sum_{\sigma} (\lambda_\sigma \psi_\sigma^\dagger - \text{h.c.}) \gamma, \end{aligned} \quad (12.50)$$

where we define

$$\delta_A(\phi) = \Delta [\cos(\phi/2) - ir \sin(\phi/2)]. \quad (12.51)$$

For a SAC decoupled from the TS wire and taken at zero field ($B_z = 0$), the ABS energy follows from Eq. (12.50) in the standard form [62]

$$E_A(\phi) = |\delta_A| = \Delta \sqrt{1 - \mathcal{T} \sin^2(\phi/2)}. \quad (12.52)$$

We emphasize that H_{at} neglects TS continuum quasiparticles as well as all types of quasiparticle poisoning processes. Let us briefly pause in order to make two remarks. First, we note that the Majorana field $\gamma = (f_\uparrow + f_\uparrow^\dagger)/\sqrt{2}$, see Eq. (12.39), couples to both spin modes ψ_σ in Eq. (12.50). The coupling $\lambda_\downarrow$ between γ and the spin- $\downarrow$ field in the SAC, $\psi_\downarrow$, is generated by crossed Andreev reflection processes, where a Cooper pair in lead S2 splits according to $\psi_{2,\uparrow}^\dagger \psi_{2,\downarrow}^\dagger \rightarrow f_\uparrow^\dagger \psi_{1,\downarrow}^\dagger$, plus the conjugate process. Second, we observe that H_{at} is invariant under a particle-hole transformation, amounting to the replacements $\psi_\sigma \rightarrow \psi_\sigma^\dagger$ and $f_\uparrow \rightarrow f_\uparrow^\dagger$, along with $B_z \rightarrow -B_z$ and $\phi_j \rightarrow 2\pi - \phi_j$.

We next notice that with $n_\sigma = \psi_\sigma^\dagger \psi_\sigma = 0, 1$ and $n_f = f_\uparrow^\dagger f_\uparrow = 0, 1$, the total fermion parity of the junction,

$$\mathcal{P}_{\text{tot}} = (-1)^{n_f + n_\uparrow + n_\downarrow} = \pm 1, \quad (12.53)$$

is a conserved quantity, $[\mathcal{P}_{\text{tot}}, H_{\text{at}}]_- = 0$. Below we restrict our analysis to the even-parity sector $\mathcal{P}_{\text{tot}} = +1$, but analogous results hold for the odd-parity case. The corresponding Hilbert subspace is spanned by four states,

$$|n_{\uparrow}, n_{\downarrow}, n_f\rangle = \left(\psi_{\uparrow}^{\dagger}\right)^{n_{\uparrow}} \left(\psi_{\downarrow}^{\dagger}\right)^{n_{\downarrow}} \left(f_{\uparrow}^{\dagger}\right)^{n_f} |0\rangle, \quad (12.54)$$

where $(n_{\uparrow}, n_{\downarrow}, n_f) \in \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ and $|0\rangle$ is the vacuum state. In this basis, the Hamiltonian (12.50) has the matrix representation

$$\mathcal{H}_{\text{at}}(\phi_1, \phi_2) = \begin{pmatrix} 0 & \delta_A^* & \lambda_{\uparrow}^*/2 & \lambda_{\downarrow}^*/2 \\ \delta_A & 0 & \lambda_{\downarrow}/2 & -\lambda_{\uparrow}/2 \\ \lambda_{\uparrow}/2 & \lambda_{\downarrow}^*/2 & B_z & 0 \\ \lambda_{\downarrow}/2 & -\lambda_{\uparrow}^*/2 & 0 & -B_z \end{pmatrix}. \quad (12.55)$$

The even-parity ground state energy, $E_G^{(e)} = \min(\varepsilon)$, follows as the smallest root of the quartic equation

$$\det(\mathcal{H}_{\text{at}} - \varepsilon) = 0. \quad (12.56)$$

In order to obtain simple results, let us now consider the special case $\lambda_2 = 0$, where the TS wire is directly coupled to lead S1 only, see Fig. 12.1(b). In that case, we also have $\lambda_{\downarrow} = 0$, see Eq. (12.49), and Eq. (12.56) implies the four eigenenergies $\pm\varepsilon_{\pm}$ with

$$\begin{aligned} \varepsilon_{\pm} &= \frac{1}{\sqrt{2}} \left(E_A^2 + B_z^2 + \frac{1}{2} |\lambda_{\uparrow}|^2 \right. \\ &\quad \left. \pm \sqrt{(E_A^2 - B_z^2)^2 + |\lambda_{\uparrow}|^2 (E_A^2 + B_z^2)} \right)^{1/2}, \end{aligned} \quad (12.57)$$

with $|\lambda_{\uparrow}|^2 = \lambda_1^2(1+r)\Delta/2$, see Eq. (12.49). The ground-state energy is thus given by $E_G^{(e)} = -\varepsilon_+$. Since E_G depends on the phases $\phi_{1,2}$ only via the Andreev level energy $E_A(\phi)$ in Eq. (12.52), the Josephson current through the SAC is given by

$$I_1 = -I_2 = \frac{2e}{\hbar} \partial_{\phi} E_G^{(e)} = -\frac{2e}{\hbar} \partial_{\phi} \varepsilon_+. \quad (12.58)$$

Note that Eq. (12.47) then implies that no supercurrent flows into the TS wire.

Next we observe that in the absence of the TS probe ($\lambda_1 = 0$), the even and odd fermion parity sectors of the SAC, $\mathcal{P}_{\text{SAC}} = (-1)^{n_{\uparrow}+n_{\downarrow}} = \pm 1$, are decoupled, see Eq. (12.55), and Eq. (12.57) yields $E_G^{(e)} = -\max(E_A, |B_z|)$. Importantly, the Josephson current is therefore fully blocked if the ground state is in the $\mathcal{P}_{\text{SAC}} = -1$ sector, i.e., for $|B_z| > E_A(\phi)$. For $\lambda_1 \neq 0$, however, $\mathcal{P}_{\text{SAC}}$ is not conserved anymore. This implies that the MBS can act as parity switch between the two Andreev sectors with parity $\mathcal{P}_{\text{SAC}} = \pm 1$. Near the level crossing point at $E_A \approx |B_z|$, i.e., assuming $|E_A^2 - B_z^2| \ll |\lambda_{\uparrow}|^2 \ll E_A^2 + B_z^2$, we obtain

$$\varepsilon_{\pm} \simeq \frac{1}{\sqrt{2}} \left(E_A^2 + B_z^2 \pm \lambda_1 \sqrt{2(1+r)\Delta(E_A^2 + B_z^2)} \right)^{1/2}, \quad (12.59)$$

which implies a nonvanishing supercurrent through the SAC even in the field-dominated regime, $|B_z| > E_A$. The MBS therefore acts as a parity switch and leaves a trace in the CPR by lifting the supercurrent blockade.

12.4.3 Another interpretation

Interestingly, for $\lambda_2 = \phi_2 = 0$, the S-TS-S setup in Fig. 12.1(b) could also be viewed as a toy model for an S-TS junction, where the TS part corresponds to a spinful model. In that analogy, the Nambu spinor $\Psi_{S,1}$ stands for the S lead while the spinful TS wire is represented by (i) the Nambu spinor $\Psi_{S,2}$ which is responsible for the residual s -wave pairing correlations, and (ii) by the MF γ (or, more generally, by the Kitaev-chain spinless boundary fermion ψ) which encodes p -wave pairing correlations. Moreover, t_0 and λ_1 should now be understood as spin-conserving phenomenological tunnel couplings acting in the s - s and s - p wave channels, respectively. The phase difference across this effective S-TS junction is $\phi = \phi_1$ and the net S-TS tunnel coupling is given by $\lambda = \sqrt{t_0^2 + \lambda_1^2}$. Putting $\lambda_1 = 0$ in the topologically trivial phase of the TS wire, the Josephson current carried by Andreev states in the s - s channel is blocked when the ground state is in the odd parity sector of the SAC. For $\lambda_1 \neq 0$, the MBS-mediated switching between odd and even parity sectors will now be activated and thereby lift the supercurrent blockade.

12.4.4 Conventional midgap level

A similar behavior as predicted above for the MBS-induced parity switch between $\mathcal{P}_{\text{SAC}} = \pm 1$ sectors could also be expected from a conventional fermionic subgap state tunnel-coupled to the SAC. Such a subgap state may be represented, e.g., by a single-level quantum dot in the Coulomb blockade regime. In particular, for a midgap (zero-energy) level with the fermion operator d , the Hamiltonian H_{at} in Eq. (12.50) has to be replaced with

$$\begin{aligned} \tilde{H}_{\text{at}} = & \sum_{\sigma=\uparrow,\downarrow=\pm} \sigma B_z \psi_\sigma^\dagger \psi_\sigma + \left(\delta_A \psi_\uparrow^\dagger \psi_\downarrow^\dagger + \text{h.c.} \right) + \\ & + \sum_{\sigma} (\lambda_\sigma \psi_\sigma^\dagger d + \text{h.c.}) . \end{aligned} \quad (12.60)$$

In the even total parity basis (12.54), the matrix representation of the Hamiltonian is then instead of Eq. (12.55) given by

$$\tilde{\mathcal{H}}_{\text{at}}(\phi_1, \phi_2) = \begin{pmatrix} 0 & \delta_A^* & 0 & 0 \\ \delta_A & 0 & \lambda_\downarrow & -\lambda_\uparrow \\ 0 & \lambda_\downarrow^* & B_z & 0 \\ 0 & -\lambda_\uparrow^* & 0 & -B_z \end{pmatrix}. \quad (12.61)$$

Assuming $|\lambda_\uparrow| = |\lambda_\downarrow| \equiv \lambda$, Eq. (12.56) then yields the eigenenergies $\pm \varepsilon_\pm$ with

$$\begin{aligned} \varepsilon_\pm = & \frac{1}{\sqrt{2}} \left(E_A^2 + B_z^2 + 2\lambda^2 \right. \\ & \left. \pm \sqrt{(E_A^2 - B_z^2)^2 + 4\lambda^2 (E_A^2 + B_z^2 + \lambda^2)} \right)^{1/2}. \end{aligned} \quad (12.62)$$

Remarkably, the ABS spectra in Eqs. (12.62) and (12.57) are rather similar for $\lambda^2 \ll \max(E_A^2, B_z^2)$. However, the MBS will automatically be located at zero energy and thus represents a generic situation.

12.5 Conclusion

We close this paper by summarizing our main findings. We have studied the Josephson effect in different setups involving both conventional s -wave BCS superconductors (S leads) and topologically nontrivial 1D p -wave superconductors (TS leads) with Majorana end states. The TS

wires have been described either by a spinless theory applicable in the deep topological regime, which has the advantage of allowing for analytical progress but makes it difficult to establish contact to experimental control parameters, or by a spinful nanowire model as suggested in Refs. [6, 7]. We have employed a unified imaginary-time Green's function approach to analyze the equilibrium properties of such devices, but a Keldysh generalization is straightforward and allows one to study also nonequilibrium applications.

For S-TS tunnel junctions, we find that in the topological phase of the TS wire, the supercurrent is mainly carried by above-gap continuum contributions. We confirm the expected supercurrent blockade [35] in the deep topological regime (where the spinless theory is fully valid and thus no residual s -wave pairing exists), while for realistic parameters, a small but finite critical current is found. To good approximation, the Josephson current obeys the usual 2π -periodic sinusoidal current-phase relation. The dependence of the critical current on the bulk Zeeman field driving the TS wire through the topological phase transition shows a kink-like feature at the critical value, which is caused by a sudden drop of the Andreev state contribution.

The supercurrent blockade in the deep topological phase could be lifted by adding a magnetic impurity to the junction, also allowing for the presence of a local magnetic field $\mathbf{B}$. Such a magnetic impurity arises from a spin-degenerate quantum dot (QD), and we have studied the corresponding S-QD-TS problem for both the spinless and the spinful TS wire model. Based on analytical results valid in the cotunneling regime as well as numerical results within the mean-field approximation, we predict φ_0 -junction behavior (anomalous Josephson effect) for the current-phase relation when the TS wire is in the topological phase.

As a final example for devices combining conventional and topological superconductors, we have shown that S-TS-S devices allow for a Majorana-induced parity switch between Andreev state sectors with different parity in a superconducting atomic contact. This observation could be useful for future microwave spectroscopy experiments of Andreev qubits in such contacts.

Part V

Old, Fundamental Theory

13 Macroscopic quantization and the proximity effect in SNS junctions by Kulik

Abstract

A microscopic theory of the stationary Josephson current in plane junctions of the superconductor normal metal - super conductor type is developed. It is shown that, owing to spatial quantization of single-particle excitation energies in the normal layer, the superconducting current contains a component that does not decrease exponentially at distances on the order of $\xi_0 \sim v_0/T_c$. The component oscillates with variation of the thickness of the normal metal layer. This is due to discontinuous variation of the number of levels within the gap at certain values of the thickness. The temperature dependence of the amplitude of the oscillating current component can be described by the expression $e^{-\pi d T/v_0}$ (d is the normal layer thickness and v_0 the Fermi velocity). The possibility of experimental observation of the effect is discussed.

13.0.1 1. Introduction

As shown by Andreev ^[1], the spectrum of the elementary excitations of a layer of normal metal in contact with super-conductors on both sides (S-N-S contact) is quantized at excitation energies not exceeding the energy gap of the superconductor Δ . Actually, this quantization implies the presence of a coherent connection between the phases of the ordering parameters of the superconductors adjacent to the normal metal, i.e., the existence of superconducting ordering for the entire S – N – S system as a whole. In the absence of phase coherence of Δ on both sides of the junction, the quantization no longer takes place. As will be shown below, the positions of the Andreev levels are determined by the relative phase of the gaps of two superconductors, $\chi = \chi_1 - \chi_2$, and a change of χ by 2π denotes a shift of the entire level system as a whole by an amount equal to their relative distance.

Thus, the existence of a discrete spectrum of single-particle excitations in the S – N – S junction is closely connected with the phase coherence for such a junction. On the other hand, phase coherence denotes the possible flow of superconducting currents (the Josephson effect) through the junction ^[2]. As is well known ^[3], the Josephson current is proportional to the derivative, with respect to x of the junction-energy term that depends on the relative phase χ :

$$I = \frac{2e}{\hbar} \frac{\partial E}{\partial \chi}. \quad (1.1)$$

It is clear that by virtue of the aforementioned sensitivity of the position of the quantized level to the phase, the energy of the junction will also depend essentially on χ , and this dependence remains in force also when the width of the normal layer greatly exceeds the dimension of the Cooper pair $\xi_0 \sim v_0/\Delta$. Thus, currents that do not have the usual exponential dependence on the thickness of the normal layer, of the type e^{-d/ξ_0} , can flow through the junction.

¹⁾ Instead, the effect attenuates exponentially at distances on the order of $\xi T \sim v_0/T$ (T -temperature, $T \ll T_c$) or l ,

where l is the mean free path in the normal metal, i.e., under conditions when "smearing" of the discrete levels takes place in the N layer as the result of collisions or finite temperatures. When $d \lesssim \xi T, l$, the phases of the gaps in the two superconductors are effectively coupled,

owing to the dependence of the energy of the system on $\chi_1 - \chi_2$, i.e., 'phase coherence'" due to the spatial quantization in the S - N - S junction, takes place.

In this paper we construct a microscopic theory of the proximity effect due to the spatial quantization in an impurity-free system ($l = \infty$). In Sec. 2 we find the energy levels and the wave functions of the singleparticle excitations and analyze their dependence on the phase. In Sec. 3 we investigate the current states in a spatially inhomogeneous system. An expression is obtained for the superconducting current in the form of a quadratic functional of the ordering parameter $\Delta(\mathbf{r})$. Finally, in Sec. 4 we calculate the Josephson current with the aid of the eigenfunction expansion of the singleparticle problem. It is shown that the maximum value of this current decreases exponentially with temperature like $\exp(-\pi d T / v_0)$, and oscillates as a function of the thickness of the normal layer with a period $\delta d = \pi v_0 / \Delta$ (v_0 - Fermi velocity). The latter effect is connected with the motion of the size-quantization E_n levels past the edge of the energy gap of the superconductor Δ .

13.0.2 2. Wave functions and energy levels of single-particle excitations

The quantum states of the elementary excitations of a superconductor are determined with the aid of the solution of the Bogolyubov-de Gennes equations ^[5,6] for two-component wave functions Ψ :

$$H\Psi = E\Psi, \quad \Psi = \begin{pmatrix} \Psi \\ \varphi \end{pmatrix}, \quad H = \begin{pmatrix} T & \Delta \\ \Delta^* & -T \end{pmatrix}, \quad (2.1)$$

$T = -\nabla^2/2m - \zeta$ is the kinetic energy operator and ζ is the chemical potential. For the one-dimensional problem $\Delta = \Delta(z)$, the dependence of Ψ on the coordinates can be chosen in the form

$$\Psi = \exp(iq_x x) \exp(iq_y y) \begin{pmatrix} \Psi(z) \\ \varphi(z) \end{pmatrix},$$

and then $\psi(z)$ and $\varphi(z)$ satisfy the equations

$$\begin{aligned} (T_z - \xi_q) \psi + \Delta(z) \varphi &= E \psi, \\ -(T'_z - \xi_q) \varphi + \Delta^*(z) \psi &= E \uparrow, \end{aligned} \quad (2.2)$$

and the quantities T_z and ξ_q are defined by

$$T_z = -\frac{1}{2m} \frac{d^2}{dz^2}, \quad \xi_q = \zeta - \frac{\mathbf{q}^2}{2m}, \quad \mathbf{q} = (q_x, q_b, 0).$$

The quantity ξ_q can be assumed to be positive. Indeed, if the momentum component parallel to the surface exceeds the Fermi momentum, $q > p_0$, then the wave functions of the

⁰¹⁾ A similar possibility was noted earlier by Aslamazov, Larkin, and Ovchinnikov ^[4] independently of the connection with the spatial quantization. According to ^[4], the Josephson current through an S-N-S junction can contain under certain conditions a component that does not vanish exponentially like $\exp(-d/\xi_0)$, this being connected with the conservation of the correlation of the Cooper pairs in the normal metal at distances that are large compared with ξ_0 . The case considered in this paper differs from that investigated in ^[4]. The model of ^[4] corresponds to a strong reflection of the electrons from the boundary of the N-layer ($1 - R \ll 1$), i.e., what was actually considered in ^[4] was not an S-N-S junction, but an S-I-N-I-S junction. The systematics of the quantum state in the N layer is different in this case, and particularly, the position of the discrete levels is practically independent of χ . Therefore the Josephson current changes in ^[4] monotonically with thickness, whereas in our case it oscillates with variation of d .

excitations (at energies $E \sim \Delta < \zeta\zeta$) will attenuate exponentially in the z direction and are of no significance in our problem.

Assume that the region $|z| < d/2$ is filled with normal metal, and in the regions $|z| > d/2$ we have superconductors, which for simplicity are assumed to be identical ($\Delta_1 = \Delta_2 = \Delta$, $T_{C1} = T_{C2} = T_c$). We assume that the transition temperature of the normal metal is equal to zero: $T_{cN} = 0$. If the thickness of the normal layer d is large compared with the coherence length ξ_0 , then $\Delta(z)$ is exponentially small everywhere inside the normal region, with the exception of a layer of width $\sim \xi_0$ near the interface with the superconductor, whereas in the superconductor $\Delta(z)$ varies over distances on the order of $\xi(T) \sim v_o/\sqrt{T_c(T_c - T)}$. If the temperature is low compared with the superconducting transition temperature T_c , then the change of $\Delta(z)$ in the superconductor also occurs over distances $\sim \xi_0$ near the interface. We considering this case ($d \gg \xi_0$, $T \ll T_c$) using the simplest model, in which $\Delta(z)$ is assumed to change jumpwise on the interface between the normal metal and superconductor (a similar model was considered in [7]) : $|\Delta(z)| = \Delta_0$ when $|z| > d/2$ and $|\Delta(z)| = 0$ when $|z| < d/2$. Here, however, it is important to take into account the fact that the phases of the ordering parameter in the left and right superconductors may differ: $\arg \Delta(z) = \chi_1$ when $z < -d/2$ and $\arg \Delta(z) = \chi_2$ when $z > d/2$. Thus, we assume for $\Delta(z)$ the dependence

$$\Delta(z) = \begin{cases} \Delta_0 e^{ix_1}, & z < -d/2 \\ 0, & -d/2 < z < d/2 \\ \Delta_0 e^{ix_2}, & z > d/2 \end{cases} \quad (2.3)$$

Substituting (2.3) in (2.2), we can easily find the wave functions of the excitations and the energy levels E . The energy spectrum is continuous when $E > \Delta$ and discrete when $E < \Delta_0$. The latter can be seen from the fact that states with energy inside the gap cannot exist in the superconductor. The excitations traveling from the normal metal towards the NS interface are therefore reflected if their energy (reckoned from the chemical potential ζ) is smaller than Δ_0 . As analyzed by Andreev [1], the reflection is connected with the small change of the quasimomentum $\delta p \sim p_0 \Delta_0 / \zeta$, and therefore proceeds via a transition for a state of the "particle" type ($p > p_0$) into a state of the "hole", type ($p < p_0$) and vice-versa. This can be readily traced in a model with a jumplike termination of the gap.

Let $\Delta = \Delta_0$ when $Z > 0$ and $\Delta = 0$ when $z < 0$. Writing the incident wave for $z < 0$ in the form

$$\Psi_0 = e^{ik_0 z} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \frac{k_0^2}{2m} = \xi_q + E \quad (2.4)$$

the reflected wave in the form of a superposition of excitations of the particle and hole type:

$$\Psi_1 = A e^{ik_1 z} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + B e^{-ik_0 z} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \frac{k_1^2}{2m} = \xi_q - E, \quad (2.5)$$

and making the wave functions and their derivatives continuous at $z = 0$ with the solutions of the system (2.2) at $z > 0$,

$$\Psi_2 = C e^{i\lambda_+ z} \begin{pmatrix} 1 \\ \gamma \end{pmatrix} + D e^{-i\lambda_- z} \begin{pmatrix} 1 \\ \gamma^* \end{pmatrix} \quad (2.6)$$

where

$$\lambda_{\pm}^2 = 2m \left(\xi_q \pm i \sqrt{\Delta_0^2 - E^2} \right), \quad \text{Im } \lambda_{\pm} \geq 0 \quad (2.7)$$

$$\gamma = \Delta_0 \left(E + i \sqrt{\Delta_0^2 - E^2} \right)^{-1}, \quad E < \Delta_0 \quad (2.8)$$

we obtain the values of the amplitudes A and B. A simple calculation then shows that $|A| \approx 1$ and $|B|$ is a small quantity of the order of $m\Delta_0/k_0^2 \sim \Delta_0/\zeta$. Consequently, ignoring small corrections of the order of $(\Delta_0/\zeta)^2$, we can assume $B = 0$ (and also $D = 0$). It is further necessary to take into account the fact that k_0 and k_1 are close in magnitude, so that $E \sim \Delta_0 \ll \xi_q \sim \zeta$ (k_0 and k_1 are quantities of the order of the Fermi momentum). Therefore, with the indicated accuracy, the continuity of the wave functions implies continuity of their derivatives at the point of contact.

Returning to the case of the S–N–S junction (2.3), and taking into account the statements made above concerning the character of the reflection of the excitations from the interface with the superconductor, we represent the expressions for the two-component wave functions Ψ in the form ($E < \Delta_0$)

$$\Psi_+ = \begin{cases} A_+ e^{ik_- z} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + B_+ e^{ik_1 z} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, & |z| < d/2, \\ C_+ e^{i\lambda_+(z-d/2)} \begin{pmatrix} e e^{ix_2} \\ \gamma \end{pmatrix}, & z > d/2, \\ D_+ e^{i\lambda_-(z+d/2)} \begin{pmatrix} \gamma \\ e^{-ix_1} \end{pmatrix}, & z < -d/2, \end{cases} \quad (2.9)$$

$$\Psi_- = \begin{cases} A_- e^{-ik_0 z} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + B_- e^{-ik_1 z} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, & |z| < d/2, \\ C_- e^{-i\gamma_-(z-d/2)} \begin{pmatrix} ix_2 \\ \gamma^* \end{pmatrix}, & z > d/2, \\ D_- e^{-i\lambda_+(z+d/2)} \begin{pmatrix} \gamma^* \\ e^{-ix_1} \end{pmatrix}, & z < -d/2, \end{cases} \quad (2.10)$$

where the plus sign corresponds to excitations moving from left to right, and the minus sign to excitations from right to left ($k_z > 0$ and $k_z < 0$, respectively).

The wave functions (2.9) and (2.10) attenuate exponentially inside the superconducting regions at distances $\sim \xi_0$ that are large compared with atomic distances. Indeed, on the basis of (2.7) we can write for $\lambda_{\pm}$:

$$\lambda_{\pm} \approx \sqrt{2m\xi_q} \pm iv^{-1} \sqrt{\Delta_0^2 - E^2} \quad (2.11)$$

where $v = \sqrt{2\xi_q/m}$ is the velocity of the excitation inside the normal layer in the direction normal to the surface of the junction: $v = m^{-1} \sqrt{2m\zeta - q^2}$. The characteristic values of the imaginary part of the wave vector (2.11) are $\Delta_0/v_0 \sim \xi_0^{-1}$. In analogy with (2.11), we present the expansions for k_0 and k_1 , which we shall need in the future

$$k_0 \approx \sqrt{2m\xi_q} + E/v, \quad k_1 \approx \sqrt{2m\xi_q} - E/v \quad (2.12)$$

Making the expressions for the wave functions (2.9) and (2.10) continuous at $z = d/2$ and $z = -d/2$, we obtain the relations between the coefficients $A_{\pm}$, $B_{\pm}$, $C_{\pm}$, and $D_{\pm}$:

$$\begin{aligned} C_{\pm} &= A_{\pm} e^{ik_0 d/2} e^{-ix_2} = B_{\pm} e^{\pm ik_1 d/2} \mathcal{Y}^{\mp 1}, \\ D_{\pm} &= A_{\pm} e^{\mp i h_0 d/2} \mathcal{Y}^{\mp 1} = B_{\pm} e^{\mp ik_1 d/2} e^{ix_1}. \end{aligned} \quad (2.13)$$

From this we readily obtain the dispersion equations that determine the allowed values of the energy E . For the states Ψ_+ and Ψ_- they take respectively the form

$$\gamma^2 e^{i(\beta_0 - k_1)d} e^{\pm i\chi} = 1, \quad \chi = \chi_1 - \chi_2 \quad (2.14)$$

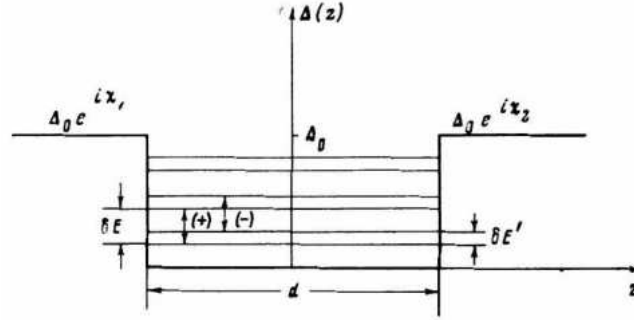
Equating the phase of the expressions in the left side of (2.15) to $2\pi n$ where $n = 0, 1, 2, \dots$ and taking (2.8) into account, we obtain for the energy levels $E_n^\pm$ the formula

$$E_{n^\pm} = (v/2d)[2(\pi n + \varphi) \mp \chi], \quad \varphi(E) = \arccos(E/\Delta_0) \cdot (2.15)$$

$\varphi(E)$ is a slow function of the energy. In particular, for states with energies $E \ll \Delta_0$, we have $\varphi \approx \pi/2$, and therefore (2.15) yields directly

$$E_{n^\pm} = (v/2d)[2\pi(n + 1/2) \mp \chi], \quad n \ll \Delta_0 d/v \quad (2.16)$$

Thus, we obtain two systems of equidistant levels (see the figure), which are degenerate in the case $\chi = 0$. It is important that the positions of the levels depend on the relative phase χ of the gaps. A change of χ by 2π corresponds to a return to the initial state. As seen from formulas (2.15) and (2.16), the distance between the levels does not depend on the phase and can be obtained from the condition of quasiclassical



Energy levels of excitations in N-layer at a fixed value of the electron velocity on the Fermi surface $v = |v_Z|$. The distance between levels is

$$\delta E = E_{n+1}^- - E_n^+ = E_{n+1}^- - E_n^- \cong \pi v/d;$$

$\delta E'$ is the splitting and is proportional to the phase difference $\chi = \chi_1 - \chi_2$:

$$\delta E' = E_n - -E_{n+} = \delta E(\chi; \pi)$$

quantization (see [8]), $\Delta E = 2\pi/T_0$ ($\hbar = 1$), where T_0 is the period of the classical motion, equal in this case to double the time of flight of the quasiparticle between the boundaries of the N layer: $T_0 = 2d/v$.²⁾

We can obtain analogously expressions for the wave functions of the continuous spectrum ($E > \Delta_0$). We present here the corresponding formulas, since they will be useful later in the calculation of the Josephson current. The states of the continuous spectrum are classified as $\Psi_\pm^0$ and $\Psi_\pm^1$, where Ψ^0 corresponds to a particle ($|k_Z| = k_0$) and Ψ^1 corresponds to a hole ($|k_Z| = k_1$), with the sign $+$ ($-$) corresponding to the case $k_z > 0$ (or $k_z < 0$). The functions $\Psi_\pm^0$ and $\Psi_\pm^1$ are analogous in form, and we therefore present an expression for only one of them where the constant is determined from the normalization condition (see below), and the quantities κ_0 , κ_1 and δ are defined as follows:

$$x_0^2 = 2m \left(\xi_q + \sqrt{E^2 - \Delta_0^2} \right), \quad x_1^2 = 2m \left(\xi_q - \sqrt{E^2 - \Delta_0^2} \right) \quad (2.18)$$

$$\delta = \Delta_0 \left(E + \sqrt{E^2 - \Delta_0^2} \right)^{-1}, \quad E > \Delta_0 \quad (2.19)$$

With quasiclassical accuracy, we can write for κ_0 and κ_1 expansions similar to (2.11) and (2.12):

$$x_0 \approx \sqrt{2m\xi_q} + v^{-1}\sqrt{E^2 - \Delta_0^2}, \quad x_1 \approx \sqrt{2m\xi_q} - v^{-1}\sqrt{E^2 - \Delta_0^2} \quad (2.20)$$

The coefficients A, B, C, and D are determined by the formulas

$$\begin{aligned} A &= \frac{e^{ik_0 d/2}}{1 - \delta^2} e^{-i\gamma_2}, \\ B &= -\frac{\delta}{1 - \delta^2} e^{ik_0 d/2} \\ C &= \frac{e^{-ik_0 d/2}}{1 - \delta^2} e^{-ix_1}, D = -\frac{\delta}{1 - \delta^2} e^{-ik_v l_2/2} \end{aligned} \quad (2.21)$$

As will be shown below, the continuous-spectrum states make their own contribution to the Josephson current, and in definite cases this contribution may even be the principal one (see Sec. 4). Nonetheless, this does not contradict the initial treatment of the effect, reported in the introduction and connected with the role of the spatial quantization. The character of the quantum states when $E > \Delta_0$ is determined essentially by the presence of discrete levels when $E < \Delta_0$. In the one-dimensional problem, the presence of a 'potential well' $\Delta(z)$ automatically implies both the occurrence of localized states with energy $E < \Delta_0$, and a change of the scattering phase shifts for the states of the continuous spectrum ($E > \Delta_0$), both effects being sensitive to the value of χ .

13.0.3 3. Superconducting currents in a spatially inhomogeneous system

We start from the well known expression for the current ^[10]

$$\mathbf{j} = \frac{ie}{m} T \sum_{\omega} [(\nabla_{\mathbf{r}'} - \nabla_{\mathbf{r}}) G_{\omega}(\mathbf{r}, \mathbf{r}')],_{\mathbf{r}' \rightarrow \mathbf{r}} \quad (3.1)$$

²⁾ This circumstance was noted in ^[9] as applied to the S-N-S junction. where $\mathbf{G}_{\omega}(\mathbf{r}, \mathbf{r}')$ are the thermodynamic Green's functions of the superconductor and satisfy, together with the functions $\mathbf{F}_{\omega}^{+}(\mathbf{r}, \mathbf{r}')$, the system of Gor'kov equations ^[10]. Taking into account the one-dimensional character of the problem, we write these equations in the form

$$\begin{aligned} (i\omega - T_z + \xi_q) G_{\omega}(z, z') + \lambda(z) F(z) F_{\omega}(z, z') &= \delta(z - z') \\ (-i\omega - T_z + \xi_q) F_{\omega}(z, z') - \lambda(z) F^{*}(z) G_{\omega}(z, z') &= 0. \end{aligned} \quad (3.2)$$

Here $\omega = (2n+1)\pi T$ are the odd frequencies, $\lambda(z)$ is the electron interaction constant, which will henceforth be assumed to be equal to $-|\lambda|$ when $|z| > d/2$ and to zero when $|z| < d/2$. $F(z)$ is the value of the Gor'kov function $F(z, z'; \tau - \tau')$ in the coinciding points ($z = z', \tau = \tau'$). The ordering parameter of the superconductor $\Delta^{*}(z)$ is defined by

$$\Delta^*(z) = |\lambda| F^*(z) \quad (3.3)$$

Introducing in (3.2) explicitly the function $U(z)$, which is equal to unity when $|z| > d/2$ and to zero when $|z| < d/2$, we rewrite these equations in the form

$$\begin{aligned} (i\omega - T_z + \xi_q) G_\omega(z, z') - \Delta(z) U(z) F_\omega + (z, z') &= \delta(z - z'), \\ (-i\omega - T_z + \xi_q) F_\omega + (z, z') + \Delta^*(z) U(z) G_\omega(z, z') &= 0 \end{aligned} \quad (3.4)$$

and by virtue of (3.3) $\Delta^*(z)$ satisfies the equation

$$\Delta^*(z) = \frac{m}{2\pi} |\lambda| T \sum_\omega \int_0^\zeta d\xi_q F_\omega^+(z, z) \quad (3.5)$$

(we shall henceforth not write out explicitly the dependence of the functions G_ω and F_ω^+ on ξ_q).

We denote by $G_\omega^0(z - z')$ the Green's function of the normal metal, satisfying the first equation of the system (3.4) with $\Delta = 0$. Eliminating F_ω^+ from (3.4), we can obtain the following closed integral equation for the function G_ω :

$$\begin{aligned} G_\omega(z, z') &= G_\omega^0(z - z') - \int_{-\infty}^\infty dz_1 \int_{-\infty}^\infty dz_2 G_\omega^0(z - z_1) \Delta(z_1) U(z_1) \\ &\quad \times G_{-\omega^0}(z_1 - z_2) \Delta^*(z_2) U(z_2) G_\omega(z_2, z'). \end{aligned} \quad (3.6)$$

Similarly, the equation for the ordering parameter (3.5) is rewritten in the form

$$\Delta^*(z) = \frac{m}{2\pi} |\lambda| T \sum_\omega \int_0^\zeta d\xi_q \int_{-\infty}^\infty dz' G_{-\omega^0}(z - z') \Delta^*(z') U(z') G_\omega(z', z) \quad (3.7)$$

We substitute (3.6) in (3.1) and calculate the superconducting current j . Naturally, for the problem of interest to us we should consider only the z -th component of the current: $j = j_z$ ($j_x = j_y = 0$), and by virtue of the continuity equation j_z does not depend on z . Therefore the current can be calculated at any point of the superconductor. From symmetry considerations it is clear that it is most convenient to carry out the calculation of the current at $z = 0$. Recognizing that in the normal state the current is equal to zero, we obtain

$$\begin{aligned} j &= -\frac{ie}{2\pi} T \sum_\omega \int_0^\zeta d\xi_q \int_{-\infty}^t dz_1 \int_{-\infty}^\infty dz_2 \left[\left(\frac{\partial}{\partial z'} - \frac{\partial}{\partial z} \right) G_\omega^0(z - z_1) \Delta(z_1) U(z_1) \right. \\ &\quad \left. \times G_{-\omega^0}(z_1 - z_2) \Delta^*(z_2) U(z_2) G_\omega(z_2, z') \right] z = z' = 0. \end{aligned} \quad (3.8)$$

We now assume for the time being that the function G_ω has been expanded in a series in powers of Δ . Such a series can be readily obtained by iterating Eq. (3.6):

$$\begin{aligned} G_\omega(z, z') &= G_\omega^0(z - z') - \int_{-\infty}^\infty dz_1 \int_{-\infty}^\infty dz_2 G_\omega^0(z - z_1) \Delta(z_1) U(z_1) \\ &\quad \times G_{-\omega'}(z_1 - z_2) \Delta^*(z_2) U(z_2) G_\omega(z_2, z') + \dots \end{aligned} \quad (3.9)$$

Substituting this expansion in (3.8) and again summing the terms flanked by the expressions $G_{\omega}^0(z - z_1) \Delta(z_1) U(z_1)$ and $\Delta^*(z_n) U(z_n) G_{\omega}^0(z_n - z')$, we can verify that (3.8) can be represented identically in the form

$$j = -\frac{ie}{2\pi} T \sum_{\omega} \int_0^{\zeta} d\xi_q \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 U(z_1) \Delta(z_1) U(z_2) \Delta^*(z_2) \times \left[\left(\frac{\partial}{\partial z'} - \frac{\partial}{\partial z} \right) G_{\omega^0}(z - z_1) G_{\omega^0}(z_2 - z') \right]_{z=z'=0} G_{-\omega}(z_2, z_1). \quad (3.10)$$

The expression in the square brackets, containing the Green's function of the normal metal $G_{\omega}^0(z - z')$, can be readily calculated on the basis of the explicit form of these functions:

$$G_{\omega\omega}^0(z - z') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ip(z-z')}}{i(1) + \xi_q - p^2/2m} dp = -\frac{m}{i\lambda_{\omega}} \exp\left(-i\hat{\lambda}_{\omega}|z - z'|\right), \quad (3.11)$$

where $\lambda_{\omega} = [2m(\xi_q + i\omega)]^{1/2}$, and the square root sign is chosen from the condition $\text{Im } \lambda_{\omega} < 0$. Since $\omega \sim T_c \ll \zeta$, we obtain

$$\lambda_{\omega} \approx -(2m\xi_q)^{1/2} \text{sign } \omega - i|\omega|/v \quad (3.12)$$

Using formula (3.11), we arrive finally at the following expression for the current:

$$j = -\frac{iem}{2\pi} T \sum_{\omega} \int_G^{\delta} d\xi_{\xi} \int_{-\infty}^{\infty} dz_1 \int_{-\infty}^{\infty} dz_2 (\text{sign } z_1 - \text{sign } z_2) U(z_1) U(z_2) \times \Delta(z_1) \Delta^*(z_2) G_{\infty}'(z_1 - z_2) G_{-\omega}(z_2, z_1). \quad (3.13)$$

Formula (3.13) was obtained earlier in connection with a microscopic calculation of the Josephson current for S-I-S junctions ^[11] (see also the review ^[12]).

The expression (3.13) is general. Thus, for example, it can be verified that for a homogeneous current state, $\Delta(z) = \Delta_0 e^{ikz}$ (in the absence of the normal metal layer), it leads to the known dependence of the superconducting current j on the parameter of "superfluid velocity" $v_s = \hbar k/2m$. The general scheme of calculating the superconducting current in an inhomogeneous system consists of specifying a current state $\Delta = \Delta(z)$ compatible with Eq. (3.7). The Green's function $G_{\omega}(z, z')$ is then determined from (3.6) and must then be substituted in (3.13).

In the case of an S - N - S junction, the current state is determined by specifying the phases χ_1 and χ_2 in accordance with (2.3)³. Substituting (2.3) in (3.13), we obtain

$$j = \frac{iem}{2\pi} \Delta_0^2 T \sum_{\omega} \int_0^{\zeta} d\xi_q \left[e^{i\chi} \int_{-\infty}^{-d/2} dz_1 \int_{d/2}^{\infty} dz_2 - e^{-i\chi} \int_{d/2}^{\infty} dz_1 \int_{-\infty}^{-d/2} dz_2 \right] \times G_{\omega}^0(z_1 - z_2) G_{-\omega}(z_2, z_1). \quad (3.14)$$

Even from this expression we see the general

character of the dependence of the current on the junction thickness d . The distance between the points z_1 and z_2 is at least equal to d . Therefore the potential term in the current is due to the product

⁰³⁾ Such an approximation is valid if the critical current of the junction is small compared with the "volume" critical current $j_{cm} \sim Ne v_c, v_c \sim \Delta_0/p_0$. In this case we can neglect the change of the phase $\chi(z)$ inside the superconductor, and consider only the jump of χ in the contact region.

$$G_{\omega}^0(d)G_{-\omega}(-d/2, d/2) \sim G_{-\omega^0}(d)G_{-\omega}^0(d)$$

and has, in accordance with (3.11) and (3.12), an asymptotic form

$$\exp(-2|\omega|d/v_0) \sim \exp(-2\pi dT/v_0)$$

(v_0 - Fermi velocity). Indeed, inside the normal metal G_{ω} has the same variation as G_{ω}^0 . Thus, at low temperatures the current does not contain the small parameter

$$\exp(-d/\xi_0) \sim \exp(-dT_c/v_0)$$

which is customarily encountered for proximity effects. As already noted, this is connected with the fact that in a pure metal at low temperatures the pairs conserve their correlation at distances that are large compared with ξ_0 .^[4] When $T \rightarrow T_c$, the current is proportional to

$$\exp(-2\pi dT_c/v_0) \sim \exp(-d/\xi_0)$$

Since in this case $\Delta(z)$ changes over distances $\sim \xi(T) \gg \xi_0$, and the remaining terms in the integrand of (3.14) change over distances of the order of ξ_0 , we can put $\Delta(z) \approx \Delta(\pm d/2)$ and take Δ outside the integral sign⁴. On the other hand, the value of the ordering parameter on the boundary with the normal metal is much smaller than its value far from the boundary, in a ratio $\Delta(d/2)/\Delta(\infty) \sim \xi_0/\xi(T)$ (see [6]). As the result, it turns out, as first noted by de Gennes [6] (see also [13]), that near T_c the Josephson current of an S-N-S junction is proportional to $[\Delta(\infty)\xi_0/\xi(T)]^2 \sim (T_c - T)^2$, i.e., to the second power of the difference ($T_c - T$), and not to the first as in the case of the Josephson effect through a dielectric gap (S-I-S junction).

13.0.4 4. Calculation of the Josephson current

The connection between the Gor'kov equations (3.4) and the Bogolyubov-de Gennes equations (2.2) is given with the aid of the following relations:

$$G_{\omega}(z, z') = \sum_{\alpha} \left[\frac{\psi_{\alpha}(z)\psi_{\alpha}^*(z')}{i\omega - E'_{\alpha}} + \frac{\varphi_{\alpha}^*(z)\varphi_{\alpha}(z')}{i\omega + E_{\alpha}} \right] \quad (4.1)$$

$$F_{\omega+}(z, z') = \sum_{\alpha} \left[\frac{\varphi_{\alpha}(z)\psi_{\alpha}^*(z')}{i\omega - E_{\alpha}} - \frac{\psi_{\alpha}^*(z)\varphi_{\alpha}(z')}{i\omega + E_{\alpha}} \right]. \quad (4.2)$$

The summation here is over the eigenstates of the single-particle problem Ψ_{α} , corresponding to positive eigenvalues E_{α} . The functions ψ_{α} and φ_{α} are assumed to be normalized in accordance with the conditions

$$\int_{-\infty}^{\infty} (|\psi_{\alpha}(z)|^2 + |\varphi_{\alpha}(z)|^2) dz = 1 \quad (4.3)$$

The formulas (4.1) and (4.2) are obtained on the basis of the completeness and orthogonality conditions

in the linear space of the functions Ψ_{α} . It must be recognized here that if

$$\Psi_{\alpha} = \begin{pmatrix} \psi_{\alpha} \\ \varphi_{\alpha} \end{pmatrix}$$

⁰⁴) We do not take into account the changes of $\Delta(z)$ near the surface of the NS junction at a distance $\sim \xi_0$, since these changes lead only a change of the numerical coefficients in the expression for j. This is analogous to specifying the gap in the form of a step function (2.3) in the case of low temperatures.

corresponds to an energy $E_\alpha > 0$, then a negative energy $-E_\alpha$ corresponds to the wave function

$$\tilde{\Psi}_\alpha = \begin{pmatrix} \varphi_\alpha^* \\ -\psi_\alpha^* \end{pmatrix}$$

This is seen directly from the equations satisfied by ψ_α and φ_α :

$$\begin{aligned} (T_z - \xi_q) \psi_\alpha + \Delta(z) U(z) \varphi_\alpha &= E_\alpha \psi_\alpha \\ -(T_z - \xi_q) \varphi_\alpha + \Delta^*(z) U(z) \psi_\alpha &= E_\alpha \varphi_\alpha \end{aligned} \quad (4.4)$$

Substituting the eigenfunction expansion (4.1) in the formula (3.13) for the Josephson current and using the explicit expression for the Green's function of the normal metal G_ω^0 (3.11), we reduce j to the form

$$j = \frac{2em^2}{\pi} \Delta_0^2 \text{Re} \int_0^\zeta d\xi_q T \sum_\omega \frac{1}{\lambda} \sum_\omega \left(\frac{\psi_{\alpha\omega+} + \psi_{\alpha\omega-}}{i\omega + E_\alpha} + \frac{\varphi_{\alpha\omega} + \varphi_{\alpha\omega-}}{i\omega - E_\alpha} \right) e^{i\chi}, \quad (4.5)$$

where

$$\begin{aligned} \psi_{\alpha\omega+} &= \int_{d/2}^\infty \psi_\alpha(z) e^{-i\lambda_{\omega^2} z} dz, \psi_{\alpha\omega-} = \int_{-\infty}^{-d/2} \psi_\alpha^*(z) e^{i\lambda_{\omega^2} z} dz \\ \varphi_{\alpha\omega+} &= \int_{d/2}^\infty \varphi_{\alpha\alpha^*}(z) e^{-i\lambda_{\omega^2} z} dz, \end{aligned} \quad (4.6)$$

$$\varphi_{\alpha\omega-} = \int_{-\infty}^{-d/2} \varphi_\alpha(z) e^{i\lambda_{\omega^2} z} dz.$$

Thus, the calculation of the Josephson current is reduced to a sum over the states of the single-particle Hamiltonian (4.4).

The remainder of the calculation, using the explicit form of the wave functions ψ_α and φ_α obtained in Sec. 2, is trivial. Leaving out this calculation, we present directly the final result. The total current is divided into two terms, $j = j_1 + j_2$, where j_1 corresponds to summation over the states of the discrete spectrum ($E < \Delta_0$), and j_2 corresponds to summation over the continuous spectrum ($E > \Delta_0$). When calculating the last term (j_2), we must replace the sum over α ($E > \Delta_0$), with allowance for the known formula for the state density in the BCS model, by the integral

$$\sum_\alpha (\dots) \rightarrow \int (\dots) \frac{L dk_z}{2\pi} \rightarrow \frac{mL}{2\pi \sqrt{2m\xi_q}} \int_{\Delta_0}^\infty (\dots) \frac{E}{\sqrt{E^2 - \Delta_0^2}} dE$$

(the wave functions of the continuous spectrum are assumed to be normalized in the segment $(-L/2, L/2)$, where $L \gg d$; naturally, the normalization length L drops out in the answer as $L \rightarrow \infty$).

The expression for j_1 takes the form

$$j_1 = \frac{2em^2}{\pi d} \Delta_0^2 T \sum_{\omega>0} \int_0^{v_0} v^2 dv e^{-\omega a/v} \quad (4.7)$$

$$\times \text{Re} \sum_n \left\{ \frac{\exp\{i(dE_n + v + \chi)\}}{(\omega + \sqrt{\Delta_0^2 - E_n^2})^2 (E_n - i\omega)} - \frac{\exp\{i(dE_n - 1v - \chi)\}}{(\omega + \sqrt{\Delta_0^2 - E_n^2})^2 (E_n - i\omega)} \right\} \text{ and for } j_2 \text{ we get accordingly}$$

$$j_2 = -\frac{4em^2}{\pi^2} \Delta_0^2 T \sum_{\omega>0} \int_0^{v_0} v dv e^{-\omega d/v} \text{Re} \int_{\Delta_0}^\infty dE \frac{\sqrt{E^2 - \Delta_0^2}}{E} \frac{(\omega + iE) e^{i d E/v}}{(\omega^2 + E^2 - \Delta_0^2)^2} \sin \chi.$$

These formulas are valid if $d \gg \xi_0$ and $T \ll T_c$, when the 'square well', approximation (2.3) can be used. It is interesting to note that the phase dependence of the part of the current corresponding to summation over the continuous spectrum has the usual form $j_2 = j_s \sin \chi$, which is typical of the Josephson effect in S-I-S junctions [23,12], whereas the current component corresponding to summation over the discrete spectrum depends on χ in a much more complicated form (it must be recognized that $E_n^\pm$ in (4.7) are functions of χ). It is clear therefore that the fact that in the Josephson effect we always have a phase dependence of the type $\sin \chi$ is connected with the absence of localized states in the problem of the passage of the particle through a potential barrier (such states can arise, of course, if the barrier has a complicated form, but it is necessary that the distance between the corresponding discrete levels be of the order of Δ , and that the levels themselves depend on the phase χ).

We proceed to investigate the obtained expressions for the current (4.7) and (4.8). The sums over the discrete levels, which enter in (4.7), can be transformed by using the known Poisson summation formula

$$\sum_{n=n_1}^{n_2} f(n) = \sum_{p=-\infty}^{\infty} \int_{n_1-\alpha_1}^{n_2+\alpha_2} f(x) e^{2\pi i p x} dx, \quad 0 < \alpha_{1,2} < 1 \quad (4.9)$$

As a result, j_1 is represented in the form of the series

$$j_1 = \sum_{p=1}^{\infty} I_p \sin p\chi, \quad (4.10)$$

where

$$I_p = \frac{8em^2}{\pi^2} \Delta_0^2 T \sum_{\omega > 0} \int_0^{v_\sigma} v dv e^{-\omega d/v} \times \operatorname{Re} \int_0^{\Delta_0} dE \frac{e^{-iEd/v} e^{2i\varphi(E)}}{\left(\omega + \sqrt{\Delta_0^2 - E^2}\right)^2 (E - i\omega)} \sin \left[2p \left(\varphi(E) - \frac{Ed}{v} \right) \right], \quad (4.11)$$

and the function $\varphi(E)$ is obtained from (2.15).

We shall assume that the thickness of the normal layer d is large not only compared with ξ_0 , but also compared with the parameter ξ_T , which is determined by the relation

$$\xi_T = v_0 / \pi T \quad (4.12)$$

If $d \gg \xi_T$, then we can retain in the sum over the frequencies $\omega = (2\nu + 1)\pi T$ only the lowest term $\omega = \omega_0 = \pi T$, corresponding to $\nu = 0$. Further, under the same condition, the integration with respect to v in (4.8) and (4.11) is carried out effectively over the region $v_0 - v \sim v_0 d / \xi_T \ll v_0$, leading to expressions of the form $(\omega_0 - imE)^{-1} \exp[-d(\omega_0 - imE)/v_0]$. As the result, only the single integral with respect to the energy remains in the formulas for j_1 and j_2 , and it can be calculated asymptotically with allowance for the condition $\omega_0 \ll \Delta_0$ ($T \ll T_c$).

Turning to formula (4.11), we see that the integrand has sharp maxima at energy values $E = 0$ ($E \sim \omega_0$) and $E = \Delta_0$ ($\Delta_0 - E \sim \omega_0^2 / \Delta_0$). Accordingly, the coefficients I_p and part of the current j_1 break up into two terms: $I_p = I_p^{(0)} + I_p^{(\Delta)}$ and $j_1 = j_1^{(0)} + j_1(\Delta)$, corresponding to integration near the points $E = 0$ and $E = \Delta_0$, respectively. In calculating $j_1^{(0)}$ it is more convenient to use (4.7). After simple manipulations we obtain

$$j_1^{(0)} = \frac{6 Nev_0}{\pi \zeta} T e^{-d \Sigma_s F(\chi)} \cos \frac{\chi}{2}, \quad (4.13)$$

where the function $F(\chi)$ has the following meaning:

$$F(\chi) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{v^2 + (n + 1/2 - \chi/2\pi)^2} \quad v = \frac{d}{\pi\xi_T}. \quad (4.14)$$

The last expression can be rewritten identically in the form

$$F(x) = -\frac{\pi i}{v} \sum_{s=-\infty}^{\infty} (-1)^s e^{i\chi(s+1/2)} e^{-2\pi v|s+1/2|} \quad (4.15)$$

from which we see that when $\nu \gg 1$ we have asymptotically

$$F(x) \approx \frac{2\pi}{v} e^{-\pi v} \sin \frac{\chi}{2} \quad (4.16)$$

We finally obtain for the part of the current $j_1^{(0)}$ the expression

$$\dot{i}_1^{(0)} \approx 6 (Nev_0/p_0d) e^{-2d/\xi_T} \sin \chi, \quad v_0/d \ll T \ll \Delta_0 \quad (4.17)$$

The exponential factor in this formula $\exp(-2d/\xi_T) = \exp(-2\pi^2 T/\delta E)$ ($\delta E = \pi v_0/d$ is the distance between levels) is exactly the same as in the theory of quantum oscillation effects [14,8], i.e., it represents the usual temperature quenching of quantum oscillations. This confirms once more our treatment assumption that the investigated effect is connected with quantization of the excitation spectrum in the N layer.

The term $j_1(\Delta)$ is a sum of the type (4.10), where $I_p^{(\Delta)}$ is estimated by

$$I_p^{(\Delta)} \approx -12 \frac{Nev_0}{p_0d} e^{-d/\xi_T} \left[\frac{Q_{2p-1}}{2p-1} \exp \left\{ i(2p-1) \frac{\Delta_0 d}{v_0} \right\} + \frac{Q_{-2p-1}}{2p+1} \exp \left\{ -i(2p+1) \frac{\Delta_0 d}{v_0} \right\} \right], \quad (4.18)$$

where

$$Q_v = T \int_0^\infty \frac{e^{-iv\epsilon d/v_0}}{(\omega_0 + \sqrt{2\Delta_0\epsilon})^2} d\epsilon. \quad (4.19)$$

For the coefficients Q_ν we obtain the following asymptotic expressions ($T_1 \ll T \ll T_c$)

$$Q_v \approx v_0/\pi i v \omega_0 d, \quad T \gg T_2, \\ Q_v \approx -\frac{\omega_0}{2\pi\Delta_0} \left[C + 2 + \ln \frac{\omega_0^2 d |v|}{2v_0\Delta_0} + \frac{\pi i}{2} \text{sign } v \right], \quad T \ll T_2, \quad (4.20)$$

where we have introduced the characteristic temperatures $T_1 \sim v_0/d$ and $T_2 \sim (v_0\Delta_0/d)^{1/2}$, with $T_1 \ll T_2 \ll T_c$ and $C = 0.577$ the Euler constant. The term j_2 in the current is calculated in similar fashion. Since the integration begins here with $E = \Delta_0$ and is carried out effectively over the region $E - \Delta_0 \sim \omega_0^2/\Delta_0 \ll \omega_0$, this term will also contain an oscillating factor $\exp(i d \Delta_0/v_0)$. In the limiting cases we obtain the following formulas

$$j_2 \approx A \cos \frac{\Delta_0 d}{v_0}, \quad T_1 \ll T \ll T_2 \ll T_c, \quad (4.21) \\ j_2 \approx A \pi^{-T_2} \left(\frac{2v_0\Delta_0}{dT^2} \right)^{3/2} \cos \left(\frac{\Delta_0 d}{v_0} + \frac{3\pi}{4} \right), \quad T_1 \ll T_2 \ll T \ll T_c,$$

where

$$A = (3Nev_0/p_0d) e^{-d/s_T} \sin \chi$$

Gathering the results together, we obtain the total junction current j . In the limit $d \gg \xi_T$, we can omit the term $j_1^{(0)}$ (formula (4.17)), since it is proportional to $\exp(-2d/\xi_T)$ and not to $\exp(-d/\xi_T)$ (compare with (4.21)). The remaining terms oscillate with the thickness of the normal metal d . Considering for simplicity the first limiting case in (4.21) ($T_1 \ll T \ll T_2$), we obtain for the critical current of the contact the estimate

$$j_c \sim (3Nev_0/p_0d) e^{-\pi d T/v_0} |\cos(\Delta_0 d/v_0)| \quad (4.22)$$

The period of the oscillations with respect to thickness is $\delta d = \pi v_0/\Delta_0$. The physical reason for the oscillating $j_c(d)$ dependence is the change of the distance between levels with increasing thickness of the normal layer d . The number of levels "accommodated" in a well of height Δ_0 then increases. At definite values of d , a jumplike change takes place in the number of levels, and this leads to oscillations of the energy, and consequently also of the Josephson current, with changing thickness. Thus, the described effect is due to the motion of the size-quantization levels past the edge of the energy gap of the superconductor Δ_0 . We note that in principle the current should oscillate also with temperature (owing to the temperature dependence of Δ_0), but in the region of temperatures $T \ll T_c$ the $\Delta(T)$ dependence is very weak.

As seen from (4.22), the ratio of the critical current of the junction j_c to the "volume" critical current $j_{cm} \sim Nev_0\Delta_0/\zeta$ (see footnote ³⁾) is a quantity of the order of $(\zeta_0/d) \exp(-d/\xi_T) \lesssim \xi_0/d \ll 1$, thereby ensuring self-consistency of the calculation (the possibility of neglecting the gradient of the phase inside the superconductors compared with its jump on the surface)⁵⁾. Just as in the case of the ordinary Josephson effect^[12], the magnetic field produced by the flowing currents can be neglected if $L \ll \lambda_J$, where L is the width of the junction and λ_J is the Josephson depth of penetration, $\lambda_J = (\hbar c^2/8\pi e \Delta j_c)^{1/2}$; the length $\Lambda = 2\lambda L + d$ is determined in this case mainly by the thickness of the layer of the normal metal d . When this condition is satisfied, the problem becomes one-dimensional.

Let us make a few concluding remarks. Although in this paper we have used the model of a rectangular gap $\Delta(z)$ (2.3), it is clear in principle that allowance for a real (continuous) variation of Δ near the boundary with the normal metal does not change the nature of the considered effect, if the width of the "well" d is large compared with the region ξ_0 where its edge is diffuse. Moreover, all the formulas of Sec. 3 actually contain not the gap $\Delta(z)$ itself, but the quantity $\Delta(z)U(z)$, which vanishes identically when $|z| < d/2$, so that in reality the presence of the jump is not connected with the chosen model (allowance for the real situation changes only the magnitude of this jump). The physics of the effect under consideration does not depend on the exact form of the boundary conditions on the surface. In particular, the fact of quantization of the energy of the excitations in the normal layer and of the jumplike change of the number of levels with thickness is not connected with the concrete form of $\Delta(z)$.

Experimental observation of the discussed effect calls for the construction of S – N – S junctions containing a flat defect-free normal layer, smoothly going over into the superconducting metal (without voids or extraneous inclusions). No strong reflection of the electrons must take place on the boundary of such a layer,

for otherwise the position of the Andreev levels becomes practically insensitive to the phase. In the foregoing calculations we assumed specular boundary conditions in the surface, but

⁰⁵⁾ It should be noted that both the expression for j_{cm} and the formula (4.22) for j_c are valid in the case when the total current is sufficiently small, so that the destruction of the superconductivity is the result of unpairing processes, and not as a result of the Meissner effect. This means that we are considering sufficiently thin films or narrow contacts having a small cross section.

actually by virtue of the specific nature of the reflection of the excitations from the NS interface the "specularity" (at low energies) is retained also if the surface is not even ^[9]. It is known that quantization appears experimentally in the so-called Tomasch effect ^[15,16], namely oscillations of the tunnel current as a function of the voltage ($\delta V = \delta E/e$). Apparently, the greatest difficulty is the determination of the transition into the superconducting phase under conditions when the normal resistance is itself extremely small (since the normal layer should be pure). In addition, sufficiently low temperatures are necessary (at large thicknesses). Thus to observe superconducting currents through a normal layer of thickness $d \sim 10^2 \xi_0$ it is necessary to have a temperature that is lower than T_c by at least two orders of magnitude.

14 Quantum point contacts in magnetic fields by Ando

Abstract

A method is described for the numerical calculations of the ballistic conductance in mesoscopic structures in magnetic fields within a lattice model. Examples of calculated results are given for the conductance across a single quantum point contact and a series of two quantum point contacts. The conductance is quantized even when the confinement is not adiabatic and interchannel scatterings are still appreciable. The quantization becomes better in magnetic fields. In a series of two quantum point contacts, the magnetic field dependence of the conductance is strongly dependent on the channel number of both emitter and collector contacts.

14.0.1 I. INTRODUCTION

Recent developments in microfabrication technology have made it possible to obtain quantum ballistic structures by introducing a confining potential in a twodimensional electron system, typically by lithography techniques on modulation-doped GaAs/Al_xGa_{1-x}As heterostructures. Various phenomena associated with the ballistic conduction have been observed at low temperatures. One of the most striking effects is the quantization of the conductance across a narrow constriction (quantum point contact). ^{1,2} Transport phenomena associated with wire junctions have also been observed, such as anomalies in the low-field Hall effect, ³⁻⁵ a bend resistance, ⁶ and a negative resistance. ⁷ The purpose of the present paper is to describe a numerical method of calculating the conductance in the ballistic regime in magnetic fields and present results for a single and a series of two point contacts.

The conductance quantization has been observed at point contacts made by a split-gate technique ^{1,2} and later also at constrictions made by focused ion beam scanning. ⁸ Effects of magnetic fields have been studied and shown to enhance the quantization. ⁹ Effects observed in combinations of several point contacts include nonadditivity of the conductance, ¹⁰⁻¹² zero-dimensional states, ^{13,14} and focusing effects. ¹⁵⁻¹⁷ Quite recently, Okada et al. ¹⁸ measured the conductance across a series of two point contacts as a function of magnetic field and found an interesting dependence on the channel number of the source contact. They have observed a single peak as a function of the magnetic field for the case of a single channel and splitting into two peaks when the channel number is 2.

The quantization of the conductance into integer multiples of $e^2/\pi\hbar$ can be explained easily in an adiabatic, i.e., slowly varying, geometry. However, the actual systems may not completely be in such a limit. Various estimates of the corrections to the adiabatic limit have been reported. ¹⁹⁻²² Calculations have also been done based on an exactly soluble model ²³ and numerically for model constrictions abruptly connected to wide regions. ²⁴⁻²⁹ These calculations have demonstrated the sensitivity of the conductance to the

form of the confinement. Effects of magnetic fields have not been studied in detail except for some analytic considerations in the adiabatic case.^{30,31}

There are various ways of calculating conductance in ballistic structures. Exact analytic solutions and their combinations²³ are possible in some cases but their applications are limited to special kinds of boundaries. A direct numerical integration of the Schrödinger equation is another candidate. The simplest way is to use a lattice model, i.e., a discretization of space. It is particularly convenient in the presence of a magnetic field which is incorporated simply in the form of a Peierls phase factor of the transfer integral. Further, it can straightforwardly be extended to the study of effects of scattering from impurities. It has been used, for example, in calculating ballistic transport associated with junctions of quantum wires.^{32,33}

In this paper, we introduce a confining potential in a square lattice and calculate the conductance across a single point contact and a series of two point contacts in magnetic fields. In Sec. II, the procedure to calculate the conductance of ballistic structures in the lattice model is briefly described. Although the method itself is well established in the absence of a magnetic field, there are some nontrivial aspects especially with respect to scattering problems in magnetic fields. This method is based on three essential ingredients: Landauer's conductance formula, the relation of transmission coefficients to the Green function, and a recursive calculation of the Green function. In Sec. III the conductance is explicitly calculated in a model quantum point contact in magnetic fields both in the absence and presence of scatterers. In Sec. IV numerical calculations are performed on a series of two point contacts. It will be demonstrated that the magnetic-field dependence of the conductivity is strongly modified by the channel number of the emitter and collector point contacts, i.e., the channel wave functions are directly reflected in the conductance. The summary and conclusion are given in Sec. V.

14.0.2 II. CONDUCTANCE CALCULATION IN A LATTICE MODEL

14.0.3 A. Hamiltonian

We consider a square lattice with lattice constant a and an isotropic nearest-neighbor transfer integral - t . The lattice point is specified as (l, j) . A magnetic field H is included in terms of a Peierls phase factor of the transfer integral. We choose the gauge such that

$$(l, j|\mathcal{H}|l+1, j) = -t \exp\left(2\pi i \tilde{H}j\right) \quad (l, j|\mathcal{H}|l, j+1) = -t \quad (2.1)$$

where $\tilde{H} = \Phi/\Phi_0$ with $\Phi = Ha^2$ being the magnetic flux passing through a unit cell and $\Phi_0 = ch/e$ being the magnetic flux quantum. The diagonal element of the Hamiltonian is chosen to be

$$(l, j|\mathcal{H}|l, j) = 4t + v(l, j) \quad (2.2)$$

where the first term on the right-hand side shifts the energy origin to the bottom of the band and the second term represents the potential energy including that of the confinement.

In the following, we measure almost all quantities in units of corresponding quantities in two dimensions, i.e., energy is measured in units of the Fermi energy E_F and length in units of the Fermi wavelength λ_F . The resonance energy appearing in the lattice model is obtained as

$$t = E_F \left(\frac{1}{2\pi}\right)^2 \left(\frac{\lambda_F}{a}\right)^2 \quad (2.3)$$

The magnetic field is given by

$$\tilde{H} = \frac{a^2}{R_c \lambda_F} = \pi \frac{\hbar \omega_c}{E_F} \left(\frac{a}{\lambda_F} \right)^2 \quad (2.4)$$

with $\omega_c = eH/mc$ the cyclotron frequency and $R_c = v_F/\omega_c$ the classical cyclotron radius, where v_F is the Fermi velocity.

Effects of scattering are introduced through randomness of site energy $v(l, j)$ distributed uniformly with width $W[v(l, j) \rightarrow v(l, j) + w(l, j)]$ with $-W/2 < w(l, j) < W/2$. The model corresponds to a system containing a high concentration of scatterers with δ potential. In terms of the mean free path Λ in two dimensions, we have

$$\frac{W}{E_F} = \left(\frac{6\lambda_F^3}{\pi^3 a^2 \Lambda} \right)^{1/2} \quad (2.5)$$

14.0.4 B. Ideal wire

First we shall consider a wire infinitely long in the x direction and consisting of M lattice sites in the y direction. For $a \ll \lambda_F$, this simulates a continuum system with width $(M+1)a$. The equation of motion can be written as

$$(E - \mathcal{H}_0) \mathbf{C}_j + tP\mathbf{C}_{j-1} + tP^*\mathbf{C}_{j+1} = \mathbf{0} \quad (2.6)$$

where $\mathbf{C}_j$ is a vector describing the amplitudes of the j th cell consisting of M sites and P is an $M \times M$ matrix consisting of phase factors:

$$P_{ll'} = \exp(2\pi i \tilde{H} l) \delta_{ll'} \quad (l, l' = 1, \dots, M) \quad (2.7)$$

The Hamiltonian $\mathcal{H}_0$ is an $M \times M$ matrix given by

$$\mathcal{H}_0 = \begin{pmatrix} v_1 + 4t & -t & 0 & \cdots & 0 \\ -t & v_2 + 4t & -t & \cdots & 0 \\ 0 & -t & v_3 + 4t & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v_M + 4t \end{pmatrix},$$

with v_l being the local potential energy at the site l .

To obtain linearly independent solutions for Eq. (2.6) we first set

$$\mathbf{C}_j = \lambda^j \mathbf{C}_0 \quad (2.9)$$

Substituting this into Eq. (2.6) we have

$$\lambda \mathbf{C}_j = t^{-1} P (\mathcal{H}_0 - E) \mathbf{C}_j - P^2 \mathbf{C}_{j-1} \quad (2.10)$$

which combined with

$$\lambda \mathbf{C}_{j-1} = \mathbf{C}_j \quad (2.11)$$

leads to the following eigenvalue problem:

$$\lambda \begin{pmatrix} \mathbf{C}_j \\ \mathbf{C}_{j-1} \end{pmatrix} = \begin{bmatrix} t^{-1} P (\mathcal{H}_0 - E) & -P^2 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \mathbf{C}_j \\ \mathbf{C}_{j-1} \end{pmatrix} \quad (2.12)$$

The wave number k is related to the eigenvalue λ through

$$\exp[ika + i\pi \tilde{H}(M+1)] = \lambda \quad (2.13)$$

This equation has $2M$ eigenvalues and $2M$ eigenvectors, which are classified into M right- and left-going waves. The M right-going solutions consist of traveling waves with velocity in the positive x direction and evanescent waves decaying exponentially in the positive x direction. Similarly, the M left-going solutions consist of traveling waves with velocity in the negative x direction and evanescent waves decaying exponentially in the negative x direction. In general, any solutions of Eq. (2.6) can be represented by a linear combination of these eigensolutions.

Let $\mathbf{u}_1(-), \dots, \mathbf{u}_M(-)$ be $\mathbf{C}_0$ of the left-going solutions corresponding to $\lambda_1(-), \dots, \lambda_M(-)$ and $\mathbf{u}_1(+), \dots, \mathbf{u}_M(+)$ be $\mathbf{C}_0$ of the right-going solutions corresponding to $\lambda_1(+), \dots, \lambda_M(+)$. Define

$$U(\pm) = (\mathbf{u}_1(\pm) \cdots \mathbf{u}_M(\pm)) \quad (2.14)$$

and

$$\Lambda(\pm) = \begin{bmatrix} \lambda_1(\pm) & & \\ & \ddots & \\ & & \lambda_M(\pm) \end{bmatrix} \quad (2.15)$$

Any left- and right-going waves are written, at $j = 0$, for example, as

$$\mathbf{C}_0(\pm) = U(\pm)\mathbf{C}(\pm) \quad (2.16)$$

where $\mathbf{C}(\pm)$ is an appropriate vector consisting of expansion coefficients. For general j we have

$$\mathbf{C}_j(\pm) = U(\pm)\Lambda(\pm)^j\mathbf{C}(\pm) \quad (2.17)$$

which leads to the relation

$$\mathbf{C}_j(\pm) = F(\pm)^{j-j'}\mathbf{C}_{j'}(\pm), \quad (2.18)$$

with

$$F(\pm) = U(\pm)\Lambda(\pm)U^{-1}(\pm). \quad (2.1)$$

Note that $U(\pm)$ is in general not a unitary matrix.

14.0.5 C. Scattering problem

Now let us consider the scattering problem for the wire with length N (from cell 1 to cell N) to both sides of which an ideal wire is attached. First, we separate the amplitude $\mathbf{C}_0$ at cell 0 into the right-going (incident) and left-going (reflected) solutions:

$$\mathbf{C}_0 = \mathbf{C}_0(+) + \mathbf{C}_0(-). \quad (2.20)$$

Using Eq. (2.19), the amplitude at cell -1 is written as

$$\mathbf{C}_{-1} = F^{-1}(+)\mathbf{C}_0(+) + F^{-1}(-)\mathbf{C}_0(-). \quad (2.21)$$

This gives

$$\mathbf{C}_{-1} = \mathbf{F}^{-1}(-)\mathbf{C}_0 + [\mathbf{F}^{-1}(+) - \mathbf{F}^{-1}(-)]\mathbf{C}_0(+).$$

Therefore, the equation of motion at cell 0 becomes

$$\begin{aligned} (E - \tilde{H}_0) \mathbf{C}_0 + tP^* \mathbf{C}_1 \\ = -tP [F^{-1}(+) - F^{-1}(-)] \mathbf{C}_0(+), \end{aligned} \quad (2.23)$$

with

$$\tilde{\mathcal{H}}_0 = \mathcal{H}_0 - tPF^{-1}(-). \quad (2.24)$$

Note that $\tilde{\mathcal{H}}_0$ is not Hermitian.

On the other hand, only right-going (out-going) waves can exist at cell $N + 1$. Therefore, we have

$$\mathbf{C}_{N+2} = \mathbf{F}(+) \mathbf{C}_{N+1}, \quad (2.25)$$

which gives the equation of motion at cell $N + 1$

$$(\mathbf{E} - \tilde{\mathcal{H}}_{N+1}) \mathbf{C}_{N+1} + tP \mathbf{C}_N = 0, \quad (2.26)$$

with

$$\tilde{H}_{N+1} = \mathcal{H}_{N+1} - tP^* F(+). \quad (2.27)$$

Note that $\mathcal{H}_{N+1} = \mathcal{H}_0$ and that $\tilde{H}_{N+1}$ is not Hermitian.

Define the Green function G as

$$G = \frac{1}{E - \tilde{\mathcal{H}}} \quad (2.28)$$

with

$$\tilde{\mathcal{H}} = \begin{pmatrix} \tilde{\mathcal{H}}_0 & -tP^* & 0 & \cdots & 0 & 0 \\ -tP & \tilde{\mathcal{H}}_1 & V & \cdots & 0 & 0 \\ 0 & -tP & \tilde{H}_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{\mathcal{H}}_N & -tP^* \\ 0 & 0 & 0 & \cdots & -tP & \tilde{\mathcal{H}}_{N+1} \end{pmatrix}, \quad (2.29)$$

where $\tilde{\mathcal{H}}_j = \mathcal{H}_j$ for $j = 1, 2, \dots, N$. Immediately, we can derive

$$\begin{aligned} \mathbf{C}_{N+1}(+) &= \mathbf{C}_{N+1} \\ &= -t(N+1|G|0)P [F^{-1}(+) - F^{-1}(-)] \mathbf{C}_0(+) \end{aligned} \quad (2.30)$$

and

$$\begin{aligned} \mathbf{C}_0(-) &= \mathbf{C}_0 - \mathbf{C}_0(+) \\ &= \{-t(0|G|0)P [F^{-1}(+) - F^{-1}(-)] - 1\} \mathbf{C}_0(+). \end{aligned} \quad (2.31)$$

From these equations we can obtain the transmission coefficient $t_{\mu\nu}$ for the incident channel ν with velocity v_ν and out-going channel μ with velocity v_μ as

$$t_{\mu\nu} = \left(\frac{v_\mu}{v_\nu}\right)^{1/2} \{-tU^{-1}(+)(N+1|G|0) \times P[F^{-1}(+) - F^{-1}(-)]U(+)\}_{\mu\nu} \quad (2.32)$$

and the reflection coefficient $r_{\mu\nu}$ for incident channel ν and out-going channel μ as

$$r_{\mu\nu} = \left(\frac{v_\mu}{v_\nu}\right)^{1/2} (U^{-1}(-) \{-t(0|G|0)P[F^{-1}(+) - F^{-1}(-)] - 1\}U(+))_{\mu\nu} \quad (2.33)$$

In the absence of a magnetic field, these formulas reduce to the simpler expressions^{34–36} widely used in various numerical studies. In terms of the transmission coefficient $t_{\mu\nu}$, the conductance G is given by the multichannel version³⁴ of Landauer's formula:³⁷

$$G = \frac{e^2}{\pi\hbar} \sum_{\mu\nu} |t_{\mu\nu}|^2 \quad (2.34)$$

To calculate the Green function, we define the diagonal and off-diagonal Green functions by

$$\begin{aligned} (j|G^{(j)}|j)^{-1} &= [j|(E - \mathcal{H}^{(j)})^{-1}|j], \\ (j|G^{(j)}|0)^{-1} &= [j|(E - \mathcal{H}^{(j)})^{-1}|0], \end{aligned} \quad (2.35)$$

where $\mathcal{H}^{(j)}$ is the total Hamiltonian for the strip comprising the 0 to j cells excluding the intercell Hamiltonian $\widetilde{\mathcal{H}}_{j,j+1}$ and $\widetilde{\mathcal{H}}_{j+1,j}$. Then, the Green function for strips with any length can be obtained by a set of recursive formulas,^{38–45}

$$\begin{aligned} (j+1|G^{(j+1)}|j+1)^{-1} &= E - \widetilde{\mathcal{H}}_{j+1} - \widetilde{\mathcal{H}}_{j+1,j} (j|G^{(j)}|j) \widetilde{\mathcal{H}}_{j,j+1}, \\ (2.36) \end{aligned}$$

$$\begin{aligned} (j+1|G^{(j+1)}|0) &= (j+1|G^{(j+1)}|j+1) \mathcal{H}_{j+1,j} (j|G^{(j)}|0), \end{aligned}$$

with the initial condition $(0|G^{(0)}|0) = (E - \widetilde{H}_0)^{-1}$. Note that $-t^{-1}F^{-1}(-)P$ and $-t^{-1}F(+)P^*$ are the diagonal Green function under the boundary condition of left- and right-going waves, respectively.⁴⁶

14.0.6 III. SINGLE POINT CONTACT

14.0.7 A. Model

We consider a wire with length L_x and width L_y ($-L_x/2 < x < L_x/2$ and $-L_y/2 < y < L_y/2$) and introduce the following potential which confines electron motion in the y direction in the vicinity of the center of the wire:

$$V(x, y) = \frac{V}{2} \left[1 + \cos \left[\frac{2\pi x}{L_x} \right] \right] + E_F \sum_{\pm} \left[\frac{y - y_{\pm}(x)}{\Delta} \right]^2 \Theta(y^2 - y_{\pm}(x)^2) \quad (3.1)$$

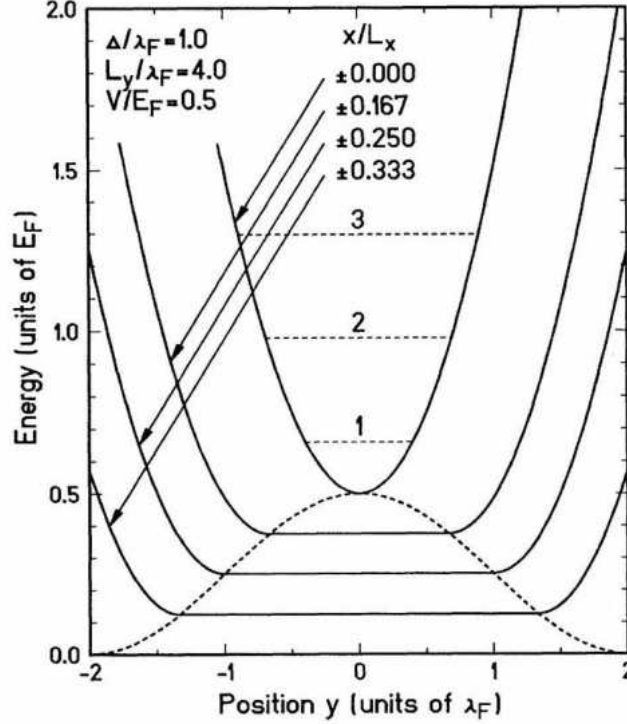


FIG. 1. An example of the potential profile at a quantum point contact for $V/E_F = 0.5$, $\Delta/\lambda_F = 1$, and $L_y/\lambda_F = 4$. The solid lines represent the potential at $x/L_x = 0, \pm 1/6, \pm 1/4$, and $\pm 1/3$ as a function of y . The horizontal dashed lines represent the bottom of the one-dimensional subbands at the contact, i.e., for the parabolic potential at $x = 0$.

where $\theta(t)$ is a step function, defined by $\theta = 1$ for $t > 0$ and $\theta = 0$ otherwise, and

$$y_{\pm}(x) = \pm \frac{L_y}{4} \left[1 - \cos \left(\frac{2\pi x}{L_x} \right) \right] \quad (3.2)$$

The parameter Δ/λ_F characterizes the width of the channel of the point contact and V represents the maximum bottom energy at $x = y = 0$. For a fixed width L_y , the wire length L_x determines the length scale over which the potential varies. The adiabatic limit corresponds to $L_x/\lambda_F \gg 1$.

At $x = 0$ the confining potential is parabolic. With increasing distance from $x = 0$, the bottom energy decreases and at the same time the effective width becomes larger. This model is expected to simulate the potential variation in the vicinity of actual quantum point contacts for appropriate values of the parameters. The potential profile for $\Delta/\lambda_F = 1$, $L_y/\lambda_F = 4$, and $V/E_F = 0.5$ is given in Fig. 1.

14.0.8 B. Numerical results

Figure 2 gives examples of the calculated conductance across the point contact for $L_x/\lambda_F = 4, 8$, and 16. The other parameters are the same as in Fig. 1. For $L_x/\lambda_F = 4$ the potential variation is far from being adiabatic and the conductance is not quantized. With the increase

of the magnetic field, however, it becomes well quantized. This is because the magnetic field creates edge

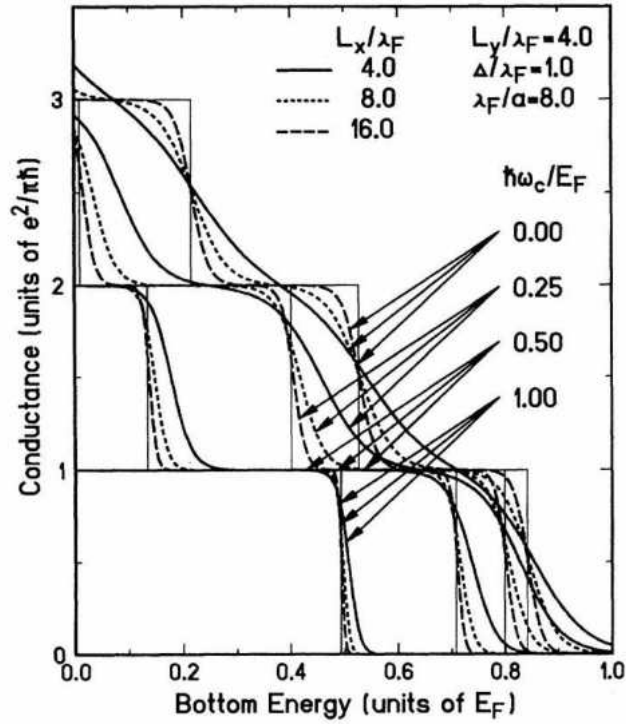


FIG. 2. Examples of the calculated conductance across a quantum point contact as a function of the bottom energy V in magnetic fields. The thin straight lines represent the channel number at the contact. When the confinement potential varies more slowly, the conductance quantization becomes better.

states and reduces backscattering considerably. For $L_x/\lambda_F = 16$ we can see a well-developed conductance quantization even in the absence of a magnetic field.

Figure 3 gives the current distribution G_μ for the outgoing or incoming channels, defined by

$$G_\mu = \frac{e^2}{\pi\hbar} \sum_v |t_{\mu\nu}|^2 = \frac{e^2}{\pi\hbar} \sum_v |t_{\mu\nu}|^2 \quad (3.3)$$

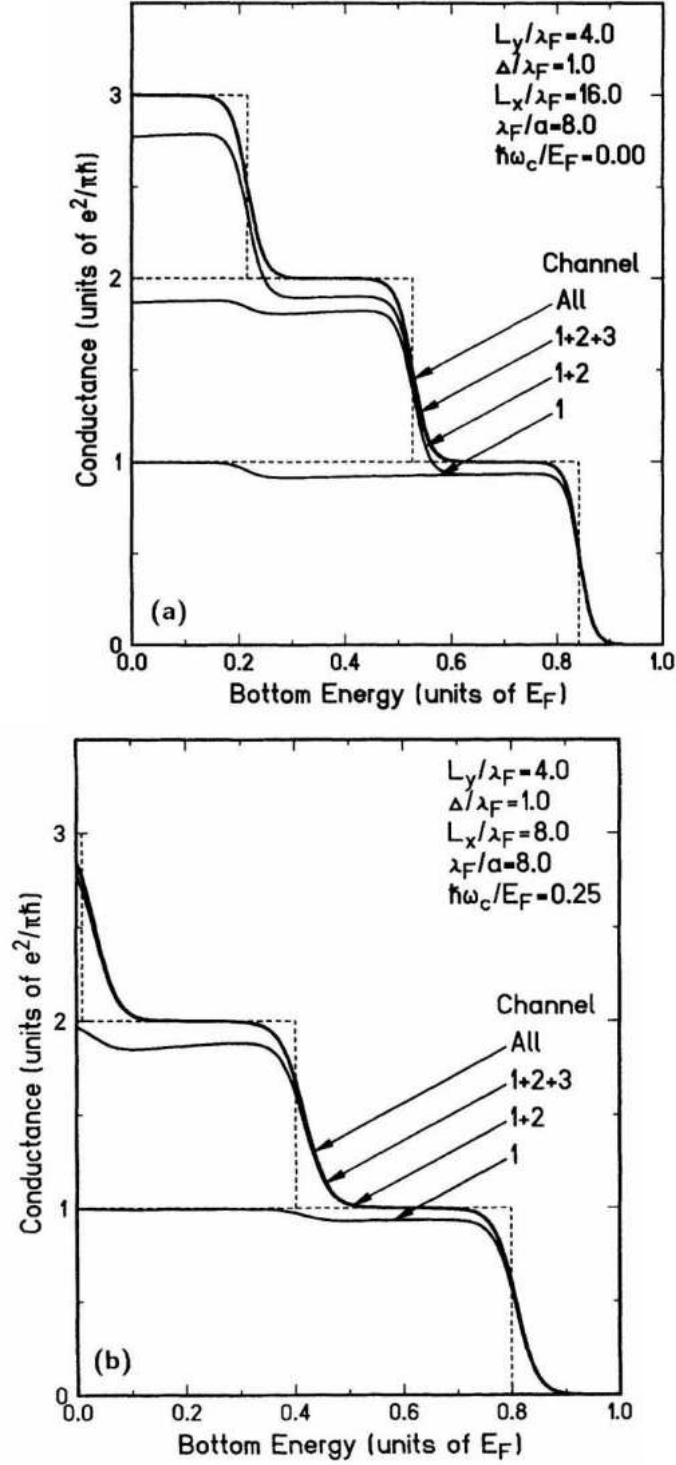


FIG. 3. Examples of the calculated current distribution among out-going channels as a function of the bottom energy V when the conductance is well quantized. Most of the current is carried by the out-going channel identical to that at the point contact because of the adiabatic nature of the potential, but contributions of other channels are still noticeable. (a) $L_x/\lambda_F = 16$ and $\hbar\omega_c/E_F = 0$, (b) $L_x/\lambda_F = 8$ and $\hbar\omega_c/E_F = 0.25$.

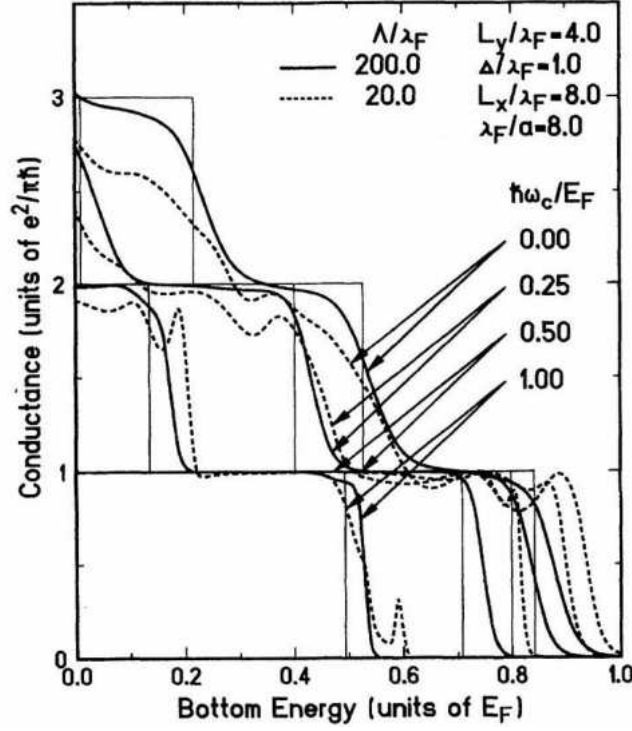


FIG. 4. Examples of the calculated conductance across a quantum point contact as a function of the bottom energy V in the presence of short-range randomness characterized by the mean free path Λ . The thin straight lines represent the channel number at the contact.

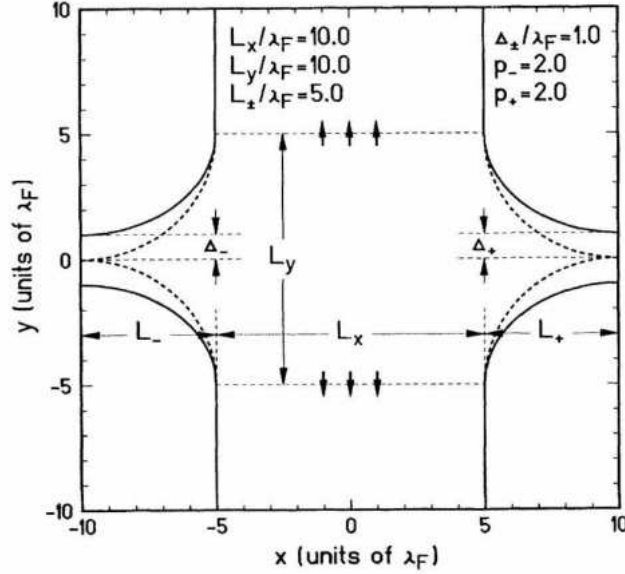


FIG. 5. A schematic illustration of a series of two quantum point contacts. Electrons are injected from the emitter point contact on the left-hand side and collected by the collector contact on the right-hand side. In the central region, both top and bottom edges are connected to an ideal and infinitely long wire. We define L_x as the width of the wire in the y direction (width of the region without confinement potential), $L_{\pm}$ as the length over which the confining potential varies in the emitter and collector contacts, and L_y as the width of the opening. At the point contacts, the bottom of the parabolic confinement potential $V_{\pm}$ can be raised in order to control the channel number. The parameters characterizing the confinement potential at the point contacts are $p_{\pm} = 2$, $\Delta_{\pm}/\lambda = 1$, $L_{\pm}/\lambda_F = 5$, $L_x/\lambda_F = 10$, and $L_y/\lambda_F = 10$.

for the two cases where the conductance is well quantized [(a) $L_x/\lambda_F = 16$ at $\hbar\omega_c/E_F = 0$ and (b) $L_x/\lambda_F = 8$ at $\hbar\omega_c/E_F = 0.25$]. It should be noted that an appreciable part of the current is carried by channels different from those at the point contact although the total conductance is well quantized. In Fig. 3(a), for example, about 90% of the current is carried by channel 1 and

the remaining $\sim 10\%$ is carried by channel 3 having the same symmetry for $0.6 \lesssim V/E_F \lesssim 0.8$. Such symmetry is destroyed by magnetic fields. Figure 3(b) shows that about 95% of the current is carried by the lowest channel and the remaining $\sim 5\%$ is carried by the first excited channel for $0.5 \lesssim V/E_F \lesssim 0.7$. Interchannel as well as intrachannel scatterings due to nonadiabaticity of the confining potential have been shown to be exponentially small in the slowly varying case.^{19,20} Therefore, the present result shows that the conductance is quantized even when the potential is far from being in the adiabatic limit.

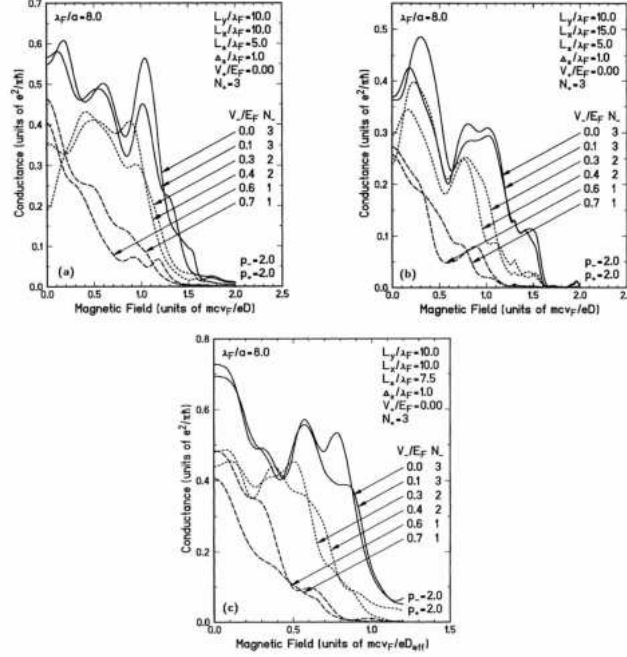


FIG. 6. Examples of the calculated conductance across a series of two quantum point contacts as a function of the magnetic field when the channel number of the collector contact is 3. The parameters are $L_y/\lambda_F = 10$, $p_{\pm} = 2$, and $\Delta_{\pm}/\lambda = 1$. (a) $L_x/\lambda_F = 10$ and $L_{\pm}/\lambda_F = 5$, (b) $L_x/\lambda_F = 15$ and $L_{\pm}/\lambda_F = 5$, (c) $L_x/\lambda_F = 10$ and $L_{\pm}/\lambda_F = 7.5$.

Examples showing the effects of disorder are given in Fig. 4 for $L_x/\lambda_F = 8$. Scattering modifies the quantization even when the mean free path is more than one order of magnitude larger than the length of the region where the confining potential is appreciable ($\Lambda/\lambda_F = 200$). When Λ and L_x become comparable, the quantization is completely destroyed and the conductance exhibits fluctuations (a precursor of the universal conductance fluctuations). The randomness often gives rise to a sharp structure due to resonant tunneling through virtual bound states.

14.0.9 IV. SERIES OF TWO POINT CONTACTS

14.0.10 A. Model

To calculate the conductance for a series of two point contacts corresponding to recent experiments,¹⁸ we shall use the model schematically shown in Fig. 5. At the boundaries $y = \pm L_y/2$, an infinitely long wire with width L_x is connected, i.e., we impose the boundary condition that only out-going waves are present at $y = \pm L_y/2$. The current is injected from the left (negative x direction) and collected by the right (positive x direction) point contact. The length of the region over which the confinement potential varies is L_- for the left and L_+ for the right contact. The outer ends of these point contacts are connected to infinitely long wires.

We first separate the region into three parts: the central region, the left emitter point-contact region, and the right collector point-contact region. The Green function is calculated

from left to right in the emitter region and from right to left in the collector region. The Green function in the central part is calculated recursively from that of a small system near the center in a manner similar to that described by Eq. (2.36). This Green function is connected to that of outgoing ones and those of the emitter and collector point-contact regions at the outmost boundaries. The resulting Green function between the left and right edges can be used to calculate the corresponding transmission coefficients under the boundary conditions that only out-going waves exist at $y = \pm L_y/2$. Because electrons flow into the wires in the y direction the conductance between the two point contacts becomes smaller than $e^2/\pi\hbar$ especially in strong magnetic fields.

The confining potential at the point contact is simulated by the parabolic potential similar to that used in the single point-contact case discussed in the previous section. For example, the potential in the left point-contact region is given by

$$V(x, y) = \frac{V_-}{2} \left[1 + \cos \left[\frac{\pi x}{L_-} \right] \right] + E_F \sum_{\pm} \left[\frac{y - y_{\pm}(x)}{\Delta(x)} \right]^2 \Theta(y^2 - y_{\pm}(x)^2), \quad (4.1)$$

with

$$y_{\pm}(x) = \pm \frac{L_y}{2} \left[1 - \left[\frac{x}{L_-} \right]^{p_-} \right]^{1/p_-}, \\ \Delta(x) = \Delta_- \left[1 - \left[\frac{x}{L_-} \right]^{p_-} \right]^{1/p_-}, \quad (4.2)$$

where the origin of x is chosen at the left edge. The similar potential is assumed in the collector region except

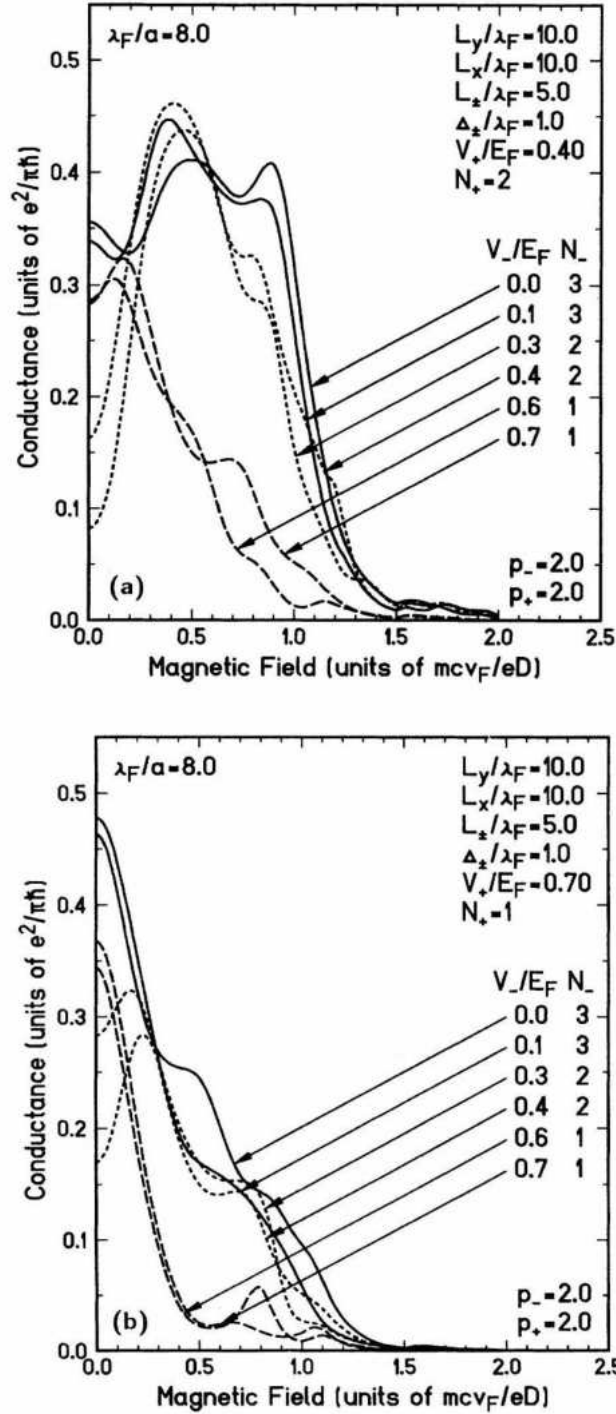


FIG. 7. Examples of the calculated conductance across a series of two quantum point contacts as a function of the magnetic field for $L_x/\lambda_F = L_y/\lambda_F = 10$, $L_{\pm}/\lambda_F = 5$, $p_{\pm} = 2$, and $\Delta_{\pm}/\lambda = 1$. (a) $N_+ = 2$ and (b) $N_+ = 1$. that Δ_- , V_- , L_- , and p_- are replaced by Δ_+ , V_+ , L_+ , and p_+ , respectively.

14.0.11 B. Numerical results

Figure 6 gives examples of the calculated results when the channel number N_+ at the collector point contact is 3. The channel number N_- of the emitter point contact is varied by raising the bottom potential V_- at the left edge. The axis of abscissa is the ratio of the distance between the emitter and collector point contacts to the classical cyclotron radius, i.e., D/R_c with $D = L_x + L_+ + L_-$. In Fig. 6(a), the width of the region without confining potential is $L_x/\lambda_F = 10$ and the length of the emitter and

collector regions is $L_{\pm}/\lambda_F = 5$. The flat region is wider in Fig. 6(b) and the point-contact region is wider in Fig. 6(c). In usual systems made on a GaAs/Al_xGa_{1-x}As single heterostructure, the Fermi

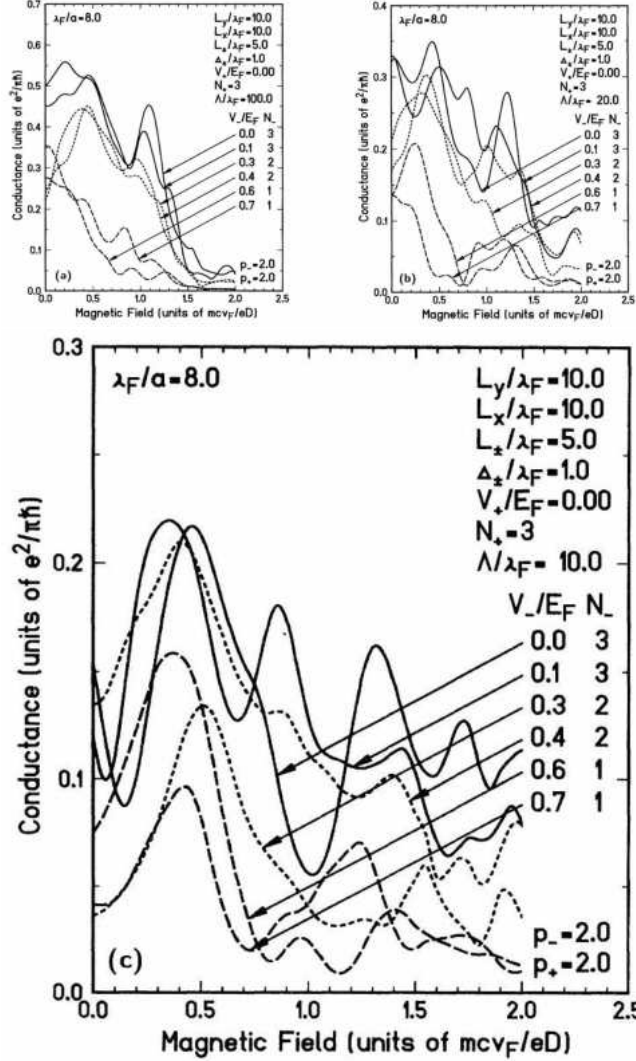


FIG. 8. Examples of the calculated conductance across a series of two quantum point contacts as a function of the magnetic field in the presence of scatterers with short-range potential. The parameters are $L_x/\lambda_F = L_y/\lambda_F = 10$, $L_{\pm}/\lambda_F = 5$, $p_{\pm} = 2$, $\Delta_{\pm}/\lambda_F = 1$, and $N_+ = 3$. (a) $\Lambda/\lambda_F = 100$, (b) 20, and (c) 10.

wavelength is typically $\lambda_F \sim 500\text{\AA}$. Therefore, we have $D \sim 1\mu\text{m}$ and $L_y \sim 0.5\mu\text{m}$, which are very close to the length scales of the point contacts used in the experiments.¹⁸

Oscillatory structures appear due to complex interferences with waves reflected from different parts of the boundaries.⁴⁷ Apart from these structures the conductance is determined essentially by the number of emitter channels. When there is a single channel in the emitter ($N_- = 1$), the conductance decays rapidly with increasing magnetic field. When the first excited channel is occupied, its contribution appears at higher magnetic fields. The difference between the conductance for $N_- = 3$ and 2 is large at weak magnetic fields, decreases, and then increases again with the magnetic field. These features are most clearly seen in Figs. 6(a) and 6(c). It can be concluded that the magnetic-field dependence of the conductance is determined essentially by the spatial (current) density distribution of the channels at the emitter point contact. This explains the recent experiments¹⁸ which have shown that, for $N_+ = 3$, the conductance exhibits a single peak at zero magnetic field for $N_- = 1$ and a double peak as a function of magnetic field for $N_- = 2$. The fluctuations due to interference effects have not clearly been observed by the experiments, however.

The conductance depends also on the channel number of the collector point contact. Examples are given in Fig. 7, where $N_+ = 2$ in 7(a) and 1 in 7(b). As is shown in Fig. 7(a), when $N_+ = 2$, the difference of the conductance for $N_- = 1$ and 2 is still appreciable but the conductance remains essentially the same for $N_- = 2$ and 3. The same feature can be seen in Fig. 7(b). When $N_+ = 1$, the conductances for different N_+ are very close to each other although there remain slight differences. This interesting dependence on the collector channel has not been reported yet.

Examples showing effects of scattering are given in Fig. 8 for the same system shown in Fig. 6(a). Scattering modifies the conductance, particularly the interference patterns, even if the mean free path is about an order of magnitude larger than the distance D between the emitter and collector contacts ($\Lambda/\lambda_F = 100$) as shown in Fig. 8(a). When the mean free path becomes comparable to the distance [Fig. 8(b)], the original interference fluctuations disappear completely and new fluctuations appear due to scatterings from impurities. When $\Lambda < D$ [Fig. 8(c)], the conductance exhibits universal conductance fluctuations. These calculations show that the amplitude of the universal conductance fluctuations is larger than that due to complex interferences of waves reflected at different boundaries. It should be noted that the latter fluctuations are continuously taken over by the universal conductance fluctuations with the increase of the randomness.

At nonzero temperatures, we have to consider various effects which destroy the phase coherence of electrons. The first is averaging due to the broadening of the electron energy distribution. Examples of the results of such energy averaging are shown in Fig. 9. With increasing temperature the amplitude of the oscillatory structures decreases gradually. At high enough temperatures the

interference fluctuations disappear completely and we can clearly identify the features corresponding to the channel wave functions. The energy broadening necessary to quench the fluctuations is estimated as $\Delta E/E_F \sim 2(\lambda_F/D)$ from the condition $\Delta k D \sim 2\pi$, which corresponds to the temperature of $k_B T/E_F \sim \Delta E/2\pi E_F \sim (1/\pi)(\lambda_F/D)$, in agreement with the numerical

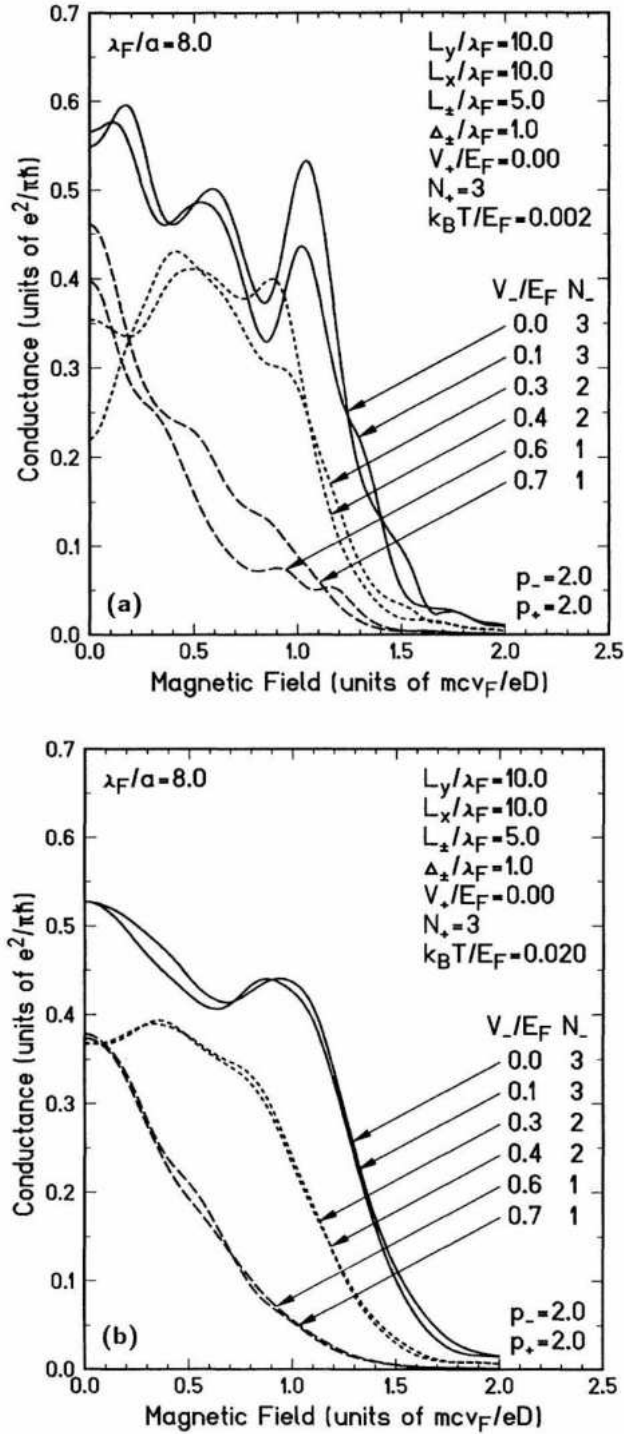


FIG. 9. Examples of the calculated conductance across a series of two quantum point contacts as a function of the magnetic field at nonzero temperatures. The parameters are $L_x/\lambda_F = L_y/\lambda_F = 10$, $L_{\pm}/\lambda_F = 5$, $p_{\pm} = 2$, $\Delta_{\pm}/\lambda_F = 1$, and $N_+ = 3$. (a) $k_B T/E_F = 0.002$ [roughly corresponding to experiments of Okada et al. (Ref. 18)] and (b) 0.02.

result. Another important effect is the inelastic scattering due to electron-electron and electron-phonon interactions. Calculations have been performed with the inclusion of a constant imaginary part in the site diagonal element of the Hamiltonian and given results quite similar to those at nonzero temperatures. Such calculations are certainly inappropriate because the imaginary part violates the current conservation, but might be sufficient for a qualitative description of the effects of inelastic scatterings on the interference fluctuations.

Numerical calculations have also been performed for $p_{\pm}$'s different from 2.0. The results show that the interference fluctuations are quite sensitive to the value of $p_{\pm}$. However, the

characteristic features determined by the channel number of the emitter and collector point contacts remain the same as long as $p_{\pm}$ is not too different from 2.0 ($p_{\pm} \lesssim 4$).

14.0.12 V. SUMMARY AND CONCLUSION

A method of calculating the conductance in ballistic structures in magnetic fields has been discussed and examples of the numerical results have been presented. The model consists of a square lattice with effects of a magnetic field being included in the form of a Peierls phase factor. The conductance across a single point contact in magnetic fields has been explicitly calculated. It has been shown that the conductance becomes quantized even if the potential is not completely adiabatic and that the magnetic field tends to make the quantization better due to a reduction in the strength of back scattering at the point contact. The condition for the quantization is not necessarily $G_{\mu} = 1$ or 0 as has frequently been discussed based on the adiabatic limit. Effects of scattering from impurities have been studied and shown to modify the quantization strongly even if the mean free path is much larger than the size of the point contact as expected.

The conductance across a series of two point contacts, similar to the system used in the experiments of Okada et al.¹⁸ has also been calculated. There appear fluctuations due to complex interferences of waves reflected from different boundaries. It has been shown that the magnetic-field dependence of the conductance is mainly determined by the density distribution of the narrow channel at the emitter point contact but depends also on the channel number of the collector point contact. The interference fluctuations are modified even in the presence of weak scattering from impurities and are gradually replaced by universal conductance fluctuations when scattering becomes strong.

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15 Quasiclassical method in the theory of SC.tex by LARKIN, OVCHINNIKOV

It is shown that replacement of quantum-mechanical averages by the average values of the corresponding classical quantities over all trajectories with a prescribed energy is not valid in the general case. The dependence of the penetration depth on the field is found without making any assumptions about the weakness of the interaction between the electrons and the field of the impurities; the case of very dirty films is also considered.

1. Use of a quasiclassical method ^[1,2] has turned out to be convenient in a number of problems in the theory of superconductivity. In this method the calculation of various characteristics of a superconductor reduces to the calculation of the average value of a product of matrix elements of single-particle operators. Usually the quantum-mechanical averages are replaced by the average values of the corresponding classical quantities over all classical trajectories of a given energy. It is shown below that in certain cases such a substitution leads to erroneous results. This is associated with the fact that the momentum operators of an electron, situated in an impurity field, do not commute at different moments of time. The problem of the dependence of the penetration depth on the magnetic field and on the impurity concentration is considered in detail in the London case. The interaction between electrons and impurities is not assumed to be weak. And in this case the dependence of the penetration depth on the field is determined not only by the cross sections but by the scattering amplitudes themselves.

The case of very dirty films ^[3] is also considered, where the interaction with impurities is not assumed to be weak or isotropic. The result only depends on the transport time τ . An equation is found for the critical field associated with an arbitrary concentration of impurities.

2. The current density is expressed in terms of the Green's function by the formula

$$\mathbf{j} = \frac{e}{m} T \text{Sp} \sum_{\omega} \delta(\mathbf{r} - \hat{\mathbf{r}}) \hat{\mathbf{p}} \tau_z \hat{G}(\hat{\mathbf{r}}, \mathbf{r}), \quad (1)$$

where the Green's function $\hat{G}(\mathbf{r}, \mathbf{r}')$ satisfies the Gor'kov equations which are conveniently written in matrix form:

$$\left[-\hat{H} + i\omega\tau_z + \begin{pmatrix} 0 & \Delta \\ -\Delta^* & 0 \end{pmatrix} + \frac{e}{m}(\hat{\mathbf{p}}\mathbf{A})\tau_z \right] \hat{G}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'),$$

$$\hat{H} = -\frac{1}{2m}\frac{\partial^2}{\partial \mathbf{r}^2} + V(\mathbf{r}) - \mu, \quad (2)$$

where $\hat{H}$ is the Hamiltonian of an electron in the field of the impurities.

In the approximation linear with respect to the field, the expression for the current in the representation of eigenfunctions of the Hamiltonian $\hat{H}$ takes the form

$$\mathbf{j}(\mathbf{r}) = -\left(\frac{e}{m}\right)^2 \text{Sp} T \sum_{\omega} \langle n | \delta(\mathbf{r} - \hat{\mathbf{r}}) \hat{\mathbf{p}} | m \rangle \langle m | (\hat{\mathbf{p}}\mathbf{A}) | n \rangle \tau_2 G_m \tau_2 G_n, \quad (3)$$

where

$$G_n = \frac{1}{\varepsilon_n^2 + \omega^2 + \Delta^2} \begin{pmatrix} -\varepsilon_n - i\omega & -\Delta \\ \Delta & -\varepsilon_n + i\omega \end{pmatrix}.$$

Changing to a time representation and using the weak dependence of the diagonal matrix element on energy near the Fermi surface, we obtain

$$\mathbf{j}(\mathbf{r}) = -2\pi v \left(\frac{e}{m}\right)^2 T \sum_{\omega} \text{Sp} \int_{-\infty}^{\infty} \langle \langle \delta(\mathbf{r} - \mathbf{r}(0)) \hat{\mathbf{p}}(0) \times (\hat{\mathbf{p}}(t_1) \mathbf{A}(t_1)) \rangle \rangle \tau_2 G(-t_1) \tau_2 G(t_1) dt_1 \quad (4)$$

where

$$\hat{\mathbf{p}}(t) = \exp(i\hat{H}t) \hat{\mathbf{p}} \exp(-i\hat{H}t), \quad v = mp_0/2\pi^2$$

and $\langle \langle \dots \rangle \rangle$ denotes the average over all states at the Fermi surface,

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G_{\xi} \exp(i\xi t) d\xi$$

$$= -\frac{i}{2\bar{E} - e^{-E|t|} (E \text{sign } t + \omega\tau_z + \Delta\tau_y)}, \quad E^2 = \omega^2 + \Delta^2 \quad (5)$$

For a Fourier component Eq. (4) takes the form

$$\mathbf{j}(\mathbf{g}) = -\frac{Ne^2}{m} Q(\mathbf{g}) \mathbf{A}(\mathbf{g}) \quad (6)$$

Below we confine our attention to consideration of the limiting London case when the change in the potential A is small over distances of the order of the size of a pair. In this case it is sufficient to find $Q(0)$. Taking formula (5) into account, we obtain

$$Q(0) = \frac{3\pi T \Delta^2}{p_0^2} \sum_{\omega} \int_{-\infty}^{\infty} E^{-2} \exp(-2E|t|) \langle \langle p_z(0) p_z(t) \rangle \rangle dt. \quad (7)$$

The average appearing in this expression may be found with the aid of the kinetic equation and is given by

$$\langle \langle p_z(0) p_z(t) \rangle \rangle = 1/3p_0^2 \exp(-|t|/\tau_{tr}) \quad (8)$$

Substituting this expression into formula (7), we obtain the usual expression for $Q^{[4]}$

$$Q(0) = \pi \Delta^2 T \sum_{\omega} [E^2 (E + 1/2\tau_{tr})]^{-1} \quad (9)$$

It is important to note that formula (8) is obtained with the aid of the kinetic equation for the pair correlation function. The collision term in this equation is expressed in terms of the scattering cross section and

in the general case has the same form both in the Born limit and in the classical limit. Therefore formula (9) may be obtained both ^[4] in the Born approximation with subsequent replacement of the Born amplitude by the exact amplitude and in the method of classical trajectories with replacement of the classical cross section by the exact cross section.

However, in the next approximation cubic in the field, both of these methods are erroneous in the general case. As shown below the Born approximation leads to the correct result only in the limit of a small mean free path, and the method of classical trajectories is not applicable even in this case.

3. One can find the next terms in an expansion of the current density in powers of the field in analogy to formulas (4) and (7). In the London limit we obtain

$$j_z = -2\pi v \left(\frac{e}{m}\right)^4 A^3 T \sum_{\omega} \text{Sp} \iint_{-\infty}^{\infty} \int dt_1 dt_2 dt_3 \langle \hat{p}_z(0) \hat{p}_z(t_1) \hat{p}_z(t_2) \hat{p}_z(t_3) \rangle \times \tau_z G(-t_1) \tau_z G(t_1 - t_2) \tau_z G(t_2 - t_3) \tau_z G(t_3) \Bigg)^4 A^3 T \sum_{\omega} \text{Sp} \iiint \int_{-\infty}^{\infty} \int \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \frac{d\omega_3}{2\pi} \quad (10)$$

where

$$P(\omega_1 \omega_2 \omega_3 \omega_4) = (2\pi)^4 \text{Sp} \left\{ \hat{p}_z \delta(\hat{H} - \omega_1) \hat{p}_z \delta(\hat{H} - \omega_2) \hat{p}_z \delta(\hat{H} - \omega_3) \hat{p}_z \delta(\hat{H} - \omega_4) \right\}, \quad (11)$$

$$\langle \langle p_z(0) p_z(t_1) p_z(t_2) p_z(t_3) \rangle \rangle = \frac{1}{v(2\pi)^4} \iint_{-\infty}^{\infty} \int P(\omega_1 \omega_2 \omega_3 0) \times \exp \{ -i [\omega_1 t_1 + \omega_2 (t_2 - t_1) + \omega_3 (t_3 - t_2)] \} d\omega_1 d\omega_2 d\omega_3 \quad (12)$$

The average value of the product of four momenta at different moments of time is not determined by the pair correlation function and cannot be found from the kinetic equation. Therefore, for its evaluation we shall use a diagram technique. ^[5] In this connection the Born approximation for the scattering of an electron by impurities will not be used.

In formula (11) it is convenient to change to the momentum representation, having written each δ -function in the form of two terms

$$2\pi i \delta(\hat{H} - \omega) = G_{\omega^A} - G_{\omega^R}, \quad G_{pp'}^R = (G_{pp}^A)^* = (-\hat{H} + \varepsilon_0 + i\delta)^{-1}. \quad (13)$$

Expanding $G_{pp'}$ in powers of the interaction V with the impurities, averaging with respect to the positions of the impurities, and neglecting, as is customary, intersecting diagrams, we obtain the following expression for the averaged Green's function:

$$G_{p^R} = (\omega - \xi - n f_{pp})^{-1}, \quad G_p^A = (G_p^R)^* \quad (14)$$

where $f_{pp'}$ satisfies the equation which is graphically depicted in Fig. 1:

$$f_{p'} = V_{pp'} + \int \frac{d\mathbf{p}_1}{(2\pi)^3} V_{\mathbf{p}\mathbf{p}_1} G_{\mathbf{p}_1} f_{\mathbf{p}_1\mathbf{p}'} \quad (15)$$

In this equation one can replace the function G_p^R by the function $G_p^0 = (\omega - \xi + i\delta)^{-1}$ since these functions differ from one another only in a narrow region near the pole, and the integrals over this region entering into formula (15) coincide for them. Therefore, the

$$\text{wavy line} = \text{dashed line} + \text{triangle} + \text{triangle} + \dots$$

FIG. 1

function $f_{pp'}$ differs from the exact scattering amplitude only by the factor $(-2\pi/m)$. The scattering amplitude at zero degrees, whose real part gives an unimportant energy shift and whose imaginary part is expressed in terms of the total cross section σ , enters into formula (14). As a result

$$G_{\mathbf{p}^R} = \left(\omega - \xi + \frac{i}{2\tau} \right)^{-1}, \quad \frac{1}{\tau} = nv\sigma, \quad \text{Im } f_{\mathbf{p}\mathbf{p}} = -\frac{v\sigma}{2}. \quad (16)$$

The diagrams shown in Fig. 2 arise upon averaging expression (11) with respect to the positions of the impurities, where each wavy line depicts a scattering amplitude, each point depicts the impurity concentration n . All solid lines on one side of a square represent either G^R or G^A . The vertex of a square corresponds to a momentum p_z if both lines coming together at this vertex represent G^R or G^A . However,



FIG. 2

if one line is G^R and the other G^A , then the vertex corresponds to an expression $\tilde{p}_z$, which satisfies the equation

$$\tilde{p}_z = p_z + \frac{n}{(2\pi)^3} \int G_{\omega_1}^A(\mathbf{p}_1) G_{\omega_2}^R(\mathbf{p}_1) |f_{\mathbf{p}_1}|^2 \tilde{p}_z(\mathbf{p}_1) d\mathbf{p}_1 \quad (17)$$

and equals

$$\tilde{p}_z = p_z \frac{\omega_2 - \omega_1 + i/\tau}{\omega_2 - \omega_1 + i/\tau_{tr}} \quad (18)$$

Each line from the sides of a square may represent either G^R or G^A ; therefore expression (11) represents a total of sixteen terms. Three of them are substantially different:

$$\begin{aligned}
P_1(\omega_1\omega_2\omega_3\omega_4) &= (2\pi)^4 \text{Sp} \{p_z G_{\omega_1}^A p_z G_{\omega_2}^A p_z G_{\omega_3}^R p_z G_{\omega_4}^R\} \\
&= -\frac{mp_0^5}{\pi} \frac{C(\omega_3 - \omega_1) + C(\omega_4 - \omega_2)}{(\omega_4 - \omega_1 + i/\tau_{tr})(\omega_3 - \omega_2 + i/\tau_{tr})}, \\
P_2(\omega_1\omega_2\omega_3\omega_4) &= (2\pi)^4 \text{Sp} \{p_z G_{\omega_1}^A p_z G_{\omega_2}^R p_z G_{\omega_3}^R p_z G_{\omega_4}^R\} \\
&= \frac{mp_0^5}{\pi} \frac{C(\omega_3 - \omega_1)}{(\omega_2 - \omega_1 + i/\tau_{tr})(\omega_4 - \omega_1 + i/\tau_{tr})}, \\
P_3(\omega_1\omega_2\omega_3\omega_4) &= (2\pi)^4 \text{Sp} \{p_z G_{\omega_1}^A p_z G_{\omega_2}^R p_z G_{\omega_3}^A p_z G_{\omega_4}^R\} \\
&= -\frac{mp_0^5}{\pi} \left\{ \frac{i}{5} (\omega_1 + \omega_3 - \omega_2 - \omega_4) + B \right\} \left\{ \left(\omega_2 - \omega_1 + \frac{i}{\tau_{tr}} \right) \right. \\
&\quad \times \left(\omega_4 - \omega_1 + \frac{i}{\tau_{tr}} \right) \left(\omega_2 - \omega_3 + \frac{i}{\tau_{tr}} \right) \left(\omega_4 - \omega_3 + \frac{i}{\tau_t} \right) \Big\}^{-1}, \tag{19}
\end{aligned}$$

where

$$\begin{aligned}
C(\omega) &= \frac{1}{\omega + i/\tau} \left[\frac{i}{5} - \frac{1}{9\tau} \frac{1}{\omega + i\delta} - \frac{4}{45} \frac{1}{\omega\tau_2 + i(\tau_2/\tau - 1)} \right] \\
B &= \frac{8}{45} \left(\frac{1}{\tau} - \frac{1}{\tau_2} \right) + \frac{npm_0}{2(2\pi)^3} \left\{ \frac{(mp_0)^2}{(2\pi)^4} \iiint \int f_{\mathbf{p}_1\mathbf{p}_2} f_{\mathbf{p}_2\mathbf{p}_1}^* f_{\mathbf{p}_3\mathbf{p}_1} f_{p_4\mathbf{p}_i}^* \right. \\
&\quad \times (\cos \theta_{\mathbf{p}_1} \cos \theta_{\mathbf{p}_3} \cos \theta_{\mathbf{p}_3} \cos \theta_{\mathbf{p}_4} - \cos^2 \theta_{\mathbf{p}_1} \cos^2 \theta_{\mathbf{p}_3}) d\Omega_{\mathbf{p}_1} d\Omega_{\mathbf{p}_2} d\Omega_{\mathbf{p}_3} d\Omega_{\mathbf{p}_s} \\
&\quad + \frac{mp_0}{\pi^2} \text{Im} \iiint f_{\mathbf{p}_1\mathbf{p}_2} f_{\mathbf{p}_2\mathbf{p}_3} f_{\mathbf{p}_3\mathbf{p}_1} \\
&\quad \times (\cos \theta_{\mathbf{p}_1} \cos \theta_{\mathbf{p}_2} - \cos^2 \theta_{\mathbf{p}_2}) \cos^2 \theta_{\mathbf{p}_3} d\Omega_{\mathbf{p}_1} d\Omega_{\mathbf{p}_2} d\Omega_{\mathbf{p}_3} \Big\}. \tag{20}
\end{aligned}$$

The remaining nonvanishing terms are expressed in terms of P_1 , P_2 , and P_3 with the aid of permutation of their arguments or complex conjugation. As a result we obtain:

$$\begin{aligned}
P(\omega_1, \omega_2, \omega_3, \omega_4) &= 2 \text{Re} \{ P_1(\omega_1\omega_2\omega_3\omega_4) + P_1(\omega_4\omega_1\omega_2\omega_3) \\
&\quad - P_2(\omega_1\omega_2\omega_3\omega_4) - P_2(\omega_4\omega_1\omega_2\omega_3) - P_2(\omega_3\omega_4\omega_1\omega_2) \\
&\quad - P_2(\omega_2\omega_3\omega_4\omega_1) + P_3(\omega_1\omega_2\omega_3\omega_4) \} \tag{21}
\end{aligned}$$

Thus, the average value of the product of four momenta is expressed not only in terms of the cross section but also in terms of the amplitude for the scattering of electrons by impurities. This means that the average under consideration cannot be obtained with the aid of a classical distribution function. In the Born approximation the terms containing the scattering amplitudes vanish. It is of more interest that they also vanish in the limit of large impurity concentrations when $\omega\tau \ll 1$. In this limit

$$P = 8/9 mp_0^5 \tau t^2 [\delta(\omega_1 - \omega_3) + \delta(\theta_2 - \omega_4)] \tag{22}$$

and the average value of a product of four momenta decomposes into a product of pair averages:

$$\begin{aligned}
\langle \langle p_z(t_1) p_z(t_2) p_z(t_3) p_z(t_4) \rangle \rangle &= \langle \langle p_z(t_1) p_z(t_2) \rangle \rangle \langle \langle p_z(t_3) p_z(t_4) \rangle \rangle \\
&\quad + \langle \langle p_z(t_1) p_z(t_k) \rangle \rangle \langle \langle p_z(t_2) p_z(t_3) \rangle \rangle. \tag{23}
\end{aligned}$$

In this case the result is expressed in terms of a transport cross section which, however, is not the classical result since it significantly depends on the order in which the momentum operators appear inside the averaging sign. This follows from the fact that the third possible product of pair averages is absent from the right-hand side of formula (23). Apparently in the limit of a large impurity concentration the product of an arbitrary number of momentum operators decomposes into a product of pair averages such that one can connect them by nonintersecting lines.

4. In order to clarify the dependence of the average on the order of the operators it is convenient to consider the special case of pairwise coincident times. From formulas (12), (19), and (20) it follows that

$$\langle p_z^2(0)p_z^2(t) \rangle = \frac{p_0^4}{9} \left[1 + \frac{4}{5} \exp \left\{ -|t| \left(\frac{1}{\tau} - \frac{1}{\tau_2} \right) \right\} \right], \quad (24)$$

$$\langle p_z(0)p_z(t)p_z(0)p_z(t) \rangle = p_0^4 \exp \left\{ -\frac{2|t|}{\tau_{tr}} \right\} \left[\frac{1}{5} + |t| \left(B - \frac{2}{5\tau_{tr}} \right) \right]. \quad (25)$$

Expression (24) agrees with the result which is obtained from the kinetic equation or by averaging over the classical trajectories. It is impossible to obtain expression (25) by such methods. This is associated with the fact that in formula (25) the momentum operators at different moments of time are arranged in an order which does not correspond to a classical trajectory. The commutator between momentum operators at different moments of time is not small even though the electrons move like free particles almost all the time.

Let us consider how this commutator changes in the quasiclassical limit

$$\langle [p_z(t)p_z(0)]^2 \rangle = h^2 \left\langle \left(\frac{\partial \mathbf{p}_z(t)}{\partial z(0)} \right)^2 \right\rangle, \quad (26)$$

where one can calculate the average in the right-hand side of formula (26) with respect to the classical trajectories. For the calculation we introduce the more general quantities

$$\begin{aligned} X_j^i &= \left\langle \left(\frac{\partial p_i(t)}{\partial r_j(0)} \right)^2 \right\rangle, & Y_j^i &= \left\langle \left(\frac{\partial r_i(t)}{\partial r_j(0)} \right)^2 \right\rangle \\ Z_j^i &= \left\langle \frac{\partial p_i(t)}{\partial r_j(0)} \frac{\partial r_i(t)}{\partial r_j(0)} \right\rangle \end{aligned} \quad (27)$$

For small times t the quantity X_j^i is proportional to the number of collisions and is determined by the change of the scattering angle associated with a change of the impact parameter. Averaging with respect to the positions of the impurities, we obtain

$$\begin{aligned} X_j^i &= t A_j^i, & A_j^i &= p_0^2 n v (a + b \delta_{ij}); \\ a &= \frac{1}{15} \int_0^\infty \left[\left(\frac{\partial \chi}{\partial \rho} \right)^2 (\cos^2 \chi + 2 \sin^2 \chi) + \frac{\sin^2 \chi}{\rho^2} \right] \rho d\rho, \\ b &= 2a - \frac{1}{3} \int_0^\infty \left(\frac{\partial \chi}{\partial \rho} \right)^2 \rho \sin^2 \chi d\rho, \end{aligned} \quad (28)$$

where χ is the scattering angle as a function of the impact parameter ρ . For a potential energy not small in comparison with the kinetic energy, the quantities a and b are of order unity.

At an arbitrary moment of time the quantities X, Y, and Z are connected by differential relationships which, for isotropic scattering, have the following form:

$$\dot{X}_j^i = A_h^i Y_j^k, \quad \dot{Y}_j^i = \frac{2}{m} Z_j^i, \quad \dot{Z}_j^i = \frac{1}{m} X_j^i \quad (29)$$

with initial conditions

$$X_{ji}(0) = 0, \quad Y_j^i(0) = \delta_{ij}, \quad Z_j^i(0) = 0.$$

Solving this system we obtain

$$X_{ji}^i = \frac{m^2}{18} \left[\frac{1}{t_0^2} f\left(\frac{t}{t_0}\right) (3\delta_{ij} - 1) + \frac{1}{t_1^2} f\left(\frac{t}{t_1}\right) \right] \quad (30)$$

where

$$f(t) = e^t + 2e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t - \frac{\pi}{6}\right)$$

$$t_0^{-1} = v(2bn)^{1/3}, \quad t_1^{-1} = v[n(6a + 2b)]^{1/3}$$

An estimate of the validity of the resulting formulas may be obtained from the condition that an initial displacement of a particle of the order of its wavelength $h p_0^{-1}$ must lead at a moment of time t to a displacement which is smaller than the interaction radius $\sim \sigma^{1/2}$:

$$\lambda^2 Y(t) \ll \sigma \quad \text{or} \quad h^2 e^{i/t_0} \ll \sigma p_0^2. \quad (31)$$

At large times the wave packet is completely washed out. In order to evaluate the average of the square of the commutator in this region, it is necessary to use not the quasiclassical formulas (26) and (30) but the difference between expressions (25) and (24).

Thus, even for quasiclassical scattering of particles by impurities, the commutator of momentum operators at different moments of time increases exponentially with the time. For electrons in a metal, we apply the method of quasiclassical trajectories only in the pure case.

The problem considered above concerning the classical and quantum motion of an electron in a field

of impurities is a one-particle problem and does not have any direct connection to superconductivity. However, it should be kept in mind that in a theory of the normal state the momentum operators must follow in the order of increasing time. One can replace the average of such a product by an average over classical trajectories. A quantum phenomenon exists in superconductors: An electron, being created of a pair in the Bose-condensate, may change into a hole, which can be regarded as an electron moving backwards in time. Therefore, just as in the example considered above, averages of products of operators which are not ordered in time arise in the problems of superconductivity theory. The quasiclassical approximation is not applicable for the calculation of such averages.

5. In order to determine the corrections to the penetration depth, let us find the dependence of the current density on the field. To third order in the field, we can obtain one of the terms in the current density by substituting expression (21) into formula (10):

$$j_1 = \frac{Ne^2}{m} \beta A^3 = \frac{2e}{\pi} \left(\frac{eA}{m} \right)^3 p_0^5 T \sum_{\omega} \frac{\Delta^2}{E^4 (E + 1/2\tau_{tr})^2} \times \left\{ \frac{\omega^2}{5(E + 1/2\tau)} + \frac{\omega^2}{9E + 18E^2\tau} + \frac{2\omega^2}{45(E + 1/2\tau) [E + 1/2(\tau_2/\tau - 1)]} - \frac{\Delta^2}{(E + 1/2\tau_{tr})^2} \left(\frac{B}{16} + \frac{E}{20} \right) \right\} \quad (32)$$

In addition, the dependence of Δ on the magnetic field should be taken into account in the first-order expression (6). In order to take this dependence into account, in the equation for Δ

$$\Delta^*(\mathbf{r}) = |\lambda|T \sum_{\omega} \text{Sp} \frac{1}{2} (\tau_x + i\tau_y) \hat{G}(\mathbf{r}, \mathbf{r}) \quad (33)$$

we expand the Green's function $\hat{G}$ to second order in the field. Changing, just as above, to a representation of eigenfunctions of the operator $\hat{H}$, we obtain

$$\Delta^* = 2\pi v |\lambda|T \sum_{\omega} \text{Sp} \frac{1}{2} (\tau_x + i\tau_y) \left\{ \hat{G}(0) + \left(\frac{e}{m} A \right)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \hat{p}_z(t_1) \hat{p}_z(t_2) \rangle \cdot G(-t_1) \tau_z G(t_1 - t_2) \tau_z G(t_2) dt_1 dt_2 \right\}. \quad (34)$$

Substituting here expression (8) for the average of a product of momenta and expression (5) for the Green's function $G(t)$, we obtain the correction of second order in the field to the absolute value of Δ :

$$\Delta_2 = - \left(6\Delta_0 \sum_{\omega} E^{-3} \right)^{-1} \left(\frac{e}{m} p_0 A \right)^2 \times \sum_{\omega} \frac{1}{E^3 (E + 1/2\tau_{tr})^2} \left(\frac{\omega^2}{E\tau_{tr}} + 2\omega^2 - \Delta^2 \right) \quad (35)$$

Taking this correction into account in expression (6) for the current to first order in the field and the thirdorder correction (32), we obtain

$$j = -\frac{Ne^2}{m} Q(0) A (1 - \alpha A^2) \quad (36)$$

where

$$\alpha = \left(\beta - \frac{\Delta_2}{A^2} \frac{\partial Q}{\partial \Delta} \right) \Big|_{Q(0)} \quad (37)$$

and the quantities $Q(0)$, β , and Δ_2 are determined by formulas (9), (32), and (35).

From here it follows that in the London case the field dependence of the penetration depth is determined by the expression

$$\delta = \delta_0 \left[1 + \frac{\alpha H^2}{4} \delta_0^2 \right]. \quad (38)$$

Thus, the dependence of the penetration depth on the magnetic field in superconducting alloys is expressed not only in terms of a cross section but also in terms of a scattering amplitude.

In the most interesting case of a large impurity concentration, when the mean free path is small in comparison with the dimensions of a pair, it is only necessary to keep the first term in formula (38); here the result agrees with that obtained by Melik-Barkhudarov ^[6] in the Born approximation.

6. In the general case it is convenient to carry out averaging of the Green's functions with respect to the positions of the impurities without isolating the average value of a product of matrix elements.

Let us expand the solution of the Gor'kov equations ^[4] in powers of the interaction $V(\mathbf{r})$ with impurities and average each term of this expansion. For a fixed difference of coordinates, the dependence of the Green's function for a pure metal in a magnetic field on the total coordinate is much weaker than the dependence on the coordinates of the interaction potential. Changing from the difference of coordinates to a momentum representation, we obtain

$$\left[-\xi - \frac{i}{m} \left(\mathbf{p} \frac{\partial}{\partial \mathbf{r}} \right) + i\omega\tau_z + \frac{e}{m} (\mathbf{p}\mathbf{A})\tau_z + \hat{\Delta} - n\Sigma_{\mathbf{p}\mathbf{p}\mathbf{r}} \right] G_{\mathbf{p}(\mathbf{r})} = 1$$

$$\xi = \frac{\mathbf{p}^2}{2m} - \mu, \quad \hat{\Delta} = \begin{pmatrix} 0 & \Delta \\ -\Delta^* & 0 \end{pmatrix}. \quad (39)$$

In all terms except the first one can regard the momenta $\mathbf{p}$ as lying on the Fermi surface. The matrix Σ is represented by the diagrams shown in Fig. 1, in which smooth lines correspond to the function $G_p(\mathbf{r})$. Summing the diagrams, we obtain

$$\Sigma_{\mathbf{p}\mathbf{p}'} = V_{\mathbf{p}'} + \frac{1}{(2\pi)^3} \int V_{\mathbf{p}_1} G_{\mathbf{p}_1}(\mathbf{r}) \Sigma_{\mathbf{p}_1\mathbf{p}'}(\mathbf{r}) d\mathbf{p}_1 \quad (40)$$

In this equation it is convenient to distinguish regions far away from the Fermi surface. For this purpose we consider a function χ satisfying the equation

$$\chi_{\mathbf{p}\mathbf{p}'} = V_{\mathbf{p}\mathbf{p}'} + \frac{1}{(2\pi)^3} \int V_{\mathbf{p}\mathbf{p}_1} \frac{1}{-\varepsilon_{\mathbf{p}_1} + \mu} \chi_{\mathbf{p}_1\mathbf{p}'} d\mathbf{p}_1 \quad (41)$$

This function differs from the scattering amplitude (15) by the fact that the integral is to be understood in the sense of a principal value. The spherical harmonic functions χ are equal to the tangents of the scattering phase shifts. The amplitude (15) is related to the function χ by the relationship

$$f_{\mathbf{p}\mathbf{p}'} = \chi_{\mathbf{p}\mathbf{p}'} - \frac{i\vartheta}{4} \int \chi_{\mathbf{p}\mathbf{p}_1} f_{\mathbf{p}_1\mathbf{p}'} d\Omega_{\mathbf{p}_1}, \quad \vartheta = \frac{mp_0}{2\pi^2} \quad (42)$$

The integral in this equation is taken over the angles of the vector $\mathbf{p}_1$, whose magnitude is equal to the momentum p_0 at the Fermi surface. Similarly Eq. (40) takes the form

$$\Sigma_{\mathbf{p}\mathbf{p}'} = \gamma_{\mathbf{p}\mathbf{p}'} - \frac{i\vartheta}{4} \int \chi_{\mathbf{p}\mathbf{p}_1} \bar{G}_{\mathbf{p}_1}(\mathbf{r}) \Sigma_{\mathbf{p}_1\mathbf{p}'}(\mathbf{r}) d\Omega_{\mathbf{p}_1} \quad (43)$$

where

$$\bar{G}_{\mathbf{p}}(\mathbf{r}) = \frac{i}{\pi} \int G_{\mathbf{p}}(\mathbf{r}) d\xi \quad (44)$$

For large values of ξ it is necessary to understand the integral (44) in the sense of a principal value, and also $\bar{G}_{\mathbf{p}}(\mathbf{r})$ only depends on the angles of the vector $\mathbf{p}$, which is situated on the Fermi surface.

The expression for Σ (formula (43)), and also all physical quantities, for example, the current density, only contain $\bar{G}_{\mathbf{p}}(\mathbf{r})$. In order to find the equation for this quantity, we write Eq. (39) in the form

$$\left[- \left(\mathbf{v} \frac{\partial}{\partial \mathbf{r}} \right) + \hat{\omega} \right] G_{\mathbf{p}}(\mathbf{r}\mathbf{r}') = 2pv\delta(\mathbf{p}, \mathbf{r} - \mathbf{r}') \quad (45)$$

where

$$G_{\mathbf{p}}(\mathbf{r}\mathbf{r}') = \frac{i}{\pi} \int G_{\mathbf{p}}(\mathbf{r}) \exp \left[-i\xi \frac{(\mathbf{p}, \mathbf{r} - \mathbf{r}')}{pv} \right] d\xi$$

$$\bar{G}_{\mathbf{p}}(\mathbf{r}) = G_{\mathbf{p}}(\mathbf{r}\mathbf{r}), \quad \hat{\omega} = \omega\tau_z - ie(\mathbf{v}\mathbf{A})\tau_z - i\hat{\Delta} + in\Sigma_{\mathbf{p}\mathbf{p}}(\mathbf{r}) \quad (46)$$

One can show that it is also possible to write Eq. (45) in the form

$$\left(\mathbf{v} \frac{\partial}{\partial \mathbf{r}'} \right) G_{\mathbf{p}}(\mathbf{r}\mathbf{r}') + G_{\mathbf{p}}(\mathbf{r}\mathbf{r}') \hat{\omega}(\mathbf{r}') = 2pv\delta(\mathbf{p}, \mathbf{r} - \mathbf{r}') \quad (47)$$

Subtracting Eq. (47) from Eq. (45) and setting $\mathbf{r} = \mathbf{r}'$, we obtain

$$- \left(\mathbf{v} \frac{\partial}{\partial \mathbf{r}} \right) \bar{G}_{\mathbf{p}}(\mathbf{r}) + \hat{\omega} \bar{G}_{\mathbf{p}}(\mathbf{r}) - \bar{G}_{\mathbf{p}}(\mathbf{r}) \hat{\omega} = 0 \quad (48)$$

In the region where the field is equal to zero or $\Delta = 0$, it follows from Eq. (39) that $\text{Tr} \bar{G}_{\mathbf{p}}(\mathbf{r}) = 0$, and from Eq. (47) it is seen that $\text{Tr} \bar{G}_p(\mathbf{r})$ does not depend on $\mathbf{r}$. Therefore it is equal to zero everywhere, and one can write the Green's function $\bar{G}_p(r)$ in the form $\bar{\mathbf{G}}_p(\mathbf{r}) = (a \cdot \tau)$. Similarly one can convince oneself that

$$\mathbf{a}^2 = 1 \quad (49)$$

The system of equations (43), (48), and (49) determines the Green's function $\mathbf{G}_{\mathbf{p}}(\mathbf{r})$ in an arbitrary magnetic field. In the Born approximation these equations were obtained in another way by Eilenberger. [7]

In the presence of a magnetic field, $\bar{G}_{\mathbf{p}}(\mathbf{r})$ depends on the angles of the vector $\mathbf{p}$; therefore the system of Eqs. (43), (48), and (49) may be solved only in limiting cases. As mentioned above, certain results may depend not only on the cross sections but also on the scattering amplitudes.

7. In the most interesting cases of the critical field, weak field, or large concentration of impurities, the physical results can be expressed in terms of a cross section. In all of these cases, in the zero-order approximation $\bar{G}_{\mathbf{p}}(\mathbf{r})$ does not depend on the angles of the vector $\mathbf{p}$, and therefore from Eq. (43) with (42) taken into account we obtain

$$\Sigma_0 = \text{Re} f_{\mathbf{p}\mathbf{p}'} + i\bar{G}_0 \text{Im} f_{\mathbf{p}\mathbf{p}'} \quad (50)$$

The first-order correction $\bar{G}_1$ anticommutes with $\bar{G}_0$, which follows from Eq. (49), and therefore from Eq. (43) we obtain the first-order correction to Σ :

$$\Sigma_{\mathbf{p}\mathbf{p}}^{(\mathbf{r})}(\mathbf{r}) = -\frac{iv}{2} \int \sigma_{\mathbf{p}\mathbf{p}_1} \bar{G}_{\mathbf{p}_1}^{(1)}(\mathbf{r}) d\Omega_{\mathbf{p}_1} \quad (51)$$

Substituting (51) into Eq. (48), we obtain formula (9) in the case of a weak field in the London limit.

In order to determine the critical field, we expand Eq. (48) to first order in Δ . As a result we obtain the following system of equations:

$$\left[\text{sign } \omega \cdot \mathbf{v} \left(\frac{\partial}{\partial \mathbf{r}} - 2ie\mathbf{A} \right) + 2|\omega| \right] K_{\mathbf{p}}(\mathbf{r}) = \Delta + nv \int \sigma_{\mathbf{p}_1} (K_{\mathbf{p}_1} - K_{\mathbf{p}}) d\Omega_{\mathbf{p}_1}$$

$$\Delta = \frac{|\lambda|mp_0}{(2\pi)^2} - T \sum_{\omega} \int K_{\mathbf{p}}(\mathbf{r}) d\Omega_{\mathbf{p}} \quad (52)$$

where

$$K_{\mathbf{p}}(\mathbf{r}) = 1/4\text{Sp}(\tau_y - i\tau_x) \bar{G}_{\mathbf{p}}^{(1)}(\mathbf{r})$$

For a thin superconducting film whose thickness is large in comparison with the mean free path but small in comparison with the dimensions of a pair, the system of equations (43), (48), and (49) may be solved for an arbitrary magnetic field. We seek the zero-order approximation to the Green's function in the form

$$\bar{G}_0 = \cos \varphi \tau_y + \sin \varphi \tau_z \quad (53)$$

In the zero-order approximation with respect to the mean free path, the angle φ is arbitrary and is determined from the condition that the average value of the correction to it is equal to zero. This correction shows up in the solution of Eq. (48) to second-order in the field and to first-order in ω and Δ . From the condition that $\bar{G}^{(2)} \sim \bar{G}_0$, we discover an equation for φ :

$$\Delta \text{tg } \varphi = \omega + 2/3e^2v^2\tau_{tr} \langle A^2 \rangle \sin \varphi \quad (54)$$

Equations (52) and (54) agree in appearance with the equations obtained previously [3,7,8] in the Born approximation.

8. Thus, one can assert that the majority of results in the theory of superconducting alloys, obtained in the Born approximation, remain valid even for a strong interaction between electrons and impurity atoms. Only in certain cases terms appear which are not present in the Born approximation and which depend not only on the cross section for the scattering of electrons by impurity atoms, but also depend on the explicit form of the amplitude.

Application of the method of classical trajectories to the theory of superconducting alloys is apparently limited to the case of weak fields. In order to calculate the dependence of the penetration depth on the field, the quantum character of the scattering of electrons by impurities turns out to be very important. Usually the method of classical trajectories has been applied in order to describe the diffuse reflection of electrons from a boundary, which is assumed to be classical. It is possible that in this case the quantum effects are less important.

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16 Three “universal” mesoscopic Josephson effects by Beenakker

A recent theory is reviewed for the sample-to-sample fluctuations in the critical current of a Josephson junction consisting of a disordered point contact or microbridge. The theory is

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based on a relation between the supercurrent and the scattering matrix in the normal state. The root-mean-square amplitude $\text{rms } I_c$ of the critical current I_c at zero temperature is given by $\text{rms } I_c \simeq e\Delta_0/\hbar$, up to a numerical coefficient of order unity (Δ_0 is the energy gap). This is the superconducting analogue of “Universal Conductance Fluctuations” in the normal state. The theory can also be applied to a ballistic point contact, where it yields the analogue of the quantized conductance, and to a quantum dot, where it describes supercurrent resonances. All three phenomena provide a measurement of the supercurrent unit $e\Delta_0/\hbar$, and are “universal” through the absence of a dependence on junction parameters.

16.1 Introduction

Nanostructures combining semiconducting and superconducting elements form a new class of systems in which to search for mesoscopic phenomena. The correspondence between transport of normal electrons and transport of Bogoliubov quasiparticles (being the elementary excitations of the superconductor) serves as a useful guide in the search. The appearance of a new length scale — the superconducting coherence length ξ — complicates the correspondence in an interesting way, by introducing two qualitatively different regimes: The *short-junction* regime (junction length $L \ll \xi$) and the *long-junction* regime ($L \gg \xi$). These two regimes appear over and above the transport regimes which depend on the relative magnitude of junction length and mean free path l : The *ballistic* regime ($l \gg L$) and the *diffusive* regime ($l \ll L$). A third transport regime in the normal state, that of *resonant tunneling*, is also subdivided into two new regimes, distinguished by the relative magnitude of ξ and the characteristic length $v_F\tau_{\text{res}}$, with τ_{res} the lifetime of the resonant state in the junction and v_F the Fermi velocity.

Our own interest in this field has focused on a set of three phenomena which provide, through the Josephson effect, a measurement of the supercurrent unit $e\Delta_0/\hbar$ (Δ_0 being the superconducting energy gap). This unit of current plays the role of the conductance quantum e^2/h in the normal state. The three phenomena are:

1. Discretization of the critical current of a ballistic point contact [17].
2. Resonant Josephson current through a quantum dot [2, 3].
3. Mesoscopic supercurrent fluctuations in a diffusive point contact [4].

These effects belong, respectively, to the ballistic, resonant-tunneling, and diffusive transport regimes, and within each regime to the short-junction limit.

Effect number 1 is the analogue of the quantized conductance of a quantum point contact [5]–[7]: The critical current I_c of a short and narrow constriction in a superconductor increases stepwise as a function of the constriction width, with step height $e\Delta_0/\hbar$ independent of the properties of the junction [17],

$$I_c = N \frac{e\Delta_0}{\hbar}. \quad (16.1)$$

The integer N is the number of transverse modes at the Fermi level which can propagate through the constriction. The short-junction limit $L \ll \xi_0$ (where $\xi_0 = \hbar v_F/\pi\Delta_0$ is the ballistic coherence length) is essential: Furusaki et al. [8] studied quantum size effects on the critical current in the long-junction limit and found a geometry-dependent behavior instead of Eq. (16.1).

Effect number 2 is described by the formula [3]

$$I_c^{\text{res}} = \frac{e}{\hbar} \frac{\Delta_0 \Gamma_0}{\Delta_0 + \Gamma_0} \quad (16.2)$$

Mesoscopic Josephson Effects

	short-junction	long-junction	E_c
ballistic	$I_c = Ne\Delta_0/\hbar$	geometry dep.	$\hbar v_F/L$
res.tunneling	$I_c^{\text{res}} = e\Delta_0/\hbar$	$I_c^{\text{res}} = e\Gamma_0/\hbar$	Γ_0
diffusive	$\text{rms } I_c \simeq e\Delta_0/\hbar$	$\text{rms } I_c \simeq eD/L^2$	$\hbar D/L^2$

Table 2: Summary of the mesoscopic Josephson effects described in the text, in the ballistic, resonant-tunneling, and diffusive transport regimes. Universal (junction-independent) behavior is obtained in the short-junction limit, which in terms of the correlation energy E_c is given by $\Delta_0 \ll E_c$.

for the critical current through a bound state at the Fermi level which is coupled with equal tunnel rate $\Gamma_0/\hbar$ to two bulk superconductors. Since $\tau_{\text{res}} = \hbar/2\Gamma_0$, the “short-junction” criterion $\xi_0 \gg v_F\tau_{\text{res}}$ is equivalent to $\Gamma_0 \gg \Delta_0$. If $\Gamma_0 \gg \Delta_0$, the critical current on resonance becomes [2, 3]

$$I_c^{\text{res}} = \frac{e\Delta_0}{\hbar}, \quad (16.3)$$

which no longer depends on the tunnel rate.

Effect number 3 is the analogue of “Universal Conductance Fluctuations” in disordered normal metals [9]–[11]. The sample-to-sample fluctuations in the critical current of a disordered point contact have root-mean-square value [4]

$$\text{rms } I_c \simeq \frac{e\Delta_0}{\hbar}, \quad (16.4)$$

up to a numerical coefficient of order unity. These mesoscopic fluctuations are *universal* in the sense that they do not depend on the size of the junction or on the degree of disorder, as long as the criteria $l \ll L \ll Nl$ and $L \ll \xi$ for the diffusive, short-junction regime are satisfied. (Here $\xi = (\xi_0 l)^{1/2}$ is the diffusive coherence length.) The short-junction limit $L \ll \xi$ is again essential for universality. The opposite long-junction limit was considered previously by Al’tshuler and Spivak [12], who calculated the supercurrent fluctuations of an SNS junction in the limit $L \gg \xi$ (S=superconductor, N=normal metal, L =separation of the SN interfaces). Their result $\text{rms } I_c \simeq ev_F l/L^2$ depends on both l and L and is therefore *not* universal in the sense of UCF.

We have summarized the three universal short-junction Josephson effects in Table 1, together with their non-universal long-junction counterparts. We have introduced the diffusion constant $D \simeq v_F l$ and the correlation energy (or Thouless energy) $E_c \simeq \hbar/\tau_{\text{dwell}}$, where τ_{dwell} is the dwell time in the junction. The distinction “short-junction” versus “long-junction” may alternatively be given in terms of the relative magnitude of E_c and Δ_0 : In the short-junction regime $\Delta_0 \ll E_c$, while in the long-junction regime $\Delta_0 \gg E_c$. The universality is lost in the long-junction regime because E_c (which depends on junction parameters) becomes the characteristic energy for the Josephson effect instead of Δ_0 . All these results hold in the zero-temperature limit, or more precisely for $k_B T \ll \min(\Delta_0, E_c)$.

In the present paper we review a scattering theory for the Josephson effect, which provides a unified treatment of the three universal (short-junction) phenomena. In Sec. 2 we show, following Ref. [13], how the supercurrent can be obtained directly from the quasiparticle excitation spectrum of the Josephson junction. This approach is equivalent to the usual method which starts from the finite-temperature Greens function, but is more convenient in the short-junction regime, where — as we shall see — the excitation spectrum has a particularly simple form. In Sec. 3 we follow Ref. [4] in relating the excitation spectrum to the normal-state scattering matrix of the junction. In Sec. 4 we take the short-junction limit and obtain a simple relation

between the supercurrent and the eigenvalues of the transmission matrix product $tt^\dagger$. This relation plays the role for the Josephson effect of the Landauer formula for the conductance [7]. We emphasize that the transmission matrix t refers to *normal* electrons. Other scattering approaches to the Josephson effect [14, 6] relate the supercurrent to the scattering matrix for Bogoliubov quasiparticles. The usefulness of the present method is that one can use directly the many properties of t known from normal-state transport theory. Indeed, as we shall see in Sec. 5, the three universal Josephson effects follow as special cases of the general scattering formula of Sec. 4.

16.2 Supercurrent from Excitation Spectrum

The quasiparticle excitation spectrum of the Josephson junction consists of the positive eigenvalues of the Bogoliubov-de Gennes (BdG) equation [16]. The BdG equation has the form of two Schrödinger equations for electron and hole wavefunctions $u(\mathbf{r})$ and $v(\mathbf{r})$, coupled by the pair potential $\Delta(\mathbf{r})$:

$$\begin{pmatrix} \mathcal{H}_0 & \Delta \\ \Delta^* & -\mathcal{H}_0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \varepsilon \begin{pmatrix} u \\ v \end{pmatrix}. \quad (16.5)$$

Here $\mathcal{H}_0 = \mathbf{p}^2/2m + V(\mathbf{r}) - E_F$ is the single-electron Hamiltonian, containing an electrostatic potential $V(\mathbf{r})$ (zero vector potential is assumed). The excitation energy ε is measured relative to the Fermi energy E_F .

In a uniform system with $\Delta(\mathbf{r}) \equiv \Delta_0 e^{i\phi}$, $V(\mathbf{r}) \equiv 0$, the solution of the BdG equation is

$$\begin{aligned} \varepsilon &= ((\hbar^2 k^2/2m - E_F)^2 + \Delta_0^2)^{1/2}, \\ u(\mathbf{r}) &= \mathcal{V}^{-1/2} (2\varepsilon)^{-1/2} e^{i\phi/2} (\varepsilon + \hbar^2 k^2/2m - E_F)^{1/2} e^{i\mathbf{k}\cdot\mathbf{r}}, \\ v(\mathbf{r}) &= \mathcal{V}^{-1/2} (2\varepsilon)^{-1/2} e^{-i\phi/2} (\varepsilon - \hbar^2 k^2/2m + E_F)^{1/2} e^{i\mathbf{k}\cdot\mathbf{r}}. \end{aligned} \quad (16.6)$$

The eigenfunction is normalized to unit probability in a volume $\mathcal{V}$,

$$\int_{\mathcal{V}} d\mathbf{r} (|u|^2 + |v|^2) = 1. \quad (16.7)$$

The excitation spectrum is continuous, with excitation gap Δ_0 . The eigenfunctions (u, v) are plane waves characterized by a wavevector $\mathbf{k}$. The coefficients of the plane waves are the two coherence factors of the BCS theory. This simple excitation spectrum is modified by the presence of the Josephson junction. The spectrum acquires a discrete part due to the non-uniformities in $\Delta(\mathbf{r})$ near the junction. The discrete spectrum corresponds to bound states in the gap ($0 < \varepsilon < \Delta_0$), localized within a coherence length from the junction. In addition, the continuous spectrum is modified. As we shall show in Sec. 3, in the short-junction limit the supercurrent is entirely determined by the *discrete* part of the excitation spectrum.

Consider an SNS junction with normal region at $|x| < L/2$. Let $\Delta(\mathbf{r}) \rightarrow \Delta_0 e^{\mp i\phi/2}$ for $x \rightarrow \pm\infty$. To determine $\Delta(\mathbf{r})$ near the junction one has to solve the self-consistency equation [16]

$$\Delta(\mathbf{r}) = |g(\mathbf{r})| \sum_{\varepsilon > 0} v^*(\mathbf{r}) u(\mathbf{r}) [1 - 2f(\varepsilon)], \quad (16.8)$$

where the sum is over all states with positive eigenvalue, and $f(\varepsilon) = [1 + \exp(\varepsilon/k_B T)]^{-1}$ is the Fermi function. The coefficient g is the interaction constant of the BCS theory of superconductivity. At an SN interface, g drops abruptly (over atomic distances) to zero. (We

assume non-interacting electrons in the normal region.) Therefore, $\Delta(\mathbf{r}) \equiv 0$ for $|x| < L/2$ regardless of Eq. (16.8). At the superconducting side of the SN interface, $\Delta(\mathbf{r})$ recovers its bulk value Δ_0 only at some distance from the interface (proximity effect). There exists a class of Josephson junctions where the suppression of $\Delta(\mathbf{r})$ on approaching the SN interface can be neglected, and one can use the step-function model

$$\Delta(\mathbf{r}) = \begin{cases} \Delta_0 e^{i\phi/2} & \text{if } x < -L/2, \\ 0 & \text{if } |x| < L/2, \\ \Delta_0 e^{-i\phi/2} & \text{if } x > L/2. \end{cases} \quad (16.9)$$

The step-function pair potential is also referred to in the literature as a “rigid boundary-condition”. Likharev [17] discusses in detail the conditions under which this model is valid:

1. If the width W of the junction is small compared to the coherence length, the non-uniformities in $\Delta(\mathbf{r})$ extend only over a distance of order W from the junction (because of “geometrical dilution” of the influence of the narrow junction in the wide superconductor). Since non-uniformities on length scales $\ll \xi$ do not affect the dynamics of the quasiparticles, these can be neglected and the step-function model holds. Constrictions in the short-junction limit belong in general to this class of junctions. Note that, if the length and width of the junction are $\ll \xi$, it is irrelevant whether it is made from a normal metal or from a superconductor. In the literature, such a junction is referred to as an ScS junction, without specifying whether the constriction (c) is N or S.
2. Alternatively, the step-function model holds if the resistivity of the junction region is much bigger than the resistivity of the bulk superconductor. This condition is formulated more precisely by Kupriyanov et al. [18], who have also studied deviations from the ideal step-function model due to the proximity effect and due to current-induced suppression of the pair potential. A superconductor – semiconductor – superconductor junction is typically in this second category.

In equilibrium, a phase difference ϕ in the pair potential induces a stationary current I through the junction (DC Josephson effect). The current–phase relationship $I(\phi)$ is 2π -periodic and antisymmetric [$I(\phi + 2\pi) = I(\phi)$, $I(-\phi) = -I(\phi)$]. The maximum $I_c \equiv \max I(\phi)$ is known as the critical current. There is a thermodynamic relation

$$I = \frac{2e}{\hbar} \frac{dF}{d\phi} \quad (16.10)$$

between the equilibrium current and the derivative of the free energy F with respect to the phase difference. To apply this relation we need to know how to obtain F from the BdG equation. The required formula was derived by Bardeen et al. [18] from the Greens function expression for F . An alternative derivation, directly from the BdG equation, has been given in Ref. [13]. The result is

$$F = -2k_B T \sum_{\varepsilon > 0} \ln [2 \cosh(\varepsilon/2k_B T)] + \int d\mathbf{r} |\Delta|^2/|g| + \text{Tr } \mathcal{H}_0. \quad (16.11)$$

The first term in Eq. (16.11) (the sum over ε) can be formally interpreted as the free energy of non-interacting electrons, all of one single spin, in a “semiconductor” with Fermi level halfway between the “conduction band” (positive ε) and the “valence band” (negative ε). This semiconductor model of a superconductor appeals to intuition, but does not give the free energy correctly. The second term in Eq. (16.11) corrects for a double-counting of the interaction energy in the semiconductor model. The third term $\text{Tr } \mathcal{H}_0$ (i.e. the sum of the single-electron eigenenergies) cancels a divergence at large ε of the series in the first term.

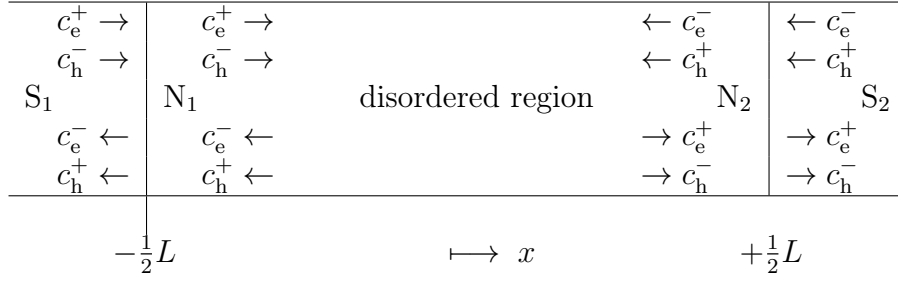


Figure 16.1: Superconductor — normal-metal — superconductor Josephson junction containing a disordered normal region. Scattering states in the normal and superconducting leads are indicated schematically. Normal scattering (disordered region) and Andreev reflection (SN interface) are spatially separated.

From Eqs. (16.10) and (16.11) one obtains [13]

$$I = -\frac{2e}{\hbar} \sum_p \tanh(\varepsilon_p/2k_B T) \frac{d\varepsilon_p}{d\phi} - \frac{2e}{\hbar} 2k_B T \int_{\Delta_0}^{\infty} d\varepsilon \ln[2 \cosh(\varepsilon/2k_B T)] \frac{d\rho}{d\phi} + \frac{2e}{\hbar} \frac{d}{d\phi} \int dx |\Delta|^2/|g|, \quad (16.12)$$

where we have rewritten $\sum_{\varepsilon>0}$ as a sum over the discrete positive eigenvalues ε_p ($p = 1, 2, \dots$), and an integration over the continuous spectrum with density of states $\rho(\varepsilon)$. The term $\text{Tr } \mathcal{H}_0$ in Eq. (16.11) does not depend on ϕ , and therefore does not contribute to I . The spatial integral of $|\Delta|^2/|g|$ does contribute in general. If the step-function model for $\Delta(\mathbf{r})$ holds, however, $|\Delta|$ is independent of ϕ so that this contribution can be disregarded. A calculation of the Josephson current from Eq. (16.12) then requires only knowledge of the eigenvalues, not of the eigenfunctions.

16.3 Excitation spectrum from scattering matrix

Following Ref. [4], we now show how to relate the excitation spectrum of Bogoliubov quasiparticles to the scattering matrix of normal electrons. The supercurrent then follows from Eq. (16.12). The model considered is illustrated in Fig. 1. It consists of a disordered normal region between two superconducting regions S_1 and S_2 . The disordered region may or may not contain a geometrical constriction. To obtain a well-defined scattering problem we insert ideal (impurity-free) normal leads N_1 and N_2 to the left and right of the disordered region. The leads should be long compared to the Fermi wavelength λ_F , but short compared to the coherence length ξ_0 . The SN interfaces are located at $x = \pm L/2$. We assume that the only scattering in the superconductors consists of Andreev reflection at the SN interfaces, i.e. we consider the case that the disorder is contained entirely within the normal region. The spatial separation of Andreev and normal scattering is the key simplification which allows us to relate the supercurrent directly to the normal-state scattering matrix.

We first construct a basis for the scattering matrix (s -matrix). In the normal lead N_1 the eigenfunctions of the BdG equation (16.5) can be written in the form

$$\begin{aligned} \Psi_{n,e}^{\pm}(N_1) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} (k_n^e)^{-1/2} \Phi_n(y, z) \exp[\pm i k_n^e (x + \frac{1}{2}L)], \\ \Psi_{n,h}^{\pm}(N_1) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} (k_n^h)^{-1/2} \Phi_n(y, z) \exp[\pm i k_n^h (x + \frac{1}{2}L)], \end{aligned} \quad (16.13)$$

where the wavenumbers k_n^e and k_n^h are given by

$$k_n^{e,h} \equiv (2m/\hbar^2)^{1/2}(E_F - E_n + \sigma^{e,h}\varepsilon)^{1/2}, \quad (16.14)$$

and we have defined $\sigma^e \equiv 1$, $\sigma^h \equiv -1$. The labels e and h indicate the electron or hole character of the wavefunction. The index n labels the modes, $\Phi_n(y, z)$ is the transverse wavefunction of the n -th mode, and E_n its threshold energy:

$$[(p_y^2 + p_z^2)/2m + V(y, z)]\Phi_n(y, z) = E_n\Phi_n(y, z). \quad (16.15)$$

The eigenfunction Φ_n is normalized to unity, $\int dy \int dz |\Phi_n|^2 = 1$. The eigenfunctions in lead N₂ are chosen similarly, but with $x + \frac{1}{2}L$ replaced by $x - \frac{1}{2}L$.

In the superconducting lead S₁, where $\Delta = \Delta_0 \exp(i\phi/2)$, the eigenfunctions are

$$\begin{aligned} \Psi_{n,e}^\pm(S_1) &= \begin{pmatrix} e^{i\eta^e/2} \\ e^{-i\eta^e/2} \end{pmatrix} (2q_n^e)^{-1/2} (\varepsilon^2/\Delta_0^2 - 1)^{-1/4} \Phi_n \exp[\pm i q_n^e (x + \frac{1}{2}L)], \\ \Psi_{n,h}^\pm(S_1) &= \begin{pmatrix} e^{i\eta^h/2} \\ e^{-i\eta^h/2} \end{pmatrix} (2q_n^h)^{-1/2} (\varepsilon^2/\Delta_0^2 - 1)^{-1/4} \Phi_n \exp[\pm i q_n^h (x + \frac{1}{2}L)]. \end{aligned} \quad (16.16)$$

We have defined

$$q_n^{e,h} \equiv (2m/\hbar^2)^{1/2}[E_F - E_n + \sigma^{e,h}(\varepsilon^2 - \Delta_0^2)^{1/2}]^{1/2}, \quad (16.17)$$

$$\eta^{e,h} \equiv \frac{1}{2}\phi + \sigma^{e,h} \arccos(\varepsilon/\Delta_0). \quad (16.18)$$

The square roots are to be taken such that $\text{Re } q^{e,h} \geq 0$, $\text{Im } q^e \geq 0$, $\text{Im } q^h \leq 0$. The function $\arccos t \in (0, \pi/2)$ for $0 < t < 1$, while $\arccos t \equiv -i \ln[t + (t^2 - 1)^{1/2}]$ for $t > 1$. The eigenfunctions in lead S₂, where $\Delta = \Delta_0 \exp(-i\phi/2)$, are obtained by replacing ϕ by $-\phi$ and L by $-L$.

The wavefunctions (16.13) and (16.16) have been normalized to carry the same amount of quasiparticle current, because we want to use them as the basis for a unitary s -matrix. Unitarity of the s -matrix is a consequence of the fact that the BdG equation conserves the quasiparticle current density

$$\mathbf{j}_{\text{qp}}(\mathbf{r}) = \frac{1}{m} \text{Re} \begin{pmatrix} u^* \\ v^* \end{pmatrix} \begin{pmatrix} \mathbf{p} & \emptyset \\ \emptyset & -\mathbf{p} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \quad (16.19)$$

The quasiparticle current I_{qp} through a lead is given by

$$I_{\text{qp}} = \int dy \int dz (\mathbf{j}_{\text{qp}})_x = \frac{\hbar}{m} \text{Re} \int dy \int dz \left(u^* \frac{\partial}{i\partial x} u - v^* \frac{\partial}{i\partial x} v \right). \quad (16.20)$$

One easily verifies that, provided the wavenumbers $k_n^{e,h}$ and $q_n^{e,h}$ are real (i.e. for propagating modes), each of the wavefunctions (16.13, 16.16) carries the same amount of quasiparticle current $|I_{\text{qp}}|$. (The direction of the current is opposite for Ψ_e^+, Ψ_h^- and Ψ_e^-, Ψ_h^+ .) We note that the wavefunctions (16.13) and (16.16) carry *different* amounts of *charge* current (i.e. electrical current), which is given by

$$I_{\text{charge}} = -e \frac{\hbar}{m} \text{Re} \int dy \int dz \left(u^* \frac{\partial}{i\partial x} u + v^* \frac{\partial}{i\partial x} v \right). \quad (16.21)$$

This does not contradict the unitarity of the s -matrix constructed from these wavefunctions, because the BdG equation does not conserve the charge of the quasiparticles. (Andreev reflection [20], i.e. the reflection of an electron into a hole at an SN interface, is an example of a non-charge-conserving scattering process.)

A wave incident on the disordered normal region is described in the basis (16.13) by a vector of coefficients

$$c_N^{\text{in}} \equiv (c_e^+(N_1), c_e^-(N_2), c_h^-(N_1), c_h^+(N_2)), \quad (16.22)$$

as shown schematically in Fig. 1. (The mode-index n has been suppressed for simplicity of notation.) The reflected and transmitted wave has vector of coefficients

$$c_N^{\text{out}} \equiv (c_e^-(N_1), c_e^+(N_2), c_h^+(N_1), c_h^-(N_2)). \quad (16.23)$$

The s -matrix s_N of the normal region relates these two vectors, $c_N^{\text{out}} = s_N c_N^{\text{in}}$. Because the normal region does not couple electrons and holes, this matrix has the block-diagonal form

$$s_N(\varepsilon) = \begin{pmatrix} s_0(\varepsilon) & \emptyset \\ \emptyset & s_0(-\varepsilon)^* \end{pmatrix}, \quad s_0 \equiv \begin{pmatrix} r_{11} & t_{12} \\ t_{21} & r_{22} \end{pmatrix}. \quad (16.24)$$

Here s_0 is the unitary and symmetric s -matrix associated with the single-electron Hamiltonian $\mathcal{H}_0$. The reflection and transmission matrices $r(\varepsilon)$ and $t(\varepsilon)$ are $N \times N$ matrices, $N(\varepsilon)$ being the number of propagating modes at energy ε . (We assume for simplicity that the number of modes in leads N_1 and N_2 is the same.) The dimension of $s_N(\varepsilon)$ is $2N(\varepsilon) + 2N(-\varepsilon)$.

We will make use of two more s -matrices. For energies $0 < \varepsilon < \Delta_0$ there are no propagating modes in the superconducting leads S_1 and S_2 . We can then define an s -matrix s_A for Andreev reflection at the SN interfaces by $c_N^{\text{in}} = s_A c_N^{\text{out}}$. The elements of s_A can be obtained by matching the wavefunctions (16.13) at $|x| = L/2$ to the decaying wavefunctions (16.16). Since $\Delta_0 \ll E_F$ one may ignore normal reflections at the SN interface and neglect the difference between $N(\varepsilon)$ and $N(-\varepsilon)$. This is known as the Andreev approximation [20]. The result is

$$s_A = \alpha \begin{pmatrix} \emptyset & r_A \\ r_A^* & \emptyset \end{pmatrix}, \quad r_A \equiv \begin{pmatrix} e^{i\phi/2} \mathbf{1} & \emptyset \\ \emptyset & e^{-i\phi/2} \mathbf{1} \end{pmatrix}, \quad (16.25)$$

where $\alpha \equiv \exp[-i \arccos(\varepsilon/\Delta_0)]$. The matrices $\mathbf{1}$ and $\emptyset$ are the unit and null matrices, respectively. Andreev reflection transforms an electron mode into a hole mode, without change of mode index. The transformation is accompanied by a phase shift, which consists of two parts: 1. A phase shift $-\arccos(\varepsilon/\Delta_0)$ due to the penetration of the wavefunction into the superconductor; 2. A phase shift equal to plus or minus the phase of the pair potential in the superconductor (*plus* for reflection from hole to electron, *minus* for the reverse process).

For $\varepsilon > \Delta_0$ we can define the s -matrix s_{SNS} of the whole junction, by $c_S^{\text{out}} = s_{\text{SNS}} c_S^{\text{in}}$. The vectors

$$c_S^{\text{in}} \equiv (c_e^+(S_1), c_e^-(S_2), c_h^-(S_1), c_h^+(S_2)), \quad (16.26)$$

$$c_S^{\text{out}} \equiv (c_e^-(S_1), c_e^+(S_2), c_h^+(S_1), c_h^-(S_2)). \quad (16.27)$$

are the coefficients in the expansion of the incoming and outgoing wave in leads S_1 and S_2 in terms of the wavefunctions (16.16) (cf. Fig. 1). By matching the wavefunctions (16.13) and (16.16) at $|x| = L/2$, we arrive after some algebra (using again $\Delta_0 \ll E_F$) at the matrix-product expression

$$s_{\text{SNS}} = U^{-1} (\mathbf{1} - M)^{-1} (\mathbf{1} - M^\dagger) s_N U, \quad (16.28)$$

$$U \equiv \begin{pmatrix} r_A & \emptyset \\ \emptyset & r_A^* \end{pmatrix}^{\frac{1}{2}}, \quad M \equiv \alpha s_N \begin{pmatrix} \emptyset & r_A \\ r_A^* & \emptyset \end{pmatrix}.$$

One can verify that the three s -matrices defined above (s_N ; s_A for $0 < \varepsilon < \Delta_0$; s_{SNS} for $\varepsilon > \Delta_0$) are *unitary* ($s^\dagger s = s s^\dagger = \mathbf{1}$) and satisfy the *symmetry relation* $s(\varepsilon, \phi)_{ij} = s(\varepsilon, -\phi)_{ji}$, as required by quasiparticle-current conservation and by time-reversal invariance, respectively.

We are now ready to relate the excitation spectrum of the Josephson junction to the s -matrix of the normal region. First the discrete spectrum. The condition $c_{\text{in}} = s_A s_N c_{\text{in}}$ for a bound state implies $\text{Det}(\mathbf{1} - s_A s_N) = 0$. Using Eqs. (16.24), (16.25), and the *folding-identity*

$$\text{Det} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{Det}(ad - ac a^{-1}b) \quad (16.29)$$

(which holds for arbitrary square matrices a, b, c, d of equal dimension, with $\text{Det } a \neq 0$), we find the equation

$$\text{Det} [\mathbf{1} - \alpha(\varepsilon_p)^2 r_A^* s_0(\varepsilon_p) r_A s_0(-\varepsilon_p)^*] = 0, \quad (16.30)$$

which determines the discrete spectrum. The density of states of the continuous spectrum is related to s_{SNS} by the general relation [21]

$$\rho = \frac{1}{2\pi i} \frac{\partial}{\partial \varepsilon} \ln \text{Det } s_{\text{SNS}} + \text{constant}, \quad (16.31)$$

where “constant” indicates a ϕ -independent term. From Eqs. (16.28) and (16.29) we find

$$\frac{\partial \rho}{\partial \phi} = -\frac{1}{\pi} \frac{\partial^2}{\partial \phi \partial \varepsilon} \text{Im} \ln \text{Det} [\mathbf{1} - \alpha(\varepsilon)^2 r_A^* s_0(\varepsilon) r_A s_0(-\varepsilon)^*], \quad (16.32)$$

which determines the ϕ -dependence of the continuous spectrum.

16.4 Short-Junction Limit

In the short-junction limit $L \ll \xi$, the determinantal equations (16.30) and (16.32) can be simplified further. As mentioned in the Introduction, the condition $L \ll \xi$ is equivalent to $\Delta_0 \ll E_c$, where the correlation energy $E_c \equiv \hbar/\tau_{\text{dwell}}$ is defined in terms of the dwell time τ_{dwell} in the junction. The elements of $s_0(\varepsilon)$ change significantly if ε is changed by at least E_c [22, 23]. We are concerned with ε of order Δ_0 or smaller (since $\rho(\varepsilon, \phi)$ becomes independent of ϕ for $\varepsilon \gg \Delta_0$). For $\Delta_0 \ll E_c$ we may thus approximate $s_0(\varepsilon) \approx s_0(-\varepsilon) \approx s_0(0) \equiv s_0$. Eq. (16.30) may now be simplified by multiplying both sides by $\text{Det } s_0$ and using $s_0^* s_0 = \mathbf{1}$ (unitarity plus symmetry of s_0), as well as the folding identity (16.29). The result can be written in the form

$$\text{Det} \left[(1 - \varepsilon_p^2 / \Delta_0^2) \mathbf{1} - t_{12} t_{12}^\dagger \sin^2(\phi/2) \right] = 0. \quad (16.33)$$

For $\varepsilon > \Delta_0$ one can see that $\text{Det}[\mathbf{1} - \alpha(\varepsilon)^2 r_A^* s_0 r_A s_0^*]$ is a real number. (Use that α is real for $\varepsilon > \Delta_0$ and that the determinant $\text{Det}(\mathbf{1} - aa^*)$ is real for arbitrary matrix a .) Eq. (16.32) then reduces to $\partial \rho / \partial \phi = 0$, from which we conclude that *the continuous spectrum does not contribute to $I(\phi)$ in the short-junction limit.*

Eq. (16.33) can be solved for ε_p in terms of the eigenvalues T_p ($p = 1, 2, \dots, N$) of the hermitian $N \times N$ matrix $t_{12} t_{12}^\dagger$,

$$\varepsilon_p = \Delta_0 [1 - T_p \sin^2(\phi/2)]^{1/2}. \quad (16.34)$$

Since $t_{12} t_{12}^\dagger = r_{11} t_{21}^\dagger t_{21} r_{11}^{-1}$ (as follows from unitarity of s_0), the matrices $t_{12} t_{12}^\dagger$ and $t_{21} t_{21}^\dagger$ have the same set of eigenvalues. We can therefore omit the indices of $tt^\dagger$. Unitarity of s_0 also implies that $0 \leq T_p \leq 1$ for all p .

Substitution of Eq. (16.34) into Eq. (16.12) yields the Josephson current

$$I(\phi) = \frac{e\Delta_0}{2\hbar} \sum_{p=1}^N \frac{T_p \sin \phi}{[1 - T_p \sin^2(\phi/2)]^{1/2}} \tanh \left(\frac{\Delta_0}{2k_B T} [1 - T_p \sin^2(\phi/2)]^{1/2} \right). \quad (16.35)$$

Eq. (16.35) holds for an arbitrary transmission matrix $tt^\dagger$, i.e. for arbitrary disorder potential. It is the *multi-channel* generalization of a formula first obtained by Haberkorn et al. [24] (and subsequently rederived by several authors [25]–[27]) for the *single-channel* case (appropriate for a geometry such as a planar tunnel barrier, where the different scattering channels are uncoupled). A formula of similar generality for the conductance is the multi-channel Landauer formula [23, 28]

$$G = \frac{2e^2}{h} \text{Tr } tt^\dagger \equiv \frac{2e^2}{h} \sum_{p=1}^N T_p. \quad (16.36)$$

In contrast to the Landauer formula, Eq. (16.35) is a *non-linear* function of the transmission eigenvalues T_p . It follows that knowledge of the conductance (i.e. of the sum of the eigenvalues) is *not sufficient* to determine the supercurrent.

16.5 Universal Josephson Effects

16.5.1 Quantum Point Contact

Consider the case that the weak link consists of a ballistic constriction ($l \gg L$) with a conductance quantized at $G = 2N_0 e^2/h$ (a *quantum point contact* [7]). The integer N_0 is the number of occupied one-dimensional subbands (per spin direction) in the constriction, or alternatively the number of transverse modes at the Fermi level which can propagate through the constriction. Note that $N_0 \ll N$. A quantum point contact is characterized by a special set of transmission eigenvalues, which are equal to either 0 or 1:

$$T_p = \begin{cases} 1 & \text{if } 1 \leq p \leq N_0, \\ 0 & \text{if } N_0 < p \leq N, \end{cases} \quad (16.37)$$

where the eigenvalues have been ordered from large to small. We emphasize that Eq. (16.37) is valid whether the transport through the constriction is adiabatic or not. In the special case of adiabatic transport, the transmission eigenvalue T_p is equal to the transmission probability $\mathcal{T}_p$ of the p -th subband. In the absence of adiabaticity there is no simple relation between T_p and $\mathcal{T}_p$.

The discrete spectrum (16.34) in the short-junction limit contains an N_0 -fold degenerate state at energy $\varepsilon = \Delta_0 |\cos(\phi/2)|$. Eq. (16.35) for the supercurrent becomes

$$I(\phi) = N_0 \frac{e\Delta_0}{\hbar} \sin(\phi/2) \tanh \left(\frac{\Delta_0}{2k_B T} \cos(\phi/2) \right), \quad |\phi| < \pi. \quad (16.38)$$

At $T = 0$ the current–phase relationship is given by $I(\phi) = N_0(e\Delta_0/\hbar) \sin(\phi/2)$ for $|\phi| < \pi$, and continued periodically for $|\phi| > \pi$. At $\phi = \pi$ the function $I(\phi)$ has a discontinuity which is smeared at finite temperatures. The ratio $I(\phi)/(\pi\Delta_0 G/e)$ at $T = 0$ is plotted in Fig. 2 (solid curve). The critical current $I_c = N_0 e\Delta_0/\hbar$ is *discretized* in units of $e\Delta_0/\hbar$. In the classical limit $N_0 \rightarrow \infty$ we recover the results of Kulik and Omel’yanchuk [29] for a *classical* ballistic point contact.

Eq. (16.38) was derived in Ref. [17] under the assumption of adiabatic transport. The present derivation (taken from Ref. [4]) does not assume adiabaticity. As discussed in Ref. [17], Eq. (16.38) breaks down if the Fermi level lies within Δ_0 of the threshold energy E_n of a subband. In the present context this breakdown can be understood from the fact that the dwell time $\tau_{\text{dwell}} = (L/v_F)(1 - E_n/E_F)^{-1/2}$ becomes infinitely long as E_F approaches E_n , so that the “short-junction” condition $\Delta_0 \ll E_c \equiv \hbar/\tau_{\text{dwell}}$ is violated.

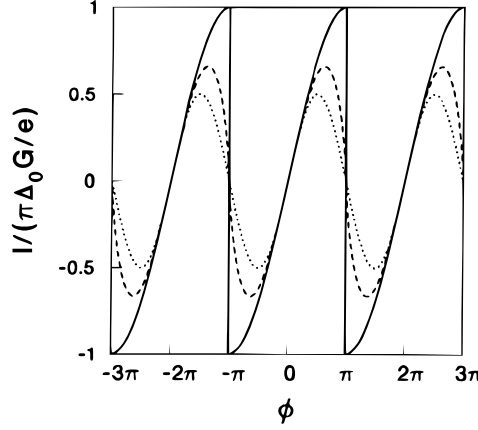


Figure 16.2: Current–phase relationship at $T = 0$ of a ballistic point contact [solid curve, Eq. (16.38)], a disordered point contact (ensemble averaged) [dashed curve, Eq. (16.49)], and a tunnel junction [dotted curve, Eq. (16.50)]. The current is normalized by the normal-state conductance of the junction.

16.5.2 Quantum Dot

Consider a small confined region (of dimensions comparable to the Fermi wavelength), which is weakly coupled by tunnel barriers to two electron reservoirs. We assume that transport through this *quantum dot* occurs via resonant tunneling through a single bound state. Let ε_{res} be the energy of the resonant level, relative to the Fermi energy E_F in the reservoirs, and let $\Gamma_1/\hbar$ and $\Gamma_2/\hbar$ be the tunnel rates through the left and right barriers. We denote $\Gamma \equiv \Gamma_1 + \Gamma_2$. If $\Gamma \ll \Delta E$ (with ΔE the level spacing in the quantum dot) and $T \ll \Gamma/k_B$, the conductance G in the case of non-interacting electrons has the form

$$G = \frac{2e^2}{h} \frac{\Gamma_1 \Gamma_2}{\varepsilon_{\text{res}}^2 + \frac{1}{4}\Gamma^2} \equiv \frac{2e^2}{h} T_{\text{BW}}, \quad (16.39)$$

where T_{BW} is the Breit-Wigner transmission probability at the Fermi level. The normal-state scattering matrix $s_0(\varepsilon)$ which yields this conductance has matrix elements [30]

$$(s_0(\varepsilon))_{nm} = \left(\delta_{nm} - \frac{i\sqrt{\Gamma_{1n}\Gamma_{2m}}}{\varepsilon - \varepsilon_{\text{res}} + i\Gamma/2} \right) e^{i(\delta_{1n} + \delta_{2m})}, \quad (16.40)$$

where $\sum_{n=1}^N \Gamma_{1n} \equiv \Gamma_1$, $\sum_{n=1}^N \Gamma_{2n} \equiv \Gamma_2$. The phases δ_{1n} , δ_{2m} , as well as the basis with respect to which the matrix elements (16.40) are calculated, need not be further specified. Büttiker [31] has shown how the conductance (16.39) follows, via the Landauer formula (16.36), from the Breit-Wigner scattering matrix (16.40).

In Ref. [3] Van Houten and the author have calculated the supercurrent through the quantum dot from the Breit-Wigner formula (16.40) in the special case of single-channel leads ($N = 1$). Just as for the conductance [31]–[33] one would expect that the results are not changed for multi-channel leads ($N > 1$). We will now demonstrate this explicitly in the limit $\Gamma \gg \Delta_0$ (which for resonant tunneling corresponds to the “short-junction” limit, cf. Sec. 1). The transmission matrix product $t_{12}t_{12}^\dagger$ (evaluated at the Fermi level $\varepsilon = 0$) following from the scattering matrix (16.40) is

$$\left(t_{12}t_{12}^\dagger \right)_{nm} = T_{\text{BW}} \gamma_n \gamma_m^*, \quad \gamma_n \equiv e^{i\delta_n} \sqrt{\Gamma_{1n}/\Gamma_1}. \quad (16.41)$$

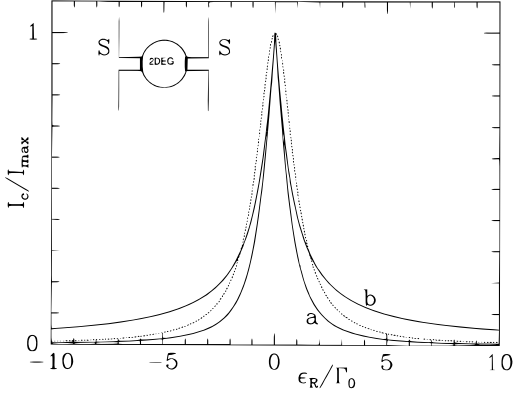


Figure 16.3: Normalized critical current versus energy of the resonant level, at zero temperature and for equal tunnel barriers ($\Gamma_1 = \Gamma_2 \equiv \Gamma_0$). The two solid curves are the results (16.44) and (16.45) for the two regimes $\Gamma_0 \gg \Delta_0$ (curve a) and $\Gamma_0, \varepsilon_{\text{res}} \ll \Delta_0$ (curve b). The dotted curve is the Breit-Wigner transmission probability (16.39). The inset shows schematically the quantum-dot Josephson junction. Taken from Ref. [3].

Its eigenvalues are

$$T_p = \begin{cases} T_{\text{BW}} & \text{if } p = 1, \\ 0 & \text{if } 2 \leq p \leq N. \end{cases} \quad (16.42)$$

The single non-zero eigenvalue has eigenvector $u_n = \gamma_n$, the other eigenvectors span the plane orthogonal to the vector γ . Substitution into Eq. (16.35) yields the current-phase relationship for a wide resonance ($\Gamma \gg \Delta_0$),

$$I(\phi) = \frac{e\Delta_0}{2\hbar} \frac{T_{\text{BW}} \sin \phi}{[1 - T_{\text{BW}} \sin^2(\phi/2)]^{1/2}} \tanh \left(\frac{\Delta_0}{2k_{\text{B}}T} [1 - T_{\text{BW}} \sin^2(\phi/2)]^{1/2} \right). \quad (16.43)$$

The critical current at zero temperature is

$$I_c = \frac{e\Delta_0}{\hbar} [1 - (1 - T_{\text{BW}})^{1/2}], \text{ if } \Gamma \gg \Delta_0, \quad (16.44)$$

in agreement with Ref. [3]. Since $T_{\text{BW}} = 1$ on resonance ($\varepsilon_{\text{R}} = 0$) in the case of equal tunnel rates ($\Gamma_1 = \Gamma_2$), we obtain the result (16.3) discussed in the Introduction.

For completeness, we also quote the formula for the current-phase relationship in the opposite regime of a narrow resonance (derived in Ref. [3]):

$$I_c = \frac{e}{\hbar} (\varepsilon_{\text{res}}^2 + \frac{1}{4}\Gamma^2)^{1/2} [1 - (1 - T_{\text{BW}})^{1/2}], \text{ if } \Gamma, \varepsilon_{\text{res}} \ll \Delta_0. \quad (16.45)$$

As shown in Fig. 3, the lineshapes (16.44) and (16.45) of a resonance in the critical current (solid curves) differ substantially from the lorentzian lineshape (16.39) of a conductance resonance (dotted curve). For $\Gamma_1 = \Gamma_2$, I_c has a *cusp* at $\varepsilon_{\text{res}} = 0$ (which is rounded at finite temperatures). On resonance, the maximum critical current I_c^{res} equals $(2e\Delta_0/\hbar\Gamma) \min(\Gamma_1, \Gamma_2)$ and $(e/\hbar) \min(\Gamma_1, \Gamma_2)$ for a wide and narrow resonance, respectively. An analytical formula, Eq. (16.2), for the crossover between these two regimes can be obtained for the case of equal tunnel rates [3]. Off-resonance, I_c has the lorentzian decay $\propto 1/\varepsilon_{\text{res}}^2$ in the case $\Gamma \gg \Delta_0$ of a wide resonance, but a slower decay $\propto 1/\varepsilon_{\text{res}}$ in the case $\Gamma, \varepsilon_{\text{res}} \ll \Delta_0$. Near $\varepsilon_{\text{res}} \simeq \Delta_0$ this linear decay of the narrow resonance crosses over to a quadratic decay (not shown in Fig. 3).

Since we have assumed non-interacting quasiparticles, the above results apply to a quantum dot with a small charging energy U for double occupancy of the resonant state. Glazman and

Matveev have studied the influence of Coulomb repulsion on the resonant supercurrent [2]. The influence is most pronounced in the case of a narrow resonance, when the critical current is suppressed by a factor Γ/Δ_0 (for $U, \Delta_0 \gg \Gamma$). In the case of a wide resonance, the Coulomb repulsion does not suppress the supercurrent, but slightly broadens the resonance by a factor $\ln(\Gamma/\Delta_0)$ (for $U, \Gamma \gg \Delta_0$). The broadening is a consequence of the Kondo effect, and occurs only for $\varepsilon_{\text{res}} < 0$, so that the resonance peak becomes somewhat asymmetric [2].

16.5.3 Disordered Point Contact

We now turn to the regime of diffusive transport through a disordered point contact. We first consider the average supercurrent (averaged over an ensemble of impurity configurations) and then the fluctuations from the average. This section corrects Ref. [4].

Average Supercurrent

The transmission eigenvalue T_p is related to a channel-dependent localization length ζ_p by

$$T_p = \cosh^{-2}(L/\zeta_p). \quad (16.46)$$

The inverse localization length is uniformly distributed between 0 and $1/\zeta_{\min} \simeq 1/l$ for $l \ll L \ll Nl$ [34]. One can therefore write

$$\frac{\langle \sum_{p=1}^N f(T_p) \rangle}{\langle \sum_{p=1}^N T_p \rangle} = \frac{\int_0^{L/\zeta_{\min}} dx f(\cosh^{-2} x)}{\int_0^{L/\zeta_{\min}} dx \cosh^{-2} x} = \int_0^\infty dx f(\cosh^{-2} x), \quad (16.47)$$

where $\langle \dots \rangle$ indicates the ensemble average and $f(T)$ is an arbitrary function of the transmission eigenvalue such that $f(T) \rightarrow 0$ for $T \rightarrow 0$. In the second equality in Eq. (16.47) we have used that $L/\zeta_{\min} \simeq L/l \gg 1$ to replace the upper integration limit by ∞ . Note that, in view of the Landauer formula (16.36), the denominator $\langle \sum_{p=1}^N T_p \rangle$ is just $h/2e^2$ times the average conductance $\langle G \rangle$.

Combining Eqs. (16.47) and (16.35) we find for the average supercurrent the expression ($0 < \phi < \pi$)

$$\begin{aligned} \langle I(\phi) \rangle &= \frac{e\Delta_0}{2\hbar} \frac{h}{2e^2} \langle G \rangle \int_0^\infty dx \frac{(\cosh x)^{-2} \sin \phi}{[1 - (\cosh x)^{-2} \sin^2(\phi/2)]^{1/2}} \\ &\quad \times \tanh \left(\frac{\Delta_0}{2k_B T} [1 - (\cosh x)^{-2} \sin^2(\phi/2)]^{1/2} \right) \\ &= \frac{\pi\Delta_0}{e} \langle G \rangle \cos(\phi/2) \int_{\cos(\phi/2)}^1 dz \frac{\tanh(\Delta_0 z / 2k_B T)}{[z^2 - \cos^2(\phi/2)]^{1/2}}. \end{aligned} \quad (16.48)$$

At $T = 0$ this integral can be evaluated in closed form, with the result

$$\langle I(\phi) \rangle = \frac{\pi\Delta_0}{e} \langle G \rangle \cos(\phi/2) \operatorname{arctanh} [\sin(\phi/2)], \quad (16.49)$$

plotted in Fig. 2 (dashed curve). Note that the derivative $dI/d\phi$ diverges at $\phi = \pi$. The critical current is $I_c^{(0)} \equiv \max \langle I(\phi) \rangle = 1.32 (\pi\Delta_0/2e) \langle G \rangle$, reached at $\phi \equiv \phi_c^{(0)} = 1.97$.

Eq. (16.49) for the average supercurrent in a disordered point contact agrees with the result of Kulik and Omel'yanchuk [35]. These authors pointed out that a disordered weak link and a tunnel junction with the *same* conductance exhibit nevertheless a quite *different* Josephson

effect. For a tunnel junction with $T_p \ll 1$ for all p one may linearize Eq. (16.35) in T_p , with the result (at zero temperature)

$$I = \frac{e\Delta_0}{2\hbar} \sin \phi \sum_{p=1}^N T_p = \frac{\pi\Delta_0}{2e} G \sin \phi, \quad (16.50)$$

which is also plotted for comparison in Fig. 2 (dotted curve). The corresponding critical current is $(\pi\Delta_0/2e)G$. The well-known result (16.50) for a tunnel junction [36] differs from the result (16.49) for a disordered weak link [35]. Zaitsev [25] has explained the difference in terms of different boundary conditions for the Greens functions at the interface between the junction and the bulk superconductor. The present work yields an alternative perspective on this issue: Although the tunnel junction and the disordered weak link may have the same conductance (i.e. the same value of $\text{Tr } tt^\dagger$), *the supercurrents differ because the distribution of transmission eigenvalues is different*. In a tunnel junction all N eigenvalues are $\ll 1$, while in the disordered system a fraction l/L of the eigenvalues is of order unity, the remainder being exponentially small [34].

Supercurrent Fluctuations

The analysis of Kulik and Omel'yanchuk is based on a diffusion equation for the *ensemble-averaged* Greens function, and can not therefore describe the mesoscopic fluctuations of $I(\phi)$ from the average. In contrast, Eq. (16.35) holds for a *specific* member of the ensemble of impurity configurations. The statistical properties of the transmission eigenvalues T_n in this ensemble are known from the theory of conductance fluctuations [11]. A general result [34, 37] is that a *linear statistic* $\mathcal{L}(f) \equiv \sum_{n=1}^N f(T_n)$ on the eigenvalues (with $f(T) \rightarrow 0$ for $T \rightarrow 0$ and $f(T) = \mathcal{O}(1)$ for $T \lesssim 1$) fluctuates with root-mean-square value of order unity

$$\text{rms } \mathcal{L} \equiv (\langle \mathcal{L}^2 \rangle - \langle \mathcal{L} \rangle^2)^{1/2} = \mathcal{O}(1), \quad (16.51)$$

independent of N , L , or l as long as $l \ll L \ll Nl$. An additional requirement is that the junction is not much wider than long: $W_y, W_z \lesssim L$ where W_y and W_z are the widths in the y - and z -direction. If the junction is much wider than long in one direction, then $\text{rms } \mathcal{L}$ is of order $(W_y W_z / L^2)^{1/2}$. It is not necessary for these results that $f(T)$ is itself a linear function of T . In the special case that $f(T) = T$, the coefficient on the right-hand-side of Eq. (16.51) is known [10]:

$$\text{rms } \sum_{n=1}^N T_n \equiv C_{\text{UCF}} = 0.365, \text{ if } W_y, W_z \ll L. \quad (16.52)$$

According to Eq. (16.35) the supercurrent $I(\phi)$ is a linear statistic on T_n , so that we conclude that (at zero temperature) $\text{rms } I(\phi) \simeq e\Delta_0/\hbar$ times a function of ϕ which is independent of the length of the junction or the degree of disorder. At small ϕ we may linearize in T and use Eq. (16.52), with the result

$$\text{rms } I(\phi) = \frac{1}{2} C_{\text{UCF}} (e\Delta_0/\hbar) \phi \tanh(\Delta_0/2k_B T) + \mathcal{O}(\phi^2). \quad (16.53)$$

The critical current $I_c \equiv \max I(\phi) \equiv I(\phi_c)$ is not by definition a linear statistic on T_n , since the phase ϕ_c at which the maximum supercurrent is reached depends itself on all the transmission eigenvalues. It is therefore not possible in general to write I_c in the form $\sum_n f(T_n)$, as required for a linear statistic. However, I_c does become a linear statistic in the limit $L/Nl \ll 1$ (or equivalently $\langle G \rangle \gg e^2/h$) of a good conductor (which is the appropriate limit for mesoscopic fluctuations). To see this, we write

$$I(\phi) = \langle I(\phi) \rangle [1 + \epsilon X(\phi)], \quad \epsilon \equiv L/Nl, \quad (16.54)$$

where the function $X(\phi)$ accounts for the sample-to-sample fluctuations of $I(\phi)$ around the ensemble average $\langle I(\phi) \rangle$. One has $\langle X \rangle = 0$, $\text{rms } X = \mathcal{O}(1)$. We now expand I_c to lowest order in ϵ . Defining $\phi_c \equiv \phi_c^{(0)} + \epsilon \phi_c^{(1)}$, where $\max \langle I(\phi) \rangle \equiv \langle I(\phi_c^{(0)}) \rangle$, one can write

$$\begin{aligned} I_c &\equiv \langle I(\phi_c) \rangle [1 + \epsilon X(\phi_c)] \\ &= \langle I(\phi_c^{(0)} + \epsilon \phi_c^{(1)}) \rangle [1 + \epsilon X(\phi_c^{(0)} + \epsilon \phi_c^{(1)})] \\ &= \langle I(\phi_c^{(0)}) \rangle [1 + \epsilon X(\phi_c^{(0)}) + \mathcal{O}(\epsilon^2)] = I(\phi_c^{(0)}). \end{aligned} \quad (16.55)$$

In the third equality we have used that, by definition, $d\langle I \rangle/d\phi = 0$ at $\phi = \phi_c^{(0)}$. Since $I(\phi_c^{(0)})$ is a linear statistic on T_n (note that $\phi_c^{(0)}$ is by definition independent of T_n), we conclude that the critical current I_c is a linear statistic on the transmission eigenvalues in the limit $\epsilon \equiv L/Nl \rightarrow 0$, and hence that $\text{rms } I_c = \text{rms } I(\phi_c^{(0)}) \simeq e\Delta_0/\hbar$. The value of the numerical coefficient remains to be calculated.

Experimentally, sample-to-sample fluctuations are not easily studied. Instead, fluctuations in a given sample as a function of Fermi energy E_F are more accessible. Josephson junctions consisting of a two-dimensional electron gas (2DEG) with superconducting contacts allow for variation of E_F in the 2DEG by means of a gate voltage [38]. Point-contact junctions can be defined in the 2DEG either lithographically or electrostatically (using split gates) [7]. For such a system one would expect that if E_F is varied on the scale of E_c , the low-temperature critical current will fluctuate by an amount of order $e\Delta_0/\hbar$, independent of the properties of the junction.

17 Quantum dynamics of a superconducting tunnel junction by Eckern

Abstract

Basing our model and method on the microscopic theory we formulate a quantum-mechanical description for the relevant variable in a superconducting tunnel junction, i.e., the phase difference across the junction. The quasiparticle degrees of freedom are responsible for dissipation and noise in the system. Because of the discreteness of the charge-transfer process, the noise is shot noise. The energy gaps in the superconductors lead to further interesting features. We discuss the consequences of these physical effects on macroscopic quantum phenomena.

17.1 I. INTRODUCTION

The dynamical variable that describes a Josephson junction is the phase difference of the order parameters of the superconductors on the two sides of the barrier. It has been known for some time that this variable is coupled to microscopic degrees of freedom, the quasiparticle excitations, which act as a heat reservoir and source of phase fluctuations. Various schemes have been put forward over the years to describe this coupling. The most physically appealing of these ideas is the so-called resistively shunted junction model.^{1,2} In its usual form this model treats the phase as a classical variable, and the barrier as a linear resistor with its associated classical thermal noise. To describe interesting physical situations that occur at very low temperatures, generalizations of this simple model are needed: A proper quantum-mechanical treatment of the dynamics of the phase and the environment of quasiparticles is required; and, one might wish to take into account the fact that junctions by themselves, i.e., without shunt resistors across them, are typically not linear resistances but display the nonlinearities associated with the energy gaps in the superconductors. A suitably general formalism, based on a microscopic

model and satisfying these requirements, has been briefly described in a note by the present authors.^{3,4} When the nonlinear conductance is ignored and the phase difference is assumed to be small the results of this model have been shown to be equivalent to those obtained from Caldeira's and Leggett's^{5,6} model treatment of linear (Ohmic) dissipation.

In this paper we describe our model and method in more detail. In addition to making connection with the case of linear dissipation, which was the useful thing to do at the time of our original publication, we emphasize the interesting aspects of our nonlinear model. We work out the consequences of the theory for switching by quantum-mechanical tunneling from a superconducting to a resistive state, we examine the question of coherent oscillations in a bistable potential well, and we display the novel Langevin equation our model leads to when the condition of least action is imposed.

In outline, the plan of this paper is as follows. In the next section we introduce the microscopic model, and show how to express its statistical properties as a functional integral over the relevant slow variable, the phase. We show how bulk energies suppress fluctuations in the magnitudes of the order parameters in the superconductors on the two sides of the barrier and also couple the time derivative of the phase to the voltage difference across the junction. These points were contained in Ref. 3. We go beyond that work by including the effect of a magnetic field and deriving the equations for statistical fluctuations in a ring containing an oxide barrier. We also show that for phase variations slow on the scale $\hbar/\Delta$ where Δ is the energy gap, the dominant effect of the quasiparticle environment at zero temperature is to renormalize the capacitance of the junction. In Sec. III we discuss the time evolution of the reduced density matrix (i.e., traced over all degrees of freedom except the phase) and obtain a functional integral form similar to that given by Feynman and Vernon,⁷ Caldeira and Leggett,⁶ and Schmid.⁸ Our formula contains the fluctuation and dissipation response functions of the system, and thus goes beyond the linear dissipation treated in Refs. 5, 6, and 8. In Sec. IV we show that in the classical limit the time evolution is determined by a Langevin equation in which the autocorrelation function of the noise current depends on the time-dependent phase, and that this corresponds to shot noise. Finally, we discuss the consequences of the physical effects we have included on macroscopic quantum effects in tunnel junctions.

17.2 II. QUANTUM STATISTICS OF A SUPERCONDUCTING TUNNEL JUNCTION

A particular virtue of superconductors and superconducting tunnel junctions is their property that they can be described by a fairly simple microscopic model, which has been shown to cover most of the essential physics. As far as the bulk superconductor is concerned the description can be based on the BCS Hamiltonian,⁹ which assumes an effective local, attractive interaction

$$\begin{aligned}
H = & \int d^3x \psi_{\sigma}^{\dagger}(x) \frac{-\hbar^2}{2m} \left(\vec{\nabla} - \frac{ie}{\hbar c} \vec{A} \right)^2 \psi_{\sigma}(x) \\
& - \frac{g}{2} \int d^3x \psi_{\sigma}^{\dagger}(x) \psi_{-\sigma}^{\dagger}(x) \psi_{-\sigma}(x) \psi_{\sigma}(x) \\
& + \int d^3x \frac{1}{8\pi} (h - h_{\text{ext}})^2.
\end{aligned} \tag{1}$$

(The notations are standard, a summation over spins is implied, and h_{ext} denotes an externally applied magnetic field.) A superconducting tunnel junction consists of two such superconductors (L and R for left and right) which are coupled. It is described by

$$H = H_L + H_R + H_T + H_Q \quad (2)$$

where, e.g., $H_L(\psi_{L\sigma})$ is of the form (1), with $\psi_{L\sigma}(x)$ referring to electrons in the left electrode. The coupling is due to the transfer of electrons between L and R and due to Coulomb interaction across the barrier. The former can be described by the tunneling Hamiltonian¹⁰

$$H_T = \int_{x \in L} d^3x \int_{x \in R} d^3x' \left[T_{xx'} \psi_{L\sigma}^\dagger(x) \psi_{R\sigma}(x') + \text{H.c.} \right].$$

The range of the tunneling matrix element $T_{xx'}$ is restricted to the vicinity of the oxide barrier. This simple form of H_T has proven to be very successful. As long as we are concerned only with time scales slow compared with the inverse plasma frequency, the Coulomb interaction becomes an effective capacitive interaction depending only on the total charges on each electrode

$$H_Q = \frac{1}{8C} (Q_L - Q_R)^2 \quad (4)$$

Here C is the usual capacitance determined by the geometry and properties of the insulator. The charge on the left side is

$$Q_L = e \int d^3x \psi_{L\sigma}^\dagger(x) \psi_{L\sigma}(x) \quad (5)$$

A superconducting ring with a weak link, i.e., a rf superconducting quantum-interference device (SQUID), in cases where this weak link is a tunnel barrier, can also be described by a Hamiltonian of the form (2) (with L and R joined in a loop).

Quantum-statistical properties of the system can be obtained from the grand canonical partition function

$$Z_G = \text{Tr}_{\psi, \vec{A}} \{ \exp[-\beta(H - \mu N)] \} \quad (6)$$

or, more generally, from a generating functional $Z(\xi)$ where ξ couples to the quantity of interest. The trace is taken over the fermion fields ψ and the vector potential $\vec{A}$. We now eliminate the quartic interactions in H by means of the Hubbard-Stratonovich procedure.¹¹ In the problem of a superconducting tunnel junction this is achieved by introducing complex order-parameter fields $\Delta_L(x\tau)$ and $\Delta_R(x\tau)$ coupling to $\psi_{L\uparrow}^\dagger(x) \psi_{L\downarrow}^\dagger(x)$ in the L electrode, and similar in R . They account for the attractive BCS interactions. Furthermore, we need a real voltage field $V(\tau)$, coupling to the charges, to account for the Coulomb interaction. The resulting effective Hamiltonian is a bilinear form in the four-dimensional space of the spinors Ψ and $\Psi^\dagger = (\psi_{L\uparrow}^\dagger, \psi_{L\downarrow}, \psi_{R\uparrow}^\dagger, \psi_{R\downarrow})$. As a result the trace over the fermion fields can be explicitly performed and we arrive at a path-integral representation (in imaginary times) of the partition function¹²

$$Z_G = \int D^2 \Delta_L(x\tau) D^2 \Delta_R(x\tau) D V(\tau) D \vec{A}(x\tau) \times e^{-S[\Delta_L, \Delta_R, V, \vec{A}]/\hbar} \quad (7)$$

where the action is

$$\begin{aligned}
-S/\hbar = & \text{Tr} \ln \overline{\widehat{G}}^{-1} - \int_0^{\hbar\beta} \frac{d\tau}{\hbar} \int_{x \in L} d^3x \frac{1}{g_L} |\Delta_L(x\tau)|^2 \\
& - (R \leftrightarrow L) - \int_0^{\hbar\beta} \frac{d\tau}{\hbar} \frac{1}{2} C V^2(\tau) \\
& - \int_0^{\hbar\beta} \frac{d\tau}{\hbar} \int d^3x \frac{1}{8\pi} [(h(x\tau) - h_{\text{ext}})]^2.
\end{aligned} \tag{8}$$

Here $\overline{\widehat{G}}$ is a 4×4 matrix Green's function in the space spanned by L and R and in the particle-hole space¹³ characteristic of superconductivity. It depends on the fields Δ_L, Δ_R , and V and its inverse is given by

$$\overline{\widehat{G}}^{-1}(x\tau, x'\tau') = \begin{bmatrix} \widehat{G}_L^{-1}(x\tau, x'\tau') & -\widehat{T}_{xx'}\delta(\tau - \tau') \\ -\widehat{T}_{xx'}^\dagger\delta(\tau - \tau') & \widehat{G}_R^{-1}(x\tau, x'\tau') \end{bmatrix}.$$

The diagonal elements in the $L - R$ space are

$$\widehat{G}_{L(R)}^{-1}(x\tau, x'\tau') = \left\{ -\hbar \frac{\partial}{\partial \tau} \hat{1} + \left[\frac{\hbar^2}{2m} \left[\vec{\nabla} - \frac{ie}{\hbar c} \vec{A} \widehat{\tau}_3 \right]^2 + \mu(\mp) \frac{ie}{2} V(\tau) \right] \widehat{\tau}_3 - \widehat{\Delta}_{L(R)}(x\tau) \right\} \delta(x - x') \delta(\tau - \tau'), \tag{10}$$

where $\widehat{\tau}_3$ is the Pauli matrix in the particle-hole space. The order-parameter field appears off-diagonal in this space,

$$\widehat{\Delta}(x\tau) = \begin{bmatrix} 0 & \Delta(x\tau) \\ \Delta^*(x\tau) & 0 \end{bmatrix} = |\Delta| e^{-i\phi \widehat{\tau}_3 \widehat{\tau}_1} \tag{11}$$

The two electrodes L and R are coupled by

$$\widehat{T}_{xx'} = \begin{bmatrix} T_{xx'} & 0 \\ 0 & -T_{xx'}^* \end{bmatrix} \tag{12}$$

At this point it is advantageous to perform a gauge transformation which makes the particle-hole offdiagonal elements real and explicitly displays the role of the phases $\phi_{L(R)}(x, \tau)$. This is achieved by

$$\overline{\widehat{U}} = \begin{pmatrix} e^{i\phi_L(x, \tau) \widehat{\tau}_3/2} & 0 \\ 0 & e^{i\phi_R(x, \tau) \widehat{\tau}_3/2} \end{pmatrix}$$

Thus the L diagonal element of $\overline{\widehat{U}} \widehat{G}^{-1} \overline{\widehat{U}}^{-1}$ becomes

$$\begin{aligned}
\widehat{G}_L^{-1} = & \left\{ -\hbar \frac{\partial}{\partial \tau} \hat{1} + i\hbar \left(\vec{v}_{SL} \cdot \vec{\nabla} \right) \hat{1} \right. \\
& + \left[\frac{\hbar^2 \nabla^2}{2m} + \mu - \frac{m}{2} \vec{v}_{SL}^2 + i \left[\frac{\hbar}{2} \frac{\partial \phi_L}{\partial \tau} - eV_L \right] \right] \widehat{\tau}_3 \\
& \left. - \widehat{\Delta}_L \right\} \delta(x - x') \delta(\tau - \tau')
\end{aligned} \tag{13}$$

whereas the right-left off-diagonal elements acquire a phase dependence

$$\widehat{T}_{xx'} = \begin{bmatrix} T_{xx'} e^{+i\phi(\tau)/2} & 0 \\ 0 & -T_{xx'}^* e^{-i\phi(\tau)/2} \end{bmatrix} \tag{14}$$

Since $T_{xx'}$ is finite only for x, x' near the junction (at $x = 0$), the phase difference across the junction

$$\phi(\tau) = \phi_L(x \rightarrow -0, \tau) - \phi_R(x \rightarrow +0, \tau)$$

enters. Above we introduced the gauge invariant superfluid velocity

$$\vec{\nabla}_{SL} = -\frac{\hbar}{2m} \left(\vec{\nabla} \phi_L + \frac{2e}{\hbar c} \vec{A} \right)$$

Also the time derivative $\partial \phi_{L(R)} / \partial \tau$ appears in a gauge invariant combination in conjunction with $V_{L(R)} = \left(\begin{smallmatrix} + \\ - \end{smallmatrix} \right) V/2$.

What we have achieved so far is only a formal rewriting of the partition function of the two-coupled superconductor as a multiple path integral. However, we have made explicit the essential variables: the order parameters, the superfluid velocity, the phase difference, and voltage across the junction. The virtue of the path-integral representation is the systematic approximation scheme it suggests. The most probable, i.e., least action path yields classical or mean-field solutions. Beyond that, it allows us to calculate correlations of fluctuations in a straightforward way.

As a first example we consider a single homogeneous superconductor and concentrate on the magnitude of the order parameter. In this case, the action simplifies to (see Ref. 4 for more details)

$$S_0/\hbar = - \sum_{k,v} e^{\omega_v 0^+} \ln(\omega_v^2 - E_k^2) + \frac{\mathcal{V}\beta}{g} \Delta^2, \quad (15)$$

where ω_v are the complex Matsubara frequencies $\omega_v = i(2v+1)\pi/\beta\hbar$ and

$$E_k^2 = \epsilon_k^2 + \Delta^2 = \left[\frac{\hbar^2 k^2}{2m} - \mu \right] + \Delta^2$$

The least-action condition $\delta S_0 / \delta \Delta = 0$ reproduces the familiar BCS gap equation⁹

$$1 = N(0)g \int_{-\hbar\omega_D}^{\hbar\omega_D} d\epsilon_k \frac{1}{2E_k(\bar{\Delta})} \tanh \frac{1}{2}\beta E_k(\bar{\Delta}), \quad (16)$$

which yields the equilibrium value $\bar{\Delta}$. For an estimate of the fluctuation of $\Delta - \bar{\Delta}$ we need

$$\begin{aligned} \left. \frac{\delta^2 S_0}{\delta \Delta^2} \right|_{\Delta=\bar{\Delta}} &= -\Delta \frac{\partial}{\partial \Delta} (N(0)\beta\mathcal{V} \\ &\quad \times \int d\epsilon_k \frac{1}{2E_k} \tanh \frac{1}{2}\beta E_k) \Big|_{\Delta=\bar{\Delta}} \end{aligned} \quad (17)$$

We thus see that bulk energies proportional to $N(0)\mathcal{V}$ suppress the fluctuations in Δ except in a miniscule critical region $(T_c - T)/T_c \leq (kT_c/\epsilon_F)^4$ below the transition temperature.¹⁴⁻¹⁷

The effect of the superfluid velocity becomes transparent if we expand $\text{Tr} \ln G^{-1}$ in v_S to second order, using the relation

$$\begin{aligned} \text{Tr} \ln \widehat{G}^{-1} &= \text{Tr} \ln \left(\widehat{G}_0^{-1} + \delta \widehat{G}^{-1} \right) \\ &= \text{Tr} \ln \widehat{G}_0^{-1} + \text{Tr} \left(\widehat{G}_0 \delta \widehat{G}^{-1} \right) \\ &\quad - \frac{1}{2} \text{Tr} \left(\widehat{G}_0 \delta \widehat{G}^{-1} \widehat{G}_0 \delta \widehat{G}^{-1} \right) + \dots \end{aligned}$$

which is valid for general matrices. As a result we find (still considering a single superconductor) a contribution to the action

$$\begin{aligned}
S_1(v_s)/\hbar &= \text{Tr} \int \frac{d^3p}{(2\pi\hbar)^3} \int \frac{d^3q}{(2\pi\hbar)^3} (\hbar\beta)^{-2} \sum_{\omega_v, \omega_0} \left[\hat{G}_0 \left[\vec{p} + \frac{\vec{q}}{2}, \omega_v + \frac{\omega_0}{2} \right] \int \hat{\mathbf{G}}_0 \left[\vec{p} - \frac{\vec{q}}{2}, \omega_v - \frac{\omega_0}{2} \right] \frac{1}{2} [\vec{p} \cdot \vec{v}_s (-) \right. \right. \\
&\quad \times [\vec{p} - \vec{v}_s (-\vec{q}, -\omega_0)] + \frac{m}{2} \hat{\tau}_3 \hat{G}_0(\vec{p}, \omega_v) \vec{v}_s(\vec{q}, \omega_0) \cdot \vec{v}_s (-) \\
&\quad \left. \left. = \frac{mN}{2} \int \frac{d^3q}{(2\pi\hbar)^3} (\hbar\beta)^{-1} \sum_{\omega_0} Q(\vec{q}, \omega_0) |\vec{v}_s(\vec{q}, \omega_0)|^2 \right] \quad (18)
\end{aligned}$$

The second form defines the response kernel $Q(q, \omega_0)$, which relates the supercurrent to the superfluid velocity or vector potential. (N is the total density of electrons.) If v_s varies slowly in space and time the action becomes

$$S/\hbar = S_0/\hbar + \int_0^{\hbar\beta} \frac{d\tau}{\hbar} \int d^3x \left(\frac{m}{2} \rho_s \vec{v}_s^2(x, \tau) + \frac{1}{8\pi} (\vec{h} - \vec{h}_{\text{ext}})^2 \right) \quad (19)$$

with the superfluid density given by $\rho_s = NQ(q \rightarrow 0, \omega_0 = 0)$.

From the condition that the action be an extremum in the vector potential $\vec{A}$ it follows that

$$0 = \frac{\delta S}{\delta \vec{A}} = -\frac{e}{c} \rho_s \vec{v}_s - \frac{1}{4\pi} \text{curl curl} (\vec{A} - \vec{A}_{\text{ext}}), \quad (20)$$

which, combined with Maxwell's equation, in regions where $j_{\text{ext}} = 0$, yields the Meissner effect

$$\Delta \vec{j}_s - \lambda^2 \vec{j}_s = 0 \quad (21)$$

From here it is a short way to flux quantization in superconducting loops of thickness much larger than the penetration depth λ . And in a SQUID geometry, where the loop contains a tunnel junction, the flux Φ through the loop and the phase difference ϕ across the junction are related by

$$\Phi + \Phi_0(\phi/2\pi) = n\Phi_0 \quad (22)$$

where Φ_0 is the flux quantum and n an integer.

Expanding the action in $[(\hbar/2)(\partial\phi_L/\partial\tau) - eV_L]$ (or R), which first contributes in second order, shows that the action is a minimum if the time derivative of the phase is pinned to the local electric potential. Thus we recover the Josephson relation

$$\hbar \frac{\partial\phi}{\partial\tau} = 2eV \quad (23)$$

Both in the case of the Meissner effect and the Josephson relation, bulk energies suppress deviations from these meanfield results, just as they did for fluctuations in $|\Delta|$.

From the previous analysis we learn that we can restrict our attention in the following to the only remaining essential variable: the phase difference across the junction ϕ , or equivalently by (22) if we consider a SQUID the flux through the superconducting loop. Hence the partition function of a junction, up to irrelevant factors, becomes

$$Z_G \sim \int D\phi \exp \left[-\frac{S_T[\phi]}{\hbar} - \int_0^{\hbar\beta} \frac{d\tau}{\hbar} \frac{C}{2} \left[\frac{\hbar}{2e} \frac{\partial\phi}{\partial\tau} \right]^2 \right] \quad (24)$$

We evaluate the action explicitly by expanding in the tunnel matrix elements, which yields
3,18

$$\frac{S_T[\phi]}{\hbar} = - \int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' \left(\alpha(\tau - \tau') \cos \frac{\phi(\tau) - \phi(\tau')}{2} - \beta(\tau - \tau') \cos \frac{\phi(\tau) + \phi(\tau')}{2} \right) \quad (25)$$

The kernels α and β are

$$\begin{aligned} \alpha(\tau) &= -\frac{2|T|^2}{\hbar^2} \int \frac{d^3 p_L}{(2\pi\hbar)^3} \int \frac{d^3 p_R}{(2\pi\hbar)^3} G_L(\tau, \vec{p}_L) G_R(-\tau, \vec{p}_R), \\ \beta(\tau) &= -\frac{2|T|^2}{\hbar^2} \int \frac{d^3 p_L}{(2\pi\hbar)^3} \int \frac{d^3 p_R}{(2\pi\hbar)^3} F_L(\tau, \vec{p}_L) F_R(-\tau, \vec{p}_R) \end{aligned} \quad (26)$$

We made the usual assumption that the tunneling matrix element depends weakly on the momenta near p_F . The Green's functions G and F are the diagonal (1,1) and off-diagonal (1,2) components of the Nambu-Green's function, whose inverse is given by Eq. (13). To the extent that the tunneling of the electrons distorts the electrodes only weakly, we can take for G and F the equilibrium functions

$$G(\omega_v, \vec{p}) = -\hbar(i\hbar\omega_v + \epsilon_p) / [(\hbar\omega_v)^2 + E_p^2]$$

and

$$F(\omega_v, \vec{p}) = \hbar\Delta / [(\hbar\omega_v)^2 + E_p^2]$$

The momentum integrations in (26) are now easily performed, which yields in the range $-\hbar\beta \leq \tau \leq \hbar\beta$

$$\int \frac{d^3 p}{(2\pi\hbar)^3} \begin{bmatrix} G \\ F \end{bmatrix}(\tau, p) = -N(0) \int_{-\infty}^{\infty} \mathcal{N}(E) \begin{bmatrix} 1 \\ \Delta/E \end{bmatrix} e^{-E\tau/\hbar} [(1 - f_E)\Theta(\tau) - f_E\Theta(-\tau)]. \quad (27)$$

Above f_E denotes the Fermi function, and $\mathcal{N}(E) = |E|\Theta(|E| - \Delta) / (E^2 - \Delta^2)^{1/2}$ is the BCS density of states.

The action (25) with α and β given by (26) provides a general description of the quantum-statistical properties of a Josephson tunnel junction. We will now proceed and in the remainder of this section discuss various limits or approximations.

At zero temperature the integration in (27) is readily performed. We find

$$\begin{aligned} \alpha(\tau) &= \frac{\hbar}{2\pi e^2 R_N} \frac{\Delta_L \Delta_R}{\hbar^2} K_1 \left(\frac{\Delta_L |\tau|}{\hbar} \right) K_1 \left(\frac{\Delta_R |\tau|}{\hbar} \right), \\ \beta(\tau) &= \frac{-\hbar}{2\pi e^2 R_N} \frac{\Delta_L \Delta_R}{\hbar^2} K_0 \left(\frac{\Delta_L |\tau|}{\hbar} \right) K_0 \left(\frac{\Delta_R |\tau|}{\hbar} \right), \end{aligned} \quad (28)$$

where K_0 and K_1 are modified Bessel functions. The prefactor involves the normal state (or high voltage) conductance of the junction $1/R_N = 4\pi e^2 |T|^2 N_L(0) N_R(0) / \hbar$. The limiting behavior of α for equal gaps $\Delta_L = \Delta_R = \Delta$ is

$$\alpha(\tau) \approx \frac{\hbar}{2\pi e^2 R_N} \begin{cases} 1/\tau^2 & \text{for } |\tau| \ll \hbar/\Delta \\ \frac{\pi}{2} \left[\frac{\Delta}{\hbar} \right]^2 \frac{\hbar}{\Delta|\tau|} e^{-(2\Delta|\tau|/\hbar)} & \text{for } |\tau| \gg \hbar/\Delta \end{cases}$$

and $\beta(\tau)$ has the same long-time expansion, but is proportional to $\ln^2 \tau$ for small $\Delta|\tau|/\hbar$. Another interesting case is the limit where Δ vanishes. Then for general temperatures

$$\alpha(\tau) = \frac{\hbar}{2\pi e^2 R_N} \frac{(\pi k T / \hbar)^2}{\sin^2(\pi k T \tau / \hbar)} \quad (\Delta = 0) \quad (29)$$

The physics of the terms involving α and β is quite different. Suppose $\phi(\tau)$ varies slowly on the time scale determined by the extent of α and β , namely $\hbar/\Delta$. This can be arranged by appropriate choice of junction parameters. In this limit the β term becomes local in time. We observe, furthermore, that β is related to the critical current of the junction $\int_{-\hbar\beta/2}^{\hbar\beta/2} d\tau \beta(\tau) = -I_c/2e$. Hence the β term (in lowest order) gives rise to the "washboard potential"

$$S_\beta/\hbar = - \int_0^{\hbar\beta} d\tau (I_c/2e) \cos \phi(\tau) \quad (30)$$

In contrast the α term has a leading ϕ dependence involving $(\partial\phi/\partial\tau)^2$. We will exploit this property further below.

If the superconducting tunnel junction is biased by an external current I , we have to add an additional potential contribution

$$S_I/\hbar = - \int_0^{\hbar\beta} d\tau (I/2e) \phi, \quad (31)$$

which tilts the washboard. This term can be justified from the kinetic energy terms $(m/2)\rho_{SL(R)}\vec{v}_{SL(R)}^2$ [see Eq. (19)] integrated over the L (or R) electrode with suitable boundary conditions "at the battery." A more precisely defined model of the current source, taking care of the boundary condition, is achieved by closing the electrodes in a large loop, with a large number of windings of the phase around the loop. This, of course, represents a SQUID with a large trapped flux. In this case the kinetic energy, integrated around the loop gives rise to the extra potential contribution S_I . Since I is the current in the bulk, it can be considered constant even if ϕ changes by amounts of order 2π .

At this level of approximation, ignoring a constant term, we can write the effective action of a current biased tunnel junction as ³

$$\begin{aligned} S[\phi]/\hbar = & \int_0^{\hbar\beta} \frac{d\tau}{\hbar} \left[\frac{C}{2} \left[\frac{\hbar}{2e} \frac{\partial\phi}{\partial\tau} \right]^2 + U(\phi) \right] \\ & + 2 \int_0^{\hbar\beta} d\tau \int_0^{\hbar\beta} d\tau' \alpha(\tau - \tau') \\ & \times \sin^2 \left\{ \frac{1}{4} [\phi(\tau) - \phi(\tau')] \right\} \end{aligned} \quad (32)$$

with a tilted washboard potential

$$U(\phi) = -\frac{I_c \hbar}{2e} \cos \phi - \frac{I \hbar}{2e} \phi$$

The action of a SQUID differs in the potential term by the magnetic field energy. Hence

$$U(\Phi) = -\frac{I_c \hbar}{2e} \cos \left[2\pi \frac{\Phi}{\Phi_0} \right] + \frac{1}{2L} (\Phi - \Phi_{\text{ext}})^2$$

where L is the geometric inductance. Also in the kinetic (capacitive) energy and the α term we should express the phase ϕ by the flux Φ using relation (22).

The action in the form (32) bears close similarity to that of Caldeira and Leggett.^{5,6} In particular, the α term represents the effect of damping. In our model it is due to quasiparticle tunneling across the barrier. We will draw further comparisons and comment on the differences in Sec. IV. In the case where the damping is due to a normal shunt tunnel junction, which we may model by setting $\Delta = 0$ in the α term, or at finite temperatures, the form (32) is about as far as we can simplify the action. In the case of an ideal superconducting tunnel junction at zero temperature, provided that the characteristic time scale of the motion of ϕ is slow compared to $\hbar/\Delta$, the short range of α as well as of β allows us further approximation. Expanding $\sin^2(\phi - \phi')/4$ leads to

$$S_\alpha/\hbar \approx \frac{1}{4} \int_0^{\hbar\beta} d\tau (\partial\phi/\partial\tau)^2 \int_0^{\hbar\beta/2} d\tau' \alpha(\tau') \tau'^2$$

In this form we see that the effects of the quasiparticles under the given conditions are to increase the effective capacitance. In a mechanical analog the particle becomes heavier. For $\Delta_L = \Delta_R$ this change is^{4,18}

$$\delta C/C = 3\pi\hbar/(32\Delta R_N C) \quad (33)$$

A similar correction results from the β term, if we include higher terms of the expansion in $(\tau - \tau')$. Combining both we find the action contains an enhanced, phase- (flux) dependent capacitance (ren is renormalized),

$$C_{\text{ren}} = C + \delta C[1 - (1/3) \cos \phi] \quad (34)$$

Above, we expressed the partition function in terms of a path integral, which is of interest if we want to calculate thermodynamic properties, including (in the limit $\beta = 1/kT \rightarrow \infty$) ground-state properties. We can easily generalize our approach to include external sources. For example, the generating functional

$$Z[\xi] = \text{Tr} \left[T \exp \left[-\frac{1}{\hbar} \int_0^{\hbar\beta} d\tau [H - I_t(\tau)\xi(\tau)] \right] \right]$$

provides a convenient, systematic formulation to express expectation values and correlation functions of the tunneling current I_t in various situations (see, e.g., Ref. 19). We will not comment on this point further since some of the results will emerge from the real-time analysis to be presented in the next section.

17.3 III. REAL-TIME DESCRIPTION

In the last section we developed a quantum-statistical description of our system of interest. The statistical Boltzmann factor $e^{-\beta H}$ leads us to consider formally path integrals in imaginary times $t = -i\tau$ where t varies between $-i\hbar\beta$ and 0. In this section we will present a path-integral description in real times. The averaging over the internal degrees of freedom (quasiparticles in the case of the tunnel junction) is done on the level of probabilities and not amplitudes. This leads us to consider double pair integrals (a forward and backward propagator), which are coupled after averaging.²⁰

As an example, we consider the time development of the density matrix $\rho(t)$. It is governed by the equation

$$\rho(t_f) = U(t_f, t_i) \rho(t_i) U(t_i, t_f), \quad (36)$$

where $U(t_f, t_i)$ -the time evolution operator-is the time-ordered exponential

$$U(t_f, t_i) = T \exp \left[-i \int_{t_i}^{t_f} dt' H(t')/\hbar \right]$$

Again our system of interest has several degrees of freedom. We want to keep track of one of them; the rest of the degrees of the reservoir remain unobserved and should be eliminated from the description. For simplicity, let us assume first that these degrees of freedom are coordinates, denoted by x and $\vec{R} = R_1, \dots, R_N$, respectively. We assume that the system is initially in "thermal equilibrium" (to be specified further in a moment) and we trace over all the final states of $\vec{R}$. We thus are lead to consider a reduced density matrix

$$\int d^N R \langle x_1 \vec{R} | \rho(t_f) | x_2 \vec{R} \rangle = \int dx'_1 dx'_2 d^N R'_1 d^N R'_2 d^N R \langle x_1 \vec{R} | U(t_f, t_i) | x'_1 \vec{R}'_1 \rangle \times \langle x'_1 \vec{R}'_1 | \rho(t_i) | x'_2 \vec{R}'_2 \rangle \langle x'_2 \vec{R}'_2 | U(t_i, t_f) | x_2 \vec{R} \rangle. \quad (37)$$

There remains some ambiguity in the way the initial state is prepared, differing (at least) in the resulting transient, short-time behavior. Feynman and Vernon,⁷ and others thereafter,^{8,21,22} assumed that at t_i the system factorizes as $\rho(t_i) = \rho_x(t_i) \cdot \rho_{\vec{R}}(t_i)$ where $\rho_{\vec{R}}$ is the thermal density matrix of the reservoir by itself. This is equivalent to assuming that there was no interaction between the reservoir and the coordinate of interest x before t_i . Alternatively, we may assume that the total system was thermalized, before, at t_i , we project out a certain state of x . Formally this means we fix the initial values of x'_1 and x'_2 by $\delta(x'_1 - x_{1i}) \delta(x'_2 - x_{2i})$ in (37). We achieve this by a measurement of x at t_i , rejecting all events which do not satisfy the initial conditions. Finally, we can imagine keeping x fixed at a value $x = x_i$ before t_i and having the reservoir thermalized in interaction with x according to this fixed value. This possibility represents those physical situations where we prepared the system in the initial state by applying a suitable field (coupling to x only) which is switched off at t_i .

We exploit the formal similarity between the statistical Boltzmann factor $\rho(t_i) = U(t_i - i\hbar\beta, t_i) / Z_G$ and the timeevaluation operator to write the reduced density matrix as

$$\int d^N R \langle x_1 \vec{R} | \rho(t_f) | x_2 \vec{R} \rangle = \int d^N R \frac{1}{Z_G} \int_C Dx D^N R e^{iS[x, R]/\hbar} \quad (38)$$

where the path is along the Kadanoff-Baym²³ contour C in the complex t plane connecting the points $t_i, t_f, t_i, t_i - i\hbar\beta$. The path integral in (38) is performed subject to the constraints $x = x_1$ at the end of the path from t_i to t_f and $x = x_2$ at the beginning of the path from t_f to $t_i - i\hbar\beta$. Also, the specification of the initial "thermal state" may be expressed by further constraints on x along the vertical part of the path.

To proceed we want to eliminate the reservoir degrees of freedom and obtain an effective description expressed in terms of the interesting variable x only. The procedure is similar in principle to that of Sec. II. But in contrast to that problem where we calculated complete traces only, we now have to specify x at various times. This is a straightforward extension, if x is a particle coordinate. It also is straightforward, in the case of a SQUID, where the interesting variable is the vector potential $\vec{A}$ (or the flux, which is a linear function of $\vec{A}$), and the reservoir variables to be integrated out are the electron degrees of freedom, i.e., $\psi_\sigma(x, t)$. However, we encounter a conceptual problem in the case of a Josephson tunnel junction, where the interesting variable is the phase difference. The problem arises from the fact that we have not identified a set of simultaneously measurable variables which together with the phase completely specify the state of the system. This prevents us from projecting out states with definite phase. We circumvent this complication by considering

the tunnel junction (again) as part of a SQUID. The Meissner effect and the flux quantization establish (on the mean-field level) a linear relation between the phase difference across the junction and the trapped flux, i.e., the vector potential $\vec{A}$. In this sense we can indeed treat the phase as an ordinary variable.

We now repeat all the steps of the last section, i.e., introduce order parameters and voltage fields; however, all the path integrals are taken along the Kadanoff-Baym contour in the complex-time plane, and we restrict ourselves again to the only strongly fluctuating variable: the phase difference in the case of the tunnel junction or the flux in the SQUID. For definiteness we concentrate on the case of the tunnel junction, where

$$\langle \phi_1 | \rho(t_f) | \phi_2 \rangle = \frac{1}{Z_G} \int_C D\phi e^{iS[\phi]/\hbar}. \quad (39)$$

The resulting effective action $S[\phi]$ for real times is obtained from the corresponding imaginary-time action by an analytic continuation $\tau \rightarrow it$. However, care has to be exercised regarding the nonlocal terms in S , e.g., those double-time integrals representing the dissipation. The kernels $\alpha(t, t')$ and $\beta(t, t')$ are discontinuous functions of their time arguments at $t = t'$ if both t and t' are on the same (either forward or backward) path. More precisely, if t and t' are on the forward (backward) path the timeordered $\alpha^c(t, t')$ [antitime ordered $\alpha^{\bar{c}}(t, t')$] enters (similarly for β), whereas for t and t' on different paths $\alpha^>(t, t')$ or $\alpha^<(t, t')$ appear, reflecting that times on the backward path are always "later" than those on the forward path.

If we express, in the usual way, the diagonal and offdiagonal Green's function G and F by the spectral densities and equilibrium distribution functions, we find

$$\begin{aligned} \alpha^{(<)}(t) &= (\pm) \frac{1}{e} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{1}{e^{(\mp)\beta\hbar\omega} - 1} I_n(\omega), \\ \beta^{(<)}(t) &= (\pm) \frac{1}{e} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{1}{e^{(\mp)\beta\hbar\omega} - 1} I_c(\omega). \end{aligned} \quad (40)$$

Again we assume that the distribution of the electrons, except for the fact that they instantaneously follow the shift of the electric potential, is little distorted by the tunneling of electrons. The quantity $I_n(\omega)$ is the quasiparticle current at voltage $\hbar\omega/e$, whereas the quantity I_c is related to the supercurrent. They are

$$\left. \frac{I_n(\omega)}{I_c(\omega)} \right\} = \frac{1}{2eR_N} \int_{-\infty}^{\infty} dE \mathcal{N} \left(E + \frac{1}{2}\hbar\omega \right) \mathcal{N} \left(E - \frac{1}{2}\hbar\omega \right) \times \left\{ \frac{1}{\frac{\Delta^2}{(E + \frac{1}{2}\hbar\omega)(E - \frac{1}{2}\hbar\omega)}} \right\} \\ \times \left\{ \tanh \left[(E + \frac{1}{2}\hbar\omega) / 2kT \right] - \tanh \left[(E - \frac{1}{2}\hbar\omega) / 2kT \right] \right\} \quad (41)$$

where $\mathcal{N}(E) = \Theta(|E| - \Delta) |E| / (E^2 - \Delta^2)^{1/2}$ is the BCS density of states. The zero voltage critical supercurrent is given as

$$I_c = P \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \frac{I_c(\omega)}{\omega}. \quad (42)$$

For real times t , α has real and imaginary parts $\alpha^{(<)}(t) = \alpha_R(t) \begin{pmatrix} + \\ - \end{pmatrix} i \alpha_I(t)$ and

$$\alpha^{c(\bar{c})}(t) = \alpha_R(t) \begin{pmatrix} + \\ - \end{pmatrix} i \operatorname{sgn}(t) \alpha_I(t).$$

Corresponding relations hold for β . We recall the fact that α and β are products of two Green's functions. From this it is apparent that α_I (β_I) has the meaning of a junction response

function, whereas α_R (β_R) has the meaning of a correlation function. Indeed, they are related by the fluctuation-dissipation theorem

$$\alpha_R(\omega) = i\alpha_I(\omega) \coth \frac{\hbar\omega}{2kT} \quad (43)$$

with a similar relation for β .

To put (39) into a more transparent form it is convenient to distinguish between the forward and backward paths connecting t_i and t_f . Calling the path function $\phi_1(t)$ on the forward segment from t_i to t_f and $\phi_2(t)$ on the backward segment we can write the real-time segments of (39) as a double path integral

$$\int D\phi_1 D\phi_2 e^{iS[\phi_1, \phi_2]/\hbar}, \quad (44)$$

where both $\phi_1(t)$ and $\phi_2(t)$ are functions over the domain $t_i \leq t \leq t_f$. Equation (44) is of the Feynman-Vernon form.⁷ Following Schmid,⁸ who discussed a similar problem of a particle coupled to harmonic oscillators, we introduce a "center-of-mass" and "relative" coordinate $\phi = \frac{1}{2}(\phi_1 + \phi_2)$ and $\chi = \phi_1 - \phi_2$. In terms of these variables the effective action, which is complex $S = S_1 + iS_2$, becomes^{4,18}

$$\begin{aligned} S_1[\phi, \chi] &= \int_{t_i}^{t_f} dt \left[C \left[\frac{\hbar\dot{\phi}}{2e} \right] \left[\frac{\hbar\dot{\chi}}{2e} \right] + I \left[\frac{\hbar\chi}{2e} \right] \right] \\ &\quad + 8\hbar \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \left(\alpha_I(t-t') \sin \frac{\phi - \phi'}{2} - \beta_I(t-t') \sin \frac{\phi + \phi'}{2} \right) \Theta(t-t') \sin \frac{\chi}{4} \cos \frac{\chi'}{4}, \\ S_2[\phi, \chi] &= 4\hbar \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt' \left[\alpha_R(t-t') \cos \frac{\phi - \phi'}{2} + \beta_R(t-t') \cos \frac{\phi + \phi'}{2} \right] \sin \frac{\chi}{4} \sin \frac{\chi'}{4}. \end{aligned} \quad (45)$$

It is possible to interpret S_2 , the imaginary part of the action, as the result of Gaussian stochastic force. In the present problem some care has to be exercised because of the factor $\cos(\phi \mp \phi')/2$ multiplying α_R (β_R). The inverse of $\alpha_R(t-t') \cos(\phi - \phi')/2$ is not a positive definite operator. We overcome this complication by splitting S_2 into two parts and introducing two independent stochastic forces ξ_1 and ξ_2 , such that

$$\begin{aligned} e^{-S_2/\hbar} &= \langle e^{iS_\xi/\hbar} \rangle_{\xi_1, \xi_2} \\ S_\xi/\hbar &= \frac{2}{e} \int_{t_i}^{t_f} dt \left[\xi_1(t) \cos \frac{\phi}{2} + \xi_2(t) \sin \frac{\phi}{2} \right] \sin \frac{\chi}{4} \end{aligned} \quad (46)$$

The average is taken over the two independent Gaussian random variables, ξ_1 and ξ_2 , with autocorrelation functions

$$\begin{aligned} \langle \xi_{1(2)}(t) \xi_{1(2)}(t') \rangle &= 2e^2 [\alpha_R(t-t') (+) \beta_R(t-t')], \\ \langle \xi_1(t) \xi_2(t') \rangle &= 0 \end{aligned} \quad (47)$$

where α_R and β_R follow from Eq. (40).

The action (45), with the possible interpretation of S_2 as given in (46), represents a general description of the dynamics of a Josephson tunnel junction (or with suitable extension of a SQUID). Frequently it is sufficient to consider simpler approximate versions. For example, in many cases the variations of the phase are slow on a time scale given by $\hbar/\Delta$. Then β_I dominates

over β_R and to lowest order can be approximated by $\beta_I = (I_c/2e) \delta(t - t')$. Substituting this into (45) yields the contribution

$$\int dt (I_c \hbar/2e) [\cos(\phi + \chi/2) - \cos(\phi - \chi/2)]$$

i.e., exactly the effect of a washboard potential $U(\phi) = -(I_c \hbar/2e) \cos \phi$. Consistently, the correlation functions of ξ_1 and of ξ_2 are identical, given by α_R only.

At $T = 0$, in the superconducting tunnel junction model, the effect of the environment for slow phase motions in real time is, as in the description of Sec. II, to modify the capacitance. Keeping the lowest nonvanishing time moments of the nonlocal terms in (45)-the zeroth and second moments of β_I and the second moment of α_I -which can be calculated exactly in this model, one finds the correction equivalent to using Eqs. (33) and (34) in the forward and backward time-evolution operators. Thus all the slow physics of the superconducting tunnel junction at zero temperature can be obtained from a Schrödinger equation.

The response function α_I and the correlation function α_R in general depend on Δ reflecting the gap dependence of the quasiparticle current. Nevertheless, it is instructive to consider them in the limit $\Delta = 0$, while retaining the washboard potential (i.e., retaining the supercurrent $I_c \neq 0$). We can think about this limit as describing a Josephson junction shunted by a normal tunnel junction with conductance much larger than that of the Josephson junction. In this limit the normal current is purely Ohmic $I_N(\omega) = \hbar\omega / (eR_N)$. Hence

$$\begin{aligned} \alpha_I^N(t) &= \hbar / (2e^2 R_N) \frac{\partial}{\partial t} \delta(t) \\ \alpha_R^N(\omega) &= \hbar\omega / (2e^2 R_N) \coth(\beta \hbar\omega/2) \end{aligned} \quad (48)$$

Within this approximation the total action becomes

$$\begin{aligned} S_1 &= \int_{t_i}^{t_f} dt \left[C \frac{\hbar \dot{\phi}}{2e} \frac{\hbar \dot{\chi}}{2e} - \frac{1}{R_N} \frac{\hbar \dot{\phi}}{2e} \frac{\hbar}{e} \sin \frac{\chi}{2} \right. \\ &\quad \left. - U(\phi + \chi/2) + U(\phi - \chi/2) \right] \\ S_2 &= 4\hbar \int_{t_i}^{t_f} \int_{t_i}^{t_f} dt dt' \sin \frac{\chi}{4} \alpha_R^N(t - t') \cos \frac{\phi - \phi'}{2} \sin \frac{\chi'}{4}. \end{aligned} \quad (49)$$

Finally, we mention that if we replace the trigonometric functions in (49) by their small argument expansions we reproduce the effective action found in Refs. 8 and 21, which describes a particle coupled linearly to a suitable set of harmonic oscillators.

17.4 IV. RESULTS, COMPARISON, AND DISCUSSION

In the previous two sections we developed the formal theory describing a superconducting tunnel junction. In the following we will first summarize what we have done so far and then draw conclusions and discuss results. We will also compare with related work on simpler systems.

17.4.1 A. Summary

We showed, starting from the microscopic theory, i.e., the BCS theory of superconductivity and a tunnel Hamiltonian for the tunnel junction, how the relevant degrees of freedom, i.e., the complex order parameter of the superconductors and the voltage difference across the junction

arise. By introducing them as variables in a functional or path integral we do not limit ourselves to a self-consistent field theory. All the other electronic degrees of freedom are assumed to adjust to these fields. We eliminate them by taking the trace over the fermion field operators. We found that fluctuations in the magnitude of the order parameter, as well as deviations from the Josephson relation $\hbar\dot{\phi} = 2eV$ and the Meissner effect are suppressed by bulk energies. In contrast the phase difference ϕ feels only a shallow potential proportional to the critical current I_c of the junction, which is small since it itself depends on the microscopic tunnel matrix element.

We obtain explicit results if we treat the tunneling in the usual approximation scheme, i.e., assume that the matrix element $T_{xx'}$ is small and weakly dependent on the momentum transfer. Our main results are the effective (imaginary time) action (25) from which the quantumstatistical properties can be derived, and the double field, real-time action (45) which governs the time evolution of the system. Our description contains the standard theory of electron tunneling in Josephson junctions, i.e., both the supercurrent, the quasiparticle tunneling current, and quasiparticle-supercurrent interference terms. We will demonstrate this below.

At this stage a comment on the physical consequence of our small T expansion is in order. We explicitly keep track of the total charge of the electrons in each electrode. However, we assume (when evaluating α and β) that the energy distribution of the electrons in the electrodes is thermal-even though the tunnel process at finite voltages may lead to a population change far away from the Fermi level. Clearly this assumption is justified if there exists a fast relaxation process. The distortion of the distribution function created by the tunneling in principle modifies the tunnel current. But we estimate this effect to be very small as long as the tunnel current is weak and the electrodes are not of microscopic dimensions. The fact that we are interested in variations on time scales short compared to inelastic lifetimes demonstrates the relevance of the second argument. On the other hand, it is the eventual thermalization of the electrons (by emission of phonons which may escape into the heat bath) that removes the energy dissipated in the charge-transfer processes from the electronic system. As is typical for transport phenomena, we can proceed, calculating response functions in low order whereas the dissipated energy becomes relevant only in higher order.

In those circumstances, where the phase varies slowly on a time scale given by $\hbar/\Delta$, the short range of $\beta(\tau)$ [or of $\beta_I(t)$] allows us to replace the β terms by a potential $-(I_c\hbar/2e)\cos\phi$. This approximation corresponds to replacing the supercurrent by $I_S = I_c \sin\phi(t)$. (The general expression will be recovered below.) Although this approximation is strictly true only for constant ϕ , experience tells us that this is a good qualitative approximation for most problems. The resulting imaginary-time action is now of a rather similar structure as that of Caldeira and Leggett.^{5,6} However, there are still-even qualitative differences in the dissipation term. The model of Caldeira and Leggett-a particle interacting linearly with a suitably chosen reservoir of harmonic oscillator [linear both in the particle (x) and reservoir (R_i) coordinates, i.e., $H_{\text{int}} = x \sum_i C_i R_i$]-was constructed to produce in the classical limit a velocity proportional, i.e., Ohmic damping. In our system the dissipation results from the tunneling of quasiparticles. The gap dependence of α and the sinusoidal phase dependence reflect the peculiarities of the quasiparticle current in a Josephson junction. Taking the $\Delta = 0$ limit of α [i.e., using (29) or (49)] models a normal tunnel junction, which has an Ohmic dissipation. In this case, at zero temperature $\alpha(\tau) \sim 1/\tau^2$. This result agrees with the kernel of Caldeira and Leggett, who obtained it by choosing the spectral density of the harmonic oscillators and coupling strength C_i suitably. Clearly, for a different choice of these parameters, within the linear coupling model, any functional dependence of $\alpha(\tau)$ can be simulated.²⁴ It is interesting to note that the dissipation of a particle due to interaction with particle-hole pairs of a Fermi fluid also results in a kernel $\alpha(\tau) \sim 1/\tau^2$ (see Refs. 25).

We furthermore draw attention to the fact that our variable is a phase which enters into our

action only as derivative or in trigonometric functions (with the exception of the approximated term representing the current source). The resulting cyclicity of the action reflects the discreteness of the underlying charge-transfer process. As a result (which will be demonstrated below) the noise of the tunnel junction, normal or superconducting, is shot noise. In this respect our result differs from that of Caldeira and Leggett even in the $\Delta = 0$ limit. Their variable is a coordinate and the noise obtained from their analysis is ordinary Gaussian, i.e., Johnson-Nyquist noise.^{8,21} Their model for dissipation describes an ordinary resistor. Formally our description reduces to theirs if, in addition to setting $\Delta = 0$, we replace in the α terms the trigonometric functions by their small argument expansions. Conversely, the phase-related features of our model can be obtained by a simple generalization of Caldeira and Leggett's model: We have to couple to two independent ensembles of harmonic oscillators $\{R_i^{(1)}\}$ and $\{R_i^{(2)}\}$ by a coupling of the form

$$H_{\text{int}} = \cos(\phi/2) \sum_i C_i^{(1)} R_i^{(1)} + \sin(\phi/2) \sum_i C_i^{(2)} R_i^{(2)}$$

The coupling strengths $C_i^{(1)}$ and $C_i^{(2)}$ and the spectral densities of the oscillators are related to $\alpha(\tau) \pm \beta(\tau)$.

Finally, we note that in our description the tunneling Hamiltonian yields both the dissipation related to the quasiparticle current and the potential corresponding to the supercurrent. Therefore, questions of potential renormalization never arise. Such questions arose in connection with the model of Caldeira and Leggett and caused some misunderstanding in the literature.²⁶ Obviously, also in their model a careful physical interpretation settles this question in an unambiguous way.⁶

17.4.2 B. Quantum Langevin equations

The time evolution of the density matrix discussed in Sec. III contains all the quantum-mechanical information on our system. It may serve as a starting point to analyze the effect of dissipation on quantum coherence (compare Chakravarty and Leggett²²) or quantum interference (compare Caldeira and Leggett²⁷). So far, this general theory has been applied only in very simple systems as, for example, a two-level system, or for a harmonic oscillator. Two approximation schemes appear useful to proceed: In certain cases, if the nonlocality of the correlation functions α_R, β_R is irrelevant, we can derive a Fokker-Planck description (see, e.g., Ref. 21). Alternatively, we may treat the variable of interest, i.e., in our case the phase ϕ , as a classical variable. The coupling to the reservoir, i.e., the quasiparticle degrees of freedom, leads to dissipation and noise. Thus in the classical limit the equation of motion is a Langevin equation. In this limit ($\hbar \rightarrow 0$) the equation of motion can be derived from the least action principle

$$\left. \frac{\delta S[\phi, \chi, \xi_i]}{\delta \chi} \right|_{\chi=0} = 0 \quad (50)$$

Here $S[\phi, \chi, \xi_i] = S_1[\phi_1 \chi] + S_\xi$ where S_1 is the real part of the action (45) (in the two fields) and S_ξ was introduced in (46) replacing the imaginary part S_2 . The fields ξ_1 and ξ_2 are Gaussian stochastic variables with correlation functions given by the $\hbar \rightarrow 0$ limit of (47). (Notice that $\chi = 0$ is a solution of $dS/d\phi = 0$.) We do not write the Langevin equation at this stage (it will be written below). Here we merely mention that the noise term is δ correlated with amplitude proportional to $2kT/R$.

There have been attempts in the past to extend phenomenologically the Langevin equation beyond the strictly classical regime.²⁸ A partial formal justification was given by Schmid.⁸ The

form of this "quantum Langevin equation" is the same as of the classical counterpart; however, the noise is "quantum noise, reflecting the quantum nature of the reservoir. In simple cases the noise correlation function is $(\hbar\omega/R) \coth(\hbar\omega/2kT)$, which in the classical limit $\hbar\omega \ll kT$ reduces to the classical white-noise spectrum. Technically, this quantum Langevin equation follows from the condition (50) generalized in the respect that the correlation functions are retained in their general $\hbar$ -dependent form. Although this extension appears suggestive, we stress that the question of its validity is not settled, unless the system is purely harmonic²⁹ (see also Refs. 30 and 31 for further comments). We will follow this procedure anyhow and present here the quantum Langevin equation of an ideal tunnel junction. We do so in order to point out the interesting differences to the ordinary quantum Langevin equation, which was applied to describe resistively shunted junctions. We will also demonstrate that in certain limits-other than $\hbar \rightarrow 0$ -the quantum form is correct and indeed necessary.

In order to avoid unnecessary complication we first discuss the limit where we express the supercurrent in a simple form and treat the quasiparticles as in the normal state [Eqs. (48) and (49)]. In this case the least-action principle (50) yields

$$C \frac{\hbar\ddot{\phi}}{2e} + \frac{1}{R_N} \frac{\hbar\dot{\phi}}{2e} - I + I_c \sin \phi = \xi_1(t) \cos \frac{\phi}{2} + \xi_2(t) \sin \frac{\phi}{2}.$$

The right-hand side reflects the current noise $I_N(t, \phi)$ of the tunnel junction. It is quantum shot noise. It appears as a sum of two independent Gaussian noise terms ξ_1 and ξ_2 , coupling to $\cos \phi/2$ and $\sin \phi/2$. Their correlation functions are

$$\langle \xi_i \xi_j \rangle_\omega = \delta_{ij} 2e^2 \alpha_R^N(\omega) = \delta_{ij} \frac{1}{R_N} \hbar\omega \coth \frac{\hbar\omega}{2kT} \quad (52)$$

In contrast the ordinary quantum Langevin equation, for example, of Schmid, which models the Johnson-Nyquist noise of a resistor, contains only one such noise term. The noise term in (51) actually emerged in this form from our formal analysis. At the same time, this form is conceptually the simplest and most accessible, for example, for a numerical analysis. In contrast the correlation function of the total noise current depends on the actual state of the system: If the phase takes the values ϕ and ϕ' at times t and t' , respectively, then

$$\langle I_N(t, \phi) I_N(t', \phi') \rangle = 2e^2 \alpha_R^N(t - t') \cos \frac{\phi - \phi'}{2} \quad (53)$$

In the special case where the junction is operated with a constant voltage bias, such that $\phi - \phi' = 2eV(t - t')/\hbar$ the power spectrum takes the familiar form obtained by Dahm et al:³²

$$\langle I_N I_N \rangle_\omega = \frac{1}{2R_N} \sum_{\pm} (\hbar\omega \pm eV) \coth \frac{\hbar\omega \pm eV}{2kT} \quad (54)$$

For voltages eV exceeding kT and $\hbar\omega$ the fluctuations in the tunnel current are proportional to the current itself. This is a characteristic feature of shot noise. The power spectrum is larger than the Johnson-Nyquist spectrum of an equivalent resistor. Notice that it is the factor $\cos(\phi - \phi')/2$ in Eq. (53), leading to the shift by $\pm eV$ in (54), combined with the fact that $\alpha_R(t - t')$ is of the quantum form which is responsible for this difference.

As a second simple example for the use of the result (53), relevant for examples in cases where the junction is used as a mixer (see Ref. 33), we consider a tunnel junction driven by an ac voltage bias at frequency ω_0 , i.e., $V = V_0 \cos(\omega_0 t)$. If the voltage is small $eV_0 \ll \hbar\omega_0$ the current noise at small frequencies $\hbar\omega \ll kT$ averaged over a period is

$$\begin{aligned}\overline{\langle I_n I_n \rangle}_\omega &= \frac{2kT}{R_N} \left[1 - \frac{1}{8} \left[\frac{2eV_0}{\hbar\omega_0} \right]^2 \right] \\ &\quad + \frac{1}{8} \left[\frac{2eV_0}{\hbar\omega_0} \right]^2 \frac{\hbar\omega_0}{R_N} \coth \frac{\hbar\omega_0}{2kT}\end{aligned}\quad (55)$$

In the limit where the relevant time scales are slow, i.e., $\hbar\omega, eV \ll kT$, the noise reduces to classical δ -correlated or white noise. In this case, the two noise sources in (51) add up and act like one classical noise term. However, if the phase evolves faster, we explore the part of the quantum power spectrum, which (for $\hbar\omega > kT$) increases proportional to ω . It is instructive to consider the correlation function in times: $\alpha_R^N(t - t')$ has a large peak at $t = t'$ [singular if $\alpha_R^N(\omega)$ did not have a high-frequency cutoff equal to ω_c], becomes negative beyond (for $|t - t'| \geq 1/\omega_c$), reflecting an anticorrelation in the noise which in part compensates for the equal-time peak, and decays on a time scale given by $\hbar/kT$ (we assume $\hbar\omega_c \gg kT$). If the phase varies on a time scale $\hbar/kT$, the fact that the noise couples to $\cos\phi$ or $\sin\phi$ can revert the effect of the anticorrelation. This leads to the increase of the shot noise over the ordinary quantum, i.e., Johnson-

Nyquist noise.

The general theory of the Josephson tunnel junction represented by the action (45) leads to a quantum Langevin equation of the form

$$\begin{aligned}C \frac{\hbar\ddot{\phi}}{2e} - I - 4e \int_{-\infty}^t dt' (\alpha_I(t - t') \sin \frac{\phi - \phi'}{2} \\ - \beta_I(t - t') \sin \frac{\phi + \phi'}{2}) \\ = \xi_1(t) \cos(\phi/2) + \xi_2(t) \sin(\phi/2)\end{aligned}\quad (56)$$

Here ξ_1 and ξ_2 are again independently Gaussian stochastic variables with different correlation functions as given by the general form (47). The integrals on the left yield the quasiparticle and supercurrent in their general nonlocal form as obtained by Werthamer.³⁴ The correlation function of the total noise current, in generalization of (53), can be written as

$$\begin{aligned}\langle I_N(t, \phi) I_N(t', \phi') \rangle &= 2e^2 (\alpha_R(t - t') \cos \frac{\phi - \phi'}{2} \\ &\quad + \beta_R(t - t') \cos \frac{\phi + \phi'}{2})\end{aligned}\quad (57)$$

which exhibits its relation to the quasiparticle and quasiparticle-pair interference current. Also, the fluctuation-dissipation theorem is apparent in this form.

In cases where the Josephson junction is operated with a fixed voltage bias the tunneling current takes the familiar form³⁵

$$\begin{aligned}I_T(V) &= I_n(eV/\hbar) + \left[\text{P} \int \frac{d\omega}{\pi} \frac{I_c(\omega)}{\omega - eV/\hbar} \right] \sin \phi(t) \\ &\quad + I_c(eV/\hbar) \cos \phi(t)\end{aligned}\quad (58)$$

The β term gives rise to the supercurrent ($\sim \sin \phi$) but also to the $\cos \phi$ or quasiparticle-pair interference current. (It has been pointed out in Ref. 36 that bulk scattering processes may

modify this interference term, which appears consistent with experiments. This is not included in our present formulation.) In the case of a constant voltage the quasiparticle part of the power spectrum (57) becomes³²

$$\langle I_N I_N \rangle_\omega^{\text{quasiparticle}} = \frac{e}{2} \sum_{\pm} I_n(\omega \pm eV/\hbar) \coth \frac{\hbar\omega \pm eV}{2kT} \quad (59)$$

For practical calculations a hybrid description between (51) and (56) is popular and qualitatively reasonable. It treats the quasiparticle (α) current in its general form but ignores the interference term and the supercurrent is set to $I_S(t) = I_c \sin \phi(t)$. Consistently the noise correlations do not involve β_R ; that means $\langle \xi_i(t) \xi_j(t') \rangle = \delta_{ij} 2e^2 \alpha_R(t-t')$.

In a different limit, namely if the phase varies slowly in time but $\dot{\phi} \neq 0$, if furthermore $T = 0$ and $\Delta \neq 0$, the quasiparticle and quasiparticle-interference term are proportional to $\ddot{\phi}_1$, exactly consistent with the capacitance renormalization (34). On this level (the classical current-phase relation), the capacitance renormalization was noticed before by Tucker.³³

In the case where the junction is operated such that the voltage does not vanish on the average (e.g., if a current I larger than I_c is imposed), the shot-noise power spectrum even at small frequencies $\hbar\omega/kT \ll 1$ is enhanced over the thermal noise [see Eq. (54)]. Although this agrees with the classical result, we stress again that we had to retain the quantum form of the correlation function to arrive at this result. Clearly this enhanced noise current enters into the current conservation relation (51) or (56). We thus see that the quantum form of the Langevin equation is correct in a nontrivial and important limit. In this context it is interesting to note the consequence of the Josephson relation $\hbar\dot{\phi} = 2eV$. Even if the voltage changes slowly on a time scale given by $\hbar/kT$, the phase may rotate fast if $eV > kT$. Whereas usually in physics we try to use the slowest variables as independent degrees of freedom, the description of a Josephson junction is simple in terms of the fast phase. On the other hand, we must sacrifice and retain the detailed structure of the correlation function.

It would be interesting to analyze the effect of shot noise in nondriven situations by evaluating Eqs. (51) or (56). It has been conjectured by Ben-Jacob et al.³⁷ that shot noise may be overall larger than an equivalent Johnson-Nyquist noise and thus enhance transition rates. However, their analysis is based on an ansatz for the noise, which was guided by the observation that the power spectrum (54) or (59) of the driven system is enhanced, but which violates the fluctuation-dissipation theorem. Consequently, their quantitative conclusions, except for the case $I \gg I_c$, are incorrect.

In a recent analysis^{19,38} the power spectrum of fluctuations in a normal tunnel junction with capacitance was analyzed. It turns out that in a closed system the discreteness of the available states reduces the current fluctuations, actually stronger than the above-mentioned enhancement. A Langevin equation of the type (51) or (56) in a continuous variable does not account for this feature; it is more appropriate for an open system where there is a continuum of available states.

17.4.3 C. Macroscopic quantum phenomena in Josephson junctions

We now turn to the question of how dissipation by quasiparticle tunneling affects macroscopic quantum phenomena in a Josephson junction. In particular, we are interested in the effect of dissipation on the transition rate from a superconducting into a resistive state. At low temperatures this transition occurs via quantum tunneling. Caldeira and Leggett^{5,6} analyzed this problem in situations where the dissipation occurs in a normal shunt resistor. They used methods developed by Langer and Callan and Coleman,³⁹ generalized to situations where the action contains nonlocal, dissipative elements. The important point is that the tunnel rate

is determined by the action of a saddle-point solution (bounce) starting and ending in the metastable potential minimum (maximum, if we work in imaginary times, where the potential is inversed) and crossing the barrier (valley) back and forth. Caldeira and Leggett found that dissipation inhibits the tunneling.^{40,41}

We obtain a qualitative estimate for the difference between Ohmic dissipation and dissipation by quasiparticle tunneling simply by comparing the effective action representing the two cases. The action of Caldeira and Leggett is obtained from our approximate version (32) which we consider first-if we ignore the effect of the energy gap, i.e., replace α by its $\Delta = 0$ limit and replace $\sin(\phi - \phi')/4$ by its argument. The fact that the phase enters only in the sine function in principle makes quasiparticle tunneling less effective as a damping mechanism than an Ohmic resistor. However, since the transition to the resistive state occurs mostly for I close to I_c , when the distance under the barrier is only a small fraction of 2π , this effect is usually of little importance. The gap dependence of the quasiparticle tunneling (α), however, leads to significant changes. A typical time scale for variations of ϕ (as relevant for the tunneling process) is given by the characteristic frequency of the potential, i.e., the Josephson frequency $\omega_J = (2e/\hbar)(I_c/C)^{1/2}$ or, in fact, if the current I is close to the critical current I_c , a small fraction of ω_J . For most experimental parameters ω_J is smaller than Δ . Hence, the exponential decay at $T = 0$ of $\alpha(\tau)$ for large τ reduces the action of the bounce and hence the effect of the dissipation below the result of Caldeira and Leggett by a factor of order $\hbar\omega_J/\Delta$ (or even smaller if I is close to I_c).

In Sec. II we showed that the effect of the quasiparticle tunneling at zero temperature for $\hbar\omega_J \ll \Delta$ is to renormalize the effective capacitance. Upon closer inspection we found that corrections from the expansion in the β term gave rise to a further phase-dependent capacitance correction. The result for C_{ren} is given in (34). It is straightforward to work out quantitatively the effect of this renormalized capacitance on the tunneling rate. If the junction is biased close to I_c the barrier is near $\phi = \pi/2$ and the effect of the β term is inefficient, i.e., $C_{\text{ren}} = C + \delta C$. In contrast, in a SQUID with small inductance in an external field, $\Phi_{\text{ext}} = \Phi_0/2$, the barrier is near $\phi = \pi$. This makes the capacitance change most efficient, i.e., $C_{\text{ren}} = C + \frac{4}{3}\delta C$. If the tunneling path extends over larger distances in ϕ (of order π) the effective capacitance depends on the tunneling path $\phi(\tau)$ and has to be determined self-consistently.

This capacitance renormalization can be significant for the claimed parameter values in the experiments of Jackel et al.⁴¹ For a junction of capacitance 10^{-14} F and critical current $I_c = 10^{-4}$ A-assuming $I_c R_N = \pi\Delta/2e$ and $\Delta/k = 15$ K-the estimate (34) yields $\delta C/C$ of order 70%. This will substantially reduce the tunneling rate. We realize, however, the experimental difficulties in determining accurately the capacitance, which makes it difficult to test our conclusion.

At finite temperatures, or for small gaps $\hbar\omega_J \geq \Delta$, or if the dissipation occurs in a normal tunnel junction, we have to find the action of the saddle-point solution of the full nonlocal action, in the same fashion as discussed, e.g., by Caldeira and Leggett in order to determine the tunnel rate. As long as there exists a gap, the effect of quasiparticle tunneling on the quantum tunneling rate of the phase will be quantitatively weaker than that of Ohmic dissipation in a resistor.

If we consider the effect of dissipation on the quantum coherence we find qualitative differences between Ohmic dissipation and quasiparticle tunneling. It has been shown by Chakravarty,⁴² by Bray and Moore,⁴³ and by others thereafter^{22,44,45} that the Ohmic dissipation gives rise to a certain phase transition at a critical coupling strength. For weak coupling an essentially quantum-mechanical behavior, though weakened or slowly decaying, is found; for strong coupling the behavior is essentially classical. Formally, this phase transition is related to the weak power-law decay of the kernel $\alpha_N(\tau) \sim 1/\tau^2$, which gives rise to logarithmic interactions in the related onedimensional model of classical particles. An energy

gap in the excitation spectrum leads to an exponential decay of $\alpha(\tau) \sim e^{-2\Delta\tau/\hbar}$ for long times. Hence the interaction will be short range and no phase transition occurs. In addition, even for weak damping, the influence of quasiparticle tunneling on the quantum coherence is quantitatively weaker than that of an Ohmic resistor with the same high-voltage resistance. If macroscopic quantum coherence is to be tested in an experiment, which obviously is at the limit of what can be achieved, it is important to take advantage of these differences.

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Part VI

Multiterminal Junctions: Well-known, Important Theories

18 Josephson effect in multiterminal topological junctions by Zazunov, Egger, Alvarado, Levy Yeyati

Abstract

We study the Josephson effect in a trijunction formed by two topological superconductor (TS) wires and a conventional *s*-wave superconductor. Using a boundary Green's function formalism, analytical results for the current-phase relation are obtained in various limiting cases by modeling the TS wires via the low-energy limit of a Kitaev chain. We show that Josephson transport critically depends on the spin canting angle θ between the boundary spin polarizations of the TS wires, which in turn suggests that the spin structure of Majorana states can be accessed through supercurrent measurements. We also extend the boundary Green's function approach to a more microscopic spinful wire model and thereby compute the dependence of θ on experimentally accessible parameters such as the Zeeman field and/or the chemical potential. Furthermore, we show that the equilibrium current-phase relation between both TS wires exhibits a robust 4π -periodicity since the conventional superconducting lead effectively locks the fermion parity of the trijunction.

18.0.1 Introduction

The possibility to induce topological superconductivity in semiconducting nanowires with strong spin-orbit coupling (e.g., InAs or InSb) by proximity coupling to a conventional superconductor has paved the way to exciting novel developments [8, 10, 2, 4, 5, 30, 15, 10, 9, 10, 11, 12, 51]. In particular, presently available devices with a hard superconducting gap [10, 9, 10, 11] and disorder levels close to the ballistic limit [12, 51] exhibit characteristic transport signatures as expected for Majorana bound states (MBSs). The latter are robust states localized at the wire ends within the topological phase. Similar experiments have also been reported for other platforms, see for instance Refs. [14, 15, 16, 17, 18], and many of our results below apply to those implementations as well. It stands to reason that the currently developed technologies will give access to quantum transport studies in a wide variety of hybrid structures involving topological superconductor (TS) wires, including multiterminal geometries [12, 51, 19]. Such devices may also be of interest for topological quantum information processing applications [10, 2, 4].

This situation claims for a flexible theoretical framework going beyond idealized models but simple enough to account in an (almost) analytical way both for subgap (Andreev and/or Majorana) bound states as well as for above-gap continuum states. In a recent work [20], an effective low-energy theory has been formulated in order to analyze nonequilibrium quantum transport in junctions of TS wires and/or conventional (normal or superconducting) terminals. Based on a tunnel Hamiltonian description valid for arbitrary junction transparency, the boundary Green's function (bGF) of the respective terminal here represents a central ingredient, see also Ref. [21]. It is well known that related approaches can successfully describe a wide variety of topologically trivial systems [47, 14]. Analytical results for the (nonlinear) conductance, noise correlations, and for the supercurrent-phase relation in topological two-terminal hybrid junctions have been reported in Ref. [20]. Moreover, a three-terminal junction with two normal leads and one TS wire has been studied within this framework [24].

The present work aims at providing a concise theoretical description of quantum transport in multiterminal junctions containing TS wires as well as conventional *s*-wave superconducting leads (hereafter referred to as S leads). We focus on the TS-S-TS setup sketched in Fig. 18.1(a), where we analyze the equilibrium supercurrent-phase relation in a phase-biased device. In addition, we go beyond previously established theories by deriving and subsequently employing the bGF for the spinful semiconductor nanowire model proposed in Refs. [6, 7]. Compared to the low-energy bGF for effectively spinless Kitaev chains used in Refs. [20, 24], this more microscopic bGF allows one to directly calculate the spin-dependent properties of a TS wire. While our results for the spinful bGF apply for arbitrary multiterminal junctions, spin-dependent properties turn out to be of special importance for the TS-S-TS setup in Fig. 18.1. Since we study equilibrium properties, we utilize the Matsubara representation throughout this paper. By instead using a Keldysh version of the theory [20], nonequilibrium features in topological multiterminal junctions, e.g., phenomena involving multiple Andreev reflection (MAR) processes, may also be studied. However, we leave this to future work.

The Josephson effect in hybrid S-TS multiterminal junctions constitutes an interesting fundamental issue. Since Majorana states have a well-defined spin polarization axis [27, 28, 29, 30, 31, 32, 33, 34], Josephson transport as well as MAR phenomena are strongly suppressed in two-terminal S-TS junctions [35, 19, 37, 38, 39, 40]. This supercurrent blockade reflects the conflicting pairing symmetries in both leads. However, the blockade could be lifted by forming a junction between an S lead and two (or more) TS leads. In this case, a spin-singlet Cooper pair in S can be split by injecting two opposite-spin electrons into different TS leads (or the reverse process). Clearly, such processes, and thus the supercurrent, will then be sensitive to the relative angle $\theta = \theta_L - \theta_R$ between the spin polarization axes of both TS wires, see Fig. 18.1(a). If the spins in the TS wires are oriented along the same direction ($\theta = 0$), the supercurrent blockade will persist, while a sizable supercurrent can flow for antiparallel alignment ($\theta = \pi$). Since each TS wire effectively acts as a spin filter, it is not possible to generate spin entanglement in this way. However, Cooper pairs can be split with high efficiency through such processes [41], and supercurrent measurements in the trijunction setup of Fig. 18.1(a) can therefore probe the spin structure associated with MBSs. We mention in passing that previous work has considered networks of TS wires [42], where TS-TS-TS trijunctions represent the basic unit [43]. Topological aspects may also become important in multiterminal junctions of conventional superconductors [11]. However, to our knowledge, multiterminal geometries containing both TS and S leads have not been discussed before.

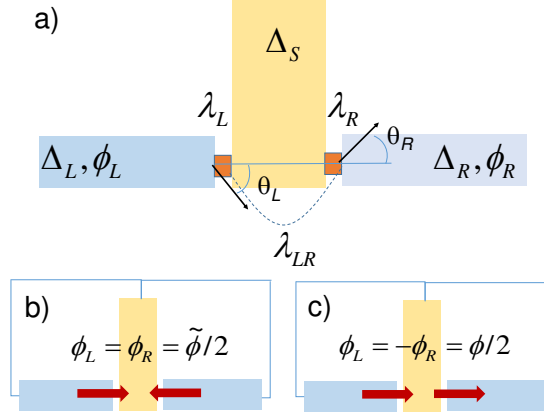


Figure 18.1: (a) *Schematic setup*: Phase-biased TS-S-TS trijunction formed from two topological superconductor (TS) wires and one conventional superconductor (S). The TS wires (blue) have the respective superconductor order parameter $\Delta_{L/R}e^{i\phi_{L/R}}$, while the S terminal has the gap Δ_S and phase $\phi_S = 0$. The S-TS tunnel couplings are denoted by $\lambda_{L/R}$, and the direct TS-TS coupling is λ_{LR} . The spin quantization axis in the respective TS wire is tilted by an angle $\theta_{L/R}$ relative to the corresponding axis in the S lead. (b) *Parallel configuration*: One chooses $\phi_L = \phi_R = \tilde{\phi}/2$ such that $\phi = 0$, see Eq. (18.2). Red arrows indicate supercurrents $I_{L/R}$ in TS wires. (c) *Serial configuration*: Here $\phi_L = -\phi_R = \phi/2$ with $\tilde{\phi} = 0$, see Eq. (18.2).

We thus consider a conventional s -wave superconductor (S) and two topological class- D superconductor (TS) wires, which are tunnel-coupled in a trijunction geometry, see Fig. 18.1(a). Here, Δ_S and $\Delta_{j=L/R}$ are real-valued superconducting gaps, where gauge invariance allows us to put the phase in the S lead to $\phi_S = 0$ without loss of generality. Taking into account current conservation,

$$I_L + I_R + I_S = 0, \quad (18.1)$$

with the convention that individual supercurrents flowing through each branch ($L/R/S$) are oriented towards the junction, it is useful to introduce the linear combinations

$$\phi = \phi_L - \phi_R, \quad \tilde{\phi} = \phi_L + \phi_R. \quad (18.2)$$

For $\phi = 0$, we have the “parallel configuration” shown in Fig. 18.1(b), where $I_L = I_R$ implies that no supercurrent flows between the TS leads. We then effectively have a Josephson junction between the s -wave superconductor and a two-channel p -wave superconductor. The phase difference across this effective junction is given by $\tilde{\phi}/2$.

For $\tilde{\phi} = 0$, we instead encounter the “serial configuration” in Fig. 18.1(c), where the supercurrent $I_R = -I_L$ flows between both TS wires but $I_S = 0$. The S lead then acts as currentless probe electrode (with gapped spectrum) coupled to an effective TS-TS junction. Remarkably, as we show in Sec. 18.1.3 below, since the S lead effectively measures the fermion parity of the trijunction, the 4π -periodic current-phase relation of a TS-TS junction [8, 10, 2, 4] is stabilized for all $\theta \neq 0$ and thereby becomes completely robust. It is worth noting that this effect cannot be captured by an approach neglecting above-gap quasiparticles. This observation suggests a practical way to experimentally observe the topological Josephson current-phase relation (CPR) in a ground state of fixed fermion parity.

In what follows, we always take $\Delta_L = -\Delta_R = \Delta$ as expected for a long TS wire interrupted by a tunnel junction [20] and mostly assume symmetric S-TS couplings, $\lambda_L = \lambda_R = \lambda$, keeping the direct TS-TS coupling λ_{LR} arbitrary. The crucial parameter controlling the Josephson effect is then given by the spin canting angle θ , see Fig. 18.1.

Our theoretical description of TS-S-TS trijunctions below proceeds along two interrelated avenues. In Sec. 18.1, we use the bGF for TS wires derived in Ref. [20], see Eq. (18.3) below,

which in turn represents the low-energy limit of the celebrated Kitaev chain [8]. In this case, the spin structure is introduced phenomenologically through the angles $\theta_{L/R}$ in Fig. 18.1(a). The advantage of this approach is that its simplicity allows for analytical results. As a second approach, in Sec. 18.2, we derive and employ the bGF for the spinful TS nanowire model in Refs. [6, 7]. This model explicitly takes into account proximity-induced pairing, spin-orbit coupling, and the magnetic Zeeman field. Details of our derivation of the bGF can be found in the Appendix. Using this bGF, the MBS spin structure no longer needs to be inserted in a phenomenological manner but emerges naturally. However, this advantage comes at the prize of increased complexity, necessitating a numerical analysis. We also study the CPR across the topological transition in the TS wires using this approach. Finally, we conclude in Sec. 18.3. Throughout, we use units with $e = \hbar = k_B = 1$.

18.1 Low-energy approach

18.1.1 Model

In this section, each TS wire is modeled by the bGF in Eq. (18.3) below, which describes the low-energy continuum limit of an effectively spinless Kitaev chain deep in the topological phase, see Ref. [20] for a detailed derivation. The spin angles $\theta_{L/R}$ are then taken into account phenomenologically in the tunnel Hamiltonian, see Eq. (18.5) below. Starting from the boundary fermion operator c_j in TS lead $j = L/R$, we define the Nambu spinor $\Phi_j = (c_j, c_j^\dagger)^T$. Using the time-ordering operator $\mathcal{T}$, the imaginary-time bGF of an uncoupled TS lead is

$$\begin{aligned} G_{j=L/R}(\tau) &= -\langle \mathcal{T} \Phi_j(\tau) \Phi_j^\dagger(0) \rangle_0 = T \sum_{\omega} e^{-i\omega\tau} G_j(\omega), \\ G_j(\omega) &= \frac{1}{i\omega} \left(\sqrt{\omega^2 + \Delta_j^2} \sigma_0 + \Delta_j \sigma_x \right), \end{aligned} \quad (18.3)$$

where Pauli matrices $\sigma_{x,y,z}$ and the identity σ_0 act in Nambu space. All frequency summations run over fermionic Matsubara frequencies, $\omega = 2\pi(n+1/2)T$, with integer n and temperature T . Similarly, we have boundary fermion operators ψ_σ for spin σ in the S lead, which are combined to form the Nambu spinor $\Psi = (\psi_\uparrow, \psi_\downarrow)^\dagger$. The respective bGF is then given by [14] (no, there is no such formula, I checked. did they put a wrong citation here?)

$$\begin{aligned} G_S(\tau) &= -\langle \mathcal{T} \Psi(\tau) \Psi^\dagger(0) \rangle_0 = T \sum_{\omega} e^{-i\omega\tau} G_S(\omega), \\ G_S(\omega) &= -\frac{i\omega\sigma_0 + \Delta_S\sigma_x}{\sqrt{\omega^2 + \Delta_S^2}}. \end{aligned} \quad (18.4)$$

In the above formulation, we have used a gauge where the superconducting phases $\phi_{L/R}$, see Fig. 18.1(a), only appear in the tunnel Hamiltonian connecting the decoupled TS and S leads in order to form a trijunction,

$$H_t = \sum_{j=L/R} \lambda_j e^{i\phi_j/2} \left(\alpha_j \psi_\uparrow^\dagger + \beta_j \psi_\downarrow^\dagger \right) c_j + \lambda_{LR} e^{-i\phi/2} c_L^\dagger c_R + \text{h.c.}, \quad (18.5)$$

where the real-valued S-TS tunnel amplitudes $\lambda_{L/R} \geq 0$ include density-of-state factors, and the prefactors $\alpha_j = e^{i\chi_j} \cos(\theta_j/2)$ and $\beta_j = e^{-i\chi_j} \sin(\theta_j/2)$ determine the weight of each spin component through the angles $\theta_{L/R}$. In addition, there are angles $\chi_{L/R}$ which, however, are not essential as shown below. Furthermore, λ_{LR} in Eq. (18.5) describes a direct coupling between

both TS leads. With the above conventions, all tunnel couplings are dimensionless quantities. In Nambu notation, Eq. (18.5) is given by

$$\begin{aligned} H_t &= \frac{1}{2} \Phi_L^\dagger W_{LR} \Phi_R + \sum_{j=L/R} \Psi^\dagger W_j \Phi_j + \text{h.c.}, \\ W_{LR} &= \lambda_{LR} e^{-i\sigma_z \phi/2} \sigma_z, \quad W_j = \lambda_j e^{i\sigma_z \phi_j/2} \text{diag}(\alpha_j, -\beta_j^*). \end{aligned} \quad (18.6)$$

Let us now take the global spin quantization axis along the spin direction of, say, the left ($j = L$) TS lead [22]. We then observe that SU(2) spin rotation invariance in the S lead implies that we can set $\alpha_L = 1$ and $\beta_L = 0$ by switching to new fermion fields for the S terminal, $\psi'_\uparrow = \alpha_L^* \psi_\uparrow + \beta_L^* \psi_\downarrow$ and $\psi'_\downarrow = -\beta_L \psi_\uparrow + \alpha_L \psi_\downarrow$. Another simplification follows by noting that χ_R can be gauged away up to a renormalization of the phase ϕ_L . We therefore put $\alpha_R = \cos(\theta/2)$ and $\beta_R = -\sin(\theta/2)$ with the angle $\theta = \theta_L - \theta_R$ between the TS spin polarization directions. Note that the direct TS-TS coupling λ_{LR} will in principle depend on θ . In particular, if only spin-conserving tunneling is possible, we expect $\lambda_{LR} = 0$ for $\theta = \pi$.

The Josephson currents $I_{L/R}$, and thus also I_S from Eq. (18.1), can then be expressed as [20]

$$I_j = -iT \text{tr} \sum_{\omega} \sigma_z [\Sigma(\omega) \mathcal{G}(\omega)]_{jj}. \quad (18.7)$$

The full bGF, $\mathcal{G}(\omega)$, follows from the uncoupled bGFs in Eq. (18.3) as a solution of the Dyson equation,

$$\mathcal{G}^{-1}(\omega) = G^{-1}(\omega) - \Sigma(\omega), \quad (18.8)$$

where the matrix $G = \text{diag}(G_L, G_R)$ acts in TS lead space and the trace operation in Eq. (18.7) is over Nambu space. The self-energy $\Sigma(\omega)$ captures the effects of integrating out the S lead. Like $\mathcal{G}(\omega)$, this self-energy is a 4×4 matrix in lead-Nambu space. Using the particle-hole symmetry relation $\Phi_j^\dagger = \Phi_j^T \sigma_x$, some algebra gives the self-energy matrix elements in lead space in terms of the Nambu matrices

$$\begin{aligned} \Sigma_{LL/RR}(\omega) &= -\frac{i\omega \lambda_{L/R}^2}{\sqrt{\omega^2 + \Delta_S^2}} \sigma_0, \\ \Sigma_{LR}(\omega) &= \Sigma_{RL}^\dagger(-\omega) = W_{LR} - \frac{\lambda_L \lambda_R}{\sqrt{\omega^2 + \Delta_S^2}} \begin{pmatrix} i\omega \cos(\theta/2) e^{-i\phi/2} & \sin(\theta/2) \Delta_S e^{-i\tilde{\phi}/2} \\ -\sin(\theta/2) \Delta_S e^{i\tilde{\phi}/2} & i\omega \cos(\theta/2) e^{i\phi/2} \end{pmatrix}, \end{aligned}$$

with ϕ and $\tilde{\phi}$ in Eq. (18.2) and W_{LR} in Eq. (18.6). Notably, the Cooper pairing term mediated between both TS wires through lead S, which corresponds to the off-diagonal Nambu component in Eq. (18.9), exhibits a pronounced θ -dependence through the factor $\sin(\theta/2)$. In particular, the induced pairing is maximal for opposite spin polarization of the TS leads ($\theta = \pi$) but vanishes for collinear polarization ($\theta = 0$).

A numerical calculation of the CPR in Eq. (18.7) poses no serious challenges, but fortunately many limiting cases of interest can also be treated analytically. In the remainder of this section, we first discuss the parallel configuration with $\phi = 0$, see Fig. 18.1(b) and Sec. 18.1.2. We then continue with the serial configuration defined by $\tilde{\phi} = 0$, see Fig. 18.1(c) and Sec. 18.1.3. These two cases are expected to capture the essential Josephson physics in our setup. We finally allow for arbitrary phase configurations in Sec. 18.1.4, where we address the atomic limit for either the S lead ($\Delta_S \rightarrow \infty$) and/or the TS leads ($\Delta \rightarrow \infty$).

18.1.2 Parallel configuration

We begin with the parallel configuration shown in Fig. 18.1(b), where $\phi = \phi_L - \phi_R = 0$ and only $\tilde{\phi} = \phi_L + \phi_R$ can change. In effect, we thus have a two-terminal Josephson junction with

phase difference $\tilde{\phi}/2$ between a two-channel TS lead and the S lead, termed “S-2TS junction” in what follows. Results for this configuration are shown in Fig. 18.2 for various values of θ .

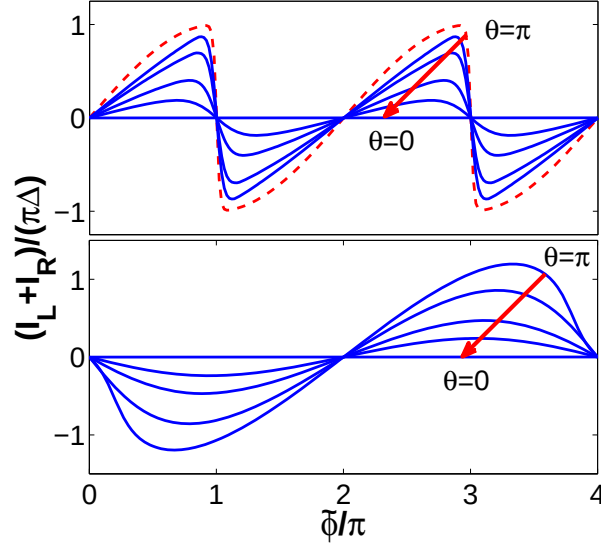


Figure 18.2: Supercurrent $I_L = I_R$ vs phase $\tilde{\phi} = \phi_L + \phi_R$ for the parallel configuration with $\phi = \phi_L - \phi_R = 0$ shown in Fig. 18.1(b), obtained numerically from Eq. (18.7) for $\lambda_L = \lambda_R = 1$, $\Delta_S = \Delta$, and $T = 0.02\Delta$. From top to bottom, the spin angle is given by $4\theta/\pi = 4, 3, 2, 1, 0$. The upper (lower) panel is for $\lambda_{LR} = 0$ ($\lambda_{LR} = 1$). The red dashed curve in the upper panel is given by Eq. (18.9) for $\Delta_S \rightarrow \infty$ and $\theta = \pi$. The blue curve for $\theta = \pi$ in the upper panel is also known analytically, see Eq. (18.11).

Without direct TS-TS coupling

Let us first discuss the case without direct TS-TS coupling, $\lambda_{LR} = 0$, cf. the upper panel in Fig. 18.2. The critical current, $I_c = \max |I_{L/R}(\tilde{\phi})|$, has a maximum for antiparallel spin alignment ($\theta = \pi$) and then monotonically decreases down to $I_c = 0$ as θ is reduced to $\theta = 0$. This behavior indicates that the supercurrent is indeed due to Cooper pair splitting processes as anticipated in Sec. 18.0.1. Such processes are favored in the antiparallel spin configuration since a Cooper pair in the S lead is built from electrons with opposite spin. On the other hand, they are ruled out for $\theta = 0$. Remarkably, the CPR is π -periodic in the phase difference $\tilde{\phi}/2$ across the S-2TS junction.

As discussed in Sec. 18.1.4 below, analytical results for the CPR follow in the atomic limit for the S lead, $\Delta_S \rightarrow \infty$, where we find

$$\begin{aligned} I_{L/R}^{(\Delta_S \rightarrow \infty)}(\tilde{\phi}) &= -\frac{\partial E_A}{\partial \tilde{\phi}} \tanh(E_A/2T), \\ E_A(\tilde{\phi}) &= \sqrt{\tau(\theta)} \Delta \cos(\tilde{\phi}/2). \end{aligned} \quad (18.9)$$

The Andreev bound state (ABS) energy $E_A(\tilde{\phi})$ depends on the spin canting angle θ via the transmission amplitude,

$$\tau(\theta) = \frac{4\lambda^4 \sin^2(\theta/2)}{[1 + \lambda^4 \sin^2(\theta/2)]^2}, \quad (18.10)$$

characterizing the transparency of the S-2TS Josephson junction with $\lambda_L = \lambda_R = \lambda$. The dependence of the ABS energy on the individual phases $\phi_{L/R}$ is 4π -periodic, reflecting the fact that single electrons are transferred into/from each TS lead. Nonetheless, the actual

CPR turns out to be π -periodic in the phase difference $\tilde{\phi}/2$ across the S-2TS junction, with jump-like behavior of the CPR near $\tilde{\phi} = \pi \pmod{2\pi}$, cf. the upper panel of Fig. 18.2. This peculiar behavior can be traced back to parity crossings and will be discussed in more detail in Sec. 18.1.4.

Another analytically accessible case is given by $\Delta_S = \Delta$. For instance taking $\theta = \pi$, where I_c will be maximal, we obtain

$$I_{L/R}^{(\Delta_S=\Delta)}(\tilde{\phi}) = - \sum_{\pm} \frac{E_A^{(\pm)}}{\partial \tilde{\phi}} \tanh \left(\frac{E_A^{(\pm)}}{2T} \right), \quad (18.11)$$

with a pair of ABS energies

$$E_A^{(\pm)}(\tilde{\phi}) = \frac{\Delta}{\sqrt{2}} \left(1 \pm \sqrt{1 - \tau^2 \cos^2(\tilde{\phi}/2)} \right)^{1/2}, \quad (18.12)$$

where $\tau = 4\lambda^2/(1 + \lambda^2)^2$. Equation (18.11) nicely matches the corresponding numerical result for $\Delta_S = \Delta$ shown in the upper panel of Fig. 18.2.

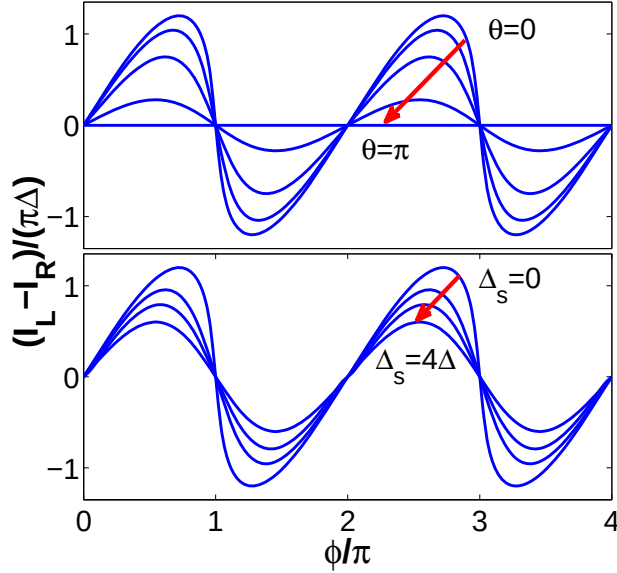


Figure 18.3: TS-TS supercurrent vs phase difference $\phi = \phi_L - \phi_R$ in the serial case $\tilde{\phi} = 0$, see Fig. 18.1(c), where $I_S = 0$ and $I_L = -I_R$. The S lead acts as currentless probe. The upper panel illustrates the supercurrent suppression with increasing spin angle θ for a normal probe ($\Delta_S = 0$), see Eq. (18.14), for $4\theta/\pi = 0, 1, 2, 3, 4$. The lower panel shows how current is suppressed with increasing Δ_S for $\theta = 0$, taking $\Delta_S/\Delta = 0, 1, 2, 4$. All other parameters are as in Fig. 18.2 with $\lambda_{LR} = 0$.

With direct TS-TS coupling

We next briefly address the effects of a direct tunnel coupling λ_{LR} between both TS wires. Typical examples for the CPR are shown in the lower panel of Fig. 18.2, where, for simplicity, we assumed the θ -independent value $\lambda_{LR} = 1$. This implies equal weights for TS-TS tunneling processes with and without spin flips. Such an assumption can be phenomenologically justified by taking into account the magnetic field, cf. the more microscopic model discussed in Sec. 18.2.

We observe from Fig. 18.2 that λ_{LR} has two important consequences as compared to the case $\lambda_{LR} = 0$ studied so far. First, the CPR becomes 2π - rather than π -periodic. Second, the CPR exhibits π -junction behavior, i.e., it appears with a phase shift of π such that $\tilde{\phi} = 0$ now

represents a maximum of the junction free energy. Apart from the above two changes, however, the overall behavior of the CPR is similar to the case $\lambda_{LR} = 0$. In particular, the critical current is maximal for $\theta = \pi$ and vanishes for $\theta = 0$.

The π -junction behavior observed numerically in Fig. 18.2 can also be captured analytically in the atomic limit $\Delta \rightarrow \infty$ for the TS wires, cf. Sec. 18.1.4. Indeed, for spin canting angle $\theta = \pi$ but arbitrary λ_{LR} , we obtain

$$I_{L/R}^{(\Delta \rightarrow \infty)} = \frac{T}{2} \sum_{\omega} \left(\lambda^2 \sin \tilde{\phi} - 2\lambda_{LR} \sqrt{1 + \frac{\omega^2}{\Delta_S^2}} \sin(\tilde{\phi}/2) \right) \frac{\lambda^2 \Delta_S^2}{\lambda^4 \omega^2 + [\lambda_{LR} \sqrt{\omega^2 + \Delta_S^2} - \lambda^2 \Delta_S \cos(\tilde{\phi}/2)]^2} \quad (18.13)$$

with $\lambda_L = \lambda_R = \lambda$. Evidently π -junction behavior, $\partial_{\tilde{\phi}} I_{L/R}(\tilde{\phi} = 0) < 0$, is predicted by Eq. (18.13) for sufficiently large direct TS-TS couplings, $\lambda_{LR} > \lambda^2$.

18.1.3 Serial configuration

We now turn to the serial configuration in Fig. 18.1(c), where $\phi_L = -\phi_R$ and hence $\tilde{\phi} = 0$. This case can be viewed as a TS-TS Josephson junction with phase difference $\phi = \phi_L - \phi_R$ in the presence of a currentless (since $I_S = 0$) probe electrode corresponding to the S lead. Figure 18.3 shows numerical results for the CPR, $I_L(\phi) = -I_R(\phi)$, in this configuration.

Without direct TS-TS coupling

As in Sec. 18.1.2, we start our analysis with the case $\lambda_{LR} = 0$. A finite supercurrent flowing between the TS leads can then only be mediated through (real or virtual) tunneling processes via states in the S lead. Note that such states are only available at energies above Δ_S . This suggests that the critical current I_c will be maximal for $\Delta_S = 0$ and then decreases with increasing Δ_S . Furthermore, since tunneling processes between the TS leads have highest amplitude for parallel spin alignment, we expect that I_c is now maximal for $\theta = 0$ but then will be suppressed with increasing θ . We emphasize that the θ -dependence of I_c is reversed with respect to the parallel configuration in Sec. 18.1.2. The current suppression with increasing θ and/or Δ_S is shown in the upper and lower panel of Fig. 18.3, respectively. It is worth mentioning that the CPR, $I_L(\phi)$, is always 2π -periodic.

In order to better understand the results in Fig. 18.3, we again turn to analytically accessible limits. In particular, when the S lead has a vanishing gap ($\Delta_S = 0$) and thus represents a normal-conducting probe, Eq. (18.7) can be simplified to the expression

$$\begin{aligned} I_L^{(\Delta_S=0)}(\phi) &= -T \sum_{\omega} \frac{2E_A \partial_{\phi} E_A}{\omega^2 + E_A^2 + \gamma |\omega| \sqrt{\omega^2 + \Delta^2}}, \\ E_A(\phi) &= \tilde{\Delta} \sqrt{1 - \tilde{\tau} \sin^2(\phi/2)}, \end{aligned} \quad (18.14)$$

with

$$\begin{aligned} \tilde{\Delta} &= \frac{\Delta}{\sqrt{1 + y^2}}, \quad \tilde{\tau} = \cos^2(\theta/2), \\ y &= \frac{1 + \lambda^4 \sin^2(\theta/2)}{2\lambda^2}. \end{aligned} \quad (18.15)$$

Importantly, the rate $\gamma = 2y/(1 + y^2)$ in Eq. (18.14) causes ABS damping even though the 2π -periodic ABS energy E_A is always detached from the TS continuum, $E_A < \Delta$. Typical CPR curves resulting from Eq. (18.14) are shown in the upper panel of Fig. 18.3.

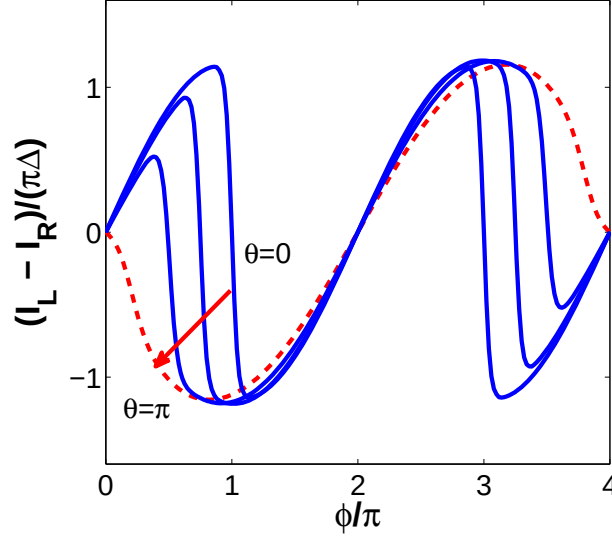


Figure 18.4: TS-TS supercurrent vs ϕ in the serial configuration ($\tilde{\phi} = 0$) obtained from Eq. (18.7) for direct TS-TS coupling $\lambda_{LR} = 1$ and several values of the spin canting angle, $4\theta/\pi = 0, 1, 2, 4$. All other parameters are as in Fig. 18.3.

Analytical progress is also possible for $\Delta_S = \Delta$, where no damping terms are encountered. Focusing on the case of maximal supercurrent, $\theta = 0$, we then obtain

$$I_L^{(\Delta_S=\Delta)}(\phi) = -\frac{\partial E_A}{\partial \phi} \tanh\left(\frac{E_A}{2T}\right), \quad (18.16)$$

with the ABS energy $E_A(\phi) = \Delta\sqrt{1 - \tau_1 \sin^2(\phi/2)}$ and $\tau_1 = 4\lambda^4/(1 + 2\lambda^2)^2$. Equation (18.16) matches the numerical curve with $\Delta_S = \Delta$ and $\theta = 0$ shown in the lower panel of Fig. 18.3, and therefore illustrates the supercurrent suppression with increasing Δ_S/Δ at fixed spin angle θ .

With direct TS-TS coupling

In the presence of a direct TS-TS coupling λ_{LR} , the 2π -periodicity of the CPR found above for $\lambda_{LR} = 0$ may turn into a 4π -periodicity. This remarkable effect is shown in Fig. 18.4. Such a periodicity is found in the absence of coupling to the S lead ($\lambda_{L/R} = 0$) when the parity of the junction is conserved [10]. In our setup, the 4π -periodicity can be understood by noting that the S lead acts as a probe electrode and thereby effectively fixes the fermion parity of the trijunction. As illustrated in Fig. 18.4, we find that the 4π Josephson effect takes place for all $\theta \neq 0 \pmod{2\pi}$.

The 4π -periodicity can also be confirmed analytically in the atomic limit $\Delta \rightarrow \infty$ for the TS wires. For $\theta = \pi$ and arbitrary λ_{LR} , we then obtain

$$I_L^{(\Delta \rightarrow \infty)} = \frac{T}{2} \sum_{\omega} \left(\lambda_{LR} \sin \phi - \frac{2\lambda^2 \Delta_S}{\sqrt{\omega^2 + \Delta_S^2}} \sin(\phi/2) \right) \frac{\lambda_{LR}(\omega^2 + \Delta_S^2)}{\lambda^4 \omega^2 + [\lambda_{LR} \sqrt{\omega^2 + \Delta_S^2} \cos(\phi/2) - \lambda^2 \Delta_S]^2}$$

with $\lambda_L = \lambda_R = \lambda$. This result explicitly shows the 4π -periodicity of the CPR. Note that in addition, π -junction behavior is predicted by Eq. (18.17) in the parameter regime $\lambda_{LR} < \lambda^2$. Let us also mention in passing that when the atomic limit is also taken for the S lead, $\Delta_S \rightarrow \infty$, Eq. (18.17) simplifies to Eq. (18.20) below (taken for $\theta = \pi$).

It is worth stressing that the TS-S-TS setup thereby provides a completely robust strategy to measure the 4π -periodic CPR expected in the ground state of a TS-TS junction with conserved

fermion parity [8, 10, 2, 4]. This setup could therefore significantly simplify future experimental studies of the topological Josephson effect in a true equilibrium state.

18.1.4 Atomic limit

We now address arbitrary phase configurations $\phi_{L/R}$ but restrict ourselves to cases where the TS gap Δ and/or the S gap Δ_S become much larger than all other energy scales of interest. Effectively taking $\Delta \rightarrow \infty$ and/or $\Delta_S \rightarrow \infty$ defines the atomic limit for the TS/S leads, where one can approximate the respective lead by a zero-bandwidth model.

First, for $\Delta_S \rightarrow \infty$, the self-energy $\Sigma(\omega)$ in Eq. (18.9) simplifies to $\Sigma_{LL/RR} = 0$ while $\Sigma_{LR}(\omega) = \Sigma_{RL}^\dagger(-\omega)$ becomes frequency independent and hence local in time,

$$\Sigma_{LR} = \begin{pmatrix} \lambda_{LR} e^{-i\phi/2} & -\lambda_L \lambda_R \sin(\theta/2) e^{-i\tilde{\phi}/2} \\ \lambda_L \lambda_R \sin(\theta/2) e^{i\tilde{\phi}/2} & -\lambda_{LR} e^{i\phi/2} \end{pmatrix}, \quad (18.17)$$

where we use Eq. (18.2).

For $\Delta \rightarrow \infty$, on the other hand, above-gap excitations in the TS wires become negligible and a projection to the respective low-energy MBS sector is sufficient. Taking this limit in Eq. (18.3) with $\Delta_L = -\Delta_R = \Delta$, we obtain the TS bGFs, $G_{L/R}(\omega) = -i(\Delta/\omega)[\sigma_0 \pm \sigma_x]$. Introducing Majorana operators $\gamma_{L,R} = \gamma_{L,R}^\dagger$ with anticommutator $\{\gamma_j, \gamma_{j'}\} = \delta_{jj'}$, this limiting form of the bGF implies a projection of the boundary spinors Φ_j in Sec. 18.1.1 onto the subspace spanned by MBSs only,

$$\Phi_L \rightarrow \sqrt{\Delta} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \gamma_L, \quad \Phi_R \rightarrow i\sqrt{\Delta} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \gamma_R. \quad (18.18)$$

Taking also the atomic limit for the S lead, the time-local form of $\Sigma(\omega)$ in Eq. (18.17) implies that the Dyson equation (18.8) is solved by diagonalizing the effective Hamiltonian

$$\begin{aligned} H_{at} &= \frac{1}{2} \Phi_L^\dagger \Sigma_{LR} \Phi_R + \text{h.c.} = iE_A(\phi, \tilde{\phi}) \gamma_L \gamma_R, \\ E_A &= 2\Delta \left(\lambda_{LR} \cos(\phi/2) + \lambda_L \lambda_R \sin(\theta/2) \cos(\tilde{\phi}/2) \right). \end{aligned} \quad (18.19)$$

Note that Eq. (18.19) recovers the ABS energy (18.9) for $\lambda_{L,R} \ll 1$ and $\lambda_{LR} = 0$. Moreover, for $\lambda_{L/R} = 0$, Eq. (18.19) reduces to the celebrated 4π -periodic ABS energy of a two-terminal TS-TS junction [8, 10, 2, 4].

Importantly, Eq. (18.19) provides insights concerning π -junction behavior. To that end, let us consider the serial configuration in Sec. 18.1.3. Putting $\tilde{\phi} = 0$ in Eq. (18.19), we obtain the $T = 0$ CPR

$$I_L(\phi) = -I_R = \frac{e\lambda_{LR}\Delta}{\hbar} \sin(\phi/2) \text{sgn}[\lambda_{LR} \cos(\phi/2) + \lambda_L \lambda_R \sin(\theta/2)]. \quad (18.20)$$

The CPR is 4π -periodic and will show a transition to the π -junction regime once the sign factor $\text{sgn}[\dots]$ becomes negative. In fact, π -junction behavior is expected to be rather common in TS-S-TS trijunctions near the atomic limit.

18.2 Boundary GF approach for spinful nanowires

So far our description of the TS leads has been based on the low-energy limit of a semi-infinite Kitaev chain, see Sec. 18.1. A more microscopic approach tailor-made for semiconductor nanowire implementations is discussed in this section. To that end, we first derive the bGF

for the spinful TS nanowire model of Refs. [6, 7], which arguably can give a rather accurate description of the band structure. This model applies for a single conduction channel but includes spin-orbit coupling, proximity-induced pairing, and a magnetic Zeeman field. In a second step, we then apply this bGF to the problem of Josephson transport in TS-S-TS junctions and compare the results to those in Sec. 18.1.

18.2.1 Derivation of the bGF

A spatially discretized version of the model in Refs. [6, 7] is given by

$$\begin{aligned} H_{\text{wire}} &= \frac{1}{2} \sum_j \left[\psi_j^\dagger \hat{h} \psi_j + \left(\psi_j^\dagger \hat{t} \psi_{j+1} + \text{h.c.} \right) \right], \\ \hat{h} &= (2t - \mu) \sigma_z \tau_0 + V_x \sigma_0 \tau_x + \Delta \sigma_x \tau_0, \\ \hat{t} &= -t \sigma_z \tau_0 + i \alpha \sigma_z \tau_z, \end{aligned} \tag{18.21}$$

where the fermion operators $c_{j\sigma}$ for a given lattice site j and spin σ are combined in the four-spinor $\psi_j^T = (c_{j\uparrow}, c_{j\downarrow}, c_{j\downarrow}^\dagger, -c_{j\uparrow}^\dagger)$. In Eq. (18.21), the Pauli matrices $\sigma_{x,y,z}$ as well as the identity σ_0 act as before in Nambu (particle-hole) space, while Pauli matrices $\tau_{x,y,z}$ and the identity τ_0 are defined in spin space.

To select reasonable model parameters for the calculations below, we first choose the lattice spacing $a = 10$ nm, which yields the nearest-neighbor hopping element $t = \hbar^2/(2m^*a^2)$ and the spin-orbit coupling $\alpha = \hbar u/a$, where u is the spin-orbit interaction parameter in the continuum model of Ref. [7]. With the effective mass $m^* \simeq 0.02m_e$ for InAs, we obtain $t = 20$ meV and estimate $\alpha \simeq 4$ meV from the measurements in Ref. [46]. The Zeeman scale V_x is determined by the magnetic field B along the wire, $V_x = \mu_B g B/2$, with Landé factor $g \approx 10$ in InAs. For the gap parameter, we assume $\Delta = 0.2$ meV. The remaining free variables are B (and hence V_x) and the chemical potential μ , which are key experimental parameters for changing the nanowire properties.

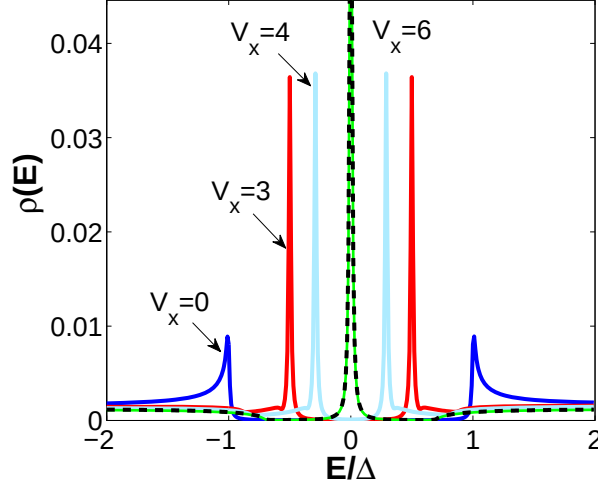


Figure 18.5: Local density of states $\rho(E)$ (in units of $1/\Delta$) at the wire boundary vs energy E (in units of Δ), using the spinful wire model (18.21). We have used a broadening parameter $\gamma_0 = 0.01\Delta$ to visualize the subgap structure. We put $\mu = 5$ meV and vary the Zeeman scale V_x from 0 to 6 meV, with all other parameters specified in the main text. The topological phase is realized for $V_x > V_c = \sqrt{\mu^2 + \Delta^2} \simeq 5.004$ meV [6, 7]. For $V_x = 6$ meV, the zero-energy Majorana peak is observed. The dashed curve gives the corresponding result obtained from the Kitaev model in Sec. 18.1, with an effective proximity gap and an effective normal density of states.

In this section, we work with the retarded GF $G^r(\omega)$ of a given TS wire, which is obtained by analytic continuation, $i\omega \rightarrow \omega + i0^+$, from the respective Matsubara GF $G(\omega)$ discussed in Sec. 18.1. To lighten notation, we drop the r superscript from now on. For an infinitely long nanowire described by Eq. (18.21), the GF in spin-Nambu space (indicated by the hat superscript) follows in frequency-momentum representation as

$$\begin{aligned}\hat{G}^{(0)}(k, \omega) &= [\omega - \hat{h} - \hat{V}(k)]^{-1}, \\ \hat{V}(k) &= -2t \cos(ka) \sigma_z \tau_0 + 2\alpha \sin(ka) \sigma_z \tau_z.\end{aligned}\tag{18.22}$$

Explicit expressions for the matrix elements $[\hat{G}^{(0)}(k, \omega)]_{lm}$ in spin-Nambu space are summarized in Appendix 18.4. The corresponding real-space representation is obtained by contour integration,

$$\hat{G}_{lm}^{(0)}(\omega) = \oint_{|z|=1} \frac{dz}{iaz} \hat{G}^{(0)}(k, \omega) z^{(l-m)}, \quad z = e^{ika}.\tag{18.23}$$

We then determine the poles of $\hat{G}^{(0)}(k, \omega)$ in the z -plane numerically, while the corresponding residues are computed analytically. Finally, the bGF for the left/right semi-infinite spinful TS wire follows by solving the respective Dyson equation for breaking the infinite chain [20],

$$\begin{aligned}\hat{G}_L &= \hat{G}_{00}^{(0)} - \hat{G}_{01}^{(0)} \left(\hat{G}_{00}^{(0)} \right)^{-1} \hat{G}_{10}^{(0)}, \\ \hat{G}_R &= \hat{G}_{00}^{(0)} - \hat{G}_{10}^{(0)} \left(\hat{G}_{00}^{(0)} \right)^{-1} \hat{G}_{01}^{(0)},\end{aligned}\tag{18.24}$$

where the indices refer to lattice sites.

As a first example for this approach, we study the local density of states at the boundary of the wire, $\rho(E) = -(1/\pi) \text{Im}[\hat{G}_L(E)]_{11}$, where the indices now refer to spin-Nambu space.

Figure 18.5 illustrates the evolution of $\rho(E)$ as one varies the Zeeman scale V_x . Below the critical value $V_c = \sqrt{\mu^2 + \Delta^2}$, the topologically trivial phase without MBSs is realized. For $V_x > V_c$, however, Majorana zero modes develop and cause a zero-energy peak in $\rho(E)$. For comparison, Fig. 18.5 also shows the limiting behavior predicted by the Kitaev model used in Sec. 18.1.

18.2.2 Spin structure of MBSs

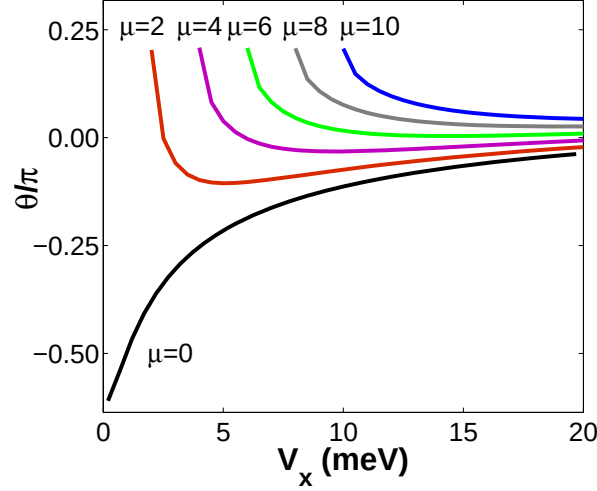


Figure 18.6: Spin canting angle $\theta = \theta_L - \theta_R$ vs Zeeman scale V_x obtained from Eq. (18.26) for different values of μ . The shown results are within the topological regime, $V_x > V_c$. All other parameters of the spinful TS wire model (18.21) are specified in the main text.

The spinful model (18.21) allows us to analyze the MBS spin structure and its evolution under changes of the magnetic field and/or the chemical potential. The spin spectral density at the boundary of the respective TS wire has the components ($\nu = x, y, z$)

$$S_{\nu,L/R}(\omega) = \frac{1}{2\pi i} \text{Tr} \left[(1 + \sigma_z) \tau_\nu (G_{L/R}^a(\omega) - G_{L/R}^r(\omega)) \right], \quad (18.25)$$

with a trace over spin-Nambu space, and the advanced GF G^a . As discussed in Refs. [27, 28, 29], the MBS spin polarization points within a plane perpendicular to the spin-orbit direction (the x - y plane in our case). We may then determine the spin angles in Fig. 18.1(a) from the zero-energy limit of Eq. (18.25),

$$\tan \theta_{L/R} = \lim_{\omega \rightarrow 0} \frac{S_{y,L/R}(\omega)}{S_{x,L/R}(\omega)}. \quad (18.26)$$

The symmetry of the setup implies $\theta_R = -\theta_L$. Hence we can compute the spin canting angle $\theta = \theta_L - \theta_R$, which was introduced in Sec. 18.1 on purely phenomenological grounds, as a function of the model parameters in Eq. (18.21).

The resulting dependence of θ on V_x and/or μ is shown in Fig. 18.6. With increasing Zeeman scale V_x (always in the topological regime $V_x > V_c$), we find that θ decreases as a power law where the exponent depends on system parameters, see also Ref. [29]. This decrease becomes less pronounced for large values of μ which may therefore be advantageous in achieving a widely tunable spin angle. In any case, the choice of V_x and/or μ will affect θ as shown in Fig. 18.6, which in turn affects Josephson transport in the trijunction as described in Sec. 18.1.

18.2.3 CPR across the topological transition

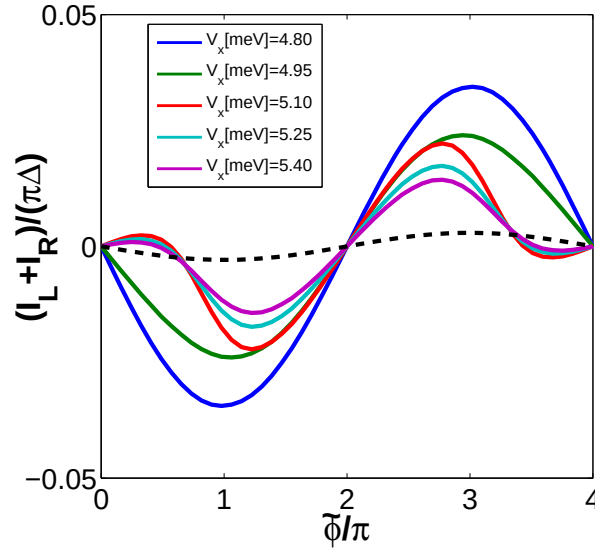


Figure 18.7: CPR for the parallel configuration from the spinful model without direct TS-TS coupling, $\lambda_{LR} = 0$. The Zeeman scale V_x is ramped through the topological transition at $V_c \simeq 5.004$ meV, with $\mu = \lambda_L = \lambda_R = 5$ meV. The dashed curve gives the CPR for $\lambda_R = 0$ and $V_x = 5.4$ meV. For all other parameters, see main text.

We next use the spinful model for analyzing the trijunction CPR across the topological transition at $V_x = V_c$. The tunnel Hamiltonian (18.5) can be directly extended to the spinful case. However, we do not absorb density of states factors in the tunnel amplitudes $\lambda_{L,R,LR}$ anymore (as in Sec. 18.1) but instead specify them in meV units.

Let us first study the parallel configuration $\phi = 0$, see Fig. 18.1(b), for the case without direct TS-TS coupling, $\lambda_{LR} = 0$. We also put $\Delta = \Delta_S$ even though the effective TS gap becomes reduced when crossing the topological transition, cf. Fig. 18.5. The resulting CPR is shown in Fig. 18.7 for $\mu = 5$ meV and $\lambda_L = \lambda_R = 5$ meV. We observe a transition from the conventional 2π -periodic behavior for $V_x < V_c$ to a distorted CPR for $V_x > V_c$ that gradually resembles the CPR predicted by the Kitaev model in Sec. 18.1, cf. the upper panel in Fig. 18.2. We emphasize that in order to match to results obtained from the Kitaev model, one should have $\mu \gg \Delta$. For $\mu \rightarrow 0$, the s -wave pairing contribution turns out to be important and masks the corresponding behavior. For reference, the dashed curve in Fig. 18.7 specifies the S-TS Josephson current for $\lambda_R = \lambda_{LR} = 0$, where the right TS wire is completely decoupled. The CPR should then vanish according to the model in Sec. 18.1, see Ref. [35]. However, when using the spinful model, we observe a finite (albeit much reduced) critical current.

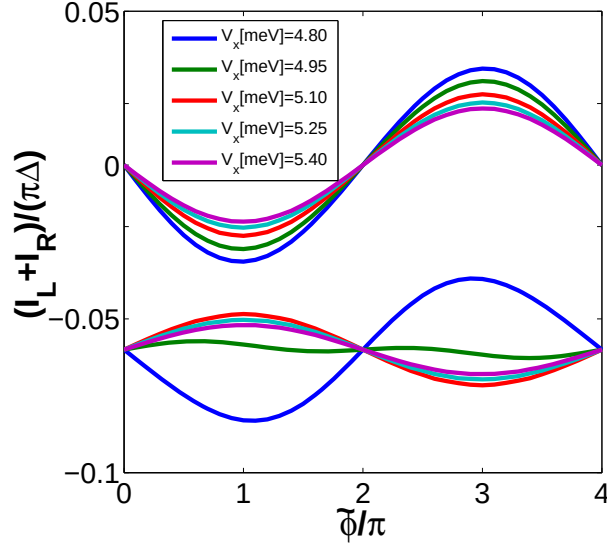


Figure 18.8: Same as in Fig. 18.7 but including a direct TS-TS coupling, $\lambda_{LR} \neq 0$. The lower (upper) set of curves corresponds to $\lambda_{LR} = 5$ meV ($\lambda_{LR} = -5$ meV), where the lower curves have been shifted by -0.06 for clarity.

Next, Fig. 18.8 shows results for the same configuration as in Fig. 18.7 but allowing for a direct TS-TS coupling λ_{LR} . We observe that the behavior of the CPR across the transition critically depends on the sign of λ_{LR} . When $\lambda_{LR} < 0$, the CPR exhibits π -junction behavior both for $V_x < V_c$ and $V_x > V_c$, and thus the evolution across the transition is smooth. However, for $\lambda_{LR} > 0$, a change from π -junction to 0-junction behavior occurs right at the topological transition. This sensitivity to the sign of the direct tunnel matrix element λ_{LR} is characteristic for our topological TS-S-TS trijunction and would not be found in topologically trivial systems.

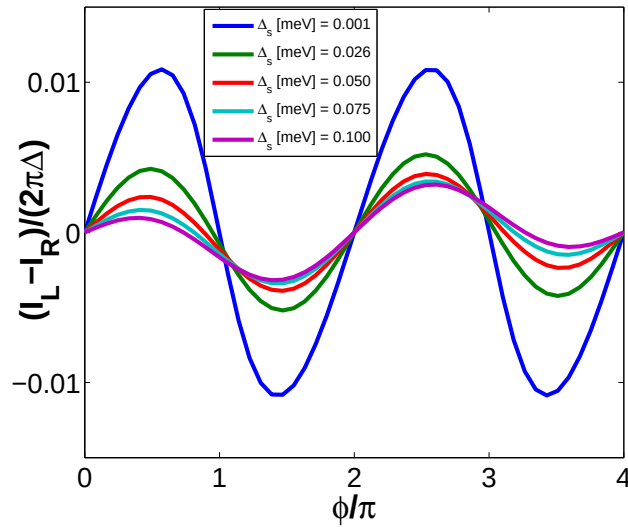


Figure 18.9: CPR for the serial configuration from the spinful model for several values of Δ_S . Parameters were chosen as $\mu = \lambda_L = \lambda_R = 5$ meV, $V_x = 7$ meV, and $\lambda_{LR} = 0$. For all other parameters, see main text.

Finally, we have also explored the CPR for the spinful model in the serial configuration, cf. Fig. 18.1(c). Our results in Fig. 18.9 correspond to a situation where both TS leads are deep in the topological phase. As a consequence, the spin angle is quite small, cf. Fig. 18.6,

and hence large supercurrents are expected from Sec. 18.1.3, cf. the upper panel of Fig. 18.3. Next, Figure 18.9 illustrates the effect of increasing the gap Δ_S in the S lead in the absence of any direct TS-TS coupling. The observed supercurrent decrease is consistent with our results in Sec. 18.1.3. This dependence of the current on Δ_S , with a full suppression for $\Delta_S \rightarrow \infty$, confirms that a finite TS-TS supercurrent must involve excited states in the S lead.

18.3 Conclusions

In this paper, we have studied Josephson transport in a trijunction geometry involving a pair of TS wires and one conventional *s*-wave (S) superconductor. One can distinguish different parameter regimes for this TS-S-TS setup. In particular, we have studied parallel vs serial configurations, see Fig. 18.1, where depending on the choice of the bias phases, one either realizes an effective Josephson junction between the S lead and a two-channel TS lead (parallel configuration), or one has a TS-TS junction with a currentless probe defined by the S lead (serial configuration). In the former case, the supercurrent is mainly carried by splitting (creating) Cooper pairs in the S lead into (from) two electrons in different TS wires. Such processes may become highly effective in the presence of Majorana states. The supercurrent is then maximized (vanishes) for antiparallel (parallel) spin alignment of both TS wires. In the serial configuration, the picture is basically the other way around since now Josephson transport involves tunneling of quasiparticles between the TS leads. In the absence of a direct coupling between both TS wires, such processes are only possible via excited states in the conventional superconductor. Interestingly, this configuration also offers a convenient way to stabilize the 4π -periodicity of the current-phase relation, see Sec. 18.1.3, and we hope that future experiments will be guided by this observation.

For understanding the physics of such devices, we found it useful to employ a comparatively simple Kitaev chain modeling, see Sec. 18.1, which allows for analytical progress in several different limits within the boundary Green's function formulation. As an alternative approach, we have also extended this formalism to the widely used spinful Majorana wire model of Refs. [6, 7]. The predictions obtained from the Kitaev model are qualitatively, and in some cases even quantitatively, reproduced from these more microscopic calculations. Most importantly, the spinful wire modeling provides access to the spin canting angle θ between the spin polarization axes in the two TS wires. We have clarified how this angle depends on two key parameters of experimental interest, namely on the chemical potential and on the Zeeman field. Although not shown explicitly, the present formalism also allows one to take into account finite-length effects in the TS wires [20], where both Majorana end states in a given TS wire will hybridize. We expect that the main consequence of this hybridization is a reduction of the critical current due to a decrease in the anomalous Green's function.

To conclude, our results show that the multiterminal Josephson effect in setups containing both conventional and topological superconductors can provide detailed information about the spin polarization of MBSs. We hope that our predictions will soon be tested experimentally.

18.4 On the boundary GF of the spinful model

Here we provide details on the derivation of the bGF for the spinful nanowire model in Sec. 18.2.1. For the infinite-wire retarded GF in frequency-momentum space, $\hat{G}^{(0)}(k, \omega)$, we write the matrix elements in spin-Nambu space in the form

$$\hat{G}_{lm}^{(0)}(k, \omega) = \frac{\tilde{G}_{lm}^{(0)}(k, \omega)}{\det(\omega - \hat{H}_k)}, \quad (18.27)$$

where the indices $l, m = 1, 2, 3, 4$ correspond to the entries in the four-spinor ψ_j in Sec. 18.2.1, and we define $\hat{H}_k = \hat{h} + \hat{V}(k)$, cf. Eqs. (18.21) and (18.22).

Using $u_k = 2\alpha \sin(ka)$ and $\epsilon_k = 2t[1 - \cos(ka)] - \mu$, we obtain the matrix elements

$$\begin{aligned}\tilde{G}_{11}^{(0)}(k, \omega) &= (\omega + \epsilon_k + u_k)[\omega^2 - \Delta^2 - (\epsilon_k - u_k)^2] + V_x^2(\epsilon_k - \omega - u_k), \\ \tilde{G}_{12}^{(0)}(k, \omega) &= V_x[\Delta^2 + (\omega + \epsilon_k)^2 - u_k^2 - V_x^2], \\ \tilde{G}_{13}^{(0)}(k, \omega) &= \Delta[\omega^2 + V_x^2 - \Delta^2 - (\epsilon_k - u_k)^2], \\ \tilde{G}_{14}^{(0)}(k, \omega) &= 2\Delta V_x(\omega + u_k),\end{aligned}$$

$$\begin{aligned}\tilde{G}_{21}^{(0)}(k, \omega) &= V_x[\Delta^2 + (\omega + \epsilon_k)^2 - u_k^2 - V_x^2], \\ \tilde{G}_{22}^{(0)}(k, \omega) &= (\omega + \epsilon_k - u_k)[\omega^2 - \Delta^2 - (\epsilon_k + u_k)^2] + \\ &\quad + V_x^2(\epsilon_k - \omega + u_k), \\ \tilde{G}_{23}^{(0)}(k, \omega) &= 2\Delta V_x(\omega - u_k), \\ \tilde{G}_{24}^{(0)}(k, \omega) &= \Delta[\omega^2 + V_x^2 - \Delta^2 - (\epsilon_k + u_k)^2],\end{aligned}$$

$$\begin{aligned}\tilde{G}_{31}^{(0)}(k, \omega) &= \Delta[\omega^2 + V_x^2 - \Delta^2 - (\epsilon_k - u_k)^2], \\ \tilde{G}_{32}^{(0)}(k, \omega) &= 2\Delta V_x(\omega - u_k), \\ \tilde{G}_{33}^{(0)}(k, \omega) &= (\omega + \epsilon_k - u_k)[-u_k^2 - V_x^2 + (\omega - \epsilon_k)^2] + \Delta^2(\epsilon_k - \omega + u_k), \\ \tilde{G}_{34}^{(0)}(k, \omega) &= V_x[\Delta^2 + (\omega - \epsilon_k)^2 - u_k^2 - V_x^2],\end{aligned}$$

$$\begin{aligned}\tilde{G}_{41}^{(0)}(k, \omega) &= 2\Delta V_x(\omega + u_k), \\ \tilde{G}_{42}^{(0)}(k, \omega) &= \Delta[\omega^2 + V_x^2 - \Delta^2 - (\epsilon_k + u_k)^2], \\ \tilde{G}_{43}^{(0)}(k, \omega) &= V_x[\Delta^2 + (\omega - \epsilon_k)^2 - u_k^2 - V_x^2], \\ \tilde{G}_{44}^{(0)}(k, \omega) &= (\omega + \epsilon_k + u_k)[-u_k^2 - V_x^2 + (\omega - \epsilon_k)^2] - \Delta^2(-\epsilon_k + \omega + u_k),\end{aligned}$$

and the determinant

$$\begin{aligned}\det(\omega - \hat{H}_k) &= \Delta^4 + 2\Delta^2(u_k^2 + \epsilon_k^2 - V_x^2 - \omega^2) + \\ &\quad + u_k^4 + 2u_k^2(V_x^2 - \omega^2 - \epsilon_k^2) + \\ &\quad + V_x^4 - 2V_x^2(\omega^2 + \epsilon_k^2) + \omega^4 - 2\epsilon_k^2\omega^2 + \epsilon_k^4.\end{aligned}$$

19 Quantum circuits with multiterminal Josephson-Andreev junctions by Matute-Cañadas, Tosi, Levy Yeyati

Abstract

We explore superconducting quantum circuits where several leads are simultaneously connected beyond the tunneling regime, such that the fermionic structure of Andreev bound states in the resulting multiterminal Josephson junction influences the states of the full circuit. Using a simple model of single channel contacts and a single level in the middle region, we discuss different circuit configurations where the leads are islands with finite capacitance and/or form loops with finite inductance. We find situations of practical interest where the circuits can be used to define noise protected qubits, which map to the bifluxon and $0-\pi$ qubits in the tunneling regime. We also point out the subtleties of the gauge choice for a proper description of these quantum circuits dynamics.

19.1 Conclusions

Mesoscopic Josephson junctions host localized states with phase-dependent energy which can be probed in spectroscopy and manipulated coherently using microwave pulses. The number of these states, their spin texture and their energy are sensitive to internal as well as control parameters such as electric fields applied with a gate voltage, magnetic fields, or strain. Compared to standard tunnel junctions, the energy of the fermionic excitations may become of the same order of the collective bosonic modes of quantum circuits that include capacitors and inductors. This “mesoscopic embedding” constitutes a promising experimental and theoretical research topic with opportunities to develop hybrid fermionic-bosonic devices more immune to decoherence than conventional ones. Additionally, on a fundamental level, these devices raise questions on the proper modeling and on the correct quantum circuit rules for their combination. In this work we have addressed these questions taking a three-terminal junction connected in different configurations as a model system.

While exploring the energy spectrum and the wavefunctions of these Josephson-Andreev junctions for different circuit configurations we have identified regimes of charge and/or flux noise protection against dephasing and/or relaxation. We have discussed different limiting cases for each configuration to recover known results, and for the multiterminal configurations we have shown their connection with other kinds of protected qubits such as the bifluxon or the $0-\pi$ qubits obtained using standard tunnel junctions arrays. This fact highlights how the multimode character of these circuits can be emulated thanks to the connectivity of the multiterminal ones, with the additional value of their higher degree of tunability.

Regarding the modeling, the nuance of the gauge *choice* for time dependent situations, which includes the considerations about the noise, emerges as an important issue to care about. Though we provide some orientation for this choice in our problem, its strong effect suggests that experimental guidance would be required for more conclusive estimations. Additionally, several open issues remain to be investigated, such as the robustness with respect to quasiparticle poisoning, which requires a more elaborate model for the superconducting leads, or the design of suitable operation protocols. We expect that the present work could motivate further experimental and theoretical efforts along these lines of research.

19.2 Introduction

Josephson junctions are essential ingredients for turning quantum circuits into artificial atoms [1, 2]. When the degrees of freedom describing the circuit are quantized, the non-linear non-dissipative Josephson element introduces the anharmonicity in the potential energy that is required to isolate a set of levels for their use as a computational basis. A conventional tunnel junction is characterized by its charging and Josephson energies, determined, respectively by its capacitance and its critical current, which depend on the tunnel barrier and the junction geometry. This description of the junction as a non-linear inductive lumped element is justified when the energy corresponding to the fermionic excitations of the junction itself is large compared to the lowest circuit levels and, thus, only bosonic excitations are considered.

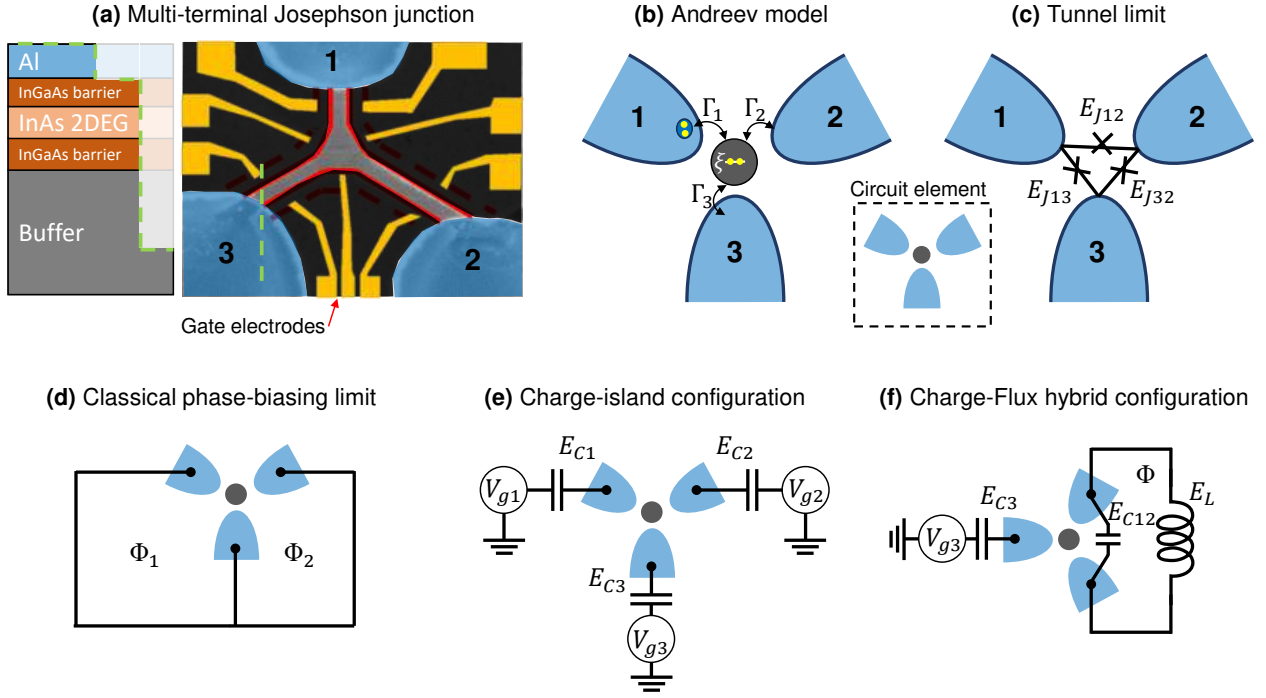


Figure 19.1: (a) Possible implementation of the trijunction in a hybrid superconductor-semiconductor heterostructure. An aluminium film epitaxially grown on top of a 2DEG (pale orange) induce superconductivity by proximity. The 2DEG can be patterned by a defining a mesa and the junctions are created by etching the Al in some regions (indicated with red lines). The gate electrodes confine the electrons of the 2DEG in the center and control the coupling with each lead. (b,c) Models describing the trijunction. The circuit element indicated in a dashed box represents both limits in (d-f). In (b), the middle region is relevant to describe how electrons move between leads. In this work, we use a single level quantum dot in the infinite gap regime at even parity, such that the relevant processes involve the exchange of Cooper pairs between the dot and the leads. In (c), the system is described by effective tunneling processes of single Cooper pairs between the leads -equivalent to 3 tunnel Josephson junctions forming a triangle-, and may reproduce the model in (b) under certain limits. (d-f) Three circuits we consider with the trijunction element. In (d), the phase is externally fixed by the external fluxes $\Phi_{1,2}$. In (e), the trijunction connects three superconducting islands with charging energies $E_{C\nu}$. In (f), two leads form a loop with inductive energy E_L , threaded by flux Φ , and the third lead is a superconducting island.

More generally, from a mesoscopic perspective, a tunnel junction is just a particular case of a weak link between two superconductors [3]. The coupling being weak, bound states with phase-dependent energies form to accommodate the phase difference between both terminals [4, 6, 15]. Solving the energy spectrum of the so-called Andreev states of this *SXS* device means the diagonalization of the Bogoliubov-de Gennes (BdG) Hamiltonian, i.e. a fermionic problem that requires a microscopic model for the *X* region. In this BdG description, the phase is a parameter determined by the circuit where the weak link is embedded, an approach justified when the phase fluctuations are small and the energy corresponding to the excited circuit levels is large compared to the Andreev states.

The hybrid situation, involving a mesoscopic junction where the phase has to be considered as an operator associated with the transfer of charge has acquired interest with the recent realizations of quantum circuits containing Josephson junctions made out of InAs-nanowires

and two-dimensional electron gases (2DEGs) [7, 8, 9, 10, 11]. These junctions may operate away from the tunneling regime and are better described in terms of high transmission conduction channels or with a quantum dot model. The number of channels, the barriers and chemical potentials can be tuned using gates, thus enabling the realization of devices where the control of the junction parameters is critical. Previous works have already considered this *Josephson-Andreev* regime at different degrees of approximation, both in situations with topological leads, where the fermionic levels correspond to Majorana states [12, 13, 14, 15, 16, 17, 18], and with trivial Andreev states [19, 15, 20, 21, 22].

These devices open up the possibility to design hybrid modes combining the fermionic structure of the weak link with the bosonic excitations associated to the electromagnetic collective modes. This is in line with nowadays efforts towards the physical realization of *protected* qubits, based on the concept of a disjoint support of the qubit wavefunctions, which enjoy protection against decoherence from one or more noise sources [23, 24, 25, 26]. Such a strategy has been pursued using multimode circuits with conventional junctions [27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37], mesoscopic junctions where the effect of the Andreev structure is approximated with a modified Josephson potential [38, 39, 40, 41, 42, 43, 44, 45], and mesoscopic junctions where the treatment of the full Andreev structure is essential, as is the case of a quantum-dot with a level in resonance with the Fermi level of the superconducting leads [7, 10, 20, 21, 22].

In this work we explore the properties of multiterminal Josephson-Andreev junctions when immersed in different kinds of quantum circuits (see Fig. 19.1). These are unique mesoscopic components where the weak-link connects several superconducting leads that have become experimentally accessible in proximitized 2D structures [42, 41]. Since there are several phase difference variables which mimic the k space in a solid, there has been considerable interest in multiterminal Josephson devices driven by the possibility of engineering topologically protected singularities in their spectrum [11, 52, 50, 24, 52, 33, 21, 17, 27, 22, 23, 20, 54, 61, 62] and, broadly, other applications that arise from the introduction of additional leads [63, 64, 65, 66, 67, 68, 42, 69, 70, 71, 72, 43, 74, 75, 76, 77, 78, 79, 80]. We find that, just at the level of a triterminal junction [Fig. 19.1a], the presence of a third lead introduces an additional degree of freedom that allows to define protected qubits.

At a more fundamental level, quantum circuits with Josephson-Andreev junctions pose basic questions on the proper quantization rules that determine the system Hamiltonian in a general, time dependent situation. For instance, if a time dependent external flux constrains the phases of some circuit elements, the phase drop distribution between and within these elements is not a gauge freedom as in the static situation. The actual drop has to be determined by the electromagnetic field established in the circuit, which depends on its structure and geometry [81, 82, 83, 84, 85] and on the currents spatial distribution [14, 87, 88, 89, 90, 91, 92]. In addition, the circuit variables bear a different nature depending on the configuration of the circuit, leading to continuous or discrete charge variables [93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103]. In the present work we analyze this problem for different circuit configurations and show its impact on the relaxation rates.

The work is organized as follows: in Section 19.3, we introduce possible experimental implementations and discuss the modeling of the junction with a single level between the leads [Fig. 19.1b], its embedding in a phase biased configuration [Fig. 19.1d] and the effect of the phase drop distribution on the relaxation rates. In Section 19.4, we consider a circuit where the leads are superconducting islands [Fig. 19.1e] and discuss a transmon-like regime that hosts two almost degenerate states. In section 19.5, we connect two leads in a loop with finite inductance with a third one, which is an island [Fig. 19.1f], and discuss a regime simultaneously protected against charge and flux noise. In section 19.6, we describe the tunnel limit of a multiterminal junction [Fig. 19.1c] and its connection to protected multimode circuits designed with several conventional elements. In section 19.7, we summarize and compare the main features of the

analyzed circuits, discuss the effect of quasiparticle poisoning and elaborate on the analysis of decoherence for feasible implementations. We finish with some conclusions in section 19.1. Additional details can be found in the Appendixes.

19.3 Modeling, phase biased configuration and gauge choice

19.3.1 The model

The basic element that we consider as a building block for quantum circuits is the Josephson-Andreev trijunction. As illustrated in Fig. 19.1(a), it consists of a normal central region coupled to three superconducting leads. Such a device can be implemented on a proximitized 2DEG defined, for instance, on a Al/InAs heterostructure [104, 11, 105, 106, 107]. We assume that the charge density in the normal region can be controlled by gates and that a situation where a few or just one conduction channel on each terminal can be reached. In such situation the Josephson coupling between the leads is mediated by a few Andreev bound states, sensitive to the phase on each superconducting contact. Notice that in the configuration of Fig. 19.1(d) two well defined phase differences between the leads can be controlled by external fluxes $\Phi_{1,2}$ through the corresponding loops, as indicated in the figure. Within this section, we assume ideal phase biasing, i.e. loops without inductance and negligible charging energy.

As commented in the introduction, determining the Andreev spectrum for such geometry requires solving the BdG equations with the corresponding boundary conditions for the superconducting phases. This is, in general, a formidable task that can be undertaken using different models and numerical techniques. For the purpose of the present work, however, we adopt several simplifying assumptions that enable a tractable but still realistic description of the device.

Assuming a single channel per lead and that the dimensions of the normal region are short compared to the superconducting coherence length, we describe it in terms of a single level model. We denote by $d_\sigma^\dagger$ the operator creating an electron with spin σ in that level. In addition, we describe the leads in the so-called infinite-gap limit, which allows us to include their effect in the normal region as a frequency independent pairing self-energy [108, 109, 110]. The effective Hamiltonian for the trijunction is given by (see Appendix 19.8)

$$H^{\text{eff}} = \sum_{\sigma} \epsilon n_{\sigma} + U n_{\uparrow} n_{\downarrow} + \sum_{\nu=1}^3 \left[\Gamma_{\nu} e^{-i\phi_{\nu}} d_{\uparrow}^{\dagger} d_{\downarrow}^{\dagger} + \text{h.c.} \right], \quad (19.1)$$

where ϵ is the position of the central level referred to the leads chemical potential, U is its charging energy, and Γ_{ν} is the tunneling rate to the lead ν , which has a phase ϕ_{ν} .

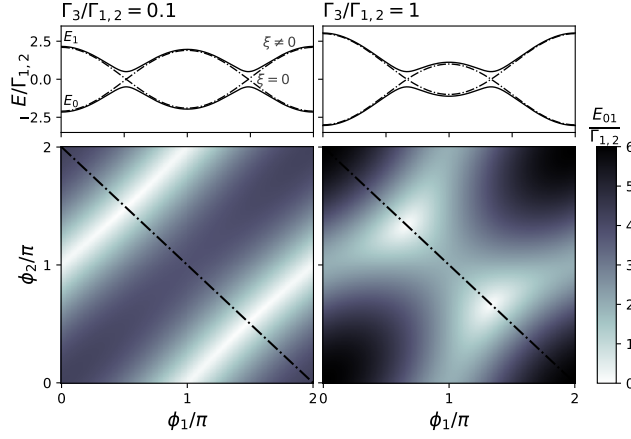


Figure 19.2: Bottom: transition energy E_{01} versus the phase differences $\phi_{1,2}$ for $\Gamma_1=\Gamma_2$ and different Γ_3 . At low coupling Γ_3 (left), the system roughly behaves as a junction between 1 and 2, and depends on the phase difference $\phi_1 - \phi_2$ between them. At comparable couplings $\Gamma_{1,2,3}$ (right), the transition landscape has a set of local minima. The level position is placed at the symmetry point $\epsilon = -U/2$, meaning $\xi=0$. Top: energies $E_{0,1}$ (up to a shift) of the Andreev states along the trajectories highlighted in the maps, with solid (dash-dotted) lines for $\xi=0.5\Gamma_{1,2}$ ($\xi=0$).

Such a model is characterized by a phase diagram with alternating even and odd parity ground states as a function of the model parameters [110, 111, 112, 52, 113, 62]. However, as in the present work we focus on the case of widely open channels with significant Josephson coupling, we assume that the trijunction remains always within the even sector, where H^{eff} is projected as

$$H_{\text{even}}^{\text{eff}} = -\xi\tau_z + \xi + \sum_{\nu} \Gamma_{\nu}(\cos \phi_{\nu}\tau_x - \sin \phi_{\nu}\tau_y), \quad (19.2)$$

with $\xi = \epsilon + U/2$ and τ_i the Pauli matrices acting on the space $\{|0\rangle_d, |\uparrow\downarrow\rangle_d\}$. The ground and excited states have energies

$$E_{0,1}(\{\phi_{\nu}\}) = -\xi \mp \sqrt{\xi^2 + \left| \sum_{\nu} \Gamma_{\nu} e^{-i\phi_{\nu}} \right|^2}. \quad (19.3)$$

Let us note that, while H^{eff} in Eq. (19.1) does depend on the phase gauge choice, its spectrum (19.3) is only sensitive to the phase differences so for simplicity we take $\phi_3 = 0$. The transition energy $E_{01} = E_1 - E_0$ is shown in Fig. 19.2 (bottom) for $\Gamma_1 = \Gamma_2$ and different ratios $\Gamma_3/\Gamma_{1,2}$. When lead 3 is weakly coupled (left panel), the energy depends mainly on the phase difference between the two strongly coupled leads, $\phi_{12} = \phi_1 - \phi_2$, which at π corresponds to a minimum in E_{01} . Indeed, in the limit $\Gamma_3=0$ we recover the familiar expression $E_{0,1}(\phi_{12}) + \xi = \mp \sqrt{\xi^2 + \Gamma^2} \sqrt{1 - T \sin^2(\phi_{12}/2)}$, where $T = (\Gamma^2 - \delta\Gamma^2)/(\xi^2 + \Gamma^2)$ is the transparency of the junction, with $\Gamma, \delta\Gamma = \Gamma_1 \pm \Gamma_2$ [20]. When the three leads are equally coupled (right panel), the landscape reveals a discrete set of extrema and saddle points. In the regime $\Gamma_3 \gg \Gamma_{1,2}$, the phase dependence is weaker and there is a minimum in E_{01} at $\phi_{1,2} \approx \pi$. On the top of Fig. 19.2, the Andreev energies $E_{0,1}$ corresponding to the trajectories in the (ϕ_1, ϕ_2) plane indicated in the bottom panels are shown in dash-dotted (solid lines) for $\xi = 0$ ($\xi = 0.5$), that is, in (out of) resonance with the dot level. These energies are plotted with a $-\xi$ shift. We note that in the tunnel (low transparency) limit $\xi \gg \Gamma_{\nu}$'s we recover the conventional sinusoidal ground state dispersion $E_0(\{\phi_{\nu}\}) \approx -\left| \sum_{\nu} \Gamma_{\nu} e^{-i\phi_{\nu}} \right|^2 / 2\xi$, which reads $-(\Gamma_1\Gamma_2/\xi) \cos \phi_{12}$ in the two terminal case.

This two-level system would be an instance of an Andreev level qubit [12, 13], a kind of qubit defined with two Andreev states of a junction [116, 117, 118, 119], and the choice of its operating point should consider aspects such as its sensitivity to noise. The noise in some parameter λ produces decoherence through two mechanisms, pure dephasing and depolarization [120]. On the one hand, pure dephasing is related with the fluctuations in the qubit energy, blurring the access to the phase of superpositions of the type $c_0|0\rangle + c_1 e^{i\varphi}|1\rangle$, $c_{0,1} \in \mathbb{R}$. For small noise amplitudes, the associated rate $T_\varphi^{-1}(\lambda)$ is proportional to $|\partial E_{01}/\partial \lambda|^2$. On the other hand, the depolarization is mainly given by the relaxation from $|1\rangle$ to $|0\rangle$, and at small amplitudes the associated rate $T_1^{-1}(\lambda)$ is proportional to $|\langle 0|\partial H/\partial \lambda|1\rangle|^2$. Several strategies aim at reducing the effect of the noise: reducing its amplitude, correcting errors, or designing the qubit such that it acquires an inherent protection [23]. The latter approach can be explored when introducing different hardware configurations, attempting to reduce the dephasing and the depolarization rates simultaneously. This is not a straightforward task and it must be noticed that simultaneous protection against noise in all parameters is impossible. In conventional qubits the noise in the charge offset and the external flux are emphasized as these are tunable parameters, but the noise in the Josephson coupling can also be important [121, 120, 122]. In the circuits we consider here, the couplings can be controlled by gate voltages, so the associated noise may be relevant too. For simplicity and to illustrate the principle, throughout this work we focus on the noise in the external flux, in the energy of the dot level and in the charge offset of the superconducting islands introduced in Sec. 19.4. We provide further analysis of decoherence in Sec. 19.7.

19.3.2 Noise protection of the Andreev qubit

The general challenge for the simultaneous reduction of dephasing and depolarization produced by noise in one parameter can be illustrated in the phase biased configuration of the present section. For example, the noise in ξ has a dephasing sweet spot at $\xi = 0$, where $E_{01}(\xi)$ is locally quadratic. However, at this spot the hybridization between the bare central level states $|0\rangle_d$ and $|\uparrow\downarrow\rangle_d$ is enhanced, so the relaxation rate increases (note that $\partial H^{\text{eff}}/\partial \xi \propto \tau_z$ and the states are $\propto |0\rangle_d \pm |\uparrow\downarrow\rangle_d$). The same occurs with the noise in the external flux, which has dephasing sweet spots where two states that were degenerate become hybridized. In the two terminal case, it occurs in the configuration of perfect transmission ($\xi=0$ and $\Gamma_1=\Gamma_2$ [20], see top panels of Fig. 19.2) at phase difference $\phi_e = \pi$, where the two degenerate states carry maximal and opposite supercurrents. There, a deviation in the energy level position or in the coupling symmetry hybridizes these states while flattening the transition energy at the same time. As will be discussed next, this analysis on the effect of flux noise must be extended to take into account the effect of the phase drop distribution.

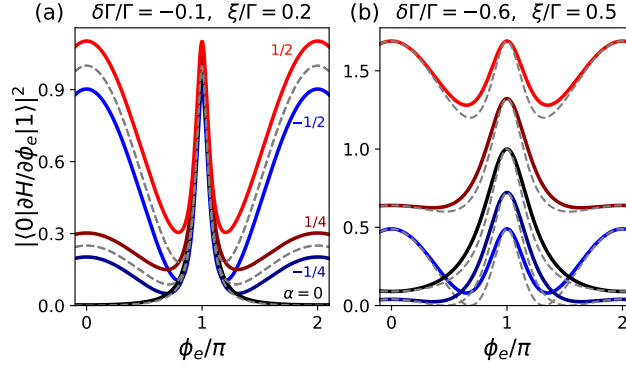


Figure 19.3: Phase dependence of the transition matrix elements (in units of $\Gamma=\Gamma_1+\Gamma_2$) for different phase drop distributions in a junction with two leads: symmetrically over the couplings ($\alpha=0$), on Γ_1 ($\alpha=-1/2$), on Γ_2 ($\alpha=1/2$) and halfway ($\alpha=\mp 1/4$). (a) Situation close to resonance. Low but finite deviation from $\xi \neq 0$ and $\Gamma_1 \neq \Gamma_2$ ($\delta\Gamma=\Gamma_1-\Gamma_2$) produces the high peak at $\delta=\pi$ due to the anticrossing between E_0 and E_1 (see Fig. 19.2 top). The slight asymmetry in the couplings produces asymmetry over the sign of α (gray dashed lines show the results for $\Gamma_{1,2}=1$). (b) Situation with larger coupling asymmetry, producing a large asymmetry over the sign of α ; dashed lines show the results for $\xi=0$.

In a static situation, the phases in Eq. (19.2) have a gauge freedom that manifests through the transformation $U=e^{if\tau_z/2}$, where the phase f redistributes the phase drop over the couplings. In fact, note that the coupling term of $H_{\text{even}}^{\text{eff}}$ in Eq. (19.2) can be written as $\sum_{\nu} \Gamma_{\nu} e^{i\phi_{\nu}\tau_z/2} \tau_x e^{-i\phi_{\nu}\tau_z/2}$, so the transformed coupling term in $U H_{\text{even}}^{\text{eff}} U^{\dagger}$ corresponds to a shift $\phi_{\nu} \rightarrow \phi_{\nu} + f$. However, if the external fluxes depend on time, such as in a situation with noise, the transformation adds to the new Hamiltonian the term $-i\hbar U(U^{-1})' = -\hbar f' \tau_z/2$, where the prime indicates time derivative and f might depend on the internal and external parameters of the circuit. This non-equivalence means that a Hamiltonian with no term $\propto f' \tau_z$ would only be correct at a certain *gauge choice* [81, 82]. Classically, the magnetic vector potential $\vec{A}$ —which determines the quantum phase drop through a Peierls substitution—may be shifted by the gradient of a scalar function F without altering the magnetic field, but if F is time dependent, the complete gauge transformation must subtract $\partial_t F$ from the scalar potential V —which determines a voltage profile—so that the electric field $\vec{E} = -\vec{\nabla} V - \partial_t \vec{A}$ is not modified.

The simplest way to notice this effect is in the flux noise matrix element of a junction with two leads. In this case, the general form of the coupling term in $H_{\text{even}}^{\text{eff}}$ is

$$(\Gamma_1 \cos(-\phi_e/2+f) + \Gamma_2 \cos(\phi_e/2+f))\tau_x \\ - (\Gamma_1 \sin(-\phi_e/2+f) + \Gamma_2 \sin(\phi_e/2+f))\tau_y,$$

where f determines how asymmetrically this phase drops on the couplings between the leads and the dot. Supposing that this asymmetry is proportional to the phase drop, i.e. $f = \alpha\phi_e$, which would include terms $\propto \phi_e' \tau_z$ in the Hamiltonian for a general gauge, the squared matrix element $|\langle 0 | \partial H / \partial \phi_e | 1 \rangle|^2$ is

$$\frac{(\Gamma_1^2 + \Gamma_2^2 + \xi^2)^2 - 4\Gamma_1^2 \Gamma_2^2}{E_{01}^2} - \frac{\xi^2}{4} + \alpha(\Gamma_2^2 - \Gamma_1^2) + \alpha^2 \left(\frac{E_{01}^2}{4} - \xi^2 \right),$$

which is illustrated in Fig. 19.3. In panel (a), the couplings are almost symmetric and the matrix element depends slightly on the sign of α (the gray dashed lines correspond to the limit of $\Gamma_1=\Gamma_2$). In panel (b), the couplings are less symmetric and the sign of α has a strong effect reducing or boosting the relaxation (the gray dashed lines correspond to the limit of $\xi=0$). The matrix element at $\phi_e=\pi$ vanishes at $\alpha=\alpha_0:=\Gamma/2\delta\Gamma$, a regime approached in the

fully asymmetric phase-drop distribution $\alpha=\pm 1/2$ at large coupling asymmetry. However, the value $|\alpha_0|\geq 1/2$ requires that the phase drop winds more than once, an unlikely situation in a superconducting loop where vortices would appear to accommodate the next flux quantum whenever the phase surpasses 2π [123, 124, 125]. The value at $\phi_e=\pi$ becomes a local minimum at $\alpha=\alpha_m$, with $2\alpha_m^2:=(\xi^2+\Gamma^2)/(\xi^2+\delta\Gamma^2)$, with $|\alpha_m|\geq 1/2$ too.

The particular f value in each configuration is related with the electromagnetic field distribution in the circuit [81, 82, 83], a calculation that is out of the scope of the present work. Our aim here is to identify the main features that determine it: the shape of the field in the central part of the junction depends on the geometry and properties of the circuit in which it is immersed [84], though in the situation we consider here we expect little variation in the field profile as the central region is small compared to a typical field wavelength and presumably homogeneous, i.e. we expect a symmetric drop of the phase difference. However, the state of the junction might carry a supercurrent, which modifies the field. A calculation in the lines of Refs. [14, 92] (see Appendix 19.8), which imposes electroneutrality in the electrodes, produces a deviation from the symmetric drop with $f=-\sum_\nu \Gamma_\nu \phi_\nu/\Delta$, where Δ is the superconducting gap. In the two terminals setup, it gives $\alpha=(\Gamma_1-\Gamma_2)/\Delta$, which vanishes in the limit of large gap.

In the setup with three leads [Fig. 19.1d], the coupling term (Eq. (19.2)) is

$$\left[\Gamma_1 \cos(-\phi_{e1}/2+f) + \Gamma_2 \cos(\phi_{e2}/2+f) + \right. \\ \left. \Gamma_3 \cos(\phi_{e1}/2-\phi_{e2}/2+f) \right] \tau_x - \left[\cos \rightarrow \sin \right] \tau_y,$$

which in the limit $\Gamma_3 \rightarrow 0$ is not equivalent to the two leads case: if the fluxes have the same source and obey $\phi_{1,2e} = \eta_{1,2}\phi_e$, then $f = (\Gamma_1\eta_1 - \Gamma_2\eta_2)\phi_e/2\Delta$. For $\eta_{1,2} = 1$, we have the same Hamiltonian as in the two leads case, but now the phase drops $-\phi_e/2 + (\Gamma_1 - \Gamma_2)\phi_e/2\Delta$ and $\phi_e/2 + (\Gamma_1 - \Gamma_2)\phi_e/2\Delta$ can go up to 2π , therefore becoming able of achieving the relaxation free point α_0 for parameters where $1/2 \leq |\alpha_0| \leq 1$.

We note that there are other transformations $U_n = e^{if_n\tau_n}$ ($n \neq z$) that generate the time dependent terms $-\hbar f'_n \tau_n/2$. The case $n=0$ ($\tau_0=1$) refers to a global energy shift and introduces a global time dependent phase with no effects. The cases $n=x, y$ have no apparent physical interpretation, but the time dependent term must be kept if the transformation has been made for example to facilitate calculations.

As argued in the previous paragraphs, the phase drop distribution is very important for the relaxation rates. In addition, from a broader perspective, its effect is pervasive on other observable quantities that depend on the matrix elements, such as the shift of a resonator coupled to the circuit or the transition rates between the states in the junction [28, 127, 128, 129, 130, 131, 132, 133].

19.3.3 Topological properties

This model of the trijunction in a perfect phase-biasing configuration can be linked with the topological singularities predicted in general multiterminal junctions [11], since there are certain configurations where E_{01} reaches 0, forming a Weyl point. In our case (see Eq. (19.3)), these points appear when the parameter ξ and the sum $\Gamma_T = \sum_\nu \Gamma_\nu e^{i\phi_\nu}$ vanish, so topological protection in this model against variations in ξ is not possible, though it is on the Γ_ν 's and the ϕ_ν 's. It is convenient to interpret Γ_T as a vector in the complex plane, formed by the addition of the constituent numbers of magnitude Γ_ν and direction $e^{i\phi_\nu}$. In the case of two terminals, the only possibility is $\Gamma_1 = \Gamma_2$ with a phase difference of π . However, with three terminals, if the Γ_ν 's satisfy the triangle inequality (each side is smaller than the sum of all the other sides) there is a set of phase differences that produce a Weyl point – a similar argument has been

recently used to map a general model of a multiterminal quantum dot Josephson junction into a two terminal one [62]. In this case, the presence of Weyl points would be protected against variations on the Γ_ν 's, as long as they continue satisfying the triangular equality. If the number of terminals is increased to $\mathcal{N}$, the generation of Weyl points becomes more probable since there are more ways of closing a polygon with $\mathcal{N}$ sides, and they live in a phase space of $\mathcal{N} - 1$ dimensions.

This configuration can be compared to a multiterminal junction in the tunnel limit, with $H = \text{Re}\left\{\sum_{\mu>\nu} E_{J_{\mu\nu}} e^{i(\phi_\mu - \phi_\nu)}\right\}$. In this case there is only one “band” but the points where $H = 0$ are interesting as they correspond to a maximum supercurrent, and the $E_{J_{\mu\nu}}$'s play a role similar to that of the Γ_ν 's for defining these points.

19.4 Charge islands' configuration

In this section we consider the situation in which the central region is connected to isolated superconducting islands [Fig. 19.1e]. In this configuration the number of Cooper pairs N_ν in each island ν is a discrete variable and the total number of pairs $N_T = \sum_\nu N_\nu + n_d$ is conserved, where $2n_d$ is the number electrons in the central region. This conservation constrains the accessible states in the basis $|\{N_\nu\}, n_d^m\rangle$ for a given value of N_T , where the $|n_d^m\rangle$ are the states in the central region. In our model, we consider only $|0\rangle_d$ ($n_d=0$) and $|\uparrow\downarrow\rangle_d$ ($n_d=1$), with $\hat{n}_d = \sum_\sigma d_\sigma^\dagger d_\sigma / 2$, so the label m will be dropped from now on. As a result of the constraint, an instance (value) of N_T can be described with a smaller basis, e.g., one lead variable can be removed using $\hat{N}_\mu = N_T - \sum_{\nu \neq \mu} \hat{N}_\nu - \hat{n}_d$.

The charging effects in and between the islands are described by a capacitance matrix $\mathcal{C}$ such that the associated Hamiltonian is $H_C = (\vec{N} - \vec{n}_g)^T \mathcal{C}^{-1} (\vec{N} - \vec{n}_g)$, where $\vec{N} = (\hat{N}_1, \hat{N}_2, \dots, \hat{N}_\mathcal{N})$ is the vector of the number operators in each of the $\mathcal{N}$ islands and $\vec{n}_g$ is composed of the corresponding effective charge offsets, which indicate the charge configuration in a situation of electrostatic equilibrium. In general, H_C can be interpreted as a confining potential in the charge basis, and it could also include capacitive couplings between the leads and the central region by increasing its size and including $\hat{n}_d$ in $\vec{N}$. The particular values of $\mathcal{C}$ depend on how the electromagnetic field accommodates into the geometry and materials of the device. We consider the regime where the absolute charge offsets are large enough so that the eigenspectrum of the $\hat{N}_\nu$'s can be extended to $N_\nu \in \mathbb{Z}$ for convenience (the n_{g_ν} 's are shifted to denote deviations from the integer set closest to equilibrium).

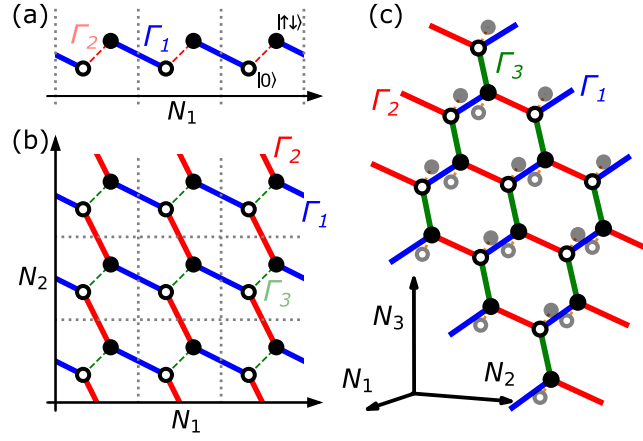


Figure 19.4: Lattice representation of the island configuration. The island charge variables N_ν define the dimensions of the *crystal* and the fermionic states are the atoms in the unit cell (dotted gray lines), with hollow (full) circles for $|0\rangle$ ($|\uparrow\downarrow\rangle$). The sets of connected states for an instance (value) of total charge N_T are the blue dimers in (a), the blue-red chains in (b) and the blue-red-green grid in (c) for the setups of 1, 2 and 3 leads, respectively. The dashed lines represent the projection of a single instance in a space with lower dimension, where the projected variable is determined by the conservation of N_T , *e.g.* a blue-red chain in (a), where $N_2 = N_T - N_1 - n_d$. The pale circles in (c) represent sites of adjacent N_T grids, which would be connected in the presence of a fourth lead.

As discussed in Ref. [20], deep inside the gap and restricted to the even parity sector, the main tunneling process is the effective interchange of Cooper pairs between the leads and the central level, though microscopically the electrons actually tunnel one by one [15]. Thus, in the infinite gap limit discussed in the previous section, the coupling Hamiltonian is $H_c = \sum_\nu \Gamma_\nu e^{-i\hat{\phi}_\nu} d_{\uparrow\downarrow}^\dagger d_{\uparrow\downarrow}^\dagger + h.c.$, where $e^{-i\hat{\phi}_\nu}$ decreases N_ν by 1 (now the phase is the generator of the charge translations with $[\hat{\phi}_\mu, \hat{N}_\nu] = i\delta_{\mu\nu}$), and $d_{\uparrow\downarrow}^\dagger d_{\uparrow\downarrow}^\dagger$ introduces both electrons of the pair in the dot. The N_T constraint in the Hilbert space $|\{N_\nu\}, n_d\rangle$, which can be described by a gauge symmetry in the phase space [20, 14], may be noticed by the fact that the states $|\{N_\nu\}, 0\rangle$ and $|\{N_\nu\}, \uparrow\downarrow\rangle$ are not connected in any way, because the coupling conserves the total charge. However, these states are hybridized when an additional island is introduced – this is the main feature explored in this section.

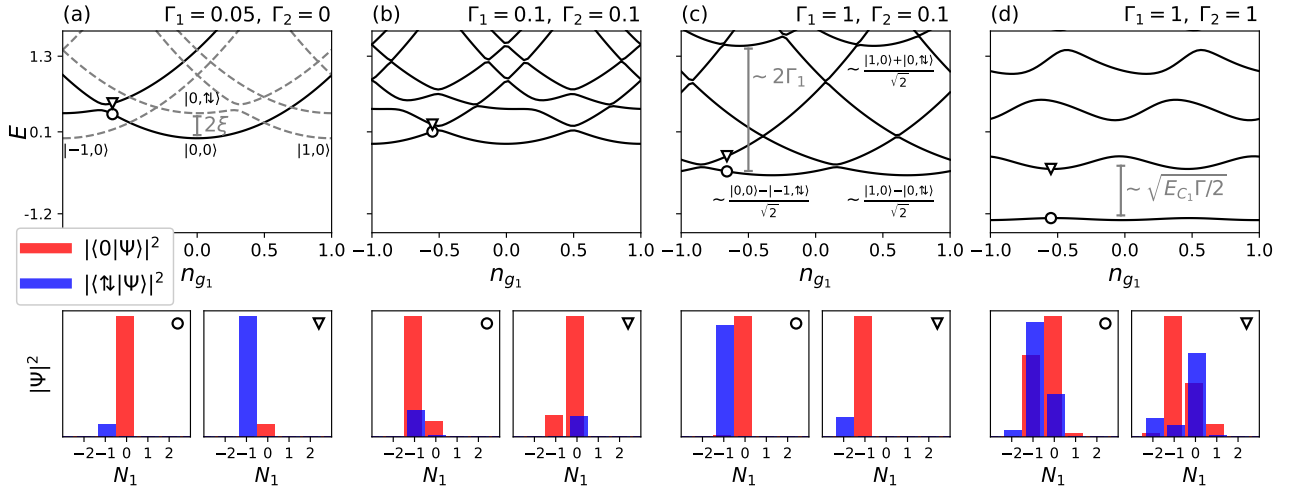


Figure 19.5: Energy dispersion over the charge offset n_{g_1} in the configurations with 1 and 2 terminals ($E_{C_2} = 0$, $\xi = 0.2$, energies in units of E_{C_1}), and wavefunctions indicated with markers (bottom panels). (a) Small coupling of the central level with island 1: the instance $N_T=0$ is in solid lines, and other instances in dashed; some bare $|N_1, n_d\rangle$ states are indicated close to the bottom of their charging parabolas. (b) Small couplings with islands 1 and 2: finite Γ_1 and Γ_2 connect all states in (a). (c) Large (small) coupling with terminal 1 (2): Γ_1 dominates the formation of symmetric and antisymmetric combinations. (d) Large couplings (more similar to a transmon except that the junction is close to resonance, not in the tunnel regime).

If the basis is portrayed as a lattice in the $\{N_\nu\}$ space with a unit cell comprising the fermionic levels, each $N_T \in \mathbb{Z}$ constitutes an independent set disjoint from the others (see Fig. 19.4). Starting with one island and the central level, these sets are the groups of two sites fulfilling $N_1 + n_d = N_T$ (the dimers connected with blue lines in Fig. 19.4a). Then, when the second island is coupled, the sets are 1D *chains* defined by $N_1 + N_2 + n_d = N_T$ (the blue and red chains in Fig. 19.4b). One of these instances of N_T is projected on panel (a), where the dashed red lines represent the coupling Γ_2 and $N_2 = N_T - N_1 - n_d$. When a third lead is connected, the chains connect between themselves through Γ_3 and form a 2D *grid* fulfilling $N_1 + N_2 + N_3 + n_d = N_T$, depicted in Fig. 19.4c and projected to $N_{1,2}$ in panel (b). Similarly, these grids may be connected when a fourth lead is included, and so on. In the following subsections, this kind of circuits is progressively explored along with some applications for the definition of protected qubits.

19.4.1 One terminal

We consider first a single superconducting island with $H_C = E_{C_1}(\hat{N}_1 - n_{g_1})^2$ connected to the central region. The relevant states $|N_1, n_d\rangle$ are

$$\begin{aligned} |0, 0\rangle &\rightarrow E = E_{C_1} n_{g_1}^2 \\ |-1, \uparrow\downarrow\rangle &\rightarrow E = E_{C_1} (-1 - n_{g_1})^2 + 2\xi, \end{aligned} \quad (19.4)$$

where a translation in $N_1 = N_T$ would refer to other instances of total charge (blue dimers in Fig. 19.4a). The energies of the $N_T=0$ instance (n_{g_1} is shifted so that it refers to the instance with lowest energy) are the solid black parabolas in Fig. 19.5a, hybridized by the tunneling Γ_1 , which opens an anticrossing at $n_{g_1} = -1/2 - \xi/E_{C_1}$ (close to the symbols that indicate the wavefunctions in the lower panels). The dashed lines in Fig. 19.5a represent other instances of N_T , which are in principle uncoupled subspaces.

As indicated in the lower panels, the two-level system in the limit $\Gamma_1 \ll E_C$ is similar to a charge qubit with a steep n_g dependence that would be detrimental for coherence in the presence of charge noise. In the opposite limit $\Gamma_1 \gg E_C$, the eigenstates approximate the symmetric and antisymmetric combinations $|N'_1, \pm\rangle = (|N_1, 0\rangle \pm |N_1 - 1, \uparrow\downarrow\rangle)/\sqrt{2}$ for a larger range in n_{g1} , with a transition energy $\sim 2\Gamma_1$. The dispersion of E_{01} over n_{g1} is reduced ($\sim E_{C1}^2 n_{g1}^2/\Gamma_1$, disregarding ξ), but now the relaxation produced by the noise in n_{g1} is larger as $\partial H/\partial n_{g1}$ is non-diagonal in the qubit basis. When compared with the transmon, this setup does not suffer the problem of reduced anharmonicity, but we note that the single level model of Eq. (19.1) would not be valid at arbitrary Γ_1 because the qubit states could approach other levels in the central region. A more microscopic model of this configuration has been studied in [134].

19.4.2 Two terminals

When a second terminal is connected to the central region, the constraint $N_1 + N_2 + n_d = N_T$ produces independent sets with an infinite number of states, the blue and red chains in Fig. 19.4b. In the extended basis $|N_1, N_2, n_d\rangle$, the coupling Γ_2 (red) connects $|0, 0, 0\rangle$ with $|0, -1, \uparrow\downarrow\rangle$, which in the projected basis $|N_1, n_d\rangle$ appears as a coupling between $|0, 0\rangle$ and $|0, \uparrow\downarrow\rangle$ (red dashed lines in Fig. 19.4a). To illustrate this we start by considering that the second terminal is very large so that there is no additional charging energy.

In this situation, we have $H_C = E_{C1}(\hat{N}_1 - n_{g1})^2$, and in the limit $\Gamma_{1,2} \ll E_{C1}$, the states (in the reduced basis) $|0, 0\rangle$ and $|-1, \uparrow\downarrow\rangle$ maintain the anticrossing given by $2\Gamma_1$ as in the previous subsection. The coupling Γ_2 does not directly open anticrossings between $|N_1, 0\rangle$ and $|N_1, \uparrow\downarrow\rangle$, because their energies never coincide. However, at $n_{g1} = -1/2$ (indicated by symbols in Fig. 19.5b) both couplings combine to hybridize $|N_1 - 1, n_d\rangle$ with $|N_1, n_d\rangle$ through an intermediate state with opposite central level occupation. For finite $|\xi| \gg \Gamma_{1,2}$ the gap opening is $\sim (\Gamma_1^2 + \Gamma_2^2)/\xi$. The states $|N_1 - 1, 0\rangle$ and $|N_1, \uparrow\downarrow\rangle$ also hybridize through two intermediate states at $n_{g1} \approx -1/2 + \xi/E_{C1}$, with an even smaller anticrossing (the size decreases with the number of intermediate states needed for the indirect coupling). In the lower panels we show the different states $|N_1\rangle$ contributing to the ground and the first excited eigenstates. As we discuss next, more and more charge states contribute as the coupling is increased.

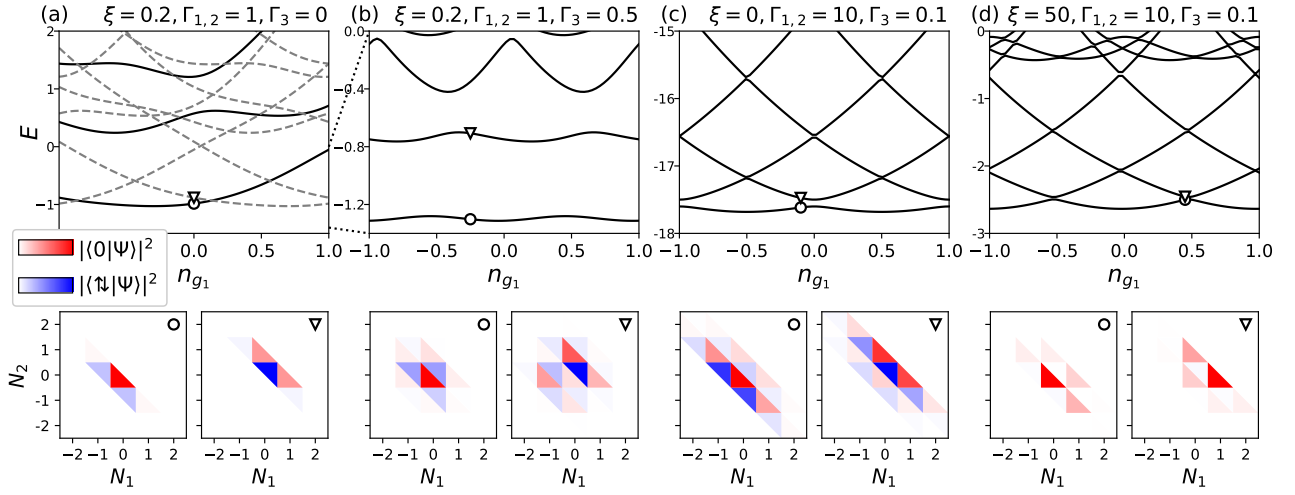


Figure 19.6: Energy dispersion over the charge offset n_{g_1} in the configurations with two and three terminals when two of them are islands with finite charging energy ($n_{g_2} = 0$, $E_{C_3}=0$, energies in units of $E_{C_1} = E_{C_2}$) and wavefunctions indicated with markers, represented in a N_1 - N_2 space where each cell is divided into a lower and an upper triangle for the level states $|0\rangle$ and $|\uparrow\downarrow\rangle$, respectively (bottom panels) (a) Only two terminals as in Fig. 19.5d but now terminal 2 has finite charging energy, which renders the spectrum aperiodic and non-equivalent between instances ($N_T=0$ in solid lines, others in dashed). (b) Introducing the coupling Γ_3 with the third terminal, which connects the different instances (chains) in (a). (c) Large couplings $\Gamma_{1,2}$ (transmon regime in the N_1 - N_2 direction) and small coupling Γ_3 (Cooper pair box regime in the orthogonal direction), central level close to resonance. (d) Same with central level far from resonance (tunneling regime).

In the limit $\Gamma_1 \gg E_{C_1} \gg \Gamma_2$ (corresponding qualitatively to Fig. 19.5c), the lowest energy states, $\sim |N'_1, -\rangle$ and $\sim |N'_1+1, -\rangle$, anticross close to $n_{g_1}=N_1$ opening a gap $\sim \Gamma_2$. Note that these two levels are completely disjoint at $\Gamma_2=0$, defining a qubit protected against relaxation by charge noise. However, as discussed in the previous subsection, the decoherence produced by the noise in n_{g_1} may be important for feasible values of Γ_1 .

We address now the regime $\Gamma_{1,2} \gg E_{C_1}$ in which the wavefunctions extend broadly over the charge states [Fig. 19.5d], thereby decreasing the modulation produced by the offset charge n_{g_1} , just like in the transmon [122]. We note that transmon implementations in hybrid semiconductor-superconductor nanostructures provide gate tunability. The resulting *gatemons* [135, 136, 39] are thus more suitable to manifest features from the Andreev structure, in particular when tuned close to resonance [7, 10]. The Josephson-Andreev junction should recover the typical tunnel junction transmon when the central level is far from resonance. In this case there is an exponential suppression of the charge dispersion with E_J^{eff}/E_{C_1} , where E_J^{eff} is the effective amplitude of Cooper pair tunneling between the leads [122]. In the resonant limit, the Josephson-Andreev transmon (*JAmom*) dispersion is even more suppressed and becomes dominated by the tunneling of pairs of Cooper pairs [20, 7, 10]. In either case, the dispersion is a non-perturbative effect.

In order to estimate the levels energies, we may disregard this dispersion and use an adiabatic approximation in the phase space, which consists in using the phase dependence $E_0(\phi_e)$ (Eq. (19.3)) of the Andreev lowest state as the effective potential in the phase variable $\phi_e \rightarrow \hat{\phi}_1$. Then, in the transmon-like limit of localized phase, it is expanded up to fourth order at $\phi_e=0$ to get the harmonic separation $\omega \approx E_{01}$ between states and the anharmonicity $\alpha_h=E_{12}-E_{01}$ at lowest order. In the tunnel limit $\xi \gg \Gamma_{1,2}$, $\omega \approx 2\sqrt{E_{C_1}\Gamma_1\Gamma_2/2\xi}$, and in the resonant limit $\delta\Gamma=0=\xi$, with $\Gamma, \delta\Gamma=\Gamma_1\pm\Gamma_2$, $\omega \approx 2\sqrt{E_{C_1}\Gamma/8}$. More details are provided in Appendix 19.9.

Two charge islands

In the situation we have analyzed so far, where there is charging energy only in one island, the spectrum is periodic in n_{g_1} , as a charge translation in N_1 equals to a charge translation in the charge offset. Moreover, all instances N_T are equivalent, because a charge translation in N_1 also corresponds to a translation in N_T . This also occurs in the situation where the charging energy only depends on the imbalance between the two islands, with $H_C = E_{C_{12}}(\hat{N}_1 - \hat{N}_2 - n_{g_{12}})^2$, which has been thoroughly analyzed in Ref. [20]. However, this is not true in general when the two leads are islands with finite charging energies.

In this case, the additional charging term in the reduced basis,

$$E_{C_2}(\hat{N}_2 - n_{g_2})^2 = E_{C_2}(N_T - \hat{N}_1 - \hat{n}_d - n_{g_2})^2,$$

removes the periodicity in the offset charges and the equivalence between N_T instances. This is shown in Fig. 19.6a, which uses the same parameters as in Fig. 19.5d except for the introduction of $E_{C_2} = E_{C_1}$. We observe in the lower panels of the figure that now the wavefunctions corresponding to two different instances delocalize along lines of fixed total charge in the $(N_1 - N_2)$ plane (see Fig. 19.4b). The transition to the three leads case (see next subsection) is similar to the previous discussion when going from one to two leads in Fig. 19.5a, where different instances become coupled by the additional lead.

19.4.3 Three terminals

When a third terminal is connected to the central region, there are two independent charge variables – we may use for example the projected basis $|N_1, N_2, n_d\rangle$ where $\hat{N}_3 = N_T - \hat{N}_1 - \hat{N}_2 - \hat{n}_d$. In this situation, if the charging potential is flat in one direction, the wavefunctions are not constrained and a continuous band is formed in that direction. Considering the charging terms as a quadratic form in the charge variables, then the previous condition occurs when one of its principal axes has a 0 eigenvalue. Therefore, in order to have a discrete set of states, we consider a fully confining potential. In particular, we select the potential for the situation where two terminals are islands with finite charging energies E_{C_1} and E_{C_2} (used in the last part of the previous subsection), and the third terminal has no charging energy.

In this case, a finite coupling Γ_3 connects the chains that were independent instances in the case with two terminals [Fig. 19.6a,b], and the wavefunctions extend in the number space. When the wavefunctions are broadly extended in charge, the system can be thought as a set of coupled harmonic oscillators, a limit which is analyzed in the tunneling regime in Sec. 19.6. Here, we focus in the limit of small Γ_3 compared with the charging energy in the transverse direction of the chains, which suggests the definition of a qubit formed by two adjacent chains [Fig. 19.6c,d]. This limit is better understood in a number basis where one degree of freedom refers to the extension of the wavefunction on the direction of a chain ($N_{12} := N_1 - N_2$, the diagonal in the wavefunction plots in Fig. 19.6) and the other one refers to the specific chain ($N_1 + N_2 + n_d = N_T - N_3$, the antidiagonal). The offset charge does not affect in the chain direction, but in the orthogonal one it produces charge parabolas. In the case $E_{C_2} = E_{C_1}$, illustrated in Fig. 19.6, the charging potential in that orthogonal direction is $\sim E_{C_1}(N_3 + n_d + N_T + n_{g_1} + n_{g_2})^2/2$. Due to the structure of the wavefunctions in the resonant and tunneling (large ξ) limits, where $\langle n_d \rangle \approx 1/2, 0$, respectively [panels c and d], the spectrum has minima in n_{g_1} at $\mathbb{Z} + 1/2$ and $\mathbb{Z}$ when setting $n_{g_2} = 0$. The structure of the wavefunctions also determines the size of the anticrossings, which is $\sim 2\Gamma_3$ in the resonant and $\sim \Gamma_3^2/\xi$ in the tunneling limit.

19.4.4 Discussion

The lattice picture used in this section of charge islands allows us to relate these systems with the definition of protected qubits that use states of disjoint support. As argued in a recent work that addresses the protection enabled by the resonant regime of a quantum-dot junction in a loop [22] (see Sec. 19.5.1), the idea of this protection can be represented in a general manner by a 1D lattice where the nearest neighbour couplings vanish. In that situation, if only next-nearest neighbour couplings are present, the lattice disconnects into two separate chains, which become the hosts of the disjoint states that define the qubit. In our case, the lattice for the Josephson-Andreev junction with three leads, where one of them profits from a controllable coupling with the central level [Fig. 19.4b,c], enjoys a similar interpretation. Here, two adjacent chains formed by the arrangement of two leads host the disjoint qubit states. The difference is that the coupling between the chains, mediated by the coupling Γ_3 with the third lead, is not completely equivalent to the reassembling of a single chain but skips alternate nearest neighbour couplings and may also involve other chains. Additionally, in this system the only parameter that controls the coupling between chains is Γ_3 , a property that seems to be model independent because no matter the particular Andreev structure, the conservation of the total charge forbids interchain coupling if the third lead is disconnected. This applies to the introduction of more levels in the modeling of the central region, quasiparticle states in the leads, or direct tunneling between the leads (probably relevant for the implementation of Fig. 19.1a).

The problem with this kind of protected Hamiltonian structures is that the parameter region where the disjointness takes place is small: the resonant regime of the quantum-dot fluxonium requires vanishing ξ and $\delta\Gamma$ [22], and the multiterminal island configuration with two independent chains requires vanishing Γ_3 for the disconnection and fine tuning of the charge offsets for the degeneracy. These parameters are also a source of dephasing, because the transition energy varies linearly from the anticrossing. In particular, the charge noise from the charge offsets $n_{g\nu}$'s is important because the chains are charge states like those in a Cooper pair box with a tunable Josephson coupling determined by Γ_3 .

Finally, as a curious property, we point out that our microscopic model of the junction bestows a *polyatomic* unit cell to the number lattice. For instance, in the case with two leads the Hamiltonian equals a Su-Schrieffer-Heeger (Rice-Mele) model for $\xi \neq 0$ [Fig. 19.4a] when there are no charging energies [137, 138]. Increasing the number of states describing the microscopic part of the junction would increase the number of sites in the unit cell, implementing different kinds of crystal Hamiltonians of $\mathcal{N}-1$ dimensions, where $\mathcal{N}$ is the number of leads. However, getting evidence of edge states in a topological phase would require the use of a charging potential able to provide a steep barrier in the number coordinate while maintaining a flat potential in the interior of the well.

19.5 Loop configuration

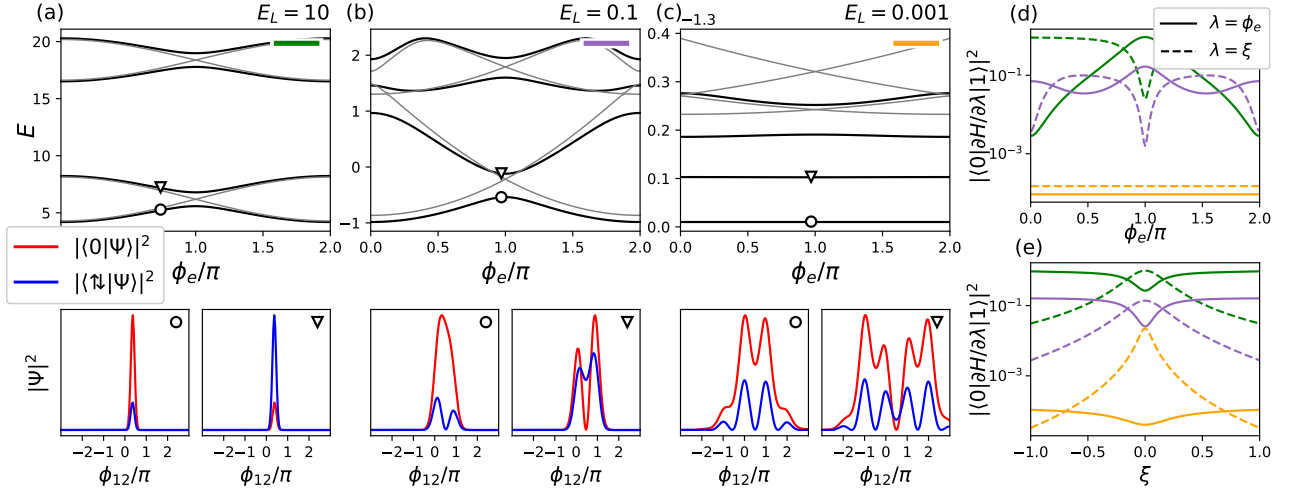


Figure 19.7: (a-c): Energy dispersion over the external phase ϕ_e for decreasing values of E_L in the loop configuration with two leads, and wavefunctions indicated with markers (bottom panels – they are centered over $\phi_{12} \sim \phi_e/2$ because of the form of the inductive term $E_L(2\phi_{12} - \phi_e)^2$). Grey lines indicate the resonant case with $\Gamma_{1,2} = 1$ and $\xi = 0$ (energies in units of $E_{C_{12}}$), and black lines indicate the case $\Gamma_1 = 0.95$, $\Gamma_2 = 1.05$ and $\xi = 0.6$ (with a shift of $-\xi$). (d,e): Matrix elements associated to noise in ϕ_e and ξ for the non-resonant parameters in a-c (setting $\phi_e = 0.95\pi$ in e) and $\alpha=0$, indicating each case with the colours in the a-c corners.

We now consider the case where two leads are connected forming a loop [Fig. 19.1f], for which a different, more macroscopic model is required. We use the variable of tunneled pairs between the leads of the loop $\hat{N}_{12} = \hat{N}_1 - \hat{N}_2$, conjugate to $\hat{\phi}_{12}$, with charging energy $E_{C_{12}} \hat{N}_{12}^2$. This model without the separate islands 1 and 2 is qualitatively different from the one derived in the previous section, because the loop disables the island nature of these leads, rendering the charge difference continuous [93] and allowing the presence of high energy persistent current states [139]. The charge in the remaining island 3 remains discrete, and the total Hamiltonian is the combination $H_{12} + H_3 + H_d + H_c$:

$$\begin{aligned}
 H_{12} &= E_{C_{12}} \hat{N}_{12}^2 + E_L (2\hat{\phi}_{12} - \phi_e)^2 \\
 H_3 &= E_{C_3} (\hat{N}_3 - n_{g_3})^2 \\
 H_d &= \epsilon \sum_{\sigma} d_{\sigma}^{\dagger} d_{\sigma} + u d_{\uparrow}^{\dagger} d_{\uparrow} d_{\downarrow}^{\dagger} d_{\downarrow} \\
 H_c &= \left(\Gamma_1 e^{-i\hat{\phi}_{12}} + \Gamma_2 e^{i\hat{\phi}_{12}} + \Gamma_3 e^{-i\hat{\phi}_3} \right) d_{\uparrow}^{\dagger} d_{\downarrow}^{\dagger} + h.c., \quad (19.5)
 \end{aligned}$$

where the Josephson processes that couple the dot with the leads in H_c take into account that a tunneling of a pair from $\nu=1(2)$ to the dot level produces a shift in $N_1 - N_2$ by ∓ 1 (translation $e^{\mp i\hat{\phi}_{12}}$) and that a tunneling from $\nu = 3$ to the dot level produces a negative shift in N_3 (translation $e^{-i\hat{\phi}_3}$). The Hamiltonian can be solved in the $|\phi_{12}, N_3, n_d\rangle$ basis with $\hat{N}_{12} = -i\partial_{\phi_{12}}$.

The phase drop $\phi_e = \Phi/\Phi_0$ is to be distributed among the inductance and the couplings $\Gamma_{1,2}$. In a dynamical situation, the correct description for a tunnel junction without Andreev structure corresponds to the phase dropping completely on the inductance [81, 83] (in the absence of additional time dependent terms). However, in our case, a freedom in the drop distribution within the junction is still present ($\Gamma_{1,2} \rightarrow e^{if} \Gamma_{1,2}$ in H_c , Eq. (19.5)). We note that the inductive potential $E_L(2\hat{\phi}_{12} - \phi_e)^2$ includes a factor of 2 as the complete transfer of a pair

from lead 1 to lead 2 requires two tunneling events. In the limit $L \rightarrow 0$ ($E_L \rightarrow \infty$), the phase becomes localized at $\hat{\phi}_{12} \rightarrow \phi_e/2$, recovering the model in section 19.3.

From a circuit perspective, the loop-island Hamiltonian contains two modes, one flux-like and one charge-like, which may be harnessed to devise protected qubits. These qubits would be similar in nature to the ones conceived in multimode circuits where several conventional elements generate the different modes [23]. In the remaining of this section, we explore the full Josephson-Andreev Hamiltonian, first in a one loop configuration, and then introducing a third floating lead. In section 19.6, we focus on the cotunneling limit, which allows a straight-forward correspondence with the protected multi-element circuits.

19.5.1 Two terminals forming one loop

This configuration is similar to a flux qubit when the Josephson-Andreev junction is in the tunneling regime [140, 141, 142, 143]. Otherwise, when the Andreev structure is close to resonance ($\Gamma_1 = \Gamma_2$, $\xi = 0$), it has been recently identified as a prospective protected qubit based on two disconnected states [21, 22].

In Fig. 19.7, we review the main features of the energy spectrum of this system, along with the corresponding wavefunctions. When E_L is large compared with E_C [Fig. 19.7a], the phase is localized and we recover the Andreev states of the central level from Section 19.3. We note that the fermionic structure already provides a difference with the flux qubits, as the first excited level is an Andreev state. The associated wavefunctions (lower panels) are hybridizations between $|0\rangle$ and $|\uparrow\downarrow\rangle$ at a localized phase. In contrast, the excited states of the flux qubit are bosonic in nature, being harmonic oscillator-like wavefunctions with finite phase fluctuations enabled by the charging term, which acts as a kinetic energy in the phase representation. This kind of states correspond to Andreev replicas such as the higher doublet in Fig. 19.7a.

When E_L is lowered, there is a competition between the inductive energy, a parabolic potential in phase representation, and the potential produced by the charge transfer terms. In the tunneling regime this potential is simply $-E_J \cos \hat{\phi}_{12}$, and the wavefunction may extend over two minima when $\phi_e = \pi$ and $E_J \sim E_L$, or several of the minima when $E_L \ll E_J$. In the latter case, if the kinetic energy E_C is large compared to E_J , the wavefunction exceeds the maxima of the cosine and the phase delocalizes broadly; otherwise, the wavefunction localizes at several of the phase potential minima, and represents a superposition of discrete current states encircling the loop. In the junction with internal structure, the Andreev level introduces an additional degree of freedom. The limit $|\xi| \gg \Gamma$ recovers the previous tunneling regime where the states involve mainly just the lower energy Andreev state and the higher one is just virtually occupied to allow the flow of charge. However, in the resonant limit, both Andreev states participate. They are always degenerate at $\phi_e = \pi$ and if E_L is lowered, their phase dependence decreases, providing a disjoint basis for a protected qubit insensitive to fluctuations in ϕ_e [21, 22]. The evolution with decreasing E_L is shown in Fig. 19.7a,b,c where the black lines refer to a situation a bit out of resonance and the gray lines to a situation of full resonance, which maintains the Andreev degeneracy for any value of E_L .

In Figs. 19.7d,e we show the matrix elements $|\langle 0 | \partial H / \partial \lambda | 1 \rangle|^2$ associated to noise in $\lambda = \phi_e$ ($\lambda = \xi$) between the two lowest energy states with solid (dashed) lines for the parameters in the panels a to c. There is a reduction in the matrix elements magnitude with the decrease of E_L as the states spread over ϕ_{12} and become less sensitive to the Andreev degrees of freedom. In general, the protection against ϕ_e noise is enhanced when approaching resonance as the states share their structure in ϕ_{12} but are orthogonal in the Andreev sectors. In contrast, the noise on the parameters $\xi, \delta\Gamma$ that move away from resonance is more harmful in that regime as it produces dephasing due to the linear splitting of the energy levels. Finally, there is the issue of

the phase drop distribution. In Fig. 19.7d,e we have used a symmetric drop in the couplings with the central level ($\alpha = 0$), as expected in an homogeneous field situation with almost symmetric coupling strengths $\Gamma_{1,2}$. However, a finite α would boost the effect of the ϕ_e noise because the pair tunneling terms would start to contribute to $\partial H/\partial \phi_e$.

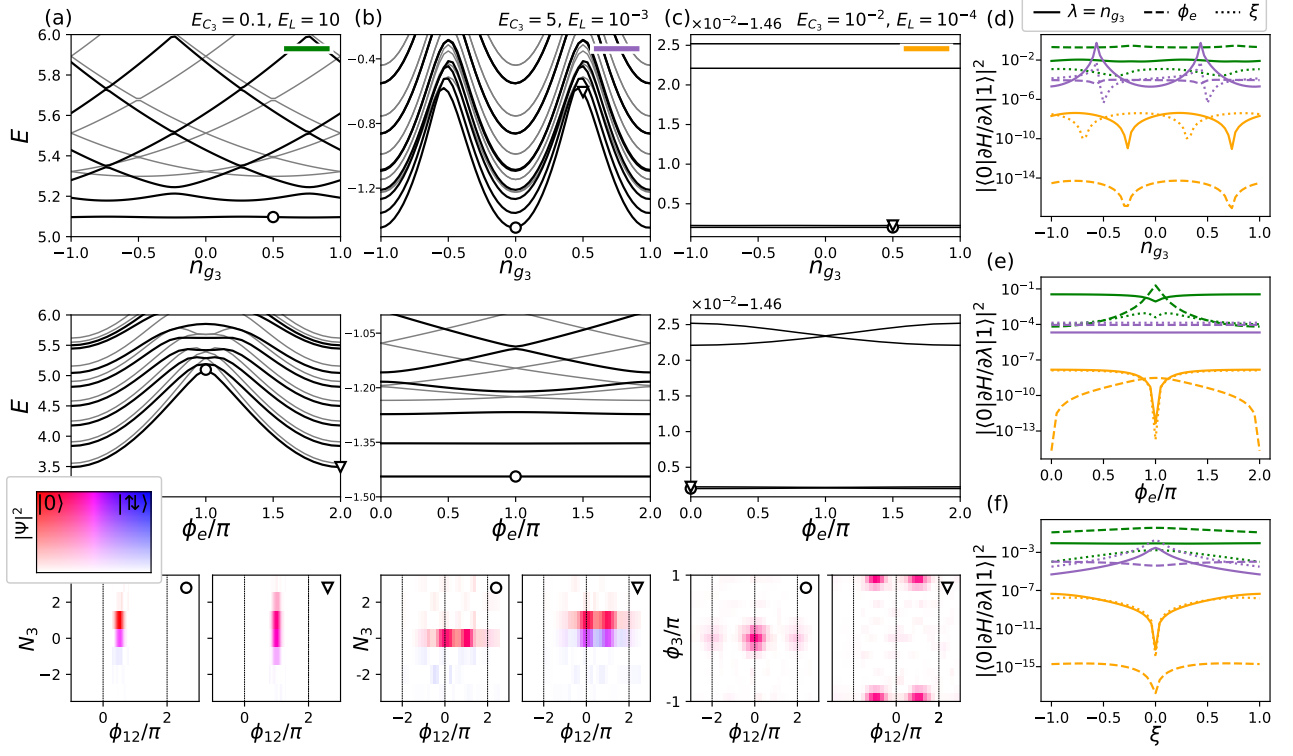


Figure 19.8: (a-c): Energy dispersion over the charge offset n_{g3} (top panels) and the external phase ϕ_e (middle panels) in the loop configuration where a third terminal is a charge island, and wavefunctions indicated with markers (bottom panels – the coloured inset is a legend that represents the participation of the central level in the wavefunction). Gray lines indicate the resonant case and black ones a situation a bit out of resonance. In panel (a), the island is in the transmon regime (delocalized $\hat{N}_3$) and the loop is close to a phase biased regime (well localized phase, parameters of Fig. 19.7a and $\Gamma_3=1$). In panel (b), the island is in the Cooper pair box regime (localized $\hat{N}_3$, with only two states participating in the global wavefunctions), and the loop in a fluxonium regime [delocalized phase, parameters of Fig. 19.7b and $\Gamma_3=1$]. This is similar to a *bifluxon*. In panel (c), the island has delocalized charge and the loop has delocalized phase (the resonant condition has a similar spectrum and it is not shown). This is similar to a $0-\pi$ qubit – the wavefunction is displayed in the $|\phi_{12}, \phi_3\rangle$ basis to show that the wavefunctions of the two lowest states are disjoint. Parameters: $\Gamma_1=0.4, \Gamma_2=0.6, \Gamma_3=1$ and $\xi=0.6$ (energies in units of E_{C12}). (d-f): Matrix elements associated to noise in n_{g3}, ϕ_e and ξ for the non-resonant parameters in (a-c) in the circle markers, and indicating each case with the colours in the (a-c) corners.

19.5.2 Three terminals: one loop and one island

In the previous subsection we have seen that a low inductive potential favours the insensitivity to dephasing produced by ϕ_e noise. However, in order to achieve the protection against decoherence it is necessary to have two states that are disjoint over a different degree of freedom, such that $\partial H/\partial \phi_e$ does not connect them. These two states were provided by the fermionic structure of the junction in the resonant limit, which fulfils this purpose but introduces new

sources of noise [21, 22]. We explore now the situation where a third terminal with charging energy is introduced (Eq. (19.5)), and how it can provide disjoint states of a different origin.

We begin in Fig. 19.8a by connecting the loop with two terminals in the phase localized regime of Fig. 19.7a to an island in the transmon regime (delocalized charge). In the phase biased limit we have $E_L \rightarrow \infty$, $\hat{\phi}_{12} \rightarrow \phi_e/2$ and the configuration is similar to the one in Sec. 19.4.2, where the central region is connected to one charge island and to one terminal without charging energy. In this analogy, Γ_3 is the coupling that shifts the occupation number of the island, and the term $\Gamma_{12}(\phi_e) = \Gamma_1 e^{-i\phi_e/2} + \Gamma_2 e^{i\phi_e/2}$ is the coupling that connects the central level states at a fixed island occupation, which is now tunable by ϕ_e . This arrangement connects all sites in the resulting $|N_3, n_d\rangle$ chain and, as a result, the occupation of the island is determined by the combination of all the couplings, not only Γ_3 . For instance, the wavefunction marked with a circle in Fig. 19.8a has a lower charge dispersion than the one with a triangle because at $\phi_e = \pi$ the effective coupling $\Gamma_{12}(\phi_e)$ is reduced.

In Fig. 19.8b we combine the phase extended regime of Fig. 19.7c in the loop with the island in the Cooper pair box limit (strongly localized charge, thus, it can be described with the two number states of lower energy). This situation benefits of the ϕ_e insensitivity already discussed in the previous subsection, but now the central region and the island hybridize into essentially 4 levels. The phase potential they generate can produce disjoint states in the limit where only one level in the island (central region) contributes, and thus the central region (island) may be set at resonance. The first case is similar to the configuration of the previous subsection, and the second one describes the tunneling regime in the central region while two island states play the role of an effective resonant dot (discussed in next section and in Appendix 19.11). The protection of this configuration is analogous to that of a bifluxon qubit [32].

In Fig. 19.8c, we set the protected regime in each degree of freedom. The wavefunctions are delocalized over $\hat{\phi}_{12}$ and $\hat{N}_3$ (fluxonium and transmon-like), producing flat transition energies over the parameters n_{g_3} and ϕ_e , so the qubit states are protected against dephasing by the associated charge and flux noise. The protection against relaxation occurs because the states have disjoint wavefunctions (see lower panels for the wavefunctions and panels (d) to (f) for the matrix elements associated to noise on n_{g_3} , ϕ_e and ξ for the settings in panels a to c). This disjointness is only observed in the space $|\phi_{12}, \phi_3\rangle$, but the fact that the wavefunctions overlap in the space $|\phi_{12}, N_3\rangle$ does not mean that it is not protected against noise on n_{g_3} , as the associated operator $\hat{N}_3 = -i\partial_{\phi_3}$ is local in the phase coordinate. The same applies to a possible charge noise in the loop with $\hat{N}_{12} \rightarrow \hat{N}_{12} - n_{g_{12}}$ and to noise in other parameters with local associated operators. Thus, the protection of this configuration is analogous to that of a $0 - \pi$ qubit [31]. This and the previous connection with protected qubits designed in multimode circuits made out from several traditional elements will be more easily discussed in the next section when focusing on the tunneling regime.

19.6 Cotunneling limit

At certain limits, a non-tunnel Josephson junction embedded in a circuit can be described as a lumped element, i.e., an effective tunnel junction [Fig. 19.1c, Sec. 19.4.2 and Appendix 19.9], which is characterized by the phase dispersion of the lowest energy Andreev state. This adiabatic approximation in a multiterminal situation provides an interpretation in terms of processes that distribute pairs between the leads, i.e. a combination of lumped junctions between each terminal. For example, in a three-terminal configuration with time reversal symmetry (e.g. in Fig. 19.2), the ground state can be written as

$$E_0(\phi_1, \phi_2) = \sum_{n_1, n_2 \in \mathbb{Z}} A_{n_1, n_2} \cos(n_1 \phi_1 + n_2 \phi_2),$$

where the form $\propto \cos(n_1\phi_1 + n_2\phi_2)$ can be associated to processes where several Cooper pairs from one lead split into the other two, and $A_{n_1, n_2} = A_{-n_1, -n_2}$. This association may be done by defining the phase variables in correspondence with their operator counterparts in Sec. 19.4. There, by using the reduced basis $|N_1, N_2\rangle$, a tunneling from terminals $1 \rightarrow 3$ ($2 \rightarrow 3$) is produced by the traslation operator $e^{-i\hat{\phi}_1}$ ($e^{-i\hat{\phi}_2}$), and from $1 \rightarrow 2$ by $e^{-i\hat{\phi}_1 + i\hat{\phi}_2}$. In this way, the phase variables recover their *operator* nature conjugate to charge-like variables in the resulting effective Hamiltonian $H_J^{\text{eff}}(\hat{\phi}_1, \hat{\phi}_2)$.

In particular, if we consider only single pair tunneling processes, the effective tunneling Hamiltonian is

$$H_J^{\text{eff}} = -E_{J_{13}} \cos \hat{\phi}_1 - E_{J_{23}} \cos \hat{\phi}_2 - E_{J_{12}} \cos (\hat{\phi}_1 - \hat{\phi}_2),$$

with $E_{J_{13}} = -2A_{1,0}$, $E_{J_{23}} = -2A_{0,1}$ and $E_{J_{12}} = -2A_{1,-1}$. We use this case to analyze the following limits.

19.6.1 Trijunction in an island configuration

In the harmonic limit of an island configuration (E_{C_ν} 's $\ll E_{J_{\mu\nu}}$'s), the wavefunctions localize in phase so we can approximate $H_J^{\text{eff}} \approx (E_{J_{13}} + E_{J_{12}})\hat{\phi}_1^2/2 + (E_{J_{23}} + E_{J_{12}})\hat{\phi}_2^2/2 - E_{J_{12}}\hat{\phi}_1\hat{\phi}_2$. The charge offsets n_{g_ν} 's have no effect in this limit (the charge becomes continuous and the n_g a gauge freedom [93]) and we can write $H_{\text{harm}}^{\text{eff}} = \vec{N}^T \mathbb{C} \vec{N} + \vec{\phi}^T \mathbb{J} \vec{\phi}$ where $\vec{N} = (\hat{N}_1, \hat{N}_2)^T$, $\vec{\phi} = (\hat{\phi}_1, \hat{\phi}_2)^T$ and

$$\mathbb{C} = \begin{pmatrix} E_{C_1} + E_{C_3} & E_{C_3} \\ E_{C_3} & E_{C_2} + E_{C_3} \end{pmatrix}, \quad \mathbb{J} = \frac{1}{2} \begin{pmatrix} E_{J_{13}} + E_{J_{12}} & -E_{J_{12}} \\ -E_{J_{12}} & E_{J_{23}} + E_{J_{12}} \end{pmatrix}, \quad (19.6)$$

such that the Hamiltonian can be diagonalized into two harmonic modes $H_{\text{harm}}^{\text{eff}} = \sum_\nu \omega_\nu (a_\nu^\dagger a_\nu + 1/2)$ (see details and extensions provided in Appendix 19.10).

In the limit where one island is in a Cooper pair box regime (e.g. $E_{C_3} \gg E_{J_{13}}, E_{J_{23}}$), that island can be described with a two level model, one for each of the lowest energy number N_3 occupations. As elaborated in Appendix 19.11, it maps to the model of the Josephson-Andreev junction with two terminals in Sec. 19.4 when $E_{J_{12}} = 0$ (the terminal 3 would play the role of the central level).

19.6.2 Trijunction with two terminals in a loop and one island configuration

If two terminals close themselves forming a loop, the model is similar to the one in Sec. 19.5, where the variables are the tunneled pairs between the leads of the loop $\hat{N}_{12}$ and the number of pairs $\hat{N}_3$ in the island. The effective Hamiltonian is $H_{12} + H_3 + H_c$, where

$$\begin{aligned} H_{12} &= E_{C_{12}} \hat{N}_{12}^2 + E_{J_{12}} \cos 2\hat{\phi}_{12} + E_L (2\hat{\phi}_{12} - \phi_e)^2 \\ H_3 &= E_{C_3} (\hat{N}_3 - n_{g_3})^2 \\ H_c &= E_{J_{13}} \cos (\hat{\phi}_{12} + \hat{\phi}_3) + E_{J_{23}} \cos (\hat{\phi}_{12} - \hat{\phi}_3), \end{aligned} \quad (19.7)$$

and the operator $e^{i2\hat{\phi}_{12}}$ transfers a pair from $1 \rightarrow 2$, $e^{i\hat{\phi}_{12} + i\hat{\phi}_3}$ from $1 \rightarrow 3$ and $e^{i\hat{\phi}_{12} - i\hat{\phi}_3}$ from $2 \rightarrow 3$. The Hamiltonian can be solved in the basis $|\phi_{12}, N_3\rangle$, and contains two modes which can be harnessed to devise protected qubits similarly to multimode circuits designed with superconductor-insulator-superconductor (SIS) tunnel junctions [23]. Two conventional limits are the following.

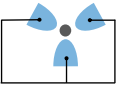
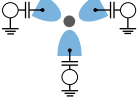
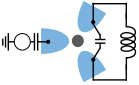
Configuration	Quantum variables	Control parameters	Circuit noise	Highlights
 Perfect phase bias	$n_d \in \mathbb{Z}_2$	ϕ_{1e}, ϕ_{2e} ξ, Γ_ν	Flux	- Sensitivity of relaxation to phase drop distribution - Appearance of Weyl points - Connection with Bi-SQUID
 Charge islands	$N_\nu \in \mathbb{Z} \quad (\phi_\nu \in [-\pi, \pi))$ $n_d \in \mathbb{Z}_2$	n_{g_ν} ξ, Γ_ν	Charge	- Mapping to a discrete lattice: platform for topology - Controllable disjointness
 Hybrid charge/flux	$N_{12} \in \mathbb{R}, N_3 \in \mathbb{Z}$ $(\phi_{12} \in \mathbb{R}, \phi_3 \in [-\pi, \pi))$ $n_d \in \mathbb{Z}_2$	ϕ_e, n_{g_3} ξ, Γ_ν	Charge, flux	- 0- π and bifluxon in the tunnel limit $\Updownarrow$ - Simultaneous noise protection to relaxation and n_g, ϕ_e -dephasing

Table 3: Summary of the circuit configurations for three terminal junctions analyzed in this work, indicating the relevant quantum variables and control parameters within our modeling, the main circuit noise sources and the most significant properties. First row for the perfect bias case (Sec. 19.3), second row for charge islands configuration (Sec. 19.4) and third row for the mixed configuration including a loop and a charge island (Sec. 19.5).

On the one hand, if $E_{J_{12}} \gg E_{J_{13}}, E_{J_{23}}$, the Hamiltonian is similar to a “ $\cos 2\varphi$ ” qubit as the potential in $\hat{\phi}_{12}$ is almost π periodic [35, 38]. Physically, the situation with only $E_{J_{12}}$ tunneling conserves the parity of tunneled pairs, generating two disconnected sets of states.

On the other hand, if $E_{J_{12}} = 0$ and $E_{J_{13}} = E_{J_{23}} := E'_J$, the coupling becomes

$$H_c = E'_J \cos \hat{\phi}_{12} \cos \hat{\phi}_3,$$

recovering the typical expression for multimode circuits, such as the bifluxon or the 0- π qubit, at the corresponding parameter regime [23].

19.7 General Discussion and Overview

In this section we summarize and compare the main features of the different circuits discussed in previous sections and provide some estimates for the expected qubit frequencies and decoherence rates. For this purpose we compile in Table 3 the relevant information regarding degrees of freedom, control parameters, and circuit noise sources for each configuration. We also highlight their correspondence to conventional superconducting qubits based on tunnel junctions or other interesting properties.

The first row in this table corresponds to the triterminal junction with perfect phase-biasing considered in Sec. 19.3. In this case the control parameters are the external fluxes Φ_{1e}, Φ_{2e} determining the phase differences between the junction terminals. In this configuration the phases have no dynamics and we deal with a purely fermionic Andreev qubit. At the central level degeneracy point, $\xi=0$, this configuration is interesting due to the appearance of protected crossings (Weyl points) (see Fig. 19.2). As the central level becomes more detuned, the energy

bands become flat at the expense of increasing its transition energy. A similar spectrum has been shown to arise in a bi-SQUID circuit based on metallic tunnel junctions [144].

The second row in Table 3 corresponds to the charge-islands configurations discussed in Sec. 19.4. The two terminal device in the tunnel regime describes the gatemon for gate tunable weak links and when the third lead is connected, the total charge delocalizes in a plane of integer quantum variables. We find an interesting regime [Figs. 19.6c,d] where the wavefunctions correspond to the charge chains of two islands coupled through the third terminal, which thus controls their disjointness. Interestingly, the mathematical (“crystalline”) structure of the charge-island configurations can be mapped into topological band models.

Finally, the third row in Table 3 corresponds to the case where two terminals are shunted by a loop with significant inductive energy. In the two-terminal case we explore the transition from an almost localized phase degree of freedom to a fluxonium-like regime. The most interesting case corresponds to this later one, where there is simultaneous protection to decoherence in flux-noise due to the phase delocalization [Fig. 19.7c] and a disjoint support with respect to the Andreev character [21, 22]. When the third lead is connected, we explored a hybrid configuration in which a charge island is coupled through the central level to a flux-loop with nonzero inductance. We compare a transmon-(flux qubit)-like regime [Fig. 19.8a], a (Cooper pair box)-fluxonium-like regime [Fig. 19.8b], and a transmon-fluxonium-like regime [Fig. 19.8c] with simultaneous delocalization in the charge and phase variables (flat bands in the parameter space) and disjoint wavefunction support. This is an interesting proposal because it can realize a $0-\pi$ qubit, whose explicit mapping to the conventional implementation we show in the tunnel limit in Sec. 19.6.

19.7.1 Poisoning

We discuss here the effects of quasiparticle (QP) poisoning, that takes place when unbound electrons with energies over the gap tunnel from one region to another [145]. Though these events may occur at long timescales compared with some of the present qubit decoherence times, they can become limiting factors when other decoherence sources are optimized, and many strategies are being developed to decrease the density of excess QPs as it follows a non thermal distribution highly dependent on the implementation. We distinguish two kinds of poisoning contributions according to the region where the quasiparticle infiltrates into. As a first scenario, poisoning on the charge islands or between capacitively coupled leads produces a shift $N_\nu \rightarrow N_\nu \pm 1/2$, which is common in two-terminal devices but more complex in multiterminal ones. As we show in Fig. 19.9, poisoning leads to the appearance of additional states for the case of three terminal devices. In the configuration with two charge islands [panel (a)], poisoning may occur in any of them and this provides four families of transitions [146]. In the situation where all terminals are islands with finite charging energy, poisoning events from the environment into the system do modify the spectrum even if they appear in pairs, because states with different total charge are not equivalent (*e.g.* Fig. 19.6a). In the configuration with a loop and an island (panel b), the states produced by poisoning in the island depend both on n_{g_3} and ϕ_e . The poisoning between the leads in the loop does not affect the spectrum as the shunting renders the charge difference continuous and an static offset charge becomes a gauge freedom [139], but it does generate relaxation [145, 147].

As a second possibility, poisoning on the junction central level, in contrast with the previous case, produces a major rearrangement of the wavefunction as the quasiparticle mediates the supercurrent. This cannot be described with a trivial shift of a parameter, but requires a model that includes the effect of the excess quasiparticles. Within our infinite gap approximation, isolated QPs at the central region are not taken into account and thus this odd state is not included. Including such effects could be possible by means of models beyond the infinite gap

approximation like the ones used in Refs. [148, 149], which can be undertaken in future studies.

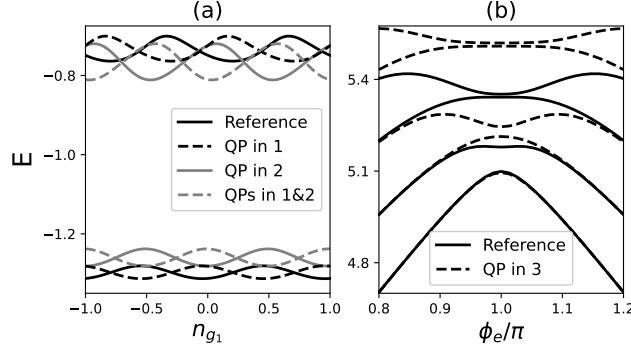


Figure 19.9: States with quasiparticle poisoning in the islands. (a) Three terminal configuration of Fig. 19.6b with two charge islands. (b) Charge-flux hybrid configuration of Fig. 19.8a at $n_{g_3} = 1/4$.

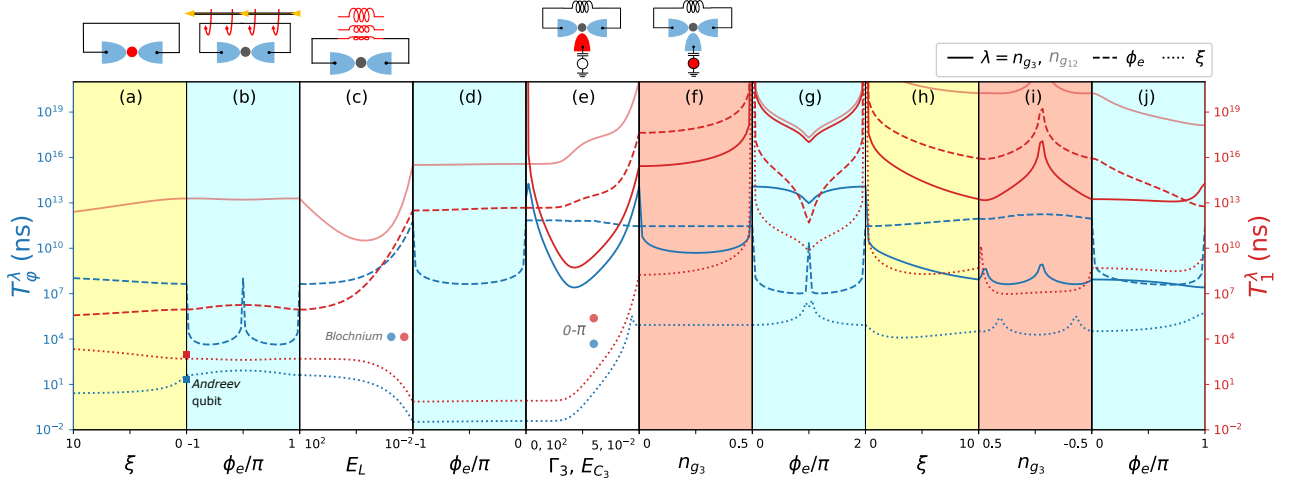


Figure 19.10: Dephasing (blue lines) and decoherence (red lines) estimated times for different configurations with a loop, while sweeping the position of the central level (light yellow background), the external flux (light cyan), the island offset charge (light orange) and other parameters to cross over between different regimes (white). (a,b) Perfect phase bias with two terminals. (c) Transition to introduce phase fluctuations until the fluxonium-like regime is reached in (d). (e) Coupling with the charge island and transition to $0-\pi$ -like protected regime in (f,g). (h) Transition deep into tunnel regime and evolution with island gate and external flux in (i,j). Parameters are given in the text, units in GHz. Decoherence times of the Andreev level qubit used for the choice of some parameters [150] are indicated with a squares; other markers indicate a Blochonium and a $0-\pi$ implementation in conventional circuits just for illustrative purposes [151, 31].

19.7.2 Relaxation and dephasing

We now discuss possible sources of decoherence and relaxation. On the one hand, the measurement device used to probe the quantum states of the circuit, which can be a read-out microwave resonator, will induce some relaxation (by means of Purcell effect, or dressed dephasing [152]) and dephasing (through measurement-induced dephasing). This contribution depends strongly on the design and can be mitigated by different well developed strategies (like Purcell filters). On the other hand, the noise on the system parameters produces an inherent

decoherence, that is to be minimized by the choice of the architecture of the device and its operation point. In circuits with conventional tunnel junctions those parameters are the offset charge, the external flux and the Josephson coupling. But in junctions with an Andreev structure, each fermionic degree of freedom (in the single level model controlled by parameters ξ and the Γ_ν 's) may also experience its own noise. Other source of decoherence is the relaxation by dielectric loss, which depends on the design and the materials [153, 147, 154, 155].

There are mainly two kinds of noise affecting a parameter: $1/f$ and quantum Johnson-Nyquist noise [156]. For instance, a background of two level systems produces $1/f$ fluctuations in the offset charges, while the coupling with a gate is responsible for its Johnson-Nyquist noise $\propto \omega \coth \omega/2k_B T$. Precise models for the spectral noise from these sources depend on the particularities of the system. Thus, here we use a simple expression able to reproduce typical behaviour of the spectral noise. As the crossover between noises occurs typically in the range of GHz both for offset charge and flux [156, 157, 158, 159] this form can be chosen as

$$S_\lambda(\omega) = 2\pi A_\lambda^2 \left(\frac{1}{\omega} + \frac{\eta_\lambda \omega}{1\text{GHz}^2} \coth \frac{\omega}{1\text{GHz}} \right),$$

where A_λ is a noise amplitude, η_λ is a factor adjusting the Johnson-Nyquist noise and we have used a temperature $T \sim 25\text{mK} \approx 0.5\text{GHz}$ and a symmetrized spectral noise such that the relaxation (depolarization) rate $1/T_1^\lambda = S_\lambda(E_{01}) |\langle 0 | \partial H / \partial \lambda | 1 \rangle|^2$ includes both emission and absorption at the qubit frequency (we disregard excitation processes to higher energy states, but in the disjoint regime these may be relevant [29]). We use typical noise amplitudes for charge and flux noise $A_{n_g} \sim 10^{-4}$, $A_{\phi_e} \sim 10^{-5}$, and $\eta_{n_g} \sim 0.01$, $\eta_{\phi_e} \sim 10$ [120, 122, 156, 116]. For the noise in the Andreev structure, we note that few experiments exist on mesoscopic Josephson junctions providing an analysis for relaxation and dephasing rates, so the accuracy of this discussion is limited. However, we try to give some reasonable estimation, for example, from the data in Ref. [150], where there is a single central gate, which mainly controls ξ (also Γ but with lower lever arm). Since there are fewer experiments which implement several gates, and none of them characterizes the decoherence, we consider just the effect on ξ . We use the same expression as for the n_g noise corresponding to a gate electrode, with the prefactor $A_\xi \sim 0.04$ GHz, which provides the order of magnitude of the dephasing and relaxation times found in nanowire junctions ($T_1 \sim 1\mu\text{s}$, $T_\varphi \sim 50\text{ns}$ in Ref. [150], $T_1 \sim 10\mu\text{s}$ in Ref. [160]), also qualitatively similar in a point contact [116]).

For the dephasing, we account only for the noise close to $\omega = 0$, *i.e.*, the $1/f$ contribution, which produces a typical time T_φ^λ

$$\left(\frac{1}{T_\varphi^\lambda} \right)^2 = c_1 A_\lambda^2 \left(\frac{\partial E_{01}}{\partial \lambda} \right)^2 + c_2 A_\lambda^4 \left(\frac{\partial^2 E_{01}}{\partial \lambda^2} \right)^2,$$

where $c_{1,2}$ are coefficients that depend on the cutoff frequencies. We use typical values $c_{1,2} \sim 30, 1200$ [29, 120] (but we do not include those cutoffs for the relaxation calculation).

We first discuss briefly the charge islands configurations (Sec. 19.4). The most amenable two terminal configuration for hosting a qubit is the *transmon* regime, when the ratio between effective Josephson coupling E_J^{eff} and the charging energy is large enough to reduce the transition dispersion with the charge offset but not as much as to lose significant anharmonicity. This suppression of charge dispersion is enhanced when the Andreev structure approaches the resonant condition [7, 10, 20]. The relaxation from charge noise is negligible [120, 122]. The Andreev structure provides E_J^{eff} (Appendix 19.9), which we analyze for simplicity at $\delta\Gamma = 0$ such that $E_J^{\text{eff}} = \Gamma^2 / \sqrt{\Gamma^2 + \xi^2}$. In conventional transmons of oxide tunnel barriers, the fluctuations of the Josephson coupling are produced by spatial reconfigurations of ions inside the junction, among other mechanisms [122]. In our model of the Josephson-Andreev junction,

we expect additional noise from the gates that control the position of the level and the couplings. At fixed Γ , dephasing by ξ behaves similarly to the dephasing in the perfect flux bias since $E_{01} \sim 2\sqrt{E_{C1}E_J^{\text{eff}}}$. It has a minimum at $\xi = 0$, then increases till $\xi \approx \Gamma/\sqrt{2}$ (using the harmonic limit) and then decreases with a tail $\sim \xi^{-4}$. Another comparison can be made by fixing E_J^{eff} by moving Γ accordingly. In this case, the ξ -dephasing maximum is at $\xi \approx 4\sqrt{3}E_J^{\text{eff}}$. Relaxation is negligible as can be noted in the harmonic and adiabatic limits $\langle 0 | \cos \hat{\phi} | 1 \rangle = 0$ by means of parity arguments. Regarding tunability, experiments on gatemons tuned close to resonance have been performed in the regime $E_{C1} < E_J^{\text{eff}} \sim \Gamma/8 \sim 5\text{GHz}$, with qubit frequencies in the order of a few GHz [7, 10]. In the case of three terminals in the charge island configuration, as discussed in Sec. 19.4.4, having a low Γ_3 is analogous to the Cooper Pair Box-regime and, when all Γ_ν are similar, to a transmon-like regime but in a larger space. This configuration is interesting for fundamental reasons more than for qubit applications.

We now discuss the loop configurations (Secs. 19.3 and 19.5), and show the estimated relaxation and dephasing times in the hybrid charge-flux configuration [Fig. 19.1f] for different regimes [Fig. 19.10]. First, we set $\Gamma_3 = 0$ and $E_L = 100\text{ GHz}$ to recover the perfect phase bias configuration with terminals [panels a and b]. When decreasing ξ towards the sweet spot [panel a], the associated dephasing rate, which dominates the decoherence, is reduced at the expense of an increase in the associated relaxation rate. The same occurs (b) with the ϕ_e noise towards its dephasing sweet spot. In this sweep, the qubit frequency oscillates between 10 GHz and 20 GHz reproducing the measurements in Ref. [150], with $\Gamma = 10\text{ GHz}$ and $\delta\Gamma = 5\text{ GHz}$. We keep these parameters whose realization is less challenging than the perfectly resonant limit. In (c), we increase (logarithmically) the inductance up to $E_L = 10\text{ MHz}$, in the order of the limit of what is feasible [151], allowing for phase fluctuations. As it is possible to control the capacitive couplings in the terminals, we use a large charging energy $E_{C12} = 50\text{ GHz}$. Thus, the result is analogous to a Blochonium [151] as $E_{C12} > E_J^{\text{eff}} \sim 1\text{ GHz}$, with ϕ_e -dephasing sweet spots at $\phi_e = 0, \pi$ and large ξ -induced decoherence (d). The qubit frequency in this panel and the following ones is reduced, in the order of 10 MHz. When we connect the third lead (e) changing Γ_3 from 0 up to $\Gamma_3 = 5\text{ GHz}$, while decreasing (logarithmically) the charging energy E_{C3} of the island down to 0.01 GHz -which was set at a large value to avoid other states (100 GHz)-, we arrive at a protected $0-\pi$ -like regime, which is tested over n_{g3} and ϕ_e (f,g). The relaxation from all parameters (even ξ) is reduced because the states become considerably disjoint, particularly in the point $n_{g3} = 1/2$, $\phi_e = 0$, though that extreme protection is lost at finite ξ (h). When the central region is brought more into the tunnel regime (h,i,j), the $0-\pi$ -like protection is maintained. We note that even if the Andreev structure confers an additional layer of complexity, the relevant parameters to achieve this kind of disjointness –that occurs in the circuit bosonic variables– are the relations between the effective Josephson couplings and the charging energies. Also, the dominant effect of slow dephasing by ξ noise can be mitigated by several techniques, such as Hahn echo and other dynamical decoupling sequences. In addition, we expect no decoherence from the coupling with other circuit -spurious- modes, which are present in some implementations of multimode protected circuits with conventional tunnel junctions but are not present in our charge-flux hybrid configuration [29, 28, 35, 23, 31]. Finally, let us mention that the rather long calculated timescales are the result of the choice of the parameters in a “hard” regime, similar to calculations for $0-\pi$ implementations in conventional circuits [29, 23].

19.8 Effective Hamiltonian in phase biased conditions

Our starting point is a single-level model coupled to superconducting leads, described by a Hamiltonian of the form

$$H = \sum_{\sigma} \epsilon d_{\sigma}^{\dagger} d_{\sigma} + H_T + \sum_{\nu} H_{\nu} ,$$

where H_ν correspond to the different terminals (ν ranging from 1 to 3 in the trijunction case) and

$$H_T = \sum_{\nu, \sigma, k} t_\nu e^{-i\phi_\nu/2} d_\sigma^\dagger c_{\nu, k\sigma} + \text{h.c.}$$

where ϕ_ν is the superconducting phase on each lead.

Using conventional field theoretical methods one can integrate out the leads, leading to an effective action on the dot

$$S_{\text{eff}} = \int dt dt' \bar{\psi}_d(t) \left[i\delta(t-t') (\partial_t - \epsilon\tau_z) - \hat{\Sigma}(t, t') \right] \psi_d(t') \quad (19.8)$$

where ψ_d is the Grassman field associated with the dot Nambu spinor $(d_\uparrow, d_\downarrow)^T$, and $\hat{\Sigma}(t, t')$ denotes the leads self-energy given by

$$\hat{\Sigma}(t, t') = \sum_\nu t_\nu^2 e^{-i\phi_\nu(t)\tau_z/2} \hat{g}(t-t') e^{i\phi_\nu(t')\tau_z/2}.$$

For the uncoupled leads Green functions $\hat{g}(t-t')$ we can use the BCS model in the wide band approximation, i.e.

$$\hat{g}(t-t') = \int \frac{d\omega}{2\pi} e^{i\omega(t-t')} \left[\frac{-\omega\tau_0 + \Delta\tau_x}{W\sqrt{\Delta^2 - \omega^2}} \right],$$

where W denotes the bandwidth. In the static case and in the absence of phase fluctuations (constant ϕ_ν) Eq. 19.8 can be written as

$$S_{\text{eff}} = \int d\omega \bar{\psi}_d(\omega) \left[\omega - \epsilon\tau_z - \sum_\nu \Gamma_\nu \left(\frac{-\omega\tau_0 + \Delta(\tau_x \cos \phi_\nu - \tau_y \sin \phi_\nu)}{\sqrt{\Delta^2 - \omega^2}} \right) \right] \psi_d(\omega), \quad (19.9)$$

where $\Gamma_\nu = t_\nu^2/W$. In the large gap limit, i.e. $\omega \ll \Delta$ the model is characterized by bound states at

$$E_A(\{\phi_\nu\}) = \sqrt{\epsilon^2 + \left| \sum_\nu \Gamma_\nu e^{-i\phi_\nu} \right|^2}.$$

On the other hand, in the presence of phase fluctuations Eq. 19.8 can be simplified following the lines of Ref. [14]. For that purpose we approximate $\hat{g}(t-t')$ as

$$\hat{g}(t-t') \simeq \delta(t-t') \frac{-i\partial_t \tau_0 + \Delta\tau_x}{W\Delta},$$

which leads to

$$S_{\text{eff}} = \int dt \bar{\psi}_d(t) \left[i\partial_t - \epsilon\tau_z - \sum_\nu \Gamma_\nu \left(\tau_x \cos \phi_\nu(t) - \tau_y \sin \phi_\nu(t) - \tau_z \frac{\dot{\phi}_\nu(t)}{2\Delta} \right) \right] \psi_d(t).$$

To remove the $\dot{\phi}_\nu$ terms we redefine $\psi_d(t) \rightarrow e^{-i\sum_\nu \tilde{\Gamma}_\nu \phi_\nu \sigma_z} \psi_d(t)$, where $\tilde{\Gamma}_\nu = \frac{\Gamma_\nu}{2\Delta}$, so that

$$S_{\text{eff}} = \int dt \bar{\psi}_d(t) \left[i\partial_t - \hat{H}_{\text{eff}} \right] \psi_d(t), \quad (19.10)$$

where

$$\hat{H}_{\text{eff}} = e^{-i\sum_\nu \tilde{\Gamma}_\nu \phi_\nu \tau_z} \left[\epsilon\tau_z + \sum_\nu \Gamma_\nu (\tau_x \cos \phi_\nu - \tau_y \sin \phi_\nu) \right] e^{i\sum_\nu \tilde{\Gamma}_\nu \phi_\nu \tau_z}. \quad (19.11)$$

This last expression can be used as the effective Hamiltonian for the multiterminal quantum dot junction. In practice only two out of the three phases $\{\phi_\nu\}$ can be considered independent. Also, in the infinite gap limit one can disregard the phase prefactors in Eq. 19.11.

19.9 Adiabatic approximation

When transforming from the number space to the phase space (for simplicity in a two-terminal configuration with one degree of freedom), it is convenient to interpret the latter as a discrete grid, that arises when the number space is truncated to an arbitrarily large interval $N \in [-N^M, N^M]$ with periodic boundary conditions, defining $|\phi_n\rangle = N_{\#}^{-1/2} \sum_N e^{iN\phi_n} |N\rangle$, with $N_{\#} = 2N^M + 1$ and $\phi_n = 2\pi n/N_{\#}$. This is justified because at finite charging energy the wavefunction has no infinite extension in N . Then, the operator $\hat{N}$ can be thought as a translation in the $|\phi_n\rangle$'s, with $\hat{N}^2 \approx 2N_{\#}^2(1 - \cos \hat{N}/N_{\#})$. In the continuum limit this converts into the second derivative $-\partial_{\phi}^2$.

At each point ϕ , the potential Hamiltonian $H_A(\phi)$ is equivalent to the one describing the phase biased configuration (Eq. 19.2), up to a *gauge* transformation. The Andreev transformation $U(\phi)H_A(\phi)U^{\dagger}(\phi) = \mathbb{E}_A$ diagonalizes the states at each phase point, but then the *kinetic* term $\sim E_C \hat{N}^2$ becomes non-diagonal in the neighbouring phase hoppings:

$$\hat{N}^2 \rightarrow \hat{N}^2 - i \left\{ U(\phi)U^{\dagger}(\phi), \hat{N} \right\} - (U(\phi)U^{\dagger}(\phi))^2,$$

where $U'(\phi) = \partial_{\phi}U(\phi)$, and it has been used that $\partial_{\phi}(U(\phi)U^{\dagger}(\phi)) = 0$. The adiabatic approximation consists in truncating the Andreev sector to its ground state (with energy equal to $\min \{\mathbb{E}_A\}$) and keeping the main kinetic term, which is the one with $\hat{N}^2$ since in the transmon-like regime the wavefunction is strongly delocalized in charge. We note that the Andreev transformation has distorted the number translation and that there is a gauge dependence similar to the one in Sec. 19.3, that arises from the choice of the reduced number basis (the complete charging term would also depend on it). When restricted to the truncated Andreev sector, it has a similar role to a phase dependent charge offset and a phase potential, which becomes less noticeable as the charge delocalizes.

In this limit, the phase localizes and we can expand the effective potential into an anharmonic oscillator

$$H_{\text{osc}}^{\text{eff}} = E_C \hat{N}^2 + E_J^{\text{eff}_2} \hat{\phi}^2 + E_J^{\text{eff}_4} \hat{\phi}^4,$$

with $E_J^{\text{eff}_n} = \partial_{\phi}^n E_0(\phi)|_{\phi=0}/n!$, which in the harmonic basis $|m\rangle = (a^{\dagger})^m |0\rangle / \sqrt{m!}$, with $a^{(\dagger)} = (E_J^{\text{eff}_2}/4E_C)^{1/4} \hat{\phi}_1 \mp i(E_C/4E_J^{\text{eff}_2})^{1/4} \hat{N}$ (see Appendix 19.10), produces states with energies E_m equal to

$$2\sqrt{E_C E_J^{\text{eff}_2}} (m + 1/2) + \frac{E_C E_J^{\text{eff}_4}}{4E_J^{\text{eff}_2}} (6m^2 + 6m + 3),$$

up to first order perturbation theory. This yields a transition E_{01} equal to the prefactors on m (the one with the square root is referred with ω in the main text) and an anharmonicity $\alpha_h := E_{12} - E_{01}$ equal to twice the prefactors on m^2 . Within the model of Sec. 19.3, $E_J^{\text{eff}_2} = (\Gamma^2 - \delta\Gamma^2)/8\sqrt{\Gamma^2 + \xi^2}$ and $\alpha_h = E_C(-1/4 + 3(\Gamma^2 - \delta\Gamma^2)/16(\Gamma^2 + \xi^2))$.

19.10 Simultaneous diagonalization in the tunnel limit

The matrix $\mathbb{C}$ in Eq. 19.6 is positive definite, hence it can be written [161]

$$\mathbb{C} = (v_C)D_C(v_C)^T = \left[(v_C)\sqrt{D_C} \right] \left[(v_C)\sqrt{D_C} \right]^T := R^T R,$$

where v_C is the matrix made of the orthonormal vectors that diagonalize $\mathbb{C}$ into D_C . Note that if we define $C_N = R^{-1}U$, with any orthogonal $U^T = U^{-1}$, we have $C_N^T \mathbb{C} C_N = \mathbb{1}$. Now, consider

that there is C_ϕ such that $C_\phi^T \mathbb{J} C_\phi = D_J$ with diagonal D_J , and that $C_\phi^T = C_N^{-1}$ for reasons that will be clear afterwards – it is allowed because in a Hamiltonian formulation the conjugate variables are independent [162]. Then the previous equation for D_J can be written as the diagonalization $U^T(R\mathbb{J}R^T)U = D_J \leftrightarrow R\mathbb{J}R^T U = U D_J$, showing that $U = (u_1, u_2, \dots)$ is made of the eigenvectors from that diagonalization, which is possible because the matrix product between parenthesis is symmetric. Multiplying by R^{-1} it is equivalent to

$$\mathbb{J}R^T U = R^{-1} U D_J = R^{-1} R^{-1T} V D_J = \mathbb{C} V D_J,$$

where $V := R^T U$ shows the structure of a generalized diagonalization into the columns of $V = (v_1, v_2, \dots)$, with eigenvalues λ satisfying $\det(\mathbb{J} - \mathbb{C}\lambda) = 0$. Note that $v_n^T \mathbb{C}^{-1} v_m = v_n^T R^{-1} R^{-1T} v_m = u_n^T u_m = \delta_{nm}$.

Then we can write $H_{\text{harm}}^{\text{eff}} = \vec{N}'^T \vec{N}' + \vec{\phi}'^T D_J \vec{\phi}'$, with $\vec{N}' = C_N^{-1} \vec{N}$ and $\vec{\phi}' = C_\phi^{-1} \vec{\phi}$. Each transformed pair $(\hat{N}'_\nu, \hat{\phi}'_\nu)$ is uncoupled from the others, and we have $[\hat{\phi}'_\mu, \hat{\phi}'_\nu] = 0$, $[\hat{N}'_\mu, \hat{N}'_\nu] = 0$ and $[\hat{\phi}'_\mu, \hat{N}'_\nu] = i \sum_\alpha (C_\phi^{-1})_{\mu\alpha} (C_N^{-1})_{\nu\alpha} = i (C_\phi^{-1} C_N^{-1T})_{\mu\nu} = i \delta_{\mu\nu}$, where the choice in the previous paragraph allows to conserve the commuting properties in the transformed operators.

Now, defining $a_\nu^{(\dagger)} = c_{1\nu}^{(*)} \hat{N}'_\nu + c_{2\nu}^{(*)} \hat{\phi}'_\nu$, imposing the commutation relations $[a_\mu, a_\nu^\dagger] = \delta_{\mu\nu}$ constrains $c_{1\nu} c_{2\nu} = -1/2$. Then, $H_{\text{harm}}^{\text{eff}}$ can be made diagonal with the choice $c_{1,2\nu} = (D_J)_\nu^{\pm 1/4} i^{-1/2 \pm 1/2} / \sqrt{2}$. Finally, $H_{\text{harm}}^{\text{eff}} = \sum_\nu 2\sqrt{D_{J\nu}} (a_\nu^\dagger a_\nu + 1/2)$, where $(D_J)_\nu = (b \pm \sqrt{b^2 - 4c})/4$, with

$$b = \sum_{\alpha=1}^3 E_{C_\alpha} (E_{J_{\alpha,\alpha-1}} + E_{J_{\alpha,\alpha+1}})$$

$$c = \left(\sum_{\alpha=1}^3 E_{C_\alpha} E_{C_{\alpha+1}} \right) \left(\sum_{\alpha=1}^3 E_{J_{\alpha-1,\alpha}} E_{J_{\alpha,\alpha+1}} \right),$$

where the indexes values 0 and 4 refer to the indexes 3 and 1, respectively, and $E_{J_{\mu\nu}} = E_{J_{\nu\mu}}$. Some particular cases are $2D_{J\nu} = \{0, (E_{C_1} + E_{C_2})E_{J_{12}}\}$ for $E_{J_{13}} = 0 = E_{J_{23}}$, and $2D_{J\nu} = \{E_{C_1}E_{J_{13}}, E_{C_2}E_{J_{23}}\}$ for $E_{J_{12}} = 0 = E_{C_3}$.

19.11 Two level island limit

In the limit $E_{C_3} \gg E_{J_{13}}, E_{J_{23}}$, we can describe island 3 with its two lowest energy levels corresponding to certain N_3 and $N_3 + 1$, with an energy difference of $2E_{C_3}(n_{g3} - 1/2)$. In this case it is useful to describe the other two terminals with $\hat{N}_{12} = \hat{N}_1 - \hat{N}_2$ (note that a third variable is not necessary - the charge conservation imposes the value of the remaining $\hat{N}_1 + \hat{N}_2$). The tunneling processes between 1 and 2 create an imbalance of two pairs in N_{12} ($-E_{J_{12}}e^{-i2\hat{\phi}_{12}} + h.c.$), while the tunneling from 1 to the island and from the island to 3 creates an imbalance of one pair ($-E_{J_{13}}e^{-i\hat{\phi}_{12}} |N_3 + 1\rangle \langle N_3| + h.c.$ and $-E_{J_{23}}e^{-i\hat{\phi}_{12}} |N_3\rangle \langle N_3 + 1| + h.c.$, respectively). Thus, in the basis $\{|N_3\rangle, |N_3 + 1\rangle\}$,

$$H_{2\text{lvl}}^{\text{eff}} = H_C - E_{J_{12}} \cos 2\hat{\phi}_{12} + \begin{pmatrix} -E_{C_3}(n_{g3} - 1/2) & -E_{J_{13}}e^{i\hat{\phi}_{12}} - E_{J_{23}}e^{-i\hat{\phi}_{12}} \\ -E_{J_{13}}e^{-i\hat{\phi}_{12}} - E_{J_{23}}e^{i\hat{\phi}_{12}} & E_{C_3}(n_{g3} - 1/2) \end{pmatrix},$$

where the correspondence with the two-terminal junction with a central level is given by $\xi = E_{C_3}(n_{g3} - 1/2)$ and $\Gamma_{1,2} = -E_{J_{13,23}}$ (at $E_{J_{12}} = 0$). From a different perspective, in the case of $E_{J_{12}} \gg E_{C_{12}}$, this tunneling term can be diagonalized with the charging term into an oscillator coupled to the two-level structure. This is similar to what occurs when coupling a circuit with a reference transmon [112].

20 Multi-terminal Josephson junctions as topological materials by Riwar, Houzet, Meyer, Nazarov

Abstract

Topological materials and their unusual transport properties are now at the focus of modern experimental and theoretical research. Their topological properties arise from the bandstructure determined by the atomic composition of a material and as such are difficult to tune and naturally restricted to ≤ 3 dimensions. Here we demonstrate that n -terminal Josephson junctions with conventional superconductors may provide a straightforward realization of tunable topological materials in $n - 1$ dimensions. For $n \geq 4$, the Andreev subgap spectrum of the junction can accommodate Weyl singularities in the space of the $n - 1$ independent superconducting phases, which play the role of bandstructure quasimomenta. The presence of these Weyl singularities enables topological transitions that are manifested experimentally as changes of the quantized transconductance between two voltage-biased leads, the quantization unit being $2e^2/(\pi\hbar)$.

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Josephson junctions created by coupling *two* superconductors through a weak link have been studied extensively for many years [2, 4, 3]. The current across a Josephson junction yields information about the Andreev bound states (ABS) forming at the junction [15, 9, 6, 11]. In turn, the ABS spectrum is determined by the properties of the junction and the superconducting leads. For instance, if the leads are topologically non-trivial, the Josephson effect may be used to probe these topological properties. In particular, the 4π -periodicity of the supercurrent indicates the presence of topologically protected zero-energy Majorana states [8, 9, 10, 11], which may arise in 1D spinless p-wave superconductors, semiconductor nanowires with proximity-induced superconductivity, or at the surface of bulk materials.

In this paper, we show that *multi*-terminal Josephson junctions may be topologically non-trivial even if the superconducting leads are topologically trivial and no exotic materials are used to make the junction. Thus, the junction itself may be regarded as an artificial topological material, which displays Weyl singularities.

The rapidly growing field of 3D Weyl semimetals deals with a bandstructure that exhibits conical energy gap closings: Weyl points [12, 13, 14, 15, 16]. Unlike the Dirac point in graphene [17] that may be gapped out through an appropriate coupling, isolated Weyl points are topologically protected. They can be regarded as monopoles with $\pm$ charge. A topological invariant – the Chern number – defined on a surface in momentum space characterizes the total charge of the monopoles it encloses. These monopoles give rise to many unusual features, such as chiral edge states and associated surface Fermi arcs [14]. Topological protection guarantees that the only way to induce a gap is either to annihilate two Weyl points of opposite charge by bringing them together, or to couple two cones at a finite distance in momentum space through breaking of momentum conservation [16].

Here, we demonstrate Weyl singularities in multi-terminal junctions with conventional s-wave superconducting leads, namely when the energy of the lowest ABS goes to zero at certain values of the superconducting phases such that the gap in the spectrum closes. Below, we also show that their topological property can be easily probed by the transconductance between two voltage-biased leads, which is proportional to the Chern number.

We consider a junction with n superconducting leads connected through a scattering region (see Fig. 20.1a). The leads $\alpha = \{0, 1, \dots, n - 1\}$ have the same gap Δ , though they may differ in the phase of the superconducting order parameter, ϕ_α . Due to gauge invariance, only $n - 1$ phases are independent, hence we may set $\phi_0 = 0$. Likewise we choose to focus on a short

scattering region, characterized by an energy- (and spin-) independent scattering matrix S in the basis of $N = \sum_{\alpha} N_{\alpha}$ transport channels, N_{α} being the number of channels in contact α . We also assume time-reversal symmetry, such that $S^T = S$.

The junction hosts a set of spin-degenerate Andreev bound states, indexed by k , whose energies $E_k \geq 0$ are determined from the equation [15]

$$\det \left[1 - e^{-2i\chi} S e^{i\hat{\phi}} S^* e^{-i\hat{\phi}} \right] = 0 \quad (20.1)$$

where $\chi = \arccos(E/\Delta)$, and $e^{i\hat{\phi}}$ is a diagonal matrix that assigns to each channel the phase factor of the corresponding terminal. The spin-degenerate ABS energies $E_k(\hat{\phi})$ (see Fig. 20.1b) are periodic in all phases with a period 2π . The total phase-dependent energy of the junction reads $E = \sum_{k\sigma} (n_{k\sigma} - 1/2) E_k$, $n_{k\sigma} = 0, 1$ being the occupation of the state k with spin σ .

If the gap closes at certain values of the phases, $\hat{\phi}^{(0)}$, the spectrum is doubly degenerate at $E = 0$. We can then describe the lowest energy band in the vicinity of the gap closing by the two-by-two Weyl Hamiltonian (Supplementary Discussion),

$$H_W = \sum_{i=x,y,z} h_i \tau_i \quad h_i = \sum_{\alpha} \delta\phi_{\alpha} M_{\alpha i} \quad (20.2)$$

with the Pauli matrices $\tau_{x,y,z}$ in the basis of the two degenerate states corresponding to eigenvalue $E = 0$. The fields h_i depend linearly on $\delta\hat{\phi} = \hat{\phi} - \hat{\phi}^{(0)}$ through the real matrix M .

The form of the Weyl Hamiltonian in Eq. (20.2) indicates that we need at least three parameters to tune the system to the degeneracy point, $h_i = 0$. Thus, for 4 terminals with 3 independent phases, the Weyl singularities appear as points in the 3D phase space. For 5 terminals, the Weyl singularities occur in general as 1D curves in the 4-dimensional space of phases. This opens up the possibility to realize a topological material in arbitrary dimensions. Note that such multi-terminal Josephson junctions cannot be characterized by means of the standard ‘‘periodic table’’ of topological semimetals [18, 19], due to the distinct behaviour of the quasimomenta $\hat{\phi}$ under particle-hole symmetry. A proper classification could be envisioned along the lines of Ref. [20].

The topology of the junction can be characterized by a set of Chern numbers. A Chern number may be defined in the 2D subspace of two phases ϕ_{α} and ϕ_{β} , through the local Berry curvature of the ABS. The Berry curvature for the bound state k with spin σ is related to its wave function $|\varphi_{k\sigma}\rangle$ through

$$B_k^{\alpha\beta} \equiv -2\text{Im} \left\langle \frac{\partial \varphi_{k\sigma}}{\partial \phi_{\alpha}} \left| \frac{\partial \varphi_{k\sigma}}{\partial \phi_{\beta}} \right. \right\rangle. \quad (20.3)$$

Note that $B_k^{\alpha\beta}$ does not depend on spin (Supplementary Discussion). The total Berry curvature of the many-body superconducting state then reads $B^{\alpha\beta} = \sum_{k\sigma} (n_{k\sigma} - 1/2) B_k^{\alpha\beta}$, in analogy to the expression for the energy. For fixed occupations $n_{k\sigma} = 0, 1$, the integral of the Berry curvature over the elementary cell yields an integer, the Chern number $C^{\alpha\beta} = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} d\phi_{\alpha} d\phi_{\beta} B^{\alpha\beta} / (2\pi)$.

Since the Weyl singularities appear as points in the 3D phase space, a third phase ϕ_{γ} may be used to tune the system through Weyl points, thus changing the Chern number. We see that a given state k only contributes to $B^{\alpha\beta}$ when it has an even occupation number, $n_{k\uparrow} = n_{k\downarrow}$. The topological signature is therefore robust provided the fermion parity is preserved, a fact which has also been pointed out for other topological systems realized in superconductors [9]. Quasiparticle poisoning, by which a single quasiparticle enters a bound state, enables transitions between even and odd ABS occupation [27], and can therefore affect the signal. In the following,

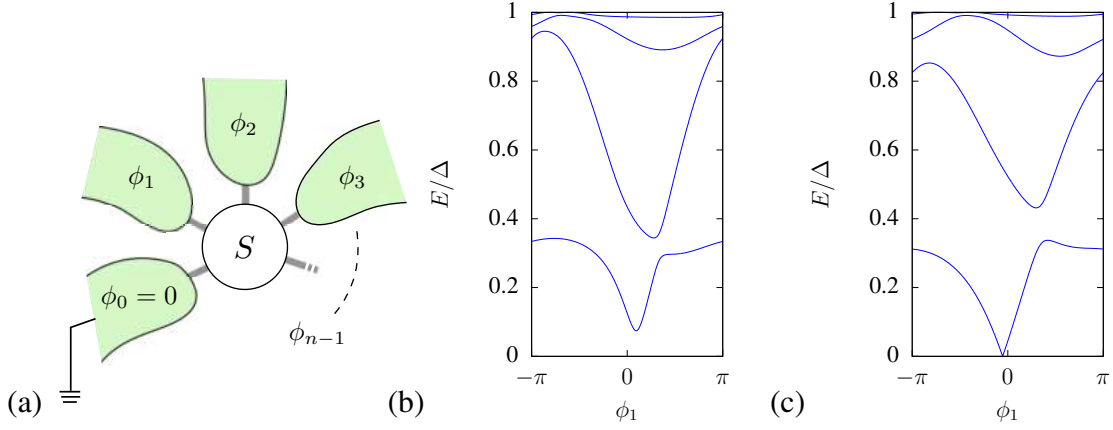


Figure 20.1: **General setup of the multi-terminal junction, and examples of typical ABS spectra.** a) The superconducting leads with phases ϕ_α , $\alpha = 0, \dots, n-1$, are connected through a scattering region described by the scattering matrix S . b) Generic ABS energy spectrum versus ϕ_1 , away from a Weyl singularity. c) ABS energy spectrum versus ϕ_1 , where the other phases are tuned to a Weyl singularity. Note the gap closing.

we focus on the ground state, $n_{k\sigma} = 0$ for all k and σ , where the Chern number may be nontrivial.

Importantly, the current response of the junction with slowly varying phases reveals the Chern number. Biasing lead β with a voltage $eV_\beta \ll \Delta$ gives rise to the instant current to contact α (Supplementary Discussion)

$$I_\alpha(t) = \frac{2e}{\hbar} \frac{\partial E}{\partial \phi_\alpha} - 2e \dot{\phi}_\beta B^{\alpha\beta}, \quad (20.4)$$

where $\dot{\phi}_\beta = 2eV_\beta/\hbar$. The first term corresponds to the adiabatic current and the second term is the first order correction in the phase velocity. Let us now apply constant voltages to two leads. For incommensurate voltages, the two phases uniformly sweep the elementary cell. In the dc limit, the adiabatic current contribution then averages out, and the Berry curvature is replaced by its average value. Thus, we find that the dc current is linear in the voltages, and the transconductance is defined by the Chern number

$$\bar{I}_\alpha = G^{\alpha\beta} V_\beta \quad \text{with} \quad G^{\alpha\beta} = -\frac{2e^2}{\pi\hbar} C^{\alpha\beta}. \quad (20.5)$$

Equation (20.5) shows that multi-terminal junctions exhibit a dc current response typical for the quantum Hall effect, although based on different physics. The transconductance quantum is four times bigger than in the quantum Hall effect, which can be traced to the $2e$ charge of the superconducting Cooper pairs. To extract the small dc signal, the averaging time needs to be sufficiently long. The relevant time scale is determined by the low-frequency current noise (Supplementary Discussion). Note that quantum Hall-like conductance quantization has also been proposed in superconducting devices with finite charging energy and hosting quantum phase slips [22]. Furthermore, superconducting junctions with a gate-tunable charging energy may realize topologically-protected discrete charge pumping [53].

We now focus on a 4-terminal junction and investigate the energy spectrum as a function of the three independent phases $\phi_{1,2,3}$. As mentioned above, such a 3D bandstructure may host Weyl points with positive or negative topological charge. The Nielsen-Ninomiya theorem [16]

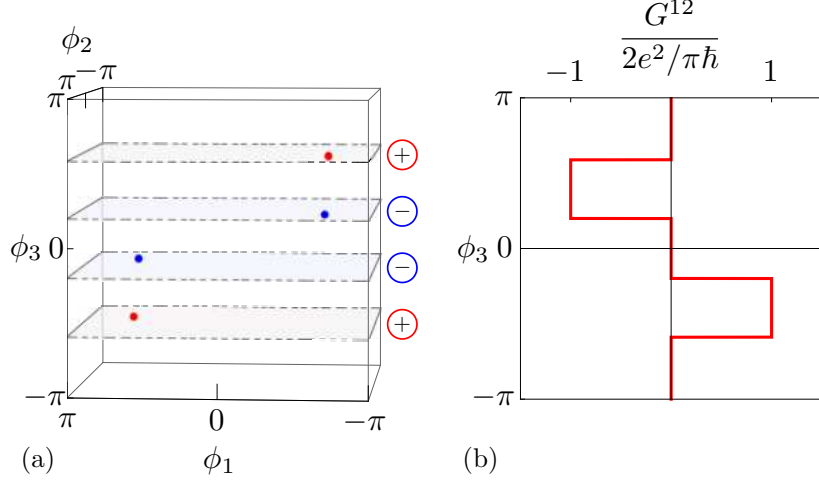


Figure 20.2: **Topological characterization of the 4-terminal junction for the single-channel case.** a) Position of the four Weyl points in the space of $\phi_{1,2,3}$ of the single-channel 4-terminal junction, the colour code indicating the respective charge. b) The resulting transconductance G^{12} indicating the Chern number, as a function of ϕ_3 for the same single channel junction as in panel a).

implies that the total topological charge of the system is zero, such that the number of Weyl points is always even. Furthermore, time-reversal invariance corresponds to a mapping from $\hat{\phi}$ to $-\hat{\phi}$, hence a Weyl point at $\hat{\phi}^{(0)}$ has a counterpart at $-\hat{\phi}^{(0)}$ with the same topological charge. Thus, Weyl points come in groups of 4.

In the simplest case, where each contact contains only one channel, the system may realize 0 or 4 Weyl points, corresponding to a topologically trivial or nontrivial 3D material, respectively. If a scattering matrix yielding Weyl points is found, small changes in the scattering matrix only modify their position, but cannot gap them. Namely, as the Weyl points carry a topological charge, individual Weyl points are stable and annihilation is possible only when two Weyl points with opposite charges coincide.

A specific example is shown in Fig. 20.2. The position and charge of the 4 Weyl points is shown in Fig. 20.2a. Without loss of generality, we fix the phase ϕ_3 and compute the transconductance G^{12} between voltage-biased contacts 1 and 2. In Fig. 20.2b, one can clearly see that the transconductance increases (decreases) by $2e^2/(\pi\hbar)$ when ϕ_3 passes through a Weyl point with positive (negative) charge. Interpreting ϕ_3 as a control parameter rather than a quasimomentum, we thus see that the 2D bandstructure of the system as a function of ϕ_1 and ϕ_2 undergoes a topological transition when ϕ_3 passes through a Weyl point. The transconductance directly measures the Chern number characterizing the corresponding 2D topological phase. Note that the transconductance satisfies the relation $G^{12}(-\phi_3) = -G^{12}(\phi_3)$ due to time-reversal symmetry.

By randomly generating scattering matrices from the circular orthogonal ensemble [S6], we find that about 5% of scattering matrices give rise to 4 Weyl points (Supplementary Discussion). More Weyl points can be obtained in multi-channel junctions where the maximal number of Weyl points is roughly proportional to the number of channels, and the probability to have no Weyl points is small. As a consequence, a greater variety of 2D topological phases with higher Chern numbers can be realized in that case. This is shown in Fig. 20.3 for a multi-channel junction hosting 36 Weyl points, where the maximal Chern number is 3. Recently realized few-channel cross junctions [26] are promising to observe the transconductance in a 4-terminal junction.

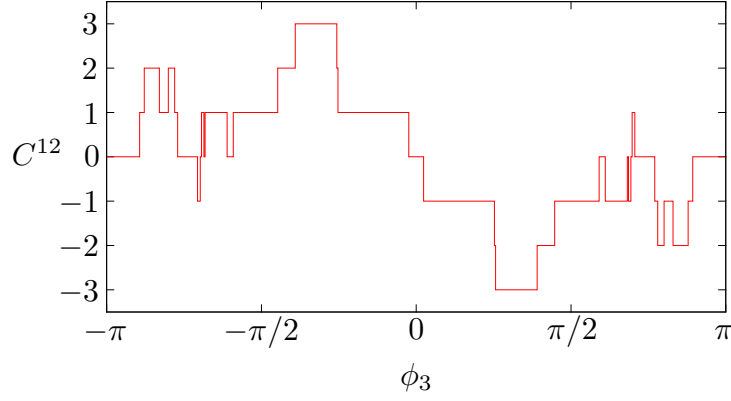


Figure 20.3: **Topological characterization of the 4-terminal junction for the multi-channel case.** Chern number as a function of ϕ_3 for a multi-channel 4-terminal junction, where the contacts 1, 2, 3, and 0 contain 12, 11, 10, and 9 channels, respectively. In this particular example, the junction hosts 36 Weyl points.

We now turn to 5-terminal junctions. In that case, the Weyl singularities appear as closed loops in the 4D space of phases. The simplest way to visualize them is to consider the additional phase ϕ_4 as a tuning parameter of the 3D system described by the phases (ϕ_1, ϕ_2, ϕ_3) . Tuning ϕ_4 the Weyl points move, but remain at zero energy. Note that in the 3D subspace, time-reversal symmetry is effectively broken, as for a fixed nonzero ϕ_4 , a Weyl point at $(\phi_1^{(0)}, \phi_2^{(0)}, \phi_3^{(0)})$ does not have a counterpart at $-(\phi_1^{(0)}, \phi_2^{(0)}, \phi_3^{(0)})$ anymore. The only constraint at a finite ϕ_4 is that the number of Weyl points is even. Once two Weyl points with opposite charge meet, they annihilate. Thus their trajectories describe closed loops in the 4D space of all phases.

As before, the presence of Weyl singularities may be probed by the transconductance between two voltage-biased terminals, say terminal 1 and 2. As a function of the two other phases ϕ_3 and ϕ_4 , we now obtain areas with different quantized values of the transconductance, corresponding to different Chern numbers. The boundaries of these areas are given by the projections of the Weyl loops on the (ϕ_3, ϕ_4) -plane. An example is shown in Fig. 20.4. Here time-reversal symmetry of the scattering matrix manifests itself in the relation $G^{12}(-\phi_3, -\phi_4) = -G^{12}(\phi_3, \phi_4)$.

The above considerations are straightforwardly extended to a larger number of terminals. Furthermore, as we saw for the 5-terminal junction, breaking time-reversal invariance does not lift the topological protection of the Weyl points at zero energy, as long as spin degeneracy is preserved. When spin degeneracy is lifted due to a Zeeman field or due to spin-orbit interactions, we expect the Weyl points to shift away from zero energy while remaining stable. The possibility of realizing zero-energy states in 3-terminal junctions with strong spin-orbit interaction was studied in Ref. [27].

To summarize, we predict the existence of topological Weyl singularities in the ABS spectrum of n -terminal superconducting junctions with $n \geq 4$. These Weyl singularities manifest themselves in a quantized transconductance between two voltage-biased contacts, when the remaining phases are tuned away from the singularities. Tuning the system through a Weyl singularity, the conductance displays a step, signaling a topological transition of the 2D system described by the two phases of the voltage-biased contacts. As with all topological properties, this signature is robust provided the system remains in its ground state. Thus, the quantized transconductance should be easily accessible experimentally at low temperatures and voltages.

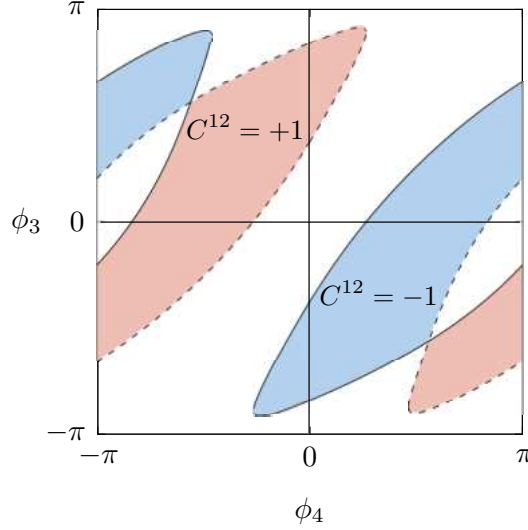


Figure 20.4: **Topological characterization of a 5-terminal junction, each contact having a single channel.** The colored areas display a nonzero Chern number C^{12} . The boundaries of these areas correspond to the projection of the Weyl singularity lines to the (ϕ_3, ϕ_4) -plane. In a 3D subspace the curves can be assigned a charge.

20.1 Supplementary information Multi-terminal Josephson junctions as topological materials

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20.1.1 Derivation of the Weyl Hamiltonian

In the following, we provide the derivation of the Weyl Hamiltonian, Eq. (2) of the main text, and associated properties of the Andreev bound state (ABS) spectrum.

The topological properties of multi-terminal junctions result from the appearance of Weyl singularities in the ABS spectrum. The ABS for an n -terminal superconducting junction are determined through [15]

$$|\psi\rangle = S_N S_A |\psi\rangle \quad \text{with} \quad S_N = \begin{pmatrix} S & 0 \\ 0 & S^* \end{pmatrix} \quad \text{and} \quad S_A = \begin{pmatrix} 0 & e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix} e^{-i\chi}. \quad (20.6)$$

Here, the two-by-two structure indicates the Nambu space of electrons and holes. The scattering matrix S_N describes the normal metal scattering region, where S and S^* provide the scattering amplitudes for electrons and holes, respectively. The matrix S_A accounts for the phase acquired in the Andreev reflection processes, where $\chi = \arccos(E/\Delta)$ and $e^{i\hat{\phi}}$ is a diagonal matrix that assigns to each channel the phase of the corresponding terminal. Note that the wave function $|\psi\rangle$ is written in terms of the outgoing states in a given spin sector $\sigma = \uparrow, \downarrow$.

To determine the ABS energy bands, it is sufficient to reduce the above equation to the electron subspace, where we find the determinant condition for the bound state energies E as

$$\det \left[1 - e^{-2i\chi} A(\hat{\phi}) \right] = 0 \quad (20.7)$$

with the unitary matrix $A(\hat{\phi}) = S e^{i\hat{\phi}} S^* e^{-i\hat{\phi}}$. The matrix A possesses the particle-hole symmetry $A^* = U^\dagger A U$, where $U(\hat{\phi}) = e^{i\hat{\phi}} S^T$. This implies that the eigenvalues of A come in pairs [18],

$e^{\pm ia_k}$, corresponding to energies $\pm E_k$, where $E_k = \Delta \cos(a_k/2)$ with $0 \leq a_k \leq \pi$. We assign eigenvectors $|\Psi_k^+\rangle$ and $|\Psi_k^-\rangle = U|\Psi_k^+\rangle^*$ to the pair of eigenvalues $e^{\pm ia_k}$, respectively.

If there is a zero-energy solution at $\hat{\phi}^{(0)}$, the spectrum of the unitary matrix $A(\hat{\phi}^{(0)})$ has a doubly degenerate eigenvalue -1 . The corresponding orthogonal eigenvectors are given as $|a^+\rangle = |\Psi_0^+(\hat{\phi}^{(0)})\rangle$ and $|a^-\rangle = |\Psi_0^-(\hat{\phi}^{(0)})\rangle$. In the vicinity of the singularity at $\hat{\phi}^{(0)}$, we expand the determinant equation (20.7) for small $\delta\hat{\phi} = \hat{\phi} - \hat{\phi}^{(0)} \ll 1$ and $E \ll \Delta$. Up to first order, we find $e^{i2\chi} \approx -1 + 2iE/\Delta$ and

$$A(\hat{\phi}) \approx A(\hat{\phi}^{(0)}) + iS\delta\hat{\phi}S^\dagger A(\hat{\phi}^{(0)}) - iA(\hat{\phi}^{(0)})\delta\hat{\phi}. \quad (20.8)$$

Projecting Eq. (20.7) onto the subspace defined by $|a^+\rangle$ and $|a^-\rangle$, and keeping only the lowest-order terms, we find the determinant equation

$$\det[E - H_W] = 0. \quad (20.9)$$

It defines the eigenvalue problem for the lowest band, described by the two-by-two Weyl Hamiltonian,

$$H_W = \frac{\Delta}{2} \begin{pmatrix} \langle a^+ | \delta\hat{\phi} - S\delta\hat{\phi}S^\dagger | a^+ \rangle & \langle a^+ | \delta\hat{\phi} - S\delta\hat{\phi}S^\dagger | a^- \rangle \\ \langle a^- | \delta\hat{\phi} - S\delta\hat{\phi}S^\dagger | a^+ \rangle & \langle a^- | \delta\hat{\phi} - S\delta\hat{\phi}S^\dagger | a^- \rangle \end{pmatrix}. \quad (20.10)$$

Using the particle-hole symmetry $|a^-\rangle = U(\hat{\phi}^{(0)})|a^+\rangle^*$, Eq. (20.10) may be simplified to

$$H_W = \sum_{i=x,y,z} h_i \tau_i \quad (20.11)$$

where $\tau_{x,y,z}$ are Pauli matrices in the basis of $\{|a^+\rangle, |a^-\rangle\}$, and the real prefactors h_i take the form $h_x + ih_y = \Delta \langle \delta\hat{\phi} \rangle_{-+}$ and $h_z = \frac{\Delta}{2} (\langle \delta\hat{\phi} \rangle_{++} - \langle \delta\hat{\phi} \rangle_{--})$. As a consequence, the energy spectrum is linear in $\delta\hat{\phi}$ and reads $E = \pm \sqrt{h_x^2 + h_y^2 + h_z^2}$. We see that 3 independent parameters are necessary to tune the energy to zero. Thus, the Weyl singularities in general appear as points in the space of three independent phases of the 4-terminal junction, and as curves in the space of four independent phases of the 5-terminal junction.

In a given 3D subspace, one may assign a charge to each Weyl point. The sign of a Weyl charge is most conveniently defined by rewriting the Weyl Hamiltonian in matrix form, $H_W = \sum_{\alpha i} \delta\phi_\alpha M_{\alpha i} \tau_i$ with $M_{\alpha i} = \partial_{\delta\phi_\alpha} h_i$. Then, in a subspace of three independent phases, M is a square matrix, and the sign of the charge is given by the sign of the determinant, $\det[M_{\alpha i}]$.

While particle-hole symmetry is local in $\hat{\phi}$, time-reversal symmetry links $\hat{\phi}$ and $-\hat{\phi}$. As a consequence, a junction with time-reversal symmetry gives rise to pairs of Weyl singularities at $\hat{\phi}^{(0)}$ and $-\hat{\phi}^{(0)}$. Indeed, using $S^T = S$, we find the relation $A(-\hat{\phi}) = e^{-i\hat{\phi}} A^\dagger(\hat{\phi}) e^{i\hat{\phi}}$. Using this relation, we find that the Weyl Hamiltonians near $\hat{\phi}^{(0)}$ and $-\hat{\phi}^{(0)}$ take the same form and hence, in a given 3D subspace, the corresponding Weyl points have the same charge.

We note that there is the possibility of Weyl singularities at finite energy. Due to particle-hole symmetry, they appear in pairs at energies $\pm E$ with the same charge. Because of particle-hole symmetry and spin degeneracy, solutions at energy $-E$ can be ascribed to solutions at energy E in the opposite spin sector [23]. Thus, zero-energy Weyl points are singly degenerate, while finite energy Weyl points are doubly degenerate. As the finite energy Weyl singularities do not affect the ground state Chern numbers used to characterize the system, we do not discuss them any further. Note that the outgoing wavefunctions in particle-hole space used to compute the Chern numbers are given as $|\psi_k^\pm\rangle = (|\Psi_k^\pm\rangle, e^{-i\chi} U^\dagger |\Psi_k^\pm\rangle)^T$. Furthermore, $|\psi_{k\sigma}\rangle$ is identified with $|\psi_k^+\rangle$ in the corresponding spin sector σ .

20.1.2 Derivation of the current

In the following, we establish the connection between the current and the Berry curvature for phases that change slowly in time.

The current operator through lead α is defined as

$$\hat{I}_\alpha = 2e \frac{\partial \hat{H}}{\partial \phi_\alpha} \quad (20.12)$$

where $\hat{H}$ is the Bogoliubov-de Gennes Hamiltonian. In order to calculate its expectation value for time-dependent phases $\hat{\phi}(t)$, we introduce the basis of instantaneous wave functions of the time-dependent Bogoliubov-de Gennes Hamiltonian $\hat{H}(t)$, such that $E_k(t)|\varphi_{k\sigma}(t)\rangle = \hat{H}(t)|\varphi_{k\sigma}(t)\rangle$. Solving the time-dependent equation $i\hbar|\dot{\varphi}\rangle = \hat{H}(t)|\varphi\rangle$ in that basis, up to first order in phase velocity $\dot{\phi}$, we obtain the current contribution from state k with spin σ as

$$I_{\alpha k}(t) \approx 2e \left[\frac{1}{\hbar} \frac{\partial E_k(t)}{\partial \phi_\alpha} - i \frac{\partial \langle \varphi_{k\sigma}(t) |}{\partial \phi_\alpha} |\dot{\varphi}_{k\sigma}(t)\rangle + i \langle \dot{\varphi}_{k\sigma}(t) | \frac{\partial |\varphi_{k\sigma}(t)\rangle}{\partial \phi_\alpha} \right]. \quad (20.13)$$

Note that $I_{\alpha k}$ does not depend on spin. The first term corresponds to the adiabatic supercurrent. Introducing the Berry curvature $B_k^{\alpha\beta} = -2\text{Im} [\partial_{\phi_\alpha} \langle \varphi_{k\sigma} | \partial_{\phi_\beta} | \varphi_{k\sigma} \rangle]$, the second term reads $-2e \sum_\beta \dot{\phi}_\beta B_k^{\alpha\beta}$. The many-body expectation value of the current is computed as

$$I_\alpha(t) = \sum_{k\sigma} I_{\alpha k}(t) \left[n_{k\sigma} - \frac{1}{2} \right] \quad (20.14)$$

where $n_{k\sigma} = 0, 1$ is the occupation of the ABS k with spin σ .

Equations (20.13) and (20.14) can be used to compute the adiabatic correction to the current, if the energy spectrum is discrete and the occupations of each state do not change with time. For a few-channel junction, the level spacing between the ABS scales as $\sim \Delta$, thus providing the adiabaticity condition $\hbar\dot{\phi} \ll \Delta$. On the other hand, defining the current contribution of the states in the continuum above the gap, as well as of ABS that reach the gap edge at some values of the phases, is problematic. In particular, adiabaticity would be broken at any phase velocity for the states in the continuum, while ABS that reach the gap edge may lead to stricter adiabaticity conditions (see next section). In the following, we argue that those states do not contribute to the topological properties considered in our work.

Indeed, multi-terminal junctions are known to be topologically trivial in certain cases. For instance, let us consider the case where terminals α and β are voltage biased. When all the other phases ϕ_γ ($\gamma \neq \alpha, \beta$) are set to zero, the junction is effectively a 3-terminal junction which is topologically trivial. In that case, the associated Chern number $C^{\alpha\beta}$ vanishes. A change of $C^{\alpha\beta}$, corresponding to a topological transition, may occur when a state crosses the Fermi level as the phases ϕ_γ are varied. As the energy spectrum is gapped, this can only occur for the lowest energy states. Thus, only those states contribute to the topological properties, and in the following we only retain their contribution in Eqs. (20.13) and (20.14). This establishes the regime of validity for the relation between the transconductance and Chern number given in the main text, see Eq. (5). To compute the Chern numbers, we use the outgoing wavefunctions $|\psi_{k\sigma}\rangle$ defined in the previous section [19].

20.1.3 Comment on Andreev bound states reaching the gap edge

Let us return to the possibility of the ABS with the highest energy reaching the gap edge. Indeed we find that, for a system with an even number of total channels, this occurs along $(n - 1)$ -dimensional subspaces in the space of n independent phases [48].

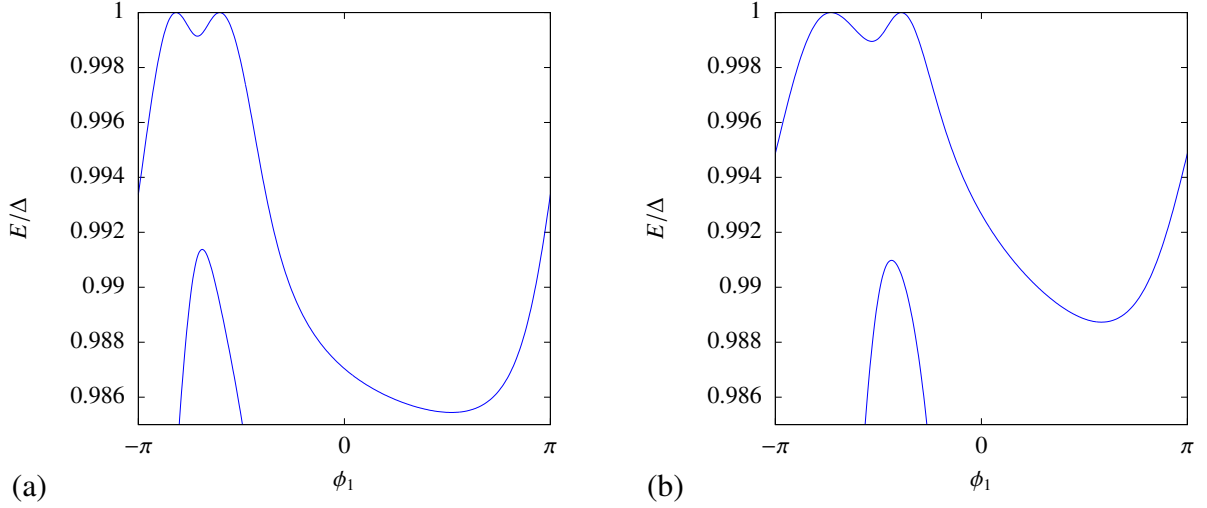


Figure 20.5: **Zoom on the highest ABS of the spectra shown in Figs. 1b and 1c in the main text.** The gap edge touchings are clearly visible.

An argument for this fact can be given in analogy with the derivation of the Weyl Hamiltonian. Namely, an ABS reaching the gap edge at some $\hat{\phi}_g$ requires that the unitary matrix $A(\hat{\phi}_g)$ has a doubly degenerate eigenvalue $+1$. We denote the corresponding orthogonal eigenvectors $|b^+\rangle$ and $|b^-\rangle = U(\hat{\phi}_g)|b^+\rangle^*$. Expanding Eq. (20.7) for small $\delta\hat{\phi} = \hat{\phi} - \hat{\phi}_g$, we can reduce the determinant equation to

$$\det \left[\text{sign}(E) \sqrt{\frac{8(\Delta - |E|)}{\Delta}} + \left(\langle \delta\hat{\phi} \rangle_{++} - \langle \delta\hat{\phi} \rangle_{--} \right) \tilde{\tau}_z \right] = 0 \quad (20.15)$$

where $\tilde{\tau}_z$ is a Pauli matrix in the basis $\{|b^+\rangle, |b^-\rangle\}$.

Thus, the solution for the ABS energies reads

$$E \approx \pm \Delta \left[1 - \frac{1}{8} \left(\langle \delta\hat{\phi} \rangle_{++} - \langle \delta\hat{\phi} \rangle_{--} \right)^2 \right]. \quad (20.16)$$

Furthermore, we see that ABS wave function has a discontinuity at $\hat{\phi}_g$. Namely, the wave function of the state at positive energy is given by $|b^+\rangle$ for $\langle \delta\hat{\phi} \rangle_{++} - \langle \delta\hat{\phi} \rangle_{--} < 0$ and by $|b^-\rangle$ for $\langle \delta\hat{\phi} \rangle_{++} - \langle \delta\hat{\phi} \rangle_{--} > 0$, and vice versa for the states at negative energies.

In contrast to the discussion for the zero-energy Weyl points, here a single parameter is sufficient to tune the ABS energy to the gap edge. Therefore, the highest ABS reaches the gap edge along $(n-1)$ -dimensional subspaces in the space of n independent phases. This is visible in Figs. 1b and 1c in the main text. A zoom is shown in Fig. 20.5.

When sweeping two phases, the curves where the highest ABS reaches the continuum cannot be avoided. If this state was occupied, this would break adiabaticity even for arbitrary slow driving. (Introducing a finite energy-dependence of the scattering matrix or spin-orbit interactions may create a finite level repulsion between the highest ABS and the continuum and, thus, restore adiabaticity, possibly under stricter conditions. This would need further investigation.) However, as far as the topological characterization of the junction is concerned, we came to the conclusion that this peculiarity should not change our results.

20.1.4 Estimation of the current noise

The topological signal of interest is a small dc contribution on top of a large ac adiabatic current. This makes it important to provide an estimate for the measurement time T_0 required

to sufficiently average the current signal. In particular, the lower bound for T_0 is provided by $S_I/|\bar{I}|^2$, where $\bar{I}$ denotes the dc current signal and S_I is the low-frequency current noise. The dominant contribution to S_I stems from fluctuations of the phases due to voltage noise and will be estimated in the following.

We define the current noise as

$$S_I = \int d\tau \langle I(t)I(t+\tau) \rangle \quad (20.17)$$

where the brackets $\langle \dots \rangle$ denote the expectation value with respect to the statistical ensemble. As the adiabatic supercurrent, i.e., the first term in Eq. (20.14), is much larger than the correction proportional to the Berry phase, it dominates the noise. We exploit the fact that this current is periodic in both phases ϕ_α and ϕ_β , such that it may be expanded in Fourier harmonics,

$$I(\phi_\alpha, \phi_\beta) = \sum_{n_\alpha, n_\beta=-\infty}^{\infty} I_{n_\alpha, n_\beta} e^{i(n_\alpha \phi_\alpha + n_\beta \phi_\beta)} . \quad (20.18)$$

Substituting Eq. (20.18) into Eq. (20.17), we obtain

$$S_I = \int d\tau \sum_{n_\alpha, n_\beta} |I_{n_\alpha, n_\beta}|^2 \langle e^{in_\alpha[\phi_\alpha(t)-\phi_\alpha(t+\tau)]} \rangle \langle e^{in_\beta[\phi_\beta(t)-\phi_\beta(t+\tau)]} \rangle \quad (20.19)$$

as the two phases fluctuate independently and the ac contribution averages to zero.

In order to evaluate the averages, we assume white noise in the voltage sources, described by

$$\frac{2e^2}{\hbar^2} \langle \delta V_\alpha(t) \delta V_\alpha(t') \rangle = \Gamma_\alpha \delta(t-t') \quad (20.20)$$

with the rate Γ_α associated to the voltage noise in lead α . Using $\phi_\alpha(t+\tau) - \phi_\alpha(t) = \omega_\alpha \tau + (2e/\hbar) \int_t^{t+\tau} ds \delta V_\alpha(s)$, where $\omega_\alpha \equiv 2eV_\alpha/\hbar$, we find

$$\langle e^{in_\alpha[\phi_\alpha(t)-\phi_\alpha(t+\tau)]} \rangle = e^{-in_\alpha \omega_\alpha \tau - \Gamma_\alpha n_\alpha^2 |\tau|} \quad (20.21)$$

and subsequently

$$S_I = 2 \sum_{n_\alpha, n_\beta} |I_{n_\alpha, n_\beta}|^2 \frac{\Gamma_\alpha n_\alpha^2 + \Gamma_\beta n_\beta^2}{(n_\alpha \omega_\alpha + n_\beta \omega_\beta)^2 + (\Gamma_\alpha n_\alpha^2 + \Gamma_\beta n_\beta^2)^2} . \quad (20.22)$$

We see that S_I depends on frequencies $\omega \sim \omega_{\alpha,\beta}$ in a quite complex fashion. In the limit of low voltage noise, $\omega \gg \Gamma$ with $\Gamma \sim \Gamma_{\alpha,\beta}$, we have to distinguish between commensurate and incommensurate voltages $V_{\alpha,\beta}$. In the commensurate case, the first term in the denominator of Eq. (20.22) vanishes for a pair (n_α, n_β) , yielding $S_I \propto 1/\Gamma$. By contrast, in the incommensurate case, the first term in the denominator of Eq. (20.22) dominates, yielding a much weaker noise, $S_I \propto \Gamma/\omega^2$. Thus, the current noise strongly varies with ω . Averaging over a window of width $\sim \omega$ yields the average noise $\overline{S_I} \propto 1/\omega$. On the other hand, in the noisy regime $\omega \ll \Gamma$, we find that $S_I \propto 1/\Gamma$, irrespective of the voltages being commensurate or not. Thus, the current noise is actually weaker than the average current noise in the opposite regime. Furthermore, incommensurability is no longer required for averaging, as the strong phase fluctuations take care of covering the entire unit cell.

This brings us to the conclusion that the strong voltage noise regime $\omega \ll \Gamma$ may be favourable for averaging – provided, of course, that the noise is still in the limit $\hbar\Gamma \ll \Delta$. To estimate the measurement time, we use $I_{n_\alpha, n_\beta} \sim e\Delta/\hbar$ and $\bar{I} \sim e^2 V/\hbar$ with $V \sim V_{\alpha,\beta}$, together with the estimate for S_I found above, to obtain

$$T_0 > \frac{\Delta^2}{(eV)^2} \frac{1}{\Gamma} . \quad (20.23)$$

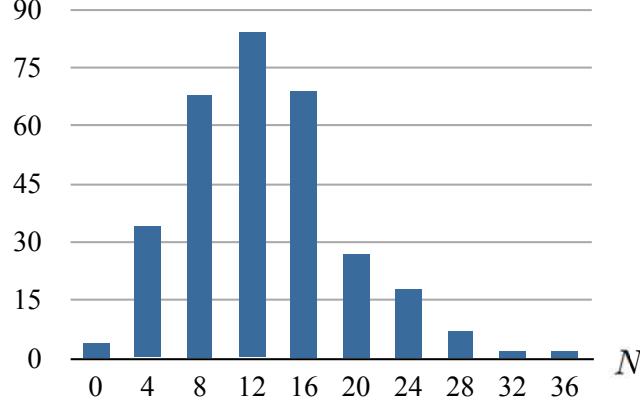


Figure 20.6: **Histogram displaying the occurrence of random scattering matrices yielding N Weyl points for the four-terminal multi-channel junction.**

20.1.5 Occurrence of Weyl points for 4-terminal junctions

Here, we estimate how often random scattering matrices give rise to Weyl points at zero energy.

As discussed in the first section of the supplementary, such Weyl singularities emerge when the matrix A has the eigenvalue -1 . Whether this occurs indeed depends on the properties of the junction, encoded in the scattering matrix S . We performed a numerical analysis for 4-terminal junctions, both in the single-channel and multi-channel cases. For this purpose, we generated symmetric scattering matrices according to the circular orthogonal ensemble using the following recipe [S6]: We randomly generated hermitian matrices $\mathcal{H}$ from a Gaussian ensemble, and numerically diagonalized them, $\mathcal{H} = \mathcal{U}^\dagger \mathcal{D} \mathcal{U}$, where $\mathcal{U}$ is a unitary matrix and $\mathcal{D}$ is a real diagonal matrix. A random symmetric scattering matrix was then generated as $S = \mathcal{U}^T \mathcal{U}$. For each S , we numerically checked the existence of zeros of the function $\left| \det \left[1 + A(\hat{\phi}) \right] \right|$ in the space of phases.

In the single-channel case, we ran the check for 965 randomly generated matrices S , out of which 46 gave rise to zero-energy solutions. Thus, we found that a total of roughly 5% of all scattering matrices yield Weyl points, while the remaining 95% provide a trivial junction.

When increasing the number of channels in each terminal, we found that the maximal number of Weyl points scales with the number of channels, and that the probability of a junction without Weyl points decreases significantly. A total of 324 random scattering matrices were generated for a junction with four terminals, where the terminals have 12, 11, 10, and 9 channels, respectively. In Fig. 20.6 we show the histogram displaying the occurrence of randomly generated scattering matrices that provide N zero-energy Weyl points. Only 4 scattering matrices gave rise to a junction without zero-energy Weyl points. Note that our algorithm has a small, but finite probability to miss some zeros. As a consequence, while the number of Weyl points has to be a multiple of 4 because of time-reversal symmetry (see main text), the algorithm misidentifies a few cases as having an odd multiple of 2 Weyl points.

Part VII

Special Effects: Selected Useful Theories

21 Josephson radiation in a superconductor-quantum dot-superconductor junction by Lamic, Meyer, Houzet

21.1 Theory

We investigate the Josephson radiation emitted by a junction made of a quantum dot coupled to two conventional superconductors. Close to resonance, the particle-hole symmetric Andreev states that form in the junction are detached from the continuum above the superconducting gap in the leads, while a gap between them opens near the Fermi level. Under voltage bias, we formulate a stochastic model that accounts for non-adiabatic processes, which change the occupations of the Andreev states. This model allows calculating the current noise spectrum and determining the Fano factor. Analyzing the finite-frequency noise, we find that the model may exhibit either an integer or a fractional AC Josephson effect, depending on the bias voltage and the size of the gaps in the Andreev spectrum. Our results assess the limitations in using the fractional Josephson radiation as a probe of topology.

The Josephson radiation is the electromagnetic signal emitted by a junction between two superconductors when it is voltage biased. Its measurement is the first experimental demonstration of the AC Josephson effect [1]. The charge of the Cooper pairs, which form the superconducting condensate, appears in the relation between the radiation frequency and voltage bias, $\omega = \omega_J$, where $\omega_J = 2eV/\hbar$ is the Josephson frequency. The coherence property of the Josephson radiation is usually limited by the electromagnetic environment of the junction. Tailoring the environment surrounding a superconducting tunnel junction recently allowed reaching the regime of Josephson lasing [2].

The interest in the Josephson radiation was revived by the prediction of a fractional AC Josephson effect at frequency $\omega = \omega_J/2$ when the superconducting leads forming the junction are topological [8, 4, 5, 6]. This fractional Josephson radiation originates from the fact that the supercurrent flowing through the junction is carried by single electrons, rather than Cooper pairs. A topological junction admits two degenerate parity sectors, carrying Josephson supercurrents with opposite values. Therefore, random parity switchings generate current noise, and eventually bring another limitation to the coherence of the Josephson radiation in the topological case [9].

On a microscopic level, the Josephson effect can be associated with the formation of subgap states, also known as Andreev bound states (ABS), in the junction. The difference between conventional and topological Josephson junctions is associated with the fact that, in equilibrium, the ABS energy depends 2π -periodically on the superconducting phase difference φ in the conventional case, while the dependence is 4π -periodic in the topological case. Then the integer or fractional Josephson radiations simply result from the substitution $\dot{\varphi} = \omega_J$ in the phase dependence of the current carried by an ABS whose occupation is fixed. However, such considerations neglect the voltage-induced non-adiabatic processes that change the ABS occupations and, therefore, reduce the coherence of the Josephson radiation, even when the effect of the external environment is negligible. Indeed, it was shown that such non-adiabatic processes in topological junctions introduce an intrinsic limitation to the visibility of the fractional Josephson radiation [8, 42, 10]. This mechanism is ultimately related with the dissipative current that flows through the junction, which is induced by these processes.

In this work, we investigate the role of non-adiabatic processes on the Josephson radiation in

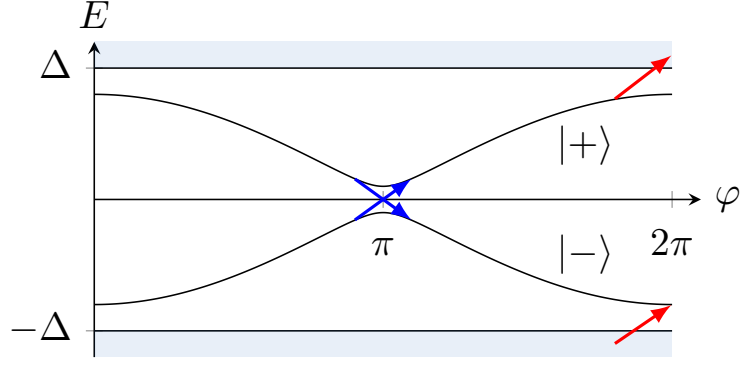


Figure 21.1: Spectrum of the Andreev bound states as a function of the superconducting phase difference. In the presence of a voltage bias, the phase increases with time and non-adiabatic transitions may occur between states $|+\rangle$ and $|-\rangle$ (blue arrows), as well as between state $|+\rangle$ and the continuum at energy $E > \Delta$ or between the continuum at energy $E < -\Delta$ and state $|-\rangle$ (red arrows).

a *non-topological* junction formed by a quantum dot connected to conventional superconducting leads. We find that such conventional junctions may display either a conventional or a fractional Josephson radiation as the voltage varies, depending on details of the ABS spectrum. In particular, a fractional Josephson radiation is predicted when the gap in the Andreev spectrum near the Fermi level is crossed diabatically, while the ABS are sufficiently detached from the continuum above the superconducting gap. Furthermore, we determine the contribution of the non-adiabatic transitions to the linewidth of the Josephson radiation. Our results assess the limitations in using the current noise spectrum to determine whether a Josephson junction is topological or not.

We consider a junction made of a spin-degenerate single-level quantum dot that is contacted to two superconducting leads. We first review the properties of its Andreev spectrum, which have been analyzed both in the presence and absence of Coulomb interaction [11]. Let us first start with the case of strong Coulomb interaction. It allows for resonant scattering of electrons between the leads in the normal state thanks to the Kondo effect. For energies smaller than the Kondo scale T_K , the transmission probability is $T_\pi = 4\Gamma_L\Gamma_R/\Gamma^2$, where $\Gamma = \Gamma_L + \Gamma_R$ and Γ_L and Γ_R are the partial level widths due to the coupling between the dot and the left and right leads. For an almost symmetric coupling, the transmission is almost ballistic, $R_\pi \equiv 1 - T_\pi \ll 1$. When the leads are superconducting and the superconducting gap is sufficiently small, $\Delta \ll T_K$, the junction accommodates two particle-hole symmetric ABS denoted $|+\rangle$ and $|-\rangle$. Their energies are approximated by

$$E_\pm(\varphi) = \pm\Delta\sqrt{T_0\cos^2(\varphi/2) + R_\pi\sin^2(\varphi/2)}. \quad (21.1)$$

In addition to the gap opening in the Andreev spectrum near $\varphi = \pi$ at $T_\pi < 1$ [12], Eq. (21.1) also shows the detachment of the ABS from the edge of the continuum spectrum near $\varphi = 0$ [13, 14], which is controlled by an effective reflection probability $R_0 \equiv 1 - T_0$ of order $(\Delta/T_K)^2 \ll 1$ (up to a logarithmic correction [15]). Interestingly, Eq. (21.1) also holds in the absence of interactions, after substituting T_K with $\Gamma \gg \Delta$ [15]. The ABS dispersion is illustrated in Fig. 21.1.

We turn now to the current flowing through the junction. Each occupied ABS carries a supercurrent

$$I_\pm(\varphi) = \frac{2e}{\hbar} \frac{\partial E_\pm(\varphi)}{\partial \varphi} \approx \mp I_J \sin \frac{\varphi}{2} \text{sign} \left(\cos \frac{\varphi}{2} \right), \quad (21.2)$$

where $I_J = e\Delta/\hbar$, and we used $R_0, R_\pi \ll 1$ in the last equality. In equilibrium, the average ABS occupations are set by the Fermi distribution, while the contribution of the continuum is negligibly small. Thus the equilibrium supercurrent at zero temperature is given by $I_-(\varphi)$.

In the presence of a dc voltage bias, the phase difference increases linearly with time, $\varphi(t) = \omega_J t + \varphi_0$ with a reference phase φ_0 . (Here we assume $V > 0$, for concreteness.) As a consequence, changes in the occupations of the ABS can occur due to non-adiabatic transitions. Using $R_0, R_\pi \ll 1$ and $V \ll \Delta/e$, we can isolate two kinds of non-adiabatic processes. Near $\varphi = \pi \bmod 2\pi$, these are the transitions between $|+\rangle$ and $|-\rangle$, which occur with the Landau-Zener probability $p_\pi = \exp(-\pi R_\pi \Delta/eV)$ [17]; p_π increases rapidly from 0 to 1 as V increases in the vicinity of $V_\pi = R_\pi \Delta/e$. Near $\varphi = 0 \bmod 2\pi$, these are non-adiabatic transitions between $|+\rangle$ and the continuum states with energy $E > \Delta$ as well as the continuum states with energy $E < -\Delta$ and $|-\rangle$, which take place with probability $p_0 = p(V/V_0)$ with $V_0 = R_0^{3/2} \Delta/e$, where the function $p(x)$ calculated in Ref. [18] (and Ref. [19] at $V \ll V_0$) is such that p_0 increases rapidly from 0 to 1 as V increases in the vicinity of V_0 .

We assume $T \ll \Delta$, so that continuum states with energy $E < \Delta$ ($E > \Delta$) are occupied (empty). We also neglect the short timescales over which the non-adiabatic processes take place on the scale of the Josephson period, $2\pi/\omega_J$. Then at each time, the state of the junction is fully characterized by the occupations $n_\pm = 0, 1$. (In particular, we ignore coherent superpositions between $|\pm\rangle$ -states.) The states $(0, 1)$ and $(1, 0)$ are the ground and first excited states in the even parity sector of the junction, respectively; the states $(0, 0)$ and $(1, 1)$ are the “poisoned” states in the odd parity sector. Within a Markov model that describes switches in their random occupations [20, 18], the average supercurrent is

$$\langle I(t) \rangle = \mathcal{I}(t) [P_{01}(\varphi(t)) - P_{10}(\varphi(t))]. \quad (21.3)$$

Here $\mathcal{I}(t) = I_-(\varphi(t))$ and $P_{n_+ n_-}(\varphi)$ with $n_+, n_- = 0, 1$ denotes the ABS occupations at a given phase. Neglecting any coupling with an external bath, these probabilities remain constant within intervals $\pi m < \varphi < \pi(m+1)$ with integer m , while their values immediately before and after the specific phases where non-adiabatic transitions can take place are related with each other through the transition probabilities p_0 and p_π ,

$$\mathbf{P}(2m\pi^+) = \mathcal{L}_0 \mathbf{P}(2m\pi^-), \quad (21.4a)$$

$$\mathbf{P}((2m+1)\pi^+) = \mathcal{L}_\pi \mathbf{P}((2m+1)\pi^-) \quad (21.4b)$$

with $\mathbf{P}(\varphi) = (P_{11}(\varphi), P_{10}(\varphi), P_{01}(\varphi), P_{00}(\varphi))^T$ and the transition matrices

$$\mathcal{L}_0 = \begin{pmatrix} 1-p_0 & p_0(1-p_0) & 0 & 0 \\ 0 & (1-p_0)^2 & 0 & 0 \\ p_0 & p_0^2 & 1 & p_0 \\ 0 & p_0(1-p_0) & 0 & 1-p_0 \end{pmatrix}, \quad (21.5a)$$

$$\mathcal{L}_\pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-p_\pi & p_\pi & 0 \\ 0 & p_\pi & 1-p_\pi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (21.5b)$$

The non-adiabatic processes accounted for by the matrix elements of $\mathcal{L}_0$ and $\mathcal{L}_\pi$ are illustrated in Fig. 21.2.

The stochastic matrix $\mathcal{L} \equiv \mathcal{L}_0 \mathcal{L}_\pi$ admits normalized right- and left-eigenvectors u_α and v_α , with a common eigenvalue λ_α , such that $\mathcal{L} u_\alpha = \lambda_\alpha u_\alpha$, $\mathcal{L}^T v_\alpha = \lambda_\alpha v_\alpha$, and $v_\alpha^T u_\beta = \delta_{\alpha\beta}$. Furthermore, the eigenvalue $\lambda_0 = 1$ is associated with the left eigenvector $v_0 = (1, \dots, 1)^T$, while other eigenvalues, $\lambda_1 = 1 - p_0$, $\lambda_2 = (1 - p_0)^2$, and $\lambda_3 = (1 - p_0)(1 - 2p_\pi)$, satisfy

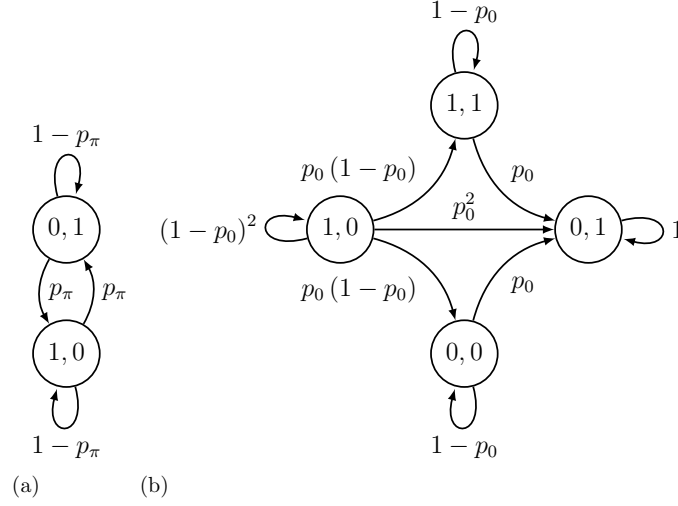


Figure 21.2: Graph of the transition matrices. Each arrow denotes a possible transition between two states (n_+, n_-) and (n'_+, n'_-) , and is labelled by the transition probability. (a) Non adiabatic processes described by $\mathcal{L}_\pi$ can exchange the occupation of states (0, 1) and (1, 0) by enabling a particle from one ABS to jump to the other ABS. (b) Non adiabatic processes described by $\mathcal{L}_0$ can populate the lower ABS, enabling transitions from (0, 0) to (0, 1), and from (1, 0) to (1, 1). They can also deplete the higher ABS state, enabling transitions from (1, 0) to (0, 0) and from (1, 1) to (0, 1).

$|\lambda_{\alpha \neq 0}| < 1$. Thus, the probability vector $\mathbf{P}$ that solves Eqs. (21.4) reaches a solution at long times that does not depend on the initial condition; it is given by the eigensolution $\alpha = 0$:

$$\mathbf{P}(\varphi) = \begin{cases} u_0, & 2m\pi < \varphi < (2m+1)\pi, \\ \mathcal{L}_\pi u_0, & (2m+1)\pi < \varphi < (2m+2)\pi, \end{cases} \quad (21.6)$$

with

$$u_0 = \frac{1}{\mathcal{N}} \begin{pmatrix} p_\pi(1 - p_0) \\ p_\pi(1 - p_0)^2 \\ \mathcal{N} - p_\pi(1 - p_0)(3 - p_0) \\ p_\pi(1 - p_0) \end{pmatrix}, \quad (21.7a)$$

$$\mathcal{L}_\pi u_0 = \frac{1}{\mathcal{N}} \begin{pmatrix} p_\pi(1 - p_0) \\ p_\pi \\ \mathcal{N} - p_\pi(3 - 2p_0) \\ p_\pi(1 - p_0) \end{pmatrix}, \quad (21.7b)$$

and $\mathcal{N} = (2 - p_0)(1 - \lambda_3)$. In particular, the ground state (0, 1) is mostly occupied with $u_0 \approx \mathcal{L}_\pi u_0 \approx (0, 0, 1, 0)^T$ at $p_\pi \ll p_0$, while all states are approximately equally occupied with $u_0 \approx \mathcal{L}_\pi u_0 \approx (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^T$ at $p_0 \ll p_\pi$.

Inserting Eq. (21.6) into (21.3), we find the average current in the long-time limit,

$$\begin{aligned} \langle I(t) \rangle &= \frac{p_0}{1 - \lambda_3} [p_\pi |\mathcal{I}(t)| + (1 - p_\pi) \mathcal{I}(t)] \\ &= I_{\text{dc}} \left\{ 1 - \sum_{n \geq 1} \frac{2}{4n^2 - 1} \left[\cos(n\varphi(t)) + 2(-1)^n n \frac{1 - p_\pi}{p_\pi} \sin(n\varphi(t)) \right] \right\}. \end{aligned} \quad (21.8)$$

The dc contribution,

$$I_{\text{dc}} = \frac{2}{\pi} \frac{p_0 p_\pi}{1 - \lambda_3} I_J, \quad (21.9)$$

relates the dissipative current with non-adiabatic processes through the gaps in the Andreev spectrum. It corresponds to the low-voltage regime of multiple Andreev reflections (MAR). It generalizes formulas derived in superconducting atomic contacts [17] at $p_0 = 1$, and in topological Josephson junctions [18] at $p_\pi = 1$. The ratio between cosine and sine harmonics of the Josephson frequency is controlled by non-adiabatic transitions between ABS with positive and negative energies. When these processes are rare, the sine harmonics dominate like in the adiabatic case.

Due to the stochastic nature of the non-adiabatic processes, the current fluctuates. We characterize the fluctuations with the current noise spectrum,

$$S(\omega) = 2 \int_0^\infty d\tau \cos(\omega\tau) \overline{S(t + \tau/2, t - \tau/2)}, \quad (21.10)$$

where the bar denotes an average over t . Within the Markov theory, we relate the current correlator,

$$S(t_1, t_2) = \langle I(t_1)I(t_2) \rangle - \langle I(t_1) \rangle \langle I(t_2) \rangle, \quad (21.11)$$

with

$$\begin{aligned} \langle I(t_1)I(t_2) \rangle &= \mathcal{I}(t_1)\mathcal{I}(t_2) \\ &\times \sum_{n_1, n_2} (-1)^{n_1+n_2} P_{n_1 \bar{n}_1 | n_2 \bar{n}_2}(\varphi_1 | \varphi_2) P_{n_2 \bar{n}_2}^\infty(\varphi_2) \end{aligned} \quad (21.12)$$

at $t_1 > t_2$. Here $P_{n_1 n'_1 | n_2 n'_2}(\varphi_1 | \varphi_2)$ with $\varphi_i = \varphi(t_i)$ is the conditional probability for the system to reside in state (n_1, n'_1) at phase φ_1 if it was in state (n_2, n'_2) at phase $\varphi_2 < \varphi_1$; it solves the same Eq. (21.4) as the probability $P_{n_1 n'_1}(\varphi_1)$, together with the initial condition $P_{n_1 n'_1 | n_2 n'_2}(\varphi_2 | \varphi_2) = \delta_{n_1, n_2} \delta_{n'_1, n'_2}$. Furthermore we used notations $\bar{0} = 1$ and $\bar{1} = 0$. Using a matrix representation [in the same basis of states as the one used in Eqs. (21.4) and (21.5)] for the closure relation, $\sum_\alpha u_\alpha v_\alpha^T = 1$, we find

$$P(\varphi_1 | \varphi_2) = \mathcal{L}_\pi^{k_1} \left(\sum_\alpha u_\alpha \lambda_\alpha^{m_1 - m_2} v_\alpha^T \right) \mathcal{L}_\pi^{-k_2} \quad (21.13)$$

for $(2m_i + k_i)\pi < \varphi_i < (2m_i + k_i + 1)\pi$ with m_i integer, $k_i = 0, 1$, and $i = 1, 2$. The second term in the r.h.s. of Eq. (21.11) compensates the contribution from the terms with $\alpha = 0$ when inserting Eq. (21.13) into (21.12). Furthermore, the expectation value of the current operator in the states with $\alpha = 1, 2$ vanishes. Therefore, only the terms with $\alpha = 3$ contribute to Eq. (21.11). As the result of a long calculation described in Sec. S1 of the supplemental material (SM) [21], we find

$$\frac{S(\omega)}{S_0} = \frac{A[1 + 4\tilde{\omega}^2 - 4\tilde{\omega} \sin(\pi\tilde{\omega})] + (B + C\tilde{\omega}^2) \cos^2(\pi\tilde{\omega})}{(1 - 4\tilde{\omega}^2)^2 [(1 + \lambda_3)^2 - 4\lambda_3 \cos^2(\pi\tilde{\omega})]} \quad (21.14)$$

with $S_0 = I_J^2 / (\pi\omega_J)$, $\tilde{\omega} = \omega / \omega_J$, and

$$A = 8p_0 p_\pi (1 - p_\pi) [1 + (1 - p_0)^2] (1 + \lambda_3) / \mathcal{N}, \quad (21.15a)$$

$$B = -16p_\pi (2 - p_\pi) \lambda_3 / \mathcal{N}, \quad (21.15b)$$

$$C = 128p_\pi (1 - p_\pi) (1 - p_0) [1 - \lambda_3 (1 - p_0)] / \mathcal{N}. \quad (21.15c)$$

Below we discuss the zero-frequency noise as well as structures related to the AC currents in the frequency dependence of the noise given by Eq. (21.14).

The zero-frequency noise is expressed in terms of an effective charge, $e^* = 2\Delta/V$, which diverges inversely with the bias voltage in the MAR regime, and a Fano factor

$$F \equiv \frac{S(0)}{e^* I_{\text{dc}}} = \frac{A + B}{8p_0 p_\pi (1 - \lambda_3)}. \quad (21.16)$$

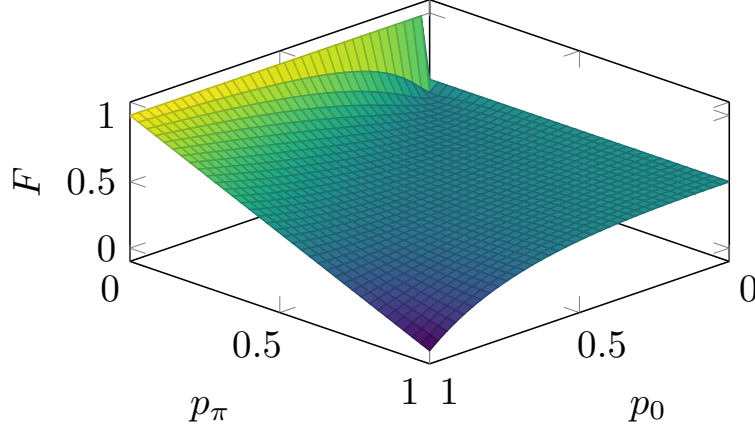


Figure 21.3: Fano factor as a function of p_0 and p_π .

In particular, $F = 1$ at $p_\pi \ll p_0$, when the bottleneck for the transfer of quasiparticles across the junction is the gap near the Fermi level [22]; $F = 1/2$ at $p_0 \ll p_\pi$, when the bottleneck consists of the two gaps (of the same width) near the edges of the continuum spectrum. In particular, at $p_\pi = 1$, one recovers the results of Ref. [18] for a topological junction (see Sec. S2 of the SM [21]). F vanishes at $1 - p_0, 1 - p_\pi \ll 1$, when quasiparticle transfer across the gap becomes deterministic. In the general case $0 < F < 1$, see Fig. 21.3.

From Eq. (21.14), we see that the noise displays sharp features when $1 \mp \lambda_3 \ll 1$. This is the case when $p_0, p_\pi \ll 1$ or $p_0, 1 - p_\pi \ll 1$. We turn now to the noise spectrum in these limiting cases for the transition probabilities.

At $p_0, p_\pi \ll 1$ the noise spectrum displays features with a narrow linewidth,

$$\gamma = \frac{1}{2\pi}(p_0 + 2p_\pi)\omega_J, \quad (21.17)$$

near $\omega = n\omega_J$ with n integer. In particular, at very low frequency, $\omega \ll \omega_J$,

$$\frac{S(\omega)}{e^* I_{\text{dc}}} = F + (F_{\text{app}} - F) \frac{\omega^2}{\omega^2 + \gamma^2}. \quad (21.18)$$

Here, Eq. (21.9) simplifies to

$$I_{\text{dc}} = \frac{2I_J}{\pi} \frac{p_0 p_\pi}{p_0 + 2p_\pi}, \quad (21.19)$$

and the Fano factor,

$$F = \frac{p_0^2 + p_0 p_\pi + 2p_\pi^2}{(p_0 + 2p_\pi)^2}, \quad (21.20)$$

is only accessible in a narrow frequency range $\omega \ll \gamma$, while an apparent Fano factor,

$$F_{\text{app}} = \frac{(12 - 4\pi + \pi^2)p_0 + 8p_\pi}{2\pi^2 p_0}, \quad (21.21)$$

characterizes the noise in a wide frequency range $\gamma \ll \omega \ll \omega_J$. Note that $F_{\text{app}} \gg F$ if $p_0 \ll p_\pi$. Furthermore

$$S(\omega) = \frac{32n^2}{(4n^2 - 1)^2 \pi} \frac{(3p_0 + 2p_\pi)p_\pi}{(p_0 + 2p_\pi)^2} \frac{\gamma I_J^2 / \pi}{(\omega - n\omega_J)^2 + \gamma^2} \quad (21.22)$$

at $|\omega - n\omega_J| \ll \omega_J$, up to a negative resonance-frequency shift of the order of $\gamma^2/\omega_J \ll \gamma$. Comparing Eq. (21.22) with the amplitude of the harmonics in Eq. (21.8), we conclude that the Josephson radiation at $p_0 \ll p_\pi \ll 1$ is dominated by the noise, Eq. (21.22); thus it

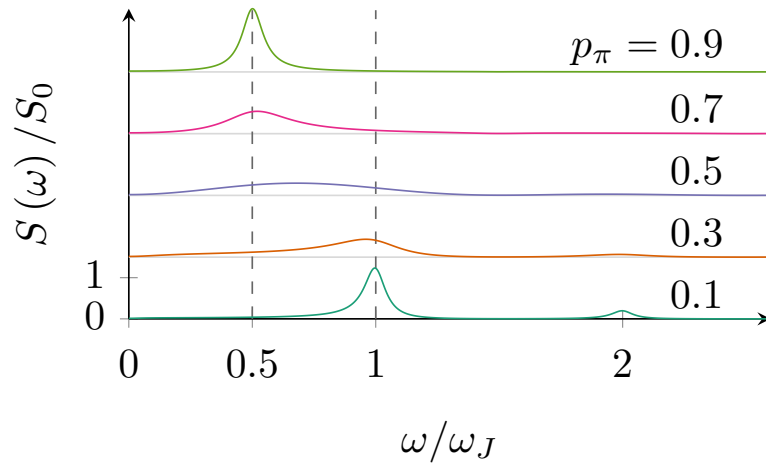


Figure 21.4: Current noise spectrum for $p_0 = 0.1$ and several equally spaced values of p_π between 0.1 and 0.9. The curves are shifted vertically for clarity.

is broadened by non-adiabatic transitions. On the other hand, the Josephson radiation at $p_\pi \ll p_0 \ll 1$ is dominated by the sine harmonics in Eq. (21.8); thus it is broadened by the environment of the junction.

At $p_0, 1 - p_\pi \ll 1$, $I_{\text{dc}} = p_0 I_J / \pi$ and $F = 1/2$; the noise spectrum displays a narrow resonance at half the Josephson frequency

$$S(\omega) = \frac{I_J^2}{4} \frac{\gamma'}{(\omega - \omega_J/2)^2 + \gamma'^2} \quad (21.23)$$

with linewidth

$$\gamma' = \frac{1}{2\pi} [p_0 + 2(1 - p_\pi)] \omega_J, \quad (21.24)$$

up to a small resonance-frequency shift, of the order of $\gamma'^2/\omega_J \ll \gamma'$, which increases as p_π decreases. Higher order resonances around $(n + 1/2)\omega_J$ are suppressed. Comparison between Eqs. (21.8) and (21.24) shows that the Josephson radiation is dominated by the noise, and it is thus broadened by the non-adiabatic processes. At $1 - p_\pi \ll 1$, transitions across the gap at π are very frequent, leading to a large, but random 4π -periodic contribution to the current. Thus, the Josephson radiation is fractional despite the junction *not* being topological. In the extreme case $p_\pi = 1$, where the transitions across the gap at π are deterministic, the system can be described as two independent topological junctions in parallel [21, 14].

The crossover between a well-resolved fractional or conventional Josephson radiation, when p_0 is small and p_π increases from 0 to 1 occurs through the gradual shift and broadening of the peaks in the noise spectrum, as illustrated in Fig. 21.4. A similar behavior has been reported in topological junctions, but with a residual coupling to the Majoranas at the far ends of the superconducting wires [42].

In a given junction, we can now distinguish two qualitatively different behaviors, depending on the ratio between crossover voltages V_0 and V_π . If $V_0 \ll V_\pi$, $F = 1$ at $V \ll V_\pi$, then it drops to 0 at $V \gg V_\pi$. Furthermore, the Josephson radiation is conventional. In the opposite case, $V_\pi \ll V_0$, $F = 1/2$ at $V \ll V_0$, then it drops to 0 at $V \gg V_0$, and there is a voltage range $V_\pi \ll V \ll V_0$ where Josephson radiation is fractional. Both the conventional and fractional Josephson radiations are broadened by the non-adiabatic processes at $V < V_0$.

In conclusion we proposed a simple model to analyze the role of non-adiabatic transitions between Andreev states in the Josephson radiation of a superconductor-quantum dot-superconductor junction. Within a simplified model of the Andreev states' dynamics, we predicted that such a conventional junction may display either fractional or conventional radiation

depending on its parameters and on the voltage bias. On the theory side, it would be interesting to extend the analysis to a more general description of the Andreev dynamics, as well as develop a theory frame to compare the interplay of the environment (neglected in our study) and the non-adiabatic transition on the loss of coherence of the Josephson radiation. On the experimental side, it would be interesting to compare our prediction with finite-frequency noise measurements in devices such as the superconductor-carbon nanotube-superconductor junction in the Kondo region, whose dc transport and shot noise have been measured recently [23].

21.2 Supplemental material

In section 21.2.1 of this supplemental material, we provide technical details on the derivation of our results for the current noise spectrum, Eqs. (14) and (15) of the main text. In section 21.2.2, we propose an alternative description of the junction at Landau-Zener probability $p_\pi = 1$ that clarifies the relation between our model and topological junctions.

21.2.1 Evaluation of the current noise spectrum

Combining Eqs. (11), (12), and (13) of the main text, one obtains the following expression for the noise:

$$S(t_1, t_2) = \mathcal{I}(t_1)\mathcal{I}(t_2) \sum_{\alpha=1}^3 v_0^T \hat{\mathcal{I}} \mathcal{L}_\pi^{k_1} u_\alpha \lambda_\alpha^{m_1-m_2} v_\alpha^T \mathcal{L}_\pi^{-k_2} \hat{\mathcal{I}} \mathcal{L}_\pi^{k_2} u_0, \quad (21.25)$$

for $(2m_i + k_i)\pi < \omega_J t_i + \varphi_0 < (2m_i + k_i + 1)\pi$. Here $\hat{\mathcal{I}}$ is the current operator, which in the basis of Eqs. (4), (5) takes the form

$$\hat{\mathcal{I}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (21.26)$$

To evaluate the noise, the left and right eigenvectors of the stochastic matrix $\mathcal{L}$ are needed. The right eigenvector u_0 , describing the stationary state is given in the main text, whereas the others read as follows:

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} p_0(1-2p_\pi) \\ -2p_\pi(1-p_0) \\ -2p_0+2p_\pi(1+p_0) \\ p_0(1-2p_\pi) \end{pmatrix}. \quad (21.27)$$

One notes that the eigenvectors u_1 and u_2 do not carry any current, $v_0^T \hat{\mathcal{I}} u_{1,2} = 0$. Furthermore, u_1 and u_2 are eigenvectors not only of $\mathcal{L}$ but also of $\mathcal{L}_\pi$, namely $\mathcal{L}_\pi u_{1,2} = u_{1,2}$, which further yields $v_0^T \hat{\mathcal{I}} \mathcal{L}_\pi u_{1,2} = 0$. As a consequence, the expression for the noise reduces to

$$S(t_1, t_2) = \mathcal{I}(t_1)\mathcal{I}(t_2) \lambda_3^{m_1-m_2} \left(v_0^T \hat{\mathcal{I}} \mathcal{L}_\pi^{k_1} u_3 \right) \left(v_3^T \mathcal{L}_\pi^{-k_2} \hat{\mathcal{I}} \mathcal{L}_\pi^{k_2} u_0 \right). \quad (21.28)$$

Only the eigenvalue λ_3 contributes to the current fluctuation. Thus, there is a unique time scale for the decay of correlations. In addition to v_0 given in the main text, the only other left eigenvector needed is v_3 , given as

$$v_3 = -\frac{1}{2(p_0 - 2p_\pi)(1 - \lambda_3)} \begin{pmatrix} p_0 \\ 2p_0 + 2p_\pi(1 - p_0) \\ -2p_\pi(1 - p_0) \\ p_0 \end{pmatrix}. \quad (21.29)$$

With the expressions provided in Eqs. (21.27) and (21.29), as well as Eqs. (5) and (7) of the main text, we can evaluate the matrix elements contributing to the noise as given in Eq. (21.28). In particular,

$$v_0^T \hat{\mathcal{I}} u_3 = 2(2p_\pi - p_0), \quad (21.30a)$$

$$v_3^T \hat{\mathcal{I}} u_0 = \frac{\lambda_3 p_\pi [(1 - p_0)(2p_\pi - p_0) - \mathcal{N}]}{\mathcal{N}(p_0 - 2p_\pi)(1 - 2p_\pi)(1 - \lambda_3)}, \quad (21.30b)$$

$$v_3^T \mathcal{L}_\pi^{-1} \hat{\mathcal{I}} \mathcal{L}_\pi u_0 = \frac{p_\pi(2p_\pi - p_0 - \mathcal{N})}{\mathcal{N}(p_0 - 2p_\pi)(1 - 2p_\pi)(1 - \lambda_3)}, \quad (21.30c)$$

and $v_0^T \hat{\mathcal{I}} \mathcal{L}_\pi u_3 = (1 - 2p_\pi) v_0^T \hat{\mathcal{I}} u_3$.

We can now turn to the computation of the noise spectrum $S(\omega)$. As a first step, we rewrite Eq. (10) of the main text as

$$S(\omega) = \lim_{T \rightarrow \infty} \frac{2}{T} \int_0^\infty d\tau \cos(\omega\tau) \int_0^T dt S(t + \tau, t). \quad (21.31)$$

The function $S(t + \tau, t)$ is periodic in t and quasi-periodic in τ with period $T = 2\pi/\omega_J$:

$$S(t + 2\pi/\omega_J + \tau, t + 2\pi/\omega_J) = S(t + \tau, t), \quad (21.32)$$

$$S(t + \tau + \frac{2\pi}{\omega_J}, t) = \lambda_3 S(t + \tau, t). \quad (21.33)$$

Using the periodicity in t , one may rewrite the time average as

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt S(t + \tau, t) = \frac{\omega_J}{2\pi} \int_0^{2\pi/\omega_J} dt S(t + \tau, t). \quad (21.34)$$

Furthermore, using the quasi-periodicity in τ , we may rewrite the Fourier transform as

$$\int_0^\infty d\tau \cos(\omega\tau) S(t + \tau, t) = \mathcal{R} \left\{ \sum_{n=0}^\infty (\lambda_3 e^{2i\pi\omega/\omega_J})^n \int_0^{2\pi/\omega_J} d\tau e^{i\omega\tau} S(t + \tau, t) \right\}. \quad (21.35)$$

Combining Eqs. (21.34) and (21.35) and performing the sum over n , one obtains

$$S(\omega) = \frac{\omega_J}{\pi} \mathcal{R} \left\{ \frac{1}{1 - \lambda_3 e^{\frac{2i\pi\omega}{\omega_J}}} \int_0^{2\pi/\omega_J} d\tau \int_0^{2\pi/\omega_J} dt e^{i\omega\tau} S(t + \tau, t) \right\}. \quad (21.36)$$

With the changes of variables $\omega_J t + \varphi_0 \rightarrow \varphi$ and $\omega_J \tau \rightarrow \delta$, we can write

$$\int_0^{2\pi/\omega_J} d\tau \int_0^{2\pi/\omega_J} dt e^{i\omega\tau} S(t + \tau, t) = \frac{1}{\omega_J^2} \int_0^{2\pi} d\delta \int_0^{2\pi} d\varphi e^{i\tilde{\omega}\tau} S(\varphi + \delta, \varphi), \quad (21.37)$$

where $\tilde{\omega} = \omega/\omega_J$. To perform the remaining double integral, one notices that $S(t, t + \tau)$ possesses discontinuities at $\delta + \varphi = n\pi$ and $\varphi = n\pi$. At these phases, the values of the indices $m_1 - m_2$, k_1 , and k_2 change in the expression for the noise, Eq. (21.28). Thus, the integration domain can be split into different segments.

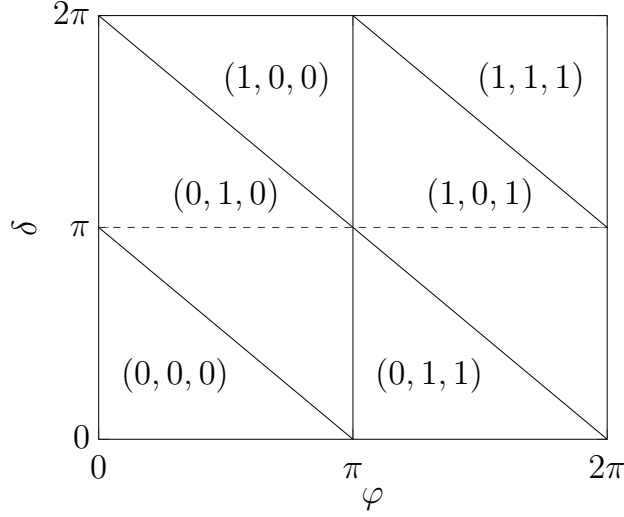


Figure 21.5: Map of the indices $(m_1 - m_2, k_1, k_2)$ determining $S(\varphi + \delta, \varphi)$ according to Eq. (21.28) as a function of φ and δ . Discontinuities in $S(\varphi + \delta, \varphi)$ appear at the boundaries of the different domains delimited by full lines. The integration scheme introduces additional segments depending on the value of δ , separated by the dashed line.

Performing the integration over φ in the eight segments shown in Fig. 21.5, we end up with a piecewise continuous function of δ . Integrating over δ and rearranging the terms, we obtain Eq. (14) of the main text with coefficients

$$A = 2(1 + \lambda_3) v_0^T \hat{\mathcal{I}} u_3 \left\{ [-2\lambda_3 + (1 - 2p_\pi)(1 - \lambda_3)] v_3^T \mathcal{L}_\pi^{-1} \hat{\mathcal{I}} \mathcal{L}_\pi u_0 + [1 - \lambda_3 + 2(1 - 2p_\pi)] v_3^T \hat{\mathcal{I}} u_0 \right\}, \quad (21.38)$$

$$B = 8v_0^T \hat{\mathcal{I}} u_3 \left[\lambda_3^2 v_3^T \mathcal{L}_\pi^{-1} \hat{\mathcal{I}} \mathcal{L}_\pi u_0 - (1 - 2p_\pi) v_3^T \hat{\mathcal{I}} u_0 \right], \quad (21.39)$$

$$C = 32\lambda_3 v_0^T \hat{\mathcal{I}} u_3 \left[v_3^T \mathcal{L}_\pi^{-1} \hat{\mathcal{I}} \mathcal{L}_\pi u_0 - (1 - 2p_\pi) v_3^T \hat{\mathcal{I}} u_0 \right]. \quad (21.40)$$

Using Eqs. (21.30), this yields the expressions for the coefficients given in Eqs. (15) of the main text.

21.2.2 Equivalent model at $p_\pi = 1$

When the Landau-Zener probability is $p_\pi = 1$, stochastic processes happen only at phases $\varphi = 2m\pi$. By contrast, at $\varphi = (2m + 1)\pi$, a deterministic exchange between the states $|\pm\rangle$ takes place. In this case, a more natural basis of states $|a\rangle, |b\rangle$ is given as

- $|a\rangle = |+\rangle$ and $|b\rangle = |-\rangle$ for $(4m - 1)\pi < \varphi < (4m + 1)\pi$,
- $|a\rangle = |-\rangle$ and $|b\rangle = |+\rangle$ for $(4m + 1)\pi < \varphi < 4(m + 3)\pi$,

which absorbs the effect of the deterministic Landau-Zener transitions due to $\mathcal{L}_\pi$ at $p_\pi = 1$.

Then the probabilities $\tilde{\mathbf{P}}(\varphi) = (P_{1a1b}(\varphi), P_{1a0b}(\varphi), P_{0a1b}(\varphi), P_{0a0b}(\varphi))^T$ are constant over the phase intervals $2m\pi < \varphi < 2(m + 1)\pi$ and evolve according to

$$\tilde{\mathbf{P}}(4m\pi^+) = \mathcal{L}_0 \tilde{\mathbf{P}}(4m\pi^-), \quad (21.41a)$$

$$\tilde{\mathbf{P}}((4m + 2)\pi^+) = \mathcal{L}_{2\pi} \tilde{\mathbf{P}}((4m + 2)\pi^-). \quad (21.41b)$$

Here the transition matrices $\mathcal{L}_\varphi$ can be decomposed into a product of Markov matrices acting on distinct ABS branches, $\mathcal{L}_\varphi = \mathcal{L}_\varphi^{(a)} \otimes \mathcal{L}_\varphi^{(b)}$. In particular,

$$\mathcal{L}_0 = \mathcal{L}_- \otimes \mathcal{L}_+, \quad \mathcal{L}_{2\pi} = \mathcal{L}_+ \otimes \mathcal{L}_-, \quad (21.42)$$

where the new transition matrices in the basis $\{|1\rangle_x, |0\rangle_x\}$ with $x = a, b$ take the form

$$\mathcal{L}_- = \begin{pmatrix} 1 & p_0 \\ 0 & 1 - p_0 \end{pmatrix}, \quad \mathcal{L}_+ = \begin{pmatrix} 1 - p_0 & 0 \\ p_0 & 1 \end{pmatrix}. \quad (21.43)$$

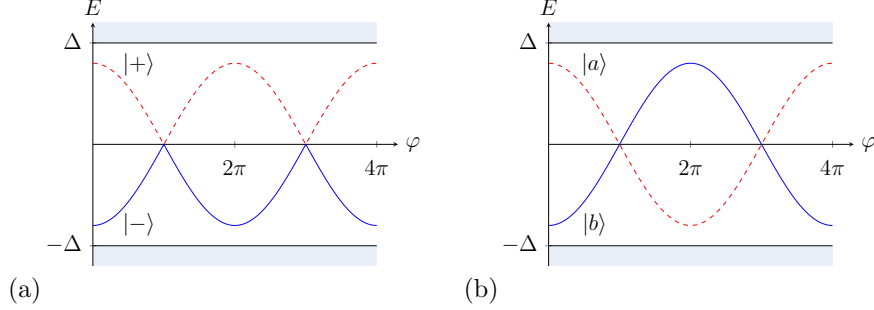


Figure 21.6: (a) Representation of the Andreev Bound states in the initial basis $|\pm\rangle$. (b) Representation of the Andreev Bound states in the basis $|a/b\rangle$. The deterministic Landau-Zener process at $\varphi = (2m + 1)\pi$ have been absorbed by the change of basis.

Thus the occupations of the two states $|a\rangle$ and $|b\rangle$ switch independently. The dynamics of each state is governed by the same equations as the dynamics of the 4π -periodic Andreev bound state appearing in a topological Josephson junction [S1]. The two states carry a current $I_{a/b}(\varphi) = \pm I_J \sin(\varphi/2)$. Furthermore, the current operator $\hat{\mathcal{I}}$ is also separable,

$$\hat{\mathcal{I}} = \frac{1}{2} (\sigma_z \otimes \mathbb{1} - \mathbb{1} \otimes \sigma_z). \quad (21.44)$$

As a consequence, at $p_\pi = 1$, the system can be described as two independent topological junctions in parallel. Thus, both the average current as well as the noise are doubled compared to the values for the topological junction reported in [S1]. As a consequence, one obtains the same Fano factor, $F = 1/2$, as well as a peak in the noise spectrum at $\omega = \omega_J/2$, corresponding to a fractional Josephson effect.

22 Phase-Controlled Force and Magnetization Oscillations in Superconducting Ballistic Nanowires by Krive, Romanovsky, Bogachek, Landman

Abstract

The emergence of superconductivity-induced phase-controlled forces in the ($10^{-2} - 10^{-1}$) nN range, and of magnetization oscillations, in nanowire junctions, is discussed. A giant magnetic response to applied weak magnetic fields, is predicted in the ballistic Josephson junction formed by a superconducting tip and a surface, bridged by a normal metal nanowire where Andreev states form.

Formation, energetics and mechanical properties of metallic nanowires (NWs), were predicted in early simulations [1], and they have been the subject of subsequent significant research endeavors [2]. For normal metals the oscillatory behavior of the elongation forces [1] with the size of a NW formed between a surface and a retracting tip have been shown to be correlated with a quantized staircase behavior of the electrical conductance [3-7]. However, the influence of superconductivity (SC) on the nanomechanical properties of such NWs has not been explored yet, and these effects are the focus of this Letter.

In normal metals the cohesive force in an atomic-scale metallic contact can be estimated as $F_n \sim \varepsilon_F/\lambda_F$, where ε_F and λ_F are the Fermi energy and wavelength. The onset of SC introduces a new energy scale, i.e. the superconducting gap $\Delta \ll \varepsilon_F$, and a new lengthscale, i.e. the SC coherence length $\xi_0 = \hbar v_F/\pi\Delta$. On first sight, the resulting SC-induced forces are expected to be of the order of $F_{sc} \sim \Delta/\xi_0$, and when added to the aforementioned normal-metal forces (F_n , which are of the order of several nN), they are estimated to be below the atomic force microscopy (AFM) detection limit [8]. However, in the superconducting regime, under certain conditions all the transverse channels ($N_\perp$) supported by the junction will contribute coherently to the free energy, and when $N_\perp \gg 1$ the above consideration may result in a gross underestimation of F_{sc} . In particular, we predict that under favorable conditions, in a nanowire connecting two superconducting electrodes, modelled here as a transparent short ($L \ll \xi_0$) superconducting-normal-superconducting (SNS) junction, a (measurable) force $F_{sc} \sim (L/\xi_0)(\varepsilon_F/\lambda_F)$ would be manifested - we refer to these forces as Andreev forces (AF); in long SNS junctions the AF are expected to be significantly smaller.

To calculate the SC - induced contribution to the total elongation force we model the junction comprised of the superconducting tip and surface and of the NW bridging them, as an SNS junction (the NW remains normal due to the suppression of superconductivity in very small structures [9]). Andreev reflections occur at the SN boundaries of the junction, and the dimensions of the NW are such that transport through it is ballistic. This holds true for short SNS junctions, while in long ones impurity scattering may occur, resulting in a reduced transmission probability.

The force equals the spatial derivative of the grand-canonical potential Ω , i.e. $F = -\partial\Omega/\partial L$. In superconducting junctions there are two contributions to the AF, $F_A = F_\perp + F_L$; $F_\perp$ is related to the dependence of the number of quantized transverse channels, $N_\perp$, on the degree of elongation, and F_L originates from the dependence of the Andreev bound states on the length of the junction.

Short ($L \ll \xi_0$) junctions. In the following we focus on the SC-induced force oscillations in short junctions. Since here the Andreev states are independent of the mode index [10], the potential can be written as $\Omega_s(\varphi, L) \approx N_\perp(L)\Omega_A(\varphi, L)$, where $\Omega_A(\varphi, L)$ corresponds to a single-channel SNS junction. In a cylindrical geometry $N_\perp(L) \simeq \pi V/\lambda_F^2 L$, where V is the volume) which remains constant during the elongation process [1]).

In general, both bound (superscript (b) below) and scattering Andreev states contribute to $\Omega_A(\varphi, L)$; for our purpose only the bound states are important (see below), and thus $F_A^{(b)} = F_\perp^{(b)} + F_L^{(b)}$, where

$$F_\perp^{(b)} = \frac{N_\perp}{L}\Omega_A^{(b)}(\varphi, L), \quad F_L^{(b)} = -N_\perp \frac{\partial\Omega_A^{(b)}(\varphi, L)}{\partial L}. \quad (22.1)$$

The spectrum of Andreev bound states in a single-mode ballistic SNS junction of length L and transparency $D \leq 1$ is found from the equation (see e.g. Ref.[11])

$$\cos\left(2 \arccos \frac{E}{\Delta} - 2 \frac{E}{\Delta_L}\right) = R + D \cos \varphi \quad (22.2)$$

where $\Delta_L \equiv \hbar v_F/L$, $D+R=1$, and φ is the superconducting phase difference. The distinction between different channels enters through Δ_L , and it is neglected for short junctions. A reduced NW transparency reflects the presence of interfacial barriers, such as those likely to form at tip-surface contacts. From Eq.(2) we obtain in the limit $L \ll \xi_0$ the bound-state energies

$$E_0^{(+)} = -E_0^{(-)} \simeq \Delta W(\varphi) \left(1 - \frac{L\sqrt{D}}{\xi_0} \left|\sin \frac{\varphi}{2}\right|\right) \quad (22.3)$$

where $W(\varphi) = \sqrt{1 - D \sin^2 \frac{\varphi}{2}}$. Using Eq.(1), with Ω_A taken simply as the thermodynamic potential for a two-level ($E_0^{(+)}, E_0^{(-)}$) system, yields

$$F_{\perp}^{(b)} \simeq -2N_{\perp} \frac{T}{L} \ln \left\{ \frac{\cosh^2(W(\varphi)\Delta/2T)}{\cosh^2(\Delta/2T)} \right\}, \quad (22.4)$$

$$F_L^{(b)} \simeq 2N_{\perp} \frac{\Delta}{\xi_0} W(\varphi) \sqrt{D \sin^2 \frac{\varphi}{2}} \tanh \left(\frac{\Delta}{2T} W(\varphi) \right). \quad (22.5)$$

For low transparency junctions ($D \ll 1$) $F_{\perp}^{(b)}$ may be approximated as

$$F_{\perp}^{(b)} \simeq \frac{N_{\perp} \Delta}{L} D \sin^2 \frac{\varphi}{2} \tanh \left(\frac{\Delta}{2T} \right) \sim \frac{N_{\perp} \Delta}{L} D \quad (22.6)$$

and since $F_L^{(b)} \sim N_{\perp} (\Delta/\xi_0) \sqrt{D}$ it is evident that $|F_{\perp}^{(b)}| \gg |F_L^{(b)}|$, provided that $D \gg (L/\xi_0)^2$. In contrast, in low transparency junctions $D \ll (L/\xi_0)^2 \ll 1$ and the $F_L^{(b)}$ contribution dominates.

For point contacts (i.e. extremely short junctions) the AF can be calculated (when $D \gg (L/\xi_0)^2$) by taking the limit $L/\xi_0 \rightarrow 0$ for the Andreev bound states (Eq.(3)). In this case, continuum states do not affect the free energy (see e.g. Ref.[11]) and, therefore, they do not contribute to the force. In contrast, for low transparency junctions the L/ξ_0 -corrections to the bound state energies determine the force oscillations. To this order the continuum states do contribute to the free energy and they can change the dependence of the AF on the phase difference. Thus, Eq.(5) can be considered as an estimate of the AF in a short junction. the contribution of the continuum states ($F_A^{(c)}$) is extremely small, i.e. $F_A^{(c)}(D \ll 1) \ll (L/\xi_0)^3 (\varepsilon_F/\lambda_F)$.

The phase-dependent force in a superconducting quantum point contact (QPC) ($D = 1$) is related to the quantized Josephson current J_s (see Ref.[10])

$$J_s = \frac{e}{\hbar} \left(-L \frac{\partial F_A^{(b)}}{\partial \varphi} \right) = N_{\perp} \frac{e\Delta}{\hbar} \sin \frac{\varphi}{2}. \quad (22.7)$$

The force oscillations (portrayed by the dependence of the force on the contact size) are determined by two distinct contributions: (i) a large phase-independent term (operative also in normal-metal NWs) of the order of $N_{\perp} \varepsilon_F/\lambda_F$ originating from incoherent contributions of all the conducting electrons to the thermodynamic potential [4-6], and (ii) a coherent SC-induced force (Eq.(4)-(6)). It is the latter, phase-dependent, term that is directly related to the quantized Josephson current.

The amplitude of the AF oscillations may be readily estimated as follows: for $D \sim 1$ and $L \ll \xi_0$ the amplitude of the Andreev force is of the order of $F_A^{(b)} \sim N_{\perp} \Delta/L \sim L \varepsilon_F/\xi_0 \lambda_F \sim (L/\xi_0) [nN]$; in the ballistic regime for a non-transition metal $\xi_0 \sim 10^{-5} - 10^{-4} \text{ cm}$. Using state-of-the-art instrumentation such forces (e.g. $10^{-2} - 10^{-1} nN$), can be measured [8].

Long ($L \gg \xi_0$) junctions. For a long transparent ($D = 1$) junction the spectrum of the Andreev-Kulik (AK) levels $|E_n| \ll \Delta$ takes the form [12]

$$E_n^{\pm} \simeq \pi \Delta_L \left(\pm \frac{\varphi}{2\pi} + \frac{1}{2} + n \right), \quad n = 0, \pm 1, \pm 2, \dots \quad (22.8)$$

Since this spectrum does not depend on the superconducting gap Δ , for temperatures $T \ll \Delta$ all the thermodynamic properties of a long ballistic junction are essentially material independent. Evaluation of the Josephson current in this case is equivalent to the calculation of the persistent

current for chiral fermions on a ring with circumference $2L$ [13]. The corresponding phase-dependent part of the thermodynamic potential $\tilde{\Omega}_A(\varphi)$ can be evaluated for the AK spectrum, yielding

$$\tilde{\Omega}_A(\varphi) = 4T \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{\cos k\varphi}{\sinh(2\pi T k / \Delta_L)}. \quad (22.9)$$

The force oscillations induced by the AK level structure in a single-channel long SNS junction are ($|\varphi| \leq \pi$)

$$\tilde{F}_A \simeq \begin{cases} \frac{\Delta_L}{2\pi L} \left(\varphi^2 - \frac{\pi^2}{3} \right) & T \ll \Delta_L \\ -16\pi \frac{T^2}{L\Delta_L} \exp\left(-\frac{2\pi T}{\Delta_L}\right) \cos \varphi & T \geq \Delta_L. \end{cases} \quad (22.10)$$

We note that the above is equivalent to the Casimir force [14]. We focus here only on the phase-dependent part of the thermodynamic potential, $\tilde{\Omega}_A(\varphi)$, and the resulting Andreev (or Casimir) force $\tilde{F}_A$, since, as aforementioned, the force in superconducting junctions (Eq.(10)) is added to a much larger phase-independent term ($\sim \varepsilon_F / \lambda_F$) that dominates the cohesive force in metallic NWs.

In a multi-channel junction the thermodynamic potential is the sum over transverse channels (ln)

$$\tilde{\Omega} = \sum_{ln} \tilde{\Omega}_A^{(ln)}(\varphi), \quad (22.11)$$

where $\tilde{\Omega}_A^{(ln)}(\varphi)$ is given by Eq.(9) with $\Delta_L^{(ln)} = \hbar v_F^{(ln)} / L$ substituted for Δ_L . For a long junction the Fermi velocity enters explicitly the expression for a single channel supercurrent, and the total current in a multi-channel junction strongly depends on the junction geometry [15].

We will model the normal part of a long SNS junction by a cylinder of length L and cross-section area $S = V/L$. Assuming hard-wall boundary conditions, the electron longitudinal velocity in the (ln)-th channel is

$$v_F^{(ln)}(L) = \sqrt{2(\varepsilon_F - a\gamma_{ln}^2) / m}, \quad (22.12)$$

where γ_{ln} are the Bessel function zeroes: $J_l(\gamma_{ln}) = 0$, and $a = \hbar^2 \pi L / 2mV$. The dependencies of the AF on the phase difference (φ) and on the length (L) of the NW are displayed, respectively, in Figs.1 and 2, where we show $\Delta F(\varphi) = F_A(\varphi) - F_A(0)$. From Fig.1 we observe that the force is enhanced at special values of the phase difference $\varphi_r = \pi(2r + 1)$, ($r = 0, \pm 1, \pm 2, \dots$). At $\varphi = \varphi_r$ one of the AK bound states coincides with the Fermi energy and, most significantly, this state is $4N_{\perp}$ -fold degenerate [16], thus amplifying its contribution. Direct observation of the SC-induced nanomechanical effect predicted here may be obtained through: (i) generation of a NW of length L via separation of an AFM tip-surface contact, using a superconducting material (e.g. Pb) at $T < T_c$, followed by (ii) measurement of the force required to maintain the NW length (L) as a function of variations of the phase-difference across the SNS junction (as seen from Fig.1 this force maximizes at $\varphi = \pi$).

The variation of the elongation force (for $\varphi = \pi$) with the NW length is shown in Fig.2. We note first that even though the number of open channels is very large for the NW junction shown in Fig.2, the magnitude of the forces is significantly smaller than in the case of short junctions (see previous subsection)[17]. The aperiodic variations of the AF originating from the change in the number of open channels upon elongation, are particularly pronounced at lower temperatures. Note however, that such aperiodic variations occur also for normal metal NW [4,6] and consequently separation of the SC-induced contribution may be difficult.

Next we consider a multichannel superconducting long junction in a weak magnetic field $\mu_B B \ll \Delta$ (where μ_B is the Bohr magneton), applied locally (i.e. only to the normal metal nanowire part of the SNS junction). Here, the only [18] influence on the AK levels is through the Zeeman coupling of the electron spin $\mathbf{s}$ to the magnetic field, $H_Z = g\mu_B \mathbf{s} \cdot \mathbf{B}$ (g is the g -factor). The thermodynamic potential $\delta\Omega_A(\varphi, B) \equiv \Omega_A(\varphi, B) - \Omega_A(0, 0)$ takes the form (see. Eq.(9))

$$\delta\Omega_A(\varphi, B) = -4T \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \frac{(1 - \cos k\varphi \cos k\chi)}{\sinh(2\pi kT/\Delta_L)}, \quad (22.13)$$

where $\chi \equiv \Delta_Z/\Delta_L$ and $\Delta_Z = g\mu_B B$ is the Zeeman energy splitting. Note that the influence of the Zeeman interaction on the thermodynamics of the SNS junction is equivalent to the influence of a gate voltage on the thermodynamic properties of quantum rings [19].

The magnetization $M_A = -\partial\delta\Omega_A(\varphi, B)/\partial B$ at low ($T \ll \Delta_L$) and high ($T > \Delta_L$) temperatures is given for a single channel junction as

$$M_A \simeq \begin{cases} -g\mu_B \left(\frac{\chi}{\pi}\right) & T = 0 ; \quad |\chi| \leq \pi \\ -8g\mu_B \frac{T}{\Delta_L} e^{-\frac{2\pi T}{\Delta_L}} \cos \varphi \sin(\chi), & T \geq \Delta_L. \end{cases} \quad (22.14)$$

Note, that the SC-induced magnetization M_A , can be of the order of several μ_B (if $g \gg 1$) even for a single-channel junction, and it is insensitive to the superconducting phase difference at low temperatures. For a multichannel quantum junction at low temperatures the dependence of $M_A(\varphi)$ exhibits typical resonant behavior at the resonant phases φ_r , as shown in Fig.3. This is a manifestation of the effect of "giant oscillations", known previously for conductance oscillations [18]. At these phases Andreev states of energies $E_A = \pm g\mu_B s B$ become $2N_{\perp}$ -fold degenerate [20], leading to giant enhancement of thermodynamic and kinetic characteristics of ballistic junctions in magnetic fields.

Since at resonance the coherent contribution ($\propto N_{\perp}$) of all transverse modes dominates the magnetization, we predict at low temperatures ($T \ll \Delta_Z$): (i) a giant response ($\propto N_{\perp}$) of an SNS junction to a magnetic field, and (ii) a step-like behaviour of the magnetization as a function of the wire diameter. At other values of the phase difference, different transverse channels contribute to $\delta\Omega_A$ with different periods (i.e. in general, incoherently), resulting in a complex structure of the magnetic oscillations.

In most cases a supercurrent is suppressed by the Zeeman interaction [21]. A magnetic field would also suppress the predicted Andreev force $\delta F_A(\varphi, B) = -\partial\delta\Omega_A(\varphi, B)/\partial L$. At low temperatures ($T \ll \Delta_L$) the force (which is periodic both in the phase, φ , and in the dimensionless Zeeman energy splitting $\chi = \Delta_Z/\Delta_L$) can be written for a single-channel junction as ($|\varphi|, |\chi| \leq \pi$): $\delta F_A \simeq (\Delta_L/2\pi L) [(\varphi)^2 - (\chi)^2]$.

In summary, we predicted and illustrated that superconductivity induces in quantum wires phase-dependent forces correlated with the supercurrent. At resonance values of the superconducting phase difference these Andreev forces become measurable (nN scale). Furthermore, we predict giant magnetization (of the order of $N_{\perp}\mu_B$) of ballistic SNS junctions in a weak magnetic field at low temperatures $T \ll \Delta_Z$. Since low-temperature STM with superconducting tips has been already demonstrated [22, 23] and used to form Josephson junction [24, 25], and in light of improved force-detection capabilities (extending to $10^{-1} - 10^{-3} nN$) [8], the above predictions provide the impetus for future experiments.

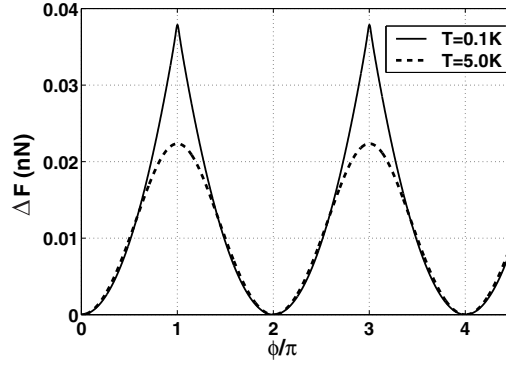


Fig. 1 Krive et. al.

Figure 22.1: $\Delta F(\varphi) = F_A(\varphi) - F_A(0)$ vs the phase difference for a NW SNS junction. We use the physical parameters for Pb, i.e. the Fermi energy is $\varepsilon_F = 1.5 \times 10^{-11} \text{erg}$, and $v_F = 1.83 \times 10^8 \text{cm/s}$. The volume $V = 5 \times 10^{-15} \text{cm}^3$, and the length of the junction $L = 10^{-4} \text{cm}$. Results are shown for two temperatures, both below $T_c(\text{Pb}) = 7.18 \text{K}$. The force was calculated as follows: first, using Eq.(9), the grand canonical potential was found for each transverse mode (with different values of v_F^{ln} and, therefore, different $\Delta_L^{ln} = \hbar v_F^{ln}/L$, see Eq.(12)). The total potential of the junction is the sum of contributions of all the transverse modes (Eq.(11)). The derivative of the potential with respect to the length of the junction was evaluated numerically.

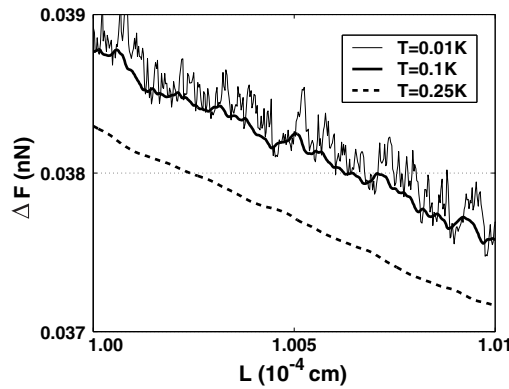


Fig. 2 Krive et. al.

Figure 22.2: $\Delta F(\varphi = \pi)$ vs the length of the junction, calculated for different temperatures. The parameters of the junction and the method of calculation are as in Fig.1.

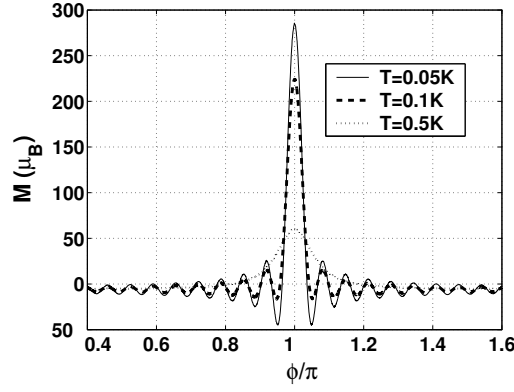


Fig. 3 Krive et. al.

Figure 22.3: Magnetization of the junction in a magnetic field of $5 Oe$, plotted vs the phase difference ϕ for several temperatures. The parameters of the junction are as in Fig.1. The magnetization was evaluated as a numerical derivative of $\delta\Omega(\varphi, B)$ (see Eq.(13)) with respect to B .

23 Josephson effect in a junction coupled to an electron reservoir by Beenakker

We extend the scattering theory of the Josephson effect to include a coupling of the Josephson junction to a gapless electron reservoir in the normal state. By opening up the system with a quasiparticle escape rate $1/\tau$, the supercurrent carried at zero temperature by an Andreev level at energy ε_A is reduced by a factor $(2/\pi) \arctan(2\varepsilon_A\tau/\hbar)$. We make contact with recent work on “non-Hermitian Josephson junctions”, by comparing this result to different proposed generalizations of the Josephson effect to non-Hermitian Hamiltonians.

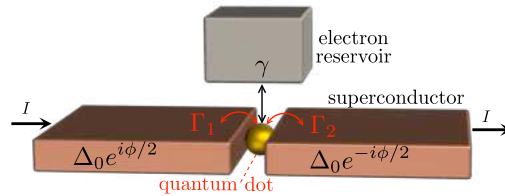


Figure 23.1: Josephson junction, formed by a quantum dot that is tunnel coupled with rates Γ_1 and Γ_2 to two superconductors (gap Δ_0 , phase difference ϕ). The quantum dot has an additional weak coupling (rate γ) to a gapless electron reservoir in the normal state. We seek the γ -dependence of the current-phase relationship $I(\phi)$.

Introduction — The thermodynamic properties of a quantum dot are affected by the coupling to a superconductor. The Josephson effect is a striking example, a current will flow through the quantum dot in equilibrium if the quantum dot forms a weak link between two superconductors [1]. This supercurrent $I(\phi)$ depends periodically on the phase difference ϕ of the pair potential in the superconductors. A dissipative electromagnetic environment may degrade the supercurrent via phase fluctuations [2, 3, 4]. Here we investigate an altogether different decay mechanism, the coupling of the quantum dot to a gapless electron reservoir (see Fig. 23.1).

The reservoir is a source of dephasing because quasiparticles that enter it from the quantum dot return without any phase coherence. Such a dephasing mechanism for persistent currents was introduced by Büttiker [5], and applied to superconducting circuits by several authors [6, 7, 8, 9, 10, 11, 12]. Our main advance is that we obtain closed-form expressions for the

supercurrent-phase relationship, for the case of a spatially uniform coupling rate to the electron reservoir.

Because the effective Hamiltonian of the quantum dot becomes non-Hermitian when we open it up to the reservoir, such a system is a simple example of a “non-Hermitian Josephson junction”, a topic of current interest [13, 14, 15, 16]. Our explicit expressions support Ref. [16] in the debate over the proper generalization of the current-phase relationship $I \propto dE/d\phi$ to complex eigenvalues E of the effective Hamiltonian.

Closed system – In a scattering formulation [17], the dependence of the density of states ρ of the Josephson junction on the superconducting phase difference ϕ is given by the determinantal expression

$$\rho(\varepsilon) = -\frac{1}{\pi} \text{Im} \frac{d}{d\varepsilon} \ln \det [1 - R_A(\varepsilon + i0^+) S_N(\varepsilon + i0^+)] + \rho_0(\varepsilon), \quad (23.1)$$

with ρ_0 the ϕ -independent density of states of the junction when it is decoupled from the superconductors. The determinant contains the product of the Andreev reflection matrix R_A from the superconductors and the scattering matrix S_N of the junction in the normal state. The energy $\varepsilon > 0$ is the excitation energy of Bogoliubov quasiparticles (electron-hole superpositions). The pair potential in the two superconductors, to the left and to the right of the junction, has amplitude Δ_0 and phase $\pm\phi/2$.

The electron and hole degree of freedom introduces a block structure in the scattering matrices. The matrix S_N is block diagonal,

$$S_N(\varepsilon) = \begin{pmatrix} s_0(\varepsilon) & 0 \\ 0 & s_0^*(-\varepsilon) \end{pmatrix}, \quad (23.2)$$

the electron and hole blocks are uncoupled and related by particle-hole symmetry.

The Andreev reflection matrix has the block structure

$$R_A(\varepsilon) = i\alpha(\varepsilon) \begin{pmatrix} 0 & r_A \\ r_A^* & 0 \end{pmatrix}, \quad r_A = \begin{pmatrix} e^{i\phi/2}\sigma_y & 0 \\ 0 & e^{-i\phi/2}\sigma_y \end{pmatrix},$$

$$\alpha(\varepsilon) = e^{-i \arccos(\varepsilon/\Delta_0)} = \varepsilon/\Delta_0 - i\sqrt{1 - \varepsilon^2/\Delta_0^2}. \quad (23.3)$$

The Pauli matrix σ_y acts on the spin degree of freedom. The blocks r_A^* and r_A describe Andreev reflection from, respectively, electron to hole and hole to electron, in opposite spin bands. Andreev reflection happens with unit probability for energies $\varepsilon < \Delta_0$; any normal reflection at the normal–superconductor (NS) interface is incorporated into S_N .

Substitution of Eqs. (24.5) and (24.4) into Eq. (24.3) gives the determinant [17]

$$\rho(\varepsilon) = -\frac{1}{\pi} \text{Im} \frac{d}{d\varepsilon} \ln \det [1 - M(\varepsilon + i0^+)] + \rho_0(\varepsilon), \quad (23.4)$$

$$M(\varepsilon) = -\alpha(\varepsilon)^2 r_A^* s_0^*(-\varepsilon) r_A s_0(\varepsilon). \quad (23.5)$$

The density of states determines the free energy F of the Josephson junction at temperature T [18]

$$F = -k_B T \int_0^\infty d\varepsilon \rho(\varepsilon) \ln [2 \cosh(\varepsilon/2k_B T)]. \quad (23.6)$$

The supercurrent through the Josephson junction then follows from the relation

$$I = \frac{2e}{\hbar} \frac{dF}{d\phi}. \quad (23.7)$$

It is convenient to extend the integration range in Eq. (24.2) to negative ε (so we no longer need to extract the imaginary part) and then to perform a partial integration,

$$F = \frac{i}{4\pi} \int_{-\infty}^{\infty} d\varepsilon \tanh(\varepsilon/2k_B T) \ln \det[1 - M(\varepsilon + i0^+)] + F_0, \quad (23.8)$$

with F_0 the ϕ -independent contribution to the free energy.

Opening up the system — We open up the Josephson junction by weakly coupling it to a gapless electron reservoir in the normal state. Quasiparticles enter the reservoir at a rate $1/\tau \equiv 2\gamma/\hbar$, assumed to be spatially uniform in the junction region. This can be modeled by coupling to the reservoir via a spatially extended tunnel barrier [19, 20]. The reservoir is closed, it does not drain any current, but particles that enter it return to the junction without any phase coherence, so they no longer contribute to the supercurrent.

The full scattering matrix of the normal region, including the scattering channels to the electron reservoir, is unitary, with $S_N(\varepsilon + i\gamma)$ a sub-unitary submatrix for the scattering channels to the superconductors. Similarly, the Andreev reflection matrix $R_A(\varepsilon + i\gamma)$ is now sub-unitary also for $\varepsilon < \Delta_0$.

Instead of Eq. (23.4) we have for the density of states the expression

$$\rho(\varepsilon) = -\frac{1}{\pi} \operatorname{Im} \frac{d}{d\varepsilon} \ln \det[1 - M(\varepsilon + i\gamma)] + \rho_0(\varepsilon). \quad (23.9)$$

The term ρ_0 now also contains the ϕ -independent contributions from the electron reservoir. The free energy then follows from

$$F = \frac{i}{4\pi} \int_{-\infty}^{\infty} d\varepsilon \tanh(\varepsilon/2k_B T) \ln \det[1 - M(\varepsilon + i\gamma)] + F_0. \quad (23.10)$$

We discuss three applications of this general formula.

Quantum dot Josephson junction — As a first application we consider the case that the two superconductors are coupled via a quantum dot containing a single resonant level, at energy ε_R relative to the Fermi level. We ignore Coulomb blockade effects, which will be less significant in the open system.

The electronic scattering matrix of the quantum dot has the form

$$s_0(\varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{i}{\varepsilon - \varepsilon_R + \frac{1}{2}i\Gamma} \begin{pmatrix} \Gamma_1 & \sqrt{\Gamma_1\Gamma_2} \\ \sqrt{\Gamma_1\Gamma_2} & \Gamma_2 \end{pmatrix}, \quad (23.11)$$

with Γ_1, Γ_2 the tunnel rates into the left and right superconductor, and $\Gamma = \Gamma_1 + \Gamma_2$. The normal-state transmission probability through the quantum dot at the Fermi level ($\varepsilon = 0$) is given by the Breit-Wigner formula [21]

$$T_{BW} = \frac{\Gamma_1\Gamma_2}{\varepsilon_R^2 + \frac{1}{4}\Gamma^2}. \quad (23.12)$$

In the weak-coupling, near-resonant regime, when $\varepsilon_R, \Gamma \ll \Delta_0$, the energy dependence of R_A can be neglected and we may substitute $\alpha(\varepsilon) \rightarrow \alpha(0) = -i$. The determinant (23.4) evaluates to

$$\ln \det[1 - M(\varepsilon)] = g \ln [1 - (\varepsilon_A/\varepsilon)^2], \quad (23.13)$$

plus ϕ -independent terms. The prefactor g accounts for spin and possibly other degeneracies. The energy

$$\varepsilon_A = \Delta_{\text{eff}} \sqrt{1 - T_{BW} \sin^2(\phi/2)}, \quad \Delta_{\text{eff}} = \sqrt{\varepsilon_R^2 + \frac{1}{4}\Gamma^2}, \quad (23.14)$$

is the energy of the Andreev level in the closed system [22, 23]. There is no dependence on Δ_0 in the weak-coupling regime.

The free energy (23.10) becomes

$$F = \frac{gi}{4\pi} \int_{-\infty}^{\infty} d\varepsilon \tanh(\varepsilon/2k_B T) \ln \left[1 - \frac{\varepsilon_A^2}{(\varepsilon + i\gamma)^2} \right] + F_0. \quad (23.15)$$

The resulting supercurrent is

$$\begin{aligned} I &= I_0 \int_{-\infty}^{\infty} \frac{d\varepsilon \varepsilon_A \tanh(\varepsilon/2k_B T)}{i\pi (\varepsilon_A)^2 - (\varepsilon + i\gamma)^2} \\ &= I_0 \frac{2}{\pi} \operatorname{Im} \psi \left(\frac{1}{2} + \frac{i\varepsilon_A + \gamma}{2\pi k_B T} \right), \end{aligned} \quad (23.16)$$

with $\psi(x)$ the digamma function and

$$I_0 = -\frac{ge}{\hbar} \frac{d\varepsilon_A}{d\phi} \quad (23.17)$$

the zero-temperature supercurrent of the closed system. The $T \rightarrow 0$ limit of Eq. (23.16) (plotted in Fig. 23.2) is

$$\lim_{T \rightarrow 0} I = I_0 \frac{2}{\pi} \arctan(\varepsilon_A/\gamma), \quad (23.18)$$

so opening up the system to an electron reservoir at $T = 0$ reduces the supercurrent by a factor $(2/\pi) \arctan(2\varepsilon_A\tau/\hbar)$.

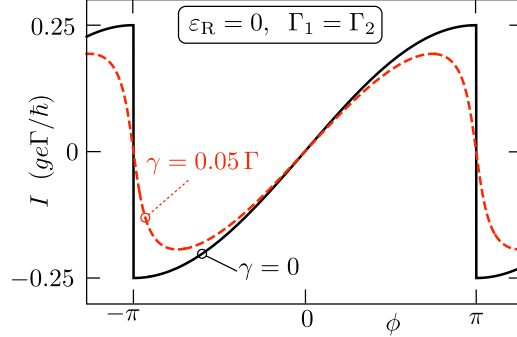


Figure 23.2: Zero-temperature current-phase relation of the quantum dot Josephson junction of Fig. 23.1, with and without the coupling to the electron reservoir. The plot is computed from Eq. (23.18) for the case $T_{BW} = 1$ of unit transmission probability at the Fermi level through the quantum dot.

Point contact Josephson junction — The second application is a point contact junction of length L short compared to the superconducting coherence length $\xi_0 = \hbar v_F/\Delta_0$. In this short-junction regime the energy dependence of S_N can be neglected relative to the energy dependence of R_A , and we may evaluate S_N at the Fermi level. We assume that time reversal symmetry is preserved, hence $\sigma_y S_N^* \sigma_y = S_N^\dagger$.

The Andreev levels in the closed system then depend on the phase difference according to [17]

$$\varepsilon_n = \Delta_0 \sqrt{1 - T_n \sin^2(\phi/2)}, \quad n = 1, 2, \dots, N, \quad (23.19)$$

in terms of the mode-dependent transmission probabilities $T_n \in [0, 1]$ in the normal state (eigenvalues of the transmission matrix product $tt^\dagger$). The number N counts the number of

propagating electron modes through the point contact. Eq. (23.19) is analagous to the result (23.14) for a quantum dot, with Δ_{eff} replaced by Δ_0 and T_{BW} replaced by T_n .

Instead of Eq. (23.13) we now have

$$\ln \det[1 - M(\varepsilon)] = g \sum_{n=1}^N \ln [1 - (\varepsilon_n/\varepsilon)^2], \quad (23.20)$$

and following the same steps as in the previous application we find the supercurrent in the open system,

$$I = \sum_{n=1}^N I_n \frac{2}{\pi} \text{Im} \psi \left(\frac{i\varepsilon_n + \gamma + \pi k_B T}{2\pi k_B T} \right), \quad (23.21)$$

$$\lim_{T \rightarrow 0} I = \sum_{n=1}^N I_n \frac{2}{\pi} \arctan(\varepsilon_n/\gamma), \quad (23.22)$$

with $I_n = -(ge/\hbar)d\varepsilon_n/d\phi$ the zero-temperature supercurrent in the n -th mode of the closed system.

Long SNS junction — For the third and final application we consider a superconductor–normal-metal–superconductor (SNS) junction of length $L \gg \xi_0$. Both states above and below Δ_0 then contribute to the supercurrent.

It is convenient to transform the slowly converging integration over energies of Eq. (23.10) into a more rapidly converging sum over Matsubara frequencies [24],

$$F = -k_B T \sum_{p=0}^{\infty} \ln \det[1 - M(i\omega_p + i\gamma)], \quad (23.23)$$

$$\omega_p = (2p + 1)\pi k_B T.$$

We will restrict ourselves for this application to zero temperature, when the sum can be replaced by an integral,

$$\lim_{T \rightarrow 0} F = -\frac{1}{2\pi} \int_0^{\infty} d\omega \ln \det[1 - M(i\omega + i\gamma)]. \quad (23.24)$$

We consider a ballistic single-mode junction. The electronic scattering matrix is

$$s_0(\varepsilon) = e^{-ik(\varepsilon)L} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (23.25)$$

We linearize the momentum near the Fermi energy, $k(\varepsilon) = k_F + \varepsilon/\hbar v_F$.

The determinant (23.5) evaluates to

$$\begin{aligned} \ln \det[1 - M(\varepsilon)] &= \\ &= g \ln [1 + \alpha^4 e^{4i\varepsilon L/\hbar v_F} - 2\alpha^2 e^{2i\varepsilon L/\hbar v_F} \cos \phi]. \end{aligned} \quad (23.26)$$

The sum over Matsubara frequencies decays on the scale of $\hbar v_F/L$, which is much smaller than Δ_0 in the long-junction regime. We may thus approximate $\alpha(i\omega) \approx \alpha(0) = -i$, when

$$\ln \det[1 - M(i\omega)] = g \ln |1 + e^{i\phi} e^{-2\omega L/\hbar v_F}|, \quad (23.27)$$

resulting in the zero-temperature free energy

$$F = \frac{g\hbar v_F}{2\pi L} \text{Re Li}_2 (-e^{i\phi - 2\gamma L/\hbar v_F}). \quad (23.28)$$

The function $\text{Li}_2(x)$ is the dilogarithm.

The corresponding supercurrent is

$$I = \frac{gev_F}{\pi L} \arctan \left[\frac{\sin \phi}{e^{2\gamma L/\hbar v_F} + \cos \phi} \right], \quad (23.29)$$

see Fig. 23.3. As a check, in the limit $\gamma \rightarrow 0$ we recover the known sawtooth ϕ -dependence [25, 26, 27],

$$\lim_{\gamma \rightarrow 0} I = \frac{gev_F}{\pi L} \arctan \tan(\phi/2) = \frac{gev_F}{\pi L} \phi, \quad -\pi < \phi < \pi. \quad (23.30)$$

Refs. [6, 7] have qualitatively similar plots to Fig. 23.3, for different models of coupling to electron reservoirs that do not allow for a closed-form solution.

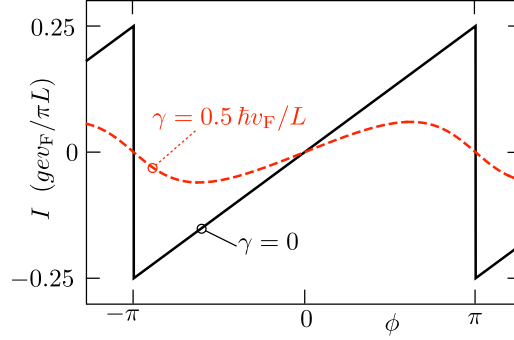


Figure 23.3: Same as Fig. 23.2, but now for a single-mode ballistic SNS junction in the long junction limit (length L large compared to the superconducting coherence length $\xi_0 = \hbar v_F / \Delta_0$). The plot is computed from Eq. (23.29).

Connection to the non-Hermitian Josephson effect — The coupling of the Josephson junction to the electron reservoir pushes the poles of its scattering matrix into the lower half of the complex plane, down to $E = \pm \varepsilon_A - i\gamma$. These poles can be considered as the complex eigenvalues of an effective non-Hermitian Hamiltonian $\mathcal{H}$. The open Josephson junction thus provides a physical realization of the non-Hermitian Josephson effect studied in Refs. [13, 14, 15, 16]. Different generalizations have been proposed for the relation $I \propto dE/d\phi$ when E is complex. Let us compare these with our findings.

To be specific, we consider the weakly coupled quantum dot Josephson junction, with effective Hamiltonian [28, 29, 30]

$$\mathcal{H} = \begin{pmatrix} \varepsilon_R - i\gamma & \frac{1}{2}e^{i\phi/2}\Gamma_1 + \frac{1}{2}e^{-i\phi/2}\Gamma_2 \\ \frac{1}{2}e^{-i\phi/2}\Gamma_1 + \frac{1}{2}e^{i\phi/2}\Gamma_2 & -\varepsilon_R - i\gamma \end{pmatrix}. \quad (23.31)$$

One checks that the eigenvalues are $\pm \varepsilon_A - i\gamma$, with ε_A given by Eq. (23.14).

Refs. [14] and [15] argue that only the real part of E contributes to the physical supercurrent, which in this case would imply no effect from the coupling to the reservoir. Shen, Lu, Lado, and Trif [16], in a remarkable recent paper, give instead the zero-temperature relation

$$I = -\frac{2e}{\pi \hbar} \frac{d}{d\phi} \text{Im Tr}(\mathcal{H} \ln \mathcal{H}), \quad (23.32)$$

which reduces precisely to our result (23.17) (with $g = 2$) [72].

Conclusion — In summary, we have calculated how the coupling to a gapless electron reservoir in the normal state reduces the supercurrent through a Josephson junction. A simple answer is obtained for the model where the escape rate $1/\tau \equiv 2\gamma/\hbar$ of quasiparticles into the reservoir is spatially uniform. At zero temperature the reduction factor for a given Andreev

level equals $(2/\pi) \arctan(2\varepsilon_A\tau/\hbar)$. This applies to a weakly coupled quantum dot Josephson junction, or to a short point contact (length L smaller than the coherence length ξ_0). A more complicated τ -dependence is obtained in a long junction ($L \gg \xi_0$), when also states above the gap contribute to the supercurrent.

24 Casimir-Josephson force on a point contact between two superconductors by Beenakker

Abstract

We calculate the elongation or contraction force F on a point contact (length L) connecting two superconductors with a phase difference ϕ . When L is small compared to the superconducting coherence length ξ_0 this force is given by $F = -(\Delta_0/\pi\xi_0) \ln(\xi_0/L) \cos\phi$ per spin-degenerate transverse mode. Quantum fluctuations in states from the continuous spectrum outside the superconducting gap Δ_0 give the dominant contribution to this force, which may be understood as the superconducting counterpart of the electromagnetic Casimir force. We compare with earlier work that only included contributions from the discrete spectrum of Andreev levels.

24.0.1 Introduction

The free energy Ω of a Josephson junction, a weak link between two superconductors, depends on the superconducting phase difference ϕ across the junction as well as on the junction length L . The derivative of Ω with respect to ϕ gives the supercurrent $I = (2e/\hbar)d\Omega/d\phi$, the derivative with respect to L gives the elongation or contraction force $F = -d\Omega/dL$ on the junction [? ? ? ? ?].

This force is the superconducting counterpart of the electromagnetic Casimir force [?], as one can see from the expression for a single-mode junction in the large- L limit [? ? ? ? ?],

$$F = -\frac{g\hbar v_F}{24\pi L^2}(\pi^2 - 3\phi^2), \quad |\phi| < \pi, \quad (24.1)$$

with v_F the Fermi velocity and g a factor that counts spin and other degeneracies. For $\phi = 0$, $g = 1$, $v_F = c$ this is precisely the attractive Casimir force produced by vacuum fluctuations of a scalar one-dimensional wave between perfectly reflecting metallic mirrors [? ? ? 72]. In the Josephson junction the superconducting condensate at zero temperature plays the role of the electromagnetic vacuum.

The result (24.1) holds if the junction is long compared to the superconducting coherence length $\xi_0 = \hbar v_F/\Delta_0$ (for a superconducting gap Δ_0 and assuming ballistic transport through the junction). The short-junction regime $L \ll \xi_0$ has no electromagnetic analogue. This regime is relevant for a point contact Josephson junction (as in Fig. 24.1). In what follows we will study that regime theoretically.

One might surmise that the point contact length L should be replaced by an effective length $L + \xi_0$, to estimate $F \simeq \hbar v_F/\xi_0^2$ in the short-junction regime. The pioneering work by Krive *et al.* [?] arrived at this expression. As we shall see, a more complete calculation introduces a logarithmic L -dependence $\propto \ln(\xi_0/L)$, due to ϕ -dependent contributions from the continuous spectrum (neglected in Ref. [?]).

The insight that the continuous spectrum gives a non-analytic contribution $\propto \ln L$ to the Josephson free energy is not new [? ? ?]. What is special about the Casimir-Josephson force is that the $\ln L$ term is the *leading contribution* in the short-junction limit — while for the supercurrent the leading contribution is L -independent.

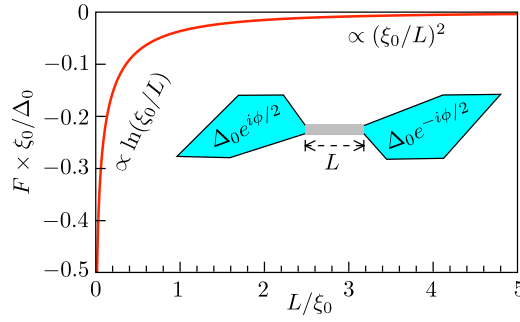


Figure 24.1: Dependence of the force on a Josephson junction on the junction length L (computed from Eq. (24.10) for superconducting phase difference $\phi = 0$). The force is negative, so it is a contraction force, crossing over to an elongation force for $\phi \gtrsim \pi/2$. The plot refers to a single transverse mode. The inverse-square large- L asymptotic is from Ref. [?], the logarithmic small- L asymptotic is obtained here.

The outline of this paper is as follows. In the next section we summarize the scattering matrix representation [? 17] of the Josephson free energy Ω , which is a convenient starting point because it treats the discrete and continuous spectrum on the same footing. We apply this to a ballistic point contact in Sec. 24.0.3, to calculate the force $F = -d\Omega/dL$ for $L \ll \xi_0$. The effect of a tunnel barrier (transmission probability Γ) in the point contact is then treated in Sec. 24.0.4. The results of these two sections differ from those of Ref. [?] in two aspects, both attributable to the continuous spectrum: an enhancement of the force by a factor $\ln(\xi_0/L)$ and a reduction of the force by a factor $\sqrt{\Gamma}$.

24.0.2 Scattering formulation

The free energy of a Josephson junction at temperature T is given by [18]

$$\Omega = -gT \int_0^\infty d\varepsilon \rho(\varepsilon) \ln[2 \cosh(\varepsilon/2T)]. \quad (24.2)$$

(We set $\hbar$ and Boltzmann's constant k_B equal to unity.) The density of states $\rho(\varepsilon)$ refers to the electron-hole symmetric spectrum of the Bogoliubov-De Gennes Hamiltonian H_{BdG} . We ignore ϕ -independent contributions to the free energy from the superconducting reservoirs [22].

The spectrum of H_{BdG} consists of bound states (Andreev levels) for $\varepsilon < \Delta_0$ and a continuous spectrum for $\varepsilon > \Delta_0$. Scattering theory [17] includes both contributions in the determinantal formula

$$\rho(\varepsilon) = -\frac{1}{\pi} \text{Im} \frac{d}{d\varepsilon} \ln \det[1 - R_A(\varepsilon + i0^+) S_N(\varepsilon + i0^+)], \quad (24.3)$$

in terms of the Andreev reflection matrix R_A and the scattering matrix S_N of the junction in the normal state.

The key physical ingredient in this representation of the density of states is the separation of length scales between Andreev reflection [20], the conversion of an electron into a hole on the length scale of the coherence length ξ_0 , and normal scattering processes on the length scale of the Fermi wave length λ_F . The ratio $\lambda_F/\xi_0 \simeq \Delta_0/E_F \ll 1$ in a superconductor.

The Andreev reflection matrix has the block structure

$$R_A(\varepsilon) = \alpha(\varepsilon) \begin{pmatrix} 0 & r_A \\ r_A^* & 0 \end{pmatrix}, \quad r_A = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}, \quad (24.4)$$

$$\alpha(\varepsilon) = \varepsilon/\Delta_0 - i\sqrt{1 - \varepsilon^2/\Delta_0^2}.$$

The blocks r_A^* and r_A describe Andreev reflection from electron to hole and from hole to electron, respectively. The diagonal elements of the submatrix r_A refer to Andreev reflection from the left and right superconductor, at a phase $\pm\phi/2$. Andreev reflection happens with unit probability for $\varepsilon < \Delta_0$, at larger energies $\alpha(\varepsilon) = \varepsilon/\Delta_0 - \sqrt{\varepsilon^2/\Delta_0^2 - 1}$ decays to zero. The normal scattering matrix S_N is block-diagonal,

$$S_N(\varepsilon) = \begin{pmatrix} s_0(\varepsilon) & 0 \\ 0 & s_0^*(-\varepsilon) \end{pmatrix}, \quad (24.5)$$

with electron scattering matrix s_0 unitary at all energies.

Substitution of Eqs. (24.4) and (24.5) into Eq. (24.3) gives the determinant [17]

$$\begin{aligned} \rho(\varepsilon) &= -\frac{1}{\pi} \text{Im} \frac{d}{d\varepsilon} \ln \det[1 - M(\varepsilon + i0^+)], \\ M(\varepsilon) &= \alpha(\varepsilon)^2 r_A^* s_0(\varepsilon) r_A s_0^*(-\varepsilon). \end{aligned} \quad (24.6)$$

As a final step, the slowly converging integration over energies in Eq. (24.2) can be transformed into a more rapidly converging sum over Matsubara frequencies. The resulting free energy is given by [?]]

$$\Omega = -gT \sum_{p=0}^{\infty} \ln \det[1 - M(i\omega_p)], \quad \omega_p = (2p+1)\pi T. \quad (24.7)$$

In the zero-temperature limit the sum over p can be replaced by an integral over ω ,

$$\lim_{T \rightarrow 0} \Omega = -\frac{g}{2\pi} \int_0^{\infty} d\omega \ln \det[1 - M(i\omega)]. \quad (24.8)$$

24.0.3 Ballistic point contact

For simplicity we consider an N -mode point contact which does not mix the transverse modes. For each mode the scattering matrix s_0 is a 2×2 matrix of reflection amplitudes (on the diagonal) and transmission amplitudes (off-diagonal).

In this section we assume ballistic transport, so only the transmission amplitudes are nonzero,

$$s_0(\varepsilon) = e^{ik(\varepsilon)L} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (24.9)$$

Here $k(\varepsilon)$ is the momentum of the n -th electron mode in the point contact region between the superconductors. For $\varepsilon \ll E_F$ we may linearize its energy dependence, $k(\varepsilon) = k_F + \varepsilon/v_F$, with k_F and v_F the momentum and velocity of the mode at the Fermi level.

The contribution of each mode to the free energy (24.8) then takes the form

$$\begin{aligned} \Omega &= -\frac{g\Delta_0}{2\pi} \int_0^{\infty} d\omega \ln[1 + \beta(\omega)^4 e^{-4\omega L/\xi_0} \\ &\quad + 2\beta(\omega)^2 e^{-2\omega L/\xi_0} \cos \phi] \\ &= -\frac{g\Delta_0}{\pi} \text{Re} \int_0^{\infty} d\omega \ln[1 + \beta(\omega)^2 e^{i\phi - 2\omega L/\xi_0}], \end{aligned} \quad (24.10)$$

$$\beta(\omega) = \omega - \sqrt{1 + \omega^2}. \quad (24.11)$$

We seek the force $F = -d\Omega/dL$.

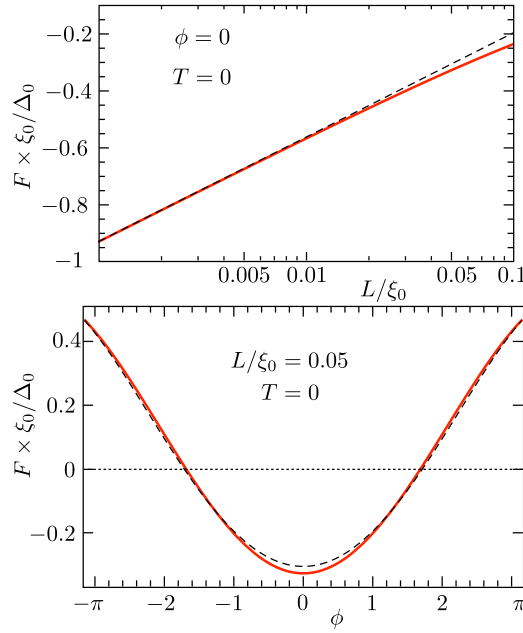


Figure 24.2: Dependence of the force on a Josephson junction on the junction length L for $\phi = 0$ (top plot, on a log-linear scale) and dependence on the phase difference ϕ for $L/\xi_0 = 0.05$ (bottom plot). The red solid curves are computed from Eq. (24.10), the black dashed curves are the small- L asymptotic (24.14). The values of the force refer to a single transverse mode. The ϕ -dependent plot repeats with 2π -periodicity outside of the interval $-\pi < \phi < \pi$. These are results at zero temperature.

For $L \gg \xi_0$ we may set the function $\beta(\omega)$ to unity in Eq. (24.10), and we recover the known result (24.1) for the force in the long-junction regime, since

$$\text{Re} \int_0^\infty dx \ln(1 + e^{i\phi - x}) = \frac{\pi^2}{12} - \frac{\phi^2}{4}, \quad |\phi| < \pi. \quad (24.12)$$

In the opposite short-junction regime $L \ll \xi_0$ the L -dependence of the integral (24.10) is governed by the large- ω range, where the integrand decays as $\omega^{-2}e^{-2\omega L/\xi_0}$. This gives a contribution to the force integral which decays as $\omega^{-1}e^{-2\omega L/\xi_0}$, producing a logarithmic L -dependence upon integration over ω ,

$$F = -\frac{g\Delta_0}{2\pi\xi_0} [\ln(\xi_0/L) \cos \phi + \mathcal{O}(1)], \quad L \ll \xi_0. \quad (24.13)$$

A calculation to higher order in L/ξ_0 in the Appendix gives

$$F = -\frac{g\Delta_0}{2\pi\xi_0} [\ln(\xi_0/L) \cos \phi - \gamma_{\text{Euler}} \cos \phi - \frac{1}{2}(1 - \phi \sin \phi) + \mathcal{O}(L)], \quad |\phi| < \pi. \quad (24.14)$$

In Fig. 24.2 we compare the full expression (24.10) with the small- L asymptotics (24.14) for the Casimir-Josephson force. This is at zero temperature. The effect of a nonzero temperature $T \lesssim \Delta_0$ is shown in Fig. 24.3: It is predominantly a rounding of the cusp in the ϕ -dependence at $\phi = \pm\pi$. The scaling with L is not significantly affected.

The logarithmic L -dependence of the force is due to the contribution from the continuous spectrum, the energy range $\varepsilon > \Delta_0$ in the free energy integral (24.2). To see this, note that the

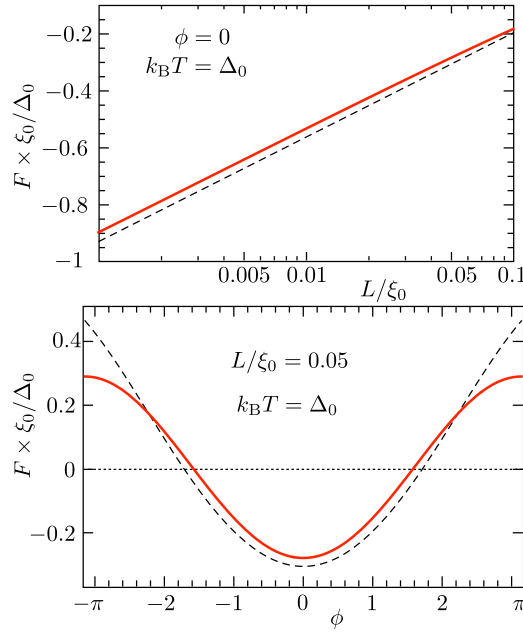


Figure 24.3: Same as Fig. 24.2, but now at temperature $T = \Delta_0$, when the integral over ω is replaced by the sum over $\omega_p = (2p + 1)\pi T$ in Eq. (24.7). The dashed curves are the zero-temperature result (24.14), included for comparison.

discrete spectrum consists for $L \ll \xi_0$ of a single excitation energy $\varepsilon_0 > 0$ per transverse mode, at [?]

$$\varepsilon_0 = \Delta_0 \left(\cos(\phi/2) - \frac{L}{2\xi_0} |\sin \phi| + \mathcal{O}(L^2) \right), \quad |\phi| < \pi. \quad (24.15)$$

This contributes to the Casimir-Josephson force an L -independent amount $d\varepsilon_0/dL \simeq \Delta_0/\xi_0$, without the $\ln(\xi_0/L)$ factor.

The integral over the continuous spectrum is cut off by the band width E_F . This cut off is ineffective if $L > \lambda_F$, but for smaller junction lengths the $\ln(\xi_0/L)$ increase saturates at $\ln(\xi_0/\lambda_F) \propto \ln(E_F/\Delta_0)$.

A final remark for this section: As one can see by comparing Eqs. (24.13) and (24.15), the continuous spectrum changes the ϕ -dependence of the force from $|\sin \phi|$ to $\cos \phi$. The removal of the cusp singularity at $\phi = 0$ changes how the force responds to the presence of a tunnel barrier in the point contact, as we shall see in the next section.

24.0.4 Point contact with a tunnel barrier

A tunnel barrier with transmission probability $\Gamma \in (0, 1)$ at the center of the point contact modifies the scattering matrix (24.9) and zero-temperature free energy (24.8) as follows:

$$s_0(\varepsilon) = e^{i(k_F + \varepsilon/v_F)L} \begin{pmatrix} \sqrt{1-\Gamma} & \sqrt{\Gamma} \\ \sqrt{\Gamma} & -\sqrt{1-\Gamma} \end{pmatrix}, \quad (24.16)$$

$$\begin{aligned} \Omega = & -\frac{g\Delta_0}{2\pi} \int_0^\infty d\omega \ln \left[1 + \beta(\omega)^4 e^{-4\omega L/\xi_0} \right. \\ & \left. + 2\beta(\omega)^2 e^{-2\omega L/\xi_0} (1 - 2\Gamma \sin^2(\phi/2)) \right]. \end{aligned} \quad (24.17)$$

So the expressions from the previous section can be used with the substitution

$$\cos \phi \mapsto 1 - 2\Gamma \sin^2(\phi/2). \quad (24.18)$$

For $\Gamma \ll 1$ the ϕ -dependence of the force on the Josephson junction is reduced by a factor Γ . In particular, in the short-junction regime $L \ll \xi_0$ one has, in view of Eq. (24.13),

$$F = \frac{g\Delta_0}{\pi\xi_0} \ln(\xi_0/L) \Gamma \sin^2(\phi/2) + \phi\text{-independent terms.} \quad (24.19)$$

Notice that the same substitution (24.18) would transform $|\sin \phi|$ into $2\sqrt{\Gamma}|\sin(\phi/2)|$ for $\Gamma \ll 1$. This explains why Ref. [?] found that a tunnel barrier reduces the ϕ -dependence of the force by a factor $\sqrt{\Gamma}$, rather than by a factor Γ .

24.0.5 Conclusion

In summary, we have calculated the Casimir-like force on a point contact Josephson junction, focusing on the regime that the length L of the junction is short compared to the superconducting coherence length ξ_0 . In the limit $L/\xi_0 \rightarrow 0$ the supercurrent through the junction is fully determined by the discrete spectrum of Andreev levels, confined to the junction by the superconducting gap Δ_0 .

We have shown that the same does not apply to the elongation or contraction force on the junction: the continuous spectrum of states outside the gap qualitatively modifies the dependence of the force on both the junction length L and the superconducting phase difference ϕ : The L -dependence acquires a logarithmic factor $\ln(\xi_0/L)$ and the ϕ -dependence becomes more sensitive to the presence of a tunnel barrier in the point contact (decreasing $\propto \Gamma$ rather than $\propto \sqrt{\Gamma}$ with the tunnel probability Γ).

In a ballistic point contact ($\Gamma = 1$) with N spin-degenerate transverse modes ($g = 2$) the difference δF between the Casimir-Josephson force at $\phi = 0$ and $\phi = \pi$ is given for $L \ll \xi_0$ by

$$\delta F = \frac{2N\Delta_0}{\pi\xi_0} \ln(\xi_0/L). \quad (24.20)$$

The logarithmic enhancement factor saturates at $\ln(E_F/\Delta_0)$ for $L \lesssim \lambda_F$. This will be at best a factor of 10, and since $\Delta_0/\xi_0 \lesssim 10^{-15}$ N one needs a large number of modes, $N \gtrsim 10^4$, to reach a measurable force in the 0.1 nN range.

An additional complication, pointed out by Krive *et al.* [?], is that N may itself be dependent on the length of the junction (decreasing with increasing L if the junction is elongated at constant volume). This will induce a ϕ -dependent contraction force $\simeq (dN/dL)\Delta_0$ of a more mundane origin than the Casimir-Josephson force originating from the L -dependence of the Andreev spectrum. A coupling between N and L would thus need to be avoided. Other elastic forces, e.g. due to a bending of the point contact, are ϕ -independent and therefore do not contribute to δF , but a successful measurement of the Casimir-Josephson force remains challenging.

In closing we remark that our result (24.14) also implies a non-analytic finite- L correction to the supercurrent of a ballistic point contact,

$$\begin{aligned} I &= \frac{2e}{\hbar} \frac{g\Delta_0}{4} \sin(\phi/2) \\ &\quad - \frac{2e}{\hbar} \frac{g\Delta_0}{2\pi} \frac{L}{\xi_0} (\ln(\xi_0/L) \sin \phi - \frac{1}{2}\phi \cos \phi), \quad |\phi| < \pi, \end{aligned} \quad (24.21)$$

which complements the known result for a diffusive point contact [? ?].

24.0.6 Short-junction limit of the force integral

In view of Eq. (24.10), the Casimir-Josephson force $F = -d\Omega/dL$ on the Josephson junction at zero temperature is given by the integral

$$F = -\frac{2g\Delta_0}{\pi\xi_0} \operatorname{Re} f(L/\xi_0),$$

$$f(L) = \int_0^\infty d\omega \frac{\omega\beta(\omega)^2 e^{i\phi-2\omega L}}{1 + \beta(\omega)^2 e^{i\phi-2\omega L}}. \quad (24.22)$$

Note that $\omega\beta(\omega)^2 \rightarrow \frac{1}{4}\omega^{-1}$ for $\omega \rightarrow \infty$, so a logarithmic singularity prevents us from directly taking the short-junction limit $L/\xi_0 \rightarrow 0$.

We isolate the logarithmic singularity by subtracting

$$h(L) = \int_0^\infty d\omega (1 + 4\omega)^{-1} e^{i\phi-2\omega L}$$

$$= -\frac{1}{4} \operatorname{Ei}(-L/2) e^{L/2+i\phi}. \quad (24.23)$$

The difference has the short-junction limit (for $|\phi| < \pi$)

$$\lim_{L \rightarrow 0} \operatorname{Re} [f(L) - h(L)] = -\frac{1}{8} (1 + (2 \ln 2) \cos \phi - \phi \sin \phi). \quad (24.24)$$

Adding the small- L expansion of the exponential integral function $\operatorname{Ei}(z) = -\int_{-z}^\infty t^{-1} e^{-t} dt$,

$$\operatorname{Re} h(L) = -\frac{1}{4} [\ln(L/2) + \gamma_{\text{Euler}}] \cos \phi + \mathcal{O}(L), \quad (24.25)$$

we arrive at

$$\operatorname{Re} f(L) = -\frac{1}{4} (\ln L + \gamma_{\text{Euler}}) \cos \phi - \frac{1}{8} (1 - \phi \sin \phi) + \mathcal{O}(L). \quad (24.26)$$

Substitution of the expansion (24.26) into Eq. (24.22) gives the result (24.14) from the main text.

Eq. (24.26) should be repeated with 2π -periodicity outside of the interval $-\pi < \phi < \pi$. The L -independent $\phi \sin \phi$ term then produces a cusp singularity at $\phi = \pm\pi$ while the term $\propto \ln L$ is smooth at $\phi = \pm\pi$. This can be understood because the cusp is due to states in the gap crossing the Fermi level, while the $\ln L$ factor is due to states outside of the gap, so the two types of singularities do not coexist in the same term.

Part VIII

Important Experiments

25 Microwave susceptibility observation of interacting many-body Andreev states by Fatemi et al.

Abstract

Electrostatic charging affects the many-body spectrum of Andreev states, yet its influence on their microwave properties has not been elucidated. We developed a circuit quantum electrodynamics probe that, in addition to transition spectroscopy, measures the microwave susceptibility of different states of a semiconductor nanowire weak link with a single dominant (spin-degenerate) Andreev level. We found that the microwave susceptibility does not exhibit a particle-hole symmetry, which we qualitatively explain as an influence of Coulomb interaction. Moreover, our state-selective measurement reveals a large, π -phase shifted contribution to the response common to all many-body states which can be interpreted as arising from a phase-dependent continuum in the superconducting density of states.

Andreev states are the supercurrent-carrying fermionic modes that govern the electro-dynamical response of Josephson devices. In devices hosting only a few transport channels, individual states may become energetically well-separated and thus addressable. This direct access to Andreev states has been leveraged to discover new phenomena in mesoscopic superconductivity and unveil applications such as Andreev qubits [19]. These experiments probed the microwave frequency response of discrete Andreev states in different regimes: from a minimal configuration of an atomic point contact with one strongly dispersing Andreev level, to multi-state configurations in long nanowire weak links where spin-orbit effects become important.

Our understanding of the microwave frequency electrodynamics involving Andreev states has so far relied on noninteracting pictures focusing on the sub-gap levels [20]. While such pictures describe atomic point contacts well [19], they have two blind spots for any finite-length weak link. First, charging energy should be present when the electrons experience a nonzero dwell time in the weak link. Yet, to our knowledge, the impact of charging energy on the microwave response of Andreev states has not been investigated. This is in contrast to measurements that have revealed a rich interplay between superconductivity and charging effects in dc transport through quantum dots [21]. Second, under the same conditions, the spectral continuum outside the superconducting gap (which we will refer to as "the continuum" for brevity) also contributes to the supercurrent [22] and therefore is electro-dynamically active in finite-length weak links. However, the dynamics of the continuum have not been isolated from the contributions of the subgap Andreev states. This requires measurements that resolve individual many-body configurations of the system.

Are charging effects and the continuum relevant for the microwave response of superconductor-semiconductor weak links? Here, we answer affirmatively by performing state-resolved microwave response measurements with a circuit quantum electrodynamics (cQED) probe. Application of cQED techniques has revealed different quasiparticle occupation configurations of Andreev levels [19], which is possible because the different states of the quantum system have different electro-dynamical susceptibility [23]. However, beyond state determination, the magnitude and dispersion of the response functions of individual states carry a wealth of physical information [24]. This information

complements that of microwave and tunneling spectroscopy which are restricted to transitions between states of the same and opposite parity, respectively.

We measured a Josephson semiconductor nanowire with a single dominant Andreev level coupled to a superconducting microwave resonator. We extracted the admittance and transition spectrum of different many-body quasiparticle configurations and find two qualitative differences to the standard theory [? ? ? ? ?]. First, the microwave responses of the even-parity states are not symmetric about the odd state, which we refer to as a violation of a particle-hole symmetry. Second, all states exhibit a common π -shifted contribution to their phase dispersion. We interpret these observations as the qualitative effects of a charging energy in the weak link and a phase-dispersing continuum, respectively. Our recently developed theory [?] can account for these discrepancies. Finally, we measured the fermion parity polarization as a function of phase and gate voltage. The switches of the polarization are consistent with $0\text{-}\pi$ transitions, while the incompleteness of the polarization indicates nonequilibrium parity dynamics. These observations lay a foundation for investigating the interplay of Coulomb interactions with superconducting pairing in conventional and topological superconducting mesoscopic devices with cQED probes.

The salient aspects of our cQED setup are depicted in Fig. 25.1(a). We grew an indium arsenide nanowire (micrograph) with two facets covered by epitaxial aluminum on which a 350 nm weak-link region was later uncovered. The nanowire weak link is embedded in a superconducting loop that partially determines the inductance of the differential mode of a superconducting microwave resonator (gray). An external coil biased with a dc current inserts magnetic flux Φ through the loop in order to phase bias the nanowire $\varphi \approx \Phi/\varphi_0$ ($\varphi_0 = \hbar/2e$ is the reduced superconducting magnetic flux quantum). The bare resonator frequency $f_{r,b} = 4,887$ GHz was measured by depleting the weak link with a negative gate voltage V_g .

Discrete Andreev states were then introduced to the weak link by opening conduction channels via increasing V_g . While doing this, we conducted two-tone spectroscopy measurements which detect microwave transitions between quantum states of the weak link. At low gate voltages, we detected gate "bias points" at which a single dispersing level was identifiable from the spectrum, such as in Fig. 25.1(b). The discrete transition frequency depends strongly on phase, consistent with a transition between Andreev states at frequency f_A , characterized as the excitation of a pair of localized quasiparticles. In similar microwave experiments, this transition is often compared with the short junction model [? 19? ?], which requires $f_A = 2\Delta_{Al}/h$ for $\Phi = 0$. However, the application of this model is inappropriate here since the measured transition frequency is well below twice the gap of the superconducting leads $f_A < 25$ GHz $\ll 2\Delta_{Al}/h \approx 100$ GHz.

Resolution of this discrepancy requires a model that includes a nonzero dwell time for electrons in the weak link. Here we focus on a minimal phenomenological model: a quantum dot with a single level coupled to two superconducting leads [? ? 33], schematically shown in Fig. 25.1(c) ¹. Within this model, dot states with zero and two electrons are hybridized due to the proximity effect between the dot and the leads, parameterized by the tunneling rates $\Gamma_{L,R}$. In the absence of the charging effect, these states would split symmetrically by $\pm E_A$ with respect to the one-electron state. This splitting leads to the microwave transition frequency $hf_A = 2E_A$. As long as the proximity effect is weak, the Andreev states formed by this hybridization remain well-detached from the superconducting gap which leads to $f_A \ll 2\Delta_{Al}/h$ consistent with the experiment. States with a single electron at the dot are also necessarily present in the system. However, when only a single level is present, we cannot probe them with two-tone spectroscopy which preserves fermion parity (inset of Fig. 25.1(b)).

We overcome this limitation with our cQED setup by measuring the discrete frequency shifts

¹Models with weak link length of order the coherence length also exhibit dwell time. Such models admit additional doublets and add theoretical complications, see supplement V.E.

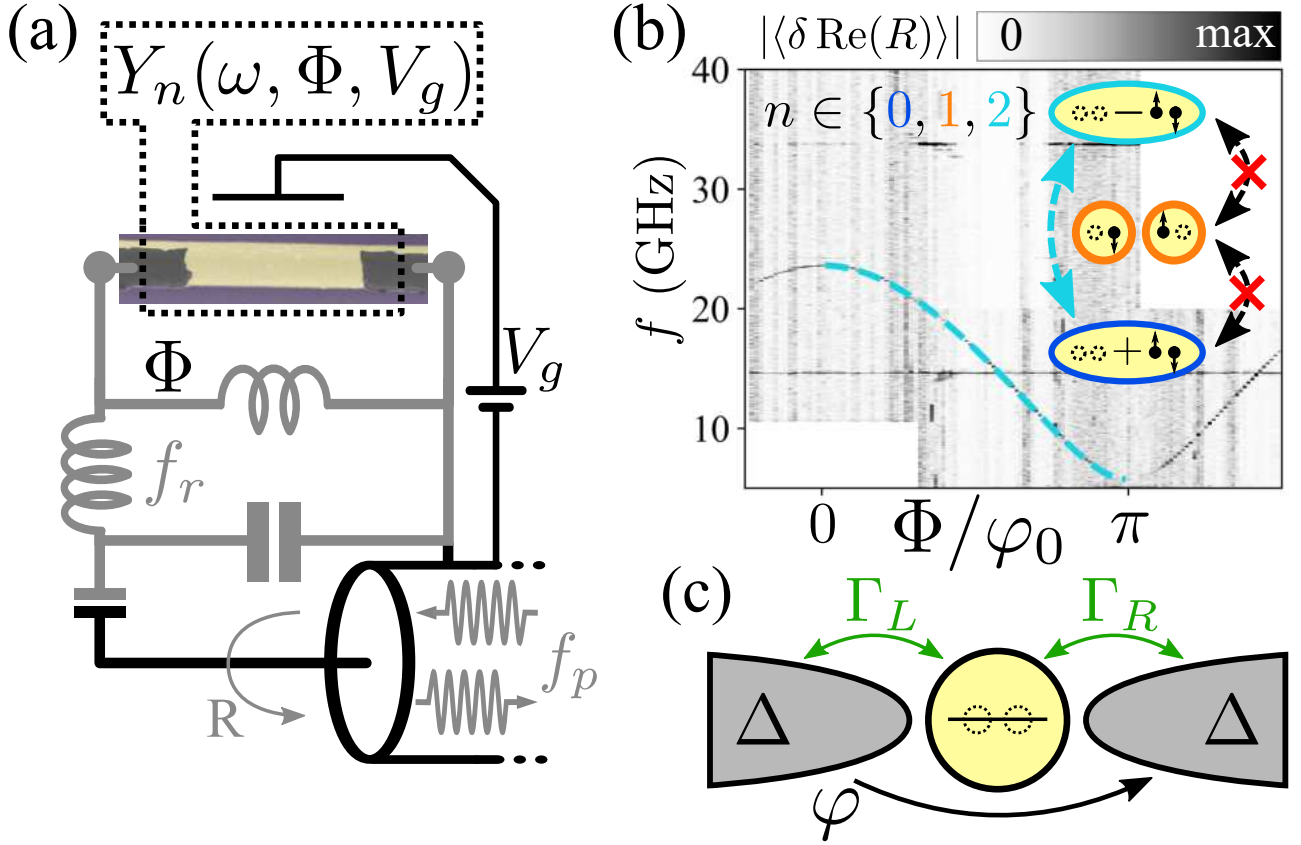


Figure 25.1: (a) Schematic of the cQED setup (description in main text), including a colorized scanning electron micrograph of a representative Josephson nanowire weak link (aluminum is dark gray, uncovered region is yellow and 500 nm long in this image, but 350 nm long in the device measured for this report). A probe tone f_p reflects with a frequency-dependent reflection amplitude R which is recorded to measure the system. (b) Two-tone spectroscopy as a function of drive frequency and phase bias revealed a single dispersing Andreev transition, with model fit (dashed cyan line, see Figure 2 for parameters) corresponding to the transition between the even parity states (inset schematic). Parity-switching transitions are forbidden (black arrows with red X). $n \in \{0, 1, 2\}$ label the eigenstates. The horizontal lines near 15 and 34 GHz are higher harmonics of the resonator. (c) Minimal picture of a single-level quantum dot coupled to superconducting reservoirs with gap Δ and phase drop φ .

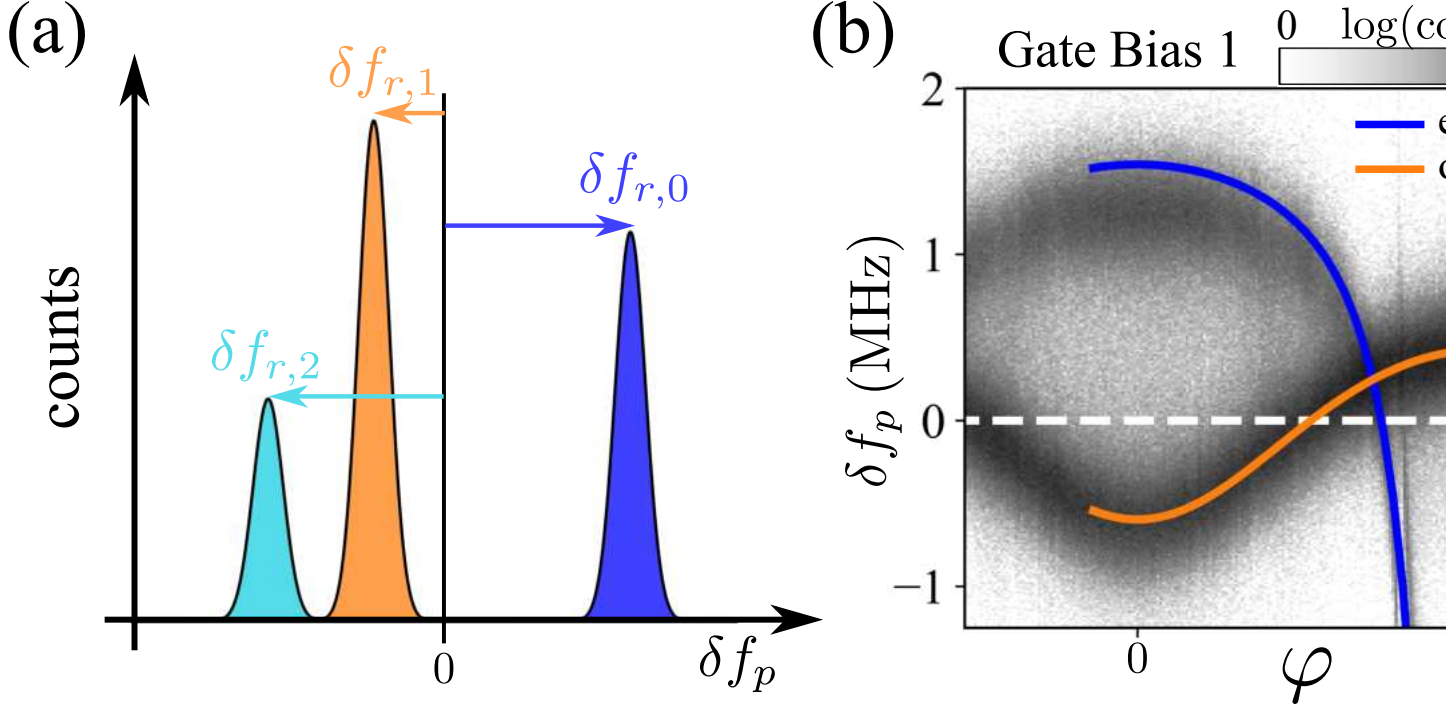


Figure 25.2: (a) Schematic of the resonator frequency shift histogram: while sweeping probe frequency relative to the bare resonator frequency $\delta f_p = f_p - f_{r,b}$, we record counts when the reflection signal R indicates a resonance. The magnitude of the frequency shift $\delta f_{r,n}$ is proportional to the admittance Y_n of state n , and normalized counts under each peak give the probability of the state. The widths of the distributions are the same, as they derive from the signal-to-noise ratio of the measurement (see supplement for details on the measurement). (b) Resonator frequency shift histogram from the same bias point as Fig. 25.1(b). Solid lines are model fits of odd (orange) and even (blue) state dispersive shifts $\delta f_{r,n}$ relative to the bare resonator frequency $f_{r,b}$ (white dashed). The fit parameters in GHz are $\Delta/h = 28.6 \pm 0.3$, $\Gamma_L/h = 6.1 \pm 0.03$, $\Gamma_R/h = 10.4 \pm 0.06$, $U/h = 21.6 \pm 0.8$, and a small offset of $\delta f_{\text{off}} = (0.045 \pm 0.004)\text{MHz}$ to the resonator which is 2% of the resonator linewidth of 2 MHz. See supplement for a discussion about the small value for Δ , which we believe to arise from finite-length effects. (c) Similar to (b) for bias point 2 in a narrow range of phase (see Fig. S8 for a larger phase range). Black circles mark extracted $\delta f_{r,n}$ at $\varphi = \pi$, and the horizontal red bar marks the average of the even-parity frequencies $\frac{1}{2}(\delta f_{r,0} + \delta f_{r,2})$ at $\varphi = \pi$, displaying clear violation of the half-sum rule $\frac{1}{2}(\delta f_{r,0} + \delta f_{r,2}) - \delta f_{r,1} = 0$ (standard deviation is smaller than the red bar thickness). The fit parameters in GHz are $\Delta/h = 25 \pm 6$, $\Gamma_L/h = 7.0 \pm 1.6$, $\Gamma_R/h = 8.1 \pm 1.8$, $U/h = 74 \pm 9$, $\delta f_{\text{off}} = (1.8 \pm 1.2) \times 10^{-4}$.

of the resonator. The frequency shifts $\delta f_{r,n}$ are determined by the state-dependent admittance Y_n of the weak link [? ?]: $\delta f_{r,n} \propto \text{Im}(Y_n(2\pi f_{r,b}))$ (see supplement for more on this). Here n labels the many-body state of the link: $n = 0, 2$ are the two even parity states and $n = 1$ is the spin-degenerate odd state, such that n can be related to the number of quasiparticles in the weak link.

The admittance is given by $Y_n(\omega) = L_n^{-1}/(i\omega) + Y_{n,\text{res}}(\omega)$. The first term describes the quasistatic response of the weak link and is determined by its inverse inductance, $L_n^{-1} = \varphi_0^{-2} \partial_\varphi^2 E_n$. This contribution carries information about the unique energy-phase relations $E_n(\varphi)$ of each microscopic state, which is out of reach of two tone spectroscopy. The second term, $Y_{n,\text{res}}(\omega)$, describes the resonant contribution to the admittance and is only appreciable close to the transition frequencies of the weak link in a given state.

Frequency shift histograms measured as a function of phase are shown for two gate bias points in Figure 25.2(b-c)² (details on the overlaid fit curves are given further below). Fig. 25.2(b) corresponds to the same bias point as Fig. 25.1(b). The lowest-energy even state $n = 0$ (blue) is identified by the strong dispersive shift of the resonator when the Andreev pair transition (Fig. 25.2(a)) approaches the resonator frequency near $\varphi = \pi$. At the second bias point, Fig. 25.2(c), the Andreev transition crosses the resonator frequency leading to an anti-crossing-like feature for the even states. There, the higher-energy even state ($n = 2$) is additionally visible (teal). At both bias points, an additional state is present in the data. The dispersive shift in this state does not have resonant signatures seen in the even states. We thus identify it as the state with odd fermion parity ($n = 1$).

These measurements exhibit two qualitative discrepancies with the phenomenological dot model in its simplest limit [? ?] of weak coupling to reservoirs and negligible charging energy. First, the odd state $n = 1$ exhibits a strong dispersion in the measurement. In contrast, in the above limit the energies E_n of the states $n \in \{0, 1, 2\}$ can be summarized by

$$E_n = (n - 1)E_A \quad (25.1)$$

which has $E_1 = 0$. Combined with the lack of transitions, this would result in $\delta f_{r,1} = 0$. Note also that $\delta f_{r,1}$ is π -phase shifted relative to $\delta f_{r,0}$ which excludes a parallel channel interpretation. The second qualitative feature is observed when all three possible occupations $n \in \{0, 1, 2\}$ are observed, as in Fig. 25.2(c). There, the average frequency shift of the even states (the "half-sum" of $n = 0, 2$) is different from that of the odd state ($n = 1$): $\frac{1}{2}(\delta f_{r,0} + \delta f_{r,2}) - \delta f_{r,1} \approx 0.6 \text{ MHz}$ ³. This observation is indicated by the red lines and does not require comparison to a model. Conventional noninteracting pictures for Andreev states (like the weak-coupling model described earlier) would predict zero such difference (i.e., the half-sum rule $\frac{1}{2}(\delta f_{r,0} + \delta f_{r,2}) - \delta f_{r,1} = 0$) due to a particle-hole symmetry, as in Eq. (25.1).

How may we reconcile these two discrepancies? First, the dispersion of the odd state may be accounted for by a phase-dispersing spectral continuum above the gap of the superconductor. The continuum is thus understood to produce a contribution E_{cont} common to all many-body states $n \in \{0, 1, 2\}$. As we describe below, the presence of the appreciable continuum contribution to energy is a consequence of intermediate-strength tunnel coupling between the dot and the superconducting leads. Second, the half-sum violation suggests that particle-hole symmetry is broken by a charging energy in the weak link. These notions prompt us to qualitatively generalize the prior formulation to

$$E_n = E_{\text{cont}} + (n - 1)E_A + (n - 1)^2 U_A \quad (25.2)$$

²Probe tone power corresponding to roughly 9 photons in the resonator.

³This naïve comparison is in principle susceptible to corrections to the dispersive shifts because they are solutions of a transcendental equation which is inherently nonlinear in $Y(\omega)$ (see supplement). We estimate the corrections to be less than 10% of the mismatch observed here. Furthermore, our probe power is small enough that frequency shifts due to nonlinearities are minimal.

where U_A is a term related to charging energy that is phase-dependent since the Andreev eigenstates are not charge eigenstates. Thus, both experimental discrepancies may be accounted for via inductive contributions to the dispersive shifts as follows from Eq. (25.2): $\partial_\varphi^2 E_1 = \partial_\varphi^2 E_{\text{cont}}$ and $\frac{1}{2}(\partial_\varphi^2 E_0 + \partial_\varphi^2 E_2) - \partial_\varphi^2 E_1 = \partial_\varphi^2 U_A$.

Now, with the aid of Fig. 25.3, we build up a detailed model based on our recent work [?] to which we will make a quantitative comparison with the data. We define a dot with a level of energy ϵ_0 with respect to the chemical potential, on-site Coulomb interaction strength U , and coupling rates $\Gamma_{L,R}$ to the superconducting leads with pair potential Δ and phase difference φ , schematically shown in Fig. 25.3(a-b). When isolated, the dot has four possible electron occupations that serve as basis states: zero and two electrons (even parity) as well as one electron (odd parity, spin degenerate). These basis states have energies 0, $2\epsilon_0 + U$, and ϵ_0 , respectively. The reservoir couplings $\Gamma_{L,R}$ hybridize the even states, which develop a strong dispersion with phase φ , while the odd states are not coupled to each other. The experimentally important configuration $\epsilon_0 = -U/2$ is depicted in Fig. 25.3(b-d). Here, the uncoupled even basis states are degenerate, and thus (provided $\Gamma_L \approx \Gamma_R$) the resulting Andreev states exhibit a weak anticrossing at phase $\varphi = \pi$, signifying resonant transmission [? ?].

The couplings $\Gamma_{L,R}$ also hybridize the dot states with the reservoir continuum (vertical green arrows in Fig. 25.3(b)). Consequently, the spectral continuum contributes $E_{\text{cont}}(\varphi)$ to the energy-phase relations of all many-body states. E_{cont} is π -phase shifted compared to the lowest even state. Notably, the odd state dispersion (orange curve in Fig. 25.3(c)) is entirely due to the continuum within this model.

Finally, nonzero charging energy $U > 0$ qualitatively influences the spectrum by raising the energy of the two-electron basis state, resulting in the term $(n-1)^2 U_A(\varphi)$ in Eq. (25.2). This term may cause an odd ground state for some or all phases (Fig. 25.3(c)). Moreover, $U_A(\varphi)$ is π -shifted like E_{cont} (Fig. 25.3(d)).

We now return to the experimental data of Fig. 25.2 to compare to this theoretical model. The model includes Coulomb interaction perturbatively, assuming $U \ll \Delta + \Gamma$ [?]. At each gate bias point, we performed a least-squares fit to all available dispersive data ($\delta f_{r,n}$ and the transition f_A) simultaneously. The bias points are gate voltage V_g “sweet spots” where the transition frequency was at a minimum, thus minimizing charge noise (see supplement). In our model, such a sweet spot corresponds to the condition $\epsilon_0 \equiv -U/2$ for a gate-controlled energy ϵ_0 .

The fitted model is overlaid on the data (Fig. 25.1(b) and Fig. 25.2(b) for gate bias point 1, Fig. 25.2(c) for gate bias point 2). We begin with bias point 1. The model quantitatively reproduces the π -phase shifted response of the odd parity state (orange) resulting from the continuum. The continuum contribution to the even state $n = 0$ is also important: it results in a lower admittance (smaller frequency shift) than would otherwise be expected, an important effect when incorporating such weak links into microwave circuits [? ? ?].

The model also accounts for the violation of the half-sum rule via the charging energy, as shown in Fig. 25.2(c). The theory correctly accounts for the sign of the effect $\frac{1}{2}(\delta f_{r,0} + \delta f_{r,2}) > \delta f_{r,1}$. However, the fit in Fig. 25.2(c) gives $U \gtrsim \Delta + \Gamma$, which is beyond the validity of our perturbative theory (see Fig. 25.2 caption and [?]). Nonetheless, our simple model allows us to see that charging effects in the weak link are appreciable ⁴.

As mentioned, charging effects of this strength can overcome superconducting pairing to produce an odd parity ground state. Within the model, the odd state is energetically favored for a range of phase and gate voltage centered at $\varphi = \pi$ and $\epsilon_0 = -U/2$. In order to check this in the experiment, we quantified the difference in the probability of the fermion parities

⁴We note that a naive estimation based on the dimensions of just the uncovered weak link region gives of order $U \sim 200$ GHz, whereas our fits produce values of U that are up to an order of magnitude smaller. This suggests strong screening by the epitaxial aluminum leads.

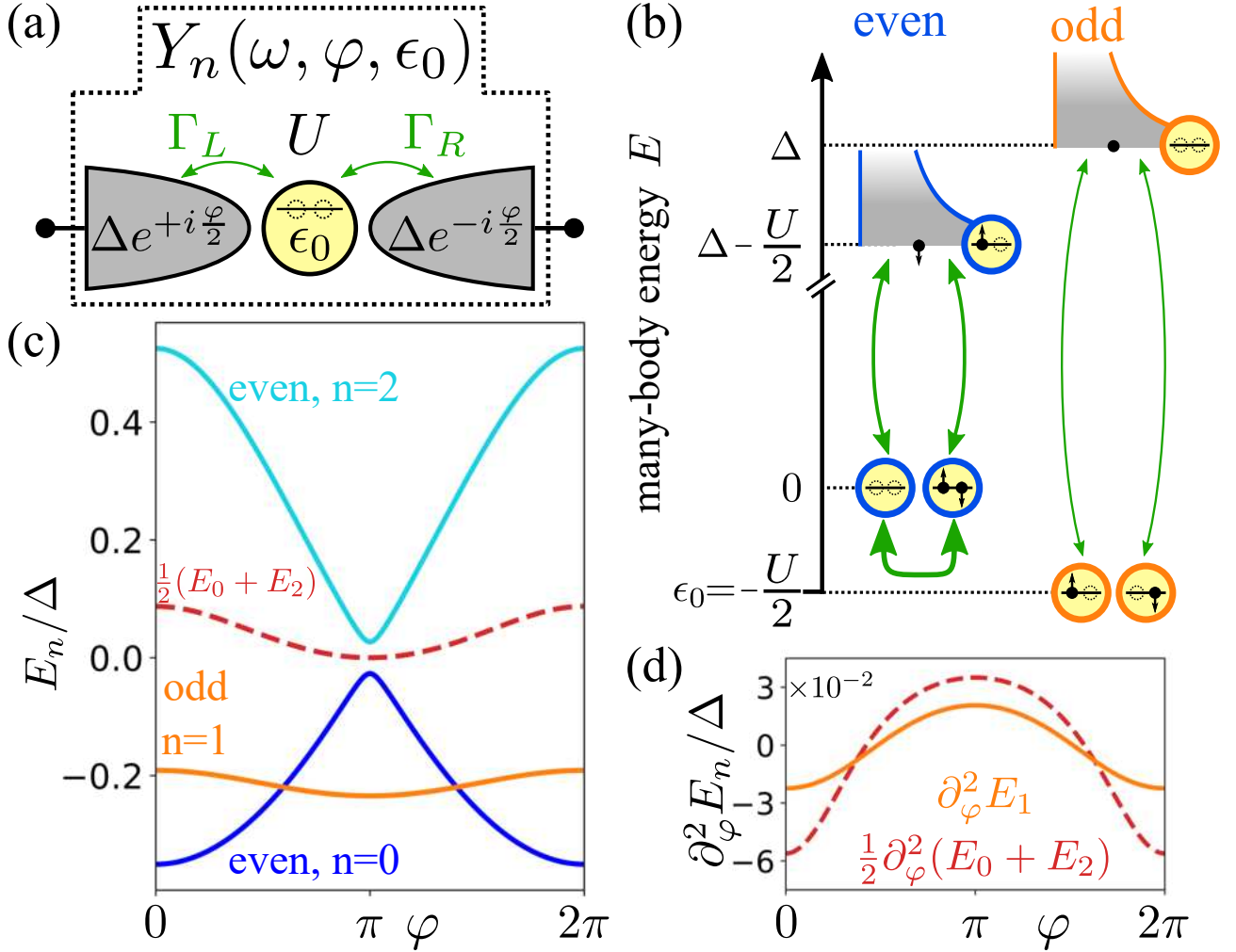


Figure 25.3: (a) Diagram for the model of a single-level quantum dot weak link with state- and frequency-dependent admittance $Y_n(\omega)$. The dot has energy offset ϵ_0 , charging energy U , and coupling rates $\Gamma_{L,R}$ to the reservoirs on the left and right. The reservoirs have the same pair potential Δ and a phase drop φ between them. (b) Energy of many-body states of the uncoupled level and reservoirs for $\epsilon_0 = -U/2$. Double-headed arrows depict hybridization due to the couplings $\Gamma_{L,R}$, with line thickness an indicator of hybridization strength. The condition $\epsilon_0 = -U/2$ allows the even manifold to exhibit resonant behavior. (c) The energy dispersion $E_n(\varphi)$ of the four discrete many-body states (recall the odd state is spin degenerate) for $\Gamma_L/\Delta = 0.32, \Gamma_R/\Delta = 0.28, U = 2(\Gamma_L + \Gamma_R), \epsilon_0 = -U/2$. Coulomb interaction causes an energetic offset between the even and odd manifolds (red dashed vs orange). (d) $\partial^2_\varphi E_n$ (proportional to inverse inductance) for the odd state is weaker than the half-sum of the even states – a measurable distinction in the microwave response caused by charging.

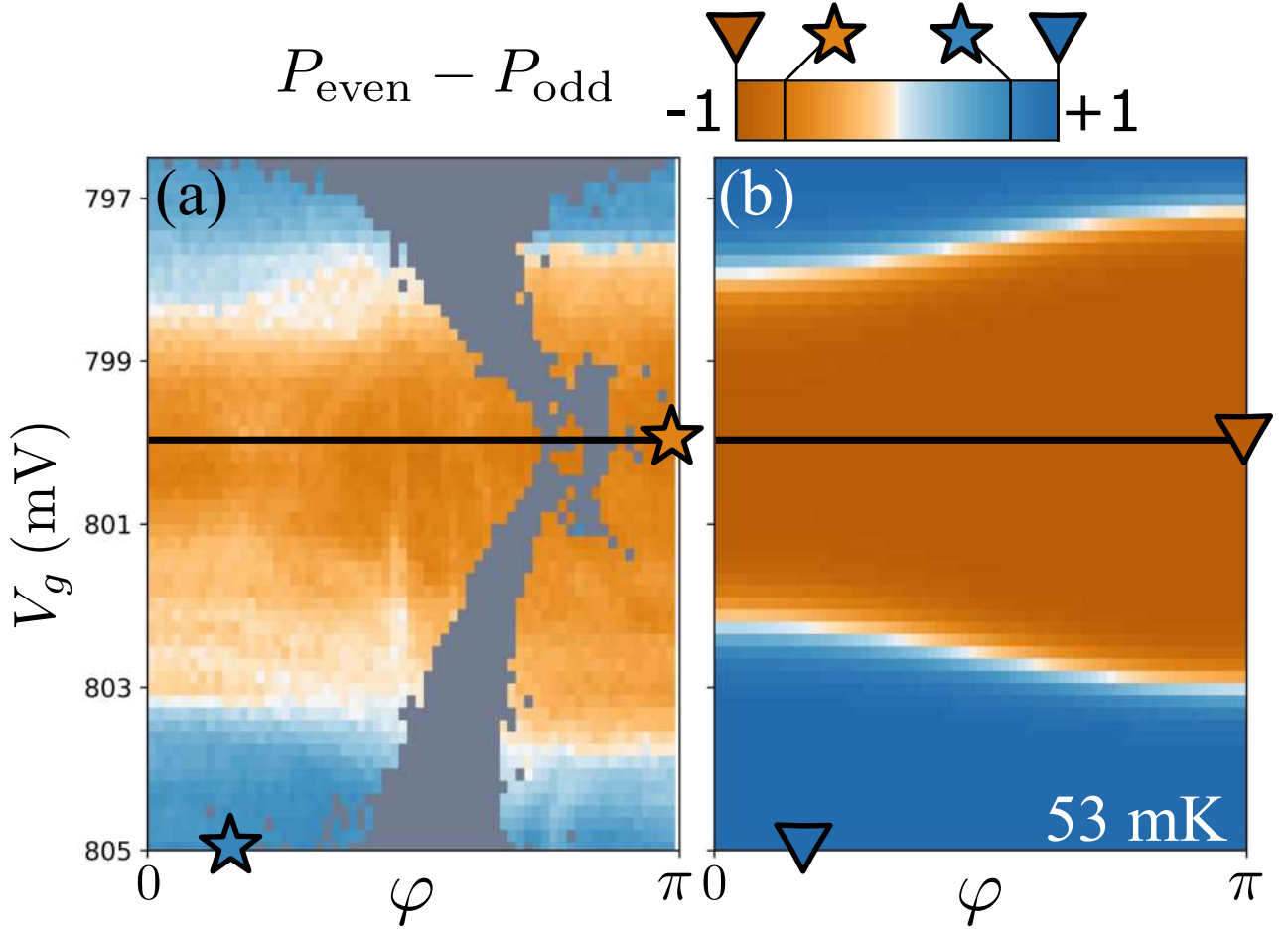


Figure 25.4: The difference in the probability of occupation of the even $n = 0$ and odd $n = 1$ states $P_{\text{even}} - P_{\text{odd}}$ (i.e. the fermion parity polarization) as a function of φ and V_g near gate bias point 2. Panel (a) is the experiment and (b) is a thermal equilibrium prediction within our model for the energies. Both panels have the same pixel size and use the same color scale. We assume $\epsilon_0 \propto V_g$ for simplicity (see supplement for calibration). (a) Gray regions denote where $\delta f_{r,n}$ were too similar to obtain reliable information. Orange and blue stars indicate the extremal experimental polarizations that are also marked on the colorbar. (b) We use $T = 53 \text{ mK}$ for the thermal equilibrium prediction (see supplement for calibration). All other parameters were taken from the fit in Fig. 25.2(c), which was taken along the black line.

$P_{\text{even}} - P_{\text{odd}}$ (the polarization) as a function of phase φ and gate voltage V_g for bias point 2, shown in Fig. 25.4(a) (see supplement for details). The sweet spot at $V_g = 800$ mV (same as Fig. 25.2(c)) exhibits majority odd population (orange) for all phases. Detuning the gate voltage V_g from this point results in a transition (white) from majority-odd (orange) to majority-even (blue). The transition voltage increases when tuning φ from 0 to π , indeed suggestive of a more stable odd state at $\varphi = \pi$. Together, these results are consistent with a $0-\pi$ transition for the weak link, as would be expected from the energies of the interacting quantum dot model. To the best of our knowledge, this is the first time that such a phase diagram has been measured in a microwave experiment.

We can further compare with the predictions of thermal equilibrium using the state energies predicted by our model, shown Fig. 25.4(b). For this, we used an independently calibrated weak link temperature of 53 mK based on the relative probability of the same-parity states $n = 0, 2$ observed near phase $\varphi = \pi$ (see supplement). We find a similar pattern for the sign of the polarization as the experiment, but a striking difference is evident in the magnitude of polarization, indicated by the darkness of the colors. Most of the theoretical phase diagram exhibits full polarization $1 - |P_{\text{even}} - P_{\text{odd}}| \ll 1$ because the typical fermion parity energy difference is much larger than the inferred temperature $|E_0 - E_1|/k_B \gg 53$ mK (k_B is Boltzmann's constant). The experimental weak link, however, never exceeds a polarization of 0.7, which, taken at face value, would require temperature > 500 mK, over an order of magnitude larger than that seen in the even-states. Quasiparticles in superconducting mesoscopic devices are known to be out of equilibrium and so parity dynamics indeed may not follow from the expectations of thermodynamic equilibrium [?]. Our data points to the need for a model of nonequilibrium fermion parity dynamics to quantitatively explain the variation of the polarization with phase bias. We leave this task for future work.

In closing, we employed a cQED setup to isolate and quantify the microwave response of the quantum many-body states in a superconducting weak link. We identified and elucidated two important features in the dispersion of the microwave response functions. One is a violation of a particle-hole symmetry that results from the influence of charging effects. The second is a substantial phase dispersion common to all states which is consistent with the presence of a phase-dispersing spectral continuum. In addition, we observed switches of the fermion parity polarization when varying gate voltage and phase bias that are consistent with a $0-\pi$ transition. Mysteries to be resolved in the future include the incompleteness of the fermion parity polarization, the consistent overestimation of the $n = 0$ admittance near $\varphi = 0$, as well as the role of yet-unidentified high-frequency transitions in spectroscopy at gate bias 2 (see Fig. (S8)). Overall, our results demonstrate that the physics of charging energy and the spectral continuum cannot be ignored in the description of weak links hosting Andreev levels in superconductor-semiconductor Josephson devices and open a path to their detailed study.

26 Spin-degeneracy breaking and parity transitions in three-terminal Josephson junctions by Coraiola, Haxell et al.

Abstract

Harnessing spin and parity degrees of freedom is of fundamental importance for the realization of emergent quantum devices. Nanostructures embedded in superconductor-semiconductor hybrid materials offer novel and yet unexplored routes for addressing and manipulating fermionic modes. Here we spectroscopically probe the two-dimensional band structure of Andreev bound states in a phase-controlled hybrid three-terminal Josephson junction. Andreev bands reveal spin-degeneracy breaking, with level splitting in excess of

~ 9 GHz, and zero-energy crossings associated to ground state fermion parity transitions, in agreement with theoretical predictions. Both effects occur without the need of external magnetic fields or sizable charging energies and are tuned locally by controlling superconducting phase differences. Our results highlight the potential of multiterminal hybrid devices for engineering quantum states.

26.1 Experiment and results

The spin of quantum particles offers an ideal basis for two-level systems, enabling spin qubit-based quantum information processing [3, 2]. Access to spin-resolved states, typically confined in semiconducting quantum dots, requires breaking of the time-reversal symmetry to lift the Kramers degeneracy, often achieved via large magnetic fields. The combination of semiconductors and superconductors into hybrid material platforms [3, 104] creates unprecedented opportunities for spin manipulation with Andreev bound states (ABSs)—fermionic modes arising in a semiconducting region bounded by superconductors [15, 6, 6, 9, 30, 11, 116, 160, 15]. Previously, resolving spin-split ABSs was attained in large magnetic fields (~ 100 mT) [51, 13] or by integration of ferromagnetic elements [16, 17, 18]. An intriguing route to locally break time-reversal symmetry without the need of these ingredients is via control over the superconducting phase difference, although it normally requires long Josephson junctions (JJs) with strong spin-orbit coupling (SOC) to lift the spin degeneracy [26, 11, 21, 127]. This led to measured level splittings up to ~ 1 GHz [117, 118, 25] and enabled the realization of Andreev spin qubits [26, 7, 13, 127, 119, 19], that leverage the advantages of both superconducting and spin qubit platforms.

Multiterminal JJs with SOC promise considerable advantages as platforms for superconducting spin manipulation [27]: in such devices, large spin splitting may be induced solely by controlling superconducting phase differences, while remaining in the short-junction limit. Concomitantly, ground state fermion parity transitions (i.e., switches between even and odd number of fermions in the superconducting condensate) are expected, marked by zero-energy Andreev level crossings in the spectrum [4]. Parity engineering is of crucial importance for realizing artificial Kitaev chains [8, 54, 2, 35] and parity-protected qubits [26, 27, 38, 39]. Unlike their two-terminal counterparts, multiterminal JJs enable parity tuning in the absence of charging energies and external magnetic fields. Recent experiments revealing ABS spectra in phase-controlled three-terminal JJs (3TJJs) [78] provided a first demonstration of higher-dimensional Andreev band structures [17, 11, 23], particularly in the context of Andreev molecules [32, 33, 44, 47], but the feasibility of spin-resolved ABSs is yet to be established.

Here we report on spin-degeneracy breaking and parity transitions controlled by superconducting phase differences in a planar 3TJJ with SOC. We achieve large ABS spin splitting (~ 40 μ eV, corresponding to a frequency in excess of 9 GHz) and level crossings at zero energy, consistent with ground state fermion parity transitions. These phenomena are realized in the absence of external magnetic fields or sizable charging energies. In situ spin and parity tuning is enabled by full phase control through integrated flux-bias lines. The spin nature of the splitting is further supported by magnetic field-dependent studies. Our results demonstrate a new approach for engineering spin and parity degrees of freedom in hybrid quantum devices.

A schematic representation of a hybrid 3TJJ is displayed in Fig. 26.1(a): three superconducting terminals, with phases ϕ_L , ϕ_R and ϕ_M , are coupled to a normal scattering region $\mathcal{S}$. Due to gauge invariance, we set $\phi_M \equiv 0$, hence ϕ_L and ϕ_R correspond to the two independent superconducting phase differences. As derived in Ref. [27] in the short-junction limit, the necessary condition for a zero-energy ABS to exist in the spectrum is that the phases of the terminals wind by 2π around the junction. This is referred to as the “discrete vortex condition” and is geometrically illustrated in Fig. 26.1(b). Assigning to each phase ϕ_α (where $\alpha \in \{L, R, M\}$)

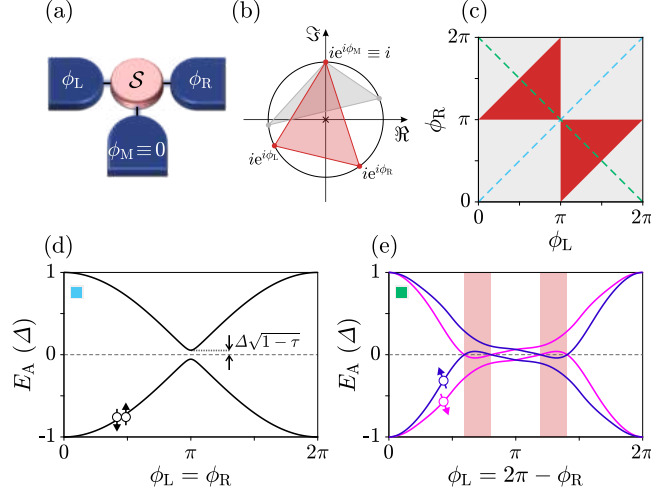


Figure 26.1: (a) Schematic representation of a hybrid three-terminal Josephson junction (3TJJ). Three superconducting leads with phases ϕ_L , ϕ_R and $\phi_M \equiv 0$ are coupled to a common scattering region $\mathcal{S}$. (b) Geometric illustration of the discrete vortex condition, i.e., the necessary condition for the occurrence of zero-energy Andreev bound states (ABSs) and ground state fermion parity transitions. A triangle with vertices $ie^{i\phi_\alpha}$ (for $\alpha \in \{L, R, M\}$) is defined on the unitary circle in the complex plane. If it covers the origin (red triangle), the phases wind by 2π around the 3TJJ and the discrete vortex condition is fulfilled. Otherwise (gray triangle), zero-energy ABSs and parity transitions are not allowed (adapted from Ref. [27]). (c) Parity diagram in the two-dimensional phase space, spanned by the superconducting phase differences ϕ_L and ϕ_R . In the red regions, the discrete vortex condition is fulfilled and parity transitions are allowed, whereas parity must stay even in the gray regions. Dashed cyan and green lines indicate the phase-space linecuts $\phi_L = \phi_R$ and $\phi_L = 2\pi - \phi_R$, respectively. (d) Energy–phase ABS dispersion along $\phi_L = \phi_R$, corresponding to a conventional, spin-degenerate ABS described by Eq. 26.1 in the text. Non-perfect transmission $\tau = 0.997$ is assumed, which prevents ABSs from reaching zero energy. (e) Dispersion of ABSs along $\phi_L = 2\pi - \phi_R$ (schematic representation adapted from Ref. [27]). Spin degeneracy is lifted due to the combination of superconducting phases and spin–orbit coupling in $\mathcal{S}$. Blue and magenta ABSs have opposite spin character. Zero-energy Andreev level crossings are allowed and designate parity transitions. Odd-parity regions are highlighted by the red shading.

the point $ie^{i\phi_\alpha}$ on the complex plane, the condition is fulfilled when the area of the resulting triangle covers the origin (as shown for example by the red triangle). In this case, the ground state fermion parity can switch from even to odd, and a zero-energy Andreev level crossing marks the transition. If the condition is not fulfilled (e.g., gray triangle), the parity of the system remains even, and the ABS energy cannot reach zero but has a finite lower bound. In the 2D phase space, values of (ϕ_L, ϕ_R) for which a discrete vortex is present in the 3TJJ (i.e., odd fermion parity is allowed) describe a pair of triangular regions [red in Fig. 26.1(c)]. We note that when any phase difference is zero ($\phi_L = 0$, $\phi_R = 0$ or $\phi_L = \phi_R$), the system behavior is reduced to that of a two-terminal JJ and its parity has to stay even, except at the points where the other phase difference is $\phi = \pi$. However, in a real junction with non-unity transmission τ , ABSs never reach zero energy, thus the gap cannot close and no parity transition may occur. Rather, ABSs described by the energy–phase dispersion relation [48]

$$E_A(\phi) = \pm \Delta \sqrt{1 - \tau \sin^2(\phi/2)}, \quad (26.1)$$

valid in the short-junction limit, have minimum energy $|E_A(\pi)| = \Delta\sqrt{1-\tau}$, where Δ is the induced superconducting gap. This corresponds to the spectrum plotted in Fig. 26.1(d) as a function of $\phi \equiv \phi_L = \phi_R$ [cyan line in (c)]. Importantly, we remark that ABSs must remain spin degenerate in this configuration, even if spin-rotation symmetry is broken by SOC in the 3TJJ. A different scenario unfolds when $\phi_L \neq \phi_R$ (both finite): in this case, if SOC is present, the spin degeneracy of the spectrum can be lifted, leading to a large spin splitting up to a significant fraction of Δ . In addition, ABSs are allowed to cross at zero energy forming extended regions of odd ground state fermion parity [27]. Both effects are visible in the spectrum of Fig. 26.1(e), shown along the phase-space linecut $\phi_L = 2\pi - \phi_R$ [green in (c)].

To investigate these phenomena, we realized a 3TJJ in an InAs/Al heterostructure [104, 48], simultaneously exploiting the strong Rashba SOC in the InAs 2D electron gas (2DEG) and the scalable, top-down patterning approach offered by the heterostructure material. The device, displayed in Fig. 26.2, features three superconducting terminals (L, M and R), defined by selective etching of the Al layer and coupled to a common scattering region, forming a 3TJJ. The dimensions of this region, $300 \text{ nm} \times 250 \text{ nm}$, were chosen to be relatively large on the scale of the spin–orbit length in InAs $l_{\text{SO}} \sim 150 \text{ nm}$ [55], yet substantially smaller than the superconducting coherence length of proximitized InAs, $\xi_{\text{InAs}} \sim 600 \text{ nm}$. As a result, the effect of SOC is expected to be appreciable in the 3TJJ (unlike the implementation described by Ref. [78]). In the middle of the scattering region, a superconducting island of diameter of 200 nm is left for two main purposes: first, it screens stray electric fields and prevents uncontrolled depletion of the 3TJJ from the gates; second, the island facilitates uniform coupling of all terminals to the scattering region, enabling ABSs to depend on all superconducting phases. The small size of the island with respect to the superconducting coherence length prevents the formation of independent two-terminal JJs between each terminal and the island. The three leads L, M and R are connected to a common node defining two closed superconducting loops, which allow for independent tuning of two phases [78]. A JJ integrated on R (referred to as “switch JJ”), with a length of 40 nm and a width of $5 \text{ }\mu\text{m}$, was designed to have a critical current much larger than between any pairs of L, M and R, therefore the phases of the 3TJJ are not influenced by it. The presence of the switch JJ does not affect the following discussion and a description of its effect is provided in the Supplementary Information. A fourth superconducting terminal (P), biased by the DC voltage V_{bias} , served as a probe to perform tunneling spectroscopy of sub-gap states in the 3TJJ. Metallic gate electrodes and flux-bias lines were patterned on top of a dielectric layer, uniformly deposited across the entire sample. Gates energized by the voltages V_{TL} and V_{TR} were both set to -1.285 V to deplete the 2DEG below and form a tunneling contact between P and the 3TJJ. In such a weak-coupling regime, the influence of the probe on the rest of the circuit can be neglected and the differential conductance G , measured between P and the common

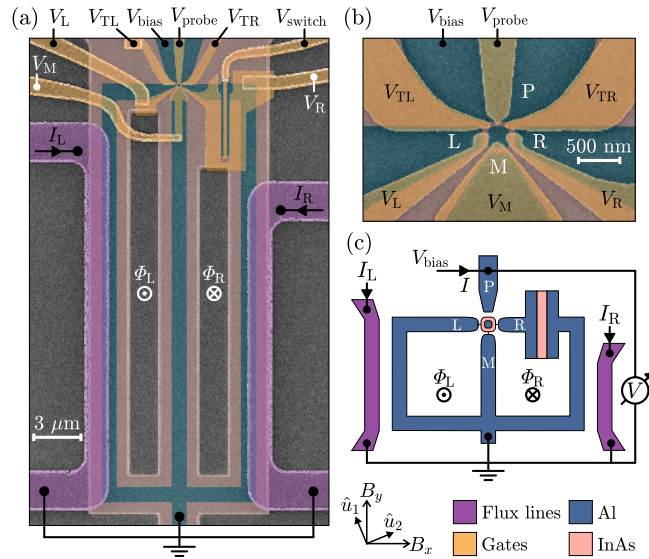


Figure 26.2: (a) False-colored scanning electron micrograph of a device identical to that under study. Selective removal of Al (blue) exposes the III–V semiconductor below (pink), before being uniformly covered by a dielectric layer (not visible). Gates (yellow) and flux-bias lines (purple) are patterned on top of the dielectric. (b) Zoom-in of (a) near the three-terminal Josephson junction (3TJJ) region. (c) Schematic of the device together with the measurement setup. Tunneling spectroscopy is performed by measuring the current I flowing through the superconducting probe P (biased by voltage V_{bias}) and the voltage V across the device with lock-in techniques, to obtain the differential conductance G . Independent control over two superconducting phase differences is enabled by the flux-line currents I_L and I_R , that generate external magnetic fluxes Φ_L and Φ_R threading the two superconducting loops. An in-plane magnetic field can be applied with a vector magnet. B_x and B_y directions are indicated, as well as $\hat{u}_2$ and $\hat{u}_1$ that are rotated by 22.5° counterclockwise.

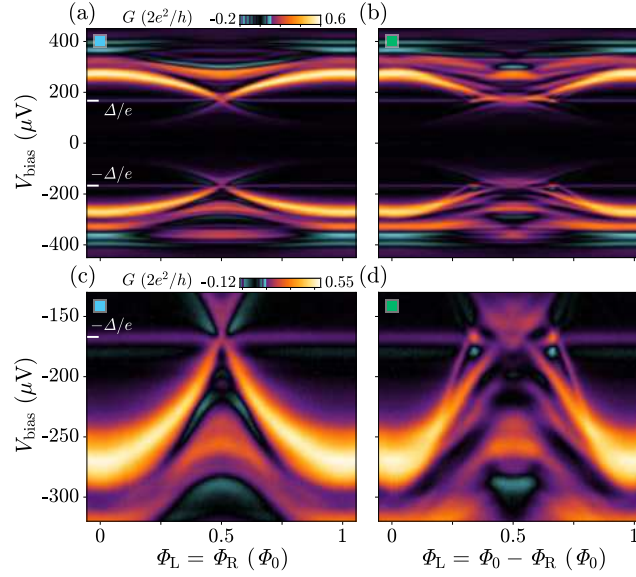


Figure 26.3: Tunneling spectroscopy of Andreev bound states (ABSs) along phase-space linecuts. (a) Differential tunneling conductance G as a function of voltage bias V_{bias} along linecut $\Phi_L = \Phi_R$, revealing a high-transmission ABS with conventional phase dispersion. Due to the superconducting probe, a transport gap between $V_{\text{bias}} = \pm\Delta/e = \pm 167 \mu\text{V}$ is present in the spectrum, delimited by conductance peaks (approximately phase-independent) attributed to multiple Andreev reflection. (b) G as a function of V_{bias} along linecut $\Phi_L = \Phi_0 - \Phi_R$. The ABS spectrum is strongly modified compared to (a), showing level splitting, compatible with spin-degeneracy breaking, and zero-energy crossings (highlighted by the conductance enhancement at $|V_{\text{bias}}| = \Delta/e$) marking ground state parity transitions in the spectrum. (c),(d) Zoom-in of (a),(b) on the ABS spectrum at negative V_{bias} .

node with low-frequency lock-in techniques [see Fig. 26.2(c)], represents the density of states (DOS) of the 3TJJ, with a bias shift $\pm\Delta/e$ introduced by the superconducting probe [51, 15]. Gates covering the normal regions between the island and the three superconducting terminals were kept at $V_L = V_M = V_R = 0$ and had the main role of screening the effect of the other gates on the scattering region, without introducing a charging energy. Two additional gates were set to $V_{\text{probe}} = 0.42 \text{ V}$ and $V_{\text{switch}} = 0$ throughout. Currents I_L and I_R injected into the flux-bias lines generated external magnetic fluxes Φ_L and Φ_R threading the superconducting loops and enabled full control over the 2D phase space. Experiments were performed in a dilution refrigerator with base temperature below 10 mK, equipped with a three-axis vector magnet. Magnetic field directions B_x and B_y are indicated in Fig. 26.2, together with the additional orientations $\hat{u}_1$ and $\hat{u}_2$ (rotated by 22.5° counterclockwise with respect to B_y and B_x , respectively). Further details regarding materials, fabrication and measurement setup are provided in the Supplementary Information.

Tunneling spectroscopy measurements of phase-dispersing ABSs are shown in Fig. 26.3. The spectra are probed as a function of V_{bias} along the two flux-space linecuts $\Phi_L = \Phi_R$ [panels (a) and (c)] and $\Phi_L = \Phi_0 - \Phi_R$ [panels (b) and (d)], obtained from linear combinations of the flux-line currents I_L and I_R [see also Fig. 26.4(a)]. Here, $\Phi_0 = h/2e$ is the superconducting flux quantum, with h the Planck constant and e the elementary charge. These linecuts correspond to those introduced in Figs. 26.1(c)–(e) for the 2D phase space (cyan and green line, respectively). All measurements show a transport gap $2\Delta/e = 334 \mu\text{V}$ related to the superconducting probe, delimited by conductance peaks at $V_{\text{bias}} = \pm 167 \mu\text{V}$ with negligible flux dependence and consistent with multiple Andreev reflection occurring at $e|V_{\text{bias}}| = \Delta$ [3]. Above the transport gap, we observe an electron-hole-symmetric spectrum dominated by flux-dependent resonances,

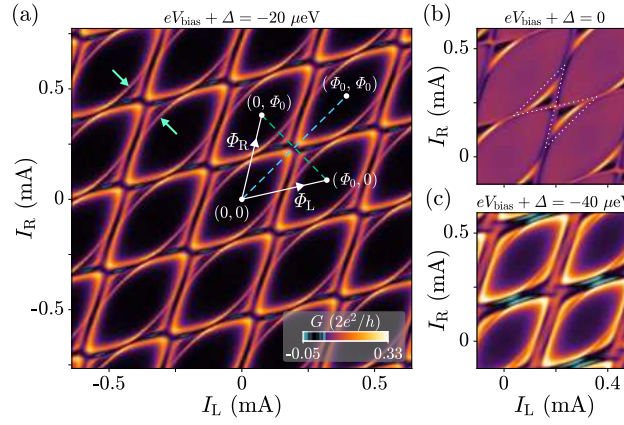


Figure 26.4: Constant-energy cut planes of the Andreev band structure measured in the two-dimensional phase space. (a) Differential tunneling conductance G as a function of flux-line currents I_L and I_R at fixed voltage bias V_{bias} , such that $eV_{\text{bias}} + \Delta = -20 \mu\text{eV}$ (with $\Delta = 167 \mu\text{eV}$), resulting in a cut plane $20 \mu\text{eV}$ below the Fermi level. Periodicity along two directions, corresponding to the external magnetic fluxes Φ_L and Φ_R , is highlighted by the white segment. Dashed cyan and green lines indicate the phase-space linecuts $\Phi_L = \Phi_R$ and $\Phi_L = \Phi_0 - \Phi_R$, shown in Fig. 26.3, and correspond to the paths $\phi_L = \phi_R$ and $\phi_L = 2\pi - \phi_R$ plotted in Fig. 26.1(c). Split lines marked by the turquoise arrows are related to the spin-split Andreev levels visible in the spectrum [Figs. 26.3(b) and 26.3(d)]. (b) As (a), but measured at $eV_{\text{bias}} + \Delta = 0$ to probe the Andreev band structure at zero energy. Conductance peaks are zero-energy Andreev level crossings in the spectrum, corresponding to transitions in the ground-state parity of the system, and thus enclose regions of odd parity. These are comprised in the phase-space regions where the discrete vortex condition is verified [red in Fig. 26.1(c), reported here as the dotted white triangles]. (c) As (a), but at $eV_{\text{bias}} + \Delta = -40 \mu\text{eV}$, showing the Andreev band structure at higher energy.

representing ABSs in the 3TJJ [6, 15]. Finite conductance features within the transport gap are ascribed to a nonzero DOS in the superconducting probe for energies $|E| \lesssim \Delta$ [49]. The spectrum shown in Fig. 26.3(a) and 26.3(c) reveals an ABS resembling the conventional energy dispersion of Eq. 26.1 with near-unity transmission, as it forms a sharp cusp that approaches the gap edge very closely. Conversely, the spectrum probed along $\Phi_L = \Phi_0 - \Phi_R$ [Figs. 26.3(b) and 26.3(d)] exhibits a striking difference from that of Eq. 26.1: as the ABS moves from high $|V_{\text{bias}}|$ towards $e|V_{\text{bias}}| = \Delta$, it markedly splits into two resonances. Both appear to fully reach $e|V_{\text{bias}}| = \Delta$, i.e., the split ABSs cross zero energy. This is more evident for the outermost ABS, that overlaps with $e|V_{\text{bias}}| = \Delta$ at a flux significantly different from $\Phi_L = 0.5\Phi_0$, resulting in a point of enhanced conductance; concomitantly, the slope of the state dispersion has a sharp change of sign at the crossing point.

Next, we map the 2D phase space by performing tunneling spectroscopy at fixed values of V_{bias} as a function of both I_L and I_R , resulting in constant-energy cut planes of the Andreev band structure. In Fig. 26.4(a), where $eV_{\text{bias}} = -\Delta - 20 \mu\text{eV} = -187 \mu\text{eV}$, the phase space is scanned over an extended region, showing periodicity along two main directions (white lines) that correspond to the external fluxes Φ_L and Φ_R , defined in Fig. 26.2. The dashed cyan and green segments represent the $\Phi_L = \Phi_R$ and $\Phi_L = \Phi_0 - \Phi_R$ linecuts displayed in Fig. 26.3. Two high-transmission ABSs, dispersing with Φ_L and Φ_R respectively, form avoided crossings around the $(\Phi_0/2 + n\Phi_0, \Phi_0/2 + m\Phi_0)$ points (with n, m integers), which indicates ABS hybridization and the formation of a phase-dependent Andreev molecule [78]. Furthermore, we note additional split lines (turquoise arrows)—only encountered along $\Phi_L = \Phi_0 - \Phi_R$ and whose paths on the phase space depend on both Φ_L and Φ_R —that are the split ABSs described for Figs. 26.3(b)

and 26.3(d). To study zero-energy Andreev level crossings in the phase space, we set $V_{\text{bias}} = -167 \mu\text{V}$ (i.e., $eV_{\text{bias}} + \Delta = 0$), resulting in the cut plane of Fig. 26.4(b). Here, we observe conductance resonances defining pairs of triangles, which constitute subregions of the phase space areas fulfilling the discrete vortex condition [red triangles in Fig. 26.1(c), whose perimeter is reported as the dotted white triangles in Fig. 26.4(b)]. Finally, Fig. 26.4(c) shows the cut plane at $eV_{\text{bias}} + \Delta = -40 \mu\text{eV}$, revealing the evolution of the high-transmission ABSs and of the split lines at higher bias, and the occurrence of an additional ABS with lower transmission in the spectrum.

The measurements presented in Fig. 26.3 are in excellent qualitative agreement with the spectra predicted by Ref. [27], schematically summarized in Figs. 26.1(d) and 26.1(e). We thus interpret the ABS splitting as spin degeneracy breaking, and the zero-energy crossings as the indication of a fermion parity transition in the ground state of the system [4]. The two effects, occurring at zero external magnetic field, are a direct manifestation of the Andreev band structure of a 3TJJ with Rashba SOC, and are enabled by control over two superconducting phase differences. Experimentally, such control via flux-bias lines is local and allows for wide tunability of the spin-splitting size and of the parity of the system. In our data, we observe a maximum energy splitting of about $40 \mu\text{eV}$, corresponding to a frequency of over 9 GHz and to 0.24Δ (for an induced superconducting gap of $\Delta = 167 \mu\text{eV}$), that is approximately one order of magnitude larger than what has been achieved in long JJs based on nanowires with SOC [117, 119] and very similar to the theoretical prediction [27]. The cut planes of the Andreev band structure at constant energy (Fig. 26.4) show the trajectories of spin-split ABSs and of zero-energy crossings in the 2D phase space. In Fig. 26.4(b), the triangles enclosed by the zero-energy conductance peaks represent regions where the ground state fermion parity has transitioned from even to odd. In the following, we will refer to these features as the “odd-parity triangles”.

To further confirm that the origin of the splitting observed in Figs. 26.3 and 26.4 is related to the spin degree of freedom, we measure additional constant-energy planes and ABS spectra along the $\Phi_L = \Phi_0 - \Phi_R$ direction while an external in-plane magnetic field $\mathbf{B}$ is applied to the device. Rashba SOC in InAs is expected to result in an in-plane spin-orbit field $\mathbf{B}_{\text{SO}}$ perpendicular to the direction of motion [54]; however, since the geometry of the 3TJJ does not impose a preferential direction for transport, we cannot make assumptions regarding the orientation of $\mathbf{B}_{\text{SO}}$. Instead, we perform a coarse angle dependence of $\mathbf{B}$ and identify directions $\hat{u}_1$ and $\hat{u}_2$ that are inferred to be approximately parallel and orthogonal to $\mathbf{B}_{\text{SO}}$ based on the symmetry of the resulting spectrum. When $\mathbf{B} \parallel \hat{u}_1$ [Figs. 26.5(a) and 26.5(d)], we observe a pronounced asymmetry in both the odd-parity triangles [panel (a)] and the spin-split ABS spectrum [panel (d)], as expected in the case that $\mathbf{B}$ has a large component along $\mathbf{B}_{\text{SO}}$ [21, 55, 117]. The spin splitting is enhanced on one side compared to the zero-field case. We also verify that all spectral features flip about $\Phi_L = \Phi_R = \Phi_0/2$ upon inversion of the magnetic field direction, as shown in Figs. 26.5(b) and 26.5(e). Conversely, an asymmetry is not expected when the field is applied orthogonal to $\mathbf{B}_{\text{SO}}$, which is approximately the situation presented in Figs. 26.5(c) and 26.5(f) for $\mathbf{B} \parallel \hat{u}_2$.

In summary, we investigated ABS spectra in a phase-controlled 3TJJ with SOC via tunneling spectroscopy experiments. Independent control over two superconducting phase differences enabled access to the Andreev band structure in the 2D phase space. We observed ABS splitting, compatible with breaking of the spin degeneracy, and zero-energy Andreev level crossings, indicating ground state fermion parity transitions, all consistent with theoretical predictions [27]. In the phase space probed at zero energy, odd-parity regions were enclosed by contours of conductance peaks forming pairs of triangles, fulfilling the necessary condition for parity transitions, namely the presence of a discrete vortex in the 3TJJ. Both effects—spin splitting and parity transitions—arise in the absence of external magnetic fields or sizable charging ener-

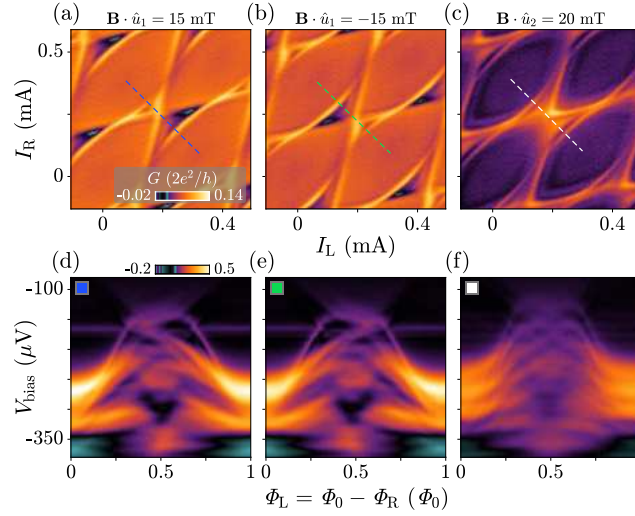


Figure 26.5: Effect of an in-plane magnetic field on the Andreev bound state (ABS) spectrum. (a) Differential tunneling conductance G as a function of flux-line currents I_L and I_R at fixed voltage bias $V_{\text{bias}} = -167 \mu\text{V}$, with an in-plane magnetic field $\mathbf{B}$ of magnitude of 15 mT applied along the direction of $\hat{u}_1$ [see Fig. 26.2]. The strong asymmetry in the ABS features is attributed to the alignment of $\mathbf{B}$ with the spin-orbit field $\mathbf{B}_{\text{SO}}$ and indicates that the split lines originate from the spin. (b) As (a), but reversing the direction of $\mathbf{B}$, which causes the inversion of all features with respect to $\Phi_L = \Phi_R = \Phi_0/2$. (c) As (a), but applying a magnetic field of 20 mT along $\hat{u}_2$ (orthogonal to $\hat{u}_1$) and at $V_{\text{bias}} = -157 \mu\text{V}$. Symmetry of the features is compatible with $\mathbf{B} \perp \mathbf{B}_{\text{SO}}$. (d) Conductance G as a function of V_{bias} along the $\Phi_L = \Phi_0 - \Phi_R$ linecut [dashed blue line in (a)], for a magnetic field of 15 mT along $\hat{u}_1$, revealing the asymmetry in the ABS spectrum. (e) As (d), but reversing $\mathbf{B}$. Spectral ABS features are reversed. (f) As (d), but with a magnetic field of 20 mT applied along $\hat{u}_2$. The spectrum symmetry is consistent with the cut plane in (c).

gies, and are tuned by controlling superconducting phase differences. The size of the observed splitting (up to $\sim 40 \mu\text{eV}$, or over 9 GHz) is substantially larger than previously reported values in nanowire-based long JJs. These results demonstrate that spin and parity degrees of freedom of quantum states are widely controllable by phase tuning in multiterminal hybrid nanostructures. An immediate application is the realization of superconducting spin qubits benefiting from a large, phase-tunable spin-level splitting at zero external fields. Parity control in open systems could be exploited for the engineering of topologically protected Kitaev chains and parity-protected qubits. The access to transition frequencies between spin-resolved levels of several gigahertz makes our devices ideal for integration with circuit quantum electrodynamics architectures, while in situ and fast frequency tuning via integrated flux-bias lines will potentially enable high-performance logic and multi-qubit coupling schemes.

26.2 Supplementary Material: Spin-degeneracy breaking and parity transitions in three-terminal Josephson junctions

26.2.1 Materials and methods

Devices were fabricated in a III–V heterostructure grown with molecular beam epitaxy techniques on an InP (001) substrate. The stack consisted of the following layers (starting from the substrate): a 1100 nm thick step-graded InAlAs buffer layer, a 6 nm thick $\text{In}_{0.75}\text{Ga}_{0.25}\text{As}$ layer, an 8 nm thick InAs layer, a 13 nm thick $\text{In}_{0.75}\text{Ga}_{0.25}\text{As}$ layer, two monolayers of GaAs and a 15 nm thick Al layer. The latter was deposited in situ without breaking vacuum. The InAs layer hosted a 2DEG that was characterized via measurements in a Hall bar geometry, yielding a peak mobility of $18000 \text{ cm}^2\text{V}^{-1}\text{s}^{-1}$ at an electron sheet density of $8 \cdot 10^{11} \text{ cm}^{-2}$. This resulted in an electron mean free path $l_e \gtrsim 260 \text{ nm}$ and a superconducting coherence length of the 2DEG proximitized by the Al sheet of $\xi_{\text{InAs}} = \sqrt{\hbar v_F l_e / (\pi \Delta)} \sim 600 \text{ nm}$, with v_F the Fermi velocity in the 2DEG and Δ the induced superconducting gap.

First, large mesa structures were isolated, suppressing parallel conduction between devices and across the middle regions of the superconducting loops. This was done by selectively etching the Al layer with Transene type D, followed by a second chemical etch to a depth of $\sim 380 \text{ nm}$ into the III–V material stack, using a $220 : 55 : 3 : 3$ solution of $\text{H}_2\text{O} : \text{C}_6\text{H}_8\text{O}_7 : \text{H}_3\text{PO}_4 : \text{H}_2\text{O}_2$. Next, Al was defined by wet etching with Transene type D at 50°C for 4 s. The dielectric, deposited on the entire chip by atomic layer deposition, consisted of a 3 nm thick layer of Al_2O_3 and a 15 nm thick layer of HfO_2 . Gate electrodes and flux-bias lines were defined by evaporation and lift-off. In a first step, 5 nm of Ti and 20 nm of Au were deposited to realize the fine features of the gates; in a second, a stack of Ti/Al/Ti/Au with thicknesses 5 nm, 340 nm, 5 nm and 100 nm was deposited to connect the mesa structure to the bonding pads and to define the flux-bias lines.

Experiments were performed in a dilution refrigerator with base temperature at the mixing chamber below 10 mK. The sample was mounted on a QDevil QBoard sample holder system, without employing any light-tight enclosure. Electrical contacts to the devices, excepts for the flux-bias lines, were provided via a resistive loom with QDevil RF and RC low-pass filters at the mixing chamber stage, and RC low-pass filters integrated on the QBoard sample holder. Currents in the flux-bias lines were injected via a superconducting loom with only QDevil RF filters at the mixing chamber stage. Signals were applied to all gates and flux-bias lines via home-made RC filters at room temperature. Electrical measurements were performed with low-frequency AC lock-in techniques, applying a fixed AC voltage $\delta V_{\text{bias}} = 3 \mu\text{V}$ at frequency 211 Hz to a contact at the superconducting probe (labeled P in Fig. 2 of the Main Text). The DC voltage bias V_{bias} was applied differentially: $V_{\text{bias}}/2$ at the superconducting probe (together with the AC signal) and $-V_{\text{bias}}/2$ at the common node [indicated as ground in the simplified

schematics of Figs. 2(a) and 2(c)]. The latter voltage was provided to the node via a current-to-voltage (I–V) converter, which also measured the AC current δI and the DC current I flowing through the device. The four-terminal AC voltage δV between the probe and the common node was measured to determine the differential conductance $G \equiv \delta I / \delta V$.

In-plane magnetic fields were applied via a superconducting vector magnet. For each field configuration (including zero field), a finite out-of-plane field component, due to offsets and hysteresis of the magnet as well as to a small misalignment of the sample axes with the magnet axes, had to be compensated. For that, the out-of-plane field was swept across a range $\geq \pm 1$ mT and the position of zero field was determined from spectral features such as the size of the superconducting gap and the sharpness of the conductance peaks (subgap states, multiple Andreev reflection peaks and high bias features). Small residual deviations from zero field, caused by hysteresis and imperfect compensation, were taken into account by offsetting the flux-bias line currents I_L and I_R in such a manner that the point where $I_L = I_R = 0$ coincided with $\Phi_L = \Phi_R = 0$ in the constant-bias maps.

26.2.2 Tunnel-gate voltage dependence of differential conductance

The gate electrodes energized by voltages V_{TL} and V_{TR} , denoted as tunnel gates (see Fig. 2 of the Main Text), allowed for electrostatic tuning of the transmission between the superconducting probe and the three-terminal Josephson junction (3TJJ). These gates were always set to a common voltage, $V_T \equiv V_{TL} = V_{TR}$. A gate covering the probe (called probe gate), slightly overlapping with the semiconducting region between the tunnel gates, was set to $V_{\text{probe}} = 0.42$ V throughout. The three gates surrounding the 3TJJ (labeled left, middle and right gate) and the gate covering the superconductor–semiconductor–superconductor Josephson junction (JJ) on the right arm of the superconducting loops (switch gate) were set to $V_L = V_M = V_R = V_{\text{switch}} = 0$, unless stated otherwise. Figure 26.6 shows the dependence on tunnel gate voltages of the differential conductance G measured as a function of the probe bias voltage V_{bias} . The conductance decreased as V_T was reduced, indicating a lower transmission between the probe and the 3TJJ, and a gradual transition from an open to a tunneling regime. In the open regime, conductance peaks were present at zero bias (remnant of a supercurrent between the probe and the leads of the 3TJJ) and at finite bias (related to multiple Andreev reflection (MAR) processes). The tunneling regime is the relevant one to perform tunneling spectroscopy ($V_T = -1.285$ V for the data of the Main Text), as the weak coupling allows for noninvasive probing of the density of states of the 3TJJ. Here, the spectrum is characterized by a transport gap of $334 \mu\text{V}$, consistent with a superconducting gap of the Al probe of $\Delta = 167 \mu\text{eV}$. Outside the probe gap, pronounced conductance resonances correspond to Andreev bound states (ABSs).

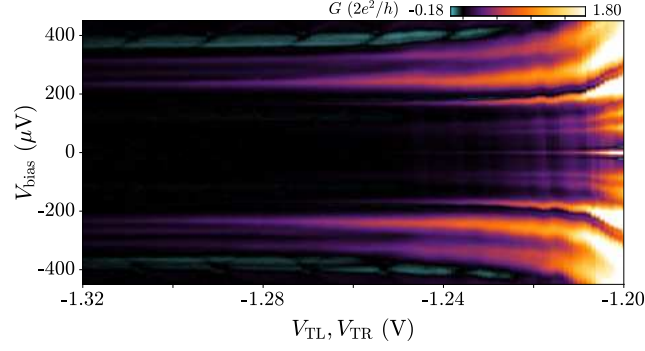


Figure 26.6: **Tunnel-gate voltage dependence.** Differential conductance G as a function of voltage bias V_{bias} and tunnel-gate voltages $V_{\text{TL}} = V_{\text{TR}}$, for $V_{\text{probe}} = 0.42$ V and $V_{\text{L}} = V_{\text{M}} = V_{\text{R}} = 0$.

26.2.3 Dependence of Andreev spectrum on individual phases

Tunneling spectroscopy measurements along the phase-space linecuts $\Phi_{\text{L}} = \Phi_{\text{R}}$ and $\Phi_{\text{L}} = \Phi_0 - \Phi_{\text{R}}$ (with Φ_{L} and Φ_{R} the external magnetic fluxes and $\Phi_0 = h/2e$ the superconducting flux quantum) were presented in Fig. 3 of the Main Text. In this section, we present measurements where each flux was varied individually while keeping the other at zero, which correspond to the white linecuts shown in Fig. 4 of the Main Text. Figures 26.7(a) and 26.7(b) show the dependence on Φ_{L} (for $\Phi_{\text{R}} = 0$) and on Φ_{R} (for $\Phi_{\text{L}} = 0$), respectively, in the same gate configuration as described in the Main Text. Each map reveals a high-transmission ABS, whose energy–phase dispersion resembles the relation $E = \pm\Delta [1 - \tau \sin^2(\pi \Phi_{\alpha}/\Phi_0)]^{1/2}$ valid for short two-terminal JJs [48], with $\tau \gtrsim 0.998$ (τ is the ABS transmission and $\alpha \in \{\text{L}, \text{R}\}$). Here, Φ_{L} - and Φ_{R} -dependent resonances correspond to distinct ABSs, that form in the L–M junction and in the R–M junction regions respectively. As for the $\Phi_{\text{L}} = \Phi_{\text{R}}$ linecut [Figs. 3(a) and 3(c)], where one phase difference was zero, neither ABS splitting nor signatures of zero-energy crossings are observed, compatible with standard two-terminal JJ physics.

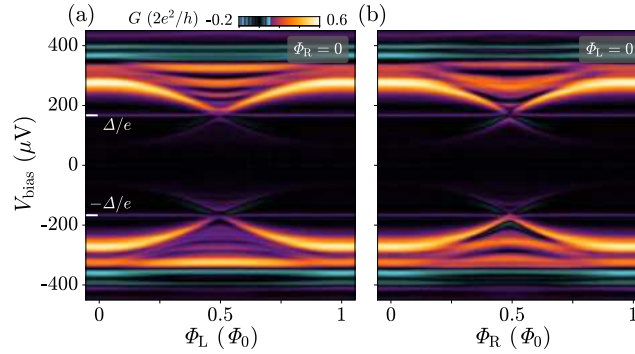


Figure 26.7: **Dependence on external magnetic fluxes.** (a) Differential conductance G as a function of voltage bias V_{bias} and flux Φ_{L} , for $\Phi_{\text{R}} = 0$. (b) As in (a), but as a function of Φ_{R} for $\Phi_{\text{L}} = 0$. Both (a) and (b) correspond to the white linecuts shown in Fig. 4(a) of the Main Text.

26.2.4 Voltage bias dependence of constant-energy cut planes

Constant-bias measurements showing the dependence on both flux-line currents—representing cut planes of the Andreev band structure at constant energy—were shown in Fig. 4 of the Main Text. There, selected values of $V_{\text{bias}} = -167, -187$ and -207 μV

were presented, corresponding to energies 0, -20 and -40 μeV with respect to the Fermi level (after subtraction of the offset $\Delta/e = 167$ μV due to the superconducting probe). In Fig. 26.8, we extend the data of Fig. 4 for several values of V_{bias} ranging from -167 μV (i.e., zero energy) through to -297 μV . Each map was measured in the same gate configuration as that of the Main Text. The smaller step size between V_{bias} values in the low-energy spectrum allowed us to follow the evolution of the split lines [Figs. 26.8(a)–(f)], until they could not be distinguished at higher $|V_{\text{bias}}|$, compatible with the smaller separation in phase between spin-split ABSs at high energies [see Figs. 1(e) and 3(d) of the Main Text]. Additional Φ_L - and Φ_R -dependent ABSs with lower transmission first appeared at $V_{\text{bias}} = -197$ μeV [Fig. 26.8(d)] and $V_{\text{bias}} = -217$ μeV [Fig. 26.8(f)] respectively, consistent with the spectra displayed in Fig. 26.7. The high energy spectrum [for $|V_{\text{bias}}| \geq 237$ μV , Figs. 26.8(g)–(j)] showed a large number of ABSs that could no longer be individually resolved.

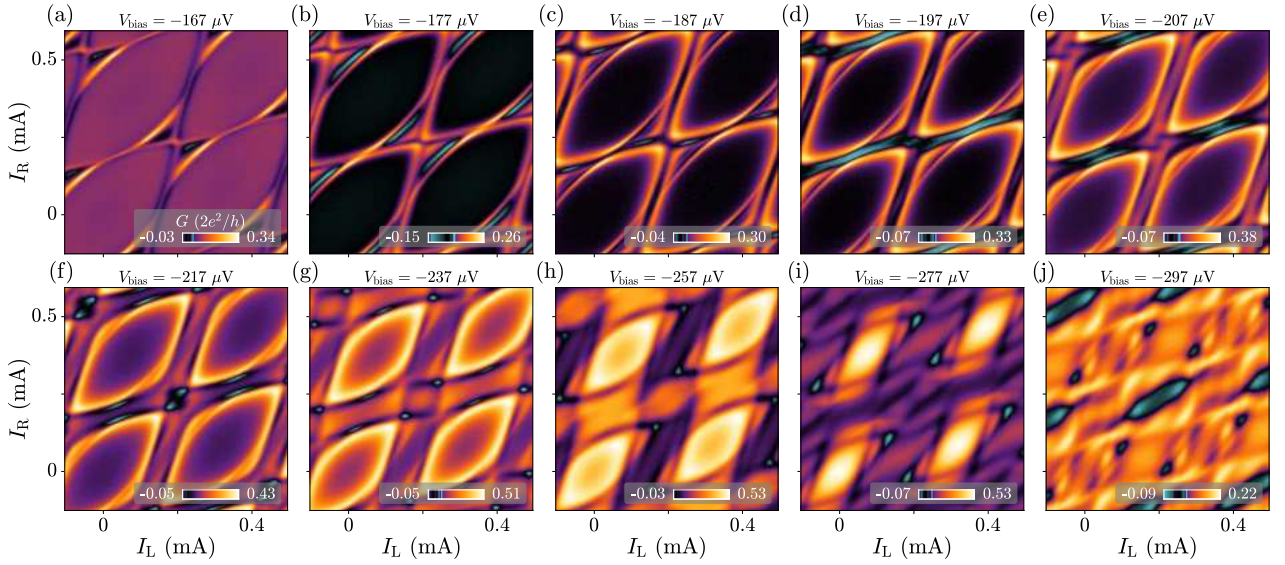


Figure 26.8: **Constant-energy cut planes in the two-dimensional phase space for varying voltage bias.** (a)–(j) Differential conductance G as a function of flux-line currents I_L and I_R . Each map is at a fixed value of V_{bias} , ranging from -167 $\mu\text{V} = -\Delta/e$ (a) through to -297 $\mu\text{V} = -\Delta/e - 130$ μV (j).

26.2.5 Effect of the switch junction

As described in the Main Text, the double-loop geometry features a superconductor–semiconductor–superconductor JJ (labeled switch JJ) on the right superconducting branch (see Fig. 2), that was tunable by the gate voltage V_{switch} . The large critical current of the switch junction makes its inductance small, hence the phase drop across the junction was small and Φ_L and Φ_R only affected the phase differences across the 3TJJ terminals L–M and R–M. Therefore, as long as the switch JJ was not depleted (which was the case with $V_{\text{switch}} = 0$, studied so far), its presence could be neglected and the device operated as if both superconducting loops were continuous. Conversely, as shown in Ref. [78], depleting the switch junction by setting V_{switch} to a sufficiently negative voltage interrupted the right loop and made the external flux Φ_R ineffective for tuning the phase difference between terminals R and M. In Fig. 26.9, we display the results for $V_{\text{switch}} = -2$ V, where the switch JJ was completely depleted. The constant-bias cut plane in panel (a), measured at $V_{\text{bias}} = -187$ μV as a function of both flux-line currents [as in Fig. 4(a) of the Main Text], confirms that the dependence on Φ_R was suppressed and no features related to the three-terminal geometry were visible. The full spectrum along Φ_L , shown in Fig. 26.9(b), was

similar to that of Fig. 26.7(a) (where $V_{\text{switch}} = 0$ and $\Phi_R = 0$) and revealed again a single high-transmission ABS as well as several states with lower transmission. Importantly, the ABS spectrum was essentially identical when measured along the $\Phi_L = \Phi_0 - \Phi_R$ linecut [see Fig. 26.9(c)], whereas splitting of ABSs and zero-energy conductance peaks was present for $V_{\text{switch}} = 0$ [Figs. 3(b) and 3(d) of the Main Text]. These results indicate that such features are specific to the three-terminal geometry with full control over two superconducting phases. We note that, when the switch JJ was depleted, terminal R was reduced to a floating superconducting island, whose phase was not controlled by external parameters. The low-energy spectrum did not show any hybridization of the Φ_L -dependent ABS with the ABS localized between terminal R (floating) and terminal M. This implies that the latter ABS could not enter the low-energy spectrum, namely that the phase difference between R and M always remained sufficiently far from π .

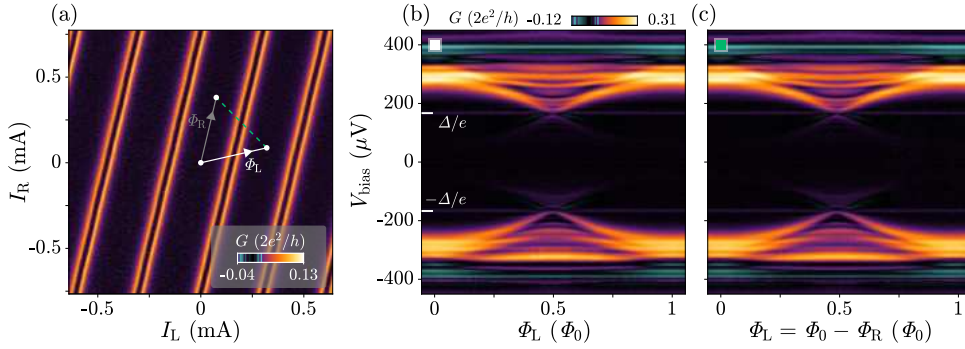


Figure 26.9: **Tunneling spectroscopy with a depleted switch junction.** (a) Differential conductance G as a function of flux-line currents I_L and I_R for $V_{\text{switch}} = -2$ V. (b) G as a function of voltage bias V_{bias} and external magnetic flux Φ_L , namely along the white linecut defined in (a), for $V_{\text{switch}} = -2$ V. (c) As in (b), but along the $\Phi_L = \Phi_0 - \Phi_R$ linecut [green in (a)].

26.2.6 Effect of V_L , V_M and V_R

The three gates covering the semiconducting regions between each terminal and the superconducting island were so far set to $V_L = V_M = V_R = 0$ and had the main role of screening the effect of the tunnel gates on the 3TJJ, hence preventing its uncontrolled depletion. In this section, we describe the individual and combined effect of finite negative voltages applied to these gates, except for the right gate that was found to be ineffective (voltage V_R had no influence on any measured quantity). In each gate configuration, the tunnel gate voltages V_{TL} and V_{TR} were adjusted to keep the transmission of the probe approximately constant (while maintaining $V_{\text{TL}} = V_{\text{TR}}$) and to perform spectroscopy in a similar regime as in the Main Text.

For Fig. 26.10, we set $V_L = -0.4$ V and adjusted V_{TL} and V_{TR} to -1.26 V to deplete the semiconducting region near terminal L and decouple it from the common scattering region. In the constant-bias cut planes [Figs. 26.10(a) and 26.10(b)], the ABS dispersing with Φ_L was substantially less visible and hybridization between ABSs near the intersection points, ABS splitting and odd-parity triangles at zero energy all disappeared. This is confirmed by the phase-space linecuts $\Phi_L = \Phi_R$ and $\Phi_L = \Phi_0 - \Phi_R$ displayed in Figs. 26.10(c) and 26.10(d), which revealed similar spectra (unlike in Fig. 3 of the Main Text) characterized by a conventional ABS dispersion.

Next, we set V_L back to 0 and applied a voltage $V_M = -0.6$ V to the middle gate to decouple terminal M from the 3TJJ region. Here, V_{TL} and V_{TR} were tuned to -1.252 V. In this new gate configuration, we performed the same four measurements as in Fig. 26.10 and show the

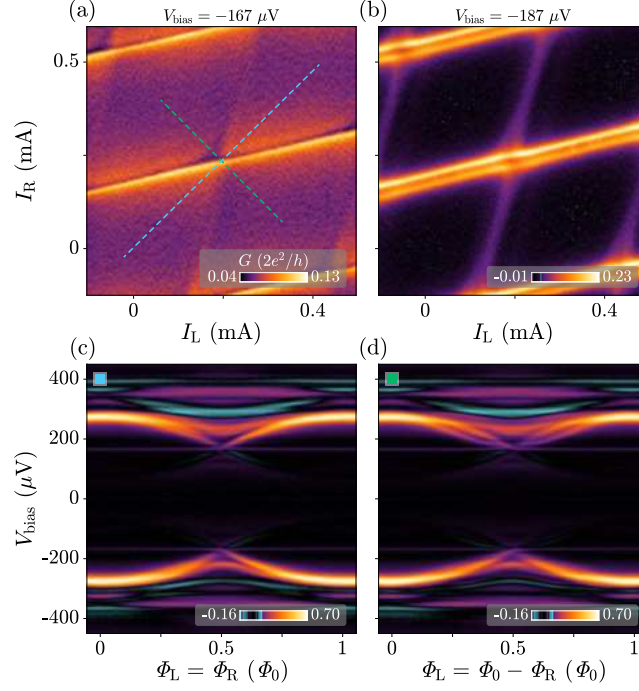


Figure 26.10: **Tunneling spectroscopy for $V_L = -0.4$ V.** (a),(b) Differential conductance G as a function of flux-line currents I_L and I_R , for $V_{\text{bias}} = -167 \mu\text{V}$ and $V_{\text{bias}} = -187 \mu\text{V}$, respectively. (c),(d) G as a function of V_{bias} along the $\Phi_L = \Phi_R$ linecut [cyan in (a)] and along the $\Phi_L = \Phi_0 - \Phi_R$ linecut [green in (a)], respectively.

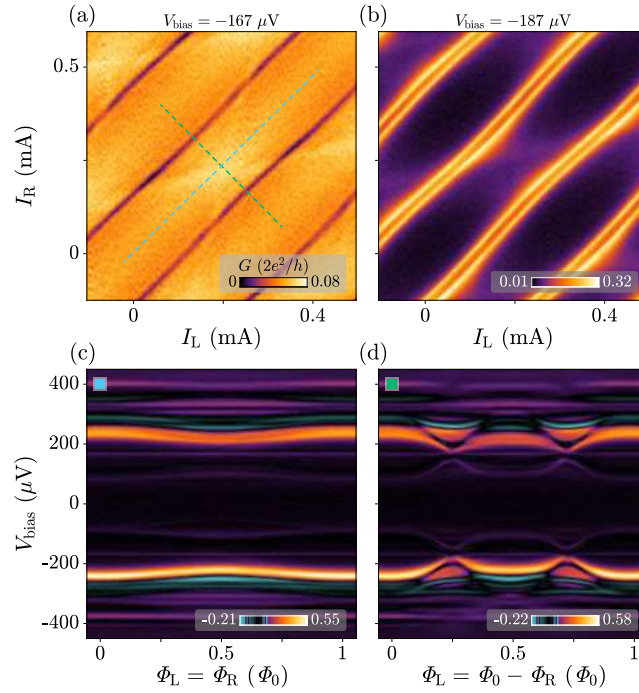


Figure 26.11: **Tunneling spectroscopy for $V_M = -0.6$ V.** (a),(b) Differential conductance G as a function of flux-line currents I_L and I_R , for $V_{\text{bias}} = -167 \mu\text{V}$ and $V_{\text{bias}} = -187 \mu\text{V}$, respectively. (c),(d) G as a function of V_{bias} along the $\Phi_L = \Phi_R$ linecut [cyan in (a)] and along the $\Phi_L = \Phi_0 - \Phi_R$ linecut [green in (a)], respectively.

result in Fig. 26.11. Interestingly, both original ABS resonances (one depending only on Φ_L , the other only on Φ_R) became extremely feeble and spectroscopy was dominated by diagonal features oriented along $\Phi_L = \Phi_R$. Consequently, these states depend on $\Phi_L - \Phi_R$, i.e., the total flux threading both superconducting loops (note that positive Φ_L and Φ_R were conventionally defined in opposite directions, see Fig. 2 of the Main Text), and they indicate the presence of a conducting channel between terminals L and R. As this was not observed in previous measurements without a superconducting island in the middle of the normal scattering region [78], we conclude that the island ensures a more uniform coupling across the three terminals, with a high-transmission channel even between the terminals with the largest separation. From the spectrum of Fig. 26.11(d), we also note that no ABS splitting nor conductance enhancement at zero energy occurred, as in Fig. 26.10 for the left gate depleted.

Figures 26.9–26.11 show that our devices support three types of ABS, depending on Φ_L , Φ_R and $\Phi_L - \Phi_R$ respectively. Each could be visualized individually by decoupling one of the three superconducting terminals. A conducting channel was present between all pairs of terminals, although none of them alone displayed level splitting and crossings at zero energy. These phenomena were exclusively possible in our devices with all three superconductors coupled to the scattering region and via control over both phase differences.

Finally, we set both V_L and V_M to -0.2 V (adjusting V_{TL} and V_{TR} to -1.25 V), for which the semiconductor below the left and middle gate was affected but not depleted. Measurements in this new configuration are reported in Fig. 26.12: the phase-space linecuts [panels (a) and (b)] correspond to Figs. 3(a) and 3(b) of the Main Text, while the constant-energy cut planes [(c)–(f)] are to be compared to Figs. 4 and 26.8. The results are qualitatively similar to those observed at $V_L = V_M = 0$, indicating that the phenomena discussed in the Main Text were relatively robust and did not depend on the specific gate configuration, nor relied on fine tuning. At relatively large $|V_{bias}|$, individual ABSs were more clearly resolved in this configuration than at $V_L = V_M = 0$ [compare for example Figs. 26.12(f) and 26.9(f)], which is compatible with a reduced number of low-transmission modes at negative gate voltages.

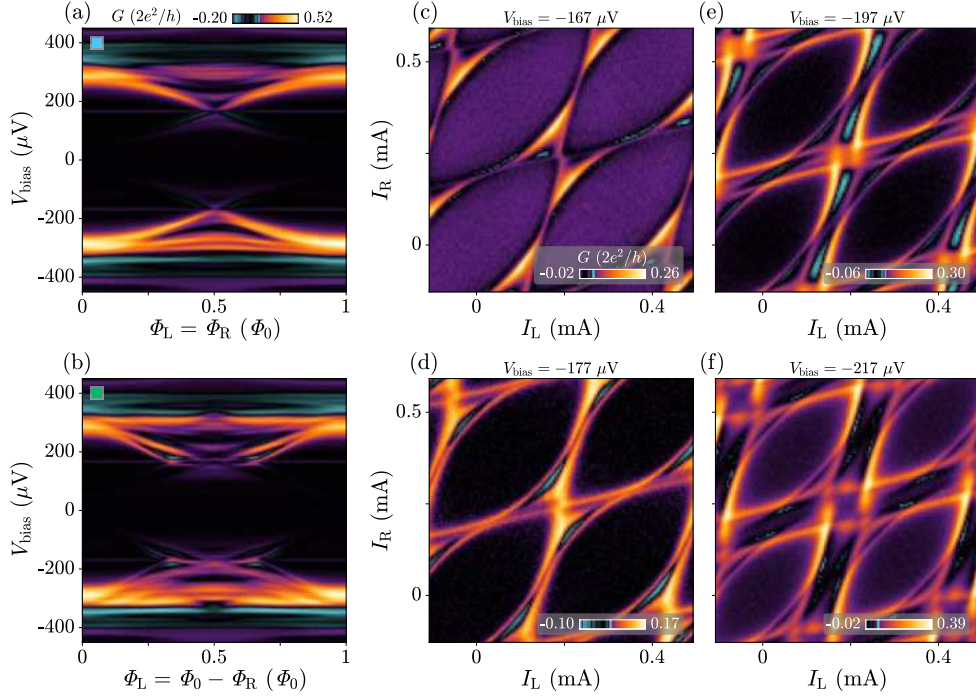


Figure 26.12: **Tunneling spectroscopy** for $V_L = V_M = -0.2 \text{ V}$. (a),(b) Differential conductance G as a function of V_{bias} along the $\Phi_L = \Phi_R$ linecut and along the $\Phi_L = \Phi_0 - \Phi_R$ linecut, respectively. (c)–(f) G as a function of flux-line currents I_L and I_R , for V_{bias} ranging from $-167 \mu\text{V}$ to $-217 \mu\text{V}$.

26.2.7 Remapping of flux-line-current space to flux space

Full control over two superconducting phase differences was enabled in our devices by superconducting flux-bias lines in combination with the double-loop geometry (see Fig. 2 of the Main Text). Currents I_L and I_R generated small local magnetic fields that threaded the two superconducting loops, resulting in the external magnetic fluxes Φ_L and Φ_R . The left flux-bias line mainly controlled Φ_L due to its proximity to the left loop, however it also produced a smaller flux that threaded the right loop and changed Φ_R . An identical argument applies to the right flux line, hence independent flux line control allowed full mapping of the two-dimensional phase space, but required cross-talk between the two lines to be taken into account. This explains the finite slope of the Φ_L and Φ_R axes on the (I_L, I_R) plane, as highlighted in Fig. 4(a) of the Main Text [plotted again in Fig. 26.13(a)]. Fluxes Φ_L and Φ_R were related to the flux-line currents I_L and I_R via the linear transformation:

$$\begin{pmatrix} \Phi_L \\ \Phi_R \end{pmatrix} = \mathbf{M} \cdot \begin{pmatrix} I_L \\ I_R \end{pmatrix} = \begin{pmatrix} M_{LL} & M_{LR} \\ M_{RL} & M_{RR} \end{pmatrix} \cdot \begin{pmatrix} I_L \\ I_R \end{pmatrix}, \quad (26.2)$$

where $\mathbf{M}$ is the mutual inductance matrix. The matrix $\mathbf{M}$ can be calculated from the constant-bias cut planes knowing both the (I_L, I_R) coordinates and the (Φ_L, Φ_R) coordinates of two points, as the origin $(0, 0)$ is the same in both current space and flux space. For instance, we consider the points $(\Phi_0, 0)$ and $(0, \Phi_0)$ [see Fig. 26.13(a)] and substitute their (I_L, I_R) coordinates in Eq. 26.2 to obtain a 4×4 equation system. As a result, we determine the mutual inductance matrix:

$$\mathbf{M} = \begin{pmatrix} 6.83 \text{ pH} & -1.34 \text{ pH} \\ -1.56 \text{ pH} & 5.74 \text{ pH} \end{pmatrix}. \quad (26.3)$$

The ratio $M_{RR}/M_{LL} \approx M_{LR}/M_{RL} \approx 0.85$ approximately corresponds to the ratio between the lengths of the vertical segments of the right and left flux line. While constant-bias cut planes

were so far plotted on the (I_L, I_R) axes to directly visualize the raw data, knowing $\mathbf{M}$ enables a conversion to the (Φ_L, Φ_R) axes using Eq. 26.2. This is illustrated in Figs. 26.13(c) and 26.13(d) for the datasets of Figs. 26.13(a) and 26.13(b), respectively. In panels (b) and (d), the dotted white triangles represent the perimeter of the region where the discrete vortex condition is fulfilled.

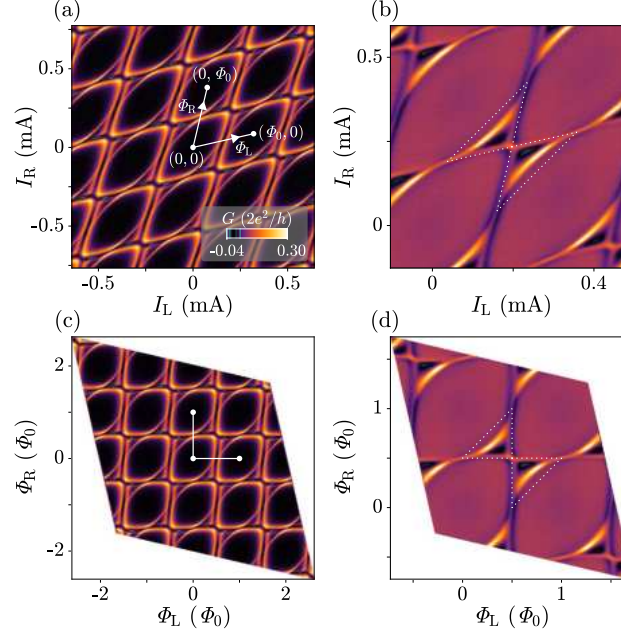


Figure 26.13: **Current-to-flux remapping.** (a),(b) Differential conductance G as a function of flux-line currents I_L and I_R , for $V_{\text{bias}} = -187 \mu\text{V}$ and $V_{\text{bias}} = -167 \mu\text{V}$, respectively [replotted from Figs. 4(a) and 4(b) of the Main Text]. (c),(d) As in (a) and (b), but remapping the (I_L, I_R) axes to the (Φ_L, Φ_R) axes [white lines in (a)], using the linear transformation described in the text. In (b) and (d), the dotted white triangles represent the perimeter of the regions where the discrete vortex condition is fulfilled.

26.2.8 Results for additional orientations of the in-plane magnetic field

The dependence on an in-plane magnetic field $\mathbf{B}$ was shown in Fig. 5 of the Main Text for the selected orientations $\hat{u}_1$ and $\hat{u}_2$, rotated by 22.5° counterclockwise with respect to the x - and y -axes (see Fig. 2) and interpreted to be roughly parallel and orthogonal to the spin-orbit field $\mathbf{B}_{\text{SO}}$, respectively. These measurements supported the interpretation that the level splitting observed in the device is a spin effect, particularly related to spin-orbit coupling (SOC). Without prior knowledge about the direction of $\mathbf{B}_{\text{SO}}$, we first determined the effect of B_x and B_y on the spectrum and inferred the direction approximately orthogonal to $\mathbf{B}_{\text{SO}}$, as described in the following. The dependence on B_y is summarized in Fig. 26.14 for fields of 20 and 40 mT applied in both the positive and negative direction. Its effect was qualitatively similar to that of the field applied along $\hat{u}_1$ [Figs. 5(a),(b),(d),(e) of the Main Text]: the odd-parity triangles [Figs. 26.14(a)–(d)] and the bias-dependent spectra [Figs. 26.14(e)–(h)] became visibly asymmetric, and flipped about $\Phi_L = \Phi_R = \Phi_0/2$ upon inversion of the field direction. The effect was more pronounced for larger B_y , with the smaller triangle almost completely disappearing. In the spectrum, the splitting was enhanced with respect to the zero-field case, as expected from $\mathbf{B}$ having a large component along $\mathbf{B}_{\text{SO}}$.

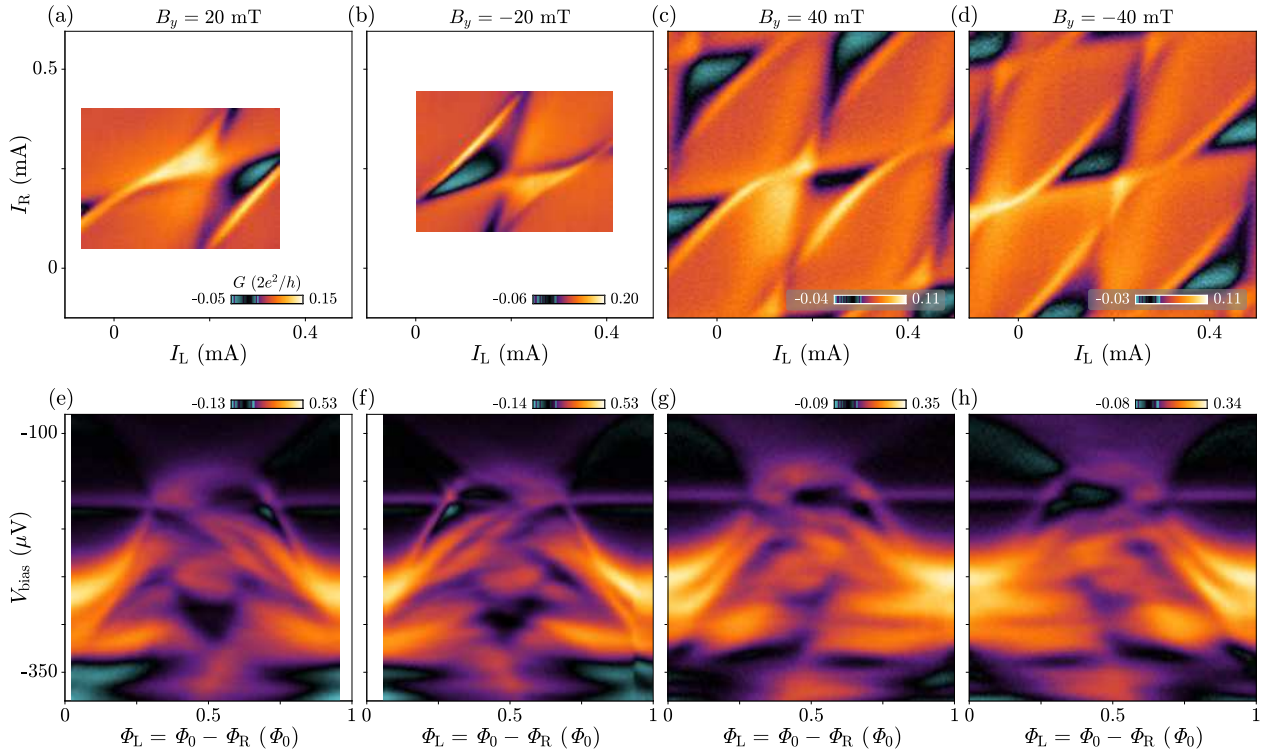


Figure 26.14: **Dependence on in-plane magnetic field B_y .** (a)–(d) Differential conductance G as a function of flux-line currents I_L and I_R , for $V_{\text{bias}} = -167 \mu\text{V}$ and $B_y = \pm 20 \text{ mT}$, $\pm 40 \text{ mT}$. (e)–(h) G as a function of V_{bias} along the $\Phi_L = \Phi_0 - \Phi_R$ linecut, at the same B_y as in (a)–(d), respectively.

Incidentally, we note that the in-plane magnetic field affected the superconducting properties of both the probe and the flux-bias lines. Since the probe was constituted by a thin film of Al and was relatively narrow along the x -direction, it was more strongly affected when $\mathbf{B}$ had a large component along x and showed enhanced resilience when $\mathbf{B}$ was largely aligned with the y -axis [51]. As a consequence, the superconducting probe gap appeared softer along B_x than along B_y (and along $\hat{u}_2$ than along $\hat{u}_1$, as seen in Fig. 5 of the Main Text). Regarding the flux-bias lines, their switching currents (well above 1 mA at $\mathbf{B} = 0$) were reduced for increasing fields, with a marginal dependence on the field orientation. For $B_y = \pm 20 \text{ mT}$, the lines remained superconducting within I_L and I_R ranges of about 400 and 350 μA [we recall that the origin of the (I_L, I_R) planes was shifted to correspond to $\Phi_L = \Phi_R = 0$, as explained in the Methods section]. At the limits of these ranges, flux-line switches to the normal state would have led to discontinuities on the constant-bias cut planes, that were hence measured limiting I_L and I_R to remain in the superconducting state. Conversely, at $B_y = \pm 40 \text{ mT}$ the flux lines were always normal, which did not cause any discontinuity but an aperiodicity in the magnitude of G on the cut planes [visible as regions of enhanced G in Figs. 26.14(c),(d) and 5(c)]. This effect is attributed to local heating of the device by the normal lines, that is dependent on I_L and I_R , and did not affect the qualitative behavior of the features observed.

Next, the in-plane field was applied along the x -direction, as displayed in Figs. 26.15(a),(b),(d),(e),(g),(h). Unlike the case of B_y , the asymmetry due to B_x was very small, although still present both in the constant-bias cut planes and in the spectra, and flipped upon field inversion. For $B_x > 0$, the asymmetry was in the same direction as for $B_y < 0$: this suggests that the direction orthogonal to $\mathbf{B}_{\text{SO}}$, where the asymmetry is expected to be completely absent, was close to the x -direction and, for a certain $B_x > 0$ component, had a smaller $B_y > 0$ component. Following this argument, the vector $\hat{u}_2$ was defined 22.5° rotated counterclockwise with respect to B_x , targeting a direction nearly orthogonal to $\mathbf{B}_{\text{SO}}$.

As shown in Figs 5(c) and 5(f) of the Main Text, the asymmetry along $\hat{u}_2$ was very small (smaller than for B_x). Consequently, the vector $\hat{u}_1$ was defined orthogonal to $\hat{u}_2$ to be approximately aligned with $\mathbf{B}_{\text{SO}}$, indeed revealing the largest asymmetry for given $|\mathbf{B}|$ [Figs. 5(a),(b),(d),(e)].

Finally, we investigated the effect of the magnetic field applied along an intermediate direction between $\hat{u}_1$ and $\hat{u}_2$, defined as $\hat{u}_3 \equiv (\hat{u}_1 + \hat{u}_2)/\sqrt{2}$ (i.e., rotated 22.5° degrees clockwise with respect to B_y). The result is shown in Figs. 26.15(c),(f),(i): the asymmetry was visible in both the constant-bias planes and in the spectrum, although smaller than for the same $|\mathbf{B}|$ applied along $\hat{u}_1$, compatible with the smaller field component along $\mathbf{B}_{\text{SO}}$. As expected, the direction of the asymmetry was the same as for $\mathbf{B} \cdot \hat{u}_1 > 0$.

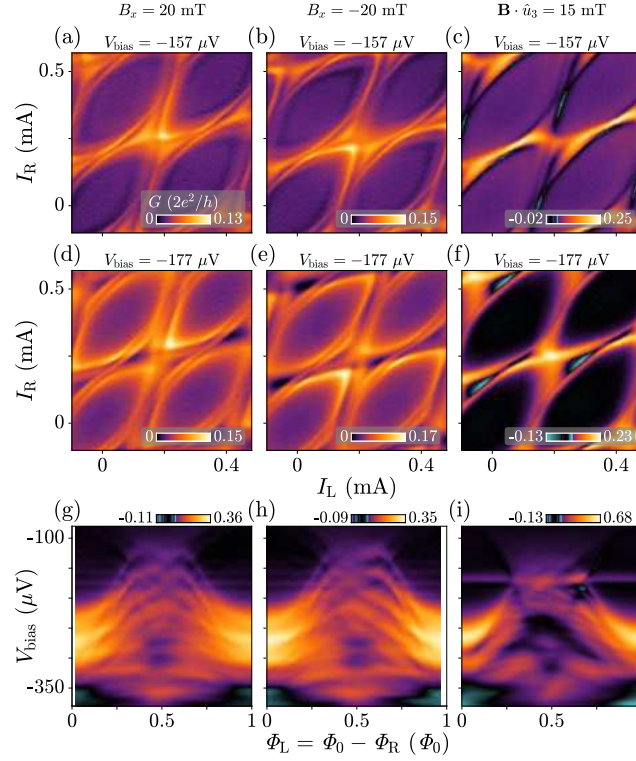


Figure 26.15: **Additional in-plane field orientations.** (a)–(f) Differential conductance G as a function of flux-line currents I_L and I_R , for $V_{\text{bias}} = -157$ μV [(a)–(c)] and $V_{\text{bias}} = -177$ μV [(d)–(f)]. (g)–(i) G as a function of V_{bias} along the $\Phi_L = \Phi_0 - \Phi_R$ linecut. Each column is characterized by a different in-plane field: 20 mT parallel to the x -axis (left), 20 mT antiparallel to the x -axis (middle), and 15 mT parallel to the $\hat{u}_3$ unit vector, that is rotated by 22.5° clockwise with respect to the y -axis.

26.2.9 Results for Device 2

Tunneling spectroscopy measurements were performed on a second device, fabricated on the same chip of the first and measured in the same cool down. The only difference between Device 1 and 2 was the size of the superconducting island in the middle of the scattering region, whose diameter was enlarged from 200 nm to 300 nm for Device 2. In Figs. 26.16 and 26.17, we report the main results obtained for Device 2 at zero magnetic field. Here, the tunnel gates and the probe gate were set to $V_{\text{TL}} = V_{\text{TR}} = -1.38$ V and $V_{\text{probe}} = 0.3$ V to operate in regime comparable to that shown for Device 1. The left, middle and right gates were set to $V_L = V_M = V_R = 0$. Bias-dependent spectroscopy, displayed in Fig. 26.16, featured a transport gap related to the superconducting probe of $2\Delta/e = 302$ μV , slightly lower than for Device 1 but still compatible with the superconducting gap of Al, and distinct flux-independent

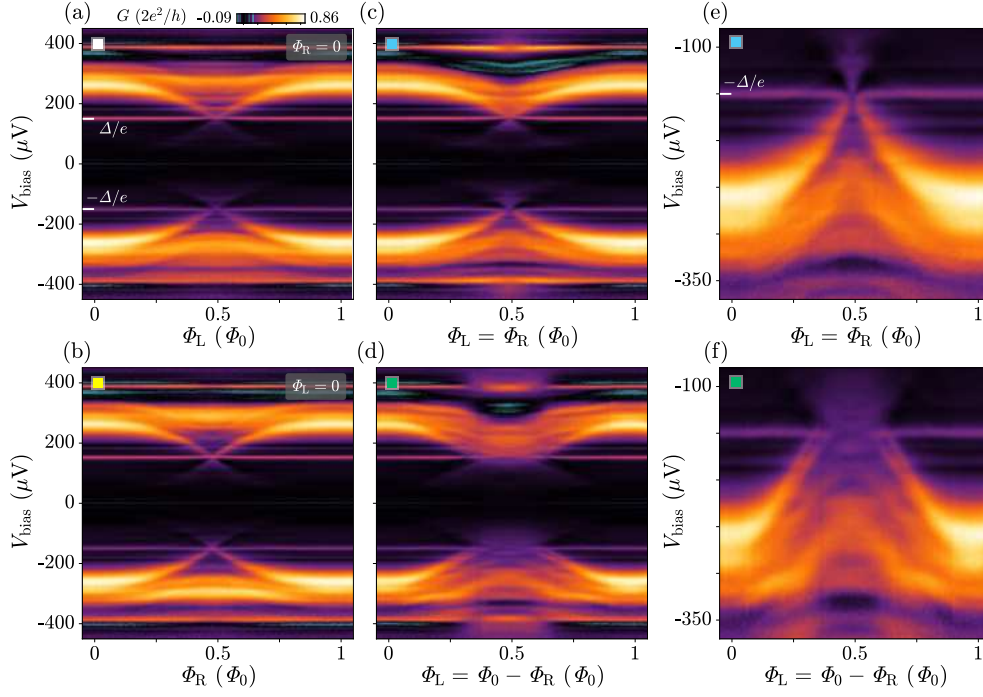


Figure 26.16: **Tunneling spectroscopy along phase-space linecuts for Device 2.** (a) Differential conductance G as a function of voltage bias V_{bias} and flux Φ_L , for $\Phi_R = 0$ [white line in Fig. 26.17(a)]. (b) As in (a), but as a function of Φ_R , for $\Phi_L = 0$ [yellow line in Fig. 26.17(a)]. (c) As in (a), but along the phase-space linecut $\Phi_L = \Phi_R$ [dashed cyan line in Fig. 26.17(a)]. (d) As in (a), but along the phase-space linecut $\Phi_L = \Phi_0 - \Phi_R$ [dashed green line in Fig. 26.17(a)]. (e),(f) Zoom-in of (c),(d).

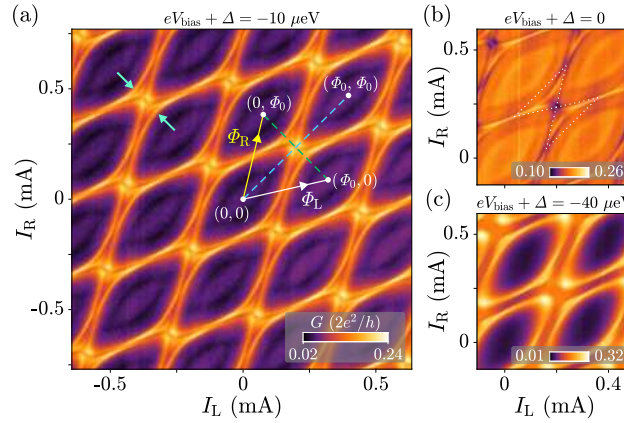


Figure 26.17: **Constant-energy cut planes in the two-dimensional phase space for Device 2.** (a) Differential conductance G as a function of flux-line currents I_L and I_R , for $V_{\text{bias}} = -161 \mu\text{V} = -\Delta/e - 10 \mu\text{V}$. Solid white and yellow lines indicate the periodicity directions, corresponding to the external magnetic fluxes Φ_L and Φ_R . Dashed cyan and green lines represent the $\Phi_L = \Phi_R$ and $\Phi_L = \Phi_0 - \Phi_R$ linecuts. Turquoise arrows indicate the split resonances discussed in the text. (b) As in (a), but for $V_{\text{bias}} = -151 \mu\text{V} = -\Delta/e$. Dotted white triangles show the perimeter of regions where the discrete vortex condition is fulfilled. (c) As in (a), but for $V_{\text{bias}} = -191 \mu\text{V} = -\Delta/e - 40 \mu\text{V}$.

conductance peaks at $\pm\Delta/e$ due to MAR. The spectrum is first shown as a function of Φ_L for $\Phi_R = 0$ [panel (a), corresponding to Fig. 26.7(a) for Device 1], as a function of Φ_R for $\Phi_L = 0$ [panel (b), corresponding to Fig. 26.7(b)] and along the $\Phi_L = \Phi_R$ linecut [panels (c) and (e),

corresponding to Figs. 3(a) and (c) of the Main Text]. In these three cases, the flux-dependent ABSs resembled the conventional energy–phase dispersion valid for two-terminal JJs [48], as one phase difference is zero. Notably, ABSs approaching the probe gap edges indicate near-unity transmission of the respective conduction channels. Spectroscopy performed along the $\Phi_L = \Phi_0 - \Phi_R$ linecut [Figs. 26.16(d) and 26.16(e)] revealed the two main features highlighted for Device 1, namely splitting of Andreev levels and conductance enhancement at the MAR peaks marking zero-energy Andreev level crossings, that were interpreted as lifting of the ABS spin degeneracy and ground state fermion parity transitions. The size of the splitting was smaller than in Device 1 by approximately a factor 2. This may be attributed to the larger size of the scattering region (here $400 \text{ nm} \times 350 \text{ nm}$) on the scale of the superconducting coherence length in InAs $\xi_{\text{InAs}} \sim 600 \text{ nm}$, and to device-to-device variability.

Selected constant-bias cut planes for Device 2 are presented in Fig. 26.17 (to compare with Fig. 4 of the Main Text). The scan over an extended region of the phase space [Fig. 26.17(a), taken at $eV_{\text{bias}} = -\Delta - 10 \text{ } \mu\text{eV} = -161 \text{ } \mu\text{eV}$] showed periodicity along the Φ_L - and Φ_R -axes. The linecuts relevant for Fig. 26.16 are plotted (color coded). Split lines along $\Phi_L = \Phi_0 - \Phi_R$, identified by the turquoise arrows, correspond to the spin-split levels observed in Figs. 26.16(d) and 26.16(f). By reducing $|V_{\text{bias}}|$ to probe the phase space at zero energy [Fig. 26.17(b)], we confirm the presence of odd-parity triangular regions whose perimeter is marked by conductance resonances (that is, zero-energy crossings). Such regions are within the portions of phase space where the discrete vortex condition is fulfilled, shown as the dotted white triangles. Finally, the constant-bias cut plane at $V_{\text{bias}} = -191 \text{ } \mu\text{V}$ ($40 \text{ } \mu\text{eV}$ below zero energy) is displayed in Fig. 26.17(c), showing additional ABSs and the split lines at a limit of visibility. The reproducibility of our experimental observations on a second device suggests the generality of these phenomena and further support our interpretation.

27 Theory of microwave spectroscopy of Andreev bound states with a Josephson junction by L Bretheau, Girit et al

Abstract

We present a microscopic theory for the current through a tunnel Josephson junction coupled to a non-linear environment, which consists of an Andreev two-level system coupled to a harmonic oscillator. It models a recent experiment [Bretheau, Girit, Pothier, Esteve, and Urbina, *Nature* **499**, 312 (2013)] on photon spectroscopy of Andreev bound states in a superconducting atomic-size contact. We find the eigenenergies and eigenstates of the environment and derive the current through the junction due to inelastic Cooper pair tunneling. The current-voltage characteristic reveals the transitions between the Andreev bound states, the excitation of the harmonic mode that hybridizes with the Andreev bound states, as well as multi-photon processes. The calculated spectra are in fair agreement with the experimental data.

27.1 Introduction

The Josephson effect, predicted and observed 50 years ago in superconducting tunnel junctions [1, 2, 3], describes the non-dissipative supercurrent that results from the coherent coupling between superconductors separated by a thin insulating barrier. Since then the supercurrent has been observed in many other weak links such as point contacts, semiconducting nanowires, carbon nanotubes, graphene sheets and thin ferromagnetic layers [4]. Microscopically the Josephson coupling is established through fermionic states whose energies depend on

the superconducting phase difference δ across the weak link. For a weak link shorter than the superconducting coherence length ξ , these so-called Andreev bound states (ABS) have energies smaller than the superconducting gap Δ and are therefore localized at the weak link over a distance of the order of ξ [5, 6, 7, 8]. In a single-channel weak link, there is only one pair of ABS, $|-\rangle$ and $|+\rangle$, with energies $\mp E_A$ (see Fig. 27.1(a)), where

$$E_A = \Delta \sqrt{1 - \tau \sin^2(\delta/2)}. \quad (27.1)$$

Here, δ is the superconducting phase difference and τ is the transmission probability for electrons in the normal state. The phase dependence gives rise to opposite supercurrents $\mp (1/\varphi_0) (\partial E_A / \partial \delta)$ in the ground and excited states (with $\varphi_0 = \hbar/2e$ the reduced flux quantum). This pair of states can be seen as a spin- $\frac{1}{2}$ and introduces an internal degree of freedom to Josephson weak links.

At zero temperature only the lower energy state $|-\rangle$ is occupied. This ground state has been probed through measurements of the current-phase relation in superconducting atomic contacts [9]. The direct observation of the higher energy state $|+\rangle$ has been achieved only recently [10, 11]. In the experiment of Ref. [10], a voltage-biased Josephson junction was used as an on-chip spectrometer (see Fig. 27.2). The dissipative subgap current through the Josephson junction is due to inelastic tunneling of Cooper pairs, the released energy being absorbed in the environment (see Fig. 27.1). Current peaks are then observed at energies corresponding to the eigenenergies of the environment.

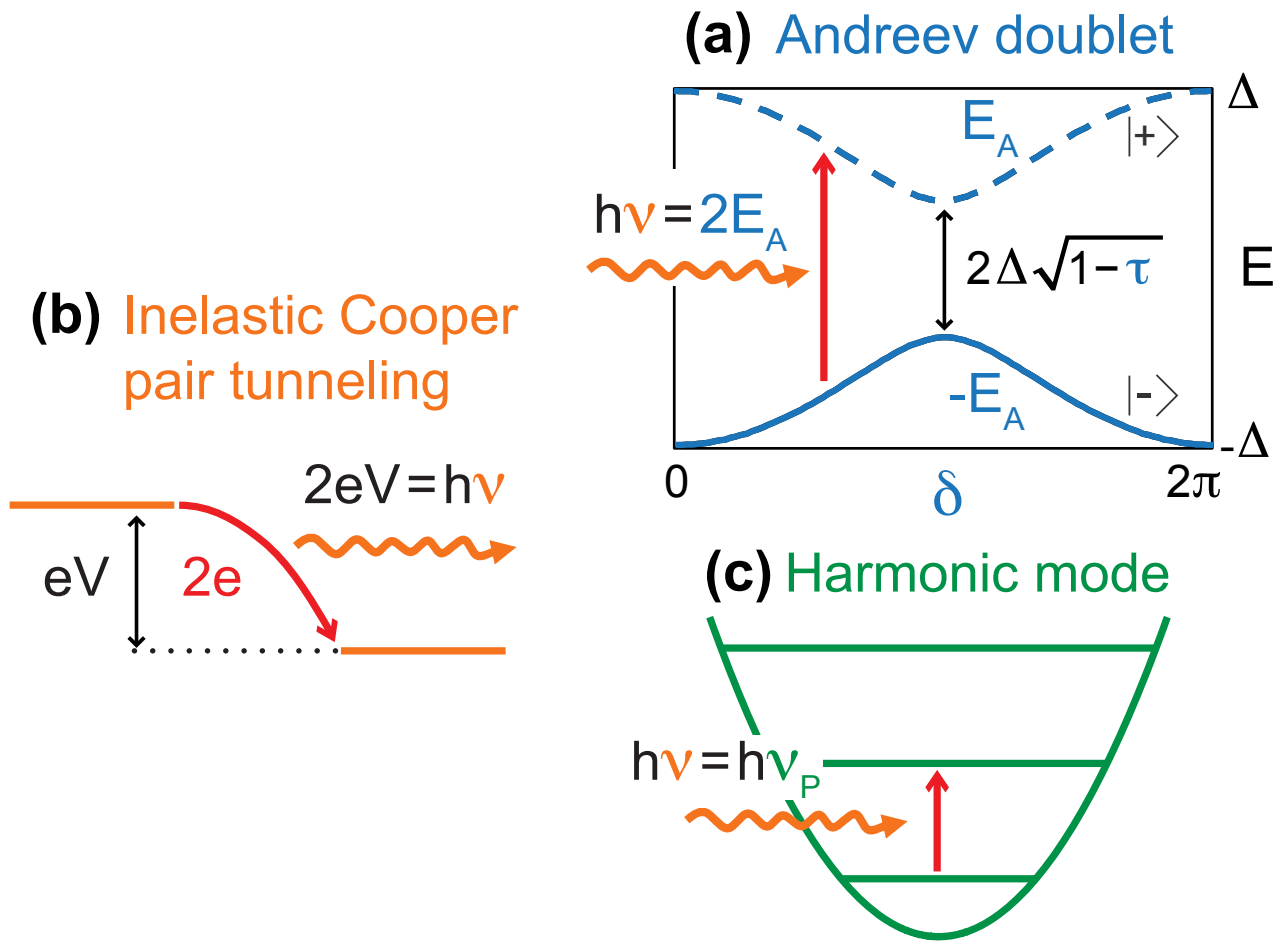


Figure 27.1: (Color online) Principle of the photon spectroscopy of Andreev bound states. **(a)** Phase (δ) dependence of the Andreev levels with energies $\mp E_A$ in a short conduction channel of transmission τ . Δ is the superconducting gap. **(b)** A Cooper pair tunneling across the spectrometer junction releases energy $2eV$ as a photon of frequency ν , which is absorbed either in the atomic contact by exciting the Andreev transition at energy $2E_A$ **(a)** or in a harmonic oscillator mode **(c)** present in the embedding circuit of the atomic contact (see Fig. 27.2).

In practice, the environment was an asymmetric SQUID formed by a superconducting atomic contact in parallel with a second Josephson junction (see Fig. 27.2). This environment can be modeled as a spin-1/2 degree of freedom (the Andreev doublet) coupled with a harmonic mode (the plasma mode of the Josephson junction) (see Fig. 27.1). The goal of the present work is to reach a quantitative understanding of the measured spectra.

The rest of the paper is organized as follows. Section 27.2 gives a Hamiltonian description of the circuit schematized in Fig. 27.2. The spectrometer Hamiltonian is then treated as a perturbation, and following previous work [12, 13, 14, 15, 16, 17], the atomic contact is treated under the assumption of small phase fluctuations. In section 27.3, the resulting spin-boson Hamiltonian of the environment is solved numerically, and the calculated current spectra are compared with the experimental ones. Finally, in section 27.4 the Hamiltonian is solved analytically in the Jaynes-Cummings approximation. Some perspectives are discussed in the conclusion.

27.2 Model

27.2.1 Setup

The setup used in the experiment [10] is shown schematically in Fig. 27.2. On the left-hand side, a voltage-biased Josephson junction (critical current $I_0 = 48$ nA) is used as a spectrometer. It is biased with a voltage source V in series with a resistor $R = 2$ k Ω . It radiates microwaves at the Josephson frequency $\nu = 2eV/h$ [18], which can be absorbed by its electromagnetic environment, an atomic-SQUID formed by an atomic point contact in parallel with a second Josephson junction. The critical current $I_L = 1.06$ μ A of this second junction is much larger than those of the spectrometer junction and of a one-atom contact. An external superconducting coil is used to apply a dc flux $\phi = \varphi_0\varphi$ through the SQUID loop. The capacitance $C = 280$ fF corresponds to the sum of the junctions capacitors. The atomic-SQUID and the spectrometer are coupled through a capacitor $\Sigma \sim 30$ pF. Whereas it behaves as an open circuit from the dc point of view and ensures that the dc voltage V falls on the spectrometer, it can be considered as a short for the Josephson radiation in the explored frequency range ($\nu \gg \nu_p(C/\Sigma)^{1/2} \sim 2$ GHz, with ν_p the plasma frequency of the SQUID junction).

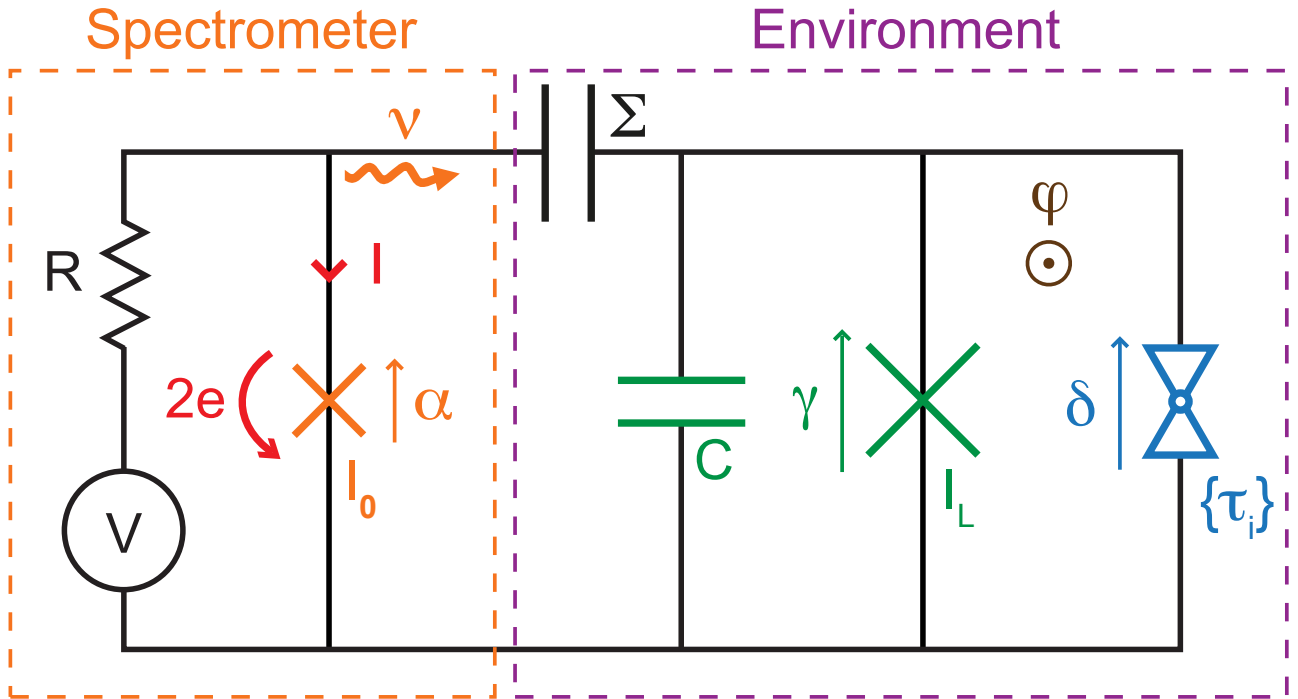


Figure 27.2: (Color online). Simplified diagram of the experimental set-up. A voltage-biased Josephson junction (orange cross) is used as a spectrometer. It emits microwaves in its environment, an atomic-SQUID formed by an ancillary Josephson junction (green cross in parallel with a capacitor C) and an atomic point contact (blue triangles) of channel transmission probabilities $\{\tau_i\}$. The absorption of a photon by the environment is accompanied by the transfer of a Cooper pair through the spectrometer. The phases γ and δ across the SQUID junction and the atomic contact are linked by the external reduced flux φ threading the loop: $\delta = \gamma + \varphi$. The phase across the spectrometer is given by $\alpha = \gamma + 2eVt$.

27.2.2 Hamiltonian of the circuit

Neglecting the coupling capacitor and the bias resistor, the circuit represented in Fig. 27.2 can be described with the Hamiltonian $H = H_{spec} + H_{SQ}$. The first term $H_{spec} = -E_J \cos(\gamma + 2eVt)$ corresponds to the spectrometer with Josephson energy $E_J = \varphi_0 I_0$. The voltage drop across the spectrometer junction induces a superconducting phase difference $\alpha = \gamma + 2eVt$, with γ the phase at the SQUID junction [19]. The coupling with the SQUID occurs through the phase operator γ .

The Hamiltonian

$$H_{SQ} = E_C N^2 - E_L \cos(\gamma) + H_A(\delta) \quad (27.2)$$

accounts for the SQUID. The operator N is the number of Cooper pairs that have crossed the tunnel junction; N and γ are conjugated operators: $[\gamma, N] = i$. $E_C = 2e^2/C$ is the charging energy for pairs and E_L the Josephson energy of the SQUID junction. The last term in Eq. (27.2) describes the atomic contact. The phase $\delta = \varphi + \gamma$ across the atomic contact differs from γ by φ , the reduced flux. For a single channel contact, and neglecting excitations which involve quasiparticles in the continuum, the Andreev Hamiltonian is [13, 14]

$$H_A = -\Delta (\text{Im}U \sigma_y + \text{Re}U \sigma_z), \quad (27.3)$$

$$\text{with } U = \left[\cos\left(\frac{\delta}{2}\right) + i\sqrt{1-\tau} \sin\left(\frac{\delta}{2}\right) \right] e^{-i\sqrt{1-\tau}\frac{\delta}{2}}$$

where the Pauli matrices $\sigma_{x,y,z}$ act in a 2×2 subspace corresponding to the two ABS. The physics of the ABS is therefore analogous to that of a spin- $\frac{1}{2}$ in a magnetic field whose magnitude $\Delta|U| = E_A$ and direction depend on the superconducting phase difference δ across the contact. The eigenstates of H_A are the ABS $|\pm\rangle$, with eigenenergies $\pm E_A$ (see Eq. (27.1) and Fig. 27.1(a)).

27.2.3 Andreev spin and plasma boson

In the experiment [10], due to the large asymmetry $E_L \gg E_A$, the phase dynamics is essentially ruled by the SQUID junction. The size of the phase fluctuations is determined by the dimensionless parameter $z = (E_C/2E_L)^{1/2} \ll 1$. Therefore, following Ref. [12], we may treat the SQUID junction as a linear inductor and retain only the lowest order coupling ($\propto \sqrt{z}$) between the SQUID junction and the atomic contact in Eq. (27.2). Namely,

$$H_{SQ} \approx E_C N^2 + E_L \gamma^2/2 + H_A(\varphi) + \varphi_0 \gamma C_A(\varphi), \quad (27.4)$$

where $C_A = \varphi_0^{-1} \partial H_A / \partial \delta$ is the Andreev current operator. Hence, the parallel combination of the SQUID junction and the capacitor C forms a harmonic oscillator of resonant frequency $\nu_p = \sqrt{2E_L E_C}/h$ (see Fig. 27.1(c)). Switching to second quantization, its Hamiltonian reads $h\nu_p(a^\dagger a + \frac{1}{2})$, where the phase $\gamma = \sqrt{z}(a + a^\dagger)$ is linked to the annihilation and creation operators a and $a^\dagger$ of the plasma mode of the SQUID.

In the basis of the Andreev states $\{|-\rangle, |+\rangle\}$, the SQUID Hamiltonian finally reads

$$H_{SQ} = h\nu_p \left(a^\dagger a + \frac{1}{2} \right) - E_A \sigma_z + (a + a^\dagger) (\Omega_x \sigma_x + \Omega_z \sigma_z), \quad (27.5)$$

where

$$\Omega_z = \Delta \sqrt{z} \frac{\tau \sin(\varphi)}{\sqrt{1 - \tau \sin^2(\frac{\varphi}{2})}} \quad (27.6)$$

and $\Omega_x = \Omega_z \sqrt{1 - \tau} \tan(\frac{\varphi}{2})$. The phase-dependence of these coupling energies is represented in Fig. 27.3.

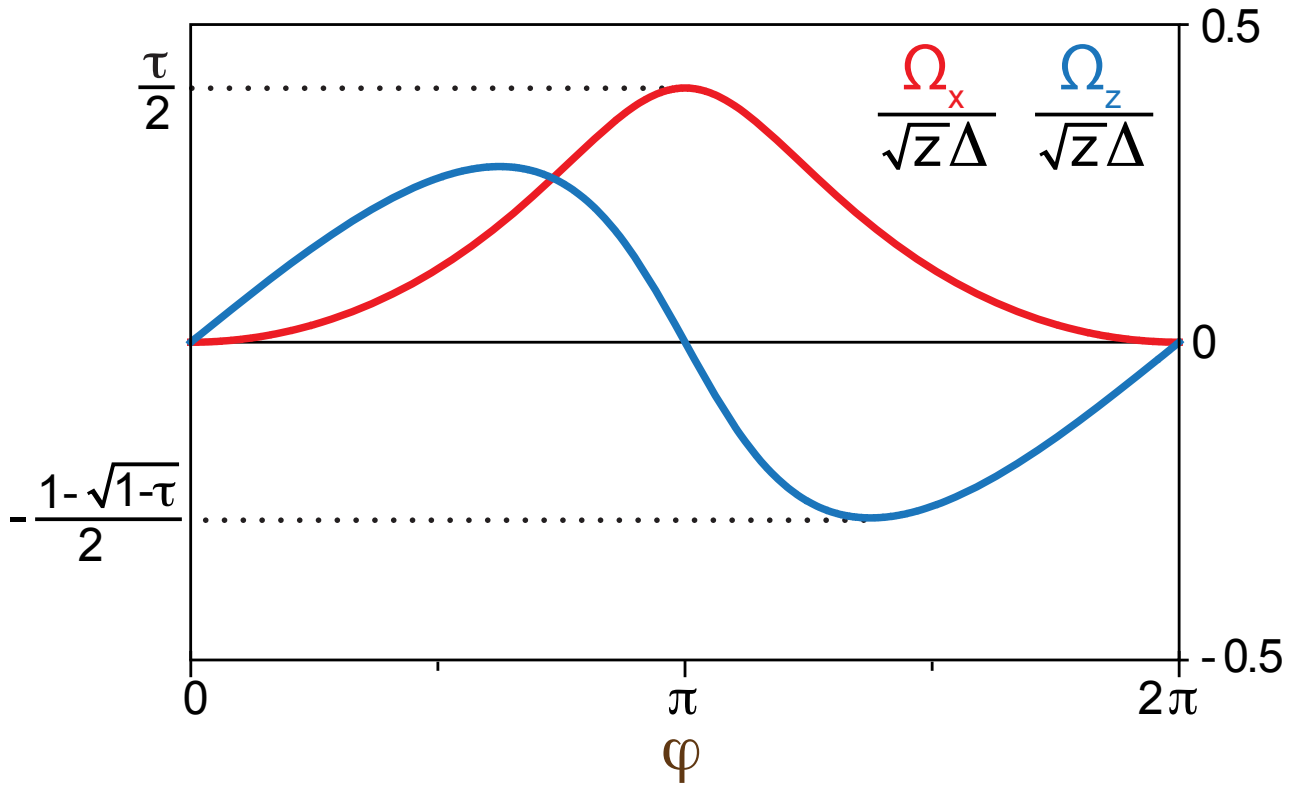


Figure 27.3: (Color online). Phase dependence of the coupling energies Ω_x (red) and Ω_z (blue) in units of $\sqrt{z}\Delta$, for $\tau = 0.8$.

The spin-boson Hamiltonian (27.5) yields a discrete eigenspectrum which results from the hybridization of the bosonic plasma mode in the SQUID junction and the Andreev spin-1/2 degree of freedom in the atomic contact (see Fig. 27.4(a)). The term $\propto \Omega_x$ allows transitions between Andreev states. It is minimum at $\varphi = 0$ and maximum at $\varphi = \pi$. Note that the flux modulation of the SQUID plasma frequency due to the contribution of the effective inductance of the atomic contact would appear as a higher order effect in z .

27.2.4 Incoherent Cooper pair current

The global circuit Hamiltonian H cannot be diagonalized analytically. Along the lines of $P(E)$ theory for dynamical Coulomb blockade [20, 21], the spectrometer may be treated as a perturbation. The dc current I flowing through the spectrometer is calculated using the current operator $I_0 \sin(\gamma + 2eVt)$ and perturbation theory up to the second order in E_J . At zero temperature and for $V > 0$,

$$\frac{I(V)}{I_0} = \frac{\pi}{2} E_J \sum_k |\langle k | e^{i\gamma} | 0 \rangle|^2 \delta((E_k - E_0) - 2eV), \quad (27.7)$$

where $|k\rangle$ are the eigenstates of H_{SQ} with energy E_k . Equation (27.7) accounts for incoherent Cooper pair tunneling through the junction's barrier at rate $I/2e$, provided that the energy $2eV$ matches an excitation energy of the junction environment (see Fig. 27.1(b)). As in $P(E)$ theory for zero temperature, we assume that, before each tunneling event, the environment has relaxed to its ground state $|0\rangle$.

In $P(E)$ theory [20, 21], the environment is purely electromagnetic, linear and is described by an impedance $Z(\omega)$. Equation (27.7) then simplifies and gives the current as a function of

$\text{Re}Z(\omega)$. In particular, this would yield current peaks at $2eV = nh\nu_p$ (n integer), with a width related to the quality factor of the bosonic plasma mode of frequency ν_p .

Such an approach is not possible here due to the strong non-linearity of the Andreev degree of freedom. As the model does not include dissipation, the spin-boson Hamiltonian H_{SQ} yields a discrete spectrum. Then, Eq. (27.7) predicts infinitely sharp dc current peaks at the excitation energies of H_{SQ} . A finite broadening of each of the peaks may be introduced phenomenologically with the substitution $\delta(E - E_0) \rightarrow (\Gamma/\pi) / [(E - E_0)^2 + \Gamma^2]$ in Eq. (27.7). While the model is unable to predict the amplitude of the linewidth Γ , it should be large enough for the incoherent Cooper pair current through the spectrometer junction to be small, $I \ll I_0$, so that perturbation theory is valid. Near voltage $V = \epsilon_k/(2e)$, where $\epsilon_k = E_k - E_0$, this condition reads $(E_J/\Gamma)P_k \ll 1$, with transition probability $P_k = |\langle k | e^{i\gamma} | 0 \rangle|^2$.

27.3 spectra calculation

27.3.1 Numerical resolution

The spectrum of Hamiltonian (27.5) can be obtained numerically. To do so, we write it as a matrix in the basis $\{|\sigma, n\rangle\}$, where $\sigma = \pm$ accounts for the Andreev spin-1/2 and n is the plasmon occupation number, and truncate to the lower energy states. By numerical resolution of a 14×14 Hamiltonian matrix ($n \leq 6$), we have computed all the resonances of energy smaller than 2Δ and the corresponding transition probabilities P_k . Figure 27.4 shows the excitation spectrum (b) and the transition probabilities (c) for the transitions indicated in (a), for a channel of transmission $\tau = 0.98$. In the experiment, the superconducting gap energy is $\Delta/h \simeq 43$ GHz, the charging energy is $E_C/h \simeq 270$ MHz and the Josephson energy of the SQUID junction is $E_L/h \simeq 900$ GHz, leading to $z \simeq 0.012$. Note that E_L is renormalized as $E_L = \varphi_0(I_L + 2\varphi_0/L)$, where $L/2 = 0.44$ nH is a parallel inductor (not shown in Fig. 27.2) accounting for a 1.35 mm-long aluminum connecting wire present in the actual geometry of the sample.

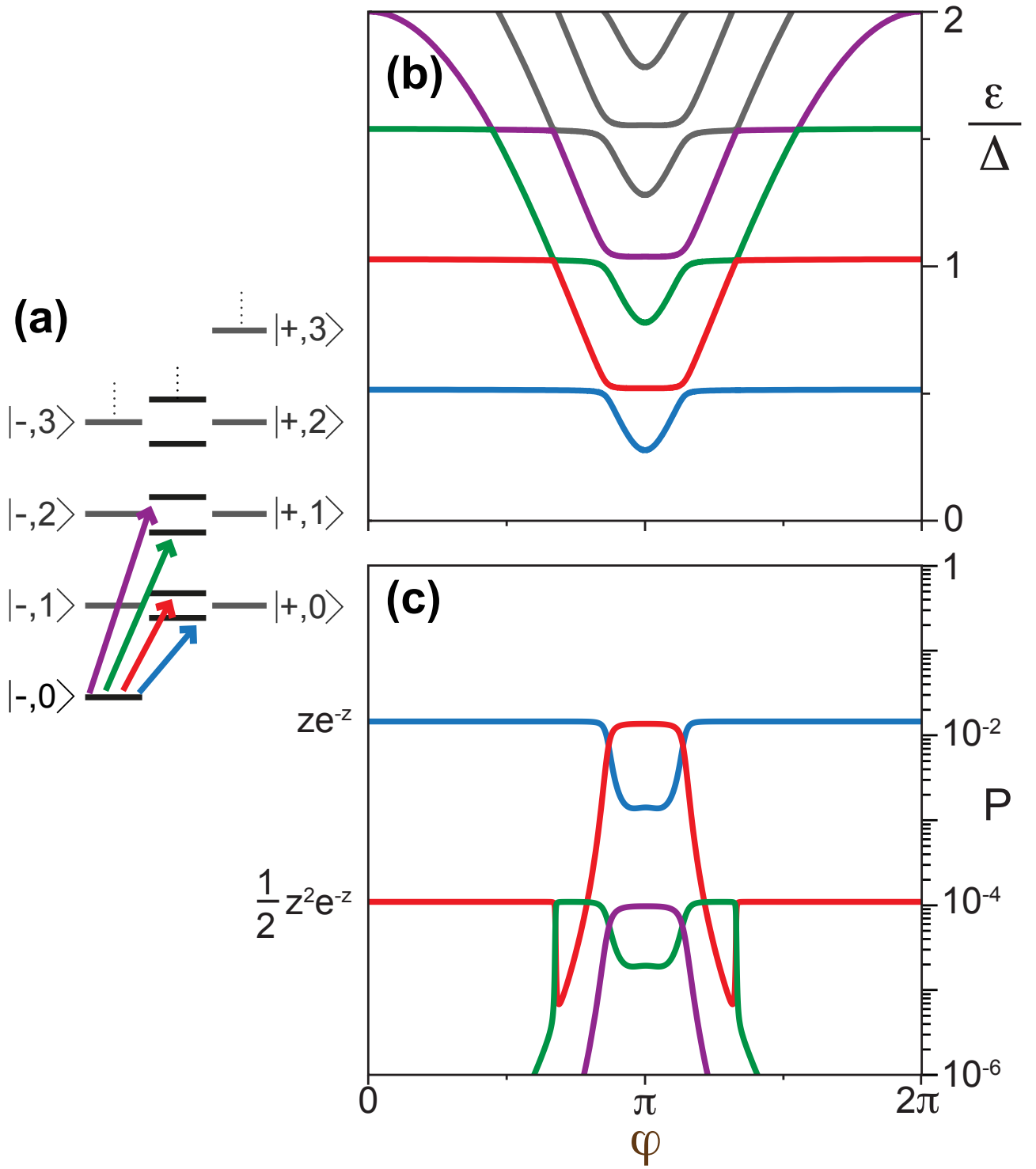


Figure 27.4: (Color online). (a) Energy spectrum diagrams of H_{SQ} for a single channel: each state is labelled $|-,n\rangle$ or $|+,n\rangle$ for the Andreev pair in the ground (−) or excited (+) state and n photons in the plasma mode. At degeneracy $2E_A = \hbar\nu_p$, the plasma and Andreev modes hybridize, which leads to an avoided crossing visible in (b). (b, c) Excitation energies ϵ (in units of Δ) (b) and transition probabilities P (c) of the first resonances, as a function of the reduced flux φ , for a channel of transmission $\tau = 0.98$. These lines are obtained by numerical resolution of a 14×14 Hamiltonian matrix. Each color encodes a transition from the ground state towards a different excited state, as represented in (a).

27.3.2 Incorporating line broadening

The experimental data displayed in the right panels of Fig. 27.5 show broad lines. One notices that the transition solely concerning the plasma mode is already broad, which indicates that this mode is subject to dissipation and/or dephasing; note also that voltage fluctuations across the spectrometer junction can limit the energy resolution [22]. To account for the broadening of the energy levels of H_{SQ} one should at least include the coupling of the plasma mode to a dissipative bath. Treating the latter as an infinite collection of harmonic oscillators, as it is usually done, would lead to a complex spin-boson problem without analytical solution. Here we simply introduce in Eq. (27.7) a phenomenological Lorentzian broadening, cf. Sec. 27.2.4. By fitting the plasma resonance peak, we found $\Gamma/h = 2$ GHz, which we used for all the calculated lines shown in the left panels of Fig. 27.5.

27.3.3 Comparison with experiment

In experiments atomic contacts have several channels. We extend the model to contacts with multiple channels by adding an Andreev term (27.3) for each channel to the Hamiltonian. This Hamiltonian describes the physics of $\mathcal{N}$ spins coupled to the same bosonic mode but not directly coupled to each other. All the approximations made before are still valid [23]. Figure 27.5 compares the calculated spectra (left) with the experimental ones from Ref. [10] (right), for two different atomic contacts AC1 (transmissions 0.942, 0.26) and AC2 (transmissions 0.985, 0.37). The theory is obtained with the numerical method sketched before, still using seven photon levels (28×28 matrix).

The model describes both the Andreev transitions $|- , 0\rangle \rightarrow |+ , 0\rangle$, of energy $2E_A$ (V-shaped lines), and the plasma transition $|- , 0\rangle \rightarrow |- , 1\rangle$, of energy $h\nu_p$ (red horizontal line at 0.51Δ). It also describes the higher harmonic transitions: $|- , 0\rangle \rightarrow |+ , 1\rangle$, of energy $2E_A + h\nu_p$ (replica of the Andreev transition, shifted up by 0.51Δ) and $|- , 0\rangle \rightarrow |- , 2\rangle$, of energy $2h\nu_p$ (white horizontal line at 1.02Δ). These processes correspond to the tunneling of one Cooper pair and emission of two photons. They are less probable and result in fainter transitions as seen both in experiment and calculation. Theory also accounts for the anti-crossings arising from the coupling between the Andreev-spin and the plasma-boson. Only crossings of transition lines involving the same number of photons show significant hybridization, in good agreement with the data.

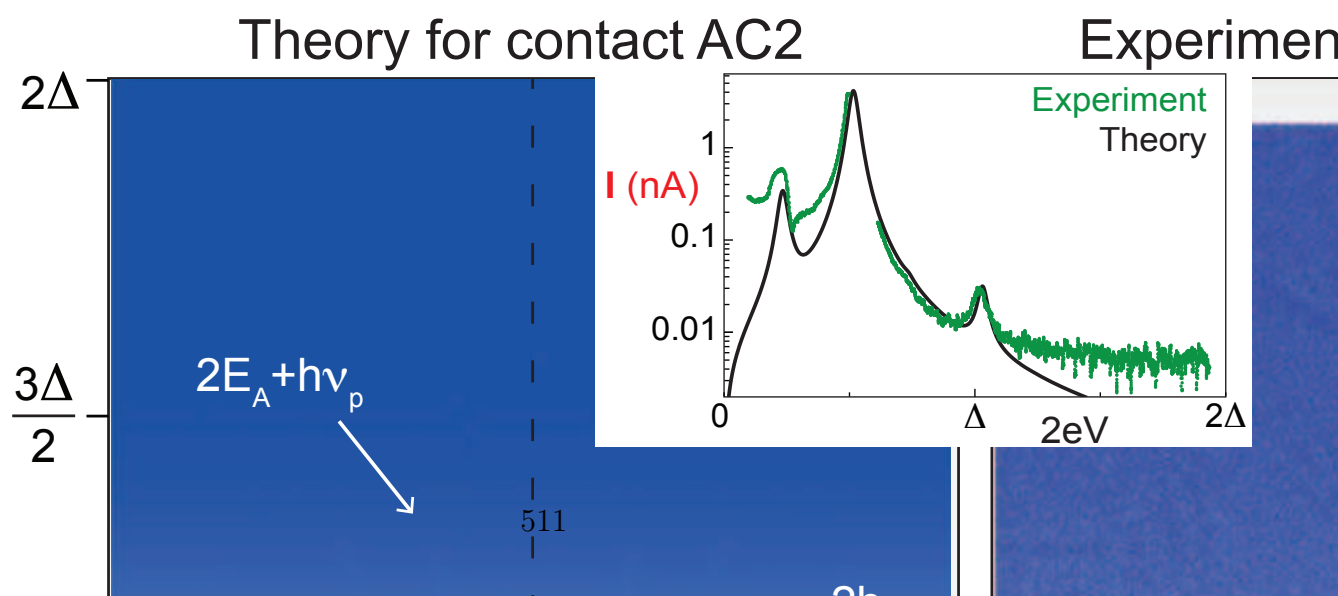
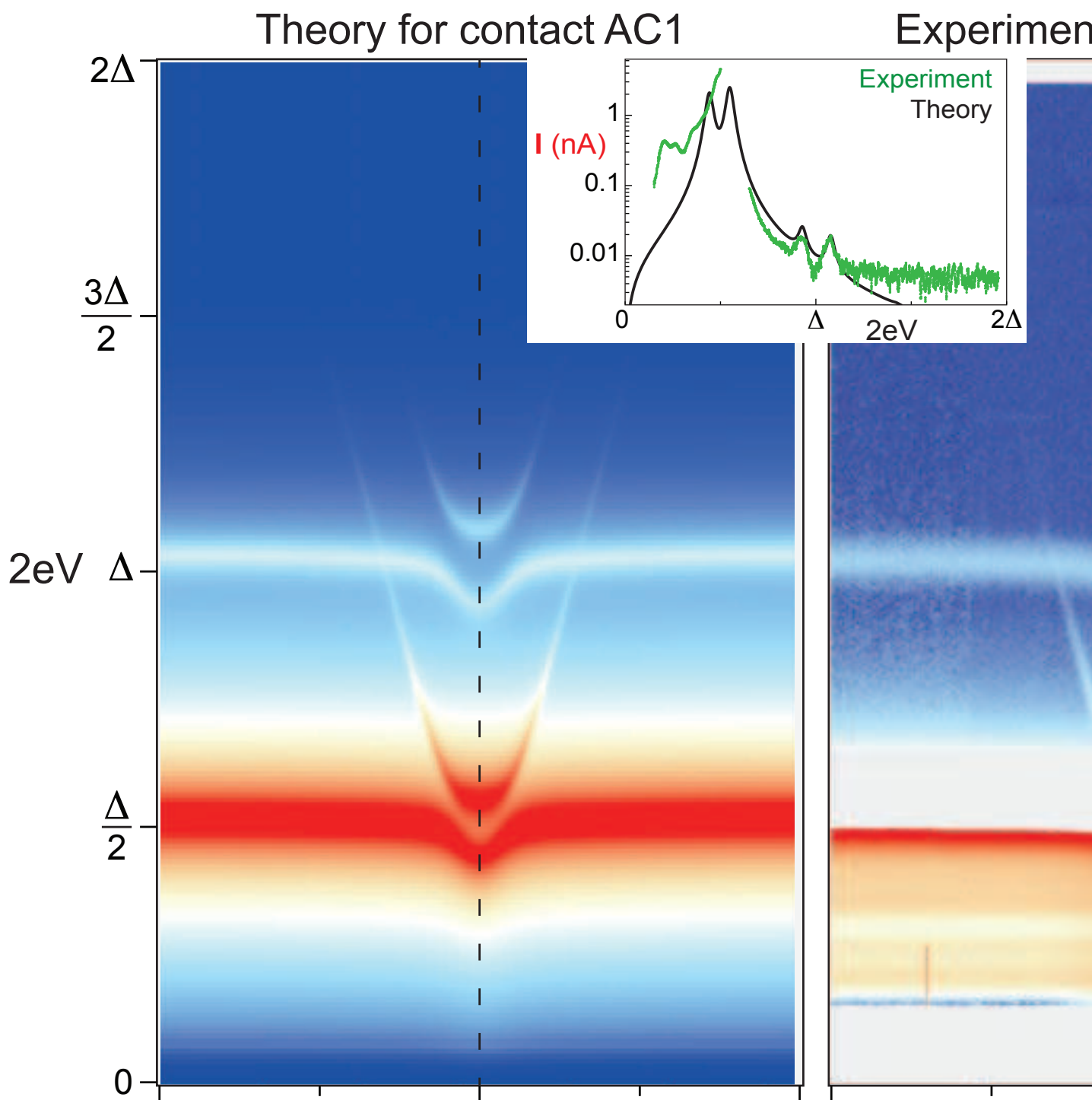
Extra phase-independent current at $2eV \lesssim \Delta/2$ in the experimental data is attributed to the coupling to uncontrolled environmental modes outside of the SQUID.

Finally, the global weakening of the signal at high V is well captured by the model. This weakening is due to the lower impedance of the SQUID capacitance compared to its inductance at frequencies larger than the plasma frequency. However, the agreement for the amplitude and width of the different peaks is not quantitative. Having set the width to 2 GHz, a correct peak amplitude is only obtained when multiplying theory by a factor 0.6. A rigorous treatment of dissipation is needed. This could be achieved by adding an electromagnetic impedance with a dissipative component in parallel with the SQUID.

27.4 Andreev-plasma mode hybridization

We use the rotating wave approximation, valid in the vicinity of the degeneracy $h\nu_p = 2E_A$, to obtain analytical expressions for the excitation energies and transition rates. The Hamiltonian can be approximated by the Jaynes-Cummings model [24, 25]:

$$H_{SQ}^{JC} = h\nu_p \left(a^\dagger a + \frac{1}{2} \right) - E_A \sigma_z + \Omega_x (a \sigma_+ + a^\dagger \sigma_-) \quad (27.8)$$



where $\sigma_{\pm} = \frac{1}{2}(\sigma_x \mp i\sigma_y)$. Then, by block-diagonalization in the subspace $\{|+, n\rangle, |-, n+1\rangle\}$, one derives the eigenstates and eigenenergies and computes the excitations energies and transition probabilities.

To fourth order in $\sqrt{z}$, the current (27.7) through the spectrometer displays four peaks

$$\frac{I(V)}{I_0} = \frac{\pi}{2} E_J \sum_{\sigma=\pm; n=1,2} P_n^{\sigma} \delta(2eV - \epsilon_n^{\sigma}) \quad (27.9)$$

at the bias voltages matching the excitation energies (see top panel of Fig. 27.6):

$$\epsilon_n^{\pm} = E_A + \left(n - \frac{1}{2}\right) h\nu_p \pm W_n \quad (27.10)$$

with $W_n = \sqrt{(E_A - \frac{1}{2}h\nu_p)^2 + n\Omega_x^2}$. The amplitudes of the current peaks are proportional to the transition probabilities (see bottom panel of Fig. 27.6):

$$\begin{cases} P_n^- = \frac{1}{n!} z^n e^{-z} \cos^2\left(\frac{\theta_n}{2}\right) \\ P_n^+ = \frac{1}{n!} z^n e^{-z} \sin^2\left(\frac{\theta_n}{2}\right) \end{cases} \quad (27.11)$$

with $\theta_n = \arctan(n\Omega_x/W_0)$.

These peaks correspond to excitations towards composite states, resulting from the hybridization of the plasma and Andreev modes (see Fig. 27.4(a)). The first two resonances in Eq. (27.9) correspond to the excitation with a single photon of the hybridized Andreev ($2E_A$) and plasma mode ($h\nu_p$); the last two resonances correspond to the excitation of higher harmonic modes at $2h\nu_p$ and $h\nu_p + 2E_A$. These two-photon processes are possible because the spectrometer, a Josephson tunnel junction, is a nonlinear emitter. They correspond to the tunneling of one Cooper pair and emission of two photons. Note that far from degeneracy, using perturbation theory with $z \ll 1$, one finds for the plasma resonance and its harmonic the transition probabilities ze^{-z} and $\frac{1}{2}z^2e^{-z}$. These amplitude are consistent with the Poisson distribution found at arbitrary z in the $P(E)$ derivation with a bosonic mode [21].

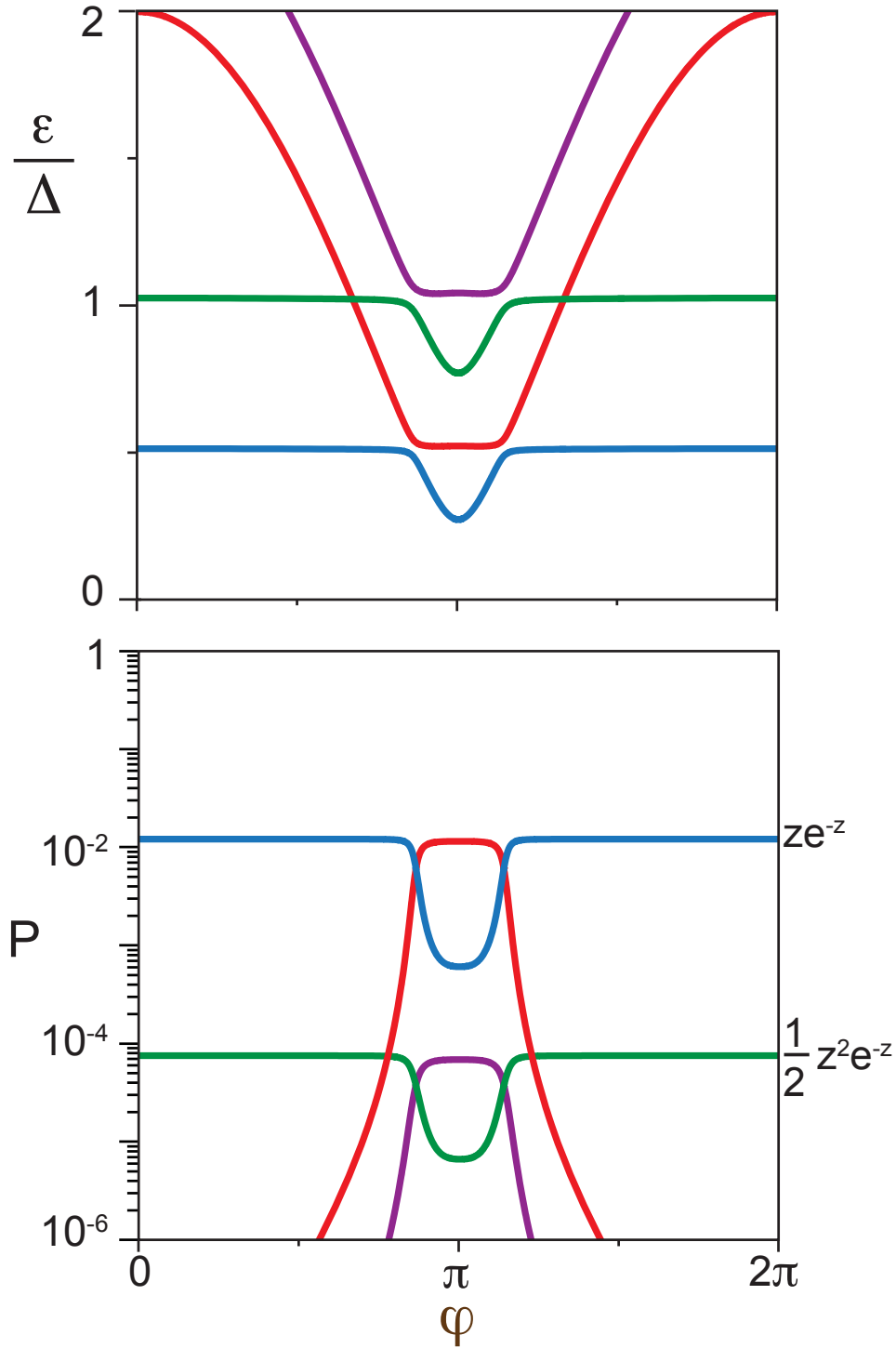


Figure 27.6: (Color online). Excitation energies ϵ (in units of Δ) (**top**) and transition probabilities P (**bottom**) of the first resonances, as a function of the reduced flux φ , in the Jaynes-Cummings approximation for a channel of transmission $\tau = 0.98$. Each color encodes a transition towards a different state, as shown in Fig. 27.4(a).

It is worth mentioning that the Jaynes-Cummings model gives an excellent description, even far from degeneracy when $|W_0| \gg \hbar\nu_p$. There, the error made in energy scales as $\Omega_x/(2E_A + \hbar\nu_p)$, and is in general negligible. However, this model fails to predict the anti-crossing between the second harmonic of the plasma resonance and the Andreev resonance at $2\hbar\nu_p = 2E_A$ (red and green lines in Fig. 27.6). In practice this correction is one order of magnitude smaller than the anti-crossing at $\hbar\nu_p = 2E_A$ and is not seen in the experiment [10].

Note however that the derivation in the case of multiple channels is much more involved when more than one Andreev transition crosses the plasma resonance.

27.5 Conclusions and perspectives

In conclusion, we have treated inelastic Copper pair tunneling through the emitter junction using perturbation theory in its coupling with the environment, and calculated the environmental transition energies and probabilities by diagonalization of the SQUID Hamiltonian. This Hamiltonian contains a spin-like term describing the atomic contact and which is coupled to a harmonic term for the plasma mode of the SQUID junction. The calculated spectra are in fair agreement with the experimental ones. However this theory could benefit from several extensions.

First of all, the atomic contact Hamiltonian was restricted to a two-level system Hamiltonian, by neglecting excitations which involve quasiparticles in the continuum. In particular, quasiparticle trapping, which was measured in a similar experiment [11], leaves the Andreev doublet in a long-lived “odd” state for which both the Andreev states are either occupied or empty [26], with time scales in the millisecond range [27]. It limits the Andreev transition rate and therefore reduces the corresponding dc current. On the other hand, transitions from the Andreev bound states to the continuum were measured on the same contacts using another detection method [11]. They do not contribute to the dc current and were not seen in the experiment described in Ref. [10] due to the long lifetime of the odd states. To deal with all the Andreev excitations on the same footing, one must consider the full Hamiltonian of the atomic contact, as done in recent works [17, 28].

Second, the description used here for the environment eigenstates and for the spectrometer is far from being complete. As discussed above, this is a difficult problem for two reasons: (i) the coupling of a two-level system to a dissipative resonator is not known analytically, and (ii) the $I(V)$ characteristic of a Josephson junction coupled to a resonator is not generically known when the perturbative expansion in E_J is not valid, despite recent progress along this direction [29]. Although the second order expansion is valid for the data discussed here since the Cooper pair current is small ($I \ll I_0$), higher order contributions corresponding to the transfer of more than one Cooper pair emitting several photons were observed in other experiments [30, 11].

Finally, it was also assumed in the model that the environment is in its ground state each time a Cooper pair tunnels. This requires the relaxation time of the excited environment states to be shorter than the inverse tunnel rates. When this is not the case, the environment modifies the tunneling process. For the harmonic oscillator bosonic mode, this could lead to stimulated emission and lasing. For the Andreev two-level system, this could saturate absorption and reduce the Cooper pair current through the spectrometer junction. A regime in which the Andreev two-level system would coherently exchange an excitation with the Josephson junction can also be envisioned.

28 The multi-terminal Josephson effect by Pankratova, Lee et al.

Establishment of phase-coherence and a non-dissipative (super)current between two weakly coupled superconductors, known as the Josephson effect [1], plays a foundational role in basic physics [2] and applications to metrology [3], precision sensing [4], high-speed digital electronics [5], and quantum computing [6]. The junction ranges from planar insulating oxides to single atoms [7], molecules [8], semiconductor nanowires [9, 10], and generally to any finite-size coherent con-

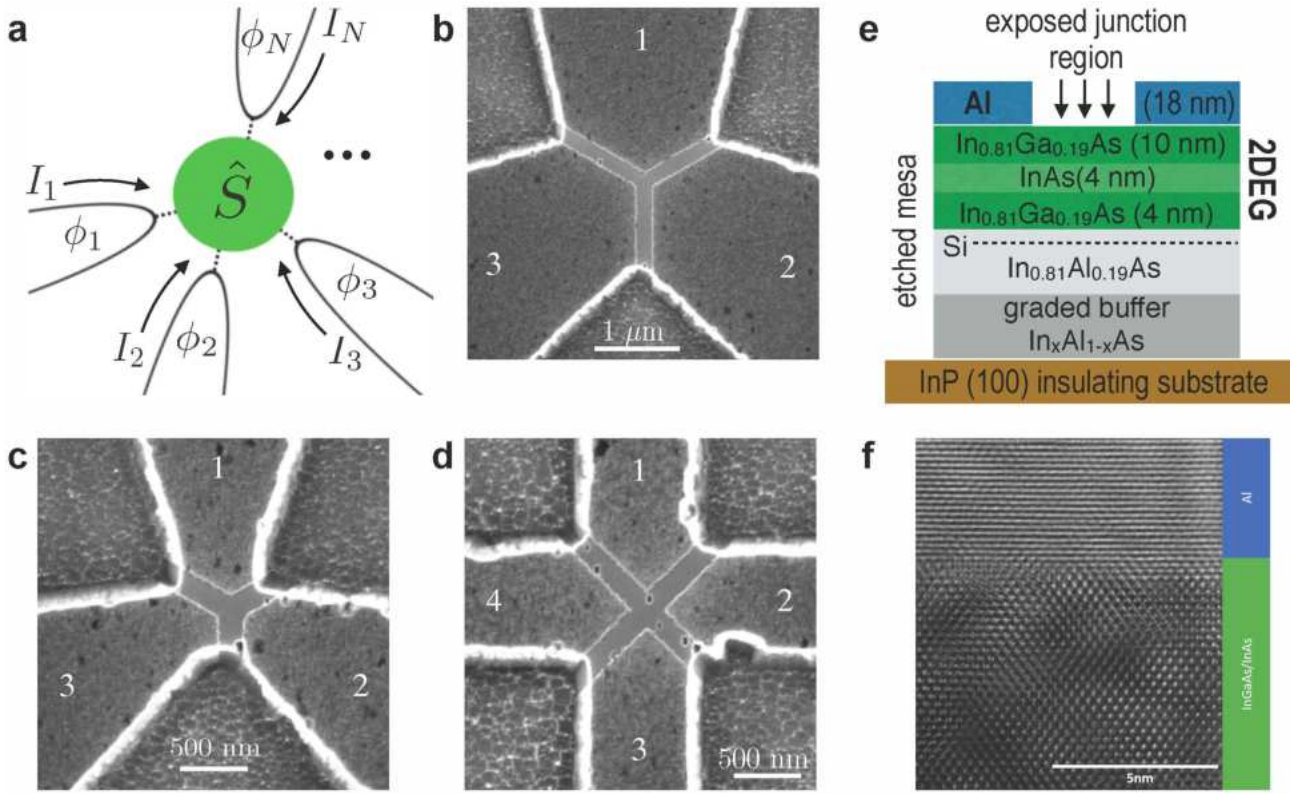


Figure 28.1: Multi-terminal Josephson junctions based on an epitaxial InAs/Al heterostructure. (a) A junction of N independent superconductors across a common normal region, characterized by a scattering matrix $\hat{S}$. Each terminal $j = 1, \dots, N$ has a superconducting phase ϕ_j and the associated supercurrent I_j . (b-d) Scanning electron microscope images of the nanofabricated junctions of $N = 3$ (b,c) and $N = 4$ (d) copies of which are used in this work. (e) Junction's schematic cross-section revealing the material stack and composition of each layer of the heterostructure. (f) Transmission electron microscope image of the semiconductor/superconductor interface.

ductor [11]. Recently, junctions of more than two superconducting terminals gained broad attention in the context of braiding of Majorana fermions in the solid state [12, 13, 14, 15, 16, 17] for fault-tolerant quantum computation [18], and accessing physics and topology in dimensions higher than three [19]. Here we report the first observation of Josephson effect in 3- and 4-terminal junctions, fabricated in a top-down fashion from a semiconductor/superconductor (InAs/Al) epitaxial two-dimensional heterostructure [20]. Due to interactions, the critical current of a N -terminal junction becomes the *boundary* of a $(N - 1)$ -dimensional manifold of simultaneously allowed supercurrents. The measured shapes of such manifolds are explained by the scattering theory of mesoscopic superconductivity, and they can be remarkably sensitive to the junction's symmetry class [21]. Furthermore, we observed a notably high-order (up to 8) multiple Andreev reflections [22] simultaneously across every terminals pair, which verifies the multi-terminal nature of normal scattering and a high interface quality in our devices. Given the previously shown gate-control of carrier density and evidence of spin-orbit scattering in InAs/Al heterostructures, and device compatibility with other 2D materials, the multi-terminal Josephson effect reported here can become a testbed for physics and applications of topological superconductivity.

In a conventional tunnel junction considered by Josephson, the supercurrent varies sinu-

soidally with the difference in phases of the complex-valued order parameter in the two superconducting leads (terminals) [2]. The amplitude of the sinusoid defines the junction's most important parameter – the critical current. A multi-terminal Josephson junction can be conceptually viewed as N independent superconductors coupled through a common element characterized by a scattering matrix $\hat{S}$ (Fig. 1a). Now a supercurrent in a given terminal periodically depends on all $N - 1$ independent phase-differences, which effectively adds an extra dimension to the problem with every new terminal. While this property was considered in the past mostly for device applications [23, 24], fundamentally new effects in junctions of three and more terminals were recently identified, such as strong breaking of Kramers degeneracy without Zeeman fields [25] and emergence of Weyl quasiparticle nodes [19, 26, 27]. In the simplest manifestation of multi-terminal Josephson effect, a supercurrent forced into one terminal influences the allowed values of supercurrent in every other terminal. Therefore, traditional measurement of individual critical currents across all possible terminal pairs is insufficient. Instead, one needs to identify all combinations of $N - 1$ independent bias currents (the N -th one is eliminated by current conservation) for which every terminal maintains zero voltage. For example, the critical current of a 3-terminal junction is a *contour* in a 2D plane.

Junctions with $N > 2$ are challenging because the central region must accommodate transparent connections to multiple finite-size leads without exceeding in size the electronic phase-breaking length. In addition, the leads must be relatively equally coupled to each other across the junction. Previous experiments on diffusive metallic systems explored out-of-equilibrium transport [28] and semi-classical topological aspects of proximity effect [29]. Notably, two-terminal supercurrents were reported in a nanocross of two crossed-grown InAs nanowires with four superconducting leads [30]. Our junctions are based on an epitaxial heterostructure of III-V materials [20] (Fig. 1b-d). An InAs quantum well is sandwiched between InGaAs barriers to confine a high-mobility 2D electron gas (2DEG) near the surface and simultaneously have a crystalline interface with a superconducting Al layer [31] (Fig. 1e,f). Three- and four-terminal junctions with sub-micron dimensions were fabricated by selectively removing the unwanted material (METHODS). The exposed 2DEG has the following parameters, obtained through transport measurements on a similar wafer [31]: Fermi wavelength $\lambda_F \approx 25$ nm, velocity $v_F \approx 10^6$ m/s, mean free path $l_e \approx 200$ nm, and phase-breaking length $l_\phi \gtrsim 1$ μ m. We thus expect a quasi-ballistic coherent transport involving a moderate range of 10 – 100 channels.

In the simpler 3-terminal case, the basic characterization consists of grounding one terminal, applying the two DC currents I_1, I_2 to the remaining terminals labeled 1, 2, and simultaneously measuring the two voltages V_1 and V_2 , and differential resistances dV_1/dI_1 and dV_2/dI_2 . The data (Fig. 2a,b) contains three generic features. First, the supercurrent state, defined by $V_2 = V_1 = 0$, consists of a simply-connected 2D region in the (I_1, I_2) -plane. It can be obtained from the data by intersecting the two individual (dark-blue) zero-resistance regions in Fig. 2a and Fig. 2b. The boundary of the supercurrent region defines a *contour* of critical current pairs $\{I_1^c, I_2^c\}$. The smooth shape of the contour is a clear evidence of interactions between the non-dissipative currents flowing in different terminals, because the fixed value of one strongly influences the maximal value of the other. Second, the three “rays” of reduced differential resistance indicate an out-of-equilibrium condition on currents where only one pair of terminals is under a zero voltage. These rays persists to large currents, at which the voltage across some terminals can exceed the gap of aluminum (Suppl. Mat.). Lastly, the multiple Andreev reflection [32, 7] (MAR) of electrons and holes off the superconducting terminals gives rise to a sharp variation of the differential resistance at sub-gap voltages. In the (I_1, I_2) -plane, the MAR appears as singularities in the differential resistance aligned along the three zero-voltage rays and around the zero-voltage boundary (Suppl. Mat.). The interference of the high-order dissipative MAR currents in the three terminals creates an intricate conductance pattern immediately outside of the supercurrent state in Fig. 2a,b.

Up to eight consecutive MAR resonances can be resolved at voltages given by $2\Delta/n$, $n = 1, 2, \dots, 8$, where Δ is interpreted as the superconducting gap induced in the region of the 2DEG directly underneath the aluminum leads [33]. The extracted value of the induced gap, $\Delta \approx 175 - 180 \mu\text{eV}$, is close to the gap of the Al film [34] ($220 \mu\text{eV}$). These observations verify a nearly ballistic transport in the exposed semiconductor junction region and a high transparency boundary with the proximity-superconducting terminals. To our knowledge, such high-order MAR sequences, without missing a single resonance, have never been resolved. The narrower leads junction (Fig. 1c) repeats all the key features of the wider leads one (Fig. 2c,d). The normal state conductance and the supercurrents are reduced roughly proportionally to the width of the leads and the high-order MAR features are smeared. Still, the $n = 1 - 4$ resonances can be resolved and they give the same value of $\Delta \approx 180 \mu\text{eV}$ as the first device. Quite interestingly, the data in Fig. 2 shows no evidence of non-local MAR [35] or quartet supercurrents [36, 37], recently explored in a InAs nanowire device with three superconducting terminals [38].

The measured 3-terminal critical current contours can be understood using the scattering theory of mesoscopic Josephson effect [39]. In the absence of a voltage bias, coherent motion of electrons and holes in the semiconductor combined with Andreev reflection off the superconducting terminals gives rise to discrete Andreev bound states (ABS) at energies below the gap. The junction can be viewed as a resonant cavity for coherent electron-hole pairs, controlled by the superconducting phase-differences across the terminals. Each ABS resonance, in general, depends on every possible phase-difference, and it is responsible for carrying supercurrents across the junction. Supercurrents interact because each contributes to shifting the energy of each individual ABS. The critical current contour can be readily calculated in terms of the normal scattering matrix $\hat{S}$ of the junction (METHODS). In the absence of microscopic details, $\hat{S}$ can be chosen randomly from a circular orthogonal ensemble. The orthogonality is required by the time-reversal symmetry and the energy-independence of matrix elements is justified by the short length of the normal region. The main property of contours modeled by orthogonal ensembles is their smooth shape, similarly to what can be seen in Fig. 2. In fact, it was straightforward to find a single matrix $\hat{S}$ which adequately matched all three critical current contours obtained by permuting the choice of the grounded terminal (Suppl. Mat.).

The critical current of a 4-terminal junction (Fig. 1d) should in general be a 3D surface. For a simpler data presentation, we switched to a balanced current bias, where terminals are paired and opposite currents are applied to terminals within each pair. For instance, the most intriguing situation is the "collision" of two supercurrents I_{24} and I_{13} flowing between the oppositely facing even (2,4) and odd (1,3) pairs. This way the junction is described by only two independent phase variables and hence the measured four-terminal transport can be interpreted by analogy to the three-terminal data (Fig. 3). Again, the interaction of supercurrents unambiguously manifests itself by the ellipse-like shape of the critical currents contour $\{I_{13}^c, I_{24}^c\}$, approximately aligned along the two current axis (Fig. 3c,d). Outside the critical current contour, there is a region of remarkable out-of-equilibrium coexistence of a non-dissipative current $-10 \text{ nA} \lesssim I_{24} \lesssim 10 \text{ nA}$ and a dissipative current $|I_{13}| \approx 100 \text{ nA}$ flowing directly across each other. In the resistive state, there are now 6 rays of enhanced conductance associated with a zero voltage across one of the 6 possible terminal pairs (Fig. 3a,b). The high-order MAR between the nearest neighbor terminals dominate the finite-voltage transport in the immediate vicinity of the supercurrent region.

Thermal cycling to ambient conditions created effectively a new 4-terminal device, which largely reproduced the main features of the original (Fig. 3e,f). The overall scale for conductance and critical currents went up by approximately a factor of two, but the induced gap (determined from MAR) remained unchanged (Suppl. Mat.). Such behavior is expected from the influence of ambient conditions onto the exposed surface 2DEG. Importantly, this device clearly shows

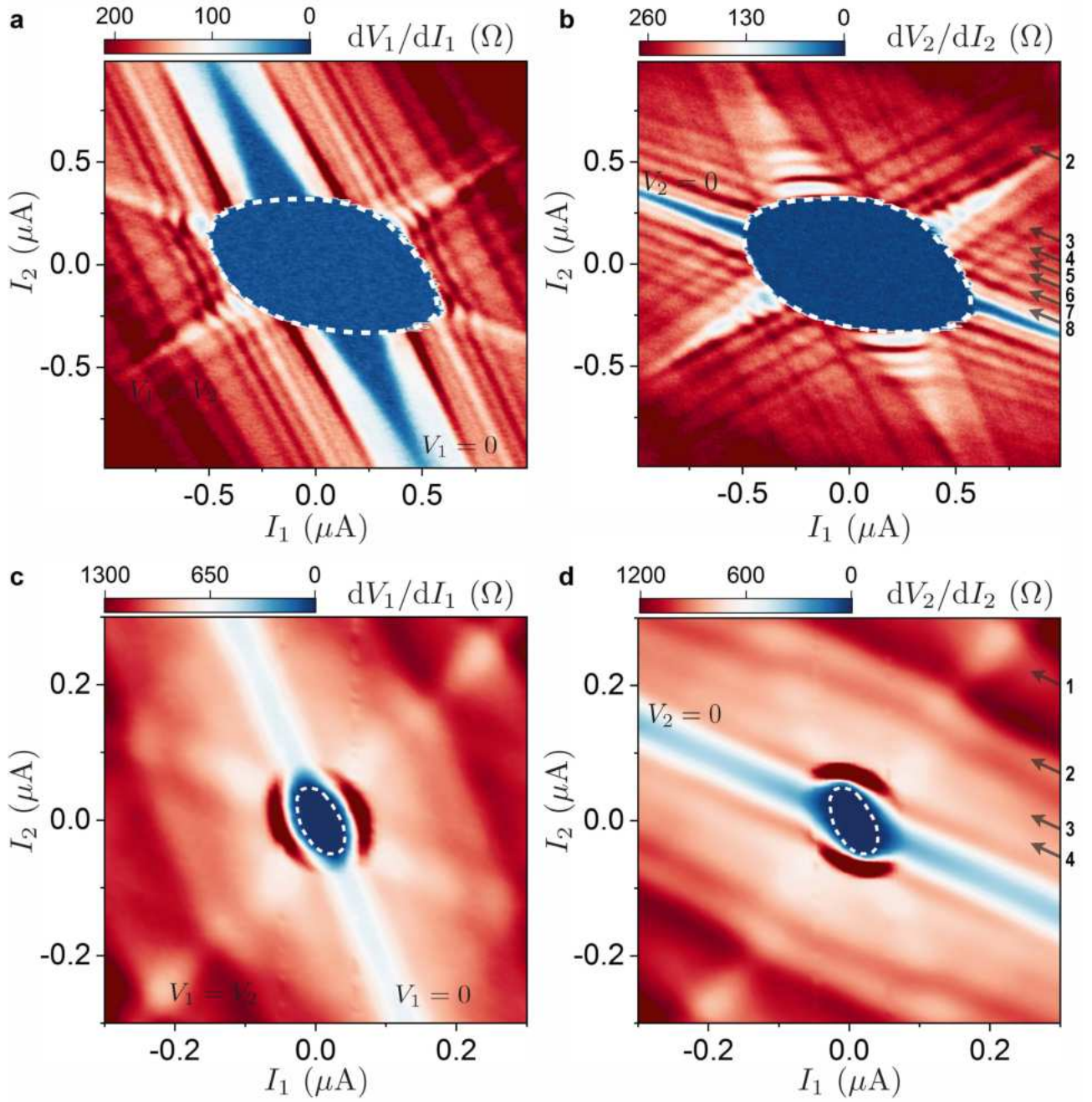


Figure 28.2: Measurement of 3-terminal critical currents. (a,b) Differential resistance maps of the device shown in Fig. 1b; terminal 3 is grounded. (c,d) Same measurement for the device shown in Fig. 1c. Critical current contours of both devices (dashed white lines) are indicated as a guide for the eye; rays of partial zero-voltage state and the example sequence of multiple Andreev reflection (MAR) between terminals 2 and 3 are labeled. More data details available in Suppl. Mat.

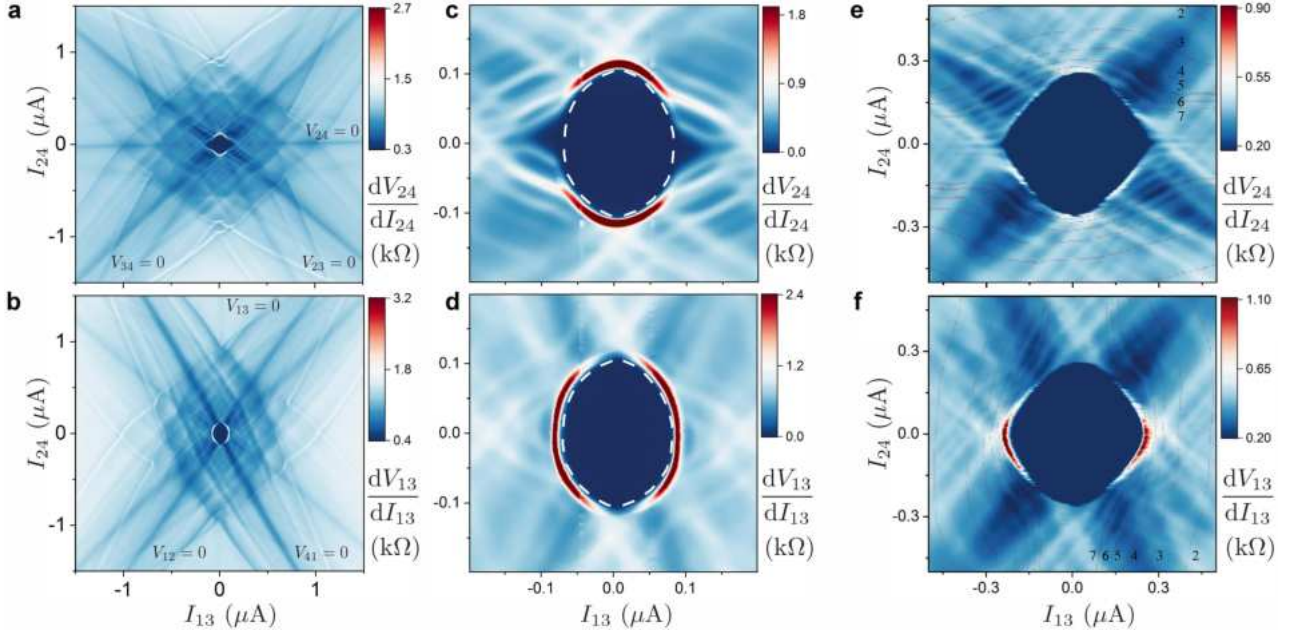


Figure 28.3: Two-current transport in a symmetrically biased (see text) 4-terminal device from Fig. 1d. (a,b) Differential resistance across the terminal pairs (2,4) (a) and (1,3) (b). The six radial rays indicate the zero voltage conditions as marked on the graphs. (c,d) Zoom-in on the corresponding supercurrent regions in (a,b). The critical current contour is emphasized by a dashed line. (e,f) Same data as in (a-d) for the same device after thermal cycling. The semi-transparent lines, obtained by measuring voltage, indicate the MAR conditions $V_{24} = 2\Delta/n$ (e) and $V_{13} = 2\Delta/n'$ (f), with $n, n' = 2, 3, \dots, 7$ and $\Delta = 160 \mu\text{eV}$.

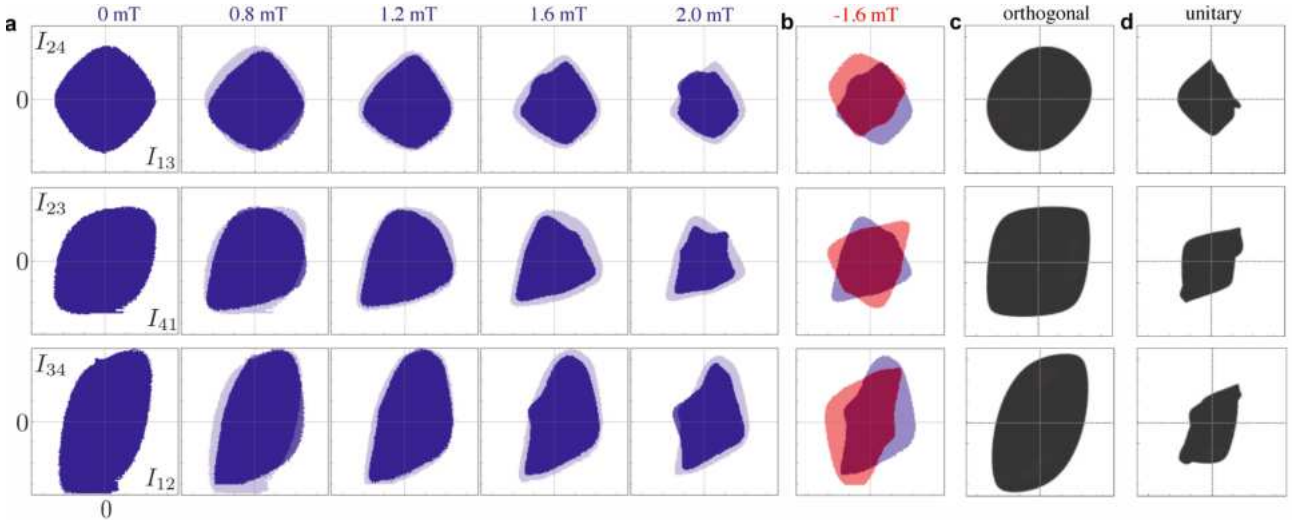


Figure 28.4: Magnetic field modulation of the critical current contours $\{I_{13}^c, I_{24}^c\}$, $\{I_{41}^c, I_{23}^c\}$, and $\{I_{12}^c, I_{34}^c\}$ of a 4-terminal junction described in Fig 1d and Fig. 3e,f. (a) [Data] the solid blue color indicates the zero-voltage condition on the two bias currents. The copy of the measured shape at a previous field is added in a semi-transparent color for ease of comparison. The range is 100 nA/div along both axis. More details available in Suppl. Mat. (b) Data at $B = 1.6 \text{ mT}$ (transparent blue) vs. $B = -1.6 \text{ mT}$ (transparent red). Note the symmetry with respect to simultaneous flipping of both axis. (c,d) Scattering theory prediction using orthogonal (c) and unitary (d) ensembles for the matrix $\hat{S}$.

the high-order MAR resonances between the physically more separated pairs (1, 3) and (2, 4) (Fig. 3c,d). In fact, the MAR features obeying to $V_{24} = 2\Delta/n$ and $V_{13} = 2\Delta/n'$ ($n, n' = 1, 2, \dots$) appear in the (I_{13}, I_{24}) -plane only upon crossing the rays $V_{12,23,34,41} = 0$. Moreover, for such conditions, the two MAR currents simultaneously flow across each other through the same physical region in the junction's center. This important observation directly confirms the presence of transparent channels across the semiconductor connecting every pair of terminals.

In the final test, we explore the effect of perpendicular magnetic field B onto all three critical current contours $\{I_{13}^c, I_{24}^c\}$, $\{I_{12}^c, I_{34}^c\}$, and $\{I_{23}^c, I_{41}^c\}$ (Fig. 4a). At $B = 0$, it was straightforward to find a matrix $\hat{S}$ from an orthogonal ensemble which can approximate all three measured contours, similarly to the three-terminal case (Fig. 4c). A magnetic field $B \lesssim 1$ mT distorts the symmetry of the contours with respect to the simultaneous flipping of both currents. In fact, individual critical currents, obtained by fixing the value of the other current in the pair, can become asymmetric and non-monotonic with field. A qualitatively new feature appears at $B \sim 2$ mT: the contours develop seemingly random small-scale deformations. The corresponding magnetic flux piercing the semiconductor is calculated to be approximately $1/5$ of the superconducting flux quantum, not taking into account flux focusing. We verified that the measured critical current contours are symmetric upon simultaneous flipping of the two currents and the direction of magnetic field (Fig. 4b).

Quite remarkably, these short-scale deformations of the contours can be explained by switching to a unitary ensemble for the scattering matrix $\hat{S}$, which is appropriate in the absence of time-reversal symmetry inside the junction (Fig. 4d). In fact, such short-scale contour fluctuations appear to be a generic feature of the unitary ensemble, which furthermore is not significantly sensitive to the number of channels (Suppl. Mat.). This observation lead us to speculate that, unlike the two-terminal currents, which are sensitive only to transmission eigenvalues, the multi-terminal currents can carry qualitatively new information on the symmetry classes of the underlying superconducting Hamiltonians [21]. In particular, it is interesting to identify the effect of spin-rotation symmetry breaking on the critical current contours [25].

In the outlook, the InAs/Al material used in this work has previously demonstrated necessary ingredients for creating solid-state Majorana fermions, such as strong spin-orbit coupling and gate control of the electron density, and possible evidence of topological superconductivity was recently reported [41]. Therefore, besides opening a new experimental direction in mesoscopic superconductivity, our realization of multi-terminal Josephson effect completes an important prerequisite for pursuing the ambitious braiding proposals [42, 16, 43], all based on junctions of multiple topological superconductors through a common coherent conductor. In the applied extreme, the interaction of multi-terminal supercurrents can form a basis for a novel transistor technology for efficient classical computing.

While preparing this manuscript, we became aware of a recent preprint reporting a multi-terminal superconducting device based on graphene [44].

We acknowledge support from NSF-PFC at JQI, NSF-EAGRE, DARPA, and ARO-LPS (NEQST), as well as useful discussions with Alex Levchenko and Manuel Houzet.

METHODS

Experimental. The hybrid quantum well heterostructure (Fig. 1e,f) is grown using a specially developed III-V molecular beam epitaxy process [31]. The three and four terminal junctions (Fig. 1b,c,d) were fabricated by a combination of electron beam lithography and wet etching (Fig. 1d). In the first step, the (Al)/(quantum well) layers are wet-etched through a resist mask using $\text{H}_2\text{O} : \text{Cl} : \text{H}_3\text{PO}_4 : \text{H}_2\text{O}_2$ (220 : 55 : 3 : 3) for 4-8 min to define multiple fork-like and cross-like geometries, electrically isolated from each other by the mesas. In the second step, Al is removed from the desired junction region using Transene Type D at 50C.

The fabrication procedure is similar to that used for two-terminal junctions [34]. The chips were wirebonded to a printed circuit board with built-in discrete-element filters, which was mounted to the Copper probe of a bottom-loading Blue Fors dilution refrigerator. The probe's temperature was between 10 – 15 mK during the experiment. Magnetic field was applied by an external hand-made superconducting coil. Differential resistance measurements were performed using the standard lock-in technique. The MAR resonances in the channel i, j were identified by plotting the lines satisfying $V_{ij} = 2\Delta/n$ ($n = 1, 2, \dots$) on top of the current-biased data (Fig. 2 and Fig. 3e,f) and adjusting Δ for the best match (Suppl. Mat.).

Theoretical. We consider a model of a general N -terminal Josephson junction (Fig. 1a) with the terminals characterized by the uniform gap Δ and phases ϕ_j , $j = 1, \dots, N$ and with the exposed semiconductor junction region characterized by a multi-channel energy-independent scattering matrix $\hat{S}$. The spectrum of discrete Andreev bound states E_n^A , $n = 1, 2, \dots$ of such a junction is given in terms of $\hat{S}$ through the Beenakker's determinant equation [39, 25, 27] on energy E :

$$\det[1 - \alpha(E/\Delta)\hat{r}\hat{S}^*\hat{r}^*\hat{S}] = 0. \quad (28.1)$$

Here $\alpha(x) = \exp(-2i \arccos x)$ defines the energy-dependent phase shift of Andreev reflection and $\hat{r} = \exp(i\hat{\phi})$, where $\hat{\phi} = \text{diag}[\phi_1, \phi_2, \dots, \phi_N]$. In the absence of other symmetries, the minimal requirement on $\hat{S}$ is unitarity, $\hat{S}^\dagger = \hat{S}^{-1}$, imposed by the conservation of normal currents in the junction. In case of a time-reversal symmetry $\hat{S}$ must be orthogonal. The ground state energy of the junction E_g and the supercurrents I_j ($j = 1, \dots, N$) are given by

$$E_g = -(1/2) \sum_n E_n^A; \quad I_j = \partial E_g / \partial \phi_j (2e/\hbar), \quad (28.2)$$

where the sum in the former equation runs over all positive ABS [45]. The effect of extended phase-sensitive states can be neglected. Assuming the $j = N$ terminal is grounded, we can set $\phi_N = 0$ and measure all other phases with respect to the grounded terminal. The Josephson energy E_J of the multi-terminal junction can be defined as $E_g = E_J(\phi_1, \dots, \phi_{N-1})$. It is a 2π -periodic function of $N - 1$ variables and the supercurrents conservation $I_N = -\sum_{j=1}^{j=N-1} I_j$ is satisfied automatically.

For $N = 3$ devices (Fig. 1b,c), the physically allowed non-dissipative currents are obtained by sweeping with variables ϕ_1 and ϕ_2 the surface of a torus. The boundary of the resulting simply-connected shape defines a 2D contour of possible critical current pairs $\{I_1^c, I_2^c\}$. The "critical current" of a three-terminal junction consists of three such contours obtained by permuting the choice of the grounded electrode. For $N = 4$ case (Fig. 1d), we use a pairwise balanced current bias without an explicit grounding. In this case it is more convenient to formally keep a four-variable Josephson energy but impose a constraint $I_i = -I_j = I_{ij}$ on the currents. Under such symmetric bias conditions, the critical current of a 4-terminal junction is also a set of three 2D contours $\{I_{13}^c, I_{24}^c\}$, $\{I_{12}^c, I_{34}^c\}$, and $\{I_{23}^c, I_{41}^c\}$.

All three critical current contours of the three-terminal device (Fig. 2a,b) can be readily matched by a randomly generated orthogonal matrix with three channels per terminal (Suppl. Mat.). The theory in Fig. 4c,d was produced using a similarly generated four-terminal orthogonal (time-reversal symmetric) and unitary (time-reversal broken) scattering matrices. We have checked that the short-scale fluctuations of the contours – the key feature of the time-reversal broken systems – can be reproduced by increasing the number of transport channels from three to ten, which is more appropriate for the experimental device parameters (Suppl. Mat.).

29 Bibliography

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- [25] For an asymmetric junction, one generally finds a shifted CPR. This can be seen by expanding to lowest nonvanishing order in Γ_j and $\tilde{\Gamma}_j$. We then find $I(\varphi) = |\Gamma_L \Gamma_R|^2 F(2\varphi) + \sum_j |\tilde{\Gamma}_j \Gamma_{-j}|^2 F_j(\varphi)$, with the π -periodic function $F(2\varphi) \sim \sin(2\varphi + 2\alpha)$, where $\alpha = \arg(\Gamma_L^* \Gamma_R)$, and 2π -periodic functions $F_{L/R}(\varphi)$.
- [26] It is straightforward to include $q \neq 0$ quasiparticles, but their effect is weak unless both E_c and Δ_w are small.

From Josephson effect in multiterminal topological junctions

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From Josephson effect in a junction coupled to an electron reservoir

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From Spin-degeneracy breaking and parity transitions in three-terminal Josephson junctions

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From Topological Superconductivity in a Planar Josephson Junction

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From Theory of non-eq noise in gener mlt-term SC hydrid devices

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From Phase-engineering the ABS of a 3T-JJ

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From Josephson Current and Noise at a Superconductor-Quantum Spin Hall Insulator-Superconductor Junction

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From Scanning Josephson spectros on the atomic scale

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From Quantum circuits with multiterminal Josephson-Andreev junctions

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From Hidden symmetry in interacting-quantum-dot-based multi-terminal Josephson junctions

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From Supplementary Material: Hidden symmetry in multi-terminal Josephson junctions by Zalom, Žonda et al.

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From Tunable pseudogaps due to non-local coherent transport in voltage-biased three-terminal Josephson junctions

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