

Superconductive Contacts, Part II

Yury Holubeu *

January 18, 2025

This note is not intended for distribution.

Superconductive contacts are discussed in details.

Additional Articles about Typical Contacts

Majorana Junctions: Main Theory

Other, Non-Typical Theories, Effects

Multiterminal Junctions: Other Theories

Other Experiments

Appendix

Contents

VI	Additional Articles about Typical Contacts	9
12	Josephson current in TS-S by Ioselevich et al.	9
12.0.1	Conclusion	9
12.0.2	Introduction	9
12.0.3	S -matrix formulation of Josephson current	11
12.0.4	Minimal model of an S-TS junction	13
12.0.5	Nanowire setup	15
12.0.6	Junction with a random scattering matrix	16
VII	Junctions with Majoranas: Main Theory	18
13	Topological Superconductivity in a Planar Josephson Junction by Pientka, Kesel-	18
	man et al.	
13.1	Introduction	18
13.2	Physical picture and summary of results	20
13.3	The $E_Z - \phi$ Phase Diagram, Topological Gap, and Majorana end modes	23
13.3.1	Class D Phase Diagram	23
13.3.2	Class BDI Phase Diagram	25
13.3.3	Topological Gap	27
13.3.4	Majorana End Modes	29
13.4	First order topological phase transitions and the critical current	30
13.5	Discussion	33
13.6	Phase diagram with normal reflection	34
13.7	Josephson current at finite temperature	37
13.8	Orbital effect of the in-plane magnetic field	37
13.9	Numerical calculations	38
13.9.1	Scattering matrix formalism	38
13.9.2	Tight binding model	38

*<https://yuriholubeu.github.io/>, yuri.holubev@gmail.com

14 Coulomb blockade of Majoranas induced transport by Zazunov et al.	39
14.1 Introduction	39
14.2 Model	40
14.2.1 Model	40
14.2.2 Strong Coulomb Blockade	41
14.3 Keldysh functional integral	42
14.3.1 AES action for Majorana induced transport	42
14.3.2 Semiclassical expansion	43
14.4 Weak Coulomb blockade limit	44
14.4.1 Langevin approach	44
14.4.2 Current-voltage relation	45
14.4.3 Retardation effects	46
14.4.4 Zero-bias anomaly	46
14.5 Concluding remarks	48
14.6 Derivation of the tunnel Hamiltonian	49
14.7 On the Keldysh functional integral	49
14.8 Derivation of Langevin equation	50
15 Robustness of Majorana bound states in the short-junction limit	51
15.1 Introduction	51
15.2 Scattering matrix formalism and the short-junction limit	53
15.3 Scaling arguments for the MBS properties in the short-junction limit	54
15.4 Model	56
15.4.1 General solution	56
15.4.2 Typical physical parameters of the heterostructure	59
15.4.3 Modeling the NS interface	59
15.4.4 Comparison with tight-binding dispersion simulations	60
15.5 Analysis of the topological phase diagram	61
15.5.1 Phases boundaries and the spectral gap	61
15.5.2 Decay length of MBS	62
15.6 Spectrum of finite length junctions	64
15.7 Orbital field effect in thin shell approximation	65
15.8 Numerical study of a three-dimensional nanowire	68
15.9 Conclusions and outlook	68
15.10 Interface transparency in a two-dimensional junction	69
15.11 Independence of critical magnetic field on the superconducting gap	71
VIII Other, Non-Typical Theories, Effects	73
16 JC and Noise at a S-QSHI-S by Fu, Kane	73
17 Quasiclassical description of transport through superconducting contacts by Cuevas, Fogelström	78
17.1 Introduction	78
17.2 Description of the approach	80
17.3 Josephson currents	83
17.3.1 Josephson current between two s-wave superconductors	84
17.3.2 Josephson current between two d-wave superconductors	85
17.3.3 Josephson current through a spin active interface	86
17.4 Solving the T-matrix equation at an applied voltage	89
17.4.1 Current at finite voltage	90
17.5 Current fluctuations	94
17.6 Conclusions	97

18 Finite frequency noise in a normal metal - topological superconductor junction by D. Bathellier, ¹ L. Raymond, ¹ T. Jonckheere, ¹ J. Rech, ¹ A. Zazunov, ² and T. Martin ¹	98
18.1 Introduction	98
18.2 Model	101
18.3 Dyson's equations and average current	102
18.3.1 Bare Green's functions	102
18.3.2 Average current	103
18.4 Finite frequency noise	104
18.4.1 Real-time noise correlator	104
18.4.2 Emission, absorption and measurable noise	105
18.5 Results	106
18.5.1 Differential conductance	107
18.5.2 Zero-frequency noise	107
18.5.3 Finite-frequency noise	107
18.5.4 Measurable noise	109
18.6 Comparison with a N-dot-S system	110
18.7 Conclusion and perspectives	112
19 OTBK excess current by Niebler et al.	121
19.1 Conclusions and outlook	121
19.2 Introduction	123
19.3 OTBK model	123
19.4 Equal barriers	125
19.4.1 Normal current	125
19.4.2 Excess current	126
19.5 Differing barrier strengths	128
19.5.1 Normal current	129
19.5.2 Excess current	129
19.6 References	131
20 Theory of non-EQ noise in general MT SC hybrid device	133
20.1 Introduction	134
20.2 Microscopic formulation	135
20.2.1 Total Hamiltonian	136
20.2.2 Green's functions in the Keldysh formalism	137
20.2.3 Self energy of the quantum dots	137
20.2.4 Current statistics	138
20.3 Application to Multiple Cooper pair resonances	140
20.3.1 Historical perspective	141
20.3.2 Short description of MCPR physics	141
20.3.3 Expressing the current and noise at a MCPR	143
20.3.4 Noise characteristics of the quartet resonance	146
20.4 Discussion	155
20.5 Conclusion	155
21 Anomalous JE induced by SO and Zeeman effect in SM nanowires by Yokoyama	157
21.0.1 Introduction	157
21.1 Model and calculations	160
21.1.1 Formulation	161
21.1.2 Scattering Matrix Approach	162
21.1.3 Tight-Binding Model	163
21.2 Numerical results	164
21.2.1 Absence of spin-orbit interaction	164

21.2.2	Presence of spin-orbit interaction	167
21.3	Conclusion and discussion	172
21.4	Calculation Method of Tight-Binding Model	174
21.5	Single Scatterer Model	176
21.6	Magnetic field in the x direction: Disappearance of $0-\pi$ transition	177
22	Nonlocal Josephson effect in Andreev molecules by J.-D. Pillet, Benzoni, J. Griesmar, J.-L. Smirr, C. O. Girit	178
IX	Multiterminal Junctions: Other Theories	186
23	Fine energy splitting of overlapping Andreev bound states in multi-terminal superconducting nanostructures by Kornich, Barakov, Nazarov	186
24	Supercurrent blockade in JJ with a Majorana wire by Zazunov et al.	195
24.0.1	Introduction	195
24.0.2	S-TS junction	196
24.0.3	S-TS-S junction	198
24.0.4	Noninteracting limit	200
24.0.5	Weak tunneling limit	201
24.0.6	Conclusions	202
25	Transport theory of multiterminal hybrid structures by R. Mélin, D. Feinberg	203
25.1	Introduction	203
25.2	Preliminaries	204
25.2.1	The Keldysh method	204
25.2.2	A useful trick on the spectral current	205
25.3	Form of the Green's functions	206
25.3.1	Green's functions in the superconductor	206
25.3.2	Green's functions in the ferromagnetic electrodes	207
25.4	Single channel electrodes: (I) Two electrodes with 100% spin polarization	208
25.4.1	Derivation of the transport formula	208
25.4.2	Exact expression of the current	209
25.4.3	Two-terminal conductance matrix	209
25.4.4	Phase resolved versus averaged conductance	211
25.5	Single channel electrodes: (II) General solution	213
25.6	Single channel electrodes: (III) Three electrodes with 100% spin polarization	215
25.6.1	Three-terminal conductance matrix	215
25.6.2	Phase resolved versus average conductance	216
25.7	Multichannel electrodes	217
25.7.1	Transport formula	217
25.7.2	Andreev reflection <i>versus</i> cotunneling	219
25.7.3	Extended contacts without phase averaging	219
25.8	Conclusions	220
25.9	A1 Expression of the Keldysh propagator: two-channel model	221
25.10A2	Derivation of the transport formula with an arbitrary number of single-channel electrodes	222
25.10.1	Solution of the Dyson equation	223
25.10.2	Solution of the Dyson-Keldysh equation	223

26 Single fermion manipulation via SC phase differences in multiterminal JJ	224
26.1 Introduction	225
26.2 General considerations	226
26.2.1 Scattering formalism and bound state equation for multiterminal Josephson junctions	226
26.2.2 Kramers degeneracy splitting	228
26.2.3 Lower bound on the energy gap and existence of zero-energy solutions	229
26.2.4 Multiterminal Josephson junction in the quantum spin Hall regime	231
26.3 Applications	231
26.3.1 Splitting of Kramers degeneracy	232
26.3.2 Andreev level crossings at zero energy	232
26.3.3 Density of states	233
26.3.4 Effect of finite junction size	235
26.4 Conclusions and discussion	236
26.5 Occurrence of a zero-energy crossing as a generalized eigenvalue problem	237
26.6 BHZ Hamiltonian	237
27 Energy spectrum and CPR of a NW JJ close to the TPT by Murthy et al.	238
27.1 Summary and conclusions	238
27.2 Introduction	239
27.3 The model	241
27.4 Spectrum and thermodynamic properties of a transparent junction	244
27.4.1 Bound states at perfect transmission	244
27.4.2 Continuum states at perfect transmission	245
27.4.3 Thermodynamic properties of the transparent junction	246
27.5 Effects of scattering at the junction	250
27.5.1 Bound states and continuum states in the presence of scattering	251
27.5.2 Thermodynamic properties in the presence of scattering	255
27.6 Spectrum of inner modes at zero backscattering, $D = 1$	257
27.6.1 Bound states	259
27.6.2 Scattering states	261
27.7 Spectrum of the junction with purely forward scattering ($D = 1$ but $\gamma \neq 0$)	261
27.7.1 The origin of the shallow states upon the topological transition	262
27.7.2 Inner-mode bound states with forward scattering in the trivial phase, $B < \Delta$	262
27.7.3 Inner-mode bound states with forward scattering in the topological phase, $B > \Delta$	262
27.7.4 Inner-mode continuum states with forward scattering	264
27.8 Spectrum at nonzero backscattering, $D < 1$	264
27.8.1 Bound states	266
27.8.2 Scattering matrix	267
27.9 Approximate results on spectrum of the junction in the presence of scattering	268
27.9.1 Hybridization of the bound states at $B < \Delta$	268
27.9.2 Hybridization between the outer-mode bound state and the states of the spectral continuum in the topological phase	269
27.9.3 Shallow bound states	270
27.10 Josephson current in the presence of scattering	270
27.10.1 General expressions for $I(\phi)$	270
27.10.2 Approximate expression for the Josephson current	272
28 Mesoscopic multiterminal Josephson structures. I Effects of nonlocal weak coupling by Amin, Omelyanchouk, Zagoskin	273
28.1 Introduction.	273
28.2 Mesoscopic Four-Terminal Junction.	274
28.2.1 System description	274
28.2.2 Circuit implementations of 4-terminal junction. Nonlocal weak coupling.	276

28.2.3	Current-phase relations. The phase dragging effect.	277
28.3	Current distribution and local density of states inside the mesoscopic weak link	279
28.4	Mesoscopic Four-Terminal SQUID	281
28.5	Conclusions	285
28.6	Appendix. Junction with arbitrary γ 's	286
29	Multi-terminal Josephson junctions: a road to topological flux networks by Lucila Peralta Gavensky, Gonzalo Usaj, C. A. Balseiro	287
29.1	Introduction	287
29.2	Multi-terminal Josephson junctions	289
29.3	Locally accessing quantum geometry in flux-space	292
29.4	On the effect of the quasiparticle continuum	292
29.5	Topological Hamiltonian	294
29.6	Experimental advances	295
29.7	Current trends	296
X	Other Experiments	297
30	Sub-Gap Structure in the Conductance of a 3TJJ by Pfeffer et al.	297
30.1	Introduction	297
30.2	The experiment	298
30.2.1	Samples and measurement process	298
30.2.2	Results	300
30.3	Interpretation	302
30.3.1	Synchronization of a.c. Josephson effects	302
30.3.2	The quartet scenario	303
30.4	conclusion	305
30.5	Analytical calculation of the quartet current	305
31	Flux-Tunable Josephson Diode Effect in a Hybrid Four-Terminal Josephson Junction by Coraiola, Svetogorov, Haxell et al.	308
31.1	Additional Phase-Space Linecuts	322
31.2	Results for Different V_L, V_R	322
31.3	Extracting the Diode Efficiency as a Function of V_L, V_R	323
31.4	Results for Device 2	323
31.5	Current-to-Flux Remapping	324
32	Phase-engineering the Andreev band structure of a three-terminal Josephson junction by Coraiola et al.	333
32.1	Introduction	333
32.2	Results	335
32.2.1	Implementation of phase-controllable 3TJJ	335
32.2.2	Probing the Andreev band structure in the 2D phase space	335
32.2.3	Andreev dispersion along phase space linecuts	337
32.2.4	Andreev molecule model and numerical simulations	337
32.2.5	Tomography of the Andreev band structure	338
32.3	Discussion	339
32.4	Methods	340
32.4.1	Materials and fabrication	340
32.4.2	Measurement techniques	341
32.5	Data availability	341
32.6	Code availability	341

32.7	Supplementary Information: Phase-engineering the Andreev band structure of a three-terminal Josephson junction	341
32.8	Supplementary Note 1: Theory	341
32.9	Supplementary Note 2: Gate dependence of differential conductance	344
32.10	Supplementary Note 3: Results for Device 2	345
32.11	Supplementary Note 4: Results for different V_G	345
32.12	Supplementary Note 5: Mutual inductance matrix	346
32.13	Supplementary Note 6: Phase shifts and inductances in the system	347
32.14	Supplementary Figures	349
33	Scanning Josephson spectroscopy on the atomic scale by Randeria, Feldman, Drozdov, Yazdani	356
33.1	Supplementary Materials: Scanning Josephson spectroscopy on the atomic scale	362
33.2	Fitting procedure for Josephson critical current	362
33.3	Suppression of the Josephson effect over Fe adatoms	364
33.4	Andreev reflections through impurity-bound Shiba states	365
34	Tunable pseudogaps due to non-local coherent transport in voltage-biased three-terminal Josephson junctions by C. Padurariu et al	368
34.1	Introduction	368
34.2	Phenomenological description	370
34.2.1	Local and non-local Josephson effect	370
34.2.2	Out-of-equilibrium dynamics	372
34.3	Microscopic model	372
34.3.1	Green's functions of superconducting terminals	373
34.3.2	Numerical implementation	374
34.3.3	The density of states (NDOS)	375
34.4	The two-terminal junction	376
34.4.1	Equilibrium	376
34.4.2	Biased junction	377
34.5	The three-terminal junction	379
34.5.1	NDOS at $\varphi_Q = 0$	379
34.5.2	NDOS: phase dependence	381
34.6	Conclusions.	384
34.7	Fourier transforms	385
XI	Adds	386
A	Введение и обзор предмета	386
A.1	Еще мотивация	386
A.2	Мышление профессионала в	386
A.2.1	Суть предмета	386
A.2.2	Отношение к предмету (!?)	386
A.2.3	Способы заработать, зная предмет	386
A.2.4	Использование предмета в обычной жизни (!)	386
A.2.5	Актуальнейшие приложения	386
A.2.6	Построение с нуля	386
A.2.7	Способы догадаться до всех главных идей	386
A.2.8	Особенности эффективного изучения	386
A.2.9	Особенности эффективного преподавания (???)	387
A.3	Acknowledgements	387
A.4	Literature	387
A.4.1	Основная	387

A.4.2	Дополнительная	387
A.5	Обзор	390
A.5.1	Популярное введение в предмет	390
A.5.2	обзор теоретических подходов	390
A.5.3	Обзор дальнейших развитий	390
A.5.4	Связи с другими науками	390
A.5.5	Описание записи	390
A.5.6	Об истории предмета	390
A.6	Головоломки на разные случаи	391
A.6.1	Головоломки для приятной беседы	391
A.6.2	Бытовые головоломки	391
A.6.3	Принципиальные головоломки	391
A.6.4	Головоломки о деталях	391
A.6.5	Головоломки для освоения типичных понятий	391
A.6.6	Головоломки для демотивации	391
A.6.7	Головоломки для доказательства, что человек не понимает предмет	391
В	Дополнения	391
В.0.1	title	391
С	Bibliography	392

Part VI

Additional Articles about Typical Contacts

12 Josephson current between topological and conventional superconductors by Ioselevich, Ostrovsky, Feigel'man

Abstract

We study the stationary Josephson current in a junction between a topological and an ordinary (topologically trivial) superconductor. Such an S-TS junction hosts a Majorana zero mode that significantly influences the current-phase relation. The presence of the Majorana state is intimately related with the breaking of the time-reversal symmetry in the system. We derive a general expression for the supercurrent for a class of short topological junctions in terms of the normal state scattering matrix. The result is strongly asymmetric with respect to the superconducting gaps in the ordinary (Δ_0) and topological (Δ_{top}) leads. We apply the general result to a simple model of a nanowire setup with strong spin-orbit coupling in an external magnetic field and proximity-induced superconductivity. The system shows parametrically strong suppression of the critical current $I_c \propto \Delta_{\text{top}}/R_N^2$ in the tunneling limit (R_N is the normal state resistance). This is in strong contrast with the Ambegaokar-Baratoff relation applicable to junctions with preserved time-reversal symmetry. We also consider the case of a generic junction with a random scattering matrix and obtain a more conventional scaling law $I_c \propto \Delta_{\text{top}}/R_N$.

12.0.1 Conclusion

In conclusion, we have studied the Josephson effect in a junction between a topological and an ordinary (topologically trivial) superconductor. Within the quasi-one-dimensional model they are characterised by an odd and even number of channels, respectively. A topological superconductor inevitably breaks time-reversal symmetry at its end, which leads to the appearance of an unpaired Majorana mode at zero energy. We have derived general expressions (12.7), (12.8) for the supercurrent for a class of short topological junctions in terms of the normal state scattering matrix S . For the minimal model of the junction with two and one channels in the leads, the current (12.13) explicitly depends on the four invariants of the scattering matrix (12.11). The result (12.13) is strongly asymmetric with respect to the two superconducting gaps Δ_0 and Δ_{top} .

We have also applied the general result to a model of a nanowire setup with parabolic spectrum and strong spin-orbit coupling in an external magnetic field and find a parametrically strong suppression of the critical current $I_c \propto \Delta_{\text{top}}/R_N^2$ in the tunneling limit. However, this behaviour is not universal and a generic perturbation of the model parametrically enhances the critical current. This is established with the help of the random matrix theory applied to a generic junction with three channels.

12.0.2 Introduction

Topological superconductors and insulators share the property of hosting a gapless edge or surface mode while having a gap in the bulk spectrum.[1, 2] The edge mode of a one-dimensional topological superconductor is a Majorana zero mode – a self-conjugate excitation

with zero energy.[3] Majorana modes are half-fermions obeying non-abelian braiding statistics. Their special properties make them promising ingredients to topological quantum computing.[4]

A lot of effort is put at the moment into the observation of Majorana excitations in condensed matter systems.[5, 6] A major route to detecting these modes is by transport measurements. When probed by a tunneling contact, the Majorana mode induces resonant Andreev reflection, leading to a robust zero-bias peak in the tunneling conductance.[7, 8, 9, 11, 10] Such a peak has been observed in several experiments [30, 15, 14] on nanowires with strong spin-orbit interaction and induced superconductivity.[15, 16] The peak is robust to perturbations and only appears in the topological phase of the system. This allows to identify it with the presence of a Majorana zero mode.

A different kind of transport experiment involves Josephson junctions between two topological superconductors. Each topological superconductor gives rise to one Majorana mode. The pair of Majorana modes from the two leads hybridise into a single Andreev bound state with energy $E \sim |\cos \varphi/2|$, where φ is the superconducting phase difference at the junction. This bound state is non-degenerate and leads to a parity switch – the fermionic parity of the systems ground state changes as φ passes π . This leads to the celebrated fractional Josephson effect [3, 17, 18, 15, 16, 19, 20, 21] where the current $I(\varphi)$ contains a 4π -periodic component directly attributable to the hybridised Majorana state.

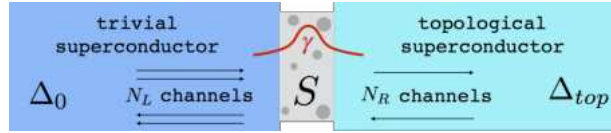


Figure 1: Schematic of the junction between an ordinary (left) and topological (right) superconductor with N_L and N_R channels, respectively. Normal transport in the junction is characterised by the $(N_L + N_R)$ -sized scattering matrix S . The Josephson junction hosts a localised Majorana zero mode γ .

In this paper we consider a Josephson junction where only one of the leads is a topological superconductor, while the other is topologically trivial, i.e. an S-TS junction. Similarly to the N-TS-junction setups used to probe topological superconductivity by normal transport, our setup deals with a single Majorana mode. This mode is topologically protected from local perturbations and its energy stays zero at any phase differences φ . Thus the Majorana zero mode does not directly contribute to the supercurrent, unlike what happens in junctions where both superconductors are topological. Nevertheless, as we show below, the stationary Josephson current in the S-TS junction is generally quite different from the current in conventional junctions even in the simplest tunnelling limit.[22]

We describe the junction between two superconductors within the general quasi-one-dimensional formalism in terms of conducting channels. Both superconducting leads are represented as wires with broken spin symmetry and induced superconductivity. The distinction between ordinary and topological superconductor is in the parity of the number of channels. When a superconducting wire with an odd number of channels is terminated, an odd number of bound states appears at its end. One of these bound states is self-conjugate and has exactly zero energy due to the mirror symmetry of the spectrum of the superconducting Hamiltonian. This is the Majorana mode characteristic to the topological superconductor.

We consider a generic junction between two superconductors with the number of channels N_L and N_R in the left and right lead being even and odd, respectively. Thus we have an ordinary (topological) superconductor as the left (right) lead of our junction, see Fig. 171. Boundary conditions at the end of a wire with an odd number of channels necessarily break the time-reversal symmetry $\mathcal{T}$. The same is true for any junction where the parity of the number

of channels changes. Thus broken time-reversal symmetry is an essential property of the S-TS junction required for an unpaired Majorana mode to form.

We derive general expressions for the supercurrent and the spectrum of Andreev bound states for a class of short S-TS junctions in terms of their scattering matrix in the normal state. This requires a generalisation of classical results [23, 24] for the Josephson current in time-reversal-symmetric junctions. While in the latter case all measurable quantities can be expressed in terms of normal transmission probabilities of the device, the former case of broken $\mathcal{T}$ symmetry leads to more complicated supertransport that requires taking into account the full scattering matrix. We further apply our general results to a particular minimal model with two and one channel in the ordinary and topological superconducting leads, respectively, as shown in Fig. 171. The expressions for the supercurrent and the excitation spectrum of such an S-TS junction appear to be strongly asymmetric in their dependence on the superconducting gaps Δ_0 and Δ_{top} of the ordinary and topological leads.

We further consider the particular case of a tunneling junction described by the one-dimensional Hamiltonian of Refs. [15, 16]. It corresponds to a nanowire with parabolic spectrum, spin-orbit coupling, and induced superconductivity in an external magnetic field. This model is widely used to describe the recent N-TS transport experiments.[30, 15, 14] We find that the critical current is suppressed in the tunneling limit and scales as the square of the normal conductance of the junction. This result is in sharp contrast with the Ambegaokar-Baratoff relation [22] that predicts linear scaling for tunnel junctions with $\mathcal{T}$ symmetry. We further explore this discrepancy by considering a generic junction with a random scattering matrix. It turns out that the typical Josephson current in such a random junction is parametrically larger and obeys the linear scaling with normal conductance similar to the Ambegaokar-Baratoff relation.

The paper is organised as follows. In Section 12.0.3 we introduce the scattering matrix formalism and derive general relations for the supercurrent. In Section 12.0.4 we solve the minimal model of an S-TS junction with two and one channels in the leads. This result is applied to the nanowire with parabolic spectrum in Section 12.0.5. A generic minimal junction with a random scattering matrix is discussed in Section 12.0.6. We summarise the results in the concluding Section 20.5.

12.0.3 S -matrix formulation of Josephson current

We consider a Josephson junction between two quasi-one-dimensional conductors with N_L and N_R channels. In the normal state, each channel hosts a pair of counter-propagating states related by time-reversal symmetry. We will describe the leads by the quasiclassical Hamiltonian $\hat{\xi} = -i\sigma_z \hat{v} \partial/\partial x$. Here $\hat{v}$ is the diagonal matrix of Fermi velocities and σ_z is a Pauli matrix in the space of left-right propagating modes. The antiunitary time-reversal operator has a simple form $\mathcal{T} = i\sigma_y \mathcal{C}$ in the channel basis, where $\mathcal{C}$ denotes complex conjugation. The Hamiltonian $\hat{\xi}$ commutes with $\mathcal{T}$. Let us note that $\mathcal{T}^2 = -1$ corresponding to the symplectic symmetry of the wires.

The properties of the junction in the normal state are encoded in the scattering matrix

$$S = \begin{pmatrix} r_L & t_{LR} \\ t_{RL} & r_R \end{pmatrix}. \quad (12.1)$$

This is a unitary matrix of the size $N_L + N_R$ that relates the amplitudes of outgoing and incoming waves. The two diagonal blocks $r_{L,R}$ of the size N_L and N_R contain reflection amplitudes in the left and right lead while the off-diagonal rectangular $N_L \times N_R$ blocks t_{LR} and t_{RL} contain transmission amplitudes. Normal transport properties of the junction can be expressed in terms of transmission probabilities T_i – eigenvalues of the matrix $t_{LR}^\dagger t_{LR}$ or $t_{RL}^\dagger t_{RL}$. Unitarity of S ensures that non-vanishing eigenvalues of these two matrices are identical.

Let us note that the time-reversal symmetry is inevitably broken already in the normal state if N_L and N_R have opposite parity. If the junction possesses time-reversal symmetry $\mathcal{T}$, its scattering matrix obeys an additional linear constraint $S = -S^T$. However, a skew-symmetric matrix of the odd size $N_L + N_R$ necessarily has a zero eigenvalue which is incompatible with the unitarity of S .

In the superconducting state, the leads are described by the Bogoliubov-de Gennes Hamiltonian

$$H = \begin{pmatrix} \hat{\xi} & \Delta \\ \Delta^* & -\hat{\xi} \end{pmatrix}. \quad (12.2)$$

We assume that the order parameter Δ takes two different values Δ_0 and Δ_{top} in the ordinary (left) and topological (right) leads of the junction. The supercurrent can be expressed as the derivative of the junction free energy F with respect to the superconducting phase difference φ via the fundamental relation $I = (2e/h)\partial F/\partial\varphi$. In terms of the Matsubara Green function, this can be represented as

$$I = -\frac{e}{h} \frac{\partial}{\partial\varphi} T \sum_{\omega} i\omega \int dx \text{Tr} G_{\omega}(x, x). \quad (12.3)$$

Here Matsubara frequencies take the discrete values $\omega = 2\pi T(n+1/2)$ and T is the temperature. The Matsubara Green function is a matrix in channel and Nambu space satisfying the equation $(i\omega - H)G_{\omega}(x, x') = \delta(x - x')$ with the Hamiltonian (12.2). Explicitly solving this equation,[25] we relate the Josephson current to the normal state scattering matrix S , Eq. (12.1),

$$I = -\frac{e}{h} \frac{\partial}{\partial\varphi} T \sum_{\omega} \ln \det[1 - S_N S_A(\omega)]. \quad (12.4)$$

Here the determinant of the size $2N_L + 2N_R$ contains the matrices of normal and Andreev scattering

$$S_N = \begin{pmatrix} S & 0 \\ 0 & -S^* \end{pmatrix}, \quad S_A = -i \begin{pmatrix} 0 & r_A \\ r_A^* & 0 \end{pmatrix}, \quad (12.5)$$

where

$$r_A = \begin{pmatrix} e^{-\theta_L + i\varphi_L} & 0 \\ 0 & e^{-\theta_R + i\varphi_R} \end{pmatrix}, \quad \theta = \text{arcsinh} \frac{\omega}{\Delta}. \quad (12.6)$$

The Andreev scattering amplitude r_A takes two different values for the left and right lead allowing for different superconducting gaps and phases. In general, the normal scattering matrix S is a function of energy, for simplicity we focus on the limit of a short junction where this energy dependence can be neglected.

We will denote the determinant in Eq. (12.4) by D . Nambu structure can be eliminated from this determinant leading to

$$D(\varphi, \omega) = \det(S - r_A^* S^T r_A). \quad (12.7)$$

Here $\varphi = \varphi_L - \varphi_R$ and an unobservable phase-independent factor is omitted. The phase difference φ and Matsubara energy ω enter D through r_A while S contains all the junction specifics. The expression (12.4) with the determinant (12.7) relates the Josephson current in an arbitrary short junction to its normal transport characteristics.

$$I = -\frac{e}{h} \frac{\partial}{\partial\varphi} T \sum_{\omega} \ln D(\varphi, \omega). \quad (12.8)$$

Let us briefly review the case of conventional Josephson junctions, where time-reversal symmetry is present in the normal state. In terms of the S -matrix, $\mathcal{T}$ symmetry means $S = -S^T$. This allows to block-diagonalise the matrix in Eq. (12.7) and express the Josephson current in terms of normal transmission probabilities T_i . For equal superconducting gaps $\Delta_L = \Delta_R = \Delta$ this leads to the classical short junction result [23]

$$I = -\frac{e}{h} \frac{\partial}{\partial \varphi} \sum_i E_i \tanh \frac{E_i}{2T}, \quad (12.9)$$

$$E_i = \Delta \sqrt{1 - T_i \sin^2 \varphi/2}. \quad (12.10)$$

We now return to the S-TS junction. The total number of channels $N_R + N_L$ is odd and the time-reversal symmetry is broken. This prevents us from block-diagonalising the matrix in Eq. (12.7) in a general case. The current is determined by the whole scattering matrix rather than by normal transmission probabilities only.

Let us demonstrate that the junction hosts a Majorana zero mode. Consider the determinant (12.7) at zero energy. Using $r_A r_A^*|_{\omega=0} = 1$, we can rewrite it as $D(\varphi, 0) = \det[(r_A S)^T - r_A S]_{\omega=0}$. The argument of the determinant here is an odd-sized skew-symmetric matrix hence $D(\varphi, 0) = 0$. We thus conclude that the junction hosts an excitation with exactly zero energy when N_L and N_R have opposite parity. In particular, $N_L = 0$, $N_R > 0$ represents a terminated superconducting wire, hosting a Majorana edge mode whenever N_R is odd. This once again proves that a topologically non-trivial (trivial) superconductor is a superconducting nanowire with an odd (even) number of channels.

12.0.4 Minimal model of an S-TS junction

The simplest model of an S-TS junction involves three channels in total with $N_L = 2$ and $N_R = 1$, see Fig. 171. The S matrix (12.1) is 3×3 in this case with r_L being a 2×2 block and r_R being just a complex number. Such a unitary matrix contains in total nine parameters. However, all observable quantities are insensitive to a particular choice of the channel basis in the leads. This means that we can perform a rotation $S \mapsto V^T S V$ with an arbitrary unitary matrix V block-diagonal in the left-right channel space. For the minimal model with three channels, matrix V contains five variables. Thus we conclude that an S-TS junction with $2 + 1$ channels is characterised by four parameters. We denote these parameters T_1 , T_A , ρ , and χ and define them as

$$T_1 = t_{RL} t_{RL}^\dagger, \quad T_A e^{i\chi} = -t_{RL} t_{LR}^*, \quad (12.11a)$$

$$\rho = \frac{1}{2} \text{tr} [r_L (r_L^\dagger - r_L^*)]. \quad (12.11b)$$

The parameter T_1 is the single non-zero transmission eigenvalue and coincides with normal-state conductance of the junction. The expression for $T_A e^{i\chi}$ can be understood as an amplitude of a fictitious Andreev reflection process where an electron from the right lead normally transmits into the left lead, gets Andreev reflected into a hole, which then transmits back into the right lead. This resembles the Andreev conductance amplitude of the junction when the right lead is in the normal state $\Delta_{\text{top}} = 0$. However T_A does not contain multiple reflection processes while the actual zero-bias Andreev conductance takes the quantised value $G_A = 2e^2/h$ due to multiple scattering.[26, 27, 28] The physical meaning of the parameter ρ will be discussed below.

A straight-forward calculation of the determinant (12.7) for a 3×3 scattering matrix yields, up to φ -independent factors

$$D(\varphi, \omega) = (2 - T_1)\omega^2 + T_1 \sqrt{(\omega^2 + \Delta_0^2)(\omega^2 + \Delta_{\text{top}}^2)} + T_A \Delta_0 \Delta_{\text{top}} \cos(\varphi - \chi) + \Delta_0^2 \rho. \quad (12.12)$$

Unitarity of S ensures that $0 \leq T_A \leq T_1 \leq 1$ as well as $0 \leq \rho \leq 2 - T_1$. With the determinant (12.12), we obtain the Josephson current from Eq. (12.4),

$$I = \frac{e\Delta_0\Delta_{\text{top}}}{h} T_A \sin(\varphi - \chi) T \sum_{\omega} \frac{1}{D(\varphi, \omega)}. \quad (12.13)$$

This general expression constitutes a central result of our paper.

Matsubara summation in Eq. (12.13) can be transformed into an integral over real energies $E = i\omega$. This allows to identify contributions to the current from discrete and continuous parts of the spectrum of the junction. A discrete sub-gap Andreev bound state appears when the integrand has a pole at $E_0 < \min\{\Delta_0, \Delta_{\text{top}}\}$. The corresponding residue yields the supercurrent carried by this bound state. There is also the Majorana zero mode in the system, but it is insensitive to φ and produces an ω factor in D , which was already dropped in Eq. (12.12). Second, the integrand has a branch cut at $\min\{\Delta_0, \Delta_{\text{top}}\} < E < \max\{\Delta_0, \Delta_{\text{top}}\}$. Integration around this cut yields the current carried by the continuous spectrum of states extending into one of the superconductors. Continuum states above $\max\{\Delta_0, \Delta_{\text{top}}\}$ do not carry a supercurrent.

Expression (12.13) greatly simplifies when the two gaps are equal, $\Delta_0 = \Delta_{\text{top}}$. In this case, the current is given by Eq. (12.9) with a single current-carrying Andreev bound state at

$$E_0 = \Delta \sqrt{\frac{\rho + T_1 + T_A}{2} - T_A \sin^2 \frac{(\varphi - \chi)}{2}}. \quad (12.14)$$

For different gaps, $\Delta_0 \neq \Delta_{\text{top}}$, the bound state energy E_0 is still straight-forward to calculate, but the resulting algebraic expression is much more cumbersome. It still has its minimum at $\varphi - \chi = \pi$, but the sub-gap level exists only in a range of phases around $\chi + \pi$ given by

$$\cos(\varphi - \chi) < \frac{(2 - T_1) \min\{\Delta_0^2, \Delta_{\text{top}}^2\}}{T_A \Delta_0 \Delta_{\text{top}}} - \frac{\rho \Delta_0}{T_A \Delta_{\text{top}}}. \quad (12.15)$$

Equation (12.15) inherits the asymmetry of Eq. (12.12) with respect to the gaps Δ_0 and Δ_{top} . In the case $\Delta_0 < \Delta_{\text{top}}$, the r.h.s. of Eq. (12.15) is proportional to $2 - T_1 - \rho > 0$, so that a sub-gap state exists at least in the interval $\pi/2 < \varphi < 3\pi/2$ where the cosine is negative. In the opposite case, $\Delta_0 > \Delta_{\text{top}}$ the bound state may be absent for all φ . The evolution of the Andreev state energy is illustrated in Fig. 172.

An important limit of Eq. (12.13) is the tunneling limit $T_1 \ll 1$. In this case we get

$$I = \frac{e\Delta_{\text{top}}}{h\sqrt{2\rho}} T_A \sin(\varphi - \chi) \quad (12.16)$$

provided $T_1 \ll \rho\Delta_0/\Delta_{\text{top}}$. A striking property of Eq. (12.16) is that the tunneling supercurrent does not depend on Δ_0 . This is in strong contrast to conventional junctions where the two leads and hence the two superconducting gaps enter on equal footing.[24]

From Eqs. (12.13) and (12.14) we can extract the physical meaning of ρ . It becomes clear if we put both T_1 and T_A to zero, which effectively disconnects the leads from each other. The bound state energy is $E_0 = \Delta_0 \sqrt{\rho/2}$ in this case. Thus, $\sqrt{\rho}$ defines the energy of the sub-gap bound state at the end of the ordinary superconductor, when it is disconnected from the topological superconductor. If $\mathcal{T}$ symmetry is preserved in the terminated ordinary superconductor, then $r_L = -r_L^T$ leading to $\rho = 2$ so that there is no sub-gap level at the end of the lead.

The asymmetry of the results with respect to $\Delta_0 \leftrightarrow \Delta_{\text{top}}$ can be traced back to the broken time-reversal symmetry. In a conventional, $\mathcal{T}$ -symmetric short junction the matrices S and S^T can be block-diagonalised simultaneously,[23] which decouples the contribution of different

channels to the supercurrent. In the S-TS junction, S and S^T are not connected by $\mathcal{T}$ -symmetry. Physically, this means that electrons and holes scatter differently at the junction – if an electron is fully reflected, the corresponding hole might still be transmitted. This prevents us from eliminating a fully reflected electron channel from the problem. As a result, all three channels in the leads carry supercurrent in combination. The different number of channels associated with the two gaps explains the strong asymmetry of Eq. (12.13) with respect to Δ_0 and Δ_{top} .

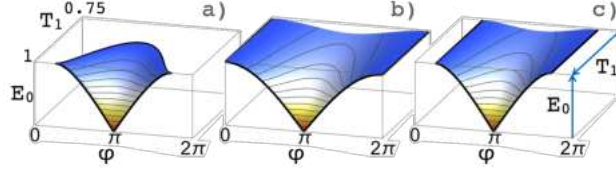


Figure 2: Bound state energy E_0 for the nanowire setup as a function of phase difference φ and normal-state conductance T_1 . The gaps are (a) $\Delta_0 = 1.5$, $\Delta_{\text{top}} = 1$, (b) $\Delta_0 = \Delta_{\text{top}} = 1$, and (c) $\Delta_0 = 1$, $\Delta_{\text{top}} = 1.5$. Front cuts at perfect transmission $T_1 = 1$ are the same for (a) and (c). As T_1 decreases, the sub-gap state in case (a) quickly becomes shallow and disappears. In cases (b) and (c), the sub-gap state survives all the way to $T_1 = 0$ (note that the plots are shown only for $0.75 < T_1 < 1$).

12.0.5 Nanowire setup

Let us now apply the general results (12.12) – (12.13) to a particular model of a junction made of a nanowire with parabolic spectrum and strong spin-orbit coupling in the presence of an external magnetic field and proximity-induced superconductivity.[15, 16] This model is widely used in literature to describe the existing experiments[30, 15, 14] on the N-TS junctions.

The ordinary and topological parts of the junction differ by the material parameters or by doping (position of the chemical potential). We consider the model with a simple spin-degenerate left lead

$$\xi_L = \frac{p^2}{2m} - \mu_L, \quad (12.17)$$

while the right lead is subject to strong spin-orbit coupling u and parallel magnetic field B (measured in energy units),

$$\xi_R = \frac{p^2}{2m} - \mu_R + up\sigma_z + B\sigma_x. \quad (12.18)$$

Here Pauli matrices σ act in spin space. While the left lead represents a wire with two spin-degenerate conducting channels, the right lead hosts just a single channel if the chemical potential is inside the Zeeman gap, $|\mu_R| < |B|$. To describe the superconductivity induced in the wires, we use the Bogoliubov-de Gennes Hamiltonian (12.2) with ξ from the above two expressions. The right lead remains in the topological state[15, 16] provided $|\Delta_{\text{top}}| < \sqrt{B^2 - \mu_R^2}$. We will assume a stronger condition $\mu_R, \Delta_{\text{top}} \ll B \ll mu^2$. This will allow us to linearise the spectrum of the single channel close to the Fermi level and completely disregard the second gapped channel in the topological superconductor. It also establishes an approximate time-reversal symmetry with $\mathcal{T} = i\sigma_y\mathcal{C}$ in the right lead.

Assuming a simple adiabatic point contact between the left and the right wire and matching

the wave functions, we calculate the scattering matrix

$$S = \begin{pmatrix} \frac{4i\alpha}{(1+\alpha)^2} & \frac{1-\alpha}{1+\alpha} & \frac{2i(1-\alpha)\sqrt{\alpha}}{(1+\alpha)^2} \\ \frac{-1+\alpha}{1+\alpha} & 0 & \frac{2\sqrt{\alpha}}{1+\alpha} \\ \frac{2i(1-\alpha)\sqrt{\alpha}}{(1+\alpha)^2} & -\frac{2\sqrt{\alpha}}{1+\alpha} & \frac{i(1-\alpha)^2}{(1+\alpha)^2} \end{pmatrix}, \quad (12.19)$$

where the only parameter is the velocity ratio $\alpha = v_L/u$ with $v_L = \sqrt{2\mu_L/m}$. For this scattering matrix, the transport parameters are

$$T_1 = \frac{8\alpha(1+\alpha^2)}{(1+\alpha)^4}, \quad T_A = \frac{16\alpha^2}{(1+\alpha)^4}, \quad (12.20)$$

$$\rho = 2\frac{(1-\alpha)^2}{(1+\alpha)^2}, \quad \chi = 0. \quad (12.21)$$

Substituting these values into Eq. (12.16), we get the following result in the tunneling limit $\alpha \ll 1$:

$$I = \frac{8\alpha^2 e \Delta_{\text{top}}}{h} \sin \varphi. \quad (12.22)$$

At the same time, the normal-state resistance is $R_N = h/(e^2 T_1) = h/(8\alpha e^2)$. Thus we can express the critical current of the junction as

$$I_c = \frac{h \Delta_{\text{top}}}{8e^3 R_N^2}. \quad (12.23)$$

We see that I_c scales as inverse normal resistance squared, in strong contrast with the Ambegaokar-Baratoff relation [22] which predicts linear scaling with inverse resistance. As mentioned earlier, I_c depends on Δ_{top} but not on Δ_0 as long as $T_1 \ll \Delta_0/\Delta_{\text{top}}$. In the case of equal gaps, the discrete level is $E_0 = \Delta\sqrt{1 - T_A \sin^2(\varphi/2)}$ which looks similar to Eq. (12.10) but is much more shallow in the tunnelling limit where $T_A \sim T_1^2$.

Instead of using two different materials for the two leads, both superconductors can be made of the same nanowire with strong spin-orbit coupling. In this case, both leads are described by Eq. (12.18) with different values of the chemical potential. In the ordinary (left) lead, we assume $\mu_L \gg B, \mu u^2$ which tunes the wire far into the topologically trivial phase. Since the two channels are only slightly split by spin-orbit interaction and the Zeeman effect, we can neglect these terms in the Hamiltonian and reduce the problem to the one studied above.

The peculiar supercurrent suppression leading to $I_c \sim \alpha^2$ is due to destructive interference of different processes contributing to the supercurrent. The parameter T_A describes the amplitude of an electron tunneling from the topological superconductor into the ordinary superconductor, then being Andreev reflected, followed by the tunneling of the hole back into the topological lead. This amplitude contains two terms, corresponding to the two intermediate states (channels or spin projections) that the particle can have in the ordinary superconductor. Each term scales as $\alpha \sim T_1$ in the tunneling limit, however, they interfere destructively with their sum scaling only as $\alpha^2 \sim T_1^2$.

12.0.6 Junction with a random scattering matrix

The nanowire model considered in the previous Section exhibits a parametrically strong suppression of the Josephson current in the tunneling limit compared to what is expected from

the Ambegaokar-Baratoff theory.[22] In order to estimate how universal this suppression is, let us consider a generic junction with a random scattering matrix.

Since time-reversal symmetry is broken, we assume that S is a random matrix from the circular unitary ensemble.[29] By calculating the joint distribution function $P(T_A, T_1, \rho, \chi)$ in this ensemble, we can find whether it is likely that a random junction has $T_A \ll T_1$ so that the supercurrent is parametrically small.

A unitary 3×3 matrix S has nine independent parameters x_i with $i = 1, \dots, 9$. Let us assume that the first four of them are T_1 , T_A , ρ , and χ , which are important for transport. The five remaining parameters amount to rotations of the channel basis in the leads, $S = V^T S_0 V$. Here the matrix S_0 depends on the four transport parameters while V is a block-diagonal (in left-right space) unitary matrix parameterised by $x_5, \dots, x_9$. The metric tensor M_{ij} of the circular unitary ensemble is defined by the uniform measure $\text{tr}(dS dS^\dagger) = M_{ij} dx_i dx_j$. The overall distribution function is $P(x_1, \dots, x_9) = \sqrt{\det \bar{M}}$. Integrating out the parameters $x_5, \dots, x_9$, we obtain the distribution function for the relevant transport characteristics.

The distribution function takes a simple form in terms of the ratio T_A/T_1 instead of T_A and an angular variable $\lambda \in (0, 2\pi)$ instead of ρ . This new variable is defined by

$$\rho = \left(1 - \frac{T_A^2}{T_1^2}\right) \left(1 - \frac{T_1}{2} + \sqrt{1 - T_1} \cos \lambda\right). \quad (12.24)$$

An explicit calculation yields the following result:

$$P\left(T_1, \frac{T_A}{T_1}, \chi, \lambda\right) = \text{const } T_1 \sqrt{\frac{2 - T_1}{1 - T_1}} \frac{T_A}{T_1}. \quad (12.25)$$

Here the constant factor provides proper normalisation. The distribution function does not depend on χ and λ and factorises as a function of T_1 and T_A/T_1 . Thus the four parameters are uncorrelated and the average T_A for a given T_1 is

$$\langle T_A \rangle = \frac{2T_1}{3}. \quad (12.26)$$

We conclude that typically $T_A \sim T_1$, so that $I_c \sim e\Delta_{\text{top}}/R_N$ in the tunneling limit and the Ambegaokar-Baratoff relation qualitatively holds.

Systems with $T_A \ll T_1$ have a very low probability within the circular unitary ensemble. The nanowire setup discussed in the previous Section represents one such rare realisation. This implies that a generic perturbation of the nanowire Hamiltonian may significantly increase the value of the critical current.

Part VII

Junctions with Majoranas: Main Theory

13 Topological Superconductivity in a Planar Josephson Junction by Pientka, Keselman et al.

Abstract

We consider a two-dimensional electron gas with strong spin-orbit coupling contacted by two superconducting leads, forming a Josephson junction. We show that in the presence of an in-plane Zeeman field the quasi-one-dimensional region between the two superconductors can support a topological superconducting phase hosting Majorana bound states at its ends. We study the phase diagram of the system as a function of the Zeeman field and the phase difference between the two superconductors (treated as an externally controlled parameter). Remarkably, at a phase difference of π , the topological phase is obtained for almost any value of the Zeeman field and chemical potential. In a setup where the phase is not controlled externally, we find that the system undergoes a first-order topological phase transition when the Zeeman field is varied. At the transition, the phase difference in the ground state changes abruptly from a value close to zero, at which the system is trivial, to a value close to π , at which the system is topological. The critical current through the junction exhibits a sharp minimum at the critical Zeeman field, and is therefore a natural diagnostic of the transition. We point out that in presence of a symmetry under a modified mirror reflection followed by time reversal, the system belongs to a higher symmetry class and the phase diagram as a function of the phase difference and the Zeeman field becomes richer.

13.1 Introduction

Since the realization of two-dimensional topological insulators a decade ago, a plethora of new phases of matter with nontrivial topology in one, two and three dimensions have been discovered in experiment. Considerable experimental and theoretical effort has been dedicated to the study of zero-energy Majorana bound states which arise in topological superconductors as edge states in one dimension or bound to vortices in two dimensions [10, 4]. Advances in nanotechnology and the prospect of using Majorana states as building blocks of topological quantum computers have triggered intense experimental efforts to realize and characterize them in one-dimensional systems [30, 15, 5, 10]. More recently, two-dimensional electron gases (2DEGs) with induced superconductivity [7, 8, 63] have emerged as a contender for topological superconductivity.

A key challenge for existing one-dimensional platforms such as proximitized semiconductor nanowires [10] or atomic chains [11] is to develop networks that allow braiding of multiple Majorana states. An alternative route towards realizing a scalable architecture is to pattern a network of one-dimensional channels into a proximitized 2DEG using gates [7, 104]. While this approach offers great flexibility in designing networks, it may be cumbersome to drive many channels individually into a topological regime via local gates employing additional local probes. Moreover, gates may change the shape of the channel or physical parameters such as spin-orbit coupling and their effect is strongly influenced by the electrostatic properties of the nearby superconductors [13].

Here we pursue a different strategy to realize Majorana bound states motivated by recent experiments on Josephson junctions in proximity-coupled 2DEGs [7, 8, 63]. Carriers with energies below the superconducting gap are trapped in the quasi one-dimensional junction

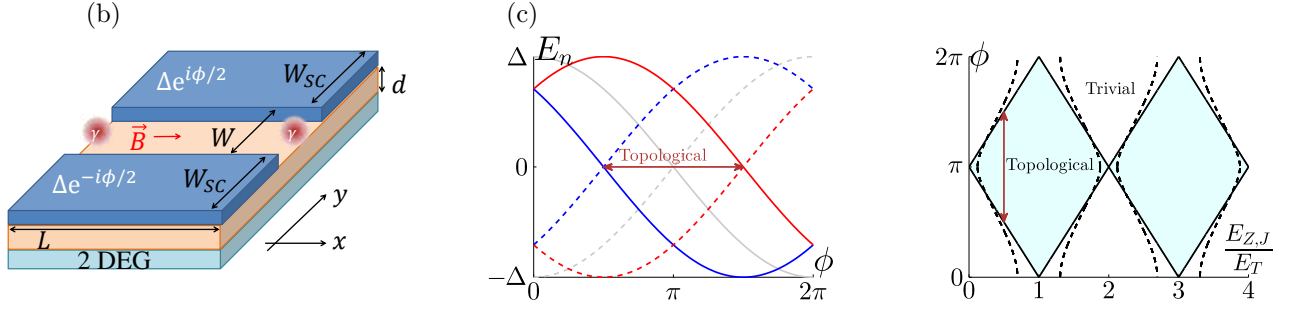


Figure 3: (a) A Josephson junction is formed in a 2DEG with Rashba spin-orbit coupling by proximity coupling it to two s-wave superconductors with relative phase difference ϕ . An in-plane magnetic field is applied parallel to the interface between the normal and the superconducting regions. (b) The bound states spectrum in a narrow junction for $k_x = 0$. The spectrum in the absence of a Zeeman field is twofold degenerate and is indicated by the grey lines. In presence of the Zeeman field the spectrum for the two spin states (plotted in red and blue) is split allowing for the appearance of a topological phase. (c) Phase diagram as a function of the Zeeman field in the junction, $E_{Z,J}$, given in units of the Thouless energy $E_T = (\pi/2) v_F/W$, and the phase difference ϕ . The solid lines are the phase boundaries in the absence of any normal backscattering at the superconducting-normal interface, while the dashed lines correspond to a junction transparency of 0.75, and a phase $k_F W + \varphi_N = 3\pi/8$ as defined in Sec. 13.3.1. The arrows indicate the range of ϕ values between the two zero energy crossings in (b) for which the system is topological.

region between two superconducting leads as depicted in Fig. 178. In the presence of a Zeeman field, the junction can enter a topological superconducting phase akin to the one in proximitized nanowires and Majorana bound states appear at the ends of the junction.

A key advantage of this setup is that the lateral dimension allows for additional experimental knobs such as a phase difference or a supercurrent across the junction. One of the central results of this work is that a phase bias can induce a robust topological phase in the junction. Most strikingly, in the absence of normal reflection, junctions at a phase difference of π host Majorana states to a large extent independently of parameters such as chemical potential, Zeeman field, width of the junction, or induced pairing strength, for as long as the gap in the bulk 2DEG does not close. Moreover, the phase difference can be used as a powerful switch that changes the topology of the entire phase space from trivial at zero to topological at π . This is in stark contrast to previous proposals which require careful gating and a Zeeman field beyond a critical value. A setup based on Josephson junctions may also facilitate the realization of topological superconductor networks. By tuning a global phase difference multiple Josephson junctions can be tuned simultaneously into a topological phase without tuning local parameters or requiring local probes.

In the presence of normal reflection in the junction or at the interface to the superconductor deviations from this ideal behavior occur. As long as normal reflection is not too strong, however, our results still hold in extended regions of the parameter space.

On the face of it, the system we consider belongs to class D in the ten-fold classification [14], since time-reversal symmetry is broken and particle-hole symmetry holds. In fact, our system has an additional symmetry given by a combination of a mirror reflection, time-reversal, and a gauge transformation, which places it in class BDI (see also Ref. [15]). Interestingly, this symmetry is present for *any* value of the phase difference between the superconductors. As a consequence, slivers with additional topological phases appear in the phase diagram as a function of the in-plane Zeeman field and the phase difference between the superconductors. The system is brought back to class D if the magnitude of the superconducting gap on the two

sides of the junction is different.

If the phase difference is not imposed externally the system can undergo a first order phase transition in which the phase jumps from a phase close to 0 to a phase close to π with increasing in-plane magnetic field. Similar transitions have previously been studied in ferromagnetic Josephson junctions [7, 16, 17, 18]. Quite remarkably, our results suggest that such a first order phase transition in the present setup is in fact a topological phase transition unique to the two dimensional geometry. The system can thus self tune into a topological phase when the magnetic field is varied and realizes a first-order topological phase transition without gap closing. Moreover, this transition is accompanied by a minimum of the critical current. Therefore, the critical current can serve as an inherent probe of the topological phase transition. Surprisingly, the contrast of the critical current modulation with the field increases with temperature. At high temperatures the critical current vanishes at the magnetic field of the underlying zero-temperature topological transition. This insight suggests that the experimental results presented by Hart *et al.* [7] indicate an underlying topological phase transition in the ground state.

This paper is organized as follows. We start by presenting the proposed setup and a summary of our results in Sec. 13.2. We then show the derivation of the phase diagram for the system as function of the phase difference and the Zeeman field, and discuss the magnitude of the topological gap and the appearance of Majorana end modes in Sec. 13.3. In Sec. 13.4 we discuss the first order topological phase transition as function of the Zeeman field and how the critical current can serve as a novel experimental probe to detect this transition in the suggested setup. We conclude with discussion of the presented results in Sec. 13.5. The paper is followed by appendices that cover several technical details.

13.2 Physical picture and summary of results

We consider a two-dimensional semiconductor with Rashba spin-orbit coupling, partially covered with two superconducting contacts in an in-plane magnetic field as depicted in Fig. 178(a). For the most part, we will be interested in the case of an infinite system, where the width of the leads and the length of the junction $W_{SC}, L \rightarrow \infty$, while the separation of the leads, W , remains finite. We describe the system by a Bogoliubov-de Gennes Hamiltonian in the Nambu basis $(\psi_\uparrow, \psi_\downarrow, \psi_\downarrow^\dagger, -\psi_\uparrow^\dagger)$

$$H = \left(\frac{k_x^2 - \partial_y^2}{2m} - \mu + \frac{m\alpha^2}{2} \right) \tau_z + \alpha(k_x \sigma_y + i\partial_y \sigma_x) \tau_z + E_Z(y) \sigma_x + \Delta(y) \tau_+ + \Delta^*(y) \tau_- . \quad (13.1)$$

Here k_x is the momentum along x which is conserved in the system (we set $\hbar = 1$ throughout the manuscript), m is the effective mass of the 2DEG, μ is the chemical potential measured from the bottom of the spin-orbit split bands, α is the strength of Rashba spin-orbit coupling and $E_Z(y) = g(y)\mu_B B/2$ is the Zeeman energy induced by an external magnetic field. We assume different g factors in the junction and underneath the superconducting leads and denote

$$E_Z(y) = E_{Z,L}\theta(|y| - W/2) + E_{Z,J}\theta(W/2 - |y|), \quad (13.2)$$

where $\theta(x)$ is a step function. For simplicity we focus on the case of zero Zeeman field underneath the leads and postpone the discussion of nonzero $E_{Z,L}$ to Sec. 13.3.1. The proximity induced pairing in the semiconductor 2DEG is accounted for by

$$\Delta(y) = \Delta e^{i\text{sgn}(y)\phi/2}\theta(|y| - W/2), \quad (13.3)$$

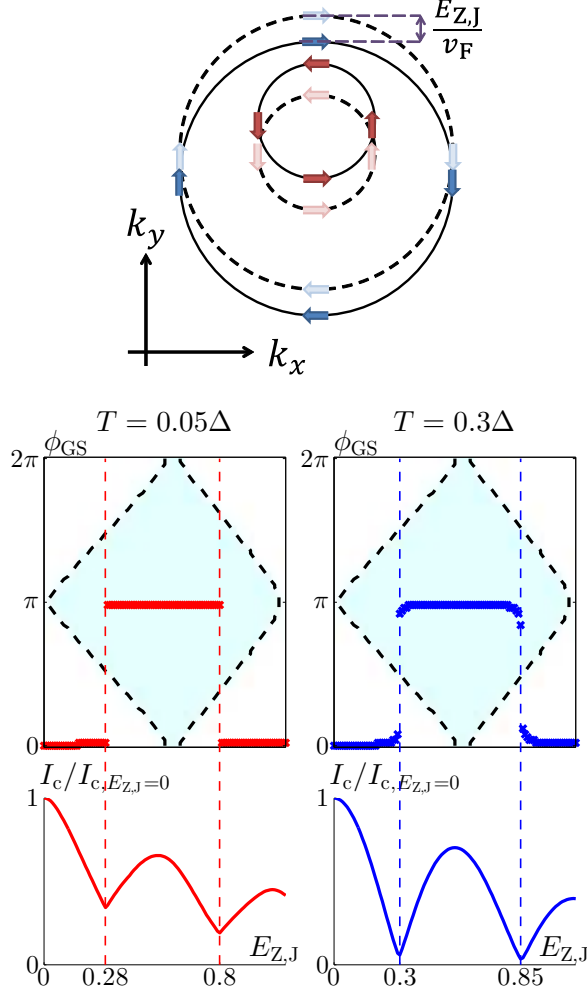


Figure 4: (a) A Zeeman field along x shifts the two Rashba-split Fermi surfaces of the 2DEG in opposite directions along y . The arrows indicate the orientation of the spin at each point on the Fermi surface. (b) The phase difference, ϕ_{GS} , that minimizes the ground state energy (upper panel) and the critical current modulation (lower panel) as function of the Zeeman field obtained numerically using a tight binding model for the system [19]. Left (right) panel corresponds to a temperature of $T = 0.05\Delta$ ($T = 0.3\Delta$). (Note that we set $k_B = 1$ throughout the manuscript.) The light blue color indicates the region in the parameter space for which the system is in the topological phase. As the Zeeman field is varied, the system undergoes a series of first order topological phase transitions, in which ϕ_{GS} changes abruptly between values lying in the topological and trivial regions of the phase diagram. The critical current exhibits minima at the points of the phase transitions. As the temperature is increased the minima become deeper.

where ϕ is the phase difference between the two superconductors. The Pauli matrices σ, τ act in the spin and particle-hole basis respectively, and $\tau_{\pm} = \tau_x \pm i\tau_y$.

States at subgap energies are confined to the quasi-one-dimensional junction between the two superconducting leads. Under suitable conditions the junction can enter a one-dimensional (1d) topological superconducting phase. In this paper we study two experimental configurations in which the model described by Eq. (13.1) and Fig. 178(a) may be realized. In the first configuration the phase across the junction is a parameter controlled externally by applying a current or a magnetic flux. In the second configuration the phase is left to self-tune so as to minimize the ground state energy. For the first configuration we calculate the phase diagram as a function of the phase across the junction and the Zeeman field, while for the second

configuration we identify the conditions under which the system self tunes to a topological phase. We find that the critical current of the junction can be used as a probe for the transitions between topological and trivial phases.

We start by evaluating the topological index for particle-hole symmetric systems in class D. As we shall see in Sec. 13.3.2, the model has a higher symmetry involving a mirror reflection followed by time reversal that places it into the BDI class. Each topological (trivial) region in the phase diagram of class D will be split into subregions with an odd (even) $\mathbb{Z}$ invariant [20, 34, 22]. Breaking this symmetry stabilizes the topological phase with a single Majorana bound state at each end.

The index in class D can be defined as the fermion parity of the ground state at $k_x = 0$ [8]. Therefore a phase transition between the trivial and topological classes must be accompanied by a single gap closing at $k_x = 0$. At this momentum spin along the x direction is conserved by the Hamiltonian in Eq. (13.1) and the spin-orbit coupling can be gauged away by substituting $\partial_y \rightarrow \partial_y + im\alpha\sigma_x$. We arrive at the effective Hamiltonian

$$H_0 = (-\partial_y^2/2m - \mu)\tau_z + E_Z(y)\sigma_x + \Delta(y)\tau_+ + \Delta(y)^*\tau_- \quad (13.4)$$

describing a one-dimensional Josephson junction in a magnetic field. Figure 178(b) shows the subgap spectrum of a narrow junction in the Andreev limit $\mu \gg \Delta$, where normal reflection is absent. At vanishing Zeeman field, the spectrum is twofold degenerate and the system is trivial for all values of ϕ . At nonzero fields the degeneracy is split opening a topological phase around $\phi = \pi$. At the zero-energy crossings the fermion parity changes and the junction undergoes topological phase transitions.

The resulting phase diagram as a function of Zeeman field and phase bias is shown in Fig. 178(c). Most strikingly the junction is in a topological phase at $\phi = \pi$ for arbitrary Zeeman fields except at isolated values given by even integer multiples of the ballistic Thouless energy of the junction $E_T = (\pi/2)v_F/W$. In contrast, at zero phase difference the system remains trivial throughout. As will be shown in Sec. 13.3.1, this result generalizes to junctions of arbitrary width as long as normal reflection can be neglected and the system remains gapped. The $\mathbb{Z}_2$ topological index cannot change at $\phi = 0, \pi$ because the spectrum is always doubly degenerate at $k_x = 0$ and topological phase transitions thus come in pairs. Hence, an externally applied phase bias is a powerful experimental knob, that allows one to tune the topology of the junction to a large extent independent of microscopic parameters.

We can qualitatively understand the effect of weak normal reflection on the phase diagram from the subgap spectra shown in Fig. 178(b). Normal backscattering couples left and right movers, lifting the degeneracy of Andreev levels at $\phi = 0, \pi$. Hence the system becomes topological (trivial) in a small range of Zeeman fields at $\phi = 0$ ($\phi = \pi$), respectively. The avoided level crossings translate to avoided crossings of phase transition lines as indicated by the dashed lines in Fig. 178(c). As long as normal reflection is not too strong, it remains possible to induce a topological phase by a phase bias in extended regions of parameter space.

We next consider the second configuration in which the phase is determined by the condition that the ground state energy is minimal. Remarkably, we shall see in Sec. 13.4 that in this case the system self tunes to the topological phase in a broad range of Zeeman fields, exhibiting a first order topological phase transition. Such a transition will be accompanied by an abrupt change in various thermodynamic quantities characterizing the system, e.g. the magnetization, as well as in the energy gap in the bulk.

The origin of the first order transition is that the phase difference, ϕ_{GS} , that minimizes the ground state energy changes abruptly between two distinct values, one in the trivial region and one in the topological region, at certain values of the Zeeman field. As a consequence, the junction is expected to show a hysteretic behavior as the Zeeman field is swept at low temperatures. Moreover, we find that the critical current exhibits a minimum at these values

of the Zeeman field as shown in Fig. 4(b). The critical current can thus be used as a novel experimental probe of the topological phase transitions in this configuration.

These findings can be understood semiclassically in the limit $E_{Z,J} \ll \alpha k_F \ll \mu$. Due to the Rashba-induced spin-momentum locking, the Zeeman field shifts the two Fermi surfaces uniformly along k_y in opposite directions as depicted in Fig. 4(a). This induces a nonzero center of mass momentum $q = 2E_{Z,J}/v_F$ in Cooper pairs traversing the junction. Thus the wavefunction of a Cooper pair leaving one superconducting lead can be described by a linear combination of singlet and triplet contributions $\cos(qy)|S\rangle + \sin(qy)|T\rangle$. For $qW > \pi/2$ (or, equivalently, $E_{Z,J} > E_T/2$) the singlet wavefunction has opposite signs at the two superconducting leads and ϕ_{GS} switches from 0 to π . As discussed above, the system is trivial at $\phi = 0$ and topological at $\phi = \pi$ in a wide range of parameters. We see therefore, that for $E_{Z,J} > E_T/2$ the system self-tunes into a topological phase via a first order phase transition. Moreover, at the $0 - \pi$ transition point ($E_{Z,J} = E_T/2$) the singlet component, which carries the supercurrent, has a node at the second interface resulting in a vanishing critical current. Beyond the semiclassical approximation, we find that the critical current remains nonzero but assumes a local minimum at the transition, as shown by the numerical results in Fig. 4(b).

When the constraint $\alpha k_F \ll \mu$ is lifted, the phase difference in the ground state is not necessarily 0 or π and varies with $E_{Z,J}$ [25]. However, generically, the system still exhibits a jump in ϕ_{GS} as a function of Zeeman field. This jump is accompanied by a change of fermion parity at $k_x = 0$ and, therefore, it coincides with a topological phase transition even in the more general case. Moreover, the critical current still exhibits a minimum at the phase transition point, $E_{Z,J} = E_T/2$. Surprisingly, the minimum grows sharper with increasing temperature, as can be seen in Fig. 4(b).

It is encouraging that the modulation of the critical current as function of an in-plane magnetic field, and in particular its revival, has been observed in experiment realizing the setup we consider [7], indicating that the topological regime in Josephson junctions is within reach of current experiments even in the absence of a phase bias. Moreover, our theoretical analysis strongly suggests that the vanishing of the critical current as a function of magnetic field, observed in Ref. [7], is indicative of a first-order topological phase transition.

A direct signature of topological superconductivity can be obtained by a straightforward extension of the setup in Fig. 178(a), when the system has a large but finite length L . Adding a tunnel probe at one end of the junction would enable the detection of Majorana bound states via tunneling conductance measurements. While the conductance should exhibit a zero-bias peak at the end of the junction, no such feature is expected when tunneling into the center of the junction.

13.3 The $E_Z - \phi$ Phase Diagram, Topological Gap, and Majorana end modes

13.3.1 Class D Phase Diagram

As was discussed in the previous section, topological phase transitions that change the parity of the number of Majorana end modes occur when there are zero-energy solutions of the model at $k_x = 0$, given by Eq. (13.4). We use scattering theory to obtain the bound state spectrum of the system and in particular to find the conditions for the formation of a zero-energy state. We work in the limit $\mu \gg \Delta$ and assume at first that there is no normal reflection at the superconducting-normal interfaces. In this case, the eigenstates decompose into those with left and right-moving currents. We denote the junction's transmission amplitude for electrons (holes) by $t_{e(h)}$ and the Andreev reflection amplitudes by $r_A^\pm = \exp(i\eta \pm i\phi/2)$,

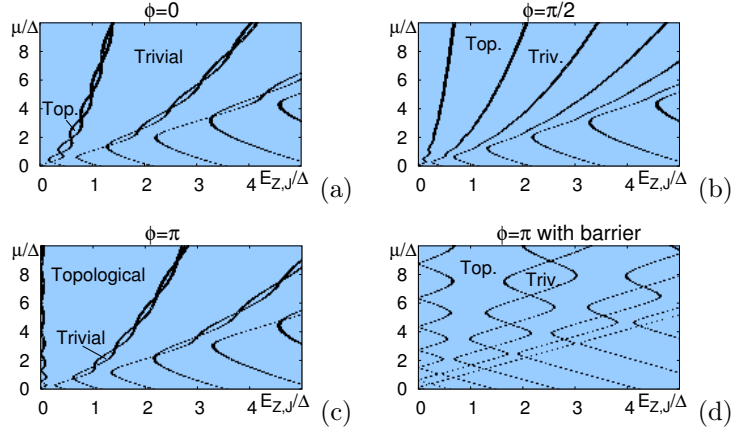


Figure 5: (a-c) Numerical results for the phase diagram as function of the Zeeman field and the chemical potential for different values of ϕ , obtained using the scattering matrix approach (see Appx. 13.9.1 for details). The width of the junction is $W = 5(m\Delta)^{-1/2}$. The phase boundaries are determined by a gap closing at $k_x = 0$. At low chemical potentials $\mu \simeq \Delta$ a finite amount of normal reflection from the superconducting leads results in oscillatory modulations of the phase boundaries as function of μ . The topological phase significantly expands as the phase difference between the leads is tuned from 0 to π . (d) Same as (c) but with a rectangular potential barrier at each interface between lead and junction. The barrier has height 200Δ and width $0.01(m\Delta)^{-1/2}$. This corresponds to a transparency of 0.82 at $\mu = 10\Delta$. The presence of a reflecting barrier leads to stronger modulations of the boundaries.

where $\eta = \cos^{-1}[(E - E_{Z,L})/\Delta]$, and the sign corresponds to the current direction¹. In the limit $\mu \gg E_{Z,J}$ we can approximate

$$t_{e(h)} = \exp \left[\pm i k_F W + i \frac{(E - E_{Z,J})}{v_F} W \right], \quad (13.5)$$

where $k_F = (2m\mu)^{1/2}$ and $v_F = k_F/m$ are the Fermi momentum and velocity respectively. The bound state spectrum can be obtained from the condition $1 = (r_A^\pm)^2 t_e t_h$ [15].

We arrive at the following condition for the subgap spectrum

$$\cos^{-1} \left(\frac{E_n - E_{Z,L}}{\Delta} \right) = \frac{\pi}{2} \frac{E_n}{E_T} - \frac{\pi}{2} \frac{E_{Z,J}}{E_T} \pm \frac{\phi}{2} + n\pi, \quad n \in \mathbb{Z}. \quad (13.6)$$

This equation implies a twofold degeneracy of the spectrum at $\phi = 0$ and $\phi = \pi$. This degeneracy is a consequence of a mirror symmetry and the absence of normal reflection from the superconducting leads. As an important consequence of this degeneracy the $\mathbb{Z}_2$ topological index cannot change at $\phi = 0, \pi$ as zero-energy crossings always come in pairs. We first consider the case $E_{Z,L} = 0$. Equation (13.6) then has zero-energy solutions for

$$\frac{\pi}{2} \frac{E_{Z,J}}{E_T} \pm \frac{\phi}{2} = \frac{\pi}{2} + \pi n. \quad (13.7)$$

This condition sets the phase boundaries for the phase diagram. It creates a diamond structure with alternating trivial and topological regions as can be seen in Fig. 178(c).

For nonvanishing normal reflection probability at the superconducting-normal interface the equation for the bound states is identical to Eq. (13.6) with ϕ replaced by

$$\tilde{\phi} = \cos^{-1} [r^2 \cos(2k_F W + 2\varphi_N) + (1 - r^2) \cos \phi]. \quad (13.8)$$

¹Note that left-moving electrons and right-moving holes acquire the same amplitude r_A^+ upon Andreev reflection.

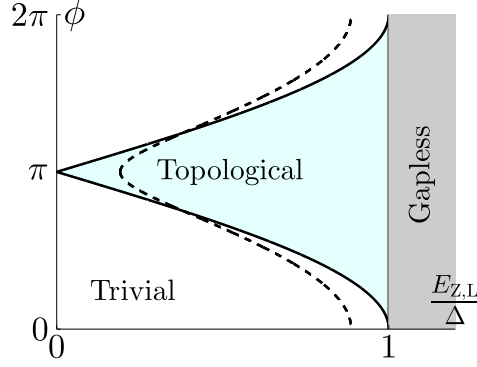


Figure 6: Phase diagram as a function of the Zeeman field in the leads $E_{Z,L}$ and the phase difference ϕ in the limit of a narrow junction with $E_{Z,J} \ll E_T$. For $E_{Z,L} > \Delta$ the system becomes gapless. The solid lines are the phase boundaries in the absence of any normal backscattering, while the dashed lines correspond to junction transparency of 0.75 and a phase $k_F W + \varphi_N = 3\pi/8$.

The phase φ_N is defined in Eq. (13.20) and depends on the details of the normal reflection. (See Appx. 13.6 for the derivation of this result). The phase boundaries are therefore given by Eq. (13.7), with $\phi \rightarrow \tilde{\phi}$. These are depicted as dashed lines in Fig. 178(c). As expected, the degeneracy at $\phi = 0$ and $\phi = \pi$ is removed, and the topological (trivial) phase can now be obtained for some range of Zeeman fields at $\phi = 0$ ($\phi = \pi$). Due to the dependence of $\tilde{\phi}$ on k_F , finite reflection amplitude will result in oscillatory modulations of the phase boundaries as function of μ as we show in Fig. 5.

We have so far neglected the effect of the Zeeman field in the lead, which limits the realization of a topological phase to junctions wide enough that $E_T \sim E_{Z,J}$. In the presence of a sizable Zeeman effect in the lead a topological phase may be accessible even in much narrower junctions where E_T greatly exceeds experimentally realizable Zeeman fields. The zero-energy solutions of Eq. (13.6) then read

$$E_{Z,L} = \Delta \cos \left(\frac{\pi}{2} \frac{E_{Z,J}}{E_T} \pm \frac{\phi}{2} \right). \quad (13.9)$$

In the limit of a narrow junction with $E_{Z,J} \ll E_T$, the first term inside the cosine can be neglected. The corresponding phase diagram is shown in Fig. 6. The topological phase is limited to $E_{Z,L} < \Delta$ as larger Zeeman fields drive the leads into a gapless regime.

Inside the gapped regime, the system remains always trivial (topological) at $\phi = 0$ ($\phi = \pi$) due to the degeneracy of the spectrum at these values of the phase difference. In the presence of normal reflection the phase difference ϕ in Eq. (13.9) is replaced by $\tilde{\phi}$ defined in Eq. (13.8). The corresponding phase boundaries are plotted in Fig. 6 as dashed lines.

13.3.2 Class BDI Phase Diagram

As mentioned in Sec. 13.2, the model in Eq. (13.1) possesses additional symmetries placing it in the BDI class. In the absence of a Zeeman field and for a phase difference of π between the superconductors, the Hamiltonian is time-reversal symmetric. In addition it commutes with a modified mirror operator with respect to the $x - z$ plane that we define as $\tilde{M}_y = (y \rightarrow -y) \times i\sigma_y \tau_z$. A Zeeman field along x , as well as a shift of the phase difference away from π , breaks both of these symmetries, but remains symmetric to their product. We can therefore define an anti-unitary effective time-reversal operator $\tilde{T} = \tilde{M}_y T$, where $T = i\sigma_y K$ is the standard time-reversal operator with K denoting complex conjugation, which commutes with the Hamiltonian. Note that $\tilde{T}^2 = 1$. The particle-hole operator, given in the basis we

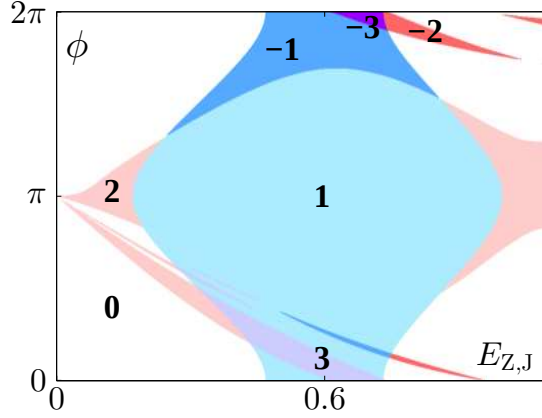


Figure 7: A phase diagram of the system as function of the Zeeman field and ϕ in presence of the mirror symmetry $\tilde{M}_y$ defined in the text. The values of the $\mathbb{Z}$ invariant corresponding to each region are indicated on the figure. Regions with odd (even) $\mathbb{Z}$ indices, corresponding to topological (trivial) regions of the D class, are filled with shades of blue (red). Note that, by our definition of ϕ , the phase diagram is not invariant under $\phi \rightarrow \phi + 2\pi$, since this operation flips the sign of the superconducting gap function. The figure was obtained using a tight binding version of the model (see Appendix 13.9.2) with the following parameters $W = 5$, $W_{SC} = 5$, $t = 1$, $\alpha = 0.5$, $\mu = -2.75$, $\Delta = 0.3$. Note that normal reflection is implicitly present in this model due to the finite width of the superconductors.

are using by $P = \sigma_y \tau_y K$ obeys $P^2 = 1$, and we therefore conclude that our model belongs to the BDI symmetry class with a $\mathbb{Z}$ topological invariant. Note that the $\mathbb{Z}_2$ invariant of class D discussed previously is determined by the parity of the $\mathbb{Z}$ invariant. We therefore expect that the topological (trivial) regions found previously will split into subregions with odd (even) $\mathbb{Z}$ indices.

To demonstrate this we use a tight binding version of the Hamiltonian in Eq. (13.1) (see Appendix 13.9.2 for details of the model) and calculate the BDI invariant following Ref. [36]. To this end we bring the chiral symmetry operator $\tilde{C} = \tilde{M}_y \tau_y$ to a diagonal form with $\mathbb{1}$ in the upper left block and $-\mathbb{1}$ in the lower right block. In this basis the Hamiltonian is purely off-diagonal and we can calculate the phase of the determinant of the off-diagonal part. The invariant is then calculated from the winding of this phase as k_x changes from 0 to π . The phase diagram obtained for a particular set of parameters is shown in Fig. 7. We note that although many additional subregions with various $\mathbb{Z}$ indices appear in the phase diagram, a large $\mathbb{Z} = 1$ gapped region is still present.

To stabilize the topological phase with a single Majorana bound state at each end, it is favorable to break this additional symmetry. To this end we introduce different magnitudes for the two superconductors, $\Delta_{1,2}$. When $|\Delta_1| \neq |\Delta_2|$, $\tilde{T}$ no longer commutes with the Hamiltonian and the symmetry class is reduced to D. We therefore expect that gap closing lines observed in Fig. 7 corresponding to phase transitions between different $\mathbb{Z}$ invariants with the same parity will no longer be present.

To verify this we employ the tight binding model and plot in Fig. 8 the bulk gap versus the phase difference ϕ at a constant Zeeman field $E_{Z,J} = 0.6$, for the same model parameters as in Fig. 7. It can be seen that when $|\Delta_1| = |\Delta_2|$, the bulk gap closes at values of ϕ corresponding to topological phase transitions between different (odd) $\mathbb{Z}$ indices. Once a different magnitude for the two superconductors is introduced, and the effective time-reversal symmetry is broken, a gap opens for all values of ϕ .

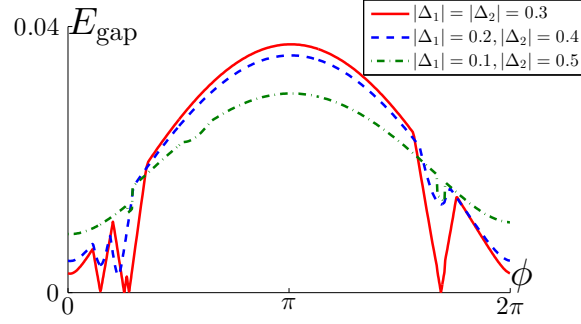


Figure 8: Bulk gap calculated along a cut in Fig. 7 with $E_{Z,J} = 0.6$. When $|\Delta_1| = |\Delta_2| = 0.3$ the system is in the BDI symmetry class. The bulk gap closes when the system undergoes topological phase transitions between regions with different (odd) $\mathbb{Z}$ indices. When $|\Delta_1| \neq |\Delta_2|$ the effective time-reversal symmetry is broken and the bulk becomes gapped for all values of ϕ .

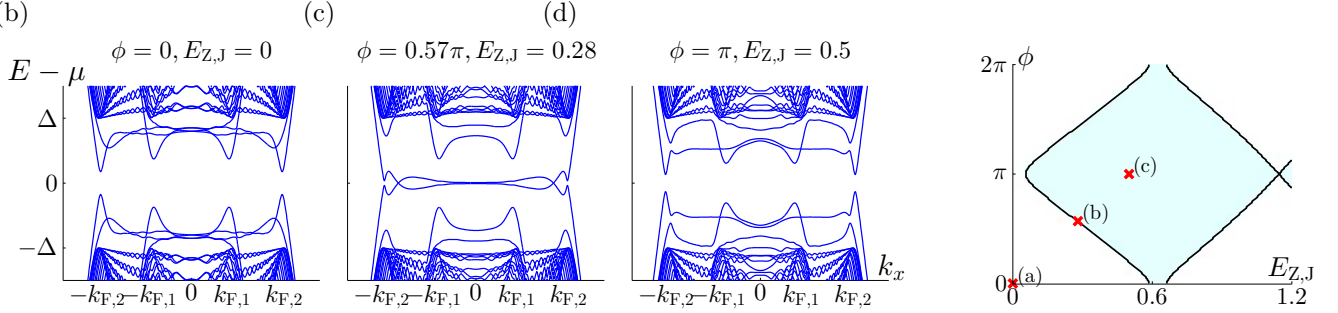


Figure 9: (a-c) Energy spectrum across the topological phase transition calculated using a tight binding model for the system (see Appendix 13.9.2 for details). The tight binding parameters used are $W = 5$, $W_{SC} = 20$, $t = 1$, $\alpha = 0.5$, $\mu = -2.8$, $\Delta = 0.3$. The Fermi momenta $k_{F,1/2}$ are calculated in the absence of a Zeeman field. The values of ϕ and $E_{Z,J}$ for which the spectra are plotted are indicated by crosses on the phase diagram shown in (d). The phase diagram is obtained by calculating the topological invariant for class D, $Q = \text{sign} [\text{Pf} (H_{k=\pi} \tau_x) / \text{Pf} (H_{k=0} \tau_x)]$ [36]. In (a) the system is in the trivial phase, in (b) the gap at $k_x = 0$ closes and the system undergoes a topological phase transition and in (c) the system is in the topological phase.

13.3.3 Topological Gap

The topological protection of the phase is governed by the size of the topological gap, which is determined by the lowest energy Andreev bound state in the junction. To estimate the magnitude of the gap, we need to consider the bound state spectrum for all k_x . Once again, we consider the case of zero normal reflection probability and no Zeeman field in the leads $E_{Z,L} = 0$.

For $k_x = 0$, the solutions of Eq. (13.6) take a simple form in the two limiting cases of a narrow and wide junction:

$$E = \begin{cases} \Delta \cos \left(\frac{\pi}{2} \frac{E_{Z,J}}{E_T} \pm \frac{\phi}{2} \right) & \Delta \ll E_T \\ E_T \left(\frac{E_{Z,J}}{E_T} \pm \frac{\phi}{\pi} + 2n + 1 \right) & \Delta \gg E_T \end{cases} \quad (13.10)$$

The largest gap in the topological region is obtained for $\phi = \pi$ and $E_{Z,J} = E_T$. For a narrow junction the gap is given by Δ , while for a wide junction the gap is smaller and given by E_T . We will consider the scenario of a narrow but finite width junction with $\Delta \lesssim E_T$, which is likely to be the most relevant experimentally. Note that in order to reach the maximal gap in the topological region in this case, a relatively large Zeeman field $E_{Z,J} > \Delta$ is required.

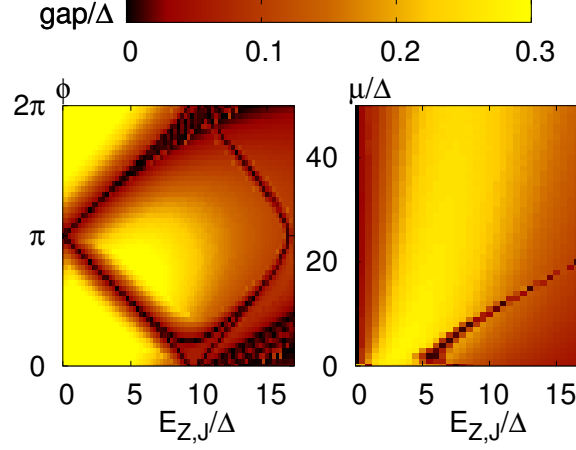


Figure 10: Induced gap as function of system parameters evaluated in the continuum model using the scattering matrix approach (see Appx. 13.9.1 for details) for $W = 1(m\Delta)^{-1/2}$, $m\alpha^2 = 9\Delta$, and $E_{Z,L} = 0$. In the left panel $\mu/\Delta = 20$. The diamond-shaped gap closing lines indicate the boundary between the trivial and the topological regions. Additional regions of small gap occur in the vicinity of BDI phase transitions, where the gap closes at nonzero momenta. In the right panel $\phi = \pi$ and a sizable topological gap is obtained in a very broad range of Zeeman fields with hardly any dependence on the chemical potential.

We next discuss the spectrum for non-zero k_x .² In this case, spin is no longer a good quantum number, and spin-orbit coupling can not be gauged out. We denote the magnitude of the Fermi momentum on the inner (outer) Fermi surface in presence of spin-orbit coupling by $k_{F,1(2)} = k_F \mp k_{SO}$, where $k_F = (2m\mu)^{1/2}$ and $k_{SO} = m\alpha$. For a given k_x , we denote the y component of the Fermi momenta on the two Fermi surfaces by $k_{F,i,y} = (k_{F,i}^2 - k_x^2)^{1/2} = k_{F,i}\sin\theta_i$, where $\theta_i = \cos^{-1}(k_x/k_{F,i})$. We note that for a given k_x , the spins of the electrons on the two Fermi surfaces are no longer orthogonal. Therefore, when, e.g., an electron in the vicinity of the inner Fermi surface is Andreev reflected from the superconductor, it will be reflected as a superposition of holes from both the inner and the outer Fermi surfaces. However, in the limit of small spin-orbit coupling, $\alpha k_F \ll \mu$, the overlap between the spins on the different Fermi surfaces remains small. (This is assuming that the Zeeman field does not alter the Rashba induced spin-momentum locking, i.e. $E_{Z,J} \ll \alpha k_F$.) In the opposite limit of large spin-orbit coupling, a large momentum transfer $\delta k_y = k_{F,2,y} - k_{F,1,y}$ is required for such a process. If $\delta k_y \gg \Delta/v_{F,1,y}$, such scattering is suppressed. We conclude that Andreev reflection between different Fermi surfaces can be neglected if $E_{Z,J}, \Delta \ll \alpha k_F$. In this case the scattering equations for the two spin species (corresponding to the two Fermi surfaces) can still be decoupled.

To write down the scattering equation we need to determine the phase shift acquired by an electron (or a hole) upon crossing the normal region of the 2DEG. To this end, we use the plane wave basis along y and diagonalize the Hamiltonian (13.1) in the normal region $|y| < W/2$. The resulting spectrum for the electrons is given by

$$E = \frac{k_x^2}{2m} + \frac{k_y^2}{2m} - \mu + \frac{m\alpha^2}{2} \pm \sqrt{\alpha^2 k_x^2 + (E_{Z,J} - \alpha k_y)^2}. \quad (13.11)$$

The spectrum for the holes can be obtained using particle-hole symmetry. We see that the energy shift of an electron (or a hole) on Fermi surface i due to the Zeeman field, to first order in $E_{Z,J}$, is given by $\Delta E_i \simeq E_{Z,J}\sin\theta_i$. Therefore, the phase accumulated when traversing

²Note that the spectrum is symmetric in k_x due to mirror-symmetry with respect to the $y-z$ plane, and hence we can consider only $k_x > 0$.

the junction, $(\Delta E_i/v_{F,i,y})W = (E_{Z,J}/v_F)W$, is the same for the two Fermi surfaces and is independent of k_x .

We conclude that the scattering equation for the bound states at nonzero k_x is given by Eq. (13.6) with $E_T \rightarrow E_{T,i}(k_x) = (\pi/2)(v_{F,i,y}/W)$ and $E_{Z,J} \rightarrow E_{Z,J} \sin \theta_i$ (such that the ratio $E_{Z,J}/E_T$ is left unchanged). Hence, the energies are given by Eq. (13.10) with the same substitution. Note that the k_x -dependent Thouless energy decreases with increasing k_x . Once $E_{T,i}(k_x)$ becomes smaller than Δ , multiple bound states appear and the gap at k_x becomes governed by $E_{T,i}(k_x)$. As k_x approaches $k_{F,i}$ the gap is reduced to be of order $1/(mW^2)$.

At high values of the Zeeman field, the gap can be further limited by the following effect. In the normal state, $\Delta = 0$, and in the presence of a non-uniform Zeeman field, $E_{Z,J} > E_{Z,L}$, a potential well is formed by the Zeeman energy in the normal region. The depth of this potential depends on k_x . In the limit $E_{Z,J} \ll \alpha k_F$, the potential at $k_x \simeq k_F$ is equal to $-E_{Z,J}^2/(\alpha k_F)$, as can be seen from Eq. (13.11). States at momenta close to k_F , bound by this potential, can lead to a suppression of the superconducting gap, once the characteristic length for the decay of their transverse wavefunction, ξ_B , becomes smaller than W . In this regime, the decay length is given by $\xi_B \simeq [2mE_{Z,J}^2/(\alpha k_F)]^{-1/2}$. The discussion above suggests, that the optimal gap in the system is obtained at $\phi = \pi$ and $E_{Z,J} = E_T$ and is equal to $\min\{\Delta, (1/mW^2)\}$. However, if the Zeeman field for which ξ_B becomes smaller than W is smaller than E_T , the optimal gap can be suppressed. The value of the Zeeman field at which $\xi_B \sim W$ is given by $E_{Z,J}^c = E_T(\alpha/v_F)^{1/2}$. This allows us to obtain a lower bound on the optimal gap in the system. In a narrow junction, $\Delta \lesssim 1/(mW^2)$, the gap at $\phi = \pi$ for $E_{Z,J} < E_{Z,J}^c$ is given by $\Delta \sin[(\pi/2)E_{Z,J}/E_T]$. At $E_{Z,J} = E_{Z,J}^c$ this gives a gap of order $\Delta(\alpha/v_F)^{1/2}$. Thus a gap of order Δ can be reached for $v_F \lesssim \alpha$ but due to the slow, power-law dependence, the system has a sizable gap also for larger values of v_F . In a wide junction, $\Delta \gg 1/(mW^2)$, the superconducting gap will, in fact, be effected by a finite ξ_B only once it becomes smaller than $\xi = 1/(mW\Delta) \ll W$. This leads to a much looser constraint on the Fermi velocity, allowing for a gap of order $1/(mW^2)$ as long as $v_F/\alpha < W/\xi$.

We conclude that a topological gap of order Δ can be obtained if the junction is narrow, $\Delta \lesssim 1/(mW^2)$, and the chemical potential is such that $v_F \lesssim \alpha$. For a wider junction the size of the topological gap is governed by $1/(mW^2)$. The optimal gap is obtained for $\phi = \pi$ and $E_{Z,J} \lesssim E_T$.

To complement this analysis we calculate the spectrum of the system as function of k_x across the phase transition, using a tight binding version of the model, given in Appendix 13.9.2, and plot it in Fig. 9. It can be seen that both in the trivial and the topological regions the smallest gap occurs at $k_x \simeq k_{F,i}$, in agreement with the discussion above. At the phase transition, the gap at $k_x = 0$ closes, and it is in fact close to zero also for other k_x .

In addition we calculate the gap numerically using the scattering matrix approach (see Fig. 10). We consider a narrow junction with $\Delta \simeq 1/(mW^2)$ and find that a sizable gap of order Δ can indeed be obtained for $\phi = \pi$ with very weak dependence on the chemical potential.

13.3.4 Majorana End Modes

In the topological phase we expect the system to host Majorana bound states at its ends. To verify the appearance of these zero-energy bound states in the proposed setup we calculate the local density of states (LDOS) close to the boundaries of the system. To this end we diagonalize a tight-binding version of the Hamiltonian in Eq. (13.1) with boundaries both along the x and the y dimensions (for details of the model see Appendix 13.9.2). The resulting LDOS as function of the phase difference is shown in Fig. 11. Indeed, a zero-energy state is present at the end of the junction in a finite range of phase differences around $\phi = \pi$.

Note that in presence of the effective time-reversal symmetry discussed in Sec. 13.3.2, mul-

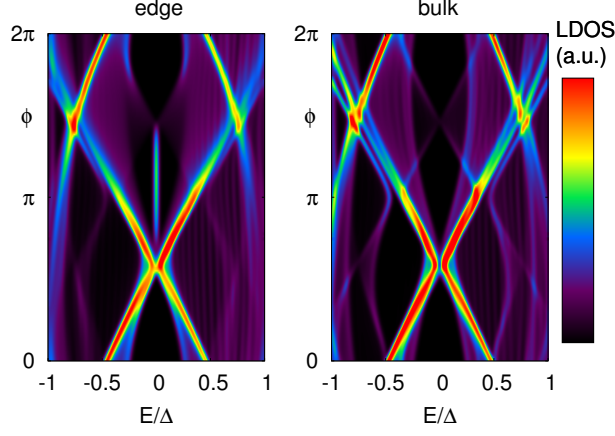


Figure 11: Local density of states at the edge (left panel) and in the center (right panel) of the junction as a function of energy and phase difference. In a range around $\phi = \pi$ a Majorana state forms at the edge. The result is obtained numerically from a tight-binding model (see Appendix 13.9.2) using the following parameters (energies and length are in units of the hopping strength and lattice spacing): $\alpha = 0.5$, $E_{Z,J} = E_{Z,L} = 0.1$, $\Delta = 0.25$, $\mu = -3.75$ (measured from the center of the tight-binding band), junction width $W = 4$, width of the superconducting leads $W_{SC} = 8$, length $L = 200$. We plot a spatial average of the density of states over a rectangle spanning the entire width of the junction in the y directions and the first 10 sites from the edge (left panel) or the most central 10 sites (right panel) in the x direction. For presentation the local density of states has been convoluted with a Gaussian with a standard deviation of 0.02Δ .

multiple Majorana bound states will appear at each end of the system. The number of zero-energy states in this case will be determined by the BDI $\mathbb{Z}$ invariant.

13.4 First order topological phase transitions and the critical current

In this section we show that if the phase difference is not imposed externally, the system will self-tune into the topological phase in a wide range of Zeeman fields. Using the bound state spectrum obtained in Sec. 13.3 we calculate the ground state energy of the system and the Josephson current in the junction. At a critical value of the Zeeman field the system undergoes a first order phase transition, in which the ground state of the junction switches between values of ϕ corresponding to the trivial and the topological phases, and that this transition is accompanied by a minimum of the critical current.

To this end, we need to sum over the contributions of all k_x to the ground state energy. In the analysis of the gap presented in Sec. 13.3.3 we found that in the limit $\Delta \ll \mu$, as well as $E_{Z,J} \ll \alpha k_{F,1/2}$, and assuming $E_{Z,L} = 0$ and no normal reflection, the spectrum for $k_x < k_{F,1}$ is given by Eq. (13.10), with $E_T \rightarrow E_{T,i}(k_x)$ and $E_{Z,J} \rightarrow E_{Z,J} \sin \theta_i$ (such that the ratio $E_{Z,J}/E_T$ is left unchanged). For $k_{F,1} < k_x < k_{F,2}$, there is only a single spin species present in the system and thus only half of the bound states remain.

We first calculate the ground state energy and the critical current in the limit $\alpha k_F \ll \mu$. In this limit $(k_{F,2} - k_{F,1})/k_F = 2k_{SO}/k_F \rightarrow 0$, and we can therefore neglect the contribution of momenta in the range $k_{F,1} < k_x < k_{F,2}$. We later relax this constraint and discuss how the results are altered.

We first focus on the limit of an ultra-narrow junction with a single bound state (for each spin species) for all k_x , i.e. $\Delta \ll 1/(mW^2)$, and consider the contribution of a single k_x to the ground

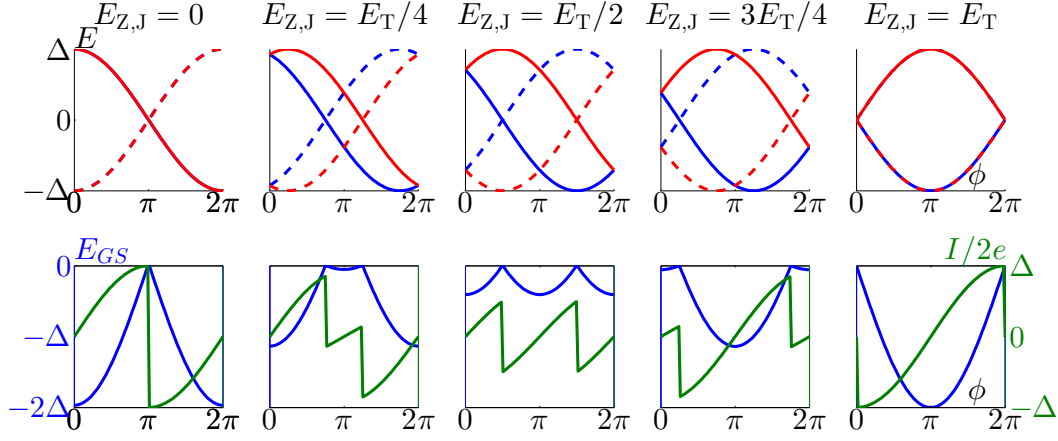


Figure 12: The upper panel shows the bound state energies of the two spin species (plotted in red and blue) and the energies of their particle-hole symmetric states (indicated by dashed lines) for a single momentum $k_x < k_{F,1}$ in a narrow junction, as the Zeeman field is varied. The contribution to the ground state energy (obtained by summing over the negative energy states) and the Josephson current are plotted for each value of Zeeman field in the lower panel in blue and green, respectively. At $E_{Z,J} = E_T/2$ the value of ϕ for which the energy is minimized shifts from 0 to π . This transition is accompanied by a minimum in the critical current.

state energy. In this limit, the dominant contribution to the ϕ -dependent part of the ground state energy (and thus also to the Josephson current) comes from the Andreev bound states [15]. We denote the bound state energies by $E_{\pm} = \Delta \cos(\phi_B \pm \phi/2)$, where $\phi_B = (\pi/2) E_{Z,J}/E_T$ is used as a shorthand notation (note that ϕ_B is independent of k_x). The ground state energy is obtained by summing over the negative energy states, i.e. $E_{GS} = -|E_+| - |E_-|$. The spectra of the bound states as well as the resulting ground state energy are plotted in Fig. 12 for different values of ϕ_B . We note that at $E_{Z,J} = (n + 1/2) E_T$ the value of ϕ for which the energy is minimized switches between $\phi = 0$ and $\phi = \pi$. Since the energy dependence on ϕ and the Zeeman field in this case is the same for all k_x , we conclude that at $E_{Z,J} = (n + 1/2) E_T$ the ground state of the entire system switches from between $\phi = 0$ and $\phi = \pi$. Note that in this transition the fermion parity of the $k_x = 0$ mode changes, indicating a transition into the topological phase. This is a first order phase transition without a gap closing.

We next calculate the critical current in the junction in the same limit. At zero temperature, the Josephson current is given by $I(\phi) = 2e \frac{d}{d\phi} E_{GS}$. We note that the maximum of the Josephson current is obtained at the same value of ϕ for all k_x . We can therefore calculate the critical current in the system, $I_c = \max |I(\phi)|$, as function of the Zeeman field, based on a single k_x . Due to the relative phase shift in the bound-states spectra of the two spins, the critical current of a single momentum is modulated as the Zeeman field is varied and is equal to $I_c = 2e\Delta \max \{\cos^2 \phi_B, \sin^2 \phi_B\}$ (see lower panel of Fig. 12). The maximal value, $I_{c,\max} = 2e\Delta$ is obtained for $\phi_B = \pi n/2$, or equivalently $E_{Z,J} = nE_T$ and the minimal one, $I_{c,\min} = I_{c,\max}/2 = e\Delta$ is obtained for $\phi_B = (\pi/2)(n + 1/2)$, or equivalently $E_{Z,J} = (n + 1/2) E_T$ [28]. Note that the minima of the critical current occur exactly at $0 - \pi$ transitions of the junction. The value of the Zeeman field at which this transition takes place is in agreement with the semiclassical argument given in Sec. 13.2. However, we see that the critical current does not vanish at these points.

When a finite temperature is considered, the Josephson current is given by $I(\phi) = 2e \frac{d}{d\phi} F$, where F is the free energy of the system. In the high temperature limit, $T \gg \Delta$, we obtain

(see Appendix 13.7)

$$I(\phi) \simeq -\frac{4e}{T} \sum_n E_n \frac{dE_n}{d\phi} = 2e \frac{\Delta^2}{T} \cos(2\phi_B) \sin(\phi). \quad (13.12)$$

We find that only the first harmonic of the Josephson current is left. The critical current is proportional to $\cos(2\phi_B)$ and is hence zero for $\phi_B = (\pi/2)(n + 1/2)$. The suppression of the minimum of the critical current with temperature can be seen in Fig. 4(b). The semiclassical result is thus recovered in the high temperature limit. This is due to the fact that higher harmonics of the critical current, which correspond to multiple Andreev reflections in the junction that are not accounted for in the semiclassical argument, are suppressed in the high temperature limit.

We next lift the constraint of an ultra-narrow junction. More specifically, we assume that for some k_x , $\Delta \ll E_{T,i}(k_x)$. In the limit of $k_{SO} \ll k_F$, we have $E_{T,1}(k_x) \simeq E_{T,2}(k_x)$ and we therefore suppress the band index below. The maximal supercurrent is still obtained at the same value of ϕ for all k_x as will be clear from the analysis below. Therefore, we can once again calculate the contribution of a single momentum k_x to the critical current of the system, by considering the supercurrent due to that momentum only. In this case the contribution of the states above the gap to the energy and the Josephson current can not in general be neglected [29]. However, taking the limit $\Delta \rightarrow \infty$ allows us to consider only the bound states. Following the derivation in Ref. [30], we find that the Josephson current in presence of the Zeeman field is given by

$$I(k_x, \phi) = 8eT \sum_{p=1}^{\infty} (-1)^{p+1} \frac{\cos(2p\phi_B) \sin(p\phi)}{\sinh(\pi^2 p T / E_T(k_x))}. \quad (13.13)$$

At zero temperature we obtain

$$I(k_x, \phi) = \frac{8e}{\pi^2} E_T(k_x) \sum_{p=1}^{\infty} (-1)^{p+1} \frac{\cos(2p\phi_B) \sin(p\phi)}{p}. \quad (13.14)$$

We note that since $E_T(k_x)$ decreases with increasing k_x the contribution of larger k_x to the critical current is smaller. For $\phi_B = \pi n/2$ the sum over p converges to a 2π periodic sawtooth function, and the critical current is maximal and equal to $I_{c,\max}(k_x) = 4eE_T(k_x)/\pi$. For $\phi_B = (\pi/2)(n + 1/2)$, all the odd harmonics are absent and we obtain a π periodic sawtooth function of half the amplitude, i.e. $I_{c,\min}(k_x) = I_{c,\max}(k_x)/2 = 2eE_T(k_x)/\pi$. Note that also in this case the minima of the critical current occur at the values of the Zeeman field for which the minimum of the energy switches between being at $\phi = 0$ and $\phi = \pi$, as can be seen by integration of the Josephson current over ϕ .

In the high temperature limit, which in this case corresponds to $T \gg E_T(k_x)$, once again only the first harmonic is left:

$$I(k_x, \phi) = 4eT e^{-\pi^2 T / E_T(k_x)} \cos(2\phi_B) \sin(\phi), \quad (13.15)$$

resulting in a vanishing current for $\phi_B = (\pi/2)(n + 1/2)$. Note also that the critical current contribution from larger k_x is suppressed more strongly at finite temperatures.

We now lift the constraint $\alpha k_F \ll \mu$, and consider the contribution of momenta $k_{F,1} < k_x < k_{F,2}$ to the ground state energy. For simplicity, in this analysis, we will once again consider the limit of an ultra-narrow junction, $\Delta \ll 1/(mW^2)$. For $k_x > k_{F,1}$, there is a single spin species in the system and the energy of the corresponding bound state is given by $E_- = \Delta \cos(\phi_B - \phi/2)$ (assuming $\alpha > 0$). Upon integration of the energy over k_x from $-k_{F,2}$ to $k_{F,2}$ we obtain

$$E_{GS} = -\frac{\Delta L}{\pi} \left(\left| \cos\left(\frac{\phi}{2} + \phi_B\right) \right| k_{F,1} + \left| \cos\left(\frac{\phi}{2} - \phi_B\right) \right| k_{F,2} \right). \quad (13.16)$$

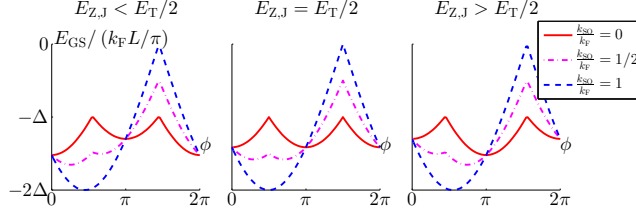


Figure 13: The ground state energy of the junction as function of the phase difference ϕ in the limit of a narrow junction with a single bound state for all k_x (see Eq. (13.16)). For $k_F > k_{SO}$, at $E_{Z,J} = E_T/2$, the system undergoes a first order phase transition as the ground state shifts between the two local minima. In the left panel $E_{Z,J} = 0.45E_T$ and the system is in the trivial phase in its ground state. In the right panel $E_{Z,J} = 0.55E_T$ and the system is in the topological phase in its ground state.

This function is depicted in Fig. 13 for several values of $E_{Z,J}$ and k_{SO}/k_F . Concentrating on $\phi_B \leq \pi/2$ and $\phi \leq \pi$, we find that this function can have two local minima at $\phi = \phi_{1,2}$ given by

$$\begin{aligned} \tan \frac{\phi_1}{2} &= \tan \phi_B \frac{k_{SO}}{k_F} & 0 \leq \phi_1 \leq \pi - 2\phi_B \\ \cot \frac{\phi_2}{2} &= \frac{k_{SO}}{k_F + (\tan \phi_B - 1) k_{SO}} & \pi - 2\phi_B \leq \phi_2 \leq \pi. \end{aligned} \quad (13.17)$$

At $\phi_B = \pi/4$, or equivalently $E_{Z,J} = E_T/2$, it can be shown that $\phi_1 + \phi_2 = \pi$, and that $E_{GS}(\phi_1) = E_{GS}(\phi_2)$. We can therefore conclude that at this value of the Zeeman field a first order phase transition occurs with the value of ϕ changing abruptly between ϕ_1 and ϕ_2 . As long as $k_{F,1} > 0$, or equivalently $k_F > k_{SO}$, we have $\phi_1 < \pi/2$ and $\phi_2 > \pi/2$ at the transition point. Therefore the system is in the trivial phase on one side of the transition and is in the topological phase on its other side.

Finite temperature will smoothen the cusp in the ground state energy of the system as function of the phase difference. However, for low enough temperatures two local minima in the free energy still exist allowing for a first order phase transition between them as the Zeeman field is varied.

We note also that in presence of normal reflection the values of the Zeeman field for which the phase transitions will occur, as well as the values for which the critical current will be minimal, will generically shift away from $E_{Z,J} = (n + 1/2) E_T$ and might no longer coincide. However, as long as the normal reflection probability is not too large, we expect these deviations to be small.

Tight-binding calculations of the critical current, complementing this analysis, were presented in Fig. 4(b). Note that these calculations are performed in a different regime, where $E_{Z,J} \gtrsim \alpha k_F$. Nevertheless, we find that a first order topological phase transition still occurs from ϕ_{GS} close to zero to ϕ_{GS} close to π .

13.5 Discussion

We have shown that one-dimensional topological superconductivity can be realized in a Josephson junction across a 2DEG with Rashba spin-orbit coupling and in-plane magnetic field. Once the phase difference between the superconductors is set to π , a ballistic junction is driven into the topological phase without any further fine tuning. If the phase is not set externally, the system can self tune into the topological phase for a range of in-plane magnetic fields. In this case the modulation of the critical current serves as a diagnostics of the phase transitions.

In practice, the system parameters should be chosen in a way to optimize the gap Δ_{top} protecting the topological phase. We find that narrow junctions with $1/(mW^2) \simeq \Delta$ allow for a gap of order Δ in the topological phase. To reach this limit the chemical potential should ideally not exceed the spin-orbit energy $m\alpha^2$, although we find sizable gaps even for larger values of μ , as the gap decays at most as $\Delta_{\text{top}} \sim \Delta(\alpha/v_F)^{1/2}$. The width of the junction also dictates the magnitude of the Zeeman field required to be close to the center of the topological phase, $E_{Z,J} \lesssim E_T$, i.e. a large Zeeman field is required if the junction is narrow. Moreover, we assume in our estimate of the gap that the Zeeman field does not interfere with the Rashba-induced spin-momentum locking, i.e. that $E_{Z,J} \ll \alpha k_F$. Thus materials with large spin-orbit coupling are favorable.

Although the orbital effects of the in-plane field have not been discussed in the manuscript, we note that nonzero magnetic field in the region between the superconducting leads and the 2DEG can give rise to a spatial modulation of the superconducting order parameter and destroy the gap in the system. Moreover, this effect can lead to oscillations of the critical current as function of the magnetic field that are not of topological origin. We further elaborate on this in Appendix 13.8.

While we expect the topological phase to be stable to a certain amount of disorder, the system will eventually enter a trivial phase at strong disorder. It would be interesting to compare the effects of disorder with topological superconductors based on semiconductor nanowires. The latter are restricted to relatively small chemical potentials, where the effects of disorder is particularly severe. This indicates that topological phases in planar Josephson junctions, for which this restriction does not exist, could be more resilient to disorder.

Note added: While we were preparing this manuscript we became aware of Ref. [31] which discusses topological superconductivity in a similar setup, as well as Ref. [32] which analyzes surface states of nanowires with some relation to our results.

13.6 Phase diagram with normal reflection

We now turn to an estimate of the phase boundaries in the presence of normal reflection. We assume that the mean free path exceeds the width of the junction so that normal reflection is limited to the superconducting regions of the 2DEG and to the superconducting-normal interface. Normal reflection can arise when μ and Δ are of the same order or when the width of the superconducting segment is comparable to the superconducting coherence length. Moreover, in experiments the proximity-providing superconductor may dope the proximitized part of the semiconductor with additional carriers due to a difference in work functions. The corresponding difference in chemical potential causes a momentum mismatch between superconducting and normal 2DEG regions which introduces normal reflection at the superconducting-normal interface.

We focus on the scattering problem at zero energy, as we are only interested in the phase diagram. In the presence of normal reflections the scattering matrix of the left (right) normal-superconducting interface $S_{L/R}$ has the form

$$S_{L/R}(\phi) = e^{\pm i\phi/2\tau_z} S e^{\mp i\phi/2\tau_z}; \quad S = \begin{pmatrix} r_e & r_A \\ r_A & r_h \end{pmatrix} \quad (13.18)$$

where $r_{e/h}$ is the normal reflection amplitude for electrons (holes). The subgap spectrum can be obtained from the condition

$$\det(\mathbb{1} - S_L T S_R T) = 0, \quad (13.19)$$

where $T = \text{diag}(t_e, t_h)$ is the transmission matrix of the junction, $t_{e/h} = \exp(ik_{e/h}W)$, with $k_{e/h}$ the electron (hole) wavevector in the normal junction. The scattering amplitudes are

constrained by unitarity, and can be parametrized as

$$\begin{aligned} r_{e/h} &= \pm r \exp(i\eta \pm i\varphi_N) \\ r_A &= (1 - r^2)^{1/2} \exp(i\eta). \end{aligned} \quad (13.20)$$

The phase η depends on the superconducting gap Δ , and the phase φ_N depends on details of the normal reflection. After a straightforward calculation we can rewrite the condition for a subgap state as

$$\cos(2\theta_- + 2\eta) = r^2 \cos(2\theta_+ + 2\varphi_N) + (1 - r^2) \cos \phi \quad (13.21)$$

where we have introduced $\theta_{\pm} = (k_e \pm k_h)W/2$. We can solve this equation in several limiting cases:

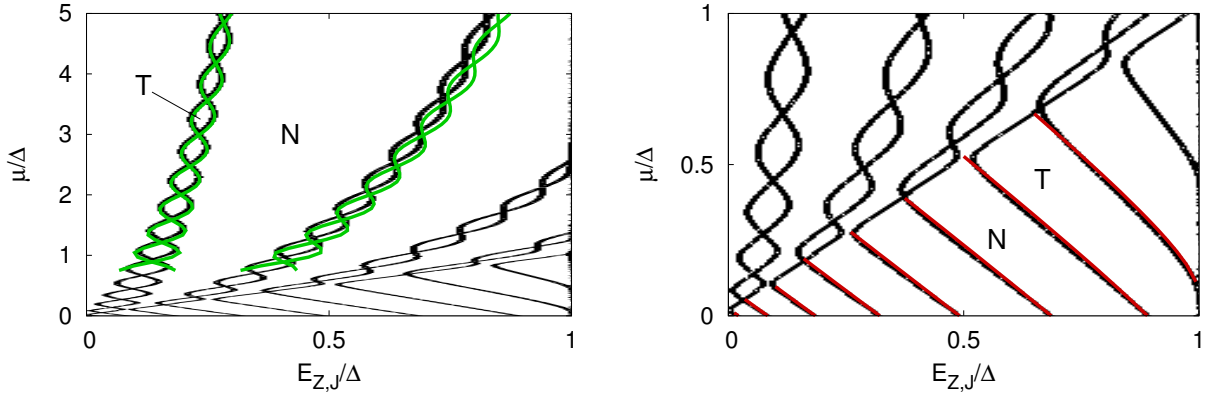


Figure 14: Numerical phase diagram and analytical estimates for two limiting cases. The green lines show the solution in the limit $B \ll \mu, \Delta$ given by Eq. (13.22) expanded to linear order in B/Δ . The normal reflection in this limit is given by $r \simeq \Delta/2\mu$ and $\varphi_N \simeq 0$. In the opposite limit $\mu \ll B < \Delta$ (red line) we use Eq. (13.28). Both panels show the same data for a width of $W = 14(m\Delta)^{-1/2}$ and $\phi = 0$.

(i) $\mu \gg E_{Z,J}$, weak normal reflection: In this limit we can use Eq. (13.5) for $t_{e/h}$. The phases $\theta_{\pm}$ are then simply given by $k_F W$ and $E_{Z,J}W/v_F = \phi_B$, respectively. The phase η can be expanded as $\eta = \arccos[(E - E_{Z,L})/\Delta] + O(r^2)$. In the case $E_{Z,L} = 0$, the condition above for the subgap states reduces to Eq. (13.6), which was used to describe the bound states in the absence of normal reflection, with ϕ replaced by $\tilde{\phi}$ defined in Eq. (13.8) of the main text.

For $\phi = 0$ and weak normal reflections $r \ll 1$ the topological phase transitions are given by

$$\phi_B = \pi n - \arccos(E_{Z,L}/\Delta) \pm 2r \sin(k_F W + \varphi_N). \quad (13.22)$$

Hence in the presence of normal reflections topological phases are possible even at zero phase bias. The analytical result agrees well with numerical results shown in Fig. 14. Similarly, for $\phi = \pi$, the phase transitions are given by

$$\phi_B = (2n + 1)\frac{\pi}{2} - \arccos(E_{Z,L}/\Delta) \pm 2r \cos(k_F W + \varphi_N). \quad (13.23)$$

The corrections to the scattering phase are linear in r only at these two special values of ϕ . At other values $0 < \phi < \pi$ the corrections are of order r^2 . This can be seen by comparing Figs. 5(a-c). The oscillations of the phase boundaries for $\phi = \pi/2$ shown in panel (b) vanish more rapidly with increasing μ than those in (a) and (c).

A topological phase may be accessible even in very narrow junctions where $E_{Z,J}, \Delta \ll E_T$ when the Zeeman field in the lead is of order of Δ . Eq. (13.21) then yields the phase boundary

$$E_{Z,L} = \Delta \sqrt{1 - r^2 \sin^2(k_F W + \varphi_N) - (1 - r^2) \sin^2 \phi/2} \quad (13.24)$$

The result plotted in Fig. 6 as dashed lines. In sufficiently short junctions when $\mu \ll E_T$ we can set $\theta_{\pm} = 0$. Moreover, one can show in this case that $\varphi_N \simeq 0$ when normal reflections are weak. The phase boundaries then follow the well-known dispersion of Andreev bound states in a short junction $E_{Z,L} = \Delta \sqrt{1 - (1 - r^2) \sin^2 \phi/2}$.

(ii) $\mu \gg E_{Z,J}$, strong normal reflection: Normal reflection should ideally be avoided as it weakens the proximity effect and reduces the overall gap of the system. To illustrate the effect of increasing normal reflections we consider the extreme case $r \rightarrow 1$. For simplicity we also set $E_{Z,L} = 0$. The phase diagram then becomes independent of ϕ and we find

$$\phi_B \simeq \pm(k_F W + \varphi_N) - \eta + n\pi \quad (13.25)$$

The phase boundaries $E_{Z,J} \simeq \pm 2\mu + 2nE_T + \text{const.}$ form diamonds in the $E_{Z,J} - \mu$ plane. This trend can already be seen for rather weak normal reflection in Fig. 5(d). Thus as normal reflection becomes stronger, the phase space decomposes into similar-sized patches of topological and trivial phase which alternate with period $k_F W$ as a function of chemical potential. Note that when the normal reflection is strong, the system may be thought of as a wire of width W weakly coupled to two superconductors. Then, the period of the oscillations corresponds to the addition of a single transverse channel to the wire.

We conclude that normal reflection is generically detrimental to topological superconductivity. Even though normal reflection increases the phase space area of the topological phase at $\phi = 0$, the small patches make the topology vulnerable to potential fluctuations. Moreover tuning the topology with a phase bias becomes less efficient in the presence of normal reflection.

(iii) $\mu < E_{Z,J}$: The phase diagram has a qualitatively different behavior when the Zeeman energy exceeds μ as illustrated in Fig. 14. The normal system becomes half metallic in the regime $\mu < E_{Z,J}$ and thus only one spin component propagates in the normal region. Similar to case (ii), superconducting correlations inside the junction are suppressed. The phase diagram becomes largely independent of the phase difference and the induced gap is reduced.

We focus on the eigenspace $\sigma_x = -1$ of the Hamiltonian at $k_x = 0$ in Eq. (13.4). In this subspace the hole part of the wavefunction is evanescent even in the normal region. For simplicity we assume a junction wider than the decay length $W \sqrt{2m(E_{Z,J} - \mu)} \gg 1$ so that the transmission of holes through the normal part is strictly zero. The scattering matrix then only involves normal reflection of electrons whose reflection amplitude $\tilde{r}$ has unit modulus while the subgap spectrum is determined by its phase.

In this case Eq. (13.19) is modified and the condition for a subgap state becomes

$$1 - \tilde{r}e^{i\theta}\tilde{r}e^{i\theta} = 0 \quad (13.26)$$

and thus

$$\tilde{r}e^{i\theta} = \pm 1, \quad (13.27)$$

where $\theta = \sqrt{2m(E_{Z,J} + \mu)}W$ is the phase shift of electrons traversing the normal region.

When assuming $\mu \ll E_{Z,J}$ we can neglect the μ dependence of $\tilde{r}$. Calculation reveals the topological phase transitions

$$\mu = E_{Z,J} + \frac{1}{2m} \left(\frac{\varphi_r + (2n+1)\pi/2}{W} \right)^2 \quad (13.28)$$

with $\varphi_r = 2 \arctan[1/(1 + \sqrt{2E_{Z,J}/\sqrt{\Delta^2 - E_{Z,L}^2}})]$. This result is in excellent agreement with numerical calculations as shown in Fig. 14.

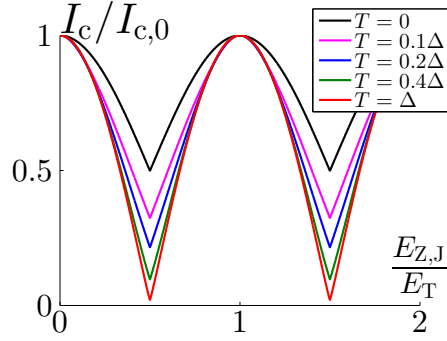


Figure 15: The critical current for the $k_x = 0$ mode in a narrow junction at different temperatures. As the temperature is increased the contrast of the modulations is increased with the minima at $E_{Z,J} = (n + 1/2) E_T$ becoming deeper.

13.7 Josephson current at finite temperature

To calculate the Josephson current at finite temperature we first calculate the many body partition function of the system. In presence of particle-hole symmetry it is given by

$$Z = \prod_n (1 + e^{-\beta E_n}) (1 + e^{\beta E_n}) = 4 \prod_n \cosh^2 \frac{\beta E_n}{2}, \quad (13.29)$$

where $\beta = T^{-1}$ and the product is taken over all the positive energy states labeled by n . The free energy is then

$$F = -T \ln Z = -\frac{8}{\beta} \sum_n \ln \left(\cosh \left(\frac{\beta E_n}{2} \right) \right) \quad (13.30)$$

and the Josephson current is

$$I(\phi) = 2e \frac{dF}{d\phi} = -8e \sum_n \tanh \left(\frac{\beta E_n}{2} \right) \frac{dE_n}{d\phi}. \quad (13.31)$$

In the high temperature limit with $\beta E_n \ll 1$ for all the bound states

$$I(\phi) = -4e\beta \sum_n E_n \frac{dE_n}{d\phi}. \quad (13.32)$$

To show the effect of finite temperature on the critical current we calculate the current for the $k_x = 0$ mode in a narrow junction with $\Delta \ll E_T$ by substituting the bound state spectrum for this case given in Eq. (13.10) into Eq. (13.31). Results are plotted in Fig. 15. It can be seen that the minima of the critical current grow deeper rapidly as the temperature is increased.

13.8 Orbital effect of the in-plane magnetic field

If the magnetic field below the superconducting leads is nonzero, it is important to examine also its orbital effect. We choose a gauge in which the vector potential is given by $\vec{A} = (0, 0, By)$. An electron tunneling between the 2DEG and the superconducting leads acquires a position dependent phase $t_\perp \sim e^{iA_z \tilde{d}} = e^{iB\tilde{d}y}$, where we denote by $\tilde{d} = d + \lambda_L$ the sum of the distance between the 2DEG and the superconductors and the London penetration depth. The induced order parameter therefore varies in space as $\Delta(y) = \Delta e^{iqy}$, where $q = 2B\tilde{d}$. If the correlation

length of the induced pairing is smaller than the width of the superconductors, W_{SC} , then the effective superconducting pairing will be

$$\bar{\Delta} = \frac{1}{W_{\text{SC}}} \int_0^{W_{\text{SC}}} \Delta e^{iqy} dy = \Delta e^{i\frac{qW_{\text{SC}}}{2}} \text{sinc}\left(\frac{qW_{\text{SC}}}{2}\right). \quad (13.33)$$

Hence for values of B equal to an integer multiple of $\pi/(\tilde{d}W_{\text{SC}})$ the superconducting gap will close, resulting, in particular, in a vanishing critical current.

13.9 Numerical calculations

13.9.1 Scattering matrix formalism

To evaluate the phase diagram and the gap of the Hamiltonian in Eq. (13.1) we employ a numerical method based on a scattering matrix approach [33, 34]. From the scattering matrix S we can obtain the bound state energies ϵ by solving $\det[\mathbb{1} - S(\epsilon)] = 0$. For the phase diagram in Fig. 5 we plot the lowest positive energy eigenvalue at $k_x = 0$ in a color scale, where energies below (above) $\epsilon = 0.01$ are plotted in black (blue). The gap in Fig. 10 is found by minimizing the lowest energy eigenvalue over all k_x .

13.9.2 Tight binding model

The tight-binding version of the Hamiltonian we use for various calculations throughout the manuscript is given by

$$H_{\text{TB}} = H_0 + H_{\text{SOC}} + H_Z + H_{\Delta} \quad (13.34)$$

$$H_0 = -\mu \sum_{sij} c_{i,j,s}^\dagger c_{i,j,s} - t \sum_{\langle ij,i'j'|ij,i'j' \rangle_s} \left[c_{i,j,s}^\dagger c_{i',j',s} + \text{h.c.} \right] \quad (13.35)$$

$$H_{\text{SOC}} = i\alpha \sum_{s,s'} \left[\sum_{i=1}^{L-1} \sum_{j=1}^{2W_{\text{SC}}+W} c_{i+1,j,s}^\dagger \sigma_y^{s,s'} c_{i,j,s'} - \sum_{i=1}^L \sum_{j=1}^{2W_{\text{SC}}+W-1} c_{i,j+1,s}^\dagger \sigma_x^{s,s'} c_{i,j,s'} - \text{h.c.} \right] \quad (13.36)$$

$$H_Z = \sum_{iss'} \left(E_{Z,J} \sum_{j=W_{\text{SC}}+1}^{W_{\text{SC}}+W} + E_{Z,L} \sum_{j=1}^{W_{\text{SC}}} + E_{Z,L} \sum_{j=W_{\text{SC}}+W+1}^{2W_{\text{SC}}+W} \right) \left(c_{i,j,s}^\dagger \sigma_x^{s,s'} c_{i,j,s'} + \text{h.c.} \right) \quad (13.37)$$

$$H_{\Delta} = \Delta e^{-i\phi/2} \sum_i \sum_{j=1}^{W_{\text{SC}}} c_{i,j,\uparrow} c_{i,j,\downarrow} + \Delta e^{i\phi/2} \sum_i \sum_{j=W_{\text{SC}}+W+1}^{2W_{\text{SC}}+W} c_{i,j,\uparrow} c_{i,j,\downarrow} + \text{h.c.}, \quad (13.38)$$

where $c_{i,j,s}$ is the annihilation operator of an electron spin s on site (i, j) with $1 \leq j \leq 2W_{\text{SC}}+W$ and $1 \leq i \leq L$ and $\langle \cdot, \cdot | \cdot, \cdot \rangle$ denotes nearest neighbors. The hopping and spin-orbit coupling strength are denoted by t, α respectively. Proximity induced pairing strength Δ is nonzero only in the leads $1 \leq j \leq W_{\text{SC}}$ and $W_{\text{SC}} + W < j \leq 2W_{\text{SC}} + W$. The Zeeman field along the x -direction has strength $E_{Z,L}$ ($E_{Z,J}$) in the leads (in the junction). This model has been used to calculate the local density of states in Fig. 11.

To describe an infinitely long junction we assume $L \rightarrow \infty$ and perform a partial Fourier transform $c_{j,k,s} = \sum_j e^{ikj} c_{j,i',s}$. The resulting Hamiltonian $H_{\text{TB}}(k)$ is used to calculate the BDI phase diagram in Fig. 7, the gap in the system when the effective time-reversal symmetry is broken in Fig. 8, the spectrum across the topological phase transition of class D in Fig. 9 and to demonstrate the first order phase transition accompanied by a minimum in the critical current in Fig. 4(b).

14 Coulomb blockade of Majorana fermion induced transport by Zazunov, Levy Yeyati, Egger

Abstract

We study Coulomb charging effects for transport through a topologically nontrivial superconducting wire, where Majorana bound states are present at the interface to normal-conducting leads. We construct the general Keldysh functional integral representation, and provide detailed results for the nonlinear current-voltage relation under weak Coulomb blockade conditions.

14.1 Introduction

Charge transport through topologically nontrivial materials is of great current interest, offering novel fundamental insights as well as potential applications in topological quantum computing.[1] The recently discovered topological insulators as well as topological superconductors (TSs)[2, 3] are predicted to exhibit spectacular nonlocal transport phenomena and have consequently attracted a lot of attention. In particular, for one-dimensional TS quantum wires, the crucial role of Majorana bound states (MBSs) located near the interface to topologically trivial regions has been emphasized.[4, 5, 6, 7, 8] Majorana fermions are special in that they are their own antiparticles, i.e., the fermion creation operator is equal to the annihilation operator, and the condensed-matter setup discussed below could offer experimental signatures for these elusive and hitherto unobserved particles.

We here study nonlinear electronic transport through a mesoscopic TS wire (also referred to as “dot”) containing a pair of MBSs. The wire is assumed to be contacted by conventional normal-conducting leads. The contacts correspond to thin insulating layers or barriers and will be modeled by a tunnel Hamiltonian. We include Coulomb interactions on the dot via the charging energy $E_c = e^2/(2C)$. The schematic setup studied in this work is shown in Fig. 171, and we briefly outline possible experimental realizations below. The noninteracting ($E_c = 0$) version of this setup was studied in detail before,[7, 9, 10, 11, 12] and one finds *resonant Andreev reflection* with the maximal (unitary) value $G_{L/R} = 2e^2/h$ of the linear conductance through the left/right contact. For the interacting case ($E_c > 0$), an important work by Fu[13] has provided a general framework to address transport in this setup. The Hilbert space constraint introduced in Ref. [13] is, however, difficult to handle in actual calculations. In fact, the concrete results presented in Ref. [13] (see also below) were mostly restricted to the strong Coulomb blockade (CB) regime realized for large E_c . We formulate an alternative Keldysh functional integral approach below, where the constraint is automatically satisfied. We give detailed results for the weak CB (small E_c) regime, where we find a pronounced interaction ($E_c \neq 0$) induced suppression of the current.

Experimental realizations for the setup shown in Fig. 171 arise under several distinct physical scenarios. For instance, a nanowire made of a 3D topological insulator (e.g., Bi_2Se_3) with proximity-induced superconducting correlations is predicted to have MBSs at both ends.[14] Similar setups can be realized in 2D topological insulators with “quantum spin Hall” edge states.[15, 13] Another example is based on the vortex core states in p -wave superconductors[9] or in 2D noncentrosymmetric superconductors.[16] Finally, the MBS should also be realizable in semiconductor quantum wires with a Zeeman field and proximity-induced superconductivity.[17] The stability of the MBS when including interactions in an isolated wire has also been addressed in several recent works.[18] However, so far no experimental signature for the MBS has been reported in any system.

The structure of this paper is as follows. In Sec. 14.2 we describe the model employed in our study and briefly show that our formulation reproduces known results[13] in the strong CB

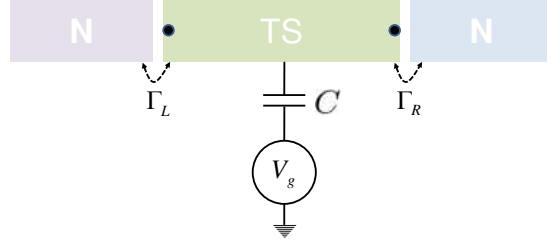


Figure 16: (Color online) Setup studied in this work: a topological superconductor (TS) dot containing Majorana bound states at the ends (indicated by black dots) is coupled via tunnel contacts (hybridizations $\Gamma_{L,R}$) to normal-conducting (N) metallic electrodes. The charging energy $E_c = e^2/(2C)$ causes phase fluctuations on the TS dot and influences the current-voltage relation. We assume that the superconducting gap Δ of the TS is large compared to the other energy scales, i.e., quasiparticle excitations on the dot can be neglected.

limit. We then construct a general Keldysh functional integral representation for the interacting problem in Sec. 14.3. This theory is employed in Sec. 14.4 to evaluate the current-voltage characteristics under weak CB conditions. We briefly summarize our main findings and provide an outlook in Sec. 24.0.5. Technical details and derivations can be found in three appendices. Note that we often use units where $\hbar = k_B = 1$.

14.2 Model

We consider transport through a mesoscopic TS quantum wire contacted by normal-conducting leads, see Fig. 171. We assume throughout this work that the superconducting gap Δ is the largest relevant energy scale, and thus no quasiparticles above the gap are involved. In particular, $\max(eV, E_c) < \Delta$. We study a “floating” (not grounded) TS dot, where current must be conserved under steady-state conditions. In some previous works, see, e.g., Ref. [9], a grounded superconductor was considered, where current need not be conserved. Note also that the situation is different for conventional (topologically trivial) superconducting dots, where transport is typically mediated by cotunneling processes and the resulting conductance is much smaller.[19] In the present case, two MBSs may host a fermion at no energy cost, and thus no even-odd parity effect[19] arises.

14.2.1 Model

For realistical TS wire length, the two MBSs located near the left/right ($j = L/R$) contact correspond to *decoupled* Majorana fermion operators, $\gamma_j = \gamma_j^\dagger$, with anticommutator relation $\{\gamma_i, \gamma_j\} = \delta_{ij}$. In the low-energy sector of interest here, only the MBSs provide available fermion states to tunnel through the superconductor. It is convenient to introduce the complex auxiliary fermion $d = (\gamma_L + i\gamma_R)/\sqrt{2}$, i.e.,

$$\gamma_L = (d + d^\dagger)/\sqrt{2}, \quad \gamma_R = -i(d - d^\dagger)/\sqrt{2}. \quad (14.1)$$

For an isolated wire, the fermion parity $(-)^n$ of the two degenerate ground states with even ($n = 2N$) or odd ($n = 2N + 1$) number of electrons fixes the occupation of the d fermion level,

$$2i\gamma_L\gamma_R = 2\hat{n}_d - 1 = (-)^{n+1}, \quad \hat{n}_d = d^\dagger d. \quad (14.2)$$

This imposes a constraint on the Hilbert space,[13] and the Majorana fermion dynamics cannot be taken as independent of the charge (or the dual phase) dynamics. Below, we instead formulate an alternative but essentially equivalent approach free of any Hilbert space constraint, see

also Ref. [20], which is technically easier to handle. To that end, consider the number operator $\hat{N}$ for Cooper pairs conjugate to the condensate phase χ on the dot, $[\chi, \hat{N}] = i$. The dot's instantaneous charge state is then fully determined by specifying the configuration (N, n_d) , where $N \in \mathbb{N}_0$ and $n_d = 0, 1$ are eigenvalues of $\hat{N}$ and $\hat{n}_d$, respectively. Importantly, Eq. (14.2) tells us that the fermion parity of the dot is determined by n_d only and not affected by changes of N . By construction, both $\hat{N}$ and χ commute with γ_j , and therefore no constraint on the Hilbert space arises.

The full Hamiltonian describing transport through the dot, $H = H_c + H_t + H_l$, contains the Coulomb charging term

$$H_c = E_c(2\hat{N} + \hat{n}_d - n_0)^2, \quad (14.3)$$

where $n_0 \in \mathbb{R}$ is tunable by the backgate voltage V_g indicated in Fig. 171. Electrons in the leads correspond to effectively spinless fermion operators $c_{j=L/R,k}$ with momentum k , where for given j , c_{jk} and $c_{jk}^\dagger$ are tunnel-coupled to the respective Majorana fermion γ_j only. For simplicity, we assume a k -independent tunnel matrix element λ_j encoding the overlap of the lead state with the respective MBS wavefunction. As detailed in App. 14.6, the tunnel Hamiltonian then takes the form

$$\begin{aligned} H_t = & \frac{1}{\sqrt{2}} \sum_k \left[\lambda_L c_{L,k}^\dagger (d + e^{-i\chi} d^\dagger) \right. \\ & \left. - i\lambda_R c_{R,k}^\dagger (d - e^{-i\chi} d^\dagger) \right] + \text{h.c.}, \end{aligned} \quad (14.4)$$

which is charge conserving. In particular, the different terms in H_t describe electron tunneling from the dot to lead j , either by destroying the d state without changing N , i.e., $(N, 1) \rightarrow (N, 0)$, or by occupying the d state together with a splitting of a Cooper pair, i.e., $(N, 0) \rightarrow (N - 1, 1)$, plus the conjugate processes. As elaborated in Appendix 14.7, we treat the lead Hamiltonian H_l within the standard wide-band approximation.[21, 22] The applied bias voltage V corresponds to the chemical potential difference in both leads, $eV = \mu_L - \mu_R$. For further reference, we also define the hybridization energy scales

$$\Gamma_j = 2\pi\nu_j|\lambda_j|^2, \quad (14.5)$$

where ν_j is the density of states in lead $j = L/R$.

It is now straightforward to rederive the well-known results[9, 10, 11, 12] for the noninteracting ($E_c = 0$) limit from the above Hamiltonian, see also Sec. 14.4. Before turning to the construction of the Keldysh functional integral for the interacting problem in Sec. 14.3, let us briefly show that the above formulation easily reproduces previous results[13] reporting electron “teleportation” under strong CB conditions, i.e., for $\max(eV, \Gamma_{L,R}) \ll E_c$.

14.2.2 Strong Coulomb Blockade

Under strong CB conditions, the charging energy (14.3) is dominant and needs to be examined first. We focus on the resonant situation realized for half-integer values for n_0 , where two different cases arise: (i) For $n_0 = 2\ell + 1/2$ with integer ℓ , degeneracy is achieved for fixed Cooper pair number $N = \ell$, where the d state is either occupied ($n_d = 1$) or empty ($n_d = 0$). (ii) When $n_0 = 2\ell - 1/2$, the degeneracy point is achieved for different Cooper pair numbers, namely for $(N, n_d) = (\ell - 1, 1)$ and $(\ell, 0)$. We conclude that the low-energy description needs to keep only two states, either of type (i) or (ii) depending on n_0 . For case (i), N remains constant and all terms $\propto e^{\pm i\chi}$ (where N is raised or lowered by one unit) can be omitted in Eq. (14.4). The tunnel Hamiltonian then takes the form

$$H_t^{(i)} = \frac{1}{\sqrt{2}} \sum_k \left(\lambda_L c_{L,k}^\dagger - i\lambda_R c_{R,k}^\dagger \right) d + \text{h.c.}$$

Under scenario (ii) only transitions $(\ell - 1, 1) \leftrightarrow (\ell, 0)$ are possible. Using $f = e^{-i\chi} d^\dagger$ as effective single-charge fermion, we arrive at

$$H_t^{(ii)} = \frac{1}{\sqrt{2}} \sum_k \left(\lambda_L c_{L,k}^\dagger + i \lambda_R c_{R,k}^\dagger \right) f + \text{h.c.}$$

In both cases, we recover the resonant tunneling model describing teleportation.[13] This leads to the unitary conductance value $G_L = G_R = e^2/h$ instead of the resonant Andreev reflection value $2e^2/h$ found when $E_c = 0$.

14.3 Keldysh functional integral

We now turn to the general Keldysh functional integral[21, 22] for the interacting problem. This corresponds to the Majorana generalization of the (real-time version of the) Ambegaokar-Eckern-Schön (AES) action for a conventional metallic grain.[23, 24, 25]

14.3.1 AES action for Majorana induced transport

The relevant low-energy degree of freedom characterizing the interacting problem is the single-electron (half Cooper pair) counting phase field $\phi(t) \equiv \chi/2$. For the nonequilibrium problem at hand, we employ the textbook Keldysh functional integral formulation[21, 22] and double the phase field $\phi(t) \rightarrow \phi_\pm(t)$ according to the forward and backward branch of the Keldysh time contour. Technically, the phase field is introduced through a Hubbard-Stratonovich transformation of the charging energy in Eq. (14.3), see App. 14.7 for details. Physically, ϕ is the dual variable to the Cooper pair charge N . We then introduce classical/quantum phase fields,[24, 25]

$$\phi_c(t) = \frac{1}{2}(\phi_+ + \phi_-), \quad \phi_q(t) = \phi_+ - \phi_-. \quad (14.6)$$

The charging energy (14.3) yields the action contribution

$$S_c = \int dt \dot{\phi}_q \left(\frac{\dot{\phi}_c}{2E_c} + n_0 \right) \quad (14.7)$$

encoding Coulomb blockade effects. By virtue of the Hubbard-Stratonovich transformation, the fields corresponding to the lead fermions and to the Majorana fermions effectively become noninteracting. In the functional integral, those fields can therefore be integrated out analytically; for technical details, see Appendix 14.7. The final (phase representation of the) functional integral for the Keldysh partition function reads

$$\mathcal{Z} = \int \mathcal{D}(\phi_c, \phi_q) e^{i(S_c + S_f)}. \quad (14.8)$$

The fermion field integrations result in the action piece

$$S_f = -i \sum_{j=L/R} \ln \text{Pf} \left[\tilde{\tau}_x i \partial_t \delta(t - t') + \tilde{\Lambda}_j(t, t') \right], \quad (14.9)$$

where “Pf” denotes the Pfaffian and $\tilde{\tau}_{x,y,z}$ are Pauli matrices in (rotated[22]) Keldysh space. Moreover, we define the antisymmetric (time and Keldysh) matrix

$$\begin{aligned} \tilde{\Lambda}_j(t, t') &= 2i\Gamma_j F(t - t') \begin{pmatrix} c(t)c(t') \cos \Phi_j(t, t') & c(t)s(t') \sin \Phi_j(t, t') \\ -s(t)c(t') \sin \Phi_j(t, t') & s(t)s(t') \cos \Phi_j(t, t') \end{pmatrix} \\ &+ i\Gamma_j \begin{pmatrix} 0 & c^2(t)\delta_-(t - t') - s^2(t)\delta_+(t - t') \\ -c^2(t)\delta_+(t - t') + s^2(t)\delta_-(t - t') & 0 \end{pmatrix} \end{aligned} \quad (14.10)$$

with $c(t) = \cos[\phi_q(t)/2]$, $s(t) = \sin[\phi_q(t)/2]$, and $\delta_{\pm}(t) \equiv \delta(t \pm 0^+)$, where the infinitesimal shifts reflect the proper causality features. At temperature T , the distribution function $F(t - t')$ has the Fourier transform

$$F(\epsilon) = \tanh[\epsilon/(2T)], \quad (14.11)$$

and the phase function Φ_j appearing in Eq. (14.10) is

$$\Phi_j(t, t') = \mu_j(t - t') + \phi_c(t) - \phi_c(t'). \quad (14.12)$$

Finally, the hybridizations Γ_j were defined in Eq. (24.23).

While Eq. (14.8), with the action in Eqs. (14.7) and (14.9), provides a general representation of the interacting nonequilibrium Majorana transport problem, approximations are necessary in order to obtain concrete analytical results.

14.3.2 Semiclassical expansion

We here are mostly interested in the weak CB regime, where standard arguments[24] imply that a semiclassical approximation in the phase representation is appropriate. In particular, fluctuations of ϕ_q around zero are small, i.e., we can expand the action in Eq. (14.8) to quadratic order in ϕ_q . Keeping the full nonlinear ϕ_c dependence, the matrix (14.10) is thus approximated by $\check{\Lambda}_{j=L/R} \simeq \check{\Lambda}_j^{(0)} + \check{\Lambda}_j^{(1)} + \check{\Lambda}_j^{(2)}$. Specifically, the matrices $\check{\Lambda}_j^{(m)}(t, t')$ of order ϕ_q^m are

$$\begin{aligned} \check{\Lambda}_j^{(0)} &= i\Gamma_j \begin{pmatrix} 2F(t - t') \cos \Phi_j(t, t') & \delta_-(t - t') \\ -\delta_+(t - t') & 0 \end{pmatrix}, \\ \check{\Lambda}_j^{(1)} &= i\Gamma_j F(t - t') \sin \Phi_j(t, t') \begin{pmatrix} 0 & \phi_q(t') \\ -\phi_q(t) & 0 \end{pmatrix}, \\ \check{\Lambda}_j^{(2)} &= -i\frac{\Gamma_j}{4} F(t - t') \cos \Phi_j(t, t') \begin{pmatrix} \phi_q^2(t) + \phi_q^2(t') & 0 \\ 0 & -2\phi_q(t)\phi_q(t') \end{pmatrix} + \frac{\Gamma_j}{2} \phi_q^2(t) \delta(t - t') \check{\tau}_y. \end{aligned}$$

At this stage, it is convenient to introduce the *interacting* Keldysh-Majorana Green's function (GF)

$$\check{G}_j(t, t') = \left(\check{\tau}_x i\partial_t + \check{\Lambda}_j^{(0)} \right)^{-1}. \quad (14.13)$$

This has a triangular representation in (rotated) Keldysh space, $\check{G}_j = \begin{pmatrix} 0 & G_j^A \\ G_j^R & G_j^K \end{pmatrix}$, with retarded/advanced GFs $G_j^{R/A}$ and Keldysh GF G_j^K . Note that in this representation, the Majorana GF has a bosonic Keldysh structure, see also Ref. [12]. According to Eq. (14.13), $G_j^{R/A}$ obeys the noninteracting ($\phi_c = 0$) equation of motion and thus has the Fourier representation $G_j^{R/A}(\epsilon) = 1/(\epsilon \pm i\Gamma_j)$. This implies the spectral function

$$A_j(\epsilon) = -\text{Im} [G_j^R(\epsilon)] = \frac{\Gamma_j}{\epsilon^2 + \Gamma_j^2}. \quad (14.14)$$

Only G_j^K is affected by interactions encoded in ϕ_c .

The action (14.9) now takes the form

$$S_f = -i \sum_j \ln \text{Pf} (\check{G}_j^{-1}) + S_f^{(1)} + S_f^{(2)} + O(\phi_q^3), \quad (14.15)$$

where the first ($\phi_q = 0$) term vanishes as a result of the unitary time evolution along the closed Keldysh contour. The first-order (in ϕ_q) contribution is

$$S_f^{(1)} = -\frac{i}{2} \sum_j \text{Tr} (\check{G}_j \check{\Lambda}_j^{(1)}) = \int dt \mathcal{I} \phi_q, \quad (14.16)$$

where the trace “Tr” extends both over time and Keldysh space, and $\mathcal{I}$ denotes the total current flowing into the dot. Some algebra yields

$$\mathcal{I}(t) = \sum_j \Gamma_j \int dt' G_j^R(t-t') \{F_{j,s}(t'-t) \cos[\phi_c(t') - \phi_c(t)] + F_{j,a}(t'-t) \sin[\phi_c(t') - \phi_c(t)]\} \quad (14.17)$$

with the time-symmetric (-antisymmetric) functions

$$F_{j,s}(t) = F(t) \sin(\mu_j t), \quad F_{j,a}(t) = F(t) \cos(\mu_j t). \quad (14.18)$$

Finally, the second-order term in Eq. (14.15) reads

$$S_f^{(2)} = -\frac{i}{2} \sum_j \text{Tr} \left[\check{G}_j \check{\Lambda}_j^{(2)} - \frac{1}{2} \left(\check{G}_j \check{\Lambda}_j^{(1)} \right)^2 \right]. \quad (14.19)$$

14.4 Weak Coulomb blockade limit

14.4.1 Langevin approach

When the charging energy E_c is sufficiently small, fluctuations of the phase fields $\phi_{c,q}$ are small and allow for a quadratic expansion of the action S_f in *both* fields.[24] This expansion is detailed in Appendix 14.8 and allows to reformulate the functional integral (14.8) in terms of the *equivalent* semiclassical Langevin equation,

$$\frac{1}{2E_c} \ddot{\phi}_c(t) + \int^t dt' \eta(t-t') \dot{\phi}_c(t') = \xi(t). \quad (14.20)$$

The *damping kernel* $\eta(t-t')$ has the Fourier representation

$$\eta(\omega) = \sum_{j=L/R} \frac{\Gamma_j}{\omega} \int \frac{d\epsilon}{2\pi} (A_j(\epsilon + \omega/2) + A_j(\epsilon - \omega/2)) [F_{j,a}(\epsilon + \omega/2) - F_{j,a}(\epsilon - \omega/2)]. \quad (14.21)$$

The Gaussian noise field $\xi(t)$ has zero mean value and the correlation function $\langle \xi(t)\xi(t') \rangle = K(t-t')$, where the *fluctuation kernel* has the Fourier representation

$$\begin{aligned} K(\omega) &= \sum_j \frac{\Gamma_j}{2} \int \frac{d\epsilon}{2\pi} (A_j(\epsilon + \omega/2) + A_j(\epsilon - \omega/2)) [1 - F_{j,a}(\epsilon + \omega/2)F_{j,a}(\epsilon - \omega/2)] \\ &- \sum_j \Gamma_j^2 \int \frac{d\epsilon}{2\pi} \text{Re} [G_j^R(\epsilon + \omega/2)G_j^R(\epsilon - \omega/2)] F_{j,s}(\epsilon + \omega/2)F_{j,s}(\epsilon - \omega/2). \end{aligned} \quad (14.22)$$

In equilibrium ($\mu_{L,R} = 0$), $F_{j,s} = 0$ and $F_{j,a} = F$, and hence the fluctuation-dissipation relation

$$K_{\text{eq}}(\omega) = \frac{\omega}{2} \coth\left(\frac{\omega}{2T}\right) \eta_{\text{eq}}(\omega) \quad (14.23)$$

is satisfied, providing an important consistency check.

In the zero-temperature limit, both $\eta(\omega)$ and $K(\omega)$ can be evaluated in closed form. Specifically, we find for the damping kernel

$$\eta_{T=0}(\omega) = \sum_{j,\pm} \frac{\Gamma_j}{\pi\omega} \tan^{-1} \left(\frac{\omega \pm \mu_j}{\Gamma_j} \right), \quad (14.24)$$

while the fluctuation kernel is

$$K_{T=0}(\omega) = \frac{|\omega|\eta(\omega)}{2} + \sum_j \frac{\Gamma_j}{2\pi} \Theta(2|\mu_j| - |\omega|) \left[\tan^{-1} \left(\frac{|\mu_j|}{\Gamma_j} \right) + \tan^{-1} \left(\frac{|\mu_j| - |\omega|}{\Gamma_j} \right) + \frac{\Gamma_j}{|\omega|} \ln \frac{(|\mu_j| - |\omega|)^2 + \Gamma_j^2}{\mu_j^2 + \Gamma_j^2} \right] \quad (14.25)$$

where Θ is the Heaviside function.

The Langevin equation (14.20) allows for an intuitive interpretation of the charge dynamics in this interacting Majorana system in terms of an effective RC circuit. Its solution is used below in order to compute the current-voltage relation.

14.4.2 Current-voltage relation

Next we outline the calculation of the dc current, $I \equiv I_L = -I_R$. By using Eq. (14.17) and the noise average $\langle \cdots \rangle_\xi = \int \mathcal{D}\xi (\cdots) e^{-\frac{1}{2}\xi K^{-1}\xi}$, we find the (indeed t -independent) result

$$I_j = \Gamma_j \int dt' G_j^R(t - t') \left[F_{j,s}(t' - t) \langle \cos[\bar{\phi}_c(t') - \bar{\phi}_c(t)] \rangle_\xi + F_{j,a}(t' - t) \langle \sin[\bar{\phi}_c(t') - \bar{\phi}_c(t)] \rangle_\xi \right].$$

$F_{j,a/s}$ is given in Eq. (14.18), and

$$\bar{\phi}_c(t) = \int^t d\tau D^R(t - \tau) \xi(\tau) \quad (14.26)$$

solves Eq. (14.20) for a given noise trajectory, where $D^R(t - \tau)$ is the retarded solution of Eq. (14.20) with $\xi(t) \rightarrow \delta(t - \tau)$. Performing the Gaussian noise average, we obtain

$$I_j = \Gamma_j \int dt' G_j^R(t - t') F_{j,s}(t' - t) e^{-J(t-t')}, \quad (14.27)$$

where the noise correlations are encoded in $J(t) = J(-t)$, similar to the standard “ $P(E)$ ” theory of dynamical CB,[26]

$$J(t - t') = \frac{1}{2} \left\langle [\bar{\phi}_c(t) - \bar{\phi}_c(t')]^2 \right\rangle_\xi. \quad (14.28)$$

Note that in order to obtain $J(t - t')$ one first has to solve the Langevin equation (14.20) for a given noise trajectory, and then average over all noise realizations.

Since $J(t - t') \geq 0$, interactions always decrease the current (14.27) with respect to the noninteracting solution, $I_j^{(0)}$. The latter follows by setting $J \rightarrow 0$ in Eq. (14.27). In fact, after Fourier transformation, we get the known result[7, 9, 10, 11, 12]

$$I_j^{(0)} = \Gamma_j \int \frac{d\epsilon}{2\pi} F(\epsilon - \mu_j) A_j(\epsilon) \quad (14.29)$$

with the Lorentzian spectral function in Eq. (14.14). In linear response ($\mu_j \rightarrow 0$) and taking the zero-temperature limit, we recover resonant Andreev reflection with quantized conductance $G_j = e\partial I_j/\partial \mu_j = 2e^2/h$.

Equation (14.27) for the current-voltage relation in the weak Coulomb blockade regime is one of the main results of this work. We use this expression later on to quantify the effect of Coulomb interactions ($E_c \neq 0$) on the transport properties in this Majorana system.

14.4.3 Retardation effects

The Langevin equation (14.20) in general contains memory effects because of the frequency-dependence of the damping kernel. These are most pronounced for $T = 0$, where $\eta(\omega)$ takes the form in Eq. (14.24). For $\omega \rightarrow 0$, this gives the finite value

$$\eta_0 = \frac{2}{\pi} \sum_j \frac{1}{1 + (\mu_j/\Gamma_j)^2}, \quad (14.30)$$

while for $|\omega| \rightarrow \infty$, we have $\eta(\omega) \simeq \Gamma/|\omega|$ with $\Gamma = \Gamma_L + \Gamma_R$. On low frequency scales, $|\omega| < \min \left(\sqrt{\mu_j^2 + \Gamma_j^2} \right)$, the kernel $\eta(\omega)$ can be approximated by $\eta(\omega) \simeq \eta_0$. This approximation corresponds to the absence of retardation in the damping kernel $\eta(t)$. This approximation works even better for finite T , where

$$\eta_0 = \sum_j \frac{\Gamma_j}{T} \int \frac{d\epsilon}{2\pi} \frac{1}{\cosh^2 \left(\frac{\epsilon}{2T} \right)} \frac{\Gamma_j}{(\epsilon - \mu_j)^2 + \Gamma_j^2} \quad (14.31)$$

yields a damping suppression, $\eta_0 \simeq \Gamma/(2T)$, in the high- T limit.

We therefore employ the simplification $\eta(\omega) = \eta_0$ from now on, which is appropriate to describe low-energy transport. This implies from Eq. (14.20) that we now have a Markovian Langevin equation,

$$\ddot{\phi}_c(t) + \Omega \dot{\phi}_c(t) = 2E_c \xi(t), \quad (14.32)$$

and $D^R(t)$ in Eq. (14.26) simplifies to

$$D^R(t) = \frac{1}{\eta_0} (1 - e^{-\Omega t}) \Theta(t). \quad (14.33)$$

The inverse “ RC time” in Eq. (14.32) is given by

$$\Omega = (RC)^{-1} = \eta_0 E_c. \quad (14.34)$$

Since $\eta_0 \leq 4/\pi$, see Eq. (14.30), and using $E_c = e^2/(2C)$, the effective resistance R in Eq. (14.34) always fulfills $R \geq h/(4e^2)$. Note that this corresponds to a parallel resistance as seen by the dot.

However, at the same time it is essential to keep the full frequency-dependence in the fluctuation kernel $K(\omega)$ when evaluating $J(t)$ in Eq. (14.28). Using Eq. (14.33), some algebra yields

$$J(t) = \frac{1}{\pi \eta_0^2} \int_0^\infty d\omega K(\omega) \frac{1 - \cos(\omega t)}{\omega^2 (1 + \omega^2/\Omega^2)}, \quad (14.35)$$

where the current $I = I_L$ follows from Eq. (14.27).

14.4.4 Zero-bias anomaly

We now present results for the IV relation in the zero-temperature limit, where MBS effects are most pronounced. We adopt the Markovian approximation described in Sec. 14.4.3 and study a symmetric system, where $\Gamma_L = \Gamma_R = \Gamma/2$ and $\mu_L = -\mu_R = eV/2$. As relevant transport quantity, we define the dimensionless nonlinear conductance

$$g(V) = \frac{I(V)}{e^2 V/h}. \quad (14.36)$$

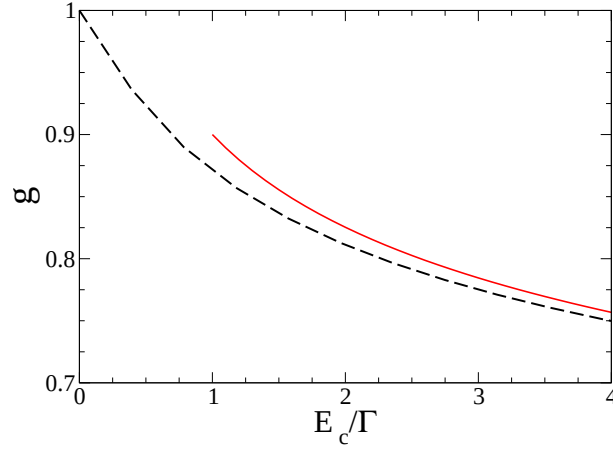


Figure 17: (Color online) Dimensionless linear conductance, $g = g(V \rightarrow 0)$, vs E_c/Γ in the weak CB regime. The black dashed curve has been obtained from numerical analysis of Eq. (14.27), with $J(t)$ in Eq. (14.35) and $K(\omega)$ in Eq. (14.25). The red solid curve gives the analytical result in Eq. (14.39) valid for $E_c \gtrsim \Gamma$.

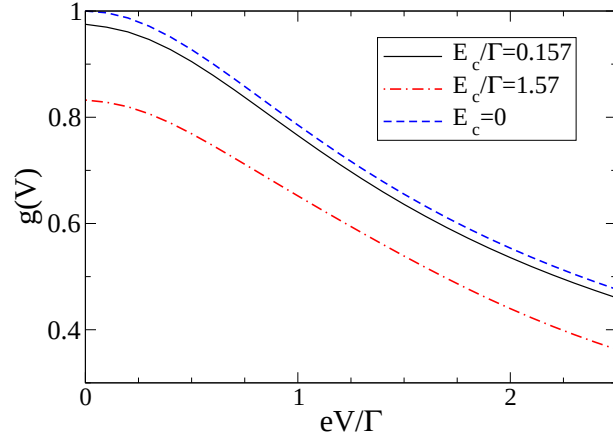


Figure 18: (Color online) Dimensionless nonlinear conductance $g(V)$ vs eV/Γ for several values of the charging energy. Results for $E_c > 0$ were obtained by numerical integration, the result for $E_c = 0$ is exact.

For $E_c = 0$, the exact solution [Eq. (16.6)] yields

$$g^{(0)}(V) = \frac{\Gamma}{eV} \tan^{-1} \left(\frac{eV}{\Gamma} \right). \quad (14.37)$$

Note that the unitary limit (resonant Andreev reflection) corresponds to $g = 1$, which is reached for $V \rightarrow 0$ in Eq. (14.37).

We now write $J(t) = J_0 + J_1$, where

$$J_0(t) = \frac{1}{2\pi\eta_0} \int_0^\infty \frac{d\omega}{\omega} \frac{1 - \cos(\omega t)}{1 + \omega^2/\Omega^2} \quad (14.38)$$

comes from the first term ($\propto \eta(\omega) = \eta_0$) in Eq. (14.25). J_1 is a pure nonequilibrium term: the relevant kernel contribution scales as $\sim V^3$. Let us then first consider the *linear response regime*, where J_1 can be discarded. To logarithmic accuracy, $J_0(t) \simeq (2\pi\eta_0)^{-1} \ln(\Omega t)$ for $\Omega t > 1$, where $\eta_0 = 4/\pi$ within linear response, see Eq. (14.30). Equation (14.27) then yields the dimensionless linear conductance in analytical form,

$$g(V \rightarrow 0) = c_0 (E_c/\Gamma)^{-1/8}, \quad c_0 \approx 0.96. \quad (14.39)$$

This result holds[27] for $\Gamma \lesssim E_c$. As a function of E_c/Γ , Eq. (14.39) reveals a *power-law suppression* of the linear conductance with the universal exponent $1/8$, reminiscent of a zero-bias anomaly. For arbitrary E_c/Γ , the current given by Eq. (14.27) can be evaluated numerically. The resulting linear conductance is shown in Fig. 172, and the analytical result (14.39) is quite accurate in describing the interaction ($E_c \neq 0$) induced suppression of the current for $E_c \gtrsim \Gamma$.

Next we turn to the nonlinear conductance $g(V)$ in Eq. (14.36). Numerical results for several values of E_c/Γ are shown in Fig. 173, where the exact result [Eq. (14.37)] for $E_c = 0$ is also depicted. We find a clear suppression of $g(V)$ when the applied bias voltage increases. This suppression becomes stronger with increasing E_c . In fact, we observe that the interaction induced suppression of the current is more pronounced for small voltage. We therefore interpret our results as a zero-bias anomaly caused by the (weak) Coulomb blockade of Majorana fermion induced transport.

14.5 Concluding remarks

In this work, we have presented a general theory of transport through a mesoscopic superconductor containing a pair of MBSs. We have studied the regime of low-energy transport, where metallic leads are attached and all relevant energy scales are small compared to the superconducting gap. The case of weak Coulomb blockade, where the charging energy E_c does not significantly exceed the typical hybridization scale Γ , has been investigated in some detail, and we found clear signatures for an interaction ($E_c \neq 0$) induced suppression of both the linear and the nonlinear conductance.

We have introduced a formulation to analyze interaction effects in electronic transport through MBSs which does not rely on the Hilbert space constraints of previous approaches and thus is easier to handle in general. We are presently extending this formulation to study several other situations of current interest. In particular, relevant questions include the full counting statistics and the nonequilibrium dephasing in multi-terminal transport geometries containing MBSs, and the Josephson effect with superconducting leads. Furthermore, the strong Coulomb blockade regime also allows for the detailed study of the full Keldysh functional integral (14.8) in terms of a rate equation approach.

To conclude, we are confident that the characteristic features reported here, i.e., the suppression of the linear conductance with increasing E_c/Γ and of the nonlinear conductance with V/Γ

in the limit of weak Coulomb blockade, can be observed experimentally once MBSs have been detected. The setup studied here may allow to find signatures of the elusive Majorana fermions in transport properties, including phenomena such as resonant Andreev reflection, teleportation, and the universal power-law suppression of the linear conductance, $g \sim (E_c/\Gamma)^{-1/8}$, for intermediate values of the ratio E_c/Γ .

14.6 Derivation of the tunnel Hamiltonian

In order to keep the paper self-contained, we here provide a derivation of the tunnel Hamiltonian (14.4). Following Flensberg,[11] the field operator $\Psi_\sigma(x)$ for electrons with spin projection $\sigma = \uparrow, \downarrow$ in a TS wire of length L (with $0 < x < L$) defines the Majorana fermion operators

$$\gamma_{j=L/R} = \sum_{\sigma} \int_0^L dx [f_{j\sigma}^*(x) \Psi_{\sigma}(x) + f_{j\sigma}(x) \Psi_{\sigma}^{\dagger}(x)]$$

with the MBS wavefunction $f_{j\sigma}(x)$. Projecting the full Nambu spinor onto the MBS subspace, one obtains[11]

$$\Psi_{\sigma}(x) \rightarrow \sum_j f_{j\sigma}(x) \gamma_j.$$

The tunnel Hamiltonian now reads

$$\begin{aligned} H_t &= \sum_{jk\sigma} \int_0^L dx t_j^*(x) C_{jk\sigma}^{\dagger} \Psi_{\sigma}(x) + \text{h.c.} \\ &= \sum_{jk\sigma} V_{j\sigma}^* C_{jk\sigma}^{\dagger} \gamma_j + \text{h.c.}, \end{aligned}$$

where $C_{jk\sigma}^{\dagger}$ creates an electron in lead $j = L/R$ with momentum k and spin projection σ . Here $t_j(x)$ is assumed k -independent for simplicity, which defines the tunnel matrix elements $V_{j\sigma} = \int dx t_j(x) f_{j\sigma}^*(x)$. Note that one can always form suitable linear combinations of $C_{jk,\uparrow}$ and $C_{jk,\downarrow}$ to form spinless lead fermions c_{jk} coupled to the MBSs,

$$\sum_{\sigma} V_{j\sigma}^* C_{jk\sigma}^{\dagger} \rightarrow \lambda_j c_{jk}^{\dagger}.$$

The other (orthogonal) linear combination then decouples from the problem. Finally, inserting the auxiliary fermion representation (14.1) and taking into account charge conservation, we arrive at the tunnel Hamiltonian quoted in Eq. (14.4).

14.7 On the Keldysh functional integral

Here we provide a detailed derivation of the AES action in Sec. 14.3.1. We start by constructing the Lagrangian L_c for the isolated dot in terms of $\phi(t)$ and the Grassmann variable $d(t)$, plus the corresponding “velocities” $\dot{\phi} = \frac{1}{2} \partial H_c / \partial N$ and $\dot{d}$. Noting that for Grassmann variables, $i\bar{d}$ is canonically dual to d , we have

$$L_c = \frac{\dot{\phi}^2}{4E_c} + n_0 \dot{\phi} + \bar{d}(i\partial_t - \dot{\phi})d.$$

Adding the tunnel contribution and performing the gauge transformation $d \rightarrow e^{-i\phi}d$, we obtain

$$\begin{aligned} L_c + L_t &= \frac{\dot{\phi}^2}{4E_c} + n_0 \dot{\phi} + \frac{i}{2} \sum_j \gamma_j \dot{\gamma}_j \\ &\quad - \sum_{jk} (\lambda_j \bar{\psi}_{jk} e^{-i\phi} \gamma_j + \text{h.c.}). \end{aligned}$$

Here we switched back from the auxiliary d fermion to the Majorana field γ_j . Note that up to a full time derivative, $i\bar{d}\dot{d} \rightarrow (i/2)\sum_j \gamma_j \dot{\gamma}_j$. The Grassmann fields $(\psi_{jk}, \bar{\psi}_{jk})$ correspond to the lead fermion operators $(c_{jk}, c_{jk}^\dagger)$. Using the Keldysh formulation, we double all fields according to the forward and backward branch of the Keldysh time contour, i.e., $\gamma_j(t) \rightarrow (\gamma_{j,+}(t), \gamma_{j,-}(t))^T$ and so on. It is also convenient to gauge out the chemical potentials μ_j in the leads, $\psi_{jk,\pm}(t) \rightarrow e^{i\mu_j t} \psi_{jk,\pm}(t)$. We now use the Keldysh matrix notation $\check{\phi}(t) = \phi_c + \tilde{\tau}_z \phi_q/2$, see Eq. (14.6). The complete Keldysh action, $S = S_\gamma + S_c + S_t + S_l$, contains S_c in Eq. (14.7) and the pieces

$$\begin{aligned} S_\gamma &= \frac{i}{2} \sum_j \int dt \gamma_j \tilde{\tau}_z \dot{\gamma}_j, \\ S_t &= - \sum_{jk} \lambda_j \int dt \bar{\psi}_{jk} \tilde{\tau}_z e^{-i[\mu_j t + \check{\phi}(t)]} \gamma_j + \text{h.c.}, \\ S_l &= \sum_{jk} \int dt \bar{\psi}_{jk} \tilde{\tau}_z (i\partial_t - \epsilon_{jk}) \psi_{jk}, \end{aligned}$$

where ϵ_{jk} refers to the dispersion relation in lead j .

The current I_j flowing from lead j into the dot follows from the Heisenberg equation of motion. The currents obey the relation $(I_L + I_R)(t) = \delta S_t / \delta \phi_q(t)$. Current conservation implies $\langle I_L \rangle = -\langle I_R \rangle$, which fixes the mean value $\langle \phi_c \rangle$ to the chemical potential μ_s of the superconducting dot. The latter has to be determined self-consistently from current conservation. We then redefine $\phi_c \rightarrow \mu_s t + \phi_c(t)$, i.e., $\phi_c(t)$ now refers to fluctuations around the mean-field value $(\mu_s t)$ of the classical phase variable. The ensuing changes in S_c and S_t can be absorbed in a renormalization of n_0 and μ_j . In particular, denoting their bare values by $\bar{n}_0$ and $\bar{\mu}_j$, respectively, we find $n_0 = \bar{n}_0 + \mu_s/(2E_c)$ and $\mu_j = \bar{\mu}_j + \mu_s$.

The next step is to integrate out the Grassmann fields $(\psi_{jk}, \bar{\psi}_{jk})$. This is a standard step[21, 22, 24, 25] and leads to the effective Majorana action S_{eff} replacing $S_l + S_t + S_\gamma$. Before turning to the result, we perform the usual rotation in Keldysh space in order to have triangular Green's function (GF) representations, using the unitary matrix[21] $\check{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. The rotated Majorana fields are $\tilde{\gamma}_j = \check{L} \gamma_j$. Evaluating the resulting momentum integrals for the leads in wide-band approximation, we find the effective Majorana action

$$\begin{aligned} S_{\text{eff}} &= \frac{1}{2} \sum_j \int dt dt' \tilde{\gamma}_j^T(t) [\tilde{\tau}_x i\partial_t \delta(t-t') + \check{Q}_j(t, t')] \tilde{\gamma}_j(t') \\ \check{Q}_j(t, t') &= i\Gamma_j e^{i\tilde{\tau}_x \phi_q(t)/2} \begin{pmatrix} \delta_-(t-t') & 2F(t-t')e^{i\Phi_j(t, t')} \\ 0 & -\delta_+(t-t') \end{pmatrix} \tilde{\tau}_x e^{-i\tilde{\tau}_x \phi_q(t')/2}. \end{aligned}$$

The hybridizations Γ_j are defined in Eq. (24.23) with lead density of states $\nu_j = \sum_k \delta(\epsilon_{jk})$, $F(t)$ has the Fourier transform (14.11), and the phase function Φ_j is specified in Eq. (14.12).

Now we are ready to also integrate out the Majorana fields. This finally yields Eq. (14.8) where

$$\check{\Lambda}_j(t, t') = \frac{1}{2} [\check{Q}_j(t, t') - \check{Q}_j^T(t', t)]$$

is the antisymmetric part of $\check{Q}_j$.

14.8 Derivation of Langevin equation

Here we provide some details concerning the derivation of the Langevin equation (14.20) in Sec. 14.4.1. We first analyze the action piece $S_f^{(1)} = \int dt \mathcal{I}\phi_q$. As mentioned in App. 14.7,

the chemical potential μ_s of the dot has to be chosen self-consistently in order to fulfill current conservation. By construction, we must therefore have the average value $\mathcal{I} = 0$ when $\dot{\phi}_c = 0$. Using Eq. (14.17), expanding to first order in ϕ_c , and performing a partial integration, we find

$$S_f^{(1)} = - \int dt \phi_q(t) \int^t dt' \eta(t-t') \dot{\phi}_c(t'),$$

where $\eta(t-t')$ is determined by Eq. (14.21). Next we note that $S_f^{(2)}$, which is already of order ϕ_q^2 , can be evaluated by replacing the interacting Keldysh GF G_j^K with its noninteracting ($\phi_c = 0$) version. After some algebra, we find from Eq. (14.19)

$$S_f^{(2)} = \frac{i}{2} \int dt dt' \phi_q(t) K(t-t') \phi_q(t')$$

with $K(t-t')$ determined by Eq. (14.22). The Fourier-transformed kernel $K(\omega)$ directly describes finite-frequency current noise correlations for $E_c = 0$, a question that has been studied for $\omega \rightarrow 0$ in Ref. [12]. Finally, we complete the derivation by Hubbard-Stratonovich transformation of the ϕ_q^2 term in the generating functional [Eq. (14.8)],

$$e^{iS_f^{(2)}} = \int \mathcal{D}\xi \, e^{-\frac{1}{2} \int dt dt' \xi(t) K^{-1}(t-t') \xi(t') + i \int dt \phi_q \xi}.$$

Functional integration over ϕ_q , which appears only linearly in the action, yields Eq. (14.20), and the Gaussian noise field $\xi(t)$ indeed has the correlation function $K(t-t')$ in Eq. (14.22).

15 Robustness of Majorana bound states in the short-junction limit

Abstract

We study the effects of strong coupling between a superconductor and a semiconductor nanowire on the creation of the Majorana bound states, when the quasiparticle dwell time in the normal part of the nanowire is much shorter than the inverse superconducting gap. This “short-junction” limit is relevant for the recent experiments using the epitaxially grown aluminum characterized by a transparent interface with the semiconductor and a small superconducting gap. We find that the small superconducting gap does not have a strong detrimental effect on the Majorana properties. Specifically, both the critical magnetic field required for creating a topological phase and the size of the Majorana bound states are independent of the superconducting gap. The critical magnetic field scales with the wire cross section, while the relative importance of the orbital and Zeeman effects of the magnetic field is controlled by the material parameters only: g factor, effective electron mass, and the semiconductor-superconductor interface transparency.

Doru Sticlet, Bas Nijholt, Anton Akhmerov

15.1 Introduction

The theory of normal conductor-superconductor (NS) hybrid systems distinguishes two limiting cases: long and short junctions. In long junctions, the dwell time τ_{dw} of a quasiparticle inside the normal region is much larger than the time $\hbar/\Delta$ it spends inside the superconductor (with Δ the superconducting gap). In this limit the induced gap inside the semiconductor is equal to $\hbar/\tau_{\text{dw}}$, and therefore it varies for different bound states. In the short-junction or

strong-coupling limit, the quasiparticles spend most of their time inside the superconductor, while the normal region effectively acts as a delta function scatterer. Then in the presence of time-reversal symmetry, the induced gap is close to Δ for every single Andreev bound state. In the short-junction limit Andreev bound states have the most weight in the superconductor, and therefore the conventional approach of integrating out the superconductor and obtaining an effective Hamiltonian of the normal system becomes inefficient due to the strong energy dependence of the effective Hamiltonian.

Systematically studying the short-junction limit is relevant for the creation of Majorana bound states (MBS) [1, 2, 10, 4, 5] in semiconductor nanowires [6, 7] partially coated with epitaxially grown aluminum that have high interface quality. These systems were observed to have a well-developed hard induced gap comparable to the gap in the bare Al [8], and subsequently showed zero bias peaks [9] and suppressed splitting of low energy states characteristic to MBS [10]. The theoretical description of the response of strongly coupled zero-dimensional NS junctions to magnetic field was analyzed in Ref. [11], where the authors report a strong suppression of the effective g factor, potentially leading to the impossibility of inducing a topological phase at magnetic fields below the Clogston limit [12].

Here we extend the analysis of Ref. [11] using the scattering formalism that allows us to capture the nonlinear features of the spectrum, and by considering higher-dimensional systems with translational invariance. The scattering formalism has been routinely applied to short junctions in mesoscopic physics [?]. Relevant works to the present study are on two-dimensional electronic gases with spin-orbit interactions [14, 14]. However, in Majorana literature the use of the scattering formalism has been limited [16]. The equivalent of the scattering formalism using the effective Hamiltonian approach amounts to introducing an effective self-energy $\Sigma(E)$ which has a proper dependence on energy E [17, 18, 19] and then neglecting the energy term in the nonlinear eigenvalue problem $[H - \Sigma(E)]\psi = E\psi$, as done in, e. g., Ref. [20].

Our overall findings are favorable for the creation of MBS in Al-based NS systems. Specifically, we find that:

- The critical magnetic field B_* required to induce a topological phase is independent of the superconducting gap. This is valid also beyond the short-junction limit, as long as penetration of magnetic field into the superconductor is negligible.
- Since B_* is inversely proportional to the wire cross section, the device design can be used to adjust B_* within a broad range.
- The localization length ξ_M of the MBS does not depend on the superconducting gap, and in optimal conditions it is proportional to the spin-orbit length l_{SO} .
- Finally, if the interface between the semiconductor and the superconductor has high transparency T , then B_* becomes only a slowly varying function of the chemical potential μ , as opposed to its usual oscillatory behavior on the scale of the mode spacing in the nanowire [21, 22].

Our analytical calculations fully coincide with the results obtained using a numerical scattering approach to short junctions and exact diagonalization of a discretized tight-binding Hamiltonian. While these conclusions are favorable for the prospect of using weak superconductors for MBS creation, we note that the effects of disorder in the superconductor are not systematically treated here. Disorder has been recently predicted to have a strong detrimental effect on the creation of MBS in systems that are in the short-junction limit [23].

The paper is organized as follows. Section 15.2 contains a pedagogical review of scattering formalism for the calculation of Andreev spectrum. The following Sec. 15.3 presents scaling arguments supporting our conclusions. In Sec. 15.4 we compute the dispersion relation of a

planar NS junction and discuss the typical device parameters. Section 15.5 investigates the Majorana phase diagram and the behavior of the MBS decay length. In Sec. 15.6 we compare the predictions of Sec. 15.5 with numerical diagonalization of finite junctions. Section 15.7 estimates the orbital effect of the magnetic field by computing the Andreev spectrum in a cylindrical geometry in a thin shell limit. In Sec. 15.8 we confirm our findings using a numerically computed Majorana phase diagram of a three-dimensional model. Lastly, section 15.9 sums up our conclusions.

15.2 Scattering matrix formalism and the short-junction limit

This section reviews the scattering approach to calculating the Andreev bound state spectrum and may be skipped by expert readers. We start by considering a general NS junction with n superconducting terminals. [24] We use the case $n = 1$ in Secs. 15.4, 15.5, 15.8 and $n = 2$ in Sec. 15.7.

The levels with $|E| < \Delta$ are Andreev bound states, i.e., coherent superpositions of electron and hole excitations which occur due to Andreev reflections [12] at the interface between the normal region and the superconducting terminals. The wave function quantization condition on the wave function requires that the total sequence of scattering events results in a phase shift of $2\pi n$. For the vector of modes ψ incoming from the superconductor to the normal region this condition reads:

$$S_A S_N \psi = \psi. \quad (15.1)$$

Here S_N is the scattering matrix of the normal region, and S_A the scattering matrix of Andreev reflection processes in the superconducting terminals. The mode vector has electron and hole components $\psi = (\psi_e, \psi_h)$.

The Andreev reflection matrix assumes a universal form when the superconductor has s -wave pairing without any sources of time-reversal symmetry breaking and additionally when the Andreev approximation holds (when the Fermi energy in the superconductor is much larger than Δ). In the literature, the Andreev spectrum is often calculated in systems where the superconductor Hamiltonian has full spin-rotation invariance (an appropriate approximation for aluminum), making the spin basis a natural choice of basis of ψ . Yet the universal structure of the Andreev reflection matrix does not change in the presence of spin-orbit coupling in the superconductor. However, in that case it is impossible to choose a spin basis due to lack of spin conservation and it is more appropriate to use a basis where the outgoing modes are time reversed of the incoming modes [26]. Throughout the paper we work in the latter basis but explain the relation to the more commonly used spin basis for reference at the end of this section. Importantly, we neglect the time-reversal symmetry breaking perturbations in the superconductor, restricting ourselves to magnetic fields much lower than critical.

The scattering matrix of the normal region is block diagonal in the Nambu space:

$$S_N(E, \mathbf{k}) = \begin{pmatrix} S_e(E, \mathbf{k}) & 0 \\ 0 & S_h(E, \mathbf{k}) \end{pmatrix}, \quad (15.2)$$

where S_e and S_h are the scattering matrices of electrons and holes. We consider NS junctions with a translational symmetry, and therefore the scattering matrices may depend on the wave vector $\mathbf{k}$ along the translationally invariant directions. We choose the hole modes ψ_h as particle-hole partners of the electron modes ψ_e . In this basis the particle-hole symmetry of the scattering matrix reads:

$$\tau_x S_N^*(E, \mathbf{k}) \tau_x = S_N(-E, -\mathbf{k}), \quad (15.3)$$

Using the block-diagonal structure of S_N it follows that the normal scattering matrix of holes is the conjugate of the scattering matrix for electrons, at opposite energy and momentum [27]:

$$S_h(E, \mathbf{k}) = S_e^*(-E, -\mathbf{k}). \quad (15.4)$$

In the same basis, the Andreev reflection matrix reads:

$$S_A = \alpha(E)R, \quad R = \begin{pmatrix} 0 & r \\ -r^* & 0 \end{pmatrix}, \quad r = \oplus_j e^{i\phi_j}, \quad (15.5)$$

where the index j runs over the terminals, ϕ_j is the superconducting phase in lead j , and $\alpha(E) = \exp(-i \arccos(E/\Delta))$. [15]

Following Ref. [15], eliminating ψ from Eq. (15.1), and using an expression for a block matrix determinant one immediately arrives to a determinantal equation for the bound state energies:

$$\det[1 + \alpha^2(E)r^*S_e(E, \mathbf{k})rS_e^*(-E, -\mathbf{k})] = 0. \quad (15.6)$$

The short-junction limit allows us to further simplify the calculation of the Andreev bound state energies when Thouless energy $E_{\text{Th}} \equiv \hbar/\tau_{\text{dw}} \gg \Delta$. Thouless energy is the typical energy scale for the matrix elements to change appreciably, therefore in the short-junction limit $S_N(E, \mathbf{k}) \approx S_N(0, \mathbf{k})$ for any $E \lesssim \Delta$. After replacing $S_e(E)$ with $S_e(0)$, the only energy-dependent term remaining in Eq. (15.1) is the coefficient $\alpha(E)$. Since the scattering matrices are invertible, Eq. (15.1) reads:

$$RS_N\psi = \alpha^{-1}(E)\psi, \text{ or } S_N^{-1}R^{-1}\psi = \alpha(E)\psi. \quad (15.7)$$

Adding the two equations yields the following energy eigenproblem:

$$\frac{1}{2}[RS_N + S_N^{-1}R^{-1}]\psi = \frac{E}{\Delta}\psi. \quad (15.8)$$

Further squaring this equation and using the unitarity of the scattering matrices S_A and S_N we arrive to the eigenproblem expression for the Andreev spectrum:

$$\left\{ \frac{1}{2} - \frac{1}{4}[S_e^\dagger(\mathbf{k})rS_e^T(-\mathbf{k})r^* + \text{H.c.}] \right\} \psi_e = \frac{E^2}{\Delta^2}\psi_e, \quad (15.9)$$

where the energy argument is suppressed, since S_e is evaluated at $E = 0$. If there is only a single superconducting terminal, the Andreev reflection matrix r reduces to a phase factor, which fully drops out from Eq. (15.9), as required by gauge invariance.

If the spin is conserved, the above derivation is nearly identical in the spin basis. The scattering matrices in the spin basis $\tilde{S}_e$ and $\tilde{r}$ are related to the basis of time-reversed modes by a transformation

$$\tilde{S}_e = -i\sigma_y S_e, \quad \tilde{r} = -i\sigma_y \oplus_j e^{i\phi_j}, \quad (15.10)$$

with the Pauli matrices σ spin operators and σ_0 an identity matrix. The symmetry condition (15.4), equation (15.6) for the Andreev spectrum, and eigenproblem for the spectrum in short-junction approximation (15.9) are identical in both bases upon replacing S_e and r with $\tilde{S}_e$ and $\tilde{r}$.

15.3 Scaling arguments for the MBS properties in the short-junction limit

The superconducting gap enters only as an overall prefactor of the Andreev state energies in Eq. (15.9), while the specific spectrum depends only on the normal state scattering matrix S_e . This simplification allows us to draw most of our conclusions about MBS properties solely using universal arguments and not by solving a specific model. For instance, since the Majorana phase transition occurs when Eq. (15.9) has a zero-energy solution, the critical field B_* does not depend on Δ . This conclusion also extends beyond the short-junction limit, since the zero

energy solutions of Eq. (15.9) always coincide with the zero energy solutions of the Eq. (15.1) and therefore with the full solutions of the Bogoliubov-de-Gennes equation.

Turning to the spatial extent of MBS in the normal region ξ_M , we observe that it is limited from below by the coherence length ξ_S in the superconductor. However ξ_S is often short: For example in aluminum films $\xi_S \sim 100$ nm due to disorder effects. If ξ_M is predominantly set by the properties of induced superconductivity in the normal region, ξ_M must also be independent of Δ , since it is a property of the eigenvectors of Eq. (15.9) at $E = 0$.

If the semiconductor has a cross section W , effective electron mass m_n , and it is coupled to the superconductor by an interface with transparency T , then the Thouless energy equals

$$E_{\text{Th}} = TN\delta, \quad \delta \equiv \frac{\hbar^2 \pi^2}{2m_n(2W)^2}, \quad (15.11)$$

where N is the number of transverse bands occupied in the semiconductor, and δ is the typical interband spacing. The denominator of the expression for δ contains $2W$ since it is the total distance traveled by a quasiparticle normal to the interface between two consecutive Andreev reflections. We focus on the experimentally relevant low-density regime when $N \sim 1$. In realistic nanowires $m_n \sim 10^{-2}m_e$, and $W \approx 100$ nm, resulting in $\delta \approx 1$ meV, much larger than the superconducting gap in aluminum $\Delta_{\text{Al}} \approx 0.2$ meV, which justifies the short-junction approximation for sufficiently transparent contacts $T \gtrsim 0.1$.

In order for the spectral gap to close in the normal region—or, in other words, for a topological phase transition to appear—the scattering matrix S_e must change by $\mathcal{O}(1)$ since, in the presence of time-reversal symmetry, all the Andreev bound states have the same energy $E = \Delta$. For the Zeeman field to cause such a perturbation, the electron spin must precess by a large angle during the propagation inside the scattering region. This results in a condition $E_Z \tau_{\text{dw}} / \hbar \sim 1$, or equivalently

$$B_* \sim E_{\text{Th}} / g\mu_B, \quad (15.12)$$

with g the effective gyromagnetic factor and μ_B the Bohr magneton.

The orbital effect of the magnetic field causes an additional time-reversal symmetry breaking perturbation to the normal scattering region. It becomes significant [causes an $\mathcal{O}(1)$ change of S_e] when the flux penetrating the scattering region becomes comparable to the flux quantum $\Phi_0 = h/e$. This defines another scale of the magnetic field, characterizing the importance of its orbital effect $B_{\text{orb}} \sim h/eW^2$. Comparing B_{orb} with B_* we get

$$\frac{B_{\text{orb}}}{B_*} \sim \frac{g\mu_B m_n}{e\hbar TN}. \quad (15.13)$$

If transparency is high and the number of modes is low, then the relative strength of the orbital and Zeeman effects of the magnetic field is a material parameter dependent on the g factor and the effective mass. For realistic materials this factor is $\mathcal{O}(1)$, which is in line with our results in Secs. 15.7 and 15.8.

Turning to the spatial extent of the MBS ξ_M , we observe that it must diverge in the topological regime in the absence of spin-orbit coupling. The spectral gap at a finite momentum appears already in the first order perturbation in spin-orbit strength α , and hence $\xi_M \sim \alpha^{-1}$. Finally, in the optimally tuned situation $N \sim 1$, and $B \sim B_*$, so that S_N only depends on two energy scales: δ and the spin-orbit energy $E_{\text{SO}} = m_n \alpha^2 / 2\hbar^2$. This means that there is only a single length scale inversely proportional to α , the spin-orbit length $l_{\text{SO}} = \hbar / m_n \alpha$, and hence $\xi_M \sim l_{\text{SO}}$.

The scattering approach highlights another important property of the Majorana phase diagram, the relation between T and the oscillatory behavior of B_* . If $T \sim 1$, there is little scattering at the NS interface, and τ_{dw} becomes a smooth function of the chemical potential

μ . Combining this with Eq. (15.12) we conclude that B_* must also depend on μ in a smooth fashion. In the opposite limit $T \ll 1$, E_{Th} reduces on resonance, when μ matches the bottom of a subband in the semiconducting region. Away from the resonance, when there are no available states at the selected energy, E_{Th} becomes very large. This behavior of Thouless energy results in the appearance of a sharp minimum in B_* whenever μ matches the bottom of a new band in the semiconductor region. In Sec. 15.4 we confirm the relation between the interface transparency and the oscillatory nature of the Majorana phase boundary.

These findings are different from the predictions of a purely 1D phenomenological model [6, 7] with the Hamiltonian

$$H_{1\text{D}} = \left(\frac{p^2}{2m_n} - \mu + \frac{\alpha}{\hbar} \sigma_y p \right) \tau_z + \Delta' \tau_x + E_Z \sigma_z, \quad (15.14)$$

where the induced superconductivity enters as a phenomenological pairing term Δ' , and the momentum p is limited to a direction along the nanowire. The induced gap follows from a perturbation theory in the weak coupling limit between the semiconductor and the superconductor. Therefore the phenomenological model is not directly applicable to the strong coupling regime for highly transparent junctions. The Hamiltonian $H_{1\text{D}}$ undergoes a topological phase transition when $E_Z^2 = \Delta'^2 + \mu^2$, and therefore B_* explicitly depends on Δ' . This difference, however, is due to the shortcomings of the effective model, and in reality our conclusions also hold in the weak-coupling/long junction limit. In the long junction limit $\Delta' \approx E_{\text{Th}}$, immediately leading us to the conclusion that B_* and ξ_M are independent of the intrinsic superconducting gap Δ . If a long junction is transparent $T \sim 1$, then the Fermi momentum drops out of the level quantization condition, hence resulting in the lack of oscillations of B_* as a function of μ . Finally, the rest of our conclusions follow in a similar fashion for the long junctions from the dimensional analysis of Eq. (15.14) after the identification $\Delta' \sim E_{\text{Th}}$.

15.4 Model

15.4.1 General solution

To verify our general arguments, we consider a specific model, a semiconductor nanowire in contact with a large superconductor. We consider the effective superconductor thickness to be infinite, unlike the typical experimental situation where the superconductor thickness is around 10 nm. This limit is nevertheless a reasonable approximation due to the large Fermi surface size mismatch between the superconductor and the semiconductor. A much larger Fermi surface in the superconductor means that most electron trajectories approaching the nanowire from the superconductor side must be reflected back. Full internal reflection in combination with diffuse scattering allows the superconductor to accommodate quasiparticle trajectories much longer than the superconducting coherence length $\xi_S = \hbar v_{s,F}/\Delta$, making the superconductor effectively infinite.

We first consider the device geometry shown in Fig. 19. The nanowire is oriented along the x axis, NS interface is at $y = 0$, and the outer boundary of the wire is at $y = W$. The superconductor occupies the half-space $y < 0$. Neglecting the orbital effect of the magnetic field makes the motion in z direction separable and reduces the problem to a purely two-dimensional geometry, shown in Fig. 19(b).

The normal state Hamiltonian of the system is that of a Rashba two-dimensional electron gas coupled to a material with negligible spin-orbit interaction and Zeeman coupling [29]:

$$H = \left[\mathbf{p} \frac{1}{2m(y)} \mathbf{p} - \mu(y) \right] \sigma_0 + \frac{1}{2\hbar} \{ \alpha(y), \boldsymbol{\sigma} \times \mathbf{p} \} \cdot \hat{z} + E_Z(y) \sigma_x. \quad (15.15)$$

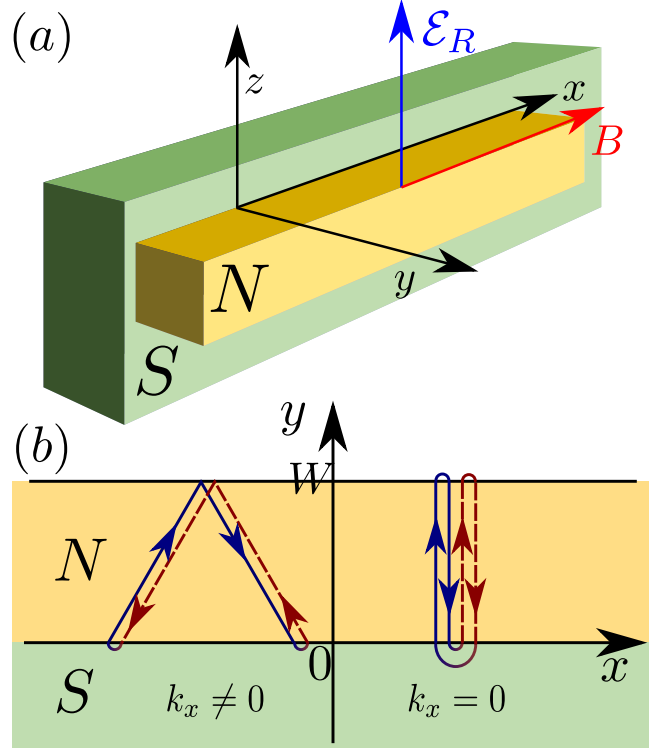


Figure 19: (Color online) (a) Nanowire with width W oriented along the x direction and coupled to a bulk superconductor. The magnetic field is parallel to the wire, while the Rashba electric field points along the z direction. (b) Semiclassical bound state trajectory in the two-dimensional nanowire. Electrons (solid blue) and holes (dashed red) specularly reflect at the boundary with vacuum and undergo Andreev reflection at the interface with the superconductor. Two types of bound states are shown: at finite longitudinal momentum k_x (left) and vanishing momentum $k_x = 0$ (right). The induced gap may only close at $k_x = 0$.

Here σ are the spin Pauli matrices, and $\mathbf{p}$ the momentum operator. The chemical potential μ and effective mass m are

$$\mu(y) = \begin{cases} \mu_n, & y \in (0, W) \\ \mu_s, & y < 0 \end{cases}, \quad m(y) = \begin{cases} m_n, & y \in (0, W) \\ m_s, & y < 0 \end{cases}. \quad (15.16)$$

Additionally, we neglect the spin-orbit coupling and the magnetic field effect in the superconductor, therefore restricting the model to $B \ll B_c$, with B_c the critical field of the superconductor:

$$\alpha(y) = \alpha\Theta(y)\Theta(W - y), \quad E_Z = \frac{g\mu_B B}{2}\Theta(y)\Theta(W - y), \quad (15.17)$$

with Θ the Heaviside step function and α , the Rashba spin-orbit coupling strength. The spin-orbit term in Eq. (15.15) is symmetrized using anticommutators to ensure current probability conservation at the interface. The Zeeman energy E_Z is due to a magnetic field of magnitude B oriented along the wire direction. The effective electric field generating the Rashba spin-orbit coupling is $\mathcal{E}_R = 2m_n\alpha/\hbar g\mu_B \hat{z}$. To compare the short-junction approximation with exact diagonalization results, we use the Bogoliubov-de Gennes (BdG) Hamiltonian:

$$H_{\text{BdG}} = \begin{pmatrix} H(B) & \Delta(y) \\ \Delta(y) & -H(-B) \end{pmatrix}, \quad (15.18)$$

with $\Delta(y) = \Delta\Theta(-y)$. The choice of a step-function pairing potential is justified due to a density of states mismatch between the superconductor and semiconductor by more than 10^6 , which renders the self-consistency condition on Δ unimportant.

To make further analytical progress, we neglect the spin-orbit coupling in the y direction $\alpha\sigma_x p_y$. This is a valid simplification since in semiconductor nanowires the spin-orbit length l_{SO} is usually larger than the nanowire width W . We later verify the validity of this approximation by computing the exact expression for the topological phase boundary and by including the transverse spin-orbit coupling in all the tight-binding simulations.

The wave function $\psi_n(k_x, y)$ in the nanowire satisfies the boundary condition $\psi_n(k_x, W) = 0$ and has the general form

$$\psi_n = u_+ c_+ \sin[k_+(W - y)] + u_- c_- \sin[k_-(W - y)], \quad (15.19)$$

with

$$\begin{aligned} u_+ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\varphi} \end{pmatrix}, & u_- &= \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i\varphi} \\ -1 \end{pmatrix}, \\ e^{i\varphi} &= \frac{E_Z - i\alpha k_x}{\sqrt{E_Z^2 + \alpha^2 k_x^2}}, \end{aligned} \quad (15.20)$$

and $c_{\pm}$ unknown amplitudes. The wave function in the superconducting lead has the form

$$\psi_s = \begin{pmatrix} a_{\text{in},\uparrow} e^{iqy} + a_{\text{out},\uparrow} e^{-iqy} \\ a_{\text{in},\downarrow} e^{iqy} - a_{\text{out},\downarrow} e^{-iqy} \end{pmatrix}, \quad (15.21)$$

with a_{in} the amplitudes of the incoming modes, a_{out} the amplitudes of the outgoing modes, and the relative signs chosen to ensure that the incoming and outgoing modes are time-reversed of each other. Finally, the momenta q and $k_{\pm}$ normal to the NS interface are fixed by the dispersion relation at energy E :

$$\begin{aligned} q &= \left[\frac{2m_s}{\hbar^2} (E + \mu_s) - k_x^2 \right]^{1/2}, \\ k_{\pm} &= \left[\frac{2m_n}{\hbar^2} (E + \mu_n \mp \sqrt{E_Z^2 + \alpha^2 k_x^2}) - k_x^2 \right]^{1/2}. \end{aligned} \quad (15.22)$$

We use the wave function continuity at $y = 0$ as well as the current conservation condition on the wave function derivative normal to the interface, in the y direction:

$$m_n^{-1} \psi'_n(k_x, 0) = m_s^{-1} \psi'_s(k_x, 0). \quad (15.23)$$

Solving for $c_{\pm}$ and a_{out} for given a_{in} we obtain the scattering matrix:

$$S_e = \frac{1}{2} \begin{pmatrix} (r_+ - r_-) e^{i\varphi} & r_+ + r_- \\ -r_+ - r_- & (r_- - r_+) e^{-i\varphi} \end{pmatrix}, \quad (15.24)$$

with the reflection phases of different spin projections given by

$$r_{\pm} = \frac{v_s - iv_{\pm} \cot(k_{\pm} W)}{v_s + iv_{\pm} \cot(k_{\pm} W)}. \quad (15.25)$$

Here we introduced the transverse velocities $v_s = \hbar q / m_s$ in the superconductor lead and $v_{\pm} = \hbar k_{\pm} / m_n$ in the nanowire.

The scattering matrix holds generally at energies below the superconducting gap $|E/\Delta| < 1$. In the short-junction approximation, S_e is evaluated at Fermi energy $E = 0$. Then the Andreev bound spectrum follows immediately upon solving the eigenvalue problem (15.9):

$$E = \pm \Delta \sqrt{1 - \frac{1}{4} |r_+(k_x) - r_-(k_x)|^2 \cos^2(\varphi)}. \quad (15.26)$$

This dispersion relation admits no zero energy solutions for $k_x \neq 0$ and $\alpha \neq 0$. The parameters values yielding $E(k_x = 0) = 0$ are the topological phase transitions, and they occur when

$$(r_+ + r_-)|_{k_x=0} = 0. \quad (15.27)$$

In the derivation of the Andreev spectrum (15.26) we neglected the effect of spin-orbit interactions in the y direction since $W \ll l_{\text{SO}}$. For completeness, we analyze the impact of this spin-orbit coupling on the condition for gap closing at $k_x = 0$. In the presence of transverse spin-orbit coupling, one needs to take into account the Hamiltonian (15.15) including the $\alpha\sigma_x p_y$ term. Then the boundary condition at the NS interface needs to be modified in order to ensure the current conservation. Integrating the Schrödinger equation near the interface $y = 0$ yields:

$$\frac{1}{m(y)} p_y \sigma_0 \psi(y) \Big|_{0^-}^{0^+} + \frac{\alpha}{\hbar} \sigma_x \psi(0) = 0. \quad (15.28)$$

Since at $k_x = 0$, the Hamiltonian (15.15) commutes with σ_x , the scattering states in the semiconductor region are also eigenstates of σ_x . Matching the wave functions at the NS interface and solving the scattering problem results in a condition for closing the excitation gap identical to Eq. (15.27), but with the modified scattering phases $r_{\pm}$ (15.25):

$$k_{\pm} = \left[\frac{2m_n}{\hbar^2} (E + \mu_n + E_{\text{SO}} \mp E_Z) \right]^{1/2}, \quad E_{\text{SO}} = \frac{m_n \alpha^2}{2\hbar^2}. \quad (15.29)$$

For our parameter choice $E_{\text{SO}} \approx 40 \mu\text{eV}$, and is more than two orders of magnitude smaller than δ . This confirms that spin-orbit dynamics in y direction is negligible.

15.4.2 Typical physical parameters of the heterostructure

To consider a specific example of the system parameters we take InSb nanowires [30] with effective mass $m_n = 0.015 m_e$ (here m_e is the free electron mass), and with spin-orbit length $l_{\text{SO}} = \hbar^2/m_n \alpha \approx 250 \text{ nm}$. The superconductor in the heterojunction is aluminum, with $m_s \approx m_e$ and chemical potential $\mu_s = 11.7 \text{ eV}$. A thin Al film has the bulk superconducting gap $\Delta = 0.25 \text{ meV}$ and the critical magnetic field B_c that varies from around 1.5 T to 2 T.

While most of our results scale trivially with W , we choose $W = 100 \text{ nm}$ whenever it is necessary to compare the magnetic field or chemical potential scales to the experimental parameters. This results in $\delta \approx 0.6 \text{ meV} \gg \Delta$, well within the requirements of the short-junction approximation.

15.4.3 Modeling the NS interface

The final crucial parameter of the hybrid system is the transparency of the NS interface. In the model Hamiltonian (15.15) the interface properties are set by the velocity ratio $v_{\pm}/v_s$, the only way the superconductor Hamiltonian parameters enter the scattering matrix (15.25). We use the v_s as a free parameter allowing us to study the effect of the interface properties on the topological phase diagram.

While the Fermi energy difference between aluminum and the semiconductor may span several orders of magnitude, the Fermi velocities do not differ so much because of a smaller effective mass in narrow band semiconductors. Specifically, the Fermi velocity in aluminum is $v_s \sim 2 \cdot 10^6 \text{ m/s}$, while $v_{\pm} \sim 2 \cdot 10^5 \text{ m/s}$ at a relatively low $\mu_n = 3 \text{ meV}$, resulting in $T \gtrsim 0.4$. In real systems, the microscopic interface properties such as coupling strength and charge accumulation further influence the interface transparency. In the absence of a Schottky barrier,

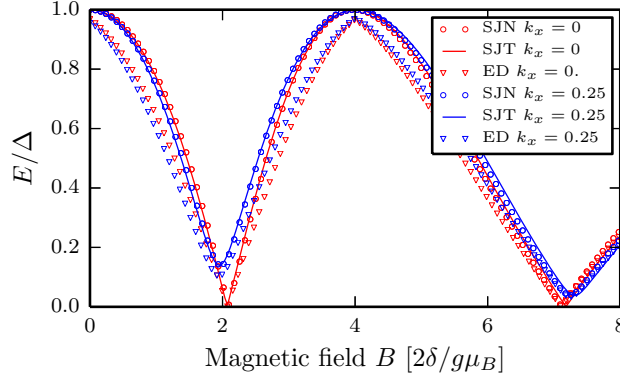


Figure 20: (Color online) Comparison of the Andreev spectra of a transparent NS junction between analytical short-junction predictions (SJT), numerical short-junction results including the spin-orbit interaction in the y direction (SJN), and exact diagonalization of the BdG Hamiltonian (15.18) (ED). The system parameters are chosen as in Sec. 15.4.2 and 15.4.3. The longitudinal momentum is either $k_x = 0$ (red), or finite such that the spectrum stays always gapped (blue). Momentum k_x is in units $k_F^0 = \sqrt{2m_n\mu_n/\hbar^2}$, and $\mu_s = \mu_n = 3 \text{ meV}$.

extra charge density at the interface smoothens the sharp change in velocity between the semiconductor and the superconductor and further enhances the transparency.

Transparency of the NS interface is hard to measure experimentally due to the complicated geometry of the normal metal-nanowire-superconductor samples. The experiments using high Δ superconductors such as NbTiN are in a long junction regime allowing us to estimate T because the induced superconducting gap is $\approx T\delta$. On the other hand, tunneling spectroscopy only provides a lower bound on the transparency: $T \gtrsim \Delta/\delta$ in the short-junction regime.

To explore the impact of interface transparency on the MBS properties we adopt two choices of v_s : the highly transparent interface corresponding to $v_s \approx v_{\pm}$ and an interface with a finite transparency where we fix the value of μ_s to a constant. For convenience we choose an anisotropic mass in the superconductor $m_{s,y} = m_n$, $m_{s,x} = m_{\parallel} \gg m_n$, so that $\mu_s = \mu_n$ results in a perfect transmission at a $k_x = 0$ and $B = 0$. The condition $m_{\parallel} \gg m_n$ ensures that v_s only weakly depends on k_x , as it should due to the Fermi surface in the superconductor being much larger than in the semiconductor. In most calculations we use $m_{\parallel} = 10 m_n$, however our conclusions are not sensitive to this choice (see Appendix 15.10 for details).

Adapting the calculations of Sec. 15.4.1 to the case of anisotropic mass and transparent limit yields the same result for the excitation spectrum (15.26), up to replacing the transverse momentum q in the superconductor with

$$q = \left[\frac{2m_n}{\hbar^2} \mu_s - \frac{m_n}{m_{\parallel}} k_x^2 \right]^{1/2}. \quad (15.30)$$

15.4.4 Comparison with tight-binding dispersion simulations

To verify the correctness of the spectrum in the short-junction limit, Eq. (15.26), we compare the analytical expressions with dispersion relations calculated using a Hamiltonian (15.15) discretized on a square lattice with lattice constant $a = 0.5 \text{ nm}$ and simulated using Kwant package[31]. We first numerically obtain the scattering matrix $S_e(k_x)$ of the normal region and use it as an input to Eq. (15.9) to obtain the dispersion relation of the hybrid system.

A further comparison is provided by modeling the hybrid system using the full BdG Hamiltonian (15.18) and calculating several eigenstates closest to the Fermi level. In this case, the

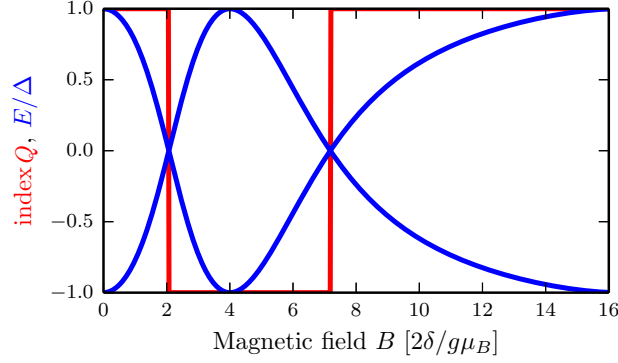


Figure 21: (Color online) The Andreev spectrum and the topological index $\mathcal{Q}$ (15.31) as a function of magnetic field. Trivial phase has $\mathcal{Q} = +1$, topological phase $\mathcal{Q} = -1$. The junction is in the transparent regime with $\mu_s = \mu_n = 3 \text{ meV}$.

junction remains infinite along the wire, but instead of a superconducting lead, we attach a large superconductor with width $W_{\text{SC}} \approx 9 \mu\text{m} \gg \xi_s$.

A comparison between analytics and the two numerical methods at a fixed chemical potential is shown in Fig. 20 and shows nearly perfect agreement between different methods. Slight deviations of exact diagonalization results occur near the bulk gap, caused by corrections to the short-junction approximation.

15.5 Analysis of the topological phase diagram

15.5.1 Phases boundaries and the spectral gap

Equation (15.27) yields the closing of the spectral gap and the topological transitions in the model. This allows us to define the topological invariant of the Hamiltonian as

$$\mathcal{Q} = \text{sign}[\text{Im}(\sqrt{-r_-^* r_+})]|_{k_x=0}, \quad (15.31)$$

where the sign of the square root is fixed by the analytic continuation and chosen such that $\mathcal{Q} = 1$ in the trivial state. A typical spectrum at $k_x = 0$ as well as $\mathcal{Q}$ for a fixed μ is shown in Fig. 21.

In addition to identifying the topological phase boundaries for each set of parameters (B, μ_n) we calculate the spectral gap

$$\Delta_{\text{spec}} = \min_{k_x} |E(k_x)|, \quad (15.32)$$

with $E(k_x)$ given by Eq. (15.26). The minimization is carried over all k_x present in the superconductor. In general, the dispersion relation has several local minima, as shown in Fig. 22, with the total number of minima approximately equal to the number of transverse modes in the normal region.

The resulting topological phase diagram of a transparent junction (with $\mu_s = \mu_n$) is shown in Fig. 23. For typical junction parameters (as described in Sec. 15.4.2) $g\mu_B B_c \approx 9\delta$, and the phase diagram for higher field values does not apply to such junctions. The minimal value of the critical field in this phase diagram corresponds to $B_* \approx 0.7 \text{ T}$. Near the topological phase transitions $\Delta_{\text{spec}} = \Delta_0$, the spectral gap at $k_x = 0$ (see Fig. 22), and it varies linearly with the distance ε from the phase transition either along the μ or E_Z axis:

$$\Delta_{\text{spec}} = \Delta_0 \sim \Delta \frac{\varepsilon}{\delta}. \quad (15.33)$$

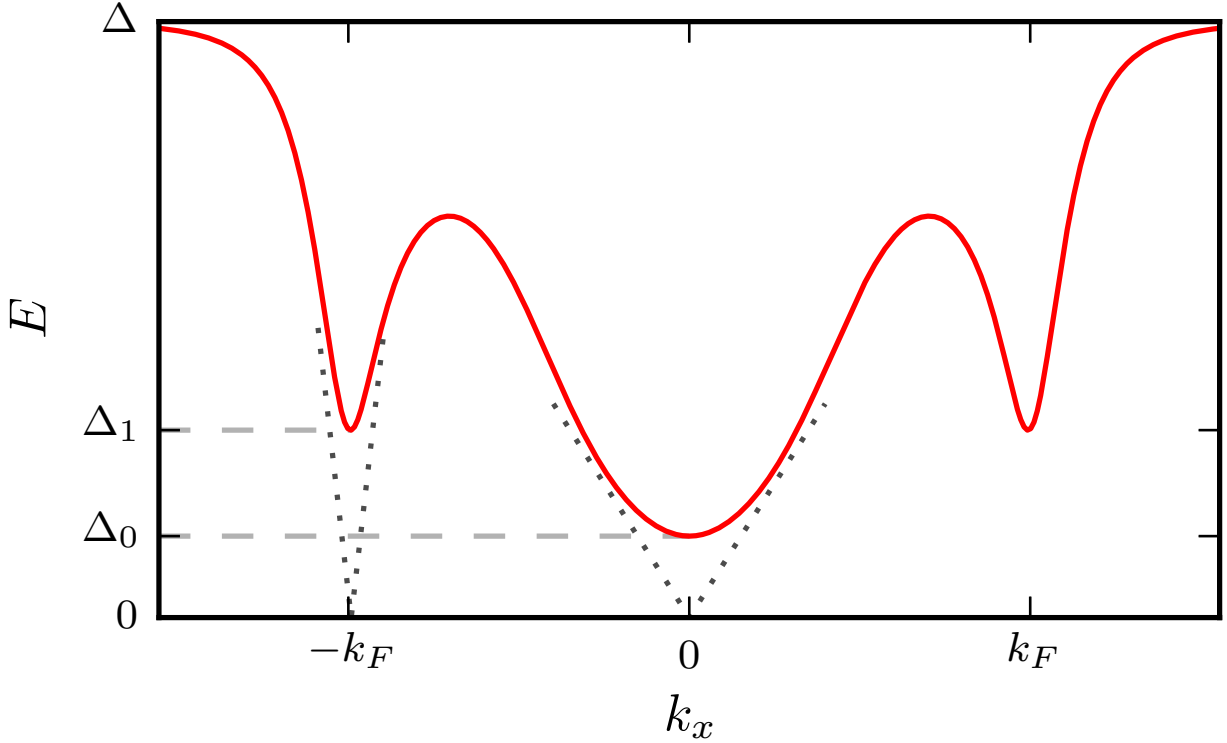


Figure 22: A schematic of the dispersion relation of the junction for typical parameters. The dispersion relation near local minima Δ_0 and Δ_1 of the Andreev state energy is well approximated with a gapped Dirac dispersion relation, with Dirac cones marked with dashed lines. When the number of available modes in the semiconductor increases, the number of minima at finite momenta grows, but the outermost minimum stays located approximately at $k_x = k_{n,F}$.

Deep in the topological phase, Δ_{spec} is limited by the gap Δ_1 at $k_x \approx k_{n,F}$ (see Fig. 22), similar to the phenomenological model of Eq. (15.14). Since Δ_{spec} must vanish linearly with α in this regime, we get

$$\Delta_{\text{spec}} = \Delta_1 \sim \Delta \sqrt{\frac{E_{\text{SO}}}{\delta}}. \quad (15.34)$$

In both estimates we assumed $\mu \sim E_Z \sim \delta$, and $T \sim 1$. Comparing Eqs. (15.33) and (15.34) we find the energy scale for the transition between the two behaviors $\varepsilon_* \sim \sqrt{E_{\text{SO}}\delta}$.

The most unusual feature of the topological phase diagram in Fig. 23 is the smooth behavior of the topological phase boundary, different from the hyperbolically-shaped boundary $E_Z^2 > \Delta'^2 + \mu^2$ of the phenomenological models[6, 7, 22]. This difference appears not due to the short-junction limit—since the magnitude of the gap does not impact the topological phase boundary—but rather because of the high interface transparency. The inset in Fig. 23 shows the shape of the topological phase boundary where we have reduced the transparency by fixing μ_s at a high value. We find that in this case the phase boundary has a hyperbolic shape predicted by the phenomenological model.

15.5.2 Decay length of MBS

Exact evaluation of the MBS decay length starting from Eq. (15.26) is not possible because the spectrum in the short-junction approximation does not correspond to a local Hamiltonian (the same fact manifests in the complex nonlinear dispersion of the Andreev states). Nevertheless, the decay length is approximated well by assessing the contributions of different local

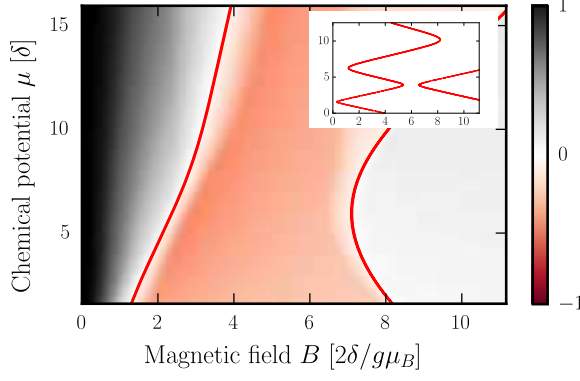


Figure 23: (Color online) The spectral gap times the topological index $Q\Delta_{\text{spec}}/\Delta \in (-1, 1)$ as a function of chemical potential μ and magnetic field B . Here we consider a transparent NS interface $\mu = \mu_s = \mu_n$ and an anisotropic mass in the superconductor $m_{\parallel} = 10 m_n$, $m_{\perp} = m_n$. The phase boundaries Eq. (15.27) are given by continuous red lines. The central region of the phase diagram is the topological phase $Q = -1$. The inset shows the phase boundaries in a similar parameter range for a junction with $\mu_s = 11.7 \text{ eV}$, $m_s = 0.015 m_n$ resulting in a low interface transparency.

minima of the dispersion relation, as shown in Fig. 22. A gapped Dirac cone with velocity v and gap Δ results in a wave function decay length $\xi = \hbar v / \Delta$ at $E = 0$. The size of the MBS ξ_M is set by the slowest decaying component of the wave function, or the largest ξ .

Once again, it is instructive to estimate ξ using scaling arguments in two regimes: near a topological transition and deep in the topological phase. At the phase transition point the slope of the Dirac cone at $k_x = 0$, $v_0 \propto \alpha$ since without spin-orbit coupling the band touching at $k_x = 0$ must have a parabolic shape. Since the bulk superconductor gap Δ must enter the spectrum only as an overall prefactor, we get

$$\hbar v_0 \sim \frac{\Delta W^2}{l_{\text{SO}}}, \quad \xi_0 = \frac{\hbar v_0}{\Delta_0} \sim \frac{W^2 \delta}{l_{\text{SO}} \varepsilon}. \quad (15.35)$$

The velocity at the outermost Dirac point must not depend on α , resulting in

$$\hbar v_1 \sim \Delta W, \quad \xi_1 = \frac{\hbar v_1}{\Delta_1} \sim l_{\text{SO}}. \quad (15.36)$$

The two length scales ξ_0 and ξ_1 become equal at $\varepsilon \sim E_{\text{SO}} \ll \sqrt{E_{\text{SO}} \delta} = \varepsilon_*$.

We obtain the behavior of ξ in the tight-binding simulations using Kwant [31] for the same parameters as in Fig. 23. In order for the self-energy to become local in the x coordinate we neglect the transverse dispersion in the superconductor and set $m_{\parallel} = \infty$. We then integrate out the superconductor and add a self-energy to the semiconductor. Finally, similar to Ref. [32] we perform an eigendecomposition of the translation operator in the x direction at zero energy to obtain the evanescent waves $\psi \propto e^{-\kappa x}$, with κ the eigenvalue of the translation operator. The largest decay length is:

$$\xi = \max \text{Re}[\kappa]^{-1}, \quad (15.37)$$

where the maximum is taken over all the eigenvalues. Then in the topological phase $\xi_M = \xi$. The results are presented in Fig. 24. The divergence in decay lengths seen in Fig. 24 corresponds to topological transitions identical to the ones found in Fig. 23. Figure 24 also confirms that ξ_M saturates at a distance $\varepsilon \sim E_{\text{SO}}$ away from that phase transition (here $E_{\text{SO}} \approx 40 \mu\text{eV}$).

We now refine the scaling arguments of Eqs. (15.35) and (15.36) by using Eq. (15.26). In particular, near the topological transition, the decay length is determined by the spectral gap

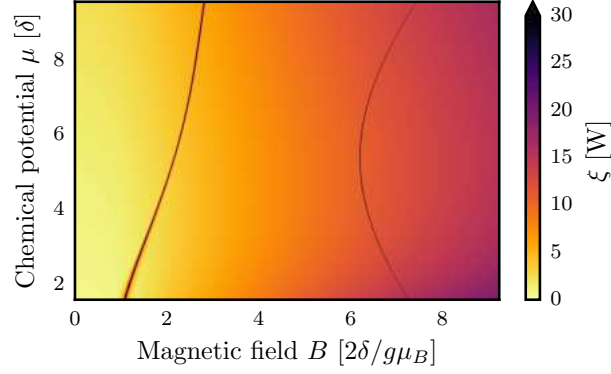


Figure 24: (Color online) The largest decay length of subgap modes in units of wire width W , as a function of chemical potential μ and magnetic field B . At the topological transitions the decay length rapidly diverges. The junction is in a fully transparent regime with $\mu_n = \mu_s$ and $m_{\parallel} = \infty$ in the superconductor.

Δ_0 and the velocity $\hbar v_0 = |\partial E / \partial k_x|_{\Delta_0=0, k_x=0}$:

$$\Delta_0 = \frac{\Delta}{2} |r_+ + r_-|_{k_x=0}, \quad (15.38)$$

with Fermi velocity

$$\hbar v_0 = \frac{\alpha \Delta}{E_Z}. \quad (15.39)$$

Therefore the MBS decay length ξ_M is inversely proportional to the magnetic field and spin-orbit length near the Majorana phase transitions.

Deep in the topological phase it is more difficult to obtain a closed form approximation for the decay length. Instead, we find the Fermi momentum and the spectral gap by performing numerical minimization of the energy dispersion (15.26). The Fermi velocity near $k_F \neq 0$ follows immediately:

$$\hbar v_1 = \frac{\Delta}{2} \frac{\partial}{\partial k_x} |r_+ + r_-|_{\alpha=0, k_x=k_F}. \quad (15.40)$$

Taking the ratio (15.36), it follows that the MBS decay length does indeed grow linearly with magnetic field and spin-orbit length deep in the topological phase [see Fig. 25], in qualitative agreement with the numerical calculation using Eq. (15.37).

Our results for the scaling of ξ_M with B and l_{SO} agree with the predictions of the phenomenological 1D model both near the topological transition or deep in the topological phase (see, e. g., Ref. [33]), but we find no dependence of ξ_M on Δ .

15.6 Spectrum of finite length junctions

To directly verify the existence of a MBS and the applicability of the short-junction limit to our system, we solve the discretized BdG Hamiltonian of a large rectangular system with a finite superconductor. The system is divided into semiconductor and superconductor regions, both modeled by the BdG Hamiltonian (15.18). The length of the system is $L = 3 \mu\text{m}$, sufficiently long to ensure that the overlap between MBS is small. Further, we choose the width of the superconductor sufficiently large $W_{\text{SC}} = 1.4 \mu\text{m} \approx 2 \xi_{\text{S}}$. The lattice constant in the tight-binding simulation is 10 nm. Finally, the remaining model parameters are chosen according to Sec. 15.4.2 and Sec. 15.4.3. We determine numerically several lowest energy states and compare them with Δ_{spec} calculated in Sec. 15.5, as shown in Fig. 26.

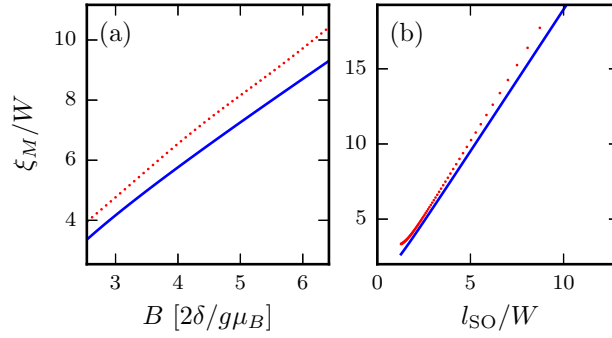


Figure 25: (Color online) Scaling behavior of the MBS decay length ξ_M deep in the topological phase. Comparison between symbolic calculation of the decay length from the linearization of the energy dispersion (15.26) (blue lines) and the numerical calculation of the slowest decaying mode Eq. (15.37) (red markers). (a) The decay length has a linear dependence with magnetic field B deep in the topological region, $\mu = 3$ meV. (b) Linear dependence with spin-orbit length $l_{\text{SO}} = \hbar^2/m\alpha$. The magnetic field is $B = 1.5$ T and chemical potential $\mu = 3$ meV.

We observe that the energy of most subgap states is bounded from below by an energy slightly lower than Δ_{spec} , as expected close to the short-junction regime. At $B > B_* \approx 1$ T the system enters a topological regime and states with $E \ll \Delta_{\text{spec}}$ formed by two coupled MBS appear. The coupling of these states decays exponentially with the size of the nanowire L . Finally after the system undergoes the second gap closing and enters the trivial phase at $B \approx 3$ T additional low energy states appear due to the presence of chiral symmetry of the Hamiltonian (15.15) [34, 35, 36]. We therefore conclude that our calculations fully apply to finite nanowires in the short-junction regime.

Furthermore, we verify in Appendix 15.11 using exact diagonalization that the critical field B_* is indeed independent on the superconducting gap Δ . With increasing Δ above $E_{\text{Th}} \approx 1$ meV the system exits the short-junction regime. Nevertheless, the critical magnetic field stays constant, in agreement with the proof of Sec. 15.3.

15.7 Orbital field effect in thin shell approximation

We now turn to evaluate the consequences of the orbital effect of the magnetic field, known to strongly influence MBS properties [37, 38, 39, 32], in the short-junction limit. This effect does not manifest in the model of Sec. 15.4 when magnetic field points in the x direction. To include the orbital effect we use a thin shell approximation, when the electron wave function in the semiconductor is confined to its surface, similar to the system studied in Ref. [38]. However unlike Ref. [38] we do not assume a constant induced gap and consider instead the nanowire contacted by a bulk superconductor, as shown in Fig. 27. We model the coupling to the superconductor as two infinite planar superconductors on each side of the 2D uncovered wire section. By doing so we neglect the possibility for electrons to tunnel through the superconductor to the other side of the uncovered section, which is justified by the density mismatch between the superconductor and the semiconductor. The thin shell limit is oversimplified and it overestimates the orbital effect of a magnetic field, however it provides an upper bound on the impact of the orbital effect and remains analytically tractable.

The superconductor covers the wire over an angle $2\pi - \theta_0$, while both the wire and the superconductor are translationally invariant in the x direction. In cylindrical coordinates (x, θ) the electron Hamiltonian on the nanowire surface reads:

$$H = \left[\frac{p_\theta^2 + p_x^2}{2m} - \mu \right] \sigma_0 - \frac{\alpha}{\hbar} p_x \sigma_y + E_Z \sigma_x, \quad (15.41)$$

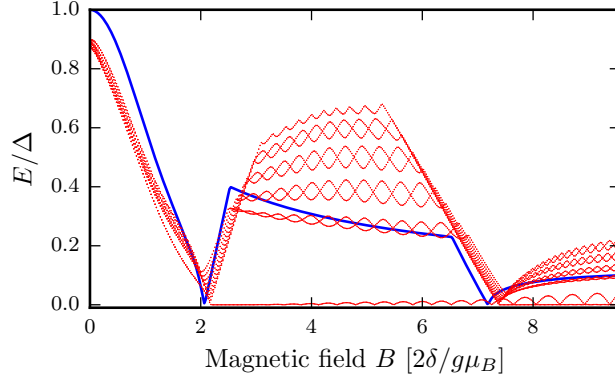


Figure 26: (Color online) Comparison between the predictions of the analytical short-junction approximation and a numerical spectrum of a finite NS junction. Solid line: Δ_{spec} calculated using Eqs. (15.26) and (15.32) as a function of magnetic field. Dotted lines: 10 lowest energy states in a finite size NS junction using the same parameters. The magnetic field is in units of $2\delta/g\mu_B$, with level spacing δ defined in Eq. (15.11). The junction is in a transparent regime, the superconductor has anisotropic mass $m_{\parallel} = 10m_n$, and the rest of parameters are as specified in Sec. 15.4.2. At a single end of the nanowire, there is only one MBS in the topological phase ($1\text{ T} \lesssim B \lesssim 3\text{ T}$, for semiconductor width $W = 100\text{ nm}$) and two in the trivial phase at high field, due to the chiral symmetry.

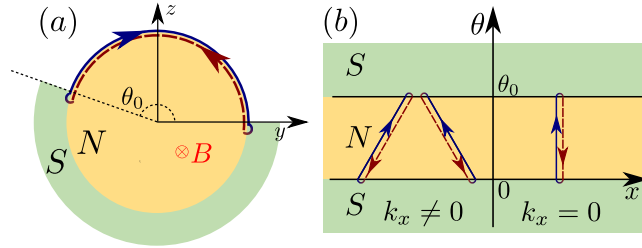


Figure 27: (Color online) (a) A cross section of a NS hybrid junction. The magnetic field is parallel to the wire axis, while the Andreev bound state trajectories are confined to the nanowire surface. (b) The equivalent two-dimensional system defined on the plane (x, θ) . Since we neglect the possibility for electrons to tunnel through the superconductor, we consider the superconducting leads at $\theta < 0$ and $\theta > \theta_0$ infinite.

with $p_{\theta} = -i\hbar R^{-1}\partial/\partial\theta$, and R the radius of the nanowire. We assume that the magnetic field is fully screened from the superconductor and choose a gauge where the vector potential $A = 0$ in the uncovered part of the surface, while the two superconducting leads have a phase difference $\phi = (2e/\hbar)\pi BR^2$. Compared to the previous sections where the treatment was more general, we assume from the start the transparent junction limit, when Fermi velocities at $k_x = 0$ are identical in the superconductor and the semiconductor and we also neglect the spin-orbit coupling in the transverse direction, as appropriate for $l_{\text{SO}} \gg R\theta_0$.

We solve the scattering problem in the basis of conserved spin projections set by Eq. (15.20) corresponding to the basis of incoming and outgoing modes:

$$\mathbf{a}^T = (a_L^+, a_L^-, a_R^+, a_R^-), \quad \mathbf{b}^T = (b_L^+, b_L^-, b_R^+, b_R^-), \quad (15.42)$$

with $\mathbf{a}$ and $\mathbf{b}$ the amplitudes of incoming and outgoing modes, R denoting the modes at $\theta \leq 0$, L the modes at $\theta \geq \theta_0$, and $\pm$ superscript corresponding to the two conserved spin directions (15.20). For each spin projection the scattering matrix is given by the classic result

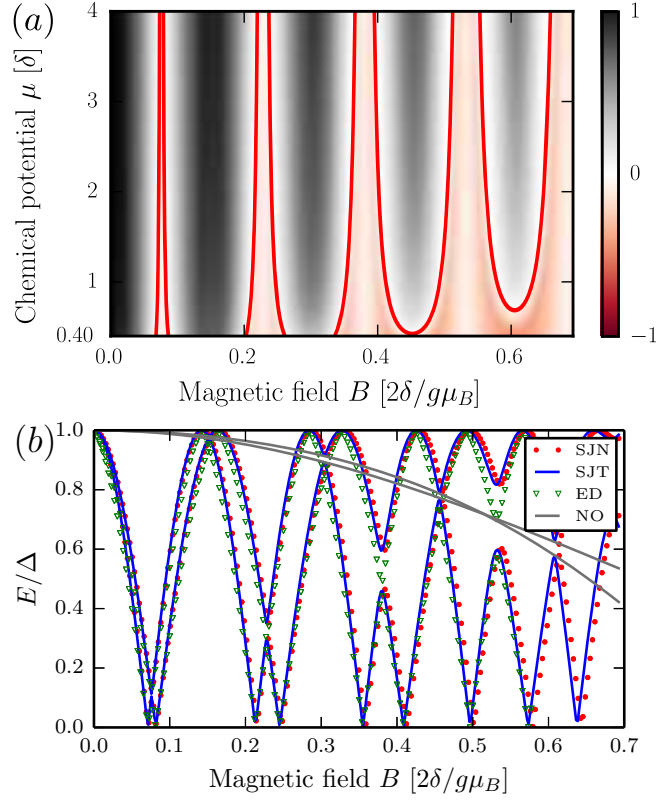


Figure 28: (Color online) (a) Majorana phase diagram $\mathcal{Q}\Delta_{\text{spec}}$ of a transparent NS junction with $\mu_s = \mu_n = \mu$, $m_{s,x} = m_{\parallel}$, $m_{s,y} = m$, $m_n = m$, and $m_{\parallel} = 10m$, as a function of chemical potential and magnetic field. The covering angle is $\theta_0 = 2$ rad, so that the width of the uncovered section $R\theta_0$ is equal to the wire diameter. (b) An example of Andreev spectrum at $k_x = 0$, $\mu_n = \mu_s = 3$ meV, and other parameters the same as in (a) in the presence and absence of the orbital effect. Panel (b) additionally presents a comparison between short junction theoretical (SJT), numerical (SJK), and exact diagonalization (ED). The theoretical spectrum without orbital effect (NO) is in gray. The magnetic field is in units of $2\delta/g\mu_B$ with $\delta = \hbar^2\pi^2/2mR^2\theta_0^2$.

for transmission through a potential barrier:

$$S_{\pm} = \begin{pmatrix} r_{\pm} & t_{\pm} \\ t_{\pm} & r_{\pm} \end{pmatrix}, \quad (15.43)$$

with

$$\begin{aligned} r_{\pm} &= \frac{(q^2 - k_{\pm}^2) \sin(k_{\pm}L)}{(q^2 + k_{\pm}^2) \sin(k_{\pm}L) + 2iqk_{\pm} \cos(k_{\pm}L)}, \\ t_{\pm} &= \frac{2iqk_{\pm}}{(q^2 + k_{\pm}^2) \sin(k_{\pm}L) + 2iqk_{\pm} \cos(k_{\pm}L)}, \end{aligned} \quad (15.44)$$

with momenta $k_{\pm}$ and q defined by Eq. (15.30). We then transform the scattering matrix to the basis of time-reversed modes (15.21) and calculate the Andreev spectrum using Eq. (15.9) with the phases of superconducting leads equal to $\phi_R = 0$ and $\phi_L = \phi$. We verify again that the dispersion relation obtained this way agrees well with two numerical tight-binding simulations at fixed chemical potential and that the difference also stays small if we include spin-orbit coupling in the transverse direction [see Fig. 28(b)]. As before, the spectrum is generically gapped except at $k_x = 0$, where topological phase transitions occur.

The resulting Majorana phase diagram is shown in Fig. 28(a), and it consists of several narrow topological regions centered around $\phi_k = (2k + 1)\pi$ with k integer. At these values of

magnetic field, the two superconducting leads have a phase difference of π , thus fully suppressing the induced gap in the transparent limit. The Zeeman field then opens a topological gap resulting in a finite extension of the topological phases around ϕ_k . We conclude that in the thin shell limit, the orbital effect of the magnetic field reduces B_* by a factor ~ 10 for typical junction parameters (we once again note that the thin shell limit overestimates the orbital effect of the magnetic field). Despite that, it is the Zeeman field responsible for opening the topological gap.

15.8 Numerical study of a three-dimensional nanowire

To confirm our findings in a model with a more realistic geometry, we numerically calculate the phase diagram of a three-dimensional nanowire in the short-junction limit. The system consists of a semiconductor nanowire infinite in the x direction and with a square cross section contacted by a bulk superconductor occupying $y < 0$ half space [see Fig. 19(a)].

Due to the large Fermi surface mismatch between the superconductor and the semiconductor we neglect the electron dispersion in the x and z directions in the superconductor. Therefore, following Sec. 15.4.2 we set $m_{s,y} = m_n = 0.015 m_e$ and $m_{s,x} = m_{s,z} = \infty$. Since the semiconductor modes with different values of k_z have different interface transparencies, we cannot ensure a transparent interface for all the modes and instead fix $\mu_s = 8 \text{ meV}$ while varying $\mu_n \equiv \mu$. The remaining system parameters are specified in Sec. 15.4.2.

The model Hamiltonian is a three-dimensional generalization of Eq. (15.15) discretized on a cubic lattice. We include the orbital effect of the magnetic field using Peierls substitution in the gauge $\mathbf{A} = (0, 0, By\Theta(y))^T$. This ensures that the vector potential is constant in the x direction and that it vanishes at the interface with the superconductor.

We calculate the excitation spectrum using Eq. (15.9) to find Δ_{spec} by minimization over k_x . The resulting phase diagram of Δ_{spec} is shown in Fig. 29. Comparing the top panels of Fig. 29 with Fig. 23 we observe the two sharp minima of $B_*(\mu)$. These correspond to the appearance of the additional bands with a different value of k_z and a minimum in the interface transparency.³

Similar to our observations from Sec. 15.7, the orbital effect of the magnetic field has a strong effect on the shape of the topological phase boundaries and reduces both Δ_{spec} and B_* similar to the thin shell simulations. Increasing the cross section of the wire [Fig. 29(a), (c), against (b), (d)] confirms that in 3D the critical fields preserve the scaling with $B_* \sim 1/W^2$ independent of the presence or absence of orbital effects.

15.9 Conclusions and outlook

We have studied the impact of a small superconducting gap on the properties of MBS in semiconductor-superconductor junctions. The short-junction formalism, appropriate for this limit, allows us to draw universal conclusions about the MBS properties. Contrary to the intuitive expectations, we show that the reduction of the superconducting gap does not alter the Majorana phase diagram and does not change the size of the MBS. We therefore conclude that in most practical systems the superconducting gap should not be used as an important parameter in optimizing MBS properties.

On the other hand, we find that the transparency of the semiconductor-superconductor boundary has an important and previously overlooked effect on the Majorana phase diagram. An interface with $T \approx 1$ produces a phase boundary between trivial and topological phases which depends weakly on the chemical potential. This is in contrast to $T \ll 1$, used in most

³The code for the tight-binding simulations of the two-dimensional and three-dimensional junctions is available in ancillary files to the manuscript.

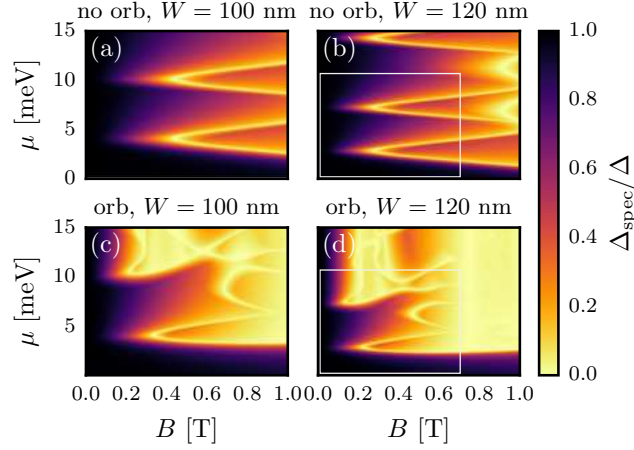


Figure 29: (Color online) Spectral gap $\Delta_{\text{spec}}/\Delta$ dependence on chemical potential μ in the nanowire and magnetic field B of a square nanowire without (top panels) and with (bottom panels) orbital effect of the magnetic field. Panels (a), (c) show results for wire section $100 \text{ nm} \times 100 \text{ nm}$, while (b), (d), for $120 \text{ nm} \times 120 \text{ nm}$. The white box in the right panels shows the same parameter range rescaled by a factor $(100/120)^2$ to highlight the W^2 scaling of the phase diagram.

prior research, that results in the critical magnetic field having an oscillatory dependence on chemical potential with minima corresponding to the opening of a new band.

Orbital effect of magnetic field plays a dual role: It reduces the critical magnetic field as well as the spectral gap in the topological regime. Contrary to the predictions of a phenomenological model that assumes a constant induced gap, we show that relative importance of magnetic field cannot be controlled by the superconducting gap or the diameter of the nanowire.

Our findings suggest that creation of MBS in proximitized two-dimensional electron gases laterally contacted by a superconductor is a promising direction of further research. In these systems the relative strength of the orbital and the Zeeman effect of magnetic field is controlled by an extra tuning parameter: the ratio between the semiconductor thickness and its width. Additionally, the critical magnetic field in such devices could be tuned using a side gate, effectively changing the semiconductor width without altering the superconductor-semiconductor interface transparency.

Another important further direction of research is the interplay between junction transparency and disorder. Since a transparent interface results in a weaker dependence of the critical magnetic field on the chemical potential, it is reasonable to conjecture that the sensitivity of MBS properties to disorder is also reduced in the transparent regime.

15.10 Interface transparency in a two-dimensional junction

The validity of the short-junction approximation depends on NS interface transparency T . In this section, we review the transparency of a sharp interface between two materials with a parabolic dispersion. We provide quantitative arguments for the choice of anisotropic mass in the superconductor in modeling a transparent interface.

We consider a planar NS interface with the boundary located at $y = 0$, and both materials occupying a half-plane, and solve the scattering problem as outlined in Sec. 15.4.1. Also following Sec. 15.4.1, we neglect the spin-orbit scattering at the interface, and use the boundary condition (15.23). At a given energy E there are generally two modes in the semiconductor and two spin-degenerate modes in the superconductor with momenta in the y direction: $k_{\pm}$ and q ,

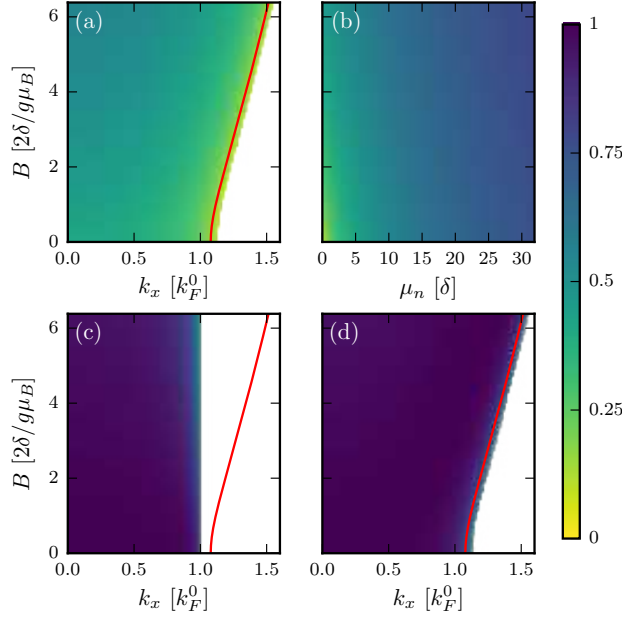


Figure 30: (Color online) Transmission probability of one of the spin polarizations T_- of an infinite NS junction. (a, c, and d) T_- as a function of magnetic field B in units of $\frac{2\delta}{g\mu_B}$, with δ defined in Eq. (15.11) and parallel momentum k_x (in units of $k_F^0 = \sqrt{2m_n\mu_n/\hbar^2}$). The momenta k_x run over the Fermi surface of the semiconductor, which is marked by the red line. (a) Using bare material parameters $\mu_n = 3$ meV, $\mu_s = 11.7$ eV, $m_s = m_e$, $m_n = 0.015 m_e$. (b) T_- versus B and μ_n at $k_x = 0$ and the same parameters as in (a). (c) T_- when the chemical potential and mass are equal in the superconductor and the semiconductor. (d) T_- for anisotropic mass in the superconductor ($m_{\parallel}, m$), $m_n \equiv m = 0.015 m_e$, with $m_{\parallel} = 10 m$, $\mu_n = \mu_s = 3$ meV. Only evanescent solutions exist in the white regions; transmission T_- becomes imaginary.

respectively:

$$k_{\pm} = \left[\frac{2m_n}{\hbar^2} (E + \mu_n \mp \sqrt{E_Z^2 + \alpha^2 k_x^2}) - k_x^2 \right]^{1/2}, \quad (15.45)$$

$$q = \left[\frac{2m_{\perp}}{\hbar^2} (E + \mu_s) - \frac{m_{\perp}}{m_{\parallel}} k_x^2 \right]^{1/2},$$

where we use the same notation as in Sec. 15.4.1 the superconductor has anisotropic mass ($m_{\perp}, m_{\parallel}$).

The transmission probabilities of two spin orientations ($\pm$) follow immediately:

$$T_{\pm} = 4 \left(\sqrt{\frac{v_{\pm}}{v_s}} + \sqrt{\frac{v_s}{v_{\pm}}} \right)^{-2}, \quad (15.46)$$

with $v_s = \hbar q/m_{\perp}$ and $v_{\pm} = \hbar k_{\pm}/m_n$ the velocities normal to the interface in two materials. Both T_+ and T_- exhibit similar behavior, except for T_+ vanishing inside the helical gap. In contrast, T_- is always well defined at $k_x = 0$ for $\mu_n > 0$. For concreteness, we illustrate the dependence of T_- on the Hamiltonian parameters.

Let us first start with realistic parameters both in the superconductor and the semiconductor. The chemical potential in the nanowire μ_n is gate tunable. We choose to fix it at 3 meV, comparable to the level spacing in a nanowire. The rest of the parameters are specified in Sec. 15.4.2. The results are plotted in Fig. 30(a), for all momenta in the semiconductor Fermi surface and an experimentally relevant range of magnetic fields. The transparency is

mostly around 40% but rapidly vanishes near the Fermi momentum. Modifying the chemical potential in the wire does not appreciably increase the transparency [see Fig. 30(b)]. The low transparency is artificial and due to the choice of a sharp change in mass and chemical potential across the interface.

Choosing $m_{\perp} = m_{\parallel} = m_n$ and $\mu_s = \mu_n$ results in a nearly perfect transmission at all angles, as shown in Fig. 30(c). However, this parameter choice is also unphysical since the semiconductor Fermi surface becomes larger than the superconductor one at any finite magnetic field. Then the interface becomes opaque for higher momenta k_x .

Finally, choosing $\mu_s = \mu_n$ and an anisotropic mass in the superconductor $(m_{s,x}, m_{s,y}) = (m_{\parallel}, m_n)$, with $m_{\parallel} \gg m_n$ results in an interface that stays transparent for all k_x [see Fig. 30(d)].

15.11 Independence of critical magnetic field on the superconducting gap

Here we verify the validity of our conclusions about the scaling of the eigenenergies and the independence of B_* on Δ using exact diagonalization of a finite BdG Hamiltonian for the semiconductor-superconductor heterostructure at different values of the superconducting gap. We use the same setup of a finite heterojunction modeled with the BdG Hamiltonian (15.18) as in Sec. 15.6. We check the behavior of the critical field and the bulk band gap by tracking the energy of the second excited state while varying B for values of Δ ranging from $40 \mu\text{eV}$ to 3 meV (almost two orders of magnitude). Our results for a heterojunction of size $3000 \text{ nm} \times 6000 \text{ nm}$ with a normal region occupying $3000 \text{ nm} \times 100 \text{ nm}$ are shown in Fig. 31.

When $\Delta \gtrsim E_{\text{Th}} \approx 1 \text{ meV}$ the system transitions to the long junction regime, so that the ratio $\Delta_{\text{spec}}/\Delta$ continues to decrease, while Δ_{spec} becomes almost independent on Δ . In the opposite limit $\Delta \ll E_{\text{Th}}$, we observe that $\Delta_{\text{spec}}/\Delta$ tends to a constant, in agreement with the short-junction limit prediction. The field values B_* where Δ_{spec} vanishes stay almost constant, with the residual variation due to the effect of a finite system size and lattice constant.

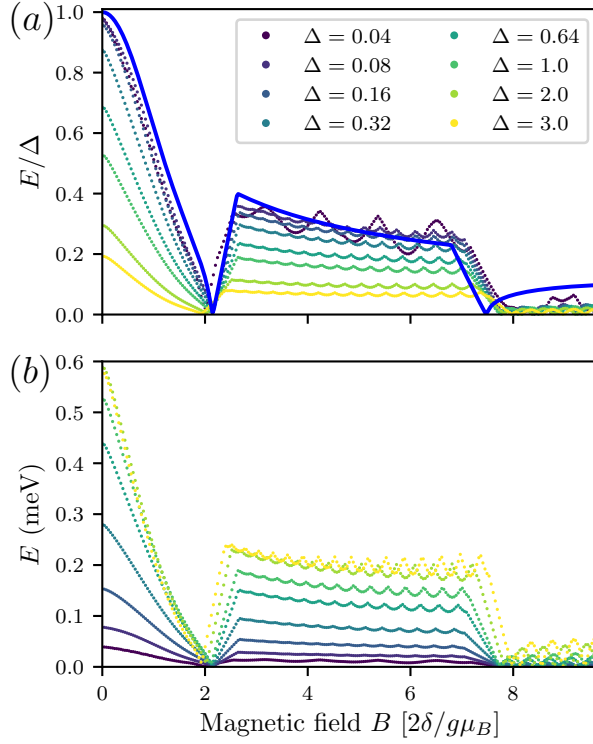


Figure 31: (Color online) The energy of the second excited state of a finite nanowire junction calculated using exact diagonalization. Up to the second topological phase transition it is a good approximation of the induced gap. The eigenenergies are normalized to Δ in panel (a) and unnormalized in panel (b). In the long junction regime the induced gap tends to a constant, while in the short-junction regime the ratio E/Δ tends to the analytical result derived for the short-junction limit. The legend applies to both panels. The superconducting gaps Δ are in meV.

Part VIII

Other, Non-Typical Theories, Effects

16 Josephson Current and Noise at a Superconductor-Quantum Spin Hall Insulator-Superconductor Junction by Fu, Kane

Abstract

We study junctions between superconductors mediated by the edge states of a quantum spin Hall insulator. We show that such junctions exhibit a fractional Josephson effect, in which the current phase relation has a 4π , rather than a 2π periodicity. This effect is a consequence of the conservation of fermion parity - the number of electrons modulo 2 - in a superconducting junction, and is closely related to the Z_2 topological structure of the quantum spin Hall insulator. Inelastic processes, which violate the conservation of fermion parity, lead to telegraph noise in the equilibrium supercurrent. We predict that the low frequency noise due these processes diverges exponentially with temperature T as $T \rightarrow 0$. Possible experiments on HgCdTe quantum wells will be discussed.

Proposals for fault tolerant topological quantum computation have motivated intense current interest in finding robust physical systems that host excitations with non-Abelian statistics [4, 2]. Recent experiments on the $\nu = 5/2$ fractional quantum Hall effect have shown encouraging indirect evidence for such excitations[3, 4], but the direct observation of non-Abelions has so far remained elusive. Recently we showed that the proximity effect between a superconductor and a three dimensional (3D) topological insulator leads to a 2D interface state that supports non-Abelian Majorana fermions[5]. A first step towards implementing this proposal would be to demonstrate experimentally the topological order responsible for Majorana fermions.

In this paper we study Josephson junctions mediated by a 2D topological insulator, also known as a quantum spin Hall insulator (QSHI)[6, 7, 8, 9, 10]. We predict that such junctions exhibit a *fractional Josephson effect*, whose origin is related to the presence of Majorana fermions. The signature of the fractional Josephson effect is that the current phase relation has a 4π rather than a 2π periodicity. This behavior was first predicted by Kitaev using an idealized model of a 1D spinless p wave superconductor[11]. Kwon et al.[12] proposed that a related effect can occur at junctions between unconventional 3D superconductors. They argued that it leads to an AC Josephson effect with half the usual Josephson frequency and that in a weak tunneling limit the Josephson current is carried by electrons rather than Cooper pairs. The 4π periodicity can occur because the junction has two states with different Josephson currents that are interchanged when the phase is advanced by 2π . At finite temperature inelastic processes can cause transitions between the states, leading to *telegraph noise* in the Josephson current. We will show that in our setup (unlike Ref. [12]) these transitions are forbidden by the local conservation of *fermion parity*, which counts the number of electrons modulo 2. This leads to an *exponential* suppression of the transition rate at low temperature. This can be experimentally probed by measuring the low frequency current noise $S(\omega \rightarrow 0)$, which we predict diverges exponentially at low temperature.

The QSHI is a time reversal invariant insulating state with a bulk energy gap generated by spin orbit interactions[6, 9]. It has recently been observed in HgCdTe quantum wells[10]. The QSHI is distinguished from an ordinary insulator by a Z_2 topological invariant[7], which necessitates the existence of gapless edge states. The states at the edge of the QSHI form a unique 1D system that is essentially *half* of an ordinary spin degenerate 1D electron gas. In the

simplest case it consists of a single band of right moving electrons paired via Kramers theorem with a left moving band with the opposite spin. These states are robust in the presence of disorder because time reversal symmetry prevents elastic backscattering. In the absence of inelastic scattering the transmission in the edge states is perfect.

Suppose the edge states are in intimate contact with an s wave superconductor. Due to the proximity effect, the tunneling of Cooper pairs will induce a pairing potential $\Delta = \Delta_0 e^{i\phi}$ in the edge states, which depends on the phase ϕ of the superconductor and the nature of the contact. Using the notation of Ref. [5] write $H = \Psi^\dagger \mathcal{H} \Psi / 2$, where $\Psi = [(\psi_\uparrow, \psi_\downarrow), (\psi_\downarrow^\dagger, -\psi_\uparrow^\dagger)]$ is express in terms of field operators $\psi_{\uparrow(\downarrow)}$ describing the right(left) movers and

$$\mathcal{H} = -iv\tau_z\sigma_z\partial_x - \mu\tau_z + \Delta_0(\cos\phi\tau_x + \sin\phi\tau_y). \quad (16.1)$$

σ_j are Pauli matrices acting in the space of right and left movers $\psi_{\uparrow,\downarrow}$ and τ_j are Pauli matrices which mix the ψ and $\psi^\dagger$ blocks of Ψ . v is the velocity of the edge states, μ is the chemical potential and we set $\hbar = 1$. The eigenstates of (16.1) come in pairs at $\pm E$. Due to the redundancy in Ψ , these states are not independent, and the Bogoliubov quasiparticle operators satisfy $\Gamma_{-E} = \Gamma_E^\dagger$.

Eq. 16.1 is similar to Kitaev's model of superconducting spinless electrons in 1D[11]. In Kitaev's model there are zero energy Majorana bound states associated with the *ends* of the sample. In our system, the edge - which is the boundary of the 2D QSHI - can not have an end. By breaking time reversal symmetry, however, a Zeeman field can introduce a mass term into H of the form

$$V_Z = M\psi^\dagger\sigma_x\psi = M\Psi^\dagger\sigma_x\Psi/2. \quad (16.2)$$

When $M > \mu$, V_Z opens an insulating gap in the edge state spectrum. V_Z could arise either from an applied magnetic field (as in Ref. [10]) or due to proximity to a magnetic material. Zero energy Majorana bound states will exist at the interface between regions with gaps dominated by Δ and M [5]. In the presence of both Δ and M the gap is the smaller of $|\Delta_0 \pm M|$. When $\Delta_0 = |M|$ a single band is gapless, and for $\Delta_0 \sim |M|$ the low energy sector of (16.1) has the form of a Su-Schrieffer-Heeger model[13], which has a well known zero energy bound state where $\Delta_0 - |M|$ changes sign. The Bogoliubov quasiparticle operator associated with this state is a Majorana fermion, which satisfies $\gamma_0 = \gamma_0^\dagger$.

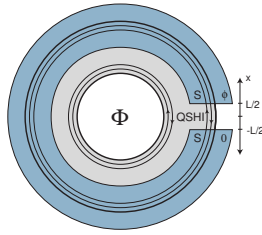


Figure 32: A S-QSHI-S junction in an RF SQUID geometry where the QSHI forms a Corbino disk.

Consider a superconductor-QSHI-superconductor (S-QSHI-S) junction in which the edge states of a QSHI connect two superconductors separated by a distance L . Fig. 1 shows an RF SQUID geometry, in which the phase difference across the junction $\phi = (2e/\hbar)\Phi$ is controlled by the magnetic flux Φ . We also assume that the QSHI forms a Corbino disk which circles the flux. As a practical matter, this geometry is not essential, but we will see that it has considerable conceptual value. We will also consider the effect of a Zeeman term in the gap between the superconductors, which will make the connection with Majorana bound states transparent. We emphasize, however, that there will be a non trivial effect even when this term

is absent. To determine the characteristics of the junction we solve the Bogoliubov de Gennes (BdG) equation with

$$\begin{aligned}\Delta(x) &= \Delta_0 [\theta(-x - L/2) + e^{i\phi}\theta(x - L/2)] \\ M(x) &= M_0\theta(x + L/2)\theta(-x + L/2).\end{aligned}\tag{16.3}$$

By matching the solutions it is straightforward to determine the spectrum of Andreev bound states in the junction. The calculation is similar to Ref. [12], as well as the theory of superconducting quantum point contacts (SQPCs) [14, 15]. However, we shall see that there is a fundamental difference with both of those theories.

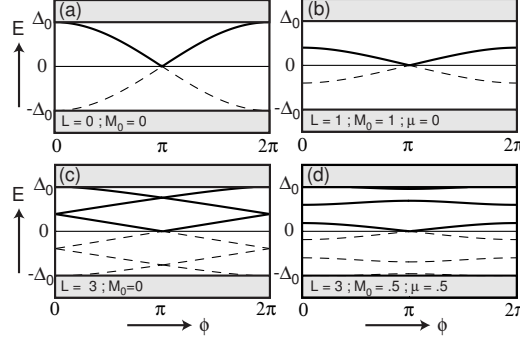


Figure 33: Spectrum of Andreev bound states in the junction as a function of phase difference ϕ for parameters indicated in each panel. L is in units of v/Δ_0 and M_0 and μ are in units of Δ_0 . (a) and (c) are independent of μ .

Fig. 2a shows the spectrum as a function of ϕ for $M_0 = 0$. For $L \lesssim v/\Delta_0$ there is a single pair of bound states $E = \pm\epsilon_0(\phi)$. For $L \ll v/\Delta_0$ our model reduces to the δ function model solved in Ref. [12], where the normal state transmission probability is $D = 1/[1 + (M_0 \sinh(\kappa L)/\kappa)^2]$, with $\kappa = \sqrt{M_0^2 - \mu^2}$. In that case

$$\epsilon_0(\phi) = \sqrt{D}\Delta_0 \cos(\phi/2).\tag{16.4}$$

Fig. 2b shows a case where $M_0 \sim \Delta_0$, so the normal state transmission $D < 1$. When $D \ll 1$ there are two weakly coupled Majorana end states at $x = \pm L/2$. When $L > v/\Delta_0$ there will be additional Andreev bound states in the junction with a level spacing of order v/L . Fig. 2c shows the case where $vL/\Delta_0 = 3$ with $M_0 = 0$, in which time reversal symmetry requires Kramers degeneracies when $\phi = 0$ or π . Fig. 2d shows the effect of finite M_0 and μ , which lifts most of the degeneracies. However, the crossing at $E = 0$ remains, and is of special significance.

To understand the level crossing consider a low energy theory for $E \ll \Delta_0$. The two eigenvectors $\xi_{\pm}$ of (16.1-16.3) with energy $\pm\epsilon_0(\phi)$ define Bogoliubov operators $\Gamma_{0\pm} = \Psi^T \xi_{0\pm}$. Due to particle-hole symmetry, $\Gamma_{0+} = \Gamma_{0-}^{\dagger} \equiv \Gamma_0$. The low energy Hamiltonian is thus

$$H = \epsilon_0(\phi)(\Gamma_0^{\dagger}\Gamma_0 - 1/2) = 2i\epsilon_0(\phi)\gamma_1\gamma_2,\tag{16.5}$$

where we introduced Majorana operators $\gamma_1 = (\Gamma + \Gamma^{\dagger})/2$, $\gamma_2 = -i(\Gamma - \Gamma^{\dagger})/2$. In the weak tunneling ($D \ll 1$) limit $\gamma_{1,2}$ describe the Majorana end states at $x = \pm L/2$, which are weakly coupled by tunneling of electrons. Eq. 16.5 describes two states distinguished by $N_0 \equiv \Gamma_0^{\dagger}\Gamma_0 = 0, 1$. Mixing these states requires an interaction that changes N_0 . Due to the pairing term in (16.1), the total charge is not conserved. However, the fermion parity, defined as the number of electrons modulo 2, is conserved in (16.1-16.3). This forbids the coupling between the two states and protects the crossing at $\epsilon_0(\phi) = 0$.

There is a problem, however, with the fermion parity. The junction Hamiltonian (16.1-16.3) is invariant under a 2π phase change, but when $\phi \rightarrow \phi + 2\pi$, the system passes through a single

level crossing and can only return to the initial state by a process which changes N_0 by 1. The fermion parity thus apparently changes when $\phi \rightarrow \phi + 2\pi$. This has to do with the unbounded spectrum as $E \rightarrow -\infty$ and reflects a *fermion parity anomaly* similar to the $SU(2)$ anomaly in 4D field theory[16]. This anomaly is related to non-Abelian statistics. When ϕ advances by 2π $\gamma_1 \rightarrow \gamma_1$ and $\gamma_2 \rightarrow -\gamma_2$. In the tunneling limit this can be interpreted as Ivanov's rule[17] for the effect of braiding a vortex between the Majorana bound states.

The physical origin of the fermion parity anomaly lies in the topological structure of the QSHI. Consider first the Corbino disk in Fig. 1 without the superconductor. In Ref. [18] we showed that the Z_2 invariant characterizing the QSHI describes the change in the Z_2 "time reversal polarization" (TRP) when flux $h/2e$ is threaded through the hole. A nonzero TRP specifies a many body Kramers degeneracy localized at either edge of the disk. Since an odd number of fermions has a Kramers degeneracy, the TRP is precisely the fermion parity. With the superconductor present, start in the groundstate at $\Phi = 0$. When flux $\Phi = h/2e$ is threaded through the hole, ϕ advances by 2π and a unit of fermion parity is transferred from the inner edge of the disk to the junction on the outer edge. The anomaly in (1-3) is similar to the chiral anomaly for edge states in the quantum Hall effect, where bulk currents violate charge conservation at the edge. Note however that though (1-3) is invariant under $\phi \rightarrow \phi + 2\pi$, the global Hamiltonian, which includes the bulk QSHI, is physically distinct when $\Phi = 0$ and $h/2e$.

The local conservation of fermion parity has important consequences for the current and noise in a S-QSHI-S junction. This is most striking near the degeneracy point for $\epsilon_0 \ll \Delta_0$ and $T \ll \Delta_0$. For the remainder of the paper we will focus on that regime. We will also consider the limit $L \ll v/\Delta_0$, where there is a single Andreev bound state and (16.4) applies, though the results can straightforwardly be generalized to the case with multiple Andreev levels, provided $T \ll v/L$. In this case, N_0 distinguishes two states, with Josephson currents $I_{\pm} = \pm I_0$ with

$$I_0(\phi) = \frac{1}{2}\sqrt{De}\Delta_0 \sin \phi/2. \quad (16.6)$$

In the absence of transitions that violate local fermion parity conservation there can be no transitions between I_+ and I_- , signaling a fractional Josephson effect.

Elastic scattering processes can be incorporated into the BdG Hamiltonian from the start, and will not lead to violations of the fermion parity. However, at finite temperature, inelastic processes[21, 22] can lead to a transition between I_+ and I_- , provided an available fermion is present to switch the fermion parity. This could be either due to a thermally excited quasiparticle or due to hopping from a bulk localized state. These processes, however, will be exponentially suppressed at low temperature. On a time scale longer than the switching time the current will thermalize, with an average value[20, 12]

$$\langle I(\phi) \rangle = I_0(\phi) \tanh \epsilon_0(\phi)/2T. \quad (16.7)$$

On shorter times, the current will exhibit telegraph noise, as it switches between $I_{\pm}$.

In order to model the inelastic processes responsible for the telegraph noise we consider the interaction of the Andreev level Γ_0 with a bath of fermions c_n (e.g. quasiparticles) and bosons b_m (e.g. phonons). We thus write

$$\begin{aligned} H = & \epsilon_0 \Gamma_0^\dagger \Gamma_0 + \sum_n E_n c_n^\dagger c_n + \sum_m \omega_m b_m^\dagger b_m \\ & + \sum_{mn} [(V_{nm}^1 c_n^\dagger b_m + V_{nm}^2 c_n b_m^\dagger) \Gamma_0 + h.c.] . \end{aligned} \quad (16.8)$$

Here $E_n, \omega_n > 0$, and we have ignored terms which create (or annihilate) both fermions and bosons. The transition rates $\tau_{\pm}^{-1}(\epsilon_0, T)$ between the states N_0 and $N_0 \pm 1$ follow from Fermi's

golden rule. For $\epsilon_0, T \ll \Delta_0$ we find

$$\tau_{\pm}^{-1} = e^{\mp\epsilon_0/2T} (w_1(T)e^{\epsilon_0/2T} + w_2(T)e^{-\epsilon_0/2T}), \quad (16.9)$$

where

$$w_{1,2}(T) = 2\pi \sum_{n,m} e^{-E_n/T} |V_{nm}^{1,2}|^2 \delta(E_n - \omega_m). \quad (16.10)$$

If either the Zeeman term vanishes ($M_0 = 0$) or the system is symmetric under $x \rightarrow -x$, then $w_1(T) = w_2(T) \equiv w(T)$. We will assume this below, though the results are only slightly modified otherwise. $w(T)$ depends on the dominant source of fermions, which we take to be either thermally activated quasiparticles or Mott variable range hopping from bulk localized states.

$$w(T) \propto \begin{cases} e^{-\Delta_0/T} & \text{quasiparticles,} \\ e^{-(T_0/T)^{1/3}} & \text{hopping.} \end{cases} \quad (16.11)$$

T_0 depends on the density of states and localization length, and we assume the hopping is 2D.

The transition rate is exponentially suppressed for $T \rightarrow 0$. At sufficiently low temperature the resulting telegraph noise could be observed in the time domain. At higher temperature there is a signature in the noise spectrum $S(\omega)$. We determine $S(\omega)$ semiclassically by solving a kinetic equation for the probability $p(t)$ that $N_0 = 1$ [21, 22]. This has the form $dp/dt = -(p - \bar{p})/\tau$, where $\tau^{-1} = \tau_+^{-1} + \tau_-^{-1} = 4w \cosh^2 \epsilon_0/2T$. $\bar{p} = (1 + \exp \epsilon_0/T)^{-1}$ follows from the detailed balance condition $\tau_+/\tau_- = e^{\epsilon_0/T}$. Temporal correlations in $I(t)$ decay exponentially on a time scale w^{-1} , and the noise spectrum $S(\omega) = 2 \int_{-\infty}^{\infty} e^{i\omega t} \langle I(t)I(0) \rangle$ is given by [19, 21]

$$S(\omega) = \frac{4I_0^2}{\cosh^2 \epsilon_0(\phi)/2T} \frac{\tau}{1 + \omega^2 \tau^2}. \quad (16.12)$$

In the zero frequency limit we have

$$S(\omega \rightarrow 0) = \frac{I_0^2}{w(T) \cosh^4 \epsilon_0(\phi)/2T}. \quad (16.13)$$

For $D = 1$, these results are similar to the theory of a SQPC [20, 21, 22, 19]. However, the current in Eq. (16.6) is *half* the value of a perfect single channel SQPC. A SQPC is like two copies of a S-QSHI-S junction. This leads to a fundamental difference, because in the SQPC there is no conservation law to prevent scattering between the $\pm I_0$ states, which can occur via low energy processes that transfer an electron between the two pairs. Elastic backscattering in the SQPC leads to an avoided crossing of the states near $E = 0$, and the Andreev states carry no current at $\phi = \pi$. Even if the transmission of the SQPC is perfect, inelastic processes will couple the states. For instance, near the degeneracy point $\epsilon_0(\phi) = 0$ spin flip scattering via the nuclear hyperfine interaction will lead to a finite lifetime for transitions, even at low temperature. It is also of interest to compare with the theory of Ref. [12]. In that work, multi channel junctions were considered. Independence of the different channels requires translational symmetry, so impurity scattering will lead to the violation of the conservation fermion parity within a given channel. Thus the low temperature behavior predicted by (16.11) and (16.13) is unique to the S-QSHI-S junction, and is a signature of the fermion parity anomaly.

We now briefly consider junctions at finite voltage bias. The analysis is similar to Ref. [19]. There are two cases to consider, depending on M_0 . For $M_0 = 0$, the perfect transmission of the edge states causes the Andreev levels to merge with the continuum levels. This leads to a finite DC current, which for $eV \ll \Delta_0$ can be understood semiclassically in terms of multiple Andreev reflections. For $w(T) \ll eV \ll \Delta_0$, the current is $I(V) = (2/\pi)I_c \text{sgn} V$, where $I_c = \sqrt{D}e\Delta_0/2$. For $M_0 > 0$, there is an energy gap δ separating the Andreev levels from the continuum, as

in Fig. 2(b,d). For $w(T) \ll eV \ll \delta$ there will be a fractional AC Josephson current with frequency $eV/\hbar$ [12]. For $eV \sim \delta$ Landau-Zener tunneling processes through the gap δ will lead to a damping of the AC Josephson current as well as a finite DC current.

We close by discussing the feasibility of experiments using the QSHI recently achieved in HgCdTe quantum wells[9, 10], which has a bulk gap of order 20 meV[23]. The desired geometry would be similar to Ref. [24], where a 2D InAs quantum well was contacted with Nb. The proximity induced gap will depend on the contact. If optimized Δ_0 could be of order the bulk gap of the superconductor. To determine the required junction size we use $v = 3.6 \text{ eV\AA}$ [23] and $\Delta_0 = .1 \text{ meV}$. Then, $L \lesssim v/\Delta_0 \sim 3 \mu\text{m}$ sets the scale for having a single Andreev level. The simplest experiment would be to study a single current biased junction, which is predicted to have a critical current $I_c = e\Delta_0/2 \sim 10 \text{ nA}$, which is half the value of a perfect single channel SQPC. Measuring the equilibrium noise at $\phi \sim \pi$ requires an inductive measurement on a ring[14]. The physics at $M_0 \neq 0$ requires a magnetic field in the junction region. For an appropriately aligned field, a field induces a gap $B \times (3.1 \text{ meV/T})$ [23] in the edge states, so a field of order $.03T$ could suppress the normal state transmission D as well as the magnitude of the Josephson current.

17 Quasiclassical description of transport through superconducting contacts by Cuevas, Fogelström

Abstract

We present a theoretical study of transport properties through superconducting contacts based on a new formulation of boundary conditions that mimics interfaces for the quasiclassical theory of superconductivity. These boundary conditions are based on a description of an interface in terms of a simple Hamiltonian. We show how this Hamiltonian description is incorporated into quasiclassical theory via a T-matrix equation by integrating out irrelevant energy scales right at the onset. The resulting boundary conditions reproduce results obtained by conventional quasiclassical boundary conditions, or by boundary conditions based on the scattering approach. This formalism is well suited for the analysis of magnetically active interfaces as well as for calculating time-dependent properties such as the current-voltage characteristics or as current fluctuations in junctions with arbitrary transmission and bias voltage. This approach is illustrated with the calculation of Josephson currents through a variety of superconducting junctions ranging from conventional to d-wave superconductors, and to the analysis of supercurrent through a ferromagnetic nanoparticle. The calculation of the current-voltage characteristics and of noise is applied to the case of a contact between two d-wave superconductors. In particular, we discuss the use of shot noise for the measurement of charge transferred in a multiple Andreev reflection in d-wave superconductors.

17.1 Introduction

At the end of the 1960's several authors demonstrated that the complete standard theory of superconductivity, both in equilibrium and out of equilibrium, can be formulated in terms of a quasiclassical transport equation [1]. This theory, known as *quasiclassical theory of superconductivity*, combines Landau's semiclassical transport equations for quasiparticles with the concept of pairing and particle-hole coherence that are the basis of the BCS theory. The quasiclassical theory provides a full description of superconducting phenomena ranging from inhomogeneous superconductors to superconducting phenomena far from equilibrium. In its traditional form, the quasiclassical theory of superconductivity is restricted to the description of weak perturbations of a superconductor: the external perturbations (magnetic field,

variations in the chemical potential, etc) should be weak ($V \ll E_F$), where E_F is the Fermi energy, of long wavelength ($q \gg k_F^{-1}$), where k_F^{-1} is the Fermi wavelength, and of low frequency ($\hbar\omega \ll E_F$) [2]. Although this limit covers many situations of interest in superconductivity, some important cases fall outside the range of validity of the usual scheme of the quasiclassical theory: strong impurities, walls, interfaces, etc. In the case of the analysis of the transport properties of superconducting contacts, the quasiclassical theory is complemented by appropriate boundary conditions, the so-called Zaitsev boundary conditions [3], or by its generalization to magnetically active interfaces by Millis, Rainer and Sauls [4]. Although these boundary conditions provide a formal solution of the problem of a strong perturbation due to interfaces, their highly non-linear form introduces many problems, e.g. these boundary conditions have spurious solutions which require special care, in particular in numerical implementations. At this point, we should mention that the recent progress in the solution of these boundary conditions make them much more tractable [5, 6, 7].

In the field of electronic transport through superconducting junctions there are many basic situations in which a complete understanding is still lacking. This is partially due to the difficulty of applying the quasiclassical theory because of the lack of simple and manageable boundary conditions. Most of these situations are related to either the current-voltage (I-V) characteristics of contacts between two superconductors with arbitrary transmission, or to situations in which there are magnetically active interfaces giving rise to spin-dependent transport. For instance, in the context of high temperature superconductors (HTS), which by now are believed to be d-wave superconductors [8], we can mention N-I-S tunnel junctions[9, 10, 11, 12], grain boundary junctions widely used for the realization of Josephson junctions [13, 14, 15], and YBCO S-N-S edge junctions [16, 17, 18]. In the two latter systems the description of the I-V characteristics requires an analysis of multiple Andreev reflections (MAR) [19], while all systems above require a careful self-consistent study of the superconducting state in the presence of pair-breaking, caused by bulk or surface impurities and by the surface itself, in order to account for results obtained in experiments [9, 10, 11, 12] by theory [20, 21, 22, 23]. Other examples, which take place in the context of conventional superconductors, are the S-N-S point contacts, where the normal region N has a length comparable to the superconducting coherent length [24, 25]. These systems demand from the theory a detailed analysis of the interplay between the proximity effect superconductivity in the normal metal and the occurrence of MAR. This analysis is currently in progress making use of the approach described in this work [26].

With respect to the transport through spin active interfaces there have been an extensive experimental effort to explore tunneling through magnetic insulator barriers[27, 28], by probing the proximity effect in S/F-structures[29, 30], and by constructing S/F-multilayers[31] (see also references therein). Andreev reflection using ferromagnet/superconductor point contacts have also been used to probe the spin polarization of the ferromagnet[32, 33]. Finally, supercurrents have been reported in S/F/S junctions by Veretennikov *et al* [34] and by Gandit *et al* [35]. To model an S/F/S-junction one can follow one of two routes. The first route is to assume an extension of a ferromagnetic metal, now characterized by a length and an exchange field, separating the two superconductors. Following this route, both critical current oscillations[36, 37] and the effect of the exchange field on the Andreev bound states[38, 39] have been studied. The limitation here is that the approach is restricted to small exchange fields in the ferromagnet, i.e. fields that are comparable to the superconducting gap. The second route, which we subscribe to, is to treat the effect of the ferromagnet as a boundary problem connecting the two superconducting half-spaces[40, 41, 4, 42]. Using Bogoliubov-deGennes equations and a WKB-approach for the ferromagnetic barrier, i.e. the scattering approach[42], Josephson current-phase relations[43] and quasiparticle tunneling[44, 45] have been studied for both conventional s-wave and unconventional d-wave superconductors.

Parallel to the quasiclassical treatment of interfaces, a different formalism referred to as the Hamiltonian approach has emerged as a new interesting tool for the analysis of transport through superconducting contacts[46, 47]. This approach is based on modeling the contact by a simple Hamiltonian in combination with non-equilibrium Green's function techniques[48]. The origin of this approach can be traced back to the work of Bardeen who introduced the tunnel Hamiltonian approximation in order to describe a tunnel junction [49]. Most of the calculations of current through superconducting contacts in the early 60's were based on this tunnel Hamiltonian, and were restricted to the lowest order transport processes such as the calculation of the Josephson current in a S-I-S junction [50, 51]. Multi-particle tunneling was first discussed by Schrieffer and Wilkins [52] in their *multiparticle tunneling theory* as a possible explanation for the observed sub-gap structure in superconducting tunnel junctions. The contributions of these higher order processes were found to be divergent, which has led to the quite extended belief that the Hamiltonian approach is pathological except for describing the lowest order tunneling processes. However, in a series of works, Caroli and coworkers in the 70's [53] and, more recently, Martín-Rodero and collaborators [46, 47] have shown that one can get rid of all the old pathologies of this approach by using a local representation and by summing the series of tunneling processes up to infinite order. In particular, within this technique a microscopic theory of Multiple Andreev reflections has been developed [47]. This theory describes quantitatively the I-V characteristic of superconducting atomic-size contact[54, 55].

In this paper we show how the Hamiltonian approach may be brought into the quasiclassical theory by integrating out large energies right at the onset. The resulting boundary condition reproduces results obtained by the conventional quasiclassical Zaitsev boundary condition[3, 5] or by boundary conditions based on the scattering approach[56, 57, 58, 59, 15]. The boundary condition is also well suited for calculating time-dependent properties such as the current-voltage characteristics or the current fluctuations of S-I-S junctions. The paper is organized in the following manner: In section II we give the energy integration of the Hamiltonian approach and state the resulting boundary condition. In section III we show how the current through a contact may be calculated from the boundary t-matrix. In this section we also calculate the Josephson current resolved in energy and on trajectory for different types of superconductors and for different types of coupling between the two superconductors. In section IV, we discuss the boundary condition at a finite bias applied between the superconductors. This is then applied to the case of two coupled d-wave superconductors. Finally, in section V, current-fluctuations are discussed and the current-current correlator is derived within our method. This is then applied to compute the trajectory resolved current-fluctuations of two coupled d-wave superconductors.

17.2 Description of the approach

The system of study is two semi-infinite superconducting electrodes coupled over some type of interface barrier. Our approach is to artificially separate the problem in to two parts in order to pose a boundary condition for the interface. The first part consists of calculating the Green's function of either conductor, extending to $\pm\infty$ respectively, in presence of a hard surface at $x = 0$. To this part of the problem, the quasiclassical theory[1] is our theory of choice. It has been shown that strong perturbations, such as rigid walls may be included into quasiclassical theory by means of effective boundary conditions posed for the quasiclassical Green's function[61, 2, 62]. To couple the two electrodes, from now denoted left (L) and right (R), we assume a phenomenological Hamiltonian as follows[46, 47]

$$\hat{H}_T = \sum_{\sigma} \hat{c}_{L,\sigma}^{\dagger} v_{LR} \hat{c}_{R\sigma} + \hat{c}_{R\sigma}^{\dagger} v_{RL} \hat{c}_{L\sigma}. \quad (17.1)$$

The potentials v_{LR} and v_{RL} , with $v_{RL}^\dagger = v_{LR} = v$, act as hopping elements connecting the two electrodes L and R . The perturbation given by $\hat{H}_T$ is short ranged ($\sim \lambda_F \ll \xi_o$) and may be strong ($v \sim E_F$). The local character of $\hat{H}_T$ allows us to view it as a single strong impurity in case of a point contact or as a line of strong impurities in case of an extended contact between the two electrodes. Following the work of Thuneberg and co-workers[63, 64] the single impurity or, in case of the line of impurities following the work of Buchholtz and Rainer[61], this strong perturbation may also be incorporated into quasiclassical theory via a T-matrix equation. Anticipating the result for the T-matrix, the effect of the contact between the two electrodes on the physical quasiclassical Keldysh-Nambu matrix Green's function, or propagator, $\check{g}_i$, in electrode $i = (L, R)$ enters as a source term in the transport equation for $\check{g}_i(\hat{\mathbf{p}}_F)$ along a trajectory $\hat{\mathbf{p}}_F$

$$\begin{aligned} i \mathbf{v}_F \cdot \nabla_{\mathbf{R}} \check{g}_i(\hat{\mathbf{p}}_F) &+ [\check{\epsilon}_i(\hat{\mathbf{p}}_F) - \check{\Delta}_i(\hat{\mathbf{p}}_F), \check{g}_i(\hat{\mathbf{p}}_F)]_{\otimes} \\ &= [\check{t}_{ii}(\hat{\mathbf{p}}_F, \hat{\mathbf{p}}_F), \check{g}_{\infty,i}(\hat{\mathbf{p}}_F)]_{\otimes} \delta(\mathbf{R} - \mathbf{R}_c). \end{aligned} \quad (17.2)$$

Here $\mathbf{R}_c$ is the position of the contact and $\mathbf{v}_F$ is the Fermi velocity at point $\hat{\mathbf{p}}_F$ on the Fermi surface. The Green's function $\check{g}_{\infty,i}$ is an intermediate Green's function obtained by solving the hard wall boundary condition of the separate electrodes, i.e. without taking the contact into account but using the self energies $\check{\epsilon}_i$ and $\check{\Delta}_i$ evaluated using the physical propagator, $\check{g}_i$, satisfying Eq. (17.2).

Our objective is to find the quasiclassical T-matrix, $\check{t}$, giving the source term in the transport equation (17.2) above. The starting point is a conventional many-body perturbation theory for the Hamiltonian $\hat{H}_T$. To proceed we artificially enlarge our Hilbert space with a "reservoir quantum number" (L, R) and the functions entering are the matrices

$$\begin{aligned} \check{G} &= \begin{pmatrix} \check{G}_{LL} & \check{G}_{LR} \\ \check{G}_{RL} & \check{G}_{RR} \end{pmatrix} & \check{T} &= \begin{pmatrix} \check{T}_{LL} & \check{T}_{LR} \\ \check{T}_{RL} & \check{T}_{RR} \end{pmatrix} \\ \check{G}_{\infty} &= \begin{pmatrix} \check{G}_{\infty,L} & 0 \\ 0 & \check{G}_{\infty,R} \end{pmatrix} & \check{v} &= \begin{pmatrix} 0 & \check{v}_{LR} \\ \check{v}_{RL} & 0 \end{pmatrix}. \end{aligned}$$

The matrix elements are the usual Keldysh-Nambu matrices of non-equilibrium superconductivity[2]. Especially, the Green's functions $\check{G}_{\infty,L}$ and $\check{G}_{\infty,R}$ are the Green's functions for the uncoupled left and right electrode. The coupling elements, $\check{v}_{LR,RL}$, between L and R are proportional to the unit matrix in the Keldysh space and in Nambu space adopt the form

$$\hat{v}_{LR} = \hat{v}_{RL}^\dagger = \begin{pmatrix} v & 0 \\ 0 & -v^\dagger \end{pmatrix}.$$

With this, we write the T-matrix equation

$$\check{T} = \check{v} + \check{v} \circ \check{G}_{\infty} \circ \check{T} \quad (17.3)$$

$$= \check{v} + \check{v} \circ \check{G} \circ \check{v}, \quad (17.4)$$

which together with the Dyson equation

$$\check{G} = \check{G}_{\infty} + \check{G}_{\infty} \circ \check{T} \circ \check{G}_{\infty} \quad (17.5)$$

$$= \check{G}_{\infty} + \check{G}_{\infty} \circ \check{v} \circ \check{G} \quad (17.6)$$

constitutes a closed set of equations that are to be brought into quasiclassical form. We have given two different ways of summing the series which correspond either to "dressing" the perturbation (equations (17.3) and (17.5)) or to "dressing" the Green's function (equations (17.4)

and (17.6)). The two sets of equations are equivalent and two useful relations

$$\tilde{v} \circ \tilde{G} = \tilde{T} \circ \tilde{G}_\infty \quad \text{and} \quad \tilde{G}_\infty \circ \tilde{T} = \tilde{G} \circ \tilde{v} \quad (17.7)$$

follow directly. Here, and above, the $\circ$ -product is short-hand for integration or summation over common arguments. Starting from equation (17.4), using equation (17.6) and the second of the two relations (17.7), it is straight forward to get the following closed set of equations, closed separately for one and each of the components, $\tilde{T}_{ij}$, of the T-matrix[47],

$$\begin{aligned} \tilde{T}_{LL} &= \tilde{v}_{LR} \circ \tilde{G}_{\infty,R} \circ \tilde{v}_{RL} \\ &\quad + \tilde{v}_{LR} \circ \tilde{G}_{\infty,R} \circ \tilde{v}_{RL} \circ \tilde{G}_{\infty,L} \circ \tilde{T}_{LL} \\ \tilde{T}_{RR} &= \tilde{v}_{RL} \circ \tilde{G}_{\infty,L} \circ \tilde{v}_{LR} \\ &\quad + \tilde{v}_{RL} \circ \tilde{G}_{\infty,L} \circ \tilde{v}_{LR} \circ \tilde{G}_{\infty,R} \circ \tilde{T}_{RR} \\ \tilde{T}_{LR} &= \tilde{v}_{LR} + \tilde{v}_{LR} \circ \tilde{G}_{\infty,R} \circ \tilde{v}_{RL} \circ \tilde{G}_{\infty,L} \circ \tilde{T}_{LR} \\ \tilde{T}_{RL} &= \tilde{v}_{RL} + \tilde{v}_{RL} \circ \tilde{G}_{\infty,L} \circ \tilde{v}_{LR} \circ \tilde{G}_{\infty,R} \circ \tilde{T}_{RL}. \end{aligned} \quad (17.8)$$

The equations above depend only on the Green's functions $\tilde{G}_{\infty,L}$ and $\tilde{G}_{\infty,R}$ of the two uncoupled systems. Since the full Green's function $\tilde{G}_{ij}$ has been eliminated from the T-matrix equations there are no Green's functions with spatial arguments in both systems. Together with the short range of $\tilde{v}_{LR,RL}$ this means that we can directly perform the quasiclassical ξ -integration on the T-matrix equations and substitute the quasiclassical Green's functions, $\check{g}_{\infty,i}$, for the full ones, $\tilde{G}_{\infty,i}$, above.

At the quasiclassical level, the Green's functions $\check{g}_{\infty,i}(\hat{\mathbf{p}}_F; t, t')$ at the interface in equations (17.8) depend on the position on the Fermi surface, $\hat{\mathbf{p}}_F$, and of two times (t, t') . The coupling elements will be assumed to be time independent but may couple different points $\hat{\mathbf{p}}_F$ and $\hat{\mathbf{p}}'_F$ on the Fermi surfaces of the two conductors. The exact form of the $(\hat{\mathbf{p}}_F, \hat{\mathbf{p}}'_F)$ -dependence of $\tilde{v}_{LR}$ is a degree of freedom in the model that allows us to consider different types of transport through the interface. We can now write down the equations for the quasiclassical $\check{t}$ -matrix components

$$\begin{aligned} \check{t}_{LL} &= \langle \tilde{v}_{LR} \otimes \check{g}_{\infty,R} \otimes \tilde{v}_{RL} \rangle_{\hat{\mathbf{p}}_F''} \\ &\quad + \langle \langle \tilde{v}_{LR} \otimes \check{g}_{\infty,R} \otimes \tilde{v}_{RL} \otimes \check{g}_{\infty,L} \otimes \check{t}_{LL} \rangle_{\hat{\mathbf{p}}_F''} \rangle_{\hat{\mathbf{p}}_F'''} \\ \check{t}_{RR} &= \langle \tilde{v}_{RL} \otimes \check{g}_{\infty,L} \otimes \tilde{v}_{LR} \rangle_{\hat{\mathbf{p}}_F''} \\ &\quad + \langle \langle \tilde{v}_{RL} \otimes \check{g}_{\infty,L} \otimes \tilde{v}_{LR} \otimes \check{g}_{\infty,R} \otimes \check{t}_{RR} \rangle_{\hat{\mathbf{p}}_F''} \rangle_{\hat{\mathbf{p}}_F'''} \\ \check{t}_{LR} &= \tilde{v}_{LR} + \langle \langle \tilde{v}_{LR} \otimes \check{g}_{\infty,R} \otimes \tilde{v}_{RL} \otimes \check{g}_{\infty,L} \otimes \check{t}_{LR} \rangle_{\hat{\mathbf{p}}_F''} \rangle_{\hat{\mathbf{p}}_F'''} \\ \check{t}_{RL} &= \tilde{v}_{RL} + \langle \langle \tilde{v}_{RL} \otimes \check{g}_{\infty,L} \otimes \tilde{v}_{LR} \otimes \check{g}_{\infty,R} \otimes \check{t}_{RL} \rangle_{\hat{\mathbf{p}}_F''} \rangle_{\hat{\mathbf{p}}_F'''}, \end{aligned} \quad (17.9)$$

where we suppressed the explicit dependence of the functions on $\hat{\mathbf{p}}_F$ and on time variables. The earlier $\circ$ -product in Eq. (17.8) is replaced by the $\otimes$ -product in the quasiclassical expression. The $\otimes$ -product stands for an integration over a common time variable together with a normal matrix multiplication in the combined Keldysh-Nambu and spin space[2]. A left-over from the ξ -integration is the intermediate averaging over position on the Fermi surface as indicated by $\langle \cdots \rangle_{\hat{\mathbf{p}}_F} = \int_{FS} \cdots d\hat{\mathbf{p}}_F$. The quasiclassical $\check{t}$ -matrix entering in to equation (17.2) is the forward scattering limit[64]

$$\check{t}_{ij} = \check{t}_{ij}(\hat{\mathbf{p}}_F, \hat{\mathbf{p}}_F; t, t')$$

of Eq. (17.9) and in general it depends on two times (t, t') .

Following Ref. [2], let us summarize the procedure for calculating the quasiclassical propagators in the presence of an interface:

1. To find $\check{g}_\infty$, we solve the conventional quasiclassical equations, the Eilenberger equation or the Usadel equation, for the uncoupled electrodes in equilibrium using hard-wall boundary conditions.

2. Use $\check{g}_\infty$ to solve the quasiclassical T-matrix equations (17.9).
3. Solve the inhomogeneous quasiclassical equation (17.2) for the physical propagator $\check{g}$.
4. Use $\check{g}$ to calculate the “smooth” self-energies $\check{\epsilon}_i$ and $\check{\Delta}_i$ which enter the quasiclassical equations for $\check{g}_\infty$ and for $\check{g}$.

The whole scheme amounts to a set of linear differential equations for $\check{g}_\infty$ and $\check{g}$, coupled in a non-linear way by the T-matrix and the self-energy equations. Substantial simplifications can be achieved in the case of low transmissive tunnel barriers or point contacts. In these cases one can neglect the influence of the neighboring electrodes in the calculation of the self-energies and then the equations for $\check{g}_\infty$ and $\check{\epsilon}_i$ and $\check{\Delta}_i$ decouple.

17.3 Josephson currents

As a first application of the boundary condition we calculate supercurrent through a variety of contacts connecting two superconducting reservoirs. The current contribution from a given trajectory, $\hat{\mathbf{p}}_F$, and at a given energy, ε , may be calculated directly by integrating the transport equation (17.2) along the direction given by $\mathbf{v}_F(\hat{\mathbf{p}}_F)$. This is easily seen as on the considered trajectory away from the contact the physical propagator $\check{g}_i(\hat{\mathbf{p}}_F)$ coincides with the intermediate propagator $\check{g}_{\infty,i}(\hat{\mathbf{p}}_F)$ calculated by the impenetrable surface condition. In absolute vicinity of the contact only the source term in Eq. (17.2), $[\check{t}_{ii}(\hat{\mathbf{p}}_F, \hat{\mathbf{p}}_F), \check{g}_{\infty,i}(\hat{\mathbf{p}}_F)]\delta(\mathbf{R} - \mathbf{R}_c)$, contributes and results in a jump in the Green’s function. The magnitude of this jump is given by integrating Eq. (17.2) over the interval $]0_-, 0_+[$. Performing the integral results in the scattered propagator

$$\check{g}_{i+}(\hat{\mathbf{p}}_F) = \check{g}_{i-}(\hat{\mathbf{p}}_F) - \frac{i}{v_F \cos \phi_i} [\check{t}_{ii}(\hat{\mathbf{p}}_F, \hat{\mathbf{p}}_F), \check{g}_{i-}(\hat{\mathbf{p}}_F)], \quad (17.10)$$

where ϕ_i is the angle $\mathbf{v}_F(\hat{\mathbf{p}}_F)$ makes with the contact normal. Note that $\check{g}_{i-}(\hat{\mathbf{p}}_F) \equiv \check{g}_{\infty,i}(\hat{\mathbf{p}}_F)$ and to calculate $\check{g}_{i+}(\hat{\mathbf{p}}_F')$, i.e. the propagator along the trajectory $(\hat{\mathbf{p}}_F')$ coupled to $(\hat{\mathbf{p}}_F)$ by pure surface scattering we must solve for $\check{g}_{i+}(\hat{\mathbf{p}}_F')$ along the path given by $\mathbf{v}_F(\hat{\mathbf{p}}_F')$. The propagator at the contact, computed in (17.10), may now be inserted in the current formula

$$j(T) = eN_F \int \frac{d\varepsilon}{4\pi i} \text{Tr} \langle \mathbf{v}_F(\hat{\mathbf{p}}_F) g^K(\hat{\mathbf{p}}_F; \varepsilon) \rangle_{\hat{\mathbf{p}}_F} = eN_F \int \frac{d\varepsilon}{4\pi i} \langle j_\varepsilon^K(\hat{\mathbf{p}}_F) \rangle_{\hat{\mathbf{p}}_F}, \quad (17.11)$$

with N_F the density of states at the Fermi level in the normal state. Since the Josephson current is an equilibrium property, the Keldysh Green’s function components of $\check{g}$ are in this case simply related to the Retarded (R) and Advanced (A) ones as $\hat{g}^K = (\hat{g}^R - \hat{g}^A) \tanh(\varepsilon/2T)$, and with $(\hat{g}^A(\hat{\mathbf{p}}_F; \varepsilon))^\dagger = \hat{\tau}_3 \hat{g}^R(\hat{\mathbf{p}}_F; \varepsilon) \hat{\tau}_3$, we get the energy and trajectory resolved current contribution across the contact, evaluated in the left superconductor, as

$$j_\varepsilon^K(\hat{\mathbf{p}}_F) = \text{Im} \left[\text{Tr} \left\{ i \hat{\tau}_3 [\hat{t}_{LL}^R, \hat{g}_{\infty,L}^R] \right\} \right] \tanh\left(\frac{\varepsilon}{2T}\right). \quad (17.12)$$

The lonely first intermediate Green’s function, $\check{g}_{\infty,i}(\hat{\mathbf{p}}_F)$, in Eq. (17.10) explicitly drop out of the current in the angle average since they obey the impenetrable surface boundary condition.

So far no reference have been made to the modeling of the contact and the $\hat{t}$ -matrix element $\hat{t}_{LL}^R$. The contact model depends on the choice of the momentum dependence of the coupling elements, $v_{RL}^\dagger = v_{LR} = v(\hat{\mathbf{p}}_F, \hat{\mathbf{p}}_F')$. Two extreme models for the $(\hat{\mathbf{p}}_F, \hat{\mathbf{p}}_F')$ -dependence will be considered: a totally disordered contact, $v(\hat{\mathbf{p}}_F, \hat{\mathbf{p}}_F') = v$, i.e. the coupling across the contact retains no memory of the momentum direction, and a momentum conserving contact

with $v(\hat{\mathbf{p}}_F, \hat{\mathbf{p}}'_F) = v\delta(\hat{\mathbf{p}}_F - \hat{\mathbf{p}}'_F)$. The $\hat{t}$ -matrix equations for the two types of contact, dropping superfluous indexing, read

$$\hat{t} = \hat{v} \langle \hat{g}_R(\hat{\mathbf{p}}_F) \rangle_{\hat{\mathbf{p}}_F} \hat{v}^\dagger + \hat{v} \langle \hat{g}_R(\hat{\mathbf{p}}_F) \rangle_{\hat{\mathbf{p}}_F} \hat{v}^\dagger \langle \hat{g}_L(\hat{\mathbf{p}}_F) \rangle_{\hat{\mathbf{p}}_F} \hat{t} \quad (17.13)$$

for the disordered contact and

$$\hat{t}(\hat{\mathbf{p}}_F) = \hat{v} \hat{g}_R(\hat{\mathbf{p}}_F) \hat{v}^\dagger + \hat{v} \hat{g}_R(\hat{\mathbf{p}}_F) \hat{v}^\dagger \hat{g}_L(\hat{\mathbf{p}}_F) \hat{t}(\hat{\mathbf{p}}_F) \quad (17.14)$$

for the momentum conserving contact. For either model, the $\hat{t}$ -matrix equation above is simple to invert after inserting the retarded Green's functions $\hat{g}_R(\hat{\mathbf{p}}_F)$ and $\hat{g}_L(\hat{\mathbf{p}}_F)$

$$\hat{g}(\hat{\mathbf{p}}_F)_{R(L)} = \begin{pmatrix} g_{R(L)} & f_{R(L)} e^{\pm i\chi/2} \\ -\tilde{f}_{R(L)} e^{\mp i\chi/2} & -g_{R(L)} \end{pmatrix}, \quad (17.15)$$

with the phase difference χ across the junction and the upper (lower) signs of the phase refer to the right (left) electrode. In equilibrium the T-matrix equation is simply an algebraic equation in energy space, which can be trivially inverted. The energy and trajectory resolved current is written

$$j_\varepsilon(\hat{\mathbf{p}}_F) = \text{Im} \left[\frac{i \mathcal{D} (f_R \tilde{f}_L e^{i\chi} - f_L \tilde{f}_R e^{-i\chi})}{2 - \mathcal{D} - \mathcal{D} g_R g_L + \frac{\mathcal{D}}{2} (f_R \tilde{f}_L e^{i\chi} + f_L \tilde{f}_R e^{-i\chi})} \right] \tanh\left(\frac{\varepsilon}{2T}\right). \quad (17.16)$$

Here we have traded in the coupling strength, v , for the transmission coefficient $\mathcal{D}$. The two are simply related as [46]

$$\mathcal{D} = \frac{4|v|^2}{(1 + |v|^2)^2}. \quad (17.17)$$

The scalar coupling element, v , is from now on a dimensionless quantity. We have normalized the original coupling element in units of $(1/\pi N_F)$ to get rid of the different prefactors that appear in the angle averages in the T-matrix equation (17.9).

17.3.1 Josephson current between two s-wave superconductors

Assuming that the two electrodes both are s-wave superconductors, we have the Green's functions on either side of the contact

$$\hat{g}(\hat{\mathbf{p}}_F)_{R(L)} = -\frac{\pi}{\Omega} \begin{pmatrix} \varepsilon & \Delta e^{\pm i\chi/2} \\ -\Delta e^{\mp i\chi/2} & -\varepsilon \end{pmatrix}, \quad (17.18)$$

where $\Omega = [\Delta^2 - \varepsilon^2]^{\frac{1}{2}}$. Since the s-wave superconductor is isotropic it does not matter which model for the coupling, (17.13) or (17.14), we choose. Additionally, in the absence of surface depairing effects it is sufficient to know the bulk Green's functions (17.18) to calculate the current contribution (17.12). Using Eq. (17.16) we find the known result that the Josephson current is carried by junction states [65, 15] located at $\varepsilon_J(\chi, T) = \pm \Delta(T) [1 - \mathcal{D} \sin^2(\chi/2)]^{\frac{1}{2}}$. The total current is the sum of all contributions and reads

$$j(T) = e N_F \mathcal{D} \frac{\pi \Delta(T) \sin \chi}{[1 - \mathcal{D} \sin^2 \frac{\chi}{2}]^{\frac{1}{2}}} \tanh\left(\frac{\varepsilon_J(\chi, T)}{2T}\right). \quad (17.19)$$

In equation (17.19) it should be noted that a second temperature dependence enters via the the temperature dependent gap, $\Delta(T)$.

17.3.2 Josephson current between two d-wave superconductors

To emphasize the modeling of the $(\hat{\mathbf{p}}_F, \hat{\mathbf{p}}'_F)$ -dependence of the coupling across the junction and the importance of using the correct surface Green's functions, we now study the current-phase relation of two d-wave superconductors. A realization of a d-wave order parameter is $\Delta_{\hat{\mathbf{p}}_F} = \Delta \cos 2(\phi_{\hat{\mathbf{p}}_F} - \alpha)$. The magnitude and sign of $\Delta_{\hat{\mathbf{p}}_F}$ depends on the position on the Fermi circle and this is measured by the angle, $\phi_{\hat{\mathbf{p}}_F}$, the angle $\hat{\mathbf{p}}_F$ makes with the crystal $\hat{a}$ -axis. The angle α tracks the relative junction-to-crystal $\hat{a}$ -axis orientation. If $\alpha = \pm\pi/4$ and specular quasiparticle scattering at the interface is assumed the order parameter seen along a trajectory changes sign at the surface and an Andreev bound state forms at zero energy for every trajectory $\hat{\mathbf{p}}_F$ [[66]]. We will stick with the junction realization $\alpha = \pm\pi/4$. To incorporate the effect of these surface states into the current-phase relation one must use the surface Green's functions[20, 67]

$$\hat{g}(\hat{\mathbf{p}}_F)_{R(L)} = \frac{\pi}{\varepsilon} \begin{pmatrix} \Omega_{\hat{\mathbf{p}}_F} & i s_R \Delta_{\hat{\mathbf{p}}_F} e^{\pm i\chi/2} \\ i s_R \Delta_{\hat{\mathbf{p}}_F} e^{\mp i\chi/2} & -\Omega_{\hat{\mathbf{p}}_F} \end{pmatrix}, \quad (17.20)$$

where $\Omega_{\hat{\mathbf{p}}_F} = [\Delta_{\hat{\mathbf{p}}_F}^2 - \varepsilon^2]^{\frac{1}{2}}$. The factor s_R discriminates between two types of junction[20, 68]: $s_R = -1$ ($\alpha_R = \alpha_L$) is referred to as a symmetric, and $s_R = 1$ ($\alpha_R = -\alpha_L$) as a mirror junction. The convention for signs of the phase, χ , are as for the s-wave superconductor. It should be said before proceeding that we are neglecting the pair-breaking effect of the surface[69] and we assume constant order parameters up to the interface. This is for the sake of simple illustration and for the comparison of analytical results with other boundary conditions.

Turning to the $\hat{t}$ -matrix equations and starting with the diffusive model of the point contact, one immediately find that the Josephson current is zero. This follows from the vanishing average, $\langle \Delta_{\hat{\mathbf{p}}_F} \rangle_{\hat{\mathbf{p}}_F} = 0$. Due to this property the anomalous propagators vanish and therefore the Josephson current as well (see Eq. (17.16)). In the opposite limit, using the momentum-conserving model (17.14), the Josephson current is not zero. After inverting the $\hat{t}$ -matrix equation and evaluating the commutator in (17.12), we find the energy resolved current at $\hat{\mathbf{p}}_F$

$$j_\varepsilon(\hat{\mathbf{p}}_F) = \pm \text{Im} \left[\mathcal{D} \Delta_{\hat{\mathbf{p}}_F}^2 \left(\frac{\sin \chi}{\varepsilon^2 - \varepsilon_J^2(\chi; \hat{\mathbf{p}}_F)} \right) \right] \tanh\left(\frac{\varepsilon}{2T}\right). \quad (17.21)$$

The sign of $j_\varepsilon(\hat{\mathbf{p}}_F)$ is (+) for a mirror and (-) for a symmetric junction. As in the case of the s-wave junction we have junction states carrying the Josephson current. The position of these states depend on the type of junction in the following way

$$\varepsilon_J(\chi; \hat{\mathbf{p}}_F) = \pm \sqrt{\mathcal{D}} |\Delta_{\hat{\mathbf{p}}_F}| \times \begin{cases} |\sin \frac{\chi}{2}|, & \text{mirror} \\ |\cos \frac{\chi}{2}|, & \text{symmetric.} \end{cases} \quad (17.22)$$

These junction states were first found by Riedel and Bagwell[70] using a scattering approach and, independently, by Barash[68] using what is known as the Zaitsev boundary condition[3]. Performing the integral over the energy we write the trajectory resolved current-phase relations

$$j(\hat{\mathbf{p}}_F) = \pm 2\pi \sqrt{\mathcal{D}} |\Delta_{\hat{\mathbf{p}}_F}| \times \begin{cases} \cos \frac{\chi}{2} \tanh\left(\frac{\sqrt{\mathcal{D}} |\Delta_{\hat{\mathbf{p}}_F}| \sin \frac{\chi}{2}}{2T}\right) \\ \sin \frac{\chi}{2} \tanh\left(\frac{\sqrt{\mathcal{D}} |\Delta_{\hat{\mathbf{p}}_F}| \cos \frac{\chi}{2}}{2T}\right) \end{cases} \quad (17.23)$$

for the two junction types, the upper being the mirror and the lower the symmetric one. The total current is the trajectory average of $j(\hat{\mathbf{p}}_F)$ multiplied by eN_F .

The main purpose of this section was to show that the quasiclassical version of the point contact coupling of two electrodes is simple to use and can recover results known in literature.

In case of unconventional superconducting electrodes results depend in a crucial way on how the contact is modeled. This gives the Hamiltonian boundary condition an advantage in flexibility to the conventional Zaitsev boundary condition which coincides with the momentum conserving contact (17.14) introduced above.

17.3.3 Josephson current through a spin active interface

As another illustration of the flexibility of this method we extend the discussion to currents through spin active interfaces, i.e. interfaces which flip the spin of the incident electrons either by spin-dependent scattering within the interface, or by a difference in spin-orbit coupling on either side of the interface. The general boundary conditions that connect the quasiclassical propagators for superconducting metals across magnetically active interfaces were introduced by Millis, Rainer and Sauls [4]. Recently one of the authors [7] derived the explicit solution of these boundary conditions for equilibrium Green functions. In order to compare with this solution, as in Ref. [[7]], we shall analyze in this section the Josephson current through a contact of two isotropic s-wave superconductors connected through a small magnetically active junction.

In order to accommodate the spin dependence we enlarge our space in such a way that every quantity is now a 2×2 matrix in spin space. In particular, the coupling elements are spin dependent and adopt the following general form

$$\hat{v}_{LR} = \hat{v}_{RL}^\dagger = \begin{pmatrix} v & 0 \\ 0 & v^\dagger \end{pmatrix} \text{ where } v = \begin{pmatrix} v_{\uparrow,\uparrow} & v_{\uparrow,\downarrow} \\ v_{\downarrow,\uparrow} & v_{\downarrow,\downarrow} \end{pmatrix}. \quad (17.24)$$

Let us stick to the case of S/F/S junction analyzed in Ref. [[7]]. In this case F stands for a small ferromagnetic particle or grain. This ferromagnetic material is treated as a partially transparent barrier which transmits the two spin projections differently. For spin-active interfaces the different components of the S -matrix, S_{ij} are 2×2 spin matrices. To proceed further a specific S -matrix was chosen in Ref. [[7]] to model the magnetic barrier

$$\hat{S} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} r & t \\ t & -r \end{pmatrix} \exp(i\Theta\sigma_3), \quad (17.25)$$

where σ_j notes the Pauli-matrices spanning spin space and parameters (t, r) are the usual transmission and reflection coefficients. The S -matrix (17.25) is one of the simplest choices that allows a variable degree of spin mixing at the interface and the spin mixing is parameterized by the spin-mixing angle Θ . By this construction $\hat{S}$ only violates spin conservation, i.e. it does not commute with the quasiparticle spin operator σ . The angle Θ will be considered as a phenomenological parameter independent of the trajectory direction (for more details see Tokuyasu *et al* Ref. [[71]]). Within the approach presented of this paper, one can easily model the previous S -matrix with a spin dependent coupling

$$\hat{v}_{LR} = \hat{v}_{RL}^\dagger = v \begin{pmatrix} \exp(i\Theta\sigma_3) & 0 \\ 0 & \exp(-i\Theta\sigma_3) \end{pmatrix}. \quad (17.26)$$

Again the quasiclassical surface Green functions are the inputs in this approach, i.e. the Green's functions at the interface calculated with impenetrable wall boundary conditions. The spin-active boundary condition must be used at the contact also for the reflecting surface[7]. In this simple case of a magnetic barrier the resulting spin-dependent propagators can be written in a 4×4 block-diagonal form

$$\hat{g}_{block}(\hat{\mathbf{p}}_F; \Theta) = \begin{pmatrix} \hat{g}(\hat{\mathbf{p}}_F; \Theta) & 0 \\ 0 & \hat{g}(\hat{\mathbf{p}}_F; -\Theta) \end{pmatrix} = \begin{pmatrix} g_{\uparrow\uparrow} & f_{\uparrow\downarrow} & 0 & 0 \\ \tilde{f}_{\downarrow\uparrow} & \tilde{g}_{\downarrow\downarrow} & 0 & 0 \\ 0 & 0 & g_{\downarrow\downarrow} & -f_{\downarrow\uparrow} \\ 0 & 0 & -f_{\uparrow\downarrow} & \tilde{g}_{\uparrow\uparrow} \end{pmatrix}, \quad (17.27)$$

where the electron and anomalous parts of $\hat{g}(\hat{\mathbf{p}}_F; \Theta)$ in the upper left corner of $\hat{g}_{block}(\hat{\mathbf{p}}_F; \Theta)$ can be written as

$$\begin{aligned} g_{\uparrow\uparrow}(\Theta) &= -\pi \frac{\varepsilon^R \cos(\Theta/2) - \Omega \sin(\Theta/2)}{\varepsilon^R \sin(\Theta/2) + \Omega \cos(\Theta/2)} \\ f_{\uparrow\downarrow}(\Theta) &= \pi \frac{\Delta e^{-i(\Theta \mp \chi)/2}}{\varepsilon^R \sin(\Theta/2) + \Omega \cos(\Theta/2)} \\ \tilde{f}_{\downarrow\uparrow}(\Theta) &= -\pi \frac{\Delta e^{i(\Theta \mp \chi)/2}}{\varepsilon^R \sin(\Theta/2) + \Omega \cos(\Theta/2)} \end{aligned} \quad (17.28)$$

for trajectories with $(\hat{\mathbf{p}}_F \cdot \hat{\mathbf{n}} > 0)$. For trajectories with reversed momentum, i.e. $(\hat{\mathbf{p}}_F \cdot \hat{\mathbf{n}} < 0)$, the phase factor $\exp[\mp i(\Theta \mp \chi)/2]$ for functions $f_{\uparrow\downarrow}(\Theta)$ and $\tilde{f}_{\downarrow\uparrow}(\Theta)$ goes to $\exp[\pm i(\Theta \pm \chi)/2]$. Above, as in the earlier examples, the phase difference, χ , between the two reservoirs is included and the upper (lower) signs refer to the right (left) reservoir. The components of the propagator in the lower right corner of $\hat{g}_{block}$ are simply related to those in the upper left corner by the replacement $\Theta \rightarrow -\Theta$.

The angle Θ induces a mixing of the two otherwise separated spin bands. It is easy to see that the resulting density of states at the interface has Andreev bound states inside the gap. These states are located at $\varepsilon_{b,\uparrow(\downarrow)} = \pm \Delta \cos(\Theta/2)$, with $+$ ($-$) for the spin-up (-down) branch. The existence of the sub-gap states alters the Josephson current-phase relation radically. The contribution to the current from the energy ε , the trajectory $\hat{\mathbf{p}}_F$, and spin band $\uparrow$ ($\downarrow$) may be calculated directly from expression (17.16). Keeping in mind that the phase that enters in Eq. (17.16) is the phase difference over the contact we write

$$j_{\varepsilon,\uparrow}(\hat{\mathbf{p}}_F; \Theta, \chi) = \text{Im} \left[\mathcal{D} \Delta^2 \left(\frac{\sin \chi}{[\Omega \cos(\Theta/2) + \varepsilon \sin(\Theta/2)]^2 - \mathcal{D} \Delta^2 \sin^2(\chi/2)} \right) \right] \tanh\left(\frac{\varepsilon}{2T}\right) \quad (17.29)$$

for the current carried by the spin-up band. The current carried by the spin-down band is given simply as $j_{\varepsilon,\downarrow}(\hat{\mathbf{p}}_F; \Theta, \chi) = j_{\varepsilon,\uparrow}(\hat{\mathbf{p}}_F; -\Theta, \chi)$. At a finite superconducting phase difference, the original two interface Andreev bound states are split up into four current carrying states located inside the gap at positions

$$\varepsilon_J = \pm \Delta \left[\cos^2(\Theta/2) - \mathcal{D} \cos(\Theta) \sin^2(\chi/2) \pm \sqrt{\mathcal{D}} \sin(\Theta) \sin(\chi/2) \sqrt{1 - \mathcal{D} \sin^2(\chi/2)} \right]^{1/2}. \quad (17.30)$$

These states give the total contribution to the current and their positions change from the tunnel regime, $\varepsilon_J = \pm \Delta \cos(\Theta/2)$, to $\varepsilon_J = \pm \Delta \cos[(\chi \pm \Theta)/2]$ at perfect transmission. In figure 34 we show the current-phase relation for a set of transparencies $\mathcal{D} = 0.1, 0.4, 0.8$, and 1.0. In each panel the spin-mixing angle Θ is varied from 0 to π in steps of $\frac{\pi}{10}$. As seen, for each value of $\mathcal{D}$ there is a range $\Theta > \Theta_c$ where the junctions are π -junctions. For small $\mathcal{D}$, the critical spin-mixing angle Θ_c is close to π and with increasing $\mathcal{D}$, Θ_c increases towards $\frac{\pi}{2}$. The magnitude of the critical current is for all but the perfect transmission junction very asymmetric for 0 and π junctions. In figure 35 we show the energy resolved spectral current (17.29) for $\mathcal{D} = 0.4$ and $\Theta = 0.7\pi$ at different phase differences, χ , over the junction. At small phase differences the junction state initially at $\varepsilon_J(\chi = 0) = \pm \Delta \cos(\Theta/2)$ splits into two states carrying current in opposite directions. This gives a small but positive current as seen in the corresponding current-phase relation in figure 34. As the phase difference is increased the two states dispersing with phase towards $\varepsilon = 0$ from either side eventually cross at $\varepsilon = 0$. Above this phase difference, both current-carrying states at $\varepsilon < 0$ ($\varepsilon > 0$) give current in the same direction and the magnitude of the current increases abruptly.

To conclude this section, let us stress that these results reproduce the result obtained in Ref. [7], showing again the versatility of the boundary conditions introduced in this work.

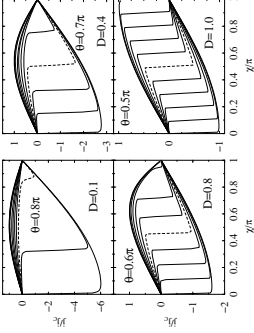


Figure 34: Zero-temperature supercurrent-phase relation for the S/F/S contact considered in this section. The four different panels correspond to $\mathcal{D} = 0.1, 0.4, 0.8$, and 1.0 . The spin-mixing angle Θ is varied from top to bottom from 0 to π in steps of $\frac{\pi}{10}$. The dashed lines indicate the value of Θ for which the contacts become π junctions. The supercurrent is normalized in units of the critical current density, j_C , for the corresponding transmission and zero spin-mixing angle.

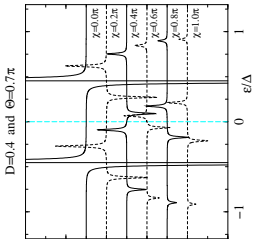


Figure 35: Energy resolved spectral current for the S/F/S contact with $\mathcal{D} = 0.4$ and $\Theta = 0.7\pi$ (see Fig. 34 right upper panel). This figure shows the total spectral current, sum of both spin contributions, as a function of energy according to Eq. (17.29), without the thermal factor. The different curves show the evolution of the four current-carrying Andreev bound states inside the gap with the superconducting phase difference. The curves are shifted vertically for clarity.

17.4 Solving the T-matrix equation at an applied voltage

In this section we shall describe how the T-matrix equation (17.9) can be solved in the case of a voltage-biased superconducting contact. As a constant bias V is applied across a junction between two superconductors, the phase difference oscillates with time according to the Josephson relation: $\phi(t) = \phi_0 + \omega_0 t$, where $\omega_0 = 2eV/\hbar$ is the Josephson frequency. This makes that every Green's function and T-matrix component depend on two time arguments. We show in this section how the time convolutions in the T-matrix equation can be handled.

As we shall show in next section the current can be expressed, for instance, only in terms of the advanced and retarded components of $\hat{t}_{LR}$. Thus, we concentrate ourselves in these components whose equations can be written as (see Eq. 9)

$$\begin{aligned} \hat{t}_{LR}^{R,A}(t, t') &= \hat{v}_{LR} + \int dt_1 \int dt_2 \hat{v}_{LR} \times \\ &\quad \hat{g}_R^{R,A}(t, t_1) \hat{v}_{RL} \hat{g}_L^{R,A}(t_1, t_2) \hat{t}_{LR}^{R,A}(t_2, t'). \end{aligned} \quad (17.31)$$

Here, we have written explicitly the time convolutions for the sake of clarity and we have omitted the $\hat{\mathbf{p}}_F$ integrations, since they do not affect the time convolutions. In this expression every quantity is a 4×4 matrix in Nambu and spin space, and from now to the end of this section we get rid of the superindexes R, A , since the equations for these two components are formally identical. We also drop the ∞ subindex, since all propagators in (17.31) are propagators of the separated electrodes. The electrode Green's functions entering Eq. (17.31) take the form: $\hat{g}_j(t, t') = \hat{U}_j^\dagger(t) \hat{g}_j(t - t') \hat{U}_j(t')$, where $j = R, L$ and $\hat{U}_j(t) = \exp[i\phi_j(t)\hat{\tau}_3/2]$, $\phi_j(t)$ being the phase of the j th superconductor. In this expression, $\hat{g}_j(t) = \int \hat{g}_j(\epsilon) \exp(-i\epsilon t) d\epsilon/2\pi$.

We use the transformation generated by $\hat{U}_j(t)$ to transfer the time dependence from the Green's functions to the hopping elements

$$\begin{aligned} \hat{t}_{LR}(t, t') &= \hat{v}_{LR}(t) \delta(t - t') + \int dt_1 \int dt_2 \hat{v}_{LR}(t) \\ &\quad \hat{g}_R(t - t_1) \hat{v}_{RL}(t_1) \hat{g}_L(t_1 - t_2) \hat{t}_{LR}(t_2, t'), \end{aligned} \quad (17.32)$$

where $\hat{v}_{LR}(t) = \hat{U}_L(t) \hat{v}_{LR} \hat{U}_R^\dagger(t) = \hat{v}_{RL}^\dagger(t) = v \exp[i\phi(t)\hat{\tau}_3/2]$. One can easily show that all physical properties of the system are invariant under this gauge transformation. Thus, we shall usually consider the T-matrix equation in this gauge, i.e. in which the hopping elements are time-dependent and the electrode Green's functions only depend on the time difference.

In order to solve Eq. (17.32) it is more convenient to work in energy space where it becomes an algebraic equation. Thus, we Fourier transform the T-matrix with respect to the temporal arguments

$$\hat{t}_{LR}(t, t') = \frac{1}{2\pi} \int d\epsilon \int d\epsilon' e^{-i\epsilon t} e^{i\epsilon' t'} \hat{t}_{LR}(\epsilon, \epsilon'). \quad (17.33)$$

It is easy to convince oneself that, due to the special time dependence of the coupling elements, the T-matrix admits a Fourier expansion of the form

$$\hat{t}_{LR}(t, t') = \sum_n e^{in\phi(t')/2} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \hat{t}_{LR}(\omega, \omega + n eV). \quad (17.34)$$

In other words, Fourier transforming Eq. (17.32) one can show that $\hat{t}_{LR}(\omega, \omega')$ satisfies the following relation

$$\hat{t}_{LR}(\epsilon, \epsilon') = \sum_n \hat{t}_{LR}(\epsilon, \epsilon + neV) \delta(\epsilon - \epsilon' + neV). \quad (17.35)$$

The problem of the calculation of the current can be reduced to the evaluation of the Fourier components $\hat{t}_{nm}(\epsilon) \equiv \hat{t}_{LR}(\epsilon + neV, \epsilon + meV)$. As it can be seen Fourier transforming Eq. (17.32), these components fulfill the following (infinite) set of algebraic linear equations

$$\hat{t}_{nm} = \hat{v}_{nm} \delta_{n,m\pm 1} + \hat{\mathcal{E}}_n \hat{t}_{nm} + \hat{\mathcal{V}}_{n,n-2} \hat{t}_{n-2,m} + \hat{\mathcal{V}}_{n,n+2} \hat{t}_{n+2,m}, \quad (17.36)$$

where $\hat{v}_{m-1,m} = v(\hat{1} + \hat{\tau}_3)/2$, $\hat{v}_{m+1,m} = v^\dagger(\hat{1} - \hat{\tau}_3)/2$, and the matrix coefficients $\hat{\mathcal{E}}_n$ and $\hat{\mathcal{V}}_{n,m}$ can be expressed in terms of the Green's functions of the uncoupled electrodes, as

$$\begin{aligned} \hat{\mathcal{E}}_n &= \begin{pmatrix} v g_{R,n+1} v^\dagger g_{L,n} & v g_{R,n+1} v^\dagger f_{L,n} \\ v^\dagger g_{R,n-1} v \tilde{f}_{L,n} & v^\dagger g_{R,n-1} v g_{L,n} \end{pmatrix} \\ \hat{\mathcal{V}}_{n,n+2} &= -v f_{R,n+1} v \begin{pmatrix} \tilde{f}_{L,n+2} & g_{L,n+2} \\ 0 & 0 \end{pmatrix} \\ \hat{\mathcal{V}}_{n,n-2} &= -v^\dagger \tilde{f}_{R,n-1} v^\dagger \begin{pmatrix} 0 & 0 \\ g_{L,n-2} & f_{L,n-2} \end{pmatrix}, \end{aligned} \quad (17.37)$$

In these equations the short-hand notation $g_{i,n} = g_i(\epsilon + neV)$ for the 2×2 spin-dependent propagators has been used. Notice that the set of linear equations (17.36) are analogous to those describing a tight-binding chain with nearest-neighbor hopping parameters $\hat{\mathcal{V}}_{n,n+2}$ and $\hat{\mathcal{V}}_{n,n-2}$. A solution can then be obtained by standard recursive techniques (see Ref. [[47]] for details).

Finally, the $\hat{\mathbf{p}}_F$ -dependence of the Green's function in Eq. (17.36) depends on our choice of the contact model. Thus for instance, for the disordered case the Green's functions appearing in Eq. (17.36) are the angle averaged ones, while for the case of a momentum conserving contact we must include the trajectory dependent Green's functions (see Eqs. 13-14).

17.4.1 Current at finite voltage

As commented in a previous section, the current contribution from a given trajectory may be calculated directly by integrating the transport equation (17.2) along the trajectory over the discontinuity given by the source term. Thus, the time-dependent current reads as

$$j(t) = e N_F \langle j(\hat{\mathbf{p}}_F, t) \rangle_{\hat{\mathbf{p}}_F}, \quad (17.38)$$

where the contribution of given trajectory with momentum $\hat{\mathbf{p}}_F$ can be written as

$$j(\hat{\mathbf{p}}_F, t) = \text{Tr} \left\{ \hat{\tau}_3 [\check{t}_{LL}, \check{g}_{\infty,L}]_{\otimes}^K \right\}. \quad (17.39)$$

This expression can be greatly simplified as follows. First, the Keldysh components of the T-matrix can be eliminated in favor of the advanced and retarded components using the relation $\check{t}^K = \check{t}^R \otimes \check{g}_{\infty}^K \otimes \check{t}^A$. On the other hand, the four elements of the enlarged space are not independent. For instance, it is easy to show the following relations: $\check{t}_{LR} = (1 + \check{t}_{LL} \otimes \check{g}_{\infty,L}) \otimes \check{v}_{LR}$ and $\check{t}_{RL} = \check{v}_{RL} \otimes (1 + \check{g}_{\infty,L} \otimes \check{t}_{LL})$. Using these relations it is rather straightforward to show that the current can be written as [47]

$$\begin{aligned} j(\hat{\mathbf{p}}_F, t) &= \text{Tr} \left[\hat{\tau}_3 \left(\hat{t}_{LR}^R \otimes \hat{g}_R^K \otimes \hat{t}_{RL}^A \otimes \hat{g}_L^A - \hat{g}_L^R \otimes \hat{t}_{LR}^R \otimes \hat{g}_R^K \otimes \hat{t}_{RL}^A \right. \right. \\ &\quad \left. \left. + \hat{g}_R^R \otimes \hat{t}_{RL}^R \otimes \hat{g}_L^K \otimes \hat{t}_{LR}^A - \hat{t}_{RL}^R \otimes \hat{g}_L^K \otimes \hat{t}_{LR}^A \otimes \hat{g}_R^A \right) \right], \end{aligned} \quad (17.40)$$

where we have dropped the symbol ∞ , since from now on in this section the only Green functions which will appear are the surface Green functions.

Taking into account the Fourier expansion of the T-matrix (see Eq. 17.34), the current adopts the form

$$j(\hat{\mathbf{p}}_F, t) = \sum_{m=-\infty}^{\infty} j_m(\hat{\mathbf{p}}_F) e^{im\phi(t)}, \quad (17.41)$$

where the different Fourier current components can be expressed in terms of the Fourier components of the harmonics $\hat{t}_{nm}(\epsilon) \equiv \hat{t}(\epsilon + neV, \epsilon + meV)$ as follows

$$\begin{aligned} j_m(\hat{\mathbf{p}}_F) = & \int d\epsilon \sum_n \text{Tr} \left[\hat{\tau}_3 \left(\hat{t}_{LR,0n}^R \hat{g}_{R,n}^K \hat{t}_{RL,nm}^A \hat{g}_{L,m}^A - \hat{g}_{L,0}^R \hat{t}_{LR,0n}^R \hat{g}_{R,n}^K \hat{t}_{RL,nm}^A \right. \right. \\ & \left. \left. + \hat{g}_{R,0}^R \hat{t}_{RL,0n}^R \hat{g}_{L,n}^K \hat{t}_{LR,nm}^A - \hat{t}_{RL,0n}^R \hat{g}_{L,n}^K \hat{t}_{LR,nm}^A \hat{g}_{R,m}^A \right) \right]. \end{aligned} \quad (17.42)$$

This is the general expression of the ac Josephson effect in a superconducting contact. This indicates that in the case of constant bias voltage there appear alternating components which oscillate not only with the Josephson frequency ω_0 , as in the case of a tunnel junction, but also with all its harmonics. When the voltage tends to zero all the harmonics sum up to yield the supercurrent.

We can further simplify the expression of the current harmonics, j_m . Using the equations of the T-matrix components it can be shown that the general relation $\hat{t}_{RL,nm}^{R,A}(\epsilon) = \hat{t}_{LR,mn}^{A,R}(\epsilon)$ holds. Thus, we can simply express the current in terms of harmonics $\hat{t}_{LR,nm}^{R,A} \equiv \hat{t}_{nm}^{R,A}$ as follows

$$\begin{aligned} j_m(\hat{\mathbf{p}}_F) = & \int d\epsilon \sum_n \text{Tr} \left[\hat{\tau}_3 \left(\hat{t}_{0n}^R \hat{g}_{R,n}^K \hat{t}_{mn}^{R\dagger} \hat{g}_{L,m}^A - \hat{g}_{L,0}^R \hat{t}_{0n}^R \hat{g}_{R,n}^K \hat{t}_{mn}^{R\dagger} \right. \right. \\ & \left. \left. + \hat{g}_{R,0}^R \hat{t}_{n0}^{A\dagger} \hat{g}_{L,n}^K \hat{t}_{nm}^A - \hat{t}_{n0}^{A\dagger} \hat{g}_{L,n}^K \hat{t}_{nm}^A \hat{g}_{R,m}^A \right) \right]. \end{aligned} \quad (17.43)$$

In order to illustrate this current formula we now investigate the dc component for a spin singlet superconductor with no spin active interface in two limiting cases: tunnel regime and perfect transmission. For the description of a poorly transmissive barrier one can solve perturbatively the equation for $\hat{t}_{nm}^{R,A}$. Up to first order in the coupling element: $\hat{t}_{LR,nm}^{R,A} \approx (v/2)(\hat{1} + \hat{\tau}_3)\delta_{m,n+1} + (v/2)(\hat{1} - \hat{\tau}_3)\delta_{m,n-1}$ and higher harmonics can be neglected. Substituting this perturbative solution into the expression of the dc current, j_0 , one finds the traditional result

$$\begin{aligned} j_0(\hat{\mathbf{p}}_F) = & \mathcal{D} \int_{-\infty}^{\infty} d\epsilon \rho_L(\epsilon - eV) \rho_R(\epsilon) \times \\ & [f_{FD}(\epsilon - eV) - f_{FD}(\epsilon)], \end{aligned} \quad (17.44)$$

where f_{FD} is the Fermi distribution function and $\rho_{L,R}$ are the local densities of states at the corresponding side of the interface: $\rho_i(\epsilon) = \frac{1}{\pi} \text{Im} \{g_i^A(\epsilon)\}$.

In the case of a perfect transmissive contact ($\mathcal{D} = 1$), the absence of backscattering greatly simplify the calculation of the different harmonics of the T-matrix components (see Ref. [[47]]). In the case of a symmetric contact the dc current for perfect transmission can be written as [72]

$$j_0(\hat{\mathbf{p}}_F) = \int_{-\infty}^{\infty} d\epsilon \sum_{m=0}^{\infty} \left[\prod_{j=1}^m |\gamma_j|^2 \right] (1 - |\gamma_0|^2 |\gamma_{m+1}|^2) \times \{f_{FD}(\epsilon) - f_{FD}(\epsilon + (m+1)eV)\}, \quad (17.45)$$

where $\gamma_j \equiv \gamma(\epsilon + jeV)$ and $\gamma(\epsilon)$ is the amplitude of an Andreev reflection in a N-S interface with perfect transmission. This amplitude is defined as $\gamma^R \equiv f^R/(g^R - i\pi)$, where the Green's functions are evaluated at the interface. This is a very appealing formula that tell us that the probability of a multiple Andreev reflection is basically the product of the different Andreev reflections that take place at each side of the interface.

In order to illustrate our approach in the case of a voltage biased contact, we consider here a junction between two d-wave superconductors. Let us analyze in particular the symmetric junction mentioned in section IIIB, whose description is based on the Green's functions of Eq. (17.20). Again, we neglect pair breaking effect and assume constant order parameter up to the interface. The proper self-consistent treatment of these junctions will be the subject of a forthcoming publication. In this contact geometry the zero-energy bound states in each side of the interface strongly affect the transport through this system. Of course, the results for the current depend on the contact model used. Let us first consider the case of a disordered contact. In this case, the propagators which enter in the current formula are the angle averaged ones. This implies that the anomalous propagators vanish, which means that the current is only due to single-quasiparticle processes. Thus, the current formula reduces to

$$j_0(V) = \int_{-\infty}^{\infty} d\epsilon \mathcal{T}(\epsilon, V) [f_{FD}(\epsilon - eV) - f_{FD}(\epsilon)], \quad (17.46)$$

where $\mathcal{T}(\epsilon, V)$ is an energy and voltage dependent transmission coefficient given by

$$\mathcal{T}(\epsilon, V) = \frac{4\pi^2 |v|^2 \langle \rho_L(\epsilon - eV) \rangle_{\hat{\mathbf{p}}_F} \langle \rho_R(\epsilon) \rangle_{\hat{\mathbf{p}}_F}}{|1 - |v|^2 \langle g_L(\epsilon - eV) \rangle_{\hat{\mathbf{p}}_F} \langle g_R(\epsilon) \rangle_{\hat{\mathbf{p}}_F}|^2}, \quad (17.47)$$

where $\langle \rho_i(\epsilon) \rangle_{\hat{\mathbf{p}}_F}$ ($i = L, R$) is the local density of states at the interface.

In Fig. 36 we show the current-voltage characteristics and differential conductance for different values of the normal transmission coefficient $\mathcal{D}$ for this disordered model. The only abrupt feature exhibited by the current inside the gap occurs in the tunnel regime. The resonant tunneling through the zero-energy bound states leads to a zero-bias anomaly and the subsequent negative differential conductance. As it is well known, the position and the height of the peak in the conductance depends on intrinsic width of zero-energy states [19, 73, 74, 75]. It is known theoretically that the elastic scattering with bulk impurities [21] or a diffusive surface layer [22] provide an intrinsic broadening, which for the case of Born scatterers is $\propto \sqrt{\Gamma\Delta}$, where $\Gamma = 1/2\tau$ is the effective pair-breaking parameter locally at the surface. In Fig. 36 we have introduced a small phenomenological broadening of $10^{-2}\Delta$ to mimic this intrinsic effect.

With this simple calculation we can already point out the following. Many calculations of the I-V curves in junctions of unconventional superconductors are limited to the tunneling regime, and make use of the traditional tunnel formula (see Eq. 17.44). However, if bound states or divergencies are present in the density of states, this perturbative expression does not even give the correct result for a low transparent contact, and one needs to include high order processes as in Eq. (17.46). Indeed, this is a well known problem in the context of s-wave superconductors (for a discussion of this problem see for instance Ref. [[47]]). In order to illustrate this fact, in Fig. 37 we compare the result of Eq. (17.46) and the tunneling result for a small transparency $\mathcal{D} = 0.01$. One can see in Fig. 37(a) that there is a clear difference between these two results. In particular, in the low voltage regime the complete expression

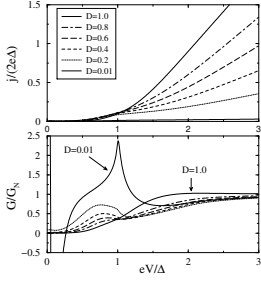


Figure 36: Zero temperature I-V characteristics (upper panel) and differential conductance for different transmissions (lower panel) of a disordered contact between two d-wave superconductors. The misorientations are $\alpha = \pi/4$. The conductance is normalized by the normal state conductance and the voltage is expressed in units of the maximum gap Δ .

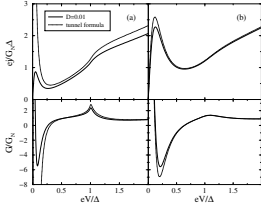


Figure 37: Comparison between the tunnel formula (Eq. 17.44) and the exact result of Eq. (17.46) for a transmission $D = 0.01$ and two different values of the broadening η : (a) $\eta = 0.01\Delta$, and (b) $\eta = 0.1\Delta$. The upper panels show the zero-temperature I-V curves and the lower ones the corresponding differential conductances.

renormalizes the unphysical divergencies obtained with the tunnel formula. As the broadening of the level is increased the difference between these two results diminishes progressively (see Fig. 37(b)). Indeed, recent experiments in *ab*-plane tunneling into YBCO [9, 10, 11, 12] suggest that the broadening of the zero bias anomaly can be as large as 25% of the gap Δ (see Ref. [[23]]). In this case, there is no significant difference in the resulting current between both expressions.

Let us now consider the case of a momentum conserving contact, which is the usual assumption in the Zaitsev boundary conditions. To gain some insight into the final result, in Fig. 38 we show the contribution of an individual trajectory $\hat{p}_r$. The current and voltage are normalized in units of the gap seen by this trajectory. As can be observed, the current exhibits a pronounced subgap structure at voltages $eV = \Delta_p/n$, where n is a integer number, together with the appearance of negative differential conductance (this can be seen better in the lower panel of this figure). These features are a simple consequence of the resonant tunneling across the zero-energy bound states. Indeed, this type of I-V has been previously obtained in the context of a junction between two conventional superconductors coupled by means of a resonant transmission (see Ref. [[76, 77]]). Notice also the presence of a zero bias peak, specially clear for low transparencies, and which is a consequence of the small broadening introduced in the calculation. The total current is obtained by averaging over the different trajectories. Thus, the final result depends on the model for the angular dependence of the normal transmission coefficient. With any reasonable reasonable model most of the features of the trajectory resolved current disappear. In particular, the subharmonic gap structure is washed out, and it only remains a peak in the conductance at $eV \approx \Delta$ (see for instance Ref. [[19]]).

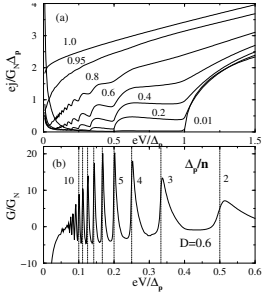


Figure 38: (a) Trajectory resolved zero-temperature I-V curves for the d-wave contact with a momentum conserving interface. The different curves correspond to different values of normal transmission coefficient. In order to see these curves in the same scale the current is normalized by the normal state conductance G_N which includes the transmission coefficient. Moreover, the current and the voltage are normalized in units of the momentum-dependent gap. (b) Differential conductance for a transmission $\mathcal{D} = 0.6$. The vertical lines indicate the positions $eV_n = \Delta_p/n$, $n = 2, \dots, 10$, as a guide for the eye.

17.5 Current fluctuations

During the last years it has become progressively clearer that a deep understanding of the electronic transport in mesoscopic systems requires the analysis of quantities which goes beyond the straightforward measurement of the current-voltage characteristics. In this sense the noise or time-dependent current fluctuations has emerged as a very useful tool which provides new information on the time correlations of the current, information about channel distributions, statistics and charge of the carriers [78]. In the case of superconducting contacts, most of the activity has been restricted to the case of s-wave superconductors [79, 80, 81, 82]. In the case of unconventional superconductors there are only a few theoretical works in the context of hybrid structures like normal metal/d-wave superconductors [83, 84]. We believe that in the next future the measurement of current fluctuations will be an important tool for a deeper understanding of the symmetry of the order parameter and origin of the superconductivity in general in the case of HTS materials. For this reason in this section we describe the calculation of the noise spectrum within our approach, valid for any type of contact between unconventional superconductors.

Let us remember that the noise is characterized by its spectral density or power spectrum $S(\omega)$, which is simply the Fourier transform at frequency ω of the current-current correlation function

$$\begin{aligned} S(\omega) &= \int d(t' - t) e^{i\omega(t' - t)} \langle \delta \hat{\mathbf{a}}(t') \delta \hat{\mathbf{a}}(t) + \delta \hat{\mathbf{a}}(t) \delta \hat{\mathbf{a}}(t') \rangle \\ &\equiv \int d(t' - t) e^{i\omega(t' - t)} K(t, t'), \end{aligned} \quad (17.48)$$

where $\delta \hat{\mathbf{a}}(t) = \hat{\mathbf{a}}(t) - \langle \hat{\mathbf{a}}(t) \rangle$ are the fluctuations in the current.

In order to obtain the expression of the current-current correlator, first we need an expression for the current operator. Within our model this operator evaluated at the interface can be written as follows

$$\hat{\mathbf{a}}(t) = ie \sum_{\sigma} \left\{ v_{LR,\sigma} \hat{c}_{L,\sigma}^{\dagger}(t) \hat{c}_{R,\sigma}(t) - v_{RL,\sigma} \hat{c}_{R,\sigma}^{\dagger}(t) \hat{c}_{L,\sigma}(t) \right\}. \quad (17.49)$$

This expression is a simple consequence of the continuity equation for the current [53].

In order to calculate the noise we need in principle to evaluate correlators of four field operators. However, we are working in the framework of a mean field theory, which means that we can decouple these correlators in terms of one-particle Green's functions using the Wick theorem. With this in mind, it is straightforward to show that the kernel $K(t, t')$ can be expressed in terms of the interface Keldysh Green's functions as follows [81]

$$K(t, t') = e^2 \left\{ \text{Tr} \left[\hat{v}_{RL} \hat{G}_{LL}^<(t, t') \hat{v}_{LR} \hat{G}_{RR}^>(t', t) \right. \right. \\ \left. \left. + \hat{v}_{LR} \hat{G}_{RR}^<(t, t') \hat{v}_{RL} \hat{G}_{LL}^>(t', t) \right. \right. \\ \left. \left. - \hat{v}_{RL} \hat{G}_{LR}^<(t, t') \hat{v}_{RL} \hat{G}_{LR}^>(t', t) \right. \right. \\ \left. \left. - \hat{v}_{LR} \hat{G}_{RL}^<(t, t') \hat{v}_{LR} \hat{G}_{RL}^>(t', t) \right] + (t \rightarrow t') \right\} \quad , \quad (17.50)$$

where the functions $\hat{G}^<$ and $\hat{G}^>$ are related to the usual advanced, retarded and Keldysh functions in the following way

$$\begin{aligned} \hat{G}^< &= \left(\hat{G}^K - \hat{G}^R + \hat{G}^A \right) / 2 \\ \hat{G}^> &= \left(\hat{G}^K + \hat{G}^R - \hat{G}^A \right) / 2. \end{aligned} \quad (17.51)$$

In order to compactify the notation, we introduce the trace $\tilde{\text{Tr}}$ and the matrix $\tilde{\tau}_3$ which act in the “reservoir” space. Then, the noise kernel reads

$$K(t, t') = -e^2 \tilde{\text{Tr}} \left[\tilde{v} \tilde{G}^<(t, t') \tilde{\tau}_3 \tilde{v} \tilde{G}^>(t', t) + \right. \\ \left. \tilde{v} \tilde{G}^>(t, t') \tilde{\tau}_3 \tilde{v} \tilde{G}^<(t', t) \right]. \quad (17.52)$$

Now, in order to eliminate the Green's functions in favor of the T-matrix elements, we use the relation

$$\tilde{G}^{<, >} = \left(\tilde{1} + \tilde{G}^R \circ \tilde{v} \right) \circ \tilde{G}_\infty^{<, >} \circ \left(\tilde{1} + \tilde{v} \circ \tilde{G}^A \right), \quad (17.53)$$

where $\tilde{G}_\infty^<(\epsilon) = \left[\tilde{G}_\infty^A(\epsilon) - \tilde{G}_\infty^R(\epsilon) \right] f_{FD}(\epsilon)$ and $\tilde{G}_\infty^>(\epsilon) = \left[\tilde{G}_\infty^A(\epsilon) - \tilde{G}_\infty^R(\epsilon) \right] (f_{FD}(\epsilon) - 1)$. Now, making use of Eqs. (3-7) it is easy to see that the following relation holds

$$\tilde{v} \circ \tilde{G}^{<, >} = \tilde{T}^R \circ \tilde{G}_\infty^{<, >} \circ \left(\tilde{1} + \tilde{T}^A \circ \tilde{G}_\infty^A \right). \quad (17.54)$$

This expression allows us to write the noise kernel as follows

$$K(t, t') = -e^2 \tilde{\text{Tr}} \left\{ \left[\tilde{T}^R \circ \tilde{G}_\infty^< \circ \left(\tilde{1} + \tilde{T}^A \circ \tilde{G}_\infty^A \right) \right] (t, t') \tilde{\tau}_3 \left[\tilde{T}^R \circ \tilde{G}_\infty^> \circ \left(\tilde{1} + \tilde{T}^A \circ \tilde{G}_\infty^A \right) \right] (t', t) + (t \rightarrow t') \right\}. \quad (17.55)$$

As explained in section II, once we have eliminated the full Green's functions in the noise kernel, we can perform the quasiclassical ξ -integration and substitute the quasiclassical Green's functions, $\tilde{g}_\infty$, for the full ones, $\tilde{G}_\infty$. Thus, the noise kernel can be expressed finally as

$$K(t, t') = -e^2 \tilde{\text{Tr}} \left\{ \left[\tilde{T}^R \otimes \tilde{g}_\infty^< \otimes \left(\tilde{1} + \tilde{T}^A \otimes \tilde{g}_\infty^A \right) \right] (t, t') \tilde{\tau}_3 \left[\tilde{T}^R \otimes \tilde{g}_\infty^> \otimes \left(\tilde{1} + \tilde{T}^A \otimes \tilde{g}_\infty^A \right) \right] (t', t) + (t \rightarrow t') \right\}. \quad (17.56)$$

Let us stick now to the case of a constant bias voltage applied across the interface. In this case, for both contact models considered in section III, we can resolve the current fluctuations in trajectories as follows

$$S(\omega, t) = e^2 N_F \langle S(\hat{\mathbf{p}}_F, \omega, t) \rangle_{\hat{\mathbf{p}}_F}, \quad (17.57)$$

where the time-dependent contribution of given trajectory with momentum $\hat{\mathbf{p}}_F$ can be written as

$$S(\hat{\mathbf{p}}_F, \omega, t) = \sum_{m=-\infty}^{\infty} S_m(\hat{\mathbf{p}}_F, \omega) e^{im\phi(t)}, \quad (17.58)$$

where the different ac-components of the noise can be expressed in terms of the Fourier component of the T-matrix elements in the following way

$$S_m(\hat{\mathbf{p}}_F, \omega) = - \int d\epsilon \sum_n \tilde{\text{Tr}} \left\{ \left[\tilde{T}^R \tilde{g}_{\infty}^< \left(\tilde{1} + \tilde{T}^A \tilde{g}_{\infty}^A \right) \right]_{0n}(\epsilon) \tilde{\tau}_3 \left[\tilde{T}^R \tilde{g}_{\infty}^> \left(\tilde{1} + \tilde{T}^A \tilde{g}_{\infty}^A \right) \right]_{nm}(\epsilon + \omega) + (\epsilon \rightarrow \epsilon + \omega) \right\}. \quad (17.59)$$

Notice that in the case of a junction between two superconductors, the noise, as the current, oscillates on time with all the harmonics of the Josephson frequency. Notice also that we have reduced the calculation of this quantity to the determination of the different Fourier components of the T-matrix elements, which has been detailed in section IV.

In order to illustrate the calculation of the current fluctuations, we consider the contact between d-wave superconductors analyzed in the previous section. In particular, we present results for the zero-frequency noise at zero temperature, S , i.e. the zero-frequency shot noise. At this point, it is worth remarking that by zero-frequency noise we mean noise at a frequency lower than any relevant energy scale in our problem, gap for instance, and high enough to neglect $1/f$ noise [14]. Again, the final result depends on the type of contact model under investigation. Let us start discussing the shot noise for the disordered contact. In this case, due to the vanishing of the anomalous Green's functions, the whole calculation reduces to the determination of the quasiparticle contribution, which in term of the transmission coefficient of Eq. (17.47) can be written as

$$S = \int_0^{eV} d\epsilon \mathcal{T}(\epsilon, V) (1 - \mathcal{T}(\epsilon, V)). \quad (17.60)$$

This is simply the result that one obtains for a normal contact with an energy and voltage dependent transmission coefficient [78]. In Fig. 39 we show the result of Eq. (17.60) for different normal transmissions. In the tunneling regime the shot noise is $S(V) \approx 2ej(V)$ and the most remarkable feature is the zero bias anomaly (see inset of Fig. 39). In the case of perfect transmission there is a non-zero noise due to the fact that the transmission coefficient $\mathcal{T}(\epsilon, V)$ is less than one in the gap region. For voltages much greater than the maximum gap, then $\mathcal{T}(\epsilon \gg \Delta, \mathcal{D} = 1.0) \rightarrow 1$, what makes that the noise at $\mathcal{D} = 1.0$ saturates in the high voltage regime.

More interesting is the case of the momentum conserving interface. In Fig. 40 we show the contribution of a trajectory of momentum $\hat{\mathbf{p}}_F$. As in the case of the current, the shot noise exhibits a rich subharmonic gap structure, which persists almost up to perfect transmission. The shape of the different curves can be understood in the same terms as the BCS case (see Ref. [81]), with the additional ingredient of the resonant tunneling through the zero energy states. Of course, as in the case of the current, most of these features disappear after performing the angular average.

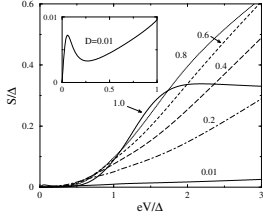


Figure 39: Zero-frequency shot noise for the disordered contact considered in Fig. 37. The inset show the low bias limit of the curve in the tunneling regime ($\mathcal{D} = 0.01$).

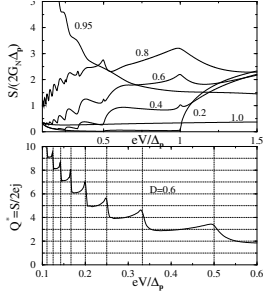


Figure 40: The upper panel shows the trajectory resolved zero-frequency shot noise for the momentum conserving case considered in Fig. 38. The shot noise and voltage are normalized by the trajectory-dependent gap Δ_p . In the lower panel one can see the effective charge defined as $Q^* = S/2ej$ as a function of voltage for a transmission $\mathcal{D} = 0.6$. The vertical lines indicate the position of the voltages $eV_n = \Delta_p/n$, $n = 2, \dots, 10$.

In the case of conventional superconductors, the shot noise has been proposed as a tool for measuring the multiple charge quanta transferred by the multiple Andreev reflections [81]. Obviously we can pose here the same question in the case of unconventional superconductors. Indeed, a noise experiment has been recently proposed by Auerbach and Altman [85] to discriminate between two possible explanations of the pronounced subharmonic gap structure observed in YBCO edge junctions [18]. Namely, usual multiple Andreev reflections in a d-wave superconductor and magnon pair creation in the context of the SO(5) theory. In this latter case the observed charge should be $Q^* = 2ne$, where $n = 1, 2, \dots$, at a voltage $eV_n = \Delta/n$. This result has to be compared with $Q^* = ne$ expected in the traditional view of MAR. In order to contribute to the solution of this puzzle, we show in Fig. 40 (lower panel) the effective charge, $Q^* = S/2ej$, for a transmission $\mathcal{D} = 0.6$. This result confirm the traditional interpretation that in the MAR process of order n a charge ne is transferred. Usually, in order to observe a clear quantization of the charge one should go to the tunneling regime [81], but in this case this is not necessary due to the resonant tunneling through the zero-energy states. Notice again that this is the contribution of a single trajectory and after angle averaging this clear quantization of the charge with voltage disappears. The exhaustive analysis of the shot noise in d-wave contacts will be presented in a forthcoming publication.

17.6 Conclusions

We have shown how a Hamiltonian approach and the quasiclassical theory of superconductivity can be combined to give a powerful tool to analyze electronic and transport properties of superconducting junctions. In particular, we have demonstrated that a simple Hamiltonian description of an interface can be used to model a great variety of contacts. This Hamiltonian description can be brought into quasiclassical theory via a T-matrix equation, resulting in a new formulation of boundary conditions. These boundary conditions do not contain any

spurious solution and can be efficiently solved to compute any transport property. The broad applicability of this formulation covers cases ranging from conventional superconductors to unconventional ones, clean systems and diffusive ones. Moreover, it can be applied to spin active interfaces and it is well suited for the description of time-dependent phenomena like the I-V characteristics and the noise properties of junctions with arbitrary transmission and bias voltage. We have illustrated this approach with the calculation of Josephson current in a great variety of situations. The calculation of I-V characteristics and the noise has been exemplified with the analysis of a contact between two d-wave superconductors. In particular, we have briefly discussed the use of shot noise as a possible tool for measuring the charge of the Andreev reflections in unconventional superconductors.

18 Finite frequency noise in a normal metal - topological superconductor junction by D. Bathellier,¹ L. Raymond,¹ T. Jonckheere,¹ J. Rech,¹ A. Zazunov,² and T. Martin¹

Abstract

A topological superconductor nanowire bears a Majorana bound state at each of its ends, leading to unique transport properties. As a way to probe these, we study the finite frequency noise of a biased junction between a normal metal and a topological superconductor nanowire. We use the non-equilibrium Keldysh formalism to compute the finite frequency emission and absorption noise to all order in the tunneling amplitude, for bias voltages below and above the superconducting gap. We observe noticeable structures in the absorption and emission noise, which we can relate to simple transport processes. The presence of the Majorana bound state is directly related to a characteristic behavior of the noise spectrum at low frequency. We further compute the noise measurable with a realistic setup, based on the inductive coupling to a resonant LC circuit, and discuss the impact of the detector temperature. We have also computed the emission noise for a non-topological system with a resonant level, exhibiting a zero-energy Andreev bound state, in order to show the specificities of the topological case. Our results offer an original tool for the further characterization of the presence of Majorana bound states in condensed matter systems.

18.1 Introduction

The search for Majorana fermions [1] – particles that are their own antiparticles – has been an active field of study in high energy physics, with no conclusive/definite evidence of their finding. Since the last decades, the Majorana paradigm has now entered condensed matter physics thanks to the pioneering work of Kitaev. [2] In this context, rather than being elementary particles, Majorana fermions emerge from the collective behavior of a many electron system. This Kitaev toy model consists in a tight-binding chain of electrons living on lattice sites, which includes hopping and superconducting (p-wave) pairing between neighboring sites. Majorana fermions exist at the boundaries of this one dimensional chain provided that certain conditions on the hopping parameter, the gap parameter, and the chemical potential are met. In this so-called topological phase, the ground state of the system is degenerate, opening the possibility to realize a quantum bit which could be in principle immune from decoherence effects, with applications to quantum information schemes.

In quantum nano-physics, a huge effort has been devoted to finding a realistic experimental equivalent of the Kitaev model. One possibility is to realize a topological superconductor

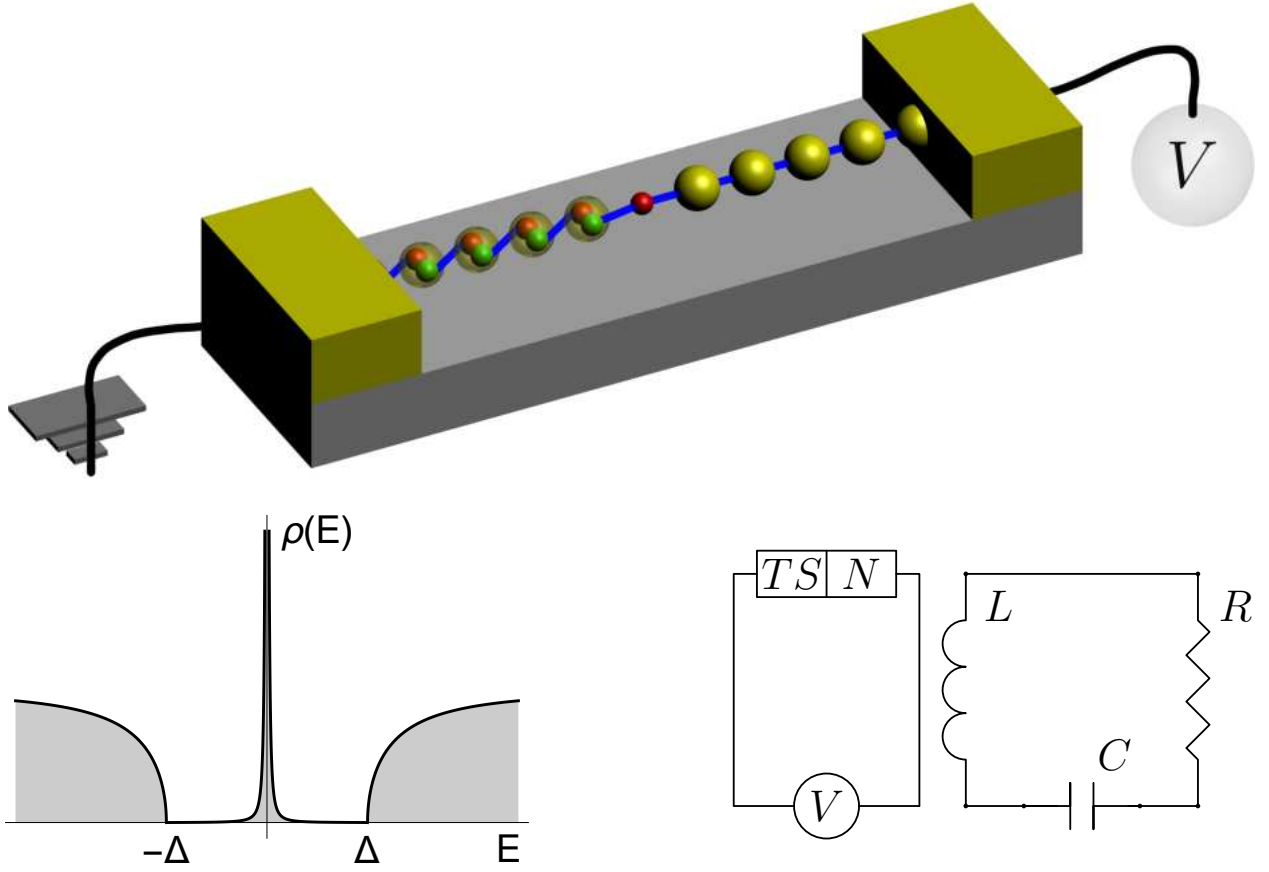


Figure 41: Top: schematic view of a junction between a normal metal (yellow dots) and a topological superconductor (red-green dots, with a red Majorana bound state at its end); the normal metal is biased at voltage V . Bottom left: topological superconductor density of states, with a central peak at zero energy due to the Majorana bound state, and square root behavior outside the gap. Bottom right: the principle of the noise detection, using the inductive coupling to a RLC circuit.[20]

(TS) nanowire by inducing s-wave superconductivity in a semiconducting nanowire subject to both Rashba spin-orbit coupling and a Zeeman magnetic field. [3, 4, 5, 6, 7, 8] Other proposals where a chain of magnetic atoms are deposited on a superconducting substrate have also been studied.[9] Many theoretical proposals for the detection of Majorana fermions relying on quantum transport setups have been published over the past few years, [10, 11, 12, 13, 14, 15] but despite the significant experimental progress, an unequivocal signature[16, 17] of these Majorana bound states (MBS) is still lacking. While some elements point in the right direction (the detection of the zero bias anomaly in the differential conductance, and quite recently a claim that the quantized zero bias conductance has been observed [18]) it is clear that the definite discovery of Majorana fermions will require more experimental tests in order for this finding to be firmly established. This is especially important in order to exploit the properties of these objects in the future, such as the generation of a Majorana qbit, and the braiding of Majorana fermions, with applications to quantum computing.

In this work, the physics of a normal metal - topological superconductor (NTS) junction (see Fig. 41) is revisited. While the non-equilibrium electronic current and the zero-frequency noise are perfectly understood by now, [19] the finite frequency noise characteristic of a NTS junction has so far not been considered. Finite frequency noise has been studied extensively in more conventional materials in the context of quantum mesoscopic physics and nano-physics,

both theoretically and experimentally. On the theoretical side, it has been argued[20, 22, 21] that since the two current operators entering the noise correlator are evaluated at different times, one can introduce two quantities - the emission and the absorption noise. A noise detector should in principle be described on the same quantum footing as the nano-device under study. In full generality, the Fourier transform of the real time noise correlators can be a complex function of the frequency. The measured signal by the quantum detector, which is a real number, should thus be a combination of such emission and absorption quantities. This has been experimentally verified in pioneering finite frequency measurement experiments performed in Delft,[23] in Orsay[24, 25, 26] and in Saclay[27, 28, 29] in different contexts. In a previous work, one of us [30, 31] computed the finite frequency noise of a normal metal - BCS superconductor junction, and found cusps of singularities at specific frequencies corresponding to different transport processes. However, these results are not expected to survive in the case of the NTS junction, as the density of states (DOS) of the topological superconductor differ substantially from its conventional BCS counterpart. The BCS density of states is zero inside the gap, and bears inverse square root divergences at the gap edges. On the opposite, the DOS of a TS wire has a square root behavior outside the gap and a so-called Majorana peak at zero energy (see Fig. 41).

Using a continuous model describing a semi-infinite Kitaev chain, we derive the finite frequency noise to all orders in the tunnel coupling parameter of the NTS junction (recovering in the process known results for the current and zero-frequency noise[19]). Within our framework, the current and noise can be expressed in terms of dressed Keldysh Green's functions (using Wick's theorem for the noise as the Hamiltonian is quadratic). Solving Dyson's equation exactly, one reaches analytical expressions for these two quantities in terms of the known bare boundary Green's functions of the two leads. The finite frequency emission and absorption noises are then computed for voltages above and below the superconducting gap, showing plateaus - with a specific frequency width - where the latter quantities are essentially constant. We interpret such features as emission and absorption processes which couple the constant density of states of a normal metal with the sharp peak in the TS density of states associated with the Majorana bound states. For completeness, we further use a generic model for the quantum detector - a LC circuit which can absorb or emit photons from or to the nano-device[20, 22, 32] - and compute the measurable noise at the resonant frequency of the LC detector.

In order to illustrate the added value of using finite frequency noise over other quantities relying on DOS, like the differential conductance, we have also computed the transport properties of a non-topological N-dot-S system, where S is a standard BCS superconductor. Indeed this system bears a zero-energy Andreev bound state, with a sharp peak in the DOS, which is susceptible to yield similar differential conductance and finite frequency noise profiles. With the right parameters, this system presents a conductance peak similar to the one of a Majorana bound state, which shows that the conductance alone is not enough to characterize a MBS. However, the emission noise shows a distinctive dip near zero-frequency in this system, which is absent for the NTS junction. We thus believe that the finite frequency noise characteristics of an NTS junction can bring unique, additional information about Majorana fermions in experiments. This constitutes the main motivation of this study.

The paper is organized as follows: Sec. 27.3 introduces the microscopic model Hamiltonian and defines the Keldysh Green's functions. The Dyson's equations which are relevant for the current and noise calculations are specified in Sec. 18.3. The noise is computed in Sec. 18.4 (real time correlator, emission and absorption noise,...) and detailed results are discussed in Sec. 18.5. Sec. 18.6 shows the calculations and the results for the emission noise of a N-dot-S system, with a BCS superconductor. We conclude this work with Sec. 18.7. In the whole following text, we fix the value of $\hbar$ to 1.

18.2 Model

A semi-infinite topological superconductor wire is a single-channel p-wave superconductor with spin-momentum locking, described from the continuous version of the Kitaev chain[2] by the helical superconductor Hamiltonian

$$H_j = \int_0^{+\infty} dx \, \psi_j^\dagger(x) (-iv_F \partial_x \sigma_3 + \Delta \sigma_2) \psi_j(x) , \quad (18.1)$$

where v_F is the Fermi velocity, Δ is the superconducting gap, and $\sigma_{1,2,3}$ are Pauli matrices in Nambu space. Here, we refer to $j = 0$ for the TS nanowire and $j = 1$ for the (spinless) normal metal lead, whose Hamiltonian has the same form, with the gap parameter set to zero. These lead Hamiltonians are conveniently written in terms of the fermionic Nambu spinor ψ_j defined as

$$\psi_j = \begin{pmatrix} c_{Rj} \\ c_{Lj}^\dagger \end{pmatrix} , \quad (18.2)$$

which combines left- and right moving fermion operators $c_{R/L,j}$.

The Hamiltonian describing the tunnel coupling between the two leads has the form (in the absence of applied bias)

$$H_T = \lambda c_0^\dagger c_1 + \text{H.c.} , \quad (18.3)$$

where c_0 and c_1 are boundary fermions built simply from the sum of the left and right fermionic fields entering the Nambu spinor, evaluated at the location $x = 0$ of the NTS junction, so that $c_j = c_{Rj}(0) + c_{Lj}(0)$. It is also convenient to write this tunnel Hamiltonian in the Nambu representation. The formalism can be generalized to an arbitrary number M of tunnel-coupled conductors as in Ref. [33]. For this, we define a tunneling matrix W in lead space, whose diagonal elements vanish: $W_{jj'} = \lambda \sigma_3 (1 - \delta_{jj'})$. Then, in Nambu representation, the tunneling Hamiltonian takes the form

$$H_T = \frac{1}{2} \sum_{jj'}^M \Psi_j^\dagger W_{jj'} \Psi_{j'} , \quad \Psi_j = \begin{pmatrix} c_j \\ c_j^\dagger \end{pmatrix} . \quad (18.4)$$

The operator accounting for the current flowing through lead j , is derived from the tunneling Hamiltonian (18.4) as

$$I_j = ie \sum_{j' \neq j} \Psi_j^\dagger \sigma_3 W_{jj'} \Psi_{j'} . \quad (18.5)$$

An essential tool for this study is the Keldysh Green's function for the boundary fermions. It is defined as $\tilde{G}_{jk}^{\eta\eta'}(t, t') = -i \langle T_K c_j(t_\eta) c_k^\dagger(t_{\eta'}) \rangle$, where $\eta = \pm$ denotes the Keldysh time index which specifies the position of times on the Keldysh contour. It is represented by the (4 by 4) matrix in Keldysh-Nambu space:

$$\tilde{G}_{jj'} = \begin{pmatrix} G_{jj'}^{++} & G_{jj'}^{+-} \\ G_{jj'}^{-+} & G_{jj'}^{--} \end{pmatrix} = \tilde{L} \begin{pmatrix} 0 & G_{jj'}^a \\ G_{jj'}^r & G_{jj'}^K \end{pmatrix} \tilde{L}^{-1} , \quad (18.6)$$

where G^a , G^r , G^K are the advanced, retarded and Keldysh components in Nambu space, and the unitary transform $\tilde{L} = (1/\sqrt{2})(\tau_0 + i\tau_2) \otimes \sigma_0$ (τ_j are Pauli matrices in Keldysh space).

We wish to express the average current in terms of such Green's functions. For this we write explicitly the expression of the off-diagonal Keldysh Green's functions:

$$G_{jj'}^{-+}(t, t') = -i \begin{pmatrix} \langle c_j(t) c_{j'}^\dagger(t') \rangle & \langle c_j(t) c_{j'}(t') \rangle \\ \langle c_j^\dagger(t) c_{j'}^\dagger(t') \rangle & \langle c_j^\dagger(t) c_{j'}(t') \rangle \end{pmatrix} \quad (18.7)$$

$$G_{jj'}^{+-}(t, t') = +i \begin{pmatrix} \langle c_{j'}^\dagger(t') c_j(t) \rangle & \langle c_{j'}(t') c_j(t) \rangle \\ \langle c_{j'}^\dagger(t') c_j^\dagger(t) \rangle & \langle c_{j'}(t') c_j^\dagger(t) \rangle \end{pmatrix} \quad (18.8)$$

The average current can then be recast[19] in the form of a Nambu trace containing the Keldysh Green's function $G_{jj'}^K(t, t') = G_{jj'}^{+-}(t, t') + G_{jj'}^{-+}(t, t')$:

$$\langle I_j \rangle = \frac{e}{2} \sum_{j' \neq j} \text{Tr}_N [\sigma_3 W_{jj'} G_{jj'}^K(t, t)] , \quad (18.9)$$

which reads, in Fourier space, for the two-lead geometry chosen here

$$I(V) = \frac{e\lambda}{2} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \text{Tr}_N G_{01}^K(\omega) , \quad (18.10)$$

where we used that, in this case, the tunneling matrix reduces to $W_{01} = W_{10} = \lambda\sigma_3$.

18.3 Dyson's equations and average current

The Dyson's equation is readily expressed in Keldysh-Nambu-lead space as

$$\check{G} = [\check{g}^{-1} - \check{W}]^{-1} , \quad (18.11)$$

where we have

$$\check{G} = \begin{pmatrix} \tilde{G}_{00} & \tilde{G}_{01} \\ \tilde{G}_{10} & \tilde{G}_{11} \end{pmatrix} , \quad \check{W} = \begin{pmatrix} \tilde{W}_{00} & \tilde{W}_{01} \\ \tilde{W}_{10} & \tilde{W}_{11} \end{pmatrix} , \quad (18.12)$$

and the tunneling self-energy $\tilde{W}_{jj'}$ is a diagonal matrix of the form $\tilde{W}_{jj'} = \text{diag}(W_{jj'}, -W_{jj'})$.

Using Eq. (18.6), this can be further expressed into a set of equations for the advanced, retarded and Keldysh Green's functions as

$$\hat{G}^{r/a} = \hat{g}^{r/a} + \hat{g}^{r/a} \hat{W} \hat{G}^{r/a} \quad (18.13)$$

$$\hat{G}^K = \left(\hat{\mathbb{1}} + \hat{G}^r \hat{W} \right) \hat{g}^K \left(\hat{\mathbb{1}} + \hat{W} \hat{G}^a \right) \quad (18.14)$$

where we used that the Keldysh component associated with the tunneling self-energy vanishes and introduced the notation $\hat{\cdot}$ for matrices in Nambu-lead space.

In order to compute the current, we thus need to know the full expression of the dressed Green's function G_{01}^K . The Dyson's equation allows us to express this Keldysh component in terms of the advanced and retarded dressed Green's functions as well as the bare Keldysh Green's function and the tunneling self-energy. One has

$$\begin{aligned} G_{01}^K &= (\sigma_0 + G_{01}^r W_{10}) g_{00}^K W_{01} G_{11}^a \\ &+ G_{00}^r W_{01} g_{11}^K (\sigma_0 + W_{10} G_{01}^a) . \end{aligned} \quad (18.15)$$

18.3.1 Bare Green's functions

As a first step, we need to specify g_{00}^K and g_{11}^K . The procedure to obtain the boundary Green's function for the uncoupled semi-infinite TS wire has been described carefully in Ref. [19]. Following this derivation, the various components of the bare Green's functions for the topological superconductor then read

$$g_{00}^K(\omega) = [1 - 2n_F(\omega)] [g_{00}^r(\omega) - g_{00}^a(\omega)] \quad (18.16)$$

$$g_{00}^{r/a}(\omega) = \frac{\sqrt{\Delta^2 - (\omega \pm i0^+)^2} \sigma_0 + \Delta \sigma_1}{\omega \pm i0^+} , \quad (18.17)$$

where $n_F(\omega)$ is the Fermi function and 0^+ an infinitesimal introduced to properly define the square root in the complex plane.

The Keldysh, retarded and advanced bare Green's functions for the metallic lead are readily obtained by setting $\Delta = 0$ in the results above, so that

$$g_{11}^K(\omega) = -2iF_1 \quad (18.18)$$

$$g_{11}^{r/a}(\omega) = \mp i\sigma_0 . \quad (18.19)$$

There we have introduced the matrix F_1 as

$$F_1 = \begin{pmatrix} 1 - 2n_F(\omega - eV) & 0 \\ 0 & 1 - 2n_F(\omega + eV) \end{pmatrix} . \quad (18.20)$$

Note that the top and bottom diagonal elements respectively have an electron-like and a hole-like character, thus leading to opposite signs for the voltage.

Introducing the Heaviside distribution $\Theta(|\omega| - \Delta)$, the boundary Keldysh Green's function on the TS side becomes

$$g_{00}^K(\omega) = 2i \tanh\left(\frac{\beta\omega}{2}\right) \times \left[\pi\Delta(\sigma_0 + \sigma_1)\delta(\omega) + \frac{\sqrt{\omega^2 - \Delta^2}}{|\omega|} \Theta(|\omega| - \Delta)\sigma_0 \right] . \quad (18.21)$$

where β is the inverse temperature.

18.3.2 Average current

In order to write the average current, we then need the expression for the dressed Green's functions $G_{00}^{r/a}$, G_{11}^a and G_{00}^r . Using Dyson's equation for the full advanced and retarded Green's function, Eq. (18.13), it is convenient to express $G_{01}^{r/a}$, $G_{10}^{r/a}$ and $G_{11}^{r/a}$ in terms of $G_{00}^{r/a}$ as

$$G_{01}^{r/a}(\omega) = \mp i\lambda G_{00}^{r/a}(\omega)\sigma_3 , \quad (18.22)$$

$$G_{10}^{r/a}(\omega) = \mp i\lambda\sigma_3 G_{00}^{r/a}(\omega) , \quad (18.23)$$

$$G_{11}^{r/a}(\omega) = \mp i\sigma_0 - \lambda^2\sigma_3 G_{00}^{r/a}(\omega)\sigma_3 . \quad (18.24)$$

It follows that the Keldysh component of the dressed Green's function which enters the current average can be written as

$$G_{01}^K = i\lambda (\sigma_0 - i\lambda^2 G_{00}^r) g_{00}^K (\sigma_0 + i\lambda^2 G_{00}^a) \sigma_3 - 2i\lambda G_{00}^r F_1 (\sigma_0 + i\lambda^2 G_{00}^a) \sigma_3 . \quad (18.25)$$

Where $G_{00}^{r/a}$ is also obtained from the Dyson's equation, and reads

$$G_{00}^{r/a} = \left[-\sigma_3 g_{00}^{r/a} \sigma_3 \pm i\lambda^2 \sigma_0 \right]^{-1} . \quad (18.26)$$

One readily sees that $G_{00}^{r/a}$ is a linear combination of σ_0 and σ_1 Pauli matrices

$$G_{00}^{r/a}(\omega) = \alpha^{r/a}(\omega)\sigma_0 + \gamma^{r/a}(\omega)\sigma_1 , \quad (18.27)$$

where, denoting $\omega_{\pm} = (\omega \pm i0^+)$,

$$\alpha^{r/a}(\omega) = \frac{\left(\sqrt{\Delta^2 - \omega_{\pm}^2} \mp i\lambda^2 \omega_{\pm}\right) \omega_{\pm}}{\Delta^2 - \left(\sqrt{\Delta^2 - \omega_{\pm}^2} \mp i\lambda^2 \omega_{\pm}\right)^2}, \quad (18.28)$$

$$\gamma^{r/a}(\omega) = \frac{\Delta \omega_{\pm}}{\Delta^2 - \left(\sqrt{\Delta^2 - \omega_{\pm}^2} \mp i\lambda^2 \omega_{\pm}\right)^2}. \quad (18.29)$$

Note that $\alpha^r = (\alpha^a)^*$ and $\gamma^r = (\gamma^a)^*$.

Keeping in mind that g_{00}^K is also a linear combination of σ_0 and σ_1 , one can significantly simplify the Nambu trace needed for the current

$$\text{Tr}_N G_{01}^K(\omega) = -2i\lambda \text{Tr}_N \left[G_{00}^r F_1 (\sigma_0 + i\lambda^2 G_{00}^a) \sigma_3 \right]. \quad (18.30)$$

Inserting all these expressions in the trace formula for the current Eq. (18.10), the result is, introducing $n_F(\omega \pm eV) \equiv n_F^{(\pm)}$ and $\delta n_F = n_F^{(+)} - n_F^{(-)}$:

$$I(V) = -2ie\lambda^2 \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \left[\alpha^r + i\lambda^2 (|\alpha^r|^2 - |\gamma^r|^2) \right] \delta n_F. \quad (18.31)$$

This is identified as a Landauer formula

$$I = \frac{e}{2\pi} \int_{-\infty}^{+\infty} d\omega R_A(\omega) \left(n_F^{(-)} - n_F^{(+)} \right). \quad (18.32)$$

with the (real valued) Andreev reflection probability

$$R_A(\omega) = -2\lambda^2 \left[\text{Im}(\alpha^r) + \lambda^2 (|\alpha^r|^2 - |\gamma^r|^2) \right], \quad (18.33)$$

whose expression matches exactly with the one of the spectral current density $J(\omega)$ obtained in Eq. (35) of Ref. [19].

18.4 Finite frequency noise

18.4.1 Real-time noise correlator

The real time noise correlator in a multi-lead geometry is defined as:

$$S_{jl}(t, t') = \langle I_j(t) I_l(t') \rangle - \langle I_j(t) \rangle \langle I_l(t') \rangle, \quad (18.34)$$

where $I_j(t)$ is defined by Eq. (18.5). As the Hamiltonian is quadratic in the fermionic degrees of freedom, we can use Wick's theorem to express the four fermion operator averages in terms of products of off-diagonal Keldysh Green's functions, again in the form of a Nambu trace:

$$\begin{aligned} S_{jl}(t, t') &= e^2 \text{Tr}_N \left[\sigma_3 \left(\hat{W} \hat{G}^{-+}(t, t') \hat{W} \right)_{jl} \sigma_3 G_{lj}^{+-}(t', t) \right] \\ &\quad - e^2 \text{Tr}_N \left[\sigma_3 \left(\hat{W} \hat{G}^{-+}(t, t') \right)_{jl} \sigma_3 \left(\hat{W} \hat{G}^{+-}(t', t) \right)_{lj} \right]. \end{aligned} \quad (18.35)$$

where we recall that $\hat{\cdot}$ corresponds to matrices in Nambu-lead space.

In our two-terminal geometry, this simplifies as

$$\begin{aligned} S_{11}(t, t') &= \lambda^2 e^2 \text{Tr}_N \left[G_{00}^{-+}(t, t') G_{11}^{+-}(t', t) \right. \\ &\quad \left. - G_{01}^{-+}(t, t') G_{01}^{+-}(t', t) \right]. \end{aligned} \quad (18.36)$$

These Green's functions are related to the retarded, advanced and Keldysh components according to

$$\hat{G}^{+-} = \frac{1}{2} \left(\hat{G}^a - \hat{G}^r + \hat{G}^K \right) , \quad (18.37)$$

$$\hat{G}^{-+} = \frac{1}{2} \left(\hat{G}^r - \hat{G}^a + \hat{G}^K \right) . \quad (18.38)$$

Writing explicitly the Keldysh component of the Dyson's equation, Eq.(18.14), for all combination of leads, one has

$$G_{00}^K = (\sigma_0 - i\lambda^2 G_{00}^r) g_{00}^K (\sigma_0 + i\lambda^2 G_{00}^a) - 2i\lambda^2 G_{00}^r F_1 G_{00}^a \quad (18.39)$$

$$G_{11}^K = (\sigma_3 - i\lambda^2 \sigma_3 G_{00}^r) (\lambda^2 g_{00}^K - 2iF_1) (\sigma_3 + i\lambda^2 G_{00}^a \sigma_3) \quad (18.40)$$

The four Green's functions G_{00}^{-+} , G_{01}^{-+} , G_{01}^{+-} , G_{11}^{+-} which enter the real-time noise correlator are ultimately obtained from the above Keldysh components

$$G_{00}^{-+} = \frac{1}{2} \left[G_{00}^r - G_{00}^a - 2i\lambda^2 G_{00}^r F_1 G_{00}^a + H \right] , \quad (18.41)$$

$$\begin{aligned} G_{11}^{+-} = & i(\sigma_0 - F_1) + \frac{\lambda^2}{2} \sigma_3 (G_{00}^r - G_{00}^a) \sigma_3 \\ & - \lambda^2 \sigma_3 G_{00}^r \sigma_3 F_1 + \lambda^2 F_1 \sigma_3 G_{00}^a \sigma_3 \\ & - i\lambda^4 \sigma_3 G_{00}^r F_1 G_{00}^a \sigma_3 + \frac{1}{2} \lambda^2 \sigma_3 H \sigma_3 , \end{aligned} \quad (18.42)$$

$$\begin{aligned} G_{01}^{-+} = & -i\frac{\lambda}{2} (G_{00}^r + G_{00}^a) \sigma_3 - i\lambda G_{00}^r \sigma_3 F_1 \\ & + \lambda^3 G_{00}^r F_1 G_{00}^a \sigma_3 + i\frac{\lambda}{2} H \sigma_3 , \end{aligned} \quad (18.43)$$

$$\begin{aligned} G_{01}^{+-} = & i\frac{\lambda}{2} (G_{00}^r + G_{00}^a) \sigma_3 - i\lambda G_{00}^r \sigma_3 F_1 \\ & + \lambda^3 G_{00}^r F_1 G_{00}^a \sigma_3 + i\frac{\lambda}{2} H \sigma_3 , \end{aligned} \quad (18.44)$$

where we introduced the new matrix

$$H(\omega) = (\sigma_0 - i\lambda^2 G_{00}^r) g_{00}^K (\sigma_0 + i\lambda^2 G_{00}^a) . \quad (18.45)$$

Substituting these back into the expression for the real-time current correlator allows one to derive an exact, albeit cumbersome, analytic expression for S_{11} .

18.4.2 Emission, absorption and measurable noise

In full generality, one may consider that the Green's functions depend on two times t and t' . In the specific setup under consideration, we are in a stationary situation as there is only one superconductor in the system. It follows that the time dependence reduces to a single variable, more precisely the time difference $\tau = t - t'$. We thus consider the two distinct correlators:

$$\begin{aligned} S^+(\tau) = S_{11}(0, \tau) = & \lambda^2 e^2 \text{Tr}_N \left[G_{00}^{-+}(-\tau) G_{11}^{+-}(\tau) \right. \\ & \left. - G_{01}^{-+}(-\tau) G_{01}^{+-}(\tau) \right] , \end{aligned} \quad (18.46)$$

$$\begin{aligned} S^-(\tau) = S_{11}(\tau, 0) = & \lambda^2 e^2 \text{Tr}_N \left[G_{00}^{-+}(\tau) G_{11}^{+-}(-\tau) \right. \\ & \left. - G_{01}^{-+}(\tau) G_{01}^{+-}(-\tau) \right] , \end{aligned} \quad (18.47)$$

Their Fourier transforms define the emission and absorption noise as

$$S^+(\Omega) = \int_{-\infty}^{+\infty} dt \langle \delta I(0) \delta I(t) \rangle e^{i\Omega t} , \quad (18.48)$$

$$S^-(\Omega) = \int_{-\infty}^{+\infty} dt \langle \delta I(t) \delta I(0) \rangle e^{i\Omega t} . \quad (18.49)$$

which are conveniently rewritten using the Fourier-transformed Green's functions as

$$S^+(\Omega) = \lambda^2 e^2 \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \text{Tr}_N [G_{00}^{-+}(\omega) G_{11}^{+-}(\omega + \Omega) - G_{01}^{-+}(\omega) G_{01}^{+-}(\omega + \Omega)] \quad (18.50)$$

$$S^-(\Omega) = \lambda^2 e^2 \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \text{Tr}_N [G_{00}^{-+}(\omega) G_{11}^{+-}(\omega - \Omega) - G_{01}^{-+}(\omega) G_{01}^{+-}(\omega - \Omega)] . \quad (18.51)$$

It is clear from these expressions that emission and absorption noises are trivially related when flipping the sign of the probing frequency, namely $S^+(\Omega) = S^-(-\Omega)$. It follows that in order to describe the whole range of physical parameters, we can safely focus on positive frequencies for both noise spectra.

In order to connect with potential experimental works looking to investigate the finite-frequency noise of the NTS junction, we also compute the noise expected to be measured from a generic quantum detector consisting of a resonant LC circuit, inductively coupled to the NTS junction (see Fig. 41). This measured noise is the result of repeated measurements of the charge at the capacitor plates. For the present setup, we use an expression formulated in Refs. [20, 32] to predict the result of such a measurement. This so-called measurable noise is given by

$$S_{\text{meas}}(\Omega) = K \{ S^+(\Omega) + N(\Omega) [S^+(\Omega) - S^-(\Omega)] \} , \quad (18.52)$$

where $N(\Omega) = [\exp(\hbar\Omega/k_B T_{LC}) - 1]^{-1}$ is the Bose-Einstein distribution associated with the oscillator modes, and K is the effective coupling constant of the quantum wire with the resonator whose expression reads

$$K = \left(\frac{\alpha}{2L} \right)^2 \frac{1}{2\eta} . \quad (18.53)$$

L and α respectively stand for the inductance of the resonant circuit and the inductive coupling, while η is the width of the resonance. This last parameter depends on the dissipative environment surrounding the LC circuit, [32] and is typically small but its inverse is kept finite. Note that in full generality, this measurable noise is a combination of emission and absorption noise, weighted by the Bose-Einstein distribution. The temperature of the noise detector T_{LC} can in general be different from that of the electrons flowing in the NTS junction.

18.5 Results

In this section, we start by briefly recalling the known results for the conductance and the zero frequency noise, as both these quantities serve as a useful benchmark for our approach. We then move on to the determination of the finite frequency emission, absorption and measurable noises at low temperature.

The NTS junction is characterized by its transparency τ ($0 \leq \tau \leq 1$), which is defined from the tunneling amplitude λ as $\tau = 4\lambda^2/(1 + \lambda^2)^2$. We use units such that $k_B = \hbar = 1$. We adopt the convention that the TS wire is grounded, the voltage drive being applied to the normal metallic lead.

18.5.1 Differential conductance

We hereby recover important results which were already obtained in a previous study.[19] At zero temperature, the differential conductance is directly given by the reflection probability $G = \frac{2e^2}{h} R_A(eV)$ [see Eq. (18.32)]. At subgap voltages, this takes a very simple form since in this case

$$R_A(eV) = \frac{1}{1 + \left(\frac{eV}{\Gamma}\right)^2}, \quad (18.54)$$

leading to the predicted Majorana zero-bias peak with quantized height $2e^2/h$ and width $\Gamma = \tau\Delta/2\sqrt{1-\tau}$ due to resonant Andreev reflexion. As one approaches perfect transmission $\tau \rightarrow 1$, the strong NTS hybridization turns this peak into a plateau at $G = 2e^2/h$ extending over the whole subgap regime, $|eV| < \Delta$ (although the Majorana state is ill-defined due to this very same hybridization). Moreover, setting $\Delta = 0$, one recovers the behavior of a spin polarized N-N junction with a Landauer conductance $G_{NN} = \frac{\tau e^2}{h}$. Adding a finite temperature to the system does not substantially change the results and only tends to smooth out the curves, as expected.

18.5.2 Zero-frequency noise

We now turn to the shot noise $S_{11} = S^\pm(\Omega = 0)$, and more specifically the Fano factor, defined as the ratio of the zero-frequency noise to the average current, $F = S_{11}/eI_1$.

In the subgap regime, as predicted in Ref. [19], this Fano factor vanishes at perfect transparency ($\tau = 1$), and increases when the transparency decreases following

$$F = 1 - \frac{1}{\arctan\left(\frac{eV}{\Gamma}\right)} \frac{\frac{eV}{\Gamma}}{1 + \left(\frac{eV}{\Gamma}\right)^2}. \quad (18.55)$$

Above the gap, even at $\tau = 1$, the Fano factor F is finite due to the concomitant presence of both Andreev reflexion and quasiparticle processes.

For a normal-normal junction, i.e. in the limit $\Delta = 0$, the zero-frequency noise at zero temperature shows, as expected,[34, 35] a linear behavior with respect to the bias voltage and reduces to $S_{11} = (e^2/h)\tau(1-\tau)eV$.

18.5.3 Finite-frequency noise

The finite-frequency noise has so far been absent in the investigations of the transport characteristics of topological superconductors. The results for the emission noise $S^+(\Omega)$ and the absorption noise $S^-(\Omega)$, obtained from a numerical evaluation of Eqs. (18.50) and (18.51), are shown in this section. Only their real parts are discussed here, as their imaginary parts yield vanishing contributions within numerical accuracy.

The noise at finite frequency Ω is created by fluctuations of current processes which are accompanied by the emission or absorption of a photon at the frequency Ω . Understanding the basic process which contributes to the current with emission or absorption of a photon, while conserving the total energy, can give us a qualitative understanding of the finite-frequency noise features.

As a warm-up, let us first recall what happens in the most simple situation of a junction between two normal metallic electrodes (corresponding to taking the limit $\Delta \rightarrow 0$). There, electronic transport occurs if occupied electron states from one lead have enough energy to reach the empty states of the other. This results in a decreasing linear behavior of the emission

noise $S^+(\Omega)$ for frequencies $\Omega \in [0, eV]$ and a vanishing emission noise for higher frequencies due to Pauli blocking.[35] In the meantime, the absorption noise $S^-(\Omega)$ increases linearly with Ω , since the bigger the frequency, the more electrons can be transferred between leads.

When we replace the normal lead by a topological superconducting wire, the noise spectrum changes drastically. These changes can be understood qualitatively by taking into account the density of states of the topological superconductor, which is gapped for energies $|E| < \Delta$, except for a single narrow peak at zero energy due to the Majorana bound state (see Fig. 41).

Qualitative picture

The qualitative picture showing the basic processes contributing to the finite-frequency noise are shown in Fig.42. The two processes of the top row (a and b) describe emission noise, while the four other processes describe absorption noise. Note that we can choose $eV > 0$ without loss of generality, thanks to the electron-hole symmetry.

When the bias voltage is smaller than the gap, $eV < \Delta$, there is only one process contributing to the emission noise, which is shown on panel (a). Electrons in the normal metal with energies in the interval $[0, eV]$ may hop to the Majorana state by emitting a photon of energy $\Omega = eV$ (see Fig. 42a). Since the width of the Majorana peak in the density of states is extremely small, and the DOS of the normal metal is constant, this emission process only shows a negligible dependence on the frequency.

For voltages larger than the gap, another emission process kicks in: electrons from the normal metal can be transferred to the empty states above the TS gap, and emit a photon with Ω in the range $[0, eV - \Delta]$ (see Fig. 42b). The frequency dependence of this process will reflect the energy dependence of the TS density of states above the gap.

Absorption processes similar to the emission processes discussed above can also be realized. These are shown on panels (c) and (d) of Fig.42, and are expected to show similar frequency dependence as their emission counterparts. In the case of panel (c), an electron from the normal metal absorbs a photon to be transmitted to the Majorana state. Panel (d) is the equivalent of panel (b) for photon absorption: an electron from the normal metal absorbs a photon and is transmitted to an empty state above the TS gap. Note however that while processes (a) and (c) exist for any voltages, the emission process of panel (b) can only occur for voltages beyond the gap.

Finally, the absorption of photons also allows processes where an electron is transmitted from the topological superconductor to the normal metal, for frequencies $\Omega \in [eV, +\infty]$. Panels (e) and (f) of Fig.42 show the corresponding processes, where the TS electron is transmitted to a normal metal empty state, either from the Majorana state [panel (e)] or from the occupied states below the gap [panel (f)].

Results for $eV < \Delta$

We now show the results for the finite frequency emission noise $S^+(\Omega)$ and absorption noise $S^-(\Omega)$ obtained at low temperature $T_e = 0.01\Delta$ for different values of the transparency τ and the voltage V (applied on the normal lead, the TS wire is assumed to be grounded).

The emission noise in the subgap regime $eV < \Delta$ is shown in Fig. 43 (top), at low transparency $\tau = 0.02$ for three different values of the voltage. For each curve, we observe a plateau that extends up to $\Omega = eV$, followed by a sharp drop to zero. This plateau corresponds to the process (a) of Fig.42, and exists only for frequencies $\Omega < eV$, hence the observed drop. The absorption noise for the same parameters is shown on Fig. 43 (bottom). The most visible feature is a sharp step at $\Omega = eV$, which corresponds to the onset of the process (e) from Fig. 42, whose contribution is independent of frequency.

Smaller structures can be observed at frequency $\Omega = \Delta - eV$ and $\Omega = \Delta + eV$, corresponding respectively to the onset of processes (d) and (f) from Fig. 42. Interestingly, a fourth process also contributes to the absorption noise, namely process (c). However, the latter cannot be associated with any visible feature as it leads to a constant contribution for all positive frequency Ω , thus resulting in a finite background contribution to the absorption noise.

Results for $eV > \Delta$

We now turn to the low-temperature results for the noise at voltages $eV > \Delta$ and in the same low transparency regime ($\tau = 0.02$).

The emission noise $S^+(\Omega)$ is shown in Fig. 44 (top), for three different voltages beyond the gap. As in the subgap regime, the main feature of the emission noise is the presence of a well-defined plateau extending from $\Omega = eV - \Delta$ all the way up to $\Omega = eV$, which ends with a sharp drop to zero. Again, this plateau is fully compatible with the process shown in Fig. 42a, as it is expected to lead to a constant contribution, and can only occur for frequencies $\Omega < eV$ (since there are no occupied electronic states at higher energy). While this contribution is still present at very low frequency, it is supplemented by another one arising from the process (b) (see Fig. 42) leading to a new structure in the noise. Indeed, for frequencies $\Omega < eV - \Delta$, electrons close to the metallic Fermi level can be transferred to the empty states above the TS gap while emitting a low-energy photon. The resulting frequency dependence of the emission noise is directly related to the energy dependence of the TS density of states above the gap.

The absorption noise for $eV > \Delta$ is shown on Fig. 44 (bottom). The main feature is again a sharp step for $\Omega = eV$, associated with the process presented in panel (e) of Fig. 42. A smaller structure is visible at $\Omega = \Delta + eV$, and can be directly tied to the onset of process (f). Processes (c) and (d) also contribute to the absorption noise, being present at all frequencies and leading respectively to a constant background contribution and to a steady increase of the noise.

It follows from the inspection of the finite-frequency noise, for voltages both above and below the TS gap, that the presence of a Majorana bound state leads to a plateau in the emission noise as a function of frequency, accompanied by a sharp drop to zero at $\Omega = eV$.

At this point, it is interesting to relax the constraint of low transparency and study the impact of a higher value of τ . This is shown in Fig. 45, where an intermediate value of the transparency, $\tau = 0.4$, has been used.

One can readily see that increasing the transparency tends to smooth out all the structures visible in the noise and discussed above. Further increasing the transparency and ultimately approaching $\tau = 1$ (not shown) makes all features in the noise spectrum disappear. The best opportunity to identify the noise plateau associated with the Majorana bound state is thus to focus on the case of tunnel junctions.

18.5.4 Measurable noise

We end this study with a short discussion of the prediction of the measurable noise. To this end, we choose the inductive coupling scheme introduced in Ref. [20] which involves an LC circuit in the vicinity of the NTS junction (see Fig. 41). This protocol relies on repeated measurements of the charge on the plates of the capacitor of the LC resonant circuit. We aim at investigating the results of the measurable noise $S_{\text{meas}}(\Omega)$ as a function of the frequency Ω , as defined in Eq. (18.52), which corresponds to the measurable response of this resonant circuit.

Following our investigations on the emission and absorption noise, we focus here on the low transparency regime, as this is the most likely candidate to observe meaningful signatures of the Majorana bound state. The new crucial parameter which dramatically influences the measurable noise is the temperature T_{LC} of the detector.

In the low temperature regime for the detector, $T_{LC} \ll \Omega$, the measurable noise, presented in Fig. 46 for voltages beyond the gap, closely resembles the emission noise obtained earlier for the same set of parameters, showing in particular the characteristic low-frequency behavior of the noise which we could associate with the presence of a Majorana bound state. This is easily understood from the very definition of the measurable noise, Eq. (18.52). Indeed, in this low temperature regime, the Bose-Einstein distribution becomes vanishingly small, so that the measurable noise reduces to

$$S_{\text{meas}}(\Omega \gg T_{LC}) \simeq K S^+(\Omega) . \quad (18.56)$$

As one increases the detector temperature (see Fig. 47 obtained for $T_{LC} = \Delta$), this connection with the emission noise is progressively altered, ultimately leading to the disappearance of any meaningful signatures. However, the typical onsets of the various processes stay unaffected leading to structures in the measurable noise at frequencies $\Omega = eV - \Delta$ and $\Omega = eV$. In particular, the measurable noise shows a sudden change of sign near $\Omega = eV$, a feature inherited from the sharp drop in the emission noise, which is also directly connected to the presence of the Majorana peak in the TS DOS. It follows that even in this high temperature regime of the detector, the measurable noise can still be used as a tool to uncover signatures of the Majorana bound state.

18.6 Comparison with a N-dot-S system

A remaining question is to inquire whether a non-topological superconductor/normal metal junction bearing zero-energy Andreev bound states (ABS), which also exhibits a zero bias peak in the differential conductance, is likely to produce the same finite-frequency noise characteristics as the NTS junction, which constitutes the focus of this study. We thus consider here a system where a standard BCS superconductor is connected to a normal lead via a single quantum dot. We compute the finite frequency-noise in a regime where the quantum dot has signatures close to a Majorana bound state when looking at the conductance only. This will allow us to point out the unique information obtained by considering the finite-frequency noise, which permits to make a clear distinction between a real Majorana bound state in a topological superconductor and the presence of an accidental bound state at zero energy close to the junction.

The Hamiltonian of the system is:

$$H = H_D + \sum_{j=S,N} H_j + H_T \quad (18.57)$$

where H_D is the Hamiltonian of the dot level at energy ϵ :

$$H_D = \epsilon \sum_{\sigma=\uparrow,\downarrow} d_{\sigma}^{\dagger} d_{\sigma}, \quad (18.58)$$

H_j ($j = S, N$) are the Hamiltonians of the BCS superconductor and of the normal lead:

$$H_j = \sum_k \Psi_{j,k}^{\dagger} (\xi_k \sigma_z + \Delta_j \sigma_x) \Psi_{j,k} \quad (18.59)$$

where $\Delta_S = \Delta$ is the superconducting gap, and $\Delta_N = 0$. $\Psi_{j,k}$ is a Nambu spinor:

$$\Psi_{j,k} = \begin{pmatrix} \psi_{j,k,\uparrow} & \psi_{j,-k,\downarrow}^{\dagger} \end{pmatrix}^T, \quad (18.60)$$

$\xi_k = k^2/(2m) - \mu$, and σ_x and σ_z are Pauli matrices in Nambu space. Finally H_T is the tunneling Hamiltonian between the leads and the dots.

$$H_T = \sum_{jk} \Psi_{jk}^\dagger \mathcal{T}_j(t) d + \text{H.c.} \quad (18.61)$$

where $d = (d_\uparrow \ d_\downarrow)^T$ is a Nambu spinor for the dot electrons. $\mathcal{T}_j$ is the tunneling amplitude between lead j and the dot, where the constant voltage V_j has been included via a Peierls substitution :

$$\mathcal{T}_j(t) = t_j \sigma_z \exp(i\sigma_z V_j t) \quad (18.62)$$

The tunneling rate between the dot and lead j is further defined as $\Gamma_j = \pi \nu_0 t_j^2$, where ν_0 is the density of states of the metal at Fermi energy.

Calculation of the average current and current correlations is carried out using the Keldysh Green function formalism, and has already been presented for a similar system in Ref. [36]. The formula for the mean current to lead j is:

$$\langle I_j \rangle = e \text{Tr}_N \left\{ \sigma_3 \int \frac{d\omega}{2\pi} \text{Re} [\mathcal{G}^r(\omega) \Sigma^K(\omega) + \mathcal{G}^K(\omega) \Sigma^a(\omega)] \right\} \quad (18.63)$$

where the trace is performed in Nambu space, $\mathcal{G}$ is the full dot Green function (including the coupling to the leads), and Σ_j is the self-energy for lead j . The superscripts r, a, K refer to the retarded, advanced and Keldysh components.

Similarly, the formula for the current autocorrelations at finite frequency Ω is:

$$\begin{aligned} S_j(\Omega) = & -\frac{e^2}{2} \text{Re} \int \frac{d\omega}{2\pi} \text{Tr}_N \left\{ \sigma_3 [\Sigma_j^K \mathcal{G}^a + \Sigma_j^r \mathcal{G}^K - \Sigma_j^a \mathcal{G}^a + \Sigma_j^r \mathcal{G}^r]_\omega \sigma_3 [\Sigma_j^K \mathcal{G}^a + \Sigma_j^r \mathcal{G}^K + \Sigma_j^a \mathcal{G}^a - \Sigma_j^r \mathcal{G}^r]_{\omega+\Omega} \right. \\ & \left. - \sigma_3 [\Sigma_j^r \mathcal{G}^r \Sigma_j^K + \Sigma_j^K \mathcal{G}^a \Sigma_j^a + \Sigma_j^r \mathcal{G}^K \Sigma_j^a - \Sigma_j^a \mathcal{G}^a \Sigma_j^a + \Sigma_j^r \mathcal{G}^r \Sigma_j^r]_\omega \sigma_3 [\mathcal{G}^K + \mathcal{G}^a - \mathcal{G}^r]_{\omega+\Omega} \right. \\ & \left. - \frac{e^2}{4} \text{Re} \int \frac{d\omega}{2\pi} \text{Tr}_N \left\{ \sigma_3 [\Sigma_j^a - \Sigma_j^r - \Sigma_j^K]_\omega \sigma_3 [\mathcal{G}^a - \mathcal{G}^r + \mathcal{G}^K]_{\omega+\Omega} + \sigma_3 [\mathcal{G}^r - \mathcal{G}^a + \mathcal{G}^K]_\omega \sigma_3 [\Sigma_j^r - \Sigma_j^a - \right. \right. \\ & \left. \left. \Sigma_j^K]_{\omega+\Omega} \right\} \right\} \quad (18.64) \end{aligned}$$

Note that there is an extra term compared to Eq. (51) of Ref. [36], as we are not computing the current cross-correlations between different leads, but instead the autocorrelations. For $\Omega > 0$ (resp. $\Omega < 0$), Eq. (18.64) gives the emission noise $S^+(\Omega)$ [resp. the absorption noise $S^-(\Omega)$] - to be compared with Eqs. (18.50)-(18.51) for the NTS case.

We now turn to the results obtained with Eqs. (18.63) and (18.64). As a first step, we need to specify the parameters of the N-dot-S system for which the conductance is similar to the one obtained for a NTS junction due to the presence of the Majorana bound state. This means a conductance peak at zero energy, as shown by Eq. (18.54). This can be obtained here if two conditions are satisfied: setting the energy ϵ of the dot to zero, and using symmetrical couplings ($\Gamma_N = \Gamma_S$). While the first condition is obvious, the second one is illustrated in Fig. 48, which shows the differential conductance $G = dI_N/dV$ as a function of the voltage V applied to the normal lead. There, we consider the case of a symmetric junction with $\Gamma_N = \Gamma_S = 0.02$ (full line), as well as the situation of an asymmetric junction with $\Gamma_N = 0.02$ and $\Gamma_S = 3\Gamma_N$ (dotted line). The other parameters are $\epsilon = 0$, and $1/(k_B T) = 300$, while the superconducting gap Δ is the unit of energy. One can see that while the symmetric junction leads to a conductance peak which is identical to the one of a NTS junction, the asymmetric one shows a splitting of the conductance peak.

Focusing then on a symmetric junction with a dot at zero energy ($\epsilon = 0$), we now plot the emission noise. Fig. 49 shows $S^+(\Omega)$ for three values of the voltage below the gap ($V =$

$0.3\Delta, 0.6\Delta$ and 0.9Δ), while Fig. 50 focuses on voltages above the gap ($V = 1.2\Delta, 1.5\Delta$ and 1.8Δ), with couplings $\Gamma_N = \Gamma_S = 0.02$ in both cases. These figures should be compared with their NTS equivalent, Fig. 43 and Fig. 44 respectively.

We see that, as in the case of the NTS junction, there is a clear plateau in the emission noise, which extends up to $\Omega = eV$. This plateau is the manifestation of the discrete level (here the dot level), which is associated with a narrow peak in the density of states. However, an important difference with respect to the NTS case is also visible in Figs. 49 and 50: at small frequency Ω , there is a dip in the emission noise, with a value at $\Omega = 0$ which is exactly half that of the plateau. When probed on a very long timescale (as in the zero frequency noise), the current fluctuations are reduced by a factor $1/2$ due to the double-barrier structure with symmetric couplings. When the timescale is reduced to a time smaller than the average transfer time of an electron between an electrode and the dot level ($\sim 1/\Gamma$), then the double-barrier nature of the setup has no impact on the current fluctuations, which explains why the dip in $S^+(\Omega)$ has a width $\sim \Gamma$. This factor $1/2$ for the contrast of the observed dip corresponds to the well-known Fano factor reduction in double-barrier symmetric junctions.[35] Interestingly, this dip in the emission noise is completely absent in the case of a NTS junction, because the Majorana bound state is an intrinsic part of the superconductor, and is not separated by a barrier. It follows that, unlike the differential conductance, the low-frequency behavior of the emission noise is able to discriminate between the NTS and the N-dot-S system. Granted, our comparison between the noise characteristics of the NTS junction and this system containing a zero energy Andreev bound state is specific to the N-dot-S model we have adopted here. More generally, zero energy Andreev bound states can be generated in the contact region due to disorder and/or to multiple subbands in the nanowire. Studying this more general situation in detail goes well beyond the scope of the present paper. At this point, we cannot absolutely rule out that such an embedded ABS in the superconductor would lead to a frequency noise characteristics similar to that of the NTS junction. We therefore cannot claim that the frequency noise characteristics obtained in this work can uniquely be attributed to the presence of the Majorana fermion in the NTS junction.

18.7 Conclusion and perspectives

The hunt for Majorana fermions continues to be an active field of research. Besides the zero voltage anomaly which was predicted, and in principle measured, by several experimental and theoretical groups, more signatures of these manifestations of collective electronic excitations are desperately needed.

In this work, we focused on a normal metal/topological superconductor junction subjected to a constant voltage bias using the non-equilibrium Keldysh formalism. Starting from the boundary Green's functions describing the two isolated parts, we were able to express the average current and noise in terms of dressed Green's functions, accounting for tunneling between the leads at all orders. This allows us to investigate the stationary current, the zero frequency noise and, the main result of this analysis, the finite-frequency emission and absorption noise.

Computed for various voltages, at small and intermediate transparency, the finite-frequency noise $S^+(\Omega)$ and $S^-(\Omega)$ for the NTS junction show various structures, which can be tied to different emission or absorption processes. In particular, the distinctive feature associated with the presence of the Majorana bound state is a plateau in the noise, extending all the way from $\Omega = 0$ up to a sharp step at frequency $\Omega = eV$ for voltages in the subgap regime. The presence of such a plateau is a general feature of a system with a discrete level, which is here the Majorana bound state. However, for a non-topological system with an accidental discrete level at zero-energy, we have shown by computing the emission noise in a N-dot-S system that the double barrier nature of the setup imposes a dip in the plateau at low frequency Ω , which is

absent for the NTS junction. In this particular N-dot-S system, the observation of the plateau for the whole frequency range could thus be used to discriminate between an accidental zero-energy Andreev bound state, and a real Majorana bound state. Nevertheless, this still leaves open the possibility that more general superconducting systems which bear ABS embedded in the superconductor would generate the same differential conductance and noise signals as the NTS junction. In any case, tunnel junctions should still be favored, as these characteristic structures are more prominent in the limit of small tunneling amplitudes, and get smoothed out when the transparency is increased.

Finally, we used a generic model for measuring noise consisting of a LC circuit inductively coupled to the normal metal/topological superconductor junction under study. In actual experiments, the temperature of this LC circuit is not necessarily the same as the electronic temperature of the junction. We see that when the temperature of the detection (LC) circuit is comparable to that of the electronic temperature of the NTS junction, we recover the same features as the ones observed in the emission and absorption noise. While higher temperatures of the detection circuit slightly blur the quality of this diagnosis, there still remains specific signatures which can be tied to tunneling processes involving the Majorana bound state. Our results therefore suggest that finite-frequency noise measurements offer a novel diagnostic of the presence of Majorana fermions in condensed matter systems. Extensions could include taking into account the finite length of the TS wire where the two Majorana bound states located at each end hybridize, or changing the chemical potential of the continuum version of the Kitaev chain model used here in order to trigger a transition to the non-topological phase. Using a more realistic model for the topological superconductor, including the electron spin, induced superconductivity, etc. is another notable extension in view of a quantitative comparison with experiments.[37]

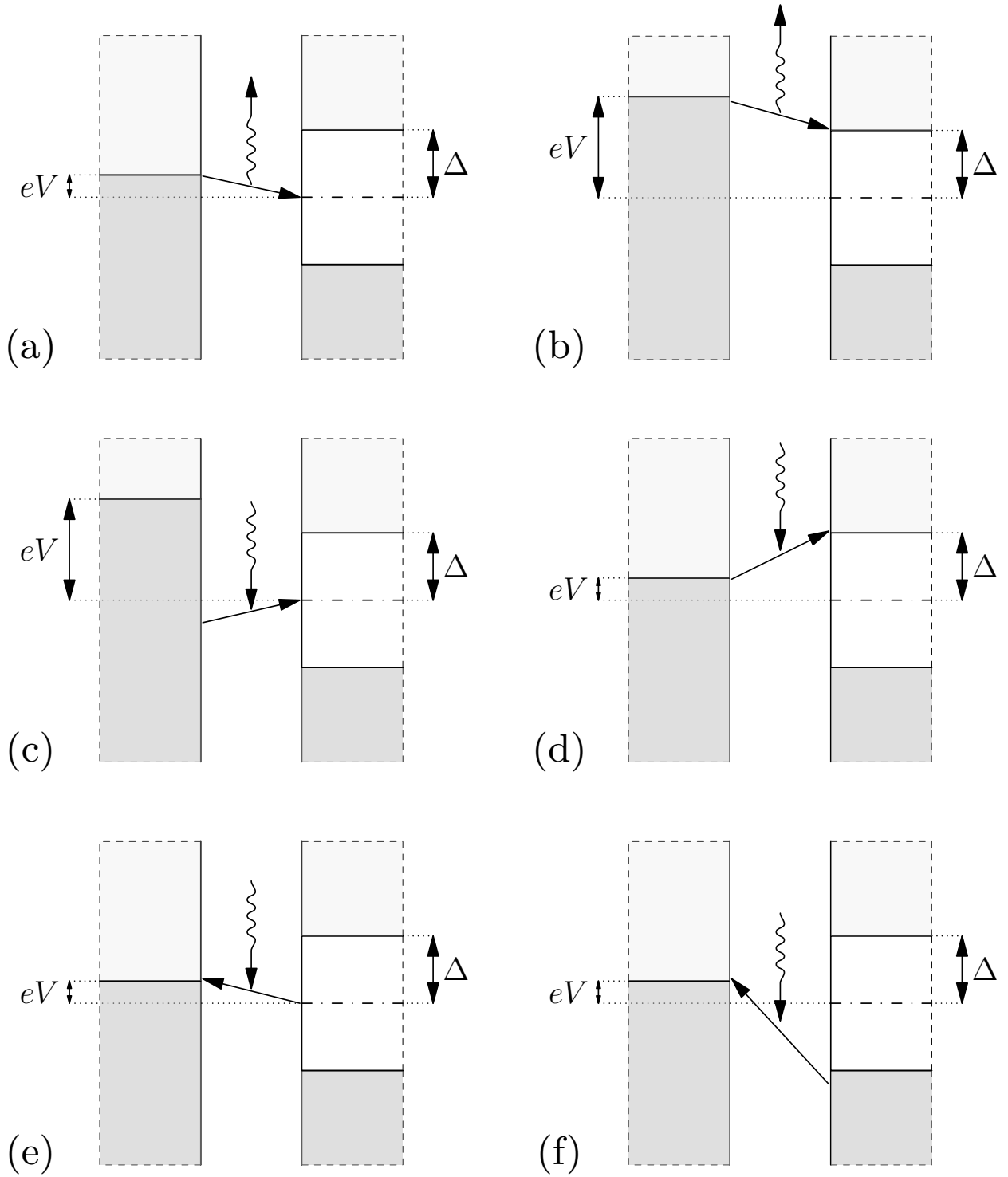


Figure 42: Energy diagram of a NTS junction. In medium grey are represented the occupied electronic states and in light grey the empty states. (a)-(b) emission processes, (b) is not available for a bias eV below the gap Δ , (c)-(d) absorption processes involving a transition from the metal to the superconductor, (e)-(f) absorption processes with transitions to the metal. Straight lines correspond to the transfer of electrons while wiggly lines are associated with the absorption or emission of photons.

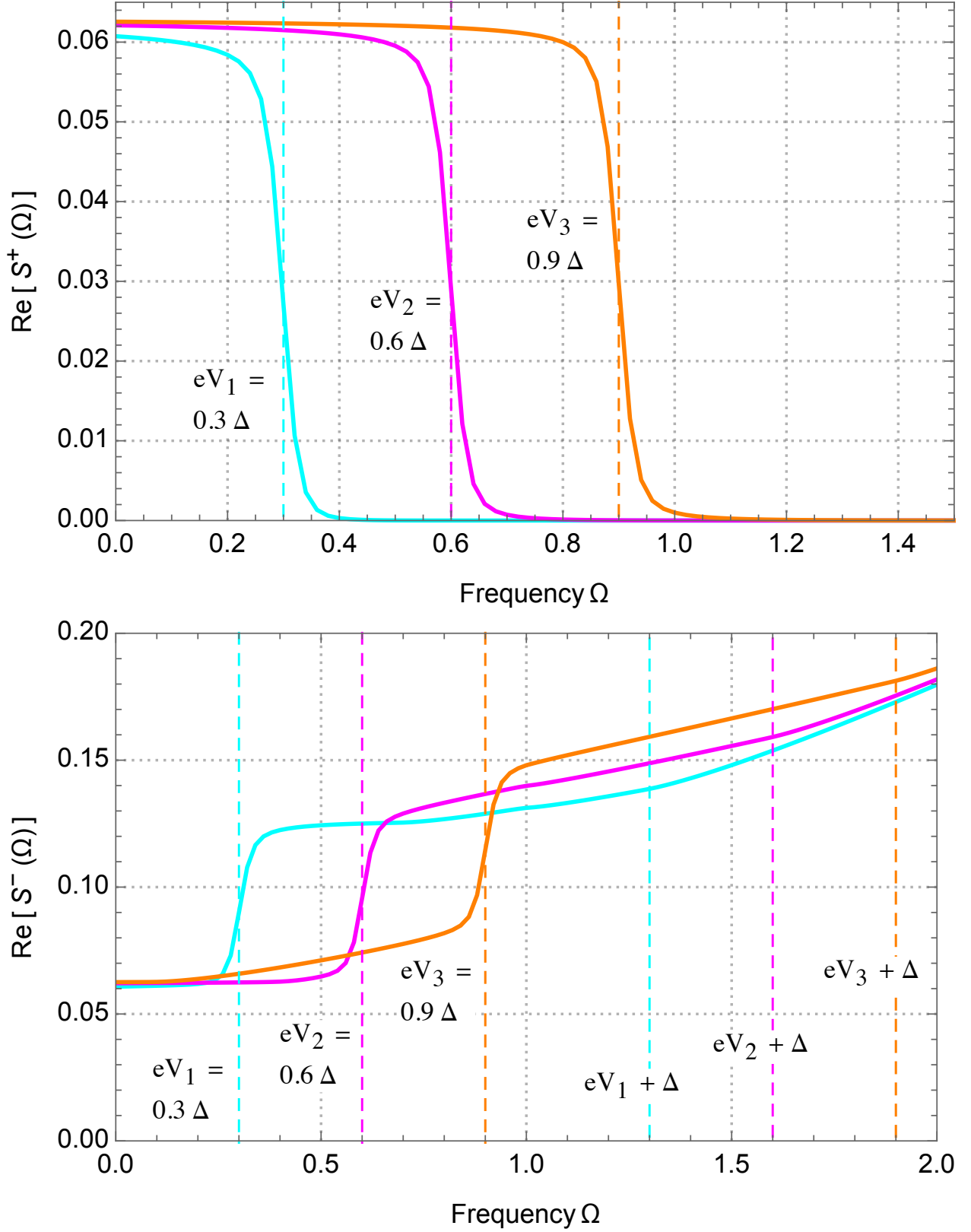


Figure 43: Emission noise $S^+(\Omega)$ (top) and absorption noise $S^-(\Omega)$ (bottom) as a function of frequency Ω (in units of Δ) at low transparency ($\tau = 0.02$) computed in the subgap regime $|eV| < \Delta$ for three different values of the voltage bias ($eV = 0.3\Delta$, 0.6Δ and 0.9Δ), and expressed in units of $e^2\Delta/h$.

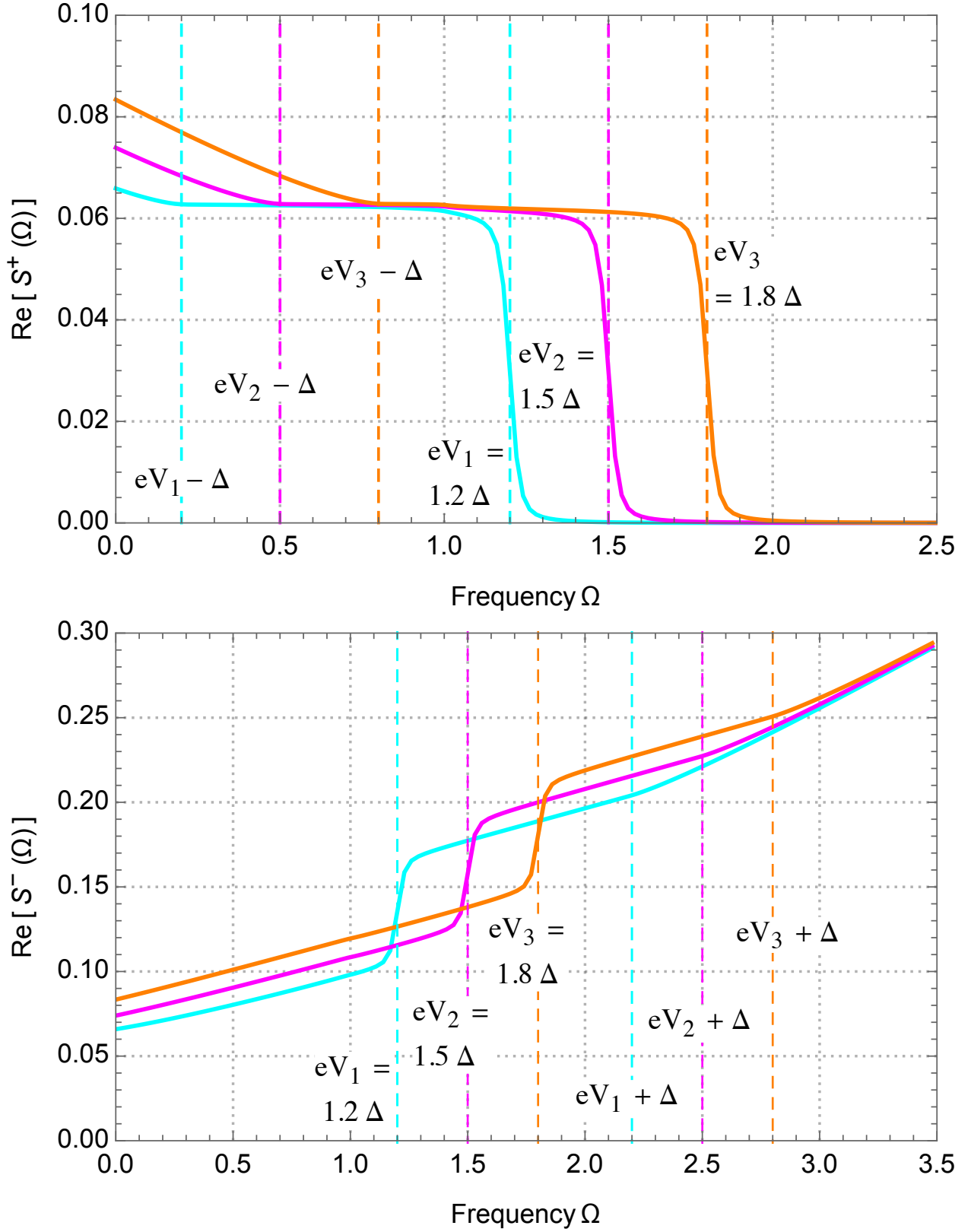


Figure 44: Emission noise $S^+(\Omega)$ (top) and absorption noise $S^-(\Omega)$ (bottom) as a function of frequency Ω (in units of Δ) at low transparency ($\tau = 0.02$) computed beyond the gap $|eV| > \Delta$ for three different values of the voltage bias ($eV = 1.2\Delta$, 1.5Δ and 1.8Δ), and expressed in units of $e^2\Delta/h$.

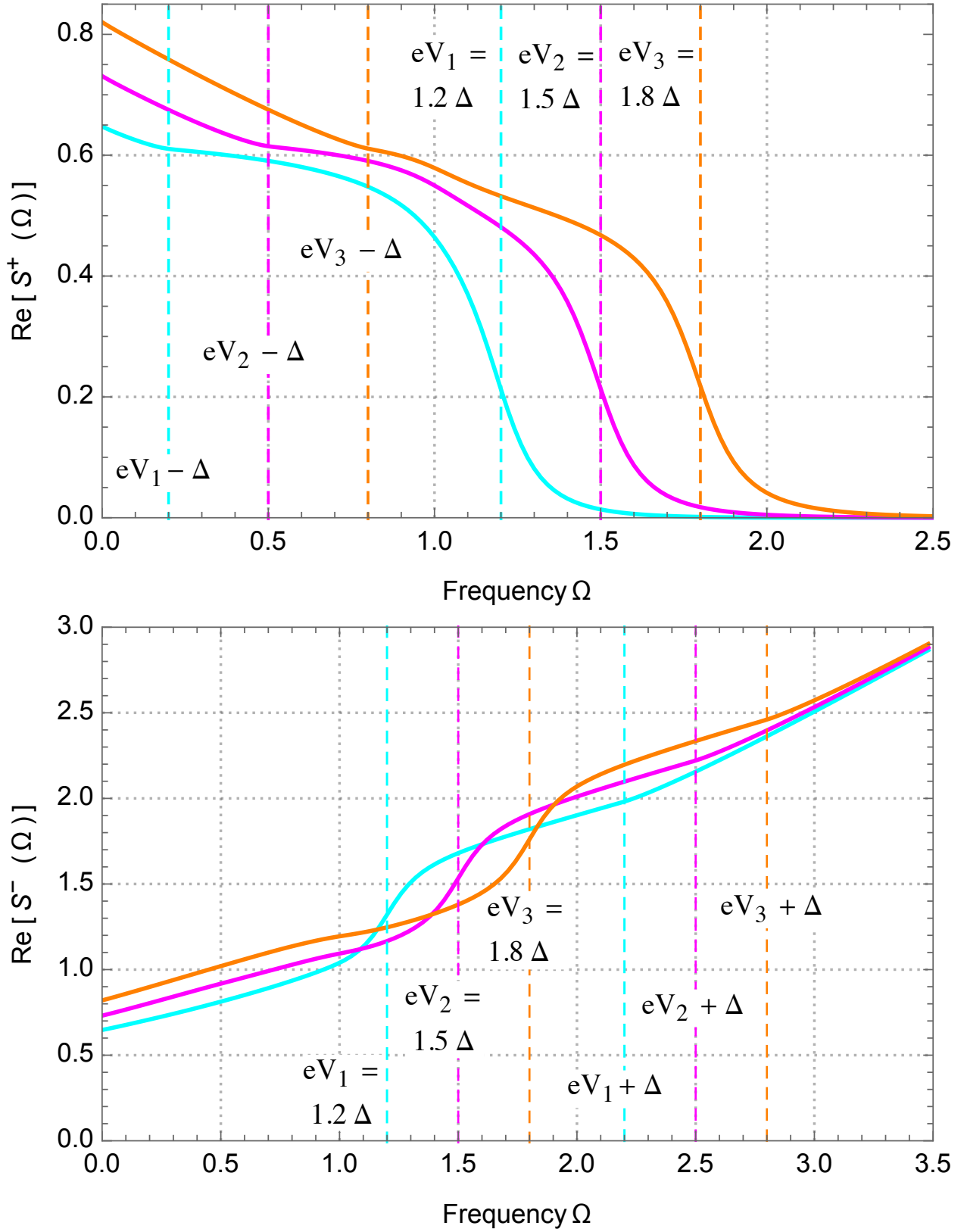


Figure 45: Emission noise $S^+(\Omega)$ and absorption noise $S^-(\Omega)$ (in units of $e^2\Delta/h$) as a function of frequency Ω (in units of Δ) at transparency $\tau = 0.4$ beyond the gap $|eV| > \Delta$, for three different values of the voltage bias ($eV = 1.2\Delta, 1.5\Delta$ and 1.8Δ).

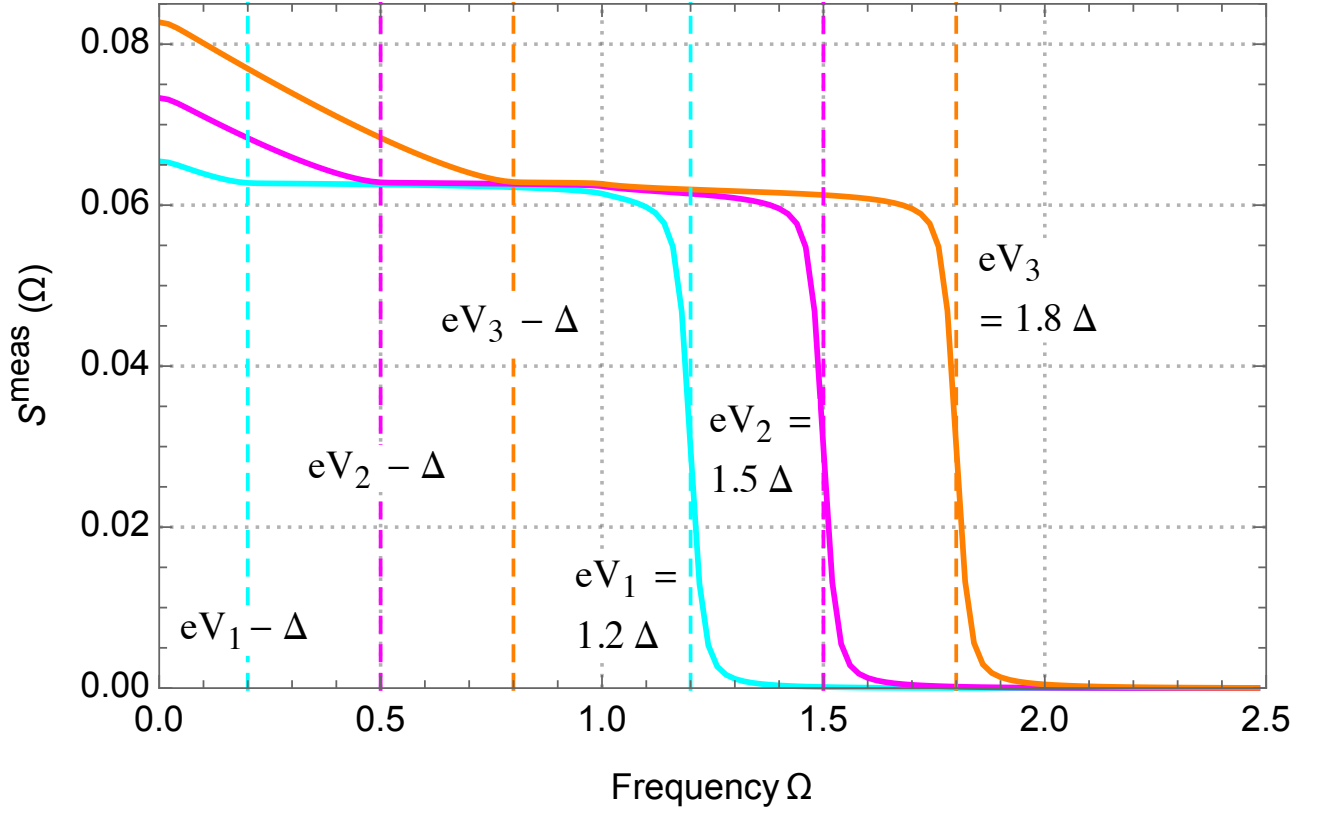


Figure 46: Measurable noise (in units of $e^2 K \Delta / h$) as a function of frequency Ω (in units of Δ) at low transparency ($\tau = 0.02$) for electron and detector temperatures $T_e = T_{LC} = 0.01\Delta$.

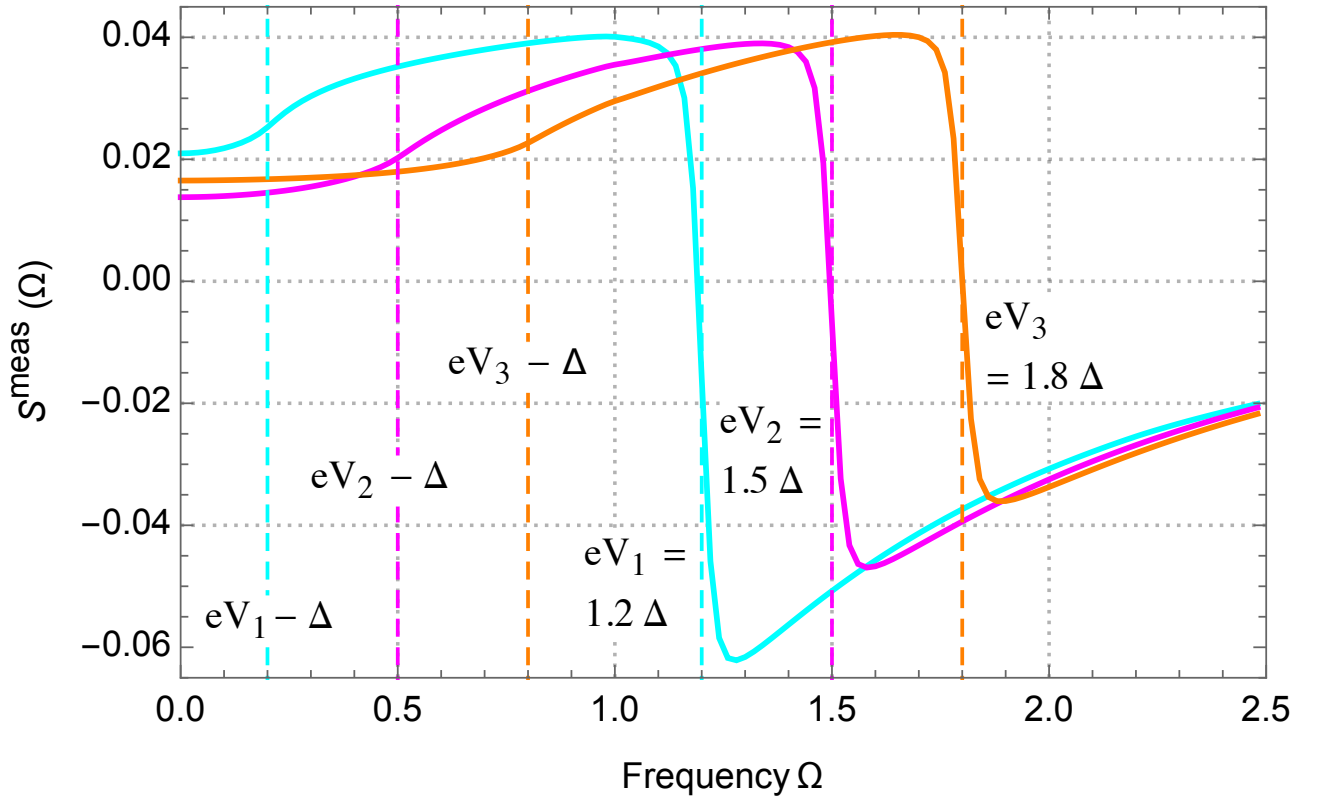


Figure 47: Measurable noise (in units of $e^2 K \Delta / h$) as a function of frequency Ω (in units of Δ) at low transparency ($\tau = 0.02$) for an electron temperature $T_e = 0.01\Delta$ and a detector temperature $T_{LC} = \Delta$.

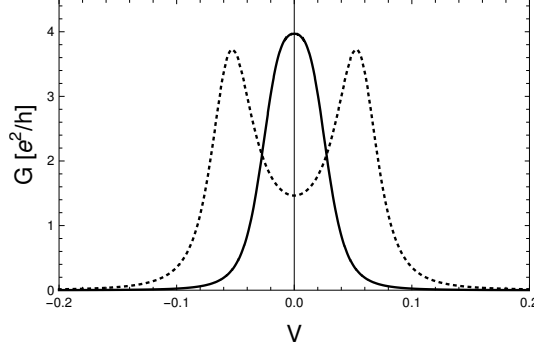


Figure 48: Differential conductance $G = dI_N/dV$, for the current to the normal lead I_N , in a N-dot-S junction, with symmetric couplings ($\Gamma_N = \Gamma_S = 0.02$, full line) and asymmetric couplings ($\Gamma_N = 0.02$, $\Gamma_S = 3\Gamma_N$, dotted line), as a function of the gap voltage of the normal lead. Other parameters : $\epsilon = 0$, $1/(k_B T) = 300$. The superconducting gap Δ is the unit of energy.

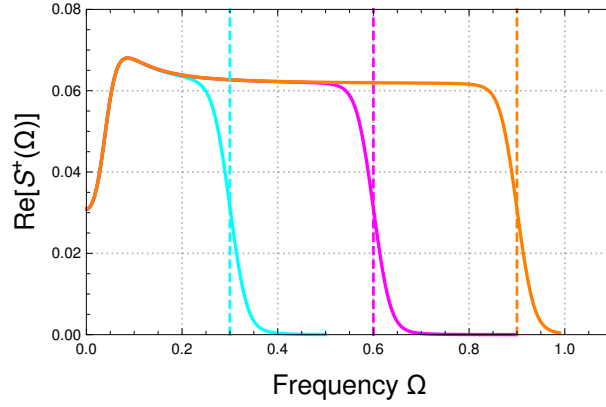


Figure 49: Emission noise $S^+(\Omega)$ for a symmetric N-dot-S junction with $\Gamma_N = 0.02$, at voltages $V = 0.3\Delta, 0.6\Delta$ and 0.9Δ (indicated by dashed vertical lines). Compare with Fig. 43 for the case of a NTS junction.

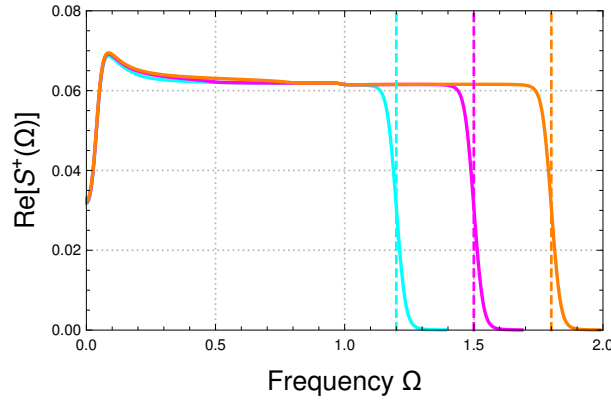


Figure 50: Emission noise $S^+(\Omega)$ for a symmetric N-dot-S junction with $\Gamma_N = 0.02$, at voltages $V = 1.2\Delta, 1.5\Delta$ and 1.8Δ (indicated by dashed vertical lines). Compare with Fig. 44 for the case of a NTS junction.

References

- [1] E. Majorana, *Il Nuovo Cimento*, **14**, 171 (1937).
- [2] A. Y. Kitaev, *Phys. Usp* **44**, 131 (2001).
- [3] R. M. Lutchyn, J. D. Sau, S. Das Sarma, *Phys. Rev. Lett.* **105**, 077001 (2010).
- [4] Y. Oreg, G. Refael, F. von Oppen, *Phys. Rev. Lett.* **105**, 177002 (2010).
- [5] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M. P. A. Fisher, *Nat. Phys.* **7**, 412 (2011).
- [6] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, *Science* **336**, 1003 (2012).
- [7] A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, and H. Shtrikman, *Nat. Phys.* **8**, 887 (2012).
- [8] R. M. Lutchyn, E. P. A. M. Bakkers, L. P. Kouwenhoven, P. Krogstrup, C. M. Marcus and Y. Oreg, *Nature Reviews Materials* **3**, 52 (2018)
- [9] S. Nadj-Perge, I. K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A. H. MacDonald, B. A. Bernevig, and A. Yazdani, *Science* **346**, 602 (2014).
- [10] J. Alicea, *Rep. Prog. Phys.* **75**, 076501 (2012).
- [11] M. Leijnse and K. Flensberg, *Semicond. Sci. Technol.* **27** 124003 (2012).
- [12] C. Beenakker, *Annu. Rev. Condens. Matter Phys.* **4**, 113 (2013).
- [13] T. D. Stanescu and S. Tewari, *Journal of Physics: Condensed Matter* **25** (23), 233201 (2013).
- [14] R. Aguado, *La Rivista del Nuovo Cimento* **40**, 523 (2017).
- [15] S. R. Elliott and M. Franz, *Rev. Mod. Phys.* **87**, 137 (2015).
- [16] S. Kashiwaya, Y. Tanaka, *Rep. Prog. Phys.* **63**, 1641 (2000).
- [17] P. Burset, B. Lu, S. Tamura, Y. Tanaka, *Phys. Rev. B* **95**, 224502 (2017) .
- [18] H. Zhang et al., *Nature* **556**, 74 (2018).
- [19] A. Zazunov, R. Egger and A. Levy Yeyati, *Phys. Rev. B* **94**, 014502 (2016).
- [20] G. B. Lesovik, R. Loosen, *JETP Lett.* **65**, 295 (1997).
- [21] R. Aguado and L. P. Kouwenhoven, *Phys. Rev. Lett* **84**, 1986 (2000).
- [22] U. Gavish, Y. Levinson, and Y. Imry, *Phys. Rev. B.* **62**, R10637 (2000).
- [23] R. Deblock, E. Ornac, L. Gurevich, and L. P. Kouwenhoven, *Science* **301**, 203 (2003).
- [24] P.-M. Billangeon, F. Pierre, H. Bouchiat, and R. Deblock, *Phys. Rev. Lett.* **96**, 136804 (2006).
- [25] P.-M. Billangeon, F. Pierre, H. Bouchiat, and R. Deblock, *Phys. Rev. Lett.* **98**, 216802 (2007).

- [26] J. Basset, H. Bouchiat, and R. Deblock, *Phys. Rev. Lett.* **105**, 166801 (2010).
- [27] E. Zakka-Bajjani, J. Segala, F. Portier, P. Roche, D. C. Glatthli, A. Cavanna, Y. Jin, *Phys. Rev. Lett.* **99**, 236803 (2007).
- [28] M. Hofheinz, F. Portier, Q. Baudouin, P. Joyez, D. Vion, P. Bertet, P. Roche, and D. Esteve, *Phys. Rev. Lett.* **106**, 217005, 2011.
- [29] C. Altimiras, O. Parlavecchio, P. Joyez, D. Vion, P. Roche, D. Esteve, F. Portier, *Phys. Rev. Lett.* **112**, 236803, 2014.
- [30] G. B. Lesovik, T. Martin, J. Torrès, *Phys. Rev. B* **60**, 11935 (1999).
- [31] J. Torrès, T. Martin, G. B. Lesovik, *Phys. Rev. B* **63**, 134517 (2001).
- [32] A. Zazunov, M. Creux, E. Paladino, A. Crépieux, and T. Martin, *Phys. Rev. Lett.* **99**, 066601 (2007).
- [33] T. Jonckheere, J. Rech, A. Zazunov, R. Egger, and T. Martin, *Phys. Rev. B* **95**, 054514 (2017).
- [34] Ya. M. Blanter, M. Büttiker, *Physics Reports* **336**, 1 (2000)
- [35] T. Martin, "Noise in Mesoscopic Physics, Nanophysics : coherence and transport : école d'été de physique des Houches, session LXXXI, 28 June-30 July 2004 / edited by H. Bouchiat, Y. Gefen, G. Montambaux and J. Dalibard ; Euro Summer School, NATO Advanced Study Institute, Ecole Thématique du CNRS / Amsterdam : Elsevier , 2005.
- [36] D. Chevallier, J. Rech, T. Jonckheere, and T. Martin, *Phys. Rev. B* **83**, 125421 (2011)
- [37] A. Zazunov, R. Egger, M. Alvarado, and A. L. Yeyati, *Phys. Rev. B* **96**, 024516 (2017)

19 Analytical calculation of the excess current in the OTBK theory by Niebler, Cuniberti, Novotný

Abstract

We present an analytical derivation of the excess current in Josephson junctions within the Octavio-Tinkham-Blonder-Klapwijk theory for both symmetric and asymmetric barrier strengths. We confirm the result found numerically by Flensberg *et al.* for equal barriers [*Physical Review B* **38**, 8707 (1988)], including the prediction of negative excess current for low transparencies, and we generalize it for differing barriers. Our analytical formulae provide for convenient fitting of experimental data, also in the less studied, but practically relevant case of the barrier asymmetry.

19.1 Conclusions and outlook

In this work we have analytically calculated the excess current within the OTBK theory describing fully incoherent SNS junctions. We have confirmed previous numerical findings [9] of negative excess current for large enough normal-state resistance in junctions with symmetric barriers. Furthermore, we extended those calculations also to the case of asymmetric barriers with qualitatively similar results, i.e. occurrence of negative excess current regardless of the

asymmetry. Our formula (19.41) can be used also in the most general case of different superconducting leads, for an experiment see, e.g. Ref. [15], where $\Delta_1 \neq \Delta_2$. The presence of two gap values prohibits further analytical treatment, however, Eq. (19.41) still holds and the integral can be easily evaluated numerically.

The numerical findings of Ref. [9] were challenged in Ref. [47] (p. 7372, paragraph below Eq. (30)) and the negativity of the excess current was interpreted as possibly stemming from a lack of convergence of the numerical study, i.e. from not reaching the true $eV \rightarrow \infty$ limit. Our study clearly demonstrates that this objection cannot hold since we explicitly work in the required limit, thus avoiding any finite- V issues. We, however, do not question the presence of non-trivial issues in the experimental determination of the excess current related to the finite voltage and possible heating effects nicely reviewed and discussed in Ref. [12]. Apparently, observations of negative excess current (so called deficit current) have been reported in experiments [13, 14].

Nevertheless, we analytically prove the correctness of the old numerical results [9] predicting negative excess current within the OTBK theory. The discrepancy with the results of Ref. [47] then must stem from the difference of the two considered models, more specifically, the role of internal coherence of the junction. While the OTBK theory only considers matching of the distribution functions between the two interfaces corresponding to fully incoherent junctions, the Hamiltonian theory of Ref. [47] matches the wavefunctions throughout the whole junction thus fully retaining the coherence within the junction. The high-voltage properties of the two models differ even qualitatively, one predicting negative excess current for small transparencies, the other one not. Another qualitative difference between OTBK and the fully coherent theory is in their dependence on the junction asymmetry: While the fully coherent results in the limit of strong coherent coupling to the leads ($\Gamma_{1,2} \gg \Delta$, relevant for many experiments, e.g. [21, 22, 23]) only depend on the asymmetry through the total junction resistance [47], it is not so in the OTBK case as we can immediately see from our result for the excess current (Eq. (19.42)) which is *not* a function of $Z_1^2 + Z_2^2$ only.

This finding shows that the coherence within the junction plays a crucial role for the superconducting transport even at finite voltage bias and therefore the level of decoherence/dephasing within a junction should be carefully considered when describing a particular experiment. This effect, i.e. nonzero dephasing within the junction, may be responsible for the experimentally observed discrepancies between the experiments [21, 22, 23] and theoretical predictions [47] systematically reported recently in Josephson junctions made of carbon nanotubes. While those discrepancies are currently interpreted as the superconducting gap renormalization this picture does not seem to be fully consistent with the positions of the subharmonic gap structure features, which appear at the positions determined by the un-renormalized gap value. The dephasing picture could capture the relevant physical mechanism instead although this remains an open issue in the currently booming field of superconducting transport in carbon-allotropes-based Josephson junctions.

Apart from the obvious usage of our newly derived analytical formulae for the OTBK excess current to fit experiments for relatively large and thus fully incoherent junctions, they can also be used as a limit benchmark of future partially-coherent theories, possibly relevant for current nanoscale experiments. These experiments as well as future devices built from novel low dimensional materials with peculiar electronic structures, such as graphene nanoribbons, could realistically be described by existing dephasing approaches for atomistic models [24, 25] coupled to non-equilibrium transport. The development of such a partially-coherent theory and the analytical evaluation of its excess current interpolating between the two limits is our next step.

19.2 Introduction

The transport in Josephson junctions with various different materials constituting the normal region has been a very active research field for decades now and continues to be one. Among the many interesting transport properties of Josephson junctions are the so-called sub-harmonic gap structure (SGS) and the excess current, both of which were accurately explained by the concept of multiple Andreev reflections (MAR). The MAR theory was first formulated for normal–superconducting (NS) interfaces by Blonder, Tinkham, and Klapwijk (BTK) [3, 2] and the BTK theory and its extensions (especially to the ferromagnetic or non-BCS superconducting contacts) are still actively used in fitting experiments [3, 4, 5] and in theoretical studies [6, 7].

The BTK theory was then extended to full SNS junctions by Octavio, Tinkham, Blonder and Klapwijk (OTBK) [4] and Flensberg, Bindslev Hansen and Octavio [9]. The OTBK approach does not keep track of the evolution of the quasiparticle phase between the interfaces and therefore assumes complete dephasing in the junction area. This assumption breaks down for sufficiently small systems such as, e.g., atomic wires and the fully coherent approach developed in mid 90's [33, 47, 12] is applicable instead. Nevertheless, the OTBK theory describes certain systems, such as microbridges, very well and keeps on being used in the literature both in experimental [13, 14, 15, 16, 17, 18] as well as theoretical [19] studies. In particular, its extension to the experimentally relevant situation of asymmetric junctions was developed and applied in Refs. [13, 14, 15].

In this work we analytically study the excess current in the OTBK theory. Although the excess current has been derived analytically in more recent coherent theories [47] it has not been reported yet in an analytic form in the older incoherent OTBK approach. We fill in this gap and provide the analytical derivation of the excess current for incoherent generally asymmetric SNS junctions described by the OTBK theory. Our formula can be used for the experimental fitting but it also has implications for the understanding of the role of coherence within the junction as discussed in more detail in the concluding section.

19.3 OTBK model

The BTK theory describes the transport through a single normal–superconducting interface, which is assumed to consist of a ballistic superconductor in contact with an equally ballistic piece of normal metal. Scattering can thus only occur at the interface at $x = 0$, which is modelled by a repulsive delta-function potential $H\delta(x)$, $H \geq 0$, with a dimensionless parameter $Z = H/\hbar v_F$ (v_F being the Fermi velocity) that represents the barrier strength [2]. Transport properties are found by matching the wave functions on either side of this barrier. The different allowed processes are identified and labelled as follows: Andreev reflection A , normal reflection B , and transmission T . The corresponding probabilities $A(E)$, $B(E)$ and $T(E)$ are expressed as functions of the quasiparticle energy E , the superconducting gap Δ , and the interface's barrier

$A(E)$	$B(E)$	$T(E)$	
$\frac{\Delta^2}{E^2 + (1+2Z^2)^2(\Delta^2 - E^2)}$	$\frac{4Z^2(1+Z^2)(\Delta^2 - E^2)}{E^2 + (1+2Z^2)^2(\Delta^2 - E^2)}$	0	for $ E < \Delta$
$\frac{\Delta^2}{(E + (1+2Z^2)\sqrt{E^2 - \Delta^2})^2}$	$\frac{4Z^2(1+Z^2)(E^2 - \Delta^2)}{(E + (1+2Z^2)\sqrt{E^2 - \Delta^2})^2}$	$\frac{2(E^2 - \Delta^2 + E(1+2Z^2)\sqrt{E^2 - \Delta^2})}{(E + (1+2Z^2)\sqrt{E^2 - \Delta^2})^2}$	for $ E > \Delta$

Table 1: The reflection and transmission probabilities for an *NS*-interface with the dimensionless barrier strength Z (after Ref. [2]; modified).

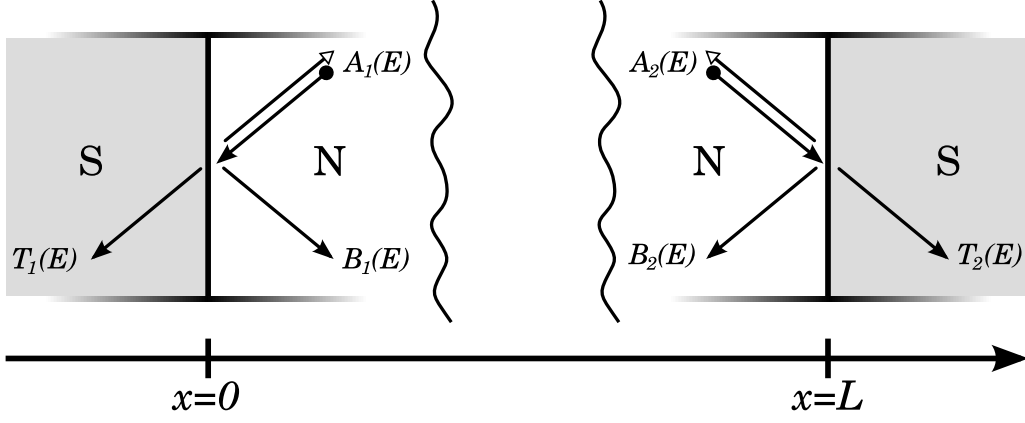


Figure 51: Schematic representation of an SNS junction, with arrows indicating the allowed processes at the interfaces: Andreev reflection A , normal reflection B and transmission T .

strength Z (cf. 1). The electrons at the normal side of the interface are separated into left- and right-moving populations, represented by the distribution functions $f_{\leftarrow}(E)$ and $f_{\rightarrow}(E)$, respectively. The current through the interface is then given by

$$I = \frac{1}{eR_0} \int_{-\infty}^{\infty} dE (f_{\rightarrow}(E) - f_{\leftarrow}(E)), \quad (19.1)$$

where $R_0 = (2N(0)e^2v_F\mathcal{A})^{-1}$ is the Sharvin resistance of the perfectly transparent interface ($Z = 0$) with $\mathcal{A}$ being the effective cross section of the contact and $N(0)$ the (single spin) density of states at the Fermi energy E_F . Blonder, Tinkham and Klapwijk showed [2] that the distribution function for the left-moving electrons is given by

$$f_{\leftarrow}(E) = A(E) [1 - f_{\rightarrow}(-E)] + B(E)f_{\rightarrow}(E) + T(E)f_0(E), \quad (19.2)$$

with $f_0(E)$ being the thermal Fermi distribution function $f_0(E) = 1/(1 + \exp(\beta(E - \mu)))$, assuming that the incoming electrons are in thermal equilibrium with their respective leads at the temperature $1/k_B\beta$ and the chemical potential μ .

This description was extended in Ref. [4] to an SN-interface followed by an NS-interface, i.e. to a full Josephson junction (cf. 51). We assume the same superconducting material on both sides, i.e. the same superconducting gap Δ , but different contacts and therefore differing barrier strengths. We have interface 1, located at $x = 0$ with barrier strength Z_1 , reflection and transmission probabilities $A_1(E)$, $B_1(E)$ and $T_1(E)$ and interface 2, at $x = L$ with Z_2 , $A_2(E)$, $B_2(E)$ and $T_2(E)$. The distribution functions $f_{\pm}(E, x)$, which are again to be taken in the normal region, are also functions of the longitudinal position within the junction, x . Now Eq. (19.2) can be applied to each of the two interfaces, which yields the following two equations [4]

$$f_{\rightarrow}(E, 0) = A_1(E) [1 - f_{\leftarrow}(-E, 0)] + B_1(E)f_{\leftarrow}(E, 0) + T_1(E)f_0(E), \quad (19.3)$$

$$f_{\leftarrow}(E, L) = A_2(E) [1 - f_{\rightarrow}(-E, L)] + B_2(E)f_{\rightarrow}(E, L) + T_2(E)f_0(E). \quad (19.4)$$

Note that we only combine distribution functions and not the quantum states (wavefunctions) at both interfaces. The relative phase of those states is therefore not considered, which is why the OTBK model only applies to incoherent junctions.

Since all energies are measured with respect to the local chemical potential, right-moving quasiparticles with energy E at $x = 0$ will arrive at $x = L$ with energy $E + eV$, while left movers

with energy E at $x = L$ will have energy $E - eV$ at $x = 0$. Thus the distribution functions at the interfaces relate to each other as

$$f_{\rightleftharpoons}(E, L) = f_{\rightleftharpoons}(E - eV, 0). \quad (19.5)$$

Eqs. (19.3)–(19.4) can be combined to eliminate, e.g., the left-moving part. Using Eq. (19.5) we can also shift all distribution functions from $x = L$ to $x = 0$ and hence omit the position argument in the following. The following equation can then be derived [13]

$$\begin{aligned} f_{\rightarrow}(E) &= A_1(E) \left\{ 1 - A_2(-E + eV) [1 - f_{\rightarrow}(E - 2eV)] \right. \\ &\quad \left. - B_2(-E + eV) f_{\rightarrow}(-E) - T_2(-E + eV) f_0(-E + eV) \right\} \\ &\quad + B_1(E) \left\{ A_2(E + eV) [1 - f_{\rightarrow}(-E - 2eV)] \right. \\ &\quad \left. + B_2(E + eV) f_{\rightarrow}(E) + T_2(E + eV) f_0(E + eV) \right\} \\ &\quad + T_1(E) f_0(E) \end{aligned} \quad (19.6)$$

that couples $f_{\rightarrow}(E)$ with $f_{\rightarrow}(-E)$, $f_{\rightarrow}(E - 2eV)$ and $f_{\rightarrow}(-E - 2eV)$ and thus gives rise to an infinite system of linear equations for, say, $f_{\rightarrow}(E)$.

19.4 Equal barriers

The simplest case, as far as the barriers are concerned, is the case in which both interfaces are characterized by the same barrier strength $Z_1 = Z_2 = Z$. For this case an additional relation $f_{\rightarrow}(E, 0) = 1 - f_{\leftarrow}(-E, L)$ was derived in Ref. [9] from Eqs. (19.3)–(19.4) and substituted into Eq. (19.5), which yields

$$f_{\rightleftharpoons}(E) = 1 - f_{\rightleftharpoons}(-E - eV) \quad (19.7)$$

and greatly simplifies the problem. As before, the suppressed position arguments imply $x = 0$. Using this result we can reformulate Eq. (19.1) to depend on right-movers only

$$I = \frac{1}{eR_0} \int_{-\infty}^{\infty} dE (f_{\rightarrow}(E) + f_{\rightarrow}(-E - eV) - 1). \quad (19.8)$$

Furthermore we can make use of Eq. (19.7) to eliminate the distribution functions for left-moving electrons in Eq. (19.3), which yields a significantly simpler equation than the fully general one from OTBK (19.6), namely

$$f_{\rightarrow}(E) = A(E) f_{\rightarrow}(E - eV) + B(E) [1 - f_{\rightarrow}(-E - eV)] + T(E) f_0(E). \quad (19.9)$$

The infinite system of linear equations generated by Eq. (19.9) was solved numerically in Ref. [9] to obtain subharmonic gap structure and excess current, but the latter can be obtained analytically [20], as we reproduce for convenience of the reader in the following.

19.4.1 Normal current

We shall first calculate the normal current to demonstrate the course of the derivation and to define some of the quantities used later on. We introduce the reflection and transmission probabilities in the normal case

$$B_n = \frac{Z^2}{1 + Z^2}, \quad T_n = 1 - B_n = \frac{1}{1 + Z^2}, \quad (19.10)$$

which are indeed the limits of $B(E)$ and $T(E)$ for vanishing Δ , as can be seen from 1.⁴ This allows us to rewrite Eq. (19.9) for the normal case as

$$f_{\rightarrow}^n(E) = B_n [1 - f_{\rightarrow}^n(-E - eV)] + T_n f_0(E), \quad (19.11)$$

where $f_{\rightarrow}^n(E)$ is the right-moving distribution function for the normal case. We rewrite the above Eq. (19.11) for the energy $-E - eV$

$$f_{\rightarrow}^n(-E - eV) = B_n [1 - f_{\rightarrow}^n(E)] + T_n f_0(-E - eV), \quad (19.12)$$

insert this again in Eq. (19.11) and solve for $f_{\rightarrow}^n(E)$, which yields

$$\begin{aligned} f_{\rightarrow}^n(E) &= \frac{T_n}{1-B_n^2} f_0(E) + \frac{B_n T_n}{1-B_n^2} f_0(E + eV) \\ &= \frac{1+Z^2}{1+2Z^2} f_0(E) + \frac{Z^2}{1+2Z^2} f_0(E + eV), \end{aligned} \quad (19.13)$$

where we have used that $f_0(-E - eV) = 1 - f_0(E + eV)$. We can write down the integrand from Eq. (19.8) with these normal-case distribution functions and simplify it to give

$$\begin{aligned} f_{\rightarrow}^n(E) + f_{\rightarrow}^n(-E - eV) - 1 &= \frac{1+Z^2}{1+2Z^2} (f_0(E) + \overbrace{f_0(-E - eV) - 1}^{-f_0(E+eV)}) \\ &\quad + \frac{Z^2}{1+2Z^2} (f_0(E + eV) + \underbrace{f_0(-E) - 1}_{-f_0(-E)}) \\ &= (f_0(E) - f_0(E + eV)) / (1 + 2Z^2), \end{aligned} \quad (19.14)$$

which is easily integrated and yields the familiar result $I_n = V/R_n$, with the normal state resistance $R_n = (1 + 2Z^2)R_0$ of the two-interface ballistic sandwich.⁵

19.4.2 Excess current

In the superconducting case we are interested in the excess current I_{exc} defined as $I_{\text{exc}} = I - I_n$ in the limit $eV \rightarrow \infty$, which is what we will assume in the rest of this section. We define $\Delta f_{\rightarrow}(E) = f_{\rightarrow}(E) - f_{\rightarrow}^n(E)$, insert this into Eq. (19.8) and subtract the normal part thus arriving at the formula for the excess current

$$I_{\text{exc}} = \frac{1}{eR_0} \int_{-\infty}^{\infty} dE (\Delta f(E) + \Delta f(-E - eV)), \quad (19.15)$$

where we have dropped the arrows from the notation, as we are only dealing with right-movers in this section. To calculate $\Delta f(E)$ we define, similarly to the above, $\Delta B(E) = B(E) - B_n$ and $\Delta T(E) = T(E) - T_n$, substitute all these definitions into Eq. (19.9), and solve for $\Delta f(E)$ to obtain

$$\begin{aligned} \Delta f(E) &= A(E) [f_n(E - eV) + \Delta f(E - eV)] \\ &\quad + \Delta B(E) [1 - f_n(-E - eV) - \Delta f(-E - eV)] \\ &\quad - B_n \Delta f(-E - eV) + \Delta T(E) f_0(E). \end{aligned} \quad (19.16)$$

Examining Eq. (19.16) and keeping in mind that $A(E)$, $\Delta B(E)$ and $\Delta T(E)$ tend toward zero for $|E| \gg \Delta$, we see that there exists only a certain energy range ϵ of the order of a few multiples

⁴In the normal state the Andreev reflection coefficient $A(E)$ is identically zero.

⁵Note, that due to the ballistic nature of the junction this resistance is *not* just the sum of the two series resistances of the individual interfaces.

of Δ where $A(E)$, $\Delta B(E)$, and $\Delta T(E)$ can be considered non-zero such that $f(E)$ will only differ significantly from $f_n(E)$ within ϵ around $E = 0$ and $E = -eV$. Since we only consider large bias those two energy regions are well separated. Therefore we can split $\Delta f(E)$ into one part which is only nonzero for $|E| < \epsilon$ and vanishes for all other energies and one part with the same properties for $|E + eV| < \epsilon$. We introduce these parts by writing

$$\Delta f(E) = \Delta \tilde{f}(E) + \Delta \tilde{f}^{-eV}(E). \quad (19.17)$$

This mathematical procedure is fully in line with the physical intuition that the only changes of the distribution functions induced by the superconductivity will occur within the few- Δ -multiples vicinity of the two Fermi energies of the leads. We can rewrite Eq. (19.16) for $\Delta f(E - eV)$ and we see that for $|E| < \epsilon$ most terms in the right hand side simply drop out as they include a vanishing multiplier. So we are left with

$$\Delta \tilde{f}^{-eV}(E - eV) = -B_n \Delta \tilde{f}(-E). \quad (19.18)$$

We insert Eq. (19.18) into Eq. (19.16), still assuming $|E| < \epsilon$, to obtain

$$\begin{aligned} \Delta \tilde{f}(E) = & A(E) [f_n(E - eV) - B_n \Delta \tilde{f}(-E)] \\ & + \Delta B(E) [1 - f_n(-E - eV) + B_n \Delta \tilde{f}(E)] \\ & + B_n^2 \Delta \tilde{f}(E) + \Delta T(E) f_0(E). \end{aligned} \quad (19.19)$$

It turns out to be convenient to consider the combination $\Delta \tilde{f}(E) + \Delta \tilde{f}(-E)$ in the following, i.e. to symmetrize the problem. Therefore, we rewrite Eq. (19.19) for $\Delta \tilde{f}(-E)$, sum the result with Eq. (19.19) and solve for $\Delta \tilde{f}(E) + \Delta \tilde{f}(-E)$ obtaining

$$\begin{aligned} & \left[1 + B_n (A(E) - B(E)) \right] (\Delta \tilde{f}(E) + \Delta \tilde{f}(-E)) = \\ & (A(E) - \Delta B(E)) [f_n(E - eV) + f_n(-E - eV)] + 2\Delta B(E) + \Delta T(E). \end{aligned} \quad (19.20)$$

Obviously, the above Eq. (19.20) again holds only for the range $|E| < \epsilon$ in which it was derived. The sum $f_n(E - eV) + f_n(-E - eV)$ that turns up on the right-hand side of Eq. (19.20) can be calculated using Eq. (19.13) and the assumption of large bias, i.e. $eV \rightarrow \infty$, so that we obtain

$$\begin{aligned} f_n(E - eV) + f_n(-E - eV) &= \frac{1+Z^2}{1+2Z^2} \overbrace{(f_0(E - eV) + f_0(-E - eV))}^{\rightarrow 2} \\ &+ \frac{Z^2}{1+2Z^2} \underbrace{(f_0(E) + f_0(-E))}_1 = \frac{2+3Z^2}{1+2Z^2}. \end{aligned} \quad (19.21)$$

From the condition of probability conservation $A(E) + B(E) + T(E) = 1$, we see that $\Delta T(E) = -A(E) - \Delta B(E)$. We substitute this into Eq. (19.20) along with Eq. (19.21) and are left with

$$\Delta \tilde{f}(E) + \Delta \tilde{f}(-E) = \frac{(A(E) - \Delta B(E))(1 + Z^2)}{\left[1 + B_n (A(E) - B(E)) \right] (1 + 2Z^2)}. \quad (19.22)$$

We now take the integrand from Eq. (19.15) and expand it by inserting Eq. (19.17) to get

$$\Delta f(E) + \Delta f(-E - eV) = \Delta \tilde{f}(E) + \Delta \tilde{f}^{-eV}(E) + \Delta \tilde{f}(-E - eV) + \Delta \tilde{f}^{-eV}(-E - eV). \quad (19.23)$$

The first and the last terms are nonzero around $E = 0$ and we can use the straightforward modification of Eq. (19.18) for the simplification $\Delta \tilde{f}(E) + \Delta \tilde{f}^{-eV}(-E - eV) = (1 - B_n) \Delta \tilde{f}(E)$. Analogously, the two middle terms in Eq. (19.23) are nonzero around $E = -eV$ and since they

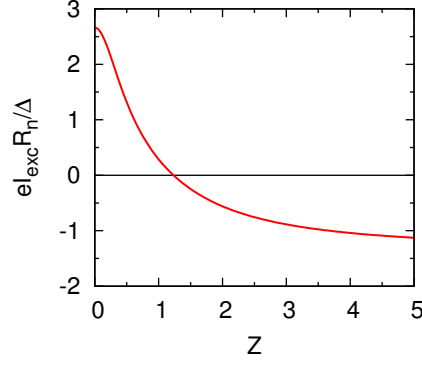


Figure 52: The excess current of a symmetric, fully incoherent Josephson junction in the OTBK model as a function of the barrier strength Z .

appear under the integral extending over the entire energy range and have strongly localized support their energy arguments can be shifted so that they are localized around $E = 0$ as well.⁶ This eventually leads to the relations $\Delta \tilde{f}^{-eV}(E) + \Delta \tilde{f}(-E - eV) \Rightarrow \Delta \tilde{f}^{-eV}(E - eV) + \Delta \tilde{f}(-E) = (1 - B_n) \Delta \tilde{f}(-E)$. Putting all the pieces together leaves us with

$$\begin{aligned} I_{\text{exc}} &= \frac{1}{eR_0} \int_{-\infty}^{\infty} dE (1 - B_n) \left(\Delta \tilde{f}(E) + \Delta \tilde{f}(-E) \right) \\ &= \frac{1}{eR_0(1+2Z^2)} \int_{-\infty}^{\infty} dE \frac{A(E) - \Delta B(E)}{[1 + B_n(A(E) - B(E))]}, \end{aligned} \quad (19.24)$$

where we used Eq. (19.22) to produce the final integrand. The analytical integration must be performed for $|E| \leq \Delta$ and $|E| \geq \Delta$, separately, because $A(E)$ and $B(E)$ take on different functional forms in these intervals. It can be evaluated using trigonometric or hyperbolic substitution for the subgap or overgap energies, respectively. Thus we find the excess current in the symmetric case is given by

$$\begin{aligned} \frac{eI_{\text{exc}} R_n}{\Delta} &= 2(1 + 2Z^2) \tanh^{-1} \left(2Z \sqrt{(1 + Z^2)/(1 + 6Z^2 + 4Z^4)} \right) \\ &\quad \times \left(Z \sqrt{(1 + Z^2)(1 + 6Z^2 + 4Z^4)} \right)^{-1} - \frac{4}{3}. \end{aligned} \quad (19.25)$$

The first (and longer) term on the right hand side of Eq. (19.25) results from the subgap integral, the $-4/3$ term is the contribution of the overgap part. This analytic result is plotted in 52 and the comparison with the earlier numerical result by Flensberg, Bindeslev Hansen and Octavio plotted in Figure 6 of Ref. [9] shows a nice agreement.

19.5 Differing barrier strengths

To obtain a similar expression for the excess current in the case of asymmetric barrier strengths we need to restart from Eq. (19.6), since Eq. (19.7) and the ensuing simplifications, in particular Eq. (19.9), cannot be used. The course of the derivation, however, is very similar to the above. Again we start by calculating the normal current.

⁶The strongly localized support of the involved terms is essential for the possibility of the variable shift and its lack can lead to seemingly paradoxical results when done formally, e.g., in Eq. (19.8).

19.5.1 Normal current

Corresponding to Eq. (19.1), the normal current is given by

$$I_n = \frac{1}{eR_0} \int_{-\infty}^{\infty} dE (f_{\rightarrow}^n(E) - f_{\leftarrow}^n(E)), \quad (19.26)$$

where $f_{\rightarrow}^n(E)$ is defined as above and $f_{\leftarrow}^n(E)$ is its left-moving counterpart. The reflection and transmission probabilities in the normal case, $B_{n,i}$ and $T_{n,i}$, are defined as above with the additional index $i \in \{1; 2\}$, which indicates the interface in question. From Eq. (19.6) we now find for the right-movers in the normal case

$$f_{\rightarrow}^n(E) = B_{n,1} [B_{n,2} f_{\rightarrow}^n(E) + T_{n,2} f_0(E + eV)] + T_{n,1} f_0(E). \quad (19.27)$$

Since we also need to consider left-movers in this section, we use Eqs. (19.3)–(19.5) to derive the left-moving counterpart to Eq. (19.6), which is not shown for reasons of length, and finally the equivalent of the above Eq. (19.27) for left movers, which reads

$$f_{\leftarrow}^n(E) = B_{n,2} [B_{n,1} f_{\leftarrow}^n(E) + T_{n,1} f_0(E)] + T_{n,2} f_0(E + eV). \quad (19.28)$$

We take the integrand from Eq. (19.26) and use Eqs. (19.27)–(19.28) to rewrite it as follows

$$\begin{aligned} f_{\rightarrow}^n(E) - f_{\leftarrow}^n(E) &= \frac{T_{n,1} T_{n,2}}{1 - B_{n,1} B_{n,2}} (f_0(E) - f_0(E + eV)) \\ &= \frac{1}{1 + Z_1^2 + Z_2^2} (f_0(E) - f_0(E + eV)), \end{aligned} \quad (19.29)$$

This is easy to integrate and yields $I_n = V/(\rho_n R_0)$, with $\rho_n = 1 + Z_1^2 + Z_2^2$. Note that $\rho_n R_0$ simply becomes R_n for $Z_1 = Z_2 = Z$, so we find the normal current from above for equal barriers again.

19.5.2 Excess current

For the calculation of the excess current we assume large bias once again and introduce $\Delta f_{\rightarrow}(E) = f_{\rightarrow}(E) - f_{\rightarrow}^n(E)$, $\Delta B_i(E) = B_i(E) - B_{n,i}$ and $\Delta T_i(E) = T_i(E) - T_{n,i}$, just like above in the case of symmetric barriers. Using these relations we can expand Eq. (19.6) and subtract Eq. (19.27) to obtain

$$\begin{aligned} \Delta f_{\rightarrow}(E) = & A_1(E) \left\{ 1 - A_2(-E + eV) [1 - f_{\rightarrow}(E - 2eV)] \right. \\ & - B_2(-E + eV) f_{\rightarrow}(-E) \\ & \left. - T_2(-E + eV) f_0(-E + eV) \right\} \\ & + B_{n,1} \left\{ A_2(E + eV) [1 - f_{\rightarrow}(-E - 2eV)] + B_{n,2} \Delta f_{\rightarrow}(E) \right. \\ & + \Delta B_2(E + eV) f_{\rightarrow}(E) + \Delta T_2(E + eV) f_0(E + eV) \left. \right\} \\ & + \Delta B_1(E) \left\{ A_2(E + eV) [1 - f_{\rightarrow}(-E - 2eV)] \right. \\ & + B_2(E + eV) f_{\rightarrow}(E) + T_2(E + eV) f_0(E + eV) \left. \right\} \\ & + \Delta T_1(E) f_0(E). \end{aligned} \quad (19.30)$$

By the same logic as before we see that $\Delta f_{\rightarrow}(E)$ is only nonzero for $|E| < \epsilon$ or $|E + eV| < \epsilon$. Therefore we split $\Delta f_{\rightarrow}(E)$ into two parts, just like we did above and with the same properties

$$\Delta f_{\rightarrow}(E) = \Delta \tilde{f}_{\rightarrow}(E) + \Delta \tilde{f}_{\rightarrow}^{-eV}(E). \quad (19.31)$$

For the remainder of the section we assume small energies ($|E| < \epsilon$), in which case Eq. (19.30) can be reduced and solved for $\Delta\tilde{f}_{\rightarrow}(E)$ to yield

$$\begin{aligned} [1 - B_1(E)B_{n,2}]\Delta\tilde{f}_{\rightarrow}(E) &= A_1(E)[1 - B_{n,2}(f_{\rightarrow}^n(-E) + \Delta\tilde{f}_{\rightarrow}(-E)) \\ &\quad - T_{n,2}f_0(-E + eV)] \\ &\quad + \Delta B_1(E)[B_{n,2}f_{\rightarrow}^n(E) + T_{n,2}f_0(E + eV)] \\ &\quad + \Delta T_1(E)f_0(E) \end{aligned} \quad (19.32)$$

We rewrite Eq. (19.32) for $\Delta\tilde{f}_{\rightarrow}(-E)$, sum the result with Eq. (19.32) and solve for $\Delta\tilde{f}_{\rightarrow}(E) + \Delta\tilde{f}_{\rightarrow}(-E)$, which gives

$$\begin{aligned} [1 + B_{n,2}(A_1(E) - B_1(E))][\Delta\tilde{f}_{\rightarrow}(E) + \Delta\tilde{f}_{\rightarrow}(-E)] \\ &= A_1(E)[2 - B_{n,2}(f_{\rightarrow}^n(E) + f_{\rightarrow}^n(-E)) \\ &\quad - T_{n,2}(f_0(E + eV) + f_0(-E + eV))] \\ &\quad + \Delta B_1(E)[B_{n,2}(f_{\rightarrow}^n(E) + f_{\rightarrow}^n(-E)) \\ &\quad + T_{n,2}(f_0(E + eV) + f_0(-E + eV))] \\ &\quad + \Delta T_1(E) \underbrace{(f_0(E) + f_0(-E))}_1 \end{aligned} \quad (19.33)$$

The sum $f_0(E + eV) + f_0(-E + eV)$ in the above becomes zero for large bias, which means that the terms explicitly involving $T_{n,2}$ drop out of Eq. (19.33). Furthermore, using Eq. (19.27) we can write

$$\begin{aligned} f_{\rightarrow}^n(E) + f_{\rightarrow}^n(-E) &= [B_{n,1}T_{n,2} \overbrace{(f_0(E + eV) + f_0(-E + eV))}^{\rightarrow 0} \\ &\quad + T_{n,1} \underbrace{(f_0(E) + f_0(-E))}_1] / (1 - B_{n,1}B_{n,2}) = \frac{T_{n,1}}{1 - B_{n,1}B_{n,2}}, \end{aligned} \quad (19.34)$$

further simplifying Eq. (19.33), which can now be written as

$$\Delta\tilde{f}_{\rightarrow}(E) + \Delta\tilde{f}_{\rightarrow}(-E) = \frac{\rho_1}{\rho_n} \frac{A_1(E) - \Delta B_1(E)}{1 + B_{n,2}(A_1(E) - B_1(E))}, \quad (19.35)$$

where $\rho_i = 1/T_{n,i} = 1 + Z_i^2$ is the dimensionless resistance of the single i -th interface in the normal state. In a similar way and using the same assumptions, i.e. $eV \rightarrow \infty$ and $|E| < \epsilon$, we can show that

$$\Delta\tilde{f}_{\rightarrow}^{-eV}(E - eV) + \Delta\tilde{f}_{\rightarrow}^{-eV}(-E - eV) = -\frac{\rho_2}{\rho_n} \frac{A_2(E) - \Delta B_2(E)}{1 + B_{n,1}(A_2(E) - B_2(E))} B_{n,1}. \quad (19.36)$$

We still need to get the left-moving equivalents of Eqs. (19.35), (19.36), so first we derive the counterpart to Eq. (19.30) for the left-movers, which is not shown, because the derivation follows the earlier pattern and does not deliver new insights. Just like above we can split $\Delta f_{\leftarrow}(E)$ up into

$$\Delta f_{\leftarrow}(E) = \Delta\tilde{f}_{\leftarrow}(E) + \Delta\tilde{f}_{\leftarrow}^{-eV}(E). \quad (19.37)$$

As for the right-movers and in much the same way we can show that for $|E| < \epsilon$ and $eV \rightarrow \infty$

$$\Delta\tilde{f}_{\leftarrow}(E) + \Delta\tilde{f}_{\leftarrow}(-E) = \frac{\rho_1}{\rho_n} \frac{A_1(E) - \Delta B_1(E)}{1 + B_{n,2}(A_1(E) - B_1(E))} B_{n,2} \quad (19.38)$$

as well as

$$\Delta \tilde{f}_{\leftarrow}^{-eV}(E - eV) + \Delta \tilde{f}_{\leftarrow}^{-eV}(-E - eV) = -\frac{\rho_2}{\rho_n} \frac{A_2(E) - \Delta B_2(E)}{1 + B_{n,1}(A_2(E) - B_2(E))}. \quad (19.39)$$

The excess current is now given by

$$\begin{aligned} I_{\text{exc}} &= \frac{1}{eR_0} \int_{-\infty}^{\infty} dE \left(\Delta f_{\rightarrow}(E) - \Delta f_{\leftarrow}(E) \right) \\ &= \frac{1}{2eR_0} \int_{-\infty}^{\infty} dE \left(\underbrace{\left(\Delta \tilde{f}_{\rightarrow}(E) + \Delta \tilde{f}_{\rightarrow}(-E) \right)}_{(19.35)} + \underbrace{\left(\Delta \tilde{f}_{\rightarrow}^{-eV}(E) + \Delta \tilde{f}_{\rightarrow}^{-eV}(-E) \right)}_{(19.36)'} \right. \\ &\quad \left. - \underbrace{\left(\Delta \tilde{f}_{\leftarrow}(E) + \Delta \tilde{f}_{\leftarrow}(-E) \right)}_{(19.38)} - \underbrace{\left(\Delta \tilde{f}_{\leftarrow}^{-eV}(E) + \Delta \tilde{f}_{\leftarrow}^{-eV}(-E) \right)}_{(19.39)'} \right). \end{aligned} \quad (19.40)$$

The braces and brackets in Eq. (19.40) indicate which terms in the expression correspond to which one of the above equations. The primed brackets are shifted in energy, which does not matter to the final result, since the integral extends over the entire energy range and the integrands have strongly localized support. Finally we can express the excess current as

$$I_{\text{exc}} = \frac{1}{2eR_0} \int_{-\infty}^{\infty} dE \left\{ \frac{\rho_1}{\rho_n} \frac{A_1(E) - \Delta B_1(E)}{1 + B_{n,2}(A_1(E) - B_1(E))} (1 - B_{n,2}) + \{1 \leftrightarrow 2\} \right\}. \quad (19.41)$$

The integral in Eq. (19.41) can be solved and the result for $Z_1 > Z_2$ is given by

$$\begin{aligned} \frac{eI_{\text{exc}}\rho_n R_0}{\Delta} &= 2\rho_n \tanh^{-1} \left(2Z_1 \sqrt{\rho_1 / (2Z_2^2 + (1 + 2Z_1^2)^2)} \right) \\ &\quad \times \left(Z_1 \sqrt{\rho_1 (2Z_2^2 + (1 + 2Z_1^2)^2)} \right)^{-1} \\ &\quad + \left[\tan^{-1} \sqrt{(Z_1^2 - Z_2^2)/\rho_n} - \tanh^{-1} \sqrt{(Z_1^2 - Z_2^2)/\rho_n} \right] \\ &\quad \times \frac{(1+2Z_1^2)(1+2Z_2^2)}{2\sqrt{\rho_n}(Z_1^2 - Z_2^2)^{\frac{3}{2}}} - 1. \end{aligned} \quad (19.42)$$

For $Z_1 < Z_2$ the excess current is obtained by exchanging Z_1 and Z_2 in Eq. (19.42) and the result is thus symmetric with respect to the interchange of the two interfaces. Again, the first term (the first line) on the right hand side of Eq. (19.42) corresponds to the sub-gap integral and it is easy to see how for $Z_1 = Z_2 = Z$ it becomes the corresponding term in Eq. (19.25). The remaining two terms (the second line), which result from the over-gap integral, converge towards $-4/3$ for $Z_1 \rightarrow Z_2$, as we now show by the Taylor expansion of $\tan^{-1}(z) = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \frac{1}{7}z^7 + \dots$ and $\tanh^{-1}(z) = z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \frac{1}{7}z^7 + \dots$ resulting in $\tan^{-1}(z) - \tanh^{-1}(z) = -\frac{2}{3}z^3 + O(z^7)$ so that the square bracket in the second line of Eq. (19.42) tends to $-\frac{2}{3}((Z_1^2 - Z_2^2)/\rho_n)^{\frac{3}{2}}$ for $Z_1 \rightarrow Z_2$. Therefore the last two lines of Eq. (19.42) reduce to $-\frac{(1+2Z_1^2)(1+2Z_2^2)}{3\rho_n^2} - 1$, which simply becomes $-4/3$ for $Z_1 = Z_2$ and we thus recover Eq. (19.25) in the symmetric case. The full result (19.42) is plotted in Figure 53 as a function of the two barrier strengths $Z_{1,2}$. The negative excess current predicted for large enough normal-state resistance persists for asymmetric junctions $Z_1 \neq Z_2$ with arbitrarily large asymmetry although its magnitude decreases (also note the prefactor ρ_n customarily multiplying the plotted excess current) and, thus, its experimental observation may be impeded by the asymmetry of real junctions.

19.6 References

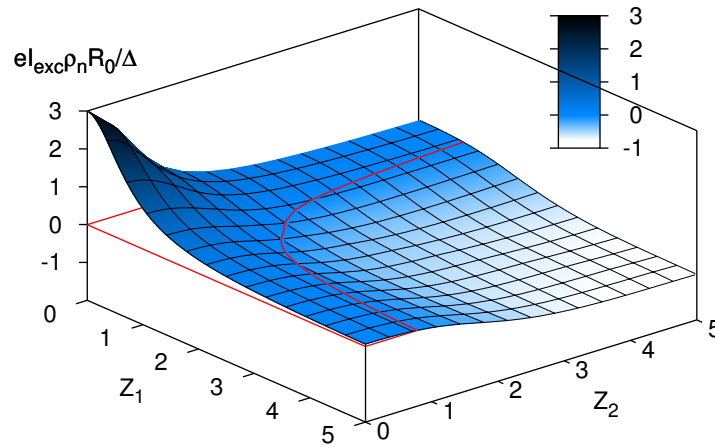


Figure 53: The excess current of an asymmetric, fully incoherent Josephson junction as a function of the two barrier strengths Z_1 and Z_2 . The isoline $I_{\text{exc}} = 0$ is shown in red.

References

- [1] Klapwijk T M, Blonder G E and Tinkham M 1982 *Physica B+C* **109-110** 1657
- [2] Blonder G E, Tinkham M and Klapwijk T M 1982 *Physical Review B* **25** 4515
- [3] Giubileo F, Aprili M, Bobba F, Piano S, Scarfato A and Cucolo A M 2005 *Physical Review B* **72** 174518
- [4] Valentine J M and Chien C L 2006 *Journal of Applied Physics* **99** 08P902
- [5] Cong Ren, Trbovic J, Kallaher R L, Braden J G, Parker J S, von Molnár S and Xiong P 2007 *Physical Review B* **75** 205208
- [6] Xia K, Kelly P J, Bauer G E W and Turek I 2002 *Physical Review Letters* **89** 166603
- [7] Linder J and Sudbø A 2008 *Physical Review B* **77** 064507
- [8] Octavio M, Tinkham M, Blonder G E and Klapwijk T M 1983 *Physical Review B*, **27** 6739
- [9] Flensberg K, Bindslev Hansen J and Octavio M 1988 *Physical Review B* **38** 8707
- [10] Bratus' E N, Shumeiko V S and Wendin G 1995 *Physical Review Letters* **74** 2110
- [11] Cuevas J C, Martín-Rodero A and Levy Yeyati A 1996 *Physical Review B* **54** 7366
- [12] Cuevas J C 1999 *PhD thesis* Universidad Autónoma de Madrid
- [13] van Hufelen W M, Klapwijk T M, Heslinga D R, de Boer M J and van der Post N 1993 *Physical Review B* **47** 5170
- [14] Kuhlmann M, Zimmermann U, Dikin D, Abens S, Keck K and Dmitriev V M 1994 *Zeitschrift für Physik B* **96** 13
- [15] Zimmermann U, Abens S, Dikin D, Keck K and Dmitriev V M 1995 *Zeitschrift für Physik B* **97** 59

- [16] Baturina T, Islamov D and Kvon Z 2002 *JETP Letters* **75** 326
- [17] Ishida H, Okanoué K, Kawakami A, Zhen Wang and Hamasaki K 2005 *IEEE Transactions on Applied Superconductivity* **15** 212
- [18] Ojeda-Aristizábal C M, Ferrier M, Guéron S and Bouchiat H 2009 arXiv:0903.2963
- [19] Pilgram S and Samuelsson P 2005 *Physical Review Letters* **94** 086806
- [20] Niebler G, Cuniberti G and Novotný T 2008 in *WDS'08 Proceedings of Contributed Papers: Part III – Physics* 124
http://www.mff.cuni.cz/veda/konference/wds/contents/pdf08/WDS08_321_f3_Niebler.pdf
- [21] Jørgensen H I, Grove-Rasmussen K, Novotný T, Flensberg K and Lindelof P E 2006 *Physical Review Letters* **96** 207003
- [22] Jørgensen H I, Grove-Rasmussen K, Flensberg K and Lindelof P E 2008 arXiv:0812.4175
- [23] Wu F, Danneau R, Queipo P, Kauppinen E, Tsuneta T and Hakonen P J 2009 *Physical Review B* **79** 073404
- [24] Pastawski H M 1991 *Physical Review B* **44** 6329
- [25] Seelig G, Büttiker M 2001 *Physical Review B* **64** 245313

20 Theory of non-equilibrium noise in general multi-terminal superconducting hybrid devices: application to multiple Cooper pair resonances by R. Jacquet, Popoff et al.

Abstract

We consider the out-of-equilibrium behavior of a general class of mesoscopic devices composed of several superconducting or/and normal metal leads separated by quantum dots. Starting from a microscopic Hamiltonian description, we provide a non-perturbative approach to quantum electronic transport in the tunneling amplitudes between dots and leads: using the equivalent of a path integral formulation, the lead degrees of freedom are integrated out in order to compute both the current and the current correlations (noise) in this class of systems, in terms of the dressed Green's function matrix of the quantum dots. In order to illustrate the efficiency of this formalism, we apply our results to the “all superconducting Cooper pair beam splitter”, a device composed of three superconducting leads connected via two quantum dots, where crossed Andreev reflection operates Cooper pair splitting. Commensurate voltage differences between the three leads allow to obtain expressions for the current and noise as a function of the Keldysh Nambu Floquet dressed Green's function of the dot system. This voltage configuration allows the occurrence of non-local processes involving multiple Cooper pairs which ultimately lead to the presence of non-zero DC currents in an out-of-equilibrium situation. We investigate in details the results for the noise obtained numerically in the specific case of opposite voltages, where the transport properties are dominated by the so called “quartet processes”, involving the coherent exchange of two Cooper pairs among all three superconducting terminals. We show that these processes are noiseless in the non-resonant case, and that this property is also observed for other voltage configurations. When the dots are in a resonant regime, the noise characteristics change qualitatively, with the appearance of giant Fano factors.

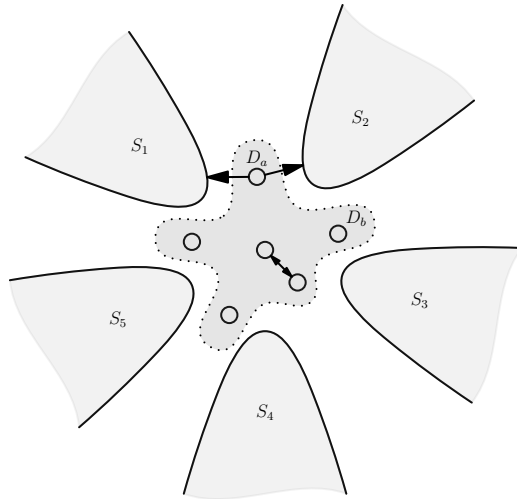


Figure 54: Sketch of a general multi-terminal mesoscopic device (with either superconducting or normal metal leads, or both) described by the present formalism: superconducting or normal metal leads are connected to an “array” of quantum dots. Electrons can either hop from a quantum dot to one of the leads, or they can alternatively tunnel to another dot. Although no direct tunneling amplitude between the leads is present in this theoretical model, we argue in the text that such processes can effectively be included by tuning appropriately both the quantum dot level and its hopping amplitude to the leads.

20.1 Introduction

In a mesoscopic device involving superconducting contacts/reservoirs, transport properties are largely influenced by Andreev reflection in the subgap voltage regime. [1] This fundamental process amounts to an electron being reflected into a hole (or vice-versa), the difference of charge being absorbed by the creation (or destruction) of a Cooper pair (CP) inside the BCS ground state of the superconductor. In a Josephson junction between two superconductors, [2] this microscopic process explains (i) the DC Josephson supercurrent, an equilibrium dissipationless phenomenon which depends on the phase difference between the two superconducting reservoirs, (ii) the AC Josephson effect which depends on the voltage applied across the junction and the Shapiro DC steps obtained when adding an RF irradiation, (iii) the pair-assisted quasiparticle transport for subgap voltages, a phase insensitive non-equilibrium dissipative phenomenon involving multiple Andreev reflections (MAR). [3, 4, 5, 6]

In more involved multi-terminal superconducting hybrid structures, a non-local version of this process may arise. This has been intensively studied in the context of the Cooper pair beam splitter (CPBS), [7, 8, 9] a three lead device composed of a single superconducting lead connected via two quantum dots to two normal metal leads. There, in addition to (direct) Andreev reflection, the two constituent electrons of a Cooper pair can be transferred to the two normal metal leads as a non-local entangled pair [10, 11] via a process called crossed Andreev reflection (CAR).

The first purpose of this paper is to present a general theoretical framework for computing the current and noise characteristics of a general class of mesoscopic devices composed of an arbitrary number of biased superconducting or normal metal leads, which are separated by quantum dots. Upon integrating the quadratic degrees of freedom of the leads and applying Wick’s theorem, analytical expressions for the noise are obtained in terms of the dressed dot Green’s function matrix in the Keldysh Nambu space. To our knowledge, such a general theoretical framework has so far only been considered for the computation of current, in a specific situation: the “all superconducting Cooper pair beam splitter” (ASCPBS), studied in Ref. [12], which is the equivalent of the CPBS albeit with all three leads being superconductors.

As is now well established, a noise diagnosis allows to gain further information compared to the current characteristics: particularly relevant is the monitoring of Fano factors (ratios between the noise and the current). We stress that this formulation of quantum transport in superconducting/normal metal hybrid devices can be readily used to study a vast class of systems, as it only requires the (numerical) solution of the Dyson equation of the dressed single particle Green's function of the quantum dot system - the quantum dot "array".

In the second part of this paper, as an application of our formalism, we consider the computation of the noise characteristics of the ASCPBS, where in the subgap regime, Andreev reflection, crossed Andreev reflection, and multiple Andreev reflections constitute the basic transport processes. In addition, the voltage differences between the three leads of this ASCPBS are tuned to a commensurate voltage configuration dubbed "Multiple Cooper pair resonance" (MCPR), [12] allowing a dissipationless Josephson-like signal for the current in the subgap regime, despite the fact that the system is driven far from equilibrium. The application of the general formalism developed here is thus to monitor the behavior of the noise as the subgap voltage differences are decreased. In some specific range of parameters where the quantum dot behaves like a quantum point contact, we find that the low-temperature noise vanishes. Conversely, in the resonant dot regime where the dots operate as energy filters, we find that the subgap voltage noise characteristics gives rise to anomalous giant Fano factors.

The paper is organized as follows. Sec. 20.2 describes the general class of models which we intend to describe, and the equivalent of a path integral formulation [13] for quantum transport is formulated, allowing us to compute both the Josephson current and the current-current correlations in a non-perturbative way in a generalized Fisher-Lee formula [14] for both the current and noise. In Sec. 20.3 we apply our formalism to MCPR in the ASCPBS. After presenting a short historical perspective of selected works dealing with multi-terminal superconducting hybrid devices, we derive the expressions for the current and noise using our Nambu Keldysh Floquet formalism, and present numerical results in two experimentally relevant regimes. The discussion and conclusions are presented in Secs. 20.4 and 20.5.

We consider units in which Planck and Boltzmann constants, together with the elementary charge are unity, *i.e.* $\hbar = 1$, $k_B = 1$, $e = 1$.

20.2 Microscopic formulation

We consider a class of mesoscopic systems composed of an arbitrary number of superconducting or normal metal leads, which are coupled to an array of quantum dots. All superconductors, labeled $j = 1, 2, \dots$ are described by BCS theory with a gap energy Δ_j (when a given lead is chosen to be a normal metal, this gap is chosen to be zero). Quantum dots are identified with the label $\alpha = a, b, \dots$ and their energy level is specified by the energy ε_α . There exists no restriction about which quantum dot is coupled to which lead or dot. The tunneling amplitude between lead j and QD α is denoted as $t_{j\alpha}$, while the direct tunneling between dots is described by the amplitude $t_{\alpha\beta} = t_{\beta\alpha}$. At first sight, it may seem that a direct coupling (not mediated by quantum dots) is ruled out in this context. This is by no means the case, as we have shown in previous work [15] that a system of two superconductors separated by a quantum dot can also describe an adjustable superconducting quantum point contact with arbitrary transmission, provided that the dot level is chosen outside the gaps of the superconductors, and that the dot-superconductor amplitudes are properly tuned. An example of such devices is depicted in Fig. 54.

20.2.1 Total Hamiltonian

The total Hamiltonian of the system can be decomposed into a contribution from the superconducting leads, the one from the QDs (including interdot tunneling) and a final one describing the tunneling between the QD and the leads, namely

$$H = \sum_j H_j + H_D + H_T(t). \quad (20.1)$$

The superconducting leads, identified by the label j , are represented by the standard BCS Hamiltonian

$$H_j = \sum_k \Psi_{jk}^\dagger \left[\left(\frac{k^2}{2m} - \mu \right) \sigma_z + \Delta_j \sigma_x \right] \Psi_{jk}, \quad (20.2)$$

where the Pauli matrices σ_x and σ_z act in Nambu space and we introduced Nambu spinors

$$\Psi_{jk} = \begin{pmatrix} \psi_{jk,\uparrow} \\ \psi_{j(-k),\downarrow}^\dagger \end{pmatrix}, \quad (20.3)$$

with $\psi_{jk\sigma}^\dagger$ the creation operator for an electron with momentum k and spin $\sigma = \uparrow, \downarrow$ in lead j .

Introducing similarly the creation operator for an electron with spin $\sigma = \uparrow, \downarrow$ on dot α as $d_{\alpha\sigma}^\dagger$, the Hamiltonian H_{D_α} of quantum dot α reads

$$H_{D_\alpha} = \epsilon_\alpha \sum_\sigma d_{\alpha\sigma}^\dagger d_{\alpha\sigma}, \quad (20.4)$$

and the tunneling Hamiltonian between the QDs a and b is given by

$$H_{D_a D_b} = t_{ab} \sum_\sigma d_{a\sigma}^\dagger d_{b\sigma} + \text{H.c.} \quad (20.5)$$

Introducing Nambu spinors of the form

$$d_\alpha = \begin{pmatrix} d_{\alpha\uparrow} \\ d_{\alpha\downarrow}^\dagger \end{pmatrix}, \quad (20.6)$$

and collecting them for each dot α into a $2N_D$ -component Nambu-dot spinor (N_D is the number of quantum dots in the system) as

$$\tilde{d} = \begin{pmatrix} d_a \\ d_b \\ d_c \\ \vdots \end{pmatrix}, \quad (20.7)$$

the Hamiltonian of the double QD can be conveniently rewritten as

$$H_D = \sum_\alpha H_{D_\alpha} + \sum_{\alpha\beta} H_{D_\alpha D_\beta} = \tilde{d}^\dagger h_D \sigma_z \tilde{d}, \quad (20.8)$$

where the Pauli matrix σ_z acts in Nambu space and the matrix

$$h_D = \begin{pmatrix} \epsilon_a & t_{ab} & t_{ac} & \dots \\ t_{ba} & \epsilon_b & t_{bc} & \dots \\ t_{ca} & t_{cb} & \epsilon_c & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (20.9)$$

is defined in dot space.

Finally, the tunneling Hamiltonian between leads and QDs is written in terms of Nambu spinors according to

$$H_T(t) = \sum_{jk\alpha} \Psi_{jk}^\dagger \mathcal{T}_{j\alpha}(t) d_\alpha + \text{H.c.} \quad (20.10)$$

where applying a Peierls substitution, we gauged the external bias away from the lead Hamiltonian and into the tunneling constants, therefore introducing $\mathcal{T}_{j\alpha}(t) = t_{j\alpha} \sigma_z e^{i\sigma_z \int_{-\infty}^t V_j dt}$.

20.2.2 Green's functions in the Keldysh formalism

In order to calculate thermodynamic averages of operators in an out-of-equilibrium system, the Keldysh time contour C is introduced: [16] it goes from $-\infty$ to $+\infty$ (+ forward branch) and goes back to $-\infty$ (− backward branch). The time ordering operator along this contour is denoted as T_C . We introduce the $4N_D$ -component Nambu-dot-Keldysh spinors collecting Nambu-dot spinors (20.7) evaluated on the two different branches of the Keldysh time contour

$$\check{d} = \begin{pmatrix} \tilde{d}^+ \\ \tilde{d}^- \end{pmatrix}. \quad (20.11)$$

The bare Green's functions of the QDs (in the absence of tunneling between dots and superconducting leads) reads

$$\check{G}_0(t, t') = -i \langle T_C \{ \check{d}(t) \check{d}^\dagger(t') \} \rangle_0. \quad (20.12)$$

The quantum mechanical averaging is performed with respect to the Hamiltonian without tunneling

$$\langle \dots \rangle_0 = \frac{\text{Tr} \{ e^{-\beta H_0} \dots \}}{\text{Tr} \{ e^{-\beta H_0} \}} \quad \text{where } H_0 = \sum_j H_j + H_D. \quad (20.13)$$

QD and superconducting degrees of freedom are coupled with the time-dependent tunneling Hamiltonian $H_T(t)$ and the Green's function dressed by this tunneling reads

$$\check{G}(t, t') = -i \langle T_C \{ S(\infty) \check{d}(t) \check{d}^\dagger(t') \} \rangle_0, \quad (20.14)$$

where $S(\infty)$ is the evolution operator along the Keldysh contour

$$S(\infty) = T_C \exp \left\{ -i \int_C dt H_T(t) \right\}. \quad (20.15)$$

Note that for the general class of systems considered here, this Greens function is a $4N_D \times 4N_D$ matrix in Nambu-dot-Keldysh space.

20.2.3 Self energy of the quantum dots

The evolution operator when averaged over the lead degrees of freedom takes the form

$$\langle S(\infty) \rangle_{\text{leads}} = T_C \exp \left[-i \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dt_1 dt_2 \right. \\ \left. \check{d}^\dagger(t_1) \check{\Sigma}_T(t_1, t_2) \check{d}(t_2) \right], \quad (20.16)$$

involving a total self-energy $\check{\Sigma}_T = \sum_j \check{\Sigma}_j$ which also takes the form of a matrix in Nambu-dot-Keldysh space. Each lead self-energy $\check{\Sigma}_j$ can be viewed as a set of Nambu-Keldysh matrices given by

$$[\check{\Sigma}_j]_{\alpha\beta}(t_1, t_2) = \mathcal{T}_{j\alpha}^\dagger(t_1) \tau_z \hat{g}_j(t_1 - t_2) \tau_z \mathcal{T}_{j\beta}(t_2), \quad (20.17)$$

and corresponding to the N_D^2 possible matrix elements in dot space. There, the new set of Pauli matrices $\tau_{x,y,z}$ acts in Keldysh space, and we introduced

$$\hat{g}_j(t-t') = -i \sum_k \left\langle T_C \left\{ \hat{\Psi}_{jk}(t) \hat{\Psi}_{jk}^\dagger(t') \right\} \right\rangle_0, \quad (20.18)$$

as the bare local Green's function of the superconducting lead j at the site of tunneling. It involves the Nambu-Keldysh spinors which collect the Nambu spinors (20.3) evaluated on the two different branches of the Keldysh time contour as

$$\hat{\Psi}_{jk} = \begin{pmatrix} \Psi_{jk}^+ \\ \Psi_{jk}^- \end{pmatrix}. \quad (20.19)$$

In order to carry out some of the upcoming calculations, it is useful to perform a rotation in Keldysh space going from the $+/-$ basis to the so-called RAK basis according to

$$\begin{pmatrix} g_j^R & g_j^K \\ 0 & g_j^A \end{pmatrix} = L \tau_z \hat{g}_j L^{-1} \quad \text{with } L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}. \quad (20.20)$$

This defines a new set of Green's functions corresponding to the retarded (R), advanced (A) and Keldysh (K) components. In the present case of BCS superconductors, these are given by

$$\begin{cases} g_j^{R,A}(\omega) = \pi \nu(0) \frac{\omega + \Delta_j \sigma_x}{i \zeta_j^{R,A}(\omega)}, \\ g_j^K(\omega) = [1 - 2f(\omega)] [g_j^R(\omega) - g_j^A(\omega)], \end{cases} \quad (20.21)$$

where $\nu(0)$ is the density of states of the lead in the normal metal regime at the Fermi level, $f(\omega)$ is the Fermi distribution and we introduced the functions

$$\begin{aligned} \zeta_j^{R,A}(\omega) &= \pm \text{sign}(\omega) \sqrt{\omega^2 - \Delta_j^2} \Theta(|\omega| - \Delta_j) \\ &\quad + i \sqrt{\Delta_j^2 - \omega^2} \Theta(\Delta_j - |\omega|). \end{aligned} \quad (20.22)$$

20.2.4 Current statistics

Current operator and average

The current operator from QD α into the lead j reads

$$I_{j\alpha}(t) = i \sum_k \Psi_{jk}^\dagger \sigma_z \mathcal{T}_{j\alpha}(t) d_\alpha + \text{H.c.} \quad (20.23)$$

As the average current does not depend on the branch of the Keldysh contour, it is convenient to introduce counting fields $\eta_{j\alpha}(t)$ in the tunneling amplitudes according to

$$\mathcal{T}_{j\alpha}(t) \rightarrow \mathcal{T}_{j\alpha}(t) e^{i\sigma_z \tau_z \eta_{j\alpha}(t)/2}, \quad (20.24)$$

so that the evolution operator becomes $S(\infty) \rightarrow S(\infty, \eta)$. The average current can then be computed through the functional differentiation

$$\langle I_{j\alpha} \rangle(t) = i \frac{1}{Z[0]} \left. \frac{\delta Z[\eta]}{\delta \eta_{j\alpha}(t)} \right|_{\eta=0}, \quad (20.25)$$

where $Z[\eta] = \langle S(\infty, \eta) \rangle_0$. Performing the differentiation explicitly, we obtain a Meir-Wingreen type formula [17] for the average current as the following $\alpha\alpha$ diagonal element in dot space

$$\langle I_{j\alpha} \rangle(t) = \frac{1}{2} \text{Tr}^{(NK)} \left\{ \sigma_z \tau_z \int_{-\infty}^{+\infty} dt' \left[\check{G}(t, t') \check{\Sigma}_j(t', t) - \check{\Sigma}_j(t, t') \check{G}(t', t) \right]^{\alpha\alpha} \right\}, \quad (20.26)$$

where $\text{Tr}^{(NK)}$ denotes the trace in Nambu-Keldysh space.

Current correlations

In full generality, we need to compute the unsymmetrized current-current correlator defined as

$$S_{i\alpha, j\beta}(t, t') = \langle I_{i\alpha}(t) I_{j\beta}(t') \rangle - \langle I_{i\alpha}(t) \rangle \langle I_{j\beta}(t') \rangle. \quad (20.27)$$

A convenient way of doing so consists in introducing new counting fields [13] $\eta_{j\alpha s}(t)$ where $s = \pm$ now specifies the branch of the Keldysh contour. The tunneling amplitudes are then redefined following this prescription as

$$\mathcal{T}_{j\alpha}(t) \rightarrow \mathcal{T}_{j\alpha}(t) e^{i\sigma_z \sum_s \pi_s \eta_{j\alpha s}(t)}, \quad (20.28)$$

where we defined the following matrices in Keldysh space to project onto a given branch of the contour

$$\pi_{\pm} = \frac{\tau_z \pm 1}{2}. \quad (20.29)$$

The current correlations are then computed through second order functional differentiation as

$$\langle I_{i\alpha}^-(t) I_{j\beta}^+(t') \rangle = -\frac{1}{Z[0]} \frac{\delta^2 Z[\eta]}{\delta \eta_{i\alpha-}(t) \delta \eta_{j\beta+}(t')} \Big|_{\eta=0}, \quad (20.30)$$

where $Z[\eta] = \langle S(\infty, \eta) \rangle_0$. Performing this differentiation, using Wick theorem, and carrying out the partial trace over Keldysh space, it can eventually be expressed in terms of RAK components as [18]

$$\begin{aligned} S_{i\alpha, j\beta}(t, t') &= -\frac{1}{2} \text{Re} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \\ &\times \text{Tr}^{(N)} \left\{ \sigma_z \left(\check{\Sigma}_i^K \check{G}^A + \check{\Sigma}_i^R \check{G}^K - \check{\Sigma}_i^A \check{G}^A + \check{\Sigma}_i^R \check{G}^R \right)_{(t, t_1) \circ (t_1, t')}^{\alpha\beta} \sigma_z \left(\check{\Sigma}_j^K \check{G}^A + \check{\Sigma}_j^R \check{G}^K + \check{\Sigma}_j^A \check{G}^A - \check{\Sigma}_j^R \check{G}^R \right)_{(t', t_2) \circ (t_2, t)}^{\beta\alpha} \right. \\ &- \sigma_z \left(\check{\Sigma}_i^R \check{G}^R \check{\Sigma}_j^K + \check{\Sigma}_i^K \check{G}^A \check{\Sigma}_j^A + \check{\Sigma}_i^R \check{G}^K \check{\Sigma}_j^A - \check{\Sigma}_i^A \check{G}^A \check{\Sigma}_j^A + \check{\Sigma}_i^R \check{G}^R \check{\Sigma}_j^R \right)_{(t, t_1) \circ (t_1, t_2) \circ (t_2, t')}^{\alpha\beta} \sigma_z \left(\check{G}^K + \check{G}^A - \check{G}^R \right)_{(t', t)}^{\beta\alpha} \\ &- \frac{\delta_{ij}}{4} \text{Tr}^{(N)} \left\{ \sigma_z \left(\check{\Sigma}_i^A - \check{\Sigma}_i^R - \check{\Sigma}_i^K \right)_{(t, t')}^{\alpha\beta} \sigma_z \left(\check{G}^A - \check{G}^R + \check{G}^K \right)_{(t', t)}^{\beta\alpha} \right. \\ &\quad \left. + \sigma_z \left(-\check{G}^A + \check{G}^R + \check{G}^K \right)_{(t, t')}^{\alpha\beta} \sigma_z \left(-\check{\Sigma}_i^A + \check{\Sigma}_i^R - \check{\Sigma}_i^K \right)_{(t', t)}^{\beta\alpha} \right\}, \quad (20.31) \end{aligned}$$

where $\circ$ stands for a convolution product in time.

Note that as the self energies are proportional to squares of tunneling amplitudes, the auto-correlation noise always dominates with respect to the crossed correlation noise at low

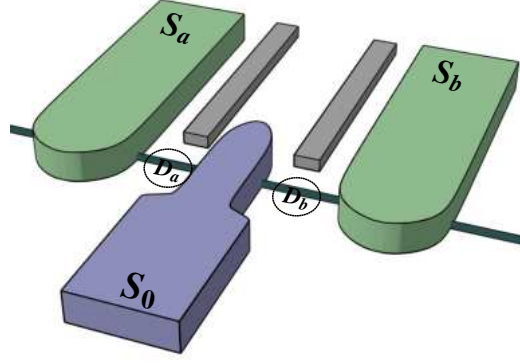


Figure 55: Three superconductors designed in an ASCPBS. [12] The central electrode S_0 is grounded while the lateral ones S_a and S_b are biased with voltages V_a and V_b . Two quantum dot nanowires D_a and D_b , with energies ϵ_a and ϵ_b which can be tuned by gates (in gray), bridge the central superconductor to the two lateral ones. The distance between the two dots is comparable to the coherence length.

transparencies: the δ_{ij} term is the dominant one in Eq. (20.31) when lowest order perturbation theory is operated.

This concludes the first task of this paper. The current and noise have been expressed in terms of the dressed Green's function matrix elements, for an arbitrary mesoscopic system composed of an array of quantum dots coupled in full generality to a set of superconducting or normal metal leads in the spirit of a generalized Fisher-Lee formula[14] applicable to (time dependent) superconducting systems. In order to make further progress, one needs to solve the corresponding Dyson's equation numerically for a specific device.

If the system contains only a single superconductor, only (single) Andreev reflection and quasiparticle transmission to/from this superconductor can occur: a stationary current flows in this system, and current correlations depend only on the time difference $t - t'$. The spectral density of noise at finite frequencies can be directly obtained by a Fourier transform with respect to $t - t'$. The results of Refs. [13, 19], which consider the noise crossed correlation is the CPBS belong to this category.

However, the next goal of this work is precisely to consider systems with 3 superconductors, where the currents flowing in the device contain ac and dc components, and where the real time current correlator depends separately on the two times t and t' .

20.3 Application to Multiple Cooper pair resonances

In this section, we study the setup presented in Fig. 55 for multiple Cooper pair splitting in a central superconductor, and recombination (as Cooper pairs) in two other (voltage biased) superconducting leads. We thus consider a central grounded superconducting electrode S_0 coupled to two lateral superconducting leads S_a and S_b via two quantum dots (QDs) D_a and D_b as illustrated in Fig. 55. All superconductors labeled $j = 0, a, b$ are described by BCS theory with a gap energy Δ_j . Each QD labeled $\alpha = a, b$ characterized by an energy level ϵ_α is coupled to the central lead and to (only) one lateral superconductor: the tunneling amplitude between lead j and QD α is denoted as $t_{j\alpha}$ and the direct inter-dot coupling is neglected ($t_{ab} = t_{ba}^* = 0$) for simplicity, and to optimize CAR processes. The two lateral leads are biased with a voltage V_j measured with respect to the chemical potential of the central superconducting electrode with $V_0 = 0$. The width of S_0 is assumed to be smaller than the superconducting coherence length, so that CAR processes can operate, splitting pairs from S_0 and distributing electrons on both QDs (as Cooper pairs).

20.3.1 Historical perspective

Early works [20, 21, 22, 23] attempting to describe the transport properties of three-terminal all-superconducting junctions have focused on the so-called incoherent regime. [24] Further investigations made clear that in the coherent regime, CAR processes [25, 26, 27, 28, 29] would allow the correlated motion of CPs originating from all three superconductors, leading to interesting signatures in subgap transport. [12] It was indeed realized that the interference of MAR processes taking place at different interfaces of the three-terminal device would lead to the intriguing possibility of phase-sensitive dissipative transport. The most fascinating feature, however, is the appearance of so-called multiple Cooper pair resonances (MCPR) which resemble some form of Andreev bound states delocalized over all superconductors, leading to a dissipation-less phase-dependent Josephson-like current in a non-equilibrium situation.

The lowest order MCPR results in the entanglement of two CPs, a process referred to as the “quartet” process and first envisioned (at least in these terms) in the equilibrium calculations of Ref. [30]. In another formulation, these resonances were predicted as voltage-induced (fractional) Shapiro steps. [31] Anomalies observed recently in the electronic subgap transport of an all-superconducting device [32, 33] could meet an interpretation in terms of quartet resonances.

Here, we wish to determine the noise characteristics of MCPR. Indeed, the correlations between currents flowing in two different leads (noise crossed correlations) of a multi-terminal setup can be measured in order to probe non-local correlations, such as the ones resulting from CAR processes. In particular, the sign of such current correlations has been used as a way to sharpen our understanding of mesoscopic devices involving superconductors from both the theoretical and experimental point of view. [26, 27, 34, 35, 20, 36, 23] In a setup where a single lead is biased, low voltage positive noise crossed correlations were predicted and ascribed to MARs. [37] In a setup consisting of a single quantum dot connected to three terminals, noise crossed correlations were also investigated using perturbative calculations [38] and quartets were shown to have a decreasing noise signal at low subgap voltages for a non-resonant dot, as opposed to the resonant case where a phase-sensitive noise was predicted. The study of out-of-equilibrium noise in multi-terminal superconducting junctions in the coherent regime is particularly interesting, as there is still a need to quantify the evolution of current correlations between different terminals, as a function of the phase and the voltage biases. In fact, the measurement of positive noise crossed correlations in three-terminal all-superconducting devices [39] has been reported recently, in agreement with the differential conductance anomaly ascribed to quartets.

20.3.2 Short description of MCPR physics

Three-terminal devices involving a superconducting (S) source connected to two normal (N) metallic leads have been extensively studied, [25, 26, 27, 28, 29, 40] mostly because they offer the possibility to generate two-particle entanglement [7, 8, 41, 42] by extracting a split Cooper pair from the BCS condensate, which granted them the name “N-S-N Cooper pair splitters”. Microscopically, Cooper pair splitting is ensured by the process of crossed Andreev reflection (CAR) which allows an ongoing hole in one normal lead to be reflected as an outgoing electron in the other N lead, making use of the (evanescent) quasiparticle states in S, provided that the separation between the metallic reservoirs is smaller than the coherence length of the superconducting material. Such devices were realized experimentally, with convincing evidence of both nonlocal current and noise. [43, 44, 45, 46, 47, 48, 49, 35]

The concept of Cooper pair splitting can be generalized to an all-superconducting device [30] where an external voltage bias is applied to the lateral superconducting leads, while the central one is grounded. In such a system, single-particle conduction inside the outgoing biased leads is prohibited for subgap voltages. However, CAR processes can still operate and, for properly

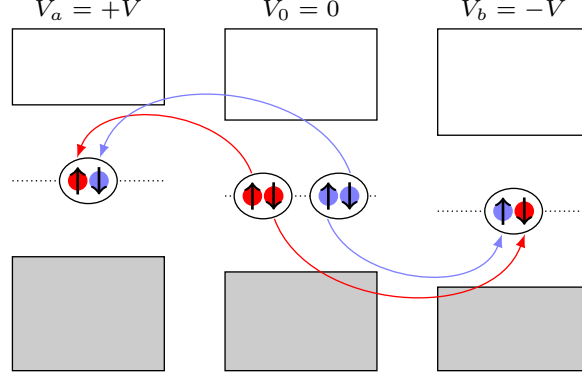


Figure 56: Energy diagram of the quartet production mechanism. For opposite applied voltages $V_a = -V_b = V$, a process involving two crossed Andreev reflections splits two Cooper pairs from the central grounded electrode, leading to the formation of a four fermion entangled state, with two pairs emitted in the lateral superconductors, one in S_a and one in S_b .

selected applied voltages, lead to interesting phenomena. Indeed, focusing first on the case of opposite potential biases for simplicity, one can envision a process by which two Cooper pairs originating from the central lead are split into entangled nonlocal pairs by virtue of a double crossed Andreev reflection, their constituent electrons ultimately recombining as newly formed Cooper pairs in the lateral leads. Such a process, shown schematically in Fig. 56, therefore leads to the coherent transfer of two Cooper pairs, relying on a combination of both direct and crossed Andreev reflections. It ultimately corresponds to the formation of a correlated four fermion state, an object sometimes referred to as “nonlocal quartet”. More importantly, this is an energy-conserving process which results in signatures in the DC current, which turn out to bear a nontrivial phase and voltage dependence.

Such a process can be extended to involve any even number of CAR processes, a key requirement for the split pairs originating from the central electrode to get recombined in the lateral leads. Quartets are thus easily generalizable to higher-order multiple Cooper pair resonances (MCPR) implicating the phase-coherent transport of $n + m$ pairs from S_0 being transferred as n pairs to S_a and m pairs to S_b . This, in turn, leads to the appearance of DC Josephson resonances at commensurate voltages, thus satisfying

$$nV_a + mV_b = 0. \quad (20.32)$$

The existence of the quartet resonance, and more generally of the MCPR, can be inferred from a simple phase argument, as proposed in Ref. [12], resulting in the commensurability condition, Eq. (20.32), to be satisfied by the applied voltages. In an equilibrium setup, the current-phase relation is obtained from differentiating the Josephson free energy with respect to the superconducting phases φ_j associated with each lead S_j .

When external voltages V_a and V_b are applied to the lateral leads S_j ($j = a, b$), the superconducting phases acquire a time dependence, namely $\varphi_a(t) = \varphi_a + 2V_a t$, and $\varphi_b(t) = \varphi_b + 2V_b t$, while we assumed $\varphi_0 = 0$ for the grounded central electrode, so that $\varphi_0(t) = 0$. It follows that focusing on the currents I_j in the lateral leads S_j ($j = a, b$), one is left with

$$I_j(t) = \sum_{p,q \in \mathbb{Z}} I_{j,pq} \sin [p\varphi_a(t) + q\varphi_b(t)]. \quad (20.33)$$

In the special case where the applied voltages are commensurate, i.e. there exists a pair of integers (n, m) such that $nV_a + mV_b = 0$, one readily sees that the currents $I_j(t)$ now contain a term which is constant in time, corresponding to the component $\{p = n, q = m\}$ along with its higher order harmonics. This results in a pure DC component, signaling the existence of a

multiple Cooper pair resonance. It depends on the combination of the bare phases $n\varphi_a + m\varphi_b$, which corresponds to the transfer of $n + m$ Cooper pairs from S_0 , outgoing as n pairs in S_a and m in S_b .

The quartet resonance discussed in this study constitutes the “lowest order” MCPR, observed for $n = m = 1$, which in a truncated perturbative treatment in the tunnel amplitudes, constitutes the dominant MCPR. Here, our application to this quartet process is by no means perturbative, and it allows to access transport regimes at high transparency through all junctions. In Ref. [12], the authors performed a systematic study of the amplitude of the DC Josephson current for such higher order processes. However, when the integers m and n identifying the MCPR are instead chosen to be large, as well as when high transparencies are specified, one is limited by numerical power. This is the reason why, as a first application of our noise diagnosis, we focus on the quartet case.

20.3.3 Expressing the current and noise at a MCPR

Double Fourier representation and Dyson equation

When arbitrary voltages V_a and V_b are applied to the lateral superconducting leads while keeping the central superconducting electrode grounded, two Josephson frequencies $2|V_a|$ and $2|V_b|$ govern the system. In general, they are independent and the QD Green’s function is a function of two times t_1 and t_2 or alternatively of $\tau = t_1 - t_2$ and $t = (t_1 + t_2)/2$. However, when the applied voltages are commensurate, $nV_a + mV_b = 0$ with n, m integers, the average time variable t becomes periodic, with period $T = \pi|m/V_a| = \pi|n/V_b| \equiv 2\pi/\omega_J$ thus defining an effective Josephson frequency ω_J . Note that this condition corresponds precisely to the one allowing for the appearance of MCPR. It then becomes possible to describe the bare and dressed QD Green’s functions, $\check{G}_0$ and $\check{G}$, along with the self energies $\check{\Sigma}_j$, using a single frequency and two harmonics indexes representing harmonics of the Josephson frequency ω_J , in the spirit of Floquet theory. [15] This specific double Fourier transform representation thus allows us to write all Green’s functions as frequency-dependent matrices in harmonics space (on top of their Nambu-dot-Keldysh structure)

$$\begin{aligned} \check{G}_{nm}(\omega) &= \frac{\omega_J}{2\pi} \int dt_1 e^{i(\omega + n\omega_J)t_1} \\ &\times \int dt_2 e^{-i(\omega + m\omega_J)t_2} \check{G}(t_1, t_2). \end{aligned} \quad (20.34)$$

This additional matrix structure in harmonics space offers an important advantage over other formulations as it allows to write the Dyson equation as a simple matrix inversion (albeit in the rather large Nambu-dot-Keldysh-harmonics space)

$$\check{G}(\omega)^{-1} = \check{G}_0(\omega)^{-1} - \check{\Sigma}_T(\omega), \quad (20.35)$$

which is translated into *RAK* components as

$$\tilde{G}^{R/A}(\omega)^{-1} = \tilde{G}_0^{R/A}(\omega)^{-1} - \tilde{\Sigma}_T^{R/A}(\omega), \quad (20.36)$$

$$\tilde{G}^K(\omega) = \tilde{G}_0^K(\omega) + \tilde{G}^R(\omega) \tilde{\Sigma}_T^K(\omega) \tilde{G}^A(\omega). \quad (20.37)$$

The bare Green’s function is not only diagonal in Keldysh space (expressed in the *RAK* basis) as $\tilde{G}_0^K = 0$, but also in harmonics space as, when expressed in the time domain, it only depends on the time difference $\tau = t_1 - t_2$. These diagonal elements can be written as Nambu-dot matrices taking the form

$$\left[\tilde{G}_0^{R/A}(\omega)^{-1} \right]_{nm} = \delta_{nm} \begin{pmatrix} \omega + n\omega_J - \epsilon_a \sigma_z & -t_{ab} \sigma_z \\ -t_{ba} \sigma_z & \omega + n\omega_J - \epsilon_b \sigma_z \end{pmatrix}. \quad (20.38)$$

Similarly, the lead self-energies can be obtained in this enlarged space using Eqs. (20.17), (20.21)-(20.34), and are given by: [50]

$$[\check{\Sigma}_j(\omega)]_{nm} = \Gamma_j \begin{pmatrix} \delta_{nm} \hat{X}_j(\omega + n\omega_J - V_j) & \delta_{n-2V_j/\omega_J, m} \hat{Y}_j(\omega + n\omega_J - V_j) \\ \delta_{n+2V_j/\omega_J, m} \hat{Y}_j(\omega + n\omega_J + V_j) & \delta_{nm} \hat{X}_j(\omega + n\omega_J + V_j) \end{pmatrix}, \quad (20.39)$$

where Γ_j is a matrix in dot space with matrix elements $\Gamma_{j\alpha\beta} = \pi\nu(0)t_{j\alpha}^*t_{j\beta}$, and where $\hat{X}_j$ and $\hat{Y}_j$ are matrices in Keldysh space, with components expressed in the RAK basis as

$$\begin{cases} X_j^{R/A}(\omega) = -\frac{\Theta(\Delta_j - |\omega|)\omega}{\sqrt{\Delta_j^2 - \omega^2}} \mp i \frac{\Theta(|\omega| - \Delta_j)|\omega|}{\sqrt{\omega^2 - \Delta_j^2}}, \\ X_j^K(\omega) = -2i \frac{\Theta(|\omega| - \Delta_j)|\omega|}{\sqrt{\omega^2 - \Delta_j^2}} \tanh \frac{\beta\omega}{2}, \\ Y_j^{R,A,K}(\omega) = -\Delta_j \frac{X_j^{R,A,K}(\omega)}{\omega}. \end{cases} \quad (20.40)$$

In all generality, an exact description of the problem would require an infinite number of harmonics. In practice, we can restrict ourselves to a finite subset by introducing a cutoff energy E_c , which needs to be much larger than any relevant energy scale of the problem (typically a few times the largest superconducting gap). The dressed dot Green's function $\tilde{G}^{R,A,K}$ is then obtained numerically from Eq. (20.36), through the inversion of large matrices. The typical size of these objects is dictated by the applied voltage biases through the Josephson frequency, as one needs $N \sim E_c/\omega_J$ frequency domains of size ω_J to properly cover the range of energy up to the cutoff E_c . This makes the handling of low-voltage situations particularly time-consuming, and numerically challenging because of resonances which require to perform integrals with a finer resolution on increasing size intervals.

Note that all analytical results can directly be applied to more general multi-terminal superconducting devices, provided that the MCPR condition is satisfied between potential differences with respect to the ground. This is also true for the results of the section below, which are not sample specific.

Current harmonics and noise in frequency space

As argued earlier, multiple Cooper pair resonances appear when commensurate voltages are applied to the lateral superconducting leads, a condition which also allows to recover a periodic behavior of the transport properties in terms of a single (effective) Josephson frequency ω_J . When the MCPR condition is satisfied, the current admits a Fourier series expansion of the form

$$\langle I_{j\alpha} \rangle(t) = \sum_{p \in \mathbb{Z}} e^{-ip\omega_J t} \mathcal{I}_{j\alpha}^p, \quad (20.41)$$

with Fourier coefficients given by [12]

$$\mathcal{I}_{j\alpha}^p = \frac{1}{2} \text{Tr}^{(NK)} \left\{ \sigma_z \tau_z \int_0^{\omega_J} \frac{d\omega}{2\pi} \sum_n \left[\check{G}(\omega) \check{\Sigma}_j(\omega) - \check{\Sigma}_j(\omega) \check{G}(\omega) \right]_{n, n-p}^{\alpha\alpha} \right\}. \quad (20.42)$$

Performing the partial trace over Keldysh space, it can be expressed in terms of RAK components as

$$\mathcal{I}_{j\alpha}^p = \frac{1}{2} \text{Tr}^{(N)} \left\{ \sigma_z \int_0^{\omega_J} \frac{d\omega}{2\pi} \sum_n \left[\tilde{G}^R(\omega) \tilde{\Sigma}_j^K(\omega) + \tilde{G}^K(\omega) \tilde{\Sigma}_j^A(\omega) - \tilde{\Sigma}_j^R(\omega) \tilde{G}^K(\omega) - \tilde{\Sigma}_j^K(\omega) \tilde{G}^A(\omega) \right]_{n, n-p}^{\alpha\alpha} \right\}. \quad (20.43)$$

From the definition of the current-current correlations, Eq. (20.27), one can introduce the finite-frequency noise, calculating the Fourier transform with respect to the time difference as

$$S_{i\alpha,j\beta}^+(\Omega, t) \equiv \int_{-\infty}^{+\infty} d\tau e^{i\Omega\tau} S_{i\alpha,j\beta}(t, t + \tau), \quad (20.44)$$

$$S_{i\alpha,j\beta}^-(\Omega, t) \equiv \int_{-\infty}^{+\infty} d\tau e^{i\Omega\tau} S_{i\alpha,j\beta}(t + \tau, t), \quad (20.45)$$

which correspond to the emission and absorption noise. In inductive coupling schemes for the detection of noise, [51, 52] at low temperatures for both the device under study and the detector, the emission noise typically dominates.

Under the MCPR condition these Fourier-transformed correlators recover a periodicity in their average time variable. Like the currents, they contain all harmonics of the Josephson frequency, which motivates the computation of the following time averages [5] corresponding to the emission and absorption noise:

$$\bar{S}_{i\alpha,j\beta}^+(\Omega) \equiv \frac{\omega_J}{2\pi} \int_0^{2\pi/\omega_J} dt S_{i\alpha,j\beta}^+(\Omega, t), \quad (20.46)$$

$$\bar{S}_{i\alpha,j\beta}^-(\Omega) \equiv \frac{\omega_J}{2\pi} \int_0^{2\pi/\omega_J} dt S_{i\alpha,j\beta}^-(\Omega, t). \quad (20.47)$$

It is clear from these expressions that emission and absorption noises are trivially related when flipping the sign of the probing frequency, namely $\bar{S}_{i\alpha,j\beta}^+(\Omega) = \bar{S}_{i\alpha,j\beta}^-(-\Omega)$. A theory for the detection of photo-assisted finite frequency shot noise has been presented in Ref. [53] with an application to the fractional quantum Hall effect, and it is directly applicable to superconducting systems.

Substituting the expression for the current correlator, Eq.(20.31), in terms of the dots Green's functions and leads self-energies back into Eqs. (20.46)-(20.47), and focusing on the behavior at zero frequency, one obtains, after performing all 4 time integrals explicitly

$$\begin{aligned} \bar{S}_{i\alpha,j\beta}(\Omega = 0) = & -\frac{(2\pi)^3}{2} \text{Re} \int_0^{\omega_J} d\omega \text{Tr}^{(NH)} \left\{ \sigma_z \left(\tilde{\Sigma}_i^K \tilde{G}^A + \tilde{\Sigma}_i^R \tilde{G}^K - \tilde{\Sigma}_i^A \tilde{G}^A + \tilde{\Sigma}_i^R \tilde{G}^R \right)_\omega^{\alpha\beta} \right. \\ & \times \sigma_z \left(\tilde{\Sigma}_j^K \tilde{G}^A + \tilde{\Sigma}_j^R \tilde{G}^K + \tilde{\Sigma}_j^A \tilde{G}^A - \tilde{\Sigma}_j^R \tilde{G}^R \right)_\omega^{\beta\alpha} \\ & - \sigma_z \left(\tilde{\Sigma}_i^R \tilde{G}^R \tilde{\Sigma}_j^K + \tilde{\Sigma}_i^K \tilde{G}^A \tilde{\Sigma}_j^A + \tilde{\Sigma}_i^R \tilde{G}^K \tilde{\Sigma}_j^A - \tilde{\Sigma}_i^A \tilde{G}^A \tilde{\Sigma}_j^A + \tilde{\Sigma}_i^R \tilde{G}^R \tilde{\Sigma}_j^R \right. \\ & \left. \left. \times \sigma_z \left(\tilde{G}^K + \tilde{G}^A - \tilde{G}^R \right)_\omega^{\beta\alpha} \right\} \right. \\ & - \frac{2\pi\delta_{ij}}{4} \text{Re} \int_0^{\omega_J} d\omega \text{Tr}^{(NH)} \left\{ \sigma_z \left(\tilde{\Sigma}_i^A - \tilde{\Sigma}_i^R - \tilde{\Sigma}_i^K \right)_\omega^{\alpha\beta} \sigma_z \left(\tilde{G}^A - \tilde{G}^R + \tilde{G}^K \right)_\omega^{\beta\alpha} \right. \\ & \left. + \sigma_z \left(-\tilde{G}^A + \tilde{G}^R + \tilde{G}^K \right)_\omega^{\alpha\beta} \sigma_z \left(-\tilde{\Sigma}_i^A + \tilde{\Sigma}_i^R - \tilde{\Sigma}_i^K \right)_\omega^{\beta\alpha} \right\}, \quad (20.48) \end{aligned}$$

where we dropped the superscript $+/-$ for $\bar{S}_{i\alpha,j\beta}$ since the two lead to the same result at zero frequency and we introduced $\text{Tr}^{(NH)}$ as the trace in Nambu-harmonics space.

It is also possible to obtain compact expressions when the noise is evaluated at harmonics

$l\omega_J$ (l integer) of the Josephson frequency

$$\begin{aligned}
\bar{S}_{i\alpha,j\beta}^-(l\omega_J) = & -\frac{(2\pi)^3}{2} \text{Re} \sum_{n,p=-\infty}^{+\infty} \int_0^{\omega_J} d\omega \text{Tr}^{(N)} \left\{ \sigma_z \left(\tilde{\Sigma}_i^K \tilde{G}^A + \tilde{\Sigma}_i^R \tilde{G}^K - \tilde{\Sigma}_i^A \tilde{G}^A + \tilde{\Sigma}_i^R \tilde{G}^R \right)_\omega^{\alpha,\beta;n,p+l} \right. \\
& \times \sigma_z \left(\tilde{\Sigma}_j^K \tilde{G}^A + \tilde{\Sigma}_j^R \tilde{G}^K + \tilde{\Sigma}_j^A \tilde{G}^A - \tilde{\Sigma}_j^R \tilde{G}^R \right)_\omega^{\beta,\alpha;p,n-l} \\
& - \sigma_z \left(\tilde{\Sigma}_i^R \tilde{G}^R \tilde{\Sigma}_j^K + \tilde{\Sigma}_i^K \tilde{G}^A \tilde{\Sigma}_j^A + \tilde{\Sigma}_i^R \tilde{G}^K \tilde{\Sigma}_j^A - \tilde{\Sigma}_i^A \tilde{G}^A \tilde{\Sigma}_j^A + \tilde{\Sigma}_i^R \tilde{G}^R \tilde{\Sigma}_j^R \right)_\omega^{\alpha,\beta;n,p+l} \\
& \left. \times \sigma_z \left(\tilde{G}^K + \tilde{G}^A - \tilde{G}^R \right)_\omega^{\beta,\alpha;p,n-l} \right\} \\
& - \frac{2\pi\delta_{ij}}{4} \text{Re} \sum_{n,p=-\infty}^{+\infty} \int_0^{\omega_J} d\omega \text{Tr}^{(N)} \left\{ \sigma_z \left(\tilde{\Sigma}_i^A - \tilde{\Sigma}_i^R - \tilde{\Sigma}_i^K \right)_\omega^{\alpha,\beta;n,p+l} \sigma_z \left(\tilde{G}^A - \tilde{G}^R + \tilde{G}^K \right)_\omega^{\beta,\alpha;p,n-l} \right. \\
& \left. + \sigma_z \left(-\tilde{G}^A + \tilde{G}^R + \tilde{G}^K \right)_\omega^{\alpha,\beta;p,n-l} \sigma_z \left(-\tilde{\Sigma}_i^A + \tilde{\Sigma}_i^R - \tilde{\Sigma}_i^K \right)_\omega^{\beta,\alpha;p,n-l} \right\}.
\end{aligned} \tag{20.49}$$

Eqs. (20.43) and (20.49) for the Josephson current and the noise deserve some attention. They are not specific to the dot geometry, lead configuration, or tunnel amplitudes whatsoever. The self energies are known [see Eq. (20.39)], and the matrix elements of the dot Green's function need “only” to be solved via Dyson's equation, with a recursive scheme, or by a direct matrix inversion.

For the application of this study, our priority is to investigate both signals at zero frequency. Nevertheless, we stress out that in past works,[15] the Josephson current harmonics have provided a useful test for the presence of decoherence effects. Indeed, in a superconducting junction with an embedded quantum dot side-coupled to a normal metal reservoir (a source of decoherence) non zero harmonics of the Josephson current are suppressed when the coupling to this normal metal reservoir is increased. Here, we point out that we can achieve a further characterization of the transport characteristics of the device via the noise harmonics, with the potential to incorporate the monitoring of decoherence effects.

20.3.4 Noise characteristics of the quartet resonance

To be more specific, we focus on the quartet resonance $V_a = -V_b = V > 0$. Moreover, we adopt an antisymmetric position for the dots $\epsilon_a = -\epsilon_b = \epsilon > 0$ to optimize CP splitting. [13] We assume identical superconducting gap for all leads, $\Delta_j = \Delta$, and symmetric couplings between superconductors and dots $\Gamma_{j\alpha\beta} = \Gamma$ while forbidding interdot tunneling, $t_d = 0$. The quartet phase $\varphi_Q = \varphi_a + \varphi_b - 2\varphi_0$ is monitored through $\varphi_b = \varphi_Q$, the two others being set to zero, $\varphi_0 = \varphi_a = 0$. We present results for the DC currents in the leads $I_a \equiv \mathcal{I}_{aa}^0$ and $I_b \equiv \mathcal{I}_{bb}^0$ (cf. Eq. (20.43)) as well as for the zero-frequency correlations of these currents $S_{aa} \equiv \bar{S}_{aa,aa}(\Omega = 0)$, $S_{bb} \equiv \bar{S}_{bb,bb}(\Omega = 0)$ and $S_{ab} \equiv \bar{S}_{aa,bb}(\Omega = 0)$ (cf. Eq. (20.48)). S_{aa} and S_{bb} are referred to as noise autocorrelations whereas S_{ab} are the noise crossed correlations. Fano factors for the lateral leads are given by the standard definition:

$$\mathcal{F}_j = \frac{S_{jj}}{2I_j} \quad \text{for } j = a, b. \tag{20.50}$$

Similarly, a Fano factor can be defined for the central lead from the sum of the currents, leading to meaningful information about noise crossed correlations

$$\mathcal{F}_0 = \frac{S_{aa} + S_{bb} + 2S_{ab}}{2(I_a + I_b)}. \tag{20.51}$$

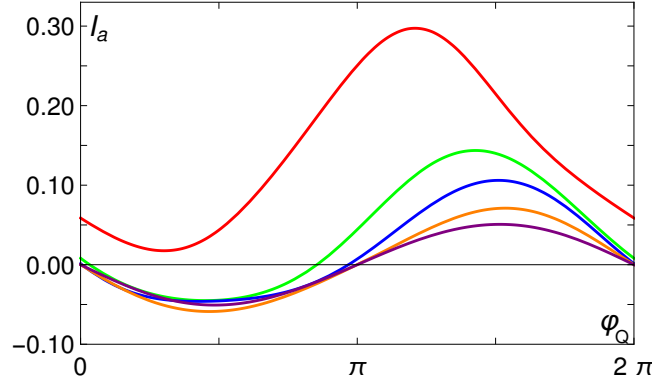


Figure 57: Current I_a (in units of Δ) as a function of the phase φ_Q in the QPC regime, for voltages $V = 0.65\Delta, 0.55\Delta, 0.45\Delta, 0.35\Delta, 0.17\Delta$ (in order of decreasing amplitude). Current I_b can be deduced from $I_b(-\varphi_Q) = -I_a(\varphi_Q)$.

The properties of the current in this system were studied by some of the authors in Ref. [[12]]. It was shown that the current can be broken down into three qualitatively different components. First the quasiparticle current I^{qp} , due to the multiple Andreev processes, which is independent of the quartet phase. It is an odd function of voltage, and thus contributes to the currents I_a and I_b with an opposite sign since the voltages of leads a and b satisfy $V_a = -V_b$. Then the “phase-MAR” component I^{phMAR} , which is due to interference effects between multiple Andreev reflection and phase dependent processes. While it is odd in voltage like the quasiparticle current, I^{phMAR} is, as suggested by its name, phase-dependent, and an even function of the quartet phase. Finally, the multipair coherent current I^{MP} , which is carried by the exchange of multiple pairs between superconducting leads, is an odd function of the quartet phase. This component is however an even function of the voltages, and thus contributes equally to I_a and I_b .

When the voltage V becomes small enough, the contributions I^{qp} and I^{phMAR} become small compared with the coherent multipair current, which thus dominates for a non-zero quartet phase. This explains why the two currents I_a and I_b become equal in the low-voltage regime, and the current phase relation $I_a(\varphi_Q)$ is that of a π junction, which can be justified by the spin singlet nature of CP pairing. When the voltage is increased, the multiple Andreev processes become more important, and the two currents I_a and I_b start deviating from each other. Within the symmetries that result from the choice of parameters we have used, we always have $I_b(-\varphi_Q) = -I_a(\varphi_Q)$.

In what follows, we investigate two different regimes: (i) the “quantum point contact” (QPC) regime which is obtained for dot energy levels placed outside the gap of the central superconductor $\epsilon > \Delta$ and for large couplings $\Gamma > \Delta$, which imply weak energy filtering from the quantum dots; (ii) the resonant dots regime which is obtained for QD energy levels within the gap $\epsilon < \Delta$ and moderate couplings $\Gamma < \Delta$ which imply efficient energy filtering by the quantum dots.

QPC regime

When the quantum dot energy levels are placed outside the gap of the central superconductor and the effective linewidths of the quantum dots are large compared with this gap, the two S-dot-S junctions behave like adjustable tunnel barriers, which justifies the nomenclature “quantum point contact regime”. This regime constitutes a natural starting point of our investigation as it provides an interesting insight into the physics at play in more complex situations. The results shown here in Figs. 57-62 were obtained for the choice of parameters $\epsilon = 6\Delta$ and $\Gamma = 4\Delta$.

The current I_a , in units of Δ , as a function of the phase φ_Q is shown in Fig. 57 for different

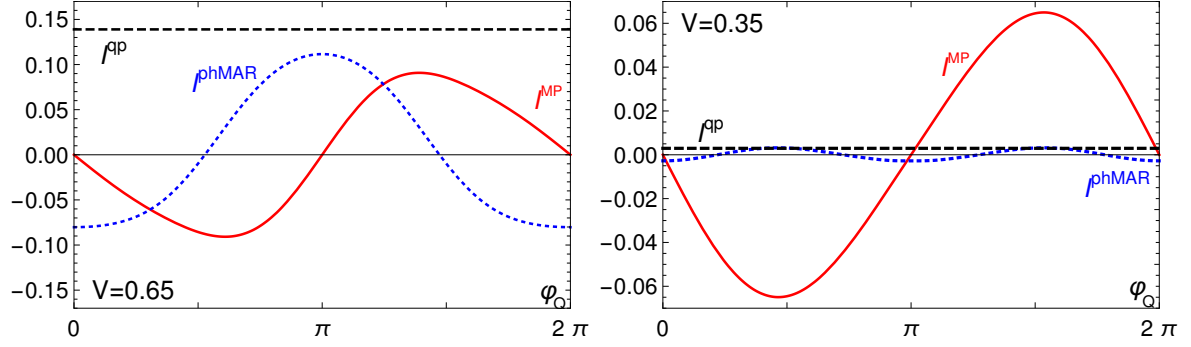


Figure 58: Decomposition of the current I_a in terms of the quasiparticle current I^{qp} , the phase-MAR current I^{phMAR} and the coherent multipair current I^{MP} (all in units of Δ), as a function of the phase φ_Q in the QPC regime, for voltages $V = 0.65\Delta$ (top) and $V = 0.35\Delta$ (bottom). For $V = 0.65\Delta$, the three contributions are of the same order of magnitude. For $V = 0.35\Delta$, the voltage is low enough to have I^{MP} strongly dominate over the other two contributions.

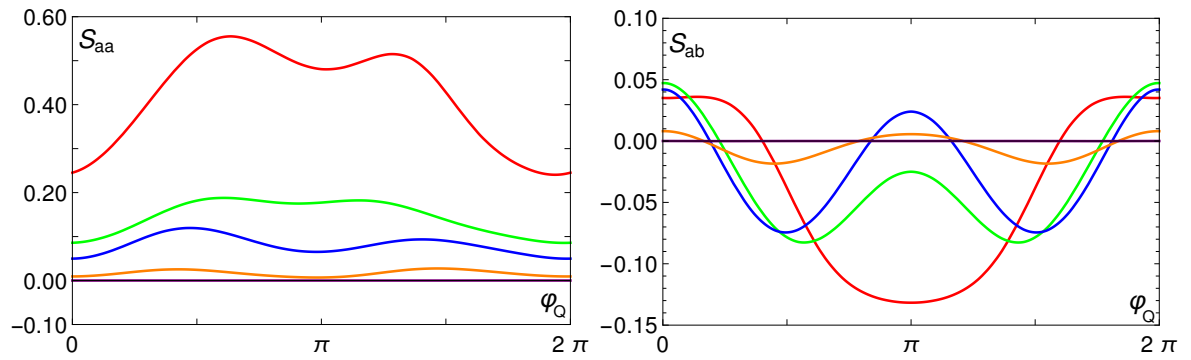


Figure 59: Autocorrelation noise S_{aa} (top) and crossed correlation noise S_{ab} (bottom), given in units of Δ , as a function of the phase φ_Q for voltages $V = 0.65\Delta, 0.55\Delta, 0.45\Delta, 0.35\Delta, 0.17\Delta$ (in order of decreasing amplitude), in the QPC regime. Autocorrelations S_{bb} can be deduced from $S_{bb}(\varphi_Q) = S_{aa}(-\varphi_Q)$.

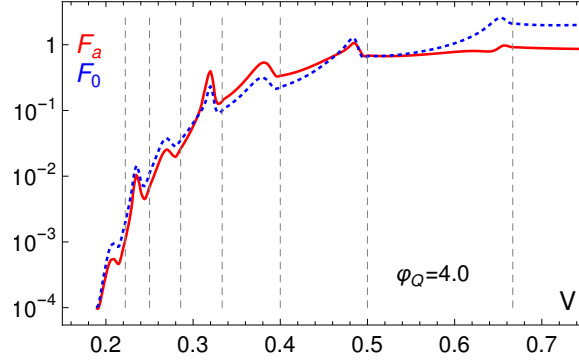


Figure 60: Fano factors $\mathcal{F}_a$ (red, full line) and $\mathcal{F}_0$ (blue, dotted line) in the QPC regime, as a function of voltage, for $\varphi_Q = 4$. Vertical lines indicate the location of the MAR onsets ($V = 2\Delta/n$).

values of the voltage V [the current I_b can be deduced from $I_b(-\varphi_Q) = -I_a(\varphi_Q)$]. For large values of the voltage (the largest here being $V = 0.65\Delta$), the current I_a is positive for all phases, with a non-symmetric phase dependence. As the voltage is lowered, the current decreases in amplitude and tends towards a sinusoidal $\sim \sin(\varphi_Q + \pi)$ dependence. This change of behavior can be understood by decomposing the current into its three contributions:[12]

$$I^{qp} = \frac{1}{2\pi} \int_0^{2\pi} d\varphi_Q I(\varphi_Q) \quad (20.52)$$

$$I^{phMAR} = \frac{1}{2} [I(\varphi_Q) + I(-\varphi_Q)] - I^{qp} \quad (20.53)$$

$$I^{MP} = \frac{1}{2} [I(\varphi_Q) - I(-\varphi_Q)], \quad (20.54)$$

where we recall that I^{qp} is the quasiparticle current due to multiple Andreev processes, I^{phMAR} is the phase-MAR current due to the interference between MAR processes and phase dependent processes, and I^{MP} is the coherent multipair current associated with multiple Cooper pair resonance. Fig. 58 shows the decomposition of the current I_a for two different values of the applied voltage: $V = 0.65\Delta$ and $V = 0.35\Delta$. For the larger applied voltage, the three components are roughly of the same magnitude, which explains the overall positive current, and its non-symmetric phase dependence. However, when V is lowered, the I^{qp} and I^{phMAR} contributions decrease rapidly to zero, while the I^{MP} component converges to a nonzero value, as shown for $V = 0.35\Delta$. Eventually, for low enough voltage, the current is fully due to the multipair component, which tends to a $\sim \sin(\varphi_Q + \pi)$ behavior. By the very nature of these multipair processes (which involve the same number of electrons being transferred from S_0 to S_a , and from S_0 to S_b) the currents I_a and I_b are simply equal.

The noise as a function of the quartet phase is shown in Fig. 59 for different voltages. The autocorrelation noise S_{aa} is naturally positive, and its amplitude decreases rapidly as the voltage is lowered. The crossed correlation noise S_{ab} is mainly negative except for φ_Q close to 0 (and π at low voltage), and its amplitude shows a similar decrease as a function of voltage. Note that the amplitude of the crossed correlations is smaller than that of the autocorrelations, a phenomenon which can be attributed to the local quasiparticle currents. Indeed, the local quasiparticle component of I_a , due to local MAR processes between electrodes a and 0, yields a finite contribution to the autocorrelations S_{aa} , while in the crossed correlations S_{ab} , the local quasiparticle current in I_a and the one in I_b are uncorrelated and thus do not contribute.

Comparing the results of Figs. 57 and 59, it is interesting to point out that as one lowers the external voltage bias, the current assumes a finite value (for non-zero phase φ_Q) while the noise correlations tend to zero, so that the DC current becomes effectively noiseless at

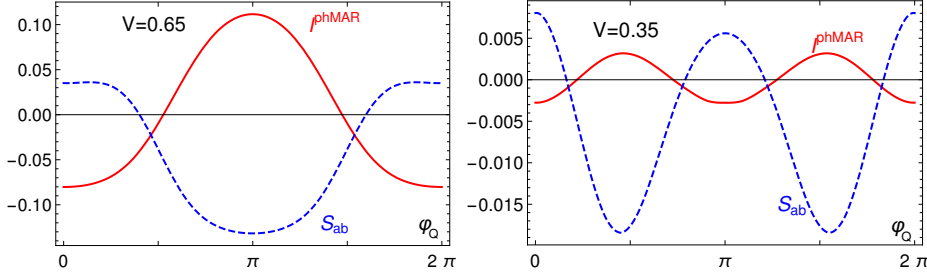


Figure 61: Current component I^{phMAR} (red, full curve), and noise crossed correlations S_{ab} (blue, dashed curve), in units of Δ , as a function of the phase φ_Q , for voltages $V = 0.65\Delta$ (top) and $V = 0.35\Delta$ (bottom).

small voltage, a feature generally attributed to Josephson physics in standard two-terminal superconducting devices at equilibrium. Indeed, as argued above when studying the current, the low-bias transport properties rely uniquely on the exchange of CPs rather than dissipative processes (such as MAR or quasiparticle tunneling) which involve the continuum spectrum of the superconductors, thus explaining the strong similarities with a conventional Josephson junction. This is better illustrated in Fig. 60 which shows the Fano factors $\mathcal{F}_a$ (for current I_a) and $\mathcal{F}_0$ (for the total current $I_a + I_b$) as a function of the voltage bias for a generic phase $\varphi_Q = 4$. The logarithmic scale clearly shows the rapid decrease of the Fano factor as $V \rightarrow 0$. Interestingly, the results of Fig. 60 also indicate the existence of peaks in the Fano factors near the MAR onsets $V = 2\Delta/n$ (vertical dashed lines). While the presence of such structures is reminiscent of what is observed in a biased two-terminal junction separated by a QPC, the overall voltage-dependence is in sharp contrast with this situation as instead of vanishing, the Fano factor of such a device increases dramatically as $1 + \text{Int}(2\Delta/V)$ at low voltages for low transparency, [5] reflecting the increase in the transmitted charge through MAR processes. The observed drop of the Fano factor in our 3-superconductor system thus constitutes direct evidence that the DC signal of quartets is noiseless when lowering the voltage.

Further understanding of the noise behavior as a function of voltage can be obtained using the current decomposition in terms of quasiparticle, phase-MAR and multipair contributions. As argued above, in the low-voltage limit, the noise (both autocorrelations and crossed correlations) goes to zero while the current reduces to its coherent multipair component I^{MP} . From this, it is clear that the multipair component of the current is noiseless. As a result, we can thus expect that the small noise contribution at small to moderate voltage is directly related to the other two components of the current. This is confirmed in Fig. 61, which shows the crossed correlations S_{ab} together with the phase-MAR component I^{phMAR} of the current, for $V = 0.65\Delta$ (top) and $V = 0.35\Delta$ (bottom). Quite strikingly, the phase dependence of the I^{phMAR} and of S_{ab} are directly correlated, which shows that the crossed correlations are indeed associated with I^{phMAR} at least as far as the phase dependence goes (since I^{qp} is phase independent). One can also see that the amplitudes of the current component and of the crossed correlations are somewhat similar. Crossed correlations decrease slower than I^{phMAR} , an effect that may be attributed to the quasiparticle current I^{qp} , which does contribute to the crossed correlations (through processes involving all three superconducting leads).

Finally, the significant drop of the Fano factor at very low voltage, which shows that the multipair process is noiseless, is not restricted to the quartet configuration. This is illustrated in Fig. 62, which presents the Fano factor $\mathcal{F}_a$ as a function of the voltage bias, for three different voltage configurations: $V_b = V_a$, $V_b = -2V_a$ and $V_b = -3V_a$. For each plot, the choice of phase is $\varphi_a = 0$ and $\varphi_b = 4$ (this choice does not impact qualitatively the results). As in Fig. 60 obtained for the quartet configuration, results are shown on a logarithmic scale, and one clearly sees that the Fano factor is strongly reduced at low enough voltage. As noticed for the quartet

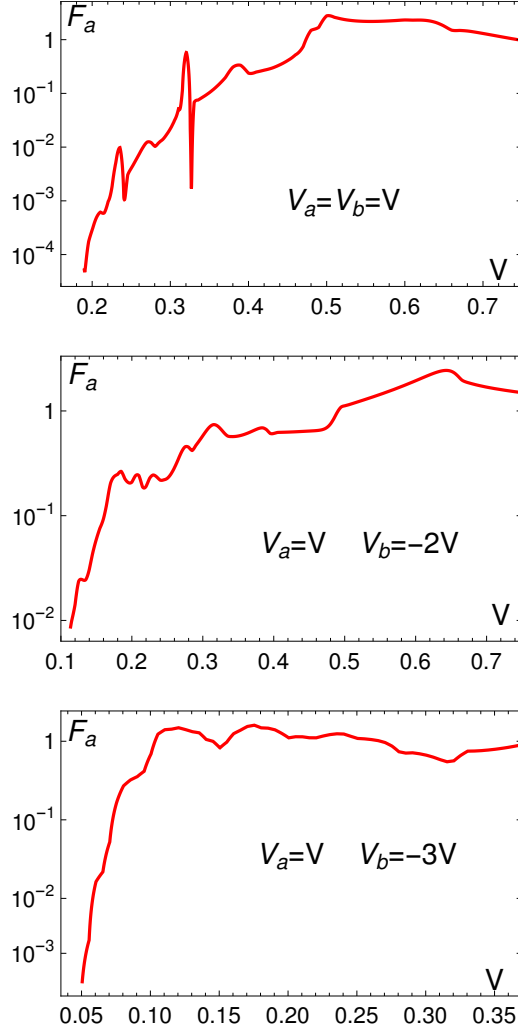


Figure 62: Fano factor $\mathcal{F}_a$ in the QPC regime, as a function of voltage, with $\varphi_a = 0$ and $\varphi_b = 4$, for three different voltages configurations.

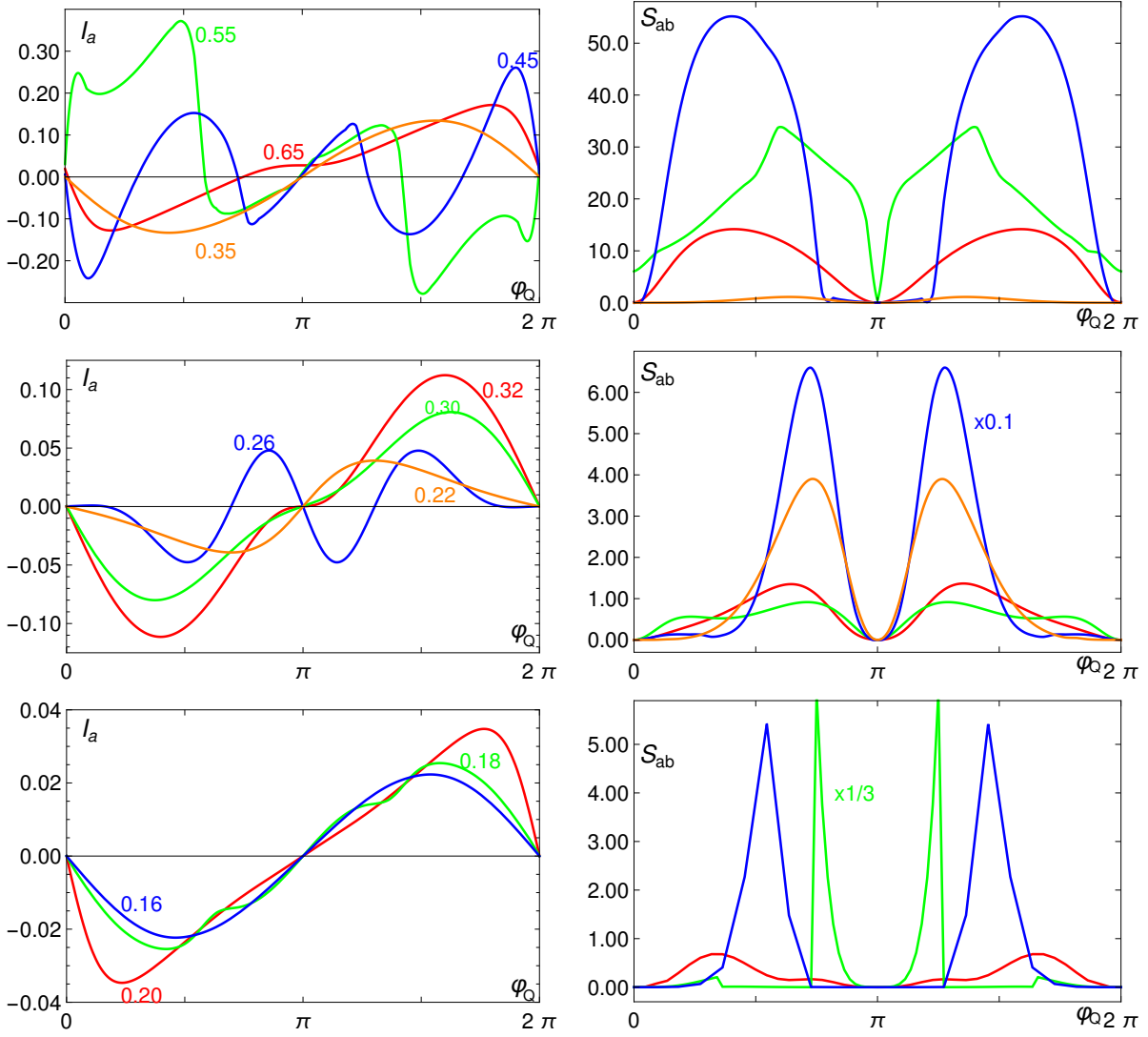


Figure 63: Current and noise correlations (in units of Δ) as a function of the phase φ_Q for different values of voltage (from $V = 0.65$ to $V = 0.16$) in the resonant dots regime. The left column shows the current I_a , with the value of the voltage noted near each curve. The right column shows the crossed correlations S_{ab} for the same values of voltages. Two curves for the S_{ab} have been scaled to fit inside the plot: the one for $V = 0.26$ (middle plot, factor 0.1) and the one for $V = 0.18$ (bottom plot, factor $1/3$).

configuration, this is due to the existence of a finite DC current as $V \rightarrow 0$, while the noise vanishes for $V \rightarrow 0$.

Resonant dots regime

We now present results obtained for the set of parameters $\epsilon = 0.6\Delta$ and $\Gamma = 0.3\Delta$ which constitutes a regime of efficient energy filtering from the dots in the gap of the central superconductor, albeit still allowing multiple Andreev processes to be identified. The main qualitative difference with the QPC regime, which is common for all the investigations led in the resonant dots regime, is the presence of phase sensitive and voltage sensitive giant Fano factors even at low voltage when the Josephson current is fully ascribed to quartet processes.

Fig. 63 shows the phase dependence of the current I_a and the noise crossed correlations S_{ab} for different voltages between $V = 0.65\Delta$ and $V = 0.16\Delta$. In all generality, the current $I_b(\varphi_Q)$ can be readily deduced from $I_b(-\varphi_Q) = -I_a(\varphi_Q)$, but except for the largest voltage considered,

our results show that $I_b(\varphi_Q) \simeq I_a(\varphi_Q)$, which means that the current is largely dominated by quartet processes. Similarly, the noises $S_{aa}(\varphi_Q)$, $S_{bb}(\varphi_Q)$ and $S_{ab}(\varphi_Q)$ are nearly equal, except again for $V = 0.65\Delta$ where there is a small but notable difference (which further increases for larger values of the voltage). In particular, this implies that the crossed correlation noise is positive for most of the voltage range, which constitutes yet another indication of the dominant character of quartet processes. This property of the currents and noises ultimately justifies our concentrating on I_a and S_{ab} in this regime.

Focusing on the results for the current (left column of Fig. 63), we see that for large and intermediate voltages (top and middle plots), the current-phase relation is quite complex and clearly non-harmonic. This is most marked around the value $V = 0.60\Delta$ which is the voltage at which the dot level positions coincide with the chemical potentials of the left and right electrodes. As the voltage is lowered, the amplitude of the current oscillations decreases, ultimately reaching a π -shifted sinusoidal relation for low-enough voltage (bottom plot), typical of a pure quartet process far from resonance.

The behavior of the noise S_{ab} in the resonant dots case (right column of Fig. 63) is qualitatively quite different from its counterpart from the QPC regime. We observe two striking characteristics. First, the noise shows a highly sensitive phase dependence, with marked peaks at values of the phase which depend on the voltage bias. Secondly, the amplitude of the noise varies quite strongly and non-monotonically with voltage, reaching huge values, order of magnitudes larger than the corresponding maximum current at the same voltage. This last property, combined with the decrease in amplitude of the current as one lowers the external voltage bias, leads to a giant Fano factor in the low-voltage regime. This behavior is in stark contrast with the one described in the previous section for the QPC regime: while in both cases the current is largely dominated by quartet processes, here it is accompanied by large current fluctuations, which are thus enhanced by the resonant nature of the quantum dots.

It is instructive to consider the overall amplitude of both current and noise, leaving aside the details of the phase dependence, as phase dependent measurements typically require a squid geometry, and we aim at computing quantities which are more directly accessible. To this aim, we consider the current amplitude at a given voltage defined as

$$I_{\max} - I_{\min} = \max_{\varphi_Q} [I_a(\varphi_Q)] - \min_{\varphi_Q} [I_a(\varphi_Q)], \quad (20.55)$$

and similarly for the noise

$$S_{\max} - S_{\min} = \max_{\varphi_Q} [S_{ab}(\varphi_Q)] - \min_{\varphi_Q} [S_{ab}(\varphi_Q)]. \quad (20.56)$$

These two quantities are shown in Fig. 64 and Fig. 65 respectively. Ignoring the fine structure, the general qualitative behavior for the current shows a broad maximum around $V = \epsilon$ (corresponding here to $V = 0.6\Delta$), and a slow decrease at lower voltage. This scale corresponds to the voltage where the energy level of both dots coincides with the chemical potential of the lateral electrode each of them is connected to. Note that the current amplitude is at most of the order of 1 (in units of Δ) and decreases to ~ 0.1 for the smallest voltages considered. The noise amplitude, however, shows a radically different behavior, with sudden bursts which become more marked as the voltage gets smaller. Remarkably, while these fluctuations can make the noise reach values as high as ~ 100 (in units of Δ), there are some wide regions of voltage (for example near $V = 0.3\Delta$) where the noise amplitude is several orders of magnitude smaller. Note that the details of the amplitude of the noise vary importantly when the parameters ϵ and Γ are varied (even within the resonant dots regime). Huge noise varying strongly with voltage has been observed [54] by some of the authors in a different context (a system of three one dimensional topological superconductors which bear Majorana fermions at their extremities) and has been attributed to a different underlying physical mechanism: the presence of a zero Majorana mode.

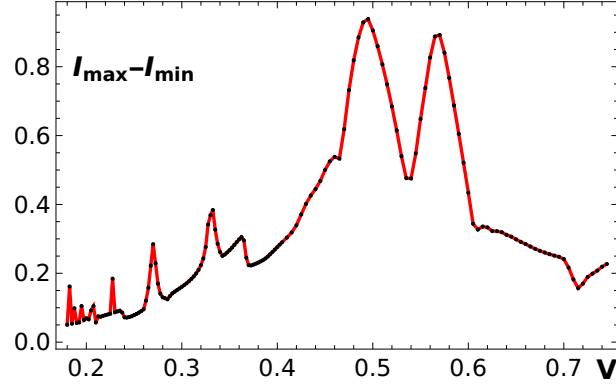


Figure 64: Amplitude of the current $I_a(\varphi_Q)$ (in units of Δ), defined in Eq. (20.55), as a function of the voltage V , in the resonant dot regime ($\epsilon = 0.6\Delta$, $\Gamma = 0.3\Delta$). The current is maximum near the voltage where the dot levels coincide with the lateral electrode chemical potentials ($V = \epsilon$).

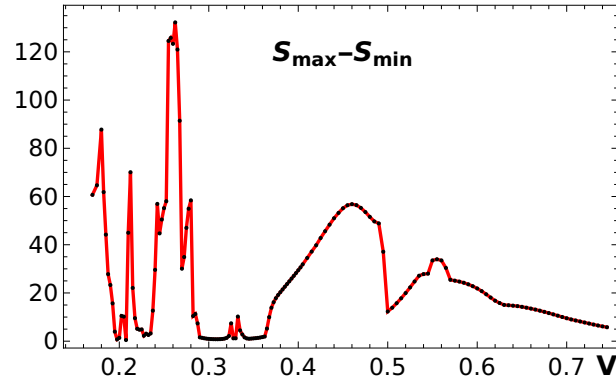


Figure 65: Amplitude of the crossed correlation noise $S_{ab}(\varphi_Q)$ (in units of Δ), defined in Eq. (20.56), as a function of the voltage V , in the resonant dot regime ($\epsilon = 0.6\Delta$, $\Gamma = 0.3\Delta$). The noise shows strong variations at small voltages, with maximum values which are several orders of magnitude larger than the current.

20.4 Discussion

One of the main goals of this application of the present formalism has been to address whether, upon decreasing the bias voltage, the noise signal of MCPR decreases, and goes to zero despite a non equilibrium situation, in the same manner that a DC Josephson junction can bear only thermal fluctuations at equilibrium. In this sense, our results seem to be in qualitative agreement with Ref. [38], which deals with a three superconducting lead device, while also surpassing it for the following reasons.

First, the formalism we have developed here is general enough to allow to treat the non equilibrium current and noise characteristics of a large class of mesoscopic systems (as exposed in the first part of this paper). Ref. [38] focused on a given geometry - a central dot system connected to all three superconducting leads - and all presented calculations were specific to this context. Moreover, the geometry chosen in Ref. [38] does not optimize CAR processes, which are indispensable for the optimal detection of MCPR. In past works on the CPBS,[13, 19] the authors showed that the optimization of CAR processes is achieved by considering a device where the two quantum dots connected to the normal metal leads have energy levels which are opposite with respect to the central grounded superconductor. This motivated, concerning the study of the ASCPBS[12] the same choice of an antisymmetric dot configuration, so that direct Andreev processes are also filtered out. In this respect, the present work achieves a more rigorous diagnosis of noise for MCPR because CAR processes are optimized from the start.

Second, our approach is non-perturbative, taking into account the coupling to/from dots and leads to all orders, for an arbitrary superconducting device. The authors of Ref. [38], however, resorted to a truncated perturbative expansion to compute the transport properties. Beyond the obvious inherent limitations of perturbation theory when it comes to dealing with high junction transmission or resonant situations, such an approach also imposes the introduction of a phenomenological dot parameter intended to regularize the perturbative treatment. This small parameter corresponds to a finite intrinsic linewidth broadening which needs to be systematically added “by hand” to the bare dot Green’s function from the start. The bare dot constitutes an isolated, coherent system, and adding such an infinitesimal in the bare dot Green’s function cannot be justified from a rigorous microscopic model. It is typically introduced to simulate the coupling to a (dissipative) electrodynamic environment, to describe a source of decoherence for the quantum dots. The addition of such an extra phenomenological parameter may have unforeseen repercussions on the transport properties at a MCPR, which were not thoroughly explored in Ref. [38], casting a shadow on the generality of their results. In our present work, an infinitesimal parameter enters the advanced/retarded lead Green’s function only, and this infinitesimal is of course rendered finite for numerical calculations. This is the norm, and subtleties about the interplay between this infinitesimal and the (adiabatic) voltage have been addressed in Ref. [4]. We checked systematically that the current and noise computed within our formalism were not affected when this infinitesimal (associated with the lead Green’s functions) was further reduced, converging toward the same results, while the bare dots Green’s functions are systematically treated as fully coherent entities, devoid of any phenomenological linewidth for the dot system.

20.5 Conclusion

We have studied a general class of multi-terminal superconducting devices composed of superconducting and normal metal leads arbitrarily connected to a quantum dot system. Expressions for the non equilibrium current and noise have been cast into matrix products of self energies and dressed single particle Green’s function. The numerical solution of the latter via Dyson’s equation allows to characterized the transport properties of a vast class of systems. This work constitutes a Hamiltonian formulation of quantum transport in the same spirit as

the Landauer-Buttiker-Imry multichannel/multi-terminal formalism,[55, 56, 57, 58] extended to superconducting hybrid systems.

We applied our formalism to the ASCPBS, a three-terminal superconducting device designed with a central grounded electrode contacted to two other leads via two QD nanowires. When CAR processes are operating on the central superconductor and the voltages applied to the lateral leads are commensurate, the partial currents are known to depend on the bare superconducting phase differences, leading to a Josephson-like signature, albeit in out-of-equilibrium conditions. [30, 12] This has been confirmed by experimental signatures for the differential conductance [32] and more recently explored for the noise. [39] Such a phenomenon is referred to as a MCPR since the underlying explanation involves the correlated motion of several CPs between the three superconducting reservoirs, with split Cooper pairs from the central lead.

Within the out-of-equilibrium Keldysh framework, the equivalent of a path integral approach leads to a Dyson equation which relates bare and dressed QD Green's functions, introducing a self-energy term which accounts for the coupling to superconductors in a non-perturbative way. The statistics of the current operator have been derived using counting fields [13] which allow generalizations to full counting statistics in principle. The commensurability of the voltages, which is assumed here for computational reasons and because it is a requirement for the observation of MCPR, allows the use of a double Fourier transformation leading to a convenient matrix representation. As a result, all observables of this system (current, noise autocorrelations and noise crossed correlations), are expressed in terms of the dressed Green's function of the dots, a procedure which requires a (large) matrix inversion of the Dyson equation.

More specifically, we have focused on the quartet resonance where the voltages imposed on the two lateral leads are opposite, and two regimes have been numerically investigated. In the QPC junction limit, where the dot energy levels are placed outside the gap of the central superconductor, and energy filtering of the dots is not effective, we have found that the noise correlations decrease when the voltage is lowered, so that the Fano factors take very low values in the adiabatic limit. We have also demonstrated that the decrease of the Fano factor is not specific to the quartet resonance, by showing similar behaviors for three other voltage configurations.

This observation is in sharp contrast with the case of a two-terminal junction where the Fano factor varies as $\sim 1/V$ at low voltage. In this regime, negative noise crossed correlations are generally observed at all voltages, except for fine-tuned values of the quartet phase. In the resonant dots regime, where the dot energy filtering is sharper and occurs within the gap of the central superconductor, only positive values of the noise crossed correlations are obtained, an indication of the dominance of the quartet process. Compared to the QPC regime, large values of the noise have been obtained together with a strong sensitivity on both phase and voltage due to the acute energy filtering. The appearance of giant Fano factors constitutes an important signature of this regime.

We therefore believe that the present noise diagnosis of our 3-superconductor setup sheds further light on the physics of multiple Cooper pair processes, with potential repercussions on experimental detection.

Possible extensions of this work could follow several directions including

(i) the investigation of lower voltages in the resonant dots regime which would require more efficient integration tools in order to further probe the noise reduction;

(ii) a thorough analysis of other voltages associated with MCPR ($mV_a + nV_b = 0$), for larger integers m and n , which differ from the specific quartet process studied in this application of our formalism. Higher fractions have been extensively studied, albeit only for the DC Josephson current, in both non-equilibrium and equilibrium setups:[12, 59] to each higher order MCPR corresponds an Andreev bound state described by a combination of CAR and direct Andreev processes in the ASCPBS which describe a closed orbit in energy space. Basic transport pro-

cesses are the same, but they involve more CAR processes necessarily. We have shown that, in the QPC regime, the Fano factor decreases for voltage configurations other than the quartet resonance. While the behavior is qualitatively the same for all voltage configurations, a detailed exploration of the latter could provide precious insight on these systems.

(iii) the influence of inter-dot tunneling, which may spoil the efficiency of CAR processes in the ASCPBS: this was indeed the case for the CPBS in Ref. [13], and we expect the MCPR signal to be decreased in the presence of such tunneling. On general grounds, strong inter-dot tunneling should be taken into account in a general, large quantum dot “array”, with the goal to describe a specific mesoscopic device or molecule with many sites/orbitals.

(iv) the inclusion of different injection regions in the leads when a superconductor is connected to several quantum dots. We aware that dimensionnality and k_F oscillations can lead to a modification of the coupling parameters that we have used here, in the same spirit as in Ref. [60], where the reduction of CAR over large points of injections was taken into account effectively.

(v) the influence of the QD Coulomb on-site energy, which could be done using either a Hubbard-Stratonovich treatment (followed by a standard approximation such as a saddle point method [60]) or a self-consistent perturbative approach; [19]

(vi) the investigation of finite frequency noise at multiples of the Josephson frequency (as well as the Josephson frequency current harmonics), for which we have available analytic expressions, but which would require a challenging experimental detection scheme similar to that of Shapiro steps. Particularly relevant would be to study the effect of decoherence (due to the coupling of the dot system to a normal metal lead for instance) on the amplitude of the higher noise harmonics.

(vii) the investigation of electron-hole decoherence effects on the zero frequency noise, using methods borrowed from quasiclassical circuit theory [19, 3, 63].

21 Anomalous Josephson effect induced by spin-orbit interaction and Zeeman effect in semiconductor nanowires by Yokoyama, Eto, Nazarov

We investigate theoretically the Josephson junction of semiconductor nanowire with strong spin-orbit (SO) interaction in the presence of magnetic field. By using a tight-binding model, the energy levels E_n of Andreev bound states are numerically calculated as a function of phase difference φ between two superconductors in the case of short junctions. The DC Josephson current is evaluated from the Andreev levels. In the absence of SO interaction, a $0-\pi$ transition due to the magnetic field is clearly observed. In the presence of SO interaction, the coexistence of SO interaction and Zeeman effect results in $E_n(-\varphi) \neq E_n(\varphi)$, where the anomalous Josephson current flows even at $\varphi = 0$. In addition, the direction-dependence of critical current is observed, in accordance with experimental results.

21.0.1 Introduction

The spin-orbit (SO) interaction in narrow-gap semiconductors, e.g., InAs and InSb, [1] attracts a lot of interest in recent studies. The SO interaction gives a possibility of electrical spin manipulation, which is a great advantage for spintronic devices. [2] For conduction electrons in direct-gap semiconductors, the SO interaction is expressed as

$$H_{\text{SO}} = \frac{\lambda}{\hbar} \boldsymbol{\sigma} \cdot [\mathbf{p} \times \nabla V(\mathbf{r})], \quad (21.1)$$

where $V(\mathbf{r})$ is an external potential and σ indicates the electron spin $\mathbf{s} = \sigma/2$. In experiments of quantum well using such materials, strong SO interaction was reported. [4, 5, 6] For an external electric field $\mathcal{E}$ perpendicular to the quantum well, the substitution of $V(\mathbf{r}) = e\mathcal{E}z$ into Eq. (21.1) yields

$$H_{\text{SO}} = \frac{\alpha}{\hbar}(p_y\sigma_x - p_x\sigma_y), \quad (21.2)$$

which is called the Rashba interaction. Here, the coupling constant $\alpha = e\mathcal{E}\lambda$ is tunable by an electric field, or a gate voltage.

The development of fabrication technique enables to construct various quantum systems with SO interaction. Particularly, semiconductor nanowires of InAs and InSb are investigated intensively, in which quantum point contacts and quantum dots can be formed. [7, 8, 9, 10, 11, 12] Indeed, the electrical manipulation of single electron spins was reported for quantum dots fabricated on the nanowires. [10, 11, 12] In recent studies, the nanowire-superconductor hybrid systems were studied for searching the Majorana fermions. [13, 14, 15, 16] The DC Josephson effect was also studied when the nanowires are connected to two superconductors (S/NW/S junctions). [17, 18, 19]

The Josephson effect is one of the most fundamental phenomena concerning quantum phase. In a Josephson junction, the supercurrent flows when the phase difference φ between two superconductors is present. For the junction using normal metals or semiconductors, the electron and hole in the normal region are coherently coupled to each other by the Andreev reflections at normal / superconductor interfaces. [12] The Andreev bound states, which have discrete energy levels E_n (Andreev levels), are formed in the normal region around the Fermi level within the superconducting energy gap Δ_0 . [22, 21] The Cooper pair transports via the Andreev bound states. For short junctions, where a distance L between two superconductors is much smaller than the coherent length ξ in the normal region, the Josephson current $I(\varphi)$ is determined by the Andreev levels. [22, 23, 24, 25] $\xi = \hbar v_F / (\pi \Delta_0) \equiv \xi_0$ for ballistic systems and $\xi = (\xi_0 l_0)^{1/2}$ for diffusive ones, where v_F is the Fermi velocity and l_0 is the mean free path. For the transmission probability T_n of conduction channel n ($= 1, 2, \dots, N$) in the normal region, the current is written as

$$I(\varphi) = \frac{e\Delta_0}{2\hbar} \sum_{n=1}^N \frac{T_n \sin \varphi}{[1 - T_n \sin^2(\varphi/2)]^{1/2}}. \quad (21.3)$$

Here, the current satisfies $I(-\varphi) = -I(\varphi)$. In the limit of low transparent junction, the current in Eq. (21.3) becomes $I(\varphi) \simeq I_0 \sin \varphi$ with $I_0 \equiv e\Delta_0 / (2\hbar) \sum_n T_n$.

In superconductor / ferromagnet / superconductor (S/F/S) junctions, the oscillation of critical current accompanying a $0-\pi$ transition was observed as a function of the thickness of ferromagnet. [26, 27, 28, 29] The $0-$ and π -states mean that the free energy is minimal at $\varphi = 0$ and π , respectively. The $0-\pi$ transition is caused by the interplay between the spin-singlet correlation of the superconductivity and the exchange interaction in ferromagnet. The exchange interaction makes the spin-dependent phase shift in the propagation through the ferromagnet. Since the Andreev bound state consists of a right-going (left-going) electron with spin σ and a left-going (right-going) hole with spin $-\sigma$, the phase shift modulates the Andreev levels. When the length of ferromagnet is increased, the $0-\pi$ transition takes place at the cusps of critical current. [29] A similar transition was observed recently in S/NW/S junctions with fixed length when the Zeeman splitting was tuned by applying a magnetic field. [30]

The effect of SO interaction in the Josephson junctions is an interesting subject, where many phenomena were predicted, e.g., fractional Josephson effect [31] and anomalous supercurrent. [32] The fractional Josephson effect is a 4π periodicity of current phase relation, $I(\varphi) \sim \sin(\varphi/2)$, which is a property of Majorana fermions induced by the SO interaction in the superconducting region. The anomalous supercurrent is a finite supercurrent at zero phase difference, $I(\varphi = 0) \neq 0$, which is induced by the breaking of symmetry of current phase

relation. The symmetry breaking is attributed to the existence of SO interaction and magnetic field in the normal region.

In the present study, we focus on the anomalous Josephson current. The DC Josephson current with SO interaction in the normal region was investigated theoretically by a lot of groups, for normal metal with magnetic impurities, [32] two-dimensional electron gas (2DEG) in semiconductor heterostructures, [27, 34, 35, 36, 37, 38, 39] open quantum dots, [40] quantum dots with tunnel barriers, [41, 43, 42, 45, 46, 13, 47] carbon nanotubes, [48] quantum wires or nanowires, [49, 50, 51, 52] quantum point contacts, [53, 54] topological insulators, [55] and others. [56] The SO interaction breaks the spin-degeneracy of Andreev levels when the time reversal symmetry is broken by the phase difference $\varphi \neq 0$ even in the absence of magnetic field. [40, 56] The splitting due to the SO interaction is obtained in the long junctions, $L \gg \xi$ (or intermediated-length junctions, $L \gtrsim \xi$). In the short junctions, however, the spin-degeneracy of Andreev levels holds. [40, 56] In both cases, the relation of $I(-\varphi) = -I(\varphi)$ is not broken, which means no supercurrent at $\varphi = 0$.

In the presence of magnetic field, the SO interaction modifies qualitatively the current phase relation. Then, the anomalous Josephson current is obtained. [32, 35, 36, 43, 47, 49, 53, 54, 52] The anomalous current flows in so-called φ_0 -state in which the free energy has a minimum at $\varphi = \varphi_0$ ($\neq 0, \pi$). [58] The anomalous Josephson current was predicted when the length of normal region L is longer than or comparable to the coherent length ξ . Krive *et al.* derived the anomalous current for long junctions with a single conduction channel. [49] Reynoso *et al.* found the anomalous current through a quantum point contact in the 2DEG for $L \gtrsim \xi$. [53, 54] They discussed an influence of spin polarization induced around the quantum point contact with SO interaction [57] on the Josephson current. They also showed the direction-dependence of critical current when a few conduction channels take part in the transport. In the experiment on the S/NW/S junctions, [30] the direction-dependent supercurrent was observed for samples of $L \gtrsim \xi$ in a parallel magnetic field to the nanowire, besides the above-mentioned $0-\pi$ transition. This should be ascribable to the strong SO interaction in the nanowires although the anomalous Josephson current was not examined.

In our previous paper, we investigated theoretically the DC Josephson effect in semiconductor nanowires with strong SO interaction in the case of short junction. [52] We examined a simple model with single scatterer to capture the physics of $0-\pi$ transition and anomalous Josephson effect. In our model, both elastic scatterings by the impurities and SO interaction in the nanowire were represented by the single scatterer. The Zeeman effect by a magnetic field shifts the wavenumber as $k_{\pm}^> = k_F + (E \pm E_Z)/(\hbar v_F)$ for $k > 0$ and $k_{\pm}^< = -k_F - (E \pm E_Z)/(\hbar v_F)$ for $k < 0$. [66] E is an energy measured from the Fermi level. $E_Z \equiv |g\mu_B B|/2$ is the Zeeman energy. The orbital magnetization is neglected in the nanowire. The Fermi velocity v_F is independent of channels. When $k > 0$ ($k < 0$), the electron and hole move to the right (left) and left (right), respectively. The propagation of electron with spin $\sigma = \pm$ and hole with $\sigma = \mp$ acquires the phase $\pm\theta_B = \pm|g\mu_B B|L/(\hbar v_F)$. The term of $2EL/(\hbar v_F)$ is safely disregarded for short junctions. The terms of k_F are canceled out by each other. The simple model showed the oscillation of critical current with increase of θ_B . The 0 - and π -states are realized when $\theta_B \sim 0$ and π , respectively. Around $\theta_B = \pi/2$, the $0-\pi$ transition takes place. In the presence of SO interaction, the anomalous Josephson current was obtained, which result means a realization of φ_0 -state. Moreover, the direction-dependence of critical current is found. The critical current indicates cusps at the local minima. The position of cusps also depends on the current direction, in accordance with the experiment. [30] Between the cusps for positive and negative current direction, the transition from $\varphi_0 \approx 0$ to $\approx \pi$ takes place.

In this paper, we study the anomalous Josephson effect numerically using a tight-binding model for the nanowire in the case of short junction. The purposes are to confirm our previous simple model and to elucidate the key ingredients of anomalous Josephson effect. First, we

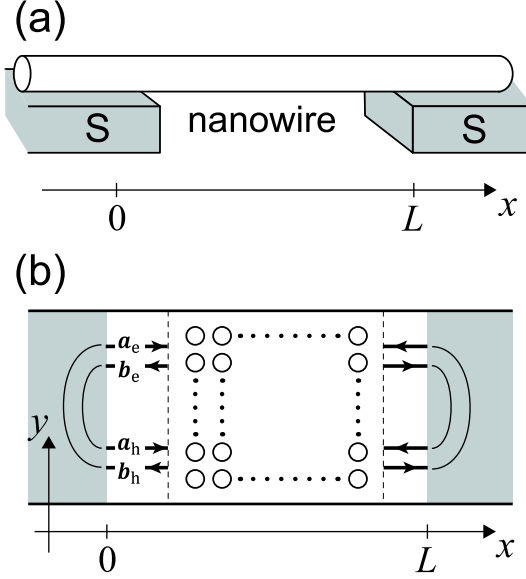


Figure 66: Our model for a quasi-one-dimensional semiconductor nanowire connected to two superconductors. The nanowire is along the x axis. (a) Schematic view of the model. The pair potential is induced in the nanowire by the proximity effect, $\Delta(x) = \Delta_0 e^{i\varphi_L}$ at $x < 0$ and $\Delta_0 e^{i\varphi_R}$ at $L < x$, whereas $\Delta(x) = 0$ at $0 < x < L$. Several impurities are present in the nanowire. The spin-orbit interaction and Zeeman effect are taken into account only in the normal region. (b) The tight-binding model gives the scattering matrix $\hat{S}_e$ ($\hat{S}_h$) for electrons (holes), which connects incoming a_e (a_h) and outgoing electrons b_e (holes b_h). At $x = 0$ and L , the electron b_e is reflected as the hole a_h by the Andreev reflection, whereas b_h is reflected as a_e .

consider the case without SO interaction. The Andreev levels are invariant against the φ -inversion, $E_n(-\varphi) = E_n(\varphi)$. As a result, the current satisfies $I(-\varphi) = -I(\varphi)$ and hence no anomalous current is found. The critical current oscillates as a function of magnetic field accompanying the $0-\pi$ transition, which is characterized by a single parameter θ_B even for $N > 1$. Next, we investigate the Josephson effect in the presence of SO interaction and Zeeman effect. The relation of $I(-\varphi) = -I(\varphi)$ is broken. As a result, the anomalous Josephson current and the direction dependence of critical current are obtained, which are qualitatively the same as those of single scatterer model. We stress that the spin-dependent channel mixing due to SO interaction plays an important role on the anomalous Josephson effect.

The organization of this paper is as follows. In Sec. II, we explain our model for the S/NW/S Josephson junction and calculation method of the Andreev levels and Josephson current. Numerical results are given in Sec. III. The last section (Sec. IV) is devoted to the conclusions and discussion.

21.1 Model and calculations

In this section, we explain our model depicted in Fig. 66. We introduce the Bogoliubov-de Gennes (BdG) equation to obtain the Andreev levels. The formulation of solving BdG equation is given in terms of scattering matrix. [22] We apply the tight-binding model to the normal region, where the scattering matrices of conduction electrons and holes are numerically calculated. [60, 61]

21.1.1 Formulation

We consider a semiconductor nanowire along the x axis connected to two superconductors at $x < 0$ and $x > L$, as shown in Fig. 66(a). The superconducting pair potential is penetrated into the nanowire by the proximity effect, whereas there is no pair potential in the normal region at $0 < x < L$. The SO interaction and Zeeman effect in a magnetic field are taken into account only in the normal region. Since InSb has a large g -factor, a large Zeeman energy is obtained for weak magnetic field, which does not break the superconductivity.

The Andreev bound states are formed in the normal region. The BdG equation to describe the Andreev bound states is written as [62, 21]

$$\begin{pmatrix} H - E_F & \hat{\Delta} \\ \hat{\Delta}^\dagger & -(H^* - E_F) \end{pmatrix} \begin{pmatrix} \psi_e \\ \psi_h \end{pmatrix} = E \begin{pmatrix} \psi_e \\ \psi_h \end{pmatrix}, \quad (21.4)$$

where $\psi_e = (\psi_{e+}, \psi_{e-})^T$ and $\psi_h = (\psi_{h+}, \psi_{h-})^T$ are the spinors for electron and hole, respectively. The energy E is measured from the Fermi level E_F . The diagonal element is the single-electron Hamiltonian $H = H_0 + H_{SO} + H_Z$ with $H_0 = \mathbf{p}^2/(2m^*) + V_{\text{conf}}(y, z) + V_{\text{imp}}$, SO interaction H_{SO} , and Zeeman effect $H_Z = g\mu_B \mathbf{B} \cdot \hat{\boldsymbol{\sigma}}/2$, using effective mass m^* , g -factor g (< 0 for InSb), Bohr magneton μ_B , and Pauli matrices $\hat{\boldsymbol{\sigma}}$. H_{SO} and H_Z are taken into account only at $0 < x < L$. V_{conf} describes the confining potential of nanowire. V_{imp} represents the potential due to the impurities. $\hat{\Delta}$ is the pair potential in the spinor space

$$\hat{\Delta} = \Delta(x)\hat{g} = \Delta(x) \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \quad (21.5)$$

where $\hat{g} = -i\hat{\sigma}_y$. [63] For simplicity, we assume that the absolute values of pair potential in left and right superconducting regions are equal to each other, $\Delta(x) = \Delta_0 e^{i\varphi_L}$ at $x < 0$ and $\Delta_0 e^{i\varphi_R}$ at $L < x$. In the normal region at $0 < x < L$, $\Delta(x) = 0$. The phase difference between two superconductors is defined as $\varphi \equiv \varphi_L - \varphi_R$. We consider a short junction, where $L \ll \xi$. No potential barrier is assumed at the boundaries between the normal and superconducting regions. The Zeeman energy $E_Z \equiv |g\mu_B B|/2$ and the pair potential Δ_0 are much smaller than the Fermi energy E_F .

The solution of BdG equation gives the Andreev levels E_n ($|E_n| < \Delta_0$) as a function of phase difference φ . When the BdG equation has an eigenenergy E_n with eigenvector $(\psi_{e,n}, \psi_{h,n})^T$, $-E_n$ is also an eigenenergy of the equation with $(\psi_{h,n}^*, \psi_{e,n}^*)^T$. In short junctions, the number of Andreev levels is given by $4N$; $2N$ positive levels and $2N$ negative ones when the number of channels is N ($2N$ if the spin degree of freedom is included). The ground-state energy of the junction is given by

$$E_{\text{gs}}(\varphi) = -\frac{1}{2} \sum_n' E_n(\varphi), \quad (21.6)$$

where the summation is taken over all the positive Andreev levels, $E_n(\varphi) > 0$. The contribution from continuous levels ($|E| > \Delta_0$) is disregarded in Eq. (21.6), which are independent of φ in the short junctions. [22] At zero temperature, the supercurrent is calculated as

$$I(\varphi) = \frac{2e}{\hbar} \frac{dE_{\text{gs}}}{d\varphi} = -\frac{e}{\hbar} \sum_n' \frac{dE_n}{d\varphi}. \quad (21.7)$$

The current is a periodic function for $-\pi < \varphi \leq \pi$. The maximum (absolute value of minimum) of $I(\varphi)$ yields the critical current $I_{c,+}$ ($I_{c,-}$) in the positive (negative) direction.

The symmetry of BdG equation should be noted here. We denote the matrix on the left side of Eq. (21.4) by $\mathcal{H}(\varphi)$. In the absence of Zeeman effect, $\mathcal{T}\mathcal{H}(\varphi)\mathcal{T}^{-1} = \mathcal{H}(-\varphi)$ with the

time-reversal operator $\mathcal{T} = -i\hat{\sigma}_y K$ for spin-1/2 particles. K is the operator to form a complex conjugate; $Kf = f^*$. If $\mathcal{H}(\varphi)$ has an eigenenergy E_n with eigenvector $(\psi_{e,n}, \psi_{h,n})^T$, $\mathcal{H}(-\varphi)$ has an eigenenergy E_n with eigenvector $\mathcal{T}(\psi_{e,n}, \psi_{h,n})^T$. Thus the Andreev levels satisfy the relation of $E_n(\varphi) = E_n(-\varphi)$. In the absence of SO interaction, $K\mathcal{H}(\varphi)K^{-1} = \mathcal{H}(-\varphi)$. Then we derive that $E_n(\varphi) = E_n(-\varphi)$ in the same way. The relation does not always hold in the presence of both SO interaction and magnetic field.

21.1.2 Scattering Matrix Approach

The BdG equation in Eq. (21.4) can be written in terms of the scattering matrix. [22]

In the normal region with SO interaction and Zeeman effect, the quantum transport of electrons (holes) is described by the scattering matrix S_e (S_h). The scattering matrix S_p ($p = e, h$) connects the amplitudes of incoming waves of N conduction channels with spin $\sigma = \pm$, $(\mathbf{a}_{pL}, \mathbf{a}_{pR})^T$, and those of outgoing waves, $(\mathbf{b}_{pL}, \mathbf{b}_{pR})^T$, as shown in Fig. 66(b),

$$\begin{pmatrix} \mathbf{b}_{pL} \\ \mathbf{b}_{pR} \end{pmatrix} = \hat{S}_p \begin{pmatrix} \mathbf{a}_{pL} \\ \mathbf{a}_{pR} \end{pmatrix}. \quad (21.8)$$

$\hat{S}_e$ and $\hat{S}_h$ are $4N \times 4N$ matrices and related to each other by $\hat{S}_e(E) = \hat{S}_h^*(-E)$. On the assumption that they are independent of energy E for $|E| < \Delta_0$, and thus $\hat{S}_e = \hat{S}_h^*$. We denote $\hat{S}_e = \hat{S}$ and $\hat{S}_h = \hat{S}^*$. $\hat{S}$ is conventionally written by reflection and transmission matrices:

$$\hat{S} = \begin{pmatrix} \hat{r}_L & \hat{t}_{LR} \\ \hat{t}_{RL} & \hat{r}_R \end{pmatrix}. \quad (21.9)$$

The scattering matrix is unitary, $\hat{S}^\dagger \hat{S} = \hat{1}$. Moreover, $\hat{r}_L^T = \hat{g}^\dagger \hat{r}_L \hat{g}$, $\hat{r}_R^T = \hat{g}^\dagger \hat{r}_R \hat{g}$, and $\hat{t}_{RL}^T = \hat{g}^\dagger \hat{t}_{LR} \hat{g}$ are satisfied if the time reversal symmetry is kept.

The Andreev reflection at $x = 0$ and L is also described in terms of scattering matrix $\hat{r}_{he}$ for the conversion from electron to hole and $\hat{r}_{eh}$ for that from hole to electron. When an electron with spin σ is reflected into a hole with $-\sigma$, it is written as [22]

$$\begin{pmatrix} \mathbf{a}_{hL} \\ \mathbf{a}_{hR} \end{pmatrix} = \hat{r}_{he} \begin{pmatrix} \mathbf{b}_{eL} \\ \mathbf{b}_{eR} \end{pmatrix}, \quad (21.10)$$

where

$$\hat{r}_{he} = e^{-i\alpha_A} \begin{pmatrix} e^{-i\varphi_L} \hat{1} \otimes \hat{g} & \\ & e^{-i\varphi_R} \hat{1} \otimes \hat{g} \end{pmatrix} \quad (21.11)$$

with $\alpha_A \equiv \arccos(E/\Delta_0)$. When a hole is reflected to an electron, it is

$$\begin{pmatrix} \mathbf{a}_{eL} \\ \mathbf{a}_{eR} \end{pmatrix} = \hat{r}_{eh} \begin{pmatrix} \mathbf{b}_{hL} \\ \mathbf{b}_{hR} \end{pmatrix} \quad (21.12)$$

with

$$\hat{r}_{eh} = e^{-i\alpha_A} \begin{pmatrix} e^{i\varphi_L} \hat{1} \otimes \hat{g}^\dagger & \\ & e^{i\varphi_R} \hat{1} \otimes \hat{g}^\dagger \end{pmatrix}. \quad (21.13)$$

We assume that the channel is conserved at the Andreev reflection in the case of $N \geq 2$. The normal reflection can be neglected in our case without potential barriers at the boundaries. [12]

The product of scattering matrices gives an equation for $(\mathbf{a}_{eL}, \mathbf{a}_{eR})^T$. The Andreev levels $E_n(\varphi)$ are calculated from this product as, [22]

$$\det \left(\hat{1} - \hat{r}_{eh} \hat{S}^* \hat{r}_{he} \hat{S} \right) = 0, \quad (21.14)$$

which is equivalent with the BdG equation in Eq. (21.4). In the absence of magnetic field, Eq. (21.14) is simply reduced to [22]

$$\det \left[1 - \left(\frac{E}{\Delta_0} \right)^2 - \tilde{t}_{\text{LR}}^\dagger \hat{t}_{\text{LR}} \sin^2 \left(\frac{\varphi}{2} \right) \right] = 0. \quad (21.15)$$

In this case, the Andreev levels are represented by the transmission eigenvalues of $\tilde{t}_{\text{LR}}^\dagger \hat{t}_{\text{LR}}$. They are two-fold degenerate reflecting the Kramers' degeneracy at $\varphi = 0$. The Andreev levels $E_n(\varphi)$ are not split by finite φ in spite of the broken time reversal symmetry.

21.1.3 Tight-Binding Model

To obtain the scattering matrix $\hat{S}$, we describe the normal region by a tight-binding model of square lattice model in two-dimensional space (xy plane), [60] as schematically shown in Fig. 66(b). We consider a quasi-one-dimensional nanowire along the x axis with width W in the y direction. The length of normal region is L . We assume hard-wall potentials at $y = 0$ and W . The Rashba-type SO interaction in Eq. (21.2) and the Zeeman effect are considered in the normal region. The Rashba interaction specifies the direction of spin quantization axis. In the experiments, the nanowire is not two-dimensional or the SO interaction may not be Rashba one. However, our model is general to represent a single or few conduction channels in the nanowire and to consider the spin mixing among channels by the SO interaction. In the following, the magnetic field is applied in the y direction, which is almost parallel to the spin quantization axis due to the Rashba interaction for the channels. The channel is split upward and downward by the Zeeman effect. The orbital magnetization is neglected.

On the tight-binding model, the Hamiltonian $H = H_0 + H_{\text{SO}} + H_Z$ is written as

$$\begin{aligned} H = & t \sum_{j,l} \mathbf{c}_{j,l}^\dagger \{ (4 + v_{j,l}) \hat{1} + \mathbf{b} \cdot \hat{\boldsymbol{\sigma}} \} \mathbf{c}_{j,l} \\ & - t \sum_{j,l} \left(\mathbf{c}_{j,l}^\dagger \hat{T}_{j,l;j+1,l} \mathbf{c}_{j+1,l} + \mathbf{c}_{j,l}^\dagger \hat{T}_{j,l;j,l+1} \mathbf{c}_{j,l+1} + \text{H. c.} \right), \end{aligned} \quad (21.16)$$

where $\mathbf{c}_{j,l} \equiv (c_{j,l,+}, c_{j,l,-})^T$ and $c_{j,l,\sigma}$ is annihilation operator of an electron at site (j, l) with spin σ . $t \equiv \hbar^2/(2m^*a^2)$ is a transfer integral with a lattice constant a . Here, j ($= 0, 1, \dots, N_x, N_x+1$) and l ($= 1, 2, \dots, N_y$) denote site labels in the x and y directions, respectively. The length is $L = N_x a$ and the width is $W = (N_y + 1)a$. At the sites of $j = 0, N_x + 1$, the SO interaction and Zeeman effect is absent. $v_{j,l} \equiv V_{j,l}/t$ is a dimensionless on-site potential by impurities. $\hat{1}$ is the unit matrix in the spinor space. $\mathbf{b} \equiv g\mu_B \mathbf{B}/(2t)$ indicates a magnetic field. The transfer term in the x direction is given by

$$\hat{T}_{j,l;j+1,l} = \hat{1} - ik_\alpha a \hat{\sigma}_y, \quad (21.17)$$

whereas that in the y direction is

$$\hat{T}_{j,l;j,l+1} = \hat{1} + ik_\alpha a \hat{\sigma}_x. \quad (21.18)$$

Here, $k_\alpha = m^* \alpha / \hbar^2$ denotes the strength of Rashba interaction. In this model, the reflection and transmission matrices are calculated by using the recursive Green's function method (see Appendix A). [61]

We set the Fermi wavelength λ_F as a parameter, which gives the Fermi energy by $E_F = 2t - 2t \cos(k_F a)$ with $k_F = 2\pi/\lambda_F$. [64] In an ideal quantum wire with width W , the dispersion relation for channel n is given by

$$E_n(k) = 4t - 2t \cos(ka) - 2t \cos \left(\frac{\pi a}{W} n \right). \quad (21.19)$$

The conduction channels satisfy $E_n(k=0) < E_F$. Then, the velocity of channel n at the Fermi energy is

$$v_{F,n} = \frac{2ta}{\hbar} \sqrt{1 - \left\{ 2 - \cos\left(\frac{\pi a}{W}n\right) - \frac{E_F}{2t} \right\}^2}. \quad (21.20)$$

We consider a nanowire with width $W = 60$ nm. The distance between left and right superconductors is $L = 1000$ nm. We set $N_x = 11$ and $N_y = 200$. The number of conduction channel is changed by tuning the Fermi wavenumber λ_F . In the following, we calculate for three cases: $\lambda_F = 90$ nm for single channel ($N = 1$), 50 nm for $N = 2$, and 25 nm for $N = 4$. A parameter of Rashba interaction is $k_\alpha/k_F = 0.15$. The on-site random potential by impurities is uniformly distributed in $-W_0/2 < V_{j,l} < W_0/2$. The mean free path l_0 is estimated as [61]

$$l_0 = \frac{6\lambda_F^3}{\pi^3 a^2} \left(\frac{E'_F}{W_0} \right)^2. \quad (21.21)$$

Here we use the modified Fermi energy $E'_F = E_F - E_1(0)$ in an one-dimensional quantum wire.

21.2 Numerical results

In this section, we present calculated results of Andreev levels and Josephson currents. First, we discuss the case without SO interaction. The critical current oscillates as a function of magnetic field and the $0-\pi$ transition is clearly found. Next, we consider the anomalous Josephson effect induced by the SO interaction.

The magnetic field is applied in the y direction. We introduce a parameter of magnetic field, $\theta_B = |g|\mu_B BL/(\hbar\bar{v}_F)$, where $\bar{v}_F \equiv \left\{ (1/N) \sum_{n=1}^N (1/v_{F,n}) \right\}^{-1}$ is the inversion average of velocity $v_{F,n}$ in Eq. (21.20). The Zeeman effect splits the dispersion relation for spin $\sigma = \pm$. The wavenumber becomes $k_{F,n,\pm} \simeq k_{F,n} \pm |g|\mu_B B/(2\hbar v_{F,n})$ for $k > 0$ and $k_{F,n,\pm} \simeq -k_{F,n} \mp |g|\mu_B B/(2\hbar v_{F,n})$ for $k < 0$. For the propagation of electron with spin $\sigma = \pm$ and hole with $\sigma = \mp$ in the normal region, the shift of phase due to the Zeeman effect is $\pm |g|\mu_B BL/(\hbar v_{F,n})$. Therefore, θ_B means the channel-average of spin-dependent phase shift of electron and hole forming the Andreev bound states.

21.2.1 Absence of spin-orbit interaction

Single conduction channel

First, we consider a sample of nanowire with a single conduction channel. Figure 67 shows the Andreev levels and Josephson currents as functions of phase difference φ between two superconductors. The magnetic field gradually increases from left-upper to right-bottom panels. In the absence of SO interaction, the spin $\sigma = \pm$ is well defined in the direction of magnetic field. In the case of single conduction channel, four Andreev levels are found in $|E| \leq \Delta_0$. The levels are denoted as $E_{\uparrow i \pm}$ and $E_{\downarrow i \pm}$, where the subscript $\uparrow(\downarrow)$ means the state of electron spin $\sigma = +1$ ($\sigma = -1$) and hole spin $\sigma = -1$ ($\sigma = +1$). $i = 1, 2, \dots$. The sign $\pm$ corresponds to the positive or negative energy at $B = 0$. We consider three regions with increasing θ_B . When $B = 0$, the levels are doubly degenerate for any φ . The ground-state energy E_{gs} becomes minimal at $\varphi = 0$, which corresponds to the 0-state [Fig. 68(a)]. The levels are split like the Zeeman splitting in the presence of magnetic field. For a weak magnetic field, E_{gs} is still minimal at $\varphi = 0$ [region (I)]. As the magnetic field is increased, the level crossing at $E = 0$ is observed, which corresponds to region (II). The crossing points move from $\varphi = \pm\pi$ to 0 with increase of θ_B . When $\theta_B \approx \pi$, no level crossing takes place in region (III). In this region, E_{gs} is minimal at $\varphi = \pi$ (π -state). The transition of 0-state to π -state takes place suddenly around $\theta_B = \pi/2$,

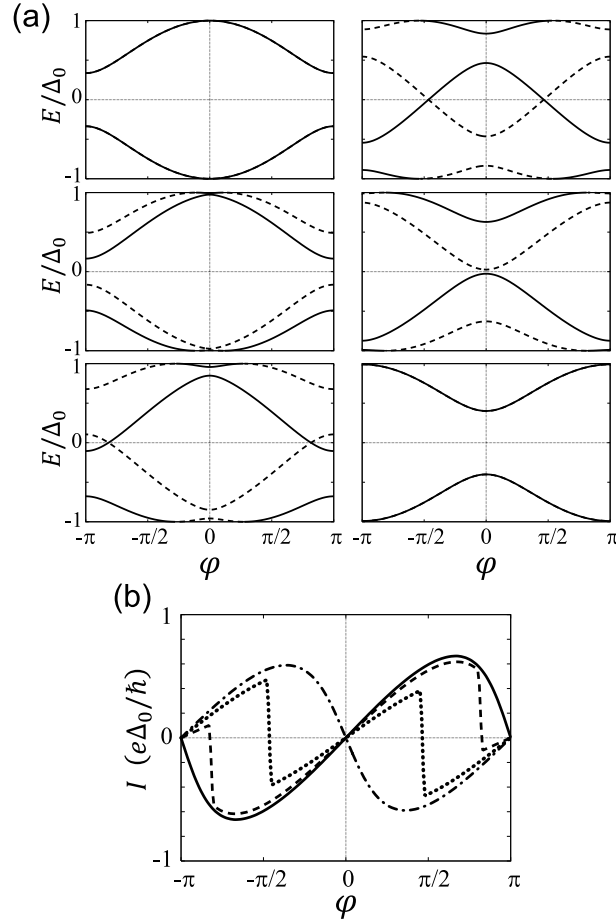


Figure 67: Calculated results for a sample when $N = 1$ and $l_0/L = 1$. The SO interaction is absent. (a) Andreev levels E_n as a function of phase difference φ between two superconductors. Solid and broken lines indicate $E_{\uparrow i\pm}$ and $E_{\downarrow i\pm}$, respectively. The magnetic field is $\theta_B = 0$ (left upper), 0.1π (left middle), 0.27π (left bottom), 0.53π (right upper), 0.79π (right middle), and π (right bottom). At $B = 0$, two lines are overlapped to each other, reflecting the Kramers' degeneracy. (b) Josephson current $I(\varphi)$ through the nanowire when $\theta_B = 0$ (solid), 0.27π (broken), 0.53π (dotted), and π (dotted broken lines).

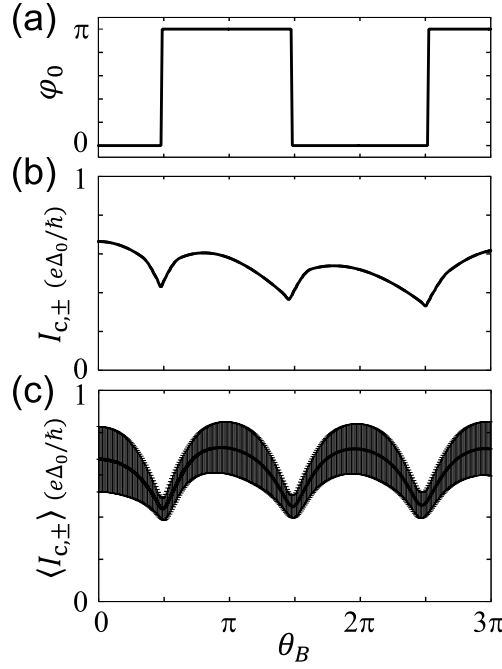


Figure 68: Calculated results for $N = 1$ and $l_0/L = 1$. The SO interaction is absent. (a) Phase difference φ_0 at the minimum of ground-state energy as a function of magnetic field, θ_B . (b) Critical current $I_{c,\pm}$. The current in the positive direction $I_{c,+}$ is identical with that in the negative direction $I_{c,-}$. The sample for (a) and (b) is same as that in Fig. 67. (c) Average of critical current, $\langle I_{c,\pm} \rangle$, with the average of fluctuation, $\sqrt{\langle [\Delta I_{c,\pm}]^2 \rangle}$, as error bars, where $\Delta I_{c,\pm} \equiv I_{c,\pm} - \langle I_{c,\pm} \rangle$. The random average is taken for 400 samples.

as shown in Fig. 68(a). This is called the 0- π transition. With increase of magnetic field, some levels go to $|E| > \Delta_0$. At the same time, another levels come into $|E| \leq \Delta_0$. Therefore, the number of Andreev levels in $|E| \leq \Delta_0$ is fixed.

The Josephson current is calculated from the sum of positive Andreev levels in Eq. (27.27). In Fig. 67(a), the Andreev levels $E_n(\varphi)$ are invariant against the inversion of φ , $E_n(-\varphi) = E_n(\varphi)$. As a result, the Josephson current satisfies $I(-\varphi) = -I(\varphi)$ in Fig. 67(b). When $\theta_B = 0$, the current $I(\varphi)$ is similar to $\sin \varphi$, which is a feature of 0-state. When the level crossing takes place at $E = 0$, the crossing results in the discontinuity in the current phase relation. Around $\theta_B = \pi/2$, a saw-tooth current phase relation is obtained. The discontinuous points move from $\pm\pi$ to 0. When $\theta_B \approx \pi$, the current is roughly $I(\varphi) \sim -\sin \varphi$, which is a feature of π -state.

Figure 68(b) shows the critical current as a function of θ_B . Since $I(-\varphi) = -I(\varphi)$, the maximum of Josephson current, $I_{c,+}$, is identical with the absolute value of minimum of current, $I_{c,-}$. When the magnetic field is stronger, the phase difference φ_0 at the minimum of ground-state energy changes between 0 and π discontinuously at $\theta_B \approx (2m+1)\pi/2$, where $m = 0, 1, 2, \dots$ [Fig. 68(a)]. The critical current oscillates with cusps around the 0- π transitions. In Fig. 68(c), we plot a random average $\langle I_{c,\pm} \rangle$ of the critical current with the fluctuation $\sqrt{\langle [\Delta I_{c,\pm}]^2 \rangle}$. The fluctuation is defined as $\sqrt{\langle [\Delta A]^2 \rangle}$ with $\Delta A \equiv A - \langle A \rangle$. $\langle I_{c,\pm} \rangle$ also exhibits the cusps at $\theta_B \approx (2m+1)\pi/2$, where its fluctuation is relatively small. When the Fermi energy is tuned, θ_B is also modified via the velocity $v_{F,1}$. However, the cusp is always found at $\theta_B \approx (2m+1)\pi/2$ (not shown).

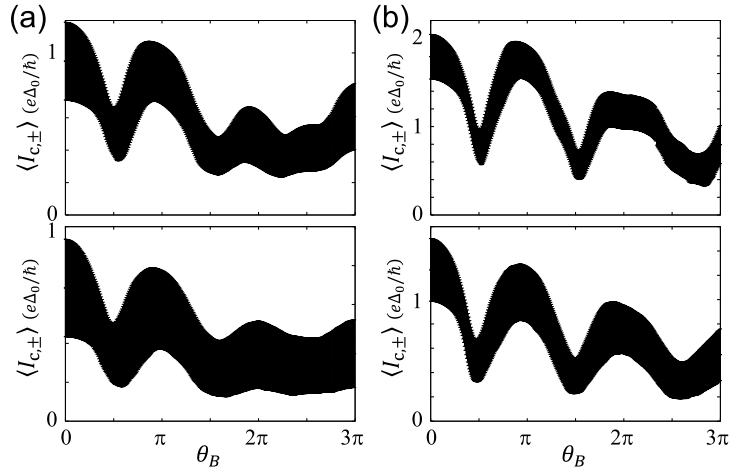


Figure 69: Calculated results of random average of critical current, $\langle I_{c,\pm} \rangle$ as a function of magnetic field, θ_B when $N = 2$ (a) and 4 (b). Error bars represent the average of fluctuation, $\sqrt{\langle [\Delta I_{c,\pm}]^2 \rangle}$, where $\Delta I_{c,\pm} \equiv I_{c,\pm} - \langle I_{c,\pm} \rangle$. The SO interaction is absent. The mean free path is $l_0/L = 1$ (upper) and 0.5 (lower panels). The random average is taken for 400 samples.

A few conduction channels

Next, we consider the case of a few conduction channels in the nanowire. For N conduction channels, $2N$ positive and $2N$ negative Andreev levels are obtained even if the channels are mixed with each other by the impurity scattering. The behavior of Andreev levels in magnetic field is qualitatively the same as in Fig. 67(a) except the number of levels. The Andreev levels keep the relation $E(-\varphi) = E(\varphi)$, which results in the current $I(-\varphi) = -I(\varphi)$. The critical current is independent of its current direction. When the magnetic field is applied, the critical current oscillates accompanying the $0-\pi$ transition around the local minima of $I_{c,\pm}$.

Upper and lower panels in Fig. 69(a) show the random average of critical current for $N = 2$ when the mean free path is $l_0/L = 1$ and 0.5, respectively. The average $\langle I_{c,\pm} \rangle$ of critical current oscillates as a function of magnetic field and indicates the first and second local minima at $\theta_B \approx \pi/2$ and $3\pi/2$, respectively. The positions of the two local minima are hardly shifted by the impurity scattering, or the mean free path l_0 . $\langle I_{c,\pm} \rangle$ becomes local minimal also around $\theta_B = 5\pi/2$ when $l_0/L = 0.5$, whereas the position of the third local minimum is shifted from $\theta_B = 5\pi/2$ in the case of $l_0/L = 1$. Figure 69(b) shows $\langle I_{c,\pm} \rangle$ in the case of $N = 4$, where $\langle I_{c,\pm} \rangle$ is local minimal at $\theta_B \approx \pi/2$ and $3\pi/2$. For both cases of $N = 2$ and 4, the local minima of $\langle I_{c,\pm} \rangle$ tend to be located at $\theta_B \approx (2m + 1)\pi/2$ when the impurity scattering is stronger. This period is the same as that of $N = 1$.

21.2.2 Presence of spin-orbit interaction

Anomalous Josephson effect

In this section, we consider the SO interaction in the nanowire. The SO interaction qualitatively modifies the Andreev levels in the presence of magnetic field. Figure 70 shows the Andreev level and the Josephson current for a sample in the case of $N = 1$. We assume that the strength of SO interaction is $k_\alpha/k_F = 0.15$. The mean free path is $l_0/L = 1$. In the absence of magnetic field, the time-reversal symmetry is kept and the Andreev levels are two-fold degenerate even when $\varphi \neq 0$ in the case of short junction. [65] The levels satisfy $E(-\varphi) = E(\varphi)$ and the ground-state energy $E_{\text{gs}}(\varphi)$ is minimum at $\varphi = 0$. As the magnetic field is increased, we find three regions as well as the case without SO interaction in Sec. 21.2.1. In region (I), for a weak magnetic field, some levels are positive for any phase difference and the others are

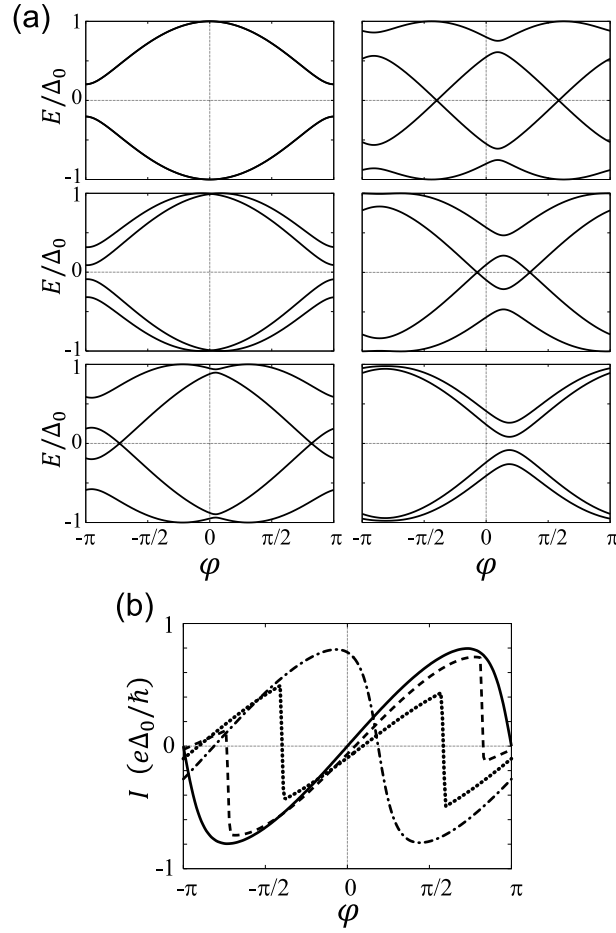


Figure 70: Calculated results for a sample when $N = 1$ and $l_0/L = 1$. The SO interaction is $k_\alpha/k_F = 0.15$. The magnetic field is applied to the y direction. (a) Andreev levels E_n as a function of phase difference φ between two superconductors. The magnetic field is $\theta_B = 0$ (left upper), 0.1π (left middle), 0.35π (left bottom), 0.7π (right upper), 1.05π (right middle), and 1.4π (right bottom). At $B = 0$, two lines are overlapped to each other, reflecting the Kramers' degeneracy. (b) Josephson current $I(\varphi)$ through the nanowire when $\theta_B = 0$ (solid), 0.35π (broken), 0.7π (dotted), and 1.4π (dotted broken lines).

negative although the levels are split by the magnetic field. With increase of θ_B , the splitting is larger and the level crossing takes place at $E = 0$ in region (II). This level crossing disappears when the magnetic field is $\theta_B \approx \pi$ [region (III)].

The finding of the regions is the same as that without SO interaction, whereas the invariance of levels against the φ -inversion is broken when $\theta_B \neq 0$. As a result, the phase difference φ_0 at the minimum of ground-state energy is deviated from 0 or π , as shown in Fig. 71(a), and the φ_0 -state is realized. [32, 53, 54] The phase difference φ_0 is almost liner to the magnetic field first, and jumps to $\varphi_0 \approx \pi$ like the $0-\pi$ transition. After the transition, φ_0 increases gradually with increase of θ_B , the slope of which is almost the same as that in the '0 like'-state. At $\theta_B \approx 2\pi$, the ' π like'-state transits back to the '0 like'-state. This behavior is understood as the $0-\pi$ transition with additional phase shift.

In Fig. 70(b), the Josephson current is calculated from the Andreev levels in Fig. 70(a). When $B = 0$, the current satisfies $I(-\varphi) = -I(\varphi)$, whereas this relation is broken in the magnetic field. With increasing magnetic field, the discontinuous points of current corresponding to the level crossings at $E = 0$ are found. $I(\varphi)$ indicates a saw-tooth behavior when $\theta_B \approx 0.7\pi$. The discontinuous points vanish at $\theta_B = 1.4\pi$. Compared with those in Fig. 67(b), the current phase relation is gradually moved to right in the panel as the magnetic field is increased.

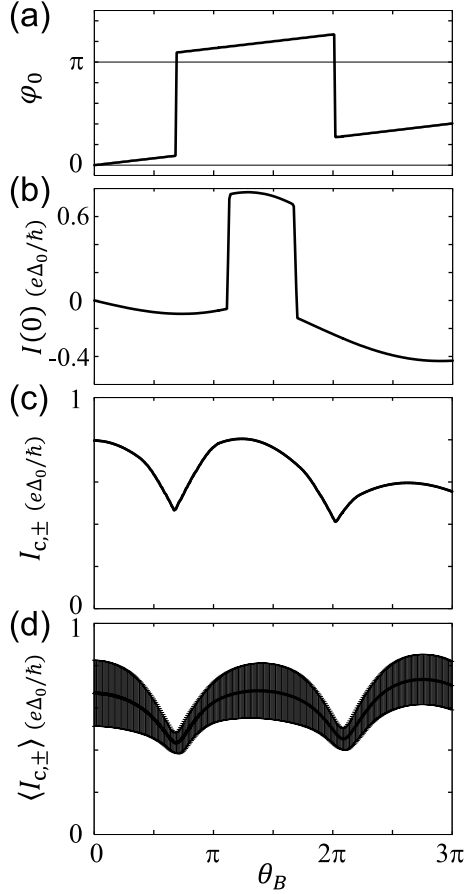


Figure 71: Calculated results for $N = 1$ and $l_0/L = 1$. The SO interaction is $k_\alpha/k_F = 0.15$. The magnetic field is applied to the y direction. (a) Phase difference φ_0 at the minimum of ground-state energy as a function of magnetic field, θ_B . (b) Anomalous Josephson current $I(\varphi = 0)$. (c) Critical current $I_{c,\pm}$. The current in the positive direction $I_{c,+}$ is identical with that in the negative direction $I_{c,-}$. The sample for (a), (b), and (c) is same as that in Fig. 70. (d) Average of critical current, $\langle I_{c,\pm} \rangle$, with the average of fluctuation, $\sqrt{\langle [\Delta I_{c,\pm}]^2 \rangle}$, as error bars, where $\Delta I_{c,\pm} \equiv I_{c,\pm} - \langle I_{c,\pm} \rangle$. The random average is taken for 400 samples.

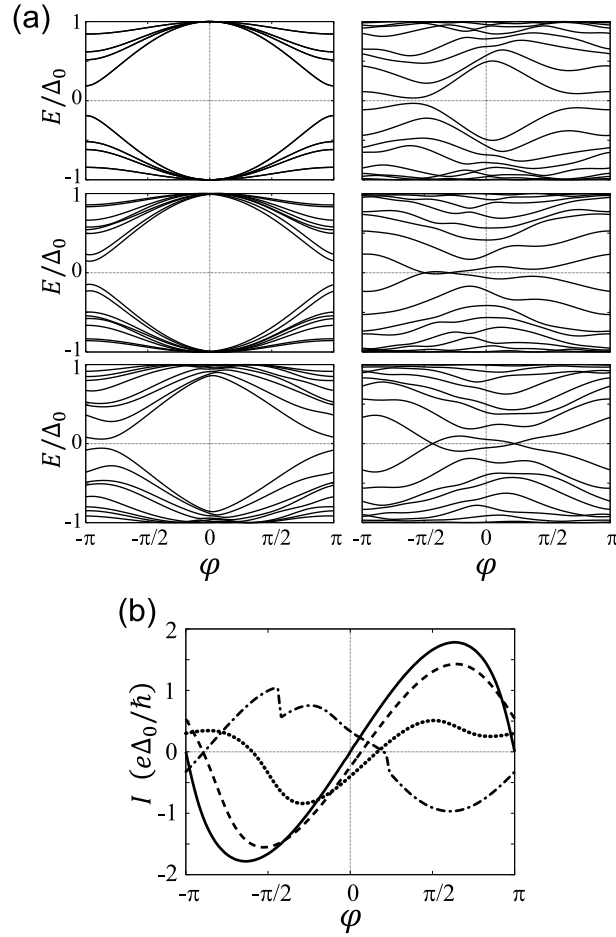


Figure 72: Calculated results for a sample when $N = 4$ and $l_0/L = 1$. The SO interaction is $k_\alpha/k_F = 0.15$. The magnetic field is applied to the y direction. (a) Andreev levels E_n as a function of phase difference φ between two superconductors. The magnetic field is $\theta_B = 0$ (left upper), 0.1π (left middle), 0.4π (left bottom), 0.8π (right upper), 1.2π (right middle), and 1.6π (right bottom). At $B = 0$, two lines are overlapped to each other, reflecting the Kramers' degeneracy. (b) Josephson current $I(\varphi)$ through the nanowire when $\theta_B = 0$ (solid), 0.4π (broken), 0.8π (dotted), and 1.6π (dotted broken lines).

As a result, a finite supercurrent at $\varphi = 0$ (anomalous Josephson current) is obtained [Fig. 71(b)]. For a weak magnetic field, the anomalous current is negative since the shift of current phase relation is positive ($\varphi_0 > 0$). In the ' π like'-state, the positive anomalous supercurrent is obtained, where $|I(\varphi = 0)|$ is enlarged up to $0.7e\Delta_0/\hbar$.

Figure 71(c) indicates the critical current $I_{c,\pm}$ as a function of θ_B . Although the relation $I(-\varphi) = -I(\varphi)$ does not hold, the critical currents for positive and negative directions are identical with each other in the case of $N = 1$. The critical current oscillates with cusps at the local minima. The distance of cusps is longer than that in Fig. 68(b), which is caused by the modification of Fermi velocity due to the SO interaction. The random average $\langle I_{c,\pm} \rangle$ of critical current indicates local minima at $\theta_B \approx 0.7\pi$ and $\approx 2.1\pi$, as shown in Fig. 71(d). The fluctuation is also small around the minima of $\langle I_{c,\pm} \rangle$, where the order of fluctuation is $0.1e\Delta_0/\hbar$. These behaviors are qualitatively the same as those in Fig. 68(c).

Direction-dependent critical current

Here, we consider the case of four conduction channels and demonstrate that the critical current depends on the current direction.

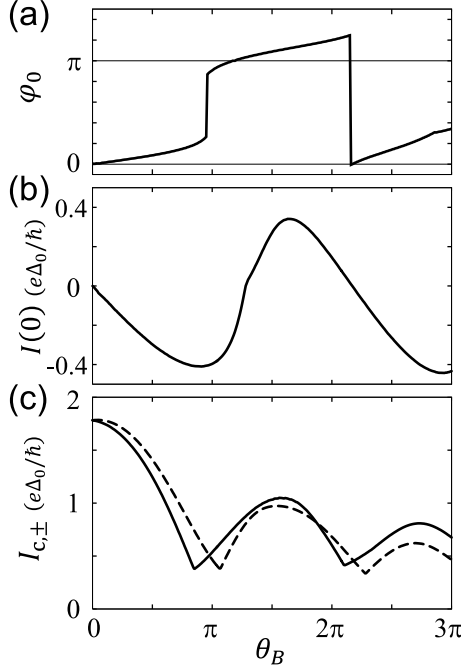


Figure 73: Calculated results for $N = 4$ and $l_0/L = 1$. The SO interaction is $k_\alpha/k_F = 0.15$. The magnetic field is applied to the y direction. The sample is same as that in Fig. 72. (a) Phase difference φ_0 at the minimum of ground-state energy as a function of magnetic field, θ_B . (b) Anomalous Josephson current $I(\varphi = 0)$. (c) Critical current in the positive $I_{c,+}$ (solid) and in the negative direction $I_{c,-}$ (broken lines).

Figures 72(a) and (b) exhibit the Andreev levels and the Josephson current, respectively, as functions of φ when the magnetic field increases. In each panel of Fig. 72(a), eight positive and eight negative levels are obtained. In the absence of magnetic field, the levels are doubly degenerate and invariant against the inversion of φ . Four channels are mixed by the impurity scattering and SO interaction, which contribute to form the Andreev bound states. In the presence of magnetic field, the Zeeman effect splits these mixed levels. Then, the Andreev levels become complicated function of φ and the symmetry of $E(-\varphi) = E(\varphi)$ is broken. As a result, the Josephson current indicates $I(-\varphi) \neq -I(\varphi)$ in Fig. 72(b), where not only the anomalous current, $I(\varphi = 0) \neq 0$, but also the difference between maximum and absolute value of minimum currents, $I_{c,+} \neq I_{c,-}$, are obtained.

We mention the three regions corresponding to region (I), (II), and (III) described in previous sections. The Josephson current roughly indicates $I(\varphi) \sim \sin \varphi$ at $\theta_B = 0$. This is the feature of 0-state in region (I) at $\theta_B \sim 0$. At $\theta_B = 1.6\pi$, the current becomes roughly $I(\varphi) \sim -\sin \varphi$, which corresponds to the feature of π -state in region (III) although the crossing of Andreev levels at $E = 0$ is found. The phase difference φ_0 at the minimum of E_{gs} also indicates the feature of these two regions: $\varphi_0 \approx 0$ at $\theta_B \sim 0$ and $\varphi_0 \approx \pi$ at $\theta_B \sim 1.5\pi$, as shown in Fig. 73(a). As the magnetic field is increased, φ_0 monotonically increases until $\theta_B \simeq 2.1\pi$. The ‘0- π like’ transition occurs at $\theta_B \approx \pi$. The boundaries between these regions and region (II) are unclear since the SO interaction tends to avoid the level crossing at $E = 0$. When θ_B is increased up to 3π , another ‘0- π like’ transition is found at $\theta_B \approx 2.1\pi$.

As shown in Fig. 72(b), the finite supercurrent at $\varphi = 0$ is induced by the interplay between SO interaction and Zeeman effect. $I(\varphi = 0)$ is plotted as a function of θ_B in Fig. 73(b). The anomalous current decreases first, and sharply increases in the ‘ π like’-state region. This behavior is qualitatively the same as that for $N = 1$ in Fig. 71(b). However, the maximum of $|I(0)|$ is smaller.

Besides the anomalous Josephson current, the direction dependence of critical current is

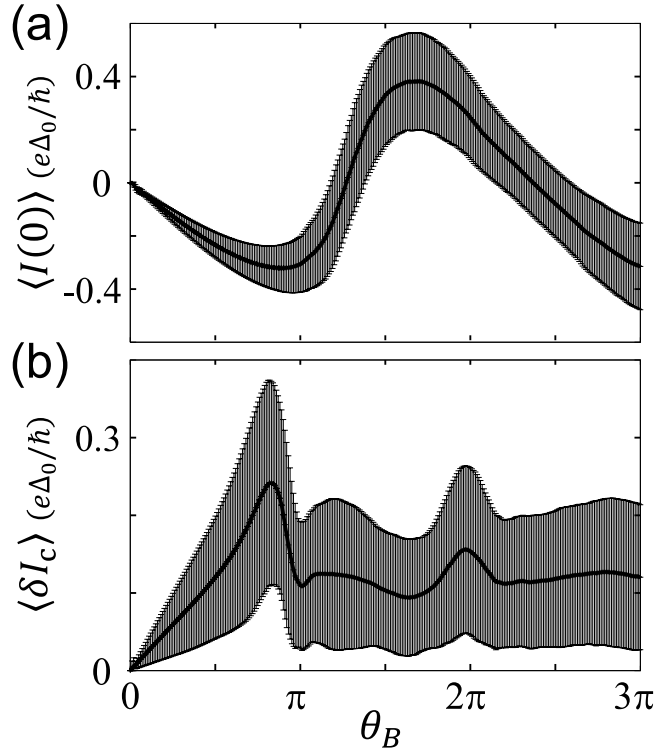


Figure 74: Calculated results of random average when $N = 4$ and $l_0/L = 1$. The SO interaction is $k_\alpha/k_F = 0.15$. The magnetic field is applied to the y direction. The random average is taken for 400 samples. (a) Average of anomalous Josephson current, $\langle I(\varphi = 0) \rangle$ as a function of magnetic field, θ_B . Error bars represent the average of fluctuation, $\sqrt{\langle [\Delta I(0)]^2 \rangle}$, where $\Delta A \equiv A - \langle A \rangle$. (b) Average of difference of critical current, $\langle \delta I_c \rangle$, where $\delta I_c \equiv |I_{c,+} - I_{c,-}|$. Error bars are $\sqrt{\langle [\Delta \{ \delta I_c \}]^2 \rangle}$.

observed in the case of $N > 1$. Figure 73(c) shows $I_{c,\pm}$ as a function of θ_B . Both critical currents oscillate with the cusps at the local minima of $I_{c,\pm}$. The position of cusps also depends on the current direction. In Fig. 73(c), $I_{c,+}$ and $I_{c,-}$ show the cusps below and above the critical points of transition in Fig. 73(a), respectively.

In Fig. 74, we examine the random average of current regarding impurity potentials. The number of samples is 400. Figure 74(a) shows the average of anomalous supercurrent, $\langle I(0) \rangle$, with its fluctuation, $\sqrt{\langle [\Delta I(0)]^2 \rangle}$. The average of anomalous current indicates negative and positive values alternatively as a function of θ_B , where $|\langle I(0) \rangle|$ is enlarged up to about $0.4e\Delta_0/\hbar$. On the other hand, $\sqrt{\langle [\Delta I(0)]^2 \rangle}$ is saturated at about $0.15e\Delta_0/\hbar$. The inversion of sign of $\langle I(0) \rangle$ attributes to the ‘0- π like’ transition. Roughly speaking, the current phase relation transits from $I(\varphi) \sim \sin(\varphi - \varphi_0)$ to $-\sin(\varphi - \varphi_0)$. Then, $I(0)$ changes the sign from negative to positive for $\varphi_0 > 0$.

In Fig. 74(b), we consider the average of $\delta I_c \equiv |I_{c,+} - I_{c,-}|$. In the absence of magnetic field, the critical current for positive and negative direction is equal to each other, $\langle \delta I_c \rangle = 0$. $\langle \delta I_c \rangle$ increases with increase of θ_B . $I_{c,\pm}$ sharply changes around the cusps. Thus $\langle \delta I_c \rangle$ becomes maximum around the ‘0- π like’ transition. In the case of $N > 1$, the oscillation of critical current is strongly affected by the impurity scattering. Then, the average and its fluctuation are saturated and almost constant for $\theta_B > \pi$ except the vicinity of critical points of transition.

21.3 Conclusion and discussion

We have studied numerically the DC Josephson effect in quasi-one-dimensional semiconductor nanowire with strong SO interaction when the Zeeman effect is present. We have examined

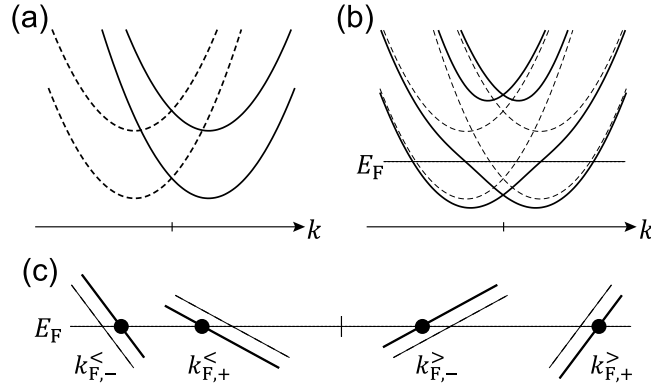


Figure 75: Schematic views of dispersion relation with SO interaction. (a) Spin-splitting due to the $p_x \sigma_y$ term in Eq. (21.2). Solid and broken lines indicate the branches with spin $\sigma = +$ and $-$, respectively. (b) Dispersion relation mixed by $p_y \sigma_x$ term (solid line). The broken line indicate the case without $p_y \sigma_x$ term. (c) Shift of wavenumber due to the Zeeman effect in the vicinity of Fermi level.

the tight-binding model to describe the electron and hole transport in the normal region in S/NW/S junction. The magnetic field and Rashba SO interaction are considered in the normal region. We have focused on the case of short junction, where the length of normal region is much smaller than the coherent length, $L \ll \xi$. In the absence of SO interaction, the Andreev levels are invariant against the inversion of phase difference φ between two superconductors. As a result, the Josephson current satisfies $I(-\varphi) = -I(\varphi)$, where no supercurrent is obtained at $\varphi = 0$. The $0-\pi$ transition accompanying an oscillation of critical current is observed when the magnetic field is increased. We have introduced a parameter θ_B for magnetic field, which describes the spin-dependent phase shift of electron and hole transport in the normal region. At $\theta_B \approx (2m+1)\pi/2$, the $0-\pi$ transition takes place and the cusp of critical current is found. In the presence of Rashba interaction, we have demonstrated the anomalous Josephson effect. The Andreev levels does not keep the relation of $E_n(-\varphi) = E_n(\varphi)$ when the magnetic field is applied. As a result, the phase difference φ_0 at the minimum of ground-state energy is deviated from 0 and π (φ_0 -state). The current phase relation becomes $I(-\varphi) \neq -I(\varphi)$, where the anomalous supercurrent at $\varphi = 0$ is obtained. In addition, the critical current depends on its current direction when more than one conduction channel is present in the nanowire. The critical current oscillates as a function of θ_B , where the position of cusps also depends on the current direction. The transition between $\varphi \approx 0$ and $\varphi \approx \pi$ takes place between the cusps of positive and negative currents.

Our calculated results have exhibited the anomalous supercurrent, $I(\varphi = 0) \neq 0$, and the direction-dependence of critical current, $I_{c,+} \neq I_{c,-}$, when $N > 1$, in accordance with the single scatterer model. [52] $I(\varphi = 0) \neq 0$ is found even when $N = 1$, which points out a role of the spin-dependent Fermi velocity on the anomalous Josephson effect. Krive *et al.* also have discussed the role of Fermi velocity in the case of long junction. [49] The dispersion relation with SO interaction is schematically shown in Fig. 75. In the nanowire, electrons are confined in the y direction. Thus, the $p_x \sigma_y$ term in Rashba interaction in Eq. (21.2) mainly contributes to the dispersion relation rather than the $p_y \sigma_x$ term. The $p_x \sigma_y$ term induces the spin-splitting at $k \neq 0$ [Fig. 75(a)]. The spins are directed in the $\pm y$ directions. The term of $p_y \sigma_x$ mixes the lowest branch with spin $\sigma = \pm$ and the second lowest one with $-\sigma$. Due to the mixing, the Fermi velocity depends on the spin direction, as shown in Fig. 75(b). Here, $\sigma = \pm$ is not good quantum number. However, since the spins are almost directed to the y axis, we use $\sigma = \pm$ to indicate the spin. We focus on the vicinity of Fermi energy in Fig. 75(c). When the magnetic field is applied in the y direction, the branches with spin $\sigma = \pm$ go downward and upward,

respectively. The wavenumbers are modified spin-dependently, $k_{F,\pm}^> = k_F \pm k_\alpha \pm E_Z/(\hbar v_{F,\pm})$ for positive wavenumber and $k_{F,\pm}^< = -k_F \pm k_\alpha \mp E_Z/(\hbar v_{F,\pm})$ for negative one. Although $k_F \pm k_\alpha$ is also modified by the channel mixing, that does not affect the following discussion. By applying these spin-dependent shifts of wavenumber, $k_{F,\pm}^>$ and $k_{F,\pm}^<$, to $\hat{\tau}_B$ in the single scatter model (see Appendix), the Andreev levels for $N = 1$ are given as

$$E_{\uparrow\pm}(\varphi) = \Delta_0 \cos \left[\frac{\theta_B}{2} + \arccos \left(\pm \sqrt{\frac{1 + \delta_B + T \cos(\varphi + \varphi_0)}{2}} \right) \right], \quad (21.22)$$

$$E_{\downarrow\pm}(\varphi) = \Delta_0 \cos \left[-\frac{\theta_B}{2} + \arccos \left(\pm \sqrt{\frac{1 + \delta_B + T \cos(\varphi + \varphi_0)}{2}} \right) \right], \quad (21.23)$$

where $\theta_B = L(k_{F,+}^> - k_{F,+}^< - k_{F,-}^> + k_{F,-}^<)/2$, $\delta_B = (1 - T) \cos \{\theta_B(2x_0 - L)/L\}$, and

$$\varphi_0 = \frac{L}{2}(k_{F,+}^> + k_{F,+}^< + k_{F,-}^> + k_{F,-}^<) = E_Z L \left(\frac{1}{\hbar v_{F,+}} - \frac{1}{\hbar v_{F,-}} \right). \quad (21.24)$$

T is a transmission probability of the scatterer at $x = x_0$ without the SO interaction. This φ_0 is proportional to the magnetic field, and results in the anomalous Josephson effect. If the $p_y \sigma_x$ term in Eq. (21.2) is disregarded, we find no anomalous Josephson effect even when $N > 1$ (not shown). In Ref. [52], the single scatterer model demonstrated the anomalous current and the direction-dependence of critical current when $N > 1$. Then, the single scatterer mixes the conduction channels spin-dependently, which effectively plays the same role as the spin-dependent Fermi velocity in the electron transport.

In this paper, we have assumed $k_\alpha/k_F = 0.15$. Typical value of Rashba constant in experiments is $\alpha = 3 - 4 \times 10^{-11} \text{ eV} \cdot \text{m}$ for InAs or InGaAs. [4, 5, 6] The SO interaction in InSb tend to be stronger than that in InAs. For $m^* = 0.014m_e$ (InSb) and $\lambda_F = 90 \text{ nm}$, the parameter $k_\alpha/k_F \simeq 0.15$ corresponds to $\alpha = 5.7 \times 10^{-11} \text{ eV} \cdot \text{m}$.

In recent experiments for InSb nanowire, the direction-dependence of critical current was observed in the magnetic field along the nanowire. [30] This situation disagrees with our results considering the Rashba interaction. Thus an actual SO interaction in the nanowire is not expressed as Eq. (21.2). In the nanowire, the direction of spin quantization axis due to the SO interaction may depend on the position x . However, our discussion can be extended to the case of general SO interaction since the anomalous Josephson effect is observed when an applied magnetic field has a parallel component to the spin quantization axis. In the experiments, a few channels may exist in the nanowire. The spacing between two superconductors is $L \simeq 500 - 1000 \text{ nm}$, whereas the coherent length in the nanowire is estimated as $\xi \sim 350 \text{ nm}$. This means $L \gtrsim \xi$. We have exhibited the anomalous Josephson effect even for $L \ll \xi$. Therefore, the long (or intermediated-length) junction is not essential condition. For the measurement, however, the long nanowire is reasonable since φ_0 in Eq. (21.24) is larger as the length L is longer. The spin relaxation length due to the SO interaction is estimated as $l_{\text{SO}} \sim 200 \text{ nm}$ ($\lesssim L$). Therefore the effect of SO interaction on the Josephson effect can be observed in experiments. The position of first cusp is located at $\theta_B \sim \pi/2$, which corresponds to $B \sim 0.2 \text{ T}$ in our situation. This order of magnitude is reasonable for the experiments.

21.4 Calculation Method of Tight-Binding Model

Here, we explain a calculation method of scattering matrix using the Green's function. [61]

We consider the matrix representation of Hamiltonian in Eq. (21.16),

$$H = \begin{pmatrix} \tilde{H}_0 & -t\tilde{T}_{0,1} & & & \\ -t\tilde{T}_{1,0} & \tilde{H}_1 & -t\tilde{T}_{1,2} & & \\ & -t\tilde{T}_{2,1} & \ddots & & \\ & & & \tilde{H}_{N_x} & -t\tilde{T}_{N_x,N_x+1} \\ & & & -t\tilde{T}_{N_x+1,N_x} & \tilde{H}_{N_x+1} \end{pmatrix}, \quad (21.25)$$

where $\tilde{H}_j$ is a $2N_y \times 2N_y$ matrix describing the j -th slice,

$$\tilde{H}_j = \begin{pmatrix} (v_{j,1} + 4t)\hat{1} & -t\hat{T}_{j,1;j,2} & & \\ -t\hat{T}_{j,2;j,1} & (v_{j,2} + 4t)\hat{1} & & \\ & & \ddots & \\ & & & (v_{j,N_y} + 4t)\hat{1} \end{pmatrix}. \quad (21.26)$$

$\hat{1}$ is a 2×2 unit matrix and $v_{j,l}$ denotes the on-site potential at (j, l) . The hopping term in Eq. (21.25) is

$$\tilde{T}_{j,j\pm 1} = \begin{pmatrix} \hat{T}_{j,1;j\pm 1,1} & & \\ & \ddots & \\ & & \hat{T}_{j,N_y;j\pm 1,N_y} \end{pmatrix}. \quad (21.27)$$

In an ideal lead, the wavefunctions of conduction channels are written as

$$\psi_\mu(j, l) = \exp(ik_\mu a j) u_\mu(l), \quad (21.28)$$

$$u_\mu(l) = \sqrt{\frac{2a}{W}} \sin\left(\frac{\pi \mu l a}{W}\right). \quad (21.29)$$

The wavenumber k_μ satisfies $E_\mu(k_\mu) = E_F$, where the dispersion relation is given by

$$E_\mu(k) = 4t - 2t \cos\left(\frac{\pi \mu a}{W}\right) - 2t \cos(ka). \quad (21.30)$$

The band edge, $E_\mu(k = 0)$, is located below E_F for the conduction modes. The wavefunction of evanescent mode is written as

$$\psi_\mu(j, l) = \exp(-\kappa_\mu a j) u_\mu(l). \quad (21.31)$$

The band edge is located above E_F and κ_μ is determined from $E_\mu(i\kappa_\mu) = E_F$. Here, we introduce some matrices for the calculation of scattering matrix. $U = (\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{N_y})$ is an unitary matrix, with $\mathbf{u}_\mu = (u_\mu(1), u_\mu(2), \dots, u_\mu(N_y))^T$ in Eq. (21.29). $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_{N_y})$, where $\lambda_\mu = \exp(ik_\mu a)$ for conduction channels and $\lambda_\mu = \exp(-\kappa_\mu a)$ for evanescent modes.

The retarded Green's function is defined as

$$G = \frac{1}{EI - H + \Sigma}, \quad (21.32)$$

where Σ is the self-energy representing the coupling with leads,

$$\Sigma = \begin{pmatrix} tF(-)^{-1} & \\ & tF(+) \end{pmatrix} \quad (21.33)$$

with $F(\pm) = U\Lambda^{\pm 1}U^{-1}$. The Green's function connects the amplitudes of incoming and outgoing waves at the slices $j = 0, N_x + 1$,

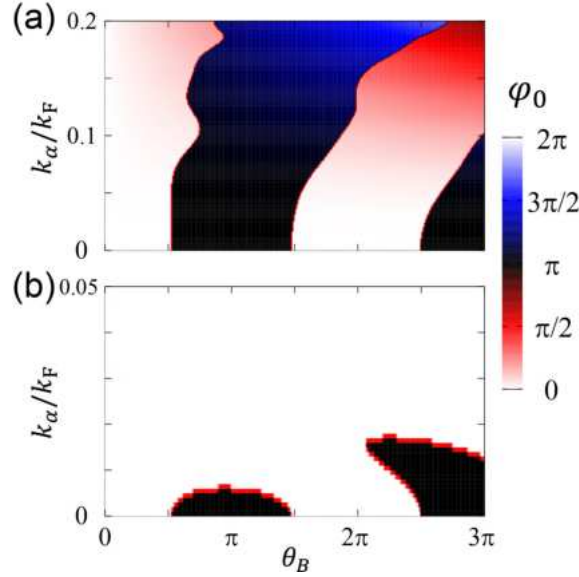


Figure 76: Grayscale plot of the phase difference φ_0 at the minimum of ground-state energy in the plane of magnetic field θ_B and SO interaction k_α/k_F when $N = 1$ and $l_0/L = 1$. The sample is same as that in Fig. 70. The magnetic field is applied in the y (a) and x directions (b).

$$\begin{pmatrix} \mathbf{C}_0(-) \\ \mathbf{C}_{N_x+1}(+) \end{pmatrix} = \begin{pmatrix} -tG_{0,0}[F^{-1}(+) - F^{-1}(-)] - 1 & -tG_{0,N_x+1}[F(+) - F(-)] \\ -tG_{N_x+1,0}[F^{-1}(+) - F^{-1}(-)] & -tG_{N_x+1,N_x+1}[F(+) - F(-)] - 1 \end{pmatrix} \begin{pmatrix} \mathbf{C}_0(+) \\ \mathbf{C}_{N_x+1}(-) \end{pmatrix}, \quad (21.34)$$

where $G_{j,j'}$ is the $2N_y \times 2N_y$ matrix for the (j, j') component of G . The vectors $\mathbf{C}_0(\pm)$ and $\mathbf{C}_{N_x+1}(\pm)$ yield the coefficients of waves of conduction channels in the ideal leads [Fig. 66(b)] as

$$\mathbf{a}_{eL} = \sqrt{V}U^{-1}\mathbf{C}_0(+), \quad (21.35)$$

$$\mathbf{a}_{eR} = \sqrt{V}U^{-1}\mathbf{C}_{N_x+1}(-), \quad (21.36)$$

$$\mathbf{b}_{eL} = \sqrt{V}U^{-1}\mathbf{C}_0(-), \quad (21.37)$$

$$\mathbf{b}_{eR} = \sqrt{V}U^{-1}\mathbf{C}_{N_x+1}(+), \quad (21.38)$$

where $\sqrt{V} \equiv \text{diag}(\sqrt{v_{F,1}}, \dots, \sqrt{v_{F,N_y}})$ is a diagonal matrix of square root of velocities in Eq. (21.20). If the channel is not conductive, the velocity is zero. By substituting these equations to Eq. (21.34), the scattering matrix $\hat{S}_e$ in Eq. (21.8) is obtained.

21.5 Single Scatterer Model

In this appendix, we explain the single scatterer model in Ref. [52]. The nanowire is along the x axis and connected with two superconductors at $x < 0$ and $x > L$. The pair potential is induced in the nanowire, the absolute value of which is constant, $\Delta(x) = \Delta_0 e^{i\varphi_L}$ at $x < 0$ and $\Delta_0 e^{i\varphi_R}$ at $L < x$, whereas $\Delta(x) = 0$ in the normal region at $0 < x < L$. We consider the short junction, $L \ll \xi$. A single scatterer describing an elastic scattering due to impurities and the SO interaction in the nanowire is introduced at $x = x_0$.

The scattering matrix for electrons by the scatterer is denoted as $\hat{S}_{\text{scatt}}$, which is given by the matrix of orthogonal ensemble in the absence of SO interaction and that of symplectic ensemble in the limit of strong SO interaction.

In the presence of magnetic field, the Zeeman effect is taken into account as the spin-dependent phase shift for electrons and holes in the propagation through the normal region. The Zeeman effect shifts the wavenumber as $k_{\pm}^{\geq} = k_F + (E \pm E_Z)/(\hbar v_F)$ for $k > 0$ and $k_{\pm}^{\leq} = -k_F - (E \pm E_Z)/(\hbar v_F)$ for $k < 0$. [66] The propagation of electron with spin $\sigma = \pm$ and hole with $\sigma = \mp$ at $0 < x < x_0$ acquires the phase $\pm\theta_{BL} = \pm 2E_Z x_0/(\hbar v_F)$, whereas that at $x_0 < x < L$ is $\pm\theta_{BR} = \pm 2E_Z(L - x_0)/(\hbar v_F)$. Here, v_F is independent of channels. The terms of $2Ex_0/(\hbar v_F)$ and $2E(L - x_0)/(\hbar v_F)$ are safely disregarded for short junctions. The terms of k_F are canceled out by each other. The phases are represented by the scattering matrix,

$$\hat{\tau}_B = \begin{pmatrix} \hat{1} \otimes \hat{\tau}_{BL} & \\ & \hat{1} \otimes \hat{\tau}_{BR} \end{pmatrix} \quad (21.39)$$

with

$$\hat{\tau}_{BL(R)} = \begin{pmatrix} e^{i\theta_{BL(R)}/2} & \\ & e^{-i\theta_{BL(R)}/2} \end{pmatrix}. \quad (21.40)$$

$\hat{1}$ is an $N \times N$ unit matrix. We fix an asymmetric parameter $\alpha_B \equiv x_0/(L - x_0) = \sqrt{2}$.

The Andreev reflection at $x = 0, L$ is also expressed in the term of scattering matrices $\hat{r}_{he}$ and $\hat{r}_{eh}$ in Sect. 21.1.2.

The product of the scattering matrices yields

$$\det \left(\hat{1} - \hat{\tau}_B \hat{r}_{eh} \hat{\tau}_B^* \hat{S}_{\text{scatt}}^* \hat{\tau}_B^* \hat{r}_{he} \hat{\tau}_B \hat{S}_{\text{scatt}} \right) = 0, \quad (21.41)$$

which determines the Andreev levels $E_n(\varphi)$.

This simple model demonstrated the anomalous Josephson effect and the direction-dependence of critical current when $N > 1$.

21.6 Magnetic field in the x direction: Disappearance of 0 - π transition

We discuss the case of magnetic field along the x direction, which is almost perpendicular to the spin quantization axis due to the Rashba interaction. In this appendix, we consider only a single conduction channel.

Figure 76 shows a grayscale plots of phase difference φ_0 at the minimum of ground-state energy in the plane of magnetic field and SO interaction. White and black regions mean the 0 - and π -state, respectively. Gray region corresponds to the φ_0 -state. In Fig. 76(a), the magnetic field is applied in the y direction, and the anomalous Josephson effect is obtained in the gray region. The critical points of transition is shifted to large θ_B when the SO interaction is stronger as mentioned in the section 21.2.2. We find the oscillation of critical points as a function of k_{α} , which maybe attributes to an interference due to the SO interaction only in the normal region.

In Fig. 76(b), only white and black regions are found. In the absence of SO interaction, the π -state is realized in $\pi/2 < \theta_B < 3\pi/2$. When k_{α}/k_F is increased, the region of π -state is narrower. Then, the π -state vanishes at $l_{\text{SO}}/L \lesssim 1$, where the SO length $l_{\text{SO}} \equiv \pi/(2k_{\alpha})$ means a distance of π rotation of spins due to the SO interaction. Figure 77(a) exhibits the phase difference φ_0 as a function of θ_B . The π -state around $\theta_B = \pi$ disappears with increase of k_{α} . The position of cusps of $I_{c,\pm}$ is also gradually closer to each other, and finally the cusps vanish [Fig. 77(b)].

The disappearance of π -state is interpreted by a spin precession due to the SO interaction. When the spin quantization axis of SO interaction is perpendicular to the magnetic field, the spin directed to the magnetic field is rotated by the SO interaction. For simple consideration, we assume that the SO interaction results in only a spin flip in electron (hole) transport.

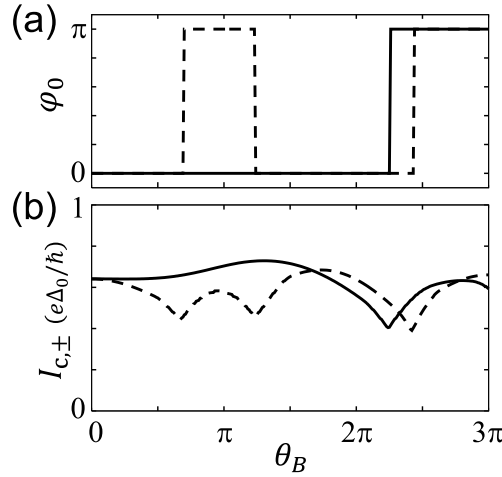


Figure 77: Calculated results for $N = 1$ and $l_0/L = 1$. The magnetic field is applied to the x direction. The sample is same as that in Fig. 70. (a) Phase difference φ_0 at the minimum of ground-state energy as a function of magnetic field, θ_B . (b) Critical current $I_{c,\pm}$. The current in the positive direction $I_{c,+}$ is identical to that in the negative direction $I_{c,-}$. Solid and broken lines in each panel indicates $k_\alpha/k_F = 0.01$ and 0.005 , respectively.

The Zeeman effect causes the spin-dependent phase shift though the shift of wavenumber. If the spin flip occurs at the middle point of normal region, the phase shift is exactly canceled out. Then, the $0-\pi$ transition can be quenched by the SO interaction. Liu *et al.* discussed a similar effect as $\pi-0$ transition by the tuning of SO interaction. [34] For in-plane magnetic field, the disappearance of π -state coincide with the anomalous Josephson effect. In our numerical calculation, we find large anomalous Josephson current even when the angle between magnetic field and SO interaction is less than $\pi/4$ (not shown). In experiments, the spin quantization axis may not be fixed. Thus, the anomalous Josephson effect is observed for arbitrary direction of magnetic field.

22 Nonlocal Josephson effect in Andreev molecules by J.-D. Pillet, Benzoni, J. Griesmar, J.-L. Smirr, C. O. Girit

Abstract

We propose the “Andreev molecule,” an artificial quantum system composed of two closely spaced Josephson junctions. The coupling between Josephson junctions in an Andreev molecule occurs through the overlap and hybridization of the junction’s “atomic” orbitals, Andreev Bound States. A striking consequence is that the supercurrent flowing through one junction depends on the superconducting phase difference across the other junction. Using the Bogolubov-de-Gennes formalism, we derive the energy spectrum and non-local current-phase relation for arbitrary separation. We demonstrate the possibility of creating a φ -junction and propose experiments to verify our predictions. Andreev molecules may have potential applications in quantum information, metrology, sensing, and molecular simulation.

Understanding and exploiting the interaction between Josephson junctions is paramount for superconducting device applications in quantum information [1], magnetometry [2], metrology [3], and quantum simulation [4]. In typical superconducting circuits, junctions interact

indirectly via electromagnetic coupling to inductors, capacitors, transmission lines, and microwave resonators. In addition to this well understood long-range interaction [5], there is a short range interaction via quasiparticle diffusion which can modify superconducting energy gaps and critical currents, but is only important close to T_c , the superconducting transition temperature, or at large bias voltages [6].

A second short-range interaction, mediated by Cooper pairs, is relevant to the majority of applications where characteristic energies are much smaller than the gap, but is still poorly understood. It becomes significant when the distance between Josephson junctions is comparable to ξ_0 , the superconducting coherence length, and can modify the electrical properties in a dramatic way.

Initially, minor effects resulting from this “order-parameter interaction” were calculated for temperatures near T_c using the Ginzburg-Landau equations [7]. More recently, theorists have investigated this problem at arbitrary temperature using Green’s function techniques. In the two-electrode geometry, where it is not possible to independently apply a phase difference to each junction, the overall current-phase relation and dc current were obtained [8, 9]. For the more relevant three-electrode geometry, non-local out-of-equilibrium supercurrents were calculated and the existence of π shifts in the current-phase relation were demonstrated [10, 11, 12, 13, 14, 15]. A remarkable phase-locking similar to Shapiro steps was predicted and subsequently measured experimentally in superconducting bi-junctions biased with commensurate voltages [16, 17]. The authors attribute these phenomena to the formation of entangled Cooper pairs called “quartets.”

Although this interpretation is intriguing, many open questions remain: what is the microscopic mechanism involved? How does the interaction depend on the distance between the junctions? Have all non-local effects been revealed? How can one detect these effects and exploit them in devices?

In this work we answer these questions and show that the phenomena can be explained by a simple analogy to the formation of a hydrogen molecule, complete with orbital hybridization and level splitting. The analog of the hydrogen atom’s atomic orbitals are Andreev Bound States (ABS), localized electronic states that are the microscopic basis for the Josephson effect [18, 19]. The equivalent of the Bohr radius is ξ_0 , the characteristic length scale for the spread of ABS wavefunctions. In our superconducting “Andreev molecule,” where two Josephson junctions are separated by a distance $l \lesssim \xi_0$, the ABS wavefunctions localized at each junction overlap and hybridize. This hybridization modifies the total energy spectrum and can result in a φ -junction where the supercurrent flowing through one junction depends on the superconducting phase difference across the other junction.

The simplest Josephson junction is composed of two superconductors connected by a short, one-dimensional quantum conductor with a single electronic channel. Experimentally, this has been realized with superconducting atomic contacts [18, 20] and in semiconductor-superconductor nanowires [21, 22, 23, 24, 25]. These experimental results are well described by the Bogolubiov-de-Gennes (BdG) formalism and motivates us to use the same formalism to calculate the junction interaction. We consider a series connection of two ideal short weak links as shown in Figure 78(a) for large separation ($l \gg \xi_0$) and in Figure 78(b) for an Andreev molecule ($l = \xi_0$). By connecting a ground electrode to the central conductor one can flow current independently through each junction or apply individual superconducting phase differences $\delta_{L(R)}$ using magnetic fields. Experimentally, achieving small separations $l \lesssim \xi_0$ and making an electrical connection to the central superconductor is feasible with microfabrication techniques since the effective superconducting coherence length is approximately 100 nm for a disordered thin film of aluminum, a typical superconductor.

Within the BdG formalism, electrons in such a circuit can be described by a 2×2 Hamil-

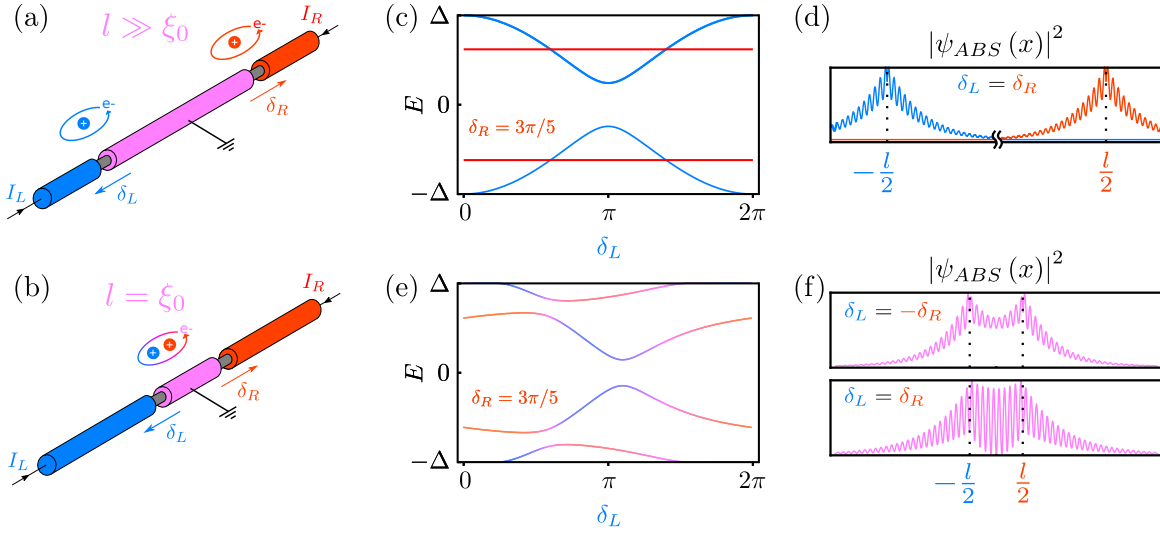


Figure 78: Two single-channel short weak links (gray) of transmissions $\tau_L = \tau_R \approx 0.94$ are connected to three superconducting regions (colored), of which the central one (pink) has length $l = 10\xi_0$ (a), or in the case of the Andreev molecule, $l = \xi_0$ (b). The ground connection allows applying phase differences δ_L and δ_R independently. (c) The Andreev Bound State (ABS) spectrum as a function of δ_L for fixed $\delta_R = 3\pi/5$ for $l = 10\xi_0$ and the corresponding wavefunctions at the two weak links (d). The degeneracies at $\delta_L = \pm\delta_R$ are lifted in the spectrum of the Andreev molecule (e) and the wavefunctions hybridize (f). Since the phases are 2π periodic, the $\delta_L = -\delta_R$ degeneracy is shown at $\delta_L = 2\pi - \delta_R = 7\pi/5$. Colors indicate relative localization of the wavefunction on the weak links (left: blue, right: red).

tonian H in Nambu space,

$$H = \begin{pmatrix} H_0 + H_{WL} & \Delta(x) \\ \Delta^*(x) & -H_0 - H_{WL} \end{pmatrix}, \quad (22.1)$$

$$\Delta(x) = \begin{cases} \Delta e^{i\delta_L} & \text{if } x < -l/2 \\ \Delta & \text{if } |x| < l/2 \\ \Delta e^{i\delta_R} & \text{if } x > l/2 \end{cases}, \quad (22.2)$$

where $H_0 = \frac{\hbar^2}{2m} \partial_x^2 - \mu$ is the single particle energy (m is the electron mass, μ the chemical potential) and $H_{WL} = U_L \delta(x + l/2) + U_R \delta(x - l/2)$ models scattering at the weak links at $x = \pm l/2$ with amplitudes $U_{L(R)}$. For simplicity we use Dirac δ -functions for the scatterers, appropriate for weak links of length shorter than ξ_0 , but expect the effects to persist in longer weak links, diminishing on a length scale given by ξ_0 . In the following, we consider symmetric scattering $U_L = U_R = U_0$ corresponding to both single-channel junctions having transmission $\tau \approx 0.94$, comparable to experimental values [24, 25].

The off-diagonal terms of the Hamiltonian, $\Delta(x)$ (Equation 22.1), describe electron pairing in each superconductor. The superconducting gap amplitude Δ is constant along the whole device and by gauge invariance we choose the phase of the central superconductor to be zero. In the one- to few-channel limit considered here the weak link supercurrents are much smaller than the critical currents of the superconducting regions and it is justified to ignore additional spatial variations in the order-parameter amplitude or phase.

Diagonalization of the BdG equation $H\psi = E\psi$ using an evanescent plane-wave basis for the wavefunctions ψ gives a discrete spectrum of Andreev Bound States with energies smaller than the superconducting gap $|E| < \Delta$ and a continuum of states for $|E| > \Delta$. For the plane-waves we make the standard approximation $\xi_0 \gg \lambda_F$ appropriate for typical superconductors, where λ_F is the Fermi wavelength. Analytic expressions for the spectrum can be obtained at arbitrary

separation for opaque junctions ($\tau \ll 1$) as well as for arbitrary transmission at large ($l/\xi_0 \gg 1$) or small separation ($l/\xi_0 \ll 1$). However the results shown here in the experimentally accessible intermediate case are calculated numerically.

Figure 78(c) shows the ABS spectrum for large separation, $l \gg \xi_0$, as a function of the left weak link's phase δ_L for fixed $\delta_R = 3\pi/5$ across the right weak link. As expected there is no interaction and the four levels are determined by the single junction expression $E_A^\pm(\delta, \tau_0) = \pm\Delta\sqrt{1 - \tau_0 \sin^2 \frac{\delta}{2}}$ by substituting $\delta = \delta_{L(R)}$ and $\tau_0 = \tau$. As a consequence of setting $\delta_R = 3\pi/5$, there are two dispersionless ABS (red lines) at energies $E_A^\pm(3\pi/5, \tau)$. Since the transmissions of both junctions are equal, $\tau_L = \tau_R = \tau$, the energy levels cross at $\delta_L = \pm\delta_R = \pm 3\pi/5$. The degeneracy at $\delta_L = -\delta_R$ is actually plotted at $\delta_L = 2\pi - \delta_R = 7\pi/5$ because the phases are 2π periodic. We confirm that the spectra correspond to distinct ABS localized at each weak link by plotting the ground-state wavefunctions in Figure 78(d) at the degeneracy point $\delta_L = \delta_R$.

The spectrum of the Andreev molecule Figure 78(e) changes drastically as a result of interactions between the junctions at small separation $l = \xi_0$. Not only do gaps open and lift the preceding degeneracies, but all spectral lines now depend on δ_L , despite the fact that δ_R remains fixed. Additional spectra for separation $l = 2\xi_0$, $l = 0.5\xi_0$, and $l \approx 0$ can be found in the Supporting Information.

The breaking of symmetry about $\delta_L = \pi$, absent in the spectra of the isolated junctions, is permitted by time-reversal invariance, which only requires $E(\delta_L, \delta_R) = E(-\delta_L, -\delta_R)$ and not $E(\delta_L, \delta_R) = E(-\delta_L, \delta_R)$. Although the continuum states are not plotted, states near the gap edge also develop a dispersion in the regions where the bound states approach the gap.

The opening of gaps is reminiscent of a coupling which hybridizes previously orthogonal states, and this picture is confirmed by a plot of the ABS wavefunctions in Figure 78(f) for $\delta_L = \pm\delta_R$. Only the wavefunction of the negative-energy state closest to zero energy is plotted in each case. The rapid oscillations here and in Figure 78(d) are due to the propagating part of the plane waves, whose wavelength $\lambda_F = \xi_0/5$ is chosen for visibility. The ABS wavefunctions are spread over both junctions with a significant weight in the central superconductor. Due to the off-diagonal term $\Delta(x)$ (Equation 22.2) the parity symmetry of the Hamiltonian is not the same for $\delta_L = \delta_R$ and $\delta_L = -\delta_R$. This results in the asymmetric spectrum as well as the differences in symmetry between the two wavefunctions, similar to bonding and anti-bonding states of a molecule.

In the context of quartets, the distinct hybridization mechanisms at $\delta_L = \delta_R$ and $\delta_L = -\delta_R$ are called double crossed Andreev reflection (dCAR) and double elastic cotunneling (dEC), respectively [10, 13]. Microscopically, dCAR corresponds to the transfer of Cooper pairs from the inner superconducting electrode to the outer ones while dEC corresponds to the transfer of Cooper pairs directly from the left to right electrode. One can link this Cooper pair transfer picture to our formalism via the scattering matrix approach as well as obtain analytical expressions for the dCAR and dEC currents in special limiting cases (see Supporting Information), but it is not necessary to invoke entanglement as with the quartet interpretation.

The transition from two independent junctions ($l \gg \xi_0$) to a single one ($l = 0$) with an intermediate molecular state is visualized in Figure 79(a) where the positive spectral lines for $\delta_L = \pm\delta_R$ are plotted as a function of l/ξ_0 for $\delta_R = 3\pi/5$ and the same transmission as before. At large separation $l/\xi_0 \gg 1$ all positive ABS converge to the same energy, $E_A^+(\delta = 3\pi/5, \tau)$, that of the upper red line in Figure 78(c).

For the hybridization mechanism at $\delta_L = -\delta_R$ (dEC, solid lines), with decreasing l/ξ_0 the two degenerate ABS of the left and right junction gradually split to form bonding and anti-bonding molecular states until the higher-energy ABS escapes into the continuum. The ABS pushed into the continuum is referred to as a “leaky” Andreev state [26] but unlike in-gap ABS are delocalized plane waves [27]. Beyond that point the device hosts only one pair of ABS, symmetric in energy, such that it behaves as a single artificial atom rather than a

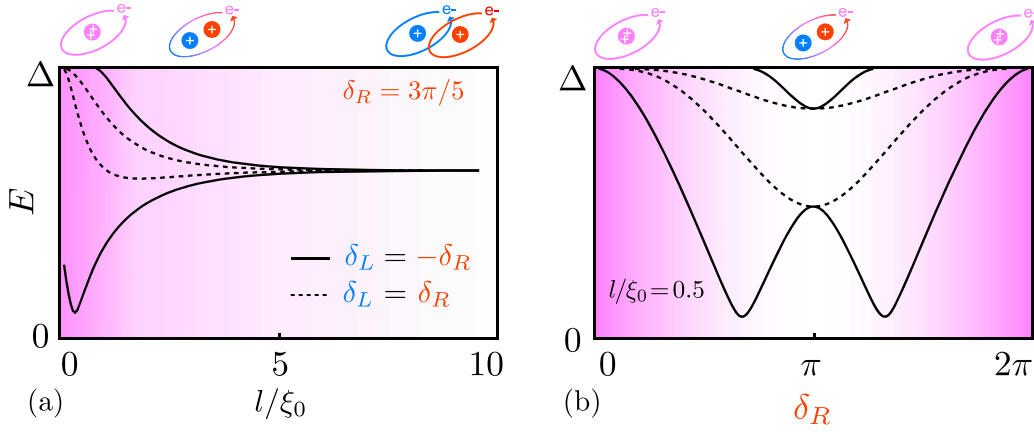


Figure 79: (a) Positive Andreev Bound State spectrum at symmetry points $\delta_L = -\delta_R$ (solid lines) and $\delta_L = \delta_R$ (dashed lines) as a function of l/ξ_0 for fixed $\delta_R = 3\pi/5$. Due to hybridization the energy levels split as the separation is reduced and some states are pushed into the continuum. (b) The level splitting, as well as the number of bound states, can also be controlled by adjusting δ_R for fixed separation, $l/\xi_0 = 0.5$. Oscillations on a scale λ_F are avoided by choosing $k_F l = \pi/2 \pmod{2\pi}$ with $k_F l \gg 1$.

molecule. Incidentally, around this transition from molecule to atom, the lower ABS reaches a minimum resulting in an overall shape which suggests the interatomic potentials used to describe the formation of molecules. Physically, this minimum occurs due to competition between hybridization which pushes the lower ABS toward zero energy and scattering which pushes it away (see Supporting Information).

At $l/\xi_0 = 0$, in contrast to a real interatomic potential, the energy of this lower state does not diverge and is less than the energy at large separation, $l/\xi_0 \gg 1$. The full spectrum for $l/\xi_0 = 0$ can be found in the Supporting Information.

For the dashed $\delta_L = \delta_R$ (dCAR) spectral lines, the level splitting is weaker than for dEC (solid lines) and both states are pushed into the continuum as l/ξ_0 approaches zero. As with the level splitting, the depth of the potential well for the lower dashed line is smaller than for dEC (lower solid line). This is because the dCAR process is in general weaker than dEC for high transmissions.

A more experimentally feasible possibility to tune the hybridization strength is adjusting δ_R at fixed separation, as shown in Figure 79(b) for $l/\xi_0 = 0.5$. Different regimes can be identified by the number of bound states. As δ_R increases from 0 to π , states gradually detach from the continuum and form bound states. The size of the level splitting of the $\delta_L = \delta_R$ line (dCAR, dashed) is maximal at $\delta_R = \pi$ whereas for $\delta_L = -\delta_R$ (dEC, solid line) this maximum occurs at an intermediate value. By changing δ_R one can control the degree of hybridization and tune the molecular nature of the system.

As a function of the transmission τ , the general tendency is for the energy splitting at $\delta_L = -\delta_R$ to monotonically increase with τ whereas the splitting at $\delta_L = \delta_R$ has a peak and goes to zero at $\tau = 1$.

The Andreev molecule, as a tunable multi-level qubit, may be useful for the preparation and manipulation of quantum states. In an Andreev molecule with moderate transmission and separation, one could envision a protocol where the system, initially in the ground state, is tuned via the phases $\delta_{L,R}$ to a level splitting in the gigahertz range. Microwave pulses could then prepare an excited state which would be protected by subsequently detuning the phases away from the splitting where transitions are less likely. Although coherence times of Andreev-based qubits [25, 28] are currently an order of magnitude lower than that of conventional superconducting qubits, the decoherence mechanisms are not understood and may well be

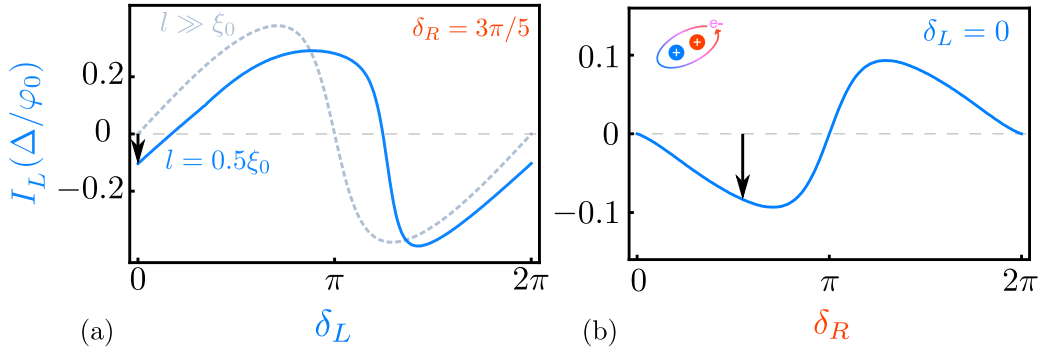


Figure 80: Andreev molecule current-phase relation and φ -junction behavior. (a) The current-phase relation of the left weak link, $I_L(\delta_L, \delta_R = 3\pi/5)$, is shifted and distorted for small separation ($l = 0.5\xi_0$, solid line) as compared to the isolated junction ($l \gg \xi_0$, dashed line). (b) The zero-phase supercurrent $I_L(\delta_L = 0)$ (black vertical arrow) is plotted as a function of δ_R , showing a tunable φ -junction effect. Currents are normalized to Δ/φ_0 , with $\varphi_0 = \hbar/2e$ the reduced flux quantum.

mitigated in the future.

The remarkable features of the spectrum of the Andreev molecule, in particular the asymmetry and additional energy dispersion in Figure 78(e), dramatically modifies the Josephson effect and gives rise to non-local Josephson supercurrents. These supercurrents are calculated from the eigenstates of the BdG equation [29, 30, 31] in a manner analogous to calculating probability currents. Both Andreev bound states and continuum states contribute significantly to the total supercurrent, unlike the case for isolated junctions where the continuum plays no role [30].

Figure 80(a) shows the supercurrent-phase relation of the left junction $I_L(\delta_L, \delta_R)$ for a fixed superconducting phase difference $\delta_R = 3\pi/5$ across the right junction. Plots are given for an isolated junction (dashed line) and for an Andreev molecule with $l = 0.5\xi_0$ (solid line). The curve for small separation is shifted with respect to that for large separation, and increases as the separation l/ξ_0 is reduced. In addition there is a distortion to the curve which breaks the odd symmetry in δ_L about the zeroes in the current. Furthermore, the positive and negative critical currents are not equal, and although not shown here, have magnitudes which depend on δ_R . These modifications, resulting from the dependence of I_L on δ_R for $l \lesssim \xi_0$, are non-local effects in that the phase drop δ_R occurs away from the left weak link, at a distance on the order of ξ_0 , which can be hundreds of nanometers depending on the superconductor.

For an isolated, time-reversal invariant Josephson junction, Figure 78(a), the supercurrent must be zero at a phase difference of 0 and π , Figure 80(a) (dashed line) [9]. But for the Andreev molecule where I_L depends on δ_R , since time-reversal symmetry only imposes $I_L(-\delta_L, -\delta_R) = -I_L(\delta_L, \delta_R)$, the current I_L can be non-zero at $\delta_L = 0$. A non-zero supercurrent at zero phase difference is the hallmark of a φ -junction, in which the current crosses zero at a phase φ instead of zero [32].

In Figure 80(b) the zero-phase supercurrent $I_L^0 = I_L(\delta_L = 0, \delta_R)$ is plotted as a function of δ_R . The sign of I_L^0 for small δ_R is negative, indicating that the current flows in the same direction as the phase difference δ_R , from left to central electrode in Figure 78(b). This is akin to a “ferromagnetic” interaction in that the current I_L^0 and phase difference δ_R are aligned. This tendency is explained by the fact that for vanishing separation l , there is only one junction with an overall phase drop given by $\delta_L - \delta_R$ and therefore $I_L(\delta_L, \delta_R) = I_L(\delta_L - \delta_R)$. In other words, the φ -junction phase offset approaches $\varphi = -\delta_R$ in the limit $l/\xi_0 \rightarrow 0$.

The two main signatures of Andreev molecules at equilibrium are the spectral avoided crossings (Figure 79) and the non-local current-phase relation (Figure 80). For large effects,

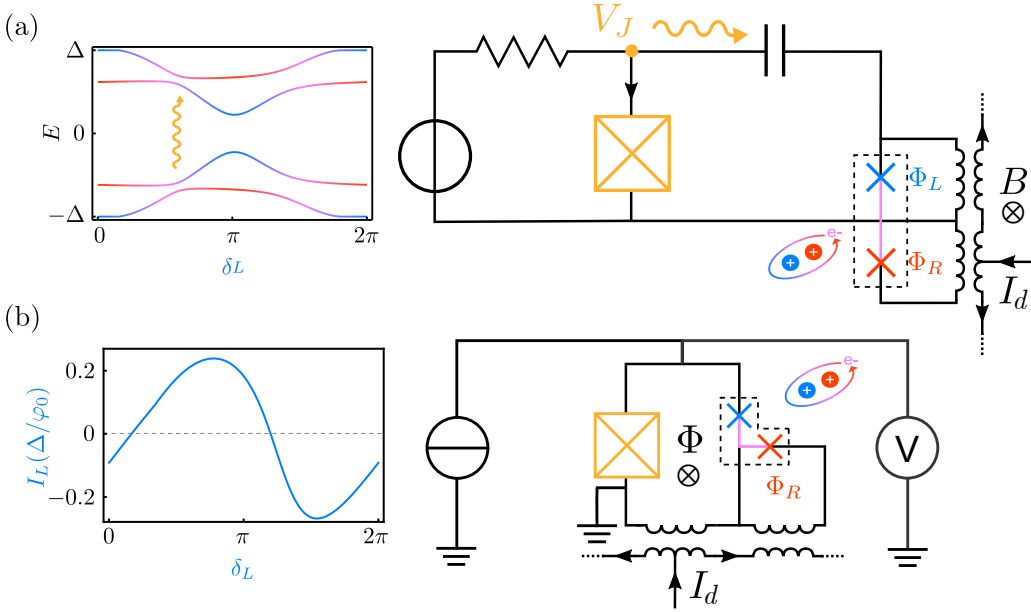


Figure 81: Experimental proposals. (a) The Josephson tunnel junction biased at V_J (boxed cross) acts as a spectrometer for transitions of the Andreev molecule (colored crosses). (b) Setup for switching-current measurements of the current-phase relation of an Andreev molecule (described in text). An overall magnetic field B and control current I_d allow independent control of $\delta_{L(R)}$ via the loop fluxes Φ , $\Phi_{L(R)}$.

the Andreev molecule should have small separation ($l \lesssim \xi_0$) and high transmission ($\tau \lesssim 1$), realistic conditions for epitaxial semiconductor-superconductor devices [24, 25]. Additional calculations have shown that these effects are robust to disorder and persist if there are multiple conduction channels. Andreev molecules can be detected with experiments at zero voltage bias, as described below. In non-equilibrium dc transport experiments, where junctions are voltage biased [10, 16, 17], these signatures are not present and dynamic processes such as mutual phase locking [5] complicate interpretation.

In Figure 81(a), we propose a setup to perform spectroscopy of an Andreev molecule in a similar manner to the spectroscopy of an Andreev “atom” in a superconducting atomic contact [18]. A large tunnel Josephson junction (yellow) is simultaneously used as an on-chip microwave source and detector. When a voltage bias V_J is applied across this spectroscopy junction, it emits microwave current at frequency $\nu = 2eV_J/h$ via the AC Josephson effect. This high-frequency current is coupled via the capacitor to the Andreev molecule, and due to inelastic Cooper-pair tunneling, will give peaks in the DC current-voltage characteristic of the spectrometer junction, $I_J(V_J)$, at voltages $h\nu/2e = E_T/2e$ corresponding to the transitions in the Andreev molecule spectrum, Figure 78(e), of energy E_T . The magnetic field B and gradiometric control current I_d allow independently tuning the phases $\delta_{L(R)}$ via the magnetic fluxes $\Phi_{L(R)}$, assuming that loop inductances are negligible compared to the Josephson inductance of the weak links. A full measurement $I_J(V_J, B, I_d)$ would allow reconstructing the spectra of Figure 78(e) as well as verifying the predictions of Figure 79(b).

Josephson spectroscopy is well suited for detecting Andreev molecules, as E_T may be comparable to Δ/h (90 GHz for aluminum), well within the spectrometer bandwidth. Conventional microwave spectroscopy [19, 25], with greater sensitivity but narrower range, may also be used, as well as tunneling spectroscopy [33, 34] instead of photonic spectroscopy.

In a second experiment, the existence of a non-local current-phase relation can be directly determined by measurements of the switching current. This type of experiment has been performed on superconducting atomic contacts [35], graphene [36] and carbon nanotubes [37].

As shown in Figure 81(b), a large Josephson junction of critical current $I_0 \gg \max(I_L, I_R)$ (boxed cross) is wired in parallel with the left weak link, L (upper black cross), of the Andreev molecule, forming an asymmetric SQUID. Similarly to the previous proposal, δ_L and δ_R are controlled independently using the flux Φ and control current I_d . Due to the asymmetry, the SQUID critical current I_{SQ} is essentially given by that of the large junction, effectively at a phase difference of $\pi/2$, modulated by the supercurrent of the L junction, $I_{SQ} \approx I_0 + I_L(\delta_L, \delta_R)$, where flux quantization constrains $\delta_L = \Phi/\varphi_0 - \pi/2$. By sending current pulses or ramps and measuring the switching of the SQUID to a non-zero voltage state, one can extract the current-phase relation $I_L(\delta_L, \delta_R)$ and demonstrate that there is a non-local component that depends on δ_R .

We have shown that two superconducting weak links separated by a distance on the order of the superconducting coherence length exhibit a non-local Josephson effect, a phenomenon attributed to the hybridization of Andreev Bound States between the two weak links and the formation of an Andreev molecule. Our work provides a microscopic understanding of the “order-parameter,” or Cooper-pair mediated, interaction between Josephson junctions. The most prominent signatures of non-local effects are a modification of the Andreev Bound State energy spectrum and the emergence of a φ -junction, a weak link with non-zero supercurrent at zero phase difference. Such signatures can be detected with well-established spectroscopy techniques and switching-current measurements.

The Andreev molecule, with tunable energy gaps, may serve as a new type of superconducting qubit, and its φ -junction behavior may find use in magnetometry. In addition the large junction interaction holds promise for applications employing arrays of coupled weak links, such as in quantum information, metrology, or sensing. Arrays of junctions in the Andreev molecule limit ($l \lesssim \xi_0$) may achieve new limits of qubit coupling, quantum amplification, voltage stability, or magnetic field sensitivity. Such arrays may also serve as quantum simulators of polymers or to implement model Hamiltonians. By controlling the separation between weak links in the array as well as the individual phase differences, the strength of the coupling between Andreev states at adjacent weak links can be varied. For example, alternating short and long separation would allow generating a Su-Schrieffer-Heeger (SSH) type Hamiltonian modeling a polyacetylene molecule. It may also be possible to introduce repulsive on-site interactions by using chains of weak links where charging energy is important.

The predicted coupling between two weak links should also exist for exotic Andreev Bound States where spin-orbit and Zeeman effects play a role, and could be exploited for topological quantum computation with Majorana bound states.

Part IX

Multiterminal Junctions: Other Theories

23 Fine energy splitting of overlapping Andreev bound states in multi-terminal superconducting nanostructures by Kornich, Barakov, Nazarov

Abstract

The recent proposals of experiments with single Andreev bound states make relevant a detailed analysis of these states in multi-terminal superconducting nanostructures. We evaluate the energy splitting of degenerate Andreev bound states, that overlap in a superconducting lead, and find that the splitting is reduced in comparison with their energy by a small factor $\sqrt{RG_Q}$, RG_Q being the dimensionless resistance of the overlap region in the normal state. This permits quantum manipulation of the quasiparticles in these states. We provide a simple scheme of such manipulation.

The superconducting nanodevices are in focus of modern experimental research, in particular because they are a promising platform for various qubit realizations, e.g. Josephson-based qubits of several kinds [1, 2, 3, 4, 5] or Majorana bound states [6, 7, 8, 9, 10]. These structures, containing superconductor-semiconductor or superconductor-insulator junctions, host Andreev bound states (ABS), which can also be used as a qubit [12, 11]. Andreev reflection between normal metal and superconductor was first discussed in Ref. [13], and has been a subject of intense theoretical and experimental research [14, 15, 16, 17, 18, 19, 20, 21, 22, 23] that spans far beyond quantum information topics.

It is known and commonly used that the ABS forming in nanostructures are defined by the properties of the nanostructure, not depending on the details of electron scattering in the adjacent superconducting leads, which is a consequence of Anderson's theorem [24]. For short nanostructures between two leads, each transport channel with transmission T gives an ABS at the energy [25] $E = \pm\Delta\sqrt{1 - T\sin^2[(\phi_1 - \phi_2)/2]}$, Δ being the superconducting gap in the leads, $\phi_{1,2}$ being the superconducting phases of the leads. ABS extend to the leads at distances of the order of the correlation length ξ_0 , this is much larger than the nanostructure size. ABS can be realized in semiconducting-nanowire-superconducting structures, the technology of those has advanced strongly owing to the applications in the field of Majorana bound states [10, 26, 27, 28, 29], and can be characterized experimentally [27, 28]. There is much recent progress in multi-terminal devices [30, 31, 32] that has been partially inspired by theoretical predictions of Weyl points and quantized transconductance [33].

Very recent experimental and theoretical developments concern so-called Andreev molecules in various layouts [34, 36, 35]. The term “molecule” refers to the situation where two single ABS have close energies, this enables their hybridization and formation of the superpositions. Refs. [34, 35] discuss ABS in quantum dots, where ABS overlap through the tunnel barrier separating the dots. An interesting alternative has been put forward in Ref. [36]. The proposed three-terminal setup encompasses two single-channel junctions that connect three superconducting leads, see Fig. 82 (a). Two single ABS may overlap and hybridize in the middle lead. The overlap and the corresponding energy splitting must cease exponentially as $\exp\{(-L/\xi_0)\}$, provided the separation of the junctions $L \gg \xi_0$. The authors of Ref. [36] indicate that Andreev molecules have potential applications in quantum information, metrology, sensing, and molecular simulation.

In this Letter we have evaluated the energy splitting of overlapping ABS and have comprehended it as an interference effect similar to mesoscopic fluctuations of conductance [37],

which develops in the lead on the scale of ABS overlap that encompasses a large number of quantum channels. We have estimated the typical energy splitting $\delta_D \sim \Delta/\sqrt{N}$, N being the number of channels, that can be estimated as the inverse of the normal dimensionless resistance of the overlap region, $N \approx (RG_Q)^{-1}$, $G_Q \equiv e^2/(\pi\hbar)$ being conductance quantum. Therefore δ_D remains fine at the energy scale of Δ . This facilitates quantum manipulations, an example of which we give. We have derived concrete expressions for $\langle|\delta_D|^2\rangle$, relating it to semiclassical propagation of an electron between the junctions, and employed it for an experimentally relevant setup. Observation of energy splitting gives an interesting and unusual way to see and explore mesoscopic fluctuations.

Let us first describe the setup under consideration (Fig. 82) in general terms specifying the details later. The setup consists of three superconducting leads, connected by two single-channel junctions, and there is an ABS formed in each junction. If one neglects their hybridization, their energies are $E_{A(B)} = \Delta\sqrt{1 - T_{A(B)} \sin^2[(\phi_{1(2)} - \phi_3)/2]}$. We plot the energies in Fig. 82 (b) for $\phi_1 - \phi_2 = \pi/2$ and $T_A = 0.95$, $T_B = 0.98$; the degeneracy at $E_A = E_B$ is avoided. The separation between the junctions is $L \gg \lambda_F$, λ_F being the Fermi wave length. This implies that the electron transport in the region covered by the ABS wave functions, occurs in a big number of transport channels. The exact number depends on the geometry of the device, material and morphology of the leads, and the detailed characteristics of electron transport, that can be ballistic, diffusive, or intermediate between the two. At the level of an estimate, all these details can be incorporated into the effective resistance R of the region spanned by the ABS wave functions, so that $N \sim (RG_Q)^{-1}$. The wave functions of the ABS overlap as shown in Fig. 82 (a) and hybridize. The hybridization is big near each avoided crossing and can be described with an off-diagonal matrix element δ_D , which is a complex number.

Let us estimate the energy splitting $2|\delta_D|$ near an avoided crossing. The energies of the states are $E_{A,B} \sim \Delta$. These states are formed by electrons coming in and out of a junction to/from the adjacent leads and returning as holes to the same junction. The electron wave function is distributed among N transport channels involved. The contributions of different channels to Andreev scattering amplitude come with the same sign and phase, this is precisely the reason for the energy of ABS not to depend on the details of the scattering in the leads. This implies that the contribution of each channel to the ABS energy can be estimated as Δ/N . As to δ_D , it is determined by the electron and hole propagation from the junction A to the junction B . Since these points are distinct and separated by $L \gg \lambda_F$ one expects the contributions of different channels to come with the different and generally random complex amplitudes. These random amplitudes may be related to mesoscopic fluctuations of electron propagation between the junctions A and B . Averaging over these random amplitudes leads to vanishing $\langle\delta_D\rangle = 0$. The average $\langle|\delta_D|^2\rangle$ is contributed by independent contributions of N channels and therefore the energy splitting can be estimated as $|\delta_D| \sim \Delta/\sqrt{N} \sim \Delta\sqrt{RG_Q}$.

The junctions between the superconducting leads may have various characteristics, such as disorder, shape, material. It is known however [38] that the only characteristic relevant for ABS is the transmission of these junctions. Therefore we are free to choose any microscopic model so we opt for a convenient resonant impurity model [40, 39], that involves a localized state of energy E_{imp} with the tunnel couplings t and t' to the leads. We label with A and B these characteristics in the corresponding junctions, see Fig. 82 (a). The width of the resonant level is given by $\Gamma = 2\pi\nu(|t|^2 + |t'|^2)$, ν being the density of states assumed equal in all leads. To model weak energy dependence of the scattering we set $\Gamma \gg \Delta$, so the transmission coefficient of the junction A is

$$T_A = \frac{4\pi^2\nu^2|t_A|^2|t'_A|^2}{(\Gamma_A/2)^2 + E_{\text{imp},A}^2}, \quad (23.1)$$

and similar for the junction B .

To find the ABS energies we derive a Dyson equation for the Green's function $G_{ij}(E)$ defined at the resonant impurities $i, j = \{A, B\}$: $G(E) = ([G^0]^{-1} - \Sigma)^{-1}$, where the blocks are the matrices in the Nambu space $G_{AA, BB}^0 = (E - E_{\text{imp}, A, B} \sigma_z)^{-1}$, and the self-energy part Σ describes the tunneling. The diagonal blocks are $\Sigma_{AA} = \mathcal{T}'_A G(\mathbf{r}'_A, \mathbf{r}'_A) \mathcal{T}_A'^* + \mathcal{T}_A G(\mathbf{r}_A, \mathbf{r}_A) \mathcal{T}_A^*$, where $\mathcal{T}_j = (t_j^*(\sigma_z + \sigma_0) + t_j(\sigma_z - \sigma_0))/2$, $j = \{A, B\}$, and $G(\mathbf{r}, \mathbf{r}')$ is a superconducting Green's function in the corresponding leads upon neglecting the tunneling, Σ_{BB} is similar. The Green's function in close points does not depend on the details of the scattering in the lead, this is a consequence of Anderson's theorem [24]. The non-diagonal blocks Σ_{AB} and Σ_{BA} are $\Sigma_{AB} = \mathcal{T}_A G(\mathbf{r}_A, \mathbf{r}_B) \mathcal{T}_B^*$ and $\Sigma_{BA} = \mathcal{T}_B G(\mathbf{r}_B, \mathbf{r}_A) \mathcal{T}_A^*$. We see that the diagonal blocks contain Green's functions in coinciding points, while non-diagonal ones contain Green's functions in two points separated by $L \gg \lambda_F$. Since Green's functions are associated with propagation amplitudes $G(\mathbf{r}_A, \mathbf{r}_B) \ll G(\mathbf{r}_A, \mathbf{r}_A), G(\mathbf{r}_B, \mathbf{r}_B)$. Thus $\Sigma_{AB}, \Sigma_{BA} \ll \Sigma_{AA}, \Sigma_{BB}$, and can be handled by means of the degenerate perturbation theory. This already implies $|\delta_D| \ll \Delta$.

The energies of ABS correspond to the poles of $G(E)$ [41], so we need to find zero eigenvalues of $G^{-1}(E)$. We find zero eigenvalues in each diagonal block and project $G^{-1}(E)$ onto the corresponding eigenvectors $|A\rangle$ and $|B\rangle$. We work near the crossing point E_0 where the unperturbed ABS energies are almost degenerate $E_A \approx E_B \approx E_0$. Expanding up to linear order near the crossing point and transforming $G^{-1}(E)$ we find that ABS energies are obtained from the effective Hamiltonian describing the level repulsion, $H_{\text{eff}} = \begin{pmatrix} E_A & \delta_D \\ \delta_D^* & E_B \end{pmatrix}$, where

$$\delta_D = -\frac{\langle A | \Sigma_{AB} | B \rangle \sqrt{\Delta^2 - E_0^2}}{\sqrt{\Gamma_A \Gamma_B}} \propto G(\mathbf{r}_A, \mathbf{r}_B). \quad (23.2)$$

The Green's function $G(\mathbf{r}_A, \mathbf{r}_B)$ changes much on a scale of λ_F upon changing the position of $\mathbf{r}_B$. This is the origin of mesoscopic fluctuations in electron transport [42]. The components of $G(\mathbf{r}_A, \mathbf{r}_B)$ can be regarded as random values with zero averages. The informative quantities are the products of these components averaged over $\mathbf{r}_B$ at the scale exceeding λ_F . These averaged products can be expressed with a normal-state quasiclassical propagator [43] $\mathcal{P}(\mathbf{r}_A, \mathbf{r}_B, t)$, that gives the probability to find an electron at $\mathbf{r}_B$ at the time moment t provided it was at $\mathbf{r}_A$ at $t = 0$ (Greek letters denote Nambu indices):

$$\begin{aligned} & \langle G(\mathbf{r}_A, \mathbf{r}_B)^{\alpha\gamma*} G(\mathbf{r}_A, \mathbf{r}_B)^{\beta\nu} \rangle = \\ & = \frac{\pi^2 \nu}{\Delta^2 - E^2} \int dt \mathcal{P}(\mathbf{r}_A, \mathbf{r}_B, t) e^{-2\sqrt{\Delta^2 - E^2}|t|} \\ & \times [\delta_{\alpha\gamma} \delta_{\beta\nu} \delta_{\alpha\beta} (2\Delta^2 - E^2) \\ & + \Delta E ((1 - \delta_{\alpha\gamma}) \delta_{\beta\nu} e^{-i\epsilon_{\alpha\gamma}\varphi_3} - \delta_{\alpha\gamma} (1 - \delta_{\beta\nu}) e^{i\epsilon_{\beta\nu}\varphi_3}) \\ & + \Delta^2 (\delta_{\alpha\beta} \delta_{\gamma\nu} (1 - \delta_{\alpha\gamma}) - \delta_{\alpha\gamma} \delta_{\beta\nu} (1 - \delta_{\alpha\beta}) \\ & + \delta_{\alpha\nu} \delta_{\gamma\beta} (1 - \delta_{\alpha\gamma}) e^{-i2\epsilon_{\alpha\gamma}\varphi_3})]. \end{aligned} \quad (23.3)$$

Let us reproduce the main estimation of the Letter, $|\delta_D| \propto 1/\sqrt{RG_Q}$, with this method. Combining Eqs. 23.2 and 23.3 we estimate $(|\delta_D|/\Delta)^2 \propto \nu^{-1} \int dt \mathcal{P}(t) e^{-\Delta|t|}$. In the course of its propagation, an electron covers the region whose spatial dimensions are defined by the dwell time $t_{\text{dw}} \simeq \Delta^{-1}$. The $\mathcal{P}(t_{\text{dw}})$ is estimated as inverse volume V of the region. With this we can estimate $(|\delta_D|/\Delta)^2 \sim t_{\text{dw}} \delta_s$, $\delta_s = (\nu V)^{-1}$. If we now compare this with the Thouless estimation [44] of the conductance of such region, $G_{\text{Th}} \simeq G_Q (\delta_s t_{\text{dw}})^{-1}$, we reproduce $(|\delta_D|/\Delta)^2 \propto R_{\text{Th}} G_Q$.

Let us specify the concrete setup. It comprises the semiconducting nanowire covered by three superconducting leads, see Fig. 83 (a); such devices have been recently fabricated [45]. The middle lead is a film of thickness d and width L . If $L \lesssim \xi_0$, the ABS wave functions overlap strongly. We assume diffusive transport in the lead, which is characterized by the resistance per

square $R_\square$. We also assume that the interface between the nanowire and the metal is sufficiently transparent, so that the electrons escape the nanowire to metal at the distances $\ll \xi_0$.

The semiclassical propagator in the film obeys diffusion equation

$$\left(\frac{\partial}{\partial t} - D\nabla^2\right) \mathcal{P}(\mathbf{r}, t) = \frac{1}{d} \delta(t) \delta(\mathbf{r} - \mathbf{r}_A). \quad (23.4)$$

This diffusion equation is subject to boundary conditions of zero probability flow across all boundaries. One satisfies these boundary conditions introducing infinite number of imaginary sources, spaced with $2L$. The propagator we obtain is

$$\mathcal{P}(\mathbf{r}_A, \mathbf{r}_B, t) = \frac{1}{dL} \sqrt{\frac{1}{\pi D|t|}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-D \frac{\pi^2 n^2}{L^2} |t|}. \quad (23.5)$$

With this we compute $|\delta_D|^2$ using Eqs. 23.2 and 23.3 to obtain

$$\frac{|\delta_D|^2}{\Delta^2} = \frac{\pi}{2} M G_Q R_{\text{eff}} F\left(\frac{L}{\xi(E_0)}\right), \quad R_{\text{eff}} = R_\square \frac{\xi(E_0)}{L}, \quad (23.6)$$

with R_{eff} being the effective resistance of the part of the lead covered by ABS wave functions, the dimensionless $F(z) = 4z/\pi \sum_{n=0}^{\infty} K_0((2n+1)z)$, K_0 being modified Bessel function of the second kind, $F(0) = 1$, incorporates information of the decay of ABS wave functions at the scale of $\xi(E) = \xi_0(1 - E^2/\Delta^2)^{-1/4}$, that is the energy-dependent correlation length. The prefactor $M \simeq 1$ incorporates information about transmissions of the junctions $M = 2|t_A|^2|t_B|^2[2\cos(\chi_A - \chi_B)E^2 + (2 + \cos(\chi_A + \chi_B) - \cos(\chi_A - \chi_B))\Delta^2 - 2E\Delta(\cos\chi_A + \cos\chi_B)]/[\Delta^2(|t_A|^2 + |t'_A|^2)(|t_B|^2 + |t'_B|^2)]$, where χ_A and χ_B are the phases of the eigenvectors $|A\rangle$ and $|B\rangle$, respectively, with $e^{i\chi_A} = [|t'_A|^2 e^{i\varphi_1} + |t_A|^2] \Delta / (E[|t_A|^2 + |t'_A|^2])$ and analogously for χ_B . Here, φ_1 and φ_2 denote phase differences with respect to φ_3 and we set $E_{\text{imp},A,B} = 0$. $M \rightarrow 1$, in the limit $E_0 \rightarrow 0$, this requires $T_{A,B} \rightarrow 1$. Thus in the limiting cases we have

$$\frac{|\delta_D|^2}{\Delta^2} = \begin{cases} \frac{\pi}{2} M G_Q R_{\text{eff}}, & L \rightarrow 0, \\ M G_Q R_{\text{eff}} \sqrt{\frac{2\pi L}{\xi(E_0)}} e^{-\frac{L}{\xi(E_0)}}, & L \rightarrow \infty, \end{cases} \quad (23.7)$$

We plot the normalized energy splitting $|\delta_D|/\delta_D(L=0)$ versus L in Fig. 83 (b), $\delta_D(L=0) = \Delta\sqrt{\pi M G_Q R_{\text{eff}}/2}$.

For the experimentally relevant parameters $\Delta = 200 \mu\text{eV}$, $\xi_0 = 96 \text{ nm}$ [27], $R_\square = 1.43 \Omega$ [46], $L = 50 \text{ nm}$, $T_A = T_B = 1$, we find the crossing point at $\varphi_2 = 2.36$, $\varphi_1 = 3.93$, and $E_0 = 76.54 \mu\text{eV}$, and obtain the splitting $2|\delta_D| = 11.26 \mu\text{eV}$. The value for $|\delta_D| \simeq \Delta/40$ even though $R_\square G_Q \approx 10^{-4}$, which seems to be small.

The separation of scales between δ_D and Δ permits interesting quantum manipulation schemes for involved states. Let us describe the simplest one: quasiparticle swap between A and B . Let us take a point in parametric space of $\varphi_{1,2,3}$, where the ABS energies are well-split (for instance, $\varphi_3 = 0$ in Fig. 82 (b)) and put a quasiparticle to the state A . We pass the avoided crossing slowly to avoid Landau-Zener tunneling in this point (for instance, changing φ_3 from 0 to $\pi/2$), this brings the quasiparticle to B . If we get back to the initial point very quickly, the quasiparticle will remain in B , this completes the manipulation protocol. The same swap occurs if the quasiparticle is in B initially.

There is an interesting case, when both junctions have almost ideal transmission $T_{A,B} = 1 - R_{A,B}$, $R_{A,B} \ll 1$, and $\varphi_{1,2} = \pi + \delta\varphi_{1,2}$, $\delta\varphi_{1,2} \ll 1$. In this case the crossing occurs at $E_0 \ll \Delta$, which can also be comparable with $|\delta_D|$. The perturbation theory does not work here,

but we can describe the situation with the following 4×4 effective Hamiltonian:

$$H_{\text{eff}} = \Delta \begin{pmatrix} 0 & h_A & g & f \\ h_A^* & 0 & f & -g \\ g & f & 0 & h_B \\ f & -g & h_B^* & 0 \end{pmatrix}, \quad (23.8)$$

where $h_{A,B} = \sqrt{R_{A,B}} + i\delta\varphi_{1,2}$. The terms f and g come from $\Sigma_{AB,BA}$, $\langle f^2 \rangle = \langle g^2 \rangle = 16\pi^3\nu^2|t_A|^2|t_B|^2G_Q R_{\text{eff}}F(L/\xi(E_0))$. In the limit $R_{A,B} = 0$ the ABS energies are given by

$$E = \sqrt{\tilde{\delta}^2 + \frac{(\delta\varphi)^2}{4} + \frac{\Phi^2}{4} \pm \Phi \sqrt{\tilde{\delta}^2 + \frac{(\delta\varphi)^2}{4}}}, \quad (23.9)$$

where $\tilde{\delta} = \sqrt{f^2 + g^2}$, $\delta\varphi_{1,2} = \Phi/2 \pm \delta\varphi/2$. Interestingly, if $|\Phi| < 2\tilde{\delta}$ the energies never cross zero, while there are two symmetric zero-energy crossings if $|\Phi| > 2\tilde{\delta}$, Fig. 84 (a), (b). In this approximation two ABS energies are precisely degenerate at $\Phi = 0$, this degeneracy is lifted upon increasing energy. At finite $R_{A,B}$ the zero-energy crossings are avoided, Fig. 84 (c), (d).

Before we conclude let us mention that the fact that the energy splitting is fine makes relevant a set of topics to research that we list here. For semiconducting nanowires the electron escape length can be $\gtrsim L$, this confines the overlap region to the nanowire and greatly enhance δ_D . The spin-orbit splitting [47] of ABS that is usually negligible can become relevant for small δ_D . Many-body effects shall provide small energy differences for doublet and singlet quasiparticle states in ABS [23]. Interestingly, the hybridization of degenerate singlet states in the setup under investigation can also occur without direct overlap of ABS wave functions, that is at $L \gg \xi_0$.

To conclude, we have investigated the energy splitting of overlapping ABS in “Andreev molecule” setup and have found it small in comparison with Δ , since it is related to mesoscopic fluctuations. This opens up possibilities for quantum manipulation and application as mentioned in Ref. [36]. The smallness of the energy splitting makes relevant the research of a variety of fine and non-trivial effects on ABS spectrum.

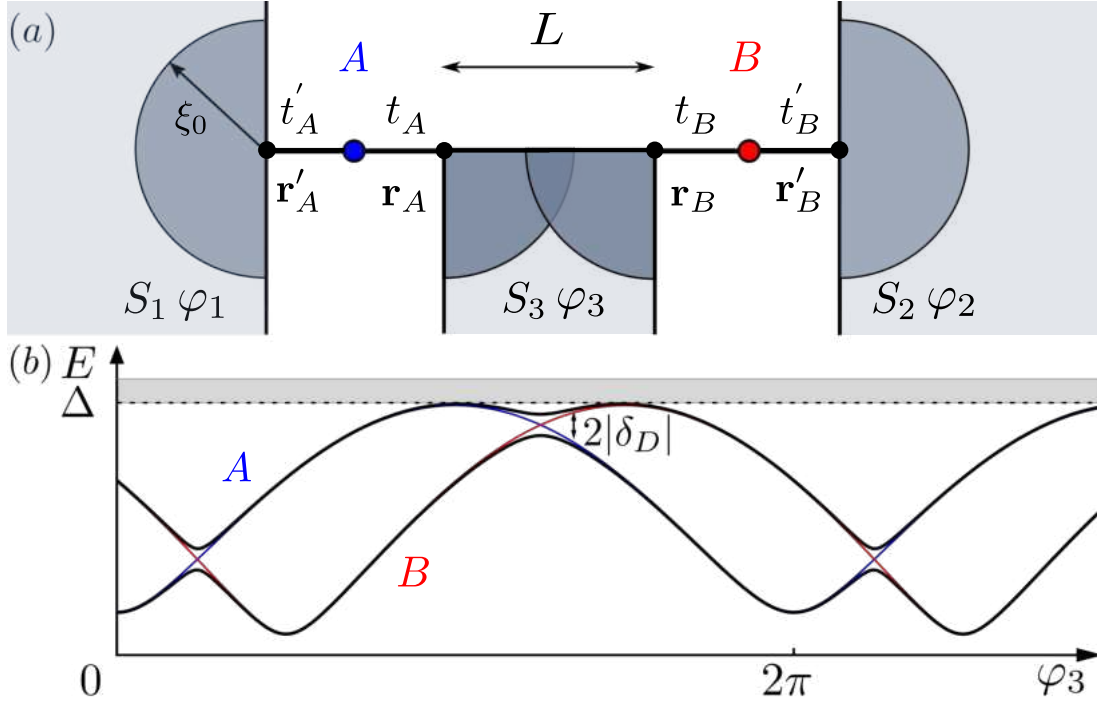


Figure 82: (a) The "Andreev molecule" setup. Two ABS are formed in the two single-channel junctions, A and B, that are separated by L and connect three superconducting leads $S_{1,2,3}$ with the same Δ . The ABS wave functions spread over the scale of ξ_0 , overlapping in S_3 . (b) The energy spectrum of ABSs versus φ_3 ($\varphi_1 = \pi$, $\varphi_2 = 3\pi/2$, the junction transmission coefficients being $T_A = 0.95$, $T_B = 0.98$), manifests avoided crossings at the degeneracy points. We show that the energy splitting at the crossings is fine even for a significant overlap, $\delta_D \propto \Delta \sqrt{RG_Q}$, where R being normal resistance of the overlap region.

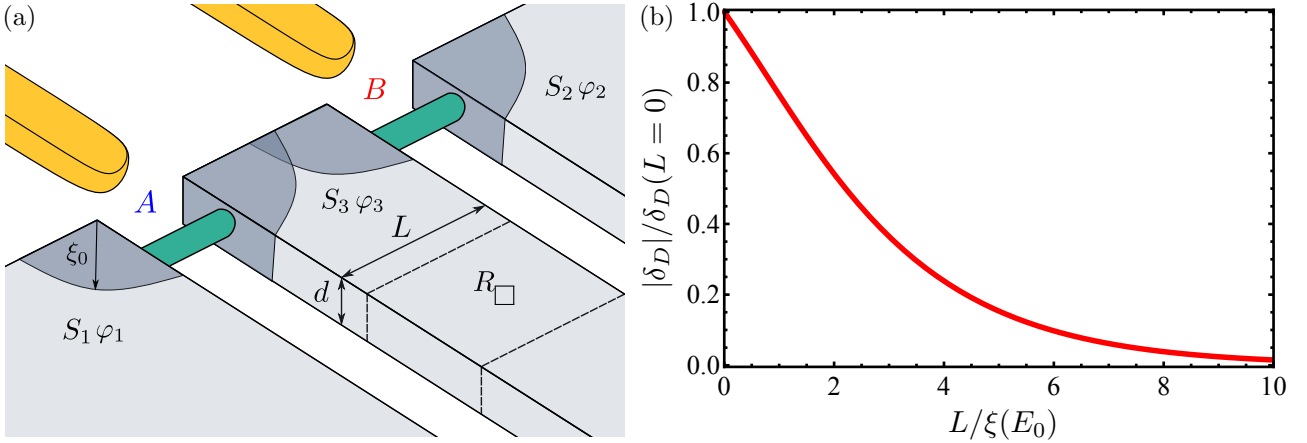


Figure 83: (a) The concrete setup under consideration comprises three superconducting leads covering a single-channel semiconducting nanowire. Two hybridizing ABS are forming at the junctions A, B. The gates (yellow) affect the potential in the wire and might be used to tune the transmission of the junctions. The middle lead is a film of thickness d and width L and is characterized by the resistance per square $R_{\square}$. The ABS wave functions overlap strongly provided $L \lesssim \xi_0$. (b) Dependence of the energy splitting $|\delta_D|/\delta_D(L=0)$ on L . The splitting vanishes exponentially upon increasing L .

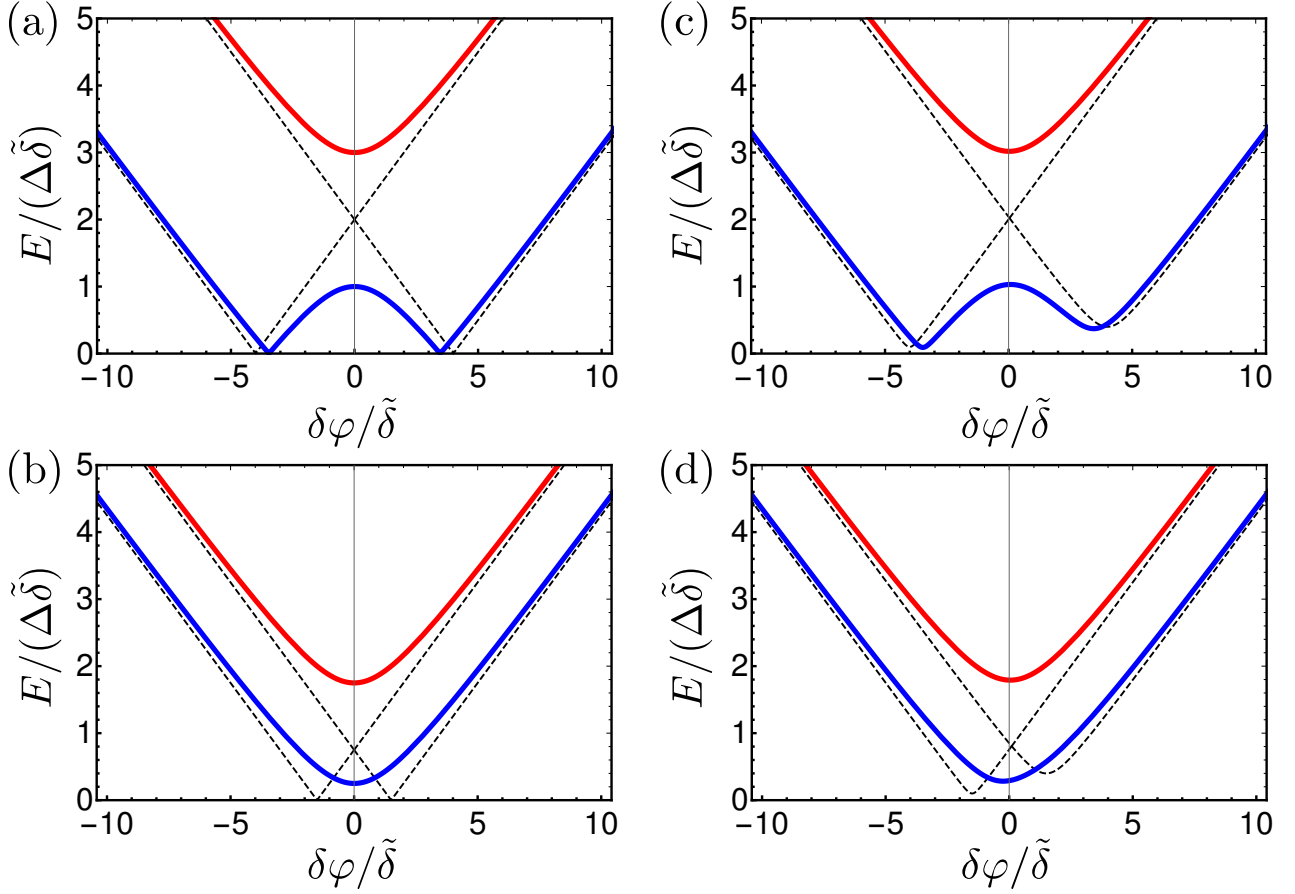


Figure 84: Spectra of the system for $E \sim 0$ for $f/\tilde{\delta} = \sqrt{3/5}$, $g/\tilde{\delta} = \sqrt{2/5}$, and (a) $\Phi/\tilde{\delta} = 4$, $R_A = R_B = 0$; (b) $\Phi/\tilde{\delta} = 1.5$, $R_A = R_B = 0$; (c) $\Phi/\tilde{\delta} = 4$ and $\sqrt{R_A}/\tilde{\delta} = 0.1$, $\sqrt{R_B}/\tilde{\delta} = 0.4$; (d) $\Phi/\tilde{\delta} = 1.5$ and $\sqrt{R_A}/\tilde{\delta} = 0.1$, $\sqrt{R_B}/\tilde{\delta} = 0.4$. The dashed line shows the case of $f = g = 0$ and for other parameters as described. The asymmetry in (c) and (d) comes from $R_A \neq R_B$.

References

- [1] Y. Nakamura, C. D. Chen, and J. S. Tsai, Phys. Rev. Lett. **79**, 2328 (1997).
- [2] S. Han, Y. Yu, X. Chu, S. Chu, and Z. Wang, Science **293**, 1457 (2001).
- [3] D. Vion, A. Aassime, A. Cottet, P. Joyez, H. Pothier *et al.*, Science **296**, 886 (2002).
- [4] J. M. Martinis, S. Nam, J. Aumentado, and C. Urbina, Phys. Rev. Lett. **89**, 117901 (2002).
- [5] A. J. Berkley, H. Xu, R. C. Ramos, M. A. Gubrud, F. W. Strauch *et al.*, Science **300**, 1548 (2003).
- [6] A. Y. Kitaev, Phys. Usp. **44**, 131 (2001).
- [7] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Phys. Rev. Lett. **105**, 077001 (2010).
- [8] Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. **105**, 177002 (2010).
- [9] J. Alicea, Rep. Prog. Phys. **75**, 076501 (2012).
- [10] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers *et al.*, Science **336**, 1003 (2012).
- [11] C. Janvier, L. Tosi, L. Bretheau, Ç. Ö. Girit, M. Stern *et al.*, Science **349**, 1199 (2015).
- [12] N. M. Chtchelkatchev and Y. V. Nazarov, Phys. Rev. Lett. **90**, 226806 (2003).
- [13] A. F. Andreev, JETP **19**, 1228 (1964).
- [14] G. E. Blonder, M. Tinkham, and T. M. Klapwijk, Phys. Rev. B **25**, 4515 (1982).
- [15] A. L. Shelankov, Sov. Phys. Solid State **26**, 981 (1984).
- [16] R. J. Soulen Jr., J. M. Byers, M. S. Osofsky, B. Nadgorny, T. Ambrose *et al.*, Science **282**, 85 (1998).
- [17] F. S. Bergeret, P. Virtanen, T. T. Heikkilä, and J. C. Cuevas, Phys. Rev. Lett. **105**, 117001 (2010).
- [18] R. S. Deacon, Y. Tanaka, A. Oiwa, R. Sakano, K. Yoshida *et al.*, Phys. Rev. Lett. **104**, 076805 (2010).
- [19] F. Chiodi, M. Ferrier, K. Tikhonov, P. Virtanen, T. T. Heikkilä *et al.*, Sci. Rep. **1**, 3 (2011).
- [20] E. J. H. Lee, X. Jiang, M. Houzet, R. Aguado, C. M. Lieber *et al.*, Nat. Nanotechnol. **9**, 79 (2013).
- [21] L. Bretheau, Ç. Ö. Girit, M. Houzet, H. Pothier, D. Esteve *et al.*, Phys. Rev. B **90**, 134506 (2014).
- [22] L. Bretheau, J. I-J. Wang, R. Pisoni, K. Watanabe, T. Taniguchi *et al.*, Nat. Phys. **13**, 756 (2017).
- [23] L. Tosi, C. Metzger, M. F. Goffman, C. Urbina, H. Pothier *et al.*, Phys. Rev. X **9**, 011010 (2019).
- [24] P. W. Anderson, J. Phys. Chem. Solids **11**, 26 (1959).

- [25] C. W. J. Beenakker, Phys. Rev. Lett. **67**, 3836 (1991).
- [26] S. M. Albrecht, A. P. Higginbotham, M. Madsen, F. Kuemmeth, T. S. Jespersen *et al.*, Nature **531**, 206 (2016).
- [27] D. J. van Woerkom, A. Proutski, B. van Heck, D. Bouman, J. I. Väyrynen *et al.*, Nat. Phys. **13**, 876 (2017).
- [28] M. F. Goffman, C. Urbina, H. Pothier, J. Nygård, C. M. Marcus *et al.*, New J. Phys. **19**, 092002 (2017).
- [29] M. Hays, G. de Lange, K. Serniak, D. J. van Woerkom, D. Bouman *et al.*, Phys. Rev. Lett. **121**, 047001 (2018).
- [30] S. R. Plissard, I. van Weperen, D. Car, M. A. Verheijen, G. W. G. Immink *et al.*, Nat. Nanotechnol. **8**, 859 (2013).
- [31] N. Pankratova, H. Lee, R. Kuzmin, K. Wickramasinghe, W. Mayer *et al.*, arXiv:1812.06017.
- [32] A. W. Draelos, M.-T. Wei, A. Seredinski, H. Li, Y. Mehta *et al.*, Nano Lett. **19**, 1039 (2019).
- [33] R.-P. Riwar, M. Houzet, J. S. Meyer, and Y. V. Nazarov, Nat. Commun. **7**, 11167 (2016).
- [34] Z. Su, A. B. Tacla, M. Hocevar, D. Car, S. R. Plissard *et al.*, Nat. Commun. **8**, 585 (2017).
- [35] Z. Scherübl, A. Pályi, and S. Csonka, Beilstein J. Nanotechnol. **10**, 363 (2019).
- [36] J.-D. Pillet, V. Benzoni, J. Griesmar, J.-L. Smirr, and Ç. Ö. Girit, arXiv:1809.11011.
- [37] S. Chakravarty and A. Schmid, Phys. Rep. **140**, 193 (1986).
- [38] C. W. J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997).
- [39] A. Levy Yeyati, J. C. Cuevas, A. López-Dávalos, and A. Martín-Rodero, Phys. Rev. B **55**, R6137(R) (1997).
- [40] J. C. Cuevas, A. Levy Yeyati, and A. Martín-Rodero, Phys. Rev. B **63**, 094515 (2001).
- [41] A. Altland and B. Simons, *Condensed Matter Field Theory*, 2nd ed. (Cambridge University Press, New York, 2010).
- [42] Y. V. Nazarov and Y. M. Blanter, *Quantum Transport: Introduction to Nanoscience* (Cambridge University Press, Cambridge, 2009).
- [43] F. W. J. Hekking and Yu. V. Nazarov, Phys. Rev. B **49**, 6847 (1994).
- [44] D. J. Thouless, Phys. Rep. **13**, 93 (1974).
- [45] C. Girit, J.-D. Pillet, V. Benzoni, J. Griesmar, J.-L. Smirr, Bulletin of the American Physical Society, (2019).
- [46] D. J. van Woerkom, A. Geresdi, and L. P. Kouwenhoven, Nat. Phys. **11**, 547 (2015).
- [47] T. Yokoyama and Y. V. Nazarov, EPL **108**, 47009 (2014).

24 Supercurrent blockade in Josephson junctions with a Majorana wire by Zazunov, Egger

We study the Josephson effect for a topologically nontrivial superconducting (TS) wire with Majorana fermion end states and tunnel-coupled to s -wave BCS superconducting (S) electrodes. In an S-TS junction, no supercurrent flow is possible under fairly general conditions. For the S-TS-S junction, bulk TS quasiparticles must be accessible to have a Josephson effect. In the noninteracting case, we derive the exact current-phase relation (CPR) and find π -periodic behavior with negative critical current for weak tunnel couplings. Charging effects then cause the anomalous CPR $I(\varphi) = I_c \cos \varphi$, where the parity-sensitive critical current I_c provides a signature for Majorana states.

24.0.1 Introduction

The exciting physics of topological insulator (TI) and topological superconductor (TS) materials currently enjoys a lot of attention.[2, 2] The zero-energy Majorana bound states (MBSs) located near the ends of a one-dimensional (1D) TS wire have sparked immense theoretical activity,[4, 15] and first reports claiming experimental evidence for MBSs have appeared.[5] Majorana fermions are special in that they are their own antiparticles: Majorana creation and annihilation operators are identical. MBSs may be useful for topological quantum computation[1] and can induce spectacular nonlocal quantum phenomena.[2, 2, 4, 15, 5, 13, 9] Possible realizations of Majorana wires include semiconductor quantum wires with proximity-induced superconductivity for strong spin-orbit and Zeeman couplings,[10, 17] nanowires made out of TIs (e.g., Bi_2Se_3) deposited on a superconductor,[12] and helical edge states in 2D hybrid HgTe/CdTe -superconductor quantum well structures.[13, 15] A Majorana wire may effectively be realized also for a hole drilled into a TI film coated by a superconductor,[14] and for the vortex core state in a 2D noncentrosymmetric[16] or p -wave[9] superconductor. For a Majorana wire contacted by normal-conducting electrodes, the conductance exhibits resonant Andreev reflection when Coulomb charging effects are negligible,[9, 10, 7, 11, 20] electron teleportation under strong Coulomb blockade conditions,[13] and universal power law scaling in the intermediate regime.[9] When two TS wires are contacted, one expects the fractional Josephson effect with 4π -periodic current-phase relation (CPR), $I(\varphi)$, due to parity conservation.[5, 15, 21, 20] If the junction also contains a topologically trivial superconductor (TS-S-TS), additional periodicities may occur.[23]

In this work, we discuss the ground-state supercurrent flowing through the S-TS and S-TS-S junctions schematically shown in Fig. 85. The left/right ($j = L/R$) electrodes correspond to standard s -wave BCS superconductors with (for simplicity identical) gap Δ and fixed phase φ_j . The 1D Majorana wire contains a pair of decoupled MBSs at its ends. We consider a finite proximity-induced TS gap Δ_w and take into account the (p -wave type) TS quasiparticles. For the S-TS-S junction with a floating (not grounded) Majorana wire, we also include a capacitive Coulomb interaction via the charging energy E_c . In practice, depending on the experimental realization, there can also be a parallel channel for Cooper pair transfer involving only the superconducting substrate, and we here focus only on the Josephson current involving the TS wire.

Our main results are as follows.

(i) For the S-TS junction, the supercurrent is completely blocked for large Δ_w and/or for pointlike tunneling contacts. We provide a general condition explaining this supercurrent blockade. This has far-reaching consequences for the Josephson current through a Majorana wire whenever s -wave superconducting electrodes are involved.

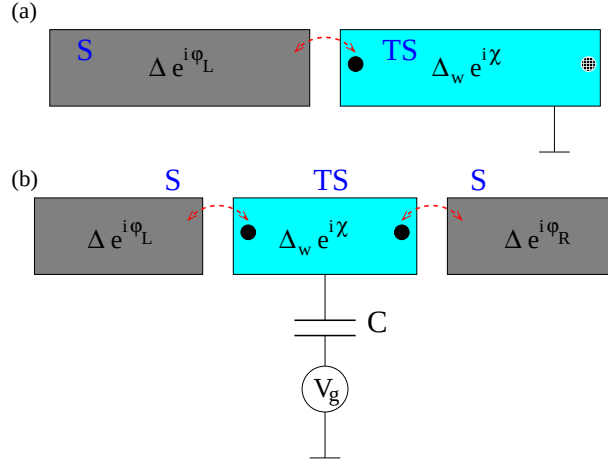


Figure 85: Schematic Josephson junction setups involving a Majorana wire (TS) with proximity-induced gap Δ_w and phase χ . Black dots stand for MBSs. The electrodes are s -wave BCS superconductors (S) with gap Δ and fixed phase $\varphi_{j=L/R}$. Tunneling processes between an S electrode and the respective MBS and/or TS quasiparticle states are indicated by red dashed arrows. (a) S-TS junction. (b) S-TS-S junction with charging energy $E_c = e^2/(2C)$.

(ii) In S-TS-S junctions, a finite supercurrent is only possible when quasiparticles (or other fermion excitations, e.g., a subgap impurity level) on the Majorana wire are accessible, at least for virtual transitions.

(iii) For the noninteracting ($E_c = 0$) S-TS-S junction, we provide the analytical solution for the CPR. This solution shows that the fractional 4π -periodic Josephson effect is absent in this setup. For weak tunnel couplings, we find a π -periodic CPR exhibiting π -junction behavior (negative critical current).

(iv) Expanding in the tunnel couplings and allowing for $E_c \neq 0$, we find the anomalous CPR

$$I(\varphi) = I_c \cos \varphi, \quad (24.1)$$

with the critical current I_c in Eq. (24.38) below. I_c is parity-sensitive and could be used to detect Majorana fermion states.

The structure of this paper is as follows. We first discuss the simplest S-TS junction in Sec. 24.0.2 and establish a general criterion for supercurrent blockade. The S-TS-S junction case is addressed in Sec. 24.0.3 by functional integral techniques. Besides allowing for some general conclusions, this leads immediately to the exact Josephson CPR in the noninteracting limit, which we discuss in Sec. 24.0.4. In Sec. 24.0.5, we then address the strongly interacting case by perturbation theory in the tunnel couplings. We finally offer some conclusions in Sec. 24.0.6. We mostly use units with $\hbar = 1$ below.

24.0.2 S-TS junction

We start with the grounded Majorana wire shown in Fig. 85(a). First, the BCS Hamiltonian describing the left S electrode is

$$H_L = \sum_{\mathbf{k}} \Psi_{\mathbf{k}}^\dagger (\xi_{\mathbf{k}} \sigma_z + \Delta \sigma_x e^{i\varphi_L \sigma_z}) \Psi_{\mathbf{k}}, \quad (24.2)$$

with Nambu spinor $\Psi_{\mathbf{k}} = (c_{\mathbf{k},\uparrow}, c_{-\mathbf{k},\downarrow}^\dagger)^T$. Here, $c_{\mathbf{k},\sigma}^\dagger$ creates an electron with momentum $\mathbf{k}$, spin projection $\sigma = \uparrow, \downarrow = \pm$ and normal-state dispersion $\xi_{\mathbf{k}}$. Pauli matrices $\sigma_{x,z}$ act in Nambu space.

The TS wire contains a pair of decoupled zero-energy MBSs near its ends. These are described by Majorana fermion operators, $\gamma_{j=L/R} = \gamma_j^\dagger$, with anticommutator algebra

$$\{\gamma_j, \gamma_{j'}\} = \delta_{jj'}. \quad (24.3)$$

It is convenient to combine the two Majorana operators into a single auxiliary complex fermion,

$$d = (\gamma_L + i\gamma_R)/\sqrt{2}. \quad (24.4)$$

We also include the bulk (p -wave type) TS quasiparticles. The wire Hamiltonian is written in terms of right/left ($\alpha = r/l$) moving fermion operators $f_{q,\alpha}^\dagger$. These describe TS quasiparticles with (discrete) 1D momentum q and Fermi velocity v_F ,

$$H_w = \sum_q \begin{pmatrix} f_{q,r}^\dagger \\ f_{-q,l} \end{pmatrix}^T \begin{pmatrix} v_F q & \Delta_w e^{-i\chi} \\ \Delta_w e^{i\chi} & -v_F q \end{pmatrix} \begin{pmatrix} f_{q,r} \\ f_{-q,l}^\dagger \end{pmatrix}. \quad (24.5)$$

Note that the zero modes $\gamma_{L,R}$ do not appear in H_w .

Following, e.g., Ref. [11], the S-TS contact is modeled by the general tunneling Hamiltonian

$$H_t = \sum_{\mathbf{k}, \sigma \sigma'} \int dx c_{\mathbf{k}, \sigma}^\dagger \mathcal{T}_{\mathbf{k}, \sigma \sigma'}(x) \psi_{\sigma'}(x) + \text{H.c.} \quad (24.6)$$

with the field operator $\psi_\sigma(x)$ for electrons with spin projection σ in the TS wire. Note that the tunnel matrix elements encoded in the kernel $\mathcal{T}_{\mathbf{k}, \sigma \sigma'}(x)$ may also describe spin-flip scattering at the interface. Expanding $\psi_\sigma(x)$ in terms of the TS quasiparticle operators $f_{q,\alpha}$ and the Majorana fermions γ_j , we obtain

$$\begin{aligned} H_t &= \sum_{\mathbf{k}, \sigma} c_{\mathbf{k}, \sigma}^\dagger \psi_{\mathbf{k}, \sigma} + \text{H.c.}, \\ \psi_{\mathbf{k}, \sigma} &= \lambda_{\mathbf{k}, \sigma} \gamma_L + \sum_{q, \alpha} t_{\mathbf{k}, \sigma; q, \alpha} f_{q, \alpha}. \end{aligned} \quad (24.7)$$

The complex-valued tunnel couplings $\lambda_{\mathbf{k}, \sigma}$ and $t_{\mathbf{k}, \sigma; q, \alpha}$ now encapsulate the overlap integrals of the kernel $\mathcal{T}_{\mathbf{k}, \sigma \sigma'}(x)$ with the Majorana and quasiparticle wavefunctions in the TS wire, respectively. We stress that the spin properties of an arbitrary TS wire (including the Majorana wavefunctions) as well as spin-flip processes in the contact are fully taken into account via the tunnel couplings $\lambda_{\mathbf{k}, \sigma}$ and $t_{\mathbf{k}, \sigma; q, \alpha}$.

Since $H = H_L + H_w + H_t$ corresponds to noninteracting fermions, the exact CPR,

$$I(\varphi) = (2e/\hbar) \partial_\varphi F(\varphi), \quad (24.8)$$

can be obtained from the partition function,

$$Z(\varphi) = \text{Tr} e^{-\beta H} = e^{-\beta F}, \quad (24.9)$$

with inverse temperature β and phase difference $\varphi = \varphi_L - \chi$. After integration over the S electrons, the φ -dependent part of the free energy comes from the action piece

$$\beta \mathcal{F}_s = \int_0^\beta d\tau_1 d\tau_2 \sum_{\mathbf{k}} \Phi_{\mathbf{k}}^\dagger(\tau_1) G_{\mathbf{k}}(\tau_1 - \tau_2) \Phi_{\mathbf{k}}(\tau_2), \quad (24.10)$$

where $\Phi_{\mathbf{k}} = (\psi_{\mathbf{k}, \uparrow}, \psi_{-\mathbf{k}, \downarrow}^\dagger)^T$ with Eq. (24.7). The (Fourier-transformed) anomalous S Green's function is

$$G_{\mathbf{k}}(\omega) = \frac{\Delta}{\omega^2 + \xi_{\mathbf{k}}^2 + \Delta^2} \sigma_x e^{i\varphi \sigma_z}. \quad (24.11)$$

Using $G_k(\tau) = G_k(-\tau)$ and $G_k \sim \sigma_{x,y}$, we find $\mathcal{F}_s = 0$ whenever the time-reversed partners $\psi_{\mathbf{k},\uparrow}$ and $\psi_{-\mathbf{k},\downarrow}$ in Eq. (24.7) are collinear,

$$\psi_{-\mathbf{k},\downarrow} = \zeta_{\mathbf{k}} \psi_{\mathbf{k},\uparrow}, \quad (24.12)$$

with some complex parameter $\zeta_{\mathbf{k}}$. [22] This is a sufficient (but not necessary) condition for supercurrent blockade in an S-TS junction.

To give some examples, consider the limit $\Delta_w \rightarrow \infty$, where no TS quasiparticles are accessible. In that case, Eq. (24.7) yields $\psi_{\mathbf{k},\sigma} = \lambda_{\mathbf{k},\sigma} \gamma_L$, which always satisfies Eq. (24.12). In the absence of quasiparticles, an S-TS junction thus never carries a supercurrent. Similarly, Eq. (24.12) trivially holds for a spin-polarized TS wire, where $\zeta_{\mathbf{k}} = 0$. However, for a point-like tunneling contact with

$$\lambda_{\mathbf{k},\sigma} \rightarrow \lambda_{\sigma}, \quad t_{\mathbf{k},\sigma;\alpha,q} \rightarrow t_{\sigma}, \quad (24.13)$$

Eq. (24.12) is generally not satisfied. Nonetheless, as we show in Sec. 24.0.4, the supercurrent also vanishes in this limit.

In order to rationalize why the supercurrent in a S-TS junction is blocked, we note that the condition (24.12) implies a suppression of the Josephson effect unless noncollinear time-reversed states are available in the TS wire. However, the point-like tunneling example [Eq. (24.13)] also indicates that the supercurrent blockade occurs even when the noncollinearity requirement is met, and a full explanation of the general supercurrent blockade is therefore more subtle.

24.0.3 S-TS-S junction

In this section, we consider the Majorana wire between two S electrodes, see Fig. 85(b), where H_w is supplemented by the interaction term [13]

$$H_c = E_c \left(2\hat{N} + \hat{n}_d - N_0 + \sum_{q;i=1,2} \hat{n}_{q,i}^{(qp)} \right)^2. \quad (24.14)$$

Cooper pairs in the wire are described by the number operator $\hat{N}$ with conjugate condensate phase χ . Note that the operator $e^{-i\chi}$ ($e^{i\chi}$) lowers (raises) the Cooper pair number (N) by one unit. The occupation of the d fermion level corresponding to the two MBSs is given by $\hat{n}_d = d^\dagger d$, and the number N_0 in Eq. (24.14) can be continuously tuned by a backgate voltage, cf. Fig. 85(b). The last term in Eq. (24.14) describes the occupation of TS quasiparticle states, $\hat{n}_{q,i}^{(qp)} = \eta_{q,i}^\dagger \eta_{q,i}$, where Eq. (24.5) is written in diagonal form,

$$H_w = \sum_q \sum_{i=1,2} E_q \eta_{q,i}^\dagger \eta_{q,i}, \quad (24.15)$$

with fermion operators $\eta_{q,1}$ (particle-like) and $\eta_{q,2}$ (hole-like) for positive energy

$$E_q = \sqrt{v_F^2 q^2 + \Delta_w^2}. \quad (24.16)$$

The $\eta_{q,i}$ are connected to $f_{q,\alpha}$ by a canonical charge-conserving transformation,

$$\begin{pmatrix} f_{q,r} \\ e^{-i\chi} f_{-q,l}^\dagger \end{pmatrix} = \begin{pmatrix} u_q & -v_q \\ v_q & u_q \end{pmatrix} \begin{pmatrix} \eta_{q,1} \\ e^{-i\chi} \eta_{q,2}^\dagger \end{pmatrix}, \quad (24.17)$$

where

$$u_q = v_{-q} = \sqrt{(1 + v_F q / E_q) / 2}. \quad (24.18)$$

The full Hamiltonian,

$$H = H_L + H_R + H_w + H_c + H_t, \quad (24.19)$$

also includes a BCS term H_R for the right S electrode, see Eq. (24.2), with $\varphi \equiv \varphi_L - \varphi_R$, and the tunneling Hamiltonian

$$H_t = H_t^{(\gamma)} + H_t^{(qp)}. \quad (24.20)$$

In what follows, point-like tunneling contacts satisfying Eq. (24.13) are assumed for simplicity. Taking into account charge conservation,[9]

$$H_t^{(\gamma)} = \frac{1}{\sqrt{2}} \sum_{\mathbf{k}} \left[\lambda_{L,\sigma} c_{\mathbf{k},L,\sigma}^\dagger (d + e^{-i\chi} d^\dagger) - i \lambda_{R,\sigma} c_{\mathbf{k},R,\sigma}^\dagger (d - e^{-i\chi} d^\dagger) \right] + \text{H.c.} \quad (24.21)$$

describes tunneling between states in lead $j = L/R$ and the corresponding Majorana fermion γ_j with amplitude $\lambda_{j,\sigma}$. Quasiparticle tunneling is contained in [see Eqs. (24.7) and (24.17)]

$$\begin{aligned} H_t^{(qp)} &= \sum_{\mathbf{k},j,\sigma} t_{j,\sigma} c_{\mathbf{k},j,\sigma}^\dagger \eta_w + \text{H.c.}, \\ \eta_w &= \sum_q \left[u_q (\eta_{q,1} + \eta_{q,2}) + v_q e^{-i\chi} (\eta_{q,1}^\dagger - \eta_{q,2}^\dagger) \right]. \end{aligned} \quad (24.22)$$

With the (normal-conducting) lead density of states ν_0 , we introduce the hybridization scales

$$\begin{aligned} \Gamma_j &= 2\pi\nu_0 \sum_{\sigma} \sigma \lambda_{j,\sigma} t_{j,-\sigma}^*, \quad \tilde{\Gamma}_j = 2\pi\nu_0 \sum_{\sigma} \lambda_{j,\sigma} t_{j,\sigma}^*, \\ \Gamma_{\lambda,j} &= 2\pi\nu_0 \sum_{\sigma} |\lambda_{j,\sigma}|^2, \quad \Gamma_t = 2\pi\nu_0 \sum_{j,\sigma} |t_{j,\sigma}|^2. \end{aligned} \quad (24.23)$$

Now we are ready to address the CPR. For $E_c \neq 0$, no exact solution can be obtained anymore, but general insights follow again by considering the partition function $Z(\varphi)$ in Eq. (24.9). Moreover, the exact CPR for $E_c = 0$ reported in Sec. 24.0.4 directly follows from our expressions below. After integration over the lead fermions,

$$Z(\varphi) = \text{Tr} \left(e^{-\beta(H_w + H_c)} \mathcal{T} e^{-(S_M + S_{qp} + S_{\text{int}})} \right), \quad (24.24)$$

where $\mathcal{T}$ denotes imaginary-time (τ) ordering and the trace indicates functional integration over the phase field $\chi(\tau)$, Majorana fields $\gamma_{L/R}(\tau)$, and quasiparticle fields $(\eta_w, \bar{\eta}_w)(\tau)$. Using the hybridization parameters (24.23) and the auxiliary function

$$f(\tau) = \int \frac{d\omega}{2\pi} f_\omega e^{-i\omega\tau}, \quad f_\omega = \frac{1}{\sqrt{\omega^2 + \Delta^2}}, \quad (24.25)$$

the action S_M in Eq. (24.24) comes from the Majorana fields only,

$$S_M = \int d\tau_1 d\tau_2 \dot{f}(\tau_1 - \tau_2) \sum_{j,j'} \Lambda_{j,j'}(\chi_1, \chi_2) \gamma_j(\tau_1) \gamma_{j'}(\tau_2), \quad (24.26)$$

where $j = L/R = +/-$ and $\chi_{i=1,2} = \chi(\tau_i)$. The matrix kernel Λ is given by

$$\begin{aligned} \Lambda_{j,j} &= \sum_{j'} \frac{\Gamma_{\lambda,j'}}{8} [1 + \cos(\delta\chi) + \sigma_{jj'} (\cos \chi_1 + \cos \chi_2)], \\ \Lambda_{j,-j} &= \pm \sum_{j'} \frac{\Gamma_{\lambda,j'}}{8} [\sin(\delta\chi) - \sigma_{jj'} (\sin \chi_1 + \sin \chi_2)], \end{aligned}$$

where $\sigma_{jj'} = \text{sgn}(jj')$ and $\delta\chi = \chi_1 - \chi_2$. Importantly, S_M in Eq. (24.26) does not depend on the phase difference φ . For $\Delta_w \rightarrow \infty$, the action piece $S_{qp} + S_{\text{int}}$ does not contribute, and therefore there is no Josephson effect in this limit.

Turning to finite Δ_w , $H_t^{(qp)}$ results in

$$S_{qp} = \Gamma_t \int d\tau_1 d\tau_2 \dot{f}(\tau_1 - \tau_2) \bar{\eta}_w(\tau_1) \eta_w(\tau_2), \quad (24.27)$$

which is also independent of φ . The only φ -dependent action piece, which ultimately can be responsible for a Josephson current, involves Majoranas and quasiparticles,

$$S_{\text{int}} = \int \frac{d\tau_1 d\tau_2}{2\sqrt{2}} \sum_j \gamma_j(\tau_1) L_j(\tau_1, \tau_2) \eta_w(\tau_2) + \text{H.c.}, \quad (24.28)$$

with the kernels ($j = L/R = +/ -$)

$$L_j(\tau_1, \tau_2) = \tilde{\Gamma}_j^* \dot{f}(\tau_1 - \tau_2) [1 + e^{-i\chi_1} \pm i(1 - e^{-i\chi_1})] + i\Delta \Gamma_j^* f(\tau_1 - \tau_2) [ie^{i\varphi_j}(1 + e^{i\chi_1}) \pm e^{i\varphi-j}(1 - e^{i\chi_1})] \quad (24.29)$$

This implies that the magnitude of the Josephson current is controlled by the ratio $|\Gamma_L \Gamma_R|/\Delta_w^2$. In particular, $I = 0$ when this ratio vanishes. Equation (24.29) also tells us how s -wave Cooper pairs are transported through the Majorana wire, namely by creation ($d^\dagger \eta_w^\dagger$) or annihilation ($e^{-i\chi} d \eta_w^\dagger$) of charge $2e$ (plus the conjugate processes). These processes simultaneously involve Majorana states and bulk TS quasiparticles.

24.0.4 Noninteracting limit

For $E_c = 0$, we again have a system of noninteracting fermions. The exact $T = 0$ CPR follows after some algebra from the expressions in Sec. 24.0.3,

$$I(\varphi) = -\frac{2e}{\hbar} \int_0^\infty \frac{d\omega}{2\pi} \partial_\varphi \ln \det M(\omega), \quad (24.30)$$

with the matrix $M_{jj'}(\omega)$ in left/right ($j = L/R = +/ -$) space. Using the notation [see also Eqs. (24.16), (24.23) and (24.25)]

$$X_\omega = \left(\sum_q \frac{1}{\omega^2 + E_q^2} \right)^{-1} + \omega^2 f_\omega \Gamma_t, \quad (24.31)$$

the diagonal elements of M are given by

$$M_{jj} = 1 + f_\omega \Gamma_{\lambda,j} + \frac{f_\omega^2}{X_\omega} \left(\Delta^2 |\Gamma_j|^2 - \omega^2 |\tilde{\Gamma}_j|^2 \right). \quad (24.32)$$

Similarly, we find for the off-diagonal elements

$$M_{j,-j} = \frac{f_\omega^2}{X_\omega} \text{Re} \left[\Delta^2 \Gamma_j^* \Gamma_{-j} e^{\pm i\varphi} - \omega^2 \tilde{\Gamma}_j^* \tilde{\Gamma}_{-j} + i\omega \Delta \left(\Gamma_j^* \tilde{\Gamma}_{-j} e^{i\varphi_j} + \tilde{\Gamma}_j^* \Gamma_{-j} e^{-i\varphi-j} \right) \right]. \quad (24.33)$$

We now draw several conclusions from Eq. (24.30).

First, for our point-like tunneling model, since only the off-diagonal matrix elements [Eq. (24.33)] depend on the phase difference φ , a single S-TS junction, where the Majorana fermion γ_R is effectively not accessible, cannot carry a supercurrent. This result was already announced in Sec. 24.0.2.

Q_1	Q_2	Q_3
$+(\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_1)$	$-(\epsilon_1, 0, \hat{\epsilon}_{-1})$	$+(\epsilon_1, 0, \hat{\epsilon}_1)$
$-(\hat{\epsilon}_{-1}, \hat{\epsilon}_2, \hat{\epsilon}_{-1})$	$+(\epsilon_3, 0, \hat{\epsilon}_1)$	$-(\epsilon_3, 0, \hat{\epsilon}_{-1})$
$+(\epsilon_1, \hat{\epsilon}_2, \epsilon_1)$	$-(\hat{\epsilon}_{-1}, 0, \epsilon_1)$	$+(\hat{\epsilon}_1, 0, \epsilon_1)$
$-(\epsilon_3, \hat{\epsilon}_2, \epsilon_3)$	$+(\hat{\epsilon}_1, 0, \epsilon_3)$	$-(\hat{\epsilon}_{-1}, 0, \epsilon_3)$
$-(\epsilon_1, \hat{\epsilon}_2, \hat{\epsilon}_1)$	$+(\epsilon_1, \Delta_w, \hat{\epsilon}_{-1})$	$-(\hat{\epsilon}_{-1}, \Delta_w, \hat{\epsilon}_{-1})$
$+(\epsilon_3, \hat{\epsilon}_2, \hat{\epsilon}_{-1})$	$-(\epsilon_3, \hat{\epsilon}_4, \hat{\epsilon}_1)$	$+(\hat{\epsilon}_1, \hat{\epsilon}_4, \hat{\epsilon}_1)$
$-(\hat{\epsilon}_1, \hat{\epsilon}_2, \epsilon_1)$	$+(\hat{\epsilon}_{-1}, \Delta_w, \epsilon_1)$	$-(\epsilon_1, \Delta_w, \epsilon_1)$
$+(\hat{\epsilon}_{-1}, \hat{\epsilon}_2, \epsilon_3)$	$-(\hat{\epsilon}_1, \hat{\epsilon}_4, \epsilon_3)$	$+(\epsilon_3, \hat{\epsilon}_4, \epsilon_3)$

Table 2: All combinations of signs $\sigma_p = \pm$ and energy arguments (E_a, E_b, E_c) summed over in Eq. (24.38) for the even-parity state, where $\hat{\epsilon}_n = \epsilon_n + \Delta_w$ and $\epsilon_n = E_n - E_0$.

Second, exploiting current conservation, some algebra reveals the periodicity

$$I(\varphi + n\pi) = I(\varphi) \quad (24.34)$$

in Eq. (24.30), with $n = 2$ or just 1 (see below). In particular, the S-TS-S junction does not allow for the fractional 4π -periodic Josephson effect omnipresent in TS-TS junctions.[15]

Third, for a symmetric junction with $\Gamma_L = \Gamma_R = \Gamma$, expanding Eq. (24.30) to lowest nontrivial order in the tunnel couplings yields[27]

$$I(\varphi) = -\frac{e\Delta}{\hbar} |\pi\nu_w\Gamma^2/\Delta_w|^2 \mathcal{G}(\Delta_w/\Delta) \sin(2\varphi) \quad (24.35)$$

with the density of states $\nu_w = L/(2\pi v_F)$ in a wire of length L and the function

$$\mathcal{G}(x) = \frac{x(2+x)}{4(1+x)^2}. \quad (24.36)$$

Equation (24.35) describes a π -periodic CPR with negative critical current, $I_c \sim -|\Gamma/\Delta_w|^4$. It is worth mentioning that I_c is suppressed by the factor $|\Gamma/\Delta_w|^2$ compared to the usual cotunneling limit. This suppression can be traced back to the destructive interference of different contributions of order Γ^2 . For finite E_c , this cancellation is incomplete and we recover $I_c \sim |\Gamma/\Delta_w|^2$, see Eq. (24.38).

24.0.5 Weak tunneling limit

In this section, we discuss the results of second-order perturbation theory in the hybridizations (24.23), valid for arbitrary charging energy E_c . In the ground state, no quasiparticles are thermally activated but virtual excitations remain possible. For simplicity, we here only keep $q = 0$ quasiparticles[26] and consider the case $\Gamma_L = \Gamma_R = \Gamma$. The ground-state energy of the isolated dot can then be expressed in terms of the energies

$$E_n = E_c(2N - N_0 + n)^2, \quad n \in \mathbb{Z}, \quad (24.37)$$

where the Cooper pair number N follows from the relation $-3/2 < 2N - N_0 < 1/2$. Using $\delta \equiv 2N - N_0$, for $-1/2 < \delta < 1/2$, the ground state $|e\rangle$ has energy E_0 and even parity, i.e., the d state is empty. Otherwise, we have an odd-parity state, $|o\rangle = d^\dagger|e\rangle$, with energy E_1 .

Some straightforward but lengthy algebra yields from Eq. (24.24) the CPR announced in Eq. (24.1). For the even-parity state $|e\rangle$, the critical current is

$$I_c = \frac{e\Delta}{\hbar} \frac{\Delta|\Gamma|^2}{2} \sum_{p=1}^{24} \sigma_p Q_{i_p} (E_{a_p}, E_{b_p}, E_{c_p}). \quad (24.38)$$

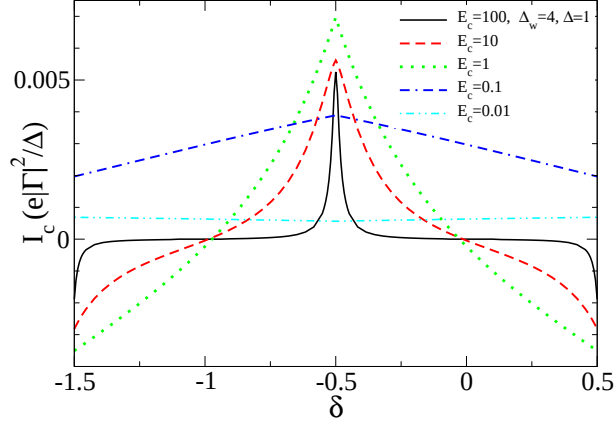


Figure 86: Anomalous current I_c in Eq. (24.1) vs $\delta = 2N - N_0$ in the cotunneling regime of an S-TS-S junction. I_c is computed from Eq. (24.38) for $\Delta_w/\Delta = 4$ and several E_c/Δ . Note that δ can be changed by varying a backgate, see Fig. 85(b). $I_c(\delta)$ is periodic; we show one fundamental interval only.

All signs $\sigma_p = \pm$ and energies (E_a, E_b, E_c) entering the real-valued Q functions are given in Table 2, where

$$Q_{i_p} = \int \frac{d\omega d\omega'}{(2\pi)^2} \frac{f_\omega f_{\omega'}}{i\omega + E_a} \times \begin{cases} \frac{1 - \delta_{E_b,0}}{E_b(i\omega' + E_c)}, & i_p = 1, \\ \frac{1}{(i\omega + E_c)(i\omega + i\omega' + E_b)}, & i_p = 2, \\ \frac{1}{(i\omega' + E_c)(i\omega + i\omega' + E_b)}, & i_p = 3. \end{cases}$$

For the odd-parity state $|o\rangle$, I_c follows from Eq. (24.38) with a particle-hole transformation, $\epsilon_n \rightarrow \epsilon_{-(n-1)}$, in Table 2.

The critical current I_c in Eq. (24.38) is shown in Fig. 86 as a function of δ . For half-integer δ , two charge states become degenerate and $|I_c|$ shows resonance enhancement. While for small E_c , we find a small positive and δ -independent I_c , these resonances become very narrow for large E_c , with I_c close to zero unless δ is nearly half-integer. The parity-sensitivity of I_c , i.e., the sign change of I_c between $\delta \approx -1/2$ and $\delta \approx 1/2 \pmod{2}$, see Fig. 86, may then provide an experimentally detectable signature for MBSs.

24.0.6 Conclusions

In this work, we have shown that under generic conditions, a supercurrent blockade occurs in S-TS Josephson junctions made by coupling an s -wave BCS superconductor and a topologically nontrivial superconducting wire with Majorana end states. In the S-TS-S junction, this blockade imposes severe restrictions; in particular, quasiparticle excitations are necessary to have a Josephson effect. For the noninteracting case, the analytical solution for the current-phase relation has been presented. It shows that the 4π -periodic Josephson effect is absent in S-TS-S junctions. For weak tunneling, the CPR is π -periodic with a negative critical current, while a finite charging energy results in the anomalous CPR $I(\varphi) = I_c \cos \varphi$. The parity sensitivity of I_c could then be used to detect the Majorana states in supercurrent measurements. We hope that our predictions will soon be tested experimentally.

25 Transport theory of multiterminal hybrid structures by R. Mélin, D. Feinberg

We derive a microscopic transport theory of multiterminal hybrid structures in which a superconductor is connected to several spin-polarized electrodes. We discuss the non-perturbative physics of extended contacts, and show that it can be well represented by averaging out the phase of the electronic wave function. The maximal conductance of a two-channel contact is proportional to $(e^2/h)(a_0/D)^2 \exp\{-D/\xi(\omega^*)\}$, where D is the distance between the contacts, a_0 the lattice spacing, $\xi(\omega)$ is the superconducting coherence length, and ω^* is the cross-over frequency between a perturbative regime ($\omega < \omega^*$) and a non perturbative regime ($\omega^* < \omega < \Delta$). The intercontact Andreev reflection and elastic cotunneling conductances are not equal if the electronic phases take a fixed value. However, these two quantities do coincide if one can average out the electronic phase. The equality between the Andreev and cotunneling conductances is also valid in the presence of at least one extended contact in which the phases take deterministic values.

25.1 Introduction

Transport of correlated pairs of electrons in multiterminal configurations has recently focused an important interest. One possible line of research is motivated by the possibility of creating entangled pairs of electrons from a superconductor [1, 2, 3, 4, 5]. This may lead to fundamental tests of quantum mechanics in solid state, or to new ways of manipulating quantum information. In a different context, the interplay between superconductivity and magnetism offers novel functionalities in multiterminal devices : the various transport channels occurring at several neighboring superconducting-ferromagnet interfaces depend in a subtle way on spin polarizations and geometry [6, 7, 8, 9, 10]. At the theoretical level, there is a need for a transport theory in multiterminal hybrid structures involving superconducting and spin-polarized elements. These structures have so far not been the object of any experiment. As a consequence, one of the objective of the present time models is to predict what should be measured in future experiments.

In ferromagnet – superconductor junctions, it is well established that Andreev reflection is suppressed by an increase of spin polarization [6]. This is because Andreev reflection can take place only in the channels having both a spin-up and a spin-down Fermi point. This theoretical prediction has been probed experimentally by two independent groups [7, 8]. Spin polarized Andreev reflection [7] and related effects [11] can even be used to measure the Fermi surface spin polarization.

We are concerned here with more sophisticated systems in which a superconductor is connected to several electrodes, which can be ferromagnetic or normal metals. In the case of ferromagnetic electrodes, it will be crucial to take into account the existence of a very small coherence length. We neglect any diffusive effect in spite of the fact that they lead to a rich physics [12, 13, 14, 15]. As a consequence, our models should apply to point contacts having a dimension much smaller than the diffusive mean free path. The fabrication of such contacts in multiterminal configurations may seem difficult in view of the present day technology, and this is why there are no available experiments on these systems. However, there are interesting phenomena taking place in these multiterminal systems. For instance, Andreev reflection can become *non local*. Namely a spin-up electron from a given electrode A can be Andreev reflected as a hole in a *different* electrode B. This effect has been studied theoretically by Byers and Flatté in Ref. [16] for normal metals, and in Ref. [3] in the ferromagnetic case which contains the richest physics. It is in our opinion crucial to develop the most general theoretical description of this phenomenon. In this respect, two approaches have been developed recently.

One is based on the analysis of the lowest order processes appearing in perturbation theory [4]. Another approach is non perturbative, but relies on effective Green's functions [5]. It is a very natural task to work out the microscopic theory of transport in ballistic multiterminal hybrid structures. Transport theory will be solved exactly by means of Green's function techniques [5]. We incorporate in our theoretical description two realistic constraints:

- (i) *Multichannel effects* which are expected to play a central role in quantum point contacts involving ferromagnetic metals. The radius of the contact can be smaller or larger than the phase coherence length of the ferromagnetic metal.
- (ii) *The strength of the tunnel amplitude* is small in low transparency contacts, and large in high transparency contacts. Our approach is non perturbative. Therefore, the tunnel matrix element can take arbitrary values. As a result, we can derive transport in the presence of arbitrary bias voltages.

An ingredient that is not incorporated at the present stage in the model is the reduction of the superconducting gap associated to the proximity effect.

We will use this non perturbative approach to address the following physical questions:

- (i) It has been already established that the multiterminal hybrid system should be described by a *conductance matrix*. The matrix elements encode all information about the current flowing in a given electrode, in response to a voltage applied in another electrode. *How does the crossed conductance behave when the voltages are close to the superconducting gap ? What is the maximal value of the crossed conductance ?*
- (ii) The superconductor Green's functions contain not only an information about non local processes, but contain also an information about the phase of electron propagation. In extended contacts, there are many phases coming into account. *Can these phases be considered as random quantities ?*
- (iii) There are two propagators associated to a superconductor: the ordinary and the anomalous propagators. One can easily realize that after phase averaging, the ordinary propagator is identical to the anomalous propagator (see section 25.3.1). As a consequence, in the tunnel approach and for unpolarized contacts, the averaged Andreev reflection conductance is equal to the the averaged elastic cotunneling conductance [4]. *Is this identity still valid in the presence of large interface transparencies ?*

The article is organized as follows. Some preliminaries are given in section 25.2. The form of the Green's functions is derived in section 25.3. The solution of the model with two single-channel electrodes is presented in section 25.4. The general solution with an arbitrary number of single-channel electrodes is presented in section 25.5. As a particular example, we discuss in section 25.6 the physics of a model with three single-channel electrodes. Multichannel electrodes are solved in section 25.7.

25.2 Preliminaries

25.2.1 The Keldysh method

We will use Green's functions techniques to solve transport theory. There is an advanced ($\hat{G}^A$), retarded ($\hat{G}^R$) and Keldysh ($\hat{G}^{+,-}$) Green's functions [18, 19]. Each of these Green's functions is a 2×2 matrix in Nambu representation. The Dyson equation for the advanced and retarded Green's functions takes the form

$$\hat{G}^{R,A} = \hat{g}^{R,A} + \hat{g}^{R,A} \otimes \hat{\Sigma} \otimes \hat{G}^{R,A}. \quad (25.1)$$

The Dyson equation for the Keldysh component is given by

$$\hat{G}^{+,-} = \left[\hat{I} + \hat{G}^R \otimes \hat{\Sigma} \right] \otimes \hat{g}^{+,-} \otimes \left[\hat{I} + \hat{\Sigma} \otimes \hat{G}^A \right]. \quad (25.2)$$

Eqs. 25.1, 25.2 are written in a compact notation in which the convolution involves a summation over time variables and space labels. $\hat{\Sigma}$ is the self energy, which contains all couplings present in the tunnel Hamiltonian. The notation $\hat{g}$ is used for the Green's functions of the disconnected system (i.e. with $\hat{\Sigma} = 0$) while $\hat{G}$ refers to the Green's functions of the connected system (i.e. with $\hat{\Sigma} \neq 0$). We will use the notation

$$\hat{g}^{A,R}(t, t') = \begin{pmatrix} g^{A,R}(t, t') & f^{A,R}(t, t') \\ f^{A,R}(t, t') & g^{A,R}(t, t') \end{pmatrix} \quad (25.3)$$

for the Nambu representation of the advanced and retarded Green's functions, with

$$g^A(t, t') = -i\theta(t - t') \langle \{c_{i,\uparrow}(t), c_{j,\uparrow}^+(t')\} \rangle \quad (25.4)$$

$$f^A(t, t') = -i\theta(t - t') \langle \{c_{i,\uparrow}(t), c_{j,\downarrow}(t')\} \rangle. \quad (25.5)$$

We will also denote by $\hat{\rho} = \frac{1}{\pi} \text{Im}(\hat{g}^A)$ the Nambu representation of the density of states:

$$\hat{\rho} = \begin{pmatrix} \rho_g & \rho_f \\ \rho_f & \rho_g \end{pmatrix}, \quad (25.6)$$

with $\rho_g = \frac{1}{\pi} \text{Im}(g^A)$ and $\rho_f = \frac{1}{\pi} \text{Im}(f^A)$. The Nambu representation of a given tunnel matrix element connecting sites a and α is $t_{a,\alpha} \hat{\sigma}^z$, where $\hat{\sigma}^z$ is one of the Pauli matrices. We use a notation in which the “sites” of the superconductor are represented by the Greek symbols $\alpha, \beta, \gamma, \dots$. The sites in the normal metal electrodes are represented by the Latin symbols $a, b, c, \dots$. The explicit form of the Keldysh Green's function connecting the two sides of a given interface is, from (25.2)

$$\hat{G}_{\alpha_k, a_k}^{+,-} = \sum_{i,j} \left[\delta_{k,i} \hat{I} + \hat{G}_{\alpha_k, a_i}^R \hat{t}_{a_i, \alpha_i} \right] \hat{g}_{\alpha_i, \alpha_j}^{+,-} \hat{t}_{\alpha_j, a_j} \hat{G}_{a_j, a_k}^A + \sum_{i,j} \hat{G}_{\alpha_k, \alpha_i}^R \hat{t}_{\alpha_i, a_i} \hat{g}_{a_i, a_j}^{+,-} \left[\delta_{k,j} \hat{I} + \hat{t}_{a_j, \alpha_j} \hat{G}_{\alpha_j, a_k}^A \right]. \quad (25.7)$$

The strategy is first to use (25.1) to calculate the advanced and retarded Green's functions and next use (25.2) to calculate the Keldysh Green's function. The current can be obtained easily from the Keldysh Green's function [19]:

$$I_{a_k, \alpha_k} = \frac{e}{h} \int d\omega \left[\hat{t}_{a_k, \alpha_k} \hat{G}_{\alpha_k, a_k}^{+,-} - \hat{t}_{\alpha_k, a_k} \hat{G}_{a_k, \alpha_k}^{+,-} \right]. \quad (25.8)$$

The spin-up (spin-down) current is given by the 11 (22) matrix element of the Nambu representation.

25.2.2 A useful trick on the spectral current

The systems of interest here are made of a single superconductor connected to an arbitrary number of external normal metal electrodes. It turns out that there exists some tricks that can be used to simplify the calculation of the current in such systems. One of these tricks is the following.

The terms in the first summation in Eq. 25.7 contain a prefactor $n_F(\omega - \mu_S)$ because α_i belongs to the superconductor. The terms in the second summation contain a prefactor $n_F(\omega - \mu_{a_i})$ because a_i belongs to a ferromagnetic electrode. The trick consists in realizing that the terms containing $n_F(\omega - \mu_S)$ coincide exactly with the terms containing $n_F(\omega - \mu_{\alpha_i})$. As a result, the total current can be written as an integral over energy of the spectral current:

$$I_k = \sum_i \int d\omega [n_F(\omega - \mu_{\alpha_i}) - n_F(\omega - \mu_S)] \mathcal{I}_{k,i}(\omega). \quad (25.9)$$

25.3 Form of the Green's functions

In this section, we present a derivation of the form of the Green's functions that will be used throughout the remainder of the article. This will give us the opportunity to discuss the relevant parameters of the model.

25.3.1 Green's functions in the superconductor

Effective Green's functions

In some cases, it will be useful to describe the superconducting and ferromagnetic reservoirs in terms of effective Green's functions. It was already shown by one of us in Ref. [5] that effective Green's functions can be used to construct a consistent non perturbative “toy model” version of transport theory. In this approach, the superconductor is viewed as zero dimensional: its dimensions are shorter than the coherence length. The superconducting effective Green's function takes the form [5]:

$$\hat{g}^{R,A}(\omega) = \frac{\pi\rho_N}{\sqrt{\Delta^2 - (\omega - \mu_S)^2}} \begin{bmatrix} -(\omega - \mu_S) \pm i\eta & \Delta \\ \Delta & -(\omega - \mu_S) \pm i\eta \end{bmatrix}, \quad (25.10)$$

and we will consider in the following the limit $\eta \rightarrow 0$. The Keldysh component is given by $\hat{g}^{+,-}(\omega) = 2i\pi n_F(\omega - \mu_S)\hat{\rho}(\omega)$, with $\hat{\rho}(\omega) = \frac{1}{\pi}\text{Im}(\hat{g}^A)$ the density of states. The ferromagnetic electrodes are described in a similar way:

$$\hat{g}^{R,A} = \mp i\pi \begin{bmatrix} \rho_{1,1} & 0 \\ 0 & \rho_{2,2} \end{bmatrix}, \quad (25.11)$$

where $\rho_{1,1}$ and $\rho_{2,2}$ are respectively the spin-up and spin-down densities of states.

Spectral representation

To address more realistic models, it is useful to restore the dependence of the Green's functions upon space coordinates. The Green's function are evaluated in terms of a spectral representation. Let us start with the pairing Hamiltonian

$$\mathcal{H} = \sum_{\vec{k},\sigma} \xi_k c_{\vec{k},\sigma}^{\dagger} c_{\vec{k},\sigma} + \Delta_k^* c_{\vec{k},\downarrow}^{\dagger} c_{\vec{k},\uparrow}^{\dagger} + \Delta_k c_{\vec{k},\uparrow} c_{\vec{k},\downarrow}, \quad (25.12)$$

with $k = |\vec{k}|$. We use the notation $\xi_{\vec{k}} = \epsilon_{\vec{k}} - \mu$, with $\epsilon_{\vec{k}} = \hbar^2 k^2 / (2m)$ for the kinetic energy. After standard manipulations, the spectral representation is found to be (i, j standing for α_i, α_j):

$$[g_{i,j}^A]_{1,1}(\omega) = \frac{1}{\mathcal{N}} \sum_k e^{i\vec{k} \cdot (\vec{x}_i - \vec{x}_j)} \left[\frac{(u_k)^2}{\omega - (\mu_S + E_k) + i\eta} + \frac{(v_k)^2}{\omega - (\mu_S - E_k) + i\eta} \right] \quad (25.13)$$

$$[g_{i,j}^A]_{1,2}(\omega) = \frac{1}{\mathcal{N}} \sum_k e^{i\vec{k} \cdot (\vec{x}_i - \vec{x}_j)} u_k v_k \left[-\frac{1}{\omega - (\mu_S + E_k) + i\eta} + \frac{1}{\omega - (\mu_S - E_k) + i\eta} \right] \quad (25.14)$$

where $\mathcal{N}$ the number of sites in the superconductor. We used the standard notation

$$E_k = \sqrt{\Delta^2 + (\xi_k)^2} \quad (25.15)$$

$$(u_k)^2 = \frac{1}{2} \left(1 + \frac{\xi_k}{E_k} \right) \quad (25.16)$$

$$(v_k)^2 = \frac{1}{2} \left(1 - \frac{\xi_k}{E_k} \right) \quad (25.17)$$

for the quasiparticle energy, and the electron and hole coherence factors.

Form of the Green's function

The spectral representation (25.13), (25.14) is valid in any dimension. We now restrict our discussion to the case of a three dimensional superconductor. We first perform the angular integration and next use the residue theorem to make the radial integration. Note that it is crucial to carry out first the angular integration. This ensures the existence of well defined convergence properties when we use the residue theorem to make the radial integration. The final result is

$$\hat{g}_{i,j}^{R,A}(\omega) = \frac{ma_0^3}{\hbar^2} \frac{1}{2\pi|\vec{x}_i - \vec{x}_j|} \exp\left\{\left(-\frac{|\vec{x}_i - \vec{x}_j|}{2\xi(\omega)}\right)\right\} \quad (25.18)$$

$$\times \left\{ \frac{\sin \varphi}{\sqrt{\Delta^2 - (\omega - \mu_S)^2}} \begin{bmatrix} -(\omega - \mu_S) \pm i\eta & \Delta \\ \Delta & -(\omega - \mu_S) \pm i\eta \end{bmatrix} - \cos \varphi \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

with $\varphi = k_F|x_i - x_j|$ and a_0 is the length of the elementary cell. The coherence length appearing in (25.18) is

$$\xi(\omega) = \begin{cases} \xi(0) \frac{\Delta}{\sqrt{\Delta^2 - \omega^2}} & \text{if } \omega < \Delta \\ +\infty & \text{if } \omega > \Delta. \end{cases} \quad (25.19)$$

We used the notation $\xi(0) = \frac{\epsilon_F}{k_F \Delta}$ for the zero-frequency coherence length, with ϵ_F the Fermi energy. In the case of two point contacts a and b treated explicitly in section 25.4, the Green's function $g_{a,b}^{R,A}$ provides a coherent coupling between charge transport at the two contacts. We end-up this section with three remarks. First, we note that with $\cos \varphi = 0$ the Green's functions are identical to the effective Green's functions given in section 25.3.1. Second, we recover the usual free-fermion Green's function in the limit $\omega \gg \Delta$:

$$g_{i,j}^R(\omega) = -i \frac{ma_0^3}{\hbar^2} \frac{1}{2\pi|\vec{x}_i - \vec{x}_j|} e^{ik_F|\vec{x}_i - \vec{x}_j|}.$$

Finally, we will discuss in detail the role played by phase averaging. Using the notation $\langle\langle \dots \rangle\rangle = \int \frac{d\varphi}{2\pi}$, one can show that $\langle\langle (g_{i,j})^2 \rangle\rangle = \langle\langle (f_{i,j})^2 \rangle\rangle$. This identity implies that in the tunnel limit the average Andreev reflection conductance is equal to the average elastic cotunneling conductance.

25.3.2 Green's functions in the ferromagnetic electrodes

The Green's function in the ferromagnetic electrodes are diagonal in Nambu space. The form of the Green's function is taken as

$$g_{i,j,\sigma}^R(\omega) = -i \frac{ma_0^3}{\hbar^2} \frac{1}{2\pi|\vec{x}_i - \vec{x}_j|} \exp\{(i\varphi^{(\sigma)})\} \exp\left\{\left(-\frac{|x_i - x_j|}{2l_\phi^{(\sigma)}}\right)\right\}, \quad (25.20)$$

where the phase is given by $\varphi^{(\sigma)} = k_F^{(\sigma)}|x_i - x_j|$ and $l_\phi^{(\sigma)}$ is the phase coherence length. There is a mismatch between the spin-up and spin-down Fermi wave vectors:

$$k_F^{(\sigma)} = \frac{\sqrt{2m}}{\hbar} \sqrt{\epsilon_F + \sigma h_{\text{ex}} + \omega},$$

and h_{ex} is the exchange field. At some point, it will be convenient to assume that the phase takes the particular value $\varphi^{(\sigma)} = 0$. With this special value of the phase, the form of the 3D Green's function (25.20) is identical to the effective Green's function (25.11). The coherence length in ferromagnetic metals is much shorter than in usual metals so that ferromagnetism is often treated in a semi-classical description (see [20, 21, 22]). For instance, the absence of Aharonov-Bohm oscillations reported in Ref. [12] shows that the coherence length in Co is smaller than

$0.3\mu m$. This can be incorporated in our model by considering that the “dissipation” η is not a small parameter. This results in a finite coherence length, which is spin-dependent, and inverse proportional to the strength of dissipation:

$$l_{\phi}^{(\sigma)} = \frac{1}{\eta} \frac{\hbar}{\sqrt{2m}} \sqrt{\epsilon_F + \sigma h_{\text{ex}} + \omega}.$$

This simple phenomenological model contains the relevant physics associated to ferromagnetic metals. For instance, the phase coherence length of spin-up electrons is larger than the spin-down coherence length. The Green’s function (25.20) is infinite when $\vec{x}_i = \vec{x}_j$, which is also the case for the superconductor Green’s function (25.18). Local quantities can be obtained by using $|\vec{x}_i - \vec{x}_j| = a_0$ instead of $\vec{x}_i = \vec{x}_j$. With this condition, the local density of states of the ferromagnet is given by

$$\rho^{(\sigma)} = \frac{1}{2\pi^2} \frac{ma_0^2}{\hbar^2} \exp \left\{ \left(-\frac{a_0}{2l_{\phi}^{(\sigma)}} \right) \right\}.$$

Spin-up electrons have thus a larger density of states than spin-down electrons.

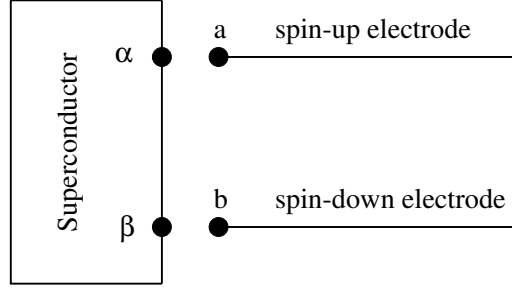


Figure 87: Schematic representation of the model considered in section 25.4. Two ferromagnetic electrodes are in contact with a superconductor. A voltage V_S is applied on the superconductor while the voltages V_a and V_b are applied on the ferromagnetic electrodes.

25.4 Single channel electrodes: (I) Two electrodes with 100% spin polarization

In this section, we consider a model in which a single channel spin-up electrode a and a single channel spin-down electrode b are in contact with a superconductor. We assume in sections 25.4.1 and 25.4.2 that a is a spin-up channel and b is a spin-down channel. The result for parallel spin orientations is given in section 25.4.3.

25.4.1 Derivation of the transport formula

Solution of the Dyson equation

Let us first calculate the Nambu representation of the propagators. The starting point is the chain of Dyson equations given by (25.1):

$$\begin{bmatrix} \hat{G}^{a,a} \\ \hat{G}^{b,a} \end{bmatrix} = \begin{bmatrix} g^{a,a} \\ 0 \end{bmatrix} + \begin{bmatrix} \hat{K}^{a,a} & \hat{K}^{a,b} \\ \hat{K}^{b,a} & \hat{K}^{b,b} \end{bmatrix} \begin{bmatrix} \hat{G}^{a,a} \\ \hat{G}^{b,a} \end{bmatrix}, \quad (25.21)$$

where we used the notation $\hat{K}^{a_i, a_j} = \hat{g}^{a_i, a_i} \hat{t}^{a_i, \alpha_i} \hat{g}^{\alpha_i, \alpha'_i} \hat{t}^{\alpha'_i, a'_i}$. The solution of Eq. 25.21 is

$$\begin{bmatrix} G_{1,1}^{a,a} & G_{1,2}^{a,b} \\ G_{2,1}^{b,a} & G_{2,2}^{b,b} \end{bmatrix} = \frac{1}{\mathcal{D}_{\text{AF}}} \begin{bmatrix} g_{1,1}^{a,a} [1 - |t^{b,\beta}|^2 g_{2,2}^{b,b} g^{\beta,\beta}] & -g_{2,2}^{b,b} t^{a,\alpha} t^{b,\beta} g_{1,1}^{a,a} f^{\alpha,\beta} \\ -g_{1,1}^{a,a} t^{a,\alpha} t^{b,\beta} g_{2,2}^{b,b} f^{\beta,\alpha} & g_{2,2}^{b,b} [1 - |t^{a,\alpha}|^2 g_{1,1}^{a,a} g^{\alpha,\alpha}] \end{bmatrix}, \quad (25.22)$$

where the determinant $\mathcal{D}_{\text{AF}}$ is given by

$$\mathcal{D}_{\text{AF}} = [1 - |t^{a,\alpha}|^2 g_{1,1}^{a,a} g^{\alpha,\alpha}] \left[1 - |t^{b,\beta}|^2 g_{2,2}^{b,b} g^{\beta,\beta} \right] - |t^{a,\alpha}|^2 |t^{b,\beta}|^2 g_{1,1}^{a,a} g_{2,2}^{b,b} f^{\alpha,\beta} f^{\beta,\alpha}, \quad (25.23)$$

and g and f have been defined as the components of the Nambu matrix in (25.3) and (25.18). If not specified, all Green's functions in a given formula stand as well for advanced and retarded functions, and similarly for the determinant $\mathcal{D}_{\text{AF}}$ given by Eq. (25.23) and the determinant $\mathcal{D}_{\text{F}}$ that will be introduced latter. The matrix in Eq. 25.22 contains the non vanishing Nambu components of the renormalized propagator. Because we assume a complete spin polarization, the other Nambu components are vanishing. For instance $G_{1,2}^{a,a} = G_{2,1}^{a,a} = G_{2,2}^{a,a} = 0$.

25.4.2 Exact expression of the current

Using the expression of the Keldysh propagator (see Appendix 25.9), we deduce the final expression of the spin-up current in electrode a :

$$I_{1,1}^{a,\alpha} = -4\pi^2 |t_{a,\alpha}|^2 \int d\omega [n_F(\omega - \mu_a) - n_F(\omega - \mu_S)] \rho_{1,1}^{a,a} \rho_g^{\alpha,\alpha} \quad (25.24)$$

$$\begin{aligned} & \times \frac{1}{\mathcal{D}^A \mathcal{D}^R} \left[1 - |t_{b,\beta}|^2 g_{2,2}^{b,b,A} g^{\beta,\beta,A} \right] \left[1 - |t_{b,\beta}|^2 g_{2,2}^{b,b,R} g^{\beta,\beta,R} \right] \\ & + 2i\pi |t_{a,\alpha}|^2 |t_{b,\beta}|^2 \int d\omega [n_F(\omega - \mu_a) - n_F(\omega - \mu_S)] \rho_{1,1}^{a,a} g_{2,2}^{b,b,A} \end{aligned} \quad (25.25)$$

$$\begin{aligned} & \times \frac{1}{\mathcal{D}^A \mathcal{D}^R} f^{\alpha,\beta,A} f^{\beta,\alpha,A} \left[1 - |t_{b,\beta}|^2 g_{2,2}^{b,b,R} g^{\beta,\beta,R} \right] \\ & - 2i\pi |t_{a,\alpha}|^2 |t_{b,\beta}|^2 \int d\omega [n_F(\omega - \mu_a) - n_F(\omega - \mu_S)] \rho_{1,1}^{a,a} g_{2,2}^{b,b,R} \end{aligned} \quad (25.26)$$

$$\begin{aligned} & \times \frac{1}{\mathcal{D}^A \mathcal{D}^R} f^{\alpha,\beta,R} f^{\beta,\alpha,R} \left[1 - |t_{b,\beta}|^2 g_{2,2}^{b,b,A} g^{\beta,\beta,A} \right] \\ & - 4\pi^2 |t_{a,\alpha}|^2 |t_{b,\beta}|^2 \int d\omega [n_F(\omega - \mu_b) - n_F(\omega - \mu_S)] \frac{1}{\mathcal{D}^A \mathcal{D}^R} \rho_{1,1}^{a,a} \rho_{2,2}^{b,b} f^{\alpha,\beta,R} f^{\beta,\alpha,A} \end{aligned} \quad (25.27)$$

which generalizes the result obtained by Cuevas *et al.* in Ref. [5] in the case of a single conduction channel. From the density of state prefactors, we see that there are two type of contributions to the current: (i) *The quasiparticle current*, which is proportional to the product of the density of state in the superconductor (ρ_g) and one of the ferromagnetic electrodes (for instance ρ_a); (ii) *The crossed Andreev current* which is proportional to the product $\rho_a \rho_b$ of the density of state in the two ferromagnetic electrodes. The term (25.24) contributes only to quasiparticle current. The term (25.27) contributes only to Andreev reflection. The mixed terms (25.25) – (25.26) contribute both to the quasiparticle and Andreev current.

25.4.3 Two-terminal conductance matrix

To understand the meaning of the transport formula (25.24), it is useful to describe transport across the multiterminal structure in terms of a differential conductance matrix:

$$\hat{\mathcal{G}} = \begin{bmatrix} \mathcal{G}_{a,a} & \mathcal{G}_{a,b} \\ \mathcal{G}_{b,a} & \mathcal{G}_{b,b} \end{bmatrix}, \quad (25.28)$$

where the matrix elements are given by

$$\mathcal{G}_{a_i, a_j}(V_a, V_b) = \frac{\partial I_{a_i}}{\partial V_{a_j}}(V_a, V_b). \quad (25.29)$$

The conductance matrix (25.28) encodes all information about transport in the two-terminal structure. The off-diagonal matrix elements should satisfy a symmetry relation: $\mathcal{G}_{a,b}(V_a, V_b) = \mathcal{G}_{b,a}(V_b, V_a)$. If the electrodes have an antiparallel spin orientation, subgap current is transported by Cooper pairs if $\omega < \Delta$, in which case we have $I_a = I_b$. This implies an additional symmetry relation: $\mathcal{G}_{a,a}(V_a, V_b) = \mathcal{G}_{b,a}(V_a, V_b)$, and $\mathcal{G}_{b,b}(V_a, V_b) = \mathcal{G}_{a,b}(V_a, V_b)$. If the electrodes have a parallel spin orientation, subgap current is due to elastic cotunneling, in which case $I_a = -I_b$. The additional symmetry relation reads $\mathcal{G}_{a,a}(V_a, V_b) = -\mathcal{G}_{b,a}(V_a, V_b)$, and $\mathcal{G}_{b,b}(V_a, V_b) = -\mathcal{G}_{a,b}(V_a, V_b)$.

Sub-gap conductance matrix: effective Green's functions

In this section as well as in section 25.4.3, we assume that $\cos \varphi = 0$ so that we can use effective Green's functions to evaluate the transport formula and work out the basic physics on simple grounds. The validity of this assumption will be discussed in section 25.4.4.

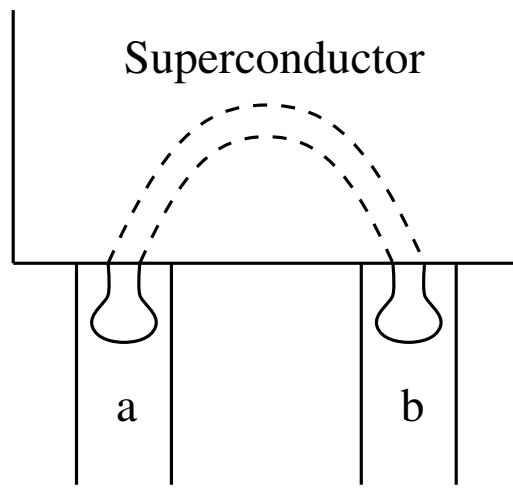


Figure 88: The diagram associated to Andreev reflection in the two-channel model.

Antiparallel magnetizations:

The sub-gap current of the two-channel model with antiparallel magnetizations originates from the non local Andreev reflections which are shown schematically on Fig. 88. This can be seen by inserting the effective Green's functions into the transport formula:

$$\mathcal{G}_{a,a} = \mathcal{G}_{b,a} = -\frac{4\Gamma_a\Gamma_b}{|\mathcal{D}_{\text{AF}}(\omega = V_a)|^2} [f^{\alpha,\beta}(\omega = V_a)]^2 \quad (25.30)$$

$$\mathcal{G}_{a,b} = \mathcal{G}_{b,b} = -\frac{4\Gamma_a\Gamma_b}{|\mathcal{D}_{\text{AF}}(\omega = V_b)|^2} [f^{\alpha,\beta}(\omega = V_b)]^2, \quad (25.31)$$

where $\Gamma_a = \pi|t_{a,\alpha}|^2\rho_a$ is the spectral line-width associated to electrode a , and a similar expression holds for Γ_b . We used the fact that $g^{\alpha,\beta}$ and $f^{\alpha,\beta}$ are real numbers below the superconducting gap. The expression of $\mathcal{D}_{\text{AF}}$ is the following:

$$|\mathcal{D}_{\text{AF}}(\omega)|^2 = \{1 - \Gamma^a\Gamma^b [g^{\alpha,\alpha}g^{\beta,\beta} - (f^{\alpha,\beta})^2]\}^2 + (\Gamma^a g^{\alpha,\alpha} + \Gamma^b g^{\beta,\beta})^2. \quad (25.32)$$

Parallel magnetizations:

The same calculation can be done if the electrodes have a parallel spin orientation. We find

$$\mathcal{G}_{a,a} = -\mathcal{G}_{b,a} = -\frac{4\Gamma_a\Gamma_b}{|\mathcal{D}_F(\omega = V_a)|^2} [g^{\alpha,\beta}(\omega = V_a)]^2 \quad (25.33)$$

$$-\mathcal{G}_{a,b} = \mathcal{G}_{b,b} = -\frac{4\Gamma_a\Gamma_b}{|\mathcal{D}_F(\omega = V_b)|^2} [g^{\alpha,\beta}(\omega = V_b)]^2, \quad (25.34)$$

with

$$|\mathcal{D}_F(\omega)|^2 = \{1 - \Gamma^a\Gamma^b [g^{\alpha,\alpha}g^{\beta,\beta} - (g^{\alpha,\beta})^2]\}^2 + (\Gamma^a g^{\alpha,\alpha} + \Gamma^b g^{\beta,\beta})^2. \quad (25.35)$$

There are two differences between the situations with antiparallel and parallel spin orientations. First, the Andreev reflection transport with antiparallel spin orientations is controlled by the anomalous propagator $f^{\alpha,\beta}$ while the elastic cotunneling transport with parallel spin orientations is controlled by the ordinary propagator $g^{\alpha,\beta}$. The second difference is in the sign of the off-diagonal conductance matrix elements. The four matrix elements have the same sign in the case of Andreev reflection because transport is mediated by Cooper pairs. The off-diagonal matrix elements have a sign opposite to the diagonal matrix elements in the case of elastic cotunneling because transport is due to single electron tunneling between the two electrodes.

Conductance matrix above the superconducting gap: effective Green's functions

Let us now assume that the voltage V_a is above the superconducting gap and that the electrodes have an antiparallel spin orientation. With the notation $g^{\alpha,\beta,A,R} = \pm i|g^{\alpha,\beta}|$, and $f^{\alpha,\beta,A,R} = \pm i|f^{\alpha,\beta}|$, the extra diagonal terms of the conductance matrix take the form

$$\mathcal{G}_{b,a} = -\frac{4\Gamma_a\Gamma_b}{\mathcal{D}_{AF}^2} |f^{\alpha,\beta}|^2,$$

which should be evaluated at the energy $\omega = V_a$. The diagonal conductance matrix element is the sum of a crossed contribution and a quasiparticle contribution: $\mathcal{G}_{a,a} = -\mathcal{G}_{b,a} - \mathcal{G}_{a,a}^{qp}$. The quasiparticle contribution is the sum of a direct and a crossed term:

$$\mathcal{G}_{a,a}^{qp} = \frac{4\pi\Gamma_a\rho_g}{\mathcal{D}_{AF}^2} \left[(1 + \Gamma_b|g^{\beta,\beta}|)^2 - \Gamma_b^2|f^{\alpha,\beta}|^2 \right],$$

where the denominator $\mathcal{D}_{AF}$ is a real number:

$$\mathcal{D}_{AF} = \mathcal{D}_{AF}^A = \mathcal{D}_{AF}^R = 1 + \Gamma^a\Gamma^b [|g^{\alpha,\alpha}g^{\beta,\beta}| - |f^{\alpha,\beta}|^2] + \Gamma^a|g^{\alpha,\alpha}| + \Gamma^b|g^{\beta,\beta}|.$$

In the limit $|\vec{x}_a - \vec{x}_b| \rightarrow +\infty$ in which the separation between the contacts becomes very large, the extra diagonal conductance matrix elements are vanishingly small. The quasiparticle term reduces to the conductance of a single channel metal – metal contact:

$$\mathcal{G}_{a,a}^{qp} = \frac{4\pi\Gamma_a\rho_g}{(1 + \Gamma_a|g^{\alpha,\alpha}|)^2}.$$

25.4.4 Phase resolved *versus* averaged conductance

Given the form (25.18) of the superconductor Green's function, we see that the conductance depends explicitly on the electronic phase difference $\varphi = k_F|\vec{x}_a - \vec{x}_b|$. This leads us to calculate the conductance in two different ways:

- (i) The phase-resolved conductance $\mathcal{G}(\varphi)$. We will focus more especially on the case $\varphi = \pi/2$. For this special value of the phase difference, the Green's function of the superconductor coincides with the effective Green's function in the limit $|\vec{x}_a - \vec{x}_b| \rightarrow a_0$.
- (ii) The averaged conductance

$$\langle\langle\mathcal{G}(\varphi)\rangle\rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{G}(\varphi) d\varphi. \quad (25.36)$$

This phase averaging is used to mimic the physics of extended contacts that will be considered later in section 25.7.

To determine the role played by phase averaging, we compare the phase-dependent and average conductances (see the end of section 25.3). It is visible on Fig. 89 that the effective Green's function conductance (i.e. with $\varphi = \pi/2$) follows closely the average conductance. This shows that the effective Green's function conductance contains already the relevant physics, as far as the two-channel problem is concerned.

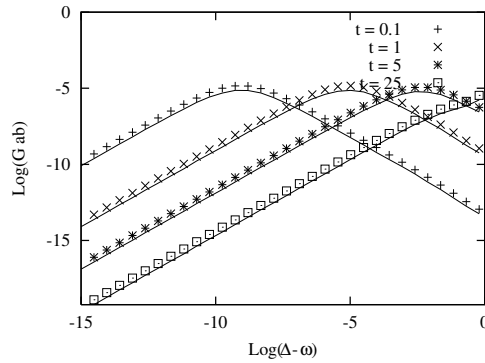


Figure 89: Variation of the logarithm of the crossed conductance $\log \mathcal{G}_{a,b}$ versus $\log(\Delta - \omega)$ ($\omega < \Delta$). $\mathcal{G}_{a,b}$ is in units of e^2/h . The points correspond to the phase-resolved conductance with $\varphi = \pi/2$, namely, to the generalized effective Green's functions. The solid lines correspond to the average conductance (25.36). We used the parameters $m = 0.01$, $k_F = 1$, $\epsilon_F = 50$, $a_0 = 1$, $\Delta = 1$. The distance between the contacts is $D = 100$ and the superconductor coherence length is $\xi_0 = \epsilon_F/(k_F \Delta) = 50$.

It is also visible on Fig. 89 that there is a cross-over energy ω^* (thus a cross-over voltage $V^* = \omega^*/e$) at which the crossed conductance reaches a maximum. If $\omega < \omega^*$ the crossed conductance behaves like $\mathcal{G}_{a,b} \sim 1/(\Delta - \omega)$. If $\omega^* < \omega < \Delta$, the crossed conductance behaves like $\mathcal{G}_{a,b} \sim \Delta - \omega$. Only when $\omega < \omega^*$ does our approach coincide with the lowest order tunnel perturbation theory. The analysis based on generalized effective Green's functions is simplified if one assumes that $D \gg a_0$, but D can be small or large compared to the superconducting coherence length $\xi(\omega)$ (see Eq. 25.19). The behavior in the energy range $\omega^* < \omega < \Delta$ is non perturbative, and can be understood by retaining in $\mathcal{D}_{AF}$ only the leading divergence, which is generated by the quartic terms (see Eq. 25.32):

$$|\mathcal{D}_{AF}|^2 \simeq \Gamma^4 \left[g^{\alpha,\alpha} g^{\beta,\beta} - (f^{\alpha,\beta})^2 \right]^2,$$

from what we deduce the expression of the crossed conductance

$$\mathcal{G}_{a,b} \simeq \frac{4}{\Gamma^2} \left(\frac{a_0}{D} \right)^2 \left(\frac{2\pi\hbar^2}{ma_0^2} \right)^2 \left(\frac{\Delta^2 - \omega^2}{\Delta^2} \right) \exp \left\{ \left(-\frac{D}{\xi(\omega)} \right) \right\}, \quad (25.37)$$

valid in the energy range $\omega > \omega^*$. The expression of the cross-over energy ω^* is obtained by equating the quadratic and quartic terms in $\mathcal{D}_{\text{AF}}$. This leads to

$$\omega^* = \Delta \sqrt{1 - \left(\Gamma \frac{ma_0^2}{2\pi\hbar^2} \right)^2}. \quad (25.38)$$

To obtain the conductance in the energy range $\omega < \omega^*$, we expand $\mathcal{D}_{\text{AF}}$ up to order Γ^2 :

$$|\mathcal{D}_{\text{AF}}|^2 \simeq 1 + 2\Gamma^2 (g^{\text{loc}})^2 + 2\Gamma^2 (f^{\alpha,\beta})^2,$$

where $g^{\text{loc}} = g^{\alpha,\alpha} = g^{\beta,\beta}$ denotes the local propagator in the superconductor. One can show easily that $f^{\alpha,\beta} \ll g^{\text{loc}}$ because $D \gg a_0$, from what we deduce

$$|\mathcal{D}_{\text{AF}}|^2 \simeq 1 + 2\Gamma^2 \left(\frac{ma_0^2}{2\pi\hbar^2} \right)^2 \frac{\omega^2}{\Delta^2 - \omega^2}. \quad (25.39)$$

$\mathcal{D}_{\text{AF}}$ is close to unity only when $\omega < \omega_0$, with

$$\omega_0 = \frac{\Delta}{\sqrt{1 + 2\Gamma^2 \left(\frac{ma_0^2}{2\pi\hbar^2} \right)^2}}. \quad (25.40)$$

Comparing (25.38) and (25.40), we see that ω^* is larger than ω_0 but ω_0 and ω^* have the same order of magnitude if Γ is small (which is the case on Fig. 89). We deduce that $\mathcal{D}_{\text{AF}} = 1$ in the relevant energy range $\omega < \omega_0 < \omega^*$. From what we obtain the conductance in the energy range $\omega < \omega_0$:

$$\mathcal{G}_{a,b} = 4\Gamma^2 \left(\frac{a_0}{D} \right)^2 \left(\frac{ma_0^2}{2\pi\hbar^2} \right)^2 \frac{\Delta^2}{\Delta^2 - \omega^2} \exp \left\{ \left(-\frac{D}{\xi(\omega)} \right) \right\}, \quad (25.41)$$

identical to the one obtained in lowest order perturbation theory. Evaluating (25.37) and (25.41) at $\omega = \omega^*$ or $\omega = \omega_0$ leads to the maximal value of the conductance:

$$\mathcal{G}_{a,b}^{\text{max}} \simeq \frac{e^2}{h} \left(\frac{a_0}{D} \right)^2 \exp \left\{ \left(-\frac{D}{\xi(\omega^*)} \right) \right\}, \quad (25.42)$$

where the numerical prefactor of order unity cannot be obtained from this simple estimate.

It is well known from the BTK scattering approach [58] that the conductance of a normal metal – superconductor contact is equal to twice the quantum of conductance e^2/h per spin channel if $\omega = \Delta$, *regardless the value of the interface scattering*. The same behavior occurs in the Keldysh formalism treatment by Cuevas *et al.* [5]. This type of resonance can be properly described only with a non perturbative approach. Eq. 25.42 constitutes a generalization of the BTK behavior to the case of spatially separated contacts having a phase difference $\varphi = \pi/2$. Given that the average and phase-resolved conductances follow closely each other (see Fig. 89), it is expected that (25.42) is also valid for the average conductance, but with an extra reduction factor.

25.5 Single channel electrodes: (II) General solution

Now we consider a model in which N spin-up electrodes and M spin-down electrodes are in contact with a superconductor (see Fig. 90). All the necessary details can be found in

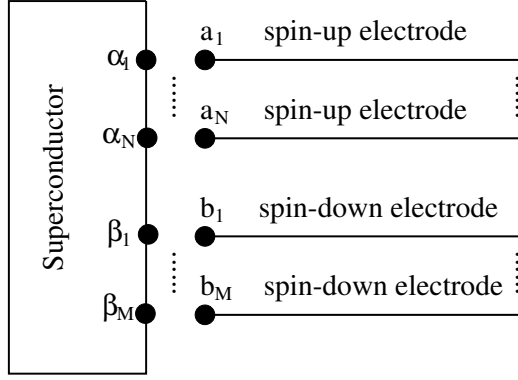


Figure 90: Schematic representation of the model with N ferromagnetic spin-up electrodes and M ferromagnetic spin-down electrodes.

Appendix 25.10. The final form of the current is

$$I^{\alpha_1, a_1} = -4\pi^2 |t^{a_1, \alpha_1}|^2 \int d\omega [n_F(\omega - \mu_{a_1}) - n_F(\omega - \mu_S)] \rho^{\alpha_1, \alpha_1} \rho_g^{\alpha_1, \alpha_1} \frac{1}{\mathcal{D}^A \mathcal{D}^R} \tilde{\mathcal{M}}_{a_1, a_1}^A \tilde{\mathcal{M}}_{a_1, a_1}^R \quad (25.43)$$

$$- 2i\pi \sum_{k=2}^N (-)^{k+1} t^{a_1, \alpha_1} t^{a_k, \alpha_k} \int d\omega [n_F(\omega - \mu_{a_1}) - n_F(\omega - \mu_S)] \times \quad (25.44)$$

$$\frac{1}{\mathcal{D}^A \mathcal{D}^R} \left(g^{\alpha_1, \alpha_k, R} \tilde{\mathcal{M}}_{a_1, a_k}^R \tilde{\mathcal{M}}_{a_1, a_1}^A - g^{\alpha_1, \alpha_k, A} \tilde{\mathcal{M}}_{a_1, a_k}^A \tilde{\mathcal{M}}_{a_1, a_1}^R \right)$$

$$- 2i\pi \sum_{k=2}^N (-)^{k+N} t^{a_1, \alpha_1} t^{b_k, \beta_k} \int_{\mu_{a_1}}^{\mu_{\alpha_k}} d\omega [n_F(\omega - \mu_{a_1}) - n_F(\omega - \mu_S)] \times \quad (25.45)$$

$$\frac{1}{\mathcal{D}^A \mathcal{D}^R} \left(f^{\alpha_1, \beta_k, R} \tilde{\mathcal{M}}_{a_1, b_k}^R \tilde{\mathcal{M}}_{a_1, a_1}^A - f^{\alpha_1, \beta_k, A} \tilde{\mathcal{M}}_{a_1, b_k}^A \tilde{\mathcal{M}}_{a_1, a_1}^R \right)$$

$$+ 4\pi^2 \sum_{k=2}^N |t^{a_1, \alpha_1}|^2 |t^{a_k, \alpha_k}|^2 \int d\omega [n_F(\omega - \mu_{a_k}) - n_F(\omega - \mu_S)] \times \quad (25.46)$$

$$\rho_{1,1}^{a_1, a_1} \rho_{1,1}^{a_k, a_k} \tilde{g}^{\alpha_1, \alpha_k, R} \tilde{g}^{\alpha_k, \alpha_1, A}$$

$$- 4\pi^2 \sum_{k=1}^M |t^{a_1, \alpha_1}|^2 |t^{b_k, \beta_k}|^2 \int d\omega [n_F(\omega - \mu_{b_k}) - n_F(\omega - \mu_S)] \times \quad (25.47)$$

$$\rho_{1,1}^{a_1, a_1} \rho_{2,2}^{b_k, b_k} \tilde{f}^{\alpha_1, \beta_k, R} \tilde{f}^{\beta_k, \alpha_1, A},$$

where we used the notation

$$\tilde{g}^{\alpha_i, \alpha_j} = \frac{\tilde{\mathcal{M}}_{a_j, a_i}}{t^{a_i, \alpha_i} t^{a_j, \alpha_j} g^{a_i, a_i} \mathcal{D}} \quad (25.48)$$

$$\tilde{f}^{\alpha_i, \beta_j} = \frac{\tilde{\mathcal{M}}_{b_j, a_i}}{t^{a_i, \alpha_i} t^{b_j, \beta_j} g^{a_i, a_i} \mathcal{D}} \quad (25.49)$$

for the renormalized propagators. There are three types of processes involved in the transport formula: (i) *The quasiparticle term* (25.43) which is proportional to $\rho^{\alpha_1, \alpha_1} \rho_g$; (ii) *The elastic cotunneling term* (25.46) in which spin-up electrons from electrode a_k are transferred into electrode 1. The elastic cotunneling terms are proportional to $\rho_{1,1}^{a_1, a_1} \rho_{1,1}^{a_k, a_k}$ (iii) *The Andreev reflection term* (25.47) which are proportional to $\rho_{1,1}^{a_1, a_1} \rho_{2,2}^{b_k, b_k}$. The mixed terms (25.44) and (25.45) contribute the three types of processes.

25.6 Single channel electrodes: (III) Three electrodes with 100% spin polarization

Let us now consider a three-terminal problem. We consider that each of the three electrodes has 100% spin polarization (see Fig. 91). The aim is to have Andreev and cotunneling processes occurring in the same multiterminal device, which allows a direct comparison of these two basic processes. The transport formula can be deduced easily from the general solution obtained in section 25.5.

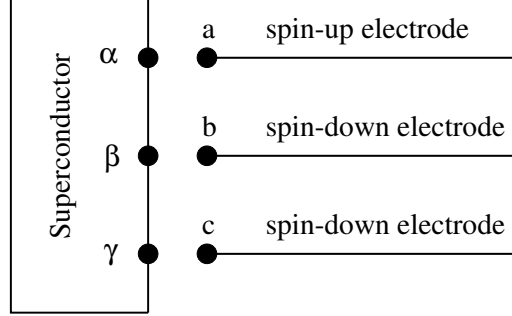


Figure 91: Schematic representation of the three channel model considered in section 25.6.

25.6.1 Three-terminal conductance matrix

The three-terminal conductance matrix generalizing (25.28) takes the form

$$\hat{\mathcal{G}} = \begin{bmatrix} \mathcal{G}_{a,a} & \mathcal{G}_{a,b} & \mathcal{G}_{a,c} \\ \mathcal{G}_{b,a} & \mathcal{G}_{b,b} & \mathcal{G}_{b,c} \\ \mathcal{G}_{c,a} & \mathcal{G}_{c,b} & \mathcal{G}_{c,c} \end{bmatrix}. \quad (25.50)$$

Let us assume that electrode a has a spin-up magnetization, and electrodes b and c have a spin-down magnetization (see Fig. 91). We use the same procedure as in section 25.4.3 to obtain the conductance matrix elements. Namely, we assume that $\cos \varphi = 0$ and replace the Green's functions by effective Green's functions.

Conductance matrix below the superconducting gap: effective Green's functions

Let us first give the form of the off diagonal matrix elements:

$$\mathcal{G}_{a,b} = \mathcal{G}_{b,a} = -\frac{4\Gamma_a\Gamma_b}{\mathcal{D}^A\mathcal{D}^R} \left[(f^{\alpha,\beta})^2 + \Gamma_c^2 (f^{\alpha,\gamma}g^{\beta,\gamma} - f^{\alpha,\beta}g^{\gamma,\gamma})^2 \right] \quad (25.51)$$

$$\mathcal{G}_{a,c} = \mathcal{G}_{c,a} = -\frac{4\Gamma_a\Gamma_c}{\mathcal{D}^A\mathcal{D}^R} \left[(f^{\alpha,\gamma})^2 + \Gamma_b^2 (f^{\alpha,\beta}g^{\beta,\gamma} - f^{\alpha,\gamma}g^{\beta,\beta})^2 \right] \quad (25.52)$$

$$\mathcal{G}_{b,c} = \mathcal{G}_{c,b} = \frac{4\Gamma_b\Gamma_c}{\mathcal{D}^A\mathcal{D}^R} \left[(g^{\beta,\gamma})^2 + \Gamma_a^2 (f^{\alpha,\beta}f^{\alpha,\gamma} - g^{\alpha,\alpha}g^{\beta,\gamma})^2 \right]. \quad (25.53)$$

From the signs of these matrix elements, and from the type of propagator involved, we see that $\mathcal{G}_{a,b}$ and $\mathcal{G}_{a,c}$ correspond to Andreev reflection while $\mathcal{G}_{b,c}$ corresponds to elastic cotunneling.

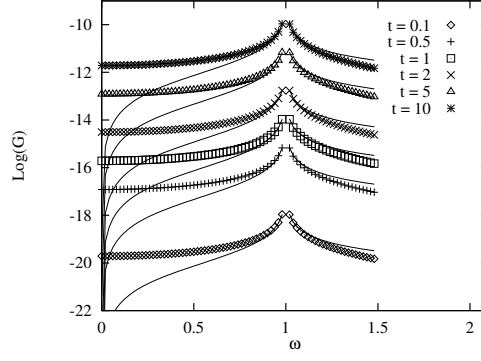


Figure 92: Variation of the logarithm of the Andreev reflection conductance $\mathcal{G}_{a,c}$ and elastic cotunneling conductance $\mathcal{G}_{b,c}$ versus reduced energy ω/Δ . The points correspond to Andreev reflection and the solid lines correspond to elastic cotunneling. We have assumed that site α coincides with site β (see Fig. 91). We used effective Green's functions with $\varphi = \pi/2$. The distance between the contacts is $D = 100$. On purpose, we did not show the behavior for $\omega \simeq \Delta$ (see Fig. 93). The parameters are identical as on Fig. 89.

Conductance matrix above the superconducting gap: effective Green's functions

Using the notation $f^{A,R} = \pm i|f|$, and $g^{A,R} = \pm i|g|$, we obtain the conductance matrix elements above the superconducting gap:

$$\mathcal{G}_{a,b} = \mathcal{G}_{b,a} = -\frac{4\Gamma_a\Gamma_b}{\mathcal{D}^A\mathcal{D}^R} [|f^{\alpha,\beta}| + \Gamma_c (|f^{\alpha,\beta}g^{\gamma,\gamma}| - |f^{\alpha,\gamma}g^{\beta,\gamma}|)]^2 \quad (25.54)$$

$$\mathcal{G}_{a,c} = \mathcal{G}_{c,a} = -\frac{4\Gamma_a\Gamma_c}{\mathcal{D}^A\mathcal{D}^R} [|f^{\alpha,\gamma}| + \Gamma_b (|f^{\alpha,\gamma}g^{\beta,\beta}| - |f^{\alpha,\beta}g^{\beta,\gamma}|)]^2 \quad (25.55)$$

$$\mathcal{G}_{b,c} = \mathcal{G}_{c,b} = -\frac{4\Gamma_b\Gamma_c}{\mathcal{D}^A\mathcal{D}^R} [|g^{\beta,\gamma}| + \Gamma_a (|g^{\beta,\gamma}g^{\alpha,\alpha}| - |f^{\alpha,\beta}f^{\alpha,\gamma}|)]^2. \quad (25.56)$$

The diagonal coefficient $\mathcal{G}_{a,a}$ takes the form

$$\mathcal{G}_{a,a} = -\mathcal{G}_{a,b} - \mathcal{G}_{a,c} + \frac{4\Gamma_a}{\mathcal{D}^A\mathcal{D}^R} \left\{ -\pi\rho_g \tilde{\mathcal{M}}^{a,a,A} \tilde{\mathcal{M}}^{a,a,R} \right. \quad (25.57)$$

$$+ \Gamma_b^2 \{ |f^{\alpha,\beta}|^2 [|g^{\beta,\beta}| + \Gamma_c (2 + \Gamma_c |g^{\gamma,\gamma}|) (|g^{\beta,\beta}g^{\gamma,\gamma}| - |G^{\beta,\gamma}|^2)] \} \quad (25.58)$$

$$+ \Gamma_c^2 \{ |f^{\alpha,\gamma}|^2 [|g^{\gamma,\gamma}| + \Gamma_b (2 + \Gamma_b |g^{\beta,\beta}|) (|g^{\beta,\beta}g^{\gamma,\gamma}| - |g^{\beta,\gamma}|^2)] \} \quad (25.59)$$

$$+ 2\Gamma_b\Gamma_c |f^{\alpha,\beta}f^{\alpha,\gamma}g^{\beta,\gamma}| [1 - \Gamma_b\Gamma_c (|g^{\beta,\beta}g^{\gamma,\gamma}| - |g^{\beta,\gamma}|^2)] \}, \quad (25.60)$$

and similar expressions can be obtained for $\mathcal{G}_{b,b}$ and $\mathcal{G}_{c,c}$.

25.6.2 Phase resolved versus average conductance

Let us now consider the special situation in which a ferromagnetic electrode with 100% spin polarization is at a distance D away from a normal metal electrode having a zero spin polarization. Namely, we assume that sites α and β coincide (see Fig. 91). This provides the simplest model containing both Andreev reflection and elastic cotunneling.

Following section 25.4.4, we evaluate the phase-resolved conductance with $\varphi_{a,b} = \varphi_{a,c} = \varphi_{b,c} = \pi/2$. With this particular value of the phases, the phase-resolved conductance coincides with the effective Green's function conductance. We have shown on Fig. 92 the energy dependence of the Andreev reflection and elastic tunneling conductances, evaluated with $\varphi = \pi/2$. It is visible that the Andreev reflection conductance is larger than the elastic cotunneling conductance by a factor Δ/ω . The Andreev conductance coincides with the elastic cotunneling

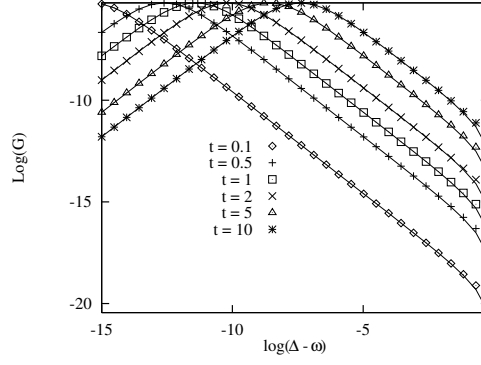


Figure 93: The same as Fig. 92 but for energies $\omega \simeq \Delta$.

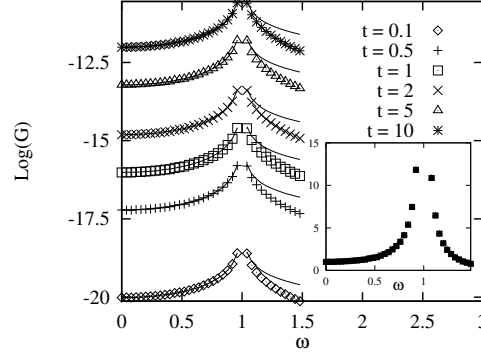


Figure 94: Variation of the logarithm of the Andreev reflection (points) and elastic cotunneling (solid lines) conductances *versus* energy ω . We averaged the conductance over all possible phase configurations (see Eq. 25.61). The insert shows the energy dependence of the rescaled conductance $\mathcal{G}(\omega)/\mathcal{G}(\omega = 0)$. The parameters are identical as on Fig. 89.

conductance only when ω is close to Δ (see Fig. 93). This behavior can be understood from the effective Green's function conductance (Eqs. 25.51 – 25.53).

To obtain the average conductance, we assume that the phase variables are independent random variables, and average the conductance over all possible values of the phases:

$$\langle\langle \mathcal{G} \rangle\rangle = \int_0^{2\pi} \frac{d\varphi_{a,b}}{2\pi} \int_0^{2\pi} \frac{d\varphi_{a,c}}{2\pi} \int_0^{2\pi} \frac{d\varphi_{b,c}}{2\pi} \mathcal{G}(\varphi_{a,b}, \varphi_{a,c}, \varphi_{b,c}). \quad (25.61)$$

It is visible on Fig. 94 that after phase averaging, the Andreev reflection conductance coincides exactly with the elastic cotunneling conductance below the superconducting gap. For energies close to Δ , we find again the existence of a maximum in the conductance at an energy ω^* . The predictions of perturbation theory are valid in the energy range $\omega < \omega^*$. In this energy range, all conductance spectra on Fig. 94 can be deduced from each other by a simple rescaling (see the insert of Fig. 94).

25.7 Multichannel electrodes

25.7.1 Transport formula

We want to determine whether extended contacts have a physics identical to the single channel contacts considered in sections 25.4, 25.5 and 25.6. We are thus lead to investigate the following situations:

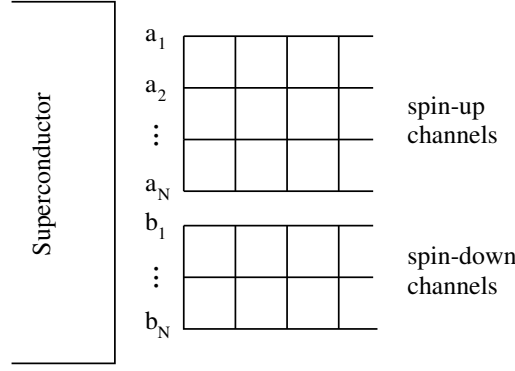


Figure 95: Schematic representation of the tight binding model. The contacts between the spin-up ferromagnet and the superconductor are noted $a_1, \dots, a_N$. The contacts between the spin-down ferromagnet and the superconductor are noted $b_1, \dots, b_N$.

- (i) *The phase averaged conductance of extended contacts.* There are $N(N - 1)/2$ phases $\varphi_{i,j}$ associated with a contact having N channels. The phase-averaged conductance is obtained by averaging the conductance over all possible values of these phases:

$$\langle\langle \mathcal{G} \rangle\rangle = \prod_{\langle i,j \rangle} \frac{d\varphi_{i,j}}{2\pi} \mathcal{G}(\{\varphi_{i,j}\}). \quad (25.62)$$

- (ii) *The phase-resolved conductance of extended contacts.* This is the conductance of an extended contact where the phases $\varphi_{i,j}$ are deterministic and take the particular value $\varphi_{i,j} = k_F |\mathbf{x}_i - \mathbf{x}_j|$ as given by Eq. 25.18.

Let us consider a model in which a multichannel fully polarized spin-up electrode is in contact with a superconductor. At a distance D , there is another fully polarized spin-down electrode. The tight binding model is represented on Fig. 95. There is one block “a” made of N fully polarized spin-up channels and another block “b” made of M fully polarized spin-down channels. The only difference with section 25.5 is the existence of a propagator g_{a_i, a_j} , g_{b_i, b_j} with $i \neq j$. The form of the Dyson matrix is still given by (25.90). Compared to (25.91), there is an additional summation in the coefficients of the Dyson matrix:

$$X^{a_i, b_j} = \sum_k g_{1,1}^{a_i, a_k} t^{a_k, \alpha_k} f^{\alpha_k, \beta_j} t^{\beta_j, b_j} \quad (25.63)$$

$$X^{b_i, a_j} = \sum_k g_{2,2}^{b_i, b_k} t^{b_k, \beta_k} f^{\beta_k, \alpha_j} t^{\alpha_j, a_j} \quad (25.64)$$

$$X^{a_i, a_j} = \sum_k g_{1,1}^{a_i, a_k} t^{a_k, \alpha_k} g^{\alpha_k, \alpha_j} t^{\alpha_j, a_j} \quad (25.65)$$

$$X^{b_i, b_j} = \sum_k g_{2,2}^{b_i, b_k} t^{b_k, \beta_k} g^{\beta_k, \beta_j} t^{\beta_j, b_j}. \quad (25.66)$$

The derivation of the transport formula is similar to section 25.5. For instance, the subgap Andreev conductance is the sum of all possible Cooper pair transmissions:

$$\mathcal{G}_{a,b}^A(\omega) = 4\pi^2 \sum_{p \in F_a} \sum_{q \in F_b} |t^{a_p, \alpha_p}|^2 |t^{b_q, \beta_q}|^2 \rho_{1,1}^{a_p, a_p}(\omega) \rho_{2,2}^{b_q, b_q}(\omega) \tilde{f}^{\alpha_p, \beta_q, R}(\omega) \tilde{f}^{\beta_q, \alpha_p, A}(\omega), \quad (25.67)$$

where “ $p \in F_a$ ” (“ $q \in F_b$ ”) means that the p runs over all possible channels in electrodes a and

b. We used the notation

$$\tilde{f}^{\alpha_p, \beta_q} = \frac{\tilde{\mathcal{M}}^{b_q, a_p}}{t^{a_p, \alpha_p} t^{b_q, \beta_q} g_{1,1}^{a_p, a_p} \mathcal{D}} \quad (25.68)$$

$$\tilde{f}^{\beta_q, \alpha_p} = \frac{\tilde{\mathcal{M}}^{a_p, b_q}}{t^{a_p, \alpha_p} t^{b_q, \beta_q} g_{2,2}^{b_q, b_q} \mathcal{D}} \quad (25.69)$$

for the renormalized propagators. The same formalism can be used to handle more complicated situations involving an arbitrary number of electrodes having arbitrary spin polarizations.

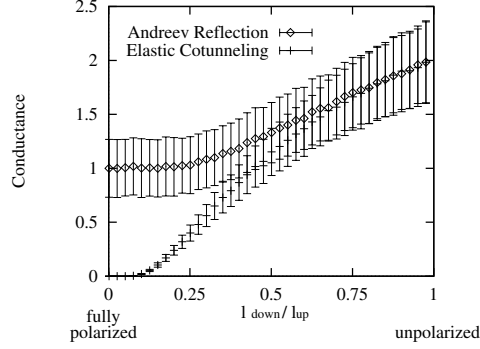


Figure 96: Variation of the $\omega = 0$ Andreev and elastic cotunneling average conductances as a function of $l^{(\downarrow)}/l^{(\uparrow)}$. The section of the two electrodes is circular, with a radius $R = 2.1$. There are 26 channels in each electrode. The average conductances are normalized with respect to the average Andreev conductance with $\omega = 0$ and $l^{(\downarrow)} = 0$. The conductance is distributed as a function of the phases. The errorbars indicate the root mean square of the conductance distribution. The parameters are identical to Fig. 89.

25.7.2 Andreev reflection versus cotunneling

Let us consider a system in which two multichannel electrodes are in contact with a superconductor: (i) a ferromagnetic electrode; and (ii) a normal metal electrode with no spin polarization.

Following the discussion in section 25.3.2, we use the ratio $l^{(\downarrow)}/l^{(\uparrow)}$ to parametrize spin polarization. There is no spin polarization if $l^{(\downarrow)}/l^{(\uparrow)} = 1$ and there is a strong spin polarization if $l^{(\downarrow)}/l^{(\uparrow)} \ll 1$. We have shown on Fig. 96 the variation of the $\omega = 0$ Andreev and cotunneling phase averaged conductances as a function of spin polarization in the ferromagnetic electrode. Elastic cotunneling at $\omega = 0$ is vanishingly small if the ferromagnet is strongly polarized. The average Andreev conductance is equal to the average elastic cotunneling conductance in the absence of spin polarization (*i. e.* with $l^{(\downarrow)} = l^{(\uparrow)}$).

Therefore, the phase averaged conductance matrix of the multichannel model behaves like the phase averaged conductance matrix of the single channel model.

25.7.3 Extended contacts without phase averaging

Now we consider extended contacts in which the phases take deterministic values. The phases are given by Eq. 25.18: $\varphi_{i,j} = k_F |\mathbf{x}_i - \mathbf{x}_j|$. We represented on Fig. 97 the dependence of the Andreev and cotunneling currents as a function of $l^{(\downarrow)}/l^{(\uparrow)}$ for $\omega = 0$. If the number of channels is sufficiently large, we see that the Andreev and elastic cotunneling conductances are almost identical for the non magnetic metal ($l^{(\downarrow)}/l^{(\uparrow)} = 1$). The behavior of extended

contacts with deterministic phases (see Fig. 97) is therefore identical to the behavior of contacts with random phases (see Fig. 96). This result, already established in the perturbative regime ($\omega \ll \Delta$) is found to be valid at any frequency and barrier transparency. We have verified that this behavior is also valid in the non perturbative regime ($\omega^* < \omega < \Delta$). Notice that the same result is found if only one contact is extended, the other one having a few channels.

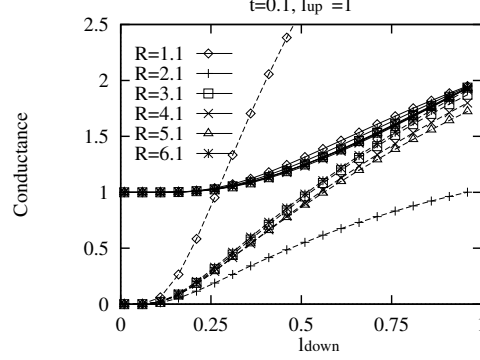


Figure 97: Variation of the $\omega = 0$ Andreev and elastic cotunneling conductances as a function of $l^{(\downarrow)}/l^{(\uparrow)}$ for an extended contact with deterministic phases. $l^{(\downarrow)}/l^{(\uparrow)} = 1$ corresponds to a non magnetic metal. The section of the two electrodes is circular, with a radius $R = 1.1$ ($\diamond$, 10 channels), $R = 2.1$ (+, 26 channels), $R = 3.1$ ($\square$, 58 channels), $R = 4.1$ ($\times$, 98 channels), $R = 5.1$ ($\triangle$, 178 channels), $R = 6.1$ (*, 242 channels). The parameters are identical to Fig. 89. The normalizations are identical to Fig. 96.

25.8 Conclusions

We have provided in this article a detailed theoretical description of ballistic transport in multiterminal hybrid structures involving a superconductor and several spin-polarized electrodes. We have performed a non perturbative calculation of the conductance matrix, using Keldysh technique, and focusing on the spin dependence, the geometry dependence and the energy behavior of the two basic processes : Crossed (intercontact) Andreev and Elastic Cotunneling. This generalizes previous perturbative calculations, valid only for low contact transparencies and at small voltages. It also generalizes non perturbative calculations made for a single contact and using effective Green's functions, which turns out to correspond to a certain choice of the phase ($\phi = \pi/2$) in the electronic propagators.

A first issue concerns the subgap voltage dependence of the conductances, compared to the usual Andreev conductance at a single contact. A maximum is found at a crossover energy (voltage), and the conductance at this maximum is reduced compared to the ideal Andreev conductance obtained exactly at $\omega = \Delta$ in the single contact case. The higher the transparency of the barriers, the lower is this crossover energy, below which the perturbative theory is essentially valid.

The other important issue concerns the phase problem. Owing to the different forms of the normal and anomalous propagators controlling (in the superconductor) the cotunneling and Andreev processes respectively, the corresponding conductances assume different values for single-channel contacts (cotunneling takes place if they have parallel spin polarization, crossed Andreev if they have antiparallel ones). This was exemplified here in a three-contact configuration with spin polarizations chosen such as both processes can be compared in equivalent geometries. On the other hand, averaging the Fermi phase oscillations present in the propagators make the two processes lead to equivalent conductance contributions. We have shown that a self-averaging effect occurs when at least one of the contacts has many channels. The resulting

symmetry (equality of cotunneling and Andreev crossed conductances) was previously demonstrated in a perturbative regime, but it now appears as much more general. As a consequence, if at least one of the contacts is not spin-polarized, the resulting intercontact conductance is zero, by compensation of cotunneling and Andreev processes [4]. Conversely, for spin-polarized contacts this offers a way of measuring, in amplitude and direction, the polarization of one contact with respect to the other through the crossed current measurement, as proposed in Ref [24]. We believe that multiterminal superconducting-ferromagnet devices have a large, yet unexplored, potential in the growing field of spintronics. Further theoretical problems concern the self-consistent calculation of the superconducting gap [10].

25.9 A1 Expression of the Keldysh propagator: two-channel model

In this appendix we derive the expression of the Keldysh propagator associated to two single-channel electrodes with 100% spin polarization (see section 25.4). We need to calculate the Keldysh component: $\hat{t}_{a,\alpha}\hat{G}_{\alpha,a}^{+,-} = \sum_{i,j} \hat{K}_{i,j}$, with $i, j \in \{a, b, \alpha, \beta\}$, and

$$\hat{K}_{\alpha,\alpha} = \hat{t}_{a,\alpha} \left[\hat{I} + \hat{G}_{\alpha,a}^R \hat{t}_{a,\alpha} \right] \hat{g}_{\alpha,\alpha}^{+,-} \hat{t}_{\alpha,a} \hat{G}_{a,a}^A \quad (25.70)$$

$$\hat{K}_{\alpha,\beta} = \hat{t}_{a,\alpha} \left[\hat{I} + \hat{G}_{\alpha,a}^R \hat{t}_{a,\alpha} \right] \hat{g}_{\alpha,\beta}^{+,-} \hat{t}_{\beta,b} \hat{G}_{b,a}^A \quad (25.71)$$

$$\hat{K}_{\beta,\alpha} = \hat{t}_{a,\alpha} \hat{G}_{\alpha,b}^R \hat{t}_{b,\beta} \hat{g}_{\beta,\alpha}^{+,-} \hat{t}_{\alpha,a} \hat{G}_{a,a}^A \quad (25.72)$$

$$\hat{K}_{\beta,\beta} = \hat{t}_{a,\alpha} \hat{G}_{\alpha,b}^R \hat{t}_{b,\beta} \hat{g}_{\beta,\beta}^{+,-} \hat{t}_{b,\beta} \hat{G}_{\beta,a}^A \quad (25.73)$$

$$\hat{K}_{a,a} = \hat{t}_{a,\alpha} \hat{G}_{\alpha,\alpha}^R \hat{t}_{\alpha,a} \hat{g}_{a,a}^{+,-} \left[\hat{I} + \hat{t}_{a,\alpha} \hat{G}_{\alpha,a}^A \right] \quad (25.74)$$

$$\hat{K}_{b,b} = \hat{t}_{a,\alpha} \hat{G}_{\alpha,\beta}^R \hat{t}_{\beta,b} \hat{g}_{b,b}^{+,-} \hat{t}_{b,\beta} \hat{G}_{\beta,b}^A. \quad (25.75)$$

We also need to calculate $\hat{t}_{\alpha,a}\hat{G}_{a,\alpha}^{+,-} = \sum_{i,j} \hat{K}'_{i,j}$, with

$$\hat{K}'_{\alpha,\alpha} = \hat{G}_{a,a}^R \hat{t}_{a,\alpha} \hat{g}_{\alpha,\alpha}^{+,-} \left[\hat{I} + \hat{t}_{\alpha,a} \hat{G}_{a,\alpha}^A \right] \quad (25.76)$$

$$\hat{K}'_{\alpha,\beta} = \hat{G}_{a,a}^R \hat{t}_{a,\alpha} \hat{g}_{\alpha,\beta}^{+,-} \hat{t}_{\beta,b} \hat{G}_{b,\alpha}^A \quad (25.77)$$

$$\hat{K}'_{\beta,\alpha} = \hat{G}_{a,b}^R \hat{t}_{b,\beta} \hat{g}_{\beta,\alpha}^{+,-} \left[\hat{I} + \hat{t}_{\alpha,a} \hat{G}_{a,\alpha}^A \right] \quad (25.78)$$

$$\hat{K}'_{\beta,\beta} = \hat{G}_{a,b}^R \hat{t}_{b,\beta} \hat{g}_{\beta,\beta}^{+,-} \hat{t}_{\beta,b} \hat{G}_{b,\alpha}^A \quad (25.79)$$

$$\hat{K}'_{a,a} = \left[\hat{I} + \hat{G}_{a,\alpha}^R \hat{t}_{\alpha,a} \right] \hat{g}_{a,a}^{+,-} \hat{t}_{a,\alpha} \hat{G}_{\alpha,\alpha}^A \quad (25.80)$$

$$\hat{K}'_{b,b} = \hat{G}_{a,\beta}^R \hat{t}_{\beta,b} \hat{g}_{b,b}^{+,-} \hat{t}_{b,\beta} \hat{G}_{\beta,\alpha}^A. \quad (25.81)$$

The expression of the four terms containing $n_F(\omega - \mu_S)$ takes the form

$$\left[\hat{K}'_{\alpha,\alpha} - \hat{K}_{\alpha,\alpha} \right]_{1,1} = 4\pi^2 n_F(\omega - \mu_S) |t_{a,\alpha}|^2 \rho_{1,1}^{a,a} \rho_g^{\alpha,\alpha} \quad (25.82)$$

$$\begin{aligned} & \times \frac{1}{\mathcal{D}^A \mathcal{D}^R} \left[1 - |t_{b,\beta}|^2 g_{2,2}^{b,b,A} g^{\beta,\beta,A} \right] \left[1 - |t_{b,\beta}|^2 g_{2,2}^{b,b,R} g^{\beta,\beta,R} \right] \\ \left[\hat{K}'_{\alpha,\beta} - \hat{K}_{\alpha,\beta} \right]_{1,1} &= 4\pi^2 n_F(\omega - \mu_S) |t_{a,\alpha}|^2 |t_{b,\beta}|^2 \rho_{1,1}^{a,a} \rho_f^{\alpha,\beta} g_{2,2}^{b,b,A} f^{\beta,\alpha,A} \\ & \times \frac{1}{\mathcal{D}^A \mathcal{D}^R} \left[1 - |t_{b,\beta}|^2 g_{2,2}^{b,b,R} g^{\beta,\beta,R} \right] \end{aligned} \quad (25.83)$$

$$\begin{aligned} \left[\hat{K}'_{\beta,\alpha} - \hat{K}_{\beta,\alpha} \right]_{1,1} &= 4\pi^2 n_F(\omega - \mu_S) |t_{a,\alpha}|^2 |t_{b,\beta}|^2 \rho_{1,1}^{a,a} \rho_f^{\beta,\alpha,A} g_{2,2}^{b,b,R} f^{\alpha,\beta,R} \\ & \times \frac{1}{\mathcal{D}^A \mathcal{D}^R} \left[1 - |t_{b,\beta}|^2 g_{2,2}^{b,b,A} g^{\beta,\beta,A} \right] \end{aligned} \quad (25.84)$$

$$\begin{aligned} \left[\hat{K}'_{\beta,\beta} - \hat{K}_{\beta,\beta} \right]_{1,1} &= 4\pi^2 n_F(\omega - \mu_S) |t_{a,\alpha}|^2 |t_{b,\beta}|^4 \rho_{1,1}^{a,a} \rho_g^{\alpha,\beta} \\ & \times \frac{1}{\mathcal{D}^A \mathcal{D}^R} g_{2,2}^{b,b,A} g_{2,2}^{b,b,R} f^{\alpha,\beta,R} f^{\beta,\alpha,A}, \end{aligned} \quad (25.85)$$

where ρ_g is one of the Nambu components of the superconductor density of states (see Eq. 25.6). The terms containing μ_a and μ_b read

$$\left[\hat{K}'_{a,a} - \hat{K}_{a,a} \right]_{1,1} = -4\pi^2 n_F(\omega - \mu_a) |t_{a,\alpha}|^2 \rho_{1,1}^{a,a} \rho_g^{\alpha,\alpha} \quad (25.86)$$

$$\begin{aligned} & \times \frac{1}{\mathcal{D}^A \mathcal{D}^R} \left[1 - |t_{b,\beta}|^2 g_{2,2}^{b,b,A} g^{\beta,\beta,A} \right] \left[1 - |t_{b,\beta}|^2 g_{2,2}^{b,b,R} g^{\beta,\beta,R} \right] \\ & + 2i\pi n_F(\omega - \mu_a) |t_{a,\alpha}|^2 |t_{b,\beta}|^2 \rho_{1,1}^{a,a} \\ & \times \frac{1}{\mathcal{D}^A \mathcal{D}^R} g_{2,2}^{b,b,A} f^{\alpha,\beta,A} f^{\beta,\alpha,A} \left[1 - |t_{b,\beta}|^2 g_{2,2}^{b,b,R} g^{\beta,\beta,R} \right] \\ & - 2i\pi n_F(\omega - \mu_a) |t_{a,\alpha}|^2 |t_{b,\beta}|^2 \rho_{1,1}^{a,a} \\ & \times \frac{1}{\mathcal{D}^A \mathcal{D}^R} g_{2,2}^{b,b,R} f^{\alpha,\beta,R} f^{\beta,\alpha,R} \left[1 - |t_{b,\beta}|^2 g_{2,2}^{b,b,A} g^{\beta,\beta,A} \right] \\ \left[\hat{K}'_{b,b} - \hat{K}_{b,b} \right]_{1,1} &= 4\pi^2 n_F(\omega - \mu_b) |t_{a,\alpha}|^2 |t_{a',\alpha'}|^2 \rho_{1,1}^{a,a} \rho_{2,2}^{b,b} \frac{1}{\mathcal{D}^A \mathcal{D}^R} f^{\alpha,\beta,R} f^{\beta,\alpha,A}. \end{aligned} \quad (25.87)$$

One arrives at the identity

$$\frac{1}{n_F(\omega - \mu_S)} \left[\hat{K}'_{\alpha,\alpha} - \hat{K}_{\alpha,\alpha} + \hat{K}'_{\alpha,\beta} - \hat{K}_{\alpha,\beta} + \hat{K}'_{\beta,\alpha} - \hat{K}_{\beta,\alpha} + \hat{K}'_{\beta,\beta} - \hat{K}_{\beta,\beta} \right]_{1,1} \quad (25.88)$$

$$= -\frac{1}{n_F(\omega - \mu_a)} \left[\hat{K}'_{a,a} - \hat{K}_{a,a} \right]_{1,1} - \frac{1}{n_F(\omega - \mu_b)} \left[\hat{K}'_{b,b} - \hat{K}_{b,b} \right]_{1,1}, \quad (25.89)$$

which constitutes for this particular system a proof of the trick on the spectral current (see section 25.2.2). The expression of the Keldysh propagators given in this appendix can be used to obtain the transport formula given in section 25.4.2.

25.10 A2 Derivation of the transport formula with an arbitrary number of single-channel electrodes

We present in this appendix the derivation of the transport formula given by Eqs. (25.43) – (25.47), associated to a situation where N ferromagnetic spin-up electrodes and M ferromagnetic spin-down electrodes are in contact with a superconductor (see Fig. 90).

25.10.1 Solution of the Dyson equation

The unknown Green's functions $\{G^{a_1,a_1}, \dots, G^{a_N,a_1}, G^{b_1,a_1}, \dots, G^{b_M,a_1}\}$ are the solution of the Dyson equation

$$\hat{\mathcal{M}} \begin{bmatrix} G^{a_1,a_1} \\ G^{a_2,a_1} \\ \dots \\ G^{a_N,a_1} \\ G^{b_1,a_1} \\ \dots \\ G^{b_M,a_1} \end{bmatrix} = \begin{bmatrix} g^{a_1,a_1} \\ 0 \\ \dots \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix},$$

where the Dyson matrix $\hat{\mathcal{M}}$ takes the form

$$\hat{\mathcal{M}} = \hat{I} + \begin{bmatrix} -\hat{Y}^{a,a} & \hat{X}^{a,b} \\ \hat{X}^{b,a} & -\hat{Y}^{b,b} \end{bmatrix}, \quad (25.90)$$

where $\hat{Y}^{a,a}$ is a $N \times N$ block, $\hat{X}^{a,b}$ is a $N \times M$ block. The matrix elements of $\hat{X}$ and $\hat{Y}$ are

$$X^{a_i,a_j} = t^{a_i,\alpha_i} t^{a_j,\alpha_j} g^{a_i,a_j} f^{\alpha_i,\alpha_j} \quad (25.91)$$

$$Y^{a_i,a_j} = t^{a_i,\alpha_i} t^{a_j,\alpha_j} g^{a_i,a_i} g^{\alpha_i,\alpha_j}. \quad (25.92)$$

The solution of the Dyson equation takes the form

$$G_{1,1}^{a_i,a_j} = \frac{(-)^{i+j}}{\mathcal{D}} g^{a_j,a_j} \tilde{\mathcal{M}}_{a_j,a_i} \quad (25.93)$$

$$G_{2,2}^{b_i,b_j} = \frac{(-)^{i+j}}{\mathcal{D}} g^{b_j,b_j} \tilde{\mathcal{M}}_{b_j,b_i} \quad (25.94)$$

$$G_{1,2}^{a_i,b_j} = \frac{(-)^{i+j+N}}{\mathcal{D}} g^{b_j,b_j} \tilde{\mathcal{M}}_{b_j,a_i} \quad (25.95)$$

$$G_{2,1}^{b_i,a_j} = \frac{(-)^{i+j+N}}{\mathcal{D}} g^{a_j,a_j} \tilde{\mathcal{M}}_{a_j,b_i}, \quad (25.96)$$

where $\mathcal{D}$ is the determinant of the Dyson matrix and $\tilde{\mathcal{M}}_{a_i,a_j}$ are the minors of this matrix.

25.10.2 Solution of the Dyson-Keldysh equation

To obtain the current through the link $a_1 - \alpha_1$, we need to evaluate the Keldysh component

$$\hat{t}^{a_1,\alpha_1} G_{\alpha_1,a_1}^{+,-} = \hat{t}^{a_1,\alpha_1} \hat{G}^{\alpha_1,\alpha_1,R} \hat{t}^{\alpha_1,a_1} \hat{g}_{a_1,a_1}^{+,-} \left[\hat{I} + \hat{t}^{a_1,\alpha_1} \hat{G}^{\alpha_1,a_1,A} \right] \quad (25.97)$$

$$+ \sum_{k=2}^N \hat{t}^{a_1,\alpha_1} \hat{G}^{\alpha_1,\alpha_k,R} \hat{t}^{\alpha_k,a_k} \hat{g}_{a_k,a_k}^{+,-} \hat{t}^{a_k,\alpha_k} \hat{G}^{\alpha_k,a_1,A} \quad (25.98)$$

$$+ \sum_{k=1}^N \hat{t}^{a_1,\alpha_1} \hat{G}^{\alpha_1,\beta_k,R} \hat{t}^{\beta_k,b_k} \hat{g}_{b_k,b_k}^{+,-} \hat{t}^{b_k,\beta_k} \hat{G}^{\beta_k,a_1,A}. \quad (25.99)$$

Let us start with (25.97). The first step is to show that

$$(25.97) = 2i\pi n_F(\omega - \mu_{a_1}) |t^{a_1,\alpha_1}|^2 \rho_{1,1}^{a_1,a_1} G_{1,1}^{\alpha_1,\alpha_1,R} \left[\hat{I} + \hat{t}^{a_1,\alpha_1} \hat{G}^{\alpha_1,\alpha_1,R} \right]_{1,1}.$$

The different terms in this equation are found to be

$$\left[\hat{I} + \hat{t}^{a_1,\alpha_1} \hat{G}^{\alpha_1,\alpha_1,R} \right]_{1,1} = \frac{\tilde{\mathcal{M}}_{a_1,a_1}}{\mathcal{D}},$$

and

$$|t^{a_1, \alpha_1}|^2 G^{\alpha_1, \alpha_1} = |t^{a_1, \alpha_1}|^2 \frac{1}{\mathcal{D}} g^{\alpha_1, \alpha_1} \tilde{\mathcal{M}}_{a_1, a_1} \quad (25.100)$$

$$+ \frac{1}{\mathcal{D}} \sum_{k \neq 1} (-)^{k+1} t^{a_1, \alpha_1} t^{a_k, \alpha_k} g^{\alpha_1, \alpha_k} \tilde{\mathcal{M}}_{a_1, a_k} \quad (25.101)$$

$$+ \frac{1}{\mathcal{D}} \sum_{k=1}^M (-)^{k+N} t^{a_1, \alpha_1} t^{b_k, \beta_k} f^{\alpha_1, \beta_k} \tilde{\mathcal{M}}_{a_1, b_k}. \quad (25.102)$$

To evaluate (25.98), we first show that

$$(25.98) = \sum_{k=2}^N 2i\pi n_F(\omega - \mu_{a_k}) t^{a_1, \alpha_1} t^{a_k, \alpha_k} \rho_{1,1}^{a_k, a_k} G_{1,1}^{\alpha_1, \alpha_k, R} [\hat{t}^{a_k, \alpha_k} G^{\alpha_k, a_1}]_{1,1}^A.$$

Using the identities

$$[\hat{t}^{a_k, \alpha_k} G^{\alpha_k, a_1}]_{1,1} = (-)^{k+1} \frac{g^{a_1, a_1}}{g^{a_k, a_k} \mathcal{D}} \tilde{\mathcal{M}}_{a_1, a_k} \quad (25.103)$$

$$[G^{\alpha_1, \alpha_k}]_{1,1} = \frac{(-)^{k+1}}{t^{a_1, \alpha_1} t^{a_k, \alpha_k} g^{a_1, a_1} \mathcal{D}} \tilde{\mathcal{M}}_{a_k, a_1}, \quad (25.104)$$

we obtain

$$(25.98) = \sum_{k=2}^N 2i\pi n_F(\omega - \mu_{a_k}) t^{a_1, \alpha_1} t^{a_k, \alpha_k} \rho_{1,1}^{a_k, a_k} g^{a_1, a_1, A} \tilde{g}_{\alpha_1, \alpha_k}^R \tilde{g}_{\alpha_k, \alpha_1}^A,$$

where $\tilde{g}_{\alpha_1, \alpha_k}$ denotes a renormalized propagator. We use a similar calculation to evaluate (25.99) and we deduce the transport formula given by Eqs. (25.43) – (25.47).

26 Single fermion manipulation via superconducting phase differences in multiterminal Josephson junctions by van Heck, Mi, Akhmerov

Abstract

We show how the superconducting phase difference in a Josephson junction may be used to split the Kramers degeneracy of its energy levels and to remove all the properties associated with time reversal symmetry. The superconducting phase difference is known to be ineffective in two-terminal short Josephson junctions, where irrespective of the junction structure the induced Kramers degeneracy splitting is suppressed and the ground state fermion parity must stay even, so that a protected zero-energy Andreev level crossing may never appear. Our main result is that these limitations can be completely avoided by using multiterminal Josephson junctions. There the Kramers degeneracy breaking becomes comparable to the superconducting gap, and applying phase differences may cause the change of the ground state fermion parity from even to odd. We prove that the necessary condition for the appearance of a fermion parity switch is the presence of a “discrete vortex” in the junction: the situation when the phases of the superconducting leads wind by 2π . Our approach offers new strategies for creation of Majorana bound states as well as spin manipulation. Our proposal can be implemented using any low density, high spin-orbit material such as InAs quantum wells, and can be detected using standard tools.

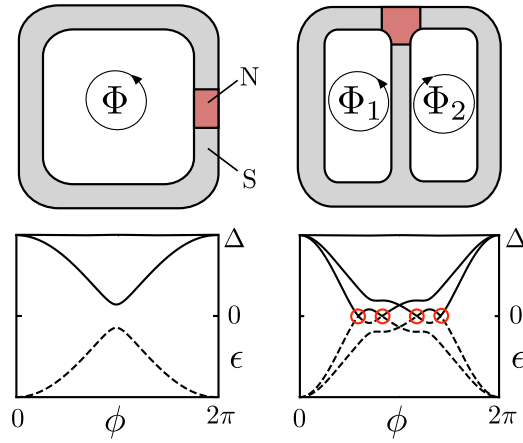


Figure 98: *Top left*: A superconducting ring (grey) allows one to concentrate the effect of a magnetic flux Φ on the small area of a Josephson junction (red). *Bottom left*: The junction has subgap Andreev levels whose energy ϵ depends on the phase difference $2e\Phi/\hbar = \phi$. Each level is doubly degenerate since in a short junction a finite phase difference does not induce a splitting of the Kramers degeneracy. *Top right*: As explained in this work, Kramers degeneracy can be efficiently removed in a three-terminal junction, even in the absence of an external magnetic field. *Bottom right*: Andreev spectrum for $2e\Phi_1/\hbar = -2e\Phi_2/\hbar = \phi$. Both the splitting of Kramers degeneracy and Andreev level crossings at zero energy (marked by red circles) appear in the spectrum.

26.1 Introduction

In quantum mechanics, Kramers' theorem guarantees that in presence of time reversal symmetry the energy levels of a system with half-integer spin are doubly degenerate even if the spin rotation symmetry is broken [1, 2]. A practical consequence of this theorem is that it is necessary to break time reversal symmetry in order to control single fermion states in a condensed matter system. The energy separation of different spin states opens the way to spin detection and manipulation and is often a necessary element for spin qubits [3] and spintronics [4, 5]. The absence of Kramers degeneracy is also a fundamental requirement for the creation of unpaired Majorana bound states in topological superconductors [10, 7].

In order to provide fine-grained manipulation of electron states, a source of time reversal symmetry breaking should be local in space and easily tunable in time. The superconducting phase difference across a Josephson junction satisfies these requirements. It allows one to concentrate the effect of a magnetic flux penetrating a large superconducting ring into the small area of the Josephson junction, whose spatial extent may be comparable to the superconducting coherence length ξ (see Fig. 98). The magnitude of the energy splitting between a Kramers pair of bound states in the junction can then be comparable to the superconducting gap Δ . The magnetic field required to control the superconducting phase difference is rather small, and may be vanishing in the junction itself. Flux bias loops applying this magnetic field allow one to address different Josephson junctions independently by tuning different fluxes, and have nanosecond response times. These features seemingly make the superconducting phase difference the perfect source of time-reversal symmetry breaking for the manipulation of single fermion states. In contrast, an external magnetic field seems to lose to phase differences in most respects: it needs to be a fraction of a Tesla to achieve a Zeeman splitting comparable to Δ . Such a field can only be tuned on the time scale of seconds and is rather hard to apply locally to only a part of a mesoscopic system.

Short Josephson junctions with Thouless energy E_T much larger than the superconducting gap Δ are the most promising for single fermion manipulation, since they have the largest

level spacing $\delta E \sim \Delta$. Unfortunately, using phase difference as a source of time reversal symmetry breaking is ineffective in short two-terminal Josephson junctions. This fact might seem surprising, since using symmetry considerations alone one would expect the spectrum of the Andreev bound states to be non-degenerate at a finite phase difference ϕ . As is well known, however, this expectation does not hold. The Andreev energy levels ϵ_k are in one-to-one correspondence with the transmission eigenvalues T_k of the scattering matrix of the junction in the normal state [15]:

$$\epsilon_k = \pm \Delta [1 - T_k \sin^2(\phi/2)]^{1/2}. \quad (26.1)$$

In the absence of time reversal symmetry breaking in the normal state, the transmission eigenvalues T_k are Kramers degenerate (see Ref. [26] for a concise proof), and hence so are the Andreev levels. Relaxing the short junction condition changes the scenario: spin-orbit coupling couples the spin of the bound states to the phase difference and lifts the Kramers degeneracy of the Andreev spectrum, albeit by a small amount of the order Δ^2/E_T [6, 11]. Therefore, time-reversal symmetry can be broken only very weakly in a two-terminal junction.

In this work, we show how this serious limitation can be removed with a simple yet crucial change in the device geometry: the addition of an extra superconducting lead, as shown in Fig. 98. Indeed, in devices with more than two superconducting terminals, the energy spectrum is not expected anymore to be in one-to-one correspondence with transmission eigenvalues. We demonstrate that in this case the effect of time reversal symmetry breaking by superconducting phase differences alone leads to large splitting of the Kramers doublets comparable to the superconducting gap Δ . Naturally, since breaking the spin-rotation symmetry remains necessary, spin-orbit coupling is still an essential ingredient. The non-degenerate Andreev spectrum makes these three-terminal junctions a promising platform for superconducting spin qubits [6, 7, 13] and the creation of Majorana bound states, as we will discuss further in Sec. 26.4.

As a consequence of the strong splitting of the Kramers degeneracy, crossings at the Fermi level can appear in the Andreev spectrum, corresponding to a switch in the ground state fermion parity [14, 4]. We find that a necessary condition for the existence of a crossing at the Fermi level is the presence of a discrete vortex in the junction. In other words, the gap in the Andreev spectrum can only close when the superconducting phases of the leads wind by 2π around the junction. If this condition is satisfied, the spectral peaks in the density of states of the junction develop at the Fermi level as expected [16, 14, 17, 18] for a superconducting quantum dot with broken time-reversal and spin-rotation symmetries (symmetry class D of the Altland-Zirnbauer classification [14]).

26.2 General considerations

26.2.1 Scattering formalism and bound state equation for multiterminal Josephson junctions

Three terminal Josephson junctions, such as the one shown in Fig. 99, are the main focus of our work. However, since most of our conclusions generalize naturally to the case of more terminals, we consider a junction with m superconducting leads. We assume that all of the leads have the same energy gap Δ and different phases $\phi_1, \dots, \phi_m$. The coupling between the superconducting leads through the normal scattering region is fully characterized by the electron scattering matrix $s(\epsilon)$, with ϵ the excitation energy. In general $s(\epsilon)$ is a $n \times n$ unitary matrix. Its size $n = n_1 + \dots + n_m$ is the sum of the number of incoming modes in the leads, counting spin. The integers $n_1, \dots, n_m$ must be even due to the fermion doubling theorem [19].

When $|\epsilon| < \Delta$, an electron escaping the scattering region must be reflected back as a hole at

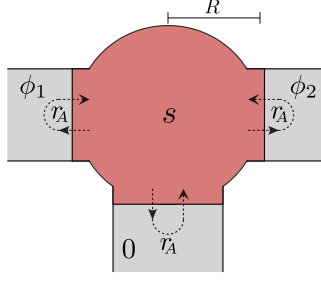


Figure 99: Three-terminal Josephson junction geometry. The scattering region (red) is a piece of a disordered two-dimensional material with spin-orbit coupling. It is connected to three superconducting leads (grey). In the normal state, the scattering region has a scattering matrix s . At energies smaller than the superconducting gap Δ , modes leaving the scattering region are reflected at the interface with the superconductor by Andreev reflection processes (black arrows), described by a scattering matrix r_A .

the interface with the superconductor⁷. Closed trajectories of electron and hole superposition form Andreev bound states in the junction, which are confined by the superconducting pairing potential in the leads. The spectrum of Andreev bound states can be expressed through two distinct scattering matrices: that of the scattering region s_N , and the scattering matrix s_A describing Andreev reflection from a superconducting interface. Both matrices are unitary and depend on the energy ϵ . As derived in Ref. [15], the condition for a presence of the bound state is given by:

$$s_A(\epsilon) s_N(\epsilon) \Psi_{\text{in}} = \Psi_{\text{in}}. \quad (26.2)$$

Here, $\Psi_{\text{in}} = (\Psi_{\text{in}}^e, \Psi_{\text{in}}^h)$ is a vector of complex coefficients describing a wave incident on the junction in the basis of the modes incoming from the superconducting leads into the normal region.

Since in the normal region electrons and holes are not coupled, s_N is block-diagonal in the electron-hole space. We choose the hole modes as particle-hole partners of the electron modes and obtain

$$s_N(\epsilon) = \begin{pmatrix} s(\epsilon) & 0 \\ 0 & s^*(-\epsilon) \end{pmatrix}. \quad (26.3)$$

For more details regarding the relation between the basis choice for a scattering matrix and its discrete symmetries, see App. A of Ref. [21]. In the same basis, the Andreev scattering matrix s_A is block off-diagonal since it couples only electron to holes and vice versa,

$$s_A(\epsilon) = \alpha(\epsilon) \begin{pmatrix} 0 & r_A^* \\ r_A & 0 \end{pmatrix}. \quad (26.4)$$

The phase factor $\alpha(\epsilon) = \sqrt{1 - \epsilon^2/\Delta^2} + i\epsilon/\Delta$ is due to the matching of the wave function at the interface between the normal region and the superconductors [15].

In the short junction limit, the energy dependence of the scattering matrix elements can be neglected,

$$s(\epsilon) \simeq s(-\epsilon) \simeq s(0) \equiv s. \quad (26.5)$$

In that case the set of discrete Andreev levels $\{\epsilon_k\}$ can be computed by substituting Eqs. (26.3, 26.4) into Eq. (26.2) and solving the resulting eigenproblem for α :

$$\begin{pmatrix} s^\dagger & 0 \\ 0 & s^T \end{pmatrix} \begin{pmatrix} 0 & r_A^* \\ r_A & 0 \end{pmatrix} \Psi_{\text{in}} = \alpha \Psi_{\text{in}}. \quad (26.6)$$

⁷In the presence of normal scattering in the superconductor, reflection from the superconductor can be represented as a combination of normal scattering followed by a subsequent perfect Andreev reflection. The former component can be combined with the scattering matrix of the normal region. Therefore, assuming perfect Andreev reflection does not reduce the generality of our results.

It is convenient to apply to the above problem the Joukowski transform

$$X \rightarrow -\frac{i}{2} (X - X^{-1}) , \quad (26.7)$$

which maps α to ϵ/Δ . In this way, we obtain an eigenproblem directly for ϵ :

$$\begin{pmatrix} 0 & -iA^\dagger \\ iA & 0 \end{pmatrix} \Psi_{\text{in}} = \frac{\epsilon}{\Delta} \Psi_{\text{in}}, \quad (26.8)$$

with

$$A \equiv \frac{1}{2} (r_A s - s^T r_A) . \quad (26.9)$$

Since A is a normal matrix ($AA^\dagger = A^\dagger A$), its eigenvalues are equal to its singular values up to a phase, and as follows from (26.8) its singular values are equal to $|\epsilon|$. We now arrive at the simplified eigenproblem for the energies of Andreev levels:

$$A \Psi_{\text{in}}^e = \frac{|\epsilon|}{\Delta} e^{i\chi} \Psi_{\text{in}}^e \quad (26.10)$$

The double degeneracy of the singular values of A is a consequence of the fact that the eigenvalues of Eq. (26.6) come in complex conjugate pairs, while only α with a positive real part are physical. The reduction of the eigenproblem to the form of Eq. (26.10) is an important simplification which allows us to derive the properties of the Andreev spectrum of the junction.

In the normal state the time-reversal symmetry is preserved in the junction and can be used to further constraint the scattering matrix s , which belongs to the circular symplectic ensemble[60] (CSE, symmetry class AII). Choosing a basis such that the outgoing modes are the time-reversed partners of the incoming ones results in s becoming an antisymmetric matrix, $s = -s^T$. Correspondingly, A becomes the anticommutator of s and r_A :

$$A = \frac{1}{2} \{s, r_A\} . \quad (26.11)$$

Moreover, in the same basis in which s is antisymmetric, the Andreev reflection matrix r_A is diagonal,

$$r_A = \begin{pmatrix} i e^{i\phi_1} \mathbf{1}_{n_1} & 0 & \dots & 0 \\ 0 & i e^{i\phi_2} \mathbf{1}_{n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & i e^{i\phi_m} \mathbf{1}_{n_m} \end{pmatrix} . \quad (26.12)$$

We are now prepared to build a theory of multiterminal Josephson junctions.

26.2.2 Kramers degeneracy splitting

For completeness, we first apply our formalism given by Eq. (26.10) to repeat the known result of the absence of the Kramers degeneracy splitting in two terminal short junctions. For $m = 2$, the Andreev reflection matrix r_A has only two distinct eigenvalues $i e^{i\phi_1}$ and $i e^{i\phi_2}$, with multiplicity n .⁸

In this case, we can use the polar decomposition of s [60]:

$$\begin{pmatrix} U_1 & 0 \\ 0 & V_1 \end{pmatrix} s \begin{pmatrix} U_2 & 0 \\ 0 & V_2 \end{pmatrix} = \begin{pmatrix} -\sqrt{1-T} & \sqrt{T} \\ \sqrt{T} & \sqrt{1-T} \end{pmatrix} . \quad (26.13)$$

⁸Generalization to unequal numbers of modes in two superconducting leads is straightforward, since in that case one of the leads will simply have several fully reflected modes leading to no extra Andreev bound states.

Here, $U_{1,2}$ and $V_{1,2}$ are $n \times n$ unitary matrices, while $T = \text{diag}(T_1, \dots, T_{n_1})$ is a $n \times n$ matrix with doubly-degenerate transmission eigenvalues T_k on its diagonal. Crucially, since

$$\begin{pmatrix} U_{1,2} & 0 \\ 0 & V_{1,2} \end{pmatrix} r_A = r_A \begin{pmatrix} U_{1,2} & 0 \\ 0 & V_{1,2} \end{pmatrix}, \quad (26.14)$$

the polar decomposition of s carries on to A :

$$\begin{pmatrix} U_1 & 0 \\ 0 & V_1 \end{pmatrix} A \begin{pmatrix} U_2 & 0 \\ 0 & V_2 \end{pmatrix} = \begin{pmatrix} -\sqrt{1-T} e^{i\phi_1} & \frac{1}{2}\sqrt{T}(e^{i\phi_1} + e^{i\phi_2}) \\ \frac{1}{2}\sqrt{T}(e^{i\phi_1} + e^{i\phi_2}) & \sqrt{1-T} e^{i\phi_2} \end{pmatrix}. \quad (26.15)$$

Diagonalization of the right hand side then immediately yields the spectrum of Eq. (26.1).

It is easy to recognize that this derivation cannot be extended to the multiterminal case. Indeed, if r_A has more than two distinct eigenvalues, Eq. (26.14) does not hold anymore and there is no polar decomposition which can be simultaneously applied to both s and A . The correspondence between Andreev levels and transmission eigenvalues of s is then lost. As a consequence, we expect the spectrum of a multiterminal junction to consist of non-degenerate levels, unless the phases in the leads are tuned in such a way that the two-terminal case of only two distinct eigenvalues of r_A is restored.

If spin-rotation symmetry is strongly broken, and the phase differences are not small, there is no small parameter in the eigenproblem of Eq. (26.10) with more than two terminals. This means that the energy splitting between Kramers partners becomes comparable to the Andreev level spacing in the junction, and scales as Δ/n , the maximal possible value. A simple estimate shows that, as one would expect, the splitting of Kramers degeneracy obtained using superconducting phase differences may never exceed the normal level spacing in the scattering region. Indeed, for the junction to be in a short junction regime, Δ should be much smaller than the Thouless energy $n\delta_0$, with δ_0 the normal level spacing in the scattering region. This immediately gives an upper bound of δ_0 on the Kramers degeneracy breaking.

26.2.3 Lower bound on the energy gap and existence of zero-energy solutions

For the two-terminal case, Eq. (26.1) implies a lower bound $|\epsilon| \geq \Delta \cos(\phi/2)$ on the energy of the Andreev states, irrespective of the junction details. Inspecting Eq. (26.11), we see that when all ϕ_i are close to each other, r_A is an almost constant matrix, so that $\{s, r_A\}/2$ is almost unitary, and consequently all of the Andreev energies are close to Δ . This suggests that it is natural to expect some lower bound for ϵ also in the multiterminal case.

To determine this lower bound, we rewrite the eigenvalue equation (26.10) as:

$$s r_A |\Psi\rangle + r_A |\Psi'\rangle = \frac{2|\epsilon|}{\Delta} e^{i\chi} |\Psi\rangle, \quad (26.16a)$$

$$|\Psi'\rangle \equiv s |\Psi\rangle, \quad \|\Psi\| = \|\Psi'\| = 1. \quad (26.16b)$$

The two above equations dictate that s is a linear mapping such that

$$|\Psi\rangle \xrightarrow{s} |\Psi'\rangle, \quad (26.17a)$$

$$r_A |\Psi\rangle \xrightarrow{s} \frac{2|\epsilon|}{\Delta} e^{i\chi} |\Psi\rangle - r_A |\Psi'\rangle. \quad (26.17b)$$

Since s is unitary, these equations may be satisfied for given Ψ and Ψ' if and only if the scalar products between the vectors on the left and right hand sides of Eqs. (26.17) are preserved. Hence, a necessary and sufficient condition for the existence of a solution is

$$\langle \Psi | r_A | \Psi \rangle + \langle \Psi' | r_A | \Psi' \rangle = \frac{2|\epsilon|}{\Delta} e^{i\chi} \langle \Psi' | \Psi \rangle. \quad (26.18)$$

Taking the absolute value on both sides and using the Cauchy-Schwarz inequality $|\langle \Psi' | \Psi \rangle| \leq \|\Psi'\| \|\Psi\| = 1$ yields the lower bound

$$|\epsilon| \geq \frac{1}{2} \Delta |\langle \Psi | r_A | \Psi \rangle + \langle \Psi' | r_A | \Psi' \rangle|. \quad (26.19)$$

We have thus reduced the problem of finding the lower bound with respect to a unitary matrix s to a problem of finding the lower bound with respect to two vectors.

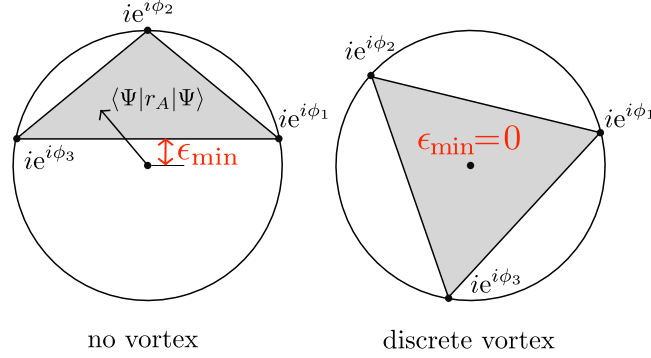


Figure 100: Geometrical illustration of Eq. (26.19) in the case of three leads. The sum of the scalar products $\frac{1}{2} \langle \Psi | r_A | \Psi \rangle$ and $\frac{1}{2} \langle \Psi' | r_A | \Psi' \rangle$ must lie within the triangle on the complex plane whose vertices are the eigenvalues $i e^{i\phi_1}, i e^{i\phi_2}, i e^{i\phi_3}$ of r_A . In the left panel, these phases do not surround the origin and the lowest allowed energy (in units of Δ) is the minimum distance between the polygon and the origin [Eq. (26.20)]. In the right panel, the phases surround the origin, a discrete vortex is present in the junction and zero-energy solutions are allowed.

The two scalar products in Eq. (26.19) are weighted sums of the eigenvalues of r_A with total weight equal to one. This means both these scalar products, as well as their averaged sum, is a point on a complex plane that must lie within a convex polygon whose vertices are the eigenvalues of r_A , see Fig. 100. We can now distinguish two possibilities, depending on whether the polygon covers the origin. If it does not, as in the left panel of Fig. 100, the energy spectrum has a lower bound $\epsilon_{\min}$ determined by the minimum distance of the polygon from the origin:

$$\epsilon_{\min} = \Delta \min_{ij} [\cos \frac{1}{2}(\phi_i - \phi_j)]. \quad (26.20)$$

On the other hand, if the polygon covers the origin, as in the right panel of Fig. 100, then a zero energy solution $\epsilon = 0$ is allowed. If we order $\phi_1 \leq \phi_2 \leq \dots \leq \phi_m$ and introduce phase differences between closest phases $\theta_i = \phi_{i+1} - \phi_i \in (-\pi, \pi]$, this happens if

$$\sum_{i=1}^m \theta_i = 2\pi. \quad (26.21)$$

We call the situation of a non-zero winding of the superconducting phases in the leads a “discrete vortex”.

Zero energy solutions are doubly degenerate and identify Andreev level crossings at Fermi energy. These crossings can be seen as topological transitions protected by fermion parity conservation. At the two sides of the gap closing point, the Pfaffian of the Hamiltonian has opposite signs, which means that energy of a single Andreev state must vanish at the transition point. Due to the number of modes in the leads being even, crossings can only occur in pairs when advancing any phase by 2π and for this reason the resulting ground state energy is 2π -periodic. Conversely, the 4π -periodic Josephson effect, a hallmark of topological superconductivity [8, 25, 9], requires an odd number of crossings in a 2π phase interval, the fermion parity anomaly.

We note that the results (26.20) and (26.21) are quite general: they hold for any number of leads and for arbitrary scattering matrices of the junction. Hence they are independent of any microscopic detail. The lower bound of Eq. (26.20) is only valid in the short junction limit, while Eq. (26.21) applies in fact to absolutely any Josephson junction since it is a Fermi level property.

26.2.4 Multiterminal Josephson junction in the quantum spin Hall regime

We observe that the lower bound (26.20) corresponds to the spectrum of a fully transmitted mode connecting two leads. This scenario can be realized in a quantum spin Hall insulator [27, 28, 29, 30]. In this case the Andreev spectrum will depend only on the phase differences between adjacent leads that are connected by topologically protected helical edge states. In fact, a straightforward generalization of the two-terminal junction of Ref. [9] yields the Andreev spectrum

$$\epsilon_i = \pm \Delta \cos \left[\frac{1}{2}(\phi_{i+1} - \phi_i) \right], \quad i = 1 \dots, m. \quad (26.22)$$

In a QSH insulator a crossing at zero energy occurs whenever one of the phase differences $\phi_{i+1} - \phi_i = \pi$ [see also the bottom left panel of Fig. (102)]. For a junction with three leads, this maximizes the region of the phase space with odd ground state fermion parity.

26.3 Applications

We now verify the results of the previous Section applied to junctions with three superconducting leads made in different physical systems. The physical systems that we study are: (i) chaotic quantum dots with random scattering matrices s uniformly sampled [31] from the circular symplectic ensemble, (ii) quantum dots made out of a quantum well with Rashba spin-orbit coupling, (iii) quantum dots made out of a quantum spin Hall insulator. In the latter two systems we obtain the scattering matrix numerically using a tight-binding simulation. We refer to these three systems as ‘RMT’, ‘Rashba’ or ‘QSH’ for brevity.

The Rashba Hamiltonian describing a 2D electron gas is given by

$$H = \frac{\mathbf{p}^2}{2m} + \alpha (p_x \sigma_y - p_y \sigma_x) - \mu + V(\mathbf{r}), \quad (26.23)$$

with $\mathbf{p} = (p_x, p_y)$ the momentum operator, σ_x and σ_y the spin Pauli matrices, α the strength of the spin-orbit coupling, and μ the chemical potential. The disordered electrostatic potential is given by $V(\mathbf{r})$. This Hamiltonian has time-reversal symmetry with operator $\Theta = i\sigma_y$.

The quantum spin Hall insulator is described by the Bernevig-Hughes-Zhang model [29], applicable to HgTe/HgCdTe and InAs/GaAs/AlSb quantum wells. For the numerical simulations, we use the extended model of Ref. [32] (see Appendix 26.6), which includes spin-orbit coupling contributions due to bulk inversion asymmetry and structural inversion asymmetry, and the material parameters reported in Ref. [33].

To extract the three-terminal scattering matrices of the normal state, we discretize the two models on a square lattice with lattice constant a . We adopt the circular dot geometry shown in Fig. 99, with a radius $R = 20a$ and three leads of width R . We consider the electrostatic disorder $V(\mathbf{r})$ to be uncorrelated and uniformly distributed in an interval $[-u, u]$. After obtaining the scattering matrix of the junction we use a gauge with $\phi_3 = 0$ and solve the eigenvalue problem (26.10) as a function of the remaining two phases ϕ_1, ϕ_2 . We perform the numerical simulations using the Kwant code [31]. The scripts with the source code are available online as ancillary files for this preprint.

26.3.1 Splitting of Kramers degeneracy

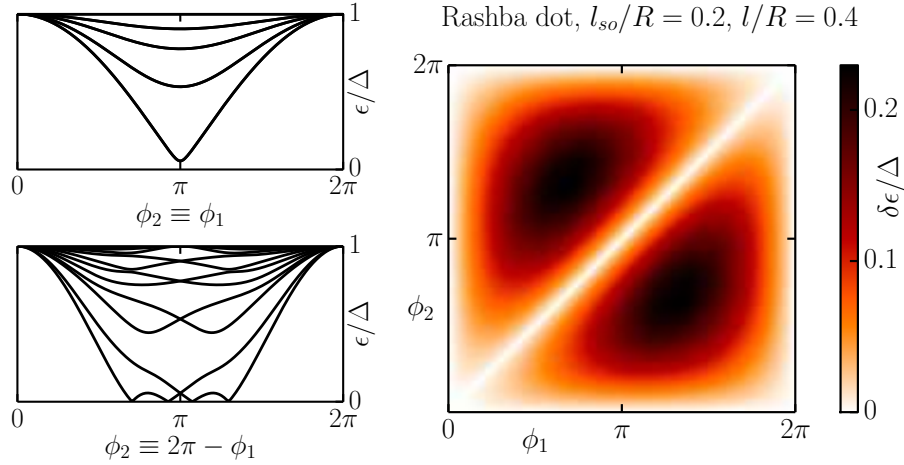


Figure 101: *Left*: Phase dependence of the Andreev levels of a Rashba dot with $\mu = 1/4ma$ for $\phi_2 = \phi_1$ (top) and $\phi_2 = 2\pi - \phi_1$ (bottom). Kramers degeneracy is present in the top panel (since one of the phase differences is zero), but not in the bottom panel. *Right*: energy difference $\delta\epsilon$ between the two lowest Andreev levels in a Rashba dot averaged over 10^2 values of $\mu \in [0, 1/2ma]$ for a fixed disorder configuration.

The first property we study is the splitting of the Andreev levels. The two-fold degenerate two-terminal junction spectrum of Eq. (26.1) should be recovered whenever any two out of three phase differences are equal, i. e. when either $\phi_1 = \phi_2$, $\phi_1 = 0$, or $\phi_2 = 0$. Away from this limit, we expect deviation from the two-terminal case and a finite splitting of the Kramers doublets.

A comparison of two typical energy spectra computed for a Rashba dot is shown in the left panel of Fig. 101 and confirms our expectations. To consider the experimentally relevant situation we choose spin-orbit interaction strength α and the disorder strength u such that the spin-orbit length $l_{so} \equiv (m\alpha)^{-1}$ and the mean free path $l \equiv 6(mau^2)^{-1} \sqrt{\mu/2ma}$ are both smaller than R , but have the same order of magnitude. We first confirm that when $\phi_1 = \phi_2$ the spectrum consists of Kramers doublets with the energies given by Eq. (26.1). On the other hand, when the two phases are opposite, $\phi_2 = 2\pi - \phi_1$, the Kramers pairs of Andreev levels have different energies, except for the time-reversal invariant points $(\phi_1, \phi_2) = 0 \bmod 2\pi$. One can also notice the presence of Andreev levels crossings at zero-energy.

To quantify the observed splitting of Kramers degeneracy, we consider the energy difference $\delta\epsilon$ between the two levels belonging to the lowest Kramers doublet. These two levels are of particular interest since they correspond to the most transparent transport channels and their energies are most sensitive to the phase differences. In the right panel of Fig. 101 the splitting $\delta\epsilon$ is computed for a Rashba dot, averaged over different values of μ in the dot. It is zero in the two-terminal limit and rises up to $\delta\epsilon \sim 0.2\Delta$ away from it. Hence, Fig. 101 confirms our conclusions that Kramers pairs of Andreev levels can be split by an energy of an order Δ solely by varying the superconducting phases. The maximal possible splitting is limited by level repulsion, and as expected, we also find that $\delta\epsilon$ is inversely proportional to the total number of Kramers doublets present in the spectrum.

26.3.2 Andreev level crossings at zero energy

By checking the Andreev level spectra of different quantum dots, we find that zero-energy crossings indeed occur for some scattering regions, as shown in the left panel of Fig. 102. A

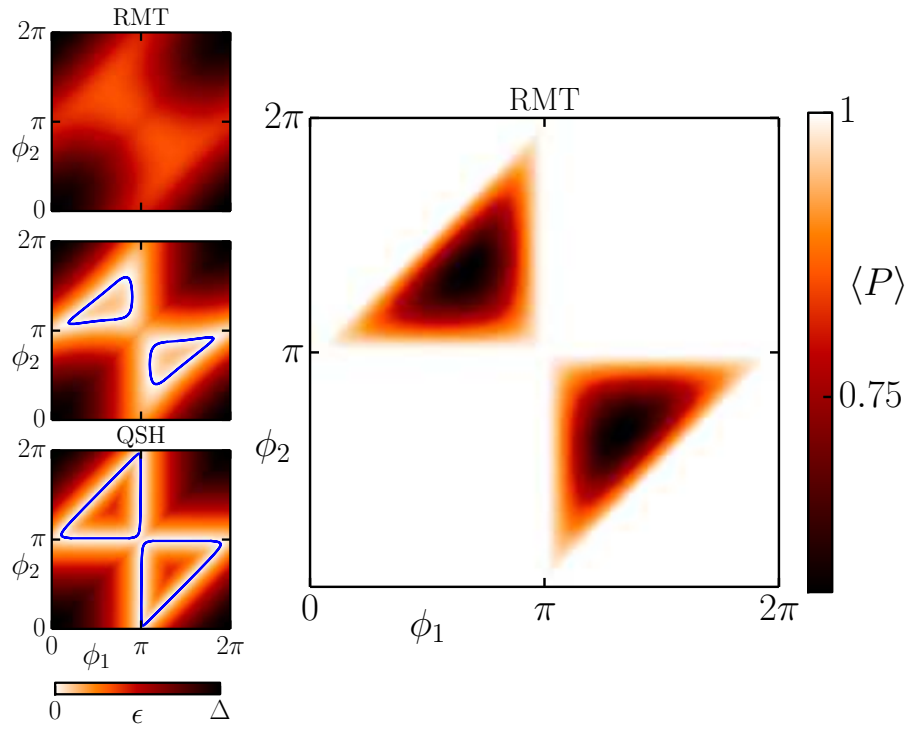


Figure 102: *Left*: Examples of the minimum energy ϵ of an Andreev bound state as a function of (ϕ_1, ϕ_2) . The first two examples are calculated using random scattering matrices, with and without with zero-energy crossings. The positions of the crossings are found numerically using method of App. 26.5, and are marked in blue. They form closed curves encircling domains of odd ground state fermion parity. The third example is for a QSH dot in the non-trivial phase, so that the fermion parity switch appears almost exactly at the boundary of the allowed zone. *Right*: ground state fermion parity $\langle P \rangle$ averaged over 10^4 random matrices of size $n = 6$, showing that fermion parity may only be odd only if the discrete vortex condition (26.21) is fulfilled.

simulation of a QSH dot⁹ also confirms the conclusion of Sec. 26.2.4 that quantum spin Hall insulators maximize the area in the phase space where the ground state fermion parity is odd. This behavior is in contrast with that of two-terminal setups, where Eq. (26.1) dictates that a Andreev level crossing at zero-energy may only occur in a time-reversal invariant system in the presence of a perfectly transmitted mode. The stringent requirement of perfect transparency is removed in a multiterminal setup.

In Section 26.2.3 we proved that zero-energy crossings occur only if a discrete vortex is present at the junction. For a more systematic study of the occurrence of the zero-energy crossings, we compute the average ground state fermion parity $\langle P \rangle$ as a function of ϕ_1 and ϕ_2 using RMT, with the results shown in the right panel of Fig. (102). The figure shows that the parity deviates from the even value, $\langle P \rangle = 1$, in exact agreement with the vortex condition, Eq. (26.21).

26.3.3 Density of states

We now study the properties of the complete Andreev spectrum. In the top panel of Fig. 103 we show the subgap density of states $\rho(\epsilon)$ of a Rashba dot, obtained for a single disorder realization while averaging over different values of μ in the dot. We observe several

⁹We use the parameters for an InAs/GaAs/AlSb quantum well, with layer thickness of 10 nm for both GaSb and InAs [33], in a dot with radius $R \simeq 200$ nm, onsite disorder strength $u = 25$ meV, and $\mu = 0$.

features of this density of states. First, when zero-energy crossings are forbidden an energy gap is present in the spectrum, in agreement with the lower bound of Eq. (26.20). Second, when crossings are allowed, a spectral peak develops at zero energy. Finally, at the time-reversal symmetric point $(\phi_1, \phi_2) = (\pi, \pi)$ there is no hard gap in the spectrum but the density of states vanishes at zero energy.

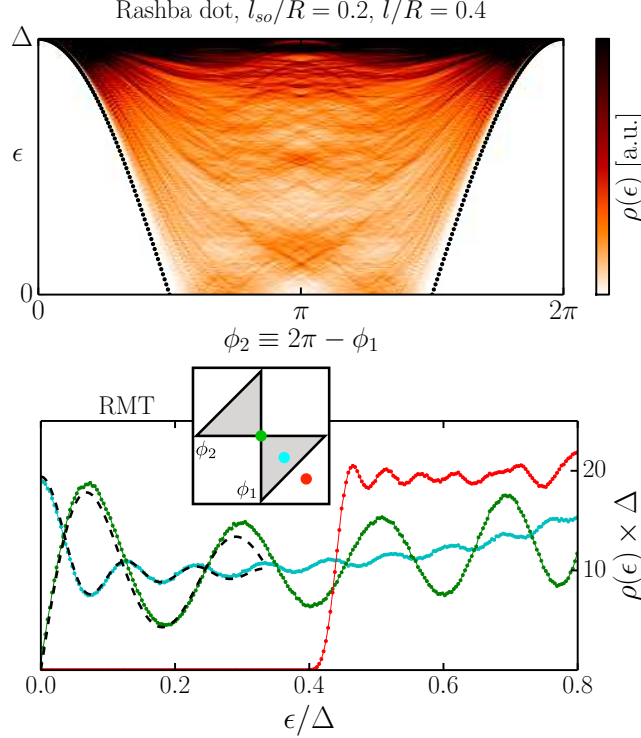


Figure 103: *Top*: density of states $\rho(\epsilon)$ of a Rashba dot, computed along the diagonal $\phi_2 = 2\pi - \phi_1$ and averaged over 10^3 values of $\mu \in [0, 1/2ma]$ for a single disorder realization. Spin-orbit coupling α and disorder strength u are the same as in Fig. 101. The dotted line shows the lower bound on the Andreev state energy (26.20). *Bottom*: Density of states obtained from 10^6 random scattering matrices with 10 modes per lead, computed for the three different values of (ϕ_1, ϕ_2) shown in the inset: in the gapped region [red, $(3\pi/4, \pi/4)$], in presence of a discrete vortex [blue, $(4\pi/3, 2\pi/3)$], and at the time-reversal invariant point [green, (π, π)]. The black dashed lines are fits of Eqs. (26.24) and (26.25), with a single free parameter δ .

The latter two features are explained by the random matrix theory of chaotic Andreev dots. The presence of a spectral peak at zero energy is expected in a chaotic superconducting dot with broken spin rotation and time-reversal symmetries (symmetry class D). In this case, the expected density of states profile is given by [16, 14, 17, 18]:

$$\rho(\epsilon) = \delta^{-1} [1 + \sin(x)/x] , \quad (26.24)$$

with $x = 2\pi\epsilon/\delta$, and δ the average level spacing at the Fermi level. At the time-reversal symmetric point (π, π) the junction has the symmetry class DIII. In this case we expect the density of states to vanish at the Fermi level [36, 14, 17], with profile

$$\rho(\epsilon) = \delta^{-1} [\pi^2 x (J_1'(x) J_0(x) + J_1^2(x)) + \pi J_1(x)] , \quad (26.25)$$

where J_0 and J_1 are Bessel functions of the first kind.

These corrections to the density of states near zero energy can be observed in our system more clearly by computing the density of states from RMT, see the bottom panel of Fig. 103. There we compare the density of states at the center of the “discrete vortex”

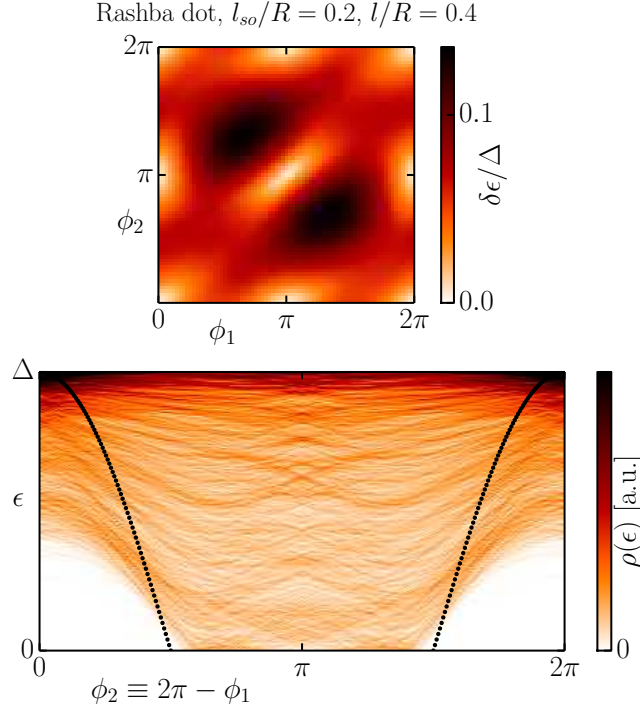


Figure 104: Spectral properties of a three-terminal junction made in a Rashba dot and with finite $\Delta = 0.01/2ma$, showing the effect of an increased size of the junction. All other parameters are as in Fig. 101. *Top*: energy difference $\delta\epsilon$ between the two lowest Andreev levels, averaged over 10 values of $\mu \in [0, 1/2ma]$. *Bottom*: density of states of the junction, obtained by averaging over 200 values of $\mu \in [0, 1/2ma]$, for a single disorder configuration and a fixed value of μ in the three arms of the junction. Black dots are the lower bound (26.20), which is valid in the limit $\Delta/E_T \rightarrow 0$.

$(\phi_1, \phi_2) = (2\pi/3, \pi/3)$ and at the time-reversal symmetric point $(\phi_1, \phi_2) = (\pi, \pi)$ to Eqs. (26.24) and (26.25) respectively, using δ as a fitting parameter. We find that close to the Fermi level the density profiles are in a good agreement with random matrix theory predictions. This result is the final confirmation that in a multiterminal short Josephson junction all the consequences of the time-reversal symmetry present in the normal state are removed in the superconducting state by the phase differences.

26.3.4 Effect of finite junction size

Most of our results are applicable in the short junction limit. If the size of the junction is increased, the short junction approximation of Eq. (26.5) gradually loses its validity. We now consider the corrections to the short junction limit. In order to do so we include the superconducting pairing explicitly in the Hamiltonian, rather than as a boundary condition for the scattering problem. We therefore compute the subgap energy spectrum by diagonalizing the Bogoliubov-de Gennes Hamiltonian

$$H_{\text{BdG}} = \begin{pmatrix} H & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H \end{pmatrix}, \quad (26.26)$$

where H is the Rashba Hamiltonian (26.23). We apply H to the geometry of Fig. 99, with $\Delta(\mathbf{r}) = 0$ in the central region and $\Delta(\mathbf{r}) = \Delta \exp(i\phi_i)$ in the three leads. We consider finite length leads, interrupted at a distance $L \gtrsim \xi$ away from the junction.

In Fig. 104 we show the results for a junction with $\Delta = 0.01/2ma$, and all other parameters the same as in Sec. 26.3.1. As expected, the subgap level spacing and hence the energy splitting

of Kramers pairs are reduced in a longer junction. In particular, the energy splitting of Kramers pair remains finite when two phases in the leads are equal and it only vanishes at time-reversal invariant points. The lower bound (26.20) on the energy gap ceases to be valid, as can be seen already from the presence of subgap states at zero phase difference. Nevertheless, in agreement with our expectations, the vortex condition (26.21) for a zero-energy crossing remains valid.

26.4 Conclusions and discussion

In conclusion, we have introduced a new method of manipulation of single electron states, which relies solely on applying the superconducting phase differences. This approach has several advantages over the standard ways that rely on the direct application of magnetic fields. It allows one to manipulate electron spin locally both in space and time, and to implement long range spin-spin coupling by using inductive coupling of the supercurrents. Finally, it is not disruptive to superconductivity, making it ideal to apply to hybrid devices.

We demonstrated that, unlike in two terminal Josephson junctions, superconducting phase difference can induce splitting of the Kramers degeneracy in the spectrum comparable to the superconducting gap when more than two superconducting leads are used. We proved that there is a universal lower bound on the induced gap in the junction, which only depends on the phases of different terminals. This lower bound vanishes when the phases of the superconducting leads form a discrete vortex. In that case the ground state fermion parity is allowed to become odd, so that the junction traps an extra fermion in its ground state.

Our findings can be directly tested experimentally using tunneling spectroscopy. This requires adding an extra normal or superconducting lead weakly coupled to the scattering region, and performing voltage bias conductance measurements. The Andreev excitation spectrum of a Josephson junction has also been studied experimentally using microwave absorption spectroscopy [37, 19] or measuring switching current probabilities [48, 10]. Either of these two methods will likewise permit to test our predictions, since both methods are equally applicable to multiterminal junctions.

We expect our results to be testable for junctions defined in any material with a sufficiently strong spin-orbit interaction. Our method of breaking Kramers degeneracy works best in materials with low effective electron mass, since that ensures large normal level spacing. For instance, for an InAs quantum dot with a radius $R \simeq 100$ nm we estimate a level spacing $\delta_0 \simeq \hbar^2 \pi^2 / 8m_{\text{eff}} R^2 = 0.5$ meV in the normal state, thus making the short junction limit $\Delta \ll n\delta_0$ within easy reach in the case of aluminum contacts. In addition to the natural candidates such as InAs, InSb quantum wells, or quantum spin Hall insulators, the recently discovered InSb nanocrosses [26] make a promising candidate for observing the physics of multiterminal SNS junction. Conventional metallic SNS junctions would not show the effects of time-reversal symmetry breaking due to the extremely small level spacing. However, superconducting break junctions [48] could potentially permit the implementation of multiterminal geometries involving a very small number of modes with a large level spacing.

There is an entirely different aspect of broken time-reversal and spin rotation symmetries in mesoscopic systems, which is beyond the scope of our investigation, but which can also be studied using our methods. If the scattering region is additionally strongly coupled to a normal lead, a persistent zero-bias peak in the Andreev conductance is formed [42, 43, 44]. In our case, we expect such a peak to develop in the presence of a discrete vortex, and to disappear in its absence.

Another venue of further investigation is to study the quantum nature of the Andreev bound states. Trapping a single Bogoliubov quasiparticle in a Josephson junction is a promising way to isolate and manipulate a spin degree of freedom - a superconducting spin qubit [6, 7, 13]. A spin- $\frac{1}{2}$ state in a Josephson junction is expected to be very stable at low temperatures, due to the

energy gap of the superconductor. These long-lived odd states have been recently observed via switching current measurements in superconducting point contacts [48, 10, 45]. The advantage of using multiterminal Josephson junctions for such qubits is that the presence of several tunable phase differences makes it possible to implement universal quantum manipulation exclusively by inductive means.

Finally, our discovery provides a better way to creating Majorana bound states in superconductor-semiconductor hybrid systems, a focus of an active experimental search [30, 15, 48, 9, 50, 51, 52]. The complication that arises in many experiments is that magnetic field required to induce a non-trivial gap in the semiconductor is too strong and spoils the properties of the superconductor. Using superconducting phases as a means of breaking time reversal symmetry and Kramers degeneracy would allow one to reach the same goal without any detrimental effect on the superconductor. Potentially it would even allow one to use aluminum, which forms high quality contacts with semiconductors and is the simplest superconducting material to use in fabrication, and whose application to Majoranas was so far limited by its extremely small critical field. One promising use of our method for creation of Majoranas is to combine multiple superconducting leads with an engineered Kitaev chain geometry of Refs. [53, 54, 55].

26.5 Occurrence of a zero-energy crossing as a generalized eigenvalue problem

Given the scattering matrices s and r_A , it is possible to determine whether zero-energy solutions exist in the (ϕ_1, ϕ_2) plane without solving for the spectrum. To do so, we can recast Eq. (26.10) at $\epsilon = 0$ as a generalized eigenvalue problem of the form

$$X \Psi_{\text{in}}^e = e^{-i\phi_1} Y \Psi_{\text{in}}^e \quad (26.27)$$

We give the explicit form of X and Y in the case of three leads. If s has the following block structure,

$$s = \begin{pmatrix} r_{11} & t_{12} & t_{13} \\ -t_{12}^T & r_{22} & t_{23} \\ -t_{13}^T & -t_{23}^T & r_{33} \end{pmatrix}, \quad (26.28)$$

then X and Y are given by

$$\begin{aligned} X &= \begin{pmatrix} 0 & -e^{-i\phi_2} t_{12} & -t_{13} \\ e^{-i\phi_2} t_{12}^T & 2e^{-i\phi_2} r_{22} & -(1 + e^{-i\phi_2}) t_{23} \\ t_{13}^T & (1 + e^{-i\phi_2}) t_{23}^T & 2r_{33} \end{pmatrix}, \\ Y &= \begin{pmatrix} 2r_{11} & t_{12} & t_{13} \\ -t_{12}^T & 0 & 0 \\ -t_{13}^T & 0 & 0 \end{pmatrix}. \end{aligned} \quad (26.29)$$

The existence of a zero-energy crossing at the position (ϕ_1, ϕ_2) can then be determined numerically by checking that Eq. (26.27) has eigenvalues with unit norm.

26.6 BHZ Hamiltonian

The BHZ Hamiltonian describing a 2D quantum spin Hall insulator reads [32]:

$$H_{\text{BHZ}} = H_0 + H_{\text{BIA}} + H_{\text{SIA}} + V(\mathbf{r}), \quad (26.30)$$

with $V(\mathbf{r})$ the electrostatic disorder, and

$$H_0 = \begin{pmatrix} h(\mathbf{p}) & 0 \\ 0 & h^*(-\mathbf{p}) \end{pmatrix}, \quad (26.31a)$$

$$H_{\text{BIA}} = \begin{pmatrix} 0 & 0 & \Delta_e p_+ & -\Delta_z \\ 0 & 0 & \Delta_z & \Delta_h p_- \\ \Delta_e p_- & \Delta_z & 0 & 0 \\ -\Delta_z & \Delta_h p_+ & 0 & 0 \end{pmatrix}, \quad (26.31b)$$

$$H_{\text{SIA}} = \begin{pmatrix} 0 & 0 & i\xi_e p_- & 0 \\ 0 & 0 & 0 & 0 \\ -i\xi_e^* p_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (26.31c)$$

and

$$h(\mathbf{p}) = (C - D\mathbf{p}^2)\sigma_0 + A(p_x\sigma_x - p_y\sigma_y) + (M - B\mathbf{p}^2)\sigma_z.$$

Here, σ are the Pauli matrices in orbital space, $\mathbf{p}$ is the momentum operator, and $p_{\pm} = p_x \pm ip_y$. The system is in a topologically nontrivial phase whenever $M < 0$.

27 Energy spectrum and current-phase relation of a nanowire Josephson junction close to the topological transition by Murthy, Kurilovich, Kurilovich, van Heck, Glazman, Nayak

Abstract

A semiconducting nanowire proximitized by an s -wave superconductor can be tuned into a topological state by an applied magnetic field. This quantum phase transition is marked by the emergence of Majorana zero modes at the ends of the wire. The fusion of Majorana modes at a junction between two nanowires results in a 4π -periodic Josephson effect. We elucidate how the 4π -periodicity arises across the topological phase transition in a highly-transparent short nanowire junction. Owing to a high transmission coefficient, Majorana zero modes coming from different wires are strongly coupled, with an energy scale set by the proximity-induced, field-independent pairing potential. At the same time, the topological spectral gap—defined by competition between superconducting correlations and Zeeman splitting—becomes narrow in the vicinity of the transition point. The resulting hybridization of the fused Majorana states with the spectral continuum strongly affects the electron density of states at the junction and its Josephson energy. We study the manifestations of this hybridization in the energy spectrum and phase dependence of the Josephson current. We pinpoint the experimentally observable signatures of the topological phase transition, focusing on junctions with weak backscattering.

27.1 Summary and conclusions

The Josephson effect is one of the most important probes of a superconductor since it directly measures the order parameter. It can also be used to distinguish topological superconductors from conventional ones, but the distinction is smeared by fluctuations of the junction's fermion parity caused by unpaired fermions in the bulk, such as those induced by disorder,

quasiparticle poisoning, or finite length of the wire. In this paper, we have shown that, even in the absence of such fermion parity fluctuations, the behavior of a highly-transparent Josephson junction is quite subtle near the magnetic-field-driven topological transition in a proximitized semiconducting nanowire. A qualitative distinction remains between the topological ($B > B_c$, where $B_c = \Delta$ in our model) and non-topological ($B < B_c$) phases: the ground state energy and the Josephson current, $I(\phi)$, are 2π -periodic in the latter case and 4π -periodic in the former. However, the 4π -periodicity of the Josephson current just above the transition emerges as a shift of the zero-crossing in $I(\phi)$ away from $\phi = \pi$ by a small amount $\propto (B - B_c)/B_c$. Consequently, the 4π -periodic component of $I(\phi)$ develops gradually on the background of a larger 2π -periodic component as the Zeeman energy B is increased past B_c . For a multi-channel junction, the 4π -periodic component of the Josephson current will be accompanied by an even larger 2π -periodic component due to the non-topological bands in the wire, further masking the topological phase in the vicinity of the transition.

Nevertheless, there is an unusual and robust signature of the topological transition in a nanowire Josephson junction at high transmission. In the topological state, the excitation spectrum (within a fixed parity sector) is gapless over a range of phase differences, $|\phi| > \pi + \phi_0^{\text{sc}}$. This feature survives in the presence of weak scattering. The gaplessness arises because, in that range of ϕ , the bulk gap is smaller than the coupling between the two Majorana modes at the junction. Consequently, the system lowers its energy by changing the fermion parity of the coupled Majorana modes (relative to their parity at $|\phi| < \pi + \phi_0^{\text{sc}}$) and creating an above-gap fermionic excitation. An additional even number of channels cannot change this conclusion: it is possible to flip the parities of channels in pairs, but there will always be one channel left over that cannot change its parity to minimize the energy without creating an above-gap excitation.

The gaplessness has observable implications for the Josephson current. It leads to kinks in the current-phase relation at the boundaries of the gapless domains, $\phi = \pm(\pi + \phi_0^{\text{sc}})$. The presence of kinks in the topological phase is in contrast to the smooth $I(\phi)$ dependence in the non-topological regime.

Finally, consistent with the notion of a phase transition, the ground state energy and critical current are non-analytic functions of $(B - B_c)$; we find a non-analytic contribution $\propto (B_c - B) \ln(B_c/|B_c - B|)$ to $I(\phi)$.

Some signatures of the 4π -periodic Josephson effect have so far been reported in dynamic properties of the junction, such as Shapiro steps [24] or the microwave emission spectrum [25]. However, the results in this paper show that a large 2π -periodic component is expected above but near the topological phase transition even for a single-channel junction. This further complicates the interpretation of these experiments, which is already non-trivial due to quasiparticle poisoning. We hope that our work will encourage further experimental activity aimed at detecting the critical point itself, in conjunction with the onset of Majorana signatures [26, 27, 28, 29]. Motivated by recent and exciting progress in the microwave spectroscopy of Andreev bound states [30, 31, 32, 23, 33], we will further investigate the signatures of the transition on the microwave absorption spectrum [34, 35] in an upcoming work. Finally, the extension of the above findings to the case of finite voltage or current bias is another interesting direction for future research.

27.2 Introduction

Single-mode semiconductor nanowires proximitized by a conventional s -wave superconductor have emerged as a leading candidate for the implementation of topological quantum computing [1, 2, 3, 4]. Due to the spin-orbit coupling in the wire, the two Kramers doublets at the Fermi energy, $\pm k_{\text{out}}$ and $\pm k_{\text{in}}$, respectively, are separated in momentum space. The supercon-

ducting proximity effect acts individually on each of the two doublets, inducing superconducting pairing gaps. The resulting state is topologically trivial. In a magnetic field, however, pairing competes with spin polarization induced by the Zeeman effect. A non-trivial topological state is formed if the Zeeman effect wins for one of the doublets. For definiteness, we concentrate on the most favorable point for the formation of a topologically non-trivial state, $k_{\text{in}} = 0$, achievable at a specific value of the Fermi energy. We also assume that the spin-orbit coupling is strong. In this case, a homogeneous magnetic field parallel to the wire induces a Zeeman splitting of the $k_{\text{in}} = 0$ “inner” doublet, while having little effect on the $\pm k_{\text{out}}$ “outer” doublet. Under these conditions, only the Cooper pairs belonging to the $\pm k_{\text{out}}$ helical modes remain intact, giving rise to a topological superconducting state which supports a Majorana zero mode (MZM) at each end of the wire.

Quantum computing operations require controllable fusion of pairs of MZMs belonging to different wires (or to different proximitized portions of the same wire) into a single Dirac fermion [5, 6]. The energy of the resulting fused state is not fixed at zero and depends on the strength and phase ϕ of the Josephson coupling between the wires. If each wire carried only the $\pm k_{\text{out}}$ helical mode, this dependence would consist of a single 4π harmonic with amplitude proportional to the transmission amplitude, $\sqrt{D}$, of the junction [7]. The fused state would be localized at the junction.

The peculiarity of the topological phase transition in a proximitized nanowire is that the natural energy scale of the coupling between the MZMs remains large (of the order of the proximity-induced pairing potential) even while the spectral gap closes at the transition. This occurs because the MZMs are formed out of the $\pm k_{\text{out}}$ helical states, while the smallest gap in the spectral continuum lies within the states adjacent to the $k_{\text{in}} = 0$ momentum. The presence of a continuum with a narrow gap strongly modifies the Dirac fermion formed by fusion at the junction. This modification affects the density of states at the junction and results in an unusual current-phase relation of the Josephson effect.

In this work, we investigate the energy spectrum and zero-temperature thermodynamic properties of the junction as it is tuned through the topological phase transition. We focus on the experimentally important case of short, almost-transparent junctions ($1 - D \ll 1$). Due to the emphasis on the topological phase transition itself, our work complements previous studies of the 4π -periodic Josephson effect in proximitized nanowires [2, 8, 9, 10, 11, 12, 13, 14, 15], and provides guidance for experiments aimed at detecting the onset of the topological phase.

27.2.0 Summary of results

In the absence of backscattering ($D = 1$), a junction does not mix the states near k_{in} with those near $\pm k_{\text{out}}$, which we will refer to as the inner and outer modes respectively, at any ϕ . If the induced gap Δ exceeds the Zeeman energy B for the inner modes, the wires are in a topologically trivial state. At finite ϕ , both inner and outer modes carry an Andreev bound state localized at the junction. These states cross zero energy and simultaneously change their ground state occupation once the phase crosses $\phi = \pm\pi$ (due to periodicity, it is sufficient to consider an interval $\phi \in [-2\pi, 2\pi]$). Such a change of the occupation is allowed, as it does not violate fermion parity conservation. The discontinuous change in the ground state leads to discontinuities of the Josephson current $I(\phi)$ at $\phi = \pm\pi$. At the critical value of the magnetic field, $B = \Delta$, the gap in the inner modes closes and then reopens at $B > \Delta$ without the bound state. The remaining bound state (associated with the outer modes) does cross zero energy at $\phi = \pm\pi$, but this time its occupation cannot change, due to fermion parity conservation. The occupation of the bound state may change only at larger phases, $\phi = \pm(\pi + \phi_0)$, at which placing a quasiparticle in the continuum above the gap in the inner-mode spectrum is energetically favorable. The peculiarity of the resulting ground state is that it contains a

quasiparticle at the edge of the continuum, and therefore is not separated by a gap from the excited states.

The appearance of $\phi_0 \neq 0$ at $B > \Delta$ signals that the properties of the junction are 4π -periodic in the topological phase. While the change of the periodicity in ϕ at $B = \Delta$ is associated with a dramatic qualitative change of the eigenstates, the quantitative changes in observables—such as the Josephson current—are more subtle. Indeed, the inner-mode spectral gap scales linearly with $|B - \Delta|$ and is small near the transition. Therefore the discontinuities in $I(\phi)$ shift from the points $\pm\pi$ by small amounts, $\phi_0 \propto (B - \Delta)\Theta(B - \Delta)$.

The effect of weak scattering on the energy spectrum and the Josephson current is different—at the qualitative as well as the quantitative level—on the two sides of the topological transition. In the topologically trivial state, backscattering couples the bound states of the inner and outer modes. As a result, the bound state energies are repelled from zero in the vicinity of $\phi = \pm\pi$. This leads to the smearing of the discontinuity in the Josephson current over a phase interval $\delta\phi_{B<\Delta} \propto \sqrt{1-D}$, similar to the standard case of a short SNS junction [16]. In the topological state the zero energy crossings at $\phi = \pm\pi$ are protected by fermion parity conservation and are thus unaffected by the scattering. At the same time, even weak scattering substantially alters the bound state energy once it approaches the edge of the inner-mode continuum. Hybridization between the bound state and the continuum states smears the discontinuity in the Josephson current over $\delta\phi_{B>\Delta} \propto (1-D)^2$. Thus on the topological side of the transition the smearing is weak compared to that on the trivial side. Furthermore, we find another spectroscopic feature of the topological transition: in the vicinity of $\phi = 0, \pm 2\pi$, weak scattering peels off shallow Andreev bound states from the continuum. These states appear only in the topologically non-trivial phase, and can thus serve as an additional signature of the transition.

Throughout the paper, we pay particular attention to the contribution of the continuous part of the spectrum to the Josephson properties. In the short junction limit, the continuum contribution to the ground state energy and to the Josephson current vanishes at zero magnetic field, as is well known, but it becomes nonzero at finite magnetic field. This contribution is 2π -periodic in both the trivial and topological phases. Yet, it carries an imprint of the transition in the form of a non-analytic dependence on the magnetic field close to the critical point.

The manuscript is structured as follows. In Section 27.3, we introduce and motivate the model of a proximitized wire used in the rest of the work. In Section 27.4, we study the properties of a perfectly transparent junction. We deal with scattering at the junction in Section 27.5, paying particular attention to the coupling it induces between the bound states and the continuum in the topological phase. We conclude in Section 27.1 with a few remarks about experiments and future directions of research. Technical calculations are left as Appendices. Throughout the work, we use units with $\hbar = 1$.

27.3 The model

We consider a Josephson junction in a proximitized nanowire with strong Rashba spin-orbit coupling. The wire is placed in a magnetic field parallel to its axis. We assume that the orbital effect of the field can be neglected, and account only for the Zeeman effect of the field. We also assume that the junction is short, i.e., its length ℓ is much smaller than the superconducting coherence length ξ (see Fig. 105 for an illustration of the setup). To model this system, we start with the mean-field many-body Hamiltonian, which takes the form [3, 2]

$$\hat{H} = \frac{1}{2} \int dx \hat{\Psi}^\dagger(x) \mathcal{H} \hat{\Psi}(x), \quad (27.1)$$

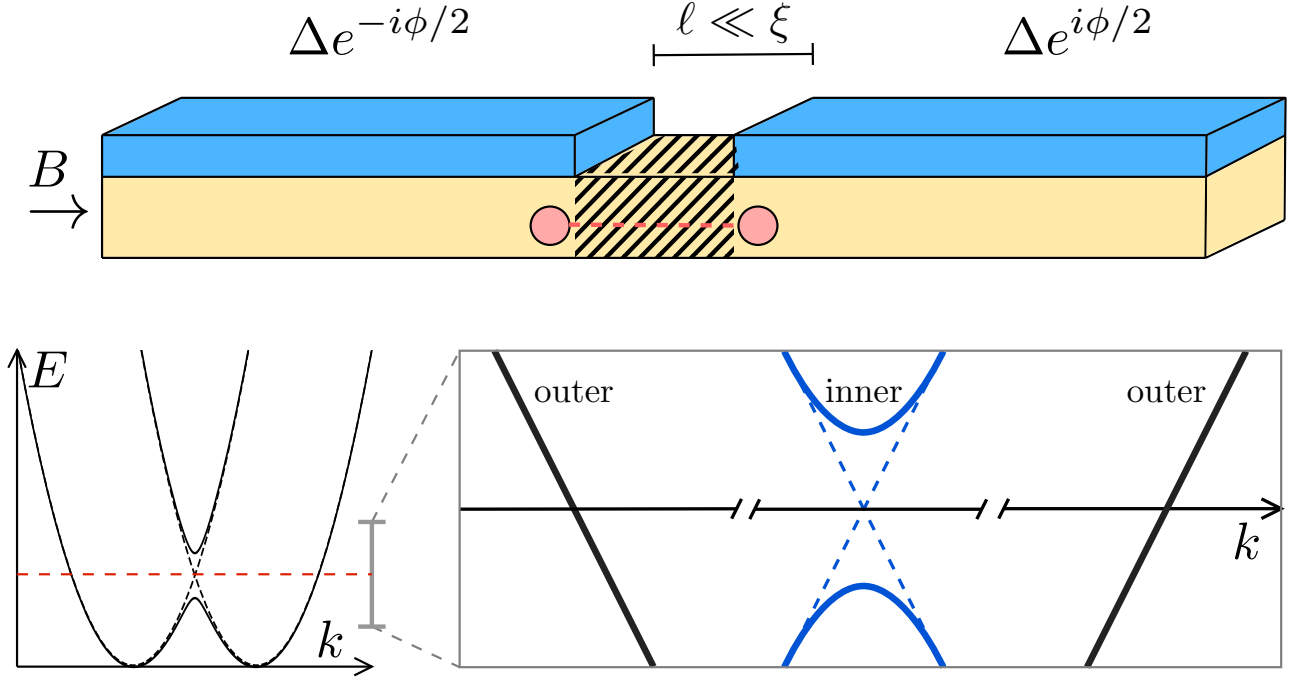


Figure 105: *Top*: Schematic layout of the system under study: a Josephson junction (shaded yellow) formed by a semiconducting wire (yellow) proximitized by an s -wave superconductor (blue). Upon applying a parallel magnetic field, Majorana zero modes may form (red dots) and couple (red dashed line) at the junction. *Bottom*: Bandstructure of a single-subband Rashba wire with an applied magnetic field. If the Fermi level is placed in the middle of the Zeeman gap (dashed red line), the low-energy linearized theory consists of a pair of inner modes and a pair of outer modes. Induced s -wave superconducting pairing acts within each pair of modes, while scattering at the junction couples inner and outer modes.

where $\hat{\Psi} = (\hat{\psi}_{\uparrow}, \hat{\psi}_{\downarrow}, \hat{\psi}_{\downarrow}^{\dagger}, -\hat{\psi}_{\uparrow}^{\dagger})^T$, $\hat{\psi}_{\sigma}$ is the annihilation operator of an electron with spin $\sigma = \uparrow$ or $\downarrow$, and the Bogoliubov-de Gennes (BdG) Hamiltonian is given by

$$\mathcal{H} = \left[-\frac{\partial_x^2}{2m} - i\alpha\partial_x\sigma_z - \mu + V(x) \right] \tau_z - B\sigma_x + \Delta(x)\tau_x. \quad (27.2)$$

Here σ_i and τ_i are Pauli matrices in spin and Nambu space, respectively; m is the effective mass, α is the spin-orbit coupling constant, μ is the chemical potential, B is the Zeeman energy (without loss of generality we take $B \geq 0$), and $\Delta(x)$ is the proximity-induced superconducting order parameter:

$$\Delta(x) = \Delta e^{i(\phi/2)\text{sgn}(x)\tau_z}, \quad (27.3)$$

where ϕ is the phase difference across the junction. The potential $V(x)$ accounts for a barrier that scatters electrons at the junction. Instead of specifying a particular functional form for $V(x)$, we will, in the following, account for its effect by imposing suitable boundary conditions at $x = 0$; these boundary conditions will be formulated directly in terms of the mesoscopic scattering parameters (transmission probability and scattering phases) of the junction in the normal state.

We study the evolution of the junction properties as the Hamiltonian is tuned across the topological phase transition by changing the magnetic field. In the model defined by Eq. (27.2), the transition from the topologically trivial ($B < B_c$) to non-trivial ($B > B_c$) phase happens at $B_c = (\Delta^2 + \mu^2)^{1/2}$ [3]. Throughout this work, we assume, for simplicity, that the chemical potential is set at the optimal point $\mu = 0$ where the critical field is minimal, $B_c = \Delta$.

Following Ref. [17], we further assume that the spin-orbit coupling is strong, $m\alpha^2 \gg \Delta, B$. The latter condition implies that the low-energy spectrum in the bulk of the nanowire consists of well-separated modes in the vicinity of the Fermi points $k = k_{\text{in}} = 0$ (the inner modes) and $k = \pm k_{\text{out}} = \pm 2m\alpha$ (the outer modes). On the technical level, the condition $m\alpha^2 \gg \Delta, B$ allows us to invoke the Andreev approximation and to expand the field operator $\hat{\Psi}$ into helical components involving modes close to the momenta k_{in} and $\pm k_{\text{out}}$ [18, 17]:

$$\hat{\Psi}(x) = e^{-im\alpha x(1+\sigma_z)} \hat{\Psi}_L(x) + e^{im\alpha x(1-\sigma_z)} \hat{\Psi}_R(x). \quad (27.4)$$

Here $\hat{\Psi}_L$ and $\hat{\Psi}_R$ denote left- and right-moving components of the field, which both have Fermi velocity given by α . Then, by inserting the decomposition (27.4) into Eq. (27.1) and averaging out rapidly oscillating terms, we obtain a set of BdG equations for the inner (Φ_i) and outer (Φ_o) mode wave functions at $x \neq 0$,

$$[-i\alpha\partial_x\tau_z\sigma_z - B\sigma_x + \Delta(x)\tau_x]\Phi_i(x) = E\Phi_i(x), \quad (27.5a)$$

$$[+i\alpha\partial_x\tau_z\sigma_z + \Delta(x)\tau_x]\Phi_o(x) = E\Phi_o(x). \quad (27.5b)$$

Notice that the Zeeman energy drops out from the BdG equations for the outer modes. This is a consequence of a large energy separation $\sim m\alpha^2 \gg B$ between spin subbands in the vicinity of $k = \pm k_{\text{out}} = \pm 2m\alpha$.

As discussed above, Eqs. (27.5) should be supplemented by a suitable boundary condition at $x = 0$. Within the Andreev approximation the boundary condition is completely determined by the scattering matrix of the junction in the normal state ($\Delta = B = 0$). Under the assumption that the scattering is spin-independent at $\Delta, B = 0$, the most general boundary condition is:

$$\Phi_i(0^+) = \frac{e^{i\gamma\sigma_z}}{\sqrt{D}} \left[\Phi_i(0^-) + e^{-i\delta\sigma_z} \sqrt{1-D} \Phi_o(0^-) \right], \quad (27.6a)$$

$$\Phi_o(0^+) = \frac{e^{-i\gamma\sigma_z}}{\sqrt{D}} \left[\Phi_o(0^-) + e^{i\delta\sigma_z} \sqrt{1-D} \Phi_i(0^-) \right]. \quad (27.6b)$$

Here, D is the normal-state transmission probability, $\gamma \in [-\pi/2, \pi/2]$ is the forward scattering phase in the normal state, and δ is the reflection phase in the normal state for a particle incoming from $x = -\infty$. Notice that the reflection phase δ can be eliminated from Eqs. (27.5), (27.6) by a unitary transformation $\Phi_o \rightarrow e^{i\delta\sigma_z} \Phi_o$. Therefore, we suppress it in the following discussions. The independence of the properties of the junction on δ is a consequence of the $m\alpha^2 \gg B, \Delta$ approximation, in which the states in the outer modes are insensitive to the magnetic field. Throughout this work, we assume that D and γ are independent of energy up to the relevant scales $|E| \sim B, \Delta$.

For future reference, we mention that in the particular case of a (repulsive) delta function barrier, $V(x) = \kappa \delta(x)$, the parameters D and γ are given by

$$D = \frac{1}{1 + (\kappa/\alpha)^2}, \quad \gamma = -\arctan \sqrt{\frac{1-D}{D}}. \quad (27.7)$$

However, in general, there is no rigid connection between D and γ like the one provided by Eq. (27.7). We treat them as independent parameters in what follows.

In the next section (Sec. 27.4) we study the spectrum of the junction and its thermodynamic properties (at $T = 0$) in the limit of perfect transmission, $D = 1$ and $\gamma = 0$. Then in Sec. 27.5 we discuss the effects induced by scattering at the junction, $D < 1$ and $\gamma \neq 0$.

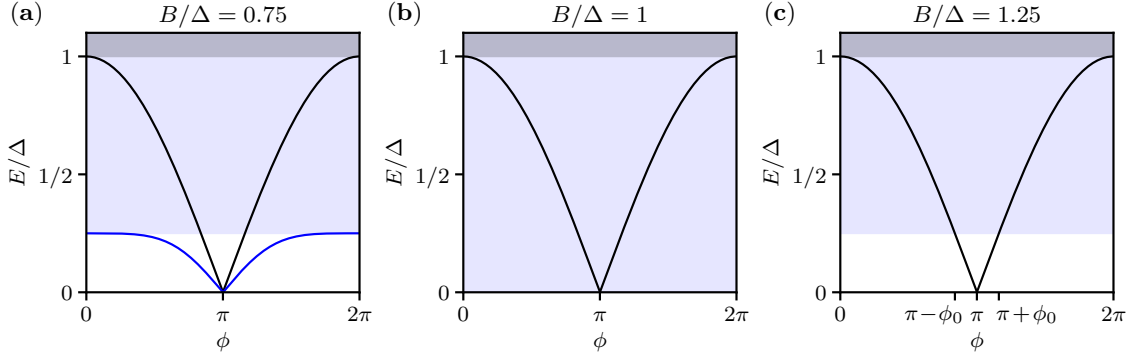


Figure 106: Evolution of the single-particle excitation spectrum of the junction with increasing Zeeman energy, in the limit of perfect transmission, $D = 1$, $\gamma = 0$. Solid curves represent the discrete energy levels. Shaded regions mark the continuous spectrum. The latter consists only of inner modes for $E < \Delta$, and of both inner and outer modes for $E \geq \Delta$. In the trivial phase, $B < \Delta$, there are bound states in the inner and outer modes (black and blue curves in panel (a), respectively). At $B = \Delta$ the gap in the inner modes closes and the corresponding bound state vanishes, see panel (b). In the topological phase, $B > \Delta$, only the outer-mode bound state remains [panel (c)]. As long as $B < 2\Delta$ the energy of the bound state in the outer modes overlaps with the inner-mode continuum in some domain of phase ϕ .

27.4 Spectrum and thermodynamic properties of a transparent junction

If $D = 1$, the inner and outer modes decouple from each other, as follows from Eq. (27.6). This simplification allows us to find a compact analytical solution of the BdG equations (27.5) at $\gamma = 0$ via a standard wave function matching procedure. The energy spectrum consists of a discrete part formed by Andreev bound states localized at the junction and a continuous part formed by extended states at energies above the gaps in their respective modes. In the outer modes the gap equals Δ , independent of B , while in the inner modes the gap is $|\Delta - B|$; it closes at the topological transition, $B = \Delta$.

27.4.1 Bound states at perfect transmission

We start by considering the bound state in the outer modes. In this case, the BdG equations are similar to those of a proximitized helical edge in a quantum spin Hall state [7]. Eq. (27.5b) decouples for left-movers ($\sigma_z = +1$) and right-movers ($\sigma_z = -1$). By matching the wavefunctions continuously across the junction (as required by Eq. (27.6) with $D = 1$, $\gamma = 0$), we obtain the equations for the bound state energies in the form $\Lambda_{o,\sigma}(E, \phi) = 0$, where

$$\Lambda_{o,\sigma}(E, \phi) = \sin[\sigma\eta(E) - \phi/2], \quad (27.8)$$

with $e^{i\eta(E)} = E/\Delta + i\sqrt{1 - (E/\Delta)^2}$ and $\sigma = \pm 1$ defining the eigenvalue of σ_z . Eq. (27.8) results in the *non-degenerate* bound state solution $E = \pm E_0(\phi)$ with

$$E_0(\phi) = \Delta \cos(\phi/2), \quad (27.9)$$

independent of B .

For the inner modes, solving the BdG equations is more cumbersome due to the presence of the Zeeman term in Eq. (27.5a). Bound states may appear at $|E| < |\Delta - B|$. The bound

state energy is a solution of the equation $\Lambda_i(E, \phi) = 0$, with (see Appendix 27.6.1)

$$\Lambda_i(E, \phi) = [(\Delta - B)^2 - E^2]^{1/2}[(\Delta + B)^2 - E^2]^{1/2} - (E^2 + \Delta^2 - B^2)F(\phi). \quad (27.10)$$

The phase dependence of this expression is encoded in the function

$$F(\phi) \equiv \frac{\sin^2(\phi/2)}{\cos^2(\phi/2) + 1}. \quad (27.11)$$

The equation $\Lambda_i(E, \phi) = 0$ admits solutions only in the topologically trivial phase, $B < \Delta$. Indeed, at $B > \Delta$ and $|E| < |\Delta - B|$, the quantity $(E^2 + \Delta^2 - B^2)$ in the second line of Eq. (27.10) is negative, and hence $\Lambda_i(E, \phi) > 0$. The bound state solution at $B < \Delta$ has energy $E = \pm E_1(\phi)$ with

$$E_1(\phi) = \frac{\Delta - \sqrt{\Delta^2 - (\Delta^2 - B^2)[1 - F^2(\phi)]}}{\text{sgn}[\cos(\phi/2)]\sqrt{1 - F^2(\phi)}}. \quad (27.12)$$

This expression has several notable features. First, $E_1(\phi) = E_0(\phi)$ at $B = 0$, i.e., the Andreev spectrum is two-fold degenerate in the absence of the magnetic field. Second, $E_1(\phi)$ crosses zero at $\phi = \pi$, simultaneously with the energy of the outer-mode bound state $E_0(\phi)$. The presence of such a crossing is a peculiarity of the perfect transmission limit, which is not robust to the presence of backscattering. Third, the energy $|E_1(\phi)|$ is bounded by $\Delta - B$, so its phase dispersion is suppressed by the magnetic field, until the bound state merges with the continuum at $B = \Delta$, in concurrence with the topological phase transition.

In the topological phase, $B > \Delta$, the bound state is no longer present in the inner modes; only the outer-mode bound state, which can be considered as two fused Majorana modes, remains at the junction. For $B < 2\Delta$ the energy of the latter, E_0 , overlaps with the inner-mode continuum, intersecting its edge at $\phi = \pi \pm \phi_0$, where

$$\phi_0 = 2 \arcsin\left(\frac{B - \Delta}{\Delta}\right). \quad (27.13)$$

This coexistence of a bound state with the continuous part of the spectrum is another peculiarity of the perfect transmission limit. The evolution of the Andreev spectrum upon increasing Zeeman energy is shown in Fig. 106.

To conclude this section, we note that for $B > 2\Delta$ the gap in the inner modes, $B - \Delta$, exceeds the gap in the outer modes, Δ . Consequently, the bound state energy E_0 is separated from the continuum at all phases (except at the points $\phi = 0, 2\pi$). In this high-magnetic field topological regime, the low-energy spectrum of the model becomes identical to that of the Fu-Kane model of a proximitized quantum spin Hall edge. Then, on a qualitative level the properties of the junction at $B > 2\Delta$ are similar to those described in Ref. [7]. In what follows we concentrate instead on the vicinity of the topological transition, $B < 2\Delta$.

27.4.2 Continuum states at perfect transmission

The continuous part of the Andreev spectrum consists of scattering states that appear at energies above the threshold values $|\Delta - B|$, Δ , $\Delta + B$. The total density of continuum states at energy E , $\rho(E, \phi)$, can be represented in the form

$$\rho(E, \phi) = L g(E) + \delta\rho(E, \phi). \quad (27.14)$$

Here L is the system size, $g(E)$ is the bulk density of states per unit length, and $\delta\rho(E, \phi)$ is the correction to the density of states due to the presence of the junction. $g(E)$ is given by a sum of three BCS-like terms with gap parameters determined by the threshold energies:

$$g(E) = g_{|\Delta - B|}(E) + 2g_\Delta(E) + g_{\Delta + B}(E), \quad (27.15)$$

where

$$g_\varepsilon(E) \equiv \Theta(E - \varepsilon) \frac{E}{\pi \sqrt{E^2 - \varepsilon^2}} \quad (27.16)$$

and $\Theta(x)$ is the Heaviside step function. The phase-dependent contribution $\delta\rho(E, \phi)$ can be recovered from the quasiparticle scattering matrix of the junction, $S(E, \phi)$, via the relation [19, 20]

$$\delta\rho(E, \phi) = \frac{1}{2\pi i} \frac{\partial}{\partial E} \ln \det S(E, \phi). \quad (27.17)$$

We note that at finite ϕ the scattering matrix is non-trivial even in the perfect transmission limit $D = 1$, $\gamma = 0$. For $|\Delta - B| < E < \Delta$ the scattering states reside in the inner modes only. In this case, the determinant of the scattering matrix $S \equiv S_i$ is given by (see Appendix 27.6.2 for details)

$$\det S_i(E, \phi) = \frac{[\Lambda_i(E, \phi)]^*}{\Lambda_i(E, \phi)}, \quad (27.18)$$

where $*$ denotes complex conjugation, and where the function Λ_i , defined in Eq. (27.10), should be analytically continued from the range $|E| < |\Delta - B|$ into the interval $E \in (|\Delta - B|, \Delta)$ by taking $[(\Delta - B)^2 - E^2]^{1/2} \rightarrow -i[E^2 - (\Delta - B)^2]^{1/2}$.

For larger energies, $E > \Delta$, there are scattering states both in the inner and in the outer modes. Owing to perfect transmission, the scattering matrix is block-diagonal in these subspaces. Therefore, its determinant is multiplicative, $\det S = (\det S_i) \cdot (\det S_{o,+}) \cdot (\det S_{o,-})$, where $\det S_i$ is given by Eq. (27.18) and $\det S_{o,\sigma}$ by

$$\det S_{o,\sigma}(E, \phi) = \frac{[\Lambda_{o,\sigma}(E, \phi)]^*}{\Lambda_{o,\sigma}(E, \phi)}. \quad (27.19)$$

From Eq. (27.8) it follows that $\Lambda_{o,+} = (\Lambda_{o,-})^*$ at $E > \Delta$. Therefore, $(\det S_{o,+}) \cdot (\det S_{o,-}) = 1$ and the outer-mode continuum states do not contribute to $\delta\rho(E, \phi)$. As a result, at all energies $E > |\Delta - B|$ the density of states correction is given by $\delta\rho(E, \phi) = (2\pi i)^{-1} \partial_E \ln \det S_i$. Explicit calculation for $|\Delta - B| < E < \Delta + B$ yields

$$\delta\rho = \frac{\partial}{\partial E} \frac{1}{\pi} \arctan \frac{(E^2 + \Delta^2 - B^2)F(\phi)}{\sqrt{[E^2 - (\Delta - B)^2][(\Delta + B)^2 - E^2]}}, \quad (27.20)$$

where $F(\phi)$ is defined in Eq. (27.11). The function $\Lambda_i(E, \phi)$ is real at $E > \Delta + B$ [see Eq. (27.10)], thus $\det S_i(E, \phi) = 1$ and $\delta\rho(E, \phi) = 0$ in that energy domain.

At zero magnetic field, the phase-dependent correction $\delta\rho(E, \phi)$ vanishes at all energies above the continuum edge, a well-known result for a short SNS junction with time-reversal symmetry [16]. At nonzero magnetic field, on the other hand, the correction is finite and it carries a signature of the topological phase transition, as will be discussed in more detail below.

27.4.3 Thermodynamic properties of the transparent junction

The detailed description of the Andreev bound states and continuum states in the nanowire junction setup, provided in Secs. 27.4.1 and 27.4.2, sets the stage for the discussion of thermodynamic properties of the system. In this section we calculate the ground state energy of the junction, $E_{\text{gs}}(\phi)$ (Sec. 27.4.3), and with its help establish the current-phase relation $I(\phi)$ of the Josephson effect at $T = 0$ (Sec. 27.4.3). In view of the periodicity of these properties, hereafter we constrain the phase difference across the junction to the interval $\phi \in [-2\pi, 2\pi]$.

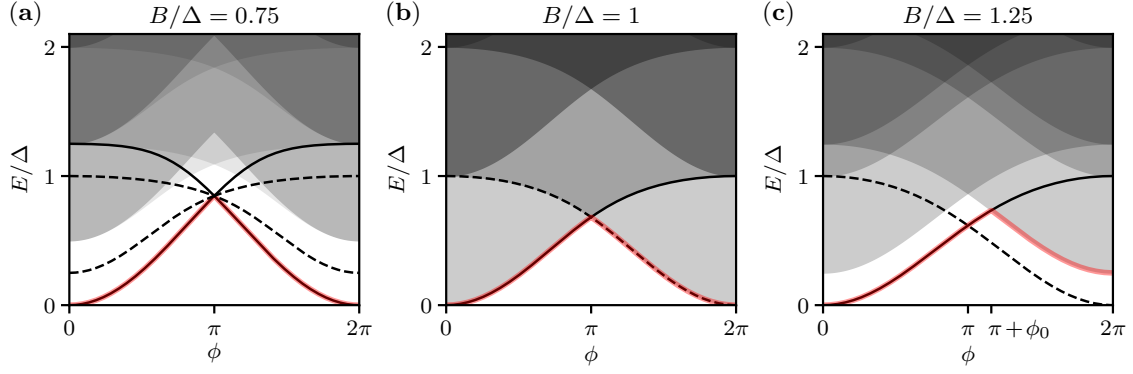


Figure 107: Evolution of the many-body spectrum with increasing Zeeman energy, in the limit of perfect transmission, $D = 1$, $\gamma = 0$. The panels are in one-to-one correspondence with the panels of Fig. 106. Solid (dashed) curves represent many-body states with even (odd) fermion parity belonging to the discrete spectrum. Shaded grey regions represent the continuous part of the spectrum, of either fermion parity. The solid red curves denote the ground state energy in the even parity sector. Note that the phase dispersion of the ground state energy includes the contribution coming from the continuous part of the BdG spectrum.

Ground state structure and many-body spectrum

We start by investigating the structure of the many-body ground state of the system. To this end we express the many-body Hamiltonian (27.1) in its eigenbasis:

$$\hat{H} = \sum_{\mathbf{b}} E_{\mathbf{b}}(\phi) (\hat{n}_{\mathbf{b}} - \frac{1}{2}) + \sum_{\mathbf{c}} E_{\mathbf{c}}(\phi) (\hat{n}_{\mathbf{c}} - \frac{1}{2}) + \dots \quad (27.21)$$

Here the sum in the first term runs over the bound states. In the trivial phase, $B < \Delta$, the index $\mathbf{b} = 0, 1$, with $E_0(\phi)$ and $E_1(\phi)$ given by Eqs. (27.9) and (27.12), respectively. In the topological phase, $B > \Delta$, a bound state is present in the outer modes only and $\mathbf{b} = 0$. $\hat{n}_{\mathbf{b}}$ are the occupation number operators of the corresponding Dirac fermions. The sum in the second term covers all continuum states with energies above the gap, $E_{\mathbf{c}} > |\Delta - B|$; $\hat{n}_{\mathbf{c}}$ are the number operators for the fermions in these states. Finally, the dots stand for a phase-independent additive term in the energy.

To find the ground state energy $E_{\text{gs}}(\phi)$ we minimize the Hamiltonian (27.21) under the constraint of a fixed fermion parity. Up to a phase-independent constant, $E_{\text{gs}}(\phi)$ can be conveniently divided into two parts:

$$E_{\text{gs}}(\phi) = E_{\text{gs}}^{(1)}(\phi) + E_{\text{gs}}^{(2)}(\phi). \quad (27.22)$$

The first part, $E_{\text{gs}}^{(1)}(\phi)$, incorporates the contributions to the ground state energy from the bound states and from the term $\sum_{\mathbf{c}} E_{\mathbf{c}} \hat{n}_{\mathbf{c}}$ in the Hamiltonian (27.21). The second part, $E_{\text{gs}}^{(2)}(\phi)$, is a residual contribution due to the quasiparticle continuum arising from the c -number term $-\frac{1}{2} \sum_{\mathbf{c}} E_{\mathbf{c}}$. By introducing the correction to the continuum density of states $\delta\rho(E, \phi)$ (see Sec. 27.4.2), we represent $E_{\text{gs}}^{(2)}(\phi)$ in the form

$$E_{\text{gs}}^{(2)}(\phi) = -\frac{1}{2} \int_{|\Delta-B|}^{+\infty} dE E [\delta\rho(E, \phi) - \delta\rho(E, 0^+)], \quad (27.23)$$

where we added a constant shift to ensure that $E_{\text{gs}}^{(2)} \rightarrow 0$ for $\phi \rightarrow 0$.

We begin the investigation of $E_{\text{gs}}(\phi)$ by considering a junction in a topologically trivial state, $B < \Delta$. For concreteness, we focus on the even fermion parity sector. As a first step, we establish the structure of the ground state in terms of occupation numbers at different phases ϕ . In the interval $\phi \in [-\pi, \pi]$ neither of the bound states is occupied in the ground state. At $\phi = \pm\pi$ the energies E_0 and E_1 of Eq. (27.21) simultaneously cross zero. Then, for larger phases $\phi \in (\pi, 2\pi]$ and $\phi \in [-2\pi, -\pi)$ it is energetically favorable for two electrons of a Cooper pair to occupy the two bound states. Such a redistribution is allowed by fermion parity conservation and results in 2π -periodicity of the thermodynamic properties of the system.

An explicit expression for $E_{\text{gs}}^{(1)}$ at $B < \Delta$ reads

$$E_{\text{gs}}^{(1)}(\phi) = \frac{1}{2} (\Delta - |E_0(\phi)|) + \frac{1}{2} (\Delta - B - |E_1(\phi)|), \quad (27.24)$$

where the bound state energies $E_{0,1}$ are given by Eqs. (27.9) and (27.12). The absolute values in Eq. (27.24) are associated with the changes in the occupation numbers of the bound states at $\phi = \pm\pi$. For convenience, we added a constant offset in Eq. (27.24) such that $E_{\text{gs}}^{(1)}(0) = 0$.

Next, we discuss the continuum contribution to the ground state energy, $E_{\text{gs}}^{(2)}$. It may be found by performing the integration in Eq. (27.23) with $\delta\rho(E, \phi)$ given by Eq. (27.20). When the magnetic field is tuned to the vicinity of the topological phase transition, the integral can be evaluated analytically. For $\Delta - B \ll \Delta$, we find:

$$E_{\text{gs}}^{(2)}(\phi) \approx \frac{\Delta}{2} \left[1 - \frac{2}{\pi} \frac{\arccos F(\phi)}{\sqrt{1 - F^2(\phi)}} \right] + \frac{F(\phi)}{2\pi} (\Delta - B) \ln \frac{\Delta}{|\Delta - B|}, \quad (27.25)$$

with $F(\phi)$ defined in Eq. (27.11). Thus, there is critical behavior in E_{gs} close to the topological transition: when B approaches Δ , the ground state energy behaves non-analytically as a function of the difference $(\Delta - B)$; see the last term in Eq. (27.25). In principle, such non-analytic behavior may be probed experimentally, for instance in the dependence of the Josephson plasma frequency on the magnetic field, and may serve as an additional signature of the topological phase transition.

In addition to the ground state energy, Eq. (27.21) contains information about the spectrum of excited states. The many-body spectrum of the transparent junction in the topologically trivial phase is shown in Fig. 107(a). Notice that for $B < \Delta$ there is always an energy gap between the ground state and the quasiparticle continuum.

The structure of the ground state in the topological regime, $B > \Delta$, is radically different from that in the trivial state. At $B > \Delta$ there is only one Andreev bound state, which comes from the outer modes. At small phase differences ϕ , this state is not filled. When the phase reaches $\phi = \pm\pi$ the bound state energy E_0 crosses zero, but the level occupation cannot change as that would violate fermion parity conservation. At $\phi = \pm(\pi + \phi_0)$, where $\phi_0 = 2 \arcsin((B - \Delta)/\Delta)$ [see Eq. (27.13)], the outer-mode bound state crosses the edge of the inner-mode continuum and, in terms of energy minimization, it becomes profitable to simultaneously occupy the Andreev state and a single quasiparticle state in the continuum. Therefore, for $\phi \in (\pi + \phi_0, 2\pi]$ and $\phi \in [-2\pi, -(\pi + \phi_0))$, there is no gap in the many-body spectrum between the ground state and the quasiparticle continuum, as depicted in Fig. 107(c). This feature is in a sharp contrast to the case of the topologically trivial junction, where a gap is present at all values of the phase difference.

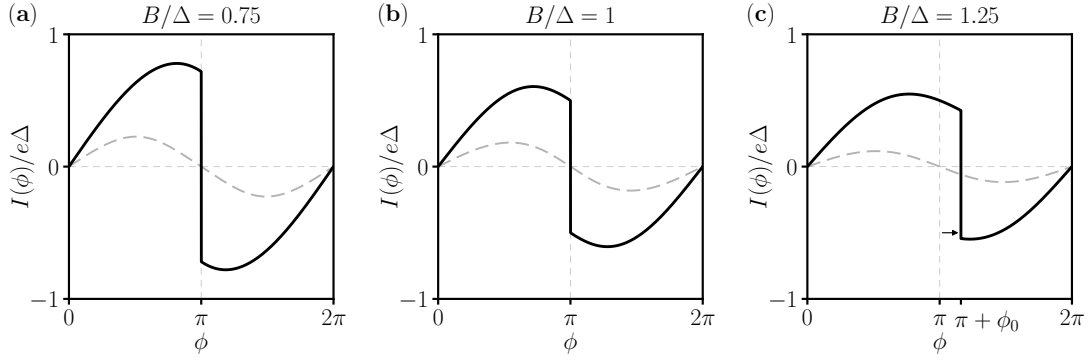


Figure 108: Evolution of the Josephson current $I(\phi)$ with increasing Zeeman energy in the limit of perfect transmission, $D = 1$, $\gamma = 0$ (solid black curves). The panels are in one-to-one correspondence with the panels of Fig. 107. In the trivial regime [$B < \Delta$, panel (a)] there is a discontinuity in the current at $\phi = \pi$. Its magnitude is determined by Eq. (27.28). Precisely at the topological transition [$B = \Delta$, panel (b)], the jump in the current decreases to $\delta I_{B=\Delta} = e\Delta$. In the topological state [$B > \Delta$, panel (c)], the discontinuity shifts from $\phi = \pi$ to $\phi = \pi + \phi_0$ [ϕ_0 is given by Eq. (27.13)], signifying the onset of the 4π -periodic Josephson effect. The magnitude of the discontinuity further decreases in accordance with Eq. (27.30). In each panel, a dashed gray curve depicts separately the continuum contribution, $I^{(2)}(\phi)$, to the Josephson current $I(\phi)$. This contribution is always 2π -periodic and smooth.

The structure of the ground state implies that in the topological state $E_{\text{gs}}^{(1)}$ is given by

$$E_{\text{gs}}^{(1)}(\phi) = \frac{\Delta}{2} \left(1 - \cos \frac{\phi}{2} \right) + \left[\Delta \cos \frac{\phi}{2} + (B - \Delta) \right] \Theta \left(\cos \frac{\pi + \phi_0}{2} - \cos \frac{\phi}{2} \right). \quad (27.26)$$

Here the second line describes the simultaneous occupation of the Andreev bound state and a state at the edge of the continuum, $E_c = B - \Delta$, occurring at $\phi = \pm(\pi + \phi_0)$. Expression (27.26) is manifestly 4π -periodic, which indicates the onset of the fractional Josephson effect on the topological side of the transition.

In contrast to $E_{\text{gs}}^{(1)}(\phi)$, the continuum contribution to the ground state energy, $E_{\text{gs}}^{(2)}$, is 2π -periodic even in the topological phase [as follows from Eq. (27.20)]. Close to the transition threshold, $B - \Delta \ll \Delta$, $E_{\text{gs}}^{(2)}$ is described by Eq. (27.25), similarly to the case $B < \Delta$. Therefore, the logarithmic behavior in Eq. (27.25) is characteristic to both sides of the topological transition.

Josephson current

Having discussed the ground state structure and $E_{\text{gs}}(\phi)$ on both sides of the topological transition, we proceed to the evaluation of the Josephson current, $I(\phi)$. At zero temperature, $I(\phi)$ is related to the ground state energy via

$$I(\phi) = 2e \frac{dE_{\text{gs}}(\phi)}{d\phi}, \quad (27.27)$$

where e is the electron charge, and thus can be calculated at arbitrary ratio B/Δ by employing the results of Sec. 27.4.3.

First, we compute $I(\phi)$ in the topologically trivial state, $B < \Delta$. An example of the current-phase relation obtained from Eqs. (27.22–27.24) and (27.27) is presented in Fig. 108(a). A

notable feature of the resulting $I(\phi)$ is the presence of a discontinuity. It occurs at $\phi = \pi$ (and, similarly, at $\phi = -\pi$) and originates from switching in the occupation numbers of the Andreev bound states. In general, such stepwise behavior of the Josephson current is common for short transparent junctions. A peculiarity of the nanowire setup is the dependence of the magnitude δI of the discontinuities on the magnetic field. Using Eq. (27.24) we find that it is given by

$$\delta I_{B<\Delta} = e\Delta \left(1 + \frac{\Delta^2 - B^2}{\Delta^2} \right). \quad (27.28)$$

The first term on the right comes from the outer-mode bound state and is independent of the Zeeman energy B . The second term originates from the inner-mode bound state. At the topological transition ($B = \Delta$) it vanishes along with the inner-mode bound state, and

$$\delta I_{B=\Delta} = e\Delta; \quad (27.29)$$

see Fig. 108(b).

In the topological regime, $B > \Delta$, the zero-temperature Josephson current can be calculated from Eq. (27.27) by using Eqs. (27.22), (27.23), and (27.26). A representative current-phase relation for $B > \Delta$ is shown in Fig. 108(c). In the topological state, the discontinuities in the Josephson current occur at $\phi = \pm(\pi + \phi_0)$ following the abrupt change in the ground state structure. The displacement of the steps in $I(\phi)$ from $\phi = \pm\pi$ is a manifestation of the 4π -periodic Josephson effect. The magnitude of the discontinuities is given by

$$\delta I_{B>\Delta} = e\Delta \sqrt{1 - (1 - B/\Delta)^2}. \quad (27.30)$$

Close to the topological transition, $B - \Delta \ll \Delta$,

$$\phi_0 \approx 2(B - \Delta)/\Delta \ll 1, \quad (27.31a)$$

$$\delta I_{B>\Delta} \approx e\Delta. \quad (27.31b)$$

Therefore, for $|\Delta - B| \ll \Delta$ the positions of the steps and their magnitude differ only by small amounts on the two sides of the transition. Consequently, in spite of the dramatic change in the ground state wave-function, the Josephson current $I(\phi)$ changes gradually across the topological phase transition. More generally, the 4π -periodic Fourier harmonics in thermodynamic quantities build up continuously at $B > \Delta$, departing from zero at $B = \Delta$.

To conclude this section, we highlight an additional interesting property of the Josephson effect in the presence of Zeeman splitting. On both sides of the transition, there exists a nonzero, 2π -periodic, contribution $I^{(2)}(\phi) = 2e dE_{\text{gs}}^{(2)}/d\phi$ that comes from the continuum states. For $B \sim \Delta$ this contribution is of the same order as the one coming from the Andreev bound states. We note, however, that $I^{(2)}(\phi)$ is smooth and thus has no effect on the discontinuities in the current (see pale dashed lines in Fig. 108). Close to the transition point, $|\Delta - B| \ll \Delta$, $I^{(2)}(\phi)$ can be computed analytically by differentiating Eq. (27.25) with respect to ϕ . This implies that $I(\phi)$ has a logarithmic contribution $\propto (\Delta - B) \ln(\Delta/|\Delta - B|)$.

27.5 Effects of scattering at the junction

In this section we discuss the influence of scattering on spectral (Sec. 27.5.1) and thermodynamic (Sec. 27.5.2) properties of the junction, focusing primarily on the case of weak backscattering. The backscattering couples inner and outer modes, modifying the structure of the Andreev states below the gap (Secs. 27.5.1, 27.5.1) as well as of the states above the continuum edge (Sec. 27.5.1). However, unlike the normal state (zero-field) conductance, which is solely determined by the backscattering strength, $1 - D$, the spectrum of the junction is

also sensitive to the forward scattering phase γ . Such sensitivity is especially prominent in the topological regime. There, even for weak backscattering, a large forward scattering phase can have a dramatic qualitative impact on the Andreev levels (see Fig. 110) and the Josephson current (see Fig. 112). Motivated by this peculiarity, we also consider analytically the energy spectrum at $D = 1$, $\gamma \neq 0$ in Appendix 27.7.

27.5.1 Bound states and continuum states in the presence of scattering

The boundary condition (27.6) indicates that, in contrast to the case of the transparent junction, the inner and outer modes cannot be considered separately at $D < 1$. In such a setting it is convenient to further employ the scattering approach to describe the spectrum of the system. A solution to the scattering problem yields scattering amplitudes which we arrange in the S -matrix, $S(E, \phi)$. In virtue of unitarity, above the gap, $E > |\Delta - B|$, the determinant of the scattering matrix can be parameterized as

$$\det S(E, \phi) = \frac{\Lambda^*(E, \phi)}{\Lambda(E, \phi)}. \quad (27.32)$$

The structure of the function $\Lambda(E, \phi)$ in the complex energy plane contains full information about the spectrum of the Josephson junction. The branch cuts of $\Lambda(E, \phi)$ situated at the real axis correspond to the states of the quasiparticle continuum with $|E| > |\Delta - B|$. In this range, the scattering induced correction to the continuum density of states $\delta\rho(E, \phi)$ can be found from $\Lambda(E, \phi)$ through Eqs. (27.17) and (27.32). Zeros of $\Lambda(E, \phi)$ on the real axis within the interval $|E| < |\Delta - B|$ correspond to the energies of bound states, i.e., the latter can be found from

$$\Lambda(E, \phi) = 0, \quad |E| < |\Delta - B|. \quad (27.33)$$

To determine $\Lambda(E, \phi)$ we solve the BdG equations (27.5) with boundary condition (27.6) and calculate the S -matrix. The resulting expression for $\Lambda(E, \phi)$, valid at arbitrary backscattering strength $1 - D$ and for any forward scattering phase γ , is explicitly presented in Appendix 27.8 [see Eq. (27.96)]. With its help the energies of the bound states and $\delta\rho(E, \phi)$ can be determined numerically at any value of B/Δ using Eqs. (27.17), (27.32), and (27.33). An example of such a numerical solution for moderately small $1 - D$ and γ is presented in Fig. 109. The figure reveals a set of interesting features of the spectrum introduced by scattering at the junction, which we discuss below. First, we concentrate on subgap energies, $|E| < |\Delta - B|$. We study the behavior of the Andreev levels on the trivial and topological sides of the transition in Secs. 27.5.1 and 27.5.1, respectively. Then, in Sec. 27.5.1 we consider the effects of scattering on the spectrum above the continuum edge, $|E| > |\Delta - B|$.

Bound states in the presence of scattering in the trivial phase, $B < \Delta$

In the topologically trivial phase, $B < \Delta$, the inner and outer-mode bound states are hybridized by the scattering. The degenerate zero-energy crossing which was present for the transparent junction at $\phi = \pi$ splits and gets pushed away from $E = 0$ at $D < 1$, $\gamma \neq 0$ [cf. Fig. 109(a) and Fig. 106(a)].

The hybridization of the bound states near $\phi = \pi$ can be addressed quantitatively in the limit

$$1 - D \ll \frac{|\Delta - B|}{\Delta} \ll 1, \quad |\gamma| \ll 1, \quad (27.34)$$

i.e., close to the topological transition threshold and for perturbatively weak scattering. In this limit, the general expression for $\Lambda(E, \phi)$ [given by Eq. (27.96)] can be substantially simplified at

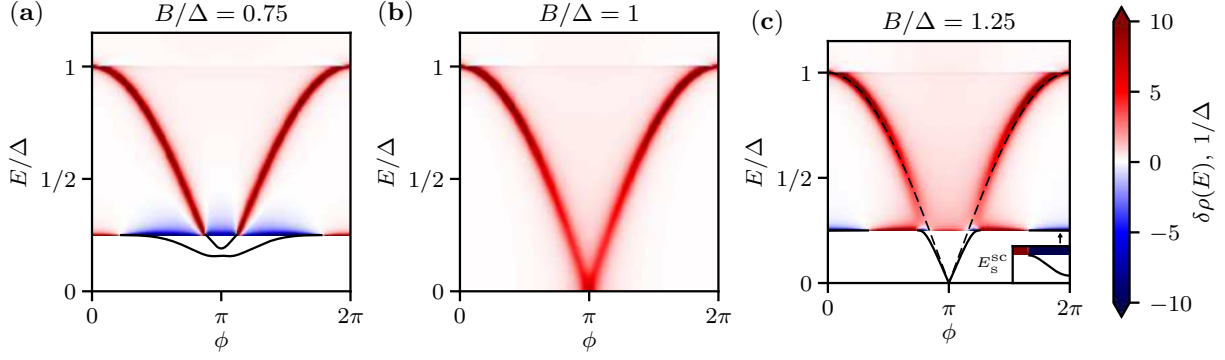


Figure 109: Evolution of the single-particle excitation spectrum of the junction with increasing Zeeman energy, in the presence of weak scattering. This is the analog of Fig. 106 for nonzero backscattering. The scattering parameters used are $D = 0.925$, $\gamma \approx -0.09\pi$ (the relation between the parameters corresponds to the case of a delta function barrier, see Eq. (27.7)). Bound states are depicted with solid black curves below the gap. The dashed curve in panel (c) shows the unperturbed bound state energy $\pm E_0(\phi)$. The inset in panel (c) is a close-up look at the vicinity of the continuum edge near $\phi = 2\pi$; there, a shallow bound state with energy $E_s^{\text{sc}}(\phi)$ is present below the continuum edge (the ϕ -axis of the inset is to scale, the energy window is $2 \cdot 10^{-3}\Delta$). A similar state is present symmetrically close to $\phi = 0$. Above the gap, $E > |\Delta - B|$, the color corresponds to the scattering-induced correction to the density of states, $\delta\rho(E, \phi)$.

$|E| \ll \Delta - B$ and $|\phi - \pi| \ll 1$ [see Eq. (27.104)]. Then, within the leading-order approximation, Eq. (27.33) yields bound states with energies $E = \pm E_{0,1}^{\text{sc}}(\phi)$ where

$$E_{0,1}^{\text{sc}}(\phi) \approx \pm \frac{E_1(\phi) - E_0(\phi)}{2} + \frac{1}{2} \sqrt{[E_1(\phi) + E_0(\phi)]^2 + 8\Delta(\Delta - B)(1 - D)}. \quad (27.35)$$

Here $E_0(\phi)$ and $E_1(\phi)$ are given by Eqs. (27.9) and (27.12), respectively. Eq. (27.35) indicates that at $\phi = \pi$ the backscattering pushes the bound state energies away from zero by an amount

$$\delta\varepsilon \approx \sqrt{1 - D} \sqrt{2\Delta(\Delta - B)}. \quad (27.36)$$

Notice that $E_{0,1}^{\text{sc}}(\phi)$ reaches the minimum $\sim (\Delta - B)\sqrt{1 - D}$ at points symmetrically shifted away from $\phi = \pi$ by $\sim \sqrt{1 - D}$ [see Fig. 109(a)]. This shift of the minimum is known to occur for a junction with strong spin-orbit coupling even at zero magnetic field, but only away from the short junction limit [21, 22, 23].

We note that within the accuracy of Eq. (27.35) the levels $E_{0,1}^{\text{sc}}(\phi)$ cross at π . This is a peculiarity of the lowest-order perturbative calculation. In a subleading order, $|\gamma| \ll 1$ results in an anticrossing between $E_0^{\text{sc}}(\phi)$ and $E_1^{\text{sc}}(\phi)$ near $\phi = \pi$ with a gap $\sim |\gamma| \delta\varepsilon$ [see Fig. 109(a)]. Taking this anticrossing into account we find the bound state energies

$$E_{\pm}^{\text{sc}}(\phi) \approx \frac{E_0^{\text{sc}}(\phi) + E_1^{\text{sc}}(\phi)}{2} \pm \frac{1}{2} \sqrt{[E_0^{\text{sc}}(\phi) - E_1^{\text{sc}}(\phi)]^2 + \gamma^2 \delta\varepsilon^2}. \quad (27.37)$$

Bound states in the presence of scattering in the topological phase, $B > \Delta$

On the topological side of the transition, $B > \Delta$, the energy of the outer-mode bound state crosses zero at $\phi = \pi$ despite the presence of scattering [see Fig. 109(c)]. The robustness of the crossing is a consequence of fermion parity conservation [7]. A strong modification of the bound state energy arises only when it approaches the edge of the continuum, driven by level repulsion between the bound state and states of the spectral continuum. As we will show in Sec. 27.5.2, this modification is important for thermodynamic properties of the junction.

To describe the repulsion of the outer-mode Andreev state from the continuum analytically, we again concentrate on the vicinity of the topological transition and assume that the scattering is perturbatively weak [Eq. (27.34)]. Then, the energy of the bound state $E_0^{\text{sc}}(\phi)$ can be obtained by solving $\Lambda(E, \phi) = 0$ approximately in the limit $B - \Delta - |E| \ll B - \Delta$ (see Appendix 27.9.2). We find that the Andreev state merges with the continuum, i.e., its energy $E_0^{\text{sc}}(\phi)$ reaches $|E| = B - \Delta$, at $\phi = \pi \pm \phi_0^{\text{sc}}$ with

$$\phi_0^{\text{sc}} \approx \phi_0 + 2(1 - D), \quad (27.38)$$

and ϕ_0 defined in Eq. (27.13). For small deviations from this point, $0 < (\pi + \phi_0^{\text{sc}}) - \phi \ll \phi_0$, we obtain

$$E_0^{\text{sc}}(\phi) \approx -(B - \Delta) \left\{ 1 - \frac{1}{2} \frac{\Delta^2}{(B - \Delta)^2} \times \left[\sqrt{(1 - D)^2 + \frac{B - \Delta}{\Delta} (\pi + \phi_0^{\text{sc}} - \phi) - (1 - D)} \right]^2 \right\}. \quad (27.39)$$

Expression (27.39) verifies that the bound state energy is almost unperturbed far from the continuum: $E_0^{\text{sc}}(\phi) \approx E_0(\phi)$ at $(\pi + \phi_0^{\text{sc}}) - \phi \gg (1 - D)^2 \Delta / (B - \Delta)$. Conversely, at $(\pi + \phi_0^{\text{sc}}) - \phi \ll (1 - D)^2 \Delta / (B - \Delta)$ the level hybridization with the continuum is effective, and $E_0^{\text{sc}}(\phi)$ reaches the continuum edge at $\phi = \pi + \phi_0^{\text{sc}}$ with a zero slope. Close to $\phi = \pi - \phi_0^{\text{sc}}$ the behavior of $E_0^{\text{sc}}(\phi)$ can be obtained from Eq. (27.39) via $E_0^{\text{sc}}(\phi) = -E_0^{\text{sc}}(2\pi - \phi)$ [see Fig. 109(c)]. Finally, we note that a small forward scattering phase $|\gamma| \ll 1$ results in a slight modification of the numeric prefactors in front of $1 - D$ in Eqs. (27.38), (27.39) (see Appendix 27.9.2). Such a modification is suppressed in these expressions.

Another interesting feature of the Andreev spectrum at $B > \Delta$ is that weak scattering induces shallow bound states at $\phi = 0$ and 2π [see inset in Fig. 109(c)]. These states merge with the edge of the continuum at a finite deviation ϕ_s of phase ϕ from $0, 2\pi$.

To quantify how these states peel off from the continuum, we compute $\Lambda(E, \phi)$ for $B - \Delta - |E| \ll B - \Delta$ in the limits $\phi \ll 1$ and $2\pi - \phi \ll 1$ (see Appendix 27.9.3) under the assumption of weak scattering, $1 - D \ll 1$, $|\gamma| \ll 1$. Then, through Eq. (27.33) we show that the shallow bound states are present for $\phi \in [0, \phi_s)$ and $\phi \in (2\pi - \phi_s, 2\pi]$, where

$$\phi_s \approx 2\sqrt{1 - D + \gamma^2}. \quad (27.40)$$

At $\phi \in [0, \phi_s)$, we find the energy of the shallow state $E = \pm E_s^{\text{sc}}(\phi)$ with

$$E_s^{\text{sc}}(\phi) \approx (B - \Delta) \left(1 - \frac{1}{128} [\phi_s^2 - \phi^2]^2 \right). \quad (27.41)$$

Near $\phi = 2\pi$ the energy of the shallow bound state satisfies $E = \pm E_s^{\text{sc}}(2\pi - \phi)$.

The shallow bound states are remarkably sensitive to the forward scattering phase γ . As one can see from Eq. (27.40), they appear in the energy spectrum even in the sole presence of forward scattering ($D = 1$, $\gamma \neq 0$). This case, for arbitrary γ , is considered in Appendix 27.7.

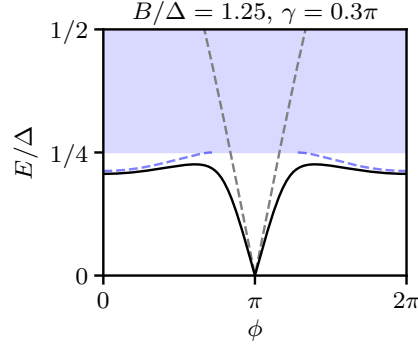


Figure 110: Single-particle excitation spectrum of the junction in the presence of a large forward scattering phase ($\gamma = 0.3\pi$) in the topological regime, $B = 1.25\Delta$. In the limit $D = 1$, two distinct shallow bound states are present near $\phi = 0, 2\pi$ (dashed blue curves). They peel off from the continuum due to $\gamma \neq 0$. At $D < 1$ these states hybridize with the outer-mode bound state (dashed gray curve), resulting in an energy level that is separated from the continuum at all ϕ (solid black curve, $D = 0.925$). The spectrum is markedly different from that at $|\gamma| \ll 1$, cf. Fig. 109(c). Note that the scattering-induced correction to the density of states above the gap is not shown here.

In addition to detailed calculations, we provide there a qualitative explanation of the shallow states' origin.

The domain of ϕ containing shallow states broadens with the increase of $|\gamma|$, as indicated by Eq. (27.40). When γ gets sufficiently close to $\pm\pi/2$, the shallow states hybridize with the outer-mode bound state due to weak backscattering (near the transition, $B - \Delta \ll \Delta$, this requires $|\gamma \pm \pi/2| \sim \sqrt{(B - \Delta)/\Delta}$, see Appendix 27.7). As a result of the hybridization, an energy level is formed that is separated from the continuum at all phase differences, see Fig. 110. We note that such a strong dependence of the Andreev spectrum on the forward scattering phase is in sharp contrast to the normal state conductance of the junction; the latter is independent of the forward scattering phase and is determined only by the transmission probability D .

Spectrum at $|E| > |\Delta - B|$ in the presence of scattering

Next, we discuss the influence of scattering on the structure of the spectrum above the continuum edge.

In Sec. 27.4.1 we observed that, in the transparent junction limit, the Andreev state in the outer modes coexists with the inner-mode continuum at $|E| > |\Delta - B|$, see Fig. 106(a)-(c). This coexistence is disrupted by weak backscattering: at $D < 1$ the outer-mode state hybridizes with the continuum states and broadens into a narrow resonance. The broadening is seen in Fig. 109 as a peak in the density of states $\delta\rho(E, \phi)$ within the energy interval $E \in (|\Delta - B|, \Delta)$.

Precisely at the transition, $B = \Delta$, the bound state is broadened at all phase differences [see Fig. 109(b)]. In this case, the broadening can be concisely addressed analytically. The shape of the associated resonance in the density of states can be determined by computing $\Lambda(E, \phi)$ via Eqs. (27.17) and (27.32). Assuming that $1 - D \ll 1$, $|\gamma| \ll 1$, $0 < E \ll \Delta$, and $|\phi - \pi| \ll 1$, we find

$$\Lambda(E, \phi) = 2iE \left([E + i\Delta(1 - D)]^2 - E_0^2(\phi) \right). \quad (27.42)$$

Then, for the corresponding contribution to the density of states we get

$$\delta\rho(E, \phi) = \frac{1}{\pi} \sum_{s=\pm 1} \frac{\Gamma}{(E + sE_0(\phi))^2 + \Gamma^2}, \quad (27.43)$$

where $\Gamma \approx (1 - D)\Delta$. This implies that $\delta\rho$ is a superposition of two Lorentzian peaks with width $\sim \Delta(1 - D)$ centered around $E = \pm E_0(\phi)$. In the subleading order, a small forward scattering phase, $|\gamma| \ll 1$, changes the prefactor in the expression for the width of the peaks, $\Gamma \approx \Delta(1 - D)(1 + \gamma^2)$. However, it does not alter $\delta\rho$ qualitatively: the broadening of the bound state into a resonance relies on a coupling between inner and outer modes, which is provided by $D < 1$. We note that expressions (27.42), (27.43) are equally applicable away from the transition at $|\Delta - B| \ll \Delta$, for energies $|\Delta - B| \ll E \ll \Delta$.

27.5.2 Thermodynamic properties in the presence of scattering

In this section we discuss how scattering affects the thermodynamic properties of the junction. We focus on the ground state energy $E_{\text{gs}}(\phi)$ (Sec. 27.5.2) and Josephson current $I(\phi)$ (Sec. 27.5.2) in the even fermion parity sector on the two sides of the topological transition.

Ground state energy and many-body spectrum

We begin by studying the influence of scattering on $E_{\text{gs}}(\phi)$ and on the many-body excitation energies of the junction. In the trivial regime, $B < \Delta$, the ground state at $D < 1$ is separated from the excitations by a gap at all phase differences. The effects of the scattering on the many-body spectrum are more intricate in the topological regime, $B > \Delta$. There, weak scattering *does not* destroy the phase domain (previously identified at $D = 1$, $\gamma = 0$) where the gap between the ground state and the excited states is absent, see Fig. 111(a). However, this domain shrinks in the presence of scattering. On the one hand, close to $\phi = \pm 2\pi$ the shallow bound states peel off from the continuum and a small gap $\propto (B - \Delta)(1 - D + \gamma^2)^2$ opens within $|\phi| \in (2\pi - \phi_s, 2\pi]$. On the other hand, backscattering shifts the points at which E_{gs} reaches the quasiparticle continuum from $\phi = \pm(\pi + \phi_0)$ to $\phi = \pm(\pi + \phi_0^{\text{sc}})$, where $\phi_0^{\text{sc}} > \phi_0$ [see Eq. (27.38)]. Overall, due to scattering the gapless domain spans the phase interval $|\phi| \in (\pi + \phi_0^{\text{sc}}, 2\pi - \phi_s)$, instead of $|\phi| \in (\pi + \phi_0, 2\pi]$ at $D = 1$, $\gamma = 0$.

This gapless domain might vanish completely if the scattering is sufficiently strong. In particular, at large forward scattering phase close to $\pm\pi/2$ the shallow bound states strongly hybridize with the outer-mode bound state, as discussed in Sec. 27.5.1. As a result, the excitation spectrum is gapped at all phases [see Fig. 111(b)].

Josephson current

The Josephson current $I(\phi)$ can be obtained from $E_{\text{gs}}(\phi)$ through Eq. (27.27) (see Appendix 27.10.1). Numerically computed examples of $I(\phi)$ at weak scattering, $1 - D$, $|\gamma| \ll 1$, are presented in Fig. 112. The plots highlight that scattering smears the discontinuities in $I(\phi)$ which were previously revealed at $D = 1$, $\gamma = 0$ (pale curves in Fig. 112). This smearing happens differently on the trivial and topological sides of the transition.

In the trivial phase, $B < \Delta$, the discontinuity at $\phi = \pi$ smears symmetrically in ϕ [see Fig. 112(a)]. The smearing can be captured analytically in the vicinity of the topological transition, for perturbatively weak scattering [Eq. (27.34)]. By using Eq. (27.37) for $|\phi - \pi| \ll 1$ we find

$$I(\phi) \approx \frac{e\Delta}{2} \frac{(\pi - \phi)}{\sqrt{(\pi - \phi)^2 + 32(\Delta - B)(1 - D)/\Delta}}. \quad (27.44)$$

Here we neglected a small contribution of the continuum states to $I(\phi)$, as it weakly alters the result close to $\phi = \pi$ (see Appendix 27.10.2 for details). Eq. (27.44) indicates that the Josephson current interpolates between $+e\Delta/2$ and $-e\Delta/2$ gradually over the phase interval

$$\delta\phi_{B < \Delta} \sim \left(\frac{\Delta - B}{\Delta}\right)^{1/2} (1 - D)^{1/2}. \quad (27.45)$$

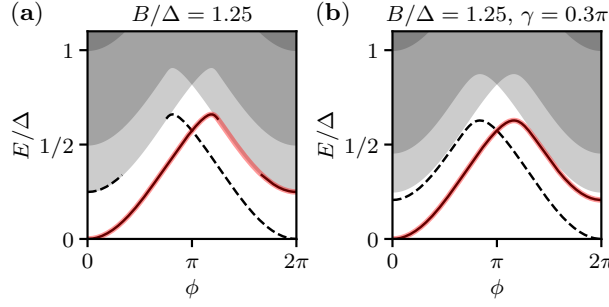


Figure 111: The influence of weak scattering on the ground state energy and many-body spectrum in the topological regime, $B = 1.25\Delta$. Solid (dashed) curves represent many-body states with even (odd) fermion parity belonging to the discrete spectrum. Shaded grey regions represent the continuous part of the spectrum, of either fermion parity. The solid red curves denote the ground state energy in the even parity sector. At weak scattering [$D = 0.925$, $\gamma \approx -0.09\pi$, panel (a)] there exists a phase region $\phi \in (\pi + \phi_0^{\text{sc}}, 2\pi - \phi_s)$ where the energy gap between the ground state and the excited states is absent. For a larger forward scattering phase [$D = 0.925$, $\gamma = 0.3\pi$, panel (b)] the gap is present at all phases ϕ .

The scaling of the smearing, $\delta\phi \propto \sqrt{1-D}$, is similar to the case of a regular short SNS junction [16]. We remark that the smearing relies on the presence of backscattering, whereas $|\gamma| \ll 1$ merely gives a subleading correction to the numeric prefactors in front of $1-D$ in Eqs. (27.44), (27.45); such corrections are omitted there.

Next, we discuss how scattering smears the discontinuity in $I(\phi)$ in the regime of the 4π -periodic Josephson effect, $B > \Delta$ [Fig. 112(c)]. The smearing can again be captured analytically in the vicinity of the transition for perturbatively weak scattering [Eq. (27.34)]. By using Eq. (27.39) for $|\pi + \phi_0^{\text{sc}} - \phi| \ll 1$ we find (see Appendix 27.10.2 for details)

$$I(\phi) \approx e\Delta \left[1 - \frac{1}{\sqrt{1 + \frac{B-\Delta}{\Delta(1-D)^2}(\pi + \phi_0^{\text{sc}} - \phi)}} \right] \times \Theta(\pi + \phi_0^{\text{sc}} - \phi) - \frac{e\Delta}{2}. \quad (27.46)$$

Backscattering replaces the discontinuity in the Josephson current at $\phi = \pi + \phi_0$ by a kink at $\phi = \pi + \phi_0^{\text{sc}}$. The resulting function $I(\phi)$ does not have a symmetry around $\phi = \pi + \phi_0$. The overall smearing of the discontinuity occurs within the interval

$$\delta\phi_{B>\Delta} \sim \frac{\Delta}{B-\Delta}(1-D)^2. \quad (27.47)$$

Comparing Eq. (27.47) with Eq. (27.45), we see that the step in the Josephson current becomes *sharper* upon the transition to the topological phase, see Fig. 112. This happens due to the difference in the underlying physical mechanisms of the smearing. In the trivial state the jump is smeared due to coupling between the bound states, whereas in the topological state the smearing is due to hybridization between the bound state and the continuum.

At small backscattering, the kink in $I(\phi)$ persists for almost all values of γ . It may vanish only at γ close to $\pm\pi/2$, when the shallow bound states merge with the outer-mode bound state and the many-body excitation spectrum becomes gapped at all phases [see Figs. 110 and 111(b)]. Then, the Josephson current turns into a smooth function of ϕ with 4π -periodicity, see Fig. 113.

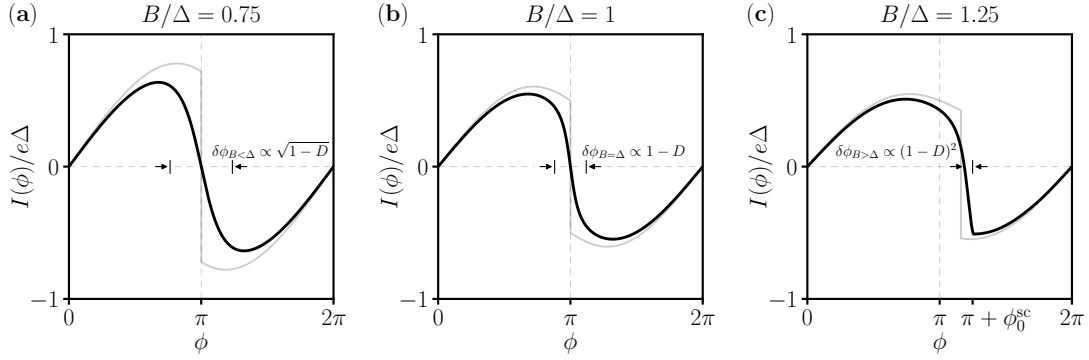


Figure 112: The influence of weak scattering on the Josephson current $I(\phi)$. Solid black curves depict $I(\phi)$ at $D = 0.925$, $\gamma \approx -0.09\pi$ (the relation between the parameters corresponds to the case of a delta function barrier, see Eq. (27.7)). Solid grey curves are provided for reference and correspond to $I(\phi)$ for the transparent junction ($D = 1$, $\gamma = 0$; cf. Fig. 108). The discontinuities present in $I(\phi)$ for the transparent junction get smeared by $D < 1$ in accordance with Eqs. (27.45), (27.47), and (27.49). In the trivial regime [$B < \Delta$, panel (a)] and at the transition threshold [$B = \Delta$, panel (b)] the jump smears symmetrically. In the topological regime [$B > \Delta$, panel (c)] the jump, displaced from $\phi = \pi$, smears asymmetrically and turns into a kink at $\phi = \pi + \phi_0^{\text{sc}}$.

Finally, it is illuminating to consider how weak scattering influences the Josephson current at the topological transition threshold, $B = \Delta$. At this point, there are no bound states in the system; only the broadened resonances are present. Thus, the behavior of the ground state energy and Josephson current is fully determined by the contribution from the continuum states. In particular, the smearing of the discontinuity in $I(\phi)$ [see Fig. 112(b)] can be studied with the help of Eq. (27.43) in the weak scattering limit, $1 - D \ll 1$, $|\gamma| \ll 1$. For $|\phi - \pi| \ll 1$ we obtain the current step

$$I(\phi) \approx \frac{e\Delta}{\pi} \arctan \frac{\pi - \phi}{2(1 - D)}, \quad (27.48)$$

smeared on the scale

$$\delta\phi_{B=\Delta} \sim 1 - D. \quad (27.49)$$

This intermediate scale matches both with Eq. (27.45) and Eq. (27.47) that describe the step width at the two sides of the transition, in its vicinity $|\Delta - B|/\Delta \sim 1 - D$.

27.6 Spectrum of inner modes at zero backscattering, $D = 1$

In this Appendix, we describe in detail the solution of the linearized BdG problem defined by Eqs. (27.5) and (27.6), for the case of zero backscattering, $D = 1$. This leads to the results presented in Sec. 27.4 of the main text. It also leads to the generalization of those results—to the case of purely forward scattering—described in Appendix 27.7. When $D = 1$ the inner and outer modes decouple from each other and can be analyzed independently for any value of γ , as follows from Eq. (27.6). Thus, in this Appendix we will fix the transmission probability $D = 1$ but leave the forward scattering phase γ arbitrary.

The BdG problem for the inner modes is defined by Eqs. (27.5a) and (27.6a) with $D = 1$. It reduces to the problem for the outer modes [Eqs. (27.5b) and (27.6b) with $D = 1$] upon taking $B \rightarrow 0$ and conjugating with σ_x . Therefore we restrict attention to the inner modes in the following.

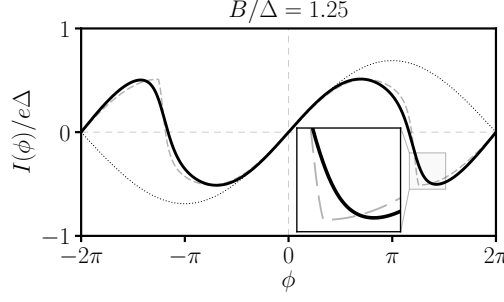


Figure 113: The influence of a large forward scattering phase γ on the Josephson current $I(\phi)$ in the topological regime, $B = 1.25\Delta$. The solid black curve depicts $I(\phi)$ at $\gamma = 0.3\pi$ and $D = 0.925$. It is a smooth 4π -periodic function of ϕ ; the kink which was present in $I(\phi)$ at smaller γ [dashed gray curve, $\gamma \approx -0.09\pi$ as in Fig. 112(c)] is smeared out. The dotted curve corresponds to the dependence $I(\phi) \sim \sin(\phi/2)$ characteristic of the 4π -periodic Josephson effect deep in the topological phase. The inset is a close-up look at how the kink in $I(\phi)$ is smeared out by sufficiently large γ .

It is convenient to remove the superconducting phase difference ϕ from the BdG equation (27.5a) and have it appear in the boundary condition (27.6a) instead. This is accomplished by the unitary gauge transformation

$$\Phi(x) \rightarrow e^{i(\phi/4)\text{sgn}(x)\tau_z}\Phi(x). \quad (27.50)$$

The transformed BdG equation and boundary condition (at $D = 1$) for the inner modes are then

$$[-i\alpha\partial_x\tau_z\sigma_z - B\sigma_x + \Delta\tau_x]\Phi(x) = E\Phi(x) \quad (27.51)$$

and

$$\Phi(0^+) = e^{-i(\phi/2)\tau_z}e^{i\gamma\sigma_z}\Phi(0^-). \quad (27.52)$$

Since we deal exclusively with the inner modes in this Appendix, we omit the subscript “i” on the wave function Φ . We solve Eqs. (27.51) and (27.52) by a standard wavefunction matching procedure, detailed below.

First consider the auxiliary translation-invariant problem

$$\mathcal{H}(-i\partial_x)\Phi(x) = E\Phi(x), \quad (27.53)$$

where

$$\mathcal{H}(k) \equiv \alpha k\tau_z\sigma_z - B\sigma_x + \Delta\tau_x. \quad (27.54)$$

For each fixed E , the space of vector-valued functions $\Phi(x)$ (in spin and Nambu space) that solve Eq. (27.53) (with no boundary conditions imposed at spatial infinity) is four-dimensional; the solutions are generalized plane waves of the form $\Phi_n(E)e^{ik_n(E)x}$, with wavevectors $k_n(E)$ that are roots of the characteristic polynomial

$$\det[E - \mathcal{H}(k)] = [E^2 - (\alpha k)^2 - (\Delta + B)^2][E^2 - (\alpha k)^2 - (\Delta - B)^2]. \quad (27.55)$$

The four roots are $\sigma k_\nu(E)$, where $\sigma, \nu = \pm 1$, and

$$k_\nu(E) = \frac{1}{\alpha} [E^2 - (\Delta + \nu B)^2]^{1/2}. \quad (27.56)$$

These are complex in general; by convention, the square root is to be taken so that $k_\nu(E)$ always lies in the upper-right quadrant of the complex k -plane ($\text{Re}k \geq 0$, $\text{Im}k \geq 0$). The associated vectors $\Phi_{\sigma,\nu}(E)$ solve

$$[E - \mathcal{H}(\sigma k_\nu(E))]\Phi_{\sigma,\nu}(E) = 0. \quad (27.57)$$

They are given by

$$\Phi_{\sigma,+}(E) = \frac{1}{2\sqrt{i \sin \eta_+(E)}} \begin{bmatrix} e^{i\sigma\eta_+(E)/2} \\ -e^{-i\sigma\eta_+(E)/2} \\ e^{-i\sigma\eta_+(E)/2} \\ -e^{i\sigma\eta_+(E)/2} \end{bmatrix}, \quad \Phi_{\sigma,-}(E) = \frac{1}{2\sqrt{i \sin \eta_-(E)}} \begin{bmatrix} e^{i\sigma\eta_-(E)/2} \\ \text{sgn}(\Delta - B) e^{-i\sigma\eta_-(E)/2} \\ \text{sgn}(\Delta - B) e^{-i\sigma\eta_-(E)/2} \\ e^{i\sigma\eta_-(E)/2} \end{bmatrix}, \quad (27.58)$$

where

$$e^{i\eta_\nu(E)} = \frac{E + \alpha k_\nu(E)}{|\Delta + \nu B|}. \quad (27.59)$$

We choose branches so that $\eta_\nu(E)$ is a continuous function of E that approaches $-i\infty$ as $E \rightarrow +\infty$. Thus, $\eta_\nu(E)$ is purely imaginary for $E > |\Delta + \nu B|$, and real for $|E| < |\Delta + \nu B|$.

The normalization factors $\frac{1}{2}(i \sin \eta_\nu(E))^{-1/2}$ in Eq. (27.58) are fixed by requiring that the associated plane waves carry unit probability current $j_P \equiv \Phi^\dagger(\tau_z \sigma_z) \Phi$ at energies $E > |\Delta + \nu B|$ (where these modes are propagating). This is necessary to obtain a unitary scattering matrix.

Any function $\Phi(x; E)$ that satisfies Eq. (27.51) can be expressed as a linear combination of the generalized plane waves at that energy E , as follows:

$$\Phi(x; E) = \begin{cases} a_{-, \nu} \Phi_{-, \nu}(E) e^{-ik_\nu(E)x} + b_{+, \nu} \Phi_{+, \nu}(E) e^{+ik_\nu(E)x}, & x > 0, \\ a_{+, \nu} \Phi_{+, \nu}(E) e^{+ik_\nu(E)x} + b_{-, \nu} \Phi_{-, \nu}(E) e^{-ik_\nu(E)x}, & x < 0, \end{cases} \quad (27.60)$$

where a summation over $\nu = \pm$ is implicit, and $k_\nu(E)$, $\Phi_{\sigma, \nu}(E)$ are given in Eqs. (27.56), (27.58) respectively. When the wavevectors are real, the coefficients $a_{\sigma, \nu}$ ($b_{\sigma, \nu}$) multiply incoming (outgoing) plane waves. When the wavevectors are imaginary, $a_{\sigma, \nu}$ ($b_{\sigma, \nu}$) multiply the generalized plane waves that grow (decay) exponentially at infinity.

The boundary condition at $x = 0$, Eq. (27.52), then yields a linear system of equations to be satisfied by the coefficients:

$$\Omega_a(E, \phi, \gamma) \vec{a} = \Omega_b(E, \phi, \gamma) \vec{b}, \quad (27.61)$$

where

$$\vec{a} = (a_{++}, a_{+-}, a_{-+}, a_{--})^T, \quad (27.62a)$$

$$\vec{b} = (b_{-+}, b_{--}, b_{++}, b_{+-})^T, \quad (27.62b)$$

and Ω_a, Ω_b are 4×4 matrices, given by

$$\Omega_a(E, \phi, \gamma) = [e^{-i(\phi/2)\tau_z} e^{i\gamma\sigma_z} \Phi_{+,+}(E), e^{-i(\phi/2)\tau_z} e^{i\gamma\sigma_z} \Phi_{+,-}(E), -\Phi_{-,+}(E), -\Phi_{-,-}(E)], \quad (27.63a)$$

$$\Omega_b(E, \phi, \gamma) = [-e^{-i(\phi/2)\tau_z} e^{i\gamma\sigma_z} \Phi_{-,+}(E), -e^{-i(\phi/2)\tau_z} e^{i\gamma\sigma_z} \Phi_{-,-}(E), \Phi_{+,+}(E), \Phi_{+,-}(E)]. \quad (27.63b)$$

Here, $[\Phi, \Psi, \dots]$ denotes the matrix whose first column is Φ , second column Ψ , and so on. The wavefunction $\Phi(x; E)$ in Eq. (27.60) represents a valid solution of the linearized inner-mode BdG problem if and only if (i) the coefficients $a_{\sigma, \nu}, b_{\sigma, \nu}$ satisfy Eq. (27.61), and (ii) $\Phi(x; E)$ satisfies appropriate conditions at $x = \pm\infty$.

27.6.1 Bound states

We first consider states belonging to the discrete spectrum (bound states). As usual, these correspond to *normalizable* wavefunctions $\Phi(x)$. Normalizability requires that the wavevectors

$\sigma k_\nu(E)$ appearing in the expansion (27.60) all have positive (negative) imaginary part for $x > 0$ ($x < 0$). It follows immediately that bound states cannot exist at energies $|E| > |\Delta + B|$, since then $k_\nu(E)$ is real according to Eq. (27.56).

In the energy range $|\Delta - B| < |E| < |\Delta + B|$, $k_-(E)$ is real and $k_+(E)$ is imaginary (with $\text{Im}k > 0$). A bound state with energy in this range would therefore have the form

$$\Phi(x; E, \phi, \gamma) = \begin{cases} b_{+,+}(E, \phi, \gamma) \Phi_{+,+}(E) e^{+ik_+(E)x}, & x > 0, \\ b_{-,+}(E, \phi, \gamma) \Phi_{-,+}(E) e^{-ik_-(E)x}, & x < 0. \end{cases} \quad (27.64)$$

However, it is easy to verify, using Eq. (27.58), that this only satisfies Eq. (27.52) when $b_{+,+} = b_{-,+} = 0$ (in which case the wavefunction vanishes identically).

Thus, bound states can exist only at energies $|E| < |\Delta - B|$. In this energy range, both $k_+(E)$ and $k_-(E)$ are imaginary (with $\text{Im}k > 0$), so the bound state wavefunction has the form (27.60) with $a_{\sigma,\nu} \equiv 0$. The $b_{\sigma,\nu}$ coefficients must still satisfy Eq. (27.61), which now reduces to:

$$0 = \Omega_b(E, \phi, \gamma) \vec{b}. \quad (27.65)$$

This equation has a nontrivial solution for $\vec{b}$ (and hence a bound state exists at the given E and ϕ) if and only if $\det\{\Omega_b(E, \phi, \gamma)\} = 0$. Explicit calculation using Eqs. (27.58) and (27.63b) yields

$$\det\{\Omega_b(E, \phi, \gamma)\} = -\frac{1}{2 \sin \eta_+ \sin \eta_-} \left([\cos^2(\phi/2) + \cos^2 \gamma] \text{sgn}(\Delta - B) \sin \eta_+ \sin \eta_- - [\sin^2(\phi/2) + \sin^2 \gamma] \text{sgn}(\Delta - B) \cos \eta_+ \cos \eta_- - [\sin^2(\phi/2) - \sin^2 \gamma] \right) \quad (27.66)$$

We now define

$$\Lambda_i(E, \phi, \gamma) = -2 \sin \eta_+ \sin \eta_- \frac{(\Delta + B)(\Delta - B)}{[\cos^2(\phi/2) + \cos^2 \gamma]} \det\{\Omega_b(E, \phi, \gamma)\}. \quad (27.67)$$

For $B \neq \Delta$, it is clear that $\det\{\Omega_b(E, \phi, \gamma)\} = 0$ if and only if $\Lambda_i(E, \phi, \gamma) = 0$. From Eq. (27.59), we have

$$|\Delta + \nu B| \cos \eta_\nu(E) = E, \quad (27.68a)$$

$$|\Delta + \nu B| \sin \eta_\nu(E) = -i\alpha k_\nu(E), \quad (27.68b)$$

Using these with Eq. (27.66) in Eq. (27.67), we obtain

$$\Lambda_i(E, \phi, \gamma) = -\alpha k_+(E) \alpha k_-(E) - F(\phi)[1 + G(\phi, \gamma)](E^2 + \Delta^2 - B^2) - G(\phi, \gamma)(E^2 - \Delta^2 + B^2). \quad (27.69)$$

with

$$F(\phi) \equiv \frac{\sin^2(\phi/2)}{\cos^2(\phi/2) + 1}, \quad (27.70a)$$

$$G(\phi, \gamma) \equiv \frac{\sin^2 \gamma}{\cos^2(\phi/2) + \cos^2 \gamma}. \quad (27.70b)$$

Thus, the inner-mode bound state energy is a solution of the equation $\Lambda_i = 0$, as stated in the main text. Since $\alpha k_\nu(E) = i[(\Delta + \nu B)^2 - E^2]^{1/2}$ for $|E| < |\Delta + \nu B|$, Eq. (27.69) reduces to Eq. (27.10) in the limit $\gamma \rightarrow 0$. For general $\gamma \neq 0$, the bound state solutions of $\Lambda_i(E, \phi, \gamma) = 0$ are analyzed in detail in Appendix 27.7.

27.6.2 Scattering states

We next consider states belonging to the continuous spectrum (scattering states). For any values of E, ϕ, γ such that $\det \Omega_b(E, \phi, \gamma) \neq 0$, we may solve Eq. (27.61) to express $\vec{b}$ as a linear function of $\vec{a}$:

$$\vec{b} = [\Omega_b(E, \phi, \gamma)]^{-1} \Omega_a(E, \phi, \gamma) \vec{a}. \quad (27.71)$$

The scattering states are then given by Eq. (27.60). In order for the wavefunctions to remain finite as $x \rightarrow \pm\infty$, the coefficient $a_{\sigma,\nu}(E)$ must vanish unless the corresponding wavevector $k_\nu(E)$ is real. This condition fixes the number of linearly independent scattering states in the inner-mode continuum at energy E .

At energies $E > \Delta + B$, all waves are propagating (the wavevectors $k_\nu(E)$ are both real). Thus, the inner-mode scattering matrix at these energies is the 4×4 matrix:

$$S_i(E, \phi, \gamma) = [\Omega_b(E, \phi, \gamma)]^{-1} \Omega_a(E, \phi, \gamma). \quad (27.72)$$

Explicit calculation shows that $\det \Omega_a(E, \phi, \gamma) = \det \Omega_b(E, \phi, \gamma)$. Therefore,

$$\det S_i(E, \phi, \gamma) = 1, \quad E > \Delta + B. \quad (27.73)$$

At energies in the range $|\Delta - B| < E < \Delta + B$, only the “ $\nu = -$ ” waves are propagating (the wavevector $k_-(E)$ is real, but $k_+(E)$ is imaginary). Thus, the inner-mode scattering matrix at these energies is a 2×2 matrix relating $(b_{+-}, b_{--})^T$ to $(a_{+-}, a_{--})^T$. It may be obtained as the appropriate sub-block of the S matrix in Eq. (27.72) (analytically continued below the threshold $E = \Delta + B$ by letting $\alpha k_+(E) \rightarrow i[(\Delta + B)^2 - E^2]^{1/2}$). In this case, the determinant of S_i is given by

$$\det S_i(E, \phi, \gamma) = \frac{-\alpha k_-(E) \alpha k_+(E) + F(\phi)[1 + G(\phi, \gamma)](E^2 + \Delta^2 - B^2) + G(\phi, \gamma)(E^2 - \Delta^2 + B^2)}{+\alpha k_-(E) \alpha k_+(E) + F(\phi)[1 + G(\phi, \gamma)](E^2 + \Delta^2 - B^2) + G(\phi, \gamma)(E^2 - \Delta^2 + B^2)}, \quad (27.74)$$

where F and G are the functions defined in Eq. (27.70). Since $k_-(E)$ is real and $k_+(E)$ is imaginary, this can be rewritten as

$$\det S_i(E, \phi, \gamma) = \frac{[\Lambda_i(E, \phi, \gamma)]^*}{\Lambda_i(E, \phi, \gamma)}, \quad (27.75)$$

where $\Lambda_i(E, \phi, \gamma)$ is given in Eq. (27.69), and $*$ denotes complex conjugation. This reduces to Eq. (27.18) in the limit $\gamma \rightarrow 0$. For general $\gamma \neq 0$, the scattering-induced contribution to the continuum density of states, $\delta\rho(E, \phi, \gamma) = (2\pi i)^{-1} \partial_E \ln \det S_i(E, \phi, \gamma)$, is analyzed in Appendix 27.7.

Finally, at energies $E < |\Delta - B|$, there are no propagating waves (the wavevectors $k_\nu(E)$ are both imaginary), and so the inner-mode scattering matrix is not defined.

27.7 Spectrum of the junction with purely forward scattering ($D = 1$ but $\gamma \neq 0$)

In Secs. 27.4.1 and 27.4.2 of the main text, we presented a detailed description of the Andreev spectrum in the nanowire junction setup in the absence of scattering ($D = 1$ and $\gamma = 0$). In this Appendix we generalize this discussion to the case of purely forward scattering ($D = 1$ but $\gamma \neq 0$). The generalization is natural on a technical level: when $D = 1$, the inner and outer modes decouple from each other [as follows from Eq. (27.6)], permitting a complete analytical solution of the BdG problem for any value of the forward scattering phase γ (see Appendix 27.6 for details of the solution). The results in this Appendix may be used to understand the Andreev spectra of short junctions that are characterized in the normal state by weak backscattering, $1 - D \ll 1$, but large forward scattering phase, $\gamma \sim 1$, following the general discussion in Sec. 27.5 of the main text.

27.7.1 The origin of the shallow states upon the topological transition

As we already mentioned, the inner and outer modes decouple from each other at $D = 1$. The Andreev spectrum of the outer modes is independent of the forward scattering phase γ ; this follows from the fact that γ can be eliminated from the outer-mode problem [Eqs. (27.5b) and (27.6b) with $D = 1$] by a unitary transformation $\Phi_o(x) \rightarrow e^{-i(\gamma/2)\text{sgn}(x)\sigma_z} \Phi_o(x)$. Thus, the contribution of the outer modes to the energy spectrum is not affected by γ .

The same unitary transformation applied to the inner modes absorbs the forward-scattering phase at the price of rotating the magnetic fields on the two sides of the junction in directions opposite to each other (the angle between the two directions is proportional to the forward-scattering phase shift). At some phase shift, the fields point in opposite directions. In the absence of superconductivity ($\Delta = 0$), the corresponding Hamiltonian is identical to a Dirac Hamiltonian in 1D with a mass $m(x)$ which changes sign at $x = 0$. That brings about a localized zero-energy state. At an arbitrary magnetic field rotation angle, the localized state moves away from zero energy, but does not vanish. Inclusion of a finite but small gap ($\Delta < B$) does not destroy this state.

27.7.2 Inner-mode bound states with forward scattering in the trivial phase, $B < \Delta$

The Andreev spectrum of the inner modes depends sensitively on γ . The inner-mode bound state energy is a solution of the equation $\Lambda_i(E, \phi, \gamma) = 0$, with Λ_i given by Eq. (27.69) (in which $\alpha k_+ \alpha k_- = -\sqrt{(\Delta + B)^2 - E^2} \sqrt{(\Delta - B)^2 - E^2} < 0$ at energies $|E| < |\Delta - B|$). In order for the equation $\Lambda_i(E, \phi, \gamma) = 0$ to admit a solution, the second term of Eq. (27.69) must be negative. Simple algebra shows that this condition is equivalent to

$$(\Delta - B)[\Delta \sin^2(\phi/2) - B \sin^2\gamma] > 0. \quad (27.76)$$

In the trivial phase, $\Delta > B$, the inner-mode bound state exists for $\phi \in (\phi_1, 2\pi - \phi_1)$, where

$$\phi_1 = 2 \arcsin(\sqrt{B/\Delta} |\sin \gamma|). \quad (27.77)$$

Its energy, $E = \pm E_1(\phi, \gamma)$, may be obtained analytically by solving $\Lambda_i(E, \phi, \gamma) = 0$ (squaring this gives a quadratic equation in E^2 , of which precisely one root satisfies $0 < E^2 < (\Delta - B)^2$ in the specified range of ϕ). The result is more complicated than Eq. (27.12) for $E_1(\phi)$, but shares many basic features with the latter. Firstly, $E_1(\phi, \gamma) = E_0(\phi)$ at $B = 0$, so the Andreev spectrum remains two-fold degenerate in the absence of the magnetic field. Secondly, $E_1(\phi, \gamma)$ still crosses zero at $\phi = \pi$, simultaneously with $E_0(\phi)$. Thirdly, $|E_1(\phi, \gamma)|$ remains bounded by $\Delta - B$. However, now $E_1(\phi_1, \gamma) = \Delta - B$ and $E_1(2\pi - \phi_1, \gamma) = B - \Delta$, so that the bound state merges with the edge of the continuum at $\phi = \phi_1$ and $2\pi - \phi_1$, and disappears when $\phi \in [0, \phi_1]$ or $\phi \in [2\pi - \phi_1, 2\pi]$.

27.7.3 Inner-mode bound states with forward scattering in the topological phase, $B > \Delta$

In the topological regime, $B > \Delta$, weak forward scattering causes shallow bound states to appear in the inner modes near $\phi = 0$ and 2π . As discussed above, the equation $\Lambda_i(E, \phi, \gamma) = 0$ admits solutions for some E in the range $|E| < |\Delta - B|$ only if Eq. (27.76) is satisfied. When $\gamma = 0$ this condition prohibits bound states in the inner modes at $B > \Delta$, but for $\gamma \neq 0$ it permits them to exist near $\phi = 0$ and 2π . For sufficiently small γ [in particular for $|\gamma| <$

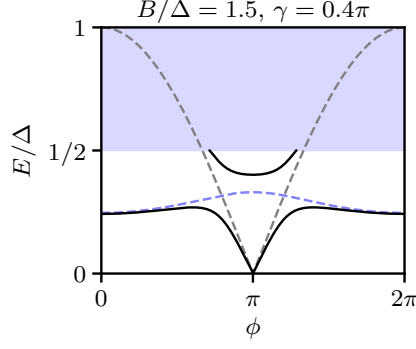


Figure 114: Single-particle excitation spectrum of the junction in the presence of a very large forward scattering phase ($\gamma = 0.4\pi$) in the topological regime, $B = 1.5\Delta$. In the limit $D = 1$, since $\gamma > \arcsin(\sqrt{\Delta/B}) \approx 0.3\pi$, a bound state is present in the inner modes at all ϕ (dashed blue curve). At $D < 1$ this state hybridizes with the outer-mode bound state (dashed gray curve) resulting in one energy level that is separated from the continuum at all phase differences, and a second level that dips below the continuum only near $\phi = \pi$ (solid black curves, $D = 0.975$). The spectrum may be compared with that at $|\gamma| \ll 1$, cf. Fig. 109(c), as well as with the spectrum at $\gamma \sim 1$ but $\gamma < \arcsin(\sqrt{\Delta/B})$, cf. Fig. 110 (note, however, that those Figures are at different values of B/Δ and D than this one).

$\arcsin(\sqrt{\Delta/B})$, cf. Eq. (27.77)], these states exist only for $\phi \in [0, \phi_1)$ and $\phi \in (2\pi - \phi_1, 2\pi]$, and have energies close to $B - \Delta$; they touch the edge of the inner-mode continuum at $\phi = \phi_1$, $2\pi - \phi_1$ and disappear when $|\phi - \pi| < \pi - \phi_1$. The shallow states have energies $E = \pm E_s^{\text{sc}}(\phi)$, for which a simple expression can be obtained in the limit $(B - \Delta) - |E| \ll (B - \Delta)$. In this limit, Eq. (27.69) reduces to

$$\begin{aligned} \frac{1}{2}\Lambda_i(E, \phi, \gamma) \approx & \sqrt{2\Delta B(B - \Delta)(B - \Delta - |E|)} \\ & - \Delta(B - \Delta)F(\phi)[1 + G(\phi, \gamma)] \\ & - B(B - \Delta)G(\phi, \gamma). \end{aligned} \quad (27.78)$$

Then for the solution to $\Lambda_i = 0$, we obtain

$$E_s^{\text{sc}}(\phi) \approx (B - \Delta) \left(1 - \frac{B}{2\Delta} \times \left[\frac{\sin^2 \gamma - (\Delta/B) \sin^2(\phi/2)}{\cos^2 \gamma + \cos^2(\phi/2)} \right]^2 \right) \quad (27.79)$$

for $\phi \in [0, \phi_1)$ and $\phi \in (2\pi - \phi_1, 2\pi]$. This approximate expression is valid as long as $B - \Delta - E_s^{\text{sc}}(\phi) \ll B - \Delta$. Note that, for $|\gamma| \ll 1$, $\phi \ll 1$ and $B - \Delta \ll \Delta$, Eq. (27.79) reduces to Eq. (27.41) [with ϕ_s given by Eq. (27.40) with $D = 1$]. As $|\gamma|$ increases, the shallow states persist over a larger range of ϕ . They merge into a single level when $|\gamma| = \arcsin(\sqrt{\Delta/B})$. This level peels away from the inner-mode continuum as $|\gamma|$ increases further, eventually becoming a zero-energy state (independent of ϕ) when $|\gamma| = \pi/2$.

As discussed in Sec. 27.5 of the main text, weak backscattering, $1 - D \ll 1$, may cause the inner-mode level(s) to hybridize with the outer-mode bound state. Recall that, at $D = 1$, the outer-mode bound state $E_0(\phi)$ reaches the edge of the inner-mode continuum at $\phi = \pi \pm \phi_0$, where ϕ_0 is given by Eq. (27.13), while the inner-mode shallow states exist for $|\phi| < \phi_1$ and

$|\phi - 2\pi| < \phi_1$. Hence if $\phi_0 + \phi_1 \ll \pi$ (i.e. for sufficiently small $|\gamma|$), the hybridization is ineffective, resulting in a spectrum similar to Fig. 109(c). As $|\gamma|$ increases past the point where $\phi_0 + \phi_1 \approx \pi$, the hybridization becomes effective, leading to a single energy level that is separated from the continuum at all ϕ , as shown in Fig. 110. Using Eqs. (27.13) for ϕ_0 and (27.77) for ϕ_1 , the condition $\phi_0 + \phi_1 \gtrsim \pi$ is equivalent to

$$|\sin \gamma| \gtrsim \sqrt{1 - (B - \Delta)/\Delta}. \quad (27.80)$$

Near the transition, $B - \Delta \ll \Delta$, this reduces to $|\gamma \pm \pi/2| \lesssim \sqrt{(B - \Delta)/\Delta}$, i.e., the condition stated in Sec. 27.5.1 of the main text.

Finally, let us consider the effect of weak backscattering when $|\gamma| > \arcsin(\sqrt{\Delta/B})$ is so large that the bound state in the inner modes is already separated from the continuum at all phases ϕ . Now hybridization with the outer-mode bound state may lead to the situation depicted in Fig. 114, in which a second energy level dips below the continuum at phases near $\phi = \pi$.

27.7.4 Inner-mode continuum states with forward scattering

At energies $E > |\Delta - B|$, the determinant of the S -matrix of the inner modes is given by (see Appendix 27.6.2)

$$\det S_i(E, \phi, \gamma) = \frac{[\Lambda_i(E, \phi, \gamma)]^*}{\Lambda_i(E, \phi, \gamma)}, \quad (27.81)$$

with Λ_i , defined in Eq. (27.69), analytically continued into the appropriate range of above-gap energies by taking $[(\Delta \pm B)^2 - E^2]^{1/2} \rightarrow -i[E^2 - (\Delta \pm B)^2]^{1/2}$. The scattering-induced contribution to the continuum density of states is then given by Eq. (27.17): $\delta\rho = (2\pi i)^{-1} \partial_E \ln \det S_i$ [since $\det S = (\det S_i) \cdot (\det S_o)$ and $\det S_o = 1$]. It vanishes for $E > \Delta + B$. Explicit calculation for $|\Delta - B| < E < \Delta + B$ yields

$$\delta\rho(E, \phi, \gamma) = \frac{\partial}{\partial E} \frac{1}{\pi} \arctan \left(\frac{(E^2 + \Delta^2 - B^2)F(\phi)[1 + G(\phi, \gamma)] + (E^2 - \Delta^2 + B^2)G(\phi, \gamma)}{\sqrt{[E^2 - (\Delta - B)^2][(\Delta + B)^2 - E^2]}} \right), \quad (27.82)$$

where $F(\phi)$ and $G(\phi, \gamma)$ are defined in Eq. (27.70). Equations (27.81) and (27.82) generalize Eqs. (27.18) and (27.20) for $\det S_i(E, \phi)$ and $\delta\rho(E, \phi)$ respectively, and reduce to the latter in the limit $\gamma \rightarrow 0$.

27.8 Spectrum at nonzero backscattering, $D < 1$

We next determine the spectrum in the presence of backscattering, $D < 1$. To streamline notation, we write

$$\Phi(x) = \begin{bmatrix} \Phi_i(x) \\ \Phi_o(x) \end{bmatrix}, \quad (27.83)$$

and introduce new Pauli matrices χ_i acting in the space of inner ($\chi_z = +1$) and outer ($\chi_z = -1$) modes. Performing the gauge transformation (27.50), the linearized BdG equations (27.5) take the form

$$\left[-i\alpha\partial_x\tau_z\sigma_z\chi_z - \frac{1}{2}B\sigma_x(1 + \chi_z) + \Delta\tau_x \right] \Phi(x) \equiv \mathcal{H}\Phi(x) = E\Phi(x), \quad (27.84)$$

where the first equality defines $\mathcal{H}$, the linearized BdG hamiltonian. The boundary condition at $x = 0$, Eq. (27.6), becomes

$$\Phi(0^+) = e^{-i(\phi/2)\tau_z} \frac{e^{i\gamma\sigma_z\chi_z}}{\sqrt{D}} \left[1 + \sqrt{1-D} \chi_x \right] \Phi(0^-) \equiv T(\phi, D, \gamma) \Phi(0^-). \quad (27.85)$$

Here the last equality defines the transfer matrix T across the junction. For convenience, we have set the reflection phase $\delta = 0$; as discussed in the main text, δ can be eliminated from the linearized BdG problem by a unitary transformation, so its actual value does not affect the spectrum of the junction in the limit we are considering. The solutions of Eqs. (27.84), (27.85) at energy E are related to those at energy $-E$ by the anti-unitary “particle-hole” operator

$$\mathcal{P} = \tau_y \sigma_y \mathcal{K}, \quad (27.86)$$

where $\mathcal{K}$ denotes the operator of complex conjugation. The operator $\mathcal{P}$ anticommutes with $\mathcal{H}$ and commutes with T , so the solutions at energies $\pm E$ are indeed mapped into one another by the action of $\mathcal{P}$.

From the analysis in Appendix 27.6, it follows that the generalized plane-wave solutions of Eq. (27.84) at energy E are $\Phi_{\sigma,\nu}^i(E) e^{i\sigma k_\nu(E)x}$ and $\Phi_{\sigma,\nu}^o(E) e^{-i\sigma k_0(E)x}$, where

$$k_\nu(E) = \frac{1}{\alpha} [E^2 - (\Delta + \nu B)^2]^{1/2}, \quad (27.87)$$

and where

$$\Phi_{\sigma,+}^i(E) = \frac{1}{2\sqrt{i \sin \eta_+(E)}} \begin{bmatrix} e^{i\sigma\eta_+(E)/2} \\ -e^{-i\sigma\eta_+(E)/2} \\ e^{-i\sigma\eta_+(E)/2} \\ -e^{i\sigma\eta_+(E)/2} \\ \mathbf{0}_4 \end{bmatrix}, \quad \Phi_{\sigma,-}^i(E) = \frac{1}{2\sqrt{i \sin \eta_-(E)}} \begin{bmatrix} e^{i\sigma\eta_-(E)/2} \\ \text{sgn}(\Delta - B) e^{-i\sigma\eta_-(E)/2} \\ \text{sgn}(\Delta - B) e^{-i\sigma\eta_-(E)/2} \\ e^{i\sigma\eta_-(E)/2} \\ \mathbf{0}_4 \end{bmatrix}, \quad (27.88a)$$

$$\Phi_{\sigma,+}^o(E) = \frac{1}{2\sqrt{i \sin \eta_0(E)}} \begin{bmatrix} \mathbf{0}_4 \\ e^{i\sigma\eta_0(E)/2} \\ -e^{-i\sigma\eta_0(E)/2} \\ e^{-i\sigma\eta_0(E)/2} \\ -e^{i\sigma\eta_0(E)/2} \end{bmatrix}, \quad \Phi_{\sigma,-}^o(E) = \frac{1}{2\sqrt{i \sin \eta_0(E)}} \begin{bmatrix} \mathbf{0}_4 \\ e^{i\sigma\eta_0(E)/2} \\ e^{-i\sigma\eta_0(E)/2} \\ e^{-i\sigma\eta_0(E)/2} \\ e^{i\sigma\eta_0(E)/2} \end{bmatrix}. \quad (27.88b)$$

Here, $\mathbf{0}_4 = (0, 0, 0, 0)^T$, and

$$e^{i\eta_\nu(E)} = \frac{E + \alpha k_\nu(E)}{|\Delta + \nu B|}. \quad (27.89)$$

As before, we choose branches so that $k_\nu(E)$ always lies in the upper-right quadrant of the complex k -plane ($\text{Re} k \geq 0$, $\text{Im} k \geq 0$), and so that $\eta_\nu(E)$ is a continuous function of E that approaches $-\infty$ as $E \rightarrow \infty$. Thus, $\eta_\nu(E)$ is purely imaginary for $E > |\Delta + \nu B|$, and real for $|E| < |\Delta + \nu B|$.

Any function $\Phi(x; E)$ that satisfies Eq. (27.84) can be expressed as a linear combination of the generalized plane waves at that energy E :

$$\Phi(x; E, \phi) = \begin{cases} a_{-, \nu}^i \Phi_{-, \nu}^i e^{-ik_\nu x} + b_{+, \nu}^i \Phi_{+, \nu}^i e^{+ik_\nu x} + a_{+, \nu}^o \Phi_{+, \nu}^o e^{-ik_0 x} + b_{-, \nu}^o \Phi_{-, \nu}^o e^{+ik_0 x}, & x > 0, \\ a_{+, \nu}^i \Phi_{+, \nu}^i e^{+ik_\nu x} + b_{-, \nu}^i \Phi_{-, \nu}^i e^{-ik_\nu x} + a_{-, \nu}^o \Phi_{-, \nu}^o e^{+ik_0 x} + b_{+, \nu}^o \Phi_{+, \nu}^o e^{-ik_0 x}, & x < 0, \end{cases} \quad (27.90)$$

where a summation over $\nu = \pm$ is implicit, $k_\nu(E)$, $\Phi_{\sigma,\nu}^\chi(E)$ ($\chi = i, o$) are given in Eqs. (27.87), (27.88) respectively, and we have suppressed all arguments for brevity. When the wavevectors are real, the coefficients $a_{\sigma,\nu}^\chi$ ($b_{\sigma,\nu}^\chi$) multiply incoming (outgoing) plane waves. When the wavevectors are imaginary, $a_{\sigma,\nu}^\chi$ ($b_{\sigma,\nu}^\chi$) multiply the generalized plane waves that grow (decay) exponentially at infinity.

The boundary condition at $x = 0$, Eq. (27.85), then yields a linear system of equations to be satisfied by the coefficients:

$$\Omega_a(E, \phi, D, \gamma) \vec{a} = \Omega_b(E, \phi, D, \gamma) \vec{b}, \quad (27.91)$$

where

$$\vec{a} = (a_{++}^i, a_{+-}^i, a_{-+}^o, a_{--}^o, a_{-+}^i, a_{--}^i, a_{++}^o, a_{+-}^o)^T, \quad (27.92a)$$

$$\vec{b} = (b_{-+}^i, b_{--}^i, b_{++}^o, b_{+-}^o, b_{++}^i, b_{+-}^i, b_{-+}^o, b_{--}^o)^T, \quad (27.92b)$$

and Ω_a, Ω_b are 8×8 matrices, given by

$$\Omega_a(E, \phi, D, \gamma) = \begin{bmatrix} T\Phi_{+,+}^i & T\Phi_{+,-}^i & T\Phi_{-,+}^o & T\Phi_{-,-}^o & -\Phi_{-,+}^i & -\Phi_{-,-}^i & -\Phi_{+,+}^o & -\Phi_{+,-}^o \end{bmatrix}, \quad (27.93a)$$

$$\Omega_b(E, \phi, D, \gamma) = \begin{bmatrix} -T\Phi_{-,+}^i & -T\Phi_{-,-}^i & -T\Phi_{+,+}^o & -T\Phi_{+,-}^o & \Phi_{+,+}^i & \Phi_{+,-}^i & \Phi_{-,+}^o & \Phi_{-,-}^o \end{bmatrix}. \quad (27.93b)$$

As before, $[\Phi, \Psi, \dots]$ denotes the matrix whose first column is Φ , second column Ψ , and so on. $T = T(\phi, D, \gamma)$ is the transfer matrix across the junction, defined in Eq. (27.85) above. The wavefunction $\Phi(x; E, \phi)$ in Eq. (27.90) represents a valid solution of the linearized BdG problem if and only if (i) the coefficients $a_{\sigma,\nu}^\chi, b_{\sigma,\nu}^\chi$ satisfy Eq. (27.91), and (ii) $\Phi(x; E, \phi)$ satisfies appropriate conditions at $x = \pm\infty$.

27.8.1 Bound states

We first consider states belonging to the discrete spectrum (bound states), which correspond to normalizable wavefunctions $\Phi(x; E, \phi)$. Normalizability requires that the wavevectors $\sigma k_\nu(E)$ appearing in the expansion (27.90) all have positive (negative) imaginary part for $x > 0$ ($x < 0$). A similar analysis to that in Appendix 27.6.1 shows that bound states can only exist at energies $|E| < \min(|\Delta - B|, \Delta)$, where all the k 's are imaginary. The bound state wavefunction then has the form (27.90) with $a_{\sigma,\nu}^\chi \equiv 0$. The $b_{\sigma,\nu}^\chi$ coefficients must still satisfy Eq. (27.91), which reduces to:

$$0 = \Omega_b(E, \phi, D, \gamma) \vec{b}. \quad (27.94)$$

This equation has a nontrivial solution for $\vec{b}$ (and hence a bound state exists at the given E and ϕ) if and only if $\det\{\Omega_b\} = 0$. Explicit calculation using Eqs. (27.88) and (27.93b) yields

$$\det\{\Omega_b\} = \frac{1}{2\Delta^2(\Delta^2 - B^2) \sin^2\eta_0 \sin\eta_- \sin\eta_+} \frac{\Lambda(E, \phi; D, \gamma)}{D^2}, \quad (27.95)$$

where

$$\Lambda(E, \phi; D, \gamma) = \Lambda_0(E, \phi) + (1 - D)\Lambda_1(E, \phi) + (1 - D)^2\Lambda_2(E, \phi) + (D \sin^2\gamma) \Lambda_\gamma(E, \phi), \quad (27.96)$$

with

$$\Lambda_0(E, \phi) = [\Delta^2 \cos^2(\phi/2) - E^2] [-\alpha k_- \alpha k_+ (1 + \cos^2(\phi/2)) - (E^2 + \Delta^2 - B^2) \sin^2(\phi/2)], \quad (27.97a)$$

$$\begin{aligned} \Lambda_1(E, \phi) = & [(E^2 + \Delta^2 - B^2)(2\Delta^2 \cos^2(\phi/2) - E^2) - 2\Delta^2 E^2] \sin^2(\phi/2) \\ & - \alpha k_0 \alpha k_- [\Delta(\Delta + B) \cos^2(\phi/2) - E^2] - \alpha k_0 \alpha k_+ [\Delta(\Delta - B) \cos^2(\phi/2) - E^2] \\ & + \alpha k_- \alpha k_+ [2\Delta^2 \cos^4(\phi/2) - E^2(1 + \cos^2(\phi/2))], \end{aligned} \quad (27.97b)$$

$$\begin{aligned} \Lambda_2(E, \phi) = & \Delta^2(\Delta^2 - B^2 + E^2) \sin^4(\phi/2) + \frac{1}{2}(\Delta^2 - B^2 - E^2) [\Delta^2 \cos \phi - E^2] - \frac{1}{2}B^2 E^2 \\ & + \frac{1}{2}\alpha k_0 \alpha k_- [\Delta(\Delta + B) \cos \phi - E^2] + \frac{1}{2}\alpha k_0 \alpha k_+ [\Delta(\Delta - B) \cos \phi - E^2] \\ & - \frac{1}{2}\alpha k_- \alpha k_+ [\Delta^2(1 - \frac{1}{2} \sin^2 \phi) - E^2], \end{aligned} \quad (27.97c)$$

$$\Lambda_\gamma(E, \phi) = [\Delta^2 \cos^2(\phi/2) - E^2] [\alpha k_- \alpha k_+ + \Delta^2 - B^2 - E^2], \quad (27.97d)$$

and

$$\alpha k_\nu(E) = i [(\Delta + \nu B)^2 - E^2]^{1/2}. \quad (27.98)$$

Note that Eq. (27.96) above reduces to $\Lambda_i(E, \phi) \cdot \Lambda_{o,+}(E, \phi) \cdot \Lambda_{o,-}(E, \phi)$ [see Eqs. (27.8), (27.10)] in the limit $D = 1$, $\gamma = 0$ (up to an overall multiplicative real, E -independent factor; the latter does not influence the spectrum of the junction).

The bound state energies $E_i^{\text{sc}}(\phi)$ (where the index i enumerates the bound states) are thus solutions of the equation $\Lambda(E, \phi; D, \gamma) = 0$ in the energy interval $|E| < \min(|\Delta - B|, \Delta)$. They can be obtained analytically only in particular limits, analyzed in detail in Appendix 27.9. In general, it is possible to determine the bound state energies $E_i^{\text{sc}}(\phi)$ numerically. To do so, we evaluate $\Lambda(E, \phi; D, \gamma)$ on a fine grid of E values spanning the range $[0, \min(|\Delta - B|, \Delta)]$, identify all intervals in which the function changes sign, and then apply a stable root-finding algorithm (Brent's method) within each sign-changing interval.

27.8.2 Scattering matrix

We next consider states belonging to the continuous spectrum (scattering states). For any E and ϕ such that $\det \Omega_b(E, \phi, D, \gamma) \neq 0$, we may solve Eq. (27.91) to express $\vec{b}$ as a linear function of $\vec{a}$:

$$\vec{b} = [\Omega_b(E, \phi, D, \gamma)]^{-1} \Omega_a(E, \phi, D, \gamma) \vec{a}. \quad (27.99)$$

The scattering states are then given by Eq. (27.90). In order for the wavefunctions to remain finite as $x \rightarrow \pm\infty$, the coefficient $\alpha_{\sigma,\nu}^x(E)$ must vanish unless the corresponding wavevector $k_\nu(E)$ is real. This condition fixes the number of linearly independent scattering states in the continuum at energy E .

The scattering matrix S at energy E is given by the block of $[\Omega_b(E, \phi, D, \gamma)]^{-1} \Omega_a(E, \phi, D, \gamma)$ corresponding to the modes that are propagating at that energy:

$$S(E, \phi, D, \gamma) = ([\Omega_b(E, \phi, D, \gamma)]^{-1} \Omega_a(E, \phi, D, \gamma))_{\text{prop}}. \quad (27.100)$$

Performing the matrix inversion and multiplication, we obtain an exact analytical expression for $S(E, \phi, D, \gamma)$. This expression is quite unwieldy at $D < 1$, so we omit it here. The determinant of the scattering matrix equals

$$\det S(E, \phi, D, \gamma) = \frac{[\Lambda(E, \phi; D, \gamma)]^*}{\Lambda(E, \phi; D, \gamma)}, \quad (27.101)$$

where $*$ denotes complex conjugation, and the function $\Lambda(E, \phi; D, \gamma)$ was defined in Eqs. (27.96) and (27.97); it is analytically continued past energy thresholds by using the appropriate complex values for the wavevectors:

$$\alpha k_\nu(E) = \begin{cases} [E^2 - (\Delta + \nu B)^2]^{1/2} & \text{if } E > |\Delta + \nu B|, \\ i [(\Delta + \nu B)^2 - E^2]^{1/2} & \text{if } E < |\Delta + \nu B|. \end{cases} \quad (27.102)$$

27.9 Approximate results on spectrum of the junction in the presence of scattering

In this Appendix we describe in detail how the approximate results on the spectrum of the system in the presence of scattering, which are presented in Sec. 27.5.1, can be obtained.

27.9.1 Hybridization of the bound states at $B < \Delta$

First, we discuss how the hybridization of the bound states, which happens near $\phi = \pi$ in the trivial phase, $B < \Delta$, can be described quantitatively by approximately solving the equation for the bound state energies, $\Lambda(E, \phi) = 0$. To begin with, we simplify the general expression for $\Lambda(E, \phi)$, Eq. (27.96), under the assumptions

$$|E| \ll \Delta - B \ll \Delta, \quad |\phi - \pi| \ll 1, \quad 1 - D \ll \frac{\Delta - B}{\Delta}, \quad |\gamma| \ll 1. \quad (27.103)$$

To leading order in the small parameters we find

$$\Lambda(E, \phi) \approx \frac{\Delta}{\Delta - B} \left[(E_1^2 - E^2)(E_0^2 - E^2) - 4(1 - D)(\Delta - B)\Delta(E^2 - E_0 E_1) + 4(1 - D)^2 \Delta^2 (\Delta - B)^2 \right]. \quad (27.104)$$

Here $E_0(\phi) \approx -\Delta(\phi - \pi)/2$, $E_1(\phi) \approx -(\Delta - B)(\phi - \pi)$ are the expressions for the energies of outer and inner-mode bound states [Eqs. (27.9) and (27.12)], evaluated at $|\phi - \pi| \ll 1$. Solving the equation $\Lambda(E, \phi) = 0$, we obtain the bound state energies $E_{0,1}^{\text{sc}}(\phi)$ that are given by Eq. (27.35).

Eq. (27.35) indicates that the two energy levels, $E_0^{\text{sc}}(\phi)$ and $E_1^{\text{sc}}(\phi)$, cross at $\phi = \pi$. This peculiarity of the lowest-order calculation is disrupted by a small forward scattering phase $|\gamma| \ll 1$. The latter produces a subleading correction $\delta\Lambda = -\Delta\gamma^2 E^4/(\Delta - B)$ to Eq. (27.104), which results in an anticrossing between the Andreev levels at $\phi = \pi$. By taking this correction into account, from $\Lambda = 0$ we find the bound state energies $E_{\pm}^{\text{sc}}(\phi)$ presented in Eq. (27.37). Equation (27.37) indicates that the anticrossing is manifested in a narrow vicinity $\sim |\gamma| \delta\varepsilon/\Delta \ll 1$ of $\phi = \pi$ only [recall, that $\delta\varepsilon$ describes the separation of the bound states' energies from zero at $\phi = \pi$ and is given by Eq. (27.36)]. It is characterized by an energy splitting $E_+^{\text{sc}}(\pi) - E_-^{\text{sc}}(\pi) = |\gamma| \delta\varepsilon \ll \delta\varepsilon$.

The anticrossing relies on the presence of a forward scattering phase. In the case $\gamma = 0$, the Andreev levels cross at $\phi = \pi$ even beyond the accuracy of Eq. (27.104). To see this, notice that the following relation between Λ_0 , Λ_1 , and Λ_2 holds at $\phi = \pi$ [see Eq. (27.97)]:

$$4\Lambda_0(E, \pi)\Lambda_2(E, \pi) = \Lambda_1^2(E, \pi). \quad (27.105)$$

Thus, at $\gamma = 0$, the expression (27.96) for $\Lambda(E, \pi)$ is a complete square; consequently, each energy level at $\phi = \pi$ is two-fold degenerate.

27.9.2 Hybridization between the outer-mode bound state and the states of the spectral continuum in the topological phase

Next, we employ the general expression for $\Lambda(E, \phi)$, Eq. (27.96), to describe how the energy of the outer-mode bound state, $E_0^{\text{sc}}(\phi)$, is affected by scattering in the topological phase, $B > \Delta$, near the continuum edge. In that, we focus on the vicinity of the topological transition and on phases ϕ close to π ,

$$0 < B - \Delta \ll \Delta, \quad |\phi - \pi| \ll 1. \quad (27.106)$$

Furthermore, we assume that the scattering is weak,

$$1 - D \ll 1, \quad |\gamma| \ll 1. \quad (27.107)$$

Given these approximations, Eq. (27.96) can be simplified considerably at $|E| \ll \Delta$. Performing a lowest-order expansion in small parameters, we find

$$\Lambda(E, \phi) \approx 2\Delta(B - \Delta) \left[E_0^2 - E^2 - 2\Delta(1 - D) \frac{E^2}{B - \Delta} - \Delta^2(1 - D)^2 + \sqrt{1 - \frac{E^2}{(B - \Delta)^2}} (E_0^2 - E^2 + \Delta^2(1 - D)^2) \right] \quad (27.108)$$

Here $E_0(\phi) \approx -\Delta(\phi - \pi)/2$ is the energy of the outer-mode bound state in a transparent junction.

By solving $\Lambda(E, \phi) = 0$ for the phase ϕ at $|E| = B - \Delta$, we establish that, in the presence of scattering, the energy E_0^{sc} of the outer-mode bound state reaches the continuum edge at points $\phi = \pi \pm \phi_0^{\text{sc}}$, where

$$\phi_0^{\text{sc}} \approx 2 \frac{B - \Delta}{\Delta} + 2(1 - D). \quad (27.109)$$

The behavior of $E_0^{\text{sc}}(\phi)$ near these points can be addressed concisely in the limit of perturbatively weak backscattering, $1 - D \ll (B - \Delta)/\Delta$. By taking

$$\delta\phi = \pi + \phi_0^{\text{sc}} - \phi \ll \phi_0^{\text{sc}}, \quad \delta E = B - \Delta + E \ll B - \Delta \quad (27.110)$$

[notice that we assume $-(B - \Delta) < E < 0$ here], we obtain the following approximate expression:

$$\Lambda(E, \phi) \approx 4\Delta(B - \Delta)^2 \left[\delta E + \Delta(1 - D) \sqrt{2\delta E/(B - \Delta)} - \Delta\delta\phi/2 \right]. \quad (27.111)$$

Then, by solving the equation for the bound state energies, $\Lambda = 0$, we arrive at expression (27.39) for $E_0^{\text{sc}}(\phi)$.

A small forward scattering phase, $|\gamma| \ll 1$, gives only a subleading correction to Eq. (27.108). This correction has a form

$$\delta\Lambda(E, \phi) \approx -2\Delta(B - \Delta)\gamma^2(E_0^2 - E^2) \left(\sqrt{1 - E^2/(B - \Delta)^2} + 1 \right). \quad (27.112)$$

It results in a shift of the points where the bound state reaches the continuum edge from $\pi \pm \phi_0^{\text{sc}}$ to $\pi \pm \phi_0^{\text{sc}, \gamma}$, where [for $1 - D \ll (B - \Delta)/\Delta$]

$$\phi_0^{\text{sc}, \gamma} \approx 2 \frac{B - \Delta}{\Delta} + 2(1 - D)(1 + \gamma^2/2). \quad (27.113)$$

This differs from ϕ_0^{sc} [see Eq. (27.109)] by a slight modification of the numeric prefactor in front of $1 - D$ only. Such prefactors experience the same modification in the expression for the bound state energy. For $\delta\phi = \pi + \phi_0^{\text{sc}, \gamma} - \phi \ll \phi_0^{\text{sc}, \gamma}$ we find [cf. Eq. (27.39)]

$$E_0^{\text{sc}, \gamma}(\phi) \approx -(B - \Delta) \left\{ 1 - \frac{\Delta^2}{2(B - \Delta)^2} \left[\sqrt{[(1 - D)(1 + \gamma^2/2)]^2 + \frac{B - \Delta}{\Delta} \delta\phi - (1 - D)(1 + \gamma^2/2)} \right]^2 \right\}. \quad (27.114)$$

27.9.3 Shallow bound states

As a next step, we use Eq. (27.96) to approximately find the energies of the shallow bound states, $E_s^{\text{sc}}(\phi)$. As stated in Sec. 27.5, such states appear near $\phi = 0, 2\pi$ in the topological phase in the presence of weak scattering, $1 - D, |\gamma| \ll 1$. First, we compute the function $\Lambda(E, \phi)$ at energies $B - \Delta - E \ll B - \Delta$ in the vicinity of the topological transition, $B - \Delta \ll \Delta$. We focus on phases $0 \leq \phi \ll 1$ [the energy of the bound state near $\phi = 2\pi$ can be obtained through $E_s^{\text{sc}}(2\pi - \phi) = E_s^{\text{sc}}(\phi)$]. Performing an expansion of Λ to leading order in $(B - \Delta)/\Delta$, ϕ , $1 - D$, γ , and $(B - \Delta - E)/(B - \Delta)$, we get

$$\Lambda(E, \phi) \approx 2\Delta^3(B - \Delta) \left\{ 2\sqrt{2\left(1 - \frac{E}{B - \Delta}\right)} - \left[1 - D + \gamma^2 - \frac{\phi^2}{4}\right] \right\}. \quad (27.115)$$

This expression demonstrates that the equation for the bound state energies, $\Lambda(E, \phi) = 0$, admits a solution $E_s^{\text{sc}}(\phi)$ in the interval $\phi \in [0, \phi_s)$ only (for $0 \leq \phi \ll 1$), where $\phi_s = 2\sqrt{1 - D + \gamma^2}$. The resulting $E_s^{\text{sc}}(\phi)$ is given by Eq. (27.41). Notice that the weakness of the scattering implies $B - \Delta - E_s^{\text{sc}}(0) \sim (B - \Delta)(1 - D + \gamma^2)^2 \ll (B - \Delta)$ and $\phi_s \ll 1$. These inequalities validate the applicability of the lowest-order expansion which was used to derive Eq. (27.115).

27.10 Josephson current in the presence of scattering

In this Appendix we describe a convenient approach to the calculation of the Josephson current (Sec. 27.10.1), which was used to produce Figs. 112 and 113. Additionally, we discuss how the approximate expressions for the Josephson current, Eqs. (27.44) and (27.46), were obtained (Sec. 27.10.2).

27.10.1 General expressions for $I(\phi)$

Trivial phase, $B < \Delta$

We first consider the junction in the trivial phase, $B < \Delta$, and discuss how the Josephson current $I(\phi)$ can be expressed directly in terms of $\Lambda(E, \phi)$. As a first step, we divide $I(\phi)$ [which is related to the ground state energy of the junction through Eq. (27.27)] into two contributions,

$$I(\phi) = I^{(1)}(\phi) + I^{(2)}(\phi). \quad (27.116)$$

The first contribution, $I^{(1)}(\phi)$, originates from the Andreev bound states. The second contribution, $I^{(2)}(\phi)$, comes from the states above the continuum edge, $E > \Delta - B$.

In the trivial phase (and in the presence of scattering) all bound states have positive energies, $E_b^{\text{sc}}(\phi) > 0$ [index “b” enumerates the bound states]. Therefore, these levels are not occupied in the ground state and the contribution $I^{(1)}(\phi)$ is given by

$$I^{(1)}(\phi) = -e \sum_b \partial_\phi E_b^{\text{sc}}(\phi). \quad (27.117)$$

The sum over the bound states in Eq. (27.117) can be expressed explicitly in terms of the function $\Lambda(E, \phi)$. Indeed, the energies of the bound states correspond to zeros of $\Lambda(E, \phi)$ in the complex plane of E (see discussion in Sec. 27.5). Thus, $I^{(1)}(\phi)$ can be represented as the following contour integral:

$$I^{(1)}(\phi) = e \int_{C_1} \frac{EdE}{2\pi i} \frac{\partial^2}{\partial E \partial \phi} \ln \Lambda(E, \phi). \quad (27.118)$$

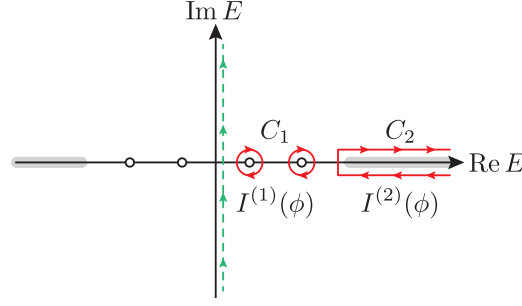


Figure 115: Integration contours C_1 and C_2 in the complex plane of energy that are featured in the expressions for $I^{(1)}(\phi)$ [Eq. (27.118)] and $I^{(2)}(\phi)$ [Eq. (27.120)], respectively. Empty circles on the real axis correspond to the bound state energies. Grey thick lines on the real axis depict branch cuts of $\Lambda(E, \phi)$ at $|E| > |\Delta - B|$. The combined contour $C_1 \cup C_2$ can be deformed into a contour running along the imaginary axis, depicted with a dashed line.

Here C_1 is a combination of contours that encircle all of E_b^{sc} (see Fig. 115). Next, we consider the contribution to the Josephson current due to the states with $E > \Delta - B$. By using Eqs. (27.17) and (27.32) we find

$$I^{(2)}(\phi) = -e \int_{\Delta-B}^{+\infty} \frac{EdE}{2\pi i} \frac{\partial^2}{\partial E \partial \phi} \ln \frac{\Lambda^*(E, \phi)}{\Lambda(E, \phi)}. \quad (27.119)$$

The function $\Lambda(E, \phi)$ has a branch cut at $E > \Delta - B$. Expressions for $\Lambda(E, \phi)$ on two sides of the cut are related by complex conjugation, $\Lambda(E + i0, \phi) = \Lambda^*(E - i0, \phi)$ [see Eq. (27.96)]. This allows us to represent Eq. (27.119) as a contour integral,

$$I^{(2)}(\phi) = e \int_{C_2} \frac{EdE}{2\pi i} \frac{\partial^2}{\partial E \partial \phi} \ln \Lambda(E, \phi), \quad (27.120)$$

where the contour C_2 is depicted in Fig. 115. Here we assume that relations (27.102) are valid at the upper side of the cut.

The two contributions to the Josephson current, Eqs. (27.118) and (27.120), can be combined into a single expression by unfolding the unified integration contour $C_1 \cup C_2$ to the imaginary axis (see dashed contour in Fig. 115). By doing this and then integrating by parts we obtain

$$I(\phi) = -\frac{e}{2\pi} \int_{-\infty}^{+\infty} d\mathcal{E} \partial_\phi \ln \Lambda(i\mathcal{E}, \phi). \quad (27.121)$$

This expression is well-suited for a numeric calculation of $I(\phi)$ at any $B \leq \Delta$, D , and γ . We used it to produce Figs. 112(a), (b).

We note that unfolding of the integration contour is well-justified mathematically. First, $\Lambda(E, \phi)$ does not have any zeros away from the real axis in the complex plane of E for the branch choice that we use [see relations (27.102) that are valid at the upper side of the positive-energy cut]. The latter corresponds to the “physical sheet” of the variable E [36]. Additionally, the integrand $\partial_\phi \ln \Lambda$ falls off like E^{-2} at $|E| \rightarrow \infty$. Therefore, the semicircular contour at infinity does not contribute to $I(\phi)$.

Topological phase, $B > \Delta$

Next, we obtain an expression for the Josephson current in the topological phase, assuming that the fermion parity of the junction is even.

To begin with, we note that at $\phi \in [-\pi, \pi]$ the Andreev bound state at the junction has positive energy $E_0^{\text{sc}}(\phi) > 0$. Consequently, it is not occupied in the ground state. Then,

following reasoning similar to that in Sec. 27.10.1, we conclude that at $|\phi| \leq \pi$ the Josephson current can be computed through Eq. (27.121).

On the other hand, Eq. (27.121) is not directly applicable at $|\phi| \in (\pi, 2\pi]$. Across $\phi = \pi$ the energy of the Andreev level $E_0^{\text{sc}}(\phi)$ becomes negative and the parity of the *global* ground state switches from even to odd [see Fig. 111]. In this case, Eq. (27.121) gives the Josephson current in the odd fermion parity sector. To get $I(\phi)$ in the even parity sector, Eq. (27.121) has to be modified to account for the difference in the bound state occupation between even and odd states. Then, for $|\phi| \in (\pi, \pi + \phi_0^{\text{sc}})$ (i.e., in the interval where the bound state is below the continuum edge) we find

$$I(\phi) = -\frac{e}{2\pi} \int_{-\infty}^{+\infty} d\mathcal{E} \partial_\phi \ln \Lambda(i\mathcal{E}, \phi) - 2e\partial_\phi E_0^{\text{sc}}(\phi). \quad (27.122)$$

At $|\phi| \in [\pi + \phi_0^{\text{sc}}, 2\pi - \phi_s]$ the energy of the bound state $E_0^{\text{sc}}(\phi)$ should be substituted in Eq. (27.122) by $-(B - \Delta)$. At $\phi \in (2\pi - \phi_s, 2\pi]$ it should be replaced by the energy of the shallow bound state, $-E_s^{\text{sc}}(\phi)$. The resulting expression was used to produce Figs. 112(c) and 113.

27.10.2 Approximate expression for the Josephson current

Estimate for the continuum contribution $I^{(2)}(\phi)$ near $\phi = \pi$ in the trivial phase, $B < \Delta$

Obtaining an approximate expression for $I(\phi)$ near $\phi = \pi$, Eq. (27.44), we calculated the contribution to the current due to the bound states $I^{(1)}(\phi)$ (with the help of Eq. (27.37)) and disregarded the continuum contribution $I^{(2)}(\phi)$. Here we confirm the validity of this approach for $|\phi - \pi| \ll 1$, $\Delta(1 - D) \ll \Delta - B \ll \Delta$, $|\gamma| \ll 1$.

To begin with, we note that both $I^{(1)}(\phi)$ and $I^{(2)}(\phi)$ vanish at $\phi = \pi$ and deviate from zero linearly away from this point. For $|\phi - \pi| \lesssim [(1 - D)(\Delta - B)/\Delta]^{1/2}$ the contribution due to the bound states can be estimated as [see Eq. (27.44)]

$$I^{(1)}(\phi) \sim e\Delta(\pi - \phi) \left(\frac{\Delta}{\Delta - B} \right)^{1/2} (1 - D)^{-1/2}. \quad (27.123)$$

To calculate the continuum contribution $I^{(2)}(\phi)$ we expand $\Lambda(E, \phi)$ to second order in $(\phi - \pi)$ and then estimate the integral in Eq. (27.119). As a result, we find

$$I^{(2)}(\phi) \sim e\Delta(\pi - \phi) \max \left\{ 1, \frac{\Delta^2}{(\Delta - B)^2} (1 - D) \right\}. \quad (27.124)$$

This estimate is valid as long as $|\phi - \pi| \ll (\Delta - B)/\Delta$. Eqs. (27.123) and (27.124) indicate that

$$I^{(2)}(\phi)/I^{(1)}(\phi) \ll 1 \quad (27.125)$$

within the whole phase interval $\delta\phi_{B < \Delta} \sim [(1 - D)(\Delta - B)/\Delta]^{1/2}$ around $\phi = \pi$, in which the smearing of the discontinuity happens. This is a result of the condition $1 - D \ll (\Delta - B)/\Delta$, which physically means that the energies of the bound states $\sim \delta\varepsilon = \sqrt{2\Delta(\Delta - B)(1 - D)}$ are well-separated from the continuum edge $E = \Delta - B$ close to $\phi = \pi$.

We note that for stronger scattering, $1 - D \sim (\Delta - B)/\Delta$, the bound states and the continuum states contribute equally to the Josephson current near $\phi = \pi$, $I^{(1)}(\phi) \sim I^{(2)}(\phi)$.

Approximate expression for the Josephson current in the topological phase, $B > \Delta$

Here we employ Eq. (27.122) to obtain an approximate expression [Eq. (27.46)] for the Josephson current under the assumptions $|\pi + \phi_0^{\text{sc}} - \phi| \ll \phi_0^{\text{sc}} \Delta(1 - D) \ll B - \Delta \ll \Delta$, $|\gamma| \ll 1$. We begin the calculation by analyzing the first term in Eq. (27.122). To compute this contribution, we employ the approximate expression for $\Lambda(E, \phi)$ given by Eq. (27.108). This expression can be further simplified in the limit we consider here. First, since $|\pi + \phi_0^{\text{sc}} - \phi| \ll \phi_0^{\text{sc}}$ and $\phi_0^{\text{sc}} \approx 2(B - \Delta)/\Delta$ we estimate $E_0 \approx (B - \Delta)$. Then, in virtue of inequality $\Delta(1 - D) \ll B - \Delta$ the terms $\Delta^2(1 - D)^2$ can be disregarded in comparison with E_0^2 . As a result, we find

$$\Lambda(i\mathcal{E}, \phi) = 2\Delta(B - \Delta) (E_0^2 + \mathcal{E}^2) \left[1 + \left(1 + \frac{\mathcal{E}^2}{(B - \Delta)^2} \right)^{\frac{1}{2}} \right]. \quad (27.126)$$

Consequently,

$$-\frac{e}{2\pi} \int_{-\infty}^{+\infty} d\mathcal{E} \partial_\phi \ln \Lambda(i\mathcal{E}, \phi) \approx -e\Delta/2. \quad (27.127)$$

Next, we account for the second term in Eq. (27.122). In the considered phase interval it is given by $-2e\Theta(\pi + \phi_0^{\text{sc}} - \phi)\partial_\phi E_0^{\text{sc}}(\phi)$. Combining this contribution with Eq. (27.127) and using the approximate expression for $E_0^{\text{sc}}(\phi)$, Eq. (27.39), we obtain Eq. (27.46).

28 Mesoscopic multiterminal Josephson structures. I Effects of nonlocal weak coupling by Amin, Omelyanchouk, Zagoskin

Abstract

We investigate nonlocal coherent transport in ballistic four-terminal Josephson structures (where bulk superconductors (terminals) are connected through a clean normal layer, e.g., a two-dimensional electron gas).

Coherent anisotropic superposition of macroscopic wave functions of the superconductors in the normal region produces phase slip lines (2D analogs to phase slip centres) and time-reversal symmetry breaking 2D vortex states in it, as well as such effects as phase dragging and magnetic flux transfer. The tunneling density of local Andreev states in the normal layer was shown to contain peaks at the positions controlled by the phase differences between the terminals.

We have obtained general dependence of these effects on the controlling supercurrent/phase differences between the terminals of the ballistic mesoscopic four-terminal SQUID.

28.1 Introduction.

The multiterminal Josephson junctions [1, 2] generalizes the usual (two-terminal) Josephson junctions [3] to the case of weak coupling between several massive superconducting banks (terminals). Compared with two-terminal junctions, such systems have additional degrees of freedom and the corresponding set of control parameters, preset transport currents and (or) applied magnetic fluxes. As a result, the current- or voltage-biased and the magnetic flux-driven regimes can be combined in one multiterminal microstructure.

One of the realizations of multiterminal coupling is a system of short dirty microbridges going from a common center to separate massive superconductors. The theory of this kind of multi-terminals was derived in [4, 5] within the phenomenological Ginzburg-Landau scheme (Aslamazov and Larkin model [6]). This approach is valid for temperatures T near the critical temperature T_c and for the local case when the characteristic spatial scale is larger then

the coherence length $\xi_0 \sim \hbar v_F / T_c$. The stationary states and the dynamical behaviour of the microbridge type multiterminals were studied for different microstructures, four-terminal SQUID controlled by the transport current, weakly coupled superconducting rings (see review of theoretical and experimental results in [7, 8]).

The Josephson effect in mesoscopic weak links with direct conductivity (S-N-S junctions, ballistic point contacts) exhibits specific features [9],[10] which are absent in conventional dirty microconstrictions near T_c [6]. As in normal metal mesoscopic structures [11], the electrodynamics of supercurrents in mesoscopic Josephson junctions is nonlocal. Supercurrent density depends on the spatial distribution of the superconducting order parameter in all points of the mesoscopic weak link region. The coherent current flow is carried by the Andreev states [12] formed inside the weak link. Nonlocal nature of mesoscopic supercurrents was demonstrated by Heida *et al.*[13], investigating the mesoscopic S-2DEG-S (superconductor-two dimensional electron gas - superconductor) Josephson junctions. They measured $2\Phi_0$ periodicity of the critical current instead of the standard Φ_0 ($\Phi_0 = hc/2e$ is the magnetic flux quantum). Theory of this effect was developed in Refs.[14, 15].

The current level of nanofabrication technology made it possible to realize multi-terminal mesoscopic Josephson junction similar to the 2-terminal junction studied in [13]. The microscopic theory of the mesoscopic ballistic Josephson multi-terminals was derived in Ref.[16]. It is valid for arbitrary temperatures $0 < T < T_c$ and describes the nonlocal coherent current states in the system. The effects of nonlocal coupling, such as phase dragging and magnetic flux transfer were obtained in Ref.[17].

In the present paper we continue study of quantum interference effects in mesoscopic multiterminals, which are related to the nonlocality of weak coupling. The paper consists of two parts. In first part (Article I) the effects of nonlocal coupling in mesoscopic multiterminal structures are studied. The general properties of Josephson multiterminals are described in Section 2. Section 3 gives the results concerning the current distribution and density of states inside weak link. In Section 3 we study specific for mesoscopic case properties of four-terminal SQUID. In the second part (Article II) a superconducting phase qubit based on mesoscopic multiterminal junction is proposed and investigated.

28.2 Mesoscopic Four-Terminal Junction.

28.2.1 System description

In a mesoscopic 4-terminal junction, the bulk superconductors (terminals) are weakly coupled to each other through a clean two-dimensional normal metal layer (2D electron gas) as it is shown in Fig.1. The pairs of terminals can be incorporated in bulk superconducting rings or in circuits with preset transport currents. In Fig.2 we show two such configurations. The first one (Fig.2a) presents two superconducting rings, each interrupted by a Josephson junction, which are at the same time weakly coupled to each other. The second configuration (Fig.2b), combines a current (or voltage) biased junction and a flux driven junction in the ring. We call this configuration the 4-terminal SQUID controlled by the transport current.

The state of the i -th terminal S_i ($i=1...4$) is determined by the phase φ_i of the complex off-diagonal potential $\Delta_0 \exp(i\varphi_i)$. Superconducting banks induce the order parameter Ψ in the normal metal region (shaded area in Fig.1). Inside this mesoscopic, fully phase coherent weak link, the supercurrent density $\mathbf{j}(\vec{\rho})$ at point $\vec{\rho}$ *nonlocally* depends on the values of the induced order parameter Ψ at all points $\vec{\rho}'$. In its turn, the order parameter $\Psi(\vec{\rho})$ depends on the phases φ_i . The total current I_i , flowing into i -th terminal depends on the phases φ_j of all the

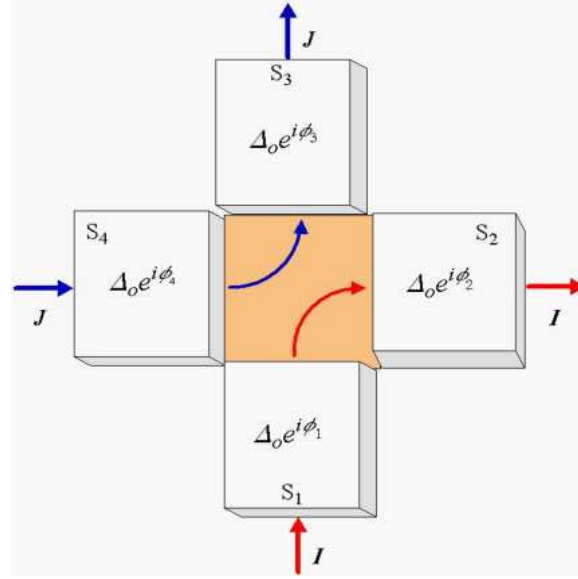


Figure 116: Mesoscopic four-terminal Josephson junction with “parallel” implementation of the supercurrents. The four bulk superconducting regions, $S_1 \dots S_4$, are weakly coupled through the thin layer of normal metal (2DEG), represented by the shaded area.

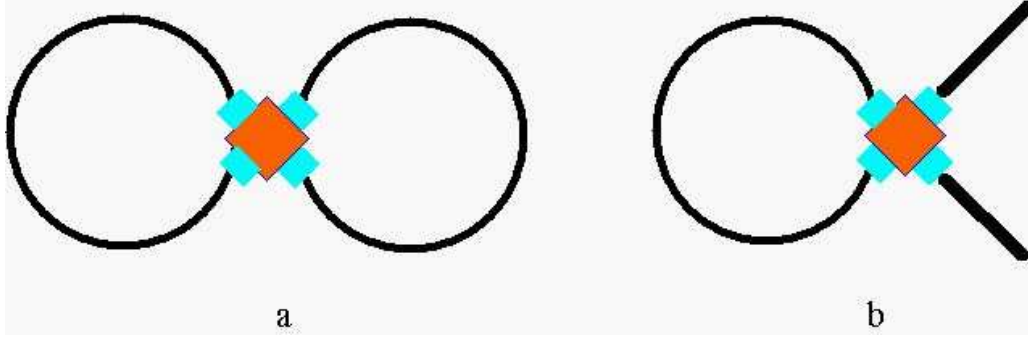


Figure 117: Superconducting microstructures based on mesoscopic four-terminal Josephson junctions. (a) Two weakly coupled superconducting rings. (b) Mesoscopic four-terminal SQUID.

banks and has the form [16] :

$$I_i = \frac{\pi \Delta_0}{e} \sum_{j=1}^4 \gamma_{ij} \sin \left(\frac{\varphi_i - \varphi_j}{2} \right) \tanh \left[\frac{\Delta_0 \cos \left(\frac{\varphi_i - \varphi_j}{2} \right)}{2T} \right]. \quad (28.1)$$

In the case of two terminals Eq. (1) reduces to the formula for ballistic point contact [10] with γ_{12} equals to Sharvin’s conductance.

Expression (1) corresponds to the case of small junction, when linear dimensions of the N-layer are smaller than the coherence length $\xi \sim \hbar v_F / \Delta_0$ (for the case of arbitrary junction’s dimensions see Ref.[16]). We are focusing here on the small junction case because the effects of nonlocality are most pronounced in this situation. The geometry dependent coefficients γ_{ij} denote the coupling between the partial Josephson currents in ballistic two-terminal S_i - S_j weak links.

Eq. (1) is simplified when $T \approx 0$, or when $T \approx T_c$. In $T = 0$ limit, it becomes

$$I_i = \frac{\pi \Delta_0(0)}{e} \sum_{j=1}^4 \gamma_{ij} \sin\left(\frac{\varphi_i - \varphi_j}{2}\right) \text{sign}\left[\cos\left(\frac{\varphi_i - \varphi_j}{2}\right)\right]. \quad (28.2)$$

Near T_c on the other hand, the order parameter is small, $\Delta_0 \rightarrow 0$ and one can write

$$I_i = \frac{\pi \Delta_0(T)^2}{4eT_c} \sum_{j=1}^4 \gamma_{ij} \sin(\varphi_i - \varphi_j). \quad (28.3)$$

Equations (2) and (3) are qualitatively similar, differing by the magnitude of critical currents and by the shape of current-phase dependencies ($\sin(\frac{\varphi}{2})\text{sign}[\cos(\frac{\varphi}{2})]$ and $\sin(\varphi)$). For definiteness, in the following we will consider the case of $T \sim T_c$, keeping in mind that the results hold qualitatively at low temperatures as well.

For the Josephson coupling energy of the junction E_J , which relates to the supercurrents I_i (3) through $I_i = (2e/\hbar)\partial E_J/\partial \varphi_i$, we have

$$E_J(\varphi_i) = \frac{\hbar}{2e} \frac{\pi \Delta_0(T)^2}{4eT_c} \sum_{j < k} \gamma_{jk} [1 - \cos(\varphi_j - \varphi_k)]. \quad (28.4)$$

Expression (1) for supercurrents I_i looks similar to Buttiker's multiprobe formula [18]

$$I_i = e \sum_j T_{ij} (\mu_i - \mu_j) \quad (28.5)$$

which relates the currents to the voltage drops between terminals in mesoscopic normal metal multiterminal system. The similarity reflects the above mentioned nonlocality of mesoscopic transport on the scale of $\xi_T \sim \hbar v_F/T$ (in the ballistic limit we are considering). The essential difference between (1) and (5) is that unlike Josephson currents of (1), the normal currents of (5) can flow only out of equilibrium; while the current-phase dependence in (1) is periodic, the current-bias dependence of (5) is certainly not.

28.2.2 Circuit implementations of 4-terminal junction. Nonlocal weak coupling.

The current-phase relations (3) determine the behaviour of the system in the presence of the transport currents and/or the diamagnetic currents induced by the magnetic fluxes through the closed superconducting rings. It is necessary to distinguish two types of circuit implementation of the mesoscopic 4-terminal junction [17]. The first one, is the "crossed" or "transverse" implementation, when the total current in one circuit goes in and out through one pair of opposite banks in Fig.1 and in the second circuit - through the other pair. In the "parallel" implementation, shown in Fig.1, the currents I and J flow through the pairs of adjacent banks. In this case, nonlocal coupling of currents inside the mesoscopic N-layer results in peculiar effect of "dragging" of the phase difference between one pair of terminals by the phase difference between another pair of terminals [17]. In the following, we consider the "parallel" implementation and study the manifestations of the phase dragging effect.

The coefficients γ_{ij} in (3,4) depend on the geometry of the weak link (the shape of the N-layer) and on the transparency of S-N interfaces. In general we have $\gamma_{ij} = \gamma_{ji}$ and $\gamma_{ii} = 0$. For the case of parallel implementation, the elements γ_{12} and γ_{34} are related to the critical currents of the individual sub-junctions S_1 - S_2 and S_3 - S_4 respectively. The matrix

$$\hat{\gamma}_{\text{coupl}} = \begin{pmatrix} \gamma_{13} & \gamma_{14} \\ \gamma_{23} & \gamma_{24} \end{pmatrix} \quad (28.6)$$

describes the coupling between these two junctions. Properties of the system (in particular the existence of the phase dragging) qualitatively depends on whether $\det(\hat{\gamma}_{\text{coupl}})$ equals to zero or not (see Appendix). In case of a conventional non-mesoscopic 4-terminal Josephson junction the coefficients γ_{ij} factorize, $\gamma_{ij} \sim (1/R_i)(1/R_j)$, where R_i are the normal resistances of dirty microbridges [7]. This yields $\det(\hat{\gamma}_{\text{coupl}}) \equiv 0$, which we call local coupling. On the other hand, in a mesoscopic system, even in a completely symmetric case of $a \times a$ square N-layer and ideal transparency ($D = 1$) of N-S_i interfaces, the coefficients γ_{ij} are given by [17]:

$$\gamma_{12} = \gamma_{34} = \gamma_0, \quad \hat{\gamma}_{\text{coupl}} = \gamma_0 \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}, \quad \gamma_0 = \frac{e^2 p_F a}{\sqrt{2} \hbar^2 \pi^2} \left(1 - \frac{1}{\sqrt{2}}\right), \quad (28.7)$$

with $\det(\hat{\gamma}_{\text{coupl}}) \neq 0$. In more general case than the completely symmetric one (Eq. (7)), we can write γ_{ij} in the form

$$\gamma_{34} = \kappa \gamma_{12} \quad \hat{\gamma}_{\text{coupl}} = \begin{pmatrix} p & q \\ q & p \end{pmatrix}. \quad (28.8)$$

This corresponds to a square N-layer, with different transparencies for junctions S₁-S₂ and S₃-S₄ and/or different width of the superconductor banks connected to the normal layer. In our numerical calculations we will use the simple form (7), i.e. $\kappa = 1$, $p = \sqrt{2}$, $q = 1$.

28.2.3 Current-phase relations. The phase dragging effect.

Let us introduce new variables:

$$\begin{aligned} \varphi_2 - \varphi_1 &= \theta, & \varphi_3 - \varphi_4 &= \phi \\ \frac{1}{2}(\varphi_1 + \varphi_2) &= \alpha, & \frac{1}{2}(\varphi_4 + \varphi_3) &= \beta \\ \alpha - \beta &= \chi, & \alpha + \beta &= \gamma. \end{aligned} \quad (28.9)$$

Without loss of generality, we can choose the phase γ equal to zero ($\sum_j \phi_j = 0$). For the circuit implementation shown in Fig.1, we have

$$I = I_2 = -I_1, \quad J = I_3 = -I_4. \quad (28.10)$$

In terms of phase differences (9) the currents I and J have the form

$$I = \sin \theta + \left[(p+q) \sin \frac{\theta}{2} \cos \frac{\phi}{2} + (p-q) \cos \frac{\theta}{2} \sin \frac{\phi}{2} \right] \cos \chi, \quad (28.11)$$

$$J = \kappa \sin \phi + \left[(p+q) \sin \frac{\phi}{2} \cos \frac{\theta}{2} + (p-q) \cos \frac{\phi}{2} \sin \frac{\theta}{2} \right] \cos \chi. \quad (28.12)$$

All γ 's (8) are normalized by γ_{12} and the currents I, J are measured in units of $I_0 = \pi \gamma_{12} \Delta_0(T)^2 / 4eT_c$.

From the current conservation (Eq.(10)), it follows that the phase χ in Eqs. (11) and (12) can take only two values, 0 or π . Minimization of E_J (4) with respect to χ also gives $\chi = 0$ or π , depending on the equilibrium values of θ and ϕ (see Appendix):

$$\cos \chi = \text{sign} \left[(p+q) \cos \frac{\phi}{2} \cos \frac{\theta}{2} - (p-q) \sin \frac{\phi}{2} \sin \frac{\theta}{2} \right]. \quad (28.13)$$

The current-phase relations (11) and (12) with the condition (13) are invariant under the transformation $\theta \rightarrow \theta + 2\pi n$, and $\phi \rightarrow \phi + 2\pi k$. The 2π periodicity of observable quantities

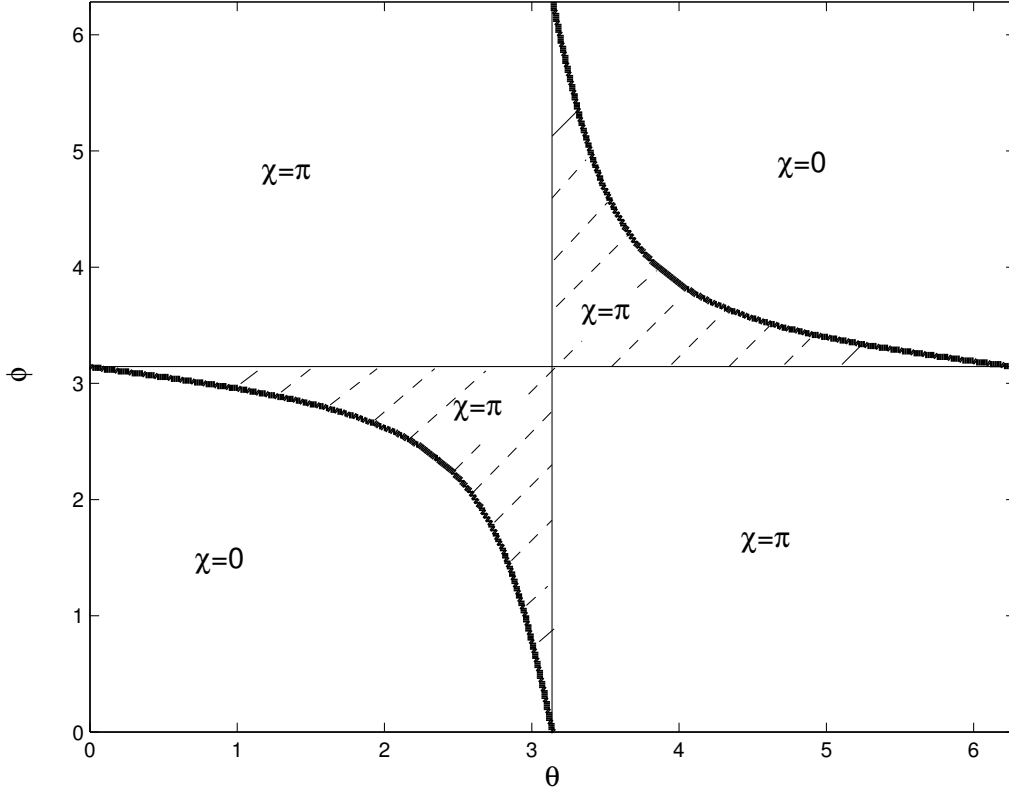


Figure 118: Phase diagram for the phase difference χ in the (θ, ϕ) -plane. Solid line separates the regions with $\chi = 0$ and $\chi = \pi$. The dashed region is absent in the case of local coupling.

is sustained by the “hidden” variable phase χ . In Fig.3 the phase diagram for χ in the (θ, ϕ) plane is presented. The solid line separates the regions with $\chi = 0$ and $\chi = \pi$. When the state of the system (θ, ϕ) crosses this line, a jump in χ occurs. Corresponding jumps take place in current-phase relations (11) and (12). The current $I(\theta)$ (11) is shown in Fig.4 for several values of the phase ϕ . Note that the function $I(\theta)$ has jumps, which for $\phi \neq 0$, are located not at $\theta = \pm\pi$, as they would be in conventional 4-terminal junctions. The jump in χ means the slippage of the phase θ (or ϕ). In the case of two-terminal or conventional 4-terminal junction the phase-slip events occur at phase difference equal to $\pi(2n + 1)$, $n = 0, \pm 1, \pm 2, \dots$. In one-dimensional structures slippage of the phase occurs at phase-slip centers (PSC), i.e. points where the order parameter equals to zero. In our case of 2D mesoscopic 4-terminal weak link, the analog of the PSC are phase-slip lines in normal metal region. They appear when the state of system (θ, ϕ) belongs to the dashed region in Fig.3. This region, which is absent in the local coupling case (it actually coincides with lines $\theta = \pi, \phi = \pi$), we call “frustrated” region for phases θ and ϕ . For states inside this region, the distribution of the supercurrent in the weak link contains 2D vortex states (see below).

Nonlocal weak coupling leads to the phase dragging effect [17]. One notices that if $p \neq q$ then putting $\theta = 0$ in (11) results in a nonzero value of the current I

$$I = (p - q) \sin \frac{\phi}{2} \operatorname{sign} \left(\cos \frac{\phi}{2} \right) \quad (28.14)$$

This current is absent in conventional 4-terminal junctions or mesoscopic four terminal junctions with crossed implementation at which $p = q$ (i.e. $\det(\gamma_{\text{coupl}}) = 0$).

Similarly if we set $I = 0$ in (11), we find a nonzero solution for θ , which again vanishes when $p = q$. This solution ($\equiv \theta_d$) is a function of ϕ and is plotted versus ϕ in Fig.5. The influence

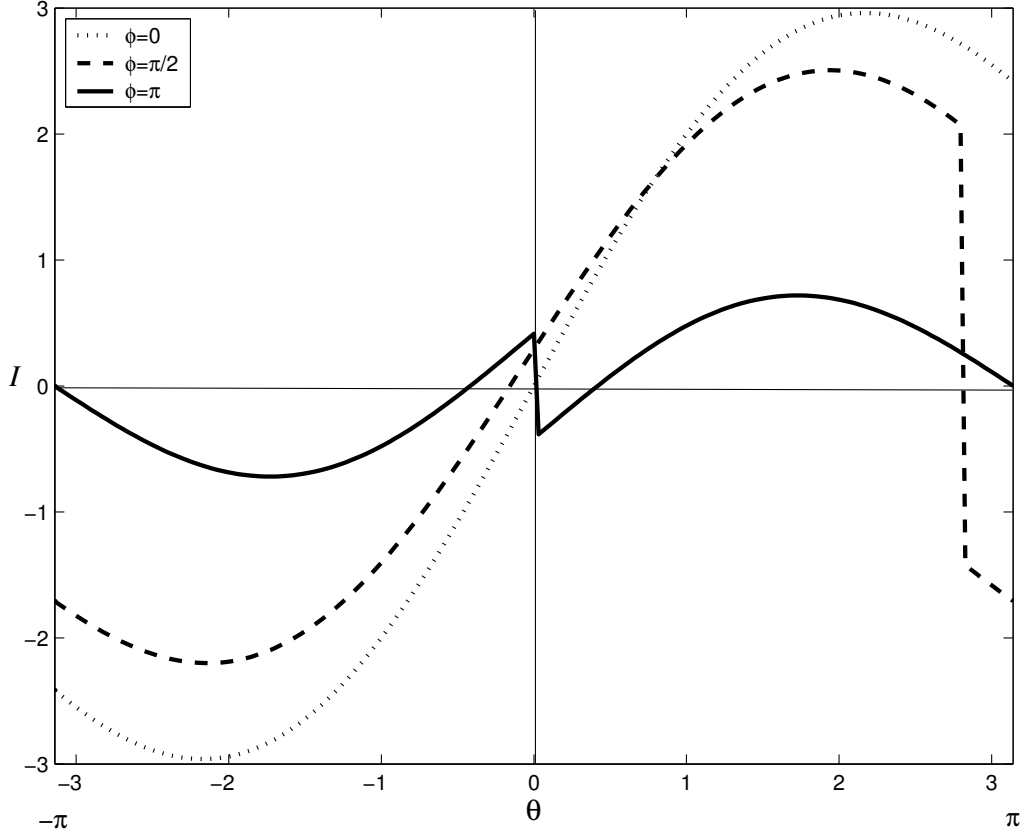


Figure 119: Current-phase relations $I(\theta)$ for different values of ϕ .

of the phase of one side of the mesoscopic 4-terminal junction on the phase of the other side is what we call phase dragging effect. This effect is one of important characteristics of the with parallel implementation.

In general current-phase relations are asymmetric, $I(-\theta) \neq -I(\theta)$, unlike in conventional cases. In another words, the presence of phase difference ϕ on the terminals S_4 - S_3 breaks the time reversal symmetry for Josephson junction S_1 - S_2 . It also follows from expression (11) that $I(\theta)$ is not only a function of $|\phi|$, as in conventional junctions, but also depends on the sign of ϕ . The phase dragging has the analogy in the normal metal mesoscopic multiterminals, described by formula (5); the normal current flowing through one pair of terminals induces a voltage difference between the other ones [18].

28.3 Current distribution and local density of states inside the mesoscopic weak link

The coupling through the normal layer determines the behaviour of the Josephson weak links S_1 - S_2 and S_4 - S_3 . On the other hand, the properties of the normal layer itself depend on the phase differences θ and ϕ on the junctions. The phases θ and ϕ can be controlled by external magnetic fluxes through the rings (Fig.2a). In this section we present the results of numerical calculations for current density distribution $\mathbf{j}(\vec{\rho})$ and density of local Andreev levels $N(\epsilon)$ inside the mesoscopic 4-terminal weak link. The expressions for $\mathbf{j}(\vec{\rho})$ and $N(\epsilon)$ as functionals of $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ were obtained in Ref.[16] by solving Eilenberger equations [19].

Fig.6 illustrates the effect of phase dragging. Two sets of phases ($\theta = -0.42$, $\phi = \pi$) and ($\theta = 0.42$, $\phi = \pi$) correspond to zero value of the current I (11) (see Fig.4) and opposite directions of the current J (12). In the absence of the current from terminal S_1 to terminal S_2 ,

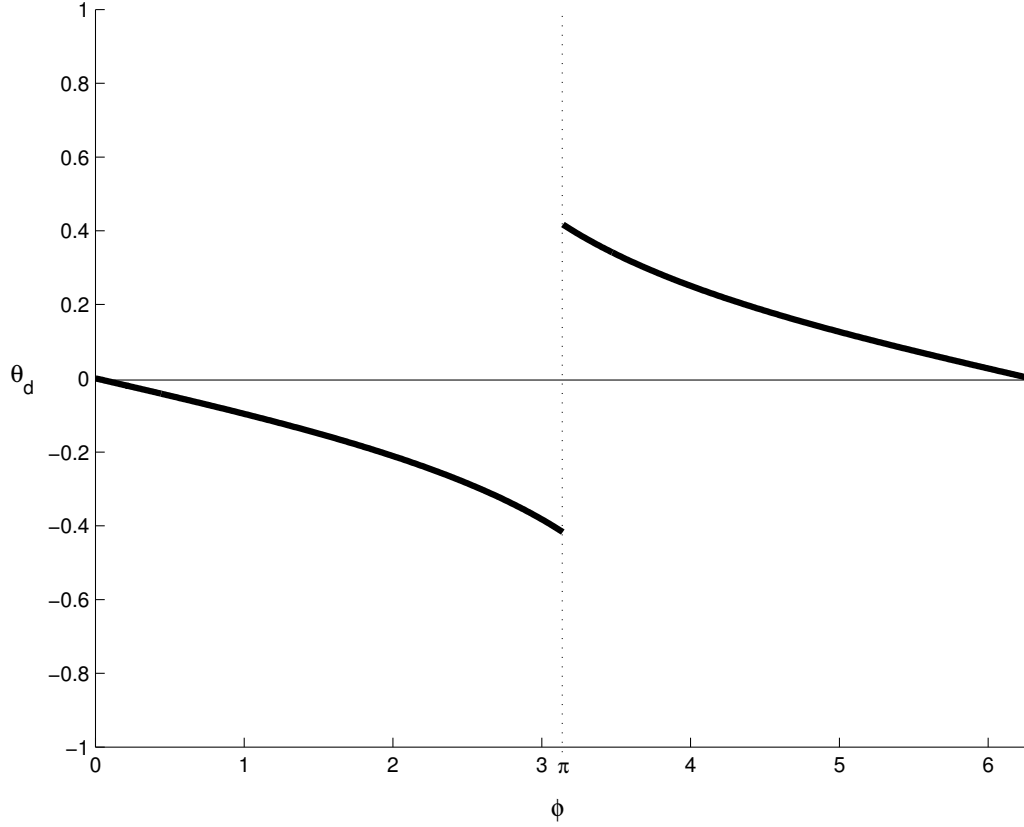


Figure 120: The dragged phase θ_d between terminals S_1 - S_2 , at zero transport current I , as a function of the phase difference ϕ between the other pair of terminals S_3 - S_4 .

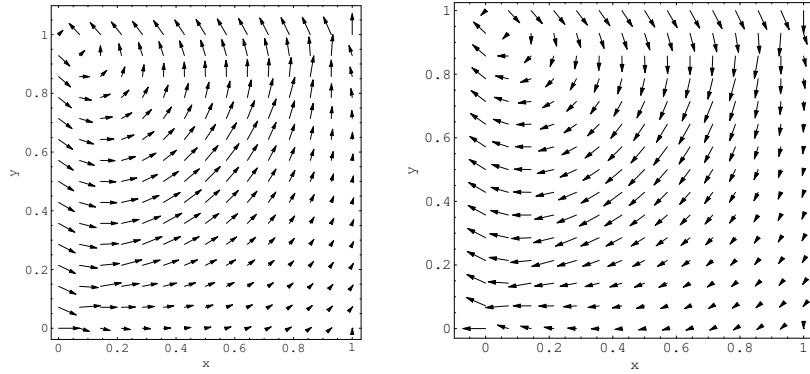


Figure 121: Distribution of the current density inside the normal layer for phase $\phi = \pi$ and two values of the phase θ at which the current $I = 0$ (Fig.4). (a) $\theta = -0.42$, (b) $\theta = 0.42$.

the phase difference on the junction S_1 - S_2 exists.

When the phases θ and ϕ lie in the “frustrated” region of the diagram Fig.3 (dashed area), the current distribution $\mathbf{j}(\vec{\rho})$ contains 2D vortex states. They are shown in Fig.7 for states $(\theta = \pi - 0.2, \phi = \pi - 0.2)$ and $(\theta = \pi + 0.2, \phi = \pi + 0.2)$. In both cases, the order parameter $\Psi(\vec{\rho})$ vanishes along the diagonal $x = y$ and its phase drops by π when crossing this 2D phase-slip line.

The Andreev scattering processes on the S_i -N interfaces lead to the appearance of the energy levels with energies ϵ inside the gap Δ_0 , $|\epsilon| < \Delta_0$, in the normal metal. The local density of

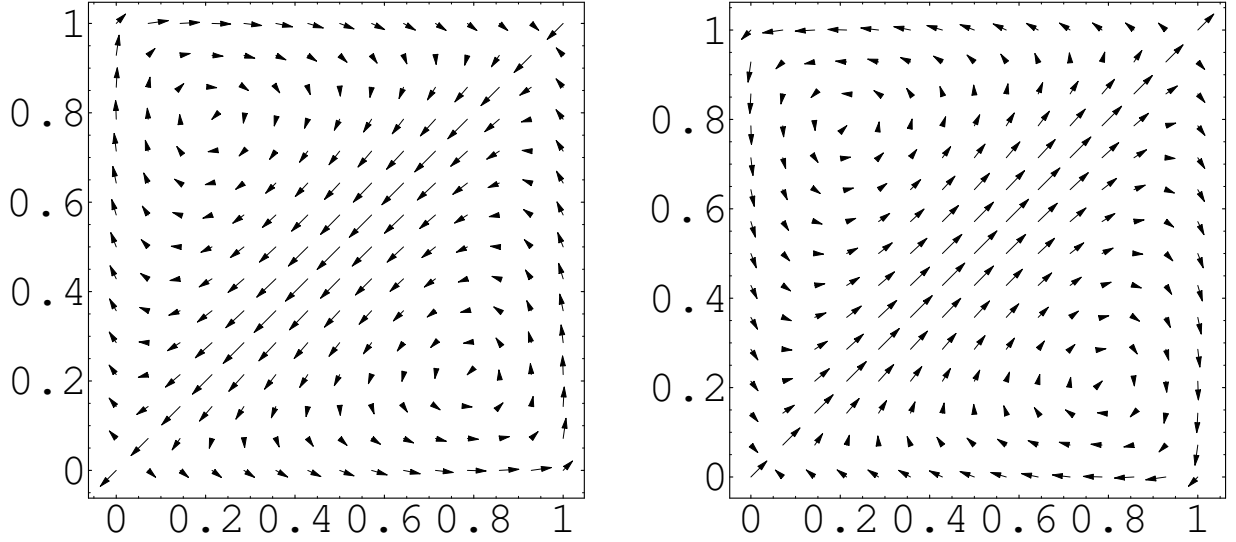


Figure 122: The vortex-like distributions of the current inside the weak link when θ and ϕ are inside the frustrated region: (a) $\theta = \phi = \pi - 0.2$, (b) $\theta = \phi = \pi + 0.2$.

electron states in the normal layer is given by the formula

$$\mathcal{N}(\epsilon, \vec{\rho}) = N(0) < \text{Re } g(\omega = -i\epsilon, \vec{\rho}, \mathbf{v}_F) >_{\mathbf{v}_F}. \quad (28.15)$$

($g(\omega, \vec{\rho}, \mathbf{v}_F)$) is Eilenberger Green's function). We have studied the dependence of the density of states, averaged over area of the N-layer, $N(\epsilon)$, on the phases θ and ϕ . This tunneling density of states can be measured by scanning tunneling microscope. It contains the spikes with intensity and position on the energy axes controlled by the phases θ and ϕ . The results are shown in Fig.8 (the δ -function singularities in $N(\epsilon)$ are smeared by introducing a small damping $\Gamma = 0.01\Delta_0$).

28.4 Mesoscopic Four-Terminal SQUID

In this section we consider the four-terminal SQUID configuration (Fig.2b). Conventional 4-terminal SQUID has been studied in detail in Ref. [5], wherein the steady states domain and dynamical properties of the system were calculated. Here we are interested in the specific features of the mesoscopic case reflected in the current-phase relations (11,12). As we have seen in the previous section, the nonlocal coupling ($p \neq q$) leads to the phase dragging effect. This dragged phase can induce a transferred magnetic flux in the ring which depends on the transport current. On the other side, the magnetic flux state in the ring influences the behaviour of the Josephson junction in the current circuit.

When the terminals 3 and 4 are short circuited by a superconducting ring with self-inductance L , the phase ϕ is related to the observable quantity, magnetic flux threading the ring Φ , $\phi = \frac{2e}{\hbar}\Phi$. The current J circulating in the ring is given by $J = (\Phi^e - \Phi)/L$, where Φ^e is the external magnetic flux threading the ring. From (11) and (12) we have

$$I = \sin \theta + \left[(p+q) \sin \frac{\theta}{2} \cos \frac{\Phi}{2} + (p-q) \cos \frac{\theta}{2} \sin \frac{\Phi}{2} \right] \cos \chi, \quad (28.16)$$

$$\frac{\Phi^e - \Phi}{\mathcal{L}} = \sin \Phi + \frac{1}{\kappa} \left[(p+q) \sin \frac{\Phi}{2} \cos \frac{\theta}{2} + (p-q) \cos \frac{\Phi}{2} \sin \frac{\theta}{2} \right] \cos \chi, \quad (28.17)$$

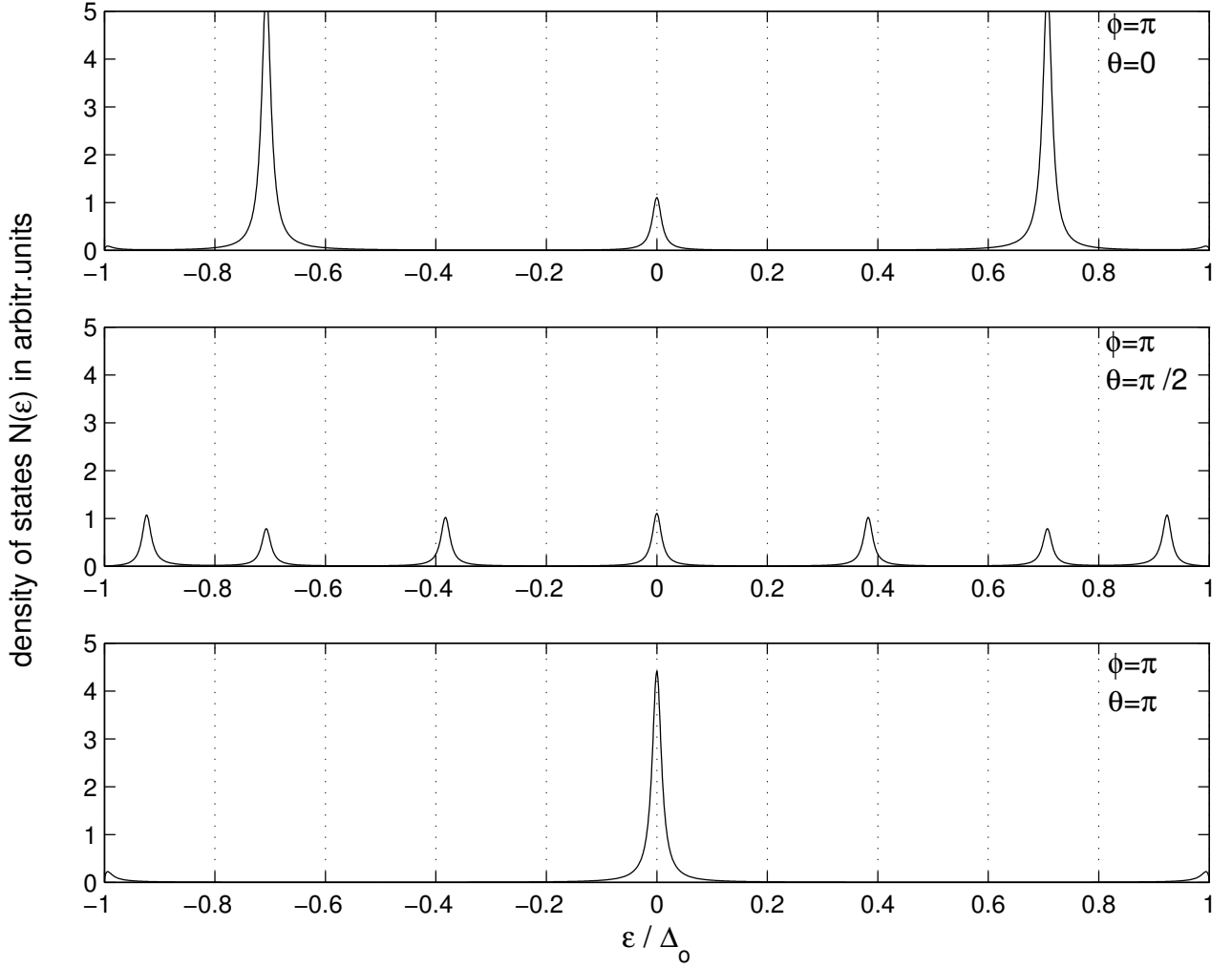


Figure 123: Density of states, $N(\epsilon)$, averaged over the normal region for different values of θ and ϕ .

where fluxes Φ, Φ^e are measured in units $\hbar/2e$, $\mathcal{L} = (2e/\hbar)LI_0\kappa$ is the dimensionless self-inductance. The parameter $\kappa = \gamma_{34}/\gamma_{12}$ is the ratio of the critical currents of the sub-junctions 3–4 and 1–2. The limiting cases of $\kappa \rightarrow \infty$ and $\kappa \rightarrow 0$ correspond to the autonomous SQUID and the current biased Josephson junction, respectively.

The transport current I and the external flux Φ^e are the external controlling parameters. The corresponding Gibbs potential for the 4-terminal SQUID takes the form

$$U(\Phi, \theta; I, \Phi^e) = \frac{\kappa(\Phi - \Phi^e)^2}{2\mathcal{L}} - I\theta - \cos(\theta) - \kappa \cos(\Phi) - 2 \left((p+q) \cos \frac{\theta}{2} \cos \frac{\Phi}{2} - (p-q) \sin \frac{\theta}{2} \sin \frac{\Phi}{2} \right) \cos(\chi) \quad (28.18)$$

The last three terms in Eq.(18) are the Josephson coupling energy (4) in terms of variables θ, Φ and χ . The minimization of U with respect to χ gives the expression (13) for $\cos(\chi)$, with ϕ replaced by Φ . At given values of the control parameters I and Φ^e , the relations (16) and (17) (together with Eq.(13)) determine the set of possible states of the system $\{\theta, \Phi\}$, among which we should choose those that correspond to the local minima of the potential U , Eq.(18). Let us consider the effect of the magnetic flux state of the ring on the behaviour of the current driven junction. The critical current of the junction, I_c , depends on the applied magnetic flux

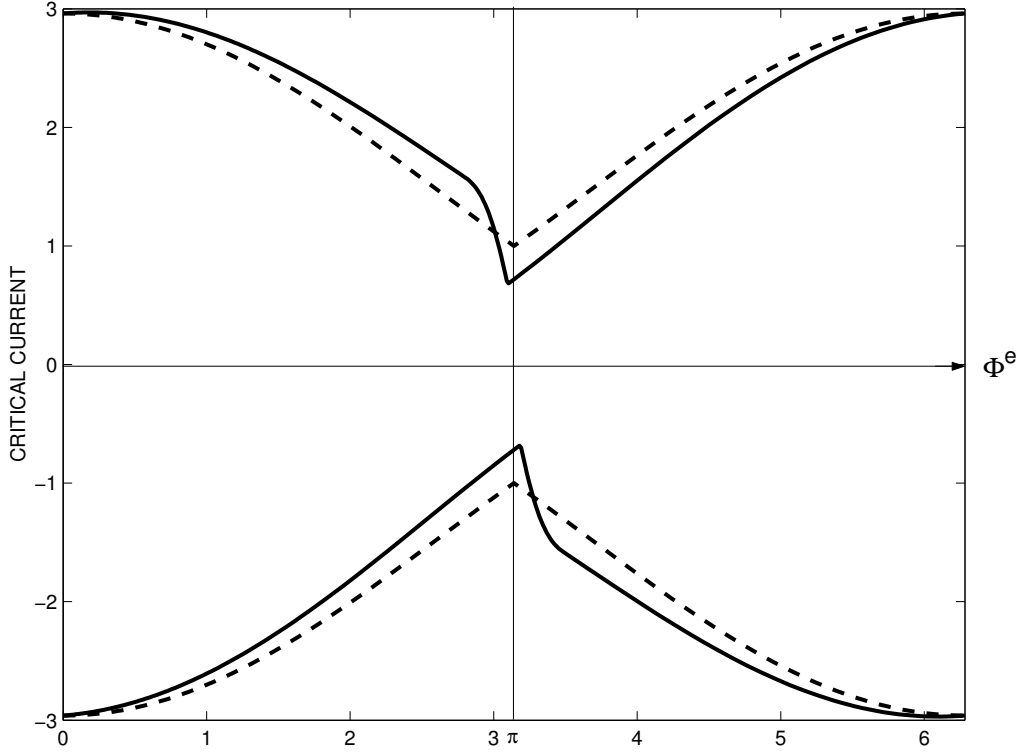


Figure 124: The steady state domain for mesoscopic four-terminal SQUID in plane (I, Φ_e) of the control parameters (solid line). Dashed line corresponds to the conventional four-terminal SQUIDs.

Φ^e . In the simplest case of small self-inductance $\mathcal{L} \ll 1$, we can neglect the difference between Φ and Φ^e in expression (16).

The maximal value of the supercurrent I (16) (with Φ replaced by Φ^e) as a function of Φ^e , $I_{max}(\Phi^e)$, is shown in Fig.9. This curve determines the boundary of the steady states domain in the (I, Φ^e) plane. The function $I_{max}(\Phi^e)$ is 2π periodic, but due to the terms proportional to $p - q$ in Eq.(16), it is not invariant under the transformation $\Phi^e \rightarrow -\Phi^e$. The symmetry is restored if we simultaneously change Φ^e on $-\Phi^e$ and I on $-I$. Note, that in conventional case ($p = q$) the boundary of the steady state domain $I_{max}(\Phi^e)$ is symmetric with respect to the axes (I, Φ^e) (dashed line in Fig.9). Thus, the critical current I_c in the transport current circuit, for a given direction of the current, depends on the sign of the magnetic flux in the ring. For finite values of self-inductance $\mathcal{L}$ equations (16) and (17) must be treated self-consistently. The critical current I_c as function of $\mathcal{L}$ is shown in Fig.10 for two values of external flux, $\Phi^e = 0$ and $\Phi^e = \pi$.

Outside the steady state domain, the stationary solutions for (θ, Φ) are absent and system goes to the nonstationary resistive regime. The simple generalization of Eqs. (16,17) in the frame of the heavily damped resistively shunted junction (RSJ) model [3] leads to equations (see [7]):

$$\frac{d\theta}{dt} = I - \sin \theta - \left[(p+q) \sin \frac{\theta}{2} \cos \frac{\Phi}{2} + (p-q) \cos \frac{\theta}{2} \sin \frac{\Phi}{2} \right] \cos \chi, \quad (28.19)$$

$$\frac{d\Phi}{dt} = \frac{\Phi^e - \Phi}{\mathcal{L}} - \sin \Phi - \frac{1}{\kappa} \left[(p+q) \sin \frac{\Phi}{2} \cos \frac{\theta}{2} + (p-q) \cos \frac{\Phi}{2} \sin \frac{\theta}{2} \right] \cos \chi, \quad (28.20)$$

$$\frac{d\chi}{dt} = -\sin(\chi) \left[(p+q) \cos \frac{\Phi}{2} \cos \frac{\theta}{2} - (p-q) \sin \frac{\Phi}{2} \sin \frac{\theta}{2} \right]. \quad (28.21)$$

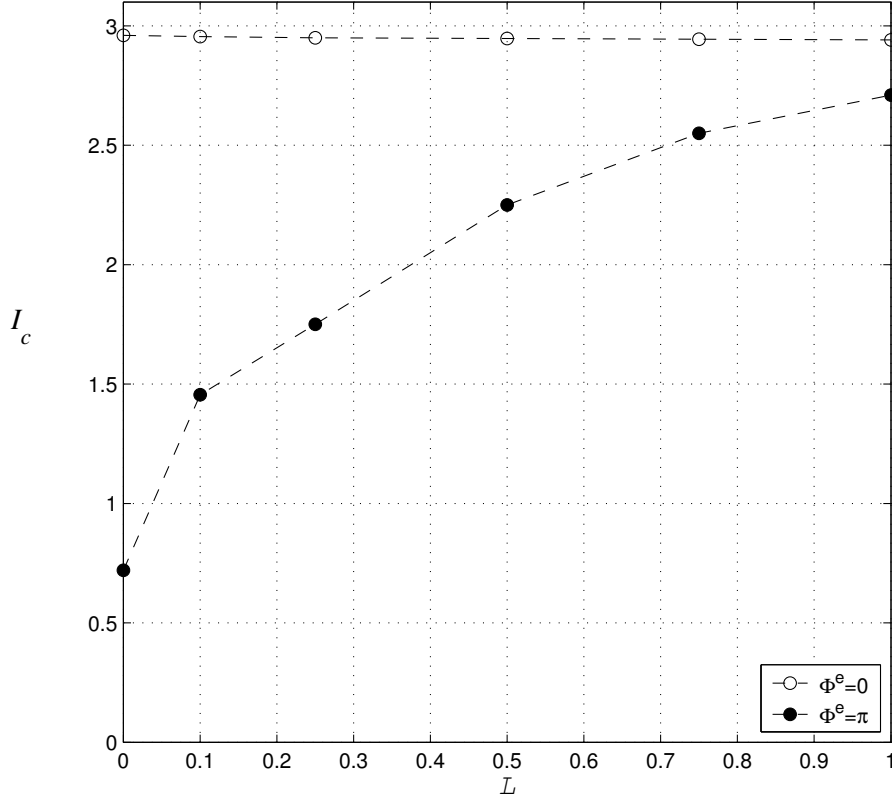


Figure 125: The critical current, I_c , between the superconductors S_1 and S_2 , as a function of $\mathcal{L}$ for $\Phi_e = 0$ and π .

They can also be presented in a form

$$\dot{\theta} = -\frac{\partial U}{\partial \theta} \quad \dot{\Phi} = -\frac{\partial U}{\partial \Phi} \quad \dot{\chi} = -\frac{1}{2} \frac{\partial U}{\partial \chi}, \quad (28.22)$$

where potential U is defined in Eq.(18). The voltages between different terminals are related to the time derivatives of the phase differences

$$V_{21} = \dot{\theta}, \quad V_{34} = \dot{\Phi}, \quad \frac{1}{2}(V_{13} + V_{24}) = \dot{\chi}. \quad (28.23)$$

The time and the voltage are measured in the units of e/I_0 and $\hbar I_0/2e^2$ respectively. Note that, inspite of the equilibrium state, the dynamical variable χ relates to an observable quantity. It's time derivative determines the voltage between the ring and the transport circuit. The features of the dynamical behaviour of the mesoscopic 4-terminal SQUID are again affected by the terms proportional to $(p - q)$, i.e. by nonlocal coupling. The current-voltage characteristics in the transport channel, $V(I)$, (the time averaged voltage V_{21} (23)), can be obtained by the numerical solution of the coupled system of nonlinear differential equations (20-22). As well as critical current I_c , the voltage $V(I)$ in applied magnetic flux Φ^e depends on the sign of the Φ^e , i.e on the direction of the external magnetic field. Full dynamical description of the mesoscopic four-terminal SQUID requires more rigorous approach than RSJ model, and will be the the subject of separate investigation. In accordance with stationary (16,17) or dynamical equations (19-21) for θ and Φ the opposite effect for influence of transport current circuit on the flux states in the ring takes place. In particular, the current I produces the flux Φ in the ring even in stationary case and in the absence of external flux Φ^e . This effect is proportional to $(p - q)$ and is absent in the conventional case. In Fig.11 we plot the dependence of the induced in the ring magnetic flux Φ on the transport current I in the case $\Phi^e = 0$.

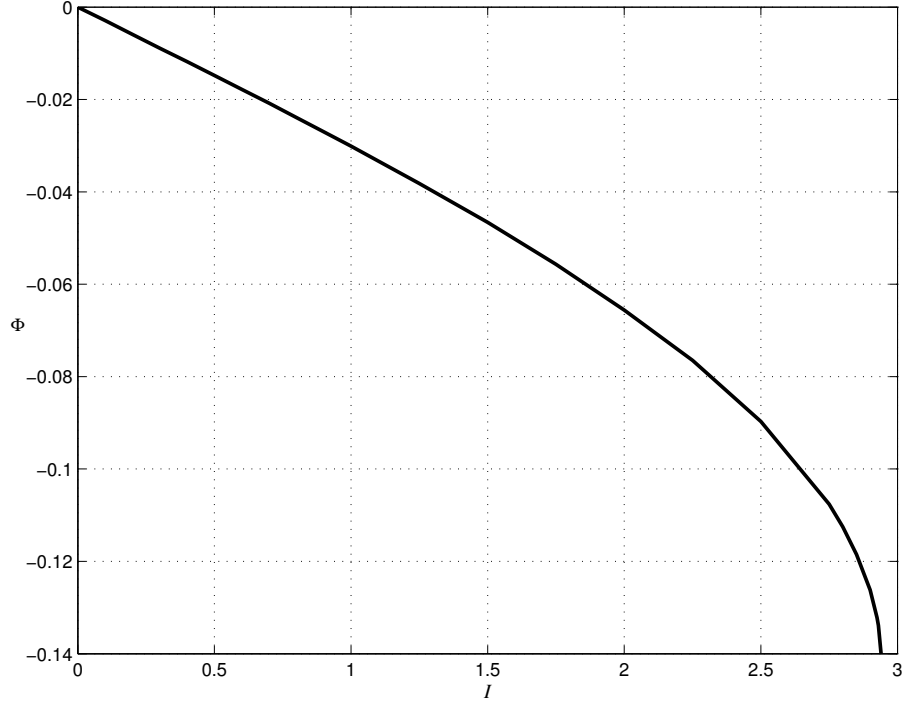


Figure 126: The flux induced inside the ring as a function of the transport current I . $\mathcal{L} = 1, \Phi_e = 0$

Special interest presents the existence of the bistable states in the system described by the potential (18). We emphasize, that in contrast to the usual SQUID, bistable states occur for any inductance $\mathcal{L}$, even for $\mathcal{L} < 1$ [5]. We will analyze the dependence of these states on the control parameters I and Φ^e in article II, when designing of the four-terminal qubit will be studied.

28.5 Conclusions

We have demonstrated that in ballistic four-terminal Josephson junctions coherent anisotropic superposition of macroscopic wave functions of the superconductors in the normal region produces formation of phase slip lines (2D analogs to phase slip centres) and time-reversal symmetry breaking 2D vortex states in it, as well as such effects as phase dragging and magnetic flux transfer. We have calculated the phase-dependent tunneling density of Andreev states in this region as well.

A degree to which the nonlocality of mesoscopic transport is manifested, depends on the characteristics of the system and is most pronounced in the ballistic case [20]. Ballistic four-terminal junctions considered here demonstrate several specific effects absent in the diffusive limit [4, 5, 6]: the phase dragging, time-reversal symmetry breaking ($I(\theta) \neq I(-\theta)$, Eq.(14)), and the vortex formation. The latter can mimic the behaviour of SNS junctions with unconventional superconductors [21]. It has indeed the same origin in direction-dependent phase of the superconducting order parameter induced in the normal part of the system, though not due to the intrinsic phase difference between different directions in a superconductor. This actually allows us more freedom in controlling the behaviour of the junction, which will be exploited in the qubit design based on such a junction in the following paper. The time-reversal symmetry breaking can be also used for direction-sensitive detection of weak magnetic fluxes.

It will be instructive to investigate the role played by finite elastic scattering in the system and look for the analogs of zero bound states, found at surfaces/interfaces of unconventional

superconductors (for a review see [22]). This, as well as vortex dynamics in the system, will be the subject of our further research.

28.6 Appendix. Junction with arbitrary γ 's

The Josephson energy of the mesoscopic Four Terminal Junction, normalized to $(\hbar/2e)(\pi\Delta_0(T)^2/4eT_c)$, is expressed by

$$E_J = -\gamma_{12} \cos \theta - \gamma_{34} \cos \phi + E_{\text{coupl}} \quad (28.24)$$

with the coupling energy E_{coupl} given by

$$\begin{aligned} E_{\text{coupl}} &= -\gamma_{13} \cos \left(\frac{-\theta - \phi}{2} + \chi \right) - \gamma_{14} \cos \left(\frac{-\theta + \phi}{2} + \chi \right) \\ &\quad - \gamma_{23} \cos \left(\frac{\theta - \phi}{2} + \chi \right) - \gamma_{24} \cos \left(\frac{\theta + \phi}{2} + \chi \right) \\ &= -(A \cos \chi + B \sin \chi) \end{aligned} \quad (28.25)$$

where

$$\begin{aligned} A &= (\gamma_{13} + \gamma_{24}) \cos \left(\frac{\theta + \phi}{2} \right) + (\gamma_{14} + \gamma_{23}) \cos \left(\frac{\theta - \phi}{2} \right) \\ B &= (\gamma_{13} - \gamma_{24}) \sin \left(\frac{\theta + \phi}{2} \right) + (\gamma_{14} - \gamma_{23}) \sin \left(\frac{\theta - \phi}{2} \right) \end{aligned} \quad (28.26)$$

Minimizing with respect to χ , we find the minimum to be

$$E_{\text{coupl}} = -\sqrt{A^2 + B^2}, \quad \chi = \cos^{-1} \left(\frac{A}{\sqrt{A^2 + B^2}} \right) \quad (28.27)$$

After some manipulations we find

$$\begin{aligned} E_{\text{coupl}} = & - \left[\gamma_{13}^2 + \gamma_{23}^2 + \gamma_{14}^2 + \gamma_{24}^2 \right. \\ & + 2(\gamma_{13}\gamma_{14} + \gamma_{23}\gamma_{24}) \cos \phi + 2(\gamma_{13}\gamma_{23} + \gamma_{14}\gamma_{24}) \cos \theta \\ & \left. + 2(\gamma_{13}\gamma_{24} + \gamma_{14}\gamma_{23}) \cos \theta \cos \phi - 2(\gamma_{13}\gamma_{24} - \gamma_{14}\gamma_{23}) \sin \theta \sin \phi \right]^{1/2} \end{aligned} \quad (28.28)$$

The last term in the bracket in (28.28) vanishes when $\det(\gamma_{\text{coupl}}) = 0$. In that case the current $I(\theta, \phi)$ will be zero at $\theta = 0$. On the other hand, if $\det(\gamma_{\text{coupl}}) \neq 0$ then $I(\theta, \phi) \neq 0$ when $\theta = 0$. This is a signature of the phase dragging effect.

In a four terminal junction with micro-bridges near T_c one has $\gamma_{ij} \sim 1/R_i R_j$. In that case the last term in (28.28) will vanish and E_{coupl} factorizes

$$E_{\text{coupl}} \sim - \left[\left(\frac{1}{R_1} - \frac{1}{R_2} \right)^2 + \frac{4 \cos^2(\theta/2)}{R_1 R_2} \right]^{\frac{1}{2}} \left[\left(\frac{1}{R_3} - \frac{1}{R_4} \right)^2 + \frac{4 \cos^2(\phi/2)}{R_3 R_4} \right]^{\frac{1}{2}} \quad (28.29)$$

In particular when $R_1 = R_2$ and $R_3 = R_4$ we find

$$E_{\text{coupl}} \sim -\frac{4}{R_1 R_3} \left| \cos \frac{\theta}{2} \right| \left| \cos \frac{\phi}{2} \right| \quad (28.30)$$

which is what one obtains from Ginzburg-Landau calculation.

In a mesoscopic four terminal junction with parallel implementation on the other hand, we have $\gamma_{13} = \gamma_{24}$ and $\gamma_{14} = \gamma_{23}$. This leads to $B = 0$ and therefore

$$E_{\text{coupl}} = -|A|, \quad \cos \chi = \text{sign}(A) \quad (28.31)$$

which gives $\chi = 0$ or π . Notice that in the general case of (28.27), χ can take other values than 0 and π .

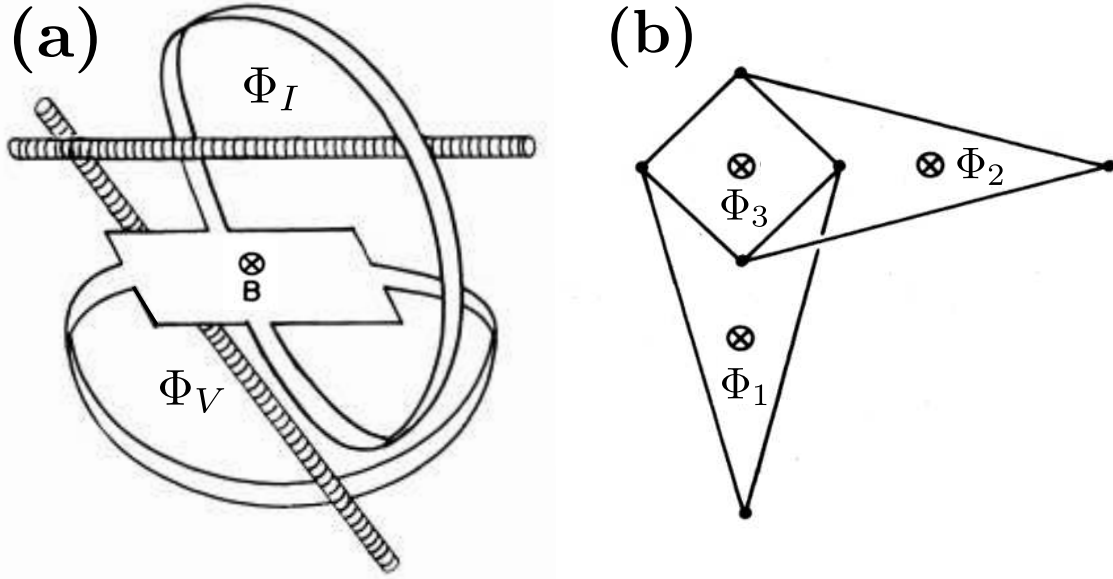


Figure 127: (a) Hall bar with an external magnetic field B in a torus geometry, adapted from Ref. [4]. The flux Φ_V is associated with an electromotive force that drives the Hall current and the flux Φ_I plays the role of an ammeter that measures the Hall current. (b) A three-loop network threaded by fluxes Φ_1, Φ_2 and Φ_3 , adapted from [7]. Its configuration mimics the one of a Hall device.

29 Multi-terminal Josephson junctions: a road to topological flux networks by Lucila Peralta Gavensky, Gonzalo Usaj, C. A. Balseiro

Multi-terminal Josephson junctions were recently proposed as a versatile and tunable platform to emulate topological Bloch-like Hamiltonians in arbitrary dimensions. In this perspective article, we will give a brief overview of the subject and recognize these mesoscopic devices as realizations of topological flux networks as the ones envisioned by J. E. Avron and coworkers in their seminal works on the early days of the quantum Hall effect. We summarize the current state-of-the-art theoretical and experimental research regarding these Josephson devices, highlighting recent developments and giving an outlook on current trends.

29.1 Introduction

Many of the foundational concepts of the field of topological condensed matter physics have been developed during the 1980's after the experimental discovery of the quantum Hall effect [1]. The remarkably robust quantization of the Hall conductance of two-dimensional electron gases at high magnetic fields in units of e^2/h was initially understood in terms of the topological properties of the one particle Bloch bands in quasimomentum space [2]. A couple of years later, these arguments were generalized to take into account the presence of many-body interactions and disorder by considering the Hall sample to live in a torus (two loop) geometry [3, 4, 5], as the one sketched in Fig. 171(a). The angular variables defining such a torus were identified as two Aharonov-Bohm fluxes (Φ_V and Φ_I) threading the physical system and the Hall conductance as the average of the Berry curvature of the ground-state wavefunction in that parameter space [6] (also known as the many-body Chern number). Soon afterwards, J. E. Avron and coworkers extended the study of topologically quantized transport coefficients to the realm of mesoscopic quantum networks[7, 8, 9]. These devices can be represented as graphs

[see for example Fig.171(b)] with D loops threaded by independent, externally controllable, flux tubes Φ_l ($l = 1, \dots, D$). The synthetic parameter space defined by these fluxes can be identified with a D -dimensional torus $\{\Phi_l\} \in \mathbb{T}^D$ with a fundamental period given by the flux unit $\Phi_0 = hc/q$, where q is the charge of the carriers in the device. These carriers are constrained to move along the links of the network, and their dynamics is governed by an appropriate Schrödinger equation defined in the corresponding multiply connected domain. In Ref. [7] concrete Hamiltonian models describing such networks were analyzed, either by considering the vertices of each graph as molecular sites with a tight-binding like description or by analyzing the dynamics of free particles moving along one-dimensional wires in the system [9].

Considering a given Hamiltonian $\hat{H}$, the operator associated to the charge current flowing through loop k is given by $\hat{I}_k = -c \frac{\partial \hat{H}}{\partial \Phi_k}$. With the aim of probing the transport properties of the network, some of its constituent fluxes can be externally driven in time. When doing so in an adiabatic manner, the expectation value of the k -th current can be obtained up to first derivatives of the perturbation as

$$\langle \hat{I}_k(t) \rangle = -c \frac{\partial E_0[\Phi(t)]}{\partial \Phi_k} + \frac{hc}{2\pi} \sum_l \mathcal{F}_{kl}[\Phi(t)] \dot{\Phi}_l, \quad (29.1)$$

where $E_0[\Phi(t)]$ is the instantaneous ground state energy of the system and

$$\mathcal{F}_{kl}[\Phi(t)] = i (\langle \partial_k \psi_0[\Phi(t)] | \partial_l \psi_0[\Phi(t)] \rangle - k \leftrightarrow l), \quad (29.2)$$

is the Berry curvature of the corresponding eigenstate $|\psi_0\rangle$ evaluated at $\Phi(t) = (\Phi_1(t), \dots, \Phi_D(t))$ —assumed to be non-degenerate. We have used here the convention $\partial_{\Phi_k} \equiv \partial_k$. The above derivation requires the adiabatic postulate to hold, which can be insured for sufficiently slow driving protocols and as long as there is a gap between the ground state of the system and the excitation spectrum. As pointed out in Ref. [7], this transport analysis strongly relies on quantum coherence of the wave function over the entire multiply connected device. Such phase rigidity is expected to be present in superconductors over macroscopic lengths and in normal metals under more stringent conditions on length and temperature scales.

The first term in Eq. (29.1) can be identified as a persistent current, which may be present even in the absence of the driving. If the fluxes vary linearly in time, a constant electromotive force (emf) $\mathcal{E}_l = -\frac{1}{c} \dot{\Phi}_l$ is generated around each loop, allowing for the definition of a conductance matrix with elements given by $G_{kl}[\Phi(t)] = \frac{\partial \langle \hat{I}_k(t) \rangle}{\partial \mathcal{E}_l}$. These transport coefficients relate the current flowing around loop k to the emf around loop l . Remarkably, when averaging them over a $\Phi_0 \times \Phi_0$ square slice in flux space, they become quantized in units of q^2/h

$$\overline{G}_{kl} = \frac{1}{\Phi_0^2} \int_0^{\Phi_0} d\Phi_k \int_0^{\Phi_0} d\Phi_l \frac{\partial \langle \hat{I}_k(t) \rangle}{\partial \mathcal{E}_l} = -\frac{q^2}{h} C_{kl}, \quad (29.3)$$

with C_{kl} the Chern number of the ground state. We have here used the Chern theorem in order to relate the integral of the Berry curvature over a closed periodic manifold with an integer topological index

$$C_{kl} = \frac{1}{2\pi} \int_0^{\Phi_0} d\Phi_k \int_0^{\Phi_0} d\Phi_l \mathcal{F}_{kl}[\Phi], \quad (29.4)$$

and that the persistent current term averages to zero. The value of this topological number depends on the remaining (unaveraged) fluxes with a Φ_0 -periodicity. Interestingly, these integers do not necessarily classify a property of a given physical sample, as in the quantum Hall setup of Fig. 171(a), but rather the topology of the whole network itself [7]: the leads connecting each vertex, the number of holes and their geometrical layout. Non trivial averaged

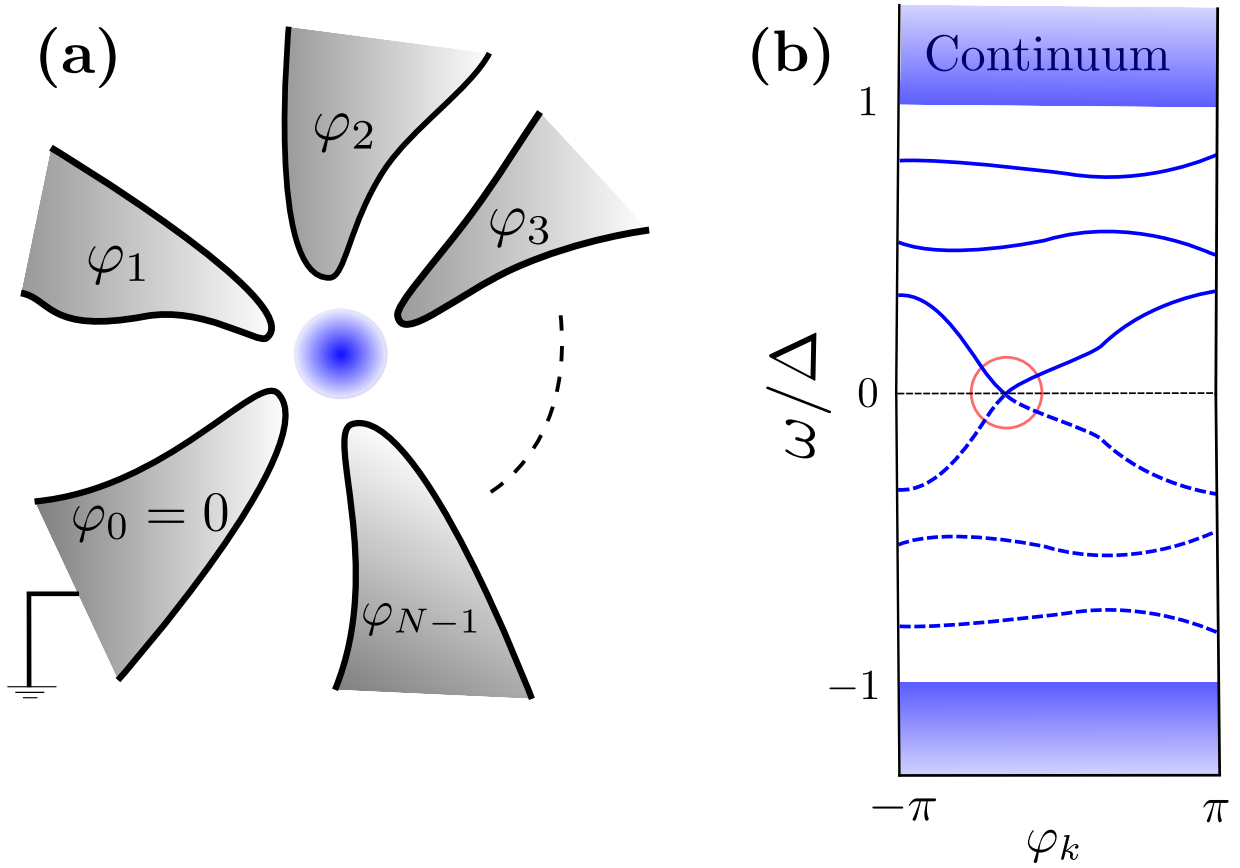


Figure 128: **(a)** Sketch of a multi-terminal Josephson junction, where N superconducting leads with phases φ_l ($l = 0, 1, \dots, N - 1$) are coupled through a central scattering region. **(b)** Illustration of a generic Bogoliubov-de Gennes quasiparticle energy spectrum of these devices as a function of one of the phase φ_k . We indicate with a red circle a zero energy crossing.

conductances generically arise in networks with at least three independent fluxes $D \geq 3$ [7], a dimensionality constraint mathematically rooted in the co-dimension theorem of Von Neumann and Wigner [10]. Indeed, three independent variables are needed to fine tune degeneracies in the energy spectrum, which are known to act as sources or sinks of Berry curvature. In the particular case of $D = 3$, the degeneracies occur at points in the parameter space, which are generically dubbed Weyl singularities. The Hamiltonian near a two-band touching point Φ_W can be approximated with a two-by-two matrix as

$$\hat{H}_W = \mathbf{h}(\Phi_W + \delta\Phi) \cdot \boldsymbol{\sigma} = \sum_{k,i} \delta\Phi_k \mathbb{M}_{ki} \sigma_i, \quad (29.5)$$

where $\mathbb{M}_{ki} = \partial_{\Phi_k} h_i|_{\Phi_W}$ and σ_i are the Pauli matrices ($i = x, y, z$). We have here neglected the terms proportional to the identity matrix, since they have no influence on the existence of the degeneracy nor on its topological classification. Weyl points can be considered as monopoles of topological charge which come in two different flavours or chiralities, determined by $\chi = \text{sign}(\det \mathbb{M})$.

29.2 Multi-terminal Josephson junctions

A few years ago, these ideas [7, 8] experienced a renaissance when multi-terminal Josephson junctions (JJs) were proposed as a novel platform for topological engineering [11]. A typical device is sketched in Fig. 172(a), where N s-wave superconducting leads (grey) are coupled through a central scattering region (blue). Each of these terminals is characterized by a complex

mean-field order parameter $\Delta_l e^{i\varphi_l}$ and gauge invariance allows to choose a gauge in which $\varphi_0 = 0$, leaving $N - 1$ free phase parameters $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \dots, \varphi_{N-1})$ in the system. For simplicity, we will take $\Delta_l = \Delta$. In these Josephson devices, the phase differences are usually controlled with superconducting loops threaded by external magnetic fluxes Φ_l so that $\varphi_l = 2\pi\Phi_l/\Phi_0^s$ with $\Phi_0^s = hc/2e$ the superconducting flux quantum. One can accordingly identify these junctions as $(N-1)$ -dimensional flux networks where the effective charge $q = 2e$ is the Cooper pair charge. The quasiparticle energy spectrum obtained when diagonalizing the Bogoliubov-de Gennes (BdG) Hamiltonian describing these junctions usually consists on a continuum spectrum above the superconducting gap Δ and a set of discrete levels (from now on indexed by the label α) living inside the gap, as illustrated in Fig. 172(b). The microscopic origin of the discreteness of this last part of the spectrum is rooted in the presence of Andreev reflection processes at the superconducting interfaces: incident electrons with energies below the superconducting gap are reflected as holes and viceversa, leading to a confined motion in the central scattering region of the nanostructure which generates the so-called Andreev bound states (ABS) [12, 13, 14]. These Bogoliubov quasiparticles bear a certain dispersion relation $\varepsilon_\alpha(\boldsymbol{\varphi})$ as a function of the phase differences of the superconducting leads, with a functional form that depends both on the properties of the terminals and the scattering region that connects them [15]. The phase differences of the leads define a closed (2π -periodic) manifold and hence play the role of an effective Brillouin zone (BZ) in the problem. Indeed, a formal analogy between the ABS of N -terminal JJs and the Bloch band structure of materials in $N - 1$ dimensions can be established, with the superconducting phases φ_l playing the role of the quasimomenta components k_l in a crystalline structure. In this way, the number of synthetic dimensions of the problem can be simply increased by adding more superconducting leads.

The key idea of Ref. [11] is that the ABS spectrum may host Weyl singularities [as the one circled in red in Fig. 172(b)] if there are at least three independent superconducting phases ($N \geq 4$), as expected from Wigner's co-dimension theorem. Note that a zero energy crossing of the eigenstates of the BdG Hamiltonian means that the excitation spectrum of the junction is gapless: there is a degeneracy between the ground state wavefunction of the junction and a state with one extra/less quasiparticle.

In the case of 4-terminal junctions, the synthetic BZ is three-dimensional and the Weyl singularities must be points in this parameter space. Due to the Nielsen-Ninomiya theorem [16], the sum of the charges of Weyl monopoles in a 3D BZ must vanish, so Weyl points can never appear in isolation: they must come in pairs with opposite charges. Furthermore, time reversal invariance in the scattering region implies that $\varepsilon_\alpha(\boldsymbol{\varphi}) = \varepsilon_\alpha(-\boldsymbol{\varphi})$, so if there is a Weyl crossing at $\boldsymbol{\varphi}_W$ there must be another one at $-\boldsymbol{\varphi}_W$. Even more, since the chirality χ of a Weyl point is even under time reversal, both must have the same topological charge. This guarantees that Weyl points must come in groups of four, as sketched in Fig. 129(a). By randomly generating scattering matrices from the circular orthogonal ensemble to describe such junctions, the authors of Ref. [11] indeed showed that a significant fraction of them yielded zero-energy Weyl points in the BdG spectrum, with the number of singularities increasing when increasing the number of channels in each terminal. A topological classification can be made by taking, without loss of generality, the phase difference φ_3 as a control parameter. In this way, it is possible to define a Chern number $C_{12}(\varphi_3)$ in the two-dimensional submanifold defined by the moving plane with coordinates (φ_1, φ_2) . By denoting the wavefunctions of the Andreev bound states as $|\phi_\alpha(\boldsymbol{\varphi})\rangle$, this topological invariant takes the form

$$C_{12}(\varphi_3) = \sum_{\alpha | \varepsilon_\alpha < 0} C_{12}^\alpha(\varphi_3), \quad (29.6)$$

with

$$C_{12}^\alpha(\varphi_3) = \frac{i}{2\pi} \int_{-\pi}^{\pi} d\varphi_1 \int_{-\pi}^{\pi} d\varphi_2 (\langle \partial_{\varphi_1} \phi_\alpha | \partial_{\varphi_2} \phi_\alpha \rangle - \varphi_1 \leftrightarrow \varphi_2). \quad (29.7)$$

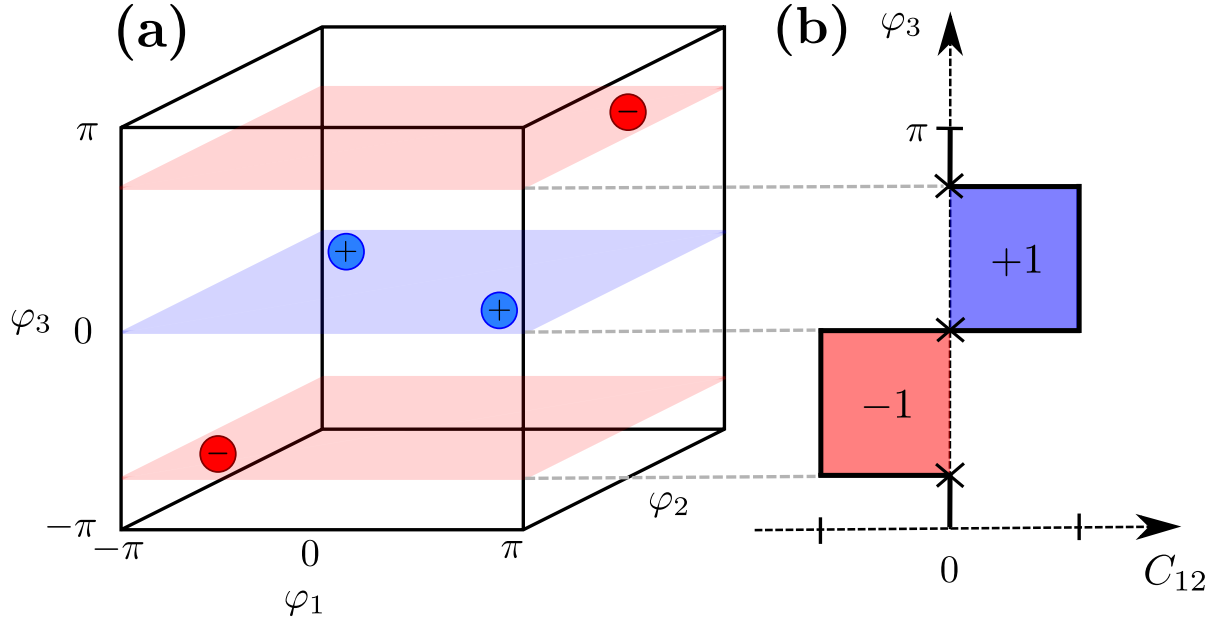


Figure 129: **(a)** Sketch of the position of four Weyl points in the effective three-dimensional Brillouin zone of a generic four-terminal Josephson junction. We mark with blue Weyl points with positive topological charge ($\chi = +1$) and with red the ones with negative topological charge ($\chi = -1$). **(b)** Chern number C_{12} defined in the two-dimensional moving plane (φ_1, φ_2) as a function of the phase difference φ_3 .

The sum of the Chern numbers C_{12}^α of the α -th ABS of the BdG Hamiltonian up to the Fermi level must be interpreted as the Chern number of a state with no quasiparticles at zero temperature. We plot in Fig. 129(b) how this topological index would look like as a function of φ_3 for a configuration of Weyl points as the one sketched in panel (a).

A natural follow up question is whether this topological classification has any observable consequences in the transport properties of these devices. Following the same line of reasoning as in the previous section, one may think of driving one of the phases φ_l in time and studying the corresponding adiabatic response in a different terminal. In these superconducting networks, this may be done by applying a bias voltage V_l in terminal l so that the corresponding superconducting phase acquires a time dependence in accordance with the Josephson relation $\dot{\varphi}_l(t) = 2eV_l/\hbar$. If the ABS spectrum is gapped and the bias voltage is small compared to this gap, with the same logic as the one used in Eq. (29.1), one can show (see Ref. [11]) that the ABS contribution to the current in terminal k at zero temperature is given by

$$\langle I_k(t) \rangle = \frac{2e}{\hbar} \sum_{\alpha | \varepsilon_\alpha < 0} \frac{\partial \varepsilon_\alpha[\boldsymbol{\varphi}(t)]}{\partial \varphi_k} - 2e \sum_{l, \alpha | \varepsilon_\alpha < 0} \mathcal{F}_{kl}^\alpha[\boldsymbol{\varphi}(t)] \dot{\varphi}_l(t), \quad (29.8)$$

with $\mathcal{F}_{kl}^\alpha$ the Berry curvature of the α -th bound state evaluated at the instantaneous phase differences $\boldsymbol{\varphi}(t)$. In this expression, spin degeneracy is assumed and a summation over spin indexes has already been made. By defining a transconductance matrix with elements $G_{kl} = \frac{\partial \langle I_k(t) \rangle}{\partial V_l}$ and phase averaging this expression one readily finds that

$$\overline{G}_{kl} = -\frac{4e^2}{h} \sum_{\alpha | \varepsilon_\alpha < 0} C_{kl}^\alpha, \quad (29.9)$$

with C_{kl}^α the Chern number of the α -th ABS in the sub-manifold defined by the phases (φ_k, φ_l) . Note that Eq. (29.9) is in complete analogy with Eq. (29.3). The above derivation can also be

generalized to the case where spin-rotation symmetry is broken due to, for example, Zeeman or spin-orbit coupling effects. In such circumstances, zero energy Weyl points split and move to finite energy while preserving their topological charge [17]. Although the gap in the spectrum is closed along a contour encircling these singularities, it was argued that the phase averaged transconductance could remain quantized provided fermion parity is preserved in the junction [17, 11].

The existence of a phase averaged topologically quantized transconductance has been subsequently tested in Ref. [18] by performing transport simulations in a voltage biased four terminal junction. These findings were followed up by numerous theoretical works that have thoroughly analyzed the emergence and evolution of Weyl nodes in the ABS spectra of a variety of JJs [19, 20, 21]. It was soon realized that three terminal Josephson devices (formally equivalent to two-dimensional bulk materials) could also host topological Andreev states when time-reversal symmetry is broken in the central scattering region [22, 23]. This may be achieved by piercing the junction area with an additional magnetic flux or by driving the system with a periodic time-dependent microwave field [24]. In this last case, the well-known tools of Floquet engineering [25] were introduced to generate topologically non-trivial Andreev states by photon dressing. Additionally, several works have also generalized the problem to heterostructures built with superconducting reservoirs in the presence of p -wave correlations, where Majorana zero modes could ultimately emerge and get coupled to each other [26, 27, 28, 29, 30, 31]. In this regime, single electrons can be transferred between terminals, resulting in a phase averaged transconductance quantized in units of $2e^2/h$ instead of $4e^2/h$.

29.3 Locally accessing quantum geometry in flux-space

An interesting advantage of these devices, compared to other solid-state platforms, is that it is conceptually much easier to probe local geometrical properties of quantum states in the artificial Brillouin zone spanned by the phase-differences. In principle, it is possible to perform measurements at each effective quasimomenta by simply manipulating the external fluxes of the junction in an independent manner. The full quantum geometry of the Andreev bound states in this synthetic flux space is encoded in the quantum geometric tensor, defined as

$$\mathcal{Q}_{kl}^\alpha[\varphi] = \langle \partial_{\varphi_k} \phi_\alpha | (1 - |\phi_\alpha\rangle\langle\phi_\alpha|) | \partial_{\varphi_l} \phi_\alpha \rangle. \quad (29.10)$$

On the one hand, the real part of this tensor provides for the Fubini-Study metric $g_{kl}^\alpha[\varphi] = \text{Re}(\mathcal{Q}_{kl}^\alpha[\varphi])$, which contains information on the “distance” between two quantum states. On the other hand, the imaginary part gives direct access to the Berry curvature $\mathcal{F}_{kl}^\alpha[\varphi] = -2\text{Im}(\mathcal{Q}_{kl}^\alpha[\varphi])$, which provides for a measure of quantum geometrical phases acquired along closed paths. In Ref. [32], a scheme to measure the components of this tensor in a generic parameter space was developed. It essentially consists on modulating the corresponding parameters periodically in time and monitoring the excitation rates between the ground and excited states. This same strategy was adapted to the case of multi-terminal Josephson devices [33, 21], where it was proposed that microwave spectroscopy could reveal all the information contained in the geometrical object of Eq. (29.10). In these nanostructures, the protocol involves applying ac-drives in the external fluxes, so as to be able to modulate in time the phase differences between different reservoirs. We note that in Ref. [24], an alternative scheme to locally probe the Berry curvature by performing synchronic transport measurements was also discussed.

29.4 On the effect of the quasiparticle continuum

The analogy between the Bloch band structure and the quasiparticle energy spectrum of multi-terminal JJs, although appealing, is incomplete. Indeed, the continuum spectrum above

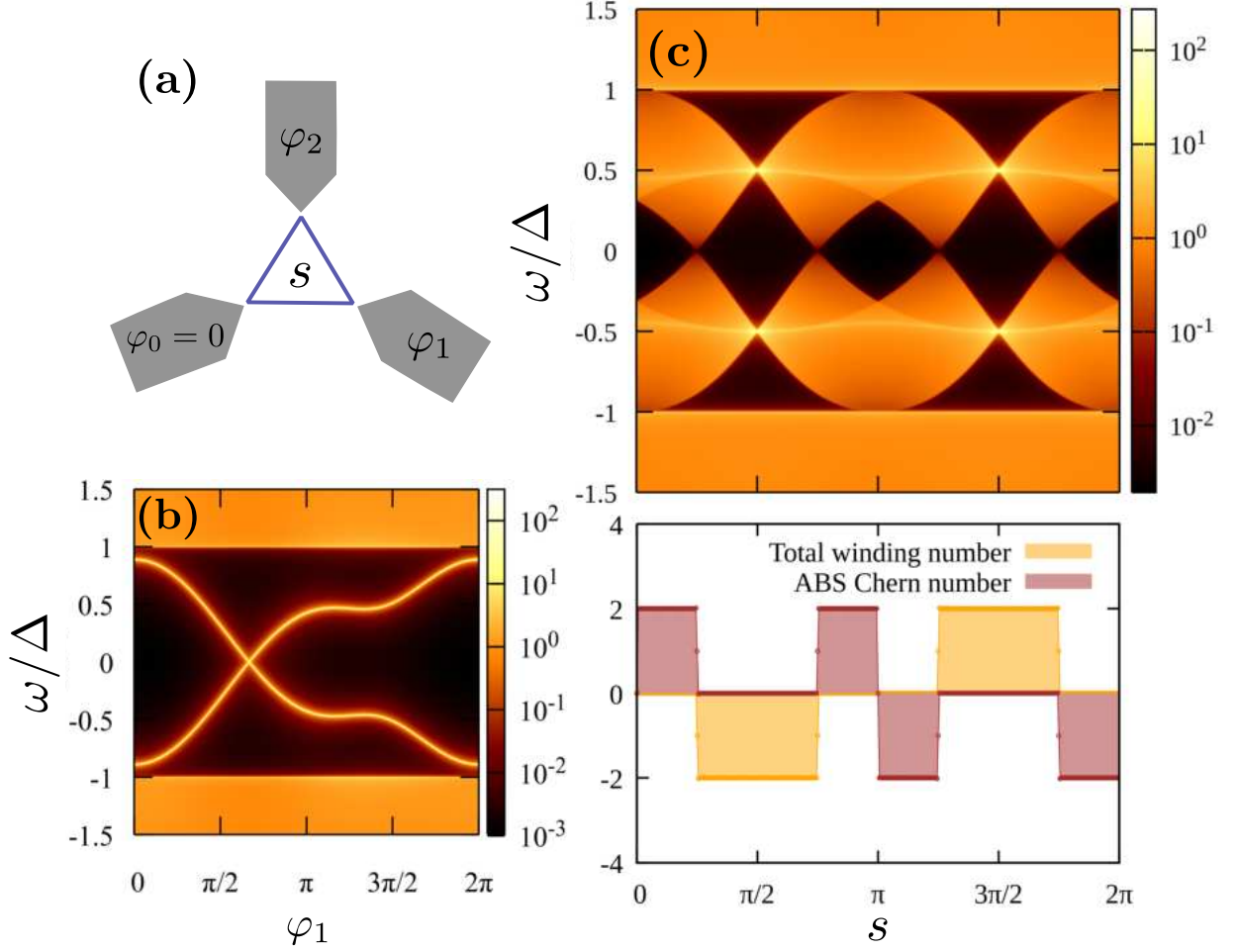


Figure 130: **(a)** Schematic setup of a three-terminal Josephson junction pierced with a central magnetic flux $\Phi_s = \Phi_0 s/2\pi$ with $\Phi_0 = hc/e$. **(b)** Spectral density of the junction $[(-1/\pi)\text{Im}(\text{Tr}\mathcal{G}_J(\omega, \varphi_1))]$ along the path $\varphi_1 = -\varphi_2$ for $s = 0.78$. **(c)** Top: Phase integrated spectral density of the junction as a function of s . Bottom: ABS Chern number and total winding number as a function of s . The difference between them corresponds to the continuum contribution.

the superconducting gap [see Fig. 172(b)] also participates in the transport properties of these devices. In this sense, Eq. (29.8) is not entirely accurate, since it disregards the contribution of these states to the current. This problem was highlighted in Ref. [22], where the authors studied a setup as the one sketched in Fig. 130(a): three superconducting reservoirs coupled via a normal region threaded by an external magnetic flux. In their particular setting, the numerically calculated phase averaged transconductance did not match the Chern number of the ABS, a fact which was attributed to a non-trivial contribution to the quantized transconductance coming from the quasiparticle continuum. In order to properly account for the continuum contribution, the adiabatic expansion of the Josephson current needs to be formulated on more general grounds. The geometrical contribution, identified as the Berry curvature of the ground state, is more conveniently written in terms of the BdG Green's functions [24, 27, 34]. Namely,

$$\langle I_k^{\text{geo}}(t) \rangle = -e \sum_{\nu'} \int_{-\infty}^{\infty} \frac{d\omega}{4\pi} \times \text{Tr} \left[\tilde{\mathcal{G}}^{-1} \partial_{\varphi_k} \tilde{\mathcal{G}} \cdot \tilde{\mathcal{G}}^{-1} \partial_{\varphi_l} \tilde{\mathcal{G}} \cdot \tilde{\mathcal{G}}^{-1} \partial_{\omega} \tilde{\mathcal{G}} - k \leftrightarrow l \right] \dot{\varphi}_l. \quad (29.11)$$

Here, $\tilde{\mathcal{G}}(\omega, t) = \tilde{\mathcal{G}}(\omega, \{\varphi_l(t)\})$ is the fully adiabatic causal Green function obtained from the instantaneous BdG Hamiltonian [27]. After averaging over the Brillouin zone as discussed before, we obtain that $\bar{G}_{kl} = -(2e^2/h) \sum_l \mathcal{N}_{kl}$ with

$$\mathcal{N}_{kl} = \frac{1}{8\pi^2} \int_{-\pi}^{\pi} d\varphi_k \int_{-\pi}^{\pi} d\varphi_l \int_{-\infty}^{\infty} d\omega \times \text{Tr} \left[\tilde{\mathcal{G}}^{-1} \partial_{\varphi_\nu} \tilde{\mathcal{G}} \cdot \tilde{\mathcal{G}}^{-1} \partial_{\varphi_{\nu'}} \tilde{\mathcal{G}} \cdot \tilde{\mathcal{G}}^{-1} \partial_{\omega} \tilde{\mathcal{G}} - k \leftrightarrow l \right], \quad (29.12)$$

the winding number of the Green's function $\tilde{\mathcal{G}}$. If the system is gapped, this quantity is known to be an integer number [35]. $\mathcal{N}_{kl}$ includes the contribution of both the ABS and the continuum spectrum on equal footing. In fact, the two contributions cannot be separated in general, except when the ABS are isolated from the continuum. Moreover, since the trace involves a sum over spin variables, whenever there is an equal spin contribution we expect the transconductance to be quantized in units of $4e^2/h$. An alternative expression was also derived in terms of the scattering matrix in Ref. [36].

For calculation purposes, it is convenient to gauge the superconducting phases to the tunneling matrix elements between the leads and the normal region. In this way, the phase dependence appears only on the tunneling Hamiltonian and the expression for the topological invariant can be written [27] in terms of the junction's Green function $\mathcal{G}_J(\omega, \{\varphi_l\})$ which involves only the region of interest. To illustrate the procedure, we consider three *s*-wave superconducting reservoirs (treated in the wide band limit) which are coupled to each other by (single site) normal hopping terms enclosing a magnetic flux Φ_s as depicted in Fig. 130(a). We assume spin degeneracy and work on the Nambu basis, so that $\mathcal{G}_J(\omega, \{\varphi_l\})$ is a 6×6 matrix corresponding to the three coupled ending sites of the superconducting leads. The spectral density, $(-1/\pi) \text{Im}[\text{Tr} \mathcal{G}_J(\omega, \{\varphi\})]$, as a function of φ_1 ($= -\varphi_2$) for a particular value of Φ_s is shown in Fig. 130(b), a situation where a Weyl singularity is present on the ABS spectrum. The phase integrated spectral density as a function of $s = 2\pi\Phi_s/\Phi_0$, and the corresponding ABS Chern number and total winding number (Eq. (29.12)), are presented in Fig. 130(c). It is important to notice here that the total winding number only changes when there is a gap closing at the Fermi energy ($\omega = 0$) while the ABS Chern number can also do it when the ABS touches the continuum ($s = 0, \pi$). It is only the former quantity that is related to the transconductance—the result in Fig. 130(c) agrees well with the numerically calculated transconductance [22].

29.5 Topological Hamiltonian

There is another way to calculate the topological invariant that, in some cases, allows to separate the contribution of the ABS and the quasiparticle continuum. As it was recently demonstrated [37, 38], under quite general assumptions, all the topological information of the Green's function $\tilde{\mathcal{G}}(\omega)$ is encoded in the so-called ‘topological Hamiltonian’ defined as $H_{\text{top}} \equiv -\tilde{\mathcal{G}}^{-1}(\omega = 0)$. The key idea is that if the system is gapped and $\tilde{\mathcal{G}}(\omega)$ well behaved, the latter can be smoothly deformed to $\mathcal{G}_{\text{eff}}(\omega) = (\omega + \tilde{\mathcal{G}}^{-1}(\omega = 0))^{-1}$ preserving its topology. In that case, the winding number of the Green's function can be calculated from the Chern numbers associated to each of the eigenstates of H_{top} .

In the case of JJs, the eigenstates of $H_{\text{top}}^J = -\mathcal{G}_J^{-1}(\omega = 0)$ should then reflect the topological nature of both the ABS and the continuum spectrum. In the example of Fig. 130, H_{top}^J has six particle-hole symmetric eigenstates but only those below the Fermi energy need to be considered. The Chern number associated to the ABS is obtained from the eigenstate whose energy is closer to the Fermi energy (Fig. 130(c)), while the one related to the continuum is just the sum of the Cherns of the two remaining modes. The sum of all these contributions is precisely the winding number of the full frequency dependent Green's function obtained before, revealing

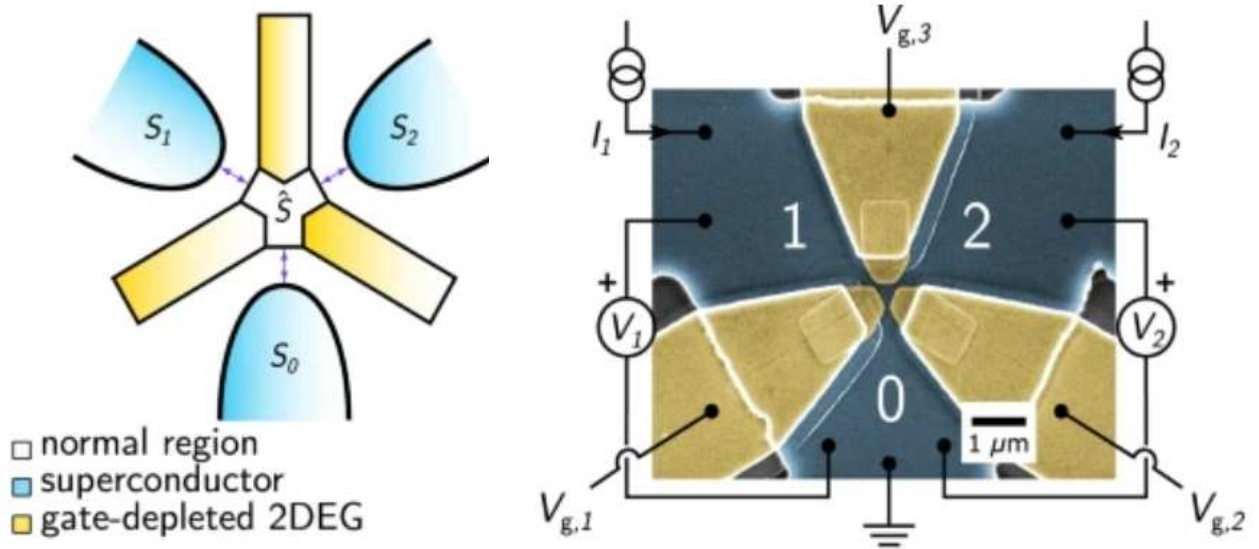


Figure 131: Left Panel: Schematic of a Y-shaped three-terminal Josephson junction in a split-gate quantum point contact geometry. Right panel: False-color SEM image of the actual device with individually tunable quantum point contacts showing measurement schematic. Extracted from Ref. [43].

the validity of the method [34]. Notably, even though the eigenenergies obtained from H_{top}^J do not represent the physical spectrum, they constitute a reduced Hilbert space that accurately describes the closing of the gaps and the topology of the original problem. Therefore, when applicable, this method provides a much simpler way of obtaining the topological invariants without the need of carrying out an integral over frequency as in Eq. (29.12).

29.6 Experimental advances

Weyl singularities in the ABS spectrum of multi-terminal Josephson devices have not been experimentally observed so far nor phase-averaged quantized transconductances measured. Contrary to the well established two-terminal JJs case, the design, fabrication and ultimate control of multi-terminal junctions is much less explored, with several experimental challenges ahead. Substantial progress has been recently made in this direction, partially fuelled by the perspective of engineering synthetic topology. Several groups have already been able to develop mesoscopic JJs with three or four terminals that exhibited superconducting phase coherent transport in all-metal junctions [39], hybrid semiconductor-superconductor heterostructures [40, 41, 42, 43] or proximitized graphene [44, 45]. Even a phase-tunable three-terminal Josephson interferometer has been realized [46, 47]. One of the challenges is to have the ability to control the number of conductance channels between the superconducting terminals, while keeping them highly transmissive—they need to be just a few in order to have enough experimental resolution to access the ABS spectrum. A step forward in this direction was recently done in Ref. [43] where the authors were able to control each channel of a three-terminal junction down to a single mode using a split-gate quantum-point contact geometry, see Fig. 131. Other aspects that need further investigation are related to the capability of independently controlling the superconducting phases, that is magnetic fluxes, while avoiding mutual inductances and also reducing quasi-particle poisoning [48] and the presence of thermal excitations. The latter is important to guarantee the ground state occupation and hence the quantization of the transconductance.

29.7 Current trends

In the search for novel approaches that could lead to the observation of more complex topology, recent theoretical works have gone beyond the original proposals and extended the study to multi-terminal JJs with degenerate ground-states, which could open the door to quantum simulation of non-Abelian physics in high dimensions and holonomic quantum computing [49, 50]. Indeed, these devices may reveal non-Abelian Berry curvatures and higher-order Chern numbers in their non-linear response or through microwave spectroscopy. Another twist might come from the recent realization of Floquet ABS in a graphene JJ [51] which could bring the possibility of Floquet engineering into the field.

Many researchers have looked for alternatives [52, 53, 54, 55] to the multi-terminal JJ setups that might show the same phenomena while being potentially more easy to control with current technologies. A promising route is the case of Weyl Josephson circuits [52]: arrays of coupled superconducting tunnel junctions that could be used to simulate Weyl band structures in both flux and charge space. By tailoring their design, synthetic Bloch Hamiltonians of different symmetries and dimensionalities can be engineered so as to obtain the desired topological properties.

A completely different approach involves using gapped superfluids of cavity exciton-polaritons in a multi-terminal setup, as proposed in Ref. [56]. The advantage of these artificial systems is that topological transitions can be controlled by pump lasers with well defined phases, while the photonic component of the eigenstates, and so their topological properties, is experimentally accessible. An interesting appeal of this particular proposal is that, being exciton-polaritons a system with intrinsic gain and losses, it opens the door to non-hermitian topology.

As we have tried to convey in this article, the study of geometrical and topological transport properties in flux networks is a research area that goes back to the early days of the quantum Hall effect [5, 7, 8, 9]. These ideas have been brought back to the table when multi-terminal JJs were proposed as a concrete platform that could artificially mimic topological matter [11]. Given the promising advances in the field, one can foresee that the experimental realization of a topological flux network is not far from reach.

Part X

Other Experiments

30 Sub-Gap Structure in the Conductance of a Three-Terminal Josephson Junction by Pfeffer et al.

Abstract

Three-terminal superconductor (S) - normal metal (N) - superconductor (S) Josephson junctions are investigated. In a geometry where a T-shape normal metal is connected to three superconducting reservoirs, new sub-gap structures appear in the differential resistance for specific combinations of the superconductor chemical potentials. Those correspond to a correlated motion of Cooper pairs within the device that persist well above the Thouless energy and is consistent with the prediction of quartets formed by two entangled Cooper pairs. A simplified nonequilibrium Keldysh Green's function calculation is presented that supports this interpretation.

30.1 Introduction

Josephson effects appear in SNS junctions where two superconductors (S) are electrically coupled through a non-superconducting material (N) [1]. The underlying mechanism is the Andreev reflection that converts a Cooper pair in S into two phase-correlated electrons in N [2].

At zero voltage, the appearance of coherent Andreev bound states leads to a non-dissipative supercurrent through the junction and a mini-gap in the density of states (DOS) of the normal metal. In the case of diffusive junctions and when the length of the normal part L is longer than the superconducting phase coherence length, both the supercurrent and the mini-gap scale with the Thouless energy given by $E_{Th} = \hbar D/L^2$ where D is the diffusive constant of the normal metal.

At finite voltage, the nonequilibrium sub-gap current is governed by Multiple Andreev Reflections (MAR). In this regime, MAR's successively raise a quasiparticle's energy until it reaches the superconducting gap Δ . Due to the superconductor density of states singularity at the gap edge, MAR's lead to a sub-gap structure in the junction differential conductance for $eV = 2\Delta/n$ [3, 4], n being an integer. This structure can be observed in diffusive SNS junctions where the diffusion time through the junction is much smaller than the inelastic scattering time.

In addition to this d.c. sub-gap quasiparticle transport, a.c. Josephson currents also appear in a diffusive SNS junction. However, during the diffusion of an Andreev pair through the junction, phase coherence is maintained only if the energy of the electron or the hole compared to the superconductor chemical potential is smaller than the Thouless energy E_{Th} [5]. The a.c. Josephson current can be indirectly revealed under microwave irradiation. Shapiro steps [1] in the d.c. current-voltage characteristics show up when the superconducting phase difference oscillation frequency $2eV/\hbar$ matches the microwave frequency or some multiple of it. The mere existence of Shapiro steps and therefore the a.c. Josephson currents essentially requires a quasi-static superconducting phase difference, i.e. a diffusion time smaller than the inverse of the Josephson frequency $2eV/\hbar$, or equivalently $eV < E_{Th} = \hbar D/L^2$.

More recently, multi-terminal junctions started to be investigated and brought a wealth of new properties, among which several remain to be experimentally uncovered. When two normal conductors are closely connected to a superconducting reservoir, Crossed Andreev Reflections (CAR) can inject two phase-correlated particles, one in each conductor, which amounts to split

a Cooper pair into two entangled electrons [6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. This only occurs when the distance between the two normal conductors is smaller than the superconducting coherence length.

Another situation is met in mesoscopic three-terminal Josephson junctions in which a single normal conductor is connected to three superconducting contacts [16, 17, 18, 19, 20]. The transport properties then depend on two independent (phase or voltage) variables. Therefore, in addition to usual Josephson processes coupling two terminals, new mechanisms are expected that connect all three reservoirs. Several theoretical predictions have been made for such systems [21, 16, 17, 36, 37]. Nonlocal MAR should show up in the so-called incoherent MAR regime where the dwell time exceeds the coherence time [17, 24]. On the other hand, the coherent regime where several MAR's can occur within the coherence time is also very interesting. Shapiro-like resonances in the absence of external microwave have been predicted whenever two a.c Josephson frequencies match [21]. On similar grounds, the production of non-local quartets, as pairs of correlated Cooper pairs, has been proposed as a new dissipationless d.c. transport mechanism, which is phase-coherent despite the non-equilibrium conditions [36, 37]. This present work reports on a new experimental study of such phenomena.

In this article, we report about electronic sub-gap transport in three-terminal Josephson junctions made upon a piece of diffusive normal metal connected to three superconducting reservoirs. The junctions are all phase-coherent as their length is smaller than the single particle phase coherence length L_Φ , and in the long junction regime, e.g. the Thouless energy is much smaller than the superconducting gap. They are also rather symmetric and with a high transparency at every SN interface leading to a large sub-gap Andreev current. Compared to a pair of two-terminal junctions, additional sub-gap structures are observed over a wide voltage range, well above the Thouless energy in a regime where one does not expect the presence of strong a.c. Josephson currents.

In the following, Section II contains the experimental details and reports the sub-gap anomalies. Section III is devoted to a physical discussion of the possible interpretations. Section IV concludes with perspectives.

30.2 The experiment

30.2.1 Samples and measurement process

The samples we have studied have been fabricated by shadow mask evaporation technique (see Scanning Electron Microscope SEM images in Fig 132 and 133). Copper and Aluminum were evaporated at different angles through a PMMA/MAA bilayer mask in an ultra-high vacuum chamber. The evaporation of a thin Cu-layer of 50 nm thickness was followed immediately by the evaporation of thick Al electrodes of thickness 500 nm without breaking the vacuum, leading to highly transparent and uniform SN interfaces. The width of the normal metal is about $0.6\,\mu m$ and its length L is around $1\,\mu m$. Using a diffusion constant for copper $D = 100\,cm^2/s$, we get a Thouless energy $E_{Th} = \hbar D/L^2 \simeq 6\,\mu eV$. This value is confirmed by fitting the temperature dependence of the critical current between two of the superconducting contacts [25]. The superconducting Aluminum energy gap is $\Delta = 170\,\mu eV$ [26]. The diffusion time is $\tau_D = L^2/D \simeq 0.1\,ns$ is much smaller than the inelastic time $\tau_{in} \simeq 1\,ns$ at $100\,mK$.

Three-terminal differential resistances were measured using an experimental set-up specially designed to perform highly-sensitive measurements of current average and fluctuations in low-impedance nano-devices at very low temperature [27], see Fig. 132. The experiment operates down to 30 mK and is equipped with 3 commercial SQUIDs as current amplifiers. Each device terminal is connected to the input coil of a SQUID in series with a macroscopic resistor with a low resistance $R_{ref} \simeq r_1 \simeq r_2 \simeq 0.09\,\Omega$.

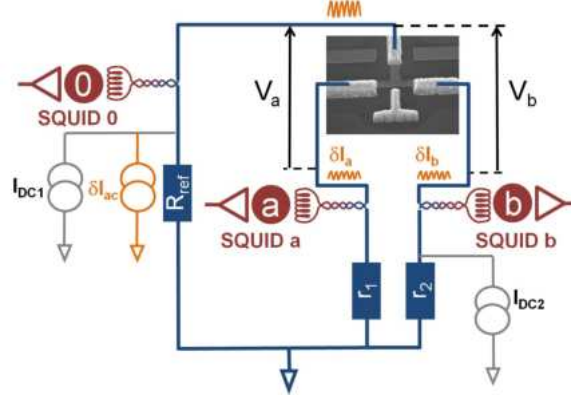


Figure 132: Experimental set-up for differential resistance measurements [27]. The three macroscopic resistors have low resistance values ($\simeq 0.1 \Omega$) allowing voltage-biasing the samples. The SEM image shows a three-terminal junction sample with a T-shape geometry.

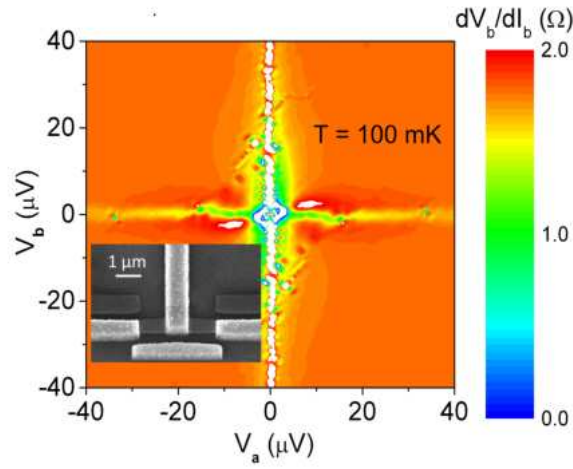


Figure 133: Differential resistance $R_{diff,b}$ of a three-terminal device with separated normal metal parts in the (V_a, V_b) plane at $T = 100$ mK. The SEM image represents a sample with such a typical geometry. In this case, only the upper half with $V_b > 0$ has been measured and the graph symmetrized.

The measurement scheme consists in sending an a.c. current modulation $\delta I_{ac} = 1 \mu A$ on the reference side and recording the current in each branch of the circuit. The differential resistance $R_{diff,a(b)}$ then reads :

$$R_{diff,a(b)} = R_{ref} \cdot (\delta I_{ac} - \delta I_0) / \delta I_{a(b)} - r_{1(2)} \quad (30.1)$$

where δI_i is the a.c. current measured in SQUID $i = 0, a$ or b . For all the samples studied here, $R_{diff,a}$ and $R_{diff,b}$ give the same behavior. In order to explore the non-linear response in the (V_a, V_b) plane, two d.c. current sources were used and the voltage differences V_a and V_b were measured with two room-temperature differential voltage amplifiers. In practice, I_{DC2} is first set to a fixed value and I_{DC1} ramped with current steps of 1 or 2 μA . When the ramp is finished, I_{DC2} is increased by a larger current step (typically 20 μA) and I_{DC1} ramped again. The density of measurement points is therefore not uniform, which explains the dotted features observed in the contour plots.

30.2.2 Results

Figure 133 shows the experimental data measured at $T = 100$ mK in a sample with separated junctions (see SEM image in Fig. 133). For this sample geometry, the separation holds as the Cu underneath the central electrode, although being continuous, is thin enough that the locally induced gap is that of the superconducting gap Δ of Aluminum. Only two anomalies corresponding to d.c. Josephson effects at $V_a = 0$ and $V_b = 0$ are detected. This confirms the absence of multi-terminal effects in the presence of a central electrode with a width (~ 900 nm) larger than the superconducting coherence length ξ_s , as already reported [18]. Such a device therefore behaves like two independent SNS junctions in parallel.

Here and in the following, the voltage range was limited to $\pm 40 \mu V$ because going beyond would require a d.c. current close to the superconducting electrodes depairing current [18]. As the investigated voltage range remains well below the superconductor energy gap, the number of multiple Andreev reflections ($\sim 2\Delta/eV$) necessary for a quasiparticle to reach the superconducting gap is more than 8, which would correspond to a total diffusion time much larger than the inelastic scattering time. This defines a strong interaction regime in which MAR cycles are interrupted by inelastic events. In such a bath of thermalized hot quasiparticles carrying an elevated effective temperature, the MAR-induced steps in the energy distribution function are completely washed out [28, 26] and hence the sub-gap structures related to the singularity of the DOS at the S/N interface cannot be observed.

We have investigated a novel type of three-terminal junction with a T-shape normal conductor connecting three superconducting electrodes S_0 , S_a and S_b whose SEM image is shown in Fig. 132. Here, S_0 corresponds to the upper central superconducting electrode, S_a and S_b the left and right superconducting electrodes respectively. The differential resistance $R_{diff,a}$ is shown in Fig. 134 at $T = 200$ mK. For this geometry, we expect three Josephson couplings J_{0a} , J_{0b} and J_{ab} , where the two indexes label the two involved superconducting terminals. In Fig. 134, the couplings J_{0a} and J_{0b} are clearly observed at $V_a = 0$ and $V_b = 0$ respectively. As expected from the definition of the differential resistance (Eq. 30.1), the Josephson coupling J_{0a} appears as a dip in the differential resistance $R_{diff,a}$ whereas the J_{0b} shows up as a peak. We have checked that the opposite behavior is observed when plotting $R_{diff,b}$.

We can see in Figure 134 that the coupling J_{ab} does not show up at $V_b - V_a = 0$. In the actual experiment, the a.c.-modulation was sent to the central electrode S_0 so that the separation of this current into the two branches S_a and S_b is not sensitive to the coupling J_{ab} . We have verified in a similar sample that the latter coupling is indeed revealed when sending the a.c.-modulation through S_a or S_b .

In addition to the two d.c. Josephson features discussed above, three other lines are clearly visible at $V_b = -V_a$, $V_b = 2V_a$ and $V_b = 1/2V_a$. Notice that in a T-shape geometry, the three superconducting reservoirs are equivalent, meaning that the voltages V_a , V_b , $V_a - V_b$ are also equivalent. We can thus state that these three lines all originate from the same type of mechanism involving the three superconducting contacts. The observation of this sub-gap structure in the low-bias differential conductance of a three-terminal superconducting hybrid device is the main experimental finding of the present work.

In a second step, we have studied the temperature dependence of the differential resistance of the T-shape device. The results are plotted at the bottom of Fig. 134. Apart from the central part that is related to the d.c. Josephson effect at very low bias, the sub-gap structure does not evolve much with temperature. All lines are found to be clearly visible up to 700 mK and $40 \mu V$. This confirms that, in the voltage range under investigation, the electronic temperature is well above the bath temperature [29].

To further investigate these new features, we have plotted some line-traces perpendicular to the $V_b = -V_a$ line (Fig. 135a), to the $V_a = 0$ line (Fig. 135b) and to the $V_b = 0$ line (Fig. 135c) for various levels of the applied voltages as indicated by the color lines in Figure 134.

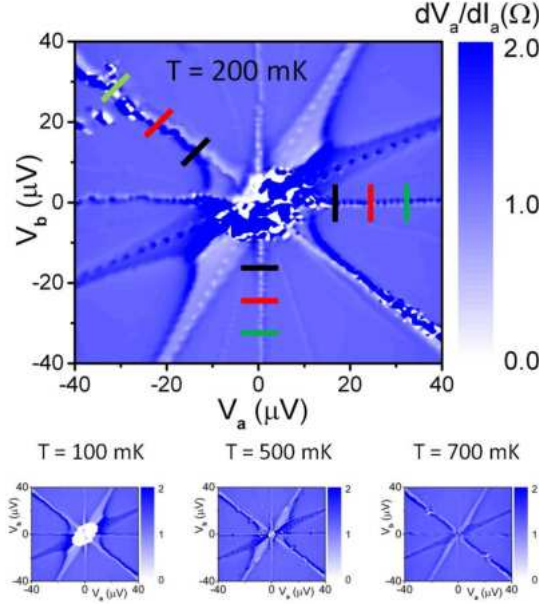


Figure 134: Differential resistance $R_{diff,a}$ of a T-shape junction in the (V_a, V_b) plane for various temperatures. At $T = 200$ mK, the data have been measured for the entire voltage range. For the other temperatures, only the upper half with $V_b > 0$ has been measured and the graph symmetrized for clarity.

As expected, the differential resistance $R_{diff,a}$ appears as a d.c. Josephson resonance around $V_a = 0$ for any value of V_b (Fig. 135b). The same type of response is observed when plotting $R_{diff,b}$ around $V_b = 0$ for any value of V_a (Fig. 135c). It turns out that when plotting the overall sample differential resistance recalculated by considering the two branches a and b as being in parallel ($R_{diff,ab} = R_{diff,a}R_{diff,b}/(R_{diff,a} + R_{diff,b})$) as a function of the voltage $V_a + V_b$, the observed profile of the sub-gap structure across the $V_b = -V_a$ line is also in striking resemblance with a Josephson resonance. This observation suggests that the additional anomalies are due to coherent effects involving the three terminals.

Moreover, it is important to notice that the features discussed here are rather robust and constant with respect to the applied voltage. More precisely, as seen in the Fig. 135, those persist at energies well above the Thouless energy. Therefore, the scheme to explain the additional features seen at non zero V_a and V_b and that involve the three terminals, must also be robust against voltage induced dephasing towards all the branches of the device.

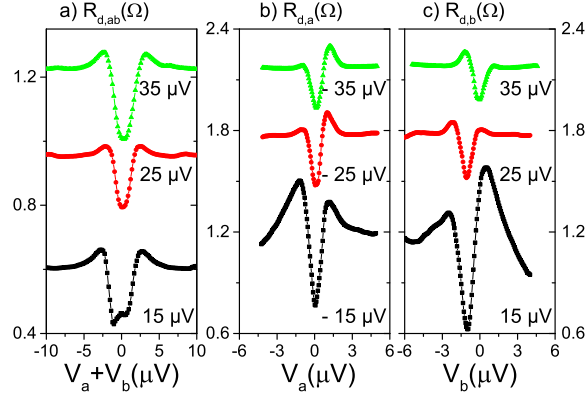


Figure 135: Line-traces at various values of the applied voltage of a) the differential resistance of the full sample considering the two branches a,b as being in parallel as a function of the voltage $V_a + V_b$, for several values of $(V_a - V_b)/2$, b) the differential resistance of branch a vs V_a for various values of V_b and c) the differential resistance of branch b vs V_b for various values of V_a . The color code follows that of the lines in Figure 134. The data are shifted for clarity except the lower ones.

30.3 Interpretation

30.3.1 Synchronization of a.c. Josephson effects

Sub-gap structures reminding the ones observed here were predicted and observed in the conductance of coupled but separated junctions [30, 31]. In this case, two a.c. Josephson currents coexist with frequencies $\nu_a = 2eV_a/h$ and $\nu_b = 2eV_b/h$. When the two frequencies match, e.g. $V_a = \pm V_b$, down-mixing through the non-linear response of the device can generate d.c. sub-gap structures similar to Shapiro steps. In our experimental scheme with a low resistive environment ($R_{ref}, r_1, r_2 \ll R_n$), this coupling could be obtained through the external circuit. Yet, no anomalies are observed in the sample with separate junctions, despite the fact that both samples have exactly the same circuit environment. In fact, due to the SQUID inductances and the wiring, the external impedances at the Josephson frequency are much larger than the resistances of the bias resistors, preventing any a.c. Josephson current to circulate in the external circuit. Therefore, the relevant coupling can only be within the sample itself.

An extended Resistively Shunted Junction (RSJ) model generalizing that of Ref. [30] could provide a phenomenological description. It involves a triangular Josephson array, shunted by the corresponding normal state resistances, that account for the quasiparticle processes within the N region. With such a model, the observation of strong resonances requires sizeable a.c. Josephson currents, whereas they are known to decrease when the voltages $eV_{a,b}$ exceed the Thouless energy E_{Th} [32]. Thus, even if the voltage decrease of a.c. Josephson currents is expected to be progressive, it should definitively lead to a reduction of the resonance for such a variation of $V_{a(b)}$. This is very much in contrast to what is observed in Fig. 135.

In addition to be quantitatively inconsistent, such an RSJ model is only phenomenological. Due to the long coherence time, transport in the present experimental conditions is truly mesoscopic and the explanation of the observations requires a phase-coherent microscopic mechanism taking place in the normal region.

One might consider a more microscopic approach and seek how the possible a.c. Josephson oscillations can synchronize together to yield a constant d.c. component. Such a problem indeed reminds the one considered in Refs. [33, 34, 25]. In a clean SNS junction polarized with a voltage V , oscillations with a frequency double of the basic Josephson frequency $\omega = \frac{2eV}{\hbar}$ are generated and show up under microwave irradiation as half-integer Shapiro steps. This was

explained by Argaman [33] within a semi-phenomenological description in which the both the Andreev levels and their steady state distribution oscillate at the Josephson frequency. The argument applies in the adiabatic regime in which the Thouless energy is much larger than the applied voltage.

The same argument could be applied to our three-terminal Josephson junctions for which the oscillating a.c. Josephson current at a frequency ν_a between two of the three terminals could be modulated by oscillation of the distribution function due to the Josephson coupling between two other terminals at a frequency ν_b . In that case, the second harmonic response obtained by Argaman et al. [33] in the case of a two terminal SNS junctions, transposes into a response at a frequency $\nu_a + \nu_b$ giving rise to DC features when $\nu_a = -\nu_b$.

Again, the conditions for such a scenario to apply are the same as in Refs. [33, 34, 25], e.g. that the voltage is small enough to allow an adiabatic approximation both in the current components and in the Andreev state distribution. The first one requires that $eV \ll E_{Th}$, the mini-gap scale, and the second is even more restrictive, $eV \ll \sqrt{E_{Th} \frac{\hbar}{\tau_i n}}$. For instance, in Lehnert et al.'s experiment, the frequency doubling is observed for $eV < 40 \mu eV$, an order of magnitude below the Thouless energy $E_{Th} = 350 \mu eV$. But in the experiment reported in our work, it is the other way around ! The sub-gap anomalies are observed for eV above E_{Th} , up to $8E_{Th}$, only limited by experimental constraints, and without any sign of decay. Thus, though qualitatively appealing, the above mechanism do not provide a good explanation for our experimental results.

30.3.2 The quartet scenario

Qualitative description

The limitation of the synchronization scenario is the voltage induced dephasing suffered by the two electrons of each of the Cooper pairs transferred between two superconducting terminals. Let us instead show that the quartet mechanism, proposed for clean bijunctions [36, 37], can be generalized to a diffusive system and is fully robust against dephasing at voltages much higher than E_{Th} .

The main idea is that two Cooper pairs are transferred in a single and fully energy-conserving quantum process in which the two pairs cross in an entangled way, by exchanging an electron between them.

To describe this mechanism, let us consider a piece of diffusive normal metal N connected to three superconducting reservoirs (S_0 , S_a and S_b) whose potentials are set to $V_0 = 0$, $V_a = +V$ and $V_b = -V$ respectively, as depicted in Fig. 136. Two Cooper pairs from S_0 can be simultaneous split in N , each of them in two electrons with opposite energies (with respect to the Fermi energy) that we define, without losing generality, as $\pm(eV + \epsilon)$ and $\pm(eV - \epsilon)$. When these energies are larger than the Thouless energy E_{Th} , the two electrons of each pair do not follow the same trajectory. Nevertheless, if the energy ϵ is small compared to E_{Th} , the electron of the first pair at $eV + \epsilon$ can follow the same phase-coherent trajectory as the electron of the second pair at $eV - \epsilon$ and for instance, reach S_a . Since $V_a = +V$, the two particles have relative opposite energies $\pm\epsilon$ and can recombine as a Cooper pair in S_a . The same mechanism holds for the two other particles at S_b . In the whole process, two Cooper pairs from S_0 are split altogether to create two spatially-separated Cooper pairs in S_a and S_b , a so-called quartet, named hereafter Q_0 as it originates from S_0 .

The key point favoring the quartet mechanism is that the coherence of each Andreev pair reaching S_a or S_b can be satisfied at any voltage $V_a = -V_b$, even when $|eV_{a(b)}| > E_{Th}$. Considering again the four electrons emitted from the two split pairs, two of them have energies $eV \pm \epsilon$ (pair (a)) and the two others $eV \pm \epsilon$ (pair (b)) (see Fig. 136). As the quartet mechanism

is a quantum process, the sum of all the possible diffusion probabilities has to be considered altogether. Among those, the situation where pair (a) propagates towards S_a and pair (b) towards S_b is phase coherent and independent of the applied voltage V . Indeed, the phase difference accumulated by the pair (a) (pair (b)) scales as $\epsilon\tau_{Da(b)}/\hbar$ where $\tau_{Da(b)}$ is the diffusion time from S_0 to $S_a(S_b)$. The quartet mode is therefore a fully coherent d.c. process taking place in the mesoscopic N region and involving four Andreev reflections.

This is very different from the scheme where two electrons of a single Cooper pair propagate towards S_a or S_b . In that case, the two electrons of a single pair have energies $\pm(eV + \epsilon)$ or $\pm(eV - \epsilon)$ and the accumulated phase difference scales with eV/E_{Th} . The effect of such trajectories has, therefore, a vanishing contribution to the electronic transport when $eV \gg E_{Th}$.

Let us note that the quartet response bears some resemblance with MAR's [17], with two important differences. First, the total energy balance of the process is zero, and second, it does not lead to quasiparticle transport above the superconducting gap.

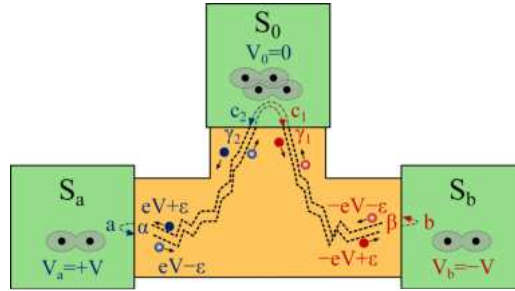


Figure 136: Schematic diagram for Q_0 quartet production, from S_0 to S_a and S_b . Two Cooper pairs are split simultaneously at S_0 with one electron of each pair propagating towards S_a and S_b where, under the appropriate energy condition ($V_a = -V_b$), they recombine to create two separated Cooper pairs.

In this quartet description, the line at $V_b = -V_a$ corresponds to the production of quartets Q_0 , whereas the line at $V_b = 2V_a$ ($V_b = 1/2V_a$) originates from quartets Q_a (Q_b) produced in S_a (S_b) towards S_0 and S_b (S_0 and S_a). At lowest order, the quartet mechanism requires only four Andreev reflections, much less than needed in the same voltage range for a quasiparticle to reach the gap edge in a usual MAR process, which makes the quartet mechanism much more robust to inelastic collisions.

Sketch of the microscopic calculation

The above arguments can be substantiated by a microscopic calculation (Appendix A), valid under the separation of energy scales $E_{Th} < eV < \Delta$. One uses non-equilibrium Keldysh Green's functions and performs a lowest order calculation in tunnel amplitudes at the different SN interfaces, inspired by Ref. [47]. The quartet current is calculated using a Hamiltonian formalism and an essential step is averaging over disorder.

The current appears as a sum of contributions, each being a product of six propagating amplitudes associated to the diagrammatic lines in Fig. 136 (Eq. 30.10 - 30.12). As a classical procedure in the treatment of diffusion in metals and in an SNS junction [36], disorder averaging takes advantage of the energy separation of these lines. In fact, two lines are correlated by disorder provided their energies are closer than E_{Th} . It results that the averaged product of six amplitudes can be decoupled into three factors. One corresponds to the diffusion of an Andreev pair from S_0 to S_a , at energies $eV \pm \epsilon$, with $\epsilon < E_{Th} < eV$, another one to the diffusion of an Andreev pair from S_0 to S_b , at energies $-eV \pm \epsilon$ and the third to the anomalous diffusion within S_0 that achieves Andreev reflection at energies $eV, -eV$ [37].

The principle of the above calculation can be benchmarked on the simpler case of an SNS junction at equilibrium for which the coherent *pair* current is proportional to the *single particle* conductance G_N times the coherent energy window given by the Thouless energy. This leads to the known scaling for the critical current $eI_c \propto G_N E_{Th}$ [25].

The main result of our calculation detailed in the Appendix, is to show that the coherent *quartet* current has a similar form and is given by the *two-particle* CAR conductance times the same energy window E_{Th} . It follows a scaling given by :

$$eI_Q \sim -G_{CAR} E_{Th} \quad (30.2)$$

The minus sign comes from the exchange and recombination process [37]. The conductance G_{CAR} is the Crossed Andreev conductance of a $N_a N S_0 N N_b$ structure in which the electrodes S_a and S_b are in the normal state and at voltages $\pm V$.

The CAR conductance can then be evaluated (see Appendix) and recast as :

$$G_{CAR} \sim \frac{G_{Na} G_{Nb}}{G_0(\xi_s)}, \quad (30.3)$$

where $G_{Na,b}$ is the conductance within each normal branch of the bijunction, and $G_0(\xi_s)$ is the normal-state conductance of a region of size ξ of the superconductor S_0 . This calculation shows that the ratio between the quartet maximum current at a bias V and the single junction critical current at zero bias is $I_Q/I_c(0) \sim G_{CAR}/G_N \sim G_N/G_0(\xi_s)$, which is not necessarily small. Based on measured sample parameters, we estimate this ratio to $\sim 0.1 - 0.5$, in fair agreement with the experiment. Notice that if $eV \ll \Delta$, G_{CAR} thus $I_{Q,max}$ does not decrease with V , in agreement with the present experiment.

30.4 conclusion

In conclusion, we reported about new sub-gap structures in the differential conductance of a metallic nanostructure with three superconducting reservoirs, a so-called bijunction. The existence of such anomalies well above the Thouless energy points towards a new and fully coherent mechanism, different from the synchronization of separated Josephson junctions, or any mesoscopic generalization of such a process. Our results are consistent with the production of non-local quartets, as a resonant pair of Cooper pairs, splitting and recombining within the N region. Therefore, our results provide a convincing experimental evidence for (double) crossed Andreev reflections in metallic superconducting / normal metal hybrid three-terminal nanostructures with a signature in the electronic response at low temperature much larger than in metallic Cooper pair splitters using only one superconducting reservoir.

The quartet mechanism carries intrinsic four-particle entanglement, generalizing two-fermion entanglement from CAR [10] that could be exploited if adding more degrees of freedom such as in quantum dots [36]. More refined probes are necessary to quantitatively study the correlated pair transport involving quartets, as well as possible other regimes, not evidenced in the present experiments, like the low-voltage adiabatic transport. An useful tool is to couple the bijunction to microwaves and study the Shapiro steps coming from deviations from the resonant situation $V_a = -V_b = V$.

30.5 Analytical calculation of the quartet current

The superconductors $S_{0,a,b}$ are described by the mean-field BCS Hamiltonian, with identical gaps Δ and phases $\varphi_0 = 0$, φ_a , φ_b . To simplify, all materials S_i, N are taken with the same bandwidth w , and they are connected by a hopping parameter τ , related to the interface

transparency T by $T = \frac{4\tau^2/w^2}{(1+\tau^2/w^2)^2}$. In Nambu notations, the hopping amplitudes take the form ($i = 0, a, b$, and α denoting the position on the interface)

$$\hat{\Sigma}_{i,\alpha}(t) = \tau \begin{pmatrix} e^{ieV_i t} & 0 \\ 0 & -e^{-ieV_i t} \end{pmatrix}. \quad (30.4)$$

The local advanced Green's functions in the superconductors are as follows in the frequency domain ($\omega_\eta = \omega - i\eta$):

$$\hat{g}_{i,i}^A(\omega) = \frac{1}{w\sqrt{\Delta^2 - \omega_\eta^2}} \begin{pmatrix} -\omega_\eta & \Delta e^{i\varphi_i} \\ \Delta e^{-i\varphi_i} & -\omega_\eta \end{pmatrix}, \quad (30.5)$$

The retarded Green's functions are obtained by changing η into $-\eta$ in the above expression. The choice of the gauge is such that the time-dependence of the phases $2eV_{a(b)}t/\hbar$ are included in the Nambu hopping amplitudes $\hat{\Sigma}_{a(b)}$ (with $\hbar = 1$). The phases $\varphi_{a(b)}$ at the origin of time are included in the off-diagonal components of the Nambu Green's functions. The local advanced Green's functions are $\hat{g}^A(\omega) \sim i\pi\rho_N$ in the normal metal, where ρ_N is the local density of states of the normal metal.

The current at some point a of the interface of the superconductor S_a is given by

$$I_a(t) = \frac{2e}{h} \mathcal{R}e \left[\hat{\Sigma}_{a,\alpha}(t) \hat{G}_{\alpha,a}^{(+ -),11}(t, t) - \hat{\Sigma}_{a,\alpha}(t) \hat{G}_{\alpha,a}^{(+ -),22}(t, t) \right] \quad (30.6)$$

where $\hat{G}_{\alpha,a}^{(+ -),11}(t, t)$ (resp. $\hat{G}_{\alpha,a}^{(+ -),22}(t, t)$) is the electron (resp. hole) Keldysh Green's function at point a .

Together with $\hat{G}^{(R,A)}$, $\hat{G}_{\alpha,a}^{(+ -)}(t, t)$ obeys a Dyson equation which allows to calculate the current as a product of Green's functions propagating electrons (holes) in the normal or superconducting regions, and hopping self-energies $\hat{\Sigma}$ at the interfaces. Stationarity allows to Fourier transform the time quantities and calculate the current contributions as a sum over the Fourier components $\hat{G}(\omega_n)$ with $\omega_n = \omega + neV$. Specifying to the voltages $V_a = V, V_b = -V$, the self-energies $\hat{\Sigma}(\omega)$ connect Green's functions with indices n differing by ± 1 .

The quartet diagram on Fig. 136 takes a typical chain form, starting at the $S_0 - N$ interface (with the frequency arguments omitted):

$$(\mathcal{A})_Q = \Sigma_{c_1,\gamma_1}^{11/00} g_{\gamma_1,\beta}^{11/00} \Sigma_{\beta,b}^{11/01} g_{bb}^{12/11} \Sigma_{b,\beta}^{22/12} g_{\beta,\gamma_1}^{22/22} \\ \Sigma_{\gamma_1,c_1}^{22/22} g_{c_1,c_2}^{21/22} \Sigma_{c_2,\gamma_2}^{11/22} g_{\gamma_2,\alpha}^{11/22} \Sigma_{\alpha,a}^{11/21} g_{aa}^{12/11} \\ \Sigma_{a,\alpha}^{22/10} g_{\alpha,\gamma_2}^{22/00} \Sigma_{\gamma_2,c_2}^{22/00} g_{02,c_1}^{21/00}. \quad (30.7)$$

The first two upper labels correspond to Nambu matrix notation and the second two to the harmonics (n, n') of half the Josephson frequency $\omega_0 = \frac{2eV}{\hbar}$. The advanced, retarded and Keldysh labels have to be inserted in this expression, resulting in eight different terms. Next, each of the eight terms is evaluated. The final expression for (A_Q) is as follows:

$$(\mathcal{A})_Q = n_F(\omega - eV)\tau^8 \\ \{g_{aa}^A g_{\alpha,\gamma_2}^A g_{\gamma_2,\gamma_1}^A g_{\gamma_1,\beta}^A g_{bb}^A g_{\beta,\gamma_1}^A g_{\gamma_1,\gamma_2}^A g_{\gamma_2,\alpha}^A - A \leftrightarrow R\} \quad (30.8)$$

where $A \leftrightarrow R$ means that “advanced” and “retarded” have been interchanged. The unperturbed Green's functions g_{ij}^A represent the amplitudes for electron and hole propagation and they are evaluated at the appropriate energies $\pm eV \pm \varepsilon$ shown on Fig. 136. Those energies correspond to the transitions between n and n' indices (see Eq. (30.7)) induced by the hopping matrix

elements. A summation over the labels α (β) and γ at the interfaces has to be carried out. This procedure is justified to describe extended contacts at lowest order in the tunnel amplitudes. As far as the applied voltages are small enough compared to the gap, the energy dependence of the Green's functions can be discarded.

The next step is to perform disorder averaging. A contribution such as $(\mathcal{A})_Q$ should be replaced by its average $\langle\langle(\mathcal{A})_Q\rangle\rangle$ over disorder in the N region and in the superconductors. Expression (30.8) contains several amplitudes, matrices in S_i , numbers in N . First, g_{aa}^A , g_{bb}^A yielding density of states factors in S_a and S_b . Second, the product $(g_{\gamma_2,\gamma_1}^A g_{\gamma_1,\gamma_2}^A)$ of amplitudes in S_0 , at energies close to eV (electrons) and $-eV$ (holes) that can be averaged separately. It describes the anomalous diffusion of a quasiparticle within S_0 , yielding Andreev-reflection at the NS_0 interface [37]. Third, the product $(g_{\alpha,\gamma_2}^A g_{\gamma_2,\alpha}^A g_{\gamma_1,\beta}^A g_{\beta,\gamma_1}^A)$ of amplitudes in N . The two first amplitudes contribute at energies $eV \pm \varepsilon$ and the two others at energies $-eV \pm \varepsilon$. We assume that $eV > E_{Th} > \varepsilon$, and use the fact that the coherence between electron and hole trajectories is limited by the Thouless energy. Then it is justified to decouple $\langle\langle g_{\alpha,\gamma_2}^A g_{\gamma_1,\beta}^A g_{\beta,\gamma_1}^A g_{\gamma_2,\alpha}^A \rangle\rangle \simeq \langle\langle g_{\alpha,\gamma_2}^A g_{\gamma_2,\alpha}^A \rangle\rangle \langle\langle g_{\gamma_1,\beta}^A g_{\beta,\gamma_1}^A \rangle\rangle$. This amounts to separately averaging the diffusive trajectories connecting S_a to S_0 , and S_b to S_0 , relying on energy rather than spatial separation. Then one obtains :

$$\begin{aligned} \langle\langle(\mathcal{A})_Q\rangle\rangle &= 2n_F(\varepsilon - eV)(\pi\rho_N)^2\tau^8 \\ &\quad \left\{ P_{\beta,\gamma_1} \tilde{P}_{c_1,c_2} P_{\gamma_2,\alpha} \right\} \sin(\varphi_a + \varphi_b). \end{aligned} \quad (30.9)$$

with

$$P_{\beta,\gamma_1} \equiv \langle\langle g_{\gamma_1,\beta}^{11}(\epsilon - eV) g_{\beta,\gamma_1}^{22}(\epsilon + eV) \rangle\rangle \quad (30.10)$$

$$P_{\gamma_2,\alpha} \equiv \langle\langle g_{\gamma_2,\alpha}^{11}(\epsilon + eV) g_{\alpha,\gamma_2}^{22}(\epsilon - eV) \rangle\rangle \quad (30.11)$$

$$\tilde{P}_{c_1,c_2} \equiv \langle\langle g_{c_1,c_2}^{12}(\epsilon - eV) g_{c_2,c_1}^{21}(\epsilon + eV) \rangle\rangle. \quad (30.12)$$

and where the characteristic phase dependence of the quartet mode stems from the four involved Andreev reflections, one at S_a , one at S_b and two at S_0 . A product of three probabilities appear : $P_{\gamma_2,\alpha}$ for electron-hole (Andreev pair) diffusion from S_a to S_0 , $\tilde{P}_{c_1,c_2}$ for the anomalous diffusion inside S_0 , and P_{β,γ_1} for the Andreev pair diffusion from S_0 to S_b . $\tilde{P}_{c_1,c_2}$ tracks the probability of two Andreev reflections at S_0 . The Andreev pair diffusion modes $P_{ij} = P(R_{ij}, \omega)$ showing out in Eq. 30.9 are obtained by a summation of the ladder diagrams [36], standard in the diffusion problem.

The diffusion probability on a distance R is $P_0(R, \omega, V) = \langle\langle g^{11,A}(R, \omega - eV) g^{22,A}(R, \omega + eV) \rangle\rangle$ and its space Fourier transform is proportional to

$$P_0(\mathbf{q}, \omega, V) \sim \frac{1}{w(i\omega + Dq^2)}. \quad (30.13)$$

Importantly, $P_0(\mathbf{q}, \omega, V)$ has no dependence on V , in the $V \ll \Delta$ limit. Most importantly, the quartet current appears even if the voltage is larger than the Thouless energy.

The above principle for the quartet current calculation can be benchmarked on the simpler case of an SNS junction at equilibrium. One considers the DC-Josephson current in a $S_a IN IS_b$ junction, and evaluate it on the same line as above, by a expansion of the current to fourth order in the transparencies. Then the Fourier transform of the diffusion probability associated to the Andreev pair modes $\langle\langle g_{\gamma_1,\beta}^{11,A}(\omega - eV) g_{\beta,\gamma_1}^{22,A}(\omega + eV) \rangle\rangle$ in N is also given by $P_0(\mathbf{q}, \omega, V)$. For comparison, in a $N_a IN IN_b$ junction with the mode $\langle\langle g_{\gamma_1,\beta}^{11,A}(\omega - eV) g_{\beta,\gamma_1}^{11,R}(\omega + eV) \rangle\rangle$, the diffusion probability is $P_0(\mathbf{q}, 0, V)$, thus without the $i\omega$ factor in the denominator of Eq. (30.13).

Depending on the diffusion taking place in N or S, this results after integration over q :

$$\begin{aligned} P_{0N}(R, \varepsilon) &\sim 2\pi\rho_N \frac{1}{2DR} \exp\left(-\sqrt{\frac{\varepsilon}{E_{Th}}}\right) \cos\left(\sqrt{\frac{\varepsilon}{E_{Th}}}\right) \\ \tilde{P}_{0S}(R, \varepsilon) &\sim 2\pi\rho_N \frac{1}{2DR} \exp\left(-\frac{R}{\xi}\right) \end{aligned} \quad (30.14)$$

where the Thouless energy for a junction of length R is $E_{Th} = \frac{\hbar D}{R^2}$. Notice that $\tilde{P}_{0S}$ decays on the effective coherence length in S_0 (taking into account disorder) [37].

The quartet current calculation follows from equations (A3-A6).

$$\begin{aligned} I_{tot} &\sim \frac{2e}{h} \mathcal{N} \frac{\tau^8}{w^2} \int d\omega P_{\beta, \gamma_1}(\omega) P_{\gamma_2, \alpha}(\omega) \\ &\int_{S_c} \frac{d^2 r}{\xi^2} \tilde{P}_{c_1, c_2}(\omega) (\delta\mathcal{V})^3 \sin(\varphi_a + \varphi_b) \end{aligned} \quad (30.15)$$

where $\mathcal{N}$ is the average number of channels, due to integration on one of the interfaces of $S_{a,b}$. The integration volume $\delta\mathcal{V} \sim \lambda_F l_e^2$ accounts for the absorbing boundary conditions for diffusion in the reservoirs [36] (λ_F is the Fermi wavelength). Integration over the surface of S_0 accounts for the range ξ of the Andreev reflection and yields a total factor $\frac{\xi}{wl_e}$ for the integrated Andreev probability in S_0 . Integration over frequency yields the factor E_{Th} , each diffusion probability contributes by a factor $\frac{l_e}{wL}$. One finally obtains:

$$eI_Q \sim -G_{CAR} E_{Th} \sin(\varphi_a + \varphi_b), \quad (30.16)$$

A yet unknown prefactor has to be added in Eq. (30.2), which is expected to be of the same order as that involved in the case of a SNS junction.

The conductance G_{CAR} refers to the Crossed Andreev Reflection (CAR) in a $N_a N S_0 N N_b$ hybrid structure. The maximum quartet current is thus naturally obtained by multiplying the CAR conductance by the Thouless energy that sets the coherence of Andreev pairs on both branches a,b of the bi-junction.

The CAR conductance is evaluated from above:

$$G_{CAR} \sim \frac{2e^2}{h} \mathcal{N} \left(\frac{\tau}{w}\right)^8 \left(\frac{l_e}{L}\right)^2 \frac{\xi}{l_e}, \quad (30.17)$$

where the ratio $\frac{\tau}{w}$ is taken from the rather good experimental conductance $T \sim 0.3$.

31 Flux-Tunable Josephson Diode Effect in a Hybrid Four-Terminal Josephson Junction by Coraiola, Svetogorov, Haxell et al.

Abstract

We investigate the direction-dependent switching current in a flux-tunable four-terminal Josephson junction defined in an InAs/Al two-dimensional heterostructure. The device exhibits the Josephson diode effect, with switching currents that depend on the sign of the bias current. The superconducting diode efficiency, reaching a maximum of $|\eta| \approx 34\%$, is widely tunable—both in amplitude and sign—as a function of magnetic fluxes and gate voltages. Our observations are supported by a circuit model of three parallel Josephson junctions with non-sinusoidal current–phase relation. With respect to

conventional Josephson interferometers, phase-tunable multiterminal Josephson junctions enable large diode efficiencies in structurally symmetric devices, where local magnetic fluxes break both time-reversal and spatial-inversion symmetries. Our work establishes a pathway to develop Josephson diodes with wide-range tunability and that do not rely on exotic materials or externally applied magnetic fields.

Nonreciprocal transport phenomena play a key role in modern electronics, with semiconductor diodes serving as the fundamental components for numerous devices [?]. In analogy to the semiconductor diode, whose electrical resistance strongly depends on the current direction, a superconducting diode allows a larger supercurrent to flow in one direction compared to the other [?]. Nonreciprocal supercurrents were recognized already in the 1970s in superconducting quantum interference devices (SQUIDs) based on superconducting bridges [?] and tunnel Josephson junctions (JJs) [? ?], arising as a consequence of the finite loop inductance. Direction-dependent switching currents were also observed in conventional superconducting thin films and interpreted as a manifestation of microscopic asymmetries in the device geometry [?]. More recently, the superconducting diode effect (SDE) has sparked renewed interest, driven by its connection to fundamental properties of a diverse range of superconducting systems, where the breaking of both inversion and time-reversal symmetries is required for the effect to occur. Since its observation in artificial superlattices [?], the SDE has been the subject of thorough experimental and theoretical investigation, both in junction-free thin films [? ? ? ? ? ?] and JJs based on semiconductors with spin-orbit coupling [? ? 107?], finite-momentum superconductors [? ? ?] or multilayered materials, realizing sizeable asymmetries even without external magnetic fields [? ? ? ? ? ? ? ?].

An alternative platform proposed to achieve the SDE in Josephson devices—where it is usually referred to as Josephson diode effect (JDE)—relies on a supercurrent interferometer, where two JJs with nonsinusoidal current-phase relations (CPRs) are combined in a SQUID [? ?]. Such CPRs, containing contributions from higher harmonics than the conventional 2π -periodic component, are routinely attained in high-quality superconductor-semiconductor planar materials [? 15? ?], where hybrid JJs host Andreev bound states (ABSs) characterized by high transmission. Key ingredients for the JDE to occur in this system are the different harmonic content between the two JJs, and a magnetic flux threading the SQUID loop [?], as recently demonstrated in two-dimensional (2D) electron [? ?] and hole [?] systems, obtaining large diode efficiencies at equilibrium up to approximately 30%. Furthermore, three-terminal Josephson devices without phase control were shown to realize the JDE by exploiting high harmonic terms in the CPR [74], multiple bias currents [? ?] and Andreev molecules [? ?].

In this work, we report on the superconducting transport properties of a superconductor-semiconductor four-terminal JJ (4TJJ) embedded in a double-loop geometry. Supercurrents are tuned by magnetic fluxes threading the superconducting loops and by gate electrodes that control the number and transmissions of ABSs. We find strong JDE, tunable in both efficiency and sign with gate voltages and two independent superconducting phases—controlled via integrated flux-bias lines, without the need for external magnetic fields—reaching a peak efficiency of $\pm 34\%$. One of the loops was further controlled by gating an additional JJ with large critical current, enabling operation of the device in a single-loop configuration. We provide an in-depth explanation of the JDE in our system by means of a simple circuit model, which maps our device to the combination of two SQUIDs, or a bi-SQUID. Simulations are performed both in an idealized case with minimal assumptions, and in an extended version that accurately captures the experimental results. We demonstrate several novelties with respect to previous work. First, we devise a geometry that combines both gate control of the current path and phase tunability in a 2D space, offering new insights into multiterminal JJs. Second, the phase-tunable bi-SQUID allows for a large diode efficiency to occur without the need of a JJ imbalance in harmonic content, which is required in conventional interferometers [?]. Third,

phase control enables large tuning of the JDE in amplitude and sign, including a vanishing diode efficiency in extended regions of the phase space. In the light of our results, multiterminal JJs in superconductor–semiconductor hybrid systems offer advantages that are pivotal for the realization of nonreciprocal transport phenomena. Thanks to the nonsinusoidal CPR and the possibility to break time-reversal symmetry solely by flux biasing, large and tunable diode efficiencies are naturally obtained without the need of an external magnetic field. Future work might further expand the study of multiterminal devices to realize nonreciprocal transport in the linear regime [? ?], presenting opportunities for innovative applications.

31.0 Results and Discussion

31.0.0 Flux-tunable multiterminal Josephson junction

The device under study, consisting of a multiterminal JJ embedded in a double-loop geometry, is displayed in Fig. 171. It was realized in an InAs/Al heterostructure [104, 48], where the epitaxial Al layer was selectively removed to expose the III–V semiconductor below. We defined four superconducting terminals, labeled S, L, M and R, coupled to a common semiconducting region. Lithographically, the minimum distances between neighboring terminals were 30 nm (for L–M and R–M) and 50 nm (for S–L and S–R), while opposite terminals had separations of 100 nm (L–R) and 120 nm (S–M). All junctions were short with respect to the superconducting coherence length in the InAs 2D electron gas, estimated to be approximately 600 nm (see Methods section). Terminals L, M and R were connected to a common node (D) forming two superconducting loops, which enabled independent control over two phase differences [78?], $\phi_L - \phi_M \equiv \phi_L$ and $\phi_R - \phi_M \equiv \phi_R$ (here, ϕ_α indicates the superconducting phase of terminal $\alpha \in \{L, M, R\}$ and ϕ_M was set to zero by convention). This was achieved by passing currents I_L and I_R through two flux-bias lines, patterned on top of a uniform dielectric layer, resulting in external magnetic fluxes Φ_L and Φ_R respectively threading the left and right loop. Gate electrodes were deposited on the same dielectric layer and energized by voltages V_α ($\alpha \in \{S, L, M, R\}$) and V_J , allowing for electrostatic tuning of the electron density in the InAs layer below. While terminals L and M were directly connected to the node D via Al strips, a planar JJ (named switch JJ) was integrated on terminal R. The switch JJ, with a length of 40 nm and a width of 5 μm , was designed to have a critical current much larger than that between any pairs of leads in the 4TJJ, and therefore the phase difference across the switch JJ can be neglected for the following discussion. Depending on the gate voltage V_J , the switch JJ was employed in two configurations: $V_J = 0$ (switch ON), where the JJ was conducting and Φ_R could be used to control ϕ_R , or $V_J = -1.5$ V (switch OFF), where the JJ was depleted, the right loop was interrupted and terminal R was reduced to a floating superconducting island. The other gate voltages were set to $V_S = 0.1$ V, $V_L = V_R = -0.1$ V and $V_M = -0.15$ V, unless stated otherwise. The device was measured in a dilution refrigerator with a base temperature of about 10 mK. Current-bias experiments were performed in a four-terminal configuration by driving a current I_{SD} between S and D and measuring the voltage drop across the device, which allowed for the measurement of the switching current I_{sw} . Along its path between S and D, the current flowed through the semiconducting region forming the four-terminal JJ, and in particular across the S–L, S–M and S–R junctions. In our geometry, the superconducting loops were designed to limit their inductance, that could in principle lead to the SDE in the system. The maximum flux variation due to the inductance of a loop, estimated to be approximately 124 pH (see details in the Supplementary Material, Section 5), for a circulating current of the order of 100 nA, is $\sim 6 \times 10^{-3} \Phi_0$. This was observed to be negligible with respect to the flux scales over which the device properties varied. Further details regarding materials, fabrication and measurement setup are provided in the Methods section. Results on a second device,

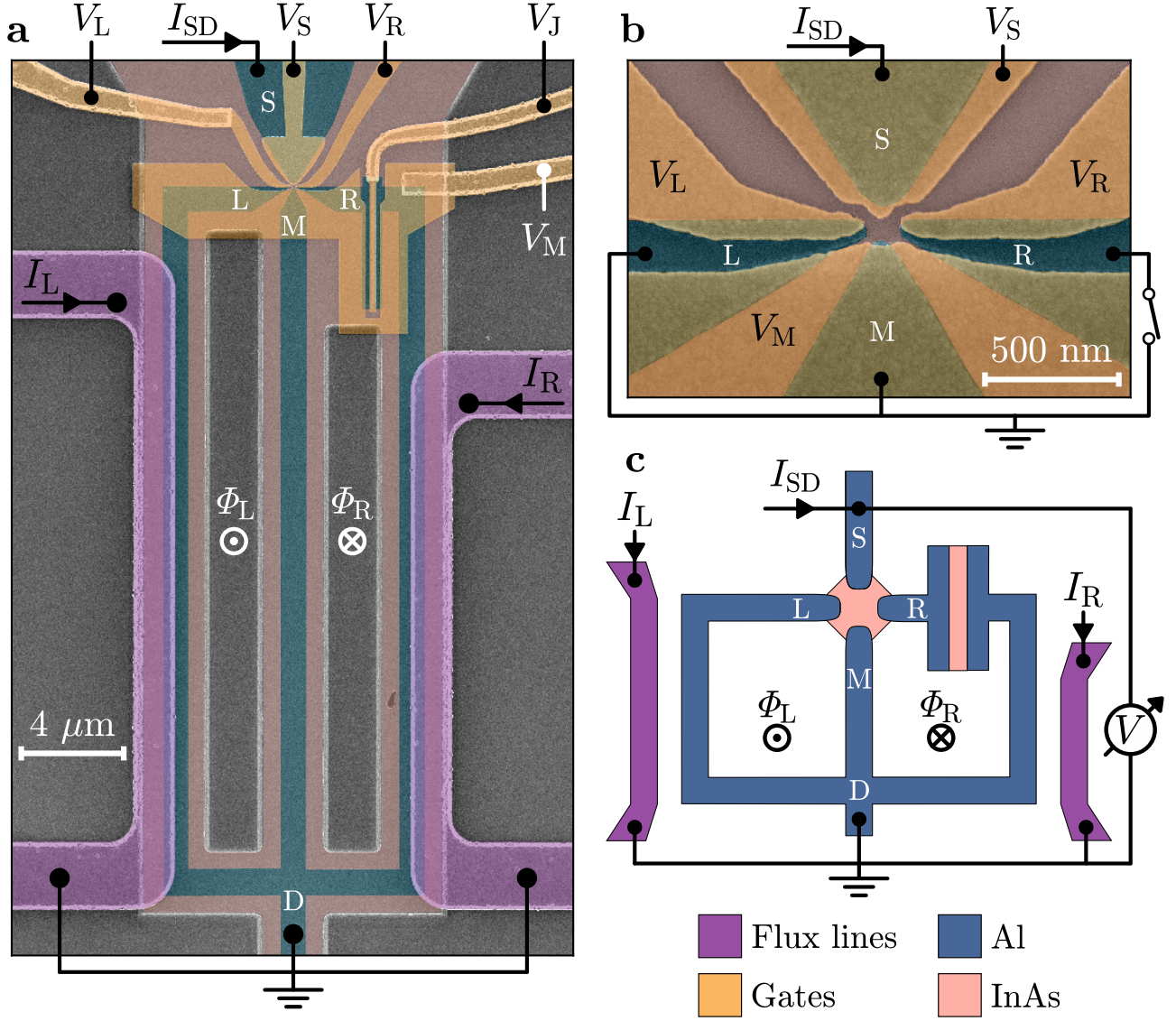


Figure 137: Device under study and measurement setup. (a) False-colored scanning electron micrograph of a device identical to that under study. Exposed III–V semiconductor is represented in pink, Al in blue, gate electrodes in yellow and flux-bias lines in purple. Bias current I_{SD} , flux-line currents I_L and I_R , magnetic fluxes threading the superconducting loops Φ_L and Φ_R , and gate voltages V_S , V_L , V_M , V_R and V_J are labeled. Superconducting terminals S, L, M, R and the common node D are also indicated. (b) Zoom-in of (a) in the vicinity of the four-terminal Josephson junction. (c) Schematic representation of the device with the measurement setup, using the same color labeling as in (a) and (b). Gate electrodes are not shown.

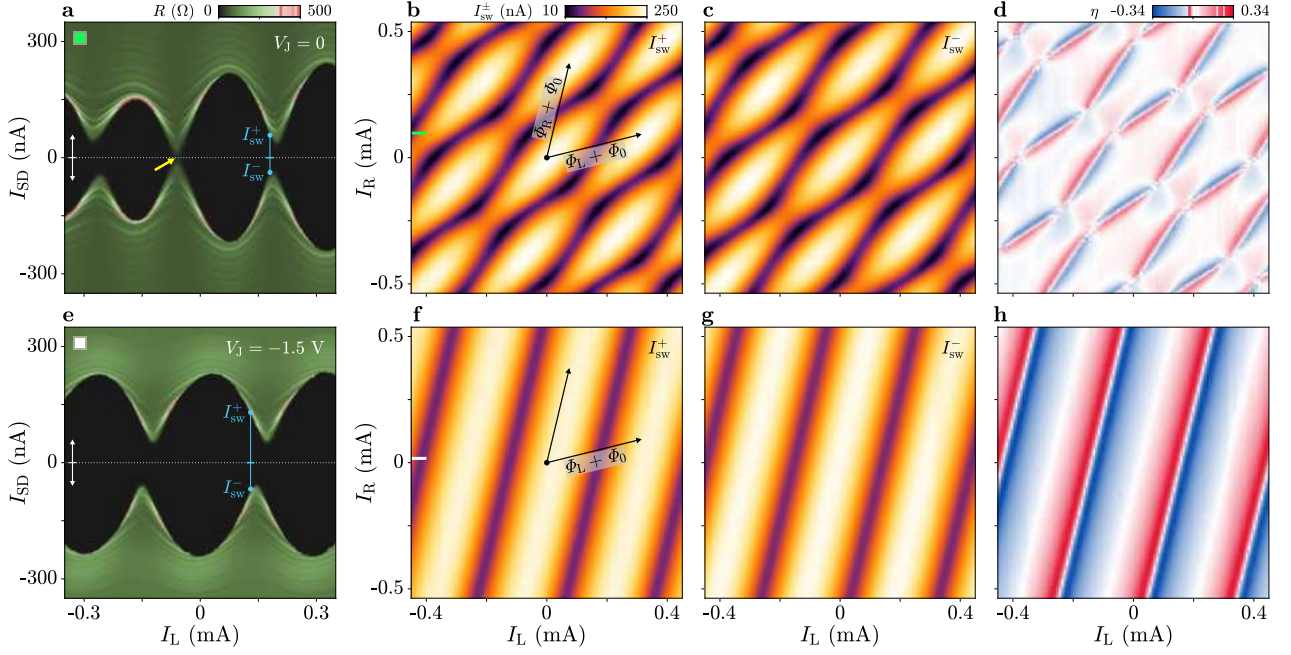


Figure 138: Phase-tunable Josephson diode effect. (a) Differential resistance R as a function of left flux-line current I_L and source-drain bias current I_{SD} , for fixed right flux-line current $I_R = 0.1$ mA. The map is obtained by merging two datasets recorded with I_{SD} ramping from 0 to either positive or negative values (see white arrows). Switching current nonreciprocities are highlighted by cyan annotations. A point where the switching current reaches zero is indicated by the yellow arrow. Gate voltages were set to $V_L = V_R = -0.1$ V, $V_S = 0.1$ V, $V_M = -0.15$ V and $V_J = 0$. (b, c) Switching currents I_{sw}^+ and I_{sw}^- , measured for positive and negative I_{SD} respectively, as functions of I_L and I_R . Arrows in (b) indicate the directions along which magnetic fluxes threading the left and right superconducting loop, Φ_L and Φ_R , vary independently. Each arrow represents the addition of one superconducting flux quantum Φ_0 to the corresponding flux. (d) Superconducting diode efficiency η calculated from (b) and (c) using Eq. 31.1 (see text), as a function of I_L and I_R . (e–h) As in (a–d), but measured for $V_J = -1.5$ V, which sets the switch JJ to the OFF state and interrupts the right loop.

similar to the one discussed in the Main Text, are presented in the Supplementary Material (see Figs. S6–S10 in Section 4). Devices studied here were employed in a previous work that investigated hybridization of ABSs in multiterminal JJs [78].

31.0.0 Nonreciprocal supercurrents in the 2D phase space

First, we present the differential resistance R as a function of the current bias I_{SD} and of the left flux-line current I_L for fixed right flux-line current $I_R = 0.1$ mA. Here, R was measured with standard lock-in techniques and I_{SD} was swept from 0 to positive or negative values to avoid retrapping effects. Figure 172a shows the result for $V_J = 0$ (switch ON): the switching current revealed oscillations of varying amplitude as a function of I_L , which, notably, were nonreciprocal at positive and negative I_{SD} . For instance, at $I_L = 0.18$ mA we measured switching currents $I_{sw}^+ = 58$ nA at $I_{SD} > 0$ and $I_{sw}^- = 38$ nA at $I_{SD} < 0$ (see cyan annotations), where $I_{sw}^\pm > 0$ by definition. This resulted in a superconducting diode efficiency η , defined as:

$$\eta = \frac{I_{sw}^+ - I_{sw}^-}{I_{sw}^+ + I_{sw}^-}, \quad (31.1)$$

of approximately 21%. We also note that the switching current vanished in a small range around $I_L \approx -60 \mu\text{A}$ (yellow arrow), namely the device had finite differential resistance at $I_{\text{SD}} = 0$. Similar maps obtained at different settings of I_R are presented in Fig. S1 of the Supplementary Material.

To efficiently measure the switching currents $I_{\text{sw}}^\pm$ and the diode efficiency η as functions of both I_L and I_R , we changed measurement technique and periodically ramped I_{SD} from zero to the amplitude $A = \pm 260 \text{ nA}$ with a repetition rate of 133 Hz, and detected when the voltage drop across the device exceeded a threshold. The time spent in the low-resistance state, averaged over 32 consecutive measurements, was converted to a current, resulting in a rapid measurement of I_{sw}^+ or I_{sw}^- (depending on the sign of A) displayed in Figs. 172b and 172c, respectively. A limitation of this measurement technique was that values of I_{sw} below $\sim 10 \text{ nA}$ could not be accurately detected due to the finite voltage threshold, which gave a finite reading of about 10 nA for small switching currents, and even when the device was resistive for zero bias current. The switching current oscillated periodically in the 2D phase space—where the periodicity axes correspond to the external magnetic fluxes Φ_L and Φ_R (black lines in Fig. 172b)—forming a pattern characterized by lobe-like features. The finite slope of the Φ_L and Φ_R axes with respect to I_L and I_R was due to the cross-coupling between the left (right) flux-bias line and the right (left) loop, as discussed in Section 5 of the Supplementary Material. The oscillations of I_{sw} exhibited maxima of approximately 250 nA where Φ_L and Φ_R were integer multiples of the superconducting flux quantum $\Phi_0 = h/2e$ (with h the Planck constant and e the elementary charge), and minima at finite phases where the limit of detection was reached, consistent with the vanishing switching current discussed for Fig. 172a. We note that the switching currents were nonreciprocal upon reversal of the current bias, while their 2D patterns were symmetric to each other with respect to the origin ($I_L = I_R = 0$, corresponding to $\Phi_L = \Phi_R = 0$). The symmetry was particularly visible in the shape of the lobes, which was inverted as the supercurrent changed sign. Figure 172d shows the superconducting diode efficiency calculated from Figs. 172b and 172c using Eq. 31.1. As expected, η reflected the 2D periodic pattern in the phase space of the switching currents, and was widely tunable as a function of I_L and I_R . We observe a fully ambipolar character and large values up to $\eta \approx \pm 21\%$ where $I_{\text{sw}}^\pm$ had a large gradient in the phase space, while the efficiency vanished in extended regions of the phase space without the need for fine tuning I_L and I_R .

31.0.0 Nonreciprocal supercurrents in single-loop configuration

Next, in Figs. 172e–h we present the measurements corresponding to those discussed in Figs. 172a–d but with the switch junction in the OFF state ($V_J = -1.5 \text{ V}$). From the differential resistance as a function of I_{SD} and I_L (Fig. 172e, here for $I_R = 17 \mu\text{A}$), we found periodic oscillations of the switching current, with I_{sw}^+ and I_{sw}^- exhibiting a phase shift from each other and opposite skewness (in the forward direction for I_{sw}^+ , backward for I_{sw}^-). Consequently, the switching currents were again nonreciprocal depending on I_{SD} , which indicates a large JDE; at $I_L = 0.13 \text{ mA}$, for example, $I_{\text{sw}}^+ = 130 \text{ nA}$ and $I_{\text{sw}}^- = 68 \text{ nA}$ (see cyan annotations), yielding $\eta \approx 31\%$. We note that, in this configuration, the switching current did not vanish for any value of I_L , with minimal values of $\approx 50 \text{ nA}$, unlike the case with the switch ON. Measurements of I_{sw}^+ and I_{sw}^- as functions of both I_L and I_R are shown in Figs. 172f and 172g. The 2D pattern observed for $V_J = 0$ was no longer present: as expected, the dependence on the flux Φ_R was suppressed when the right superconducting loop was interrupted, and periodicity remained along a single direction. In agreement with Fig. 172e, the switching current oscillations were shifted in phase and their skewness was reversed depending on the sign of the current bias. The superconducting diode efficiency, displayed in Fig. 172h for the data of panels f and g, was also characterized by a periodic behavior as a function of Φ_L and reached maxima of approximately

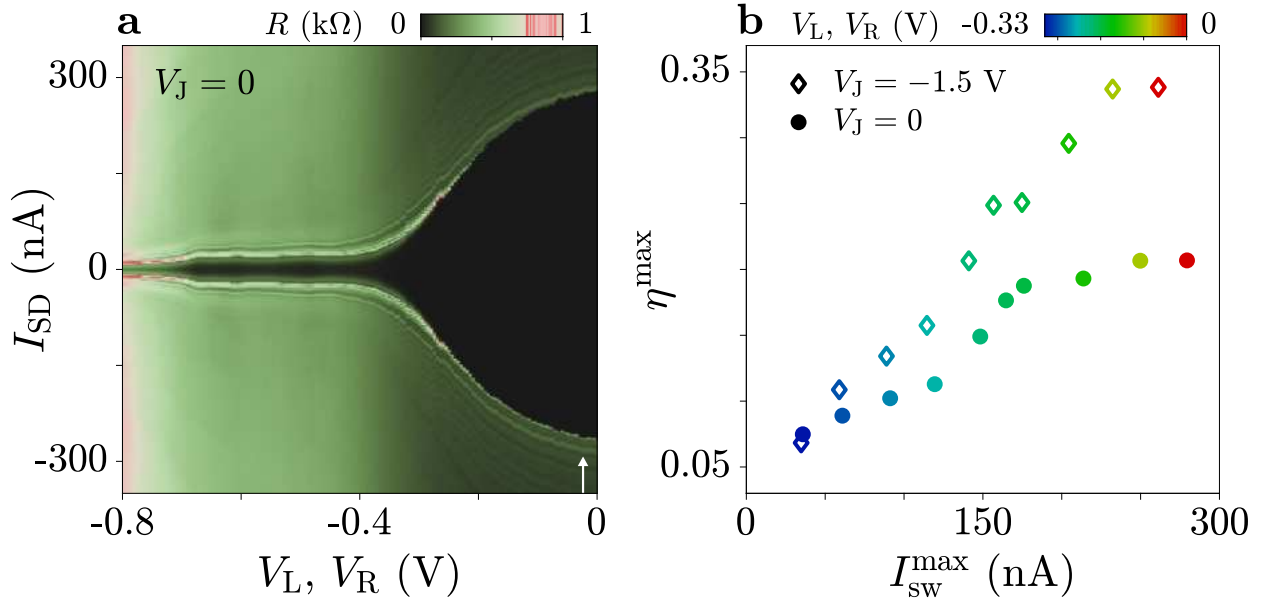


Figure 139: Gate-tuning of the maximum switching current and diode efficiency. (a) Differential resistance R as a function of gate voltages $V_L = V_R$ and bias current I_{SD} (swept from negative to positive values, see white arrow). (b) Maximum Josephson diode efficiency $\eta^{\max}$ as a function of the maximum switching current $I_{sw}^{\max}$ for multiple values of $V_L = V_R$ (see colorscale). Each point is obtained from maps similar to Figs. 172b and 172c (see Supplementary Material, Section 3 for more details). Full circles refer to the situation with $V_J = 0$, empty diamonds to the situation with $V_J = -1.5$ V.

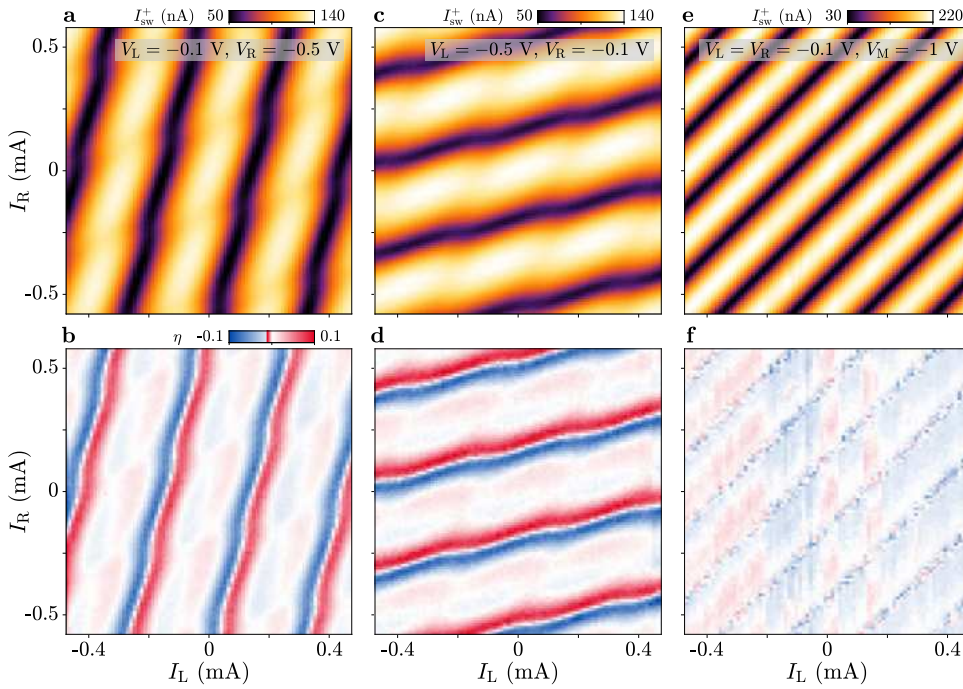


Figure 140: Routing of the supercurrent. (a) Switching current I_{sw}^+ measured for positive bias current I_{SD} , as a function of flux-line currents I_L and I_R . Measurements are performed with $V_L = -0.1$ V and $V_R = -0.5$ V. (b) Diode efficiency η for the configuration of (a). (c, d) As in (a, b), but for $V_L = -0.5$ V and $V_R = -0.1$ V. (e, f) As in (a, b), but for $V_L = -0.1$ V, $V_R = -0.1$ V, and $V_M = -1$ V.

34%.

31.0.0 Gate-tunable Josephson diode effect

Electrostatic tunability over the supercurrents and the JDE was enabled by gates controlling the electron density in the semiconducting region of the 4TJJ. In Fig. 173a, we show the differential resistance as a function of the current bias (here swept from negative to positive values, thus displaying both retrapping and switching currents) while the gate voltages V_L and V_R varied simultaneously, for $V_J = 0$ and $I_L = I_R = 0$. The switching current and the retrapping current, respectively 280 nA and -265 nA at $V_L = V_R = 0$, decreased for more negative voltages, until a finite resistance was measured at $I_{SD} = 0$ for $V_L = V_R \approx -0.35$ V. To investigate the voltage-tunability of the JDE, we measured the switching currents $I_{sw}^\pm$ as functions of I_L and I_R (as in Figs. 172b, c, f, g) for varying V_L and V_R , and, in each configuration, we extracted the peak values of I_{sw}^+ , named $I_{sw}^{\max}$, and of the superconducting diode efficiency, $\eta^{\max}$ (see Supplementary Material, Section 3 for the details of the extraction procedure). The result is presented in Fig. 173b, where $\eta^{\max}$ is plotted as a function of $I_{sw}^{\max}$ for the two settings of the switch JJ $V_J = 0$ and $V_J = -1.5$ V. In both cases, we observed an increasing trend of $\eta^{\max}$ as $I_{sw}^{\max}$ increased. For any gate setting, the diode efficiency was larger at $V_J = -1.5$ V than at $V_J = 0$ V, up to 34% and 21% respectively (at $V_L = V_R = 0$), while $I_{sw}^{\max}$ was slightly smaller in the former case (up to 260 nA, instead of 280 nA for $V_J = 0$).

We further characterized the gate dependence of the device by allowing an asymmetric tuning of V_L and V_R (at $V_J = 0$), as shown in Figs. 157a, b and 157c, d for the configurations $V_L = -0.1$ V, $V_R = -0.5$ V and $V_L = -0.5$ V, $V_R = -0.1$ V. In each case, the switching current measured as a function of I_L and I_R for positive current bias is displayed in the first panel, while the second panel presents the diode efficiency extracted from I_{sw}^+ and I_{sw}^- . The two configurations revealed a complementary behavior: the switching current oscillations and the diode efficiency were almost completely suppressed as a function of $\Phi_{L(R)}$ when $V_{L(R)}$ was set to sufficiently negative value, depleting the semiconducting region between terminals S and L (R). This highlights the possibility of routing the supercurrents flowing in our device by gating, which enabled electrostatic control over the phase dependence of the JDE. The results obtained for $V_R = -0.5$ V (panels a and b) were reminiscent of those previously observed for $V_J = -1.5$ V (Figs. 172f–h), where data were independent of Φ_R .

Finally, we restored the symmetric gate configuration $V_L = V_R = -0.1$ V and studied the effect of depleting the middle gate V_M , set to -1 V (see Fig. 157e and 157f). Here, we observed periodic oscillations of the switching current along a single direction of the phase space, corresponding to the $(\Phi_L - \Phi_R)$ -axis. The frequency of these oscillations was doubled compared to the case where V_M was not depleted (e.g., Fig. 172b), consistent with the exclusion of terminal M from the current path. As a consequence, screening currents induced by the flux-bias lines only circulated in the perimeter of the double-loop geometry, leading I_L and I_R to control the total flux $\Phi_L - \Phi_R$ (note that Φ_L and Φ_R were defined with opposite signs in Figs. 171a and 171c). Notably, in this symmetric gate configuration where no current flowed into terminal M, the JDE was essentially suppressed (see Fig. 157f). Results for additional gate settings are shown in the Supplementary Material, see Figs. S2–S5.

31.0.0 Minimal-model description of JDE

To understand the behavior of our device in more depth, and the underlying origin of the JDE, we introduce a simple circuit model that describes the supercurrents of the 4TJJ. In the switch-ON configuration, terminals L, M and R are all connected to the same node D, therefore the current can flow from S to D (or from D to S) via three available JJs S–L, S–M and S–R. That is, the 4TJJ is mapped onto a bi-SQUID as three distinct JJs are connected in parallel.

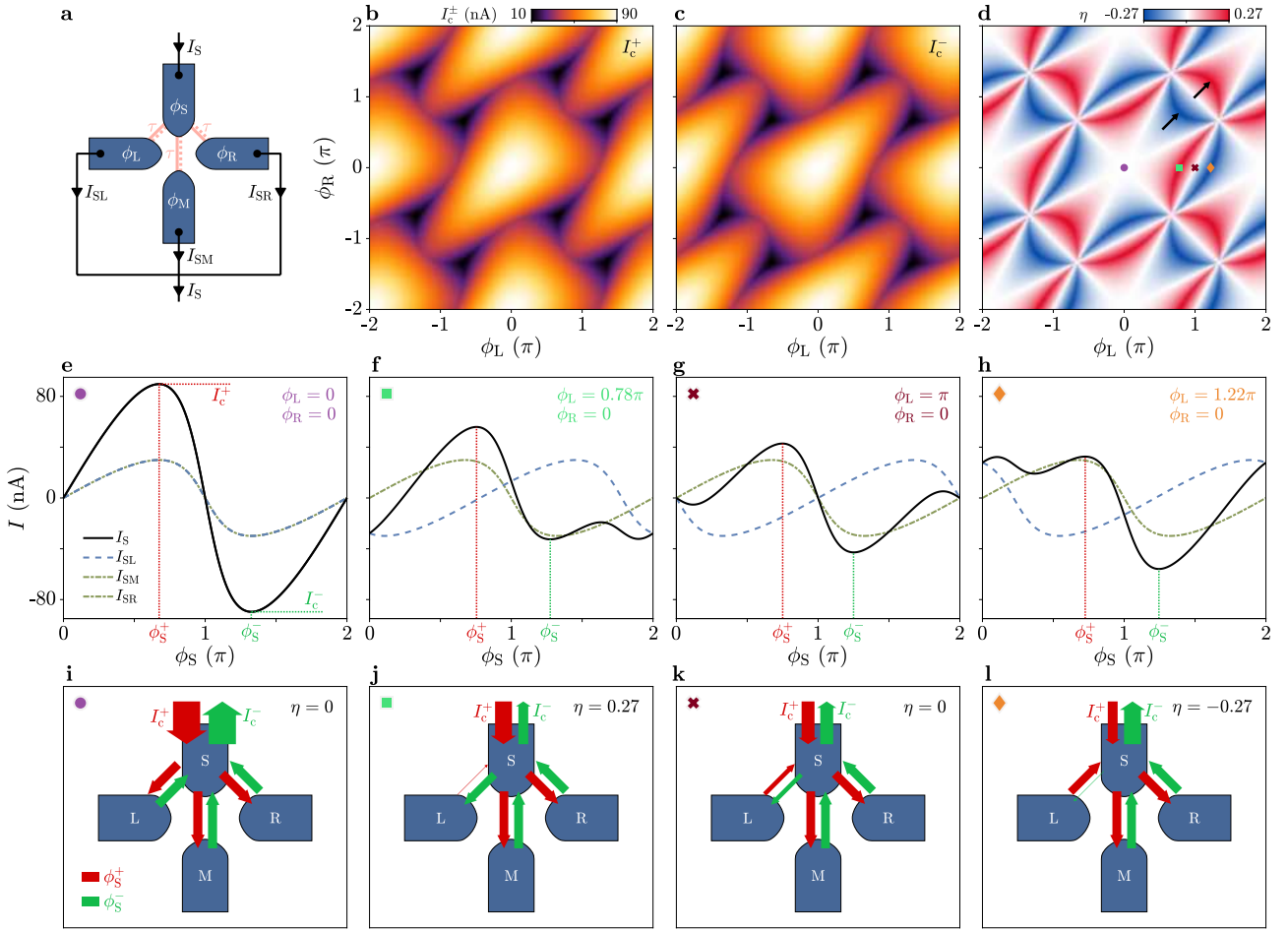


Figure 141: Minimal-model description of the Josephson diode effect. (a) Schematic representation of the four-terminal Josephson junction and circuit parameters. (b, c) Simulated critical currents I_c^+ and I_c^- for positive and negative I_{SD} , respectively, as functions of the phase differences ϕ_L and ϕ_R . The transmission of the three channels is $\tau = 0.9$. (d) Diode efficiency η derived from (b) and (c) as a function of ϕ_L and ϕ_R . (e–h) Supercurrents in the four leads as functions of phase ϕ_S . The phases of the other leads are indicated in each panel. The four cases correspond to the colored markers in (d). The value of ϕ_S where I_S has its maximum (I_c^+) (minimum I_c^-), labeled ϕ_S^+ (ϕ_S^-), is highlighted by the red (green) dotted line. (i–l) Schematic representation of the supercurrent flow in the phase configurations shown in (e–h). Red and green arrows indicate supercurrents for ϕ_S^+ and ϕ_S^- , respectively. The wider the arrow, the larger the supercurrent.

The total current flowing into lead S is thus expressed as:

$$I_S = I_{SL} + I_{SM} + I_{SR} = \sum_{\alpha} I_{S\alpha}, \quad (31.2)$$

where $I_{S\alpha}$ is the current flowing from terminal S to terminal α via the corresponding JJ. First, we consider the minimal model of a single numerical parameter, schematically shown in Fig. 141a. Each of the three JJs, that are identical to each other, is described by one conductive channel with transmission τ , resulting in the CPR [48]:

$$I_{S\alpha} = \frac{e\Delta}{2\hbar} \frac{\tau \sin(\phi_S - \phi_{\alpha})}{\sqrt{1 - \tau \sin^2 \frac{\phi_S - \phi_{\alpha}}{2}}}. \quad (31.3)$$

with Δ the superconducting gap ($\Delta = 180 \mu\text{eV}$ is used considering Al leads) and $\hbar = h/2\pi$. We assume a high transmission $\tau = 0.9$, for which the CPR of Eq. 31.3 has a significantly

nonsinusoidal character, as harmonics higher than the first provide a sizable contribution. The independent variables in the model are the three superconducting phase differences ϕ_L , ϕ_R and ϕ_S , defined with respect to $\phi_M \equiv 0$. The first two phases are related to the external magnetic fluxes by $\phi_{L(R)} = 2\pi \Phi_{L(R)}/\Phi_0$ (neglecting the inductance of the loops, see discussion in the Supplementary Material, Section 5), whereas ϕ_S varies depending on current bias I_{SD} . The critical currents for the two bias directions are then obtained as:

$$I_c^\pm = \max_{\phi_S} [\pm I_S(\phi_S)]. \quad (31.4)$$

In Figs. 141b and 141c, we show $I_c^\pm$ computed as functions of ϕ_L and ϕ_R , while the diode efficiency calculated with Eq. 31.1 (where $I_{sw}^\pm$ are substituted by $I_c^\pm$) is displayed in Fig. 141d. The critical currents, fulfilling the condition $I_c^+(\phi_L, \phi_R) = I_c^-(-\phi_L, -\phi_R)$, exhibit patterns that are prominently asymmetric with respect to $\phi_L = \phi_R = 0$ (modulo 2π), which leads to a strong JDE with η up to approximately 27%. The dependence of η on ϕ_L and ϕ_R reflects the triangular shapes observed for $I_c^\pm$, with features arranged according to three main orientations in the phase space. The origin of the JDE is investigated by fixing ϕ_L and ϕ_R and computing the CPR of Eq. 31.2 as a function of ϕ_S , $I_S(\phi_S)$, and its components $I_{SL}(\phi_S)$, $I_{SM}(\phi_S)$ and $I_{SR}(\phi_S)$, all obtained from Eq. 31.3. For simplicity we always keep $\phi_R = 0$ and select four values of ϕ_L (colored markers in Fig. 141d), where $|\eta|$ is either zero ($\phi_L = 0, \pi$) or maximal ($\phi_L = 0.78\pi, 1.22\pi$). The individual and combined CPRs at these phase-space points are plotted in Fig. 141e–h. In each case, we identify the values of ϕ_S that maximize the total current I_S (ϕ_S^+ , red dotted lines) and its inverse $-I_S$ (ϕ_S^- , green dotted lines), such that $I_S(\phi_S^\pm) = I_c^\pm$. The currents flowing to and from terminal S are schematically depicted in Fig. 141i–l for the same ϕ_L and ϕ_R values of panels e–h. In the schematics, red (green) arrows show the situation at $\phi_S^{+(-)}$, their width and direction indicate the magnitude and direction of the current. We note that all individual CPRs $I_{S\alpha}(\phi_S)$ have the same amplitude ≈ 30 nA and skewness, both given by the transmission τ (identical for the three channels), but notably $I_{SL}(\phi_S)$ is phase-shifted by ϕ_L . When $\phi_L = 0$ (Fig. 141e, i), all components are in-phase and $I_S(\phi_S) = 3I_{S\alpha}(\phi_S)$, hence a standard nonsinusoidal CPR is obtained. Positive and negative critical currents are identical, and all currents are simply reversed between ϕ_S^+ and ϕ_S^- . In contrast, when $I_{SL}(\phi_S)$ is shifted by $\phi_L = 0.78\pi$ (Fig. 141f, j), the total CPR becomes nonreciprocal for positive and negative currents. The I_{SL} -component is very small at ϕ_S^+ , but comparable with $I_{SM,SR}$ at ϕ_S^- ; since in both cases I_{SL} has opposite sign with respect to $I_{SM,SR}$, this asymmetry leads to $I_c^+ > I_c^-$. A symmetric scenario is recovered for $\phi_L = \pi$, when I_{SL} is shifted by half period from the other components. Here, I_S always has the opposite sign to $I_{SM,SR}$, but same absolute value at $\phi_S^\pm$, such that $I_c^+ = I_c^-$ and no JDE is present. Finally, the results obtained for $\phi_L = 1.22\pi$ (Fig. 141h, l), symmetric about $\phi_L = \pi$ to $\phi_L = 0.78\pi$, show the same CPRs discussed for Fig. 141f, j upon sign inversion of both current and ϕ_S , confirming that here the JDE is strong and has opposite direction compared to the previous case.

We note that the JDE has been derived within our minimal model despite the presence of three identical channels, i.e., of same harmonic content, whereas in a conventional SQUID comprising two JJs the harmonic content must be different between the two JJs [?]. The multiterminal geometry that we discuss can also be reduced to a conventional SQUID, where two JJs and a phase degree of freedom (for example, the S–M and S–R junctions and ϕ_R) are replaced by an effective JJ of tunable harmonic content. The effective JJ, together with the third JJ and the remaining phase difference (in the same example, S–L and ϕ_L), realizes the proposal of Ref. [?]. An important advantage offered by our platform lies in the possibility of phase-tuning the harmonic content of the effective junction, establishing wide and flexible control over the JDE. The nonsinusoidal character of the individual CPRs, which is still a requirement, is obtained in high-transmission hybrid JJs (as those realized in this work), while

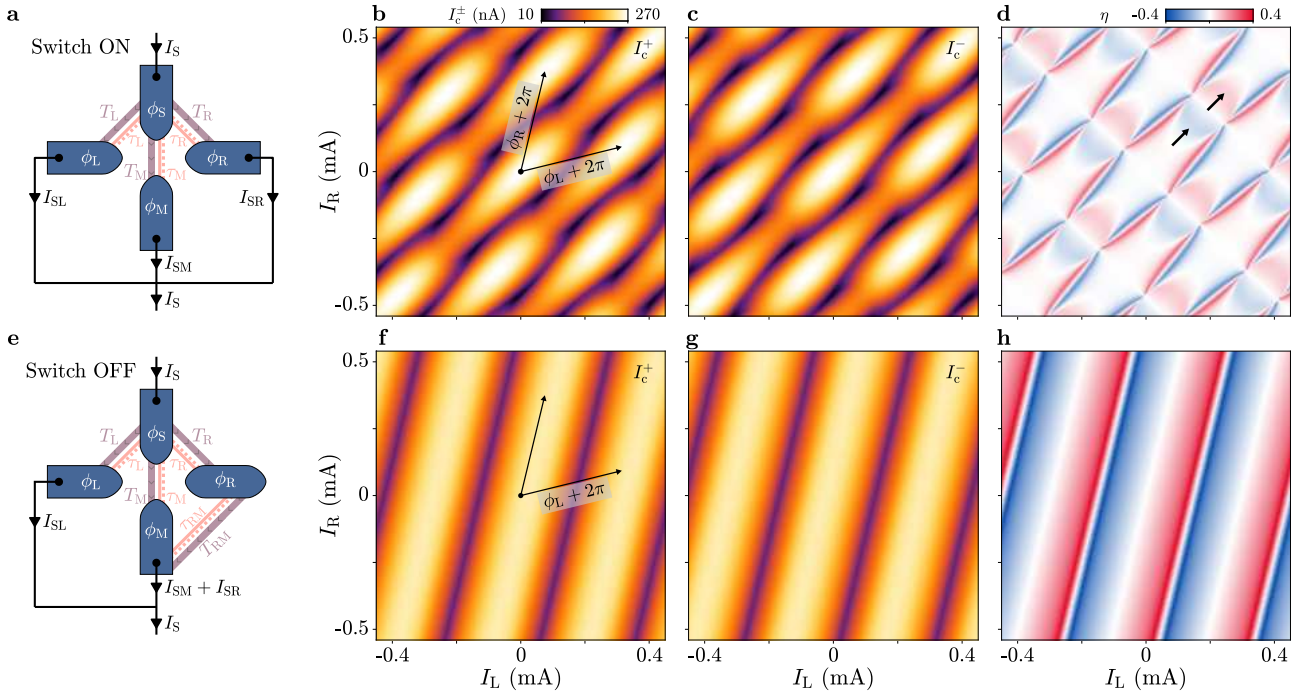


Figure 142: Josephson diode effect in the extended model. (a) Schematic representation of the four-terminal Josephson junction and circuit parameters for the switch-ON configuration (see text for details). (b, c) Simulated critical currents I_c^+ and I_c^- for positive and negative current bias (respectively), as functions of the flux-line currents I_L and I_R . Currents I_L and I_R are calculated from the superconducting phase differences ϕ_L , ϕ_R and the mutual inductance matrix (see discussion in the Supplementary Material, Section 5). Black arrows, whose directions indicate the periodicity axes ϕ_L and ϕ_R , represent the corresponding phase winding by 2π . (d) Diode efficiency η obtained from (b) and (c) as a function of I_L and I_R . (e) Schematic representation of the four-terminal Josephson junction and circuit parameters for the switch-OFF configuration. (f, g) As in (b, c), but for the case with switch-OFF. (h) Diode efficiency η obtained from (f) and (g) as a function of I_L and I_R .

our novel geometry eliminates the need for precise control over the transmissions of the single junctions after device fabrication.

31.0.0 Simulations with the extended model

After discussing the origin of the JDE by means of a minimal, single-parameter model, we extend the latter to better capture the experimental data presented in Fig. 172. Again, we consider the three JJs S-L, S-M and S-R that contribute to the total current according to Eq. 31.2. For each junction we consider two contributions to the current $I_{S\alpha}$: in addition to a high-transparency channel, with transmission τ_α , a component with conventional sinusoidal CPR [62], associated to a large number of low-transmission channels, is included:

$$I_{S\alpha} = \frac{e\Delta}{2\hbar} \left[\frac{\tau_\alpha \sin(\phi_S - \phi_\alpha)}{\sqrt{1 - \tau_\alpha \sin^2 \frac{\phi_S - \phi_\alpha}{2}}} + T_\alpha \sin(\phi_S - \phi_\alpha) \right], \quad (31.5)$$

where $e\Delta T_\alpha/2\hbar$ is the critical current of the sinusoidal component and T_α the sum of the transmissions of all low-transmission channels. We note that τ_α and T_α may vary depending on the junction S- α . With this extended model, that is schematically represented in Fig. 142a, we compute the critical currents using Eq. 31.4 for any settings of ϕ_L and ϕ_R . The simulated

I_c^+ , I_c^- and η are shown in Fig. 142b–d for parameters $\tau_L = \tau_R = 0.92$, $\tau_M = 0.89$, $T_L = 3.5$, $T_M = 1.5$ and $T_R = 3.6$. For a better comparison to the experimental results, the three quantities are plotted as functions of the flux-bias line currents I_L and I_R , calculated from the phases ϕ_L and ϕ_R by applying a linear transformation (see Supplementary Material, Section 5 for more details). Simulations reproduced the measurements displayed in Fig. 172b–d to a good degree. Calculated critical currents were between 10 and 270 nA, with diode efficiencies up to $\eta^{\max} \approx 25\%$ and patterns in the 2D phase space closely resembling the experimental data. By comparing the simulation of η in Fig. 142d to the result previously obtained with the minimal model (Fig. 141d), we note a reduction of $|\eta|$ and broadening of the features located near $(\phi_L, \phi_R) = (\pi, \pi)$, modulo 2π (see black arrows in both figures). This effect, also clearly visible in Fig. 172d, is mainly related to the sinusoidal components to the CPR of Eq. 31.5, where T_M is substantially smaller than $T_{L,R}$, and marginally related to τ_M , only slightly smaller than $\tau_{L,R}$. This is expected in the device under study, as the larger length of the S–M JJ (lithographically of 120 nm) compared to S–L and S–R (50 nm) reduced both the transmission of the most transmissive modes and the number of channels with low transmission.

The switch-OFF configuration (Fig. 172e–h) is investigated by further adjusting the numerical model based on the following considerations. When the right superconducting loop is interrupted, the current flowing from terminal S to R does not have a direct path to D, but it must flow across the R–M junction. Thus, we introduce this junction in the model, with CPR I_{RM} also assumed to have the form of Eq. 31.5. Here the phase difference $\phi_R - \phi_M = \phi_R$ is used instead of $\phi_S - \phi_\alpha$ and parameters τ_{RM} and T_{RM} substitute τ_α and T_α (see schematic of Fig. 142e). For these parameters we choose the values $\tau_{RM} = 0.97$ and $T_{RM} = 3.2$. The other consequence of interrupting the right loop is that ϕ_R is not controlled externally with a magnetic flux, hence it is first calculated imposing the condition $I_{SR}(\phi_S - \phi_R) = I_{RM}(\phi_R)$ (i.e., the current flowing from S to R equals that flowing from R to M), for any value of ϕ_S . Once ϕ_R is determined, $I_c^\pm(\phi_L)$ is computed using Eqs. 31.2, 31.4 and 31.5, leading to the result shown in Figs. 142f and 142g. In agreement with Figs. 172f and 172g, the model produces oscillations of $I_c^\pm$ as a function of ϕ_L between 50 nA and 240 nA, with a phase shift when reversing the current direction. This results in a diode efficiency (see Fig. 142h) up to 40%, comparable to the measured value of $\approx 34\%$ and exhibiting an oscillating behavior depending on ϕ_L , similar to that in Fig. 172h.

The higher η obtained in the switch-OFF case is understood by considering the higher asymmetry in the supercurrent distribution obtained in this setting. In fact, the supercurrent flowed directly from S to the common node D only via S–L and S–M, which had largely different transmissions due to the device geometry, while it had to traverse both S–R and R–M to reach D via R. This realized a strongly asymmetric situation, where junctions with different harmonic components led to large diode efficiencies [?]. When the switch was ON (Figs. 172d and 142d) and junction S–R directly connected S to D, the supercurrent distribution became more symmetric, and spatial-inversion symmetry was broken by the different phases tuned with the loops: that is, local magnetic fluxes broke both time-reversal and spatial-inversion symmetry. Similar arguments apply to the case of Figs. 157b and 157d, however setting gates to negative values to deplete parts of the semiconducting region reduced the maximum switching current, which also resulted in a decrease of the JDE efficiency (see Fig. 173). The absence of JDE for the situation of Fig. 157f is explained by considering that, with terminal M blocked, the supercurrent flowed in S–L and S–R only, which were almost balanced channels. Furthermore, phase tuning could not break spatial-inversion symmetry with M blocked, effectively resulting in a single superconducting loop between L and R. This situation therefore realized the balanced SQUID device of Ref. [?], showing no JDE despite the nonsinusoidal CPR of the individual junctions.

We finally note that our model captured the main experimental results well without invok-

ing hybridized ABSs in the semiconducting region, even though such states were measured in the same devices by means of local tunneling spectroscopy [78]. There, the terminal S was weakly coupled to the multiterminal region and ABSs arising in the L–M and M–R junctions overlapped. In the present case, these JJs were not required to describe the experimental data (except for the switch-OFF configuration, where the M–R junction was not phase-controllable), since the supercurrents flowing from S to L, M and R had direct superconducting paths to D, without traversing L–M or M–R; this also excluded the contribution of hybridized modes between the two JJs. Furthermore, supercurrents were transported by several ABSs present in our devices, most of which solely depended on phase differences between pairs of superconducting contacts. Presumably, there were very few hybridized ABSs and they were localized close to the center of the multiterminal region. Thus, in the geometry studied here, they did not provide a significant contribution to the total supercurrent.

31.0Conclusions

We reported switching current measurement of a 4TJJ in a InAs/Al heterostructure hosting ABSs with large transmission probabilities, resulting in nonsinusoidal CPRs between pairs of terminals. The switching current measured between two contacts showed a strong dependence on the bias current direction, resulting in a JDE. The JDE efficiency could be widely controlled—both in amplitude and sign—by magnetic fluxes, independently tuned via integrated flux-bias lines, and gate electrodes, which routed the supercurrent to different transport channels. No external magnetic fields were required. In a first gate setting, where transport through the entire semiconductor region was allowed, the JDE efficiency was periodically modulated by magnetic fluxes, with peak values reaching $\eta \approx \pm 21\%$, including large regions in parameter space with $\eta \approx 0$. When a superconducting arm was interrupted, introducing a larger asymmetry in the supercurrent distribution, a peak efficiency $\eta \approx \pm 34\%$ was reached. The 4TJJ was mapped onto a simple bi-SQUID geometry, with three parallel JJs containing ABSs with large transmission probability. A theoretical model reproduced the experimental observations to a good degree, including switching current and diode efficiency patterns. In our devices, the JDE is a consequence of the nonsinusoidal CPR and the multiterminal geometry, which allows breaking of spatial-inversion symmetry by controlling the magnetic fluxes in the loops. Unlike realizations based on a single loop, an asymmetry between junctions is not required. Our work highlights the potential of phase-tunable multiterminal JJs to engineer JDE with large and widely controllable efficiencies, without the need for exotic materials or external magnetic fields, and underscores the role of these devices as a versatile platform for upcoming applications.

31.0Methods

31.0.0Materials and Fabrication

Devices were realized in a III–V heterostructure grown by molecular beam epitaxy on an InP (001) substrate [48]. The semiconducting stack (starting from the substrate) consisted of a 1100 nm thick step-graded InAlAs buffer layer, a 6 nm thick $\text{In}_{0.75}\text{Ga}_{0.25}\text{As}$ layer, an 8 nm thick InAs layer, a 13 nm thick $\text{In}_{0.75}\text{Ga}_{0.25}\text{As}$ layer and two monolayers of GaAs. On top, an epitaxial 15 nm thick Al layer was deposited in situ without breaking vacuum. A two-dimensional electron gas (2DEG) was confined in the InAs and its properties were characterized via measurements performed in a Hall bar geometry, which gave an electron peak mobility of $18000 \text{ cm}^2\text{V}^{-1}\text{s}^{-1}$ at an electron sheet density of $8 \times 10^{11} \text{ cm}^{-2}$. This resulted in an electron mean free path $l_e \gtrsim 260 \text{ nm}$ and a superconducting coherence length of the 2DEG proximitized by the Al sheet

of $\xi_{\text{InAs}} = \sqrt{\hbar v_F l_e / (\pi \Delta)} \sim 600$ nm, with v_F the Fermi velocity in the 2DEG and Δ the induced superconducting gap.

In the fabrication process, large mesa structures were first isolated, suppressing parallel conduction between devices and across the middle regions of the superconducting loops. This was done by selectively etching the Al layer with Transene type D, followed by a second chemical etch to a depth of ~ 380 nm into the III–V material stack, using a 220 : 55 : 3 : 3 solution of $\text{H}_2\text{O} : \text{C}_6\text{H}_8\text{O}_7 : \text{H}_3\text{PO}_4 : \text{H}_2\text{O}_2$. Next, features were defined in the Al layer by wet etching with Transene type D at 50°C for 4 s. The dielectric, uniformly deposited on the entire chip by atomic layer deposition, consisted of a 3 nm thick layer of Al_2O_3 and a 15 nm thick layer of HfO_2 . Gate electrodes and flux-bias lines were defined by evaporation and lift-off. In a first step, 5 nm of Ti and 20 nm of Au were deposited to realize the fine features of the gates; in a second step, a stack of Ti/Al/Ti/Au with thicknesses 5 nm, 340 nm, 5 nm and 100 nm was deposited to connect the mesa structure to the bonding pads and to define the flux-bias lines.

31.0.0 Measurements Techniques

Experiments were performed in a dilution refrigerator with a base temperature at the mixing chamber below 10 mK. The sample was mounted on a QDevil QBoard sample holder system, without employing any light-tight enclosure. Electrical contacts to the devices, excepts for the flux-bias lines, were provided via a resistive loom with QDevil RF and RC low-pass filters at the mixing chamber stage, and RC low-pass filters integrated on the QBoard sample holder. Currents were passed through the flux-bias lines via a superconducting loom with only QDevil RF filters at the mixing chamber stage. Signals were applied to all gates and flux-bias lines via home-made RC filters at room temperature.

In all electrical measurements, a bias current I_{SD} was driven between terminal S and node D of the device by applying equal and opposite voltages to S and D via bias resistors, whose resistance was much larger than that of the device under study. The voltage drop across S and D was detected in a four-terminal configuration. Measurements of the differential resistance were performed with lock-in-amplifier techniques, by applying a fixed AC current $\delta I = 2.5$ nA to D in addition to the DC bias I_{SD} and detecting the AC voltage δV between S and D, thus obtaining the differential resistance $R \equiv \delta V / \delta I$. Measurements of $I_{\text{sw}}^\pm$ were done by periodically ramping I_{SD} from 0 to an amplitude A , where A was positive or negative depending on whether I_{sw}^+ or I_{sw}^- was measured; the absolute value of A was adjusted depending on the gate configuration to be slightly larger than $I_{\text{sw}}^\pm$. The signal form was a sawtooth wave, applied at a frequency $f = 133$ Hz using a waveform generator. The voltage drop across S and D was measured with an oscilloscope (averaging 32 measurements), which detected the time interval Δt where the voltage was below a threshold, hence allowing for the calculation of the switching current as $I_{\text{sw}}^\pm = |A| f \Delta t$.

The dilution refrigerator was equipped with a superconducting vector magnet which, despite not being utilized for the experiments, produced a small magnetic field offset. Therefore, small offsets in the flux-bias line currents I_L and I_R (up to ~ 100 μA) were considered in datasets, in such a manner that the point where $I_L = I_R = 0$ corresponded to a point of the phase space where $\eta = 0$ and $I_{\text{sw}}^\pm$ were maximal, as expected when no magnetic fluxes thread the superconducting loops.

31.0 Data availability

Data presented in this work will be available on Zenodo. The data that support the findings of this study are available upon request from the corresponding author.

31.0 Supplementary Material

31.1 Additional Phase-Space Linecuts

The switching current measured for positive bias current I_{sw}^+ in the phase space (i.e., as a function of both flux-line currents I_L and I_R), shown in Fig. 2b of the Main Text, is plotted again in Fig. S.1a over extended ranges of I_L and I_R . Here, in addition to the phase-space linecut $I_R = 100 \mu\text{A}$ displayed in the Main Text (green marker), we consider the linecuts $I_R = 20 \mu\text{A}$ (white marker) and $I_R = -60 \mu\text{A}$ (blue marker). Along these directions, we measured the differential resistance R across the device as a function of I_L and of the bias current I_{SD} (always swept from 0 to positive or negative values), and show the result in Figs. S.1b–d. The three linecuts revealed oscillations of the switching current as a function of I_L , with maximum switching currents of approximately 240 nA. In all measurements, the supercurrent was nonreciprocal at positive and negative I_{SD} , confirming the presence of Josephson diode effect (JDE) with phase-dependent efficiency η , consistent with Fig. 2d of the Main Text.

31.2 Results for Different V_L , V_R

Figures S.2a–c show three phase-space linecuts, indicated by the colored markers in Fig. S.1a, where the gate voltages V_L and V_R were set to -0.2 V (while $V_L = V_R = -0.1 \text{ V}$ in Fig. 2 of the Main Text and in Fig. S.1). The other gate voltages were kept to $V_S = 0.1 \text{ V}$, $V_M = -0.15 \text{ V}$ and $V_J = 0$. The switching current oscillations qualitatively resembled those described for $V_L = V_R = -0.1 \text{ V}$ (Fig. S.1) but with reduced amplitude, as the maximum switching current was approximately 170 nA in agreement with Fig. 3b of the Main Text. Regions of vanishing supercurrent (see for example the yellow arrow in Fig. S.2a) were more prominent in this gate configuration, consistent with the larger normal-state resistance. The linecut of Fig. S.2b ($I_R = 20 \mu\text{A}$) was also measured with the switch junction voltage set to $V_J = -1.5 \text{ V}$ (switch OFF, Fig. S.2d), yielding a picture very similar to Fig. 2e of the Main Text. With respect to the case where $V_L = V_R = -0.1 \text{ V}$, we confirm a reduction of the maximum supercurrent (up to about 170 nA) and diode efficiency (up to approximately 25%).

The results shown thus far were acquired with symmetric gate voltages $V_L = V_R$, except in Fig. 4a–d of the Main Text where either gate was strongly depleted. In these configurations, the currents flowing into terminals L and R were relatively symmetric, as supported by our simulations (Fig. 6), where a good fit was found for $\tau_L = \tau_R$ and $T_L \approx T_R$. For $V_L = V_R = -0.3 \text{ V}$, we observed switching current patterns that were slightly asymmetric along the two periodicity directions, as shown in Figs. S.3a and S.3b (see cyan arrows in panel a). This was likely due to different lever arm of the gates at voltages V_L and V_R , which created an imbalance in the device for intermediate gate voltages. By setting $V_L = -0.28 \text{ V}$ and $V_R = -0.31 \text{ V}$, a more balanced situation was restored (see Figs. S.3d and S.3e). This effect was also visible in the superconducting diode efficiencies, shown in Figs. S.3c and S.3f for the two cases: while the maximum $|\eta|$ slightly varied depending on the periodicity axis at $V_L = V_R = -0.3 \text{ V}$, it was substantially more symmetric in the two directions at $V_L = -0.28 \text{ V}$, $V_R = -0.31 \text{ V}$ (see black arrows in panels c and f, respectively).

Staying in the gate configuration with $V_L = -0.28 \text{ V}$, $V_R = -0.31 \text{ V}$, we defined two phase-space linecuts (colored markers in Fig. S.3d, corresponding to $I_R = 20 \mu\text{A}$ and $I_R = -140 \mu\text{A}$) and measured R along these directions as a function of I_L and I_{SD} . The result is presented in Figs. S.4a and S.4b, showing switching current oscillations with maxima of about 70 nA and minima where the supercurrent vanished. Nonreciprocity was still present in the switching current depending on the sign of I_{SD} , despite less markedly than for the previous gate configurations (where V_L and V_R were set to less negative values), consistent with diode efficiencies up to approximately 10% reported in Fig. S.3f. Finally, the gate voltage V_J was set

to -1.5 V to operate with the switch OFF, and supercurrent oscillations were measured along the linecut at $I_R = 20$ μ A (see Fig. S.4c), also revealing maximum switching currents of close to 70 nA.

31.3 Extracting the Diode Efficiency as a Function of V_L , V_R

In Fig. 3b of the Main Text, we showed the gate-tunability of the JDE in our devices by presenting the dependence of the maximum diode efficiency $\eta^{\max}$ as a function of the gate voltages $V_L = V_R$. In the switch-ON configuration ($V_J = 0$), each data point was extracted from measurements of I_{sw}^+ and I_{sw}^- taken as a function of both I_L and I_R . Figures S.5a–e and S.5f–j display I_{sw}^+ and η obtained using Eq. 1 of the Main Text for five selected gate configurations ranging between $V_L = V_R = -0.3$ V and $V_L = V_R = 0$. Since values close to the limit of detection (~ 10 nA) gave large variability of the extracted η (see for example Figs. S.3c and S.3f in the regions of phase space where I_{sw}^+ was $\gtrsim 10$ nA), regions where I_{sw}^+ was lower than a threshold of 20 nA were not considered. We note that the exact choice of the threshold did not significantly alter the result of the extraction. In each gate configuration, $I_{\text{sw}}^{\max}$ and $\eta^{\max}$ were obtained by considering the 99.9th percentile of I_{sw}^+ and η , respectively, throughout the corresponding maps. In switch-OFF case ($V_J = -1.5$ V), included in Fig. 3b of the Main Text, the extraction was simplified for two reasons: first, since only one periodicity axis remained in the phase space, $I_{\text{sw}}^\pm$ could be measured along a single direction (e.g., as a function of I_L for any fixed I_R), thus improving both speed and resolution of the measurement; second, $I_{\text{sw}}^\pm$ did not approach the limit of detection very closely, therefore a threshold was not required to select the data. Similar to the switch-ON case, $I_{\text{sw}}^{\max}$ and $\eta^{\max}$ were obtained by considering the 99.9 percentile of I_{sw}^+ and η for each gate setting.

31.4 Results for Device 2

To further study the JDE arising in a four-terminal Josephson junction (4TJJ) embedded in a double-loop geometry, we characterized a second device fabricated on the same chip. The circuit layout of Device 2 and the geometry of the two superconducting loops were lithographically identical to Device 1 (see Figs. 1a and 1c of the Main Text). The 4TJJ region, displayed in Fig. S.6, featured a different layout of terminal S (significantly wider than for Device 1), whereas terminals L, M and R were designed to be identical between the two devices. The shape of the gates was also varied, in particular for the gate energized by the voltage V_S .

Current-bias measurements were performed as described for Device 1. Figure S.7 shows the switching current $I_{\text{sw}}^\pm$ and the extracted diode efficiency η as functions of the flux-line currents I_L and I_R for three configurations of the gate voltages V_L and V_R (set to a common value): -0.4 V (a–c), -0.5 V (d–f) and -0.6 V (g–i). The other gate voltages were $V_S = 0.15$ V, $V_M = -0.15$ V and $V_J = 0$ (switch ON). Switching current oscillations in the 2D phase space at $V_L = V_R = -0.4$ V qualitatively resemble those observed for Device 1 in Figs. 2b and 2c of the Main Text. The larger current maxima and minima, occurring despite voltages were applied in a more negative range than in Device 1, are understood by considering the different geometry of terminal S, that is compatible with a higher number of conduction channels in the S–L and S–R junctions, and a stronger screening of the electric field generated by the gates. Another feature of Device 2, that was particularly visible at $V_L = V_R = -0.5$ V, -0.6 V, was the large asymmetry between the two periodicity directions Φ_L and Φ_R . This is attributed to a smaller supercurrent flowing from S to L than from S to R, likely due to a combination of a smaller number and transmission of the modes on the left side of the device, even for symmetrically applied gate voltages. We note that the case with $V_L = V_R = -0.6$ V is qualitatively similar to Figs. 4c and 4d of the Main Text, where only V_L was depleted. The asymmetry was reflected

in the diode efficiency η : while oscillations of η in the phase space had some resemblance with those of Device 1 (see Figs. 2d of the Main Text), and also reached large maxima up to $|\eta| \approx 28\%$, they showed different amplitude and features depending on which flux was varied. A prominent example is visible in Fig. S.7c, where features of η near the phase-space points $(\Phi_L, \Phi_R) = (0, \Phi_0/2)$ (modulo Φ_0), marked by the orange arrow, have different shape and smaller η than those near $(\Phi_L, \Phi_R) = (\Phi_0/2, 0)$ (purple arrow). Conversely, for $V_L = V_R = -0.5$ V, -0.6 V, η was larger in proximity $(\Phi_L, \Phi_R) = (0, \Phi_0/2)$, consistent with the earlier suppression of Φ_L dependence as the gate voltages were lowered.

Switching currents and diode efficiency of Device 2 were also measured in the switch-OFF configuration ($V_J = -1.5$ V), as shown in Fig. S.8 for $V_L = V_R = -0.5$ V. Oscillations of both $I_{\text{sw}}^\pm$ and η , suppressed along Φ_R , were similar to those reported for Device 1 in Figs. 2f–h of the Main Text. Interestingly, the maximum diode efficiencies of approximately 12% were significantly lower than in the corresponding switch-ON case (Fig. S.7f), where $\eta^{\text{max}} \approx 25\%$. Again, this is attributed to the asymmetric supercurrent distribution in Device 2, which led to small diode efficiencies along the Φ_L -axis and required tuning of Φ_R to reach larger values in the case of Fig. S.7f. When the switch was OFF, control over Φ_R was disabled, hence reducing η^{max} .

The results shown for Device 2 were qualitatively captured by our extended theoretical model, with parameters $(\tau_L, \tau_M, \tau_R, T_L, T_M, T_R)$ set to $(0.91, 0.87, 0.93, 3.5, 2, 6.5)$ for Figs. S.9a–c (Configuration 1), to $(0.44, 0.65, 0.87, 1, 1, 3.3)$ for Figs. S.9d–f (Configuration 2), and to $(0, 0.53, 0.67, 0.2, 0.2, 1)$ for Figs. S.9g–i (Configuration 3). Furthermore, we simulated the device in the switch-OFF case using the same parameters as in Configuration 2 and, in addition, $\tau_{\text{RM}} = 0.75$, $T_{\text{RM}} = 4$; the result is displayed in Fig. S.10. In all cases, the phase axes (ϕ_L, ϕ_R) were converted to flux-line-current axes (I_L, I_R) using the transformation described in Section 31.5. We note that the chosen parameters were such that $\tau_L < \tau_R$ and $T_L < T_R$, supporting our interpretation of the asymmetry between Φ_L and Φ_R as a result of asymmetric supercurrent flow in the device between terminals L and R. In Configuration 3, the data was best described by using $\tau_L = 0$, suggesting that the high-transmission channel between S and L was depleted. The measurements performed on Device 2 and the simulations done within the same model introduced for Device 1 support the generality of the observed phenomena, and in particular corroborate the presence and origin of the JDE in our device.

31.5 Current-to-Flux Remapping

The b-SQUID geometry of our devices enables control over two superconducting phase differences (between terminals L and M, and between R and M), tuned by the currents in the two flux-bias lines, I_L and I_R . These currents generated magnetic fluxes threading the two superconducting loops, Φ_L and Φ_R , with a cross-coupling leading to an effect of $I_{L(R)}$ on the flux through the opposite loop, $\Phi_{R(L)}$. As a consequence, the Φ_L - and Φ_R -axes had a finite slope with respect to the I_L - and I_R -axes, as visible in Fig. 2b of the Main Text. We describe the cross-coupling by considering a mutual inductance matrix $\mathbf{M}$ that relates flux Φ_L and Φ_R to flux-line currents I_L and I_R :

$$\begin{pmatrix} \Phi_L \\ \Phi_R \end{pmatrix} = \mathbf{M} \cdot \begin{pmatrix} I_L \\ I_R \end{pmatrix} = \begin{pmatrix} M_{LL} & M_{LR} \\ M_{RL} & M_{RR} \end{pmatrix} \cdot \begin{pmatrix} I_L \\ I_R \end{pmatrix}. \quad (31.6)$$

To quantify $\mathbf{M}$ (for Device 1), we consider the (Φ_L, Φ_R) and (I_L, I_R) coordinates of two points of the phase space [in addition to the origin, $(\Phi_L, \Phi_R) = (I_L, I_R) = (0, 0)$], such as $(\Phi_0, 0)$ and $(0, \Phi_0)$, and substitute them in Eq. 32.13 [78]. The resulting 4×4 equation system leads to the mutual inductance matrix:

$$\mathbf{M} = \begin{pmatrix} 6.98 \text{ pH} & -1.40 \text{ pH} \\ -1.73 \text{ pH} & 5.86 \text{ pH} \end{pmatrix} \quad (31.7)$$

for Device 1, and

$$\mathbf{M} = \begin{pmatrix} 7.03 \text{ pH} & -1.39 \text{ pH} \\ -1.67 \text{ pH} & 5.80 \text{ pH} \end{pmatrix}$$

for Device 2. We note that $\mathbf{M}$ is very similar between the two devices, as the loop and flux-line geometry was lithographically identical. In Fig. S.11, we apply $\mathbf{M}$ from Eq. 32.14 to perform a basis transformation and plot the data presented in Figs. 2b–d (shown again in Figs. S.11a–c) as a function of the magnetic fluxes Φ_L and Φ_R (see Fig. S.11d–f).

The inverse transformation, obtained by inverting the mutual inductance matrix, was employed to map the phase space (ϕ_L, ϕ_R) to the flux-line-current space (I_L, I_R) for the simulated data of Figs. 6 of the Main Text, S.9 and S.10. In this transformation, the phase differences were considered to be linearly related to the fluxes, thus $\Phi_{L(R)} = \Phi_0 \times \phi_{L(R)}/2\pi$ were substituted in Eq. 32.13 to obtain a direct relation between phases and currents. This assumption is justified by noting that the inductance of the superconducting loops, comprising geometric and kinetic contributions $L_{\text{loop}} = L_{\text{geom}} + L_k$, is negligible compared to the Josephson inductance L_J existing between any pairs of terminals in the 4TJJ. For each superconducting loop, the geometric inductance is estimated as $L_{\text{geom}} \approx 24 \text{ pH}$ [?], while the kinetic inductance is calculated using the expression [67]:

$$L_k = \frac{l}{w} \frac{h}{2\pi^2} \frac{R_{\square}}{\Delta} \approx 100 \text{ pH}, \quad (31.8)$$

where $l = 57 \text{ }\mu\text{m}$ is the length of the loop perimeter, $w = 1 \text{ }\mu\text{m}$ the width of the Al strip forming the loop, h the Planck's constant, $R_{\square} \approx 1.5 \text{ }\Omega$ the normal state resistivity of the heterostructure stack (measured in Hall bar geometry where the Al was not removed) and $\Delta \approx 180 \text{ }\mu\text{eV}$ is the superconducting gap of Al. Geometric and kinetic contributions lead to a total inductance $L_{\text{loop}} \approx 124 \text{ pH}$. The Josephson inductance between two superconducting terminals (for example, L and M when considering the left loop) is estimated as $L_J = \Phi_0/2\pi I_c \sim 3 \text{ nH}$, where $I_c \sim 100 \text{ nA}$ gives the order of magnitude of the junction's critical current.

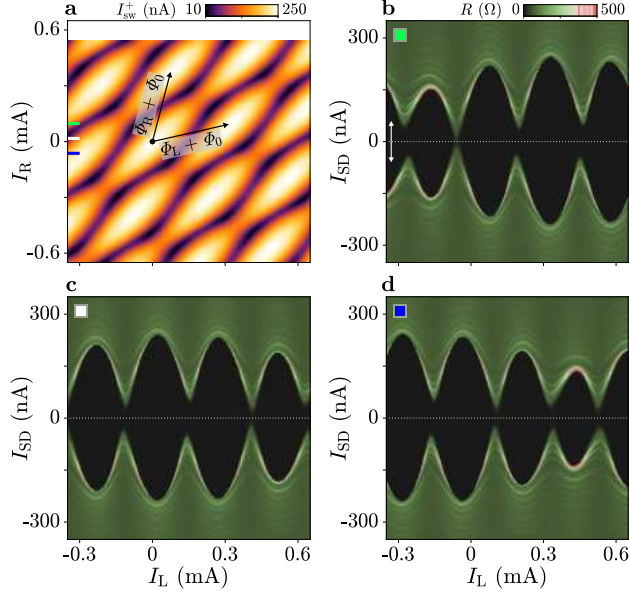


Figure S.1: Phase-space linecuts at $V_L = V_R = -0.1$ V. (a) Switching current I_{sw}^+ , measured for $I_{SD} > 0$, as a function of flux-line currents I_L and I_R , as in Fig. 2b of the Main Text. Colored markers indicate the position of $I_R = 100 \mu A$ (green), $I_R = 20 \mu A$ (white) and $I_R = -60 \mu A$ (blue). (b–d) Differential resistance R as a function of I_L and I_{SD} for $I_R = 100 \mu A$ (b), $I_R = 20 \mu A$ (c) and $I_R = -60 \mu A$ (d), as indicated in (a). Each map is obtained by merging two datasets recorded with I_{SD} ramping from 0 to either positive or negative values [see white arrows in (b)].

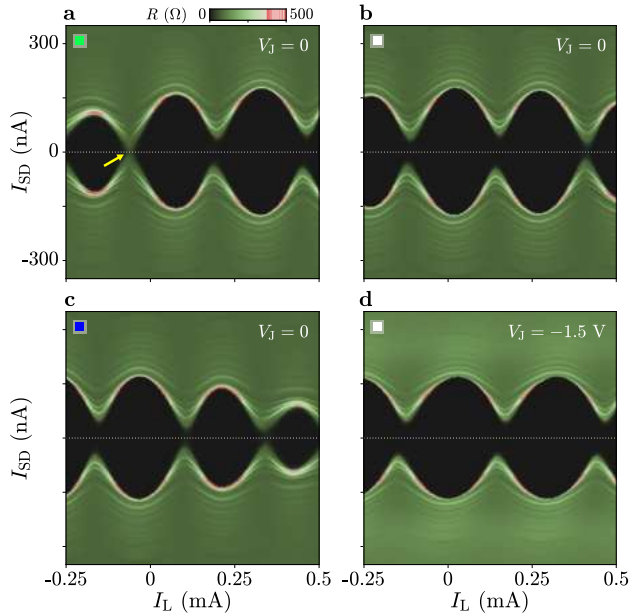


Figure S.2: Phase-space linecuts at $V_L = V_R = -0.2$ V. (a–c) Differential resistance R as a function of I_L and I_{SD} , for $V_J = 0$ (switch ON) and $I_R = 100 \mu A$ (a), $I_R = 20 \mu A$ (b) and $I_R = -60 \mu A$ (c), as indicated in Fig. S.1a. In (a), a point where the switching current reaches zero is indicated by the yellow arrow. (d) As in (b), but for $V_J = -1.5$ V (switch OFF).

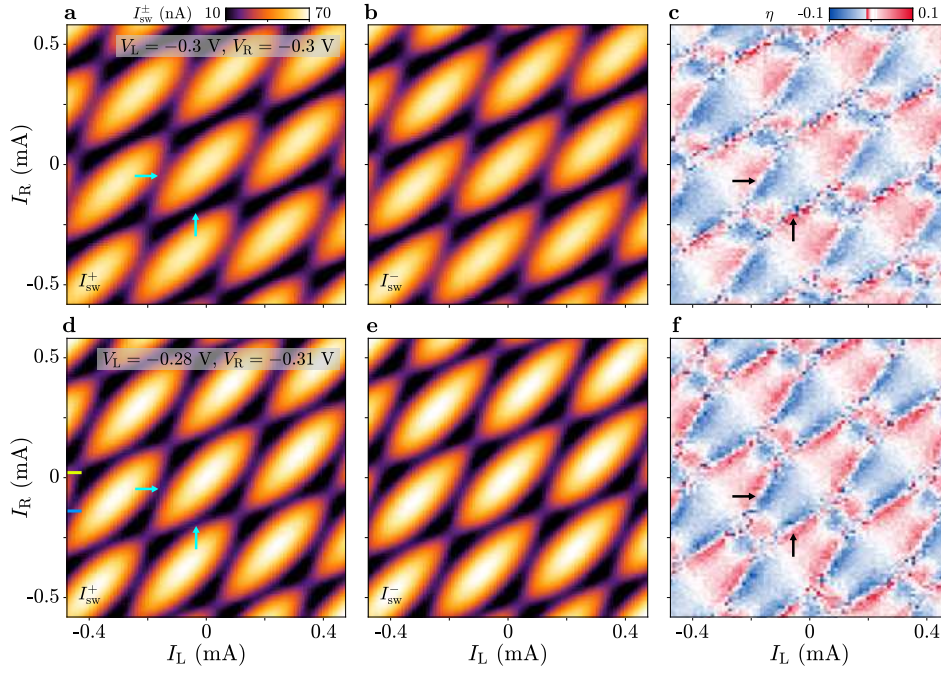


Figure S.3: Comparing symmetric and asymmetric gate configurations. (a, b) Switching currents I_{sw}^+ and I_{sw}^- (respectively) as functions of flux-line currents I_L and I_R for symmetric gate voltages $V_L = V_R = -0.3$ V. The other gate voltages were kept to $V_S = 0.1$ V, $V_M = -0.15$ V and $V_J = 0$. The cyan arrows indicate the switching current minima described in the text. (c) Superconducting diode efficiency η extracted from (a) and (b) as a function of I_L and I_R . Regions indicated by the black arrows are discussed in the text. (d-f) As in (a-c), but for asymmetric gate voltages $V_L = -0.28$ V and $V_R = -0.31$ V. In (d), yellow and blue markers indicate the position of $I_R = 20 \mu A$ and $I_R = -140 \mu A$, respectively.

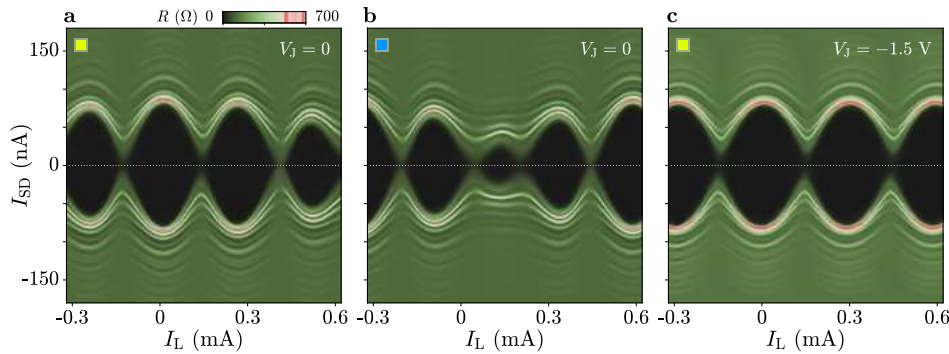


Figure S.4: Phase-space linecuts at $V_L = -0.28$ V and $V_R = -0.31$ V. (a) Differential resistance R as a function of I_L and I_{SD} for $V_J = 0$ and $I_R = 20 \mu A$ (yellow marker in Fig. S.3d). (b) As in (a), but for $I_R = -140 \mu A$ (blue marker in Fig. S.3d). (c) As in (a), but for $V_J = -1.5$ V (switch OFF). Each map is obtained by merging two datasets recorded with I_{SD} ramping from 0 to either positive or negative values.

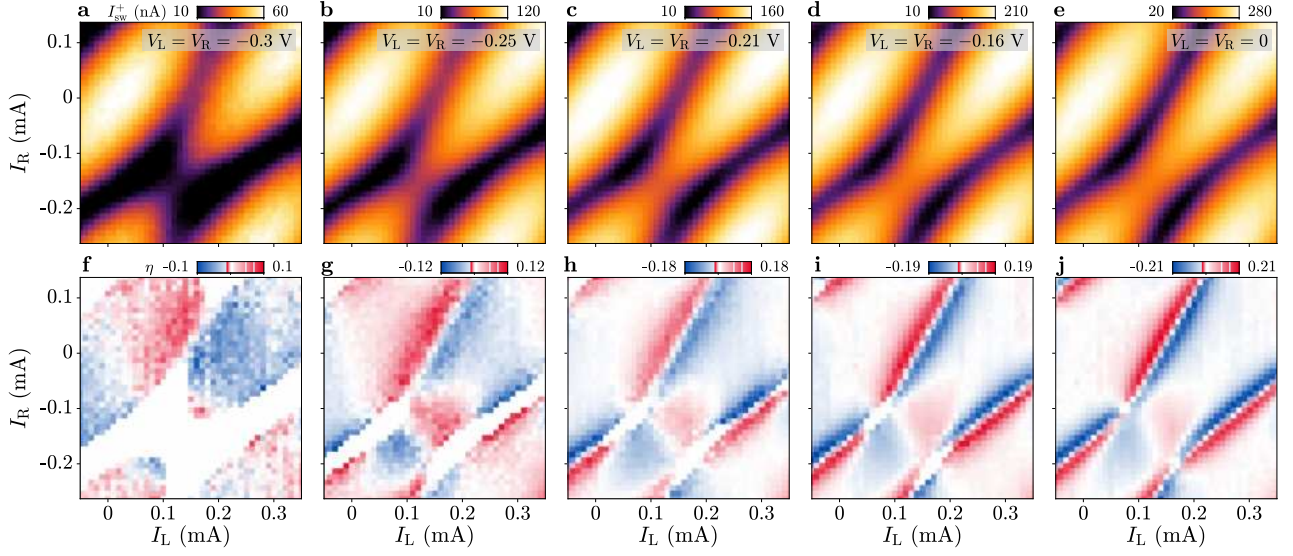


Figure S.5: Results for varying V_L , V_R and extraction of diode efficiencies. (a–e) Switching current I_{sw}^+ , measured for $I_{SD} > 0$, as a function of flux-line currents I_L and I_R for five settings of $V_L = V_R$, indicated on the subfigures. (f–j) Diode efficiency extracted from (a–e) and the corresponding measurements of I_{sw}^- (not shown), respectively, as a function of I_L and I_R . The other gate voltages were kept to $V_S = 0.1$ V, $V_M = -0.15$ V and $V_J = 0$.

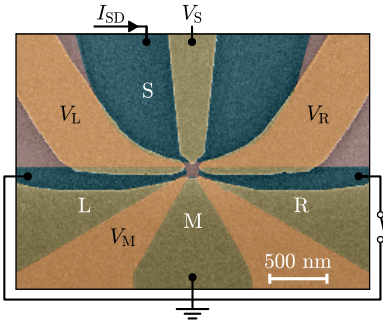


Figure S.6: Device 2: false-colored scanning electron micrograph of the four-terminal Josephson-junction region. Color legend and labeled quantities are as in Figs. 1a and 1b of the Main Text. Circuit layout and loop geometry of Device 2 are lithographically identical to Device 1 (see Figs. 1a and 1b).

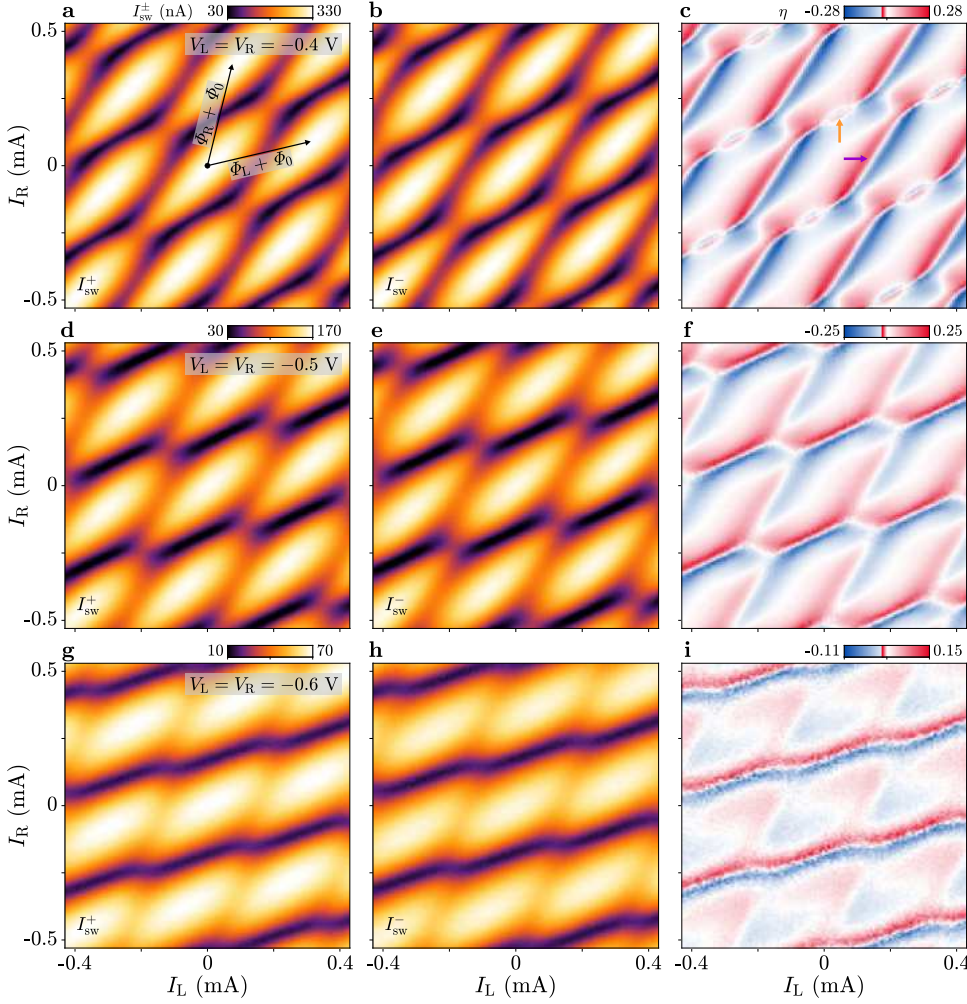


Figure S.7: Phase- and gate-tunable Josephson diode effect in Device 2. (a, b) Switching currents I_{sw}^+ and I_{sw}^- , measured for $I_{SD} > 0$ and $I_{SD} < 0$ respectively, as functions of flux-line currents I_L and I_R , for $V_L = V_R = -0.4$ V. Directions of the black arrows in (a) indicate the periodicity axes, corresponding to the external magnetic fluxes Φ_L and Φ_R threading the two superconducting loops, while their length (one period) indicates the addition of one superconducting flux quantum Φ_0 to the corresponding flux. (c) Superconducting diode efficiency η obtained from (a) and (b) as a function of I_L and I_R . Features indicated by the orange and purple arrows are discussed in the text. (d–f) As in (a–c), but for $V_L = V_R = -0.5$ V. (g–i) As in (a–c), but for $V_L = V_R = -0.6$ V. In all cases, the other gate voltages were set to $V_S = 0.15$ V, $V_M = -0.15$ V and $V_J = 0$.

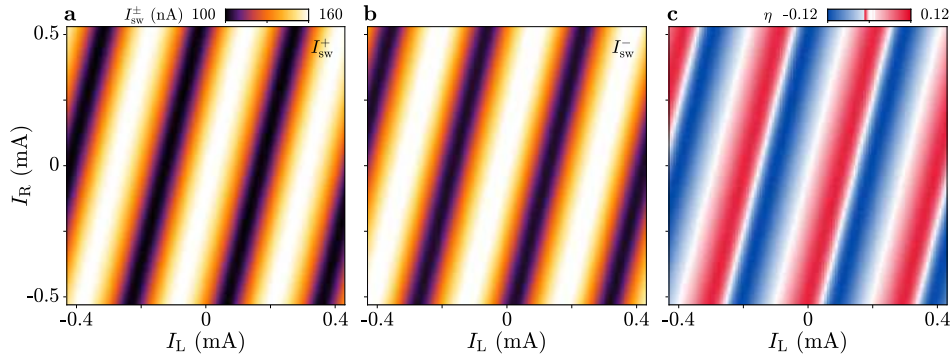


Figure S.8: Josephson diode effect in Device 2 for $V_J = -1.5$ V (switch OFF). (a, b) Switching currents I_{sw}^+ and I_{sw}^- , measured for $I_{SD} > 0$ and $I_{SD} < 0$ respectively, as functions of flux-line currents I_L and I_R , for $V_L = V_R = -0.4$ V. Here, $V_L = V_R = -0.5$ V, $V_S = 0.15$ V and $V_M = -0.15$ V. (c) Superconducting diode efficiency η obtained from (a) and (b) as a function of I_L and I_R .

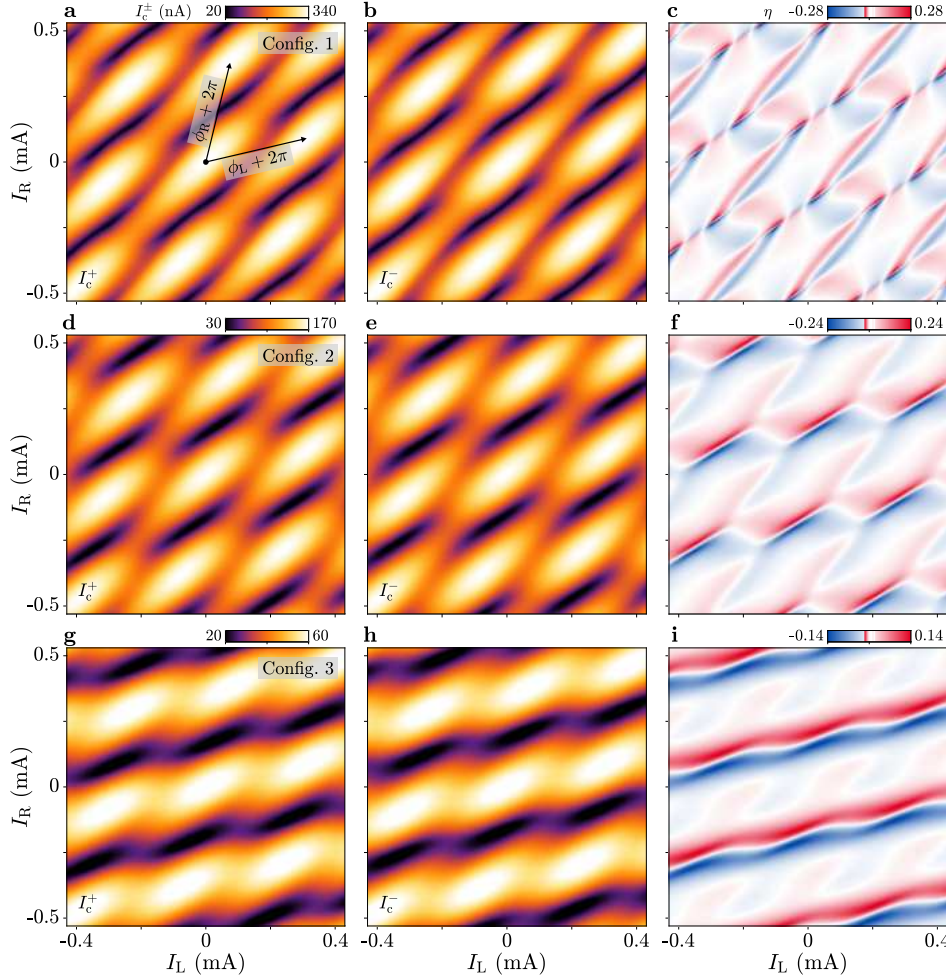


Figure S.9: Simulations with the extended model for Device 2. (a, b) Critical currents I_c^+ and I_c^- , simulated for positive and negative current bias (respectively), as functions of flux-line currents I_L and I_R , using the parameters of Configuration 1 (see text). The model is described in the Main Text. Currents I_L and I_R were obtained from the superconducting phase differences ϕ_L (between terminals S and L) and ϕ_R (between S and R) by applying a linear transformation (see Section 31.5 for additional details). Phase axes ϕ_L and ϕ_R , that are the periodicity directions, are indicated by the black arrows, whose length represents winding of the corresponding phase by 2π . (c) Diode efficiency η extracted from (a) and (b) as a function of I_L and I_R . (d–f) As in (a–c), but using the parameters of Configuration 2. (g–i) As in (a–c), but using the parameters of Configuration 3.

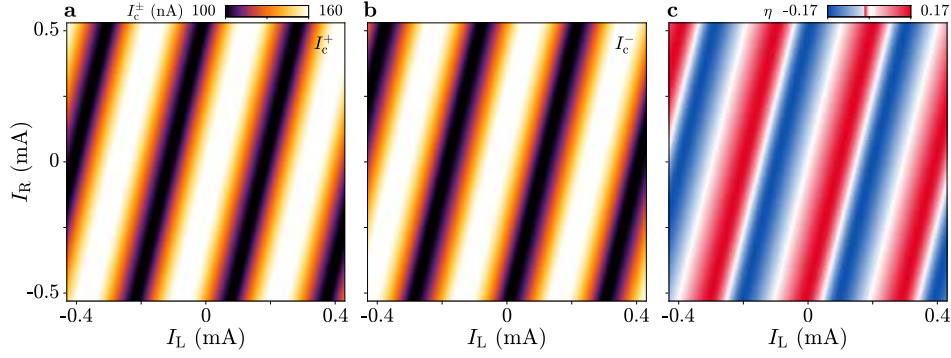


Figure S.10: Simulations with the extended model for Device 2 in the switch-OFF case. (a, b) Critical currents I_c^+ and I_c^- , simulated for positive and negative current bias (respectively), as functions of flux-line currents I_L and I_R , using the parameters of Configuration 2 (see text) and additional parameters $\tau_{RM} = 0.75$ and $T_{RM} = 4$. The model is described in the Main Text. (c) Diode efficiency η extracted from (a) and (b) as a function of I_L and I_R .

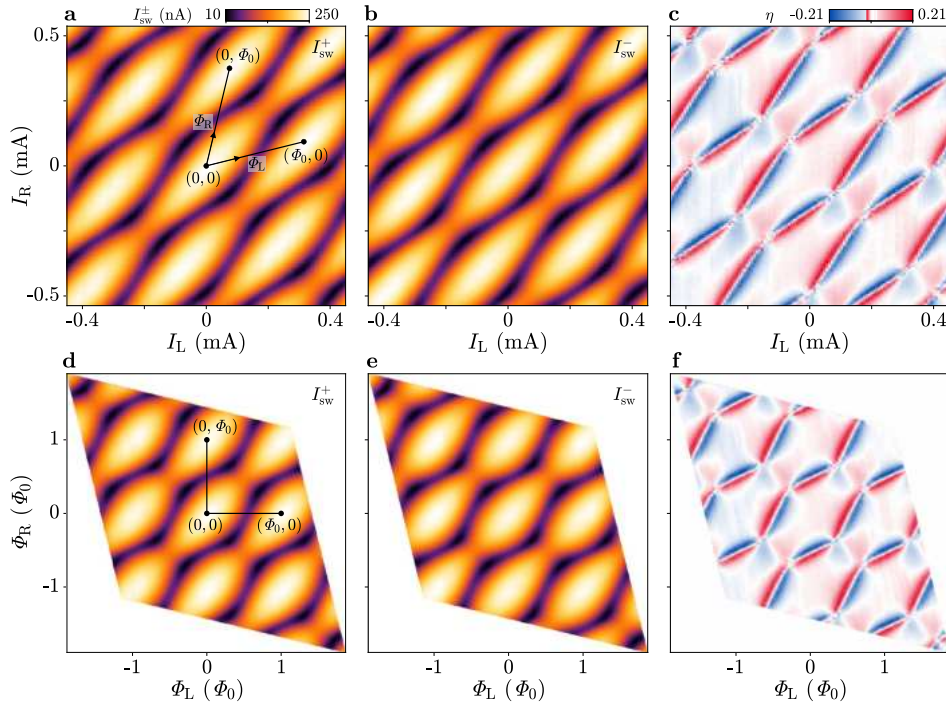


Figure S.11: Linear transformation mapping the flux-line-current space into the flux space. (a–c) Switching currents I_{sw}^+ and I_{sw}^- and diode efficiency η as functions of flux-line currents I_L and I_R , as in Figs. 2b–d of the Main Text. (d–f) Same datasets of (a–c) plotted as functions of the external magnetic fluxes Φ_L and Φ_R , upon applying the linear transformation described in the text.

32 Phase-engineering the Andreev band structure of a three-terminal Josephson junction by Coraiola et al.

Abstract

In hybrid Josephson junctions with three or more superconducting terminals coupled to a semiconducting region, Andreev bound states may form unconventional energy band structures, or Andreev matter, which are engineered by controlling superconducting phase differences. Here we report tunnelling spectroscopy measurements of three-terminal Josephson junctions realised in an InAs/Al heterostructure. The three terminals are connected to form two loops, enabling independent control over two phase differences and access to a synthetic Andreev band structure in the two-dimensional phase space. Our results demonstrate a phase-controlled Andreev molecule, originating from two discrete Andreev levels that spatially overlap and hybridise. Signatures of hybridisation are observed in the form of avoided crossings in the spectrum and band structure anisotropies in the phase space, all explained by a numerical model. Future extensions of this work could focus on addressing spin-resolved energy levels, ground state fermion parity transitions and Weyl bands in multiterminal geometries.

32.1 Introduction

In a normal conductor interfacing two or more superconductors, charge carriers at energies within the superconducting gap are confined by Andreev reflection processes occurring at the interfaces [12]. As a result, resonant sub-gap electronic excitations known as Andreev bound states (ABSs) arise in the normal region, enabling transport of a Josephson supercurrent between the superconducting terminals [15, 5]. These discrete ABS levels were proposed as a basis for quantum computing applications [114, 115, 6, 7]. More recently, ABSs have been the subject of intense experimental investigation in various material platforms [6, 9, 10, 7, 12, 13, 117, 15], culminating in the coherent control of Andreev pair [116, 160] and spin [119, 19] qubits. While these studies focused on Josephson junctions (JJs) with two superconducting terminals, where the ABS energies depend on a single superconducting phase difference, multiterminal JJs (MTJJs) have also emerged as a promising alternative. In the presence of $N \geq 3$ terminals, the ABS band structure spanned by the $N - 1$ independent phase differences was predicted to exhibit a plethora of phenomena, including lifting of the spin degeneracy [27], ground state fermion parity transitions [27], Weyl singularities [17, 11, 18, 22, 23] and other topological properties [64, 28, 36, 29, 26, 33]. Moreover, MTJJs were proposed to realise Andreev molecules [32, 33, 34, 35]—a system where single ABSs overlap and form hybridised energy levels—and explored as a platform to generate Cooper quartets [36, 37, 39, 40, 40]. Extensive experimental work was also conducted on MTJJs in the presence of DC current bias [44, 41, 42, 44] and microwave irradiation [45]. While tunnelling spectroscopy of metallic three-terminal JJs (3TJJs) was performed previously [46], establishing phase control over the superconducting proximity gap, further investigation is needed to understand the properties of ABS bands in multiterminal devices. The perspective of engineering synthetic band structures by exploiting the higher dimensionality of the phase space is particularly attractive, as it could enable effects unattainable in two-terminal geometries.

Here, we report on an experimental realisation of superconductor–semiconductor 3TJJs and study ABSs in the system with tunnelling spectroscopy. Owing to the independent control over two superconducting phase differences, we probe the Andreev band structure in the two-dimensional (2D) phase space and find signatures of hybridisation between highly transmissive ABSs, resulting in the formation of an Andreev molecule. Our measurements are supported by a theoretical model and demonstrate the feasibility of Andreev matter and phase-engineering of Andreev bands in hybrid nanostructures.

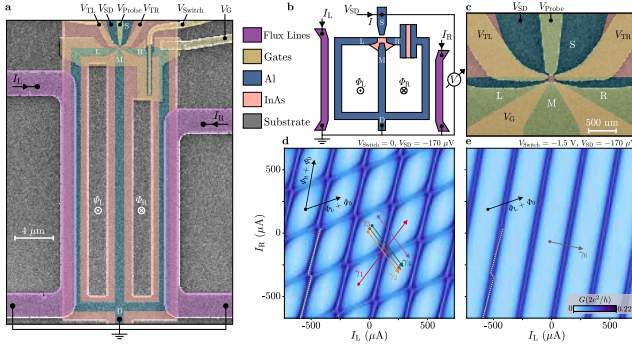


Figure 154: **Experimental setup and tunnelling conductance in the two-dimensional phase space.** **a**, False-coloured scanning electron micrograph of a device identical to that under study, defined by selective removal of the Al (blue), exposing the semiconductor below (pink). Gates (yellow) and flux-bias lines (purple) were deposited on a uniform dielectric layer (not visible). Bias voltage V_{SD} , gate voltages V_α ($\alpha \in \{\text{TL}, \text{TR}, \text{Probe}, \text{Switch}, \text{G}\}$), left (right) flux-bias line current $I_{L(R)}$ and external magnetic flux threading the left (right) superconducting loop $\Phi_{L(R)}$ are indicated. **b**, Schematic representation of the device and the measurement setup. **c**, Zoom-in of **a** near the three-terminal Josephson junction (3TJJ) region. **d,e**, Differential conductance G between tunnelling probe and 3TJJ measured as a function of the currents I_L and I_R injected into the flux-bias lines, at fixed voltage bias $V_{SD} = -170 \mu\text{V}$. In **d** (**e**), the switch junction is in the ON (OFF) state, $V_{\text{Switch}} = 0$ ($V_{\text{Switch}} = -1.5 \text{ V}$). The directions of the black arrows indicate the periodicity axes, along which the external magnetic fluxes Φ_L and Φ_R vary independently. Each black arrow represents the addition of one superconducting flux quantum $\Phi_0 = h/2e$ (where h is the Planck constant and e the elementary charge) to the corresponding flux. The dotted yellow line follows a Φ_L -dependent resonance in **d** and is replicated in **e** to highlight the slope difference. The coloured arrows labelled γ_{1-6} define the linecuts shown in Fig. 172.

32.2 Results

32.2.1 Implementation of phase-controllable 3TJJ

The device under investigation, shown in Fig. 171a–c, was realised in an InAs/Al heterostructure [104, 48]. Selective etching of the Al layer defined three superconducting terminals (labelled L, M and R) coupled to a normal region, constituting a hybrid 3TJJ. The three terminals were connected to a common node (D) forming two closed loops; while leads L and M were directly connected to D via Al strips, a superconductor–normal–superconductor (SNS) JJ was integrated on terminal R. The junction, with a length of 40 nm and a width of 5 μm , was designed to have a critical current much larger than that existing between any pairs of L, M and R, hence the superconducting phase difference across the junction is neglected for the following discussion. A fourth superconducting lead (S) was employed as a probe to enable DC tunnelling spectroscopy of the sub-gap states in the 3TJJ. Metallic gate electrodes and flux-bias lines were patterned on top of an insulating layer uniformly deposited across the entire sample. The transmission between the probe and the 3TJJ was tuned via two gates energised by the common voltage $V_T \equiv V_{TL} = V_{TR}$, which was set to -1.07 V to enter the tunnelling regime: in this configuration, the probe was weakly coupled to the 3TJJ and its influence on the rest of the circuit was limited. Gate tuning of the SNS junction enabled its operation as a switch, with the ON (OFF) state defined by $V_{\text{Switch}} = 0$ ($V_{\text{Switch}} = -1.5$ V). This allowed for the connection or disconnection of terminal R from D, hence electrostatically selecting between a three-terminal (switch ON) and a two-terminal configuration (switch OFF). Two additional gate voltages were kept to $V_{\text{Probe}} = 150$ mV and $V_G = 50$ mV. A current $I_{L(R)}$ injected into the left (right) flux-bias line generated an external magnetic flux $\Phi_{L(R)}$ threading the left (right) superconducting loop, thus tuning the phase difference between L (R) and M, and enabling control over the whole 2D phase space. As schematically depicted in Fig. 171b, a DC voltage bias V_{SD} was applied to the probe S and the differential tunnelling conductance G was measured between S and D with standard lock-in techniques. Experiments were performed in a dilution refrigerator with base temperature below 10 mK. Further details about materials, fabrication and measurement are provided in the Methods section.

32.2.2 Probing the Andreev band structure in the 2D phase space

In Fig. 171d,e, the voltage bias was set to -170 μV and the tunnelling conductance was measured as a function of the currents I_L and I_R injected into the flux-bias lines, resulting in a scan over an extended region of the 2D phase space at constant energy. Resonances in conductance correspond to peaks in the density of states (DOS) of the normal region under study [6, 15] and represent ABSs in the 3TJJ. Each state is intersected twice per period at $V_{SD} = -170$ μV (i.e., at an energy below its maximum), giving rise to the characteristic appearance of pairs of lines in these maps. When the switch is set to the ON state (Fig. 171d), we observe periodic features as a function of both I_L and I_R , attributed to the presence of two distinct ABSs whose energies disperse with Φ_L and Φ_R , respectively. The cross-dependence between the flux-bias lines accounts for the finite slope of the (Φ_L, Φ_R) axes, as indicated by the black arrows. Remarkably, resonances associated to different ABSs are connected to each other in proximity of the intersections, forming closed diamond-like loops and avoided crossings at the corners of the diamonds. Each state undergoes a phase shift when intersecting the other, defining a zig-zag trend. Next, we set the switch junction voltage to -1.5 V (OFF, Fig. 171e) and observe that the ABS resonances depending on Φ_R disappear, while the complex 2D periodic pattern is transformed into a simple structure depending on a single flux. Notably, the slope of the Φ_L -dependent resonances differs between panels d and e (see dotted yellow lines), which is compatible with the periodic phase shift present only when the switch is ON.

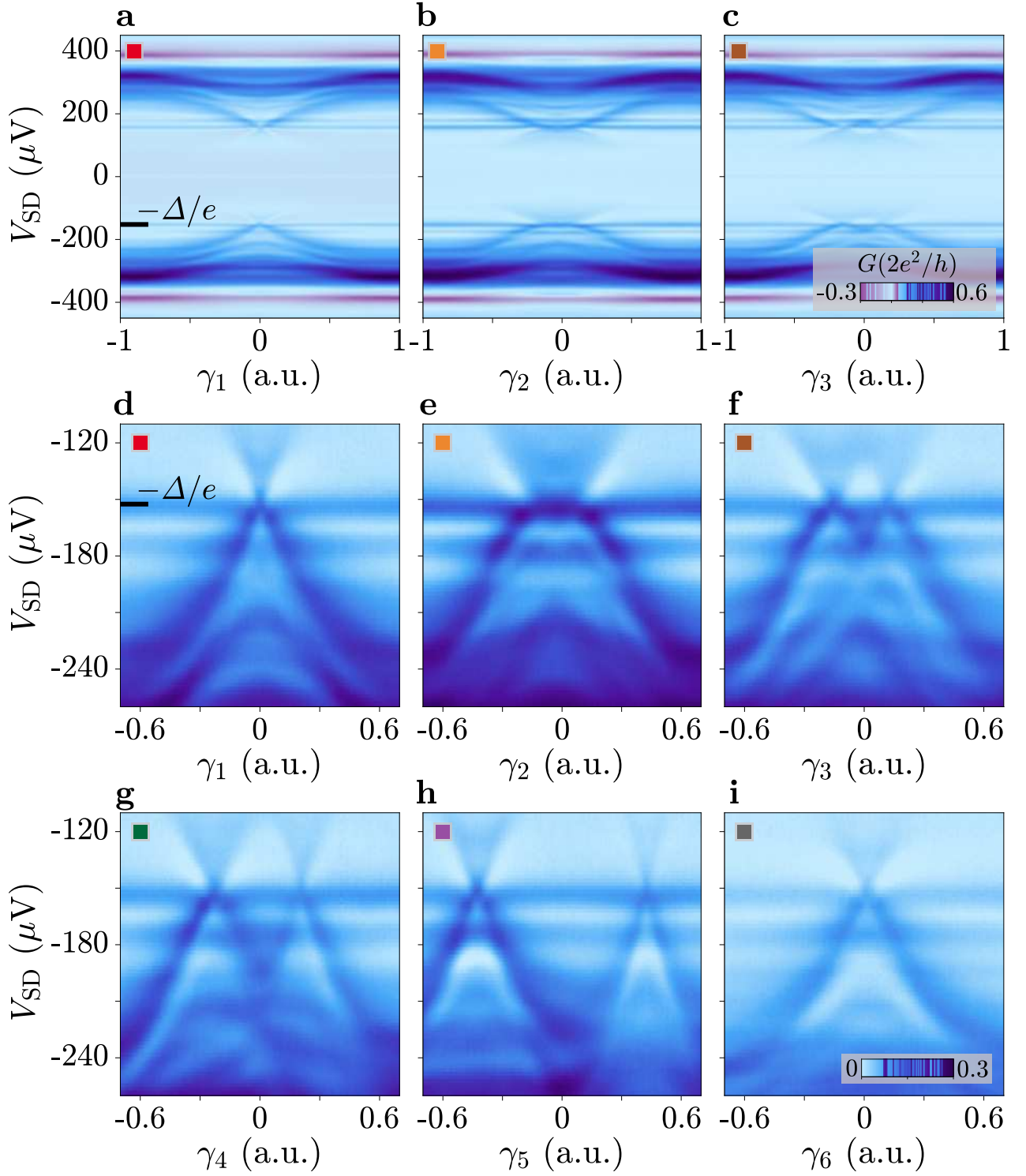


Figure 155: **Tunnelling conductance spectra along phase space linecuts.** **a–c**, Differential tunnelling conductance G measured as a function of voltage bias V_{SD} along the linecuts γ_i (coloured arrows in Fig. 171d), with $V_{Switch} = 0$. The lower edge of the transport gap $-\Delta/e$, due to the superconducting tunnelling probe, is indicated by the black marker. **d–h**, As **a–c**, but plotted over restricted ranges of V_{SD} and γ_i . **i**, As **d–h**, but along linecut γ_6 (defined in Fig. 171e), for $V_{Switch} = -1.5$ V. The colourbar in **i** applies to **d–i**.

32.2.3 Andreev dispersion along phase space linecuts

With the phase space overview acquired at constant voltage bias, we select cut lines γ_{1-6} (arrows in Fig. 171d,e) along which the tunnelling conductance is measured as a function of V_{SD} . The full datasets along γ_1 , γ_2 and γ_3 are displayed in Fig. 172a–c. Each shows a transport gap $2\Delta/e \approx 310 \mu\text{V}$ (e is the elementary charge), consistent with a superconducting gap of the Al probe $\Delta \approx 155 \mu\text{V}$, and an electron–hole-symmetric spectrum revealing phase-dependent ABSs. The presence of regions with finite conductance at $e|V_{SD}| \lesssim \Delta$ is ascribed to broadened features in the DOS of the superconducting probe for energies $|E| \sim \Delta$. This could be due to a combination of quasiparticle-lifetime broadening [49] and additional subgap bound states forming between probe and 3TJJ. On either side of the spectrum, we notice two differential conductance peaks at $V_{SD} = \pm 155 \mu\text{V}$ and $\pm 175 \mu\text{V}$, whose position in bias does not vary appreciably with γ_i . The first is attributed to the multiple Andreev reflection peak at $\pm\Delta/e$, while the second, specific to this device, might be related to mesoscopic defects in the tunnelling probe or to a region in the device with a larger superconducting gap. These peaks provide a contribution to the measured differential conductance, adding to that of ABS resonances and spectroscopy background. This accounts for the intensity modulation of these peaks depending on γ_i . Further supported by measurements at different tunings of the probe (see Supplementary Note 4), we do not observe a distortion of the ABS dispersion when the states cross the peaks. In Fig. 172d–i, all six linecuts are plotted in restricted V_{SD} and γ_i ranges for better clarity. In each case, ABSs approach the transport gap edge very closely, which indicates near-unity transmission. We also note that such highly transmissive ABSs intersect $V_{SD} = -170 \mu\text{V}$ twice per period, thus accounting for the pairs of resonances in Fig. 171d,e. Interestingly, when comparing the γ_1 and γ_2 linecuts (Fig. 172d,e), we find anisotropic ABS phase dispersion in the vicinity of (π, π) phase ($\gamma_{1,2} = 0$), with a narrow, cusp-like shape versus a broader and flatter peak, respectively. In the spectroscopy measurements performed along γ_{3-5} , i.e., lines parallel to γ_2 but offset from an intersection point of Fig. 171d, two distinct highly transmissive ABSs appear. When their separation in phase is small, the states partially mix and an avoided crossing is observed (Fig. 172f), an effect which is weaker at larger separation (Fig. 172g), until it is completely suppressed (Fig. 172h). Finally, linecut γ_6 , where the switch junction is kept in the OFF state, reveals a single highly transmissive ABS (Fig. 172i).

32.2.4 Andreev molecule model and numerical simulations

To simulate ABSs arising in a 3TJJ, we study a minimal theoretical model comprising three superconducting terminals with phases ϕ_L , ϕ_R and ϕ_M , as schematically sketched in the inset of Fig. 173a. Due to gauge invariance, only two phases are independent, therefore we set $\phi_M \equiv 0$. Between lead L (R) and M, we assume a highly transmissive channel, fully described by two coupling parameters to the leads and hosting a spin-degenerate ABS, whose energy $E_{L(R)}$ depends on $\phi_{L(R)} - \phi_M \equiv \phi_{L(R)}$. The choice of the coupling parameters led to transmissions $T_1 \approx 0.998$ and $T_2 \approx 0.992$ for the left and right ABS, respectively. Coupling between the ABSs is described by the parameter t , resulting in a minimal set of five numerical parameters, plus a broadening parameter (more details of the model are explained in Supplementary Note 1). When the two ABSs are isolated ($t = 0$), their energy dispersions solely depend on the transmission of the respective channels [15]. However, for finite coupling $t \neq 0$, hybridisation between the states is enabled in the regions of (ϕ_L, ϕ_R) where the isolated energies E_L and E_R are comparable, resulting in an Andreev molecule [32, 33]. To study the experimentally relevant case, we assume $t = 1.1\Delta$, indicating ABSs strongly coupled to each other. In Fig. 173a, we display a numerical simulation of the Andreev energy bands as a function of ϕ_L and ϕ_R for negative energies (i.e., below the Fermi level $E_F \equiv 0$), noting that specular bands exist at positive energies owing to electron–hole symmetry. The finite coupling between the ABSs

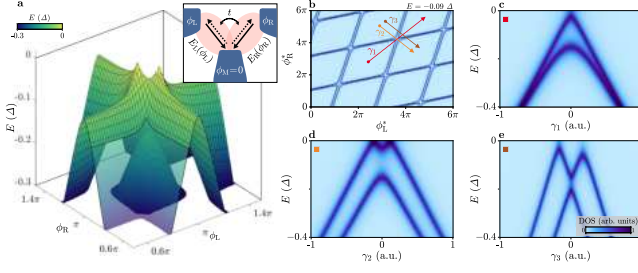


Figure 156: **Theoretical model of coupled Andreev bound states and Andreev molecule.** **a**, Simulated Andreev bound state (ABS) energy bands as a function of the superconducting phase differences ϕ_L and ϕ_R for negative energies $E \leq 0$. The band structure at positive energies (not shown) is specular due to electron–hole symmetry. Inset: simplified schematic of the model. Three superconducting terminals (blue), with phases ϕ_L , ϕ_R and $\phi_M = 0$, are interconnected via two Andreev channels (pink) that are located between the middle lead and either the left or the right lead. The energy $E_{L(R)}$ of the left (right) ABS depends on the phase difference $\phi_{L(R)} - \phi_M \equiv \phi_{L(R)}$. The two ABSs are coupled to each other, enabling hybridisation of their energy levels, described by the parameter t . A full description of the model is provided in the Supplementary Information. **b**, Simulated density of states (DOS) at fixed energy $E = -0.09\Delta$ as a function of the cross-coupled superconducting phase differences ϕ_L^* and ϕ_R^* , defined as linear combinations of ϕ_L and ϕ_R to better represent the experimental data (see Supplementary Information for more details). **c–e**, Simulated DOS as a function of energy along the linecuts of the phase space γ_{1-3} , defined in **b** (coloured arrows).

results in hybridised bands with a pronounced splitting. In the region with both phases tuned near to π , the band closer to zero energy shows a striking anisotropy in the phase space and the dispersion is strongly modified compared to that of high-transmission ABSs in a ballistic two-terminal JJ. To better compare experimental and numerical results, we calculate the DOS at fixed energy $E = -0.09\Delta$ as a function of the cross-coupled phases ϕ_L^* and ϕ_R^* (Fig. 173b), and as a function of energy along the three phase space linecuts γ_{1-3} (Fig. 173c–e). All simulations qualitatively reproduce the key features observed in the measurements of Figs. 171(d) and 172, and lead to the same order of magnitude for the avoided crossings. In the constant-energy simulation (Fig. 173b), we confirm the presence of a periodic pattern characterised by avoided crossings and phase shifts of the ABS resonances near the intersection points, where individual ABSs are connected to each other forming closed loops. Further, the spectra presented in Fig. 173c–e resemble the results shown in Fig. 172d–f: the ABS dispersion approaching zero energy has a sharp peak along γ_1 and is broader along γ_2 . The individual ABSs reappear with a significant separation when probed along γ_3 , while maintaining a sizeable avoided crossing. The experimentally measured phase shifts are larger than those calculated theoretically. The enhanced shift is likely produced by the finite inductive coupling between the loops, which is not included in the numerical model and implies a coupling between the fluxes Φ_L and Φ_R . The change of slope of the ABS resonances between Figs. 171d and 171e is consistent with the phase shift and is related to the same coupling mechanism [50, 51]. These effects are discussed in more detail in Supplementary Note 6.

32.2.5 Tomography of the Andreev band structure

Another visualisation of the ABS energy bands is provided by combining multiple constant-energy cut planes—each showing the dependence on both superconducting phase differences—to achieve a tomographic representation of the band structure. For this purpose, we measured the tunnelling conductance as a function of I_L and I_R for several values of V_{SD} . The outcome

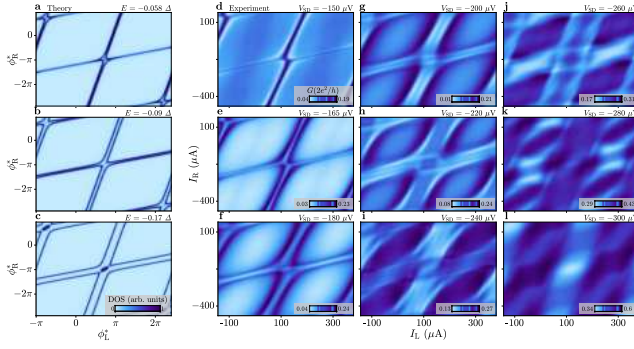


Figure 157: **Constant-energy planes as a function of the two phases for different energies.** **a–c**, Simulated density of states as a function of the cross-coupled superconducting phase differences ϕ_L^* and ϕ_R^* at fixed values of energy E . **d–l**, Differential tunnelling conductance G measured as a function of the currents I_L and I_R injected into the flux-bias lines at fixed values of voltage bias V_{SD} , for $V_{Switch} = 0$.

is summarised in Fig. 157, where panels a–c display three theoretically calculated planes in the low-energy spectrum and d–l report experiments for nine values of V_{SD} . At $V_{SD} \approx -\Delta/e$ (i.e., near zero energy), where the ABSs are probed in proximity of their maximum, we observe single resonances (Fig. 157a,d), which split into pairs at more negative bias (Fig. 157b,e). We confirm the presence of the features described for Fig. 171d and Fig. 173b. Notably, in Fig. 157c,f resonances related to different states are connected to each other by arcs enclosing an oval region. At even more negative bias, additional resonances arise, compatible with low-transmission ABSs appearing in the spectrum from $V_{SD} \approx -200 \mu\text{V}$ (see Fig. 172). Similar to the high-transmission ABSs discussed previously, low-transmission states first occur as single lines (Fig. 157g) and then split into pairs (Fig. 157h,i). Several additional modes emerge at larger $|V_{SD}|$, making it difficult to resolve individual states while approaching the continuum (Fig. 157j–l). We remark that the theoretical model assumes only two high-transmission modes and does not include any additional states with lower transmission, unlike the experimentally measured spectrum. Therefore, a direct comparison of the spectrum at high $|E|$ is beyond the scope of the model.

Our main experimental findings, including avoided crossings and phase shifts in the constant-bias planes of Figs. 171d and 157, as well as the anisotropy highlighted in Figs. 172d,e and 173c,d, were qualitatively reproduced on a second device (see Supplementary Note 3). These measurements suggest the generality of the phenomena observed.

32.3 Discussion

Supported by our theoretical model of coupled ABSs, we interpret the experimental results summarised in Figs. 171, 172 and 157 as evidence of coupling and hybridisation between two highly transmissive ABSs in the normal region of the 3TJJ. Thus, our devices constitute an implementation of an Andreev molecule, comparable to Refs. [32, 33]. Avoided crossings in both the phase space and the energy spectrum, together with the anisotropic ABS dispersion motivate our interpretation. Unlike previous experimental studies of Andreev molecules in two-terminal geometries [52, 53], this realisation is in an open system and is a direct manifestation of the phase-controlled, multidimensional Andreev band structure.

Our system can be considered the magnetic dual of a double quantum dot [54], where electric fields controlled by gate voltages, capacitance and charge on the dots (quantised in units of e) are substituted by magnetic fields controlled by currents in flux-bias lines, inductance and magnetic flux threading superconducting loops (quantised in units of the superconducting flux

quantum $\Phi_0 = h/2e$, with h the Planck constant). Both in a double quantum dot and in the present Andreev molecule, overlapping wave functions of two discrete and localised states, coupling in a middle region, result in avoided crossings between their otherwise degenerate energy levels. In our device structure, two discrete levels, namely high-transmission ABSs, form in the short L–M and R–M junctions (whose minimum length is lithographically 30 nm) and are coupled to each other by their close proximity.

Spin-resolved ABSs are not observed in the experiments, well described by a spin degenerate model, despite the presence of spin–orbit coupling in our system. We note that here the spin–orbit length, $l_{\text{SO}} \sim 150$ nm for InAs [55], is larger than the separation between pairs of terminals of the 3TJJ, resulting into a relatively weak strength of spin–orbit coupling. Enlarging the size of the 3TJJ would thus be required to resolve spin–orbit splitting of ABSs.

In conclusion, ABSs in hybrid 3TJJs were investigated with tunnelling spectroscopy measurements. Owing to the individual control over two superconducting phase differences, we explored a synthetic Andreev band structure and found signatures of coupling and hybridisation between two highly transmissive ABSs, consistent with their overlap in the 3TJJ region and the formation of an Andreev molecule. In the 2D phase space probed at constant voltage bias, we observed periodic patterns with avoided crossings and phase shifts near the intersections between ABS resonances. We measured the spectrum along selected linecuts of the phase space, finding a strong anisotropy of the ABS band structure and avoided crossings between the states. The experiments are well described by a theoretical model of two coupled ABSs. Our results provide new insights into the physics of multiterminal devices, establish phase control over the ABS band structure and demonstrate the feasibility of realising exotic Andreev matter. Future studies of multidimensional band structures could focus on phase-engineering spin-resolved Andreev levels [27], ground state fermion parity transitions [27, 56, 112] and topological bands, including Weyl singularities [17, 11, 18, 22, 23].

Note. We recently became aware of the unpublished data of Refs. [58] and [59], where three-terminal devices were investigated.

32.4 Methods

32.4.1 Materials and fabrication

Devices were fabricated in a III–V heterostructure grown with molecular beam epitaxy techniques on an InP (001) substrate. The stack consisted of a step-graded InAlAs buffer layer covered by an $\text{In}_{0.75}\text{Ga}_{0.25}\text{As}/\text{InAs}/\text{In}_{0.75}\text{Ga}_{0.25}\text{As}$ quantum well and two monolayers of GaAs. The InAs layer, hosting a two-dimensional electron gas (2DEG), was 8 nm thick and buried 13 nm below the surface. On top of the III–V stack, a 15 nm thick Al layer was deposited *in situ* without breaking vacuum. Characterisation of the 2DEG in a gated Hall bar revealed a peak mobility of $18000 \text{ cm}^2\text{V}^{-1}\text{s}^{-1}$ at an electron sheet density of $8 \cdot 10^{11} \text{ cm}^{-2}$. This resulted in an electron mean free path $l_e \gtrsim 260$ nm, indicating that both the three-terminal Josephson junction and the two-terminal switch junction were in the ballistic regime.

First, large mesa structures were isolated, suppressing parallel conduction between devices and across the middle regions of the superconducting loops. This was done by selectively etching the Al layer with Transene type D, followed by a second chemical etch to a depth of ~ 380 nm into the III–V material stack, using a 220 : 55 : 3 : 3 solution of $\text{H}_2\text{O} : \text{C}_6\text{H}_8\text{O}_7 : \text{H}_3\text{PO}_4 : \text{H}_2\text{O}_2$. Next, Al was defined by wet etching with Transene type D at 50°C for 4 s. The dielectric, deposited on the entire chip by atomic layer deposition, consisted of a 3 nm thick layer of Al_2O_3 and a 15 nm thick layer of HfO_2 . Gate electrodes and flux-bias lines were defined by evaporation and lift-off. In a first step, 5 nm of Ti and 20 nm of Au were deposited to realise the fine features of the gates; in a second, a stack of Ti/Al/Ti/Au with thicknesses 5 nm, 340 nm,

5 nm and 100 nm was deposited to connect the mesa structure to the bonding pads and to define the flux-bias lines.

32.4.2 Measurement techniques

Experiments were performed in a dilution refrigerator with base temperature at the mixing chamber below 10 mK. The sample was mounted on a QDevil QBoard sample holder system, without employing any light-tight enclosure. Electrical contacts to the devices, excepts for the flux-bias lines, were provided via a resistive loom with QDevil RF and RC low-pass filters at the mixing chamber stage, and RC low-pass filters integrated on the QBoard sample holder. Currents in the flux-bias lines were injected via a superconducting loom with only QDevil RF filters at the mixing chamber stage. Signals were applied to all gates and flux-bias lines via home-made RC filters at room temperature. Electrical transport measurements were performed with low-frequency AC lock-in techniques. A fixed AC voltage $\delta V_{SD} = 5 \mu\text{V}$ at frequency 211 Hz and a variable DC voltage V_{SD} were applied to a contact at the superconducting probe (labelled S in Fig. 171a). The AC current δI and the DC current I_{SD} flowing in the grounded terminal D were measured via a current-to-voltage (I–V) converter. By measuring the AC voltage δV between terminals S and D in a four-terminal configuration, the differential conductance $G \equiv \delta I / \delta V$ was determined. The refrigerator was equipped with a vector magnet which, despite not being utilised for the experiments, produced a small magnetic field offset. Hence, arbitrary offsets in the flux-bias line currents I_L and I_R of $-18 \mu\text{A}$ and $74 \mu\text{A}$ were considered in datasets, in such a manner that the point where $I_L = I_R = 0$ was at the centre of a diamond-like region in the constant-bias maps.

32.5 Data availability

The data presented in this study have been deposited in Zenodo [<https://zenodo.org/record/8360770>]. Further data that support the findings of this study are available upon request from the corresponding author.

32.6 Code availability

Computer code used to perform the numerical simulations presented in this work has been deposited in Zenodo [<https://zenodo.org/record/8360770>].

32.7 Supplementary Information: Phase-engineering the Andreev band structure of a three-terminal Josephson junction

32.8 Supplementary Note 1: Theory

In the following, we discuss hybridisation of two Andreev bound states (ABSs) between three superconducting leads. The problem has been investigated extensively in the context of the Andreev molecule [32, 33, 34, 35]. Here, we treat hybridisation empirically with a coupling parameter γ introducing an avoided crossing in the spectrum. Assuming relatively weak coupling between the two ABSs, the spectrum resembles the energy levels of the independent states except for the points where they would cross, where hybridisation leads to the anticrossing. A simple perturbation theory can be applied if the energies $E_{L,R}$ are sufficiently larger than γ :

$$E_{\pm} = \frac{E_L(\phi_L) + E_R(\phi_R)}{2} \pm \sqrt{\left(\frac{\gamma}{2}\right)^2 + \left[\frac{E_L(\phi_L) - E_R(\phi_R)}{2}\right]^2}, \quad (32.1)$$

where

$$E_\alpha = -\Delta \sqrt{1 - \tau_\alpha^2 \sin^2 \frac{\phi_\alpha}{2}}, \quad \alpha \in \{L, R\} \quad (32.2)$$

is the energy of an independent ABS between two superconducting leads with phase difference ϕ_α and energy gap Δ in a channel with transmission τ_α^2 . We recall that the perturbation theory is applicable for $|E_\alpha(\phi_\alpha)| \gg |\gamma|$: this condition does not hold if the transmissions of the junctions are large ($1 - \tau_\alpha \ll 1$) and the phase differences are close to $(2n + 1)\pi$, which is the regime studied in the experiment. Nevertheless, in such experimentally relevant limit we follow the approach of Refs. [33, 34] and expand the independent ABS energies around $\phi_\alpha = \pi$:

$$E_\alpha \approx -\Delta \sqrt{1 - \tau_\alpha^2 \left[1 - \frac{(\phi_\alpha - \pi)^2}{4} \right]} \approx -\frac{\Delta}{2} \sqrt{(\phi_\alpha - \pi)^2 + (2r_\alpha)^2}, \quad (32.3)$$

with $r_\alpha^2 = 1 - \tau_\alpha^2 \ll 1$ the reflection amplitude squared. In the limit of perfect transmission $r_L = r_R = 0$, the equations derived in Refs. [33, 34] for the ballistic regime can be employed:

$$E_\pm \approx -\frac{\Delta}{2} \sqrt{\delta^2 + \frac{(\varphi_L + \varphi_R)^2}{4} + \frac{(\varphi_L - \varphi_R)^2}{4} \pm (\varphi_L - \varphi_R) \sqrt{\delta^2 + \frac{(\varphi_L + \varphi_R)^2}{4}}}, \quad (32.4)$$

where $\varphi_\alpha = \phi_\alpha - \pi$ and $\delta = \gamma/\Delta$ is the dimensionless coupling parameter. The formula is applicable for any points close to $(\phi_L, \phi_R) = [\pi(2n + 1), \pi(2m + 1)]$, with n and m integers. The corresponding constant-energy cut plane of the spectrum at $E = -0.01\Delta$ is shown in Supplementary Fig. 158a as a function of the phase differences. Here, hybridisation of purely ballistic channels introduces avoided crossings along the anti-diagonal direction $\phi_L = 2\pi - \phi_R$.

Moreover, we consider small but finite reflection amplitudes $r_\alpha \ll 1$ for each channel and use the effective Hamiltonian introduced in [33] (by projecting on the low-energy states) to derive the spectrum:

$$E_\pm \approx -\frac{\Delta}{2\sqrt{2}} \times \sqrt{2\delta^2 + \varphi_L^2 + (2r_L)^2 + \varphi_R^2 + (2r_R)^2 \pm \sqrt{[\varphi_L^2 + (2r_L)^2 - \varphi_R^2 - (2r_R)^2]^2 + 4\delta^2 [(\varphi_L - \varphi_R)^2 + (2r_L)^2 + (2r_R)^2]}} \quad (32.5)$$

The constant-energy plane at $E = -0.01\Delta$ is displayed in Supplementary Fig. 158b, where we observe that non-zero reflection amplitudes introduce additional avoided crossings along the diagonal direction $\phi_L = \phi_R$.

In order to go beyond the limiting case accessible analytically, valid for weak coupling between the ABSs, we perform numerical simulations of a comparable system. Andreev states are modelled as two single-level quantum dots (QDs) coupled to three superconducting terminals. A coupling between the QDs accounts for hybridisation of the ABSs in the three-terminal junction. The model is schematically shown in Supplementary Fig. 159. For each ABS, the only relevant parameter is the transmission of the corresponding junction [60], hence we can arbitrarily opt for any microscopic model. The QD model offers a flexible tool to describe a scattering region and is well suited for numerical analyses of the system, even for strong coupling between the ABSs, where the analytical approach cannot be used. The coupling between each QD and the superconducting terminals can be arbitrarily strong, hence this model is also suited to describe ABSs in the open regime, rather than isolated dots weakly coupled to the leads [92].

The total Hamiltonian of the system is expressed as:

$$H = H_{\text{DD}} + \sum_{\alpha=\text{M,L,R}} H_{\text{S},\alpha} + H_{\text{DS}}. \quad (32.6)$$

Here, H_{DD} is the Hamiltonian of the double QD:

$$H_{\text{DD}} = \epsilon_1 \sum_{\sigma} d_{1\sigma}^{\dagger} d_{1\sigma} + \epsilon_2 \sum_{\sigma} d_{2\sigma}^{\dagger} d_{2\sigma} + t \sum_{\sigma} d_{1\sigma}^{\dagger} d_{2\sigma} + \text{h.c.}, \quad (32.7)$$

where the creation operator $d_{j\sigma}^{\dagger}$ of the j -th QD, its energy ϵ_j and the coupling parameter t are defined. The superconducting terminals are described by Bardeen–Cooper–Schriber Hamiltonians [62]:

$$H_{\text{S},\alpha} = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c_{\mathbf{k}\sigma,\alpha}^{\dagger} c_{\mathbf{k}\sigma,\alpha} + \sum_{\mathbf{k}} \left(\Delta e^{i\phi_{\alpha}} c_{\mathbf{k}\uparrow,\alpha}^{\dagger} c_{-\mathbf{k}\downarrow,\alpha}^{\dagger} + \Delta e^{-i\phi_{\alpha}} c_{-\mathbf{k}\downarrow,\alpha} c_{\mathbf{k}\uparrow,\alpha} \right), \quad (32.8)$$

where $c_{\mathbf{k}\sigma,\alpha}^{\dagger}$ is the creation operator of an electron with momentum $\mathbf{k}$ and spin σ in lead $\alpha \in \{\text{M,L,R}\}$, Δ is the superconducting gap (assumed to be the same for all leads), ϕ_{α} the superconducting phase of lead α and $\xi_{\mathbf{k}}$ the normal state dispersion in the leads. The tunnel coupling between each QD and the superconductors is expressed by the term H_{DS} , having the form:

$$H_{\text{DS}} = v_{\text{M},1} \sum_{\mathbf{k},\sigma} d_{1\sigma}^{\dagger} c_{\mathbf{k}\sigma,\text{M}} + v_{\text{M},2} \sum_{\mathbf{k},\sigma} d_{2\sigma}^{\dagger} c_{\mathbf{k}\sigma,\text{M}} + v_{\text{L},1} \sum_{\mathbf{k},\sigma} d_{1\sigma}^{\dagger} c_{\mathbf{k}\sigma,\text{L}} + v_{\text{R},2} \sum_{\mathbf{k},\sigma} d_{2\sigma}^{\dagger} c_{\mathbf{k}\sigma,\text{R}} + \text{h.c.}, \quad (32.9)$$

where $v_{\alpha,j}$ denotes the coupling of the j -th QD to lead α .

Two superconductor–QD–superconductor junctions are identified: between leads L and M through QD₁ and between leads R and M through QD₂. In the absence of coupling between the QDs ($t = 0$), the ABS energy of QD₁₍₂₎ depends only on $\phi_{\text{L(R)}} - \phi_{\text{M}}$. For finite coupling ($t \neq 0$) the ABSs hybridise, which is the case presented in the Main Text. We do not consider a direct ballistic channel between terminals L and R due to their larger separation in the device compared to the L–M and R–M junctions. Nevertheless, the energy of a hypothetical ABS in such channel would depend on $\phi_{\text{L}} - \phi_{\text{R}}$, hence it would be close to the superconducting gap in proximity of the points where $\phi_{\text{L}} = \phi_{\text{R}} = \pi$. As a consequence, any hybridisation of the additional state with the two included in the model would be suppressed near those regions of phase space. To evaluate this model in the limit of strong coupling between superconducting leads and QDs, i.e., $\Gamma_{\alpha,j} \equiv \pi N_0 v_{\alpha,j}^2 > \Delta$ (where N_0 is the normal density of states in the leads), we calculate the ABS density of states by determining the Green’s function of the coupled system with the Dyson equation:

$$\mathbf{G}_{\text{DD}} = \mathbf{g}_{\text{DD}} + \mathbf{g}_{\text{DD}} \mathbf{V}_{\text{DS}} \mathbf{g}_{\text{SS}} \mathbf{V}_{\text{DS}}^{\dagger} \mathbf{G}_{\text{DD}}, \quad (32.10)$$

where $\mathbf{G}_{\text{DD}}$ is the dressed Green’s function of the double QD, $\mathbf{g}_{\text{DD}}$ the unperturbed Green’s function of the two QDs, $\mathbf{V}_{\text{DS}}$ the coupling between the QDs and the superconducting leads and $\mathbf{g}_{\text{SS}} = \text{diag}(\mathbf{g}_{\text{SS,M}}, \mathbf{g}_{\text{SS,L}}, \mathbf{g}_{\text{SS,R}})$ the unperturbed Green’s function of the three leads. From this expression, the ABS density of states is calculated as $\rho = -\frac{1}{\pi} \text{Im} \{ \text{tr}(\mathbf{G}_{\text{DD}}) \}$ and the eigenenergies are obtained from the poles of $\mathbf{G}_{\text{DD}}$, yielding the results shown in Figs. 3 and 4 of the Main Text. As previously discussed, the only relevant parameters for the hybridised ABSs are the inter-QD coupling t and the transmission amplitudes, which are related to the QD–lead couplings $\Gamma_{\alpha,j}$ via the expressions:

$$T_1 = \frac{4 \Gamma_{\text{M},1} \Gamma_{\text{L},1}}{(\Gamma_{\text{M},1} + \Gamma_{\text{L},1})^2}, \quad T_2 = \frac{4 \Gamma_{\text{M},2} \Gamma_{\text{R},2}}{(\Gamma_{\text{M},2} + \Gamma_{\text{R},2})^2}, \quad (32.11)$$

valid in the limit of strong lead–QD coupling. These parameters influence the region of the avoided crossing between the two ABS, as illustrated in Supplementary Fig. 160. In all simulations, a broadening parameter η was assumed in the Green’s functions. We observe qualitative agreement between the numerical simulations and the analytical study previously discussed (see Supplementary Fig. 158).

The parameters used for the simulations in the Main Text were: $t = 1.1\Delta$, $\Gamma_{L,1} = 5.5\Delta$, $\Gamma_{M,1} = \Gamma_{R,2} = 6\Delta$, $\Gamma_{M,2} = 5\Delta$, corresponding to transmissions $T_1 \approx 0.998$ and $T_2 \approx 0.992$, and broadening $\eta = 0.02\Delta$. Moreover, we recall that the constant-energy planes of Figs. 3b and 4a–c were plotted by introducing a cross-dependence between the phase differences $\phi_L - \phi_M \equiv \phi_L$ and $\phi_R - \phi_M \equiv \phi_R$ to better represent the experimental data, where a cross-dependence between the two flux-bias lines was present (see also discussion in Section 32.12). In particular, the cross-coupled phases ϕ_L^* and ϕ_R^* were defined as linear combinations of ϕ_L and ϕ_R with the transformation:

$$\begin{pmatrix} \phi_L^* \\ \phi_R^* \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \phi_L \\ \phi_R \end{pmatrix}, \quad (32.12)$$

where the coefficients $a = 0.9708$, $b = 0.2400$, $c = 0.2832$ and $d = 1.122$ were used.

32.9 Supplementary Note 2: Gate dependence of differential conductance

The transmission between the superconducting probe and the three-terminal Josephson junction (3TJJ) was controlled with two gate electrodes, denoted tunnel gates and set to the same voltage $V_T \equiv V_{TL} = V_{TR}$, and had a finite dependence on the gates energised by the voltages V_{Probe} and V_G , denoted probe gate and global gate respectively (see Fig. 1a,c). We always kept $V_{\text{Probe}} = 150$ mV. The effect of the gate at voltage V_{Switch} (switch gate) on the probe transmission was negligible. The differential conductance G measured as a function of V_T and V_{SD} , with V_{SD} the DC voltage bias applied to the probe, is shown in Supplementary Fig. 161a,b for $V_G = 50$ mV and $V_G = -150$ mV respectively. In both cases the conductance, hence the probe transmission, decreases with V_T , transitioning from open to tunnelling regime. In the former, we observe a zero-bias conductance peak, corresponding to a remnant of supercurrent between the probe and the superconducting leads, and several peaks at finite bias, related to multiple Andreev reflection (MAR) processes. In the tunnelling regime, where the conductance at large $|V_{SD}|$ is substantially lower than the conductance quantum $G_0 = 2e^2/h$, a transport gap of ≈ 310 μV is present in the spectrum, consistent with a superconducting gap of Al $\Delta \approx 155$ μeV . Pronounced features appearing at the edges of the gap correspond to Andreev bound states (ABSs). As shown in Supplementary Fig. 161b, by further lowering V_T the probe could be completely pinched off. We remark that tunnelling spectroscopy in our devices was performed in a superconductor–insulator–superconductor (SIS) configuration, hence the $G(V_{SD})$ traces result from a convolution product between two relatively complex densities of states [6, 15] and do not provide a direct measurement of the gap hardness. Nevertheless, since $G \approx 0$ across a significant voltage range around $V_{SD} = 0$ (for any flux-line currents I_L and I_R , as seen in Fig. 2 of the Main Text), we consider the gap hardness to be comparable to previous reports [63], where also a density-of-states broadening to what we noted in the Main Text was observed.

The dependence on the global gate is presented in Supplementary Fig. 161c for $V_T = -1.07$ V. We find that its main effect on the spectrum is also to change the transmission of the probe, which is progressively reduced for decreasing V_G until the pinch off is reached. Notably, in both the V_T and the V_G dependence, no sharp conductance peaks relating to resonant transport via spurious quantum dots are observed.

Tunnelling spectroscopy measurements were performed on a second device, fabricated on the same chip of the first and measured in the same cool down. Device 2 was lithographically similar to Device 1, except for the width of the superconducting probe and the shape of the tunnel and probe gates (see Supplementary Fig. 161d). The dependences on the tunnel gate voltage V_T and on the global gate voltage V_G are plotted in Supplementary Fig. 161e and f respectively, showing features qualitatively very similar to Device 1.

32.10 Supplementary Note 3: Results for Device 2

The main experimental results shown in the Main Text for Device 1 were qualitatively reproduced in Device 2, as illustrated in Supplementary Figs. 162 and 163. Here, the tunnel gates were set to $V_T = -1.395$ V and $V_T = -1.42$ V respectively, with the global gate voltage set to $V_G = -150$ mV and the probe gate voltage to $V_{\text{Probe}} = 100$ mV. In these configurations, the probe was in the tunnelling regime and its transmission was comparable to that of Device 1 for the measurements presented in the Main Text.

In Supplementary Fig. 162, we show the constant-bias planes (i.e., G as a function of I_L and I_R at fixed values of V_{SD}) corresponding to Figs. 1d,e, and 4d-l of the Main Text. When the switch junction is in the ON state (defined by $V_{\text{Switch}} = 0$), we confirm the presence of avoided crossings between a Φ_L -dependent ABS and a Φ_R -dependent ABS, with the resonances associated to one state connecting to those of the other, and phase shifts occurring near the avoided crossings. The 2D pattern is strongly simplified when the switch is OFF ($V_{\text{Switch}} = -1.5$ V, Supplementary Fig. 162b), as the Φ_R -dependent ABS disappears and no sign of hybridization is observed. The band structure tomography is displayed in Supplementary Figs. 162c-k and is compatible with the results presented in the Main Text (see Fig. 4).

Further, we select linecuts of the phase space, indicated by the coloured arrows in Supplementary Figs. 162a,b, along which we perform bias-dependent spectroscopy (Supplementary Fig. 163, to be compared with Fig. 2). Again, we observe strong dispersion anisotropy when comparing linecut γ_1 to γ_2 . Incidentally, while we still note a conductance peak at $V_{\text{SD}} = \pm 155$ μV , whose position in bias does not vary appreciably with γ_i and which is attributed to MAR processes, this device does not reveal a second peak at larger $|V_{\text{SD}}|$. This shows that the peak at $V_{\text{SD}} = \pm 175$ μV in Device 1 is a device-specific feature. Since the ABS dispersion of Device 1 is qualitatively reproduced in Device 2, we corroborate that the peak at ± 175 μV in the former does not interact with the ABSs or affect their main properties.

32.11 Supplementary Note 4: Results for different V_G

The measurements of Device 1 shown in the Main Text were obtained in a single gate configuration (except for the switch gate voltage, set to -1.5 V for Figs. 1e, 2i and to 0 V elsewhere). The main results were reproduced in different gate configurations, defined by setting a new value of V_G and adjusting V_T to maintain a comparable probe transmission, crucially remaining in the tunnelling regime. In Supplementary Figs. 164–169 we show the three cases $V_G = -66$ mV, $V_G = -150$ mV and $V_G = -250$ mV, as described in the following. Remarkably, all the key features discussed in the Main Text were qualitatively reproduced in these configurations, highlighting the generality of our results and further supporting our conclusions. Incidentally, we did not observe a variation in the transmission of the highly transmissive ABSs as a function of the global gate voltage, namely they approached the edges of the transport gap at $|V_{\text{SD}}| \approx 155$ μV very closely for any V_G . This was verified in the range -300 mV $\leq V_G \leq 150$ mV, noting that at $V_G = -300$ mV the sub-gap states were at a limit of visibility but still preserved the same dispersion with phase.

- $V_G = -66$ mV: Supplementary Figs. 164a,b, 164c-k and 165 correspond to Figs. 1d,e, 4

and 2a–h respectively. V_T is adjusted to -1.038 V. Interestingly, we observe that the two ABSs have very similar visibility in this regime (namely, the resonances associated with each state have similar conductance), in contrast to the regime shown in the Main Text ($V_G = 50$ mV), where the ABS dispersing with Φ_L had higher conductance. As a possible explanation, we hypothesise that, when V_G is reduced, the electrostatic potential profile at the tunnelling barrier shifts slightly, in such a manner that its maximum displaces from left to right. At the optimised value $V_G = -66$ mV, transport between the probe and either side of the three-terminal region is symmetric, resulting in similar conductance when tunnelling into either ABS. Furthermore, in the constant-bias planes of Supplementary Fig. 164, and particularly in panel b, we notice an additional set of resonances with the same dependence on I_L and I_R as the left ABS. This is attributed to a very small phase modulation of the MAR peak, i.e., the nearly flat line at $V_{SD} = -155$ μ V that is visible in Supplementary Fig. 165. These peak, particularly prominent in the present regime, accounts for a conductance background in the constant-bias maps and its modulation translates into a phase-periodic background.

- $V_G = -150$ mV: Supplementary Figs. 166a,b, 166c–k and 167 correspond to Figs. 1d,e, 4 and 2 respectively. V_T is adjusted to -1.02 V, where the probe transmission is smaller than in the previous regimes: the differential conductance at large V_{SD} (near the range limits) is reduced by approximately a factor 2. This results in a lower signal-to-noise ratio, due to which periodic noise features become visible in the constant-bias measurements (Supplementary Fig. 166), as long as G is relatively small. At the same time, the conductance resonances at $V_{SD} = \pm 155$ μ V and $V_{SD} = \pm 175$ μ V in the spectra of Supplementary Fig. 167 become substantially less prominent, and so is their modulation as a function of γ_i . Visibly, the ABS dispersion is not distorted when the states cross the horizontal peaks, remaining qualitatively very similar to that observed in Fig. 2. This provides further support to the absence of an interaction between the peaks at $V_{SD} = \pm 155$ μ V, ± 175 μ V and the ABSs. Finally, in this gate configuration, the ABS depending on Φ_R shows the highest conductance, consistent with the argument presented for the case $V_G = -66$ mV.
- $V_G = -250$ mV: Supplementary Figs. 168a,b, 168c–k and 169 correspond to Figs. 1d,e, 4 and 2a,b,d,e,i respectively. V_T is adjusted to -984 mV, where the probe transmission is comparable to the regime $V_G = -150$ mV.

32.12 Supplementary Note 5: Mutual inductance matrix

As described in the Main Text, our devices feature two flux bias lines where currents I_L and I_R are injected. Since I_L generates a magnetic field that is stronger over the left superconducting loop than over the right one, it controls mainly the external magnetic flux Φ_L threading the left loop, associated to the superconducting phase difference between the terminals L and M (see Fig. 1a). Similarly, I_R tunes mostly Φ_R , thus the phase difference between R and M, in such a manner that the combination of the two flux lines enables full phase control over a two-dimensional space. Nevertheless, each flux line has also a finite coupling to the opposite loop, thus Φ_L and Φ_R depend on both I_L and I_R according to the linear relation:

$$\begin{pmatrix} \Phi_L \\ \Phi_R \end{pmatrix} = \mathbf{M} \cdot \begin{pmatrix} I_L \\ I_R \end{pmatrix} = \begin{pmatrix} M_{LL} & M_{LR} \\ M_{RL} & M_{RR} \end{pmatrix} \cdot \begin{pmatrix} I_L \\ I_R \end{pmatrix}, \quad (32.13)$$

where $\mathbf{M}$ is the mutual inductance matrix. We calculate $\mathbf{M}$ from the constant-bias conductance measurement of Fig. 1d, plotted again in Supplementary Fig. 170a. First, we observe that the origin $(I_L, I_R) = (0, 0)$ corresponds to the origin of the flux space $(\Phi_L, \Phi_R) = (0, 0)$. We associate the centres of the adjacent diamond-like regions to the addition or subtraction of one

superconducting flux quantum $\Phi_0 = h/(2e)$ to either Φ_L or Φ_R . By evaluating Eq. 32.13 in two centre points, for example those related to $(\Phi_0, 0)$ and $(0, \Phi_0)$, we write a 4×4 equation system and determine:

$$\mathbf{M} = \begin{pmatrix} 6.96 \text{ pH} & -1.43 \text{ pH} \\ -1.69 \text{ pH} & 5.79 \text{ pH} \end{pmatrix}. \quad (32.14)$$

The ratio $M_{RR}/M_{LL} \approx M_{LR}/M_{RL} \approx 0.8$ approximately corresponds to the ratio between the length of the vertical segments of the right and left flux line. Knowing $\mathbf{M}$, we can apply the linear transformation of Eq. 32.13 to convert the (I_L, I_R) axes to the (Φ_L, Φ_R) axes, as illustrated in Supplementary Fig. 170b for the dataset of panel a.

32.13 Supplementary Note 6: Phase shifts and inductances in the system

In the constant-bias measurements showing the dependence on I_L and I_R with the switch junction ON, such as in Figs. 1d and 4d–i, we remarked the presence of phase shifts occurring when resonances associated to different states (i.e., the Φ_L - and the Φ_R -dispersing ABS) intersect each other. Concomitantly, we noted a slope variation of the Φ_L -dispersing ABS depending on the ON/OFF state of the switch junction (yellow dashed line in Fig. 1e). In this section, we quantify the shifts and relate them to the different inductive contributions in the device. We find that a Josephson-like mutual inductive coupling between the two loops accounts for the presence of both the shifts and the slope difference.

Since a shift is present for both ABSs, it corresponds to a vector $(\Delta\Phi_L, \Delta\Phi_R)$ in the space of the two external magnetic fluxes Φ_L and Φ_R . For the extraction of this vector, we consider again the dataset of Fig. 4e from the Main Text, shown also in Supplementary Fig. 170c and, upon applying the basis transformation described in Section 32.12, in Supplementary Fig. 170d as a function of Φ_L and Φ_R . As a guide for the eye, we overlay lines following the dips between pairs of ABS resonances. In panel c, we mark the (vector) shifts $\Delta\mathbf{I}_L^*$ and $\Delta\mathbf{I}_R^*$ along the periodicity directions, while in panel d the shifts $\Delta\Phi_L$ and $\Delta\Phi_R$ are by construction parallel to the axes. These quantities are linked by the following relations:

$$\begin{pmatrix} \Delta\Phi_L \\ 0 \end{pmatrix} = \mathbf{M} \cdot \Delta\mathbf{I}_L^*, \quad \begin{pmatrix} 0 \\ \Delta\Phi_R \end{pmatrix} = \mathbf{M} \cdot \Delta\mathbf{I}_R^*. \quad (32.15)$$

From the data, we find $\Delta\Phi_L \approx \Delta\Phi_R \approx 0.092 \cdot \Phi_0 \equiv \Delta\Phi$, showing that the effect is symmetric for the two ABSs.

This phase shift is then expressed as $\Delta\Phi = M_{\text{cpl}} \cdot I_{\text{circ}}$, namely as the product between a mutual inductance, accounting for the coupling between two fluxes, and a circulating current in either loop. Neither M_{cpl} nor I_{circ} can be directly determined from the experimental data (only their product $\Delta\Phi$), therefore we estimate the inductive coupling term considering different origins. The geometric mutual inductance between the two loops of our devices is $M_{\text{geom}} \approx 7.3 \text{ pH}$ [64]. Since the loops share a strip of epitaxial Al, its inductance, dominated by the kinetic inductance L_k , has to be added to M_{geom} [65, 66]. The kinetic inductance of the strip is estimated as [67]:

$$L_k = \frac{l}{w} \frac{h}{2\pi^2} \frac{R_{\square}}{\Delta} \approx 44 \text{ pH}, \quad (32.16)$$

where $l = 25 \text{ }\mu\text{m}$ and $w = 1 \text{ }\mu\text{m}$ are the length and width of the strip, $R_{\square} \approx 1.5 \text{ }\Omega$ the normal state resistivity of the heterostructure stack measured in a Hall bar geometry (where the Al film was not removed) and $\Delta \approx 180 \text{ }\mu\text{eV}$ the superconducting gap of Al. The geometric and kinetic contribution lead to a combined mutual inductance $M_{\text{loops}} \approx 51.3 \text{ pH}$. If we assume that $M_{\text{cpl}} = M_{\text{loops}}$, we require a circulating current $I_{\text{circ}} \approx 3.7 \text{ }\mu\text{A}$ for the shift $\Delta\Phi = 0.092 \cdot \Phi_0$. However, this value is much larger than any reasonable estimate of the critical current between

each pair of terminals, which constitutes an upper bound to the supercurrent circulating in the loops. As a consequence, M_{cpl} must be substantially larger than M_{loops} to limit I_{circ} .

The only contribution that we have not discussed so far is a Josephson-like coupling due to the 3TJJ. Based on its geometry, we expect our 3TJJ to have a critical current of the order of 100 nA and a Josephson inductance of a few nH, which would explain the phase shift to a good degree. We deduce that, in our devices, the phase shifts derive from a Josephson-like coupling term rather than the geometric and kinetic inductive couplings, in analogy with existing couplers between superconducting qubits containing Josephson elements [50, 51].

This mutual coupling between Φ_L and Φ_R is also responsible for the slope difference depending on the state of the switch junction (see Fig. 1d,e). In the OFF state, no circulating current can flow in the right loop, thus no mutual effect is present. In the (I_L, I_R) plane (Fig. 1e of the Main Text), the ABS resonance is parallel to the Φ_R axis (indicated in Fig. 1d). When the switch junction is ON, the slope of the ABSs deviates from the directions of Φ_L and Φ_R . This is clearly visible on the (Φ_L, Φ_R) axes (Supplementary Fig. 170b,d), as the resonances are not vertical or horizontal. In fact, due to the mutual coupling, the variation of a flux (for example Φ_L) from 0 to $\Phi_0/2$ causes a gradual change in the other (Φ_R), up to a maximum shift. Upon crossing $\Phi_L = \Phi_0/2$, the circulating current in the left loop reverses direction, consequently the induced current in the right loop and the Φ_R -shift also flip sign. Therefore, the finite ABS slopes on the (Φ_L, Φ_R) axes (corresponding to the slope difference depending on the state of the switch junction) and the phase shifts are consistent. Both result from the finite Josephson-like mutual inductive coupling between the two loops.

32.14 Supplementary Figures

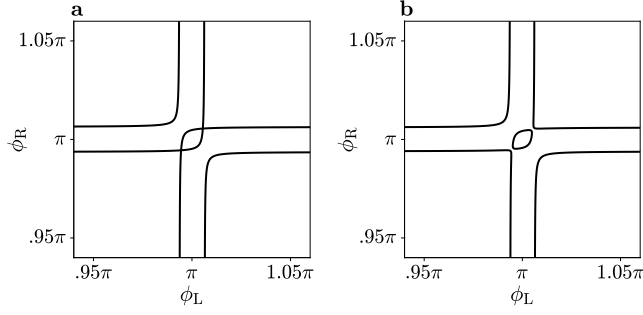


Figure 158: **Analytical calculations.** **a,b**, Constant-energy planes at $E = -0.01\Delta$ as a function of the superconducting phase differences ϕ_L and ϕ_R for coupling $\delta = 0.01$, in the fully ballistic regime $r_L = r_R = 0$ (**a**) and with finite normal reflection amplitudes $r_L = 0.003$, $r_R = 0.002$ (**b**).

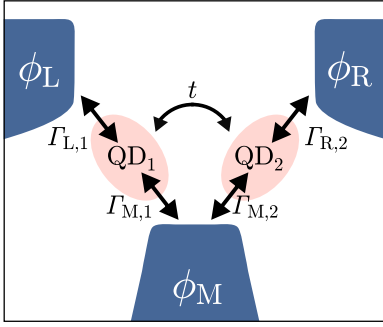


Figure 159: **Schematic of the numerical model.** Andreev bound states in the three-terminal Josephson junction are modelled as two single-level quantum dots (QDs) coupled to three superconducting leads with phases ϕ_L , ϕ_R and ϕ_M . $\text{QD}_{1(2)}$ is coupled to leads L (R) and M, as represented by the parameters $\Gamma_{L,1}$ and $\Gamma_{M,1}$ ($\Gamma_{R,2}$ and $\Gamma_{M,2}$) respectively. The parameter t accounts for the coupling between the QDs, enabling their hybridisation.

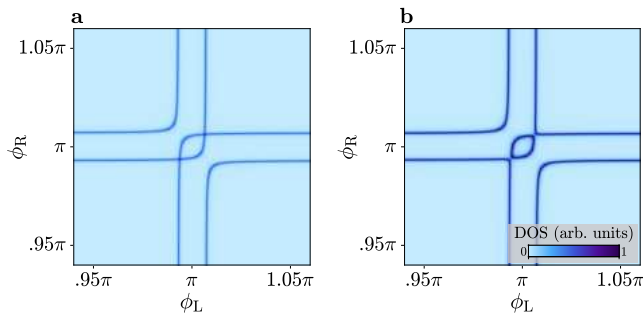


Figure 160: **Numerical simulations.** **a,b**, Density of states at fixed energy $E = -0.01\Delta$ as a function of the superconducting phase differences $\phi_L - \phi_M \equiv \phi_L$ and $\phi_R - \phi_M \equiv \phi_R$, for coupling energy $t = 0.05\Delta$ and broadening $\eta = 5 \cdot 10^{-4}\Delta$. In **a**, $\Gamma_{M,1} = \Gamma_{M,2} = \Gamma_{L,1} = \Gamma_{R,2} = 5\Delta$, namely perfect transmissions are assumed. In **b**, $\Gamma_{L,1} = 4.98\Delta$, $\Gamma_{M,1} = \Gamma_{R,2} = 5\Delta$ and $\Gamma_{M,2} = 4.97\Delta$, hence finite reflection amplitudes are considered. In both cases, the effective coupling $t/\Gamma \approx 0.01$ is similar to that introduced for the analytical case (see Fig. 158).

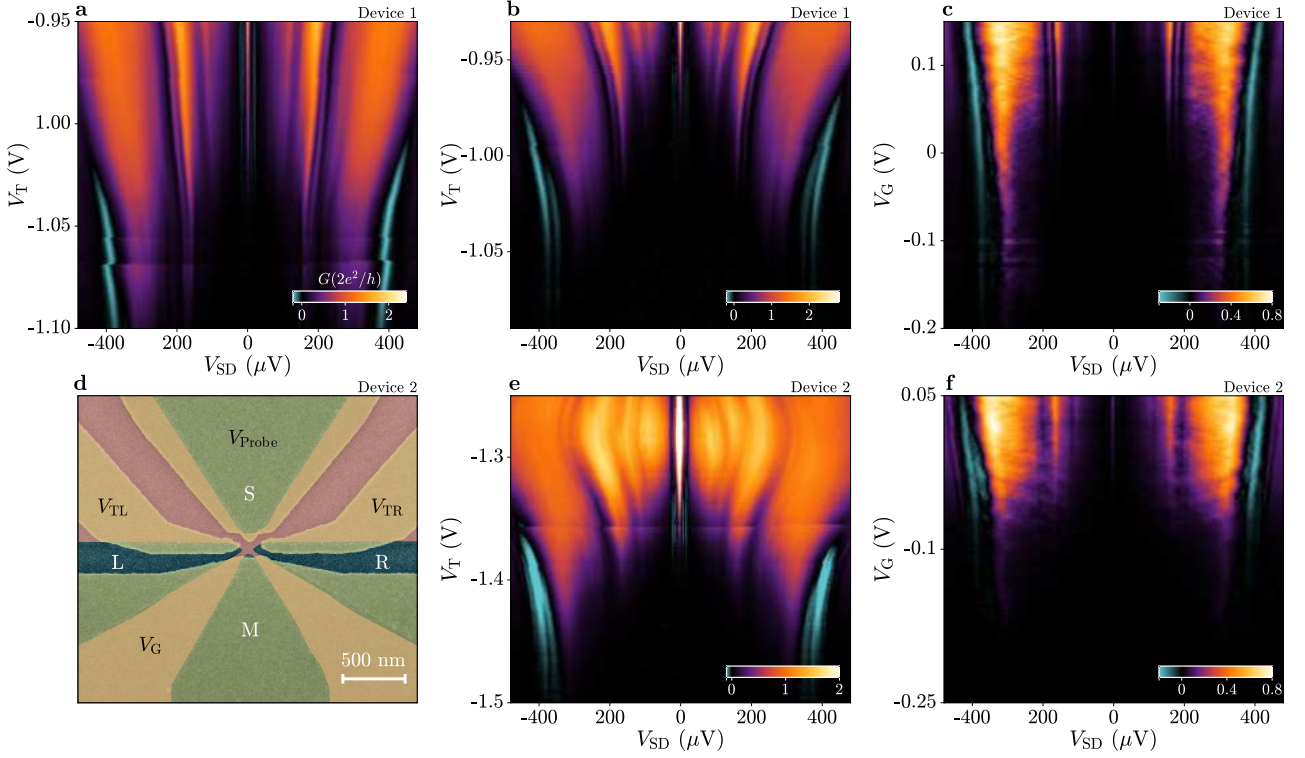


Figure 161: **Gate dependence of differential conductance.** **a,b**, Differential conductance G as a function of voltage bias V_{SD} and tunnelling gates voltage $V_T \equiv V_{TL} = V_{TR}$ in Device 1, for global gate voltage $V_G = 50$ mV in **a** and $V_G = -150$ mV in **b**. **c**, G as a function of V_{SD} and V_G in Device 1, for $V_T = -1.07$ V. **d**, False-coloured scanning electron micrograph of Device 2 in proximity of the three-terminal junction region (see Fig. 1a–c of the Main Text for the colour legend). **e**, G as a function of V_{SD} and V_T in Device 2, for $V_G = -150$ mV and $V_{Probe} = 100$ mV. **f**, G as a function of V_{SD} and V_G in Device 2, for $V_T = -1.5$ V and $V_{Probe} = 100$ mV.

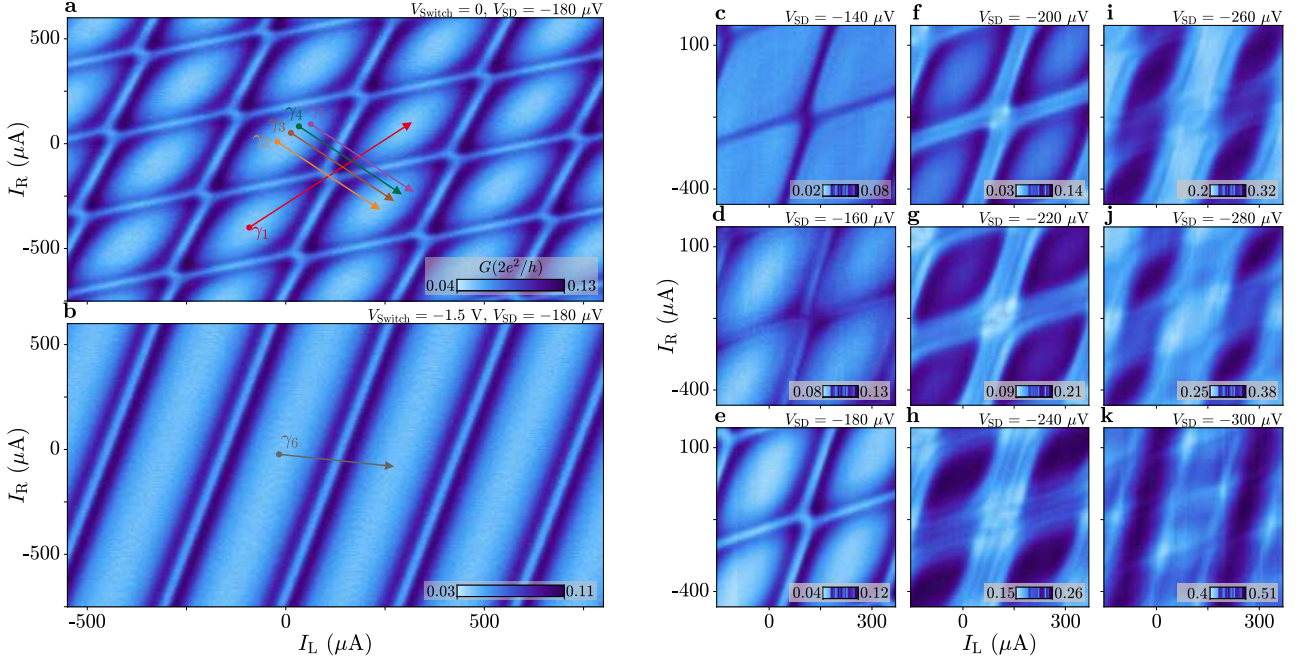


Figure 162: **Constant-bias planes as a function of the two phases in Device 2.** **a–k**, Differential conductance G as a function of the currents I_L and I_R injected into the flux-bias lines at fixed values of voltage bias V_{SD} . **a**, $V_{\text{Switch}} = 0$, $V_{SD} = -180 \mu\text{V}$. **b**, $V_{\text{Switch}} = -1.5 \text{ V}$, $V_{SD} = -180 \mu\text{V}$. **c–k**, $V_{\text{Switch}} = 0$, V_{SD} varied between $-140 \mu\text{V}$ (**c**) and $-300 \mu\text{V}$ (**k**).

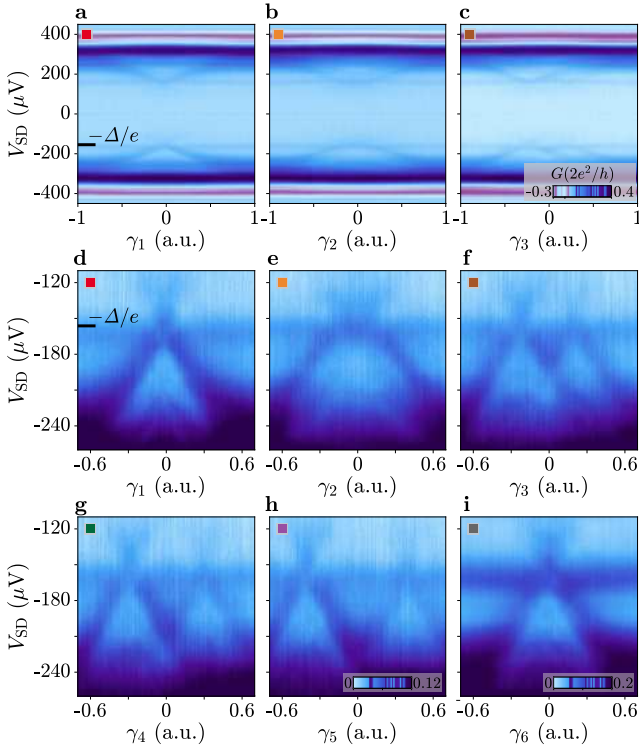


Figure 163: **Tunnelling conductance along phase space linecuts in Device 2.** **a–c**, Differential conductance G as a function of voltage bias V_{SD} along the linecuts γ_i (coloured arrows in Fig. 162a), for $V_{\text{Switch}} = 0$. **d–h**, As **a–c**, but plotted over restricted ranges of V_{SD} and γ_i . **i**, As **d–h**, but along linecut γ_6 (defined in Fig. 163), for $V_{\text{Switch}} = -1.5 \text{ V}$. The colourbar in **h** applies to **d–h**.

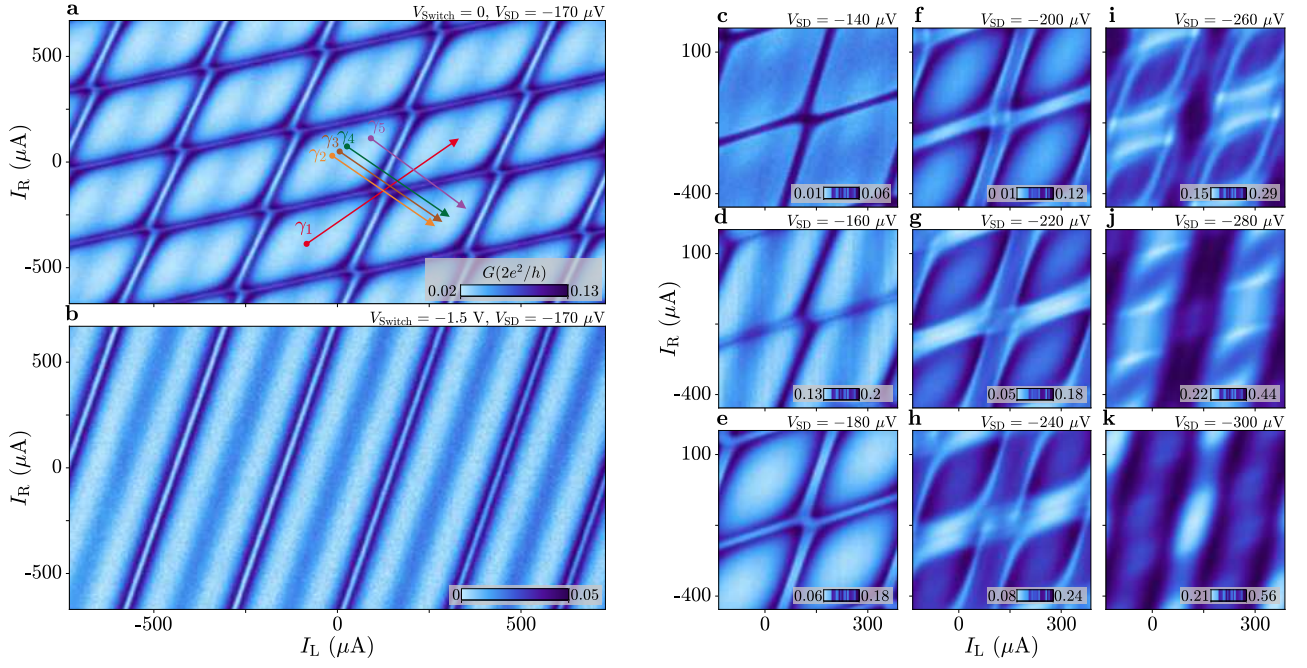


Figure 164: **Constant-bias planes as a function of the two phases at $V_G = -66 \text{ mV}$.** **a–k**, Differential conductance G as a function of the currents I_L and I_R injected into the flux-bias lines at fixed values of voltage bias V_{SD} . **a**, $V_{\text{Switch}} = 0$, $V_{\text{SD}} = -170 \mu\text{V}$. **b**, $V_{\text{Switch}} = -1.5 \text{ V}$, $V_{\text{SD}} = -170 \mu\text{V}$. **c–k**, $V_{\text{Switch}} = 0$, varying V_{SD} .

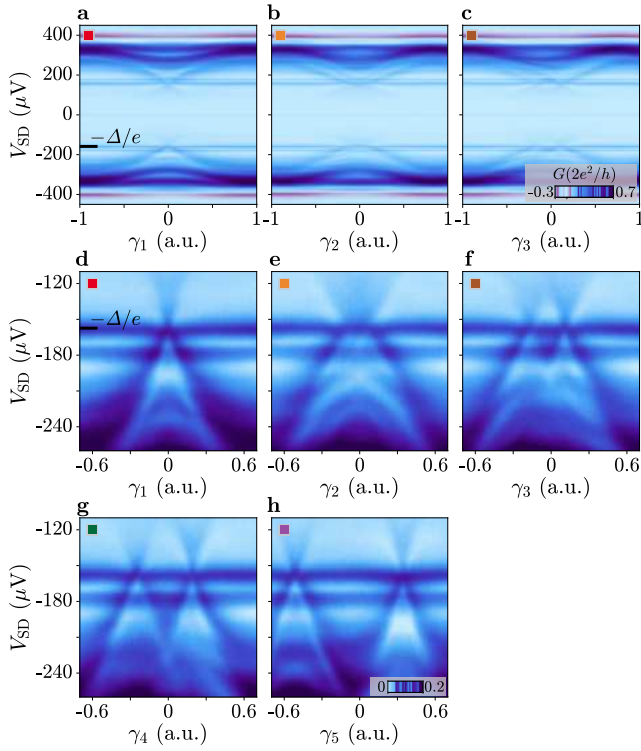


Figure 165: **Tunnelling conductance along phase space linecuts at $V_G = -66 \text{ mV}$.** **a–c**, Differential conductance G as a function of voltage bias V_{SD} along the linecuts γ_i (coloured arrows in Fig. 164a), for $V_{\text{Switch}} = 0$. **d–h**, As **a–c**, but plotted over restricted ranges of V_{SD} and γ_i .

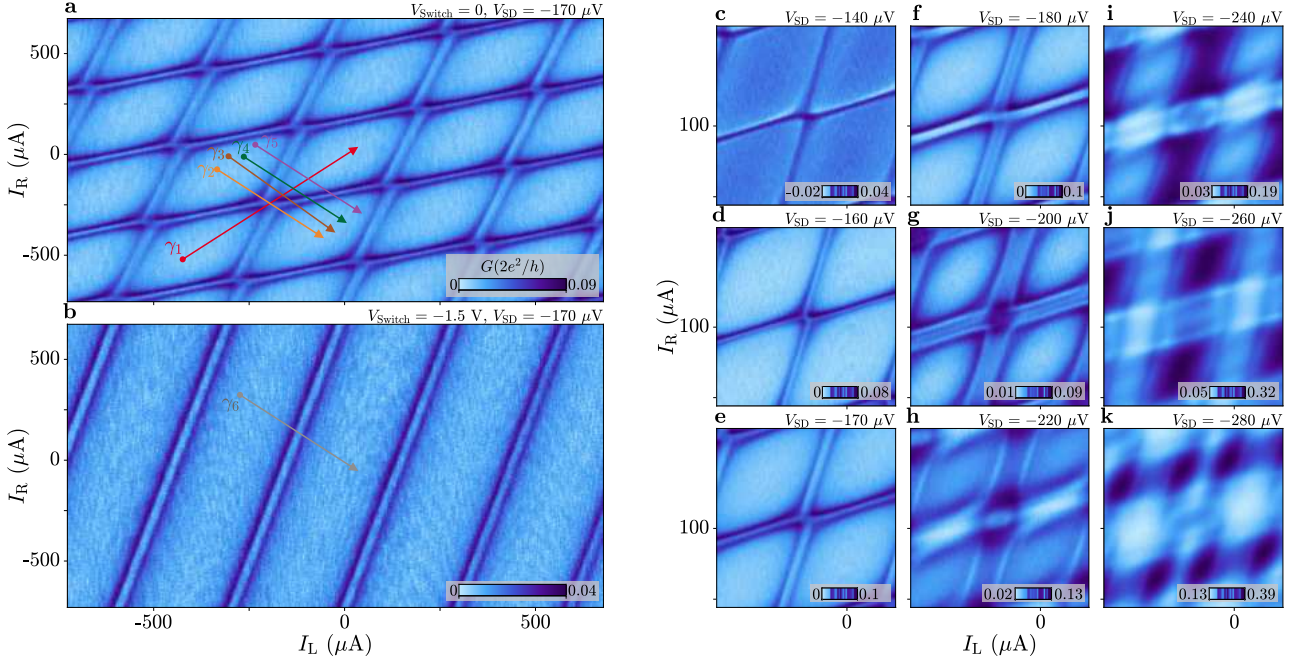


Figure 166: **Constant-bias planes as a function of the two phases at $V_G = -150$ mV.** **a–k**, Differential conductance G as a function of the currents I_L and I_R injected into the flux-bias lines at fixed values of voltage bias V_{SD} . **a**, $V_{\text{Switch}} = 0$, $V_{SD} = -170$ μV . **b**, $V_{\text{Switch}} = -1.5$ V, $V_{SD} = -170$ μV . **c–k**, $V_{\text{Switch}} = 0$, varying V_{SD} .

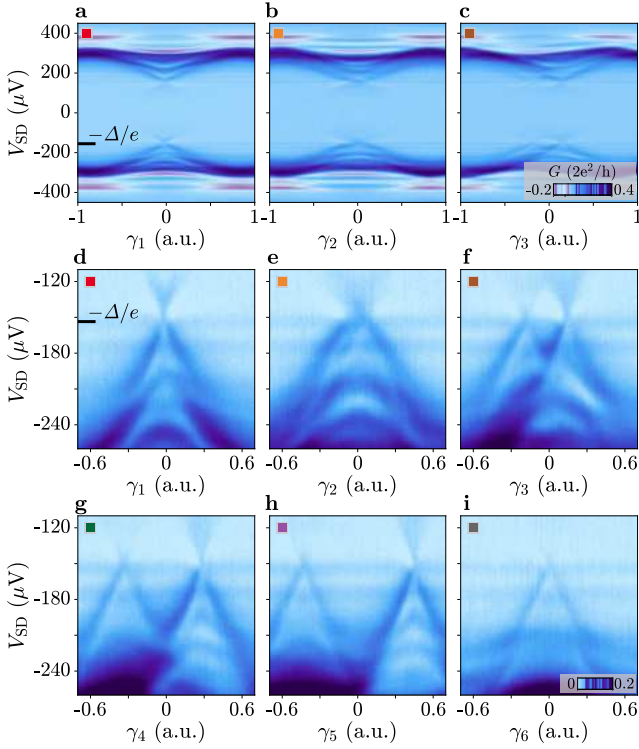


Figure 167: **Tunnelling conductance along phase space linecuts at $V_G = -150$ mV.** **a–c**, Differential conductance G as a function of voltage bias V_{SD} along the linecuts γ_i (coloured arrows in Fig. 166a), for $V_{\text{Switch}} = 0$. **d–h**, As **a–c**, but plotted over restricted ranges of V_{SD} and γ_i . **i**, As **d–h**, but along linecut γ_6 (defined in Fig. 166b), for $V_{\text{Switch}} = -1.5$ V.

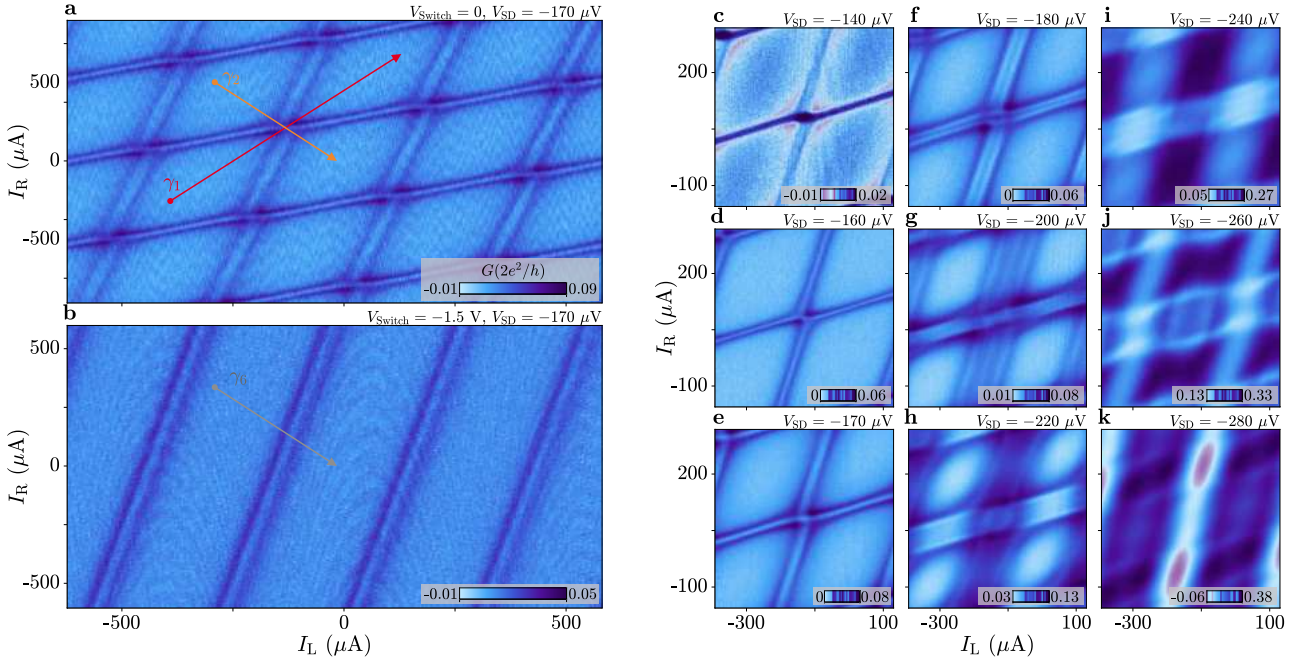


Figure 168: **Constant-bias planes as a function of the two phases at $V_G = -250$ mV.** **a–k**, Differential conductance G as a function of the currents I_L and I_R injected into the flux-bias lines at fixed values of voltage bias V_{SD} . **a**, $V_{\text{Switch}} = 0$, $V_{SD} = -170 \mu\text{V}$. **b**, $V_{\text{Switch}} = -1.5$ V, $V_{SD} = -170 \mu\text{V}$. **c–k**, $V_{\text{Switch}} = 0$, varying V_{SD} .

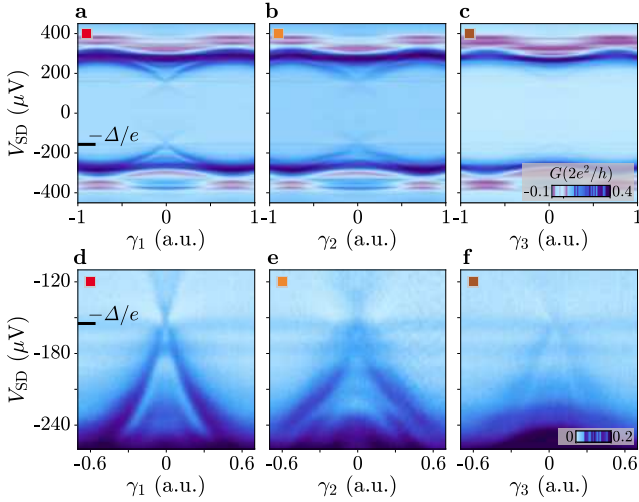


Figure 169: **Tunnelling conductance along phase space linecuts at $V_G = -250$ mV.** **a–c**, Differential conductance G as a function of voltage bias V_{SD} along the linecuts γ_i (coloured arrows in Fig. 168a,b). $V_{\text{Switch}} = 0$ in **a**, **b** and $V_{\text{Switch}} = -1.5$ V in **c**. **d–f**, As **a–c**, but plotted over restricted ranges of V_{SD} and γ_i .

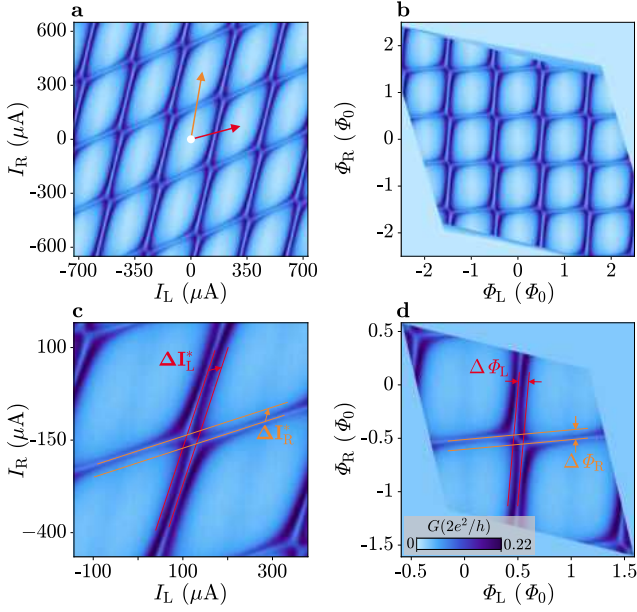


Figure 170: **Current-to-flux remapping and phase shift analysis.** **a**, Differential conductance G as a function of the currents I_L and I_R injected into the flux-bias lines at fixed voltage bias $V_{SD} = -170 \mu\text{V}$ (repetition of Fig. 1d from the Main Text). The red and orange arrows indicate the periodicity axes, namely the directions of the external magnetic flux axes Φ_L and Φ_R . The white dot and the end points of the red and orange arrows correspond to the (Φ_L, Φ_R) points $(0, 0)$, $(\Phi_0, 0)$ and $(0, \Phi_0)$, respectively. **b**, As **a**, but remapping the (I_L, I_R) axes to the (Φ_L, Φ_R) axes by using Eqs. 32.13 and 32.14 (see text for details). **c**, G as a function of I_L and I_R at $V_{SD} = -165 \mu\text{V}$ (repetition of Fig. 4e from the Main Text). The red (orange) lines follow the middle dip between the two resonances of the $\Phi_{L(R)}$ -dependent Andreev bound state, on both sides of the intersection point. The $\Phi_{L(R)}$ shift is indicated by the vector $\Delta\mathbf{I}_{L(R)}^*$, whose direction is parallel to the $\Phi_{L(R)}$ axis on the (I_L, I_R) plane (see **a**). **d**, As **c**, but remapping the (I_L, I_R) axes to the (Φ_L, Φ_R) axes. The phase shifts $\Delta\Phi_L$ and $\Delta\Phi_R$, by construction parallel to the axes, are indicated by the arrows.

33 Scanning Josephson spectroscopy on the atomic scale by Randeria, Feldman, Drozdov, Yazdani

Abstract

The Josephson effect provides a direct method to probe the strength of the pairing interaction in superconductors. By measuring the phase fluctuating Josephson current between a superconducting tip of a scanning tunneling microscope (STM) and a BCS superconductor with isolated magnetic adatoms on its surface, we demonstrate that the spatial variation of the pairing order parameter can be characterized on the atomic scale. This system provides an example where the local pairing potential suppression is not directly reflected in the spectra measured via quasiparticle tunneling. Spectroscopy with such superconducting tips also show signatures of previously unexplored Andreev processes through individual impurity-bound Shiba states. The atomic resolution achieved here establishes scanning Josephson spectroscopy as a promising technique for the study of novel superconducting phases.

A number of novel superconducting states of matter such as those appearing in disordered superconductors, heavy fermion materials, and high- T_c superconductors have been predicted to have pairing order parameters that are spatially modulated on atomic length scales. These short range spatial modulations can occur due to different mechanisms such as the inhomogeneous material properties in disordered superconductors [1, 2, 3, 4], a momentum dependent pairing interaction, such as the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state proposed for heavy fermion materials [5, 6, 7], or the interplay between different forms of electronic ordering in the pair density waves proposed for high- T_c cuprates [8, 9, 10]. Although spectroscopic mapping with a scanning tunneling microscope (STM) can provide evidence for variations in the local density of states (LDOS) through quasi-particle tunneling, such measurements probe the superconducting order parameter only indirectly. If the Josephson effect can be measured and mapped on the atomic scale, then it would allow for direct characterization of the local pairing order parameter and high-resolution studies of novel superconducting phases [11].

This goal has motivated previous efforts in the use of superconducting tips in STM [12] and has led to the local observation of thermal phase fluctuating Josephson supercurrent close to the point contact regime [13, 14, 15]. Subsequent measurements have mapped the Josephson effect on the nanometer scale, applying this technique to vortices [16, 17] and high- T_c cuprates [18, 19]. A major challenge in improving the resolution of these experiments has been satisfying the competing requirements of a high junction impedance necessary for imaging and a low junction impedance allowing for the strong tip-sample coupling necessary to observe the Josephson effect despite thermal fluctuations. Extending the Josephson STM measurements to millikelvin temperatures allows for mapping of the Cooper pair current at junction resistances that are compatible with atomic resolution imaging.

In this letter, we use scanning Josephson spectroscopy to probe variations of the superconducting order parameter on the scale of a single atom. We map the strength of the phase fluctuating Josephson current between a superconducting Pb tip and a Pb(110) surface with a dilute concentration of magnetic impurities using a dilution refrigerator STM system. By modeling the interaction of the electromagnetic environment with our Josephson STM setup, we can understand the spectroscopic data taken with a superconducting tip, including the signatures of photon-assisted inelastic Cooper pair tunneling at non-zero bias. Our measurements show a 10-15% reduction of the Josephson critical current I_c over a few Angstrom length scale in the vicinity of the magnetic adatoms, thereby demonstrating a local suppression of the order parameter. We do not observe a commensurate shift in the coherence peak energies in the single particle spectrum, which is consistent with theoretical calculations [20, 21], highlighting the

ability of the Josephson STM technique to probe physics inaccessible through traditional STM quasiparticle tunneling. Additionally, spatially resolved spectroscopy with a superconducting tip allows us to detect novel Andreev tunneling processes through impurity-bound Shiba states of the individual adatoms.

Our measurements have been carried out using a home-built dilution refrigerator STM system, with a base temperature of 20 mK and a spectroscopic resolution that corresponds to an effective electron temperature of 250 mK [22]. For the present experiments, we used a Pb(110) single crystal that was prepared in-situ with several cycles of Ar sputtering and annealing to produce an atomically ordered flat surface. Fig. 171(a) shows spectroscopic measurements of the atomic Pb(110) surface [inset of Fig. 171(a)] measured using a normal W tip at base temperature. The two coherence peaks in the spectrum are indicative of two pairing gaps in bulk Pb associated with two different Fermi surfaces. As shown in this figure, the data can be modeled using a sum of two BCS densities of states with corresponding gaps, $\Delta_1 = 1.26$ meV and $\Delta_2 = 1.42$ meV, including an energy broadening associated with finite temperature (250 mK) and a quasi-particle lifetime ($8 \mu\text{eV}$). Although the two gaps in Pb have been previously detected in planar junctions [23, 24] and in STM studies using superconducting tips [25], our ability to resolve them with a normal tip demonstrates the high energy resolution afforded by the low temperature operation of our system.

To create a Josephson junction in our STM setup, we prepare a superconducting Pb tip by indenting a W tip into the Pb(110) substrate until the spectra measured with such a tip exhibit features indicative of quasi-particle tunneling between two superconductors [12, 25]. As seen in Fig. 171(b), the tunneling spectra with such tips show sharp coherence peaks at voltages corresponding to the sum of the tip and substrate superconducting gaps, with a tip gap Δ_{tip} ranging from 1.3-1.4 meV (approximately that of bulk Pb), depending on the tip. We do not resolve additional structure in the coherence peaks arising from multiple gaps in the tip, which suggests that the superconducting apex of the tip is amorphous in nature, consistent with previous measurements [25]. For simplicity, we assume below that the gap of the tip and sample are the same and define $\Delta_{\text{Pb}} \simeq (\Delta_1 + \Delta_2)/2 \simeq \Delta_{\text{tip}} \approx 1.35$ meV.

While spectroscopy at high junction resistances with the Pb tips show only signatures of quasi-particle tunneling, even at junction resistances of about $1 \text{ M}\Omega$, we begin to resolve features associated with Andreev reflections and Cooper pair tunneling between the tip and the sample [Fig. 171(c)]. Successively decreasing the junction impedance highlights the evolution of distinct features at the characteristic energy of $\pm\Delta_{\text{Pb}}$, arising from Andreev processes in the STM junction [26] as illustrated in the inset of Fig. 171(c). The prominent peak in conductance at zero bias corresponds to the IV characteristic shown in Fig. 171(d), where the maximum Cooper pair current I_{max} occurs at a voltage near zero bias ($V_p \approx 40 \mu\text{V}$) due to a phase fluctuating Josephson supercurrent, discussed in more detail below. Moreover, we resolve periodic features in both current and conductance that appear at remarkably regular voltage intervals ($95 \mu\text{eV}$), which are related to the interaction of the STM Josephson junction with its electromagnetic environment [Fig. 171(e)]. Finally, we also note that at low junction impedances, the large injection of quasi-particles results in an increased broadening of the coherence peaks compared to spectra at higher junction resistance [Fig. 171(c)].

We can understand the spectroscopic features of our STM Josephson junction by comparing the magnitude of the three relevant energy scales: (i) the Josephson coupling energy between the tip and sample, $E_J = (\Delta_{\text{Pb}}/R_N) \cdot \pi\hbar/(4e^2) = \hbar/(2e) \cdot I_c$, which depends on the normal state junction resistance R_N and the pairing gap Δ_{Pb} , (ii) the thermal energy corresponding to our electron temperature, $k_B T = 22 \mu\text{eV}$, and (iii) the charging energy $E_C = (2e)^2/2C$, which depends on the capacitance C of the junction. For our STM junction with a typical $C \approx 4 \text{ fF}$, most measurements are performed in a regime where the charging energy dominates the behavior of the Josephson junction, $E_J \lesssim k_B T \ll E_C$. In this case, the thermal fluctuation of

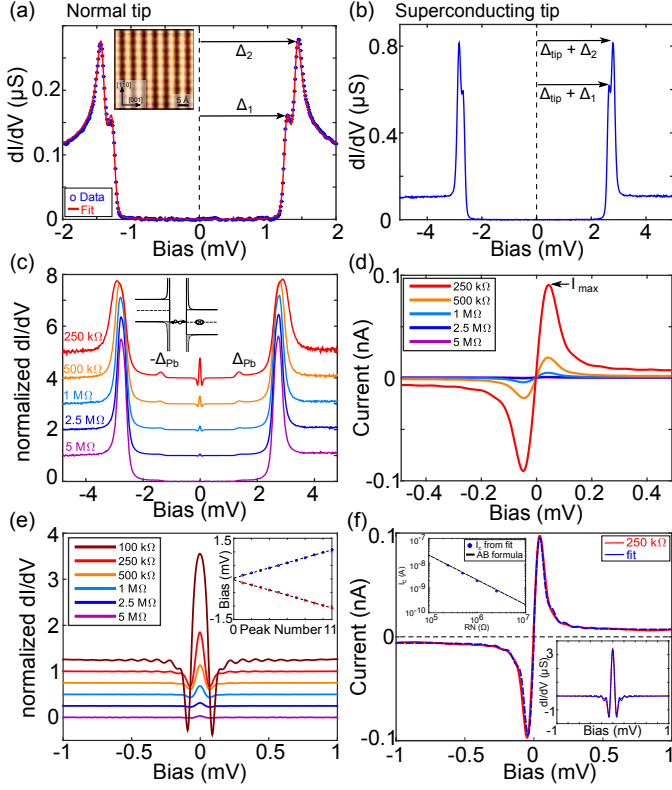


Figure 171: (a) dI/dV of Pb(110) with a normal tip resolving two superconducting gaps. Inset: atomic scale topography of the Pb(110) surface. (b) dI/dV of Pb(110) with a superconducting Pb tip. (c) Superconducting tip spectroscopy on Pb(110), normalized to R_N . Curves are offset for clarity. As junction resistance is decreased, features due to Andreev reflections ($V = \pm\Delta_{Pb}$) and the Josephson effect ($V = 0$) become more pronounced. Due to the quality of the tip, the two superconducting gaps of Pb are not clearly resolved. Inset: schematic of an Andreev process between two superconductors with the same gap, occurring at a threshold $e|V| = \Delta_{Pb}$. (d) IV characteristics of STM Josephson junction around zero bias, for different normal state resistance R_N . (e) Oscillations in dI/dV (offset for clarity, normalized to R_N) due to photon-assisted Cooper pair tunneling with characteristic frequency of $\nu \approx 23$ GHz, determined by fitting the bias of the peaks as a function of oscillation number (inset). (f) Fit to IV and dI/dV (bottom right inset) data for $R_N = 250$ k Ω using the P(E) theory. Top left inset: Values of critical current extracted from P(E) fits (blue dots), in agreement with the Ambegaokar-Baratoff formula (black). All point spectra in this figure were acquired at a parking bias of $V = -5$ mV at which the normal state junction resistance was determined (well outside the superconducting gap).

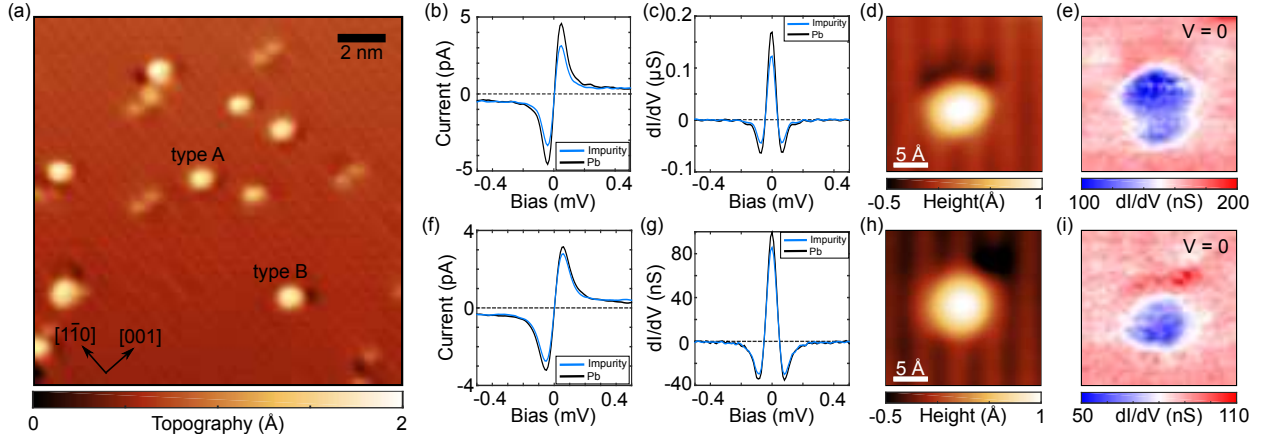


Figure 172: (a) Topography of Pb after evaporating a submonolayer of Fe. Individual Fe adatoms tend to be centered on a trench (type A) or row (type B) of the Pb(110) surface as marked. Suppression of pair current [(b),(f)] and dI/dV [(c),(g)] on the Fe adatom (blue) compared to on Pb (black) at 1 M Ω junction impedance for type A [(b)-(e)] and type B [(f)-(i)] adatoms. Simultaneous topographies [(d),(h)] and differential conductance maps [(e),(i)] at $V = 0$ for $R_N = 1$ M Ω demonstrate the spatial suppression of the Josephson effect over the Fe adatoms. Parking conditions $V = -5$ mV and $I = 5$ nA for data in (b)-(i), ensuring that R_N is the same over both the adatom and bare Pb. Different superconducting tips were used for the point spectra and the conductance maps, but the I_c suppression is consistent across measurements.

the phase across the junction are enhanced by quantum fluctuations, which together result in a shift of the Josephson pair current to non-zero bias [27], as we observe in our data shown in Fig. 171(d). The oscillations in the conductance at higher bias [Fig. 171(e)] arise from the interaction between such a phase incoherent Josephson junction and the standing electromagnetic modes associated with the STM tip. Photon absorption or emission with energy $E = \hbar\nu = 2eV$ facilitate Cooper pair tunneling across the STM junction at non-zero bias, a signature similar to the AC Josephson effect [28, 29]. Treating our STM setup as an open-ended $\lambda/4$ antenna with resonances at $\nu_n = (2n + 1) \cdot c/l$ [28] allows us to fit the voltage spacing of the oscillations to extract a characteristic frequency $\nu_0 \approx 23$ GHz, which is intrinsic to our instrument, corresponding to an approximate tip length $l = 3.3$ mm. The high quality factor of these oscillations highlight the sensitivity of the STM Josephson junction to its electromagnetic environment.

A quantitative understanding of our Josephson STM characteristics can be obtained by using the P(E) theory to model the probability of Cooper pair tunneling across the junction mediated by its electromagnetic environment [30, 27]. The energy exchange with the environment as well as thermal effects contribute to inelastic pair tunneling across the junction. As shown in the fit in Fig. 171(f), we can accurately capture the particular shape of our spectra using this theory, the details of which are discussed in the Supplementary Materials. Within this model, both the maximal phase-fluctuating pair current I_{max} and the differential conductance at zero bias $dI/dV(V = 0)$ are proportional to the square of the intrinsic Josephson critical current, I_c^2 [30, 27]. The values that we obtain for the intrinsic I_c as a function of the normal state junction resistance R_N are in good agreement with calculations from the Ambegaokar-Baratoff formula, $I_c = \pi/(2e) \cdot \Delta_{Pb}/R_N$ [31], as shown in the top inset of Fig. 171(f), justifying the use of this model to fit the data. Although the small capacitance of a typical STM junction makes such Josephson junctions phase incoherent, mapping of the phase-fluctuating pair current at low bias in the STM setup still provides a direct method for the spatial characterization of the

pairing amplitude in a superconducting sample.

To demonstrate that the Josephson STM technique can probe variations of the order parameter on the atomic scale, we investigate individual magnetic atoms deposited on the surface of our Pb(110) substrate. Magnetic impurities on an s-wave superconductor are the simplest example of pair-breaking defects that are well known to suppress superconductivity, for instance seen in the suppression of T_c with increasing impurity concentration [32, 33]. Previous STM spectroscopy has used quasiparticle tunneling to show that magnetic atoms on a BCS superconductor induce in-gap Shiba states [34, 35, 36, 37, 38]; however such LDOS measurements do not directly probe the superconducting order parameter. To probe the spatial variation of the pairing strength for this model system, we deposit a sub-monolayer of Fe adatoms on Pb(110) in situ, at a temperature of about 20 K. As shown in a typical STM topography [Fig. 172(a)], this low temperature deposition results in the appearance of features with Angstrom height, which are consistent with individual Fe atoms residing in two different atomic sub-lattice binding sites (types A and B) on the Pb(110) substrate [Figs. 172(d),(h)].

To probe the pairing amplitude near these magnetic Fe atoms, we perform spectroscopic measurements using superconducting Pb tips at junction resistances that are low enough to detect the phase fluctuating Josephson current but high enough to perform STM imaging without disrupting the adatoms on the surface. As shown in Figs. 172(b),(f), IV measurements of the two different types of Fe adatoms sites show a suppression of the peak in the pair current I_{max} near zero bias as compared to the bare substrate, also seen in an analogous measurement of differential conductance over the magnetic impurities [Figs. 172(c),(g)]. This reduction of the phase-fluctuating pair current and conductance is a direct signature of the local suppression of the pairing amplitude caused by the magnetic impurities, corresponding to a 10-15% reduction of the critical current I_c on an Fe adatom based on fits to the spectra (see Supplementary Materials). The precise value of I_{max} and the zero bias conductance depend on the superconducting tips and the strength of the exchange coupling between the impurity and underlying Pb substrate (as is evident from the difference between type A and B adatoms), but the suppression of the intrinsic critical current is consistent across dozens of adatoms measured with various different Pb tips.

Another measurement that reflects the local pairing suppression is mapping the differential conductance at zero bias in the vicinity of the Fe defects to probe the spatial variations of the order parameter. These measurements, shown in Figs. 172(e),(i), demonstrate our key finding that this Josephson STM technique can resolve spatial variations of the pairing amplitude on the atomic scale. Data from additional adatoms showing a suppression of the order parameter, as well as control experiments that rule out artifacts of tip height variations are detailed in the Supplementary Materials. Our observation that the pairing amplitude recovers back to its unperturbed value for the Pb within Angstroms of the magnetic adatoms, and not on the scale of the coherence length is consistent with theoretical predictions [39, 40]. Theory also predicts an oscillating power-law for the order parameter, $\Delta(r) \sim \sin^2(k_F r)/(k_F r)^2$ as a function of the distance r from a magnetic impurity, where k_F is the Fermi wavevector [40]. However, both the adatom geometry on the surface of a 3D superconductor and the short Fermi wavelength in Pb give rise to the short-range decay of the order parameter, making it difficult to detect these oscillations in the current experiment.

In contrast, the suppression of the local order parameter is not prominently seen in quasiparticle tunneling measurements over the magnetic impurity. Superconducting tip spectra [Fig. 173(a)] over an Fe adatom show a suppression in the intensity of the coherence peaks with no appreciable shift in their energy as compared to the bare substrate. In the presence of bound states, the spectral weight is redistributed from the coherence peaks to the bound state energies, thus the suppression of the local pairing amplitude does not translate into a shift in the energies of the coherence peaks. Self-consistent calculations confirm this distinction

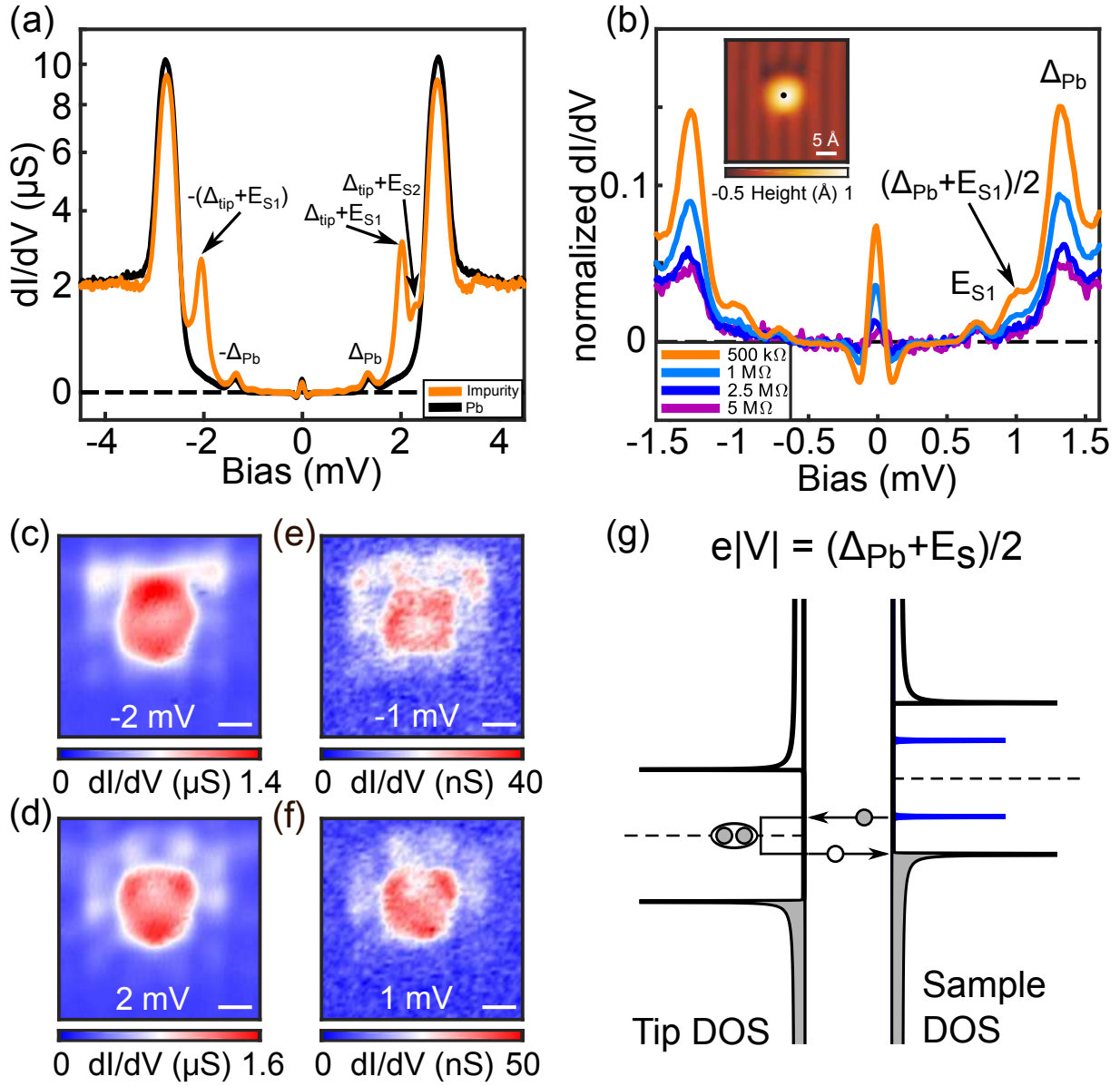


Figure 173: (a) dI/dV with a superconducting tip ($V = -5$ mV, $I = 10$ nA) on a type A impurity, plotted on a log scale, showing the appearance of Shiba states at $\Delta_{Pb} + E_{S1} = \pm 2.06$ meV and $\Delta_{Pb} + E_{S2} = +2.28$ meV (marked by arrows). (b) A zoom-in of dI/dV at lower junction resistances highlights multiple Andreev reflections through the Shiba state observed at energies of E_{S1} and $(\Delta_{Pb} + E_{S1})/2$. Parking bias $V = -5$ mV for all spectra. [(c)-(f)] Conductance maps at $eV = \pm(\Delta_{Pb} + E_{S1})$ and $eV = \pm(\Delta_{Pb} + E_{S1})/2$ taken simultaneously with topograph in inset of (b). Corresponding spatial patterns between the Shiba state ($V = 2$ mV) and the Andreev reflection state ($V = 1$ mV), and similarly for states at negative bias are observed. Scale bar is 5 $\text{\AA}$. (g) Schematic for Andreev reflection process with threshold voltage $eV = \pm(\Delta_{Pb} + E_{S1})/2$, where the relevant energy gap is that of the sample.

between measurements of the order parameter and the local density of states [20, 21]. Thus, our measurements demonstrate the utility of scanning Josephson spectroscopy in directly extracting local variations of the superconducting order parameter.

Spectroscopic measurements with a superconducting tip can also be used to probe previously unexplored Andreev reflection processes through the in-gap Shiba states localized near individual magnetic adatoms. Typically, impurity-bound Shiba states are detected in measurements with superconducting tips through quasi-particle or Andreev tunneling processes at an energy of $eV = \Delta_{\text{tip}} + E_S$, where E_S is the energy of the Shiba state [38, 41]. Performing similar measurements on our Fe defects, we find the expected signatures of the Shiba states, an example of which is shown in Fig. 173(a) for a type A Fe defect, resolving two different Shiba states at $e|V| = \Delta_{\text{tip}} + E_{S1}$ and $e|V| = \Delta_{\text{tip}} + E_{S2}$. Examining the dependence of such measurements on the coupling between the superconducting tip and sample, we find that at lower junction resistance, there are additional subgap features in the spectra at lower biases [Fig. 173(b)]. In particular, we find peaks in conductance at $e|V| = E_{S1}$ and $e|V| = (\Delta_{Pb} + E_{S1})/2$, which correspond to previously undetected Andreev reflections through an impurity-bound Shiba state. The feature at $e|V| = (\Delta_{Pb} + E_{S1})/2$, which is due to a sub-gap tunneling process, cannot occur due to quasiparticle tunneling but rather arises from an Andreev process through the Shiba state, as illustrated in Fig. 173(g). Further corroboration that such Andreev processes involve the localized Shiba state can be obtained by comparing the spatial patterns of maps over the impurities at the Andreev reflection energy ($e|V| = (\Delta_{Pb} + E_{S1})/2$) with that of tunneling at $e|V| = \Delta_{Pb} + E_{S1}$. The excellent correspondence between the hole-like (electron-like) conductance measurements shown in Figs. 173(c),(e) (Figs. 173(d),(f)) for the type A impurity demonstrates the role of Shiba states in such Andreev processes. (See also Supplementary Material for other examples and a discussion of possible complications due to imperfect superconducting tips).

In conclusion, we demonstrate that combining Josephson spectroscopy with atomic resolution STM imaging provides a method to directly probe the local superconducting order parameter, which is not directly accessible through traditional quasiparticle measurements. Despite the challenge of operating such Josephson junctions in a fully phase coherent regime due to their ultra small dimensions, the fluctuating pair current is still a powerful tool to examine spatial variations of the pairing amplitude. Our key accomplishment, demonstrating that these measurements can be performed with atomic resolution, paves the way for using scanning Josephson techniques to study an inhomogeneous or spatially modulated order parameter in novel superconducting materials.

33.1 Supplementary Materials: Scanning Josephson spectroscopy on the atomic scale

33.2 Fitting procedure for Josephson critical current

The general tunneling properties of ultrasmall junctions are well described by the P(E) theory, which can be tailored to the case of Cooper pair tunneling between two superconducting junctions by treating the Josephson energy as a perturbation to the charging energy ($E_J \ll E_C$) [1, 2, 3]. This theory considers a voltage-biased junction coupled to an arbitrary environment, where inelastic tunneling processes mediated by the external environment are characterized by a probability distribution $P(E)$. For STM junctions with a capacitance on the order of 1 fF, the dynamics are dominated by the charging energy, which gives rise to quantum fluctuations of the phase. However, photon assisted tunneling of Cooper pairs is still possible if the environment is able to compensate for the bias by emitting or absorbing a photon of energy $E = 2eV = \hbar\nu$ as a Cooper pair tunnels across the junction. Except in the case of simple environmental impedance

functions, it is difficult to obtain an analytic expression for $P(E)$. We follow a numerical method to compute the probability of Cooper pair tunneling due to interactions with an environment modeled by a complex impedance $Z(\omega)$, as discussed in [2], using the convolution

$$P_Z(E) = I(E) + \int_{-\infty}^{+\infty} d\omega K(E, \omega) P_Z(E - \hbar\omega). \quad (33.1)$$

The inhomogeneity $I(E)$ is given by a Lorentzian representing the effect of the Ohmic resistance of the environment and serves as an initial guess for $P(E)$. Detailed forms of $I(E)$ and $K(E, \omega)$ are given in [2]. We solve this convolution iteratively until self consistency is reached.

The data show periodic oscillations in the differential conductance [Fig. 174(a)], which we presume arise from standing modes of the tip. We model the STM tip geometry as an open-ended finite transmission line, formed between the tip holder and the sample, with eigenfrequencies corresponding to $\nu_n = (2n + 1) \cdot c/l$, for an STM tip of length l , where c is the speed of light [4]. The total impedance seen by the junction is given according to [3]

$$\frac{Z_t(\omega)}{R_Q} = \rho \frac{1 + \frac{i}{\alpha} \tan\left(\frac{\pi}{2} \frac{\omega}{\omega_0}\right)}{\left[1 - \delta \frac{\omega}{\omega_0} \tan\left(\frac{\pi}{2} \frac{\omega}{\omega_0}\right)\right] + i\alpha \left[\delta \frac{\omega}{\omega_0} + \tan\left(\frac{\pi}{2} \frac{\omega}{\omega_0}\right)\right]}, \quad (33.2)$$

where $\rho = R_L/R_Q$ for a load resistance R_L and the quantum of resistance $R_Q = h/(2e)^2 = 6.45k\Omega$, $\omega_0 = 2\pi\nu_0$ is the characteristic $\lambda/4$ frequency, $1/\alpha$ is an effective quality factor of the resonator and $\delta = \omega_0 RC$, the ratio of the $\lambda/4$ frequency to the cutoff frequency $1/RC$. The load resistance provides a measure of the effective ohmic impedance seen by the junction, which is on the order of the vacuum impedance $Z_0 = 377\Omega$, largely determined by the wiring.

The dissipative interaction of a quantum system with the environment opens up one path for photon assisted tunneling as discussed above. Thermal fluctuations on the capacitive junction create an additional channel for inelastic tunneling of Cooper pairs, where the voltage noise across the capacitor has an rms $\sqrt{\bar{v}^2} = \sqrt{k_B T/C}$, corresponding to a gaussian distribution [4]

$$P_C(E) = \frac{1}{\sqrt{4\pi E_C k_B T}} \exp\left[-\frac{E^2}{4E_C k_B T}\right]. \quad (33.3)$$

The total $P(E)$ is the convolution of the probability distributions from these two channels $P_Z(E)$ and $P_C(E)$. The net Cooper pair current is given by

$$I(V) = \frac{\pi e E_J^2}{\hbar} [P(2eV) - P(-2eV)] = \frac{\pi \hbar I_c^2}{4e} [P(2eV) - P(-2eV)], \quad (33.4)$$

where the Josephson tunneling determines the overall amplitude which is proportional to I_c^2 [2, 3]. The fitting parameters we include in our numerical model are (i) a junction capacitance C , (ii) a load resistance R_L , (iii) a characteristic frequency ν_0 , which is intrinsic to the system and can be used to determine the length of the STM tip (iv) α , which is inversely proportional to the quality factor of the resonator and (v) an overall amplitude $A = (\pi \hbar)/(4e) \cdot I_c^2$ from which the critical current can be extracted. We fix the temperature to the electron temperature $T = 250$ mK, which accounts for extrinsic sources of noise (e.g. from room temperature electronics; for details of the measurement circuit, see [5]). Since the resonances of the tip eigenmodes are more clearly resolved in the differential conductance for tunneling between a Pb tip and a Pb(110) sample, we fit dI/dV to the derivative of current calculated from the $P(E)$ theory [Fig. 174(a)]. In order to fit the IV characteristics of the junction, an additional constant term must be added to account for the background [Fig. 174(b)], likely arising from quasiparticle poisoning.

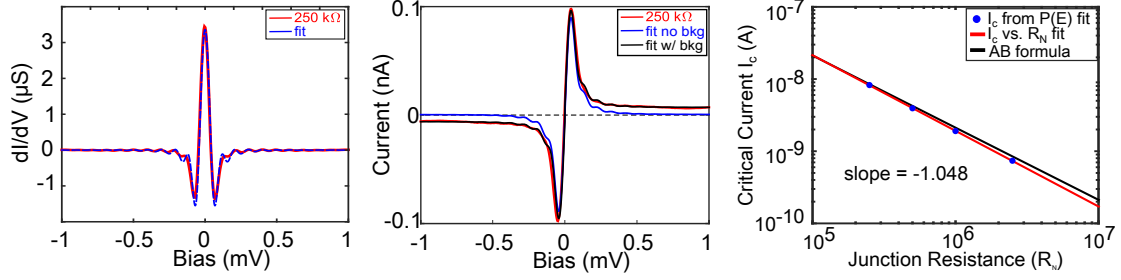


Figure 174: Here we provide additional details of the fit shown in Fig. 1(f) of the main text. (a) Fit to dI/dV with a Pb tip on a Pb(110) sample for $R_N = 250 \text{ k}\Omega$ based on the numerical implementation of the P(E) theory. Fitting parameters are $C = 4.3 \text{ fF}$, $R_L = 502 \text{ }\Omega$ and $\alpha = 0.15$ with the temperature fixed at $T = 250 \text{ mK}$. From the overall amplitude of the fit, we obtain a critical current $I_c = 8.25 \text{ nA}$ compared to $I_c = 8.5 \text{ nA}$ from the Ambegaokar-Baratoff formula for $\Delta_{Pb} = 1.35 \text{ meV}$. (b) Calculated $I(V)$ from the dI/dV fitting parameters (blue) demonstrates the need to include an additional term to account for a quasiparticle background in order to match the $250 \text{ k}\Omega$ $I(V)$ data. A constant term, which is of the form $c \cdot 2(\Theta(V) - 0.5)$ where $\Theta(x)$ is the Heaviside function, is added with $c = 6.5 \text{ pA}$ (black). (c) For critical currents extracted from the fitting procedure, we find $I_c \propto R_N^{-1.048}$, with the values of I_c in good agreement with the Ambegaokar Baratoff formula.

Fig. 174(c) shows the linear dependence of the critical current values extracted from the fit of dI/dV over a range of junction resistances R_N . This is in remarkable agreement with the Ambegaokar Baratoff formula [6]

$$I_c = \frac{\pi}{2e} \frac{\Delta_{Pb}}{R_N}, \quad (33.5)$$

assuming the gap is the same for the tip and the sample with $\Delta_{Pb} = 1.35 \text{ meV}$. Thus, even though we are in the phase fluctuating regime, the incoherent Cooper pair tunneling across our Josephson STM junction provides a measure of the local order parameter of the system. The maps in Figs. 2 (main text) and 176 show the zero bias conductance on Pb to be constant to within 2-3% (1 standard deviation) of the average value. For measurements performed with the same superconducting tip, the values of the remaining fit parameters, C, R_L, ν_0, α , have a less than 5% variation at different junction resistances between $2.5 \text{ M}\Omega$ and $250 \text{ k}\Omega$. For different superconducting tips, the capacitance and resistance show a 10-15% variation, whereas α can vary by up to 30%.

33.3 Suppression of the Josephson effect over Fe adatoms

A quantitative treatment of the suppression of the Josephson critical current over an Fe adatom can be obtained by comparing fits to the conductance over the impurity to that on clean Pb. Fig. 175(a)-(b) shows an example fit to dI/dV for Pb and a type A impurity measured with the same superconducting tip, demonstrating a 13% suppression of I_c . Overall, we find a 10-15% suppression of the critical current over the Fe impurity for dozens of adatoms (both type A and type B) measured with various different superconducting tips at several junction resistances.

Next, we present data from two additional adatoms and discuss several measurements that rule out variations in tip-sample distance as a cause for the suppression of the critical current. We simultaneously establish the junction resistance well outside the gap and perform spectroscopic measurements of the Josephson effect on the Fe adatoms. The parking conditions (setpoint current and voltage) that determine R_N are identical both on the bare Pb as well as

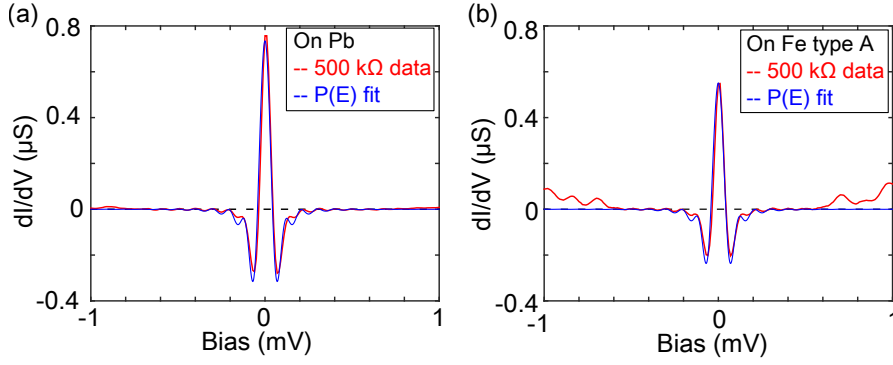


Figure 175: Fits to differential conductance on (110) surface of Pb compared to that on an Fe adatom showing a suppression of the Josephson effect. Data measured at a junction resistance $R_N = 500 \text{ k}\Omega$ with the same superconducting Pb tip. Fit parameters for both panels: $C = 4.1 \text{ fF}$, $R_L = 550 \text{ }\Omega$ and $\alpha = 0.18$. From the overall fit parameter, we obtain values for the critical current, $I_c = 4.15 \text{ nA}$ on Pb compared to $I_c = 3.60 \text{ nA}$ on the Fe adatom, indicating a 13% suppression over the magnetic impurity.

over the Fe adatoms, ensuring there is no positional variation in R_N . Moreover, we note that the magnitude of suppression we observe makes it highly unlikely that the change in z-height is responsible for the reduction of the Josephson effect. Since the tunneling current is exponentially sensitive to tip-sample distance, a $1 \text{ \AA}$ increase in tip height would lead to an order of magnitude decrease in the intrinsic Josephson critical current. In contrast, we observe only a 10-15% suppression of the Josephson critical current over the Fe adatoms, which shows that it is not related to an increased distance from the Pb(110) surface.

Spectroscopic maps of the zero bias differential conductance at $R_N = 1 \text{ M}\Omega$, taken with the same superconducting tip at two different parking biases of $V = -5 \text{ mV}$ and $V = -20 \text{ mV}$ are shown in Fig. 176(a)-(d). In order to maintain the same junction resistance, the setpoint current is four times as large in the second measurement ($I = 5 \text{ nA}$ and $I = 20 \text{ nA}$, respectively). Both data sets show similar conductance values and identical spatial patterns for zero bias suppression, confirming that the I_c suppression is not due to spurious signals arising from differences in the density of states outside the gap over the bare Pb compared to the adatoms. The slight discrepancy in absolute conductance values is likely due to the fact that the $V = -20 \text{ mV}$ parking bias is well outside strong-coupling energy regime in Pb, whereas the phonon modes are still present at $V = -5 \text{ mV}$. Fig. 176(e)-(h) compares zero bias differential conductance maps taken at the same parking bias of $V = -5 \text{ mV}$ for two different junction resistances, $R_N = 500 \text{ k}\Omega$ and $R_N = 250 \text{ k}\Omega$, corresponding to a difference in tip-sample distance between the two data sets, and a decrease in tip height relative to the $1 \text{ M}\Omega$ measurements. The two measurements show similar spatial patterns and a similar percentage suppression of the zero bias conductance over the adatom, though the absolute conductance values differ by the expected amount based on R_N . Given the exponential sensitivity of tunneling current to tip height, a factor of four variation in the junction impedance is considerable. Therefore, we rule out measurement artifacts from z-height variations and attribute the reduction in I_c to the suppression of the superconducting order parameter over the magnetic impurities.

33.4 Andreev reflections through impurity-bound Shiba states

Here we provide additional data on Andreev reflections through the localized Shiba states of Fe adatoms on a Pb(110) surface. The Shiba states measured with a superconducting tip for the type B Fe adatom are shown in Fig. 177(a), where the presence of the impurity also

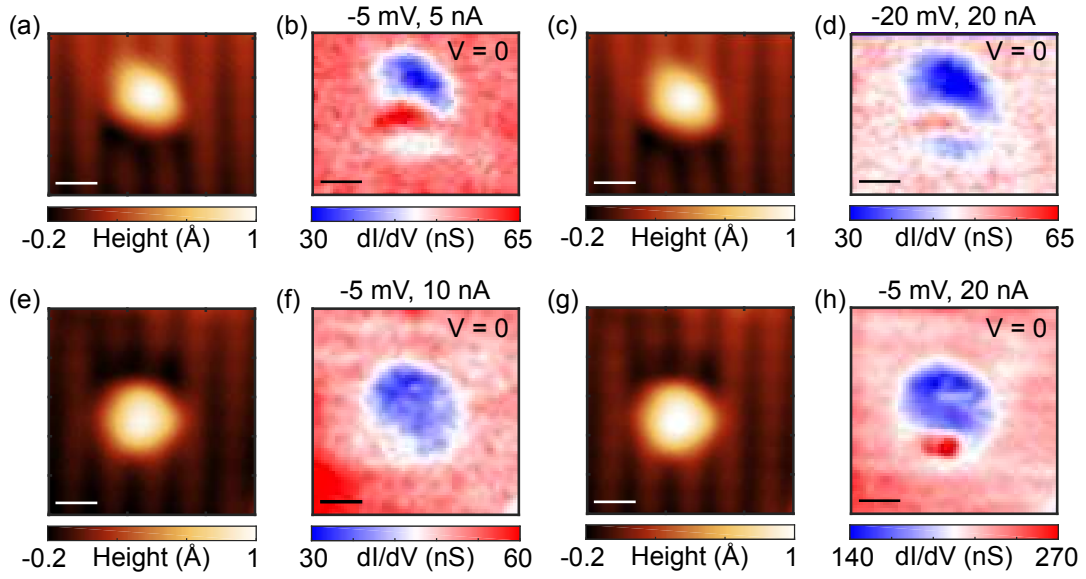


Figure 176: (a)-(d) Simultaneous topographies and differential conductance maps at $V = 0$ for $R_N = 1 \text{ M}\Omega$ over an Fe adatom with the same tip for different parking conditions: $V = -5 \text{ mV}$, $I = 5 \text{ nA}$ [(a)-(b)] and $V = -20 \text{ mV}$, $I = 20 \text{ nA}$ [(c)-(d)]. In both cases, the spatial pattern of the suppression of the zero bias conductance is identical, and the slight discrepancies in absolute conductance values is likely due to the phonon modes in Pb. (e)-(h) Simultaneous topographies and differential conductance maps over a second Fe adatom with the same tip (but different than that used for (a)-(d)) for different values of junction resistance, showing similar spatial patterns of suppression. (e)-(f) Parking conditions of $V = -5 \text{ mV}$, $I = 10 \text{ nA}$ corresponding to $R_N = 500 \text{ k}\Omega$. (g)-(h) Parking conditions of $V = -5 \text{ mV}$, $I = 20 \text{ nA}$ corresponding to $R_N = 250 \text{ k}\Omega$. Scale bar in all panels is $5 \text{ \AA}$.

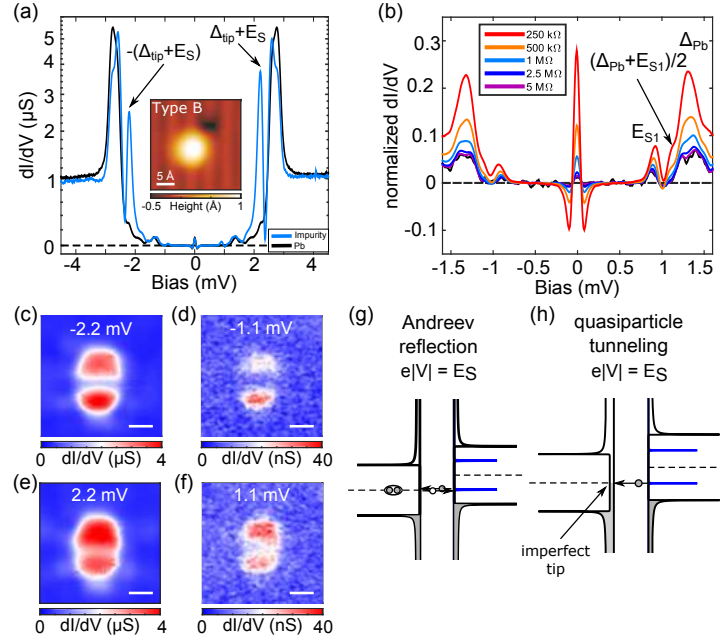


Figure 177: (a) Superconducting tip spectra on a type B Fe impurity demonstrating the presence of Shiba states. Measured at $R_N = 1 M\Omega$ ($V = -5$ mV, $I = 5$ nA), plotted on a log scale. (b) Point spectra on a type B Fe adatom showing sub- Δ features at E_S and $(\Delta_{Pb} + E_S)/2$, which increase in strength as junction resistance is decreased. Parking bias $V = -5$ mV, normalized to R_N . [(c)-(f)] Conductance maps at $eV = \pm(\Delta_{Pb} + E_S)$ and $eV = \pm(\Delta_{Pb} + E_S)/2$ with corresponding spatial patterns between the Shiba state ($V = 2.2$ mV) and the sub gap Andreev reflection feature ($V = 1.1$ mV), and similarly for negative bias. Scale bar is 5 Å. (g) Andreev reflection process which would give rise to a peak in dI/dV at the bare Shiba state energy E_S that is expected to get stronger for lower junction resistance. (h) Direct quasiparticle tunneling into the Shiba state due to an imperfect tip could also give rise to a peak in dI/dV at E_S , which would be independent of junction resistance.

changes the coupling of the tip into the two gaps of Pb(110). Fig. 177(b) shows the evolution of subgap Andreev processes as the junction resistance is decreased. The Andreev reflections that give rise to the peaks at $eV = \pm\Delta_{Pb}$ and $eV = \pm(\Delta_{Pb} + E_S)/2$ are discussed in the main text [inset of Fig. 1(c) and Fig. 3(g)].

The schematic in Fig. 177(g)-(h) depicts two processes which could give rise to the feature in the conductance at $e|V| = E_S$. An Andreev process at the Shiba state energy would involve an electron (hole) sourced from the Shiba state to be retroreflected as a hole (electron) back into the same state. On the other hand, an imperfect tip with some non-zero DOS up to E_F could give rise to direct quasiparticle tunneling into the Shiba state. We expect that an Andreev process should get stronger for larger tip-sample coupling (smaller junction resistance), but quasiparticle tunneling is expected to be independent of R_N . For the type A adatom [main text Fig. 3(b)], the peak in dI/dV at E_S persists even for high resistances ($R_N > 1 M\Omega$), an indication that the peak arises from a combination of quasiparticle and Andreev tunneling processes. However it is not possible to obtain direct quasiparticle tunneling at a threshold bias of $e|V| = (\Delta_{Pb} + E_S)/2$, which can arise only through Andreev reflections where an electron from the Shiba state is reflected as a hole into the continuum. Fig. 177(c)-(f) show that the spatial patterns of the Shiba states for the Type B adatom at a bias of $eV = \pm(\Delta_{Pb} + E_{S1})$ have a striking similarity to ones at the Andreev reflection process of $eV = \pm(\Delta_{Pb} + E_{S1})/2$, corroborating the role of the Shiba state in the Andreev process.

34 Tunable pseudogaps due to non-local coherent transport in voltage-biased three-terminal Josephson junctions by C. Padurariu et al

Abstract

We investigate the proximity effect in junctions between $N = 3$ superconductors under commensurate voltage bias. The bias is chosen to highlight the role of transport processes that exchange multiple Cooper pairs coherently between more than two superconductors. Such non-local processes can be studied in the dc response, where local transport processes do not contribute. We focus on the proximity-induced normal density of states that we investigate in a wide parameter space. We reveal the presence of deep and highly tunable pseudogaps and other rich structures. These are due to a static proximity effect that is absent for $N = 2$ and is sensitive to an emergent superconducting phase associated to non-local coherent transport. In comparison with results for $N = 2$, we find similarities in the signature peaks of multiple Andreev reflections. We discuss the effect of electron-hole decoherence and of various types of junction asymmetries. Our predictions can be investigated experimentally using tunneling spectroscopy.

34.1 Introduction

Quantum transport in Josephson junctions has been the focus of extensive research, predominantly studying junctions between $N = 2$ superconductors. Short junctions exhibit a strong proximity effect that manifests in equilibrium as an induced minigap in the density of states. A finite minigap is accompanied by a non-dissipative superconducting current. On the contrary, out of equilibrium dynamics due to voltage bias leads to entirely dissipative quasiparticle transport in two-terminal junctions. When the bias voltage V is below the superconducting gap of the leads Δ , the dissipative quasiparticle motion is described by multiple Andreev reflections (MAR) [1]. Electrons and holes cross the structure, being Andreev-reflected at each junction interface. Each crossing provides the energy eV , giving rise to features in the $I(V)$ curve [2] at integer fractions of $2\Delta/e$. In the regime dominated by MAR, the density of states no longer manifests a clear minigap, [3, 4] instead exhibiting peaks located at energy intervals separated by eV .

Recently, the study of junctions between $N \geq 3$ superconductors has attracted considerable interest, both theoretical [5, 6, 7, 8] and experimental [9, 12, 11, 10]. Unique features have been revealed, that do not manifest in the $N = 2$ junctions. In equilibrium, mapping the subgap Andreev spectrum has revealed level crossings at zero energy for non-trivial phase values [14, 13, 15, 12]. For $N \geq 4$, the crossing point was shown to have analogous topological properties to Weyl points in topological semi-metals [15, 16].

Voltage bias further emphasizes the complex phenomenology of $N \geq 3$ junctions. When the voltages are chosen such that the ratio of any two is a rational number (commensurate bias), the transport is no longer entirely dissipative as is the case in $N = 2$ junctions. Previous works have shown that a non-dissipative dc current component [5, 7, 8] arises in the junction due to coherent exchange of multiple Cooper pairs non-locally between three or more superconductors. The non-local current is sensitive to bias, as well as an emerging *stationary* phase that is obtained by combining the phases of multiple superconductors.

The simplest setup consists of the three-terminal Josephson junction (TTJ) where the non-dissipative current is expected to be largest when the two independent phases are affected by opposite voltage bias, V and $-V$, as shown in Fig. 178. Under these conditions the elementary non-local transport process has been termed the quartet process. It corresponds to the exchange of two Cooper pairs, four quasiparticles, between the three superconductors, as shown in Fig.

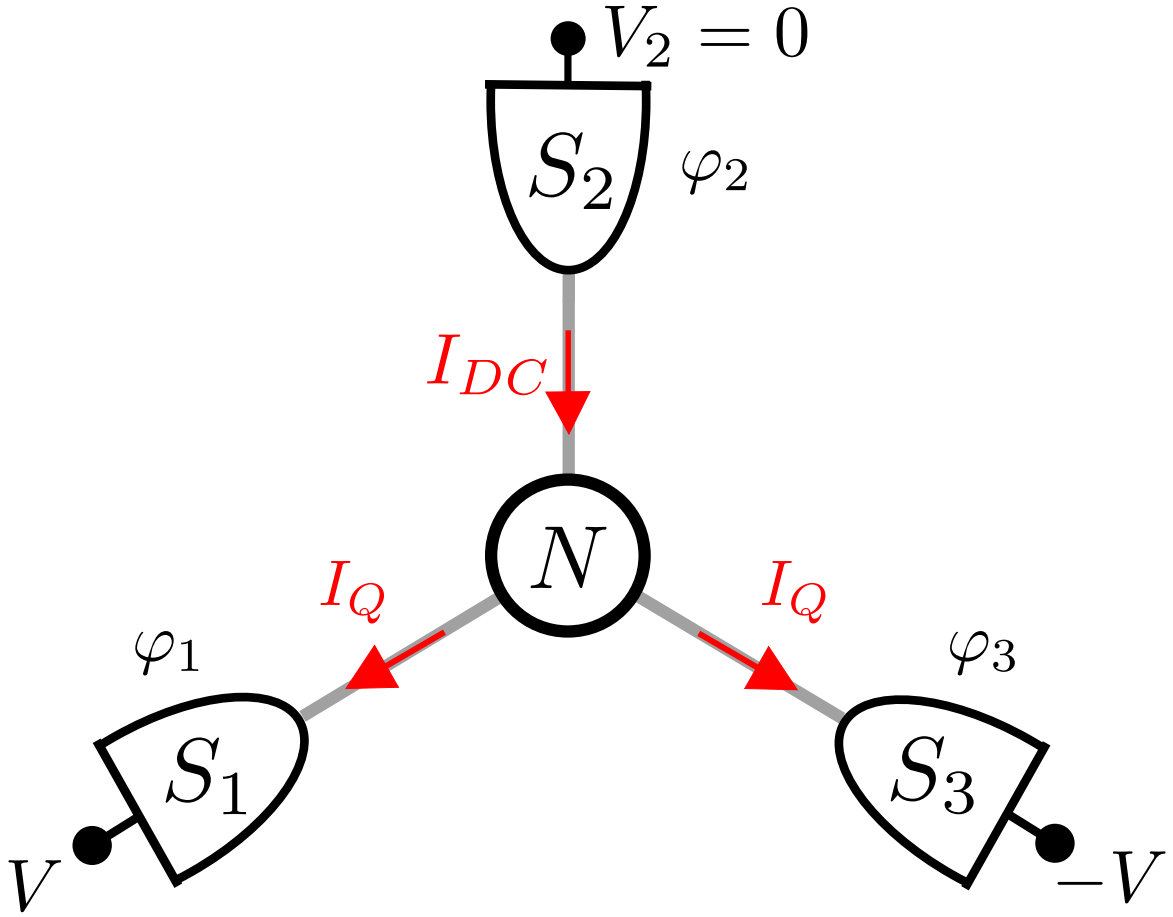


Figure 178: (Color online.) Schematic of the setup. Three superconducting electrodes S_1 , S_2 , and S_3 at voltages $V_1 = V$, $V_2 = 0$, and $V_3 = -V$ and with superconducting phases φ_1 , φ_2 , and φ_3 are connected to a short, diffusive metallic region. The dc current, I_{DC} , in S_2 drives equal dc currents I_Q in S_1 and S_3 , as explained in Sec. 34.2.

179. The situation has been recently investigated in Refs. [9], [10], and [11]. Two of the first experiments study a diffusive TTJ, where robust transport anomalies[9] and Shapiro steps[10] were observed as a function of two applied voltages $V_{1,3}$, that have been interpreted in terms of three quartet modes. The third experiment studies a phase-coherent TTJ realized in a semiconducting nanowire[11], showing positive current cross-correlation that are interpreted as evidence of the non-local quartet processes.

Motivated by these recent experiments, in this paper we describe the proximity effect in a short, metallic TTJ under voltage bias, V and $-V$, as shown in Fig. 178. We argue that driving a dc current in terminal 2, that is assumed at zero voltage, enables the control of the static non-local phase governing the quartet process, φ_Q . We calculate the normal density of states (NDOS) in a wide parameter regime by employing the quantum circuit formulation of the quasiclassical Usadel equation [17, 18, 19]. For comparison we study the NDOS in the biased two-terminal junction. We reveal the characteristic rich structure of the NDOS originating from MAR, that is similar between two- and three-terminal junctions. We additionally reveal features characteristic only to the three-terminal junction. The most striking of these are the pseudogaps appearing in the NDOS in the regime where coherent non-local processes give rise to bound states. pseudogaps differ from the proximity-induced minigap in that their edges are not as sharp, they do not in all regimes resemble the edges of the bulk gap, and may be less pronounced. What makes pseudogaps unique is the combination of properties: i. they are tunable by the quartet phase, and ii. they depend strongly on voltage bias.

Our study includes the importance of electron-hole decoherence, introduced phenomenologically using the quasiparticle dwell time in the normal region, τ_d . Despite describing a short junction on the scale of the coherence length, the dwell time can become appreciable compared to $\hbar/\Delta$ if the contact resistance at the SN interfaces, R_b , is much larger than the intrinsic resistance of the junction G_N^{-1} . The Thouless energy [20] is proportional to the inverse dwell time and can be decreased by a factor $R_b G_N \gg 1$. For this reason the Thouless energy can become comparable to or smaller than the superconducting gap, Δ , even in short junctions. The magnitude of the proximity-induced minigap in the NDOS is drastically modified by decoherence effects in a large variety of Josephson junctions [21, 22, 23, 24].

We begin our presentation in Section 34.2 with a phenomenological description of dynamics in a TTJ under voltage bias. The theoretical method and equations of quasiclassical circuit theory are presented in Section 34.3. Section 34.4 discusses the NDOS of a voltage-biased two-terminal Josephson junction, with peaks interpreted in terms of MAR processes. Section 34.5 discusses the NDOS of a biased TTJ, revealing the signature of MAR processes as well as pseudogaps originating from non-local processes. Section 34.6 presents our conclusions.

34.2 Phenomenological description

34.2.1 Local and non-local Josephson effect

The Josephson effect in an N -terminal Josephson junction is governed by $N - 1$ independent superconducting phase differences. Due to 2π -periodicity, the phase-dependent part of the junction energy can be expanded in harmonics. For $N = 3$ we choose the gauge $\varphi_2 = 0$ and express E_J as a Fourier series in φ_1 and φ_3 ,

$$E_J(\varphi_1, \varphi_3) = \sum_{m_1, m_3} E_{(m_1, m_3)} e^{i(m_1 \varphi_1 + m_3 \varphi_3)}, \quad (34.1)$$

where m_1 and m_3 are integers running along the entire real axis, and the Fourier coefficients $E_{(m_1, m_3)}$ are generally complex energies chosen such that E_J is real.

We explore non-local transport by choosing to evaluate the current flowing from terminal 2 into terminals $j = \{1, 3\}$, given by $I_j = (2e/\hbar) \partial E_J(\varphi_1, \varphi_3) / \partial \varphi_j$,

$$I_j(\varphi_1, \varphi_3) = \sum_{m_1, m_3} I_{j, (m_1, m_3)} e^{i(m_1 \varphi_1 + m_3 \varphi_3)}, \quad (34.2)$$

where $I_{j, (m_1, m_3)} = (2e/\hbar) i m_j E_{(m_1, m_3)}$. The total current flowing into terminal 2 is obtained from current conservation, $I_1 + I_2 + I_3 = 0$. Any possible current flowing from terminal 1 into terminal 3 does not modify the discussion.

We define the non-local component of the current flowing from terminal 2 into terminals $j = \{1, 3\}$ by $I_{j, \text{NL}} = \partial^2 I_j / \partial \varphi_1 \partial \varphi_3$. The harmonic structure of the Josephson current permits identification of local terms, giving $I_{j, \text{NL}} = 0$, and non-local terms, giving rise to a finite $I_{j, \text{NL}}$. Three contributions correspond to the local Josephson effect between terminals: 1 and 2 given by harmonics $(m_1, 0)$; 2 and 3 given by harmonics $(0, m_3)$, and 1 and 3 given by harmonics $(-m, m)$, with $-m_1 = m_3 = m$. All other pairs of harmonics correspond to the non-local Josephson effect.

The non-local Josephson term lowest in the order of harmonics corresponds to $(m_1, m_3) = (1, 1)$. It has been named the quartet term, as it implies a coherent exchange of two Cooper pairs, four quasiparticles, between the superconductors as shown in Fig. 179. In the following we show how the quartet term can be filtered from terms corresponding to the rest of the harmonics when driving the junction under commensurate voltage bias, $V_1 = -V_3 = V$.

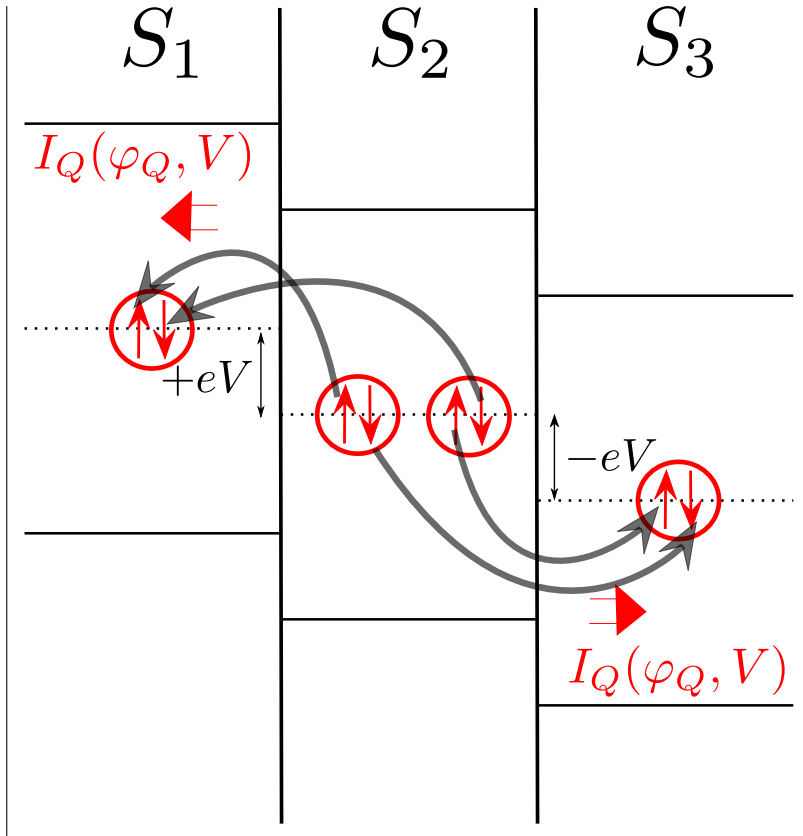


Figure 179: (Color online.) Diagram of the four particle quartet process, i.e. the lowest order non-dissipative transport process. The resulting current, I_Q , is a dc current sensitive to voltage, V , as well as the quartet phase, $\varphi_Q = (\varphi_1 - \varphi_2) + (\varphi_3 - \varphi_2)$.

34.2.2 Out-of-equilibrium dynamics

Under commensurate voltage bias, $V_1 = -V_3 = V$, $V_2 = 0$, the phases are given by: $\varphi_2 = 0$, $\varphi_1 = \varphi_{10} + \omega t$, and $\varphi_3 = \varphi_{30} - \omega t$, where $\omega = 2eV/\hbar$ is the Josephson frequency. The effect of biasing is to separate the harmonics of the Josephson energy in frequency space.

Under these biasing conditions, the quartet term and its harmonics (m, m) give rise to dc current in terminal $j = \{1, 3\}$,

$$I_{j,\text{DC}}(\varphi_Q, V) = I_Q = \sum_m I_m(V) \exp(im\varphi_Q), \quad (34.3)$$

where $\varphi_Q = \varphi_{10} + \varphi_{30}$ is the quartet phase and $I_m = (2e/\hbar)imE_{(m,m)}$. A detailed discussion of the coefficients I_m and their dependence on the bias voltage will be presented elsewhere.

The quartet phase can be tuned independently of the bias voltage by imposing an external current in terminal 2. Current conservation leads to a current-phase dependence, $I_{DC}(\varphi_Q)$, similar to the dc Josephson effect, $I_{DC} = -I_2 = 2I_Q(V, \varphi_Q)$. The indirect control of the quartet phase by current bias is similar to the control of the phase drop in a two-terminal Josephson junction by dc current bias. In analogy, the dc current is 2π -periodic in the quartet phase. If I_{DC} surpasses a certain critical value, depending on the details of the junction, the dc behavior of the junction becomes resistive. This situation, together with a discussion of the current flowing between terminals 1 and 3, will be presented in detail elsewhere. For discussing the proximity-induced normal density of states (NDOS) in the junction, we will use V and φ_Q as independent control parameters.

34.3 Microscopic model

We describe transport in a metallic TTJ using quasiclassical equations of non-equilibrium superconductivity. These take the form of a diffusive equation for the quasiclassical Keldysh-Nambu Green's function, [25] also known as the Usadel equation (see also Ref. [18])

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}} \left(\mathcal{D}(\mathbf{x}) \check{G} \frac{\partial}{\partial \mathbf{x}} \check{G} \right) - i [\check{H}, \check{G}] &= 0, \\ \check{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}, \quad \check{G}^2 = \check{1}, \quad \check{H} = \begin{pmatrix} \hat{H} & 0 \\ 0 & \hat{H} \end{pmatrix}; \\ \hat{H} = E\hat{\sigma}_z + \frac{1}{2}\Delta(\mathbf{x})(i\hat{\sigma}_y + \hat{\sigma}_x) + \frac{1}{2}\Delta^*(\mathbf{x})(i\hat{\sigma}_y - \hat{\sigma}_x). \end{aligned} \quad (34.4)$$

In addition to Keldysh-Nambu space (denoted with a check hat, $\check{G}$), the quasiclassical Green's function generally depends on two times (or energies $\mathbf{E}$) and on spatial coordinates $\check{G}(\mathbf{E}, \mathbf{x})$. The Pauli matrices are defined in Nambu space (denoted with a hat) $\vec{\sigma} = \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$, and $\mathcal{D}(\mathbf{x})$ denotes the diffusion coefficient. Matrix products in the Usadel equation are understood as convolutions of the quantities in the double time (or energy) representation, as detailed in the Appendix.

The Usadel equation applies to the most common experimental situation where the junction dimensions are larger than the elastic mean-free path. It is a conservation equation for the Keldysh-Nambu current density, $\check{j}(\mathbf{x})$,

$$\frac{\partial}{\partial \mathbf{x}} \check{j}(\mathbf{x}) + \frac{ie^2\nu}{\hbar} [\check{H}, \check{G}] = 0; \quad \check{j} = -\sigma(\mathbf{x}) \check{G} \frac{\partial}{\partial \mathbf{x}} \check{G}. \quad (34.5)$$

Here, ν is the electronic NDOS and $\sigma(\mathbf{x})$ is the conductivity. The two quantities are related by $\sigma = e^2\mathcal{D}\nu$.

Hereafter we employ a discretized version of the Usadel equation that describes the system in terms of finite quantum circuit elements [17, 18]. The bulk superconducting terminals S_i are described by coordinate-independent Keldysh-Nambu Green's functions $\check{G}_i$. The junction area is represented by a single node described by the unknown Green's function $\check{G}_c$. The node is separated from each terminal S_i by a connector that models the transparency of the contact via transmission coefficients $T_n^{(i)}$ corresponding to channel n in contact i . The Keldysh-Nambu matrix current flowing between terminal S_i and the node takes the compact form,[17]

$$I_{ic} = \frac{2e^2}{\pi\hbar} \sum_n \frac{T_n^{(i)} [\check{G}_i, \check{G}_c]}{4 + T_n^{(i)} (\{\check{G}_i, \check{G}_c\} - 2)}. \quad (34.6)$$

The fraction notation for matrix inversion is justified since $\check{G}_i$ and $\check{G}_c$ commute with $\{\check{G}_i, \check{G}_c\}$.

Decoherence between electrons and holes is accounted for phenomenologically by connecting the node to a fictitious terminal [17]. In contrast to the other three terminals, that correspond to the superconductors, the Keldysh-Nambu current flowing in the fictitious terminal does not contain particle or energy currents. The Green's function of the fictitious terminal is chosen such that the corresponding Keldysh-Nambu current describes only the leakage of electron-hole coherence. The Keldysh-Nambu current matrix to the fictitious terminal is given by,

$$I_{fc} = \frac{2e^2}{\pi\hbar} \sum_i \sum_n \frac{T_n^{(i)}}{4} [\check{G}_f, \check{G}_c], \quad (34.7)$$

$$\check{G}_f = -i \frac{E\tau_d}{\hbar} \begin{pmatrix} \hat{\sigma}_z & 0 \\ 0 & \hat{\sigma}_z \end{pmatrix},$$

where τ_d is the dwell time of quasiparticles in the junction, including the connectors. By including I_{fc} , the transport equation can be written as a conservation of the current of coherences,

$$\sum_i \check{I}_{ic} + \check{I}_{fc} = 0. \quad (34.8)$$

Since each of the currents are given by a commutation relation between the unknown Green's function of the central node $\check{G}_c$ and a matrix defined by Eqs. (34.6) and (34.7), it is convenient to rewrite the current conservation as a commutation relation $[\check{G}_c, \check{M}] = 0$, where the matrix denoted by $\check{M}$ adds up the terms corresponding to the four currents,

$$\check{M} = \sum_{i,n} T_n^{(i)} \left(\frac{\check{G}_i}{1 + \frac{T_n^{(i)}}{4} (\{\check{G}_i, \check{G}_c\} - 2)} + \check{G}_f \right). \quad (34.9)$$

It is important to note that matrix $\check{M}$ depends non-linearly on the unknown Green's function of the central node $\check{G}_c$, as well as on the known Green's functions of the terminals. The relation $[\check{G}_c, \check{M}(\check{G}_c)] = 0$ is a non-linear equation to be solved numerically for $\check{G}_c$.

34.3.1 Green's functions of superconducting terminals

In equilibrium, transport is stationary and the Green's functions depend on a single energy (or, in time representation, on the difference of the two times and independent of their sum). As a function of energy, the Green's functions of the superconducting terminals are given by,

$$G_i^R = \frac{1}{\xi} \begin{pmatrix} \epsilon & \Delta_i \\ -\Delta_i^* & -\epsilon \end{pmatrix}; \quad G_i^A = -\frac{1}{\xi^*} \begin{pmatrix} \epsilon^* & \Delta_i \\ -\Delta_i^* & -\epsilon^* \end{pmatrix}, \quad (34.10)$$

where complex energies have been introduced $\epsilon = E + i0^+$ and $\xi = \sqrt{\epsilon + |\Delta|}\sqrt{\epsilon - |\Delta|}$. Here, $\Delta_i = |\Delta|e^{i\varphi_i}$. The positive, vanishing imaginary part of ϵ specifies the position with respect to the branch cut of the square root function in the complex plane.

The advanced and retarded Green's functions are related by $G^A = -\hat{\sigma}_z (G^R)^\dagger \hat{\sigma}_z$ and the Keldysh Green's function G^K is obtained from:

$$G^K = (G^R - G^A) \tanh(\beta E/2), \quad (34.11)$$

where $\beta = (k_B T_e)^{-1}$ (T_e is the temperature).

We consider voltage-biased terminals, $V_1 = -V_3 = V, V_2 = 0$. According to the second Josephson relation, $\dot{\varphi}_i = 2eV_i/\hbar$, constant voltage bias gives rise to time-dependent superconducting phase differences that in general give rise to non-stationary transport. As a result, Green's functions acquire a non-trivial dependence on both energies, or equivalently in time representation, on both the difference, $(t_1 - t_2)$, as well as the sum, $(t_1 + t_2)$ of the two times. We relate the out-of-equilibrium Green's function of terminal i to its equilibrium value by the following gauge transformation,

$$\check{G}_i(t_1, t_2) = e^{i\hat{\sigma}_z e V_i t_1} \check{G}_i(t_1 - t_2) e^{-i\hat{\sigma}_z e V_i t_2}. \quad (34.12)$$

34.3.2 Numerical implementation

The theoretical framework outlined so far is sufficiently general to describe out-of-equilibrium transport for arbitrary bias. However, the non-linear equations that determine the unknown Green's function of the node, $\check{G}_c(t_1, t_2)$, are very difficult to solve in general. The dependence on two times (or two energies) must be solved on a discrete grid, where each grid point corresponds to an entry of the unknown matrix $\check{G}_c(t_1, t_2)$ (keeping in mind that each entry is a 4×4 matrix in Keldysh-Nambu space). In the general case the size of matrices involved grows quickly giving rise to an overwhelming computational problem.

To proceed, we use the properties of commensurate bias. In general, transport is governed by two Josephson frequencies corresponding to the two independent voltage differences. For commensurate bias, the two Josephson frequencies are harmonics of a single frequency ω_0 , the greatest common divisor. For the specific bias $V_1 = -V_3 = V, V_2 = 0$, the greatest common divisor is the Josephson frequency $\omega_0 = 2eV/\hbar$. We take advantage of this property by performing a double-time Fourier transform, (detailed in the Appendix) previously used in a different context in Ref. [30]. In the transformed representation the Green's functions depend on a single energy (as in equilibrium) and on the harmonics of ω_0 counted by two indices, $\check{G}(E, n, m)$. The definition contains redundancy in the indices, $\check{G}(E, n, m) = \check{G}(E - p\omega_0, n + p, m + p)$, therefore the Green's functions are determined by the value in the energy interval $[-\omega_0/2, \omega_0/2]$. (here we have set $\hbar = 1$) An alternative representation with only one harmonic index has been used in Ref. [28] for a two-terminal Josephson junction in the tunnel limit.

The practical numerical implementation involves the truncation of the harmonics by a value N_m , whereby the Green's functions are square matrices of dimension $4(2N_m + 1)$ defined on a one-dimensional grid in the energy interval $[-\omega_0/2, \omega_0/2]$. The matrix entries decay quickly at large harmonics. It is sufficient to truncate the harmonic expansion at $N_m = (2\Delta/eV)$.

We take the following steps to solve for the unknown Green's function of the central node, $\check{G}_c$, iteratively at each energy: (i) we start with a guess value for $\check{G}_{c,n}$, (ii) obtain and diagonalize $\check{M}_n(\check{G}_{c,n})$, (iii) obtain a new value, $\check{G}_{c,\text{new}}$ that commutes with $\check{M}_n$ and has as eigenvalues the signs of the real part of the eigenvalues of $\check{M}_n$, (iv) we check if $\check{G}_{c,\text{new}}$ is within a certain tolerance of $\check{G}_{c,n}$, to ascertain convergence, and finally, (v) if convergence was not achieved, we use a modified matrix as the guess of the next iteration step, $\check{G}_{c,n+1} = ||\check{G}_{c,n} + \alpha \check{G}_{c,\text{new}}||$, where $\alpha < 1$ is a convergence parameter that must be reduced at energies where the transport depends sharply

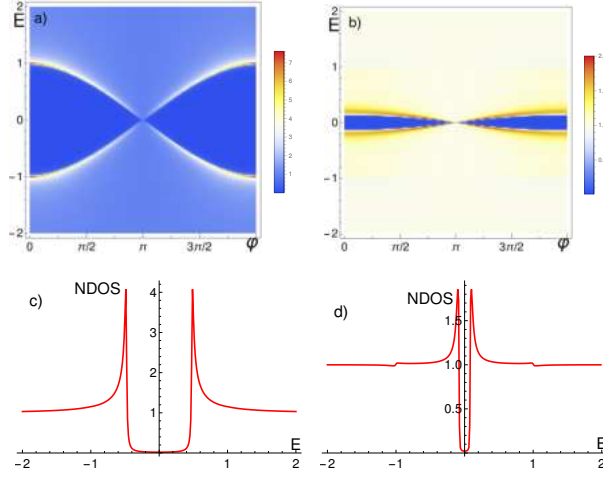


Figure 180: (Color online.) Normal density of states (NDOS) for a symmetric SNS junction ($T = 0.3$), for the small decoherence case (left column, $\tau_d = 0.05$), and for the large decoherence case (right column, $\tau_d = 5.0$). (a) NDOS as a function of the phase φ and of the energy E , for $\tau_d = 0.05$. (b) Same as (a), with $\tau_d = 5.0$. (c) Cut of (a) for the phase $\varphi = 2\pi/3$. (d) Cut of (b) for the phase $\varphi = 2\pi/3$. Energy E is measured in units of Δ and τ_d in units of $\hbar/\Delta$. ($\Delta = \hbar = e = 1$.)

on energy, and $||\dots||$ denotes the normalization that ensures $\check{G}_{c,n+1}^2 = 1$. In the calculation, a finite imaginary part is added to the energy, $\varepsilon = E + i\eta$, with $\eta/\Delta \ll 1$, to generate numerically smooth transport resonances. The parameter η may be understood as a phenomenological description of weak inelastic effects. Convergence to the $\eta = 0+$ limit is especially slow for all superconducting multi-terminal calculations [26], requiring small convergence parameters of the order $\alpha \simeq \eta/\Delta$. In the numerical calculation we have used $\eta = 0.01$.

34.3.3 The density of states (NDOS)

The NDOS can be measured using a tunnel probe, as has been already realized for a three terminal junction in equilibrium [12]. We model the tunnel probe by adding a normal terminal tunnel coupled to the junction. The current to the tunnel probe is given by,

$$I_t = \frac{e^2}{2\pi\hbar} T_{\text{tun}} \text{Tr} \left(\hat{\sigma}_z [\check{G}_t, \check{G}_c]^K \right), \quad (34.13)$$

where K denotes the Keldysh part of the matrix and $T_{\text{tun}} \ll 1$ describes the coefficient of the tunnel contact. Given that $G_t^{R,A} = \pm \hat{\sigma}_z$ and $G_t^K = 4 \tanh \left(\left(\frac{\beta}{2} (\varepsilon \hat{\sigma}_z + V_t) \right) \right)$, one can rewrite I_t as:

$$I_t = \frac{e^2}{2\pi\hbar} T_{\text{tun}} \text{Tr} \left[2G_c^K + 4 \tanh \left(\frac{\beta}{2} (\varepsilon + \hat{\sigma}_z V_t) \right) (G_c^A - G_c^R) \right]. \quad (34.14)$$

For small tunnel coefficients T_{tun} , the effect of the probe voltage V_t on $\check{G}_c$ can be neglected. In this case, the tunnel conductance, $\mathcal{G}_t = dI_t/dV_t$, is given by,

$$\mathcal{G}_t = \frac{8e^2}{\pi\hbar} T_{\text{tun}} \frac{d}{dV_t} \text{Tr} \left[\tanh \left(\frac{\beta}{2} (\varepsilon + \hat{\sigma}_z V_t) \right) (G_c^A - G_c^R) \right]. \quad (34.15)$$

The expression of the tunnel conductance is identical to its expression in equilibrium, with the exception that here the term $(G_c^A - G_c^R)$ includes the dynamics driven by the bias V . We conclude that the out-of-equilibrium NDOS can be probed by tunneling spectroscopy, with

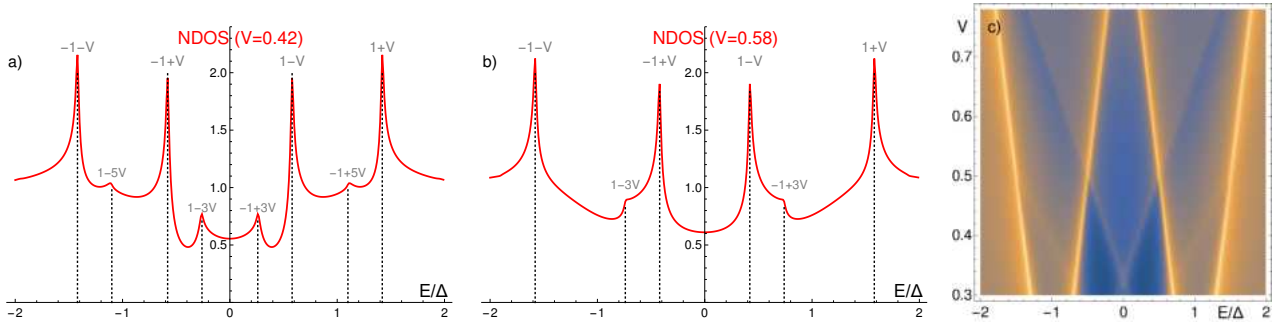


Figure 181: (Color online.) Normal density of states (NDOS) as a function of energy E in a biased junction at transparency $T = 0.1$, for small decoherence ($\tau_d = 0.05$). (a) and (b) show the NDOS for $V = 0.42$ and $V = 0.58$, and (c) shows the density plot of the NDOS as a function of the energy E and the voltage V . In (a) and (b), the vertical dotted lines show the positions of the expected peaks at voltages $\pm(\Delta \pm (2p + 1)V)$ due to MAR processes. ($\Delta = \hbar = e = 1$.)

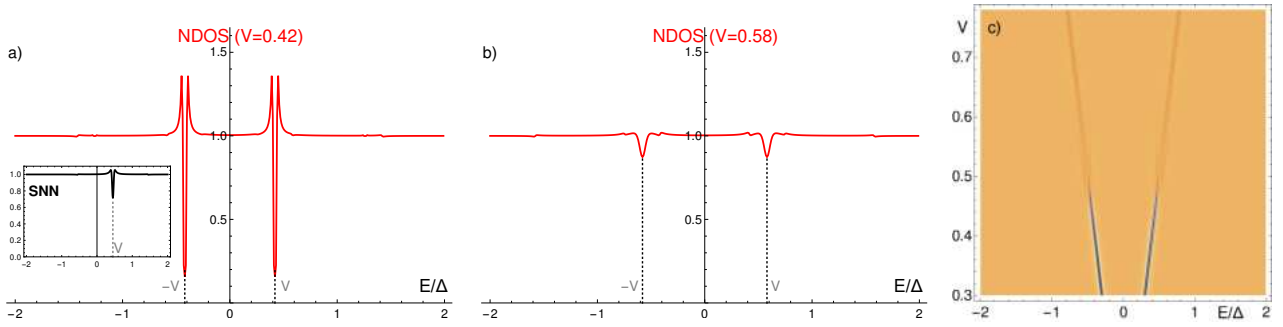


Figure 182: (Color online.) Same as Fig. 181, but for large decoherence ($\tau_d = 5.0$). In (a) and (b), the vertical dotted lines show the positions of the expected minigaps peaks at voltages $\pm V$ corresponding to the chemical potentials of the two electrodes. The inset in (a) shows the NDOS computed for a SNN junction where the S electrode is biased at voltage $V = 0.45$, with $\tau_d = 5$.

the following observation. Care must be taken at energies where the NDOS presents sharp structures that result from the divergence of the bulk superconductor NDOS. At these energies the perturbative treatment of the tunnel probe may fail. For this reason tunnel measurement of the NDOS may give rise to a rounding of the sharpest features of the NDOS that we predict in absence of the probe. In the following we will neglect the effect of the probe on transport and calculate the NDOS given by $\mathcal{N}(E) = \text{Re } \tilde{G}_c^{11}(E, 0, 0)$.

34.4 The two-terminal junction

As a preliminary, we present the results of quantum circuit theory for the conventional, two-terminal Josephson junction, both in equilibrium as well as in a symmetric junction biased at $(V, -V)$, for arbitrary transparency. This last problem has been considered by Bezuglyi et al., in the tunneling limit [27, 28]. The choice of bias, $2V$, helps us make a direct comparison with the biasing conditions of the $N = 3$ terminals junction.

34.4.1 Equilibrium

As a reference, the NDOS of a symmetric equilibrium SNS junction (with transparencies $T = 0.3$) is represented in Fig. 180, for small ($\tau_d \ll 1$) and large ($\tau_d > 1$) decoherence. (Throughout the paper we have set $\hbar = 1$, the elementary charge $e = 1$ and, unless explicitly

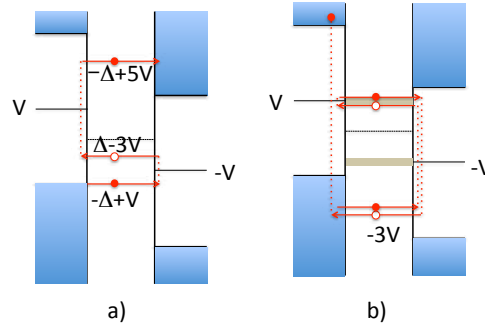


Figure 183: (Color online.) Schematic diagrams showing the dominant MAR processes at lowest order in the number of Andreev reflections. (a) No dephasing case ($n = 3$), dominated by the gap edges ($eV = 0.4\Delta$). (b) With dephasing ($n = 4$), the (brown online) shaded areas denote the pseudogaps at $\pm eV$ ($eV = 0.4\Delta$).

shown, $\Delta = 1$. The dwell time τ_d is presented in units of $\hbar/\Delta$.) The NDOS vanishes within the well-known minigap. The minigap persists for all parameters, except at $\varphi = \pi$ for the symmetric case, where the resonance gives rise to perfect transmission. For strong decoherence, the NDOS exhibits a sharp minigap that scales as the Thouless energy.

34.4.2 Biased junction

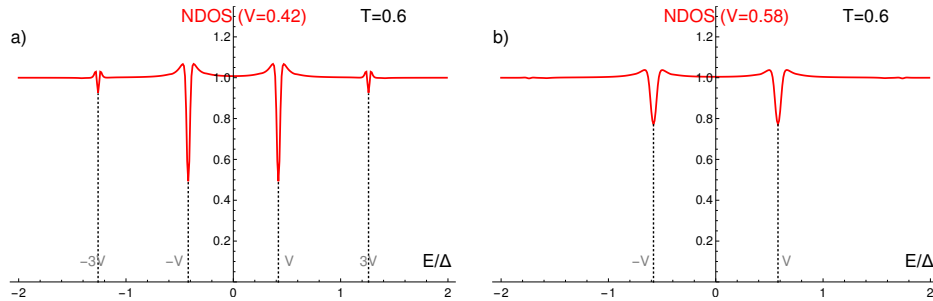


Figure 184: (Color online.) Same as Figs. 182(a) and 182(b), but with transparency $T = 0.6$.

When the two-terminal junction is biased, the NDOS shows a series of sharp resonances that are strongly affected by decoherence, as can be seen in Fig. 181 ($\tau_d = 0.05$) and in Fig. 182 ($\tau_d = 5.0$) for small transparency, $T = 0.1$.

For weak decoherence, Figs. 181(a) and 181(b) show the peaks of the NDOS for voltages $V = 0.42$ and $V = 0.58$. The sharp peaks resemble the divergence of the density of states of the bulk BCS-superconductor at the edges of the gap. The position of the peaks corresponds to the voltages, $-V$ and V . The transport in this regime is well described by coherent MAR. The peaks can be explained by a MAR diagram as in Fig. 183(a). The bulk gap edges of the two superconductors induce by proximity the peak structure of the junction NDOS. For instance, a quasiparticle leaving S_1 at energy $-\Delta + V$, is reflected as a hole in S_3 at energy $\Delta - 3V$. Higher p -order MAR processes give rise to peaks at $\pm(\Delta \pm (2p+1)V)$. The dotted lines in Figs. 181(a) and 181(b) correspond to the position of the peaks predicted by the MAR diagram. They agree well with the computed NDOS. Larger structures appear, centered at $\pm(\Delta \pm V)$, weaker ones

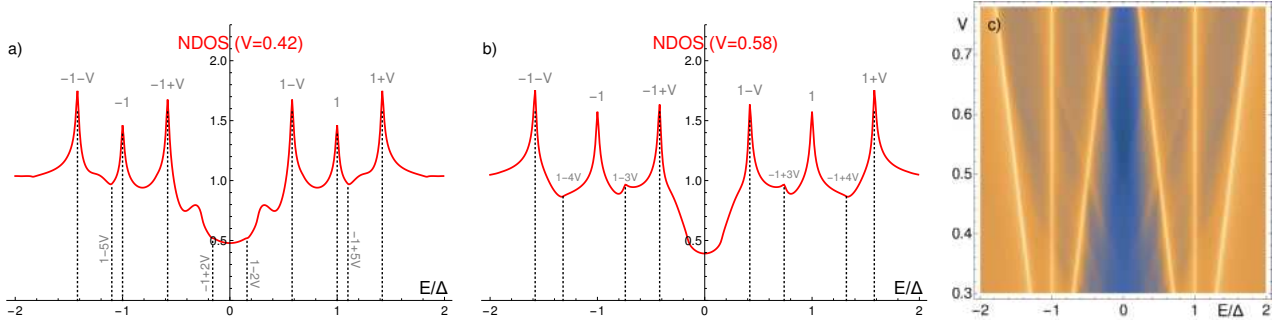


Figure 185: (Color online.) Normal density of states (NDOS) as a function of energy E in a biased TJJ with three electrodes at voltages $-V, 0, V$, at transparency $T = 0.1$, for small decoherence ($\tau_d = 0.05$). (a) and (b) show the NDOS for $V = 0.42$ and $V = 0.52$, and (c) shows the density plot of the NDOS as a function of the energy E and the voltage V . In (a) and (b), the vertical dotted lines show the positions of the expected peaks at voltages $\pm(\Delta \pm (2p+1)V)$ or $\pm(\Delta \pm (2p)V)$ due to MAR processes. ($\Delta = \hbar = e = 1$.)

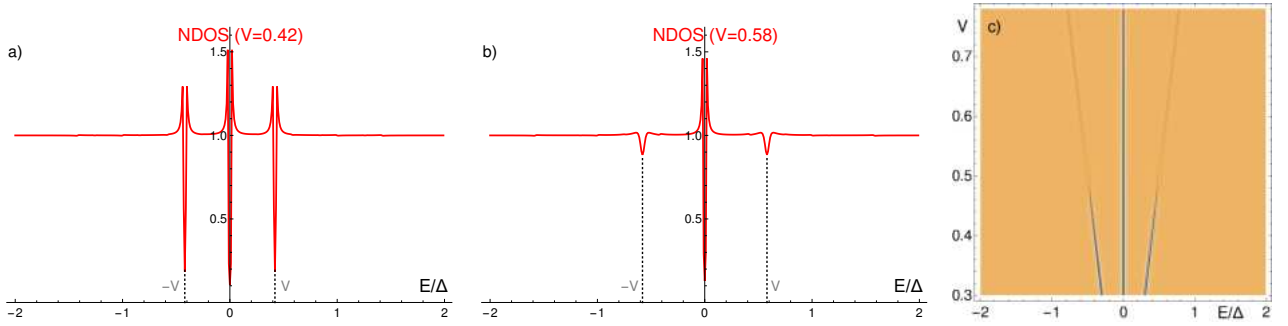


Figure 186: (Color online.) Same as Fig. 185, but for large decoherence ($\tau_d = 5.0$). In (a) and (b), the vertical dotted lines show the positions of the expected minigap peaks at voltages $\pm V$ and 0 corresponding to the chemical potentials of the two electrodes.

at $\pm(\Delta - 3V)$, and increasingly weaker ones at $\pm(\Delta - 5V)$, corresponding to the reduction of the MAR amplitude for transparency $T < 1$. The linear dependence of the peaks position as a function of the bias voltage V is shown in Fig. 181(c). Four lines in the E - V plane correspond to peaks of the NDOS at $\pm(\Delta \pm V)$, while weaker lines correspond to the higher order MAR at $\pm(\Delta - 3V)$ and $\pm(\Delta - 5V)$.

For strong decoherence, Fig. 182 shows a different structure of the NDOS. The peaks due to MAR processes are washed away and minigaps appear around energy $\pm V$, similar to the equilibrium minigap in Fig. 180(d). A similar minigap appears in diffusive SN junctions, with size given by the Thouless energy, where it is attributed to reflectionless tunneling [29, 22]. The inset of Fig. 182(a) shows the NDOS of an SNN interface, calculated using circuit theory, with voltage $V = 0.45$ applied to the S electrode. The result shows the proximity minigap developing at the chemical potential of the S electrode, $E = V$. Due to strong decoherence, the NDOS of the SNS junction exhibits the separate signature of each NS interface.

Two qualitatively different regimes are observed in Fig. 182(c) for $V < \Delta/2$ and $V > \Delta/2$. At the threshold, the lower edge of the bulk gap of the terminal biased at $+V = \Delta/2$ coincides with the chemical potential of the terminal biased at $-V = -\Delta/2$. The same condition marks the threshold of the $n = 2$ MAR process, as seen in Fig. 183(b). The regimes differ in the coupling strength to the continuum of quasiparticle states, the regime $V > 0.5\Delta$ marking the stronger coupling giving rise to rounded features of the NDOS.

In Fig. 184(a), we observe small corrections to the structure of the NDOS, visible at large transparency, $T = 0.6$. The feature can be ascribed to the $n = 4$ MAR process and corresponds

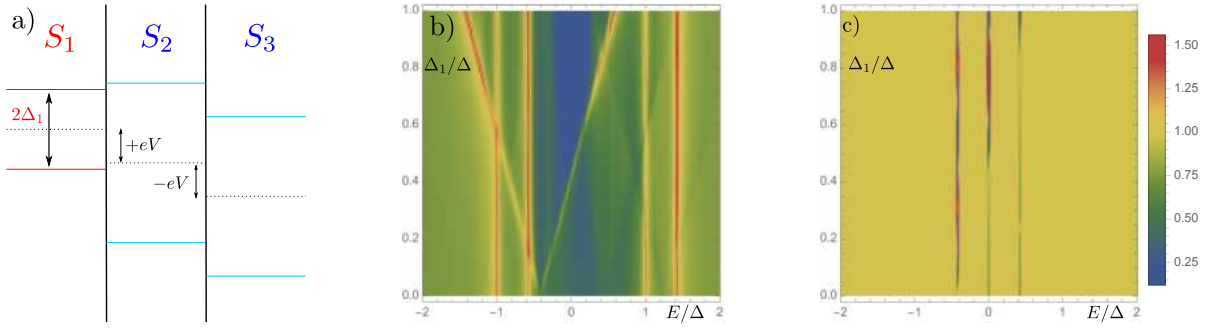


Figure 187: (Color online.) NDOS as a function of energy E/Δ for asymmetric bulk superconducting gaps, at transparency $T = 0.1$ and voltage $V = 0.42$, (a) schematic illustration of asymmetric gap Δ_1 , with $\Delta_2 = \Delta_3 = \Delta$, (b) at $\tau_d = 0.05$ as a function of Δ_1 , (c) at $\tau_d = 5$ as a function of Δ_1 . ($\Delta = \hbar = e = 1$.)

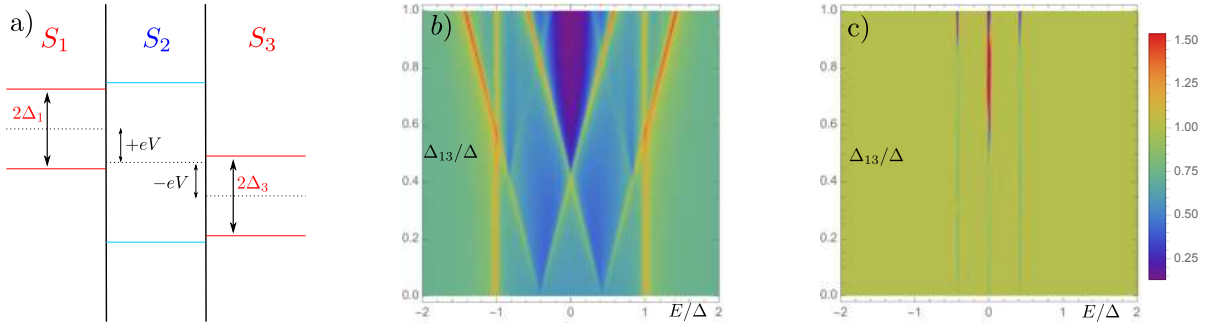


Figure 188: (Color online.) Same as Fig. 187, (a) schematic illustration of asymmetric gaps $\Delta_1 = \Delta_3 = \Delta_{13}$, with $\Delta_2 = \Delta$, (b) at $\tau_d = 0.05$ as a function of Δ_{13} , and (c) at $\tau_d = 5$ as a function of Δ_{13} .

to energy $E = \pm 3V$. In contrast, the transparency does not significantly modify the NDOS in the case of small decoherence (for this reason we only show the results for $T = 0.1$). Indeed, for a symmetric two-terminal junction and for vanishing τ_d , the small transparency limit of the circuit theory equations has been previously shown to be identical to the exact result [31].

34.5 The three-terminal junction

We now consider a metallic TTJ with superconducting terminals S_i , $i = \{1, 2, 3\}$, biased at voltages $V_1 = -V_3 = V$, $V_2 = 0$. Below, we set all superconducting phases to zero and discuss the effect of bias, transparency and bulk gap asymmetry on the structure of the NDOS. In the following subsection, we discuss the dependence of the NDOS on the phase φ_Q .

34.5.1 NDOS at $\varphi_Q = 0$

For a symmetric junction, $\Delta_i = \Delta$ and $T_i = T$, we find that the junction transparency does not significantly modify the spectral structure of the NDOS, with few exceptions that we will point out. For this reason, we choose $T = 0.1$ to produce the plots. The NDOS of the TTJ in the limits of small and large decoherence, Figs. 185 and 186, respectively, are similar to those for the conventional, two-terminal junction, presented in Figs. 181 and 182, with additional structures emerging from the presence of a third terminal, biased at $V_2 = 0$.

For small decoherence, Fig. 185, the NDOS presents an additional structure originating from the gap edges of the superconducting electrode at voltage $V_2 = 0$ at $\pm(\Delta + V_2)$. New MAR channels develop: quasiparticles with energy near the gap edges in $S_{1,3}$ scatter at energies

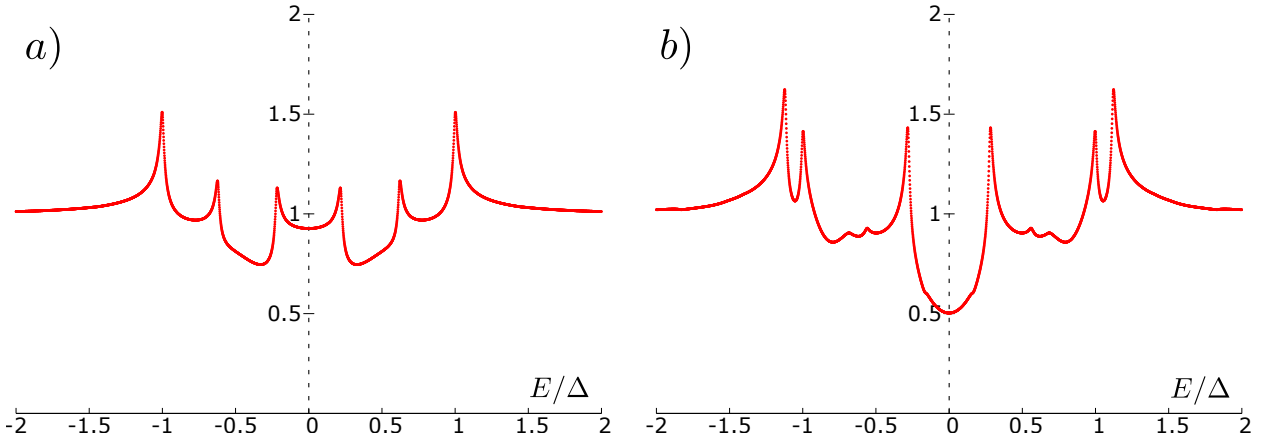


Figure 189: Line cut of Fig. 188(b) at fixed Δ_{13} for (a) $\Delta_{13} = 0.2\Delta$ and (b) $\Delta_{13} = 0.7\Delta$.

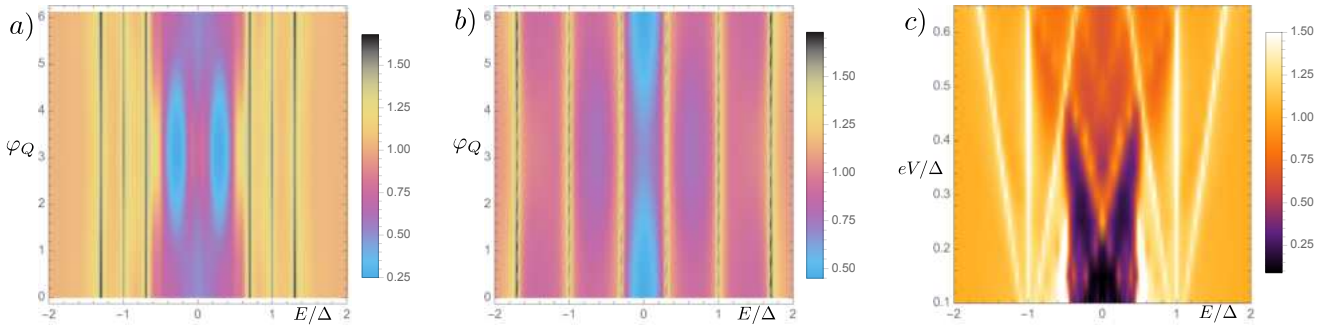


Figure 190: (Color online.) NDOS as a function of energy E/Δ in a biased TTJ with three electrodes at voltages $-V, 0, V$, for small decoherence $\tau_d = 0.05$ and transparency $T = 0.1$, (a) at $V = 0.3$ as a function of φ_Q , (b) at $V = 0.7$ as a function of φ_Q , and (c) at $\varphi_Q = \pi$ as a function of V . ($\Delta = \hbar = e = 1$.)

$\pm(\Delta \pm (2p+1)V)$, while those with energy near the gap edge in S_2 scatter at energies $\pm(\Delta \pm 2pV)$. Corresponding structures of the NDOS can be seen in Fig. 185 for all MAR processes of order n smaller than $n < (V/\Delta)$. The NDOS presents six large peaks, at energies $\pm(\Delta \pm V)$ and $\pm\Delta$ that correspond to the gap edges, and smaller structures corresponding to higher-order MAR processes. The position of these peaks depends linearly on the voltage as seen in Fig. 185(c), in agreement with the interpretation in terms of MAR diagrams.

For large decoherence, Fig. 186, the NDOS presents an additional (third) minigap at $E = V_2 = 0$, compared to the two-terminal junction. The third minigap is unaffected by the two voltage regimes $V \lesssim \Delta/2$ that determine the shape of the minigap edges at $E = \pm V$. The edges of the minigap at $E = 0$ remain sharp irrespective of V , since the chemical potential of the corresponding superconductor, S_2 , does not change with V .

It is interesting to track the position of the resonant structures of the NDOS by introducing an asymmetry in the bulk superconducting gaps. If S_1 is a weaker superconductor with $\Delta_1 < \Delta$, Fig. 187(b) shows the linear dependence of the corresponding MAR resonance on Δ_1 , confirming its relation to the gap edge of S_1 , according to the expression $\pm(\Delta_1 \pm 2pV)$. In the large decoherence regime, the position of minigaps does not change with Δ_1 . However, the shape of the minigap corresponding to the NS_1 interface is strongly affected by Δ_1 , showing the transition between sharp and smoothed out features at the threshold $V = \Delta_1/2$. The gap Δ_1 has an effect on the shape of the other two minigaps as well, signaling the persistence of weak non-local transport in the large decoherence regime.

Similarly, if both biased superconductors, S_1 and S_3 , have a smaller bulk gap, $\Delta_1 = \Delta_3 \equiv$

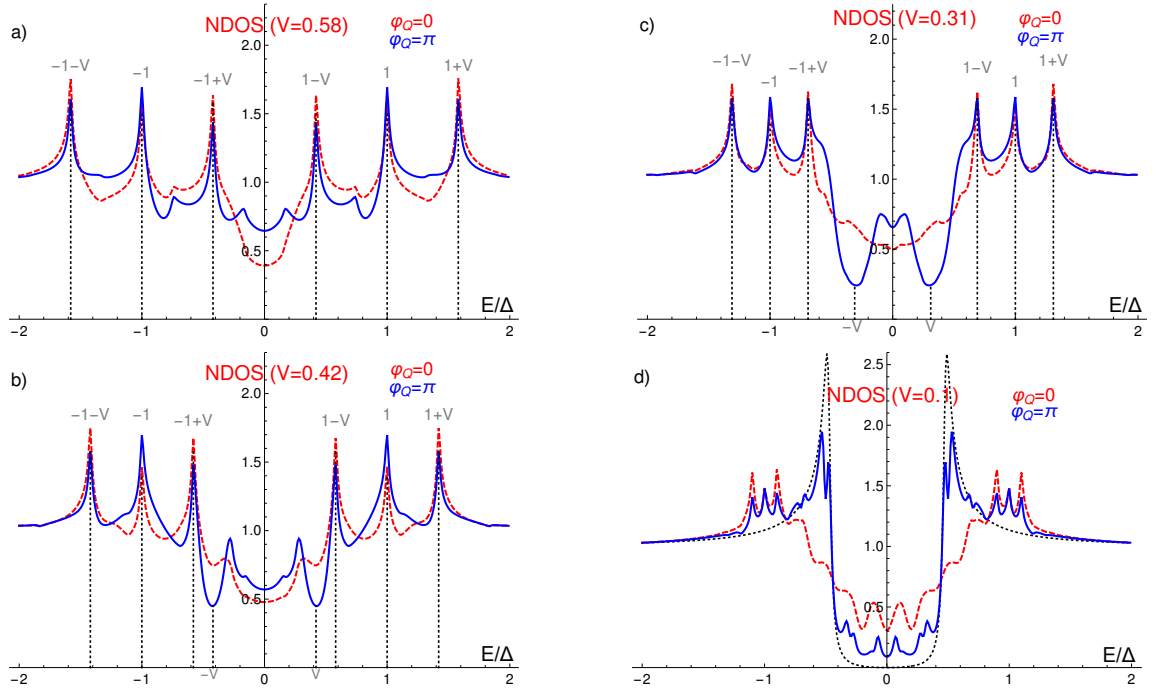


Figure 191: (Color online.) NDOS as a function of energy E in a biased TTJ with three electrodes at voltages $-V, 0, V$, for small decoherence $\tau_d = 0.05$ and transparency $T = 0.1$, for (a) $V = 0.58$, (b) $V = 0.42$, (c) $V = 0.31$, and (d) $V = 0.1$. The red (dashed) curves correspond to $\varphi_Q = 0$, while the blue (full) curves correspond to $\varphi_Q = \pi$. ($\Delta = \hbar = e = 1$.)

Δ_{13} , with $\Delta_{13} < \Delta_2 = \Delta$, the corresponding resonances shift linearly with Δ_{13} , as seen in Fig. 188. For low decoherence, the asymmetric setup shares similarities with the well-known Cooper pair splitter (CPS) setup [32, 33], where a junction between a superconductor and two normal metals is formed, the metals acting as collectors for electrons resulting from splitting Cooper pairs. In our setup, the collectors are the weaker superconductors S_1 and S_3 , and the CPS device is recovered in the limit $\Delta_{13} \rightarrow 0$. From Fig. 188 we observe that in the regime where $\Delta_{13} < V$, the modification to the CPS NDOS mainly consists of small pseudogaps located at $\pm V$, of width $2\Delta_{13}$. These are similar to the proximity-induced minigap at a biased SN interface, see the inset of Fig. 182(a), and their positions match the chemical potential of superconductors S_1 and S_3 .

A qualitatively different regime is obtained in the opposite limit $\Delta > \Delta_{13} > V$, where a pronounced pseudogap in the NDOS opens at zero energy, with width $2(\Delta_{13} - V)$. The gap corresponds to the energy window where coherent transport processes of the quartet type participate in the transport. In this regime, non-local processes of the quartet type coexist with Cooper pair splitting processes, both mechanisms contributing to non-local correlations in the currents I_1 and I_3 . The comparison is shown in Fig. 189 where (a) shows a line cut at small Δ_{13} , while (b) shows the gap around $E = 0$ that develops for large Δ_{13} . The complex statistical properties of transport in this regime will be discussed elsewhere.

34.5.2 NDOS: phase dependence

Unique to multi-terminal junctions is the strong dependence of the NDOS on the stationary phases of the superconductors. In particular, for the TTJ under the commensurate bias chosen, the NDOS depends on the quartet phase, φ_Q , introduced in Sec. 34.2. Here we discuss at length the effect of φ_Q , that we consider the central finding of this study.

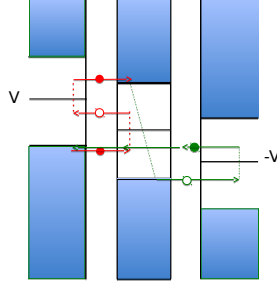


Figure 192: (Color online.) Diagram of a typical phase-MAR process in the TTJ. The thick red arrow shows a quasiparticle path for the 3rd-order MAR between S_1 and S_2 . The thin green arrow shows a multi-particle path for a multi-pair process that involves all superconductors by a cross-Andreev reflection in S_2 . The resulting interference gives rise to phase-dependent MAR.

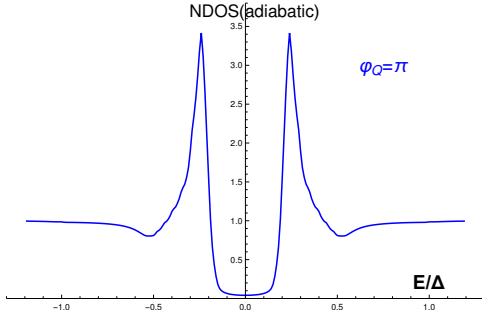


Figure 193: (Color online.) NDOS computed using an adiabatic approximation for small voltage, for a quartet phase $\varphi_Q = \pi$ (see text for detail)

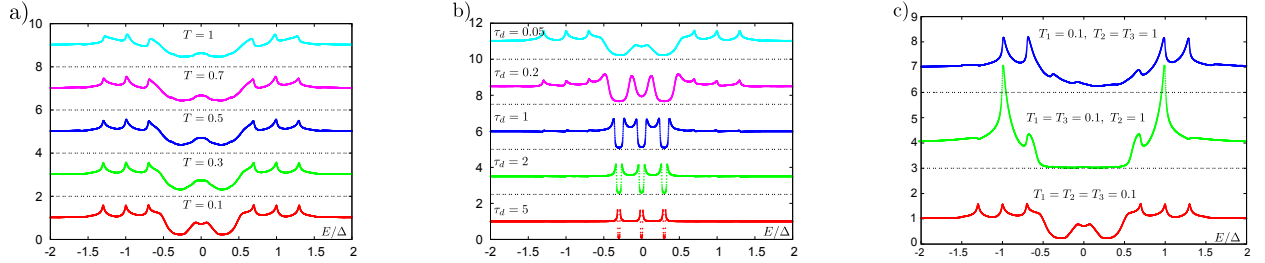


Figure 194: (Color online.) Discussion of the pseudogap features of the NDOS for voltage $V = 0.3$ and $\varphi_Q = \pi$, (a) for different transparencies in the symmetric junction, from the top $T = 1, 0.7, 0.5, 0.3, 0.1$, (b) for different strengths of decoherence, from the top $\tau_d = 0.05, 0.2, 1, 2, 5$, and (c) for different transparency symmetries: top $T_1 = 0.1, T_2 = T_3 = 1$; middle $T_1 = T_3 = 0.1, T_2 = 1$; and bottom $T_1 = T_2 = T_3 = 0.1$. The different curves are offset for clarity and the dotted horizontal lines indicate the offset, $\mathcal{N}(E) = 0$.

As predicted in Sec. 34.2, the NDOS is a periodic function of φ_Q , with period 2π , as can be seen in Figs. 190(a) and 190(b). Pronounced differences between the NDOS at $\varphi_Q = 0$ and $\varphi_Q = \pi$ can be seen in Fig. 191. The height of the large peaks at $E = \pm\Delta, \pm(\Delta + V), \pm(\Delta - V)$ depends on φ_Q . While the position of these MAR resonances remains unchanged, the structure of the NDOS in-between the peaks is dramatically affected. In particular, we focus on two dips in the NDOS, centered at $E = \pm V$, shown in Figs. 191(b) and 191(c). As a function of the quartet phase, the dips are shown in Fig. 190(a) to be fully formed at $\varphi_Q = \pi$, while disappearing at $\varphi_Q = 0$. Fixing the quartet phase at $\varphi_Q = \pi$, we show the dependence of the

NDOS on voltage in Fig. 190(c). Upon lowering the voltage below the threshold $V = \Delta/2$, the dips emerge at $E = \pm V$. Upon further lowering the voltage, the dips merge into a large, well defined pseudogap with sharp edges, seen in Fig. 191(d). Within the pseudogap the NDOS is not flat. An interference pattern develops characterized by small, isolated peaks in the NDOS that depend strongly on the voltage. These small structures are diminished by decoherence and may be ascribed to local as well as non-local MAR processes.

The shape of the structures at $E = \pm V$ for $V \lesssim \Delta/2$ is similar to that of other dips of the NDOS and does not remind of the sharp-edged shape of the bulk gap. However, they are unique in that they emerge upon the sharp threshold $V \leq \Delta/2$. The same threshold marks the limit where the quartet process can give rise to bound states in the junction, that do not couple to the continuum. Such bound states support a non-dissipative transport of quasiparticle quartets. The structures at $E = \pm V$ are absent for $V = 0.7$, see Fig. 190(b). Despite their shape, we consider it justified to refer to these dips at $E = \pm V$ as pseudogaps as they signal the presence of non-dissipative coherent processes.

The variation with the phase φ_Q of the amplitude and widths of the peaks in the NDOS can be attributed to phase-dependent MAR processes and to the quartet process. The phase-dependent MAR processes [8] can be understood by an interference between two distinct paths that transport a quasiparticle across the energy gap. An example is given in Fig. 192, where the third-order MAR process between S_1 and S_2 interferes with the process involving an Andreev reflection at the S_2 - S_3 interface and a cross-Andreev reflection in S_2 . The interference gives rise to the phase factor $e^{i(\varphi_1 + \varphi_3)}$. The phase-MAR processes can be interpreted as binding, not only Cooper pairs to quasiparticles (as in usual MAR), but also quartets to quasiparticles.

The NDOS for $\varphi_Q = \pi$ at low voltage can be fitted by the NDOS of a single superconducting electrode with a gap δ , taken as the fitting parameter: $\mathcal{N}(E) = |x|/\sqrt{x^2 - \delta^2}$ for $|x| > \delta$. The fit for $\delta = 0.5$ is shown as a thin black dotted line in Fig. 191(d). The fit reproduces the shape of the NDOS, except at the peaks, $E = \pm\Delta, \pm(\Delta + V), \pm(\Delta - V)$, that are due to MAR processes. The curves for $V = 0.1$ show that the NDOS resembles the NDOS of a junction at equilibrium when $\varphi_Q = \pi$.

A qualitative understanding can be obtained by appealing to the equilibrium NDOS, as calculated by the same model in Ref. [13] as a function of the two phases φ_1, φ_3 , and measured in Ref. [12]. It is found that at low transparency, for a symmetric TTJ, the Andreev spectrum crosses zero energy in two points situated on the line $\varphi_Q = 0$. Those regions extend in the phase plane for finite transparency, and the feature is robust against weak asymmetry. The proximity-induced minigap closes in some regions of the phase plane, particularly along the curve $\varphi_Q = 0$, but remains open for $\varphi_Q = \pi$. At small voltage, in the adiabatic regime, we expect a minigap at $\varphi_Q = \pi$ and not at $\varphi_Q = 0$. Qualitatively, the equilibrium Green's function of the node is given by $\check{G}_c(\varphi_1, \varphi_3) = (\omega - \check{\Sigma}(\omega))^{-1}$. The self-energy can then be averaged to account for the rotation of the phase, $\varphi_1 - \varphi_3 = \frac{4eV}{\hbar}t$. This effective node Green's function $\langle \check{G}_c \rangle(\varphi_Q)$ depends only on the quartet phase. The corresponding NDOS is plotted in Fig. 193, and matches qualitatively the exact calculation shown in Fig. 191(d). The sharp edges of the minigap are reproduced. The approximation is valid when the Andreev bound states change adiabatically with the variation of the phase, while preserving their occupation. For this reason, it cannot recover the exact peak amplitudes and the oscillations that reflect non-adiabatic features, e.g. Landau-Zener transitions between Andreev bound states and MAR processes.

The effect of transmission, decoherence and asymmetry on the pseudogaps is explored in Fig. 194. Fig. 194(a) shows the relative robustness of the pseudogap structures as a function of transparency in the symmetric junction. The position of the pseudogaps remains unchanged and only the depth reduces slightly as the transmission increases.

The dependence of the pseudogaps on the strength of decoherence is illuminating. Fig. 194(b) shows that at large decoherence the NDOS presents three minigaps, corresponding to

three decoupled NS interfaces. As the decoherence strength decreases, the three minigaps broaden, and eventually, at $\tau_d \ll 1$, the three structures merge forming the two broad pseudogaps observed for the coherent junction at $V = 0.3$.

The origin of the pseudogap structures in the NDOS can be further probed by introducing an asymmetry in the transmission of the contacts. Fig. 194(c) shows the situation when one of the contacts, or two of them, are fully transparent, while the other contacts have low transparency $T = 0.1$. In this case, the two pseudogaps at $V = 0.3$ are washed away. This is an additional argument for the importance of non-local coherent transport processes in determining the structure of the NDOS. The dependence on symmetry suggests that pseudogaps originate from transport processes that involve all three superconductors.

34.6 Conclusions.

We have presented a detailed study of the NDOS in a three-terminal Josephson junction under commensurate bias, $(V_1, V_2, V_3) = (V, 0, -V)$.

Using a phenomenological argument, we have shown that in general, the transport in this regime depends periodically on the phase combination, $\varphi_Q = (\varphi_1 - \varphi_2) + (\varphi_3 - \varphi_2)$, that is a constant of motion. The phase governs non-local transport of quartets, where the elementary transport process exchanges four quasiparticles between all three superconductors. We argue that the quartet phase, φ_Q , can be tuned by the current flow in S_2 .

Using the circuit theory formulation of non-equilibrium equations of superconducting transport, we reveal the NDOS, first for a conventional two-terminal junction, for comparison and as a benchmark of the method, and subsequently for the three-terminal junction, discussing its features for a wide range of parameters.

The complicated structure of the NDOS is greatly affected by electron-hole decoherence. For small decoherence, the two- and three-terminal junctions have in common a complicated structure of peaks that we have shown to correspond to MAR resonances. We resolve the position of resonances in terms of bias and the order of MAR processes. The additional terminal gives rise to additional MAR channels in the three-terminal junction.

For large decoherence, we have shown that each NS interface contributes a minigap structure positioned around the chemical potential of the terminal. The interfaces appear decoupled in first approximation, with corrections arising due to finite decoherence. We have identified two transport regimes separated by the bias threshold $V = \Delta/2$. For low bias, the weaker coupling to the quasiparticle continuum gives rise to sharp features, while for high bias the coupling to the continuum is strong and the sharp edges are rounded.

The central result of our study is the dependence of the NDOS in the three-terminal junction on φ_Q . At $\varphi_Q = \pi$ and at small bias voltage the NDOS presents a sharp-edged pseudogap that forms around $E = 0$. The sharp pseudogap bifurcates into a pair of pseudogaps for increasing bias, before vanishing for $V > \Delta/2$. Our predictions for the NDOS can be verified experimentally by tunneling spectroscopy. The presence of pseudogaps and the possibility to tune the NDOS by controlling the stationary phase in the presence of voltage bias has no equivalent in the two-terminal device, as it originates from non-local transport processes involving all three superconductors. In a subsequent contribution we will supplement this analysis of the NDOS by a discussion of the current flowing in the junction, focusing on the non-local and non-dissipative current that is discussed as a function of phase φ_Q and voltage V . The tunable and non-local properties of the coherent transport supported by the three-terminal junction recommend it as an interesting superconducting circuit element that may inspire quantum engineering applications.

34.7 Fourier transforms

Green's functions generally depend on two time-space coordinates $\check{G}(\mathbf{x}_1, \mathbf{x}_2, t_1, t_2)$. In the circuit theory model the junction is separated into regions where transport is uniform and can be characterized by a coordinate independent Green's function, $\check{G}(t_1, t_2)$. These are the Green's functions that describe the terminals and the Green's function of the central node.

The Green's functions are related by equations originating from the Usadel equation where matrix products are defined in the Keldysh-Nambu- t_1 - t_2 space. Since the time indices are continuous, matrix products take the form of a convolution in time, $\check{G} = \check{G}_1 \circ \check{G}_2$,

$$\check{G}(t_1, t_2) = \int dt \check{G}_1(t_1, t) \check{G}_2(t, t_2). \quad (34.16)$$

In energy representation,

$$\check{G}(E_1, E_2) = \int dt_1 dt_2 e^{i(E_1 t_1 - E_2 t_2)} \check{G}(t_1, t_2), \quad (34.17)$$

$$\check{G}(E_1, E_2) = \int \frac{dE}{2\pi} \check{G}_1(E_1, E) \check{G}_2(E, E_2). \quad (34.18)$$

Under stationary transport conditions the Green's function depends only on $(t_1 - t_2)$, or alternatively, only on one energy $G(E_1, E_2) = G(E_1) \delta(E_1 - E_2)$. In this case, the convolution reduces to a simple matrix product of Keldysh-Nambu Green's functions defined at a given energy. The equations are therefore easily implemented numerically.

Our work addresses transport in non-stationary conditions. As discussed in Sec. 34.3, the transport is periodic and described by the Josephson frequency $\omega_0 = 2eV/\hbar$. We define the following double time Fourier transform to a representation in energy and harmonics of ω_0 , (here we have set $\hbar = 1$)

$$\check{G}(E, n, m) = \frac{1}{T} \int_{-\infty}^{\infty} dt_1 \int_{-T-t_1}^{T-t_1} dt_2 e^{i(E+n\omega_0)t_1} e^{-i(E+m\omega_0)t_2} \check{G}(t_1, t_2), \quad T = \frac{2\pi}{\omega_0} \quad (34.19)$$

The restriction in integration over t_2 implements the condition, $-T/2 < (t_1 + t_2)/2 < T/2$, valid due to periodicity, $\check{G}(t_1, t_2) = \check{G}(t_1 + T, t_2 + T)$.

With the above transformation it can be shown that the convolution product becomes a product of matrices defined at a given energy E , with size corresponding to the Keldysh-Nambu- n - m space, $\check{G} = \check{G}_1 \circ \check{G}_2$,

$$\check{G}(E, n, m) = \sum_p \check{G}_1(E, n, p) \check{G}_2(E, p, m). \quad (34.20)$$

We have used this expression to find the numerical solution of the non-stationary transport problem, further detailed in Sec. 34.3.

Part XI

Adds

A Введение и обзор предмета

Приведу общие важные соображения о том, как я вижу предмет.

(все это уже проявлено выше, тут лишь четкие указания для полноты и для тех, у кого вопросы в том, что мне итак очевидно.)

A.1 Еще мотивация

Обсудим все, что нас мотивирует для изучения предмета.

A.2 Мышление профессионала в

(потом раскрою)

A.2.1 Суть предмета

A.2.2 Отношение к предмету (!?)

A.2.3 Способы заработать, зная предмет

A.2.4 Использование предмета в обычной жизни (!)

Укажем, как предмет используется в обычной жизни.

A.2.5 Актуальнейшие приложения

A.2.6 Построение с нуля

(потом раскрою, еще я не профессионал, а вопрос этот самый профессиональный)

A.2.7 Способы догадаться до всех главных идей

незаменимая часть нормального понимания предмета.

(потом раскрою)

A.2.8 Особенности эффективного изучения

Осудим, что лучше понимать, чтобы эффективно учить предмет.

Способы изучения предмета

(потом раскрою)

Необходимые темы для

(потом раскрою)

Дополнительные темы для

(потом раскрою)

A.2.9 Особенности эффективного преподавания (???)

Обсудим, каким образом наиболее эффективно преподавать.

A.3 Acknowledgements

Currently, no one except me has worked on the sections of this note (with the exception of sections taken from books).

A.4 Literature

A.4.1 Основная

Основная обучающая

Задачники

Литература крепкого минимума

A.4.2 Дополнительная

Дополнительная по теории

В помощь

Статьи о теоретических методах

Книги из статьи Беенаккера про силу казимира в Джозефсоне

References

- [1] I. V. Krive, I. A. Romanovsky, E. N. Bogachek, and U. Landman, *Phase-controlled force and magnetization oscillations in superconducting ballistic nanowires*, Phys. Rev. Lett. **92**, 126802 (2004).
- [2] I. V. Krive, S. I. Kulinich, R. I. Shekhter, and M. Jonson, *Charge and spin effects in mesoscopic Josephson junctions*, Low Temp. Phys. **30**, 554 (2004).
- [3] A.V. Parafilo, I. V. Krive, E. N. Bogachek, and U. Landman, *Superconducting phase-dependent force in SNS junctions with a movable scatterer*, Low Temp. Phys. **38**, 446 (2012).
- [4] A. Bulgac, P. Magierski, and A. Wirzba, *Fermionic Casimir effect in case of Andreev reflection*, Europhys. Lett. **72**, 327 (2005).
- [5] A. Wirzba, A. Bulgac, and P. Magierski, *Casimir interaction between normal or superfluid grains in the Fermi sea*, J. Phys. A **39**, 6815 (2006).
- [6] H. B. G. Casimir, *On the attraction between two perfectly conducting plates*, Proc. Royal Netherl. Acad. Arts Sciences **51**, 793 (1948).

- [7] J.-S. Caux, H. Saleur, and F. Siano, *Josephson current in Luttinger liquid-superconductor junctions*, Phys. Rev. Lett. **88**, 106402 (2002).
- [8] J. N. Fuchs, A. Recati, and W. Zwerger, *Oscillating Casimir force between impurities in one-dimensional Fermi liquids*, Phys. Rev. A **75**, 043615 (2007).
- [9] D. Zhabinskaya, J. M. Kinder, and E. J. Mele, *Casimir effect for massless fermions in one dimension: A force-operator approach*, Phys. Rev. A **78**, 060103(R) (2008).
- [10] G. Plunien, B. Müller, and W. Greiner, *The Casimir Effect*, Phys. Rep. **134**, 87 (1986).
- [11] V. M. Mostepanenko and N. N. Trunov, *The Casimir Effect and its Applications* (Clarendon Press, Oxford, 1997).
- [12] For the electromagnetic Casimir effect the case $\phi = \pi$ in Eq. (??) appears if one imposes Dirichlet boundary conditions on one mirror and Neumann boundary conditions on the other, see T. Boyer, *Casimir forces and boundary conditions in one dimension: Attraction, repulsion, Planck spectrum, and entropy*, Am. J. Phys. **71**, 990 (2003).
- [13] The ϕ -dependence of the Casimir-Josephson force (??) can be checked with the corresponding expression for the supercurrent, $dF/d\phi = -(\hbar/2e)dI/dL$, in agreement with the known result [26] $I = (gev_F/L)\phi/2\pi$ in the long-junction limit.
- [14] J. Bardeen and J. L. Johnson, *Josephson current flow in pure superconducting-normal-superconducting junctions*, Phys. Rev. B **5**, 72 (1972).
- [15] L. G. Aslamasov, A. I. Larkin, and Yu. N. Ovchinnikov, *Josephson effect in superconductors separated by a normal metal*, [Sov. Phys. JETP **28**, 171 (1969)].
- [16] P. W. Brouwer and C. W. J. Beenakker, *Anomalous temperature dependence of the supercurrent through a chaotic Josephson junction*, Chaos, Solitons & Fractals **8**, 1249 (1997).
- [17] A. Levchenko, A. Kamenev, and L. Glazman, *Singular length dependence of critical current in superconductor/normal-metal/superconductor bridges*, Phys. Rev. B **74**, 212509 (2006).
- [18] C. W. J. Beenakker, *Universal limit of critical-current fluctuations in mesoscopic Josephson junctions*, Phys. Rev. Lett. **67**, 3836 (1991); *Three “universal” mesoscopic Josephson effects*, arXiv:cond-mat/0406127.
- [19] J. Bardeen, R. Kümmel, A. E. Jacobs, and L. Tewordt, *Structure of vortex lines in pure superconductors*, Phys. Rev. **187**, 556 (1969).
- [20] C. W. J. Beenakker and H. van Houten, *The superconducting quantum point contact*, arXiv:cond-mat/0512610.
- [21] A. F. Andreev, *Thermal conductivity of the intermediate state of superconductors*, Sov. Phys. JETP **19**, 1228 (1964).
- [22] In a diffusive point contact (mean free path $\ell \ll L$) the force is dominated by the L -dependence of the conductance: $F \simeq -\Delta_0(\hbar/e^2)dG/dL \simeq N\Delta_0\ell/L^2$.

From phase-Controlled Force and Magnetization Oscillations in Superconducting Ballistic Nanowires.

References

- [1] U.Landman et al., Science **248**, 454 (1990).
- [2] see review in: N.Agrait, A.L. Yeyati, and J.M. van Ruitenbeek, Phys. Reports 377,8103 (2003).
- [3] G.Rubio, N.Agrait and S.Vieira, Phys.Rev.Lett. **76**, 2302 (1996).
- [4] C.Yannouleas and U.Landman, J.Phys.Chem.B **101**, 5780 (1997).
- [5] C.A. Stafford, D.Baeriswyl, and J.Bürki, Phys.Rev.Lett. **79**, 2863 (1997).
- [6] C.Yannouleas, E.N.Bogachek, and U.Landman, Phys. Rev. B **57**, 4872 (1998).
- [7] E.N.Bogachek, A.M.Zagoskin, and I.O.Kulik, Fiz. Nizk. Temp. **16**, 1484 (1990); [Sov.J.Low Temp.Phys. **16**, 796 (1990).]; J.I. Pascual et al. Science 267, 1793 (1995).
- [8] D.Leckband and J. Israelachvili, Q.Rev.Biophys. **34**, 105 (2001).
- [9] C.N.Lau et al., Phys.Rev.Lett. **87**, 217003 (2001).
- [10] C.W.J.Beenakker and H.van Houten, Phys.Rev.Lett. **66**, 3056 (1991).
- [11] P.Samuelsson et al., Phys. Rev.B **62**, 1319 (2000).
- [12] I.O.Kulik, Zh.Eksp. Teor. Fiz. **57**, 1745 (1969) [Sov. Phys. JETP **30**, 944 (1970)].
- [13] D.L.Maslov et al., Phys. Rev. B **53**, 1548 (1996).
- [14] The connection between the supercurrent in a long ballistic SNS junction and the Casimir energy was considered by J.-S.Caux, H.Saleur, and F.Siano, Phys.Rev.Lett. **88**, 106402 (2002).
- [15] A.Furusaki, H.Takayanagi, and M.Tsukuda, Phys. Rev. Lett.**67**, 132 (1991).
- [16] H.A.Blom et al., Phys.Rev.B **57**, 9995 (1998).
- [17] For a long junction ($L \gg \xi_0$) with only a very small number of channels ($N_{\perp} \sim 1$), the force is negligibly small $\tilde{F}_A \sim \hbar v_F / L^2 \sim (\varepsilon_F / \lambda_F)(\lambda_F / L)^2$. On the other hand, for a relatively short) i.e. $L \sim 0.1\xi_0 \sim 10$ nm) ballistic NW, the SC-induced contribution to the force can become quite large. For such a NW with a diameter $2R = L$ we estimate) using the Sharvin expression $k_F^2 / 4\pi$, where $k_F(Pb) \sim 10^8 \text{ cm}^{-1}$) that the number of channels is ~ 1500 . From Eqs. (4) and (5) we obtain an AF (evaluated at $\varphi = \pi$) of 10^{-1} nN for $T = 0.1 \text{ K}$, and half this force for $T = 6 \text{ K}$.
- [18] When the magnetic field influences both the normal and the superconducting parts of a SNS junction the supercurrent may be suppressed even for weak fields. In addition under such circumstances the AK level structure may be globally destroyed since the phase difference in the presence of a magnetic field parallel to the SNS interface is changed along the SN interfaces. (see G.A. Gogadze and I.O. Kulik, Zh. Eksp. Teor. Fiz. **60**, 1819 (1971) [Sov. Phys. JETP **33**, 984 (1971)]). The above motivates the restriction to a magnetic field applied only to the normal part of the SNS junction.
- [19] P.Sandström and I.V.Krive, Ann.Phys. **257**, 18 (1997).

- [20] A.Kadigrobov et al., Phys.Rev.B **60**, 14593 (1999).
- [21] A.I.Buzdin, L.N.Bulayevskii, and S.V.Panyukov, Pis'ma Zh. Eksp. Teor. Fiz. **35**, 147 (1982) [JETP Lett. **35**, 178 (1982)].
- [22] S.H.Pan, E.W.Hudson, and J.C.Davis, Appl. Phys. Lett. **73**, 2992 (1998).
- [23] Rubio-Bollinger et al. Phys.Rev.B **67**, 121407 (2003).
- [24] O.Naaman, W.Teizer, and R.C.Dynes Phys. Rev. Lett. **87**, 09704 (2001).
- [25] Y.De Wilde et al., Phys. Rev. Lett. **80**, 153 (1998).

О приложениях

А.5 Обзор

(потом раскрою)

А.5.1 Популярное введение в предмет

Обсудим, что из себя представляет предмет наиболее кратко, выделяя самую суть.

А.5.2 обзор теоретических подходов

такие-то есть, такие полезные, такие - нет.

А.5.3 Обзор дальнейших развитий

А.5.4 Связи с другими науками

Обсудим связи с разделами
(потом раскрою)

А.5.5 Описание записи

Общее описание записи

Общие особенности записи

Особенности глав и разделов

Первая часть про предмет в двух словах

Вторая часть

Часть про приложения

какие вообще приложения я разбирал?

Обозначения и константы

А.5.6 Об истории предмета

Обсудим вкратце историю развития...

А.6 Головоломки на разные случаи

Тут я приведу вопросы, которые очень интересно использовать в разных случаях в жизни.

А.6.1 Головоломки для приятной беседы

А.6.2 Бытовые головоломки

А.6.3 Принципиальные головоломки

А.6.4 Головоломки о деталях

А.6.5 Головоломки для освоения типичных понятий

Допустим, видим человека, который в принципе хотел бы понять суть, опишем, какие в какой последовательности будем задавать, чтобы в типичном случае ему было бы интереснее въезжать. В частности это могут быть младшие студенты или редко школьники.

Головоломки для освоения школьного уровня

Головоломки для освоения уровня теормина

Головоломки для освоения других особенностей

А.6.6 Головоломки для демотивации

А.6.7 Головоломки для доказательства, что человек не понимает предмет

(тоже такие можно собрать, иногда очень и полезно это понимать, но очень редко.)

В Дополнения

В.0.1 title

C Bibliography

References

Bibliography from “Josephson current between topological and conventional superconductors”

References

- [1] X.-L. Qi, S.-C. Zhang, Rev. Mod. Phys. **83**, 1057 (2011).
- [2] M. Z. Hasan, C. L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
- [3] A. Yu. Kitaev, Physics-Uspekhi **44**, 131 (2001).
- [4] C. Nayak, S. H. Simon, A. Stern, M. Freedman, S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).
- [5] J. Alicea, Rep. Prog. Phys. **75**, 076501 (2012).
- [6] C. W. J. Beenakker, Annu. Rev. Con. Mat. Phys. **4**, 113 (2013).
- [7] L. Fu and C. L. Kane, Phys. Rev. Lett. **102**, 216403 (2009).
- [8] A. R. Akhmerov, J. Nilsson, and C. W. J. Beenakker, Phys. Rev. Lett., **102**, 216404 (2009).
- [9] K. T. Law, P. A. Lee, and T. K. Ng, Phys. Rev. Lett., **103**, 237001 (2009).
- [10] P. A. Ioselevich, M. V. Feigelman, New J. Phys. **15**, 055011 (2013).
- [11] P. A. Ioselevich, P. M. Ostrovsky, M. V. Feigelman, Phys. Rev. B **86**, 035441 (2012).
- [12] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Science **336**, 1003 (2012).
- [13] A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, and H. Shtrikman, Nature Physics **8**, 887 (2012).
- [14] M. T. Deng, C. L. Yu, G. Y. Huang, M. Larsson, P. Caroff, H. Q. Xu, Nano Lett. **12**, 6414 (2012).
- [15] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Phys. Rev. Lett. **105**, 77001 (2010).
- [16] Y. Oreg, G. Refael, F. von Oppen, Phys. Rev. Lett. **105**, 177002 (2010).
- [17] H.-J. Kwon, K. Sengupta, V. M. Yakovenko, Low Temp. Phys. **30**, 613 (2004).
- [18] L. Fu and C. L. Kane, Phys. Rev. B. **79**, 161408 (2009).
- [19] P. A. Ioselevich, M. V. Feigelman, Phys. Rev. Lett. **106**, 077003 (2011).
- [20] L. Jiang, D. Pekker, J. Alicea, G. Refael, Y. Oreg, and F. von Oppen, Phys. Rev. Lett., **107**, 236401 (2011).
- [21] Y. Ran, P. Hosur, and A. Vishwanath, Phys. Rev. B **84**, 184501 (2011).
- [22] V. Ambegaokar, A. Baratoff, Phys. Rev. Lett. **10**, 486 (1963).

- [23] C. W. J. Beenakker and H. van Houten, *Phys. Rev. Lett.* **66**, 3056 (1991).
- [24] L. -F. Chang and P. F. Bagwell, *Physical Review B*, **49**, 22 (1994).
- [25] P. W. Brouwer, C. W. J. Beenakker, *Chaos, Solitons and Fractals* **8**, 1249 (1997).
- [26] M. Wimmer, A. R. Akhmerov, J. P. Dahlhaus, and C. W. J. Beenakker, *New J. Phys.* **13** 053016 (2011).
- [27] A. R. Akhmerov, J. P. Dahlhaus, F. Hassler, M. Wimmer, and C. W. J. Beenakker, *Phys. Rev. Lett.* **106**, 057001 (2011).
- [28] D. I. Pikulin and Yuli V. Nazarov, *JETP Letters*, **94**, 9, 693-697 (2011).
- [29] F. M. Dyson, *J. Math. Phys.* **3**, 1199 (1962).

Bibliography from “Andreev to Majorana bound states in hybrid superconductor-semiconductor nanowires”

References

- [1] H. Kamerlingh Onnes, *Comm. Phys. Lab. Univ. Leiden* **122**, 122 (1911).
- [2] D. van Delft and P. Kes, *Phys. Today* **63**, 38 (2010).
- [3] P.-G. De Gennes, *Superconductivity of metals and alloys* (CRC Press, 2018).
- [4] M. Tinkham, *Introduction to superconductivity* (Courier Corporation, 2004).
- [5] J. D. Martin, *Phys. Today* **72**, 30 (2019).
- [6] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Phys. Rev.* **106**, 162 (1957).
- [7] L. N. Cooper, *Phys. Rev.* **104**, 1189 (1956).
- [8] V. L. Ginzburg and L. D. Landau, *Zh. Eksp. Teor. Fiz.* **20**, 1064 (1950).
- [9] M. Cyrot, *Rep. Prog. Phys.* **36**, 103 (1973).
- [10] C. Caroli, P. D. Gennes, and J. Matricon, *Phys. Lett.* **9**, 307 (1964).
- [11] L. Yu, *Acta. Phys. Sin.* **21**, 75 (1965).
- [12] H. Shiba, *Prog. Theor. Phys.* **40**, 435 (1968).
- [13] A. Rusinov, *Sov. Phys. JETP* **9**, 85 (1969).
- [14] G. E. Blonder, M. Tinkham, and T. M. Klapwijk, *Phys. Rev. B* **25**, 4515 (1982).
- [15] A. F. Andreev, *Sov. Phys. JETP* **19**, 1228 (1964).
- [16] A. F. Andreev, *Sov. Phys. JETP* **22**, 18 (1966).
- [17] P. De Gennes and D. Saint-James, *Phys. Lett.* **4** (1963).
- [18] I. O. Kulik, *Sov. Phys. JETP* **30**, 944 (1970).

- [19] J. A. Sauls, *Philos. Trans. Royal Soc. A* **376** (2018).
- [20] M. Z. Hasan and C. L. Kane, *Rev. Mod. Phys.* **82**, 3045 (2010).
- [21] X.-L. Qi and S.-C. Zhang, *Rev. Mod. Phys.* **83**, 1057 (2011).
- [22] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, *Phys. Rev. Lett.* **49**, 405 (1982).
- [23] M. Leijnse and K. Flensberg, *Semicond. Sci. Technol.* **27**, 124003 (2012).
- [24] J. Alicea, *Rep. Prog. Phys.* **75**, 076501 (2012).
- [25] C. Beenakker, *Annu. Rev. Cond. Mat. Phys.* **4**, 113 (2013).
- [26] M. Sato and S. Fujimoto, *J. Phys. Soc. Jpn* **85**, 072001 (2016).
- [27] R. Aguado, *Riv. Nuovo Cimento* **40**, 523 (2017).
- [28] M. Sato and Y. Ando, *Rep. Prog. Phys.* **80**, 076501 (2017).
- [29] M. Salomaa and G. Volovik, *Phys. Rev. B* **37**, 9298 (1988).
- [30] G. E. Volovik and G. Volovik, *The universe in a helium droplet*, Vol. 117 (Oxford University Press New York, 2009).
- [31] N. Read and D. Green, *Phys. Rev. B* **61**, 10267 (2000).
- [32] A. Y. Kitaev, *Phys. Usp.* **44**, 131 (2001).
- [33] M. Sato and S. Fujimoto, *Phys. Rev. B* **79**, 094504 (2009).
- [34] E. Majorana, *Il Nuovo Cimento* **14**, 171 (1937).
- [35] Y. Nishida, L. Santos, and C. Chamon, *Phys. Rev. B* **82**, 144513 (2010).
- [36] R. Jackiw and P. Rossi, *Nucl. Phys. B* **190**, 681 (1981).
- [37] T. Fukui, K. Shiozaki, T. Fujiwara, and S. Fujimoto, *J. Phys. Soc. Jpn.* **81**, 114602 (2012).
- [38] A. Kitaev, *Ann. Phys.* **303**, 2 (2003).
- [39] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, *Rev. Mod. Phys.* **80**, 1083 (2008).
- [40] L. Fu and C. L. Kane, *Phys. Rev. Lett.* **100**, 096407 (2008).
- [41] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, *Phys. Rev. Lett.* **105**, 077001 (2010).
- [42] Y. Oreg, G. Refael, and F. von Oppen, *Phys. Rev. Lett.* **105**, 177002 (2010).
- [43] P. Středa and P. Šeba, *Phys. Rev. Lett.* **90**, 256601 (2003).
- [44] T. D. Stanescu and S. Tewari, *J. Phys.: Condens. Matter* **25**, 233201 (2013).
- [45] R. M. Lutchyn, E. P. A. M. Bakkers, L. P. Kouwenhoven, P. Krogstrup, C. M. Marcus, and Y. Oreg, *Nat. Rev. Mater.* **3**, 52 (2018).

- [46] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, *Science* **336**, 1003 (2012).
- [47] D. J. van Woerkom, A. Proutski, B. van Heck, D. Bouman, J. I. Väyrynen, L. I. Glazman, P. Krogstrup, J. Nygård, L. P. Kouwenhoven, and A. Geresdi, *Nat. Phys.* **13**, 876 EP (2017).
- [48] L. Tosi, C. Metzger, M. F. Goffman, C. Urbina, H. Pothier, S. Park, A. Levy Yeyati, J. Nygård, and P. Krogstrup, *Phys. Rev. X* **9**, 011010 (2019).
- [49] E. J. H. Lee, X. Jiang, R. Aguado, G. Katsaros, C. M. Lieber, and S. De Franceschi, *Phys. Rev. Lett.* **109**, 186802 (2012).
- [50] E. J. H. Lee, X. Jiang, M. Houzet, R. Aguado, C. M. Lieber, and S. De Franceschi, *Nat. Nanotechnol.* **9**, 79 (2014).
- [51] E. J. H. Lee, X. Jiang, R. Žitko, R. Aguado, C. M. Lieber, and S. De Franceschi, *Phys. Rev. B* **95**, 180502 (2017).
- [52] K. Grove-Rasmussen, G. Steffensen, A. Jellinggaard, M. H. Madsen, R. Žitko, J. Paaske, and J. Nygård, *Nat. Commun.* **9**, 2376 (2018).
- [53] Z. Su, A. Zarassi, J.-F. Hsu, P. San-Jose, E. Prada, R. Aguado, E. J. H. Lee, S. Gazibegovic, R. L. M. Op het Veld, D. Car, S. R. Plissard, M. Hocevar, M. Pendharkar, J. S. Lee, J. A. Logan, C. J. Palmstrøm, E. P. A. M. Bakkers, and S. M. Frolov, *Phys. Rev. Lett.* **121**, 127705 (2018).
- [54] C. Jünger, A. Baumgartner, R. Delagrangé, D. Chevallier, S. Lehmann, M. Nilsson, K. A. Dick, C. Thelander, and C. Schönenberger, *Commun. Phys.* **2**, 76 (2019).
- [55] A. C. Potter and P. A. Lee, *Phys. Rev. Lett.* **105**, 227003 (2010).
- [56] A. C. Potter and P. A. Lee, *Phys. Rev. B* **83**, 094525 (2011).
- [57] R. M. Lutchyn, T. D. Stanescu, and S. Das Sarma, *Phys. Rev. Lett.* **106**, 127001 (2011).
- [58] R. M. Lutchyn and M. P. A. Fisher, *Phys. Rev. B* **84**, 214528 (2011).
- [59] W. S. Cole, S. Das Sarma, and T. D. Stanescu, *Phys. Rev. B* **92**, 174511 (2015).
- [60] C. Reeg, D. Loss, and J. Klinovaja, *Phys. Rev. B* **96**, 125426 (2017).
- [61] C. Reeg, D. Loss, and J. Klinovaja, *Phys. Rev. B* **97**, 165425 (2018).
- [62] S. Vaitiekenas, M.-T. Deng, J. Nygård, P. Krogstrup, and C. M. Marcus, *Phys. Rev. Lett.* **121**, 037703 (2018).
- [63] A. E. Antipov, A. Bargerbos, G. W. Winkler, B. Bauer, E. Rossi, and R. M. Lutchyn, *Phys. Rev. X* **8**, 031041 (2018).
- [64] O. Dmytruk, D. Chevallier, D. Loss, and J. Klinovaja, *Phys. Rev. B* **98**, 165403 (2018).
- [65] H. Pan, J. D. Sau, T. D. Stanescu, and S. Das Sarma, *Phys. Rev. B* **99**, 054507 (2019).
- [66] A. Vuik, D. Eeltink, A. R. Akhmerov, and M. Wimmer, *New J. Phys.* **18**, 033013 (2016).

- [67] F. Domínguez, J. Cayao, P. San-Jose, R. Aguado, A. Levy Yeyati, and E. Prada, *npj Quant. Mater.* **2**, 13 (2017).
- [68] A. E. G. Mikkelsen, P. Kotetes, P. Krogstrup, and K. Flensberg, *Phys. Rev. X* **8**, 031040 (2018).
- [69] M. W. A. de Moor, J. D. S. Bommer, D. Xu, G. W. Winkler, A. E. Antipov, A. Bargerbos, G. Wang, N. van Loo, R. L. M. O. het Veld, S. Gazibegovic, D. Car, J. A. Logan, M. Pendharkar, J. S. Lee, E. P. A. M. Bakkers, C. J. Palmstrøm, R. M. Lutchyn, L. P. Kouwenhoven, and H. Zhang, *New J. Phys.* **20**, 103049 (2018).
- [70] B. D. Woods, T. D. Stanescu, and S. Das Sarma, *Phys. Rev. B* **98**, 035428 (2018).
- [71] S. D. Escribano, A. Levy Yeyati, and E. Prada, *Beilstein J. Nanotechnol.* **9**, 2171 (2018).
- [72] G. W. Winkler, A. E. Antipov, B. van Heck, A. A. Soluyanov, L. I. Glazman, M. Wimmer, and R. M. Lutchyn, *Phys. Rev. B* **99**, 245408 (2019).
- [73] E. Prada, P. San-Jose, and R. Aguado, *Phys. Rev. B* **86**, 180503(R) (2012).
- [74] G. Kells, D. Meidan, and P. W. Brouwer, *Phys. Rev. B* **86**, 100503 (2012).
- [75] T. D. Stanescu and S. Tewari, *Phys. Rev. B* **87**, 140504(R) (2013).
- [76] D. Roy, N. Bondyopadhyaya, and S. Tewari, *Phys. Rev. B* **88**, 020502 (2013).
- [77] C. Fleckenstein, F. Domínguez, N. Traverso Ziani, and B. Trauzettel, *Phys. Rev. B* **97**, 155425 (2018).
- [78] T. D. Stanescu and S. Tewari, *Phys. Rev. B* **89**, 220507 (2014).
- [79] C.-X. Liu, J. D. Sau, T. D. Stanescu, and S. Das Sarma, *Phys. Rev. B* **96**, 075161 (2017).
- [80] F. Peñaranda, R. Aguado, P. San-Jose, and E. Prada, *Phys. Rev. B* **98**, 235406 (2018).
- [81] C. Reeg, O. Dmytruk, D. Chevallier, D. Loss, and J. Klinovaja, *Phys. Rev. B* **98**, 245407 (2018).
- [82] C.-X. Liu, J. D. Sau, and S. Das Sarma, *Phys. Rev. B* **97**, 214502 (2018).
- [83] C. Moore, T. D. Stanescu, and S. Tewari, *Phys. Rev. B* **97**, 165302 (2018).
- [84] C. Moore, C. Zeng, T. D. Stanescu, and S. Tewari, *Phys. Rev. B* **98**, 155314 (2018).
- [85] A. Vuik, B. Nijholt, A. R. Akhmerov, and M. Wimmer, *SciPost Phys.* **7**, 61 (2019).
- [86] O. A. Awoga, J. Cayao, and A. M. Black-Schaffer, *Phys. Rev. Lett.* **123**, 117001 (2019).
- [87] J. Avila, F. Peñaranda, E. Prada, P. San-Jose, and R. Aguado, *Commun. Phys.* **2**, 133 (2019).
- [88] T. D. Stanescu and S. Tewari, *Phys. Rev. B* **100**, 155429 (2019).
- [89] B. D. Woods, J. Chen, S. M. Frolov, and T. D. Stanescu, *Phys. Rev. B* **100**, 125407 (2019).
- [90] E. Prada, R. Aguado, and P. San-Jose, *Phys. Rev. B* **96**, 085418 (2017).

- [91] M.-T. Deng, S. Vaitiekėnas, E. Prada, P. San-Jose, J. Nygård, P. Krogstrup, R. Aguado, and C. M. Marcus, [Phys. Rev. B **98**, 085125 \(2018\)](#).
- [92] P. San-Jose, J. Cayao, E. Prada, and R. Aguado, [Sci. Rep. **6**, 21427 \(2016\)](#).
- [93] G. Goldstein and C. Chamon, [Phys. Rev. B **84**, 205109 \(2011\)](#).
- [94] J. C. Budich, S. Walter, and B. Trauzettel, [Phys. Rev. B **85**, 121405 \(2012\)](#).
- [95] D. Rainis and D. Loss, [Phys. Rev. B **85**, 174533 \(2012\)](#).
- [96] M. Trif and Y. Tserkovnyak, [Phys. Rev. Lett. **109**, 257002 \(2012\)](#).
- [97] M. J. Schmidt, D. Rainis, and D. Loss, [Phys. Rev. B **86**, 085414 \(2012\)](#).
- [98] T. L. Schmidt, A. Nunnenkamp, and C. Bruder, [Phys. Rev. Lett. **110**, 107006 \(2013\)](#).
- [99] M. S. Scheurer and A. Shnirman, [Phys. Rev. B **88**, 064515 \(2013\)](#).
- [100] F. L. Pedrocchi and D. P. DiVincenzo, [Phys. Rev. Lett. **115**, 120402 \(2015\)](#).
- [101] O. Dmytruk, M. Trif, and P. Simon, [Phys. Rev. B **92**, 245432 \(2015\)](#).
- [102] M. Sekania, S. Plugge, M. Greiter, R. Thomale, and P. Schmitteckert, [Phys. Rev. B **96**, 094307 \(2017\)](#).
- [103] C. Knapp, T. Karzig, R. M. Lutchyn, and C. Nayak, [Phys. Rev. B **97**, 125404 \(2018\)](#).
- [104] P. P. Aseev, J. Klinovaja, and D. Loss, [Phys. Rev. B **98**, 155414 \(2018\)](#).
- [105] H.-L. Lai, P.-Y. Yang, Y.-W. Huang, and W.-M. Zhang, [Phys. Rev. B **97**, 054508 \(2018\)](#).
- [106] P. P. Aseev, P. Marra, P. Stano, J. Klinovaja, and D. Loss, [Phys. Rev. B **99**, 205435 \(2019\)](#).
- [107] M. T. Deng, S. Vaitiekėnas, E. B. Hansen, J. Danon, M. Leijnse, K. Flensberg, J. Nygård, P. Krogstrup, and C. M. Marcus, [Science **354**, 1557 \(2016\)](#).
- [108] F. Nichele, A. C. C. Drachmann, A. M. Whiticar, E. C. T. O’Farrell, H. J. Suominen, A. Fornieri, T. Wang, G. C. Gardner, C. Thomas, A. T. Hatke, P. Krogstrup, M. J. Manfra, K. Flensberg, and C. M. Marcus, [Phys. Rev. Lett. **119**, 136803 \(2017\)](#).
- [109] H. Zhang, Ö. Gül, S. Conesa-Boj, M. Nowak, M. Wimmer, K. Zuo, V. Mourik, F. K. de Vries, J. van Veen, M. W. A. de Moor, J. D. S. Bommer, D. J. van Woerkom, D. Car, S. R. Plissard, E. P. A. M. Bakkers, M. Quintero-Pérez, M. C. Cassidy, S. Koelling, S. Goswami, K. Watanabe, T. Taniguchi, and L. P. Kouwenhoven, [Nat. Commun. **8**, 16025 EP \(2017\)](#).
- [110] S. Nadj-Perge, I. K. Drozdov, B. A. Bernevig, and A. Yazdani, [Phys. Rev. B **88**, 020407 \(2013\)](#).
- [111] S. Nadj-Perge, I. K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A. H. MacDonald, B. A. Bernevig, and A. Yazdani, [Science **346**, 602 \(2014\)](#).
- [112] G. C. Ménard, S. Guissart, C. Brun, R. T. Leriche, M. Trif, F. Debontridder, D. Demaille, D. Roditchev, P. Simon, and T. Cren, [Nat. Commun. **8**, 2040 \(2017\)](#).

- [113] G. C. Ménard, A. Mesaros, C. Brun, F. Debontridder, D. Roditchev, P. Simon, and T. Cren, *Nat. Commun.* **10**, 2587 (2019).
- [114] A. Palacio-Morales, E. Mascot, S. Cocklin, H. Kim, S. Rachel, D. K. Morr, and R. Wiesendanger, *Sci. Adv.* **5** (2019).
- [115] L. Fu and C. L. Kane, *Phys. Rev. B* **79**, 161408 (2009).
- [116] J. Wiedenmann, E. Bocquillon, R. S. Deacon, S. Hartinger, O. Herrmann, T. M. Klapwijk, L. Maier, C. Ames, C. Brüne, C. Gould, A. Oiwa, K. Ishibashi, S. Tarucha, H. Buhmann, and L. W. Molenkamp, *Nat. Commun.* **7**, 10303 (2016).
- [117] H. J. Suominen, M. Kjaergaard, A. R. Hamilton, J. Shabani, C. J. Palmstrøm, C. M. Marcus, and F. Nichele, *Phys. Rev. Lett.* **119**, 176805 (2017).
- [118] L. Bretheau, C. O. Girit, H. Pothier, D. Esteve, and C. Urbina, *Nature* **499**, 312 (2013).
- [119] C. Janvier, L. Tosi, L. Bretheau, Ç. Ö. Girit, M. Stern, P. Bertet, P. Joyez, D. Vion, D. Esteve, M. F. Goffman, H. Pothier, and C. Urbina, *Science* **349**, 1199 (2015).
- [120] J.-D. Pillet, C. H. L. Quay, P. Morfin, C. Bena, A. Levy Yeyati, and P. Joyez, *Nat. Phys.* **6**, 965 (2010).
- [121] A. Eichler, M. Weiss, S. Oberholzer, C. Schönenberger, A. Levy Yeyati, J. C. Cuevas, and A. Martín-Rodero, *Phys. Rev. Lett.* **99** (2007).
- [122] T. Dirks, T. L. Hughes, S. Lal, B. Uchoa, Y.-F. Chen, C. Chialvo, P. M. Goldbart, and N. Mason, *Nat. Phys.* **7**, 386 (2011).
- [123] R. S. Deacon, Y. Tanaka, A. Oiwa, R. Sakano, K. Yoshida, K. Shibata, K. Hirakawa, and S. Tarucha, *Phys. Rev. Lett.* **104**, 076805 (2010).
- [124] J. C. Estrada Saldaña, A. Vekris, G. Steffensen, R. Žitko, P. Krogstrup, J. Paaske, K. Grove-Rasmussen, and J. Nygård, *Phys. Rev. Lett.* **121**, 257701 (2018).
- [125] M. Hays, G. de Lange, K. Serniak, D. J. van Woerkom, D. Bouman, P. Krogstrup, J. Nygård, A. Geresdi, and M. H. Devoret, *Phys. Rev. Lett.* **121**, 047001 (2018).
- [126] S. Plugge, A. Rasmussen, R. Egger, and K. Flensberg, *New J. Phys.* **19**, 012001 (2017).
- [127] I. C. Fulga, A. Haim, A. R. Akhmerov, and Y. Oreg, *New J. Phys.* **15**, 045020 (2013).
- [128] R. Kümmel, *Z. Phys. A* **218**, 472 (1969).
- [129] J. Xiang, A. Vidan, M. Tinkham, R. M. Westervelt, and C. M. Lieber, *Nature Nanotechnol.* **1**, 208 (2006).
- [130] J. Ridderbos, M. Brauns, A. Li, E. P. A. M. Bakkers, A. Brinkman, W. G. van der Wiel, and F. A. Zwanenburg, *Phys. Rev. Mater.* **3**, 084803 (2019).
- [131] T. S. Jespersen, M. L. Polianski, C. B. Sørensen, K. Flensberg, and J. Nygård, *New J. Phys.* **11**, 113025 (2009).
- [132] Y.-J. Doh, J. A. van Dam, A. L. Roest, E. P. A. M. Bakkers, L. P. Kouwenhoven, and S. De Franceschi, *Science* **309**, 272 (2005).

- [133] H. Y. Günel, I. E. Batov, H. Hardtdegen, K. Sladek, A. Winden, K. Weis, G. Panaitov, D. Grützmacher, and T. Schäpers, *J. App. Phys.* **112**, 034316 (2012).
- [134] M. F. Goffman, C. Urbina, H. Pothier, J. Nygård, C. M. Marcus, and P. Krogstrup, *New J. Phys.* **19**, 092002 (2017).
- [135] H. A. Nilsson, P. Samuelsson, P. Caroff, and H. Q. Xu, *Nano Lett.* **12**, 228 (2012).
- [136] M. T. Deng, C. L. Yu, G. Y. Huang, M. Larsson, P. Caroff, and H. Q. Xu, *Nano Lett.* **12**, 6414 (2012).
- [137] C. Beenakker, in *Transport phenomena in mesoscopic systems: proceedings of the 14th Taniguchi symposium, Shima, Japan, November 10-14, 1991* (Springer-Verlag, 1992).
- [138] C. W. J. Beenakker, *Phys. Rev. B* **46**, 12841 (1992).
- [139] K. K. Likharev, *Rev. Mod. Phys.* **51**, 101 (1979).
- [140] A. Furusaki and M. Tsukada, *Phys. Rev. B* **43**, 10164 (1991).
- [141] C. W. J. Beenakker and H. van Houten, *Phys. Rev. Lett.* **66**, 3056 (1991).
- [142] P. F. Bagwell, *Phys. Rev. B* **46**, 12573 (1992).
- [143] A. Furusaki, *Superlattices Microstruct.* **25**, 809 (1999).
- [144] R. Landauer, *Phys. Lett. A* **85**, 91 (1981).
- [145] B. D. Josephson, *Phys. Lett.* **1**, 251 (1962).
- [146] B. D. Josephson, *Adv. Phys.* **14**, 419 (1965).
- [147] F. Kos, S. E. Nigg, and L. I. Glazman, *Phys. Rev. B* **87**, 174521 (2013).
- [148] M. Hofheinz, F. Portier, Q. Baudouin, P. Joyez, D. Vion, P. Bertet, P. Roche, and D. Esteve, *Phys. Rev. Lett.* **106**, 217005 (2011).
- [149] T. Holst, D. Esteve, C. Urbina, and M. H. Devoret, *Phys. Rev. Lett.* **73**, 3455 (1994).
- [150] A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, *Phys. Rev. A* **69**, 062320 (2004).
- [151] D. Car, S. Conesa-Boj, H. Zhang, R. L. M. Op het Veld, M. W. A. de Moor, E. M. T. Fadaly, Ö. Gül, S. Kölling, S. R. Plissard, V. Toresen, M. T. Wimmer, K. Watanabe, T. Taniguchi, L. P. Kouwenhoven, and E. P. A. M. Bakkers, *Nano Lett.* **17**, 721 (2017).
- [152] W. Chang, S. M. Albrecht, T. S. Jespersen, F. Kuemmeth, P. Krogstrup, J. Nygård, and C. M. Marcus, *Nat. Nanotechnol.* **10**, 232 (2015).
- [153] G. L. R. Anselmetti, E. A. Martinez, G. C. Ménard, D. Puglia, F. K. Malinowski, J. S. Lee, S. Choi, M. Pendharkar, C. J. Palmstrøm, C. M. Marcus, L. Casparis, and A. P. Higginbotham, *Phys. Rev. B* **100**, 205412 (2019).
- [154] E. M. Spanton, M. Deng, S. Vaitiekėnas, P. Krogstrup, J. Nygård, C. M. Marcus, and K. A. Moler, *Nat. Phys.* **13**, 1177 (2017).
- [155] S. Hart, Z. Cui, G. Ménard, M. Deng, A. E. Antipov, R. M. Lutchyn, P. Krogstrup, C. M. Marcus, and K. A. Moler, *Phys. Rev. B* **100**, 064523 (2019).

- [156] R. Rifkin and B. S. Deaver, *Phys. Rev. B* **13**, 3894 (1976).
- [157] A. Jellinggaard, K. Grove-Rasmussen, M. H. Madsen, and J. Nygård, *Phys. Rev. B* **94**, 064520 (2016).
- [158] W. Chang, V. E. Manucharyan, T. S. Jespersen, J. Nygård, and C. M. Marcus, *Phys. Rev. Lett.* **110**, 217005 (2013).
- [159] R. Zitko, J. S. Lim, R. López, and R. Aguado, *Phys. Rev. B* **91**, 045441 (2015).
- [160] S. De Franceschi, L. Kouwenhoven, C. Schönenberger, and W. Wernsdorfer, *Nat. Nanotechnol.* **5**, 703 (2010).
- [161] A. C. Hewson, *The Kondo Problem to Heavy Fermions*, Cambridge Studies in Magnetism (Cambridge University Press, 1993).
- [162] M. R. Buitelaar, T. Nussbaumer, and C. Schönenberger, *Phys. Rev. Lett.* **89**, 256801 (2002).
- [163] T. Sand-Jespersen, J. Paaske, B. M. Andersen, K. Grove-Rasmussen, H. I. Jørgensen, M. Aagesen, C. B. Sørensen, P. E. Lindelof, K. Flensberg, and J. Nygård, *Phys. Rev. Lett.* **99**, 126603 (2007).
- [164] K. Grove-Rasmussen, H. I. Jørgensen, B. M. Andersen, J. Paaske, T. S. Jespersen, J. Nygård, K. Flensberg, and P. E. Lindelof, *Phys. Rev. B* **79**, 134518 (2009).
- [165] A. Kumar, M. Gaim, D. Steininger, A. L. Yeyati, A. Martín-Rodero, A. K. Hüttel, and C. Strunk, *Phys. Rev. B* **89**, 075428 (2014).
- [166] S. Li, N. Kang, P. Caroff, and H. Q. Xu, *Phys. Rev. B* **95**, 014515 (2017).
- [167] J. O. Island, R. Gaudenzi, J. de Bruijckere, E. Burzurí, C. Franco, M. Mas-Torrent, C. Rovira, J. Veciana, T. M. Klapwijk, R. Aguado, and H. S. J. van der Zant, *Phys. Rev. Lett.* **118**, 117001 (2017).
- [168] B. M. Andersen, K. Flensberg, V. Koerting, and J. Paaske, *Phys. Rev. Lett.* **107**, 256802 (2011).
- [169] Z. Su, A. B. Tacla, M. Hocevar, D. Car, S. R. Plissard, E. P. A. M. Bakkers, A. J. Daley, D. Pekker, and S. M. Frolov, *Nat. Commun.* **8**, 585 (2017).
- [170] J. C. E. Saldaña, A. Vekris, R. Žitko, G. Steffensen, P. Krogstrup, J. Paaske, K. Grove-Rasmussen, and J. Nygård, *arXiv:1812.09303* (2018).
- [171] B. W. Heinrich, J. I. Pascual, and K. J. Franke, *Prog. Surf. Sci.* **93**, 1 (2018).
- [172] J. Chen, B. D. Woods, P. Yu, M. Hocevar, D. Car, S. R. Plissard, E. P. A. M. Bakkers, T. D. Stanescu, and S. M. Frolov, *Phys. Rev. Lett.* **123**, 107703 (2019).
- [173] J. A. van Dam, Y. V. Nazarov, E. P. A. M. Bakkers, S. De Franceschi, and L. P. Kouwenhoven, *Nature* **442**, 667 (2006).
- [174] R. Delagrangé, D. J. Luitz, R. Weil, A. Kasumov, V. Meden, H. Bouchiat, and R. Deblock, *Phys. Rev. B* **91**, 241401 (2015).
- [175] R. Maurand, T. Meng, E. Bonet, S. Florens, L. Marty, and W. Wernsdorfer, *Phys. Rev. X* **2**, 011009 (2012).

- [176] J. C. Estrada Saldaña, R. Žitko, J. P. Cleuziou, E. J. H. Lee, V. Zannier, D. Ercolani, L. Sorba, R. Aguado, and S. De Franceschi, *Sci. Adv.* **5** (2019).
- [177] M. T. Deng, C. L. Yu, G. Y. Huang, M. Larsson, P. Caroff, and H. Q. Xu, *Sci. Rep.* **4**, 7261 (2014).
- [178] M. Cheng and R. M. Lutchyn, *Phys. Rev. B* **86**, 134522 (2012).
- [179] S. Park and A. Levy Yeyati, *Phys. Rev. B* **96**, 125416 (2017).
- [180] B. van Heck, J. I. Väyrynen, and L. I. Glazman, *Phys. Rev. B* **96**, 075404 (2017).
- [181] M. Hays, V. Fatemi, K. Serniak, D. Bouman, S. Diamond, G. de Lange, P. Krogstrup, J. Nygård, A. Geresdi, and M. H. Devoret, *arXiv:1908.02800* (2019).
- [182] H.-J. Kwon, V. M. Yakovenko, and K. Sengupta, *Low Temp. Phys.* **30**, 613 (2004).
- [183] D. I. Pikulin and Y. V. Nazarov, *Phys. Rev. B* **86**, 140504 (2012).
- [184] P. San-Jose, E. Prada, and R. Aguado, *Phys. Rev. Lett.* **108**, 257001 (2012).
- [185] J. Klinovaja and D. Loss, *Phys. Rev. B* **86**, 085408 (2012).
- [186] R. V. Mishmash, D. Aasen, A. P. Higginbotham, and J. Alicea, *Phys. Rev. B* **93**, 245404 (2016).
- [187] A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, and H. Shtrikman, *Nat. Phys.* **8**, 887 (2012).
- [188] J. Chen, P. Yu, J. Stenger, M. Hocevar, D. Car, S. R. Plissard, E. P. A. M. Bakkers, T. D. Stanescu, and S. M. Frolov, *Sci. Adv.* **3** (2017).
- [189] Ö. Gül, H. Zhang, J. D. S. Bommer, M. W. A. de Moor, D. Car, S. R. Plissard, E. P. A. M. Bakkers, A. Geresdi, K. Watanabe, T. Taniguchi, and L. P. Kouwenhoven, *Nat. Nanotechnol.* **13**, 192 (2018).
- [190] A. Grivnin, E. Bor, M. Heiblum, Y. Oreg, and H. Shtrikman, *Nat. Commun.* **10**, 1940 (2019).
- [191] K. T. Law, P. A. Lee, and T. K. Ng, *Phys. Rev. Lett.* **103**, 237001 (2009).
- [192] K. Flensberg, *Phys. Rev. B* **82**, 180516 (2010).
- [193] M. Wimmer, A. R. Akhmerov, J. P. Dahlhaus, and C. W. J. Beenakker, *New J. Phys.* **13**, 053016 (2011).
- [194] F. Setiawan, C.-X. Liu, J. D. Sau, and S. Das Sarma, *Phys. Rev. B* **96**, 184520 (2017).
- [195] S. M. Albrecht, A. P. Higginbotham, M. Madsen, F. Kuemmeth, T. S. Jespersen, J. Nygård, P. Krogstrup, and C. M. Marcus, *Nature* **531**, 206 (2016).
- [196] J. Shen, S. Heedt, F. Borsoi, B. van Heck, S. Gazibegovic, R. L. M. Op het Veld, D. Car, J. A. Logan, M. Pendharkar, S. J. J. Ramakers, G. Wang, D. Xu, D. Bouman, A. Geresdi, C. J. Palmstrøm, E. P. A. M. Bakkers, and L. P. Kouwenhoven, *Nat. Commun.* **9**, 4801 (2018).

- [197] S. Vaitiekėnas, G. W. Winkler, B. van Heck, T. Karzig, M.-T. Deng, K. Flensberg, L. I. Glazman, C. Nayak, P. Krogstrup, R. M. Lutchyn, and C. M. Marcus, [Science](#) **367** (2020).
- [198] D. Laroche, D. Bouman, D. J. van Woerkom, A. Proutski, C. Murthy, D. I. Pikulin, C. Nayak, R. J. J. van Gulik, J. Nygård, P. Krogstrup, L. P. Kouwenhoven, and A. Geresdi, [Nat. Commun.](#) **10**, 245 (2019).
- [199] L. P. Rokhinson, X. Liu, and J. K. Furdyna, [Nat. Phys.](#) **8**, 795 (2012).
- [200] B. van Heck, F. Hassler, A. R. Akhmerov, and C. W. J. Beenakker, [Phys. Rev. B](#) **84**, 180502 (2011).
- [201] M. Houzet, J. S. Meyer, D. M. Badiane, and L. I. Glazman, [Phys. Rev. Lett.](#) **111**, 046401 (2013).
- [202] W. H. Parker, B. N. Taylor, and D. N. Langenberg, [Phys. Rev. Lett.](#) **18**, 287 (1967).
- [203] S. Shapiro, [Phys. Rev. Lett.](#) **11**, 80 (1963).
- [204] F. Domínguez, F. Hassler, and G. Platero, [Phys. Rev. B](#) **86**, 140503 (2012).
- [205] J. D. Sau and F. Setiawan, [Phys. Rev. B](#) **95**, 060501 (2017).
- [206] H. Kamata, R. S. Deacon, S. Matsuo, K. Li, S. Jeppesen, L. Samuelson, H. Q. Xu, K. Ishibashi, and S. Tarucha, [Phys. Rev. B](#) **98**, 041302 (2018).
- [207] A. De Cecco, K. Le Calvez, B. Sacépé, C. B. Winkelmann, and H. Courtois, [Phys. Rev. B](#) **93**, 180505 (2016).
- [208] K. Le Calvez, L. Veyrat, F. Gay, P. Plaindoux, C. B. Winkelmann, H. Courtois, and B. Sacepe, [Commun. Phys.](#) **2**, 4 (2019).
- [209] J. Picó-Cortés, F. Domínguez, and G. Platero, [Phys. Rev. B](#) **96**, 125438 (2017).
- [210] P. Virtanen and P. Recher, [Phys. Rev. B](#) **88**, 144507 (2013).
- [211] F. Domínguez, O. Kashuba, E. Bocquillon, J. Wiedenmann, R. Deacon, T. Klapwijk, G. Platero, L. Molenkamp, B. Trauzettel, and E. Hankiewicz, [Phys. Rev. B](#) **95**, 195430 (2017).
- [212] J. I. Väyrynen, G. Rastelli, W. Belzig, and L. I. Glazman, [Phys. Rev. B](#) **92**, 134508 (2015).
- [213] P. San-Jose, E. Prada, and R. Aguado, [Phys. Rev. Lett.](#) **112**, 137001 (2014).
- [214] J. Tiira, E. Strambini, M. Amado, S. Roddaro, P. San-Jose, R. Aguado, F. S. Bergeret, D. Ercolani, L. Sorba, and F. Giazotto, [Nat. Commun.](#) **8**, 14984 (2017).
- [215] J. Cayao, P. San-Jose, A. M. Black-Schaffer, R. Aguado, and E. Prada, [Phys. Rev. B](#) **96**, 205425 (2017).
- [216] Y. Peng, F. Pientka, E. Berg, Y. Oreg, and F. von Oppen, [Phys. Rev. B](#) **94**, 085409 (2016).
- [217] B. Nijholt and A. R. Akhmerov, [Phys. Rev. B](#) **93** (2016).

- [218] H. A. Nilsson, P. Caroff, C. Thelander, M. Larsson, J. B. Wagner, L.-E. Wernersson, L. Samuelson, and H. Q. Xu, [Nano Lett.](#) **9**, 3151 (2009).
- [219] G. W. Winkler, D. Varjas, R. Skolasinski, A. A. Soluyanov, M. Troyer, and M. Wimmer, [Phys. Rev. Lett.](#) **119**, 037701 (2017).
- [220] S. Takei, B. M. Fregoso, H.-Y. Hui, A. M. Lobos, and S. Das Sarma, [Phys. Rev. Lett.](#) **110**, 186803 (2013).
- [221] P. Krogstrup, N. L. B. Ziino, W. Chang, S. M. Albrecht, M. H. Madsen, E. Johnson, J. Nygård, C. M. Marcus, and T. S. Jespersen, [Nat. Mater.](#) **14**, 400 (2015).
- [222] S. Gazibegovic, D. Car, H. Zhang, S. C. Balk, J. A. Logan, M. W. A. de Moor, M. C. Cassidy, R. Schmits, D. Xu, G. Wang, P. Krogstrup, R. L. M. Op het Veld, K. Zuo, Y. Vos, J. Shen, D. Bouman, B. Shojaei, D. Pennachio, J. S. Lee, P. J. van Veldhoven, S. Koelling, M. A. Verheijen, L. P. Kouwenhoven, C. J. Palmstrøm, and E. P. A. M. Bakkers, [Nature](#) **548**, 434 EP (2017).
- [223] S. D. Escribano, A. L. Yeyati, and E. Prada, [arXiv:2001.04375](#) (2020).
- [224] J. S. Lim, L. m. c. Serra, R. López, and R. Aguado, [Phys. Rev. B](#) **86**, 121103 (2012).
- [225] S. Das Sarma, J. D. Sau, and T. D. Stanescu, [Phys. Rev. B](#) **86**, 220506 (2012).
- [226] D. Rainis, L. Trifunovic, J. Klinovaja, and D. Loss, [Phys. Rev. B](#) **87**, 024515 (2013).
- [227] G. Sharma, C. Zeng, T. D. Stanescu, and S. Tewari, [arXiv:2001.10551](#) (2020).
- [228] O. Dmytruk and J. Klinovaja, [Phys. Rev. B](#) **97**, 155409 (2018).
- [229] C.-X. Liu, J. D. Sau, and S. Das Sarma, [Phys. Rev. B](#) **95**, 054502 (2017).
- [230] J. Danon, E. B. Hansen, and K. Flensberg, [Phys. Rev. B](#) **96**, 125420 (2017).
- [231] Z. Cao, H. Zhang, H.-F. Lü, W.-X. He, H.-Z. Lu, and X. C. Xie, [Phys. Rev. Lett.](#) **122**, 147701 (2019).
- [232] T. D. Stanescu, S. Tewari, J. D. Sau, and S. Das Sarma, [Phys. Rev. Lett.](#) **109**, 266402 (2012).
- [233] Y. Huang, H. Pan, C.-X. Liu, J. D. Sau, T. D. Stanescu, and S. Das Sarma, [Phys. Rev. B](#) **98**, 144511 (2018).
- [234] D. Bagrets and A. Altland, [Phys. Rev. Lett.](#) **109**, 227005 (2012).
- [235] C. W. J. Beenakker, [Rev. Mod. Phys.](#) **87**, 1037 (2015).
- [236] A. Altland and M. R. Zirnbauer, [Phys. Rev. B](#) **55**, 1142 (1997).
- [237] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, [Phys. Rev. B](#) **78**, 195125 (2008).
- [238] D. Chevallier, D. Sticlet, P. Simon, and C. Bena, [Phys. Rev. B](#) **85**, 235307 (2012).
- [239] J. Cayao, E. Prada, P. San-Jose, and R. Aguado, [Phys. Rev. B](#) **91**, 024514 (2015).
- [240] A. Ptok, A. Kobińska, and T. Domański, [Phys. Rev. B](#) **96**, 195430 (2017).

- [241] D. I. Pikulin and Y. V. Nazarov, *Phys. Rev. B* **87**, 235421 (2013).
- [242] D. Leykam, K. Y. Bliokh, C. Huang, Y. D. Chong, and F. Nori, *Phys. Rev. Lett.* **118**, 040401 (2017).
- [243] H. Shen, B. Zhen, and L. Fu, *Phys. Rev. Lett.* **120**, 146402 (2018).
- [244] Z. Gong, Y. Ashida, K. Kawabata, K. Takasan, S. Higashikawa, and M. Ueda, *Phys. Rev. X* **8**, 031079 (2018).
- [245] M. McGinley and N. R. Cooper, *Phys. Rev. B* **99**, 075148 (2019).
- [246] P. Yu, J. Chen, M. Gomanko, G. Badawy, E. P. A. M. Bakkers, K. Zuo, V. Mourik, and S. M. Frolov, [arXiv:2004.08583](#) (2020).
- [247] P. Szumniak, D. Chevallier, D. Loss, and J. Klinovaja, *Phys. Rev. B* **96**, 041401 (2017).
- [42] As nanowire screening lengths are often in the hundreds of nanometers, it is plausible to expect smoother confinement potentials, and hence suppressed splittings, as the nanowire length is increased.
- [249] C.-K. Chiu, J. D. Sau, and S. Das Sarma, *Phys. Rev. B* **96**, 054504 (2017).
- [250] S. Hoffman, C. Schrade, J. Klinovaja, and D. Loss, *Phys. Rev. B* **94**, 045316 (2016).
- [251] R. Wakatsuki, M. Ezawa, and N. Nagaosa, *Phys. Rev. B* **89**, 174514 (2014).
- [252] T. Karzig, C. Knapp, R. M. Lutchyn, P. Bonderson, M. B. Hastings, C. Nayak, J. Alicea, K. Flensberg, S. Plugge, Y. Oreg, C. M. Marcus, and M. H. Freedman, *Phys. Rev. B* **95**, 235305 (2017).
- [253] D. J. Clarke, *Phys. Rev. B* **96**, 201109 (2017).
- [254] A. Schuray, L. Weithofer, and P. Recher, *Phys. Rev. B* **96**, 085417 (2017).
- [255] G. C. Ménard, G. L. R. Anselmetti, E. A. Martinez, D. Puglia, F. K. Malinowski, J. S. Lee, S. Choi, M. Pendharkar, C. J. Palmstrøm, K. Flensberg, C. M. Marcus, L. Casparis, and A. P. Higginbotham, *Phys. Rev. Lett.* **124**, 036802 (2020).
- [256] D. Puglia, E. A. Martinez, G. C. Ménard, A. Pöschl, S. Gronin, G. C. Gardner, R. Kallagher, M. J. Manfra, C. M. Marcus, A. P. Higginbotham, and L. Casparis, [arXiv:2006.01275](#) (2020).
- [257] H. Zhang, D. E. Liu, M. Wimmer, and L. P. Kouwenhoven, *Nat. Commun.* **10**, 5128 (2019).
- [258] S. M. Frolov, M. J. Manfra, and J. D. Sau, *Nature Physics* **16**, 718 (2020).
- [259] R. Aguado and L. P. Kouwenhoven, *Phys. Today* **73**, 44 (2020).
- [260] F. Peñaranda, R. Aguado, P. San-Jose, and E. Prada, *Phys. Rev. Research* **2**, 023171 (2020).
- [261] S. Vaitiekėnas, Y. Liu, P. Krogstrup, and C. M. Marcus, [arXiv:2004.02226](#) (2020).
- [262] T. W. Larsen, K. D. Petersson, F. Kuemmeth, T. S. Jespersen, P. Krogstrup, J. Nygård, and C. M. Marcus, *Phys. Rev. Lett.* **115**, 127001 (2015).

- [263] G. de Lange, B. van Heck, A. Bruno, D. J. van Woerkom, A. Geresdi, S. R. Plissard, E. P. A. M. Bakkers, A. R. Akhmerov, and L. DiCarlo, *Phys. Rev. Lett.* **115**, 127002 (2015).
- [264] D. Sabonis, O. Erlandsson, A. Kringhøj, B. van Heck, T. W. Larsen, I. Petkovic, P. Krogstrup, K. D. Petersson, and C. M. Marcus, *arXiv:2005.01748* (2020).
- [265] A. Bargerbos, W. Uilhoorn, C.-K. Yang, P. Krogstrup, L. P. Kouwenhoven, G. de Lange, B. van Heck, and A. Kou, *Phys. Rev. Lett.* **124**, 246802 (2020).
- [266] A. Kringhøj, B. van Heck, T. W. Larsen, O. Erlandsson, D. Sabonis, P. Krogstrup, L. Casparis, K. D. Petersson, and C. M. Marcus, *Phys. Rev. Lett.* **124**, 246803 (2020).
- [267] E. Ginossar and E. Grosfeld, *Nature communications* **5**, 4772 (2014).
- [268] M. Trif, O. Dmytruk, H. Bouchiat, R. Aguado, and P. Simon, *Phys. Rev. B* **97**, 041415 (2018).
- [269] A. Keselman, C. Murthy, B. van Heck, and B. Bauer, *SciPost Phys.* **7**, 50 (2019).
- [270] J. Avila, E. Prada, P. San-Jose, and R. Aguado, *arXiv:2003.02852* (2020).
- [271] J. Avila, E. Prada, P. San-Jose, and R. Aguado, *arXiv:2003.02858* (2020).
- [272] F. Finocchiaro, F. Guinea, and P. San-Jose, *Phys. Rev. Lett.* **120**, 116801 (2018).
- [273] M. Thakurathi, P. Simon, I. Mandal, J. Klinovaja, and D. Loss, *Phys. Rev. B* **97**, 045415 (2018).
- [274] A. F. Young, J. D. Sanchez-Yamagishi, B. Hunt, S. H. Choi, K. Watanabe, T. Taniguchi, R. C. Ashoori, and P. Jarillo-Herrero, *Nature* **505**, 528 (2014).
- [275] G.-H. Lee, K.-F. Huang, D. K. Efetov, D. S. Wei, S. Hart, T. Taniguchi, K. Watanabe, A. Yacoby, and P. Kim, *Nat Phys* **13**, 693 (2017).
- [276] P. San-Jose, J. L. Lado, R. Aguado, F. Guinea, and J. Fernández-Rossier, *Phys. Rev. X* **5**, 041042 (2015).

Bibliography from “Josephson effect in junctions of conventional and topological superconductors”

References

- [1] A.Yu. Kitaev, *Usp. Fiz. Nauk (Suppl)* **171**, 131 (2001).
- [2] R.M. Lutchyn, J.D. Sau, and S. Das Sarma, *Phys. Rev. Lett.* **105**, 077001 (2010).
- [3] Y. Oreg, G. Refael, and F. von Oppen, *Phys. Rev. Lett.* **105**, 177002 (2010).
- [4] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M.P.A. Fisher, *Nat. Phys.* **7**, 412 (2011).
- [5] J. Alicea, *Rep. Prog. Phys.* **75**, 076501 (2012).
- [6] M. Leijnse and K. Flensberg, *Semicond. Sci. Techn.* **27**, 124003 (2012).
- [7] C.W.J. Beenakker, *Annu. Rev. Con. Mat. Phys.* **4**, 113 (2013).

- [8] D. Aasen, M. Hell, R.V. Mishmash, A. Higginbotham, J. Danon, M. Leijnse, T.S. Jespersen, J.A. Folk, C.M. Marcus, K. Flensberg, and J. Alicea, *Phys. Rev. X* **6**, 031016 (2016).
- [9] L.A. Landau, S. Plugge, E. Sela, A. Altland, S.M. Albrecht, and R. Egger, *Phys. Rev. Lett.* **116**, 050501 (2016).
- [10] S. Plugge, L.A. Landau, E. Sela, A. Altland, K. Flensberg, and R. Egger, *Phys. Rev. B* **94**, 174514 (2016).
- [11] S. Plugge, A. Rasmussen, R. Egger, and K. Flensberg, *New J. Phys.* **19**, 012001 (2017).
- [12] R. Aguado, *Rivista Nuovo Cimento* **11**, 523 (2017).
- [13] V. Mourik, K. Zuo, S.M. Frolov, S.R. Plissard, E.P.A. Bakkers, and L.P. Kouwenhoven, *Science* **336**, 1003 (2012).
- [14] A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, and H. Shtrikman, *Nat. Phys.* **8**, 887 (2012).
- [15] S.M. Albrecht, A.P. Higginbotham, M. Madsen, F. Kuemmeth, T.S. Jespersen, J. Nygård, P. Krogstrup, and C.M. Marcus, *Nature* **531**, 206 (2016).
- [16] M.T. Deng, S. Vaitiekenas, E.B. Hansen, J. Danon, M. Leijnse, K. Flensberg, J. Nygård, P. Krogstrup, and C.M. Marcus, *Science* **354**, 1557 (2016).
- [17] Ö. Gül, H. Zhang, F.K. de Vries, J. van Veen, K. Zuo, V. Mourik, S. Conesa-Boj, M.P. Nowak, D.J. van Woerkom, M. Quintero-Pérez, M.C. Cassidy, A. Geresdi, S. Koelling, D. Car, S.R. Plissard, E.P.A.M. Bakkers, and L.P. Kouwenhoven, *Nano Lett.* **17**, 2690 (2017).
- [18] S.M. Albrecht, E.B. Hansen, A.P. Higginbotham, F. Kuemmeth, T.S. Jespersen, J. Nygård, P. Krogstrup, J. Danon, K. Flensberg and C.M. Marcus, *Phys. Rev. Lett.* **118**, 137701 (2017).
- [19] H. Zhang, Ö. Gül, S. Conesa-Boj, M.P. Nowak, M. Wimmer, K. Zuo, V. Mourik, F.K. de Vries, J. van Veen, M.W.A. de Moor, J.D.S. Bommer, D.J. van Woerkom, D. Car, S.R. Plissard, E.P.A.M. Bakkers, M. Quintero-Pérez, M.C. Cassidy, S. Koelling, S. Goswami, K. Watanabe, T. Taniguchi, and L.P. Kouwenhoven, *Nat. Commun.* **8**, 16025 (2017).
- [20] F. Nichele, A.C.C. Drachmann, A.M. Whiticar, E.C.T. O’Farrell, H.J. Suominen, A. Fornieri, T. Wang, G.C. Gardner, C. Thomas, A.T. Hatke, P. Krogstrup, M.J. Manfra, K. Flensberg, and C.M. Marcus, *Phys. Rev. Lett.* **119**, 136803 (2017).
- [21] H.J. Suominen, M. Kjaergaard, A.R. Hamilton, J. Shabani, C.J. Palmstrøm, C.M. Marcus, and F. Nichele, *Phys. Rev. Lett.* **119**, 176805 (2017).
- [22] S. Gazibegovich, D. Car, H. Zhang, S.C. Balk, J.A. Logan, M.W.A. de Moor, M.C. Cassidy, R. Schmits, D. Xu, G. Wang, P. Krogstrup, R.L.M. Op het Veld, J. Shen, D. Bouman, B. Shojaei, D. Pennachio, J.S. Lee, P.J. van Veldhoven, S. Koelling, M.A. Verheijen, L.P. Kouwenhoven, C.J. Palmstrøm, and E.P.A.M. Bakkers, *Nature* **548**, 434 (2017).
- [23] H. Zhang, C.X. Liu, S. Gazibegovic, D. Xu, J.A. Logan, G. Wang, N. van Loo, J.D.S. Bommer, M.W.A. de Moor, D. Car, R.L.M. Op het Veld, P.J. van Veldhoven, S. Koelling, M.A. Verheijen, M. Pendharkar, D.J. Pennachio, B. Shojaei, J.S. Lee, C.J. Palmstrom, E.P.A.M. Bakkers, S. Das Sarma, and L.P. Kouwenhoven, *arXiv:1710.10701*.

- [24] M.T. Deng, S. Vaitiekėnas, E. Prada, P. San-Jose, J. Nygård, P. Krogstrup, R. Aguado, and C.M. Marcus, arXiv:1712.03536.
- [25] D. Laroche, D. Bouman, D.J. van Woerkom, A. Proutski, C. Murthy, D.I. Pikulin, C. Nayak, R.J.J. van Gulik, J. Nygård, P. Krogstrup, L.P. Kouwenhoven, and A. Geresdi, arXiv:1712.08459.
- [26] S. Nadj-Perge, I.K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A.H. MacDonald, B.A. Bernevig, and A. Yazdani, *Science* **346**, 602 (2014).
- [27] M. Ruby, F. Pientka, Y. Peng, F. von Oppen, B.W. Heinrich, and K.J. Franke, *Phys. Rev. Lett.* **115**, 197204 (2015).
- [28] H.H. Sun, K.W. Zhang, L.H. Hu, C. Li, G.Y. Wang, H.Y. Ma, Z.A. Xu, C.L. Gao, D.D. Guan, Y.Y. Li, C. Liu, D. Qian, Y. Zhou, L. Fu, S.C. Li, F.C. Zhang, and J.F. Jia, *Phys. Rev. Lett.* **116**, 257003 (2016).
- [29] B.E. Feldman, M.T. Randeria, J. Li, S. Jeon, Y. Xie, Z. Wang, I.K. Drozdov, B. Andrei Bernevig, and A. Yazdani, *Nat. Phys.* **13**, 286 (2017).
- [30] R.S. Deacon, J. Wiedenmann, E. Bocquillon, F. Domínguez, T.M. Klapwijk, P. Leubner, C. Brüne, E.M. Hankiewicz, S. Tarucha, K. Ishibashi, H. Buhmann, and L.W. Molenkamp, *Phys. Rev. X* **7**, 021011 (2017).
- [31] A. Zazunov and R. Egger, *Phys. Rev. B* **85**, 104514 (2012).
- [32] H.-J. Kwon, K. Sengupta, V.M. Yakovenko, *Eur. Phys. J. B* **37**, 349 (2004).
- [33] E.N. Bratus, V.S. Shumeiko, and G. Wendin, *Phys. Rev. Lett.* **74**, 2110 (1995).
- [34] D. Averin and A. Bardas, *Phys. Rev. Lett.* **75**, 1831 (1995).
- [35] J.C. Cuevas, A. Martín-Rodero, and A. Levy Yeyati, *Phys. Rev. B* **54**, 7366 (1996).
- [36] Yu.V. Nazarov and Ya.M. Blanter, *Quantum Transport: Introduction to Nanoscience* (Cambridge University Press, Cambridge, UK, 2010).
- [37] Y. Peng, F. Pientka, Y. Vinkler-Aviv, L.I. Glazman, and F. von Oppen, *Phys. Rev. Lett.* **115**, 266804 (2015).
- [38] P.A. Ioselevich, P.M. Ostrovsky, and M.V. Feigelman, *Phys. Rev. B* **93**, 125435 (2016).
- [39] G. Sharma and S. Tewari, *Phys. Rev. B* **93**, 195161 (2016).
- [40] A. Zazunov, R. Egger, and A. Levy Yeyati, *Phys. Rev. B* **94**, 014502 (2016).
- [41] F. Setiawan, W.S. Cole, J.D. Sau, and S. Das Sarma, *Phys. Rev. B* **95**, 174515 (2017).
- [42] F. Setiawan, W.S. Cole, J.D. Sau, and S. Das Sarma, *Phys. Rev. B* **95**, 020501(R) (2017).
- [43] A. Zazunov, R. Egger, M. Alvarado, and A. Levy Yeyati, *Phys. Rev. B* **96**, 024516 (2017).
- [44] O. Deb, K. Sengupta, and D. Sen, arXiv:1712.03726.
- [45] A. Buzdin, *Phys. Rev. Lett.* **101**, 107005 (2008).
- [46] A. Zazunov, R. Egger, T. Jonckheere, and T. Martin, *Phys. Rev. Lett.* **103**, 147003 (2009).

- [47] C. Schrade, S. Hoffman, and D. Loss, Phys. Rev. B **95**, 195421 (2017).
- [48] T. Yokoyama, M. Eto and Y.V. Nazarov, Phys. Rev. B **89**, 195407 (2014).
- [49] A. Camjayi, L. Arrachea, A. Aligia and F. von Oppen, Phys. Rev. Lett. **119**, 046801 (2017).
- [50] J. Cayao, P. San-Jose, A.M. Black-Schaffer, R. Aguado and E. Prada, Phys. Rev. B **96**, 205425 (2017).
- [51] J. Cayao, P. San-Jose, A.M. Black-Schaffer, E. Prada and R. Aguado, arXiv:1712.08127.
- [52] L. Bretheau, C.Ö. Girit, H. Pothier, D. Esteve, and C. Urbina, Nature **499**, 312 (2013).
- [53] L. Bretheau, C.Ö. Girit, C. Urbina, D. Esteve, and H. Pothier, Phys. Rev. X **3**, 041034 (2013).
- [54] C. Janvier, L. Tosi, L. Bretheau, C.Ö. Girit, M. Stern, P. Bertet, P. Joyez, D. Vion, D. Esteve, M.F. Goffman, H. Pothier, and C. Urbina, Science **349**, 1199 (2015).
- [55] B. Tarasinski, D. Chevallier, J.A. Hutasoit, B. Baxevanis, and C.W.J. Beenakker, Phys. Rev. B **92**, 144306 (2015).
- [56] L.I. Glazman and K.A. Matveev, JETP Lett. **49**, 659 (1989).
- [57] A.V. Rozhkov and D.P. Arovas, Phys. Rev. Lett. **82**, 2788 (1999).
- [58] E. Vecino, A. Martín-Rodero, and A.L. Yeyati, Phys. Rev. B **68**, 035105 (2003).
- [59] F. Siano and R. Egger, Phys. Rev. Lett. **93**, 047002 (2004).
- [60] M.S. Choi, M. Lee, K. Kang, and W. Belzig, Phys. Rev. B **70**, 020502 (2004).
- [61] C. Karrasch, A. Oguri, and V. Meden, Phys. Rev. B **77**, 024517 (2008)
- [62] A. Martín-Rodero and A.L. Yeyati, Adv. Phys. **60**, 899 (2011).
- [63] D.J. Luitz, F.F. Assaad, T. Novotný, C. Karrasch, and V. Meden, Phys. Rev. Lett. **108**, 227001 (2012).
- [64] A.Y. Kasumov, R. Deblock, M. Kociak, B. Reulet, H. Bouchiat, I.I. Khodos, Y.B. Gorbатов, V.T. Volkov, C. Journet, and M. Burghard, Science **284**, 1508 (1999).
- [65] J.A. van Dam, Y.V. Nazarov, E.P.A.M. Bakkers, S. De Franceschi, and L.P. Kouwenhoven, Nature **442**, 7103 (2006).
- [66] J.P. Cleuziou, W. Wernsdorfer, V. Bouchiat, T. Ondarcuhu, and M. Monthieux, Nature Nanotech. **1**, 53 (2006).
- [67] H. Ingerslev Jørgensen, T. Novotný, K. Grove-Rasmussen, K. Flensberg, and P.E. Lindelof, Nano Lett. **7**, 2441 (2007).
- [68] A. Eichler, R. Deblock, M. Weiss, C. Karrasch, V. Meden, C. Schönenberger, and H. Bouchiat, Phys. Rev. B **79**, 161407 (2009).
- [69] R. Delagrangé, D.J. Luitz, R. Weil, A. Kasumov, V. Meden, H. Bouchiat, and R. Deblock, Phys. Rev. B **91**, 241401 (2015).

- [70] A. Zazunov, A. Levy Yeyati, and R. Egger, Phys. Rev. B **84**, 165440 (2011).
- [71] Y. Peng, Y. Bao, and F. von Oppen, Phys. Rev. B **95**, 235143 (2017).
- [72] A. Brunetti, A. Zazunov, A. Kundu, and R. Egger, Phys. Rev. B **88**, 144515 (2013).
- [73] D.B. Szombati, S. Nadj-Perge, D. Car, S.R. Plissard, E.P.A.M. Bakkers, and L.P. Kouwenhoven, Nature Physics **12**, 568 (2016).
- [74] M. Leijnse and K. Flensberg, Phys. Rev. Lett. **107**, 210502 (2011).
- [75] D. Sticlet, C. Bena, and P. Simon, Phys. Rev. Lett. **108**, 096802 (2012).
- [76] E. Prada, P. San-Jose, and R. Aguado, Phys. Rev. B **86**, 180503(R) (2012).
- [77] E. Prada, R. Aguado, and P. San-Jose, Phys. Rev. B **96**, 085418 (2017).
- [78] S. Hoffman, D. Chevallier, D. Loss, and J. Klinovaja, Phys. Rev. B **96**, 045440 (2017).
- [79] A. Martin-Rodero and A. Levy Yeyati, J. Phys. Cond. Matt. **24**, 385303 (2012).
- [80] A.V. Balatsky, I. Vekhter, and J.X. Zhu, Rev. Mod. Phys. **78**, 373 (2006).
- [81] A. Zazunov, V.S. Shumeiko, E. N. Bratus', J. Lantz, and G. Wendin, Phys. Rev. Lett. **90**, 087003 (2003).

Bibliography from “Anomalous Josephson effect induced by spin-orbit interaction and Zeeman effect in semiconductor nanowires”

References

- [1] R. Winkler, *Spin-Orbit Coupling Effects in Two-Dimensional Electron and Hole Systems* (Springer, Berlin Heidelberg, 2003).
- [2] I. Žutić, J. Fabian, and S. Das Sarma, Rev. Mod. Phys. **76**, 323 (2004).
- [3] E. I. Rashba, Fiz. Tverd. Tela (Leningrad) **2**, 1224 (1960); Yu. A. Bychkov and E. I. Rashba, J. Phys. C **17**, 6039 (1984).
- [4] J. Nitta, T. Akazaki, H. Takayanagi, and T. Enoki, Phys. Rev. Lett. **78**, 1335 (1997).
- [5] D. Grundler, Phys. Rev. Lett. **84**, 6074 (2000).
- [6] Y. Sato, T. Kita, S. Gozu, and S. Yamada, J. Appl. Phys. **89**, 8017 (2001).
- [7] C. Fasth, A. Fuhrer, L. Samuelson, V. N. Golovach, and D. Loss, Phys. Rev. Lett. **98**, 266801 (2007).
- [8] A. Pfund, I. Shorubalko, K. Ensslin, and R. Leturcq, Phys. Rev. B **79**, 121306(R) (2009).
- [9] S. Nadj-Perge, S. M. Frolov, J. W. W. van Tilburg, J. Danon, Yu. V. Nazarov, R. Algra, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Phys. Rev. B **81**, 201305(R) (2010).
- [10] S. Nadj-Perge, S. M. Frolov, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Nature **468**, 1084 (2010).

- [11] S. Nadj-Perge, V. S. Pribiag, J. W. G. van den Berg, K. Zuo, S. R. Plissard, E. P. A. M. Bakkers, S. M. Frolov, and L. P. Kouwenhoven, Phys. Rev. Lett. **108**, 166801 (2012).
- [12] M. D. Schroer, K. D. Petersson, M. Jung, and J. R. Petta Phys. Rev. Lett. **107**, 176811 (2011).
- [13] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Science **336**, 1003 (2012).
- [14] L. P. Rokhinson, X. Liu, and J. K. Furdyna, Nature Phys. **8**, 795 (2012).
- [15] A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, and H. Shtrikman, Nature Phys. **8**, 887 (2012).
- [16] M. T. Deng, C. L. Yu, G. Y. Huang, M. Larsson, P. Caroff, and H. Q. Xu, Nano Lett. **12**, 6414 (2012).
- [17] Y.-J. Doh, J. A. van Dam, A. L. Roest, E. P. A. M. Bakkers, L. P. Kouwenhoven, and S. De Franceschi, Science **309**, 272 (2005).
- [18] J. A. van Dam, Yu. V. Nazarov, E. P. A. M. Bakkers, S. De Franceschi, and L. P. Kouwenhoven Nature **442**, 667 (2006).
- [19] H. A. Nilsson, P. Samuelsson, P. Caroff, and H. Q. Xu, Nano Lett. **12**, 228 (2011).
- [20] A. F. Andreev, Zh. Eksp. Teor. Fiz. **46**, 1823 (1964); **49**, 655 (1965) [Sov. Phys. JETP **19**, 1228 (1964); **22**, 455 (1966)].
- [21] Y. V. Nazarov and Y. M. Blanter, *Quantum Transport: introduction to nanoscience*, (Cambridge University Press, Cambridge, 2009).
- [22] C. W. J. Beenakker, Phys. Rev. Lett. **67**, 3836 (1991); Phys. Rev. Lett. **68**, 1442(E) (1992).
- [23] J. Bardeen and J. L. Johnson, Phys. Rev. B **5**, 72 (1972).
- [24] A. Furusaki and M. Tsukada, Solid State Commun. **78**, 299 (1991).
- [25] A. Furusaki, H. Takayanagi, and M. Tsukada, Phys. Rev. B **45**, 10563 (1992).
- [26] A. I. Buzdin and M. Y. Kupriyanov, Pis'ma Zh. Eksp. Teor. Fiz. **53**, 308 (1991) [JETP Lett. **53**, 321 (1991)].
- [27] T. Kontos, M. Aprili, J. Lesueur, and X. Gison, Phys. Rev. Lett. **86**, 304 (2001).
- [28] V. V. Ryazanov, V. A. Oboznov, A. Yu. Rusanov, A. V. Veretennikov, A. A. Golubov, and J. Aarts, Phys. Rev. Lett. **86**, 2427 (2001).
- [29] V. A. Oboznov, V. V. Bol'ginov, A. K. Feofanov, V. V. Ryazanov, and A. I. Buzdin, Phys. Rev. Lett. **96**, 197003 (2006).
- [30] L. P. Kouwenhoven, S. M. Frolov, V. Mourik, and K. Zuo, private communications.
- [31] L. Fu and C. L. Kane, Phys. Rev. B **79**, 161408(R) (2009).
- [32] A. Buzdin, Phys. Rev. Lett. **101**, 107005 (2008).
- [33] E. V. Bezuglyi, A. S. Rozhavsky, I. D. Vagner, and P. Wyder, Phys. Rev. B **66**, 052508 (2002).

- [34] J.-F. Liu, K. S. Chan, and J. Wang, Appl. Phys. Lett. **96**, 182505 (2010).
- [35] J.-F. Liu and K. S. Chan, Phys. Rev. B **82**, 125305 (2010).
- [36] J.-F. Liu, K. S. Chan, and J. Wang, J. Phys. Soc. Jpn. **80**, 124708 (2011).
- [37] A. G. Mal'shukov and C. S. Chu, Phys. Rev. B **78**, 104503 (2008).
- [38] A. G. Mal'shukov, S. Sadjina, and A. Brataas, Phys. Rev. B **81**, 060502(R) (2010).
- [39] A. G. Mal'shukov and C. S. Chu, Phys. Rev. B **84**, 054520 (2011).
- [40] B. Bèri, J. H. Bardarson, and C. W. J. Beenakker, Phys. Rev. B **77**, 045311 (2008).
- [41] L. Dell'Anna, A. Zazunov, R. Egger, and T. Martin, Phys. Rev. B **75**, 085305 (2007).
- [42] F. Dolcini and L. Dell'Anna, Phys. Rev. B **78**, 024518 (2008).
- [43] A. Zazunov, R. Egger, T. Jonckheere, and T. Martin, Phys. Rev. Lett. **103**, 147004 (2009).
- [44] C. Padurariu and Yu. V. Nazarov, Phys. Rev. B **81**, 144519 (2010).
- [45] C. Karrasch, S. Andergassen, and V. Meden Phys. Rev. B **84**, 134512 (2011).
- [46] S. Droste, S. Andergassen, and J. Splettstoesser, J. Phys.: Condens. Matter **24**, 415301 (2012).
- [47] A. Brunetti, A. Zazunov, A. Kundu, and R. Egger, Phys. Rev. B **88**, 144515 (2013).
- [48] J. S. Lim, R. López, and R. Aguado, Phys. Rev. Lett. **107**, 196801 (2011).
- [49] I. V. Krive, L. Y. Gorelik, R. I. Shekhter, and M. Jonson, Low. Temp. Phys. **30**, 398 (2004).
- [50] I. V. Krive, A. M. Kadigrobov, R. I. Shekhter, and M. Jonson, Phys. Rev. B **71**, 214516 (2005).
- [51] M. Cheng and R. M. Lutchyn, Phys. Rev. B **86**, 134522 (2012).
- [52] T. Yokoyama, M. Eto, and Yu. V. Nazarov, J. Phys. Soc. Jpn. **82**, 054703 (2013).
- [53] A. A. Reynoso G. Usaj, C. A. Balseiro, D. Feinberg, and M. Avignon, Phys. Rev. Lett. **101**, 107001 (2008).
- [54] A. A. Reynoso G. Usaj, C. A. Balseiro, D. Feinberg, and M. Avignon, Phys. Rev. B **86**, 214519 (2012).
- [55] Y. Tanaka, T. Yokoyama, and N. Nagaosa, Phys. Rev. Lett. **103**, 107002 (2009).
- [56] N. M. Chtchelkatchev and Yu. V. Nazarov, Phys. Rev. Lett. **90**, 226806 (2003).
- [57] M. Eto, T. Hayashi, and Y. Kurotani, J. Phys. Soc. Jpn. **74**, 1934 (2005).
- [58] The φ_0 -state is used for the $0 - \pi$ junctions, where the normal and ferromagnetic metals are arranged in parallel without SO interaction. The ground-state energy indicates two minima at $\varphi = \pm\varphi_0$. See Ref. [59] and cited references therein.
- [59] H. Sickinger, A. Lipman, M. Weides, R. G. Mints, H. Kohlstedt, D. Koelle, R. Kleiner, and E. Goldobin, Phys. Rev. Lett. **109**, 107002 (2012).

- [60] S. Datta, *Electronic Transport in Mesoscopic Systems* (Cambridge University Press, Cambridge, 1995).
- [61] T. Ando, Phys. Rev. B **44**, 8017 (1991).
- [62] G. E. Blonder, M. Tinkham, and T. M. Klapwijk, Phys. Rev. B **25**, 4515 (1982).
- [63] The Hamiltonian for the holes would be given by $-\mathcal{T}H\mathcal{T}^{-1}$, using the time reversal operator $\mathcal{T} = -i\hat{\sigma}_y K = \hat{g}K$, for $(\phi_{h-}, \phi_{h+})^T$. In our definition of ψ_h , $\hat{g}$ appears in the off-diagonal part in Eq. (21.4).
- [64] Here, E_F is the Fermi energy in the absence of SO interaction. In the presence of SO interaction, we shift the energy by $-\hbar^2 k_\alpha^2 / (2m^*)$ to keep the electron density constant. [57]
- [65] If the length of junction or the energy-dependence of scattering matrix are taken into account, the Andreev levels are split by the SO interaction at $\varphi \neq 0, \pm\pi$ since a finite phase difference breaks the time-reversal symmetry, mentioned in Ref. [40].
- [66] N. M. Chtchelkatchev, W. Belzig, Yu. V. Nazarov, and C. Bruder, JETP Lett. **74**, 323 (2001).

Bibliography from “Andreev bound states and their signatures J. A. Sauls”

References

- [1] *Theory of Superconductivity*, Phys. Rev., 108, pp. 1175–1204, (1957), J. Bardeen, L. N. Cooper, and J. R. Schrieffer.
- [2] *Thermal Conductivity of the Intermediate State in Superconductors*, Sov. Phys. JETP, 19, 1228, (1964), A. F. Andreev.
- [3] *On the Energy Spectrum on Superconductors*, Zh. Eksp. Teor. Fiz., 34, pp. 735–738, (1958), L. P. Gorkov.
- [4] *New Method in the Theory of Superconductivity*, Zh. Eksp. Teor. Fiz., 34, pp. 58–65, (1958), N. N. Bogoliubov.
- [5] *Andreev Scattering in Superfluid ^3He* , in ‘Helium Three’ (edited by W. P. Halperin and L. P. Pitaevskii, eds.), pp. 313–352 Elsevier Science Publishers, Amsterdam, (1990), J. Kurkijärvi and D. Rainer.
- [6] *Transformation of Gorkov’s Equation for Type II Superconductors into Transport-Like Equations*, Zeit. f. Physik, 214, 195, (1968), G. Eilenberger.
- [7] *Quasiclassical Method in the Theory of Superconductivity*, Sov. Phys. JETP, 28, 1200, (1969), A. I. Larkin and Y. N. Ovchinnikov.
- [8] *The Quasiclassical Approach to ^3He* , Phys. Rep., 101, 221, (1983), J. W. Serene and D. Rainer.
- [9] *On the derivation of the Quasiclassical Equations for Superconductors*, J. Low Temp. Phys., 60, 29, (1985), A. Shelankov.
- [10] *Quasiclassical Theory of Superconductivity Near Magnetically Active Interfaces*, Phys. Rev. B, 38, 4504, (1988), A. Millis, D. Rainer, and J. A. Sauls.

- [11] *Current carried by bound states of a superconducting vortex*, Phys. Rev. B, 54, pp. 10094–10106, (1996), D. Rainer, J. A. Sauls, and D. Waxman.
- [12] *Vortices in Chiral, Spin-Triplet Superconductors and Superfluids*, New J. Phys., 11, 075008, (2009), J. A. Sauls and M. Eschrig.
- [13] *Electron Spectrum of the Intermediate State of Superconductors*, Sov. Phys. JETP, 22, pp. 455–458, (1966), A. F. Andreev.
- [14] *Solitons in Polyacetylene*, Phys. Rev. Lett., 42, pp. 1698–1701, (1979), W. P. Su, J. R. Schrieffer, and A. J. Heeger.
- [15] *Midgap states as a novel signature of $d_{x^2-y^2}$ -wave superconductivity*, Phys. Rev. Lett., 72, 1526, (1994), C. R. Hu.
- [16] *Solitons with fermion number 1/2*, Phys. Rev. D, 13, pp. 3398–3409, (1976), R. Jackiw and C. Rebbi.
- [17] *Spectral asymmetry and Riemannian Geometry. I*, Camb. Phil. Soc., 77, 43, (1975), M. Atiyah, V. Patodi, and I. Singer.
- [18] *Properties of Superconducting Microbridges in the Pure Limit*, Sov. J. Low Temp. Phys., 3, 459, (1977), I. Kulik and A. Omelyanchuk.
- [19] *Surface states, Edge Currents, and the Angular Momentum of Chiral p -wave Superfluids*, Phys. Rev. B, 84, 214509, (2011), J. A. Sauls.
- [20] *Tunneling Between Superconductors*, Phys. Rev. Lett., 10, 486, (1963), V. Ambegaokar and A. Baratoff.
- [21] *Suppression of the Josephson current through a narrow, mesoscopic, semiconductor channel by a single impurity*, Phys. Rev. B, 46, pp. 12573–12586, (1992), P. F. Bagwell.
- [22] *Superfluid ^3He and Heavy Fermion Superconductors Near Surfaces and Interfaces*, Can. Journ. of Phys., 65, 1440, (1987), J. Kurkijärvi, D. Rainer, and J. A. Sauls.
- [23] *Distribution functions in nonequilibrium theory of superconductivity and Andreev spectroscopy in unconventional superconductors*, Phys. Rev. B, 61, pp. 9061–9076, (2000), M. Eschrig.
- [24] *Theory of Nonequilibrium Spin Transport and Spin Transfer Torque in Superconducting-Ferromagnetic Nanostructures*, Phys. Rev. B, 78, 174511, (2008), E. Zhao and J. A. Sauls.
- [25] *Theory of Andreev bound states in S - F - S junctions and S - F proximity devices*, Phil. Trans. R. Soc. A, 376, 20150149, (2018), M. Eschrig.
- [26] *Phase Modulated Thermal Conductance of Josephson Weak Links*, Phys. Rev. Lett., 91, 077003, (2003), E. Zhao, T. Löfwander, and J. A. Sauls.
- [27] *Edge modes, edge currents, and gauge invariance in $p_x + ip_y$ superfluids and superconductors*, Phys. Rev. B, 69, 184511, (2004), M. Stone and R. Roy.
- [28] *Edge mass current and the role of Majorana fermions in A -phase superfluid ^3He* , Phys. Rev. B, 85, 100506, (2012), Y. Tsutsumi and K. Machida.

- [29] *Thermoelectric and thermophase effects in Josephson junctions*, Phys. Rev. B, 55, pp. 12691–12700, (1997), G. D. Guttman, B. Nathanson, E. Ben-Jacob, and D. J. Bergman.
- [30] *Heat transport through a Josephson junction*, Phys. Rev. B, 87, 094522, (2013), D. Golubev, T. Faivre, and J. P. Pekola.
- [31] *Entropy Transport Between Two Superconductors by Electron Tunneling*, Phys. Rev. Lett., 15, pp. 921–923, (1965), K. Maki and A. Griffin.
- [32] *Phase-dependent thermal transport in Josephson junctions*, Phys. Rev. B, 55, pp. 3849–3855, (1997), G. D. Guttman, B. Nathanson, E. Ben-Jacob, and D. J. Bergman.
- [33] *Interference effect heat conductance in a Josephson junction and its detection in an rf SQUID*, Phys. Rev. B, 57, pp. 2717–2719, (1998), G. D. Guttman, E. Ben-Jacob, and D. J. Bergman.
- [34] *Possible New Effects in Superconductive Tunnelling*, Phys. Lett., 1, pp. 251–253, (1962), B. D. Josephson.
- [35] *Heat transport through Josephson point contacts*, Phys. Rev. B, 69, 134503, (2004), E. Zhao, T. Löfwander, and J. A. Sauls.
- [36] *Theory of Half-Metal-Superconductor Heterostructures*, Phys. Rev. Lett., 90, 137003, (2003), M. Eschrig, J. Kopu, J. C. Cuevas, and G. Schön.
- [37] *Dynamics of Spin Transport in Voltage-biased Josephson Junctions*, Phys. Rev. Lett., 98, 206601, (2007), E. Zhao and J. A. Sauls.
- [38] *Non-equilibrium charge and spin transport in superconducting-ferromagnetic-superconducting point devices*, Phil. Trans. R. Soc. A, 376, 20150229, (2018), C. Holmqvist, M. Fogelström, and W. Belzig.
- [39] *Josephson Current and Thermoelectric Phenomena in Superconducting Point Contacts*, Sov. J. Low Temp. Phys., 18, pp. 819–821, (1992), I. Kulik and A. Omelyanchuk.
- [40] *Quantitative measurements of the thermal resistance of Andreev interferometers*, Phys. Rev. B, 72, 020502, (2005), Z. Jiang and V. Chandrasekhar.
- [41] *The Josephson heat interferometer*, Nature, 492, pp. 401–405, (2012), F. Giazotto, M. Pèrez, and M. Josè.
- [42] *Opportunities for mesoscopics in thermometry and refrigeration: Physics and applications*, Rev. Mod. Phys., 78, pp. 217–274, (2006), F. Giazotto, T. T. Heikkilä, A. Luukanen, A. M. Savin, and J. P. Pekola.
- [43] *Nanoscale phase engineering of thermal transport with a Josephson heat modulator*, Nature Nanotechnology, 11, pp. 258–262, (2016), A. Fornieri, C. Blanc, R. Bosisio, S. D’Ambrosio, and F. Giazotto.

References

- [1] I. Yanson, V. Svistunov, and I. Dmitrenko, *J. Exp. Theor. Phys.* **21**, 650 (1965).
- [2] M. C. Cassidy, A. Bruno, S. Rubbert, M. Irfan, J. Kammhuber, R. N. Schouten, A. R. Akhmerov, and L. P. Kouwenhoven, *Science* **355**, 939 (2017).
- [8] A. Y. Kitaev, *Phys. Uspekhi* **44**, 131 (2001), [arXiv:0010440 \[cond-mat\]](#) .
- [4] H. J. Kwon, K. Sengupta, and V. M. Yakovenko, *Eur. Phys. J. B* **37**, 349 (2004), [arXiv:0210148 \[cond-mat\]](#) .
- [5] R. S. Deacon, J. Wiedenmann, E. Bocquillon, F. Domínguez, T. M. Klapwijk, P. Leubner, C. Brüne, E. M. Hankiewicz, S. Tarucha, K. Ishibashi, H. Buhmann, and L. W. Molenkamp, *Phys. Rev. X* **7**, 21011 (2017).
- [6] D. Laroche, D. Bouman, D. J. van Woerkom, A. Proutski, C. Murthy, D. I. Pikulin, C. Nayak, R. J. J. van Gulik, J. Nygård, P. Krogstrup, L. P. Kouwenhoven, and A. Geresdi, *Nat. Commun.* **10**, 245 (2019).
- [9] L. Fu and C. L. Kane, *Phys. Rev. B* **79**, 161408(R) (2009), [arXiv:0804.4469](#) .
- [8] D. M. Badiane, M. Houzet, and J. S. Meyer, *Phys. Rev. Lett.* **107**, 177002 (2011), [arXiv:1108.3870](#) .
- [42] D. I. Pikulin and Y. V. Nazarov, *Phys. Rev. B* **86**, 140504(R) (2012).
- [10] P. San-Jose, E. Prada, and R. Aguado, *Phys. Rev. Lett.* **108**, 257001 (2012), [arXiv:1112.5983](#) .
- [11] A. Martín-Rodero and A. Levy Yeyati, *Adv. Physique* **60**, 899 (2011), [arXiv:1111.4939](#) .
- [12] L. I. Glazman and A. Matveev, *J. Exp. Theor. Phys.*, *JETP Lett.* **49**, 659 (1989).
- [13] E. Vecino, A. Martín-Rodero, and A. Levy Yeyati, *Phys. Rev. B* **68**, 035105 (2003), [arXiv:0212375 \[cond-mat\]](#) .
- [14] A. Zazunov, S. Plugge, and R. Egger, *Phys. Rev. Lett.* **121**, 207701 (2018), [arXiv:1809.06892](#) .
- [15] T. Matsuura, *Prog. Theor. Phys.* **57**, 1823 (1977).
- [15] C. W. J. Beenakker and H. van Houten, in *Single-Electron Tunneling and Mesoscopic Devices. Springer Series in Electronics and Photonics*, edited by H. Koch and H. Lübbig (Springer, Berlin, 1992) pp. 175–179, [arXiv:0111505 \[cond-mat\]](#) .
- [17] D. Averin and A. Bardas, *Phys. Rev. Lett.* **75**, 1831 (1995).
- [18] M. Houzet, J. S. Meyer, D. M. Badiane, and L. I. Glazman, *Phys. Rev. Lett.* **111**, 046401 (2013), [arXiv:1303.4909](#) .
- [19] A. Levy Yeyati, A. Martín-Rodero, and E. Vecino, *Phys. Rev. Lett.* **91**, 266802 (2003).
- [20] D. Averin and H. T. Imam, *Phys. Rev. Lett.* **76**, 3814 (1996).

- [21] See supplemental material for the derivation of equation (14) and (15) and an alternative description of the junction at $p_\pi = 1$, which clarifies the relation between our model and topological junctions.
 - [22] Y. Naveh and D. V. Averin, *Phys. Rev. Lett.* **82**, 4090 (1999), [arXiv:9902202 \[cond-mat\]](#) .
 - [23] T. Hata, R. Delagrangé, T. Arakawa, S. Lee, R. Deblock, H. Bouchiat, K. Kobayashi, and M. Ferrier, *Phys. Rev. Lett.* **121**, 247703 (2018), [arXiv:1805.07853](#) .
- Из дополнения одна ссылка:
 Ref: Driss Manuel and Julia 2013 M. Houzet, J.S. Meyer, D.M. Badiane, and L.I. Glazman, *Phys. Rev. Lett.* **111**, 046401 (2013).

From Supercurrent blockade in Josephson junctions with a Majorana wire

References

- [1] M.Z. Hasan and C.L. Kane, *Rev. Mod. Phys.* **82**, 3045 (2010).
- [2] X.L. Qi and S.C. Zhang, *Rev. Mod. Phys.* **83**, 1057 (2011).
- [3] A.Yu. Kitaev, *Phys. Usp.* **44**, 131 (2001).
- [4] For a review, see C.W.J. Beenakker, [arXiv:1112.1950](#).
- [5] S. Sasaki, M. Kriener, K. Segawa, K. Yada, Y. Tanaka, M. Sato, and Y. Ando, *Phys. Rev. Lett.* **107**, 217001 (2011).
- [6] C. Nayak, S.H. Simon, A. Stern, M. Freedman, and S. Das Sarma, *Rev. Mod. Phys.* **80**, 1083 (2008).
- [7] L. Fu and C.L. Kane, *Phys. Rev. Lett.* **100**, 096407 (2008).
- [8] L. Fu, *Phys. Rev. Lett.* **104**, 056402 (2010).
- [9] A. Zazunov, A.L. Yeyati, and R. Egger, *Phys. Rev. B* **84**, 165440 (2011).
- [10] R.M. Lutchyn, J.D. Sau, and S. Das Sarma, *Phys. Rev. Lett.* **105**, 077001 (2010).
- [11] Y. Oreg, G. Refael, and F. von Oppen, *Phys. Rev. Lett.* **105**, 177002 (2010).
- [12] A. Cook and M. Franz, *Phys. Rev. B* **84**, 201105(R) (2011).
- [13] L. Fu and C.L. Kane, *Phys. Rev. B* **79**, 161408(R) (2009).
- [14] P.A. Ioselevich and M.V. Feigel'man, *Phys. Rev. Lett.* **106**, 077003 (2011).
- [15] M. Sato and S. Fujimoto, *Phys. Rev. B* **79**, 094504 (2009).
- [16] C.J. Bolech and E. Demler, *Phys. Rev. Lett.* **98**, 237002 (2007).
- [17] J. Nilsson, A.R. Akhmerov, and C.W.J. Beenakker, *Phys. Rev. Lett.* **101**, 120403 (2008).
- [18] K.T. Law, P.A. Lee, and T.K. Ng, *Phys. Rev. Lett.* **103**, 237001 (2009).
- [19] K. Flensberg, *Phys. Rev. B* **82**, 180516(R) (2010).

- [20] M. Wimmer, A.R. Akhmerov, J.P. Dahlhaus, and C.W.J. Beenakker, New J. Phys. **13**, 053016 (2011).
- [21] K.T. Law and P.A. Lee, Phys. Rev. B **84**, 081304 (2011).
- [22] B. van Heck, F. Hassler, A.R. Akhmerov, and C.W.J. Beenakker, Phys. Rev. B **84**, 180502(R) (2011).
- [23] L. Jiang, D. Pekker, J. Alicea, G. Refael, Y. Oreg, and F. von Oppen, Phys. Rev. Lett. **107**, 236401 (2011).
- [24] The collinearity condition (24.12) gives $\Phi_{\mathbf{k}} = S_{\mathbf{k}} F_{\mathbf{k}}$, with $S_{\mathbf{k}} = \text{diag}(1, \zeta_{\mathbf{k}}^*)$ and $F_{\mathbf{k}} = (\psi_{\mathbf{k},\uparrow}, \psi_{\mathbf{k},\uparrow}^\dagger)^T$. Now we have $(F_{\mathbf{k}}^\dagger)^T = \sigma_x F_{\mathbf{k}}$, which (using a matrix notation in time and Nambu space) yields with $\tilde{G}_k = S_{\mathbf{k}}^\dagger G_k S_{\mathbf{k}} \sim \sigma_{x,y}$ the relation $\Phi_{\mathbf{k}}^\dagger G_k \Phi_{\mathbf{k}} = F_{\mathbf{k}}^\dagger \tilde{G}_k F_{\mathbf{k}} = F_{\mathbf{k}}^T \sigma_x \tilde{G}_k \sigma_x (F_{\mathbf{k}}^\dagger)^T = -F_{\mathbf{k}}^\dagger (\sigma_x \tilde{G}_k \sigma_x)^T F_{\mathbf{k}} = -F_{\mathbf{k}}^\dagger \tilde{G}_k F_{\mathbf{k}} = 0$.
- [25] For an asymmetric junction, one generally finds a shifted CPR. This can be seen by expanding to lowest nonvanishing order in Γ_j and $\tilde{\Gamma}_j$. We then find $I(\varphi) = |\Gamma_L \Gamma_R|^2 F(2\varphi) + \sum_j |\tilde{\Gamma}_j \Gamma_{-j}|^2 F_j(\varphi)$, with the π -periodic function $F(2\varphi) \sim \sin(2\varphi + 2\alpha)$, where $\alpha = \arg(\Gamma_L^* \Gamma_R)$, and 2π -periodic functions $F_{L/R}(\varphi)$.
- [26] It is straightforward to include $q \neq 0$ quasiparticles, but their effect is weak unless both E_c and Δ_w are small.

From Josephson effect in multiterminal topological junctions

References

- [1] A.Yu. Kitaev, Usp. Fiz. Nauk (Suppl) **171**, 131 (2001).
- [2] J. Alicea, Rep. Prog. Phys. **75**, 076501 (2012).
- [3] M. Leijnse and K. Flensberg, Semicond. Sci. Techn. **27**, 124003 (2012).
- [4] C.W.J. Beenakker, Annu. Rev. Con. Mat. Phys. **4**, 113 (2013).
- [5] D. Aasen *et al.*, Phys. Rev. X **6**, 031016 (2016).
- [6] V. Mourik, K. Zuo, S.M. Frolov, S.R. Plissard, E.P.A. Bakkers, and L.P. Kouwenhoven, Science **336**, 1003 (2012).
- [7] A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, and H. Shtrikman, Nat. Phys. **8**, 887 (2012).
- [8] S.M. Albrecht, A.P. Higginbotham, M. Madsen, F. Kuemmeth, T.S. Jespersen, J. Nygård, P. Krogstrup, and C.M. Marcus, Nature **531**, 206 (2016).
- [9] M.T. Deng, S. Vaitiekenas, E.B. Hansen, J. Danon, M. Leijnse, K. Flensberg, J. Nygård, P. Krogstrup, and C.M. Marcus, Science **354**, 1557 (2016).
- [10] Ö. Gül *et al.*, Nano Lett. **17**, 2690 (2017).
- [11] S.M. Albrecht, E.B. Hansen, A.P. Higginbotham, F. Kuemmeth, T.S. Jespersen, J. Nygård, P. Krogstrup, J. Danon, K. Flensberg and C.M. Marcus, Phys. Rev. Lett. **118**, 137701 (2017).

- [12] H. Zhang *et al.*, arXiv:1603.04069.
- [13] H.J. Suominen, M. Kjaergaard, A.R. Hamilton, J. Shabani, C.J. Palmstrøm, C.M. Marcus, and F. Nichele, arXiv:1703.03699.
- [14] S. Nadj-Perge, I.K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A.H. MacDonald, B.A. Bernevig, and A. Yazdani, *Science* **346**, 602 (2014).
- [15] M. Ruby, F. Pientka, Y. Peng, F. von Oppen, B.W. Heinrich, and K.J. Franke, *Phys. Rev. Lett.* **115**, 197204 (2015).
- [16] H.H. Sun *et al.*, *Phys. Rev. Lett.* **116**, 257003 (2016).
- [17] B.E. Feldman, M.T. Randeria, J. Li, S. Jeon, Y. Xie, Z. Wang, I.K. Drozdov, B. Andrei Bernevig, and A. Yazdani, *Nat. Phys.* **13**, 286 (2017).
- [18] R.S. Deacon *et al.*, *Phys. Rev. X* **7**, 021011 (2017).
- [19] S. Gazibegovich *et al.*, arXiv:1705.01480.
- [20] A. Zazunov, R. Egger, and A. Levy Yeyati, *Phys. Rev. B* **94**, 014502 (2016).
- [21] Y. Peng, Y. Bao, and F. von Oppen, *Phys. Rev. B* **95**, 235143 (2017).
- [22] J.C. Cuevas, A. Martín-Rodero, and A. Levy Yeyati, *Phys. Rev. B* **54**, 7366 (1996).
- [23] Yu.V. Nazarov and Ya.M. Blanter, *Quantum Transport: Introduction to Nanoscience* (Cambridge University Press, Cambridge, UK, 2010).
- [24] T. Jonckheere, J. Rech, A. Zazunov, R. Egger, and T. Martin, *Phys. Rev. B* **95**, 054514 (2017).
- [25] R.M. Lutchyn, J.D. Sau, and S. Das Sarma, *Phys. Rev. Lett.* **105**, 077001 (2010).
- [26] Y. Oreg, G. Refael, and F. von Oppen, *Phys. Rev. Lett.* **105**, 177002 (2010).
- [27] M. Leijnse and K. Flensberg, *Phys. Rev. Lett.* **107**, 210502 (2011).
- [28] D. Sticlet, C. Bena, and P. Simon, *Phys. Rev. Lett.* **108**, 096802 (2012).
- [29] E. Prada, P. San-Jose, and R. Aguado, *Phys. Rev. B* **86**, 180503(R) (2012).
- [30] D. Rainis, L. Trifunovic, J. Klinovaja, and D. Loss, *Phys. Rev. B* **87**, 024515 (2013).
- [31] L. Jiang, D. Pekker, J. Alicea, G. Refael, Y. Oreg, A. Brataas, and F. von Oppen, *Phys. Rev. B* **87**, 075438 (2013).
- [32] J.J. He, T.K. Ng, P.A. Lee, and K.T. Law, *Phys. Rev. Lett.* **112**, 037001 (2014).
- [33] E. Prada, R. Aguado, and P. San-Jose, arXiv:1702.02525.
- [34] S. Hoffman, D. Chevallier, D. Loss, and J. Klinovaja, arXiv:1705.03002.
- [35] A. Zazunov and R. Egger, *Phys. Rev. B* **85**, 104514 (2012).
- [36] Y. Peng, F. Pientka, Y. Vinkler-Aviv, L.I. Glazman, and F. von Oppen, *Phys. Rev. Lett.* **115**, 266804 (2015).

- [37] P.A. Ioselevich, P.M. Ostrovsky, and M.V. Feigelman, Phys. Rev. B **93**, 125435 (2016).
- [38] G. Sharma and S. Tewari, Phys. Rev. B **93**, 195161 (2016).
- [39] F. Setiawan, W.S. Cole, J.D. Sau, and S. Das Sarma, Phys. Rev. B **95**, 174515 (2017).
- [40] F. Setiawan, W.S. Cole, J.D. Sau, and S. Das Sarma, Phys. Rev. B **95**, 020501(R) (2017).
- [41] J. Nilsson, A.R. Akhmerov, and C.W.J. Beenakker, Phys. Rev. Lett. **101**, 120403 (2008).
- [42] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M.P.A. Fisher, Nat. Phys. **7**, 412 (2011).
- [43] L. Jiang, D. Pekker, J. Alicea, G. Refael, Y. Oreg, and F. von Oppen, Phys. Rev. Lett. **107**, 236401 (2011).
- [44] R.P. Riwar, M. Houzet, J.S. Meyer, and Y. Nazarov, Nat. Comm. **7**, 11167 (2016).
- [45] Our model describes a spin-polarized p -wave TS wire as well as a helical TS wire, where right- and left-movers have opposite spin. These two case imply $\theta_j = 0$ and $\theta_j = \pi/2$, respectively. For pointlike tunneling contacts, see Eq. (??), left- and right-movers enter with equal amplitude, and both two cases are indistinguishable due to the SU(2) spin symmetry of the S lead.
- [46] S. Estévez Hernández, M. Akabori, K. Sladek, Ch. Volk, S. Alagha, H. Hardtdegen, M.G. Pala, N. Demarina, D. Grützmacher, and Th. Schäpers, Phys. Rev. B **82**, 235303 (2010).

From Transport theory of multiterminal hybrid structures

References

- [1] G.B. Lesovik, T. Martin, and G. Blatter, arXiv:cond-mat/0009193.
- [2] M. S. Choi, C. Bruder and D. Loss, Phys. Rev. B **62**, 13569 (2000);
P. Recher, E. V. Sukhorukov and D. Loss, Phys. Rev. B **63**, 165314 (2001).
- [3] G. Deutscher and D. Feinberg, App. Phys. Lett. **76**, 487 (2000).
- [4] G. Falci, D. Feinberg, and F.W.J. Hekking, Europhysics Letters **54**, 255 (2001).
- [5] R. Mélin, J. Phys.: Condens. Matter **13**, 6445 (2001);
R. Mélin, to appear in the Proceedings of the XXXVIth Rencontres de Moriond, T. Martin and G. Montambaux Eds., EDP Sciences (2001).
- [6] M.J.M. de Jong and C.W.J. Beenakker, Phys. Rev. Lett. **74**, 1657 (1995).
- [7] R.J. Soulen *et al.*, Science **282**, 85 (1998).
- [8] S.K. Upadhyay *et al.*, Phys. Rev. Lett. **81**, 3247 (1998).
- [9] I. Baladié, A. Buzdin, N. Ryzhanova, and A. Vedyayev, Phys. Rev. B **63**, 054518 (2001).
A. Buzdin, A.V. Vedyayev, and N. Ryzhanova, Europhys. Lett. **48**, 686 (1999).
- [10] V. Apinyan and R. Mélin, cond-mat/0107038.
- [11] R. Mélin, Europhysics Letters **51**, 202 (2000).

- [12] M. Giroud *et al.*, Phys. Rev. B **58**, R11872 (1998).
- [13] V.T. Petrashov, I.A. Sosnin, and C. Troadec, arXiv:cond-mat/0007278.
- [14] J. Aumentado and V. Chandrasekhar, arXiv:cond-mat/0007433.
- [15] W. Belzig, A. Brataas, Yu. V. Nazarov, and G.E. Bauer, arXiv:cond-mat/0005188.
- [16] J.M. Byers and M.E. Flatté, Phys. Rev. Lett. **74**, 306 (1995).
- [17] J.C. Cuevas, A. Martin-Rodero, and A. Levy Yeyati, Phys. Rev. B **54**, 7366 (1996).
- [18] L.V. Keldysh, Sov. Phys. JETP **20**, 1018 (1965).
- [19] C. Caroli, R. Combescot, P. Nozières and D. Saint-James, J. Phys. C: Solid St. Phys. **4**, 916 (1971); *ibid.* **5**, 21 (1972).
- [20] T. Valet and A. Fert, Phys. Rev. B **48**, 7099 (1993);
T. Valet and A. Fert, J. Magn. Magn. Mater. **121**, 378 (1993);
A. Fert, T. Valet, and J. Barnas, J. Appl. Phys. **75**, 6693 (1994);
A. Fert, J.L. Duvail, and T. Valet, Phys. Rev. B **52**, 6513 (1995).
- [21] M.A.M. Gijs and G. Bauer, Adv. in Physics **46**, 285 (1997).
- [22] R. Mélin and D. Denaro, Eur. Phys. J. B **18**, 149 (2000).
- [23] G.E. Blonder, M. Tinkham, and T.M. Klapwijk, Phys. Rev. B **25**, 4515 (1982).
- [24] D. Feinberg, G. Deutscher, G. Falci and F. Hekking, to appear in the Proceedings of the XXXVIth Rencontres de Moriond, T. Martin and G. Montambaux Eds., EDP Sciences (2001).

From Three “universal” mesoscopic Josephson effects by Beenakker

References

- [1] C. W. J. Beenakker and H. van Houten, Phys. Rev. Lett. **66**, 3056 (1991).
- [2] L. I. Glazman and K. A. Matveev, Pis'ma Zh. Eksp. Teor. Fiz. **49**, 570 (1989) [JETP Lett. **49**, 659 (1989)].
- [3] C. W. J. Beenakker and H. van Houten, in: Proc. 4th Int. Conf. on Superconducting and Quantum Effect Devices and their Applications (*SQUID'91*), ed. by H. Koch and H. Lübbig (Springer, Berlin, to be published); cond-mat/0111505
- [4] C. W. J. Beenakker, Phys. Rev. Lett. **67**, 3836 (1991); Erratum (ibidem, to be published). The assertion made in this Ref. (directly above Eq. 16) that $T_p \ll 1$ for $l \ll L$ is incorrect (see Sec. 5.3 of the present paper).
- [5] B. J. van Wees, H. van Houten, C. W. J. Beenakker, J. G. Williamson, L. P. Kouwenhoven, D. van der Marel, and C. T. Foxon, Phys. Rev. Lett. **60**, 848 (1988).
- [6] D. A. Wharam, T. J. Thornton, R. Newbury, M. Pepper, H. Ahmed, J. E. F. Frost, D. G. Hasko, D. C. Peacock, D. A. Ritchie, and G. A. C. Jones, J. Phys. C **21**, L209 (1988).

- [7] A review of quantum transport in semiconductor nanostructures is: C. W. J. Beenakker and H. van Houten, *Solid State Phys.* **44**, 1 (1991); cond-mat/0412664
- [8] A. Furusaki, H. Takayanagi, and M. Tsukada, *Phys. Rev. Lett.* **67**, 132 (1991); *Phys. Rev. B* (to be published).
- [9] B. L. Al'tshuler, *Pis'ma Zh. Eksp. Teor. Fiz.* **41**, 530 (1985) [*JETP Lett.* **41**, 648 (1985)].
- [10] P. A. Lee and A. D. Stone, *Phys. Rev. Lett.* **55**, 1622 (1985).
- [11] A review of mesoscopic fluctuation phenomena is: B. L. Al'tshuler, P. A. Lee, and R. A. Webb, eds., *Mesoscopic Phenomena in Solids* (North-Holland, Amsterdam, 1991).
- [12] B. L. Al'tshuler and B. Z. Spivak, *Zh. Eksp. Teor. Fiz.* **92**, 609 (1987) [*Sov. Phys. JETP* **65**, 343 (1987)].
- [13] C. W. J. Beenakker and H. van Houten, in: *Nanostructures and Mesoscopic Systems*, ed. by W. P. Kirk and M. A. Reed (Academic, New York, to be published); cond-mat/0512610
- [14] A. Furusaki and M. Tsukada, *Solid State Comm.* **78**, 299 (1991).
- [15] B. J. van Wees, K.-M. H. Lenssen, and C. J. P. M. Harmans, *Phys. Rev. B* **44**, 470 (1991).
- [16] P. G. de Gennes, *Superconductivity of Metals and Alloys* (Benjamin, New York, 1966).
- [17] K. K. Likharev, *Rev. Mod. Phys.* **51**, 101 (1979).
- [18] M. Yu. Kupriyanov and V. F. Lukichev, *Fiz. Nizk. Temp.* **8**, 1045 (1982) [*Sov. J. Low Temp. Phys.* **8**, 526 (1982)]; A. A. Zubkov and M. Yu. Kupriyanov, *Fiz. Nizk. Temp.* **9**, 548 (1983) [*Sov. J. Low Temp. Phys.* **9**, 279 (1983)].
- [19] J. Bardeen, R. Kümmel, A. E. Jacobs, and L. Tewordt, *Phys. Rev.* **187**, 556 (1969).
- [20] A. F. Andreev, *Zh. Eksp. Teor. Fiz.* **46**, 1823 (1964); **49**, 655 (1965) [*Sov. Phys. JETP* **19**, 1228 (1964); **22**, 455 (1966)].
- [21] E. Akkermans, A. Auerbach, J. E. Avron, and B. Shapiro, *Phys. Rev. Lett.* **66**, 76 (1991).
- [22] A. D. Stone and Y. Imry, *Phys. Rev. Lett.* **56**, 189 (1986).
- [23] Y. Imry, in: *Directions in Condensed Matter Physics*, Vol. 1, ed. by G. Grinstein and G. Mazenko (World Scientific, Singapore, 1986).
- [24] W. Haberkorn, H. Knauer, and J. Richter, *Phys. Status Solidi A* **47**, K161 (1978).
- [25] A. V. Zaïtsev, *Zh. Eksp. Teor. Fiz.* **86**, 1742 (1984) [*Sov. Phys. JETP* **59**, 1015 (1984)].
- [26] G. B. Arnold, *J. Low Temp. Phys.* **59**, 143 (1985).
- [27] A. Furusaki and M. Tsukada, *Physica B* **165&166**, 967 (1990).
- [28] D. S. Fisher and P. A. Lee, *Phys. Rev. B* **23**, 6851 (1981).
- [29] I. O. Kulik and A. N. Omel'yanchuk, *Fiz. Nizk. Temp.* **3**, 945 (1977); **4**, 296 (1978) [*Sov. J. Low Temp. Phys.* **3**, 459 (1977); **4**, 142 (1978)].
- [30] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, 1977).

- [31] M. Büttiker, IBM J. Res. Dev. **32**, 63 (1988).
- [32] V. Kalmeyer and R. B. Laughlin, Phys. Rev. B **35**, 9805 (1987).
- [33] W. Xue and P. A. Lee, Phys. Rev. B **38**, 3913 (1988).
- [34] A. D. Stone, P. A. Mello, K. A. Muttalib, and J.-L. Pichard, in Ref. [11].
- [35] I. O. Kulik and A. N. Omel'yanchuk, Pis'ma Zh. Eksp. Teor. Fiz. **21**, 216 (1975) [JETP Lett. **21**, 96 (1975)].
- [36] V. Ambegaokar and A. Baratoff, Phys. Rev. Lett. **10**, 486 (1963); **11**, 104(E) (1963).
- [37] Y. Imry, Europhys. Lett. **1**, 249 (1986).
- [38] H. Takayanagi and T. Kawakami, Phys. Rev. Lett. **54**, 2449 (1985).

From Josephson effect in a junction coupled to an electron reservoir

References

- [1] A. A. Golubov, M. Yu. Kupriyanov, and E. Ilichev, *The current-phase relation in Josephson junctions*, Rev. Mod. Phys. **76**, 411 (2004).
- [2] D. V. Averin, *Adiabatic quantum computation with Cooper pairs*, Solid State Comm. **105**, 659 (1998).
- [3] J. P. Pekola and J. J. Toppari, *Decoherence in circuits of small Josephson junctions*, Phys. Rev. B **64**, 172509 (2001).
- [4] Y. Makhlin, G. Schön, and A. Shnirman, *Dissipative effects in Josephson qubits*, Chem. Phys. **296**, 315 (2004).
- [5] M. Büttiker, *Small normal-metal loop coupled to an electron reservoir*, Phys. Rev. B **32**, 1846(R) (1985).
- [6] B. J. van Wees, K.-M. H. Lenssen, and C. J. P. M. Harmans, *Transmission formalism for supercurrent flow in multiprobe superconductor-semiconductor-superconductor devices*, Phys. Rev. B **44**, 470(R) (1991).
- [7] L.-F. Chang and P. F. Bagwell, *Control of Andreev-level occupation in a Josephson junction by a normal-metal probe*, Phys. Rev. B **55**, 12678 (1997).
- [8] N. A. Mortensen, A.-P. Jauho, and K. Flensberg, *Dephasing in semiconductor-superconductor structures by coupling to a voltage probe*, Superlattice Microstr. **28**, 67 (2000).
- [9] T. Gramspacher and M. Büttiker, *Distribution functions and current-current correlations in normal-metal-superconductor heterostructures*, Phys. Rev. B **61**, 8125 (2000).
- [10] M. Belogolovskii, A. Golubov, M. Grajcar, M. Yu. Kupriyanov, and P. Seidel, *Charge transport across a mesoscopic superconductor-normal metal junction: coherence and decoherence effects*, Physica C **357–360**, 1592 (2001).

- [11] M. Belogolovskii, *Phase-breaking effects in superconducting heterostructures*, Phys. Rev. B **67**, 100503(R) (2003).
- [12] B. Béri, *Dephasing-enabled triplet Andreev conductance*, Phys. Rev. B **79**, 245315 (2009).
- [13] V. Kornich, *Current-voltage characteristics of the normal metal-insulator-PT-symmetric non-Hermitian superconductor junction as a probe of non-Hermitian formalisms*, Phys. Rev. Lett. **131**, 116001 (2023).
- [14] C.-A. Li, H.-P. Sun, and B. Trauzettel, *Anomalous Andreev spectrum and transport in non-Hermitian Josephson junctions*, arXiv:2307.04789.
- [15] J. Cayao and M. Sato, *Non-Hermitian phase-biased Josephson junctions*, arXiv:2307.15472.
- [16] P.-X. Shen, Z. Lu, J. L. Lado, and M. Trif, *Non-Hermitian persistent current transport*, arXiv:2403.09569.
- [17] C. W. J. Beenakker, *Universal limit of critical-current fluctuations in mesoscopic Josephson junctions*, Phys. Rev. Lett. **67**, 3836 (1991); extended version at arXiv:cond-mat/0406127.
- [18] J. Bardeen, R. Kümmel, A. E. Jacobs, and L. Tewordt, *Structure of vortex lines in pure superconductors*, Phys. Rev. **187**, 556 (1969).
- [19] M. R. Zirnbauer, *Conductance fluctuations of disordered mesoscopic devices with many weakly coupled probes*, Nucl. Phys. A **560**, 95 (1993).
- [20] P. W. Brouwer and C. W. J. Beenakker, *Voltage-probe and imaginary potential models for dephasing in a chaotic quantum dot*, Phys. Rev. B **55**, 4695 (1997).
- [21] G. Breit and E. P. Wigner, *Capture of slow neutrons*, Phys. Rev. **49**, 519 (1936).
- [22] C. W. J. Beenakker and H. van Houten, *Resonant Josephson current through a quantum dot*, in: *Single-Electron Tunneling and Mesoscopic Devices*, ed. by H. Koch and H. Lübbig (Springer, Berlin, 1992). Online at arXiv:cond-mat/0111505.
- [23] I. A. Devyatov and M. Yu. Kupriyanov, *Resonant Josephson tunneling through S-I-S junctions of arbitrary size*, JETP **85**, 189 (1997).
- [24] P. W. Brouwer and C. W. J. Beenakker, *Anomalous temperature dependence of the supercurrent through a chaotic Josephson junction*, Chaos, Solitons & Fractals **8**, 1249 (1997). Several misprints are corrected in the online version at arXiv:cond-mat/9611162.
- [25] C. Ishii, *Josephson currents through junctions with normal metal barriers*, Prog. Theor. Phys. **44**, 1525 (1970).
- [26] J. Bardeen and J. L. Johnson, *Josephson current flow in pure superconducting-normal-superconducting junctions*, Phys. Rev. B **5**, 72 (1972).
- [27] A. V. Svidzinsky, T. N. Antsygina, and E. N. Bratus, *Concerning the theory of the Josephson effect in pure SNS junctions*, J. Low Temp. Phys. **10**, 131 (1973).
- [28] T. Meng, S. Florens, and P. Simon, *Self-consistent description of Andreev bound states in Josephson quantum dot devices*, Phys. Rev. B **79**, 224521 (2009).
- [29] P. Recher, Yu. V. Nazarov, and L. P. Kouwenhoven, *Josephson light-emitting diode*, Phys. Rev. Lett. **104**, 156802 (2010).

- [30] D. O. Oriekhov, Y. Cheipesh, and C. W. J. Beenakker, *Voltage staircase in a current-biased quantum-dot Josephson junction*, *Phys. Rev. B* **103**, 094518 (2021).
- [31] To evaluate Eq. (??) one can work in a basis where the Hamiltonian (??) is diagonal, with eigenvalues $\pm\varepsilon_A - i\gamma$, and note that $(d/d\phi)\text{Tr}\mathcal{H} = 0$ while $d\mathcal{H}/d\phi$ is real, so that $(d/d\phi)\text{Im}\text{Tr}(\mathcal{H}\ln\mathcal{H}) = \text{Tr}(d\mathcal{H}/d\phi)\text{Im}\ln\mathcal{H} = (d\varepsilon_A/d\phi)\text{Im}\ln\frac{\varepsilon_A - i\gamma}{-\varepsilon_A - i\gamma} = 2(d\varepsilon_A/d\phi)\arctan(\varepsilon_A/\gamma)$. This is at zero temperature. We have checked that the finite- T expression (??) also agrees with Ref. [16].

From Spin-degeneracy breaking and parity transitions in three-terminal Josephson junctions

References

- [1] Loss, D. & DiVincenzo, D. P. Quantum computation with quantum dots. *Phys. Rev. A* **57**, 120–126 (1998).
- [2] Burkard, G., Ladd, T. D., Pan, A., Nichol, J. M. & Petta, J. R. Semiconductor spin qubits. *Rev. Mod. Phys.* **95**, 025003 (2023).
- [3] Krogstrup, P. *et al.* Epitaxy of semiconductor–superconductor nanowires. *Nat. Mater.* **14**, 400–406 (2015).
- [4] Shabani, J. *et al.* Two-dimensional epitaxial superconductor–semiconductor heterostructures: A platform for topological superconducting networks. *Phys. Rev. B* **93**, 155402 (2016).
- [5] Beenakker, C. W. J. & van Houten, H. Josephson current through a superconducting quantum point contact shorter than the coherence length. *Phys. Rev. Lett.* **66**, 3056–3059 (1991).
- [6] Furusaki, A. & Tsukada, M. Current-carrying states in Josephson junctions. *Phys. Rev. B* **43**, 10164–10169 (1991).
- [7] Pillet, J.-D. *et al.* Andreev bound states in supercurrent-carrying carbon nanotubes revealed. *Nat. Phys.* **6**, 965–969 (2010).
- [8] Chang, W., Manucharyan, V. E., Jespersen, T. S., Nygård, J. & Marcus, C. M. Tunneling spectroscopy of quasiparticle bound states in a spinful Josephson junction. *Phys. Rev. Lett.* **110**, 217005 (2013).
- [9] Bretheau, L., Girit, Ç. Ö., Pothier, H., Esteve, D. & Urbina, C. Exciting Andreev pairs in a superconducting atomic contact. *Nature* **499**, 312–315 (2013).
- [10] Bretheau, L., Girit, Ç. Ö., Urbina, C., Esteve, D. & Pothier, H. Supercurrent spectroscopy of Andreev states. *Phys. Rev. X* **3**, 041034 (2013).
- [11] Janvier, C. *et al.* Coherent manipulation of Andreev states in superconducting atomic contacts. *Science* **349**, 1199–1202 (2015).
- [12] Hays, M. *et al.* Direct microwave measurement of Andreev-bound-state dynamics in a semiconductor-nanowire Josephson junction. *Phys. Rev. Lett.* **121**, 047001 (2018).

- [13] Nichele, F. *et al.* Relating Andreev bound states and supercurrents in hybrid Josephson junctions. *Phys. Rev. Lett.* **124**, 226801 (2020).
- [14] Lee, E. J. H. *et al.* Spin-resolved Andreev levels and parity crossings in hybrid superconductor–semiconductor nanostructures. *Nat. Nanotechnol.* **9**, 79–84 (2014).
- [15] van Woerkom, D. J. *et al.* Microwave spectroscopy of spinful Andreev bound states in ballistic semiconductor Josephson junctions. *Nat. Phys.* **13**, 876–881 (2017).
- [16] Sau, J. D., Lutchyn, R. M., Tewari, S. & Das Sarma, S. Generic new platform for topological quantum computation using semiconductor heterostructures. *Phys. Rev. Lett.* **104**, 040502 (2010).
- [17] Escribano, S. D., Prada, E., Oreg, Y. & Yeyati, A. L. Tunable proximity effects and topological superconductivity in ferromagnetic hybrid nanowires. *Phys. Rev. B* **104**, L041404 (2021).
- [18] Vaitiekėnas, S. *et al.* Evidence for spin-polarized bound states in semiconductor–superconductor–ferromagnetic-insulator islands. *Phys. Rev. B* **105**, L041304 (2022).
- [19] Chtchelkatchev, N. M. & Nazarov, Y. V. Andreev quantum dots for spin manipulation. *Phys. Rev. Lett.* **90**, 226806 (2003).
- [20] Béni, B., Bardarson, J. H. & Beenakker, C. W. J. Splitting of andreev levels in a josephson junction by spin-orbit coupling. *Phys. Rev. B* **77**, 045311 (2008).
- [21] Reynoso, A. A., Usaj, G., Balseiro, C. A., Feinberg, D. & Avignon, M. Spin-orbit-induced chirality of Andreev states in Josephson junctions. *Phys. Rev. B* **86**, 214519 (2012).
- [22] Park, S. & Yeyati, A. L. Andreev spin qubits in multichannel Rashba nanowires. *Phys. Rev. B* **96**, 125416 (2017).
- [23] Tosi, L. *et al.* Spin-orbit splitting of Andreev states revealed by microwave spectroscopy. *Phys. Rev. X* **9**, 011010 (2019).
- [24] Hays, M. *et al.* Continuous monitoring of a trapped superconducting spin. *Nat. Phys.* **16**, 1103–1107 (2020).
- [25] Bargerbos, A. *et al.* Spectroscopy of spin-split andreev levels in a quantum dot with superconducting leads. arXiv:2208.09314 (2022).
- [26] Padurariu, C. & Nazarov, Y. V. Theoretical proposal for superconducting spin qubits. *Phys. Rev. B* **81**, 144519 (2010).
- [27] Padurariu, C. & Nazarov, Y. V. Spin blockade qubit in a superconducting junction. *EPL (Europhysics Letters)* **100**, 57006 (2012).
- [28] Hays, M. *et al.* Coherent manipulation of an Andreev spin qubit. *Science* **373**, 430–433 (2021).
- [29] Pita-Vidal, M. *et al.* Direct manipulation of a superconducting spin qubit strongly coupled to a transmon qubit. *Nat. Phys.* (2023).
- [30] van Heck, B., Mi, S. & Akhmerov, A. R. Single fermion manipulation via superconducting phase differences in multiterminal Josephson junctions. *Phys. Rev. B* **90**, 155450 (2014).

- [31] Beenakker, C. W. J. *et al.* Wigner-Poisson statistics of topological transitions in a Josephson junction. *Phys. Rev. Lett.* **111**, 037001 (2013).
- [32] Kitaev, A. Y. Unpaired Majorana fermions in quantum wires. *Phys.-Uspekhi* **44**, 131–136 (2001).
- [33] Sau, J. D. & Sarma, S. D. Realizing a robust practical Majorana chain in a quantum-dot-superconductor linear array. *Nat. Commun.* **3** (2012).
- [34] Leijnse, M. & Flensberg, K. Parity qubits and poor man’s Majorana bound states in double quantum dots. *Phys. Rev. B* **86**, 134528 (2012).
- [35] Dvir, T. *et al.* Realization of a minimal kitaev chain in coupled quantum dots. *Nature* **614**, 445–450 (2023).
- [36] Douçot, B. & Ioffe, L. B. Physical implementation of protected qubits. *Rep. Prog. Phys.* **75**, 072001 (2012).
- [37] Brooks, P., Kitaev, A. & Preskill, J. Protected gates for superconducting qubits. *Phys. Rev. A* **87**, 052306 (2013).
- [38] Larsen, T. W. *et al.* Parity-protected superconductor-semiconductor qubit. *Phys. Rev. Lett.* **125**, 056801 (2020).
- [39] Gyenis, A. *et al.* Experimental realization of a protected superconducting circuit derived from the $0-\pi$ qubit. *PRX Quantum* **2**, 010339 (2021).
- [40] Coraiola, M. *et al.* Hybridisation of Andreev bound states in three-terminal Josephson junctions. arXiv:2302.14535 (2023).
- [41] Yokoyama, T. & Nazarov, Y. V. Singularities in the Andreev spectrum of a multiterminal Josephson junction. *Phys. Rev. B* **92**, 155437 (2015).
- [42] Riwar, R.-P., Houzet, M., Meyer, J. S. & Nazarov, Y. V. Multi-terminal Josephson junctions as topological matter. *Nat. Commun.* **7**, 11167 (2016).
- [43] Xie, H.-Y., Vavilov, M. G. & Levchenko, A. Topological Andreev bands in three-terminal Josephson junctions. *Phys. Rev. B* **96**, 161406 (2017).
- [44] Pillet, J.-D., Benzoni, V., Griesmar, J., Smir, J.-L. & Girit, Ç. O. Nonlocal Josephson effect in Andreev molecules. *Nano Lett.* **19**, 7138–7143 (2019).
- [45] Kornich, V., Barakov, H. S. & Nazarov, Y. V. Fine energy splitting of overlapping Andreev bound states in multiterminal superconducting nanostructures. *Phys. Rev. Res.* **1**, 033004 (2019).
- [46] Matsuo, S. *et al.* Observation of nonlocal Josephson effect on double InAs nanowires. *Commun. Phys.* **5**, 221 (2022).
- [47] Haxell, D. Z. *et al.* Demonstration of nonlocal Josephson effect in Andreev molecules. arXiv:2306.00866 (2023).
- [48] Beenakker, C. W. J. Universal limit of critical-current fluctuations in mesoscopic Josephson junctions. *Phys. Rev. Lett.* **67**, 3836–3839 (1991).
- [49] Cheah, E. *et al.* Control over epitaxy and the role of the InAs/Al interface in hybrid two-dimensional electron gas systems. arXiv:2301.06795 (2023).

- [50] Fan, F., Chen, Y., Pan, D., Zhao, J. & Xu, H. Q. Measurements of spin–orbit interaction in epitaxially grown InAs nanosheets. *Appl. Phys. Lett.* **117**, 132101 (2020).
- [51] Suominen, H. J. *et al.* Zero-energy modes from coalescing Andreev states in a two-dimensional semiconductor-superconductor hybrid platform. *Phys. Rev. Lett.* **119**, 176805 (2017).
- [52] Klapwijk, T., Blonder, G. & Tinkham, M. Explanation of subharmonic energy gap structure in superconducting contacts. *Physica B+C* **109–110**, 1657–1664 (1982).
- [53] Dynes, R. C., Narayanamurti, V. & Garno, J. P. Direct measurement of quasiparticle-lifetime broadening in a strong-coupled superconductor. *Phys. Rev. Lett.* **41**, 1509–1512 (1978).
- [54] Ihn, T. *Semiconductor Nanostructures: Quantum States and Electronic Transport* (Oxford University Press, 2010), 2 edn.
- [55] Yokoyama, T., Eto, M. & Nazarov, Y. V. Anomalous Josephson effect induced by spin-orbit interaction and Zeeman effect in semiconductor nanowires. *Phys. Rev. B* **89**, 195407 (2014).

From Single fermion manipul via SC phase diff in multiterminal JJ

References

- [1] H. A. Kramers, *Proc. Amsterdam Acad.* **33**, 959 (1930).
- [2] E. Wigner, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* **1932**, 546 (1932).
- [3] D. Loss and D. P. DiVincenzo, *Phys. Rev. A* **57**, 120 (1998).
- [4] S. A. Wolf, D. D. Awschalom, R. A. Buhrman, J. M. Daughton, S. v. Molnár, M. L. Roukes, A. Y. Chtchelkanova, and D. M. Treger, *Science* **294**, 1488 (2001).
- [5] I. Žutić, J. Fabian, and S. Das Sarma, *Rev. Mod. Phys.* **76**, 323 (2004).
- [10] J. Alicea, *Rep. Prog. Phys.* **75**, 076501 (2012).
- [7] C. W. J. Beenakker, *Ann. Rev. Cond. Mat. Phys.* **4**, 113 (2013).
- [15] C. W. J. Beenakker, *Phys. Rev. Lett.* **67**, 3836 (1991).
- [26] J. H. Bardarson, *J. Phys. A* **41**, 405203 (2008).
- [6] N. M. Chtchelkatchev and Y. V. Nazarov, *Phys. Rev. Lett.* **90**, 226806 (2003).
- [11] B. Béri, J. H. Bardarson, and C. W. J. Beenakker, *Phys. Rev. B* **77**, 045311 (2008).
- [7] C. Padurariu and Y. V. Nazarov, *Phys. Rev. B* **81**, 144519 (2010).
- [13] C. Padurariu and Y. V. Nazarov, *Europhys. Lett.* **100**, 57006 (2012).
- [14] A. Altland and M. R. Zirnbauer, *Phys. Rev. B* **55**, 1142 (1997).

- [4] C. W. J. Beenakker, J. M. Edge, J. P. Dahlhaus, D. I. Pikulin, S. Mi, and M. Wimmer, *Phys. Rev. Lett.* **111**, 037001 (2013).
- [16] M. L. Mehta, *Random Matrices* (Elsevier S&T, 2004).
- [17] D. A. Ivanov, *J. Math. Phys.* **43**, 126 (2002).
- [18] D. Bagrets and A. Altland, *Phys. Rev. Lett.* **109**, 227005 (2012).
- [19] C. Wu, B. A. Bernevig, and S.-C. Zhang, *Physical Review Letters* **96**, 106401 (2006).
- [42] In the presence of normal scattering in the superconductor, reflection from the superconductor can be represented as a combination of normal scattering followed by a subsequent perfect Andreev reflection. The former component can be combined with the scattering matrix of the normal region. Therefore, assuming perfect Andreev reflection does not reduce the generality of our results.
- [21] I. C. Fulga, F. Hassler, and A. R. Akhmerov, *Phys. Rev. B* **85**, 165409 (2012).
- [60] C. W. J. Beenakker, *Rev. Mod. Phys.* **69**, 731 (1997).
- [45] Generalization to unequal numbers of modes in two superconducting leads is straightforward, since in that case one of the leads will simply have several fully reflected modes leading to no extra Andreev bound states.
- [8] A. Y. Kitaev, *Phys. Uspekhi* **44**, 131 (2001).
- [25] H.-J. Kwon, K. Sengupta, and V. M. Yakovenko, *Euro. Phys. J. B* **37**, 349 (2003).
- [9] L. Fu and C. L. Kane, *Phys. Rev. B* **79**, 161408 (2009).
- [27] C. L. Kane and E. J. Mele, *Phys. Rev. Lett.* **95**, 146802 (2005).
- [28] C. L. Kane and E. J. Mele, *Phys. Rev. Lett.* **95**, 226801 (2005).
- [29] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, *Science* **314**, 1757 (2006).
- [30] M. König, S. Wiedmann, C. Brüne, A. Roth, H. Buhmann, L. W. Molenkamp, X.-L. Qi, and S.-C. Zhang, *Science* **318**, 766 (2007).
- [31] F. Mezzadri, *Notices Am. Math. Soc.* **54**, 592 (2007).
- [32] C. Liu, T. L. Hughes, X.-L. Qi, K. Wang, and S.-C. Zhang, *Phys. Rev. Lett.* **100**, 236601 (2008).
- [33] C. Liu and S. Zhang, in *Contemporary Concepts of Condensed Matter Science*, Topological Insulators, Vol. 6, edited by M. Franz and L. Molenkamp (Elsevier, 2013) pp. 59–89.
- [31] C. W. Groth, M. Wimmer, A. R. Akhmerov, and X. Waintal, *New J. Phys.* **16**, 063065 (2014).
- [47] We use the parameters for an InAs/GaAs/AlSb quantum well, with layer thickness of 10 nm for both GaSb and InAs [33], in a dot with radius $R \simeq 200$ nm, onsite disorder strength $u = 25$ meV, and $\mu = 0$.
- [36] T. Nagao and K. Slevin, *J. Math. Phys.* **34**, 2075 (1993).

- [37] L. Bretheau, Ç. Ö. Girit, H. Pothier, D. Esteve, and C. Urbina, [Nature](#) **499**, 312 (2013).
- [19] L. Bretheau, Ç. Ö. Girit, M. Houzet, H. Pothier, D. Esteve, and C. Urbina, [Physical Review B](#) **90**, 134506 (2014).
- [48] M. Zgirski, L. Bretheau, Q. Le Masne, H. Pothier, D. Esteve, and C. Urbina, [Phys. Rev. Lett.](#) **106**, 257003 (2011).
- [10] L. Bretheau, Ç. Ö. Girit, C. Urbina, D. Esteve, and H. Pothier, [Phys. Rev. X](#) **3**, 041034 (2013).
- [26] S. R. Plissard, I. van Weperen, D. Car, M. A. Verheijen, G. W. G. Immink, J. Kammhuber, L. J. Cornelissen, D. B. Szombati, A. Geresdi, S. M. Frolov, L. P. Kouwenhoven, and E. P. A. M. Bakkers, [Nat. Nano.](#) **8**, 859 (2013).
- [42] D. I. Pikulin and Y. V. Nazarov, [JETP Lett.](#) **94**, 693 (2012).
- [43] D. I. Pikulin, J. P. Dahlhaus, M. Wimmer, H. Schomerus, and C. W. J. Beenakker, [New J. Phys.](#) **14**, 125011 (2012).
- [44] S. Mi, D. I. Pikulin, M. Marciiani, and C. W. J. Beenakker, [arXiv:1405.6896](#) (2014).
- [45] D. G. Olivares, A. L. Yeyati, L. Bretheau, Ç. Ö. Girit, H. Pothier, and C. Urbina, [Phys. Rev. B](#) **89**, 104504 (2014).
- [30] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. a. M. Bakkers, and L. P. Kouwenhoven, [Science](#) **336**, 1003 (2012).
- [15] A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, and H. Shtrikman, [Nat. Phys.](#) **8**, 887 (2012).
- [48] I. Knez, R.-R. Du, and G. Sullivan, [Phys. Rev. Lett.](#) **109**, 186603 (2012).
- [9] W. Chang, V. E. Manucharyan, T. S. Jespersen, J. Nygård, and C. M. Marcus, [Phys. Rev. Lett.](#) **110**, 217005 (2013).
- [50] E. J. H. Lee, X. Jiang, R. Aguado, G. Katsaros, C. M. Lieber, and S. De Franceschi, [Phys. Rev. Lett.](#) **109**, 186802 (2012).
- [51] E. J. H. Lee, X. Jiang, M. Houzet, R. Aguado, C. M. Lieber, and S. De Franceschi, [Nat. Nano.](#) **9**, 79 (2014).
- [52] S. Hart, H. Ren, T. Wagner, P. Leubner, M. Mühlbauer, C. Brüne, H. Buhmann, L. W. Molenkamp, and A. Yacoby, [arXiv:1312.2559](#) (2013).
- [53] A. Romito, J. Alicea, G. Refael, and F. von Oppen, [Phys. Rev. B](#) **85**, 020502 (2012).
- [54] J. D. Sau and S. D. Sarma, [Nat. Comm.](#) **3**, 964 (2012).
- [55] I. C. Fulga, A. Haim, A. R. Akhmerov, and Y. Oreg, [New J. Phys.](#) **15**, 045020 (2013).

References

- [10] J. Alicea, Reports on Progress in Physics **75**, 076501 (2012), URL <http://stacks.iop.org/0034-4885/75/i=7/a=076501>.
- [4] C. Beenakker, Annual Review of Condensed Matter Physics **4**, 113 (2013), URL <http://dx.doi.org/10.1146/annurev-conmatphys-030212-184337>.
- [30] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Science **336**, 1003 (2012), ISSN 0036-8075, URL <http://science.sciencemag.org/content/336/6084/1003>.
- [15] A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, and H. Shtrikman, Nat. Phys. **8**, 887 (2012), ISSN 1745-2473, URL <http://www.nature.com/nphys/journal/v8/n12/full/nphys2479.html>.
- [5] S. Nadj-Perge, I. K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A. H. MacDonald, B. A. Bernevig, and A. Yazdani, Science **346**, 602 (2014), ISSN 0036-8075, URL <http://science.sciencemag.org/content/346/6209/602>.
- [10] S. M. Albrecht, A. P. Higginbotham, M. Madsen, F. Kuemmeth, T. S. Jespersen, J. Nygård, P. Krogstrup, and C. M. Marcus, Nature **531**, 206 (2016), ISSN 0028-0836, URL <http://dx.doi.org/10.1038/nature17162>.
- [7] S. Hart, H. Ren, M. Kosowsky, G. Ben-Shach, P. Leubner, C. Brüne, H. Buhmann, L. W. Molenkamp, B. I. Halperin, and A. Yacoby, arXiv:1509.02940 (2015).
- [8] Z. Wan, A. Kazakov, M. J. Manfra, L. N. Pfeiffer, K. W. West, and L. P. Rokhinson, Nat Commun **6**, (2015), URL <http://dx.doi.org/10.1038/ncomms8426>.
- [63] M. Kjaergaard, H. J. Suominen, M. P. Nowak, A. R. Akhmerov, J. Shabani, C. J. Palmstrøm, F. Nichele, and C. M. Marcus (2016), [arXiv:1607.04164](https://arxiv.org/abs/1607.04164).
- [10] T. Hyart, B. van Heck, I. C. Fulga, M. Burrello, A. R. Akhmerov, and C. W. J. Beenakker, Phys. Rev. B **88**, 035121 (2013), URL <http://link.aps.org/doi/10.1103/PhysRevB.88.035121>.
- [11] J. Li, T. Neupert, B. A. Bernevig, and A. Yazdani, Nature Communications **7**, 10395 (2016), URL <http://dx.doi.org/10.1038/ncomms10395>.
- [104] J. Shabani, M. Kjaergaard, H. J. Suominen, Y. Kim, F. Nichele, K. Pakrouski, T. Stankevici, R. M. Lutchyn, P. Krogstrup, R. Feidenhans'l, et al., Phys. Rev. B **93**, 155402 (2016), URL <http://link.aps.org/doi/10.1103/PhysRevB.93.155402>.
- [13] A. Vuik, D. Eeltink, A. R. Akhmerov, and M. Wimmer, New Journal of Physics **18**, 033013 (2016), URL <http://stacks.iop.org/1367-2630/18/i=3/a=033013>.
- [14] A. Altland and M. R. Zirnbauer, Physical Review B **55**, 1142 (1997), URL <http://link.aps.org/doi/10.1103/PhysRevB.55.1142>.
- [15] T. Mizushima and M. Sato, New Journal of Physics **15**, 075010 (2013), URL <http://stacks.iop.org/1367-2630/15/i=7/a=075010>.

- [16] V. V. Ryazanov, V. A. Oboznov, A. Y. Rusanov, A. V. Veretennikov, A. A. Golubov, and J. Aarts, Phys. Rev. Lett. **86**, 2427 (2001), URL <http://link.aps.org/doi/10.1103/PhysRevLett.86.2427>.
- [17] T. Kontos, M. Aprili, J. Lesueur, F. Genêt, B. Stephanidis, and R. Boursier, Phys. Rev. Lett. **89**, 137007 (2002), URL <http://link.aps.org/doi/10.1103/PhysRevLett.89.137007>.
- [18] S. M. Frolov, D. J. Van Harlingen, V. A. Oboznov, V. V. Bolginov, and V. V. Ryazanov, Phys. Rev. B **70**, 144505 (2004), URL <http://link.aps.org/doi/10.1103/PhysRevB.70.144505>.
- [19] See Appendix 13.9.2 for details of the model. The parameters used are: $W = 5$, $W_{SC} = 10$, $t = 1$, $\alpha = 0.1$, $\mu = -2.4$, $\Delta = 0.3$. The phase diagram is obtained by calculating the topological invariant for class D, $Q = \text{sign}[\text{Pf}(H_{k=\pi}\tau_x)/\text{Pf}(H_{k=0}\tau_x)]$ [36]. To obtain the phase difference which minimizes the free energy of the system, as well as the critical current across the junction, a summation over the contribution of all k_x momenta to the free energy was performed.
- [20] A. Y. Kitaev, AIP Conf. Proc. **1134**, 22 (2009).
- [34] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, AIP Conference Proceedings **1134**, 10 (2009), ISSN 0094243X, URL http://proceedings.aip.org/resource/2/apcpcs/1134/1/10_1?isAuthorized=no.
- [22] In the presence of interactions, the $\mathbb{Z}$ classification is modified to $\mathbb{Z}_8$ [23].
- [23] L. Fidkowski and A. Kitaev, Phys. Rev. B **81**, 134509 (2010), URL <http://link.aps.org/doi/10.1103/PhysRevB.81.134509>.
- [8] A. Y. Kitaev, Physics-Uspekhi **44**, 131 (2001), URL <http://stacks.iop.org/1063-7869/44/i=10S/a=S29>.
- [25] F. Dolcini, M. Houzet, and J. S. Meyer, Phys. Rev. B **92**, 035428 (2015), URL <http://link.aps.org/doi/10.1103/PhysRevB.92.035428>.
- [15] C. W. J. Beenakker, Phys. Rev. Lett. **67**, 3836 (1991), URL <http://link.aps.org/doi/10.1103/PhysRevLett.67.3836>.
- [36] S. Tewari and J. D. Sau, Phys. Rev. Lett. **109**, 150408 (2012), URL <http://link.aps.org/doi/10.1103/PhysRevLett.109.150408>.
- [28] T. Yokoyama and Y. V. Nazarov, EPL (Europhysics Letters) **108**, 47009 (2014), URL <http://stacks.iop.org/0295-5075/108/i=4/a=47009>.
- [29] C. Ishii, Progress of theoretical Physics **44**, 1525 (1970).
- [30] A. M. Zagoskin, *Quantum theory of many-body systems* (Springer International Publishing, 2014).
- [31] M. Hell, M. Leijnse, and K. Flensberg, arXiv:1608.08769 (2016).
- [32] D. Sticlet, B. Nijholt, and A. Akhmerov, arXiv:1609.00637 (2016).
- [33] J. H. Bardarson, J. Tworzydło, P. W. Brouwer, and C. W. J. Beenakker, Phys. Rev. Lett. **99**, 106801 (2007), URL <http://link.aps.org/doi/10.1103/PhysRevLett.99.106801>.

- [34] P. W. Brouwer, M. Duckheim, A. Romito, and F. von Oppen, *Phys. Rev. B* **84**, 144526 (2011), URL <http://link.aps.org/doi/10.1103/PhysRevB.84.144526>.

From Robustness of Majorana bound states in the short-junction limit

References

- [1] X.-L. Qi and S.-C. Zhang, *Rev. Mod. Phys.* **83**, 1057 (2011).
- [2] M. Leijnse and K. Flensberg, *Semicond. Sci. Technol.* **27**, 124003 (2012).
- [10] J. Alicea, *Rep. Prog. Phys.* **75**, 076501 (2012).
- [4] C. Beenakker, *Annu. Rev. Condens. Matter Phys.* **4**, 113 (2013).
- [5] S. R. Elliott and M. Franz, *Rev. Mod. Phys.* **87**, 137 (2015).
- [6] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, *Phys. Rev. Lett.* **105**, 077001 (2010).
- [7] Y. Oreg, G. Refael, and F. von Oppen, *Phys. Rev. Lett.* **105**, 177002 (2010).
- [8] W. Chang, S. M. Albrecht, T. S. Jespersen, F. Kuemmeth, P. Krogstrup, J. Nygård, and C. M. Marcus, *Nat. Nanotechnol.* **10**, 232 (2015).
- [9] M. T. Deng, S. Vaitiekėnas, E. B. Hansen, J. Danon, M. Leijnse, K. Flensberg, J. Nygård, P. Krogstrup, and C. M. Marcus, *Science* **354**, 1557 (2016).
- [10] S. M. Albrecht, A. P. Higginbotham, M. Madsen, F. Kuemmeth, T. S. Jespersen, J. Nygård, P. Krogstrup, and C. M. Marcus, *Nature* **531**, 206 (2016).
- [11] W. S. Cole, S. Das Sarma, and T. D. Stanescu, *Phys. Rev. B* **92**, 174511 (2015).
- [12] A. M. Clogston, *Phys. Rev. Lett.* **9**, 266 (1962).
- [13] C. W. J. Beenakker, in *Quantum Dots: a Doorway to Nanoscale Physics*, Lecture Notes in Physics No. 667, edited by W. D. Heiss (Springer Berlin Heidelberg, 2005) pp. 131–174.
- [14] E. V. Bezuglyi, A. S. Rozhavsky, I. D. Vagner, and P. Wyder, *Phys. Rev. B* **66**, 052508 (2002).
- [14] O. V. Dimitrova and M. V. Feigel'man, *J. Exp. Theor. Phys.* **102**, 652 (2006).
- [16] M. Cheng and R. M. Lutchyn, *Phys. Rev. B* **86**, 134522 (2012).
- [17] J. Liu, A. C. Potter, K. T. Law, and P. A. Lee, *Phys. Rev. Lett.* **109**, 267002 (2012).
- [18] T. D. Stanescu, R. M. Lutchyn, and S. Das Sarma, *Phys. Rev. B* **87**, 094518 (2013).
- [19] Y. Peng, F. Pientka, L. I. Glazman, and F. von Oppen, *Phys. Rev. Lett.* **114**, 106801 (2015).
- [20] K. Pöyhönen, A. Westström, and T. Ojanen, *Phys. Rev. B* **93**, 014517 (2016).
- [21] M. Wimmer, A. R. Akhmerov, M. V. Medvedyeva, J. Tworzydło, and C. W. J. Beenakker, *Phys. Rev. Lett.* **105**, 046803 (2010).

- [22] R. M. Lutchyn, T. D. Stanescu, and S. Das Sarma, *Phys. Rev. Lett.* **106**, 127001 (2011).
- [23] W. S. Cole, J. D. Sau, and S. Das Sarma, *Phys. Rev. B* **94**, 140505 (2016).
- [24] B. van Heck, S. Mi, and A. R. Akhmerov, *Phys. Rev. B* **90**, 155450 (2014).
- [12] A. F. Andreev, *Zh. Eksp. Teor. Fiz.* **46**, 1823 (1964), [*Sov. Phys. JETP* 19, 1228 (1964)].
- [26] J. H. Bardarson, *J. Phys. A-Math. Theor.* **41**, 405203 (2008).
- [27] C. W. J. Beenakker, *Rev. Mod. Phys.* **87**, 1037 (2015).
- [15] C. W. J. Beenakker, *Phys. Rev. Lett.* **67**, 3836 (1991).
- [29] D. J. BenDaniel and C. B. Duke, *Phys. Rev.* **152**, 683 (1966).
- [30] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. a. M. Bakkers, and L. P. Kouwenhoven, *Science* **336**, 1003 (2012).
- [31] C. W. Groth, M. Wimmer, A. R. Akhmerov, and X. Waintal, *New J. Phys.* **16**, 063065 (2014).
- [32] B. Nijholt and A. R. Akhmerov, *Phys. Rev. B* **93**, 235434 (2016).
- [33] J. Klinovaja and D. Loss, *Phys. Rev. B* **86**, 085408 (2012).
- [34] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, *AIP Conf. Proc.* **1134**, 10 (2009).
- [35] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, *New J. Phys.* **12**, 065010 (2010).
- [36] S. Tewari and J. D. Sau, *Phys. Rev. Lett.* **109**, 150408 (2012).
- [37] J. S. Lim, R. López, and L. Serra, *EPL (Europhysics Letters)* **103**, 37004 (2013).
- [38] J. Osca, R. López, and L. Serra, *Eur. Phys. J. B* **87**, 84 (2014).
- [39] J. Osca and L. Serra, *Phys. Rev. B* **91**, 235417 (2015).
- [42] The code for the tight-binding simulations of the two-dimensional and three-dimensional junctions is available in ancillary files to the manuscript.

From Sub-Gap Structure in the Conductance of a Three-Terminal Josephson Junction

References

- [1] K. Likharev, Gordon and Breach, New York (1991).
- [2] A. F. Andreev, *Sov. Phys. JETP* **22**, 455 (1966).
- [3] T. M. Klapwijk, G. E. Blonder, and M. Tinkham, *Physica* **109/110B**, 1657 (1982).
- [4] M. Octavio, M. Tinkham, G. E. Blonder, and T. M. Klapwijk, *Phys. Rev. B* **27**, 6739 (1983).

- [5] H. Courtois, P. Gandit, B. Pannetier, and D. Maily, *Superlattices and Microstructures* **25**, 721 (1999).
- [6] J. M. Byers and M. E. Flatté, *Phys. Rev. Lett.* **74**, 306 (1995).
- [7] M. P. Anantram and S. Datta, *Phys. Rev. B* **53**, 16390 (1996).
- [8] J. Torrès and T. Martin, *The European Physical Journal B - Condensed Matter and Complex Systems* **12**, 319 (1999).
- [9] G. Deutscher and D. Feinberg, *Applied Physics Letters* **76**, 487 (2000).
- [10] P. Recher, E. V. Sukhorukov, and D. Loss, *Phys. Rev. B* **63**, 165314 (2001).
- [11] D. Beckmann, H. B. Weber, and H. v. Löhneysen, *Phys. Rev. Lett.* **93**, 197003 (2004).
- [12] P. Cadden-Zimansky and V. Chandrasekhar, *Phys. Rev. Lett.* **97**, 237003 (2006).
- [13] L. Hofstetter, J. Csonka, S. Nygård, and C. Schönenberger, *Nature* **461**, 960 (2009).
- [14] L. G. Herrmann, F. Portier, P. Roche, A. L. Yeyati, T. Kontos, and C. Strunk, *Phys. Rev. Lett.* **104**, 026801 (2010).
- [15] A. Das, Y. Ronen, M. Heiblum, D. Mahalu, A. V. Kretinin, and H. Shtrikman, *Nature Com.* **3**, 1165 (2012).
- [16] S. Duhot, F. Lefloch, and M. Houzet, *Phys. Rev. Lett.* **102**, 086804 (2009).
- [17] M. Houzet and P. Samuelsson, *Phys. Rev. B* **82**, 060517 (2010).
- [18] B. Kaviraj, O. Coupiac, H. Courtois, and F. Lefloch, *Phys. Rev. Lett.* **107**, 077005 (2011).
- [19] F. Giazotto and F. Taddei, *Phys. Rev. B* **84**, 214502 (2011).
- [20] A. V. Galaktionov, A. D. Zaikin, and L. S. Kuzmin, *Phys. Rev. B* **85**, 224523 (2012).
- [21] J. C. Cuevas and H. Pothier, *Physical Review B* **75**, 174513 (2007).
- [36] A. Freyn, B. Douçot, D. Feinberg, and R. Mélin, *Phys. Rev. Lett.* **106**, 257005 (2011).
- [37] T. Jonckheere, J. Rech, T. Martin, B. Douçot, D. Feinberg, and R. Mélin, *Phys. Rev. B* **87**, 214501 (2013).
- [24] N. M. Chtchelkatchev, T. I. Baturina, A. Glatz, and V. M. Vinokur, *Phys. Rev. B* **82**, 024526 (2010).
- [25] P. Dubos, H. Courtois, O. Buisson, and B. Pannetier, *Phys. Rev. Lett.* **87**, 206801 (2001).
- [26] C. Hoffmann, F. Lefloch, M. Sanquer, and B. Pannetier, *Phys. Rev. B* **70**, 180503 (2004).
- [27] A. H. Pfeffer, B. Kaviraj, O. Coupiac, and F. Lefloch, *Review of Scientific Instruments* **83**, 115107 (2012).
- [28] F. Pierre, A. Anthore, H. Pothier, C. Urbina, and D. Esteve, *Phys. Rev. Lett.* **86**, 1078 (2001).
- [29] H. Courtois, M. Meschke, J. T. Peltonen, and J. P. Pekola, *Phys. Rev. Lett.* **101**, 067002 (2008).

- [30] M. A. H. Nerenberg, J. A. Blackburn, and D. W. Jillie, *Phys. Rev. B* **21**, 118 (1980).
- [31] D. W. Jillie, M. A. H. Nerenberg, and J. A. Blackburn, *Phys. Rev. B* **21**, 125 (1980).
- [32] A. F. Volkov and H. Takayanagi, *Phys. Rev. Lett.* **76**, 4026 (1996).
- [33] N. Argaman, *Superlattices and Microstructures* **25**, 861 (1999).
- [34] K. W. Lehnert, N. Argaman, H.-R. Blank, K. C. Wong, S. J. Allen, E. L. Hu, and H. Kroemer, *Phys. Rev. Lett.* **82**, 1265 (1999).
- [47] J. C. Cuevas, A. Martin-Rodero, and A. L. Yeyati, *Phys. Rev. B* **54**, 7366 (1996).
- [36] E. Akkermans and G. Montambaux, *Mesoscopic Physics of Electrons and Photons* (Cambridge University Press, 2007).
- [37] D. Feinberg, *European Physical Journal B* **36**, 419 (2003).

From The multi-terminal Josephson effect by Pankratova et al

References

- [1] Brian David Josephson, “Possible new effects in superconductive tunnelling,” *Phys. Lett.* **1**, 251–253 (1962).
- [2] Antonio Barone and Gianfranco Paterno, *Physics and applications of the Josephson effect* (Wiley, 1982).
- [3] Hansjörg Scherer and Benedetta Camarota, “Quantum metrology triangle experiments: a status review,” *Measurement Science and Technology* **23**, 124010 (2012).
- [4] John Clarke and Alex I Braginski, *The SQUID handbook: Applications of SQUIDs and SQUID systems* (John Wiley & Sons, 2006).
- [5] Konstantin K. Likharev and Vasilii K. Semenov, “Rsfq logic/memory family: A new josephson-junction technology for sub-terahertz-clock-frequency digital systems,” *IEEE Transactions on Applied Superconductivity* **1**, 3–28 (1991).
- [6] Michel H. Devoret and Robert J. Schoelkopf, “Superconducting circuits for quantum information: an outlook,” *Science* **339**, 1169–1174 (2013).
- [7] Elke Scheer, Nicolás Agraït, Juan Carlos Cuevas, Alfredo Levy Yeyati, Bas Ludoph, Alvaro Martín-Rodero, Gabino Rubio Bollinger, Jan M van Ruitenbeek, and Cristián Urbina, “The signature of chemical valence in the electrical conduction through a single-atom contact,” *Nature* **394**, 154 (1998).
- [8] Clemens B Winkelmann, Nicolas Roch, Wolfgang Wernsdorfer, Vincent Bouchiat, and Franck Balestro, “Superconductivity in a single-c 60 transistor,” *Nature Physics* **5**, 876 (2009).
- [9] Pablo Jarillo-Herrero, Jorden A Van Dam, and Leo P Kouwenhoven, “Quantum super-current transistors in carbon nanotubes,” *Nature* **439**, 953 (2006).

- [10] Yong-Joo Doh, Jorden A van Dam, Aarnoud L Roest, Erik PAM Bakkers, Leo P Kouwenhoven, and Silvano De Franceschi, “Tunable supercurrent through semiconductor nanowires,” *Science* **309**, 272–275 (2005).
- [11] Hélène Le Sueur, P Joyez, H Pothier, C Urbina, and Daniel Esteve, “Phase controlled superconducting proximity effect probed by tunneling spectroscopy,” *Phys. Rev. Lett.* **100**, 197002 (2008).
- [12] Liang Fu and Charles L Kane, “Superconducting proximity effect and majorana fermions at the surface of a topological insulator,” *Physical Review Letters* **100**, 096407 (2008).
- [13] Roman M Lutchyn, Jay D Sau, and S Das Sarma, “Majorana fermions and a topological phase transition in semiconductor-superconductor heterostructures,” *Phys. Rev. Lett.* **105**, 077001 (2010).
- [14] Yuval Oreg, Gil Refael, and Felix von Oppen, “Helical liquids and majorana bound states in quantum wires,” *Phys. Rev. Lett.* **105**, 177002 (2010).
- [15] Vincent Mourik, Kun Zuo, Sergey M Frolov, SR Plissard, Erik PAM Bakkers, and Leo P Kouwenhoven, “Signatures of majorana fermions in hybrid superconductor-semiconductor nanowire devices,” *Science* **336**, 1003–1007 (2012).
- [16] T Hyart, B Van Heck, I. C. Fulga, M Burrello, A. R. Akhmerov, and C. W. J. Beenakker, “Flux-controlled quantum computation with majorana fermions,” *Physical Review B* **88**, 035121 (2013).
- [17] David Aasen, Michael Hell, Ryan V Mishmash, Andrew Higginbotham, Jeroen Danon, Martin Leijnse, Thomas S Jespersen, Joshua A Folk, Charles M Marcus, Karsten Flensberg, *et al.*, “Milestones toward majorana-based quantum computing,” *Physical Review X* **6**, 031016 (2016).
- [18] A. Yu. Kitaev, “Fault-tolerant quantum computation by anyons,” *Ann. Phys.* **303**, 2–30 (2003).
- [19] Roman-Pascal Riwar, Manuel Houzet, Julia S Meyer, and Yuli V Nazarov, “Multi-terminal josephson junctions as topological matter,” *Nature communications* **7**, 11167 (2016).
- [20] J Shabani, M Kjaergaard, HJ Suominen, Younghyun Kim, F Nichele, K Pakrouski, T Stankevic, Roman M Lutchyn, P Krogstrup, R Feidenhans, *et al.*, “Two-dimensional epitaxial superconductor-semiconductor heterostructures: A platform for topological superconducting networks,” *Physical Review B* **93**, 155402 (2016).
- [21] Alexander Altland and Martin R Zirnbauer, “Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures,” *Physical Review B* **55**, 1142 (1997).
- [22] G. E. Blonder, M Tinkham, and T. M. Klapwijk, “Transition from metallic to tunneling regimes in superconducting microconstrictions: Excess current, charge imbalance, and supercurrent conversion,” *Physical Review B* **25**, 4515 (1982).
- [23] Yu G Bevza, VI Karamushko, EA Kel’man, GG Tsach, and IM Dmitrenko, “Electrically controlled josephson elements,” *Sov. J. Low Temp. Phys.(Engl. Transl.);(United States)* **5** (1979).
- [24] MHS Amin, AN Omelyanchouk, and AM Zagorskin, “Dc squid based on the mesoscopic multiterminal josephson junction,” *Physica C: Superconductivity* **372**, 178–180 (2002).

- [25] B Van Heck, S Mi, and A. R. Akhmerov, “Single fermion manipulation via superconducting phase differences in multiterminal josephson junctions,” *Physical Review B* **90**, 155450 (2014).
- [26] Julia S Meyer and Manuel Houzet, “Nontrivial chern numbers in three-terminal josephson junctions,” *Phys. Rev. Lett.* **119**, 136807 (2017).
- [27] Hong-Yi Xie, Maxim G Vavilov, and Alex Levchenko, “Topological andreev bands in three-terminal josephson junctions,” *Physical Review B* **96**, 161406 (2017).
- [28] Andreas H Pfeffer, Jean Eudes Duvauchelle, Hervé Courtois, Régis Mélin, Denis Feinberg, and Francois Lefloch, “Subgap structure in the conductance of a three-terminal josephson junction,” *Physical Review B* **90**, 075401 (2014).
- [29] E Strambini, S D’Ambrosio, F Vischi, FS Bergeret, Yu V Nazarov, and F Giazotto, “The ω -squirr as a tool to phase-engineer josephson topological materials,” *Nature nanotechnology* **11**, 1055 (2016).
- [30] Sébastien R Plissard, Ilse Van Weperen, Diana Car, Marcel A Verheijen, George WG Immink, Jakob Kammhuber, Ludo J Cornelissen, Daniel B Szombati, Attila Geresdi, Sergey M Frolov, *et al.*, “Formation and electronic properties of insb nanocrosses,” *Nature nanotechnology* **8**, 859 (2013).
- [31] Kaushini Wickramasinghe, William Mayer, Joseph Yuan, Tri Nguyen, Vladimir Manucharyan, and Javad Shabani, “High mobility surface inas two-dimensional heterostructures for hybrid superconductor-semiconductor systems,” *arXiv preprint arXiv:1802.09569* (2018).
- [32] D Averin and A Bardas, “ac josephson effect in a single quantum channel,” *Phys. Rev. Lett.* **75**, 1831 (1995).
- [33] Morten Kjærgaard, Henri Juhani Suominen, MP Nowak, AR Akhmerov, J Shabani, CJ Palmstrøm, Fabrizio Nichele, and Charles M Marcus, “Transparent semiconductor-superconductor interface and induced gap in an epitaxial heterostructure josephson junction,” *Physical Review Applied* **7**, 034029 (2017).
- [34] William Mayer, Joseph Yuan, Kaushini Wickramasinghe, Tri Nguyen, Matthieu Dartiaillh, and Javad Shabani, “Superconducting proximity effect in epitaxial al-inas heterostructures,” *arXiv preprint arXiv:1810.02514* (2018).
- [35] M Houzet and Peter Samuelsson, “Multiple andreev reflections in hybrid multiterminal junctions,” *Physical Review B* **82**, 060517 (2010).
- [36] JC Cuevas and H Pothier, “Voltage-induced shapiro steps in a superconducting multiterminal structure,” *Physical Review B* **75**, 174513 (2007).
- [37] Axel Freyn, Benoît Douçot, Denis Feinberg, and Régis Mélin, “Production of nonlocal quartets and phase-sensitive entanglement in a superconducting beam splitter,” *Phys. Rev. Lett.* **106**, 257005 (2011).
- [38] Yonatan Cohen, Yuval Ronen, Jung-Hyun Kang, Moty Heiblum, Denis Feinberg, Régis Mélin, and Hadas Shtrikman, “Nonlocal supercurrent of quartets in a three-terminal josephson junction,” *Proceedings of the National Academy of Sciences* , 201800044 (2018).

- [39] C. W. J. Beenakker, “Universal limit of critical-current fluctuations in mesoscopic josephson junctions,” *Physical Review Letters* **67**, 3836 (1991).
- [40] M Büttiker, “Four-terminal phase-coherent conductance,” *Phys. Rev. Lett.* **57**, 1761 (1986).
- [41] Antonio Fornieri, Alexander M Whiticar, F Setiawan, Elías Portolés Marín, Asbjørn CC Drachmann, Anna Keselman, Sergei Gronin, Candice Thomas, Tian Wang, Ray Kallagher, *et al.*, “Evidence of topological superconductivity in planar josephson junctions,” arXiv preprint arXiv:1809.03037 (2018).
- [42] Jason Alicea, Yuval Oreg, Gil Refael, Felix Von Oppen, and Matthew P A Fisher, “Non-abelian statistics and topological quantum information processing in 1d wire networks,” *Nature Physics* **7**, 412 (2011).
- [43] Torsten Karzig, Christina Knapp, Roman M Lutchyn, Parsa Bonderson, Matthew B Hastings, Chetan Nayak, Jason Alicea, Karsten Flensberg, Stephan Plugge, Yuval Oreg, *et al.*, “Scalable designs for quasiparticle-poisoning-protected topological quantum computation with majorana zero modes,” *Physical Review B* **95**, 235305 (2017).
- [44] Anne W Draelos, Ming-Tso Wei, Andrew Seredinski, Hengming Li, Yash Mehta, Kenji Watanabe, Takashi Taniguchi, Ivan V Borzenets, Francois Amet, and Gleb Finkelstein, “Supercurrent flow in multi-terminal graphene josephson junctions,” arXiv preprint arXiv:1810.11632 (2018).
- [45] Pierre-Gilles De Gennes, *Superconductivity of metals and alloys* (CRC Press, 2018).

From Theory of non-eq noise in gener mlt-term SC hydrid devices

References

- [1] A. F. Andreev, *Soviet Physics JETP* **19**, 1228 (1964).
- [2] M. Tinkham, *Introduction to Superconductivity: Second Edition* (Dover Publications, 2004).
- [3] M. Octavio, M. Tinkham, G. E. Blonder, and T. M. Klapwijk, *Phys. Rev. B* **27**, 6739 (1983).
- [4] J. C. Cuevas, A. Martín-Rodero, and A. L. Yeyati, *Phys. Rev. B* **54**, 7366 (1996).
- [5] J. C. Cuevas, A. Martín-Rodero, and A. L. Yeyati, *Phys. Rev. Lett.* **82**, 4086 (1999).
- [6] E. N. Bratus’, V. S. Shumeiko, and G. Wendin, *Phys. Rev. Lett.* **74**, 2110 (1995).
- [7] G. B. Lesovik, T. Martin, and G. Blatter, *Eur. Phys. J. B* **24**, 287 (2001).
- [8] P. Recher, E. V. Sukhorukov, and D. Loss, *Phys. Rev. B* **63**, 165314 (2001).
- [9] O. Sauret, D. Feinberg, and T. Martin, *Phys. Rev. B* **70**, 245313 (2004).
- [10] N. M. Chtchelkatchev, G. Blatter, G. B. Lesovik, and T. Martin, *Phys. Rev. B* **66**, 161320 (2002).
- [11] O. Sauret, T. Martin, and D. Feinberg, *Phys. Rev. B* **72**, 024544 (2005).

- [12] T. Jonckheere, J. Rech, T. Martin, B. Douçot, D. Feinberg, and R. Mélin, *Phys. Rev. B* **87**, 214501 (2013).
- [13] D. Chevallier, J. Rech, T. Jonckheere, and T. Martin, *Phys. Rev. B* **83**, 125421 (2011).
- [14] D. S. Fisher and P. A. Lee, *Phys. Rev. B* **23**, 6851 (1981).
- [15] T. Jonckheere, A. Zazunov, K. V. Bayandin, V. Shumeiko, and T. Martin, *Phys. Rev. B* **80**, 184510 (2009).
- [16] J. Rammer and H. Smith, *Rev. Mod. Phys.* **58**, 323 (1986).
- [17] Y. Meir and N. S. Wingreen, *Phys. Rev. Lett.* **68**, 2512 (1992).
- [18] D. Bathellier, L. Raymond, T. Jonckheere, J. Rech, A. Zazunov, and T. Martin, *Phys. Rev. B* **99**, 104502 (2019).
- [19] J. Rech, D. Chevallier, T. Jonckheere, and T. Martin, *Phys. Rev. B* **85**, 035419 (2012).
- [20] S. Duhot, F. Lefloch, and M. Houzet, *Phys. Rev. Lett.* **102**, 086804 (2009).
- [21] M. Houzet and P. Samuelsson, *Phys. Rev. B* **82**, 060517(R) (2010).
- [22] N. M. Chtchelkatchev, T. I. Baturina, A. Glatz, and V. M. Vinokur, *Phys. Rev. B* **82**, 024526 (2010).
- [23] B. Kaviraj, O. Coupiac, H. Courtois, and F. Lefloch, *Phys. Rev. Lett.* **107**, 077005 (2011).
- [24] E. V. Bezuglyi, E. N. Bratus', V. S. Shumeiko, G. Wendin, and H. Takayanagi, *Phys. Rev. B* **62**, 14439 (2000).
- [25] J. M. Byers and M. E. Flatté, *Phys. Rev. Lett.* **74**, 306 (1995).
- [26] T. Martin, *Physics Letters A* **220**, 137 (1996).
- [27] M. P. Anantram and S. Datta, *Phys. Rev. B* **53**, 16390 (1996).
- [28] G. Deutscher and D. Feinberg, *Appl. Phys. Lett.* **76**, 487 (2000).
- [29] R. Mélin, *J. Phys.: Condens. Matter* **13**, 6445 (2001).
- [30] A. Freyn, B. Douçot, D. Feinberg, and R. Mélin, *Phys. Rev. Lett.* **106**, 257005 (2011).
- [31] J. C. Cuevas and H. Pothier, *Phys. Rev. B* **75**, 174513 (2007).
- [32] A. H. Pfeffer, J. E. Duvauchelle, H. Courtois, R. Mélin, D. Feinberg, and F. Lefloch, *Phys. Rev. B* **90**, 075401 (2014).
- [33] J. E. Duvauchelle, A. H. Pfeffer, H. Courtois, and F. Lefloch, *IEEE Transactions on Applied Superconductivity* **26**, 1 (2016).
- [34] Torrès, J. and Martin, T., *Eur. Phys. J. B* **12**, 319 (1999).
- [35] A. Das, Y. Ronen, M. Heiblum, D. Mahalu, A. V. Kretinin, and H. Shtrikman, *Nature Communications* **3**, 1165 (2012).
- [36] A. Freyn, M. Flöser, and R. Mélin, *Phys. Rev. B* **82**, 014510 (2010).

- [37] R.-P. Riwar, D. M. Badiane, M. Houzet, J. S. Meyer, and Y. V. Nazarov, *Physica E: Low-dimensional Systems and Nanostructures* **82**, 92 (2016).
- [38] R. Mélin, M. Sotto, D. Feinberg, J.-G. Caputo, and B. Douçot, *Phys. Rev. B* **93**, 115436 (2016).
- [39] Y. Cohen, Y. Ronen, J.-H. Kang, M. Heiblum, D. Feinberg, R. Mélin, and H. Shtrikman, *Proceedings of the National Academy of Sciences* **115**, 6991 (2018), <https://www.pnas.org/content/115/27/6991.full.pdf>.
- [40] P. Burset, W. J. Herrera, and A. L. Yeyati, *Phys. Rev. B* **84**, 115448 (2011).
- [41] V. Bouchiat, N. Chtchelkatchev, D. Feinberg, G. B. Lesovik, T. Martin, and J. Torrès, *Nanotechnology* **14**, 77 (2002).
- [42] P. Samuelsson, E. V. Sukhorukov, and M. Büttiker, *Phys. Rev. Lett.* **91**, 157002 (2003).
- [43] D. Beckmann, H. B. Weber, and H. v. Löhneysen, *Phys. Rev. Lett.* **93**, 197003 (2004).
- [44] S. Russo, M. Kroug, T. M. Klapwijk, and A. F. Morpurgo, *Phys. Rev. Lett.* **95**, 027002 (2005).
- [45] P. Cadden-Zimansky and V. Chandrasekhar, *Phys. Rev. Lett.* **97**, 237003 (2006).
- [46] J.-P. Cleuziou, W. Wernsdorfer, V. Bouchiat, T. Ondarcuhu, and M. Monthieux, *Nature Nanotech.* **1**, 53 (2006).
- [47] L. Hofstetter, S. Csonka, J. Nygård, and C. Schönenberger, *Nature* **461**, 960 (2009).
- [48] L. G. Herrmann, F. Portier, P. Roche, A. L. Yeyati, T. Kontos, and C. Strunk, *Phys. Rev. Lett.* **104**, 026801 (2010).
- [49] L. Hofstetter, S. Csonka, A. Baumgartner, G. Fülöp, S. d'Hollosy, J. Nygård, and C. Schönenberger, *Phys. Rev. Lett.* **107**, 136801 (2011).
- [50] A. Zazunov, R. Egger, C. Mora, and T. Martin, *Phys. Rev. B* **73**, 214501 (2006).
- [51] G. B. Lesovik and R. Loosen, *JETP Lett.* **65**, 295 (1997).
- [52] A. Zazunov, M. Creux, E. Paladino, A. Crépieux, and T. Martin, *Phys. Rev. Lett.* **99**, 066601 (2007).
- [53] D. Chevallier, T. Jonckheere, E. Paladino, G. Falci, and T. Martin, *Phys. Rev. B* **81**, 205411 (2010).
- [54] T. Jonckheere, J. Rech, A. Zazunov, R. Egger, A. L. Yeyati, and T. Martin, *Phys. Rev. Lett.* **122**, 097003 (2019).
- [55] M. Büttiker, Y. Imry, R. Landauer, and S. Pinhas, *Phys. Rev. B* **31**, 6207 (1985).
- [56] M. Büttiker, *Phys. Rev. Lett.* **57**, 1761 (1986).
- [57] T. Martin and R. Landauer, *Phys. Rev. B* **45**, 1742 (1992).
- [58] M. Büttiker, *Phys. Rev. B* **46**, 12485 (1992).
- [59] J. Rech, T. Jonckheere, T. Martin, B. Douçot, D. Feinberg, and R. Mélin, *Phys. Rev. B* **90**, 075419 (2014).

- [60] R. Jacquet, J. Rech, T. Jonckheere, A. Zazunov, and T. Martin, *Phys. Rev. B* **92**, 235429 (2015).
- [19] Y. V. Nazarov, *Superlattices and Microstructures* **25**, 1221 (1999).
- [3] Y. V. Nazarov and Y. M. Blanter, *Quantum Transport: Introduction to Nanoscience* (Cambridge University Press, 2009).
- [63] C. Padurariu, T. Jonckheere, J. Rech, T. Martin, and D. Feinberg, *Phys. Rev. B* **95**, 205437 (2017).

From Phase-engineering the ABS of a 3T-JJ

References

- [1] Andreev, A. F. Thermal conductivity of the intermediate state of superconductors. *Sov. Phys. JETP* **19**, 1228–1231 (1964).
- [2] Beenakker, C. W. J. & van Houten, H. Josephson current through a superconducting quantum point contact shorter than the coherence length. *Phys. Rev. Lett.* **66**, 3056–3059 (1991).
- [3] Furusaki, A. & Tsukada, M. Current-carrying states in Josephson junctions. *Phys. Rev. B* **43**, 10164–10169 (1991).
- [4] Despósito, M. A. & Levy Yeyati, A. Controlled dephasing of Andreev states in superconducting quantum point contacts. *Phys. Rev. B* **64**, 140511 (2001).
- [5] Zazunov, A., Shumeiko, V. S., Bratus', E. N., Lantz, J. & Wendin, G. Andreev level qubit. *Phys. Rev. Lett.* **90**, 087003 (2003).
- [6] Chtchelkatchev, N. M. & Nazarov, Y. V. Andreev quantum dots for spin manipulation. *Phys. Rev. Lett.* **90**, 226806 (2003).
- [7] Padurariu, C. & Nazarov, Y. V. Theoretical proposal for superconducting spin qubits. *Phys. Rev. B* **81**, 144519 (2010).
- [8] Pillet, J.-D. *et al.* Andreev bound states in supercurrent-carrying carbon nanotubes revealed. *Nat. Phys.* **6**, 965–969 (2010).
- [9] Chang, W., Manucharyan, V. E., Jespersen, T. S., Nygård, J. & Marcus, C. M. Tunneling spectroscopy of quasiparticle bound states in a spinful Josephson junction. *Phys. Rev. Lett.* **110**, 217005 (2013).
- [10] Bretheau, L., Girit, Ç. Ö., Pothier, H., Esteve, D. & Urbina, C. Exciting Andreev pairs in a superconducting atomic contact. *Nature* **499**, 312–315 (2013).
- [11] Bretheau, L., Girit, Ç. Ö., Urbina, C., Esteve, D. & Pothier, H. Supercurrent spectroscopy of Andreev states. *Phys. Rev. X* **3**, 041034 (2013).
- [12] Bretheau, L. *et al.* Tunnelling spectroscopy of Andreev states in graphene. *Nat. Phys.* **13**, 756–760 (2017).
- [13] van Woerkom, D. J. *et al.* Microwave spectroscopy of spinful Andreev bound states in ballistic semiconductor Josephson junctions. *Nat. Phys.* **13**, 876–881 (2017).

- [14] Tosi, L. *et al.* Spin-orbit splitting of Andreev states revealed by microwave spectroscopy. *Phys. Rev. X* **9**, 011010 (2019).
- [15] Nichele, F. *et al.* Relating Andreev bound states and supercurrents in hybrid Josephson junctions. *Phys. Rev. Lett.* **124**, 226801 (2020).
- [16] Janvier, C. *et al.* Coherent manipulation of Andreev states in superconducting atomic contacts. *Science* **349**, 1199–1202 (2015).
- [17] Hays, M. *et al.* Direct microwave measurement of Andreev-bound-state dynamics in a semiconductor-nanowire Josephson junction. *Phys. Rev. Lett.* **121**, 047001 (2018).
- [18] Hays, M. *et al.* Coherent manipulation of an Andreev spin qubit. *Science* **373**, 430–433 (2021).
- [19] Pita-Vidal, M. *et al.* Direct manipulation of a superconducting spin qubit strongly coupled to a transmon qubit. *Nat. Phys.* **19**, 1110–1115 (2023).
- [20] van Heck, B., Mi, S. & Akhmerov, A. R. Single fermion manipulation via superconducting phase differences in multiterminal Josephson junctions. *Phys. Rev. B* **90**, 155450 (2014).
- [21] Yokoyama, T. & Nazarov, Y. V. Singularities in the Andreev spectrum of a multiterminal Josephson junction. *Phys. Rev. B* **92**, 155437 (2015).
- [22] Riwar, R.-P., Houzet, M., Meyer, J. S. & Nazarov, Y. V. Multi-terminal Josephson junctions as topological matter. *Nat. Commun.* **7**, 11167 (2016).
- [23] Eriksson, E., Riwar, R.-P., Houzet, M., Meyer, J. S. & Nazarov, Y. V. Topological transconductance quantization in a four-terminal Josephson junction. *Phys. Rev. B* **95**, 075417 (2017).
- [24] Meyer, J. S. & Houzet, M. Nontrivial Chern numbers in three-terminal Josephson junctions. *Phys. Rev. Lett.* **119**, 136807 (2017).
- [25] Xie, H.-Y., Vavilov, M. G. & Levchenko, A. Topological Andreev bands in three-terminal Josephson junctions. *Phys. Rev. B* **96**, 161406 (2017).
- [26] Deb, O., Sengupta, K. & Sen, D. Josephson junctions of multiple superconducting wires. *Phys. Rev. B* **97**, 174518 (2018).
- [27] Xie, H.-Y. & Levchenko, A. Topological supercurrents interaction and fluctuations in the multiterminal Josephson effect. *Phys. Rev. B* **99**, 094519 (2019).
- [28] Repin, E. V., Chen, Y. & Nazarov, Y. V. Topological properties of multiterminal superconducting nanostructures: Effect of a continuous spectrum. *Phys. Rev. B* **99**, 165414 (2019).
- [29] Peralta Gavensky, L., Usaj, G. & Balseiro, C. A. Topological phase diagram of a three-terminal Josephson junction: From the conventional to the Majorana regime. *Phys. Rev. B* **100**, 014514 (2019).
- [30] Houzet, M. & Meyer, J. S. Majorana-Weyl crossings in topological multiterminal junctions. *Phys. Rev. B* **100**, 014521 (2019).

- [31] Klees, R. L., Rastelli, G., Cuevas, J. C. & Belzig, W. Microwave spectroscopy reveals the quantum geometric tensor of topological Josephson matter. *Phys. Rev. Lett.* **124**, 197002 (2020).
- [32] Pillet, J.-D., Benzoni, V., Griesmar, J., Smirr, J.-L. & Girit, Ç. O. Nonlocal Josephson effect in Andreev molecules. *Nano Lett.* **19**, 7138–7143 (2019).
- [33] Kornich, V., Barakov, H. S. & Nazarov, Y. V. Fine energy splitting of overlapping Andreev bound states in multiterminal superconducting nanostructures. *Phys. Rev. Res.* **1**, 033004 (2019).
- [34] Kornich, V., Barakov, H. S. & Nazarov, Y. V. Overlapping Andreev states in semiconducting nanowires: Competition of one-dimensional and three-dimensional propagation. *Phys. Rev. B* **101**, 195430 (2020).
- [35] Pillet, J.-D., Benzoni, V., Griesmar, J., Smirr, J.-L. & Girit, Ç. Scattering description of Andreev molecules. *SciPost Phys. Core* **2**, 009 (2020).
- [36] Freyn, A., Douçot, B., Feinberg, D. & Mélin, R. Production of nonlocal quartets and phase-sensitive entanglement in a superconducting beam splitter. *Phys. Rev. Lett.* **106**, 257005 (2011).
- [37] Jonckheere, T. *et al.* Multipair dc Josephson resonances in a biased all-superconducting bijunction. *Phys. Rev. B* **87**, 214501 (2013).
- [38] Pfeffer, A. H. *et al.* Subgap structure in the conductance of a three-terminal Josephson junction. *Phys. Rev. B* **90**, 075401 (2014).
- [39] Cohen, Y. *et al.* Nonlocal supercurrent of quartets in a three-terminal Josephson junction. *Proc. Natl. Acad. Sci. U.S.A.* **115**, 6991–6994 (2018).
- [40] Huang, K.-F. *et al.* Evidence for $4e$ charge of Cooper quartets in a biased multi-terminal graphene-based Josephson junction. *Nat. Commun.* **13**, 3032 (2022).
- [41] Draelos, A. W. *et al.* Supercurrent flow in multiterminal graphene Josephson junctions. *Nano Lett.* **19**, 1039–1043 (2019).
- [42] Graziano, G. V., Lee, J. S., Pendharkar, M., Palmstrøm, C. J. & Pribiag, V. S. Transport studies in a gate-tunable three-terminal Josephson junction. *Phys. Rev. B* **101**, 054510 (2020).
- [43] Pankratova, N. *et al.* Multiterminal Josephson effect. *Phys. Rev. X* **10**, 031051 (2020).
- [44] Matsuo, S. *et al.* Observation of nonlocal Josephson effect on double InAs nanowires. *Commun. Phys.* **5**, 221 (2022).
- [45] Arnault, E. G. *et al.* Multiterminal inverse AC Josephson effect. *Nano Lett.* **21**, 9668–9674 (2021).
- [46] Strambini, E. *et al.* The ω -SQUIPT as a tool to phase-engineer Josephson topological materials. *Nat. Nanotechnol.* **11**, 1055–1059 (2016).
- [47] Shabani, J. *et al.* Two-dimensional epitaxial superconductor-semiconductor heterostructures: A platform for topological superconducting networks. *Phys. Rev. B* **93**, 155402 (2016).

- [48] Cheah, E. *et al.* Control over epitaxy and the role of the InAs/Al interface in hybrid two-dimensional electron gas systems. *Phys. Rev. Mater.* **7**, 073403 (2023).
- [49] Dynes, R. C., Narayanamurti, V. & Garno, J. P. Direct measurement of quasiparticle-lifetime broadening in a strong-coupled superconductor. *Phys. Rev. Lett.* **41**, 1509–1512 (1978).
- [50] van der Ploeg, S. H. W. *et al.* Controllable coupling of superconducting flux qubits. *Phys. Rev. Lett.* **98**, 057004 (2007).
- [51] Menke, T. *et al.* Demonstration of tunable three-body interactions between superconducting qubits. *Phys. Rev. Lett.* **129**, 220501 (2022).
- [52] Su, Z. *et al.* Andreev molecules in semiconductor nanowire double quantum dots. *Nat. Commun.* **8**, 585 (2017).
- [53] Kürtössy, O. *et al.* Andreev molecule in parallel InAs nanowires. *Nano Lett.* **21**, 7929–7937 (2021).
- [54] Ihn, T. *Semiconductor Nanostructures: Quantum States and Electronic Transport* (Oxford University Press, 2010), 2 edn.
- [55] Fan, F., Chen, Y., Pan, D., Zhao, J. & Xu, H. Q. Measurements of spin–orbit interaction in epitaxially grown InAs nanosheets. *Appl. Phys. Lett.* **117**, 132101 (2020).
- [56] Whiticar, A. M. *et al.* Zeeman-driven parity transitions in an Andreev quantum dot. *Phys. Rev. B* **103**, 245308 (2021).
- [57] Bargerbos, A. *et al.* Singlet-doublet transitions of a quantum dot Josephson junction detected in a transmon circuit. *PRX Quantum* **3**, 030311 (2022).
- [58] Lee, H. *Supercurrent and Andreev bound states in multi-terminal Josephson junctions*. Ph.D. thesis, University of Maryland (2022).
- [59] Matsuo, S. *et al.* Phase-dependent Andreev molecules and superconducting gap closing in coherently coupled Josephson junctions. arXiv:2303.10540 (2023).
- [60] Beenakker, C. W. J. Random-matrix theory of quantum transport. *Rev. Mod. Phys.* **69**, 731–808 (1997).
- [61] Kurilovich, P. D., Kurilovich, V. D., Fatemi, V., Devoret, M. H. & Glazman, L. I. Microwave response of an Andreev bound state. *Phys. Rev. B* **104**, 174517 (2021).
- [62] Tinkham, M. *Introduction to Superconductivity* (Dover Publications, 2004), 2 edn.
- [63] Kjaergaard, M. *et al.* Quantized conductance doubling and hard gap in a two-dimensional semiconductor–superconductor heterostructure. *Nat. Commun.* **7**, 12841 (2016).
- [64] M_{geom} was calculated analytically as the mutual inductance between two rectangular loops of width $a = 3.5 \mu\text{m}$ and height $b = 25 \mu\text{m}$ separated along the direction of their width by the distance $c = 0.5 \mu\text{m}$ (i.e., half the width of the Al strips constituting the loops in our devices), using $M_{\text{geom}} = \frac{\mu_0 b}{2\pi} \ln \frac{(a+c)^2}{c(2a+c)}$. Here, μ_0 is the vacuum magnetic permeability.
- [65] Ter Haar, A. C. J. *Single and coupled Josephson junction quantum bits*. Ph.D. thesis, TU Delft (2005).

- [66] Paauw, F. G. *Superconducting flux qubits: Quantum chains and tunable qubits*. Ph.D. thesis, TU Delft (2009).
- [67] Annunziata, A. J. *et al.* Tunable superconducting nanoinductors. *Nanotechnology* **21**, 445202 (2010).

From Josephson Current and Noise at a Superconductor-Quantum Spin Hall Insulator-Superconductor Junction

References

- [1] A. Kitaev, Ann. Phys. **303**, 2 (2003).
- [2] C. Nayak, et al., arXiv:0707.1889 (2008).
- [3] M. Dolev, et al., arXiv:0802.0930 (2008).
- [4] I. P. Radu, et al., arXiv:0803.3530 (2008).
- [5] L. Fu and C.L. Kane, Phys. Rev. Lett. **100**, 096407 (2008).
- [6] C.L. Kane and E.J. Mele, Phys. Rev. Lett. **95** 226801 (2005).
- [7] C.L. Kane and E.J. Mele, Phys. Rev. Lett. **95** 146802 (2005).
- [8] B.A. Bernevig and S.C. Zhang, Phys. Rev. Lett. **96**, 106802 (2006).
- [9] A. Bernevig, T. Hughes and S.C. Zhang, Science **314**, 1757 (2006).
- [10] M. König, et al., Science **318**, 766 (2007).
- [11] A. Kitaev, ArXiv:cond-mat/0010440 (unpublished).
- [12] H.J. Kwon, K. Sengupta and V.M. Yakovenko, Eur. Phys. J. B **37**, 349 (2004); Low Temp. Phys. **30**, 613 (2004).
- [13] W.P. Su, J.R. Schrieffer and A.J. Heeger, Phys. Rev. B **22**, 2099 (1980).
- [14] A.A. Golubov, M.. Kupriyanov and E. Il'ichev, Rev. Mod. Phys. **76**, 411 (2004).
- [15] C.W.J. Beenakker, Phys. Rev. Lett. **67**, 3836 (1991).
- [16] E. Witten, Phys. Lett. **117B**, 324 (1982).
- [17] D.A. Ivanov, Phys. Rev. Lett. **86**, 268 (2001).
- [18] L. Fu and C.L. Kane, Phys. Rev. B **74**, 195312 (2006).
- [19] D. Averin and A. Bardas, Phys. Rev. Lett. **75**, 1831 (1995).
- [20] I.O. Kulik and A.N. Omel'yanchuk, Sov. J. Low Temp. Phys. **4**, 142 (1978).
- [21] D. Averin and A. Bardas, Phys. Rev. B **53**, R1705 (1996).
- [22] D. Averin and H.T. Imam, Phys. Rev. Lett. **76**, 3814 (1996).
- [23] M. König et al., J. Phys. Soc. Jpn. **77**, 031007 (2008).
- [24] H. Takayanagi, T. Akazaki and J. Nitta, Phys. Rev. Lett. **75**, 3533 (1995).

From Josephson Current through a Semiconductor Nanowire: effect of strong spin-orbit coupling and Zeeman splitting

References

- [1] A. A. Golubov, M. Y. Kupriyanov, and E. Il'ichev, *Rev. Mod. Phys.* **76**, 411 (2004).
- [2] A. I. Buzdin, *Rev. Mod. Phys.* **77**, 935 (2005).
- [3] F. S. Bergeret, A. F. Volkov, and K. B. Efetov, *Rev. Mod. Phys.* **77**, 1321 (2005).
- [4] S. De Franceschi, L. Kouwenhoven, C. Schoenenberger, and W. Wernsdorfer, *Nat. Nanotechnol.* **5**, 703 (2010).
- [5] J. A. van Dam, Y. V. Nazarov, E. P. A. M. Bakkers, S. D. Franceschi, and L. P. Kouwenhoven, *Nature* **442**, 667 (2006).
- [6] G. Katsaros, *Nat. Nanotechnology* **5**, 458 (2010).
- [7] T. Dirks, T. L. Hughes, S. Lal, B. Uchoa, Y.-F. Chen, C. Chialvo, P. M. Goldbart, and N. Mason, *Nat. Phys.* **7**, 386 (2011).
- [8] A. Y. Kasumov, R. Deblock, M. Kociak, B. Reulet, H. Bouchiat, I. I. Khodos, Y. B. Gorbatov, V. T. Volkov, C. Journet, and M. Burghard, *Science* **284**, 1508 (1999).
- [9] A. F. Morpurgo, J. Kong, C. M. Marcus, and H. Dai, *Science* **286**, 263 (1999).
- [10] Y. Doh, J. van Dam, A. Roest, E. Bakkers, L. Kouwenhoven, and S. De Franceschi, *Science* **309**, 272 (2005).
- [11] J. Xiang, *Nat. Nanotechnology* **1**, 208 (2006).
- [12] P. Štředa and P. Šeba, *Phys. Rev. Lett.* **90**, 256601 (2003).
- [13] We note that helical modes at the edge of two-dimensional quantum spin Hall insulator [28] are fundamentally different from the semiconductor nanowire setup. First of all, helical edge states are time-reversal invariant whereas it is necessary to break time-reversal symmetry in the present setup in order to drive the nanowire into the helical state. As a result, normal reflection, which is prohibited at the interface between helical edge and s-wave superconductor, is allowed in the case SC-SM-SC system. Second, helical edge states having Dirac spectrum and semiconductor nanowires with parabolic spectrum are described by the different boundary conditions at the interface with a SC. Thus, transport properties of these two systems are very different.
- [14] O. V. Dimitrova and M. V. Feigel'man, *JETP* **102**, 652 (2006).
- [15] Y. S. Barash and I. V. Bobkova, *Phys. Rev. B* **65**, 144502 (2002).
- [16] A. Buzdin, L. Bulaevskii, and S. Panyukov, *JETP Lett.* **35**, 178 (1982).
- [17] V. V. Ryazanov, V. A. Oboznov, A. Y. Rusanov, A. V. Veretennikov, A. A. Golubov, and J. Aarts, *Phys. Rev. Lett.* **86**, 2427 (2001).
- [18] N. M. Chtchelkatchev and Y. V. Nazarov, *Phys. Rev. Lett.* **90**, 226806 (2003).
- [19] B. Béri, J. H. Bardarson, and C. W. J. Beenakker, *Phys. Rev. B* **77**, 045311 (2008).

- [20] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Phys. Rev. Lett. **105**, 077001 (2010).
- [21] Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. **105**, 177002 (2010).
- [22] A. M. Black-Schaffer and J. Linder, Phys. Rev. B **84**, 180509 (2011).
- [23] A. Buzdin, Phys. Rev. Lett. **101**, 107005 (2008).
- [24] I. Kulik, Sov. Phys. JETP **30**, 944 (1970).
- [25] C. W. J. Beenakker, Phys. Rev. Lett. **67**, 3836 (1991).
- [26] C. W. J. Beenakker, *Transport Phenomena in Mesoscopic Systems* (Springer, 1992).
- [27] D. Zhang, J. Wang, A. M. DaSilva, J. S. Lee, H. R. Gutierrez, M. H. W. Chan, J. Jain, and N. Samarth, Phys. Rev. B **84**, 165120 (2011).
- [28] L. Fu and C. L. Kane, Phys. Rev. B **79**, 161408 (2009).
- [29] E. V. Bezuglyi, A. S. Rozhavsky, I. D. Vagner, and P. Wyder, Phys. Rev. B **66**, 052508 (2002).

From Scanning Josephson spectros on the atomic scale

References

- [1] B. Sacepe, T. Dubouchet, C. Chapelier, M. Sanquer, M. Ovadia, D. Shahar, M. Feigel'man, and L. Ioffe, [Nat. Phys. **7**, 239 \(2011\)](#).
- [2] K. Bouadim, Y. L. Loh, M. Randeria, and N. Trivedi, [Nat. Phys. **7**, 884 \(2011\)](#).
- [3] K. K. Gomes, A. N. Pasupathy, A. Pushp, S. Ono, Y. Ando, and A. Yazdani, [Nature **447**, 569 \(2007\)](#).
- [4] A. N. Pasupathy, A. Pushp, K. K. Gomes, C. V. Parker, J. Wen, Z. Xu, G. Gu, S. Ono, Y. Ando, and A. Yazdani, [Science **320**, 196 \(2008\)](#).
- [5] P. Fulde and R. A. Ferrell, [Phys. Rev. **135**, A550 \(1964\)](#).
- [6] A. Larkin and Y. Ovchinnikov, Soviet Physics-JETP **20**, 762 (1965).
- [7] Y. Matsuda and H. Shimahara, [Journal of the Physical Society of Japan **76**, 051005 \(2007\)](#).
- [8] H.-D. Chen, O. Vafek, A. Yazdani, and S.-C. Zhang, [Phys. Rev. Lett. **93**, 187002 \(2004\)](#).
- [9] E. Berg, E. Fradkin, and S. A. Kivelson, [Nat. Phys. **5**, 830 \(2009\)](#).
- [10] P. A. Lee, [Phys. Rev. X **4**, 031017 \(2014\)](#).
- [11] J. Šmakov, I. Martin, and A. V. Balatsky, [Phys. Rev. B **64**, 212506 \(2001\)](#).
- [12] J. Rodrigo and S. Vieira, [Physica C: Superconductivity **404**, 306 \(2004\)](#).
- [13] O. Naaman, W. Teizer, and R. C. Dynes, [Phys. Rev. Lett. **87**, 097004 \(2001\)](#).
- [14] Rodrigo, J. G., Suderow, H., and Vieira, S., [Eur. Phys. J. B **40**, 483 \(2004\)](#).

- [15] N. Levy, T. Zhang, J. Ha, F. Sharifi, A. A. Talin, Y. Kuk, and J. A. Stroscio, *Phys. Rev. Lett.* **110**, 117001 (2013).
- [16] T. Proslier, A. Kohen, Y. Noat, T. Cren, D. Roditchev, and W. Sacks, *Europhys. Lett.* **73**, 962 (2006).
- [17] N. Bergeal, Y. Noat, T. Cren, T. Proslier, V. Dubost, F. Debontridder, A. Zimmers, D. Roditchev, W. Sacks, and J. Marcus, *Phys. Rev. B* **78**, 140507 (2008).
- [18] H. Kimura, R. P. Barber, S. Ono, Y. Ando, and R. C. Dynes, *Phys. Rev. B* **80**, 144506 (2009).
- [19] M. H. Hamidian, S. D. Edkins, S. H. Joo, A. Kostin, H. Eisaki, S. Uchida, M. J. Lawler, E. A. Kim, A. P. Mackenzie, K. Fujita, J. Lee, and S. J. C. Davis, (2015), [arXiv:1511.08124](https://arxiv.org/abs/1511.08124).
- [20] M. E. Flatté and J. M. Byers, *Phys. Rev. Lett.* **78**, 3761 (1997).
- [21] M. E. Flatté and J. M. Byers, *Phys. Rev. B* **56**, 11213 (1997).
- [22] S. Misra, B. B. Zhou, I. K. Drozdov, J. Seo, L. Urban, A. Gyenis, S. C. J. Kingsley, H. Jones, and A. Yazdani, *Rev. Sci. Inst.* **84**, 103903 (2013), <http://dx.doi.org/10.1063/1.4822271>.
- [23] P. Townsend and J. Sutton, *Phys. Rev.* **128**, 591 (1962).
- [24] B. L. Blackford and R. H. March, *Phys. Rev.* **186**, 397 (1969).
- [25] M. Ruby, B. W. Heinrich, J. I. Pascual, and K. J. Franke, *Phys. Rev. Lett.* **114**, 157001 (2015).
- [26] M. Ternes, W.-D. Schneider, J.-C. Cuevas, C. P. Lutz, C. F. Hirjibehedin, and A. J. Heinrich, *Phys. Rev. B* **74**, 132501 (2006).
- [27] G.-L. Ingold, H. Grabert, and U. Eberhardt, *Phys. Rev. B* **50**, 395 (1994).
- [28] B. Jaeck, M. Eltschka, M. Assig, A. Hardock, M. Etzkorn, C. R. Ast, and K. Kern, *App. Phys. Lett.* **106**, 013109 (2015), <http://dx.doi.org/10.1063/1.4905322>.
- [29] A. Roychowdhury, M. Dreyer, J. R. Anderson, C. J. Lobb, and F. C. Wellstood, *Phys. Rev. Applied* **4**, 034011 (2015).
- [30] G.-L. Ingold and H. Grabert, *EPL (Europhysics Letters)* **14**, 371 (1991).
- [31] V. Ambegaokar and A. Baratoff, *Phys. Rev. Lett.* **10**, 486 (1963).
- [32] P. Anderson, *Journal of Physics and Chemistry of Solids* **11**, 26 (1959).
- [33] A. A. Abrikosov and L. P. Gor'kov, *Sov. Phys. JETP* **12**, 1243 (1961).
- [34] H. Shiba, *Prog. Theor. Phys.* **40**, 435 (1968).
- [35] L. Yu, *Acta Phys. Sin.* **21**, 75 (1965).
- [36] A. I. Rusinov, *Sov. Phys. JETP* **9**, 85 (1969).
- [37] A. Yazdani, B. A. Jones, C. P. Lutz, M. F. Crommie, and D. M. Eigler, *Science* **275**, 1767 (1997).

- [38] S.-H. Ji, T. Zhang, Y.-S. Fu, X. Chen, X.-C. Ma, J. Li, W.-H. Duan, J.-F. Jia, and Q.-K. Xue, *Phys. Rev. Lett.* **100**, 226801 (2008).
- [39] J. Heinrichs, *Phys. Rev.* **168**, 451 (1968).
- [40] P. Schlottmann, *Phys. Rev. B* **13**, 1 (1976).
- [41] M. Ruby, F. Pientka, Y. Peng, F. von Oppen, B. W. Heinrich, and K. J. Franke, *Phys. Rev. Lett.* **115**, 087001 (2015).
From appendix:

References

- [1] M. H. Devoret, D. Esteve, H. Grabert, G.-L. Ingold, H. Pothier, and C. Urbina, *Phys. Rev. Lett.* **64**, 1824 (1990).
- [2] G.-L. Ingold and H. Grabert, *EPL (Europhysics Letters)* **14**, 371 (1991).
- [3] G.-L. Ingold, H. Grabert, and U. Eberhardt, *Phys. Rev. B* **50**, 395 (1994).
- [4] B. Jaeck, M. Eltschka, M. Assig, A. Hardock, M. Etzkorn, C. R. Ast, and K. Kern, *App. Phys. Lett.* **106**, 013109 (2015), <http://dx.doi.org/10.1063/1.4905322>.
- [5] S. Misra, B. B. Zhou, I. K. Drozdov, J. Seo, L. Urban, A. Gyenis, S. C. J. Kingsley, H. Jones, and A. Yazdani, *Rev. Sci. Inst.* **84**, 103903 (2013), <http://dx.doi.org/10.1063/1.4822271>.
- [6] V. Ambegaokar and A. Baratoff, *Phys. Rev. Lett.* **10**, 486 (1963).

From Quantum circuits with multiterminal Josephson-Andreev junctions

References

- [1] J. Clarke, A. N. Cleland, M. H. Devoret, D. Esteve, and J. M. Martinis, Quantum Mechanics of a Macroscopic Variable: The Phase Difference of a Josephson Junction, *Science* **239**, 992 (1988).
- [2] A. Blais, A. L. Grimsmo, S. M. Girvin, and A. Wallraff, Circuit quantum electrodynamics, *Rev. Mod. Phys.* **93**, 025005 (2021).
- [3] Y. V. Nazarov and Y. M. Blanter, *Quantum Transport: Introduction to Nanoscience* (Cambridge University Press, 2009).
- [4] I. O. Kulik, Macroscopic quantization and the proximity effect in S-N-S junctions, *Pis'ma Zh. Eksp. Teor. Fiz [Sov. Phys. JETP]* **30**, 944 (1970) **57** (1969).
- [5] A. Furusaki, H. Takayanagi, and M. Tsukada, Theory of quantum conduction of supercurrent through a constriction, *Physical Review Letters* **67**, 132 (1991).
- [15] C. W. J. Beenakker and H. van Houten, Josephson current through a superconducting quantum point contact shorter than the coherence length, *Physical Review Letters* **66**, 3056 (1991).

- [7] A. Bargerbos, W. Uilhoorn, C.-K. Yang, P. Krogstrup, L. P. Kouwenhoven, G. de Lange, B. van Heck, and A. Kou, Observation of Vanishing Charge Dispersion of a Nearly Open Superconducting Island, *Phys. Rev. Lett.* **124**, 246802 (2020).
- [8] M. Pita-Vidal, A. Bargerbos, R. Žitko, L. J. Splitthoff, L. Grünhaupt, J. J. Wesdorp, Y. Liu, L. P. Kouwenhoven, R. Aguado, B. van Heck, A. Kou, and C. K. Andersen, Direct manipulation of a superconducting spin qubit strongly coupled to a transmon qubit, *Nature Physics* **19**, 1110–1115 (2023).
- [9] A. Kringhøj, L. Casparis, M. Hell, T. W. Larsen, F. Kuemmeth, M. Leijnse, K. Flensberg, P. Krogstrup, J. Nygård, K. D. Petersson, and C. M. Marcus, Anharmonicity of a superconducting qubit with a few-mode Josephson junction, *Phys. Rev. B* **97**, 060508 (2018).
- [10] A. Kringhøj, B. van Heck, T. W. Larsen, O. Erlandsson, D. Sabonis, P. Krogstrup, L. Casparis, K. D. Petersson, and C. M. Marcus, Suppressed Charge Dispersion via Resonant Tunneling in a Single-Channel Transmon, *Phys. Rev. Lett.* **124**, 246803 (2020).
- [11] M. Coraiola, D. Z. Haxell, D. Sabonis, H. Weisbrich, A. E. Svetogorov, M. Hinderling, S. C. ten Kate, E. Cheah, F. Krizek, R. Schott, W. Wegscheider, J. C. Cuevas, W. Belzig, and F. Nichele, Phase-engineering the andreev band structure of a three-terminal josephson junction, *Nature Communications* **14**, 10.1038/s41467-023-42356-6 (2023).
- [12] L. Fu, Electron Teleportation via Majorana Bound States in a Mesoscopic Superconductor, *Physical Review Letters* **104**, 10.1103/physrevlett.104.056402 (2010).
- [13] B. van Heck, F. Hassler, A. R. Akhmerov, and C. W. J. Beenakker, Coulomb stability of the 4π -periodic Josephson effect of Majorana fermions, *Physical Review B* **84**, 10.1103/physrevb.84.180502 (2011).
- [14] D. B. Karki, K. A. Matveev, and I. Martin, Physics of the majorana superconducting qubit hybrids, *Phys. Rev. B* **109**, 085410 (2024).
- [15] A. Keselman, C. Murthy, B. van Heck, and B. Bauer, Spectral response of Josephson junctions with low-energy quasiparticles, *SciPost Physics* **7**, 10.21468/scipostphys.7.4.050 (2019).
- [16] J. Ávila, E. Prada, P. San-Jose, and R. Aguado, Superconducting islands with topological Josephson junctions based on semiconductor nanowires, *Physical Review B* **102**, 10.1103/physrevb.102.094518 (2020).
- [17] J. Ávila, E. Prada, P. San-Jose, and R. Aguado, Majorana oscillations and parity crossings in semiconductor nanowire-based transmon qubits, *Phys. Rev. Res.* **2**, 033493 (2020).
- [18] D. M. Pino, R. S. Souto, and R. Aguado, Minimal kitaev-transmon qubit based on double quantum dots, *Phys. Rev. B* **109**, 075101 (2024).
- [19] L. Bretheau, Ç. O. Girit, M. Houzet, H. Pothier, D. Esteve, and C. Urbina, Theory of microwave spectroscopy of Andreev bound states with a Josephson junction, *Phys. Rev. B* **90**, 134506 (2014).
- [20] T. Vakhtel and B. van Heck, Quantum phase slips in a resonant Josephson junction, *Phys. Rev. B* **107**, 195405 (2023).

- [21] J. J. Cáceres, *Qubit protegido contra errores y compuertas cuánticas rápidas basadas en la transición Landau-Zener-Stuckelberg*, Master's thesis, Centro Atómico Bariloche, Instituto Balseiro (2022).
- [22] T. Vakhtel, P. D. Kurilovich, M. Pita-Vidal, A. Bargerbos, V. Fatemi, and B. van Heck, Tunneling of fluxons via a Josephson resonant level (2023), [arXiv:2310.03102 \[cond-mat.mes-hall\]](#) .
- [23] A. Gyenis, A. Di Paolo, J. Koch, A. Blais, A. A. Houck, and D. I. Schuster, Moving beyond the Transmon: Noise-Protected Superconducting Quantum Circuits, [PRX Quantum](#) **2**, 030101 (2021).
- [24] A. Calzona and M. Carrega, Multi-mode architectures for noise-resilient superconducting qubits, [Superconductor Science and Technology](#) **36**, 023001 (2022).
- [25] J. Danon, A. Chatterjee, A. Gyenis, and F. Kuemmeth, Protected solid-state qubits, [Applied Physics Letters](#) **119**, 260502 (2021).
- [26] B. Douçot and L. B. Ioffe, Physical implementation of protected qubits, [Reports on Progress in Physics](#) **75**, 072001 (2012).
- [27] P. Brooks, A. Kitaev, and J. Preskill, Protected gates for superconducting qubits, [Phys. Rev. A](#) **87**, 052306 (2013).
- [28] J. M. Dempster, B. Fu, D. G. Ferguson, D. I. Schuster, and J. Koch, Understanding degenerate ground states of a protected quantum circuit in the presence of disorder, [Phys. Rev. B](#) **90**, 094518 (2014).
- [29] P. Groszkowski, A. D. Paolo, A. L. Grimsmo, A. Blais, D. I. Schuster, A. A. Houck, and J. Koch, Coherence properties of the $0-\pi$ qubit, [New Journal of Physics](#) **20**, 043053 (2018).
- [30] A. D. Paolo, A. L. Grimsmo, P. Groszkowski, J. Koch, and A. Blais, Control and coherence time enhancement of the $0-\pi$ qubit, [New Journal of Physics](#) **21**, 043002 (2019).
- [31] A. Gyenis, P. S. Mundada, A. Di Paolo, T. M. Hazard, X. You, D. I. Schuster, J. Koch, A. Blais, and A. A. Houck, Experimental Realization of a Protected Superconducting Circuit Derived from the $0-\pi$ Qubit, [PRX Quantum](#) **2**, 010339 (2021).
- [32] K. Kalashnikov, W. T. Hsieh, W. Zhang, W.-S. Lu, P. Kamenov, A. Di Paolo, A. Blais, M. E. Gershenson, and M. Bell, Bifluxon: Fluxon-Parity-Protected Superconducting Qubit, [PRX Quantum](#) **1**, 010307 (2020).
- [33] M. T. Bell, W. Zhang, L. B. Ioffe, and M. E. Gershenson, Spectroscopic Evidence of the Aharonov-Casher Effect in a Cooper Pair Box, [Phys. Rev. Lett.](#) **116**, 107002 (2016).
- [34] B. Douçot and J. Vidal, Pairing of Cooper Pairs in a Fully Frustrated Josephson-Junction Chain, [Phys. Rev. Lett.](#) **88**, 227005 (2002).
- [35] W. C. Smith, A. Kou, X. Xiao, U. Vool, and M. H. Devoret, Superconducting circuit protected by two-Cooper-pair tunneling, [npj Quantum Information](#) **6**, 10.1038/s41534-019-0231-2 (2020).
- [36] I. V. Protopopov and M. V. Feigel'man, Anomalous periodicity of supercurrent in long frustrated Josephson-junction rhombi chains, [Phys. Rev. B](#) **70**, 184519 (2004).

- [37] I. V. Protopopov and M. V. Feigel'man, Coherent transport in Josephson-junction rhombi chain with quenched disorder, *Phys. Rev. B* **74**, 064516 (2006).
- [38] T. W. Larsen, M. E. Gershenson, L. Casparis, A. Kringhøj, N. J. Pearson, R. P. G. McNeil, F. Kuemmeth, P. Krogstrup, K. D. Petersson, and C. M. Marcus, Parity-Protected Superconductor-Semiconductor Qubit, *Phys. Rev. Lett.* **125**, 056801 (2020).
- [39] R. Aguado, A perspective on semiconductor-based superconducting qubits, *Applied Physics Letters* **117**, 240501 (2020).
- [40] L. Casparis, M. R. Connolly, M. Kjaergaard, N. J. Pearson, A. Kringhøj, T. W. Larsen, F. Kuemmeth, T. Wang, C. Thomas, S. Gronin, G. C. Gardner, M. J. Manfra, C. M. Marcus, and K. D. Petersson, Superconducting gatemon qubit based on a proximitized two-dimensional electron gas, *Nature Nanotechnology* **13**, 915 (2018).
- [41] L. Chirolli and J. E. Moore, Enhanced Coherence in Superconducting Circuits via Band Engineering, *Physical Review Letters* **126**, 10.1103/physrevlett.126.187701 (2021).
- [42] A. Maiani, M. Kjaergaard, and C. Schrade, Entangling Transmons with Low-Frequency Protected Superconducting Qubits, *PRX Quantum* **3**, 030329 (2022).
- [43] C. Schrade, C. M. Marcus, and A. Gyenis, Protected Hybrid Superconducting Qubit in an Array of Gate-Tunable Josephson Interferometers, *PRX Quantum* **3**, 030303 (2022).
- [44] G.-L. Guo, H.-B. Leng, and X. Liu, Parity-protected superconducting qubit based on topological insulators (2023), [arXiv:2308.12027 \[cond-mat.mes-hall\]](https://arxiv.org/abs/2308.12027).
- [45] H. Patel, V. Pathak, O. Can, A. C. Potter, and M. Franz, *d*-mon: A transmon with strong anharmonicity based on planar *c*-axis tunneling junction between *d*-wave and *s*-wave superconductors, *Phys. Rev. Lett.* **132**, 017002 (2024).
- [42] N. Pankratova, H. Lee, R. Kuzmin, K. Wickramasinghe, W. Mayer, J. Yuan, M. G. Vavilov, J. Shabani, and V. E. Manucharyan, Multiterminal Josephson Effect, *Phys. Rev. X* **10**, 031051 (2020).
- [41] G. V. Graziano, J. S. Lee, M. Pendharkar, C. J. Palmstrøm, and V. S. Pribiag, Transport studies in a gate-tunable three-terminal Josephson junction, *Phys. Rev. B* **101**, 054510 (2020).
- [11] R.-P. Riwar, M. Houzet, J. S. Meyer, and Y. V. Nazarov, Multi-terminal Josephson junctions as topological matter, *Nature Communications* **7**, 11167 (2016).
- [52] V. Fatemi, A. R. Akhmerov, and L. Bretheau, Weyl Josephson circuits, *Phys. Rev. Res.* **3**, 013288 (2021).
- [50] L. P. Gavensky, G. Usaj, and C. A. Balseiro, Multi-terminal Josephson junctions: A road to topological flux networks, *Europhysics Letters* **141**, 36001 (2023).
- [24] L. Peralta Gavensky, G. Usaj, D. Feinberg, and C. A. Balseiro, Berry curvature tomography and realization of topological haldane model in driven three-terminal josephson junctions, *Phys. Rev. B* **97**, 220505 (2018).
- [52] L. Teshler, H. Weisbrich, J. Sturm, R. L. Klees, G. Rastelli, and W. Belzig, Ground state topology of a four-terminal superconducting double quantum dot, *SciPost Phys.* **15**, 214 (2023).

- [33] R. L. Klees, G. Rastelli, J. C. Cuevas, and W. Belzig, Microwave Spectroscopy Reveals the Quantum Geometric Tensor of Topological Josephson Matter, *Phys. Rev. Lett.* **124**, 197002 (2020).
- [21] R. L. Klees, J. C. Cuevas, W. Belzig, and G. Rastelli, Ground-state quantum geometry in superconductor–quantum dot chains, *Phys. Rev. B* **103**, 014516 (2021).
- [17] T. Yokoyama and Y. V. Nazarov, Singularities in the Andreev spectrum of a multiterminal Josephson junction, *Phys. Rev. B* **92**, 155437 (2015).
- [27] B. van Heck, S. Mi, and A. R. Akhmerov, Single fermion manipulation via superconducting phase differences in multiterminal Josephson junctions, *Phys. Rev. B* **90**, 155450 (2014).
- [22] J. S. Meyer and M. Houzet, Nontrivial Chern Numbers in Three-Terminal Josephson Junctions, *Phys. Rev. Lett.* **119**, 136807 (2017).
- [23] H.-Y. Xie, M. G. Vavilov, and A. Levchenko, Topological Andreev bands in three-terminal Josephson junctions, *Phys. Rev. B* **96**, 161406 (2017).
- [20] H.-Y. Xie, M. G. Vavilov, and A. Levchenko, Weyl nodes in andreev spectra of multiterminal josephson junctions: Chern numbers, conductances, and supercurrents, *Phys. Rev. B* **97**, 035443 (2018).
- [54] L. Peyruchat, J. Griesmar, J.-D. Pillet, and Ç. O. Girit, Transconductance quantization in a topological Josephson tunnel junction circuit, *Phys. Rev. Res.* **3**, 013289 (2021).
- [61] A. Mukhopadhyay, U. Khanna, and S. Das, Anomalous topology and synthetic flat band in multi-terminal Josephson Junctions (2023), [arXiv:2309.15159 \[cond-mat.mes-hall\]](#) .
- [62] P. Zalom, M. Žonda, and T. Novotný, Hidden symmetry in interacting-quantum-dot-based multiterminal josephson junctions, *Phys. Rev. Lett.* **132**, 126505 (2024).
- [63] R. Mélin, J.-G. Caputo, K. Yang, and B. Douçot, Simple Floquet-Wannier-Stark-Andreev viewpoint and emergence of low-energy scales in a voltage-biased three-terminal Josephson junction, *Phys. Rev. B* **95**, 085415 (2017).
- [64] O. Deb, K. Sengupta, and D. Sen, Josephson junctions of multiple superconducting wires, *Phys. Rev. B* **97**, 174518 (2018).
- [65] R. Mélin, Ultralong-distance quantum correlations in three-terminal Josephson junctions, *Phys. Rev. B* **104**, 075402 (2021).
- [66] A. Melo, V. Fatemi, and A. R. Akhmerov, Multiplet supercurrent in Josephson tunneling circuits, *SciPost Phys.* **12**, 017 (2022).
- [67] R. Mélin and D. Feinberg, Quantum interferometer for quartets in superconducting three-terminal Josephson junctions, *Phys. Rev. B* **107**, L161405 (2023).
- [68] J. Cayao, P. Burset, and Y. Tanaka, Controllable odd-frequency cooper pairs in multisu-perconductor josephson junctions, *Phys. Rev. B* **109**, 205406 (2024).
- [69] C. Padurariu, T. Jonckheere, J. Rech, R. Mélin, D. Feinberg, T. Martin, and Y. V. Nazarov, Closing the proximity gap in a metallic Josephson junction between three superconductors, *Phys. Rev. B* **92**, 205409 (2015).

- [70] A. Zazunov, R. Egger, M. Alvarado, and A. L. Yeyati, Josephson effect in multiterminal topological junctions, *Phys. Rev. B* **96**, 024516 (2017).
- [71] M. H. S. Amin, A. N. Omelyanchouk, and A. M. Zagoskin, Mesoscopic multiterminal Josephson structures. I. Effects of nonlocal weak coupling, *Low Temperature Physics* **27**, 616 (2001).
- [72] M. Amin, A. Omelyanchouk, A. Blais, A. M. van den Brink, G. Rose, T. Duty, and A. Zagoskin, Multi-terminal superconducting phase qubit, *Physica C: Superconductivity* **368**, 310 (2002).
- [43] G. V. Graziano, M. Gupta, M. Pendharkar, J. T. Dong, C. P. Dempsey, C. Palmstrøm, and V. S. Pribiag, Selective control of conductance modes in multi-terminal Josephson junctions, *Nature Communications* **13**, 10.1038/s41467-022-33682-2 (2022).
- [74] M. Gupta, G. V. Graziano, M. Pendharkar, J. T. Dong, C. P. Dempsey, C. Palmstrøm, and V. S. Pribiag, Gate-tunable superconducting diode effect in a three-terminal Josephson device, *Nature Communications* **14**, 10.1038/s41467-023-38856-0 (2023).
- [75] R. Mélin, C. B. Winkelmann, and R. Danneau, Magnetointerferometry of multiterminal josephson junctions, *Phys. Rev. B* **109**, 125406 (2024).
- [76] C. G. Prosko, W. D. Huisman, I. Kulesh, D. Xiao, C. Thomas, M. J. Manfra, and S. Goswami, Flux-tunable Josephson Effect in a Four-Terminal Junction (2023), [arXiv:2311.17158 \[cond-mat.mes-hall\]](#) .
- [77] I. A. Day, K. Vilkelis, A. L. R. Manesco, A. M. Bozkurt, V. Fatemi, and A. R. Akhmerov, Chiral adiabatic transmission protected by Fermi surface topology (2023), [arXiv:2311.17160 \[cond-mat.mes-hall\]](#) .
- [78] M. Coraiola, D. Z. Haxell, D. Sabonis, M. Hinderling, S. C. ten Kate, E. Cheah, F. Krizek, R. Schott, W. Wegscheider, and F. Nichele, Spin-degeneracy breaking and parity transitions in three-terminal Josephson junctions (2023), [arXiv:2307.06715 \[cond-mat.mes-hall\]](#) .
- [79] D. C. Ohnmacht, M. Coraiola, J. J. García-Esteban, D. Sabonis, F. Nichele, W. Belzig, and J. C. Cuevas, Quartet Tomography in Multiterminal Josephson Junctions (2023), [arXiv:2311.18544 \[cond-mat.supr-con\]](#) .
- [80] M. Coraiola, A. E. Svetogorov, D. Z. Haxell, D. Sabonis, M. Hinderling, S. C. ten Kate, E. Cheah, F. Krizek, R. Schott, W. Wegscheider, J. C. Cuevas, W. Belzig, and F. Nichele, Flux-tunable josephson diode effect in a hybrid four-terminal josephson junction, *ACS Nano* **18**, 9221–9231 (2024).
- [81] X. You, J. A. Sauls, and J. Koch, Circuit quantization in the presence of time-dependent external flux, *Phys. Rev. B* **99**, 174512 (2019).
- [82] R.-P. Riwar and D. P. DiVincenzo, Circuit quantization with time-dependent magnetic fields for realistic geometries, *npj Quantum Information* **8**, 10.1038/s41534-022-00539-x (2022).
- [83] J. Bryon, D. Weiss, X. You, S. Sussman, X. Croot, Z. Huang, J. Koch, and A. A. Houck, Time-dependent magnetic flux in devices for circuit quantum electrodynamics, *Phys. Rev. Appl.* **19**, 034031 (2023).

- [84] A. Kenawy, F. Hassler, and R.-P. Riwar, Electromotive force in driven topological quantum circuits, *Physical Review B* **106**, [10.1103/physrevb.106.035430](#) (2022).
- [85] A. Kenawy, F. Hassler, and R.-P. Riwar, Time-dependent driving and topological protection in the fractional Josephson effect (2023), [arXiv:2306.09440 \[cond-mat.supr-con\]](#) .
- [86] A. Zazunov, V. S. Shumeiko, G. Wendin, and E. N. Bratus', Dynamics and phonon-induced decoherence of andreev level qubit, *Phys. Rev. B* **71**, 214505 (2005).
- [87] D. A. Ivanov and M. V. Feigel'man, Two-level Hamiltonian of a superconducting quantum point contact, *Phys. Rev. B* **59**, 8444 (1999).
- [88] V. Ambegaokar, U. Eckern, and G. Schön, Quantum Dynamics of Tunneling between Superconductors, *Physical Review Letters* **48**, 1745 (1982).
- [89] U. Eckern, G. Schön, and V. Ambegaokar, Quantum dynamics of a superconducting tunnel junction, *Physical Review B* **30**, 6419 (1984).
- [90] D. N. Pham, W. Fan, M. G. Scheer, and H. E. Türeci, Flux-based three-dimensional electrodynamic modeling approach to superconducting circuits and materials, *Physical Review A* **107**, [10.1103/physreva.107.053704](#) (2023).
- [91] A. Grankin, A. J. Kollár, and M. Hafezi, Extended Josephson junction qubit system (2023), [arXiv:2309.05212 \[quant-ph\]](#) .
- [92] P. D. Kurilovich, V. D. Kurilovich, V. Fatemi, M. H. Devoret, and L. I. Glazman, Microwave response of an andreev bound state, *Phys. Rev. B* **104**, 174517 (2021).
- [93] M. H. Devoret, Does Brian Josephson's Gauge-Invariant Phase Difference Live on a Line or a Circle?, *Journal of Superconductivity and Novel Magnetism* **34**, 1633 (2021).
- [94] E. B. Sonin, Quantum rotator and Josephson junction: Compact vs. extended phase and dissipative quantum phase transition, *Low Temperature Physics* **48**, 400 (2022).
- [95] A. Osborne, T. Larson, S. G. Jones, R. W. Simmonds, A. Gyenis, and A. Lucas, Symplectic geometry and circuit quantization, *PRX Quantum* **5**, 020309 (2024).
- [96] N. Crescini, S. Cailleaux, W. Guichard, C. Naud, O. Buisson, K. W. Murch, and N. Roch, Evidence of dual Shapiro steps in a Josephson junction array, *Nature Physics* **19**, 851 (2023).
- [97] A. Murani, N. Bourlet, H. le Sueur, F. Portier, C. Altimiras, D. Esteve, H. Grabert, J. Stockburger, J. Ankerhold, and P. Joyez, Absence of a Dissipative Quantum Phase Transition in Josephson Junctions, *Physical Review X* **10**, [10.1103/physrevx.10.021003](#) (2020).
- [98] G. Schön and A. Zaikin, Distinguishing phases in Josephson junctions and the choice of states, *Physica B: Condensed Matter* **152**, 203 (1988).
- [99] S. Apenko, Environment-induced decompactification of phase in Josephson junctions, *Physics Letters A* **142**, 277 (1989).
- [100] G. Schön and A. Zaikin, Quantum coherent effects, phase transitions, and the dissipative dynamics of ultra small tunnel junctions, *Physics Reports* **198**, 237 (1990).

- [101] R.-P. Riwar, Charge quantization and detector resolution, *SciPost Phys.* **10**, 093 (2021).
- [102] C. Koliofoti and R.-P. Riwar, Compact description of quantum phase slip junctions, *npj Quantum Information* **9**, [10.1038/s41534-023-00790-w](https://doi.org/10.1038/s41534-023-00790-w) (2023).
- [103] T. Herrig, J. H. Pixley, E. J. König, and R.-P. Riwar, Quasiperiodic circuit quantum electrodynamics, *npj Quantum Information* **9**, [10.1038/s41534-023-00786-6](https://doi.org/10.1038/s41534-023-00786-6) (2023).
- [104] J. Shabani, M. Kjaergaard, H. J. Suominen, Y. Kim, F. Nichele, K. Pakrouski, T. Stankevic, R. M. Lutchyn, P. Krogstrup, R. Feidenhans'l, S. Kraemer, C. Nayak, M. Troyer, C. M. Marcus, and C. J. Palmstrøm, Two-dimensional epitaxial superconductor-semiconductor heterostructures: A platform for topological superconducting networks, *Phys. Rev. B* **93**, 155402 (2016).
- [105] C. Baumgartner, L. Fuchs, L. Frész, S. Reinhardt, S. Gronin, G. C. Gardner, M. J. Manfra, N. Paradiso, and C. Strunk, Josephson Inductance as a Probe for Highly Ballistic Semiconductor-Superconductor Weak Links, *Phys. Rev. Lett.* **126**, 037001 (2021).
- [106] C. Baumgartner, L. Fuchs, A. Costa, S. Reinhardt, S. Gronin, G. C. Gardner, T. Lindemann, M. J. Manfra, P. E. Faria Junior, D. Kochan, J. Fabian, N. Paradiso, and C. Strunk, Supercurrent rectification and magnetochiral effects in symmetric Josephson junctions, *Nature Nanotechnology* **17**, 39–44 (2021).
- [107] A. Costa, C. Baumgartner, S. Reinhardt, J. Berger, S. Gronin, G. C. Gardner, T. Lindemann, M. J. Manfra, J. Fabian, D. Kochan, N. Paradiso, and C. Strunk, Sign reversal of the Josephson inductance magnetochiral anisotropy and $0-\pi$ -like transitions in supercurrent diodes, *Nature Nanotechnology* **18**, 1266–1272 (2023).
- [108] I. Affleck, J.-S. Caux, and A. M. Zagoskin, Andreev scattering and Josephson current in a one-dimensional electron liquid, *Phys. Rev. B* **62**, 1433 (2000).
- [109] T. Meng, S. Florens, and P. Simon, Self-consistent description of Andreev bound states in Josephson quantum dot devices, *Phys. Rev. B* **79**, 224521 (2009).
- [110] A. Martín-Rodero and A. L. Yeyati, Josephson and Andreev transport through quantum dots, *Advances in Physics* **60**, 899 (2011).
- [111] V. Fatemi, P. D. Kurilovich, M. Hays, D. Bouman, T. Connolly, S. Diamond, N. E. Frattini, V. D. Kurilovich, P. Krogstrup, J. Nygård, A. Geresdi, L. I. Glazman, and M. H. Devoret, Microwave Susceptibility Observation of Interacting Many-Body Andreev States, *Phys. Rev. Lett.* **129**, 227701 (2022).
- [112] A. Bargerbos, M. Pita-Vidal, R. Žitko, J. Ávila, L. J. Splitthoff, L. Grünhaupt, J. J. Wesdorp, C. K. Andersen, Y. Liu, L. P. Kouwenhoven, R. Aguado, A. Kou, and B. van Heck, Singlet-Doublet Transitions of a Quantum Dot Josephson Junction Detected in a Transmon Circuit, *PRX Quantum* **3**, 030311 (2022).
- [113] V. V. Baran, E. J. P. Frost, and J. Paaske, Surrogate model solver for impurity-induced superconducting subgap states, *Phys. Rev. B* **108**, L220506 (2023).
- [114] M. A. Despósito and A. Levy Yeyati, Controlled dephasing of Andreev states in superconducting quantum point contacts, *Phys. Rev. B* **64**, 140511 (2001).
- [115] A. Zazunov, V. S. Shumeiko, E. N. Bratus', J. Lantz, and G. Wendin, Andreev Level Qubit, *Phys. Rev. Lett.* **90**, 087003 (2003).

- [116] C. Janvier, L. Tosi, L. Bretheau, Ç. O. Girit, M. Stern, P. Bertet, P. Joyez, D. Vion, D. Esteve, M. F. Goffman, H. Pothier, and C. Urbina, Coherent manipulation of Andreev states in superconducting atomic contacts, [Science](#) **349**, 1199 (2015).
- [117] L. Tosi, C. Metzger, M. F. Goffman, C. Urbina, H. Pothier, S. Park, A. L. Yeyati, J. Nygård, and P. Krogstrup, Spin-Orbit Splitting of Andreev States Revealed by Microwave Spectroscopy, [Phys. Rev. X](#) **9**, 011010 (2019).
- [118] M. Hays, V. Fatemi, K. Serniak, D. Bouman, S. Diamond, G. de Lange, P. Krogstrup, J. Nygård, A. Geresdi, and M. H. Devoret, Continuous monitoring of a trapped superconducting spin, [Nature Physics](#) **16**, 1103–1107 (2020).
- [119] M. Hays, V. Fatemi, D. Bouman, J. Cerrillo, S. Diamond, K. Serniak, T. Connolly, P. Krogstrup, J. Nygård, A. L. Yeyati, A. Geresdi, and M. H. Devoret, Coherent manipulation of an Andreev spin qubit, [Science](#) **373**, 430 (2021).
- [120] G. Ithier, E. Collin, P. Joyez, P. J. Meeson, D. Vion, D. Esteve, F. Chiarello, A. Shnirman, Y. Makhlin, J. Schrieffer, and G. Schön, Decoherence in a superconducting quantum bit circuit, [Phys. Rev. B](#) **72**, 134519 (2005).
- [121] D. J. Van Harlingen, T. L. Robertson, B. L. T. Plourde, P. A. Reichardt, T. A. Crane, and J. Clarke, Decoherence in Josephson-junction qubits due to critical-current fluctuations, [Phys. Rev. B](#) **70**, 064517 (2004).
- [122] J. Koch, T. M. Yu, J. Gambetta, A. A. Houck, D. I. Schuster, J. Majer, A. Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, Charge-insensitive qubit design derived from the Cooper pair box, [Phys. Rev. A](#) **76**, 042319 (2007).
- [123] B. S. Deaver and W. M. Fairbank, Experimental Evidence for Quantized Flux in Superconducting Cylinders, [Phys. Rev. Lett.](#) **7**, 43 (1961).
- [124] R. Doll and M. Näbauer, Experimental Proof of Magnetic Flux Quantization in a Superconducting Ring, [Phys. Rev. Lett.](#) **7**, 51 (1961).
- [125] A. Kenawy, W. Magnus, and B. Sorée, Flux Quantization and Aharonov-Bohm Effect in Superconducting Rings, [Journal of Superconductivity and Novel Magnetism](#) **31**, 1351 (2017).
- [126] D. G. Olivares, A. L. Yeyati, L. Bretheau, i. m. c. O. Girit, H. Pothier, and C. Urbina, Dynamics of quasiparticle trapping in Andreev levels, [Phys. Rev. B](#) **89**, 104504 (2014).
- [127] S. Park and A. L. Yeyati, Andreev spin qubits in multichannel Rashba nanowires, [Phys. Rev. B](#) **96**, 125416 (2017).
- [128] M. Trif, O. Dmytruk, H. Bouchiat, R. Aguado, and P. Simon, Dynamic current susceptibility as a probe of Majorana bound states in nanowire-based Josephson junctions, [Phys. Rev. B](#) **97**, 041415 (2018).
- [129] S. Park, C. Metzger, L. Tosi, M. F. Goffman, C. Urbina, H. Pothier, and A. L. Yeyati, From Adiabatic to Dispersive Readout of Quantum Circuits, [Phys. Rev. Lett.](#) **125**, 077701 (2020).
- [130] C. Metzger, S. Park, L. Tosi, C. Janvier, A. A. Reynoso, M. F. Goffman, C. Urbina, A. Levy Yeyati, and H. Pothier, Circuit-QED with phase-biased Josephson weak links, [Phys. Rev. Res.](#) **3**, 013036 (2021).

- [131] C. Hermansen, A. L. Yeyati, and J. Paaske, Inductive microwave response of Yu-Shiba-Rusinov states, *Phys. Rev. B* **105**, 054503 (2022).
- [132] N. Ackermann, A. Zazunov, S. Park, R. Egger, and A. L. Yeyati, Dynamical parity selection in superconducting weak links, *Phys. Rev. B* **107**, 214515 (2023).
- [133] Y. Fauvel, J. S. Meyer, and M. Houzet, Opportunities for the direct manipulation of a phase-driven Andreev spin qubit (2023), [arXiv:2312.14865](#) .
- [134] L. Pavešić, D. Bauernfeind, and R. Žitko, Subgap states in superconducting islands, *Phys. Rev. B* **104**, L241409 (2021).
- [135] T. W. Larsen, K. D. Petersson, F. Kuemmeth, T. S. Jespersen, P. Krogstrup, J. Nygård, and C. M. Marcus, Semiconductor-Nanowire-Based Superconducting Qubit, *Phys. Rev. Lett.* **115**, 127001 (2015).
- [136] G. de Lange, B. van Heck, A. Bruno, D. J. van Woerkom, A. Geresdi, S. R. Plissard, E. P. A. M. Bakkers, A. R. Akhmerov, and L. DiCarlo, Realization of microwave quantum circuits using hybrid superconducting-semiconducting nanowire josephson elements, *Phys. Rev. Lett.* **115**, 127002 (2015).
- [137] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Solitons in Polyacetylene, *Phys. Rev. Lett.* **42**, 1698 (1979).
- [138] M. J. Rice and E. J. Mele, Elementary Excitations of a Linearly Conjugated Diatomic Polymer, *Phys. Rev. Lett.* **49**, 1455 (1982).
- [139] J. Koch, V. Manucharyan, M. H. Devoret, and L. I. Glazman, Charging Effects in the Inductively Shunted Josephson Junction, *Phys. Rev. Lett.* **103**, 217004 (2009).
- [140] T. P. Orlando, J. E. Mooij, L. Tian, C. H. van der Wal, L. S. Levitov, S. Lloyd, and J. J. Mazo, Superconducting persistent-current qubit, *Phys. Rev. B* **60**, 15398 (1999).
- [141] C. H. van der Wal, A. C. J. ter Haar, F. K. Wilhelm, R. N. Schouten, C. J. P. M. Harman, T. P. Orlando, S. Lloyd, and J. E. Mooij, Quantum Superposition of Macroscopic Persistent-Current States, *Science* **290**, 773 (2000).
- [142] J. R. Friedman, V. Patel, W. Chen, S. K. Tolpygo, and J. E. Lukens, Quantum superposition of distinct macroscopic states, *Nature* **406**, 43–46 (2000).
- [143] V. E. Manucharyan, J. Koch, L. I. Glazman, and M. H. Devoret, Fluxonium: Single Cooper-Pair Circuit Free of Charge Offsets, *Science* **326**, 113–116 (2009).
- [144] L. Peyruchat, R. H. Rodriguez, J. L. Smir, R. Leone, and Ç. Ö. Girit, Spectral signatures of non-trivial topology in a superconducting circuit (2024), [arXiv:2401.10876 \[cond-mat.mes-hall\]](#) .
- [145] L. I. Glazman and G. Catelani, Bogoliubov quasiparticles in superconducting qubits, *SciPost Phys. Lect. Notes* **31** (2021).
- [146] J. Wills, G. Campanaro, S. Cao, S. Fasciati, P. Leek, and B. Vlastakis, Spatial charge sensitivity in a multimode superconducting qubit, *Phys. Rev. Appl.* **17**, 024058 (2022).
- [147] I. M. Pop, K. Geerlings, G. Catelani, R. J. Schoelkopf, L. I. Glazman, and M. H. Devoret, Coherent suppression of electromagnetic dissipation due to superconducting quasiparticles, *Nature* **508**, 369–372 (2014).

- [148] L. Pavešić and R. Žitko, Generalized transmon hamiltonian for andreev spin qubits, [Phys. Rev. B **109**, 155164 \(2024\)](#).
- [149] V. V. Baran and J. Paaske, Bcs surrogate models for floating superconductor-semiconductor hybrids (2024), [arXiv:2402.18357 \[cond-mat.supr-con\]](#) .
- [150] C. Metzger, *Spin & charge effects in Andreev Bound States*, [Theses](#), Université Paris-Saclay (2022).
- [151] I. V. Pechenezhskiy, R. A. Mencia, L. B. Nguyen, Y.-H. Lin, and V. E. Manucharyan, The superconducting quasicharge qubit, [Nature **585**, 368–371 \(2020\)](#).
- [152] L. Tosi, I. Lobato, M. F. Goffman, C. Metzger, C. Urbina, and H. Pothier, Effects of the measurement power on states discrimination and dynamics in a circuit-QED experiment, [arXiv:2310.04556 \(2024\)](#).
- [153] J. M. Martinis, K. B. Cooper, R. McDermott, M. Steffen, M. Ansmann, K. D. Osborn, K. Cicak, S. Oh, D. P. Pappas, R. W. Simmonds, and C. C. Yu, Decoherence in josephson qubits from dielectric loss, [Phys. Rev. Lett. **95**, 210503 \(2005\)](#).
- [154] H. Sun, F. Wu, H.-S. Ku, X. Ma, J. Qin, Z. Song, T. Wang, G. Zhang, J. Zhou, Y. Shi, H.-H. Zhao, and C. Deng, Characterization of loss mechanisms in a fluxonium qubit, [Phys. Rev. Appl. **20**, 034016 \(2023\)](#).
- [155] H. Zhang, S. Chakram, T. Roy, N. Earnest, Y. Lu, Z. Huang, D. K. Weiss, J. Koch, and D. I. Schuster, Universal fast-flux control of a coherent, low-frequency qubit, [Phys. Rev. X **11**, 011010 \(2021\)](#).
- [156] P. Krantz, M. Kjaergaard, F. Yan, T. P. Orlando, S. Gustavsson, and W. D. Oliver, A quantum engineer’s guide to superconducting qubits, [Applied Physics Reviews **6**, 10.1063/1.5089550 \(2019\)](#).
- [157] O. Astafiev, Y. A. Pashkin, Y. Nakamura, T. Yamamoto, and J. S. Tsai, Quantum noise in the josephson charge qubit, [Phys. Rev. Lett. **93**, 267007 \(2004\)](#).
- [158] F. Yan, S. Gustavsson, A. Kamal, J. Birenbaum, A. P. Sears, D. Hover, T. J. Gudmundsen, D. Rosenberg, G. Samach, S. Weber, J. L. Yoder, T. P. Orlando, J. Clarke, A. J. Kerman, and W. D. Oliver, The flux qubit revisited to enhance coherence and reproducibility, [Nature Communications **7**, 10.1038/ncomms12964 \(2016\)](#).
- [159] C. M. Quintana, Y. Chen, D. Sank, A. G. Petukhov, T. C. White, D. Kafri, B. Chiaro, A. Megrant, R. Barends, B. Campbell, Z. Chen, A. Dunsworth, A. G. Fowler, R. Graff, E. Jeffrey, J. Kelly, E. Lucero, J. Y. Mutus, M. Neeley, C. Neill, P. J. J. O’Malley, P. Roushan, A. Shabani, V. N. Smelyanskiy, A. Vainsencher, J. Wenner, H. Neven, and J. M. Martinis, Observation of classical-quantum crossover of $1/f$ flux noise and its paramagnetic temperature dependence, [Phys. Rev. Lett. **118**, 057702 \(2017\)](#).
- [160] M. Hays, G. de Lange, K. Serniak, D. J. van Woerkom, D. Bouman, P. Krogstrup, J. Nygård, A. Geresdi, and M. H. Devoret, Direct microwave measurement of andreev-bound-state dynamics in a semiconductor-nanowire josephson junction, [Phys. Rev. Lett. **121**, 047001 \(2018\)](#).
- [161] A. Eremenko, [Simultaneous diagonalization of two quadratic forms and a generalized eigenvalue problem \(2019\)](#).

- [162] D. Han, Y. S. Kim, and M. E. Noz, Illustrative example of Feynman's rest of the universe, *American Journal of Physics* **67**, 61 (1999).

From Energy spectrum and current-phase relation of a nanowire Josephson junction close to the topological transition by Murthy, Kurilovich, Kurilovich, van Heck, Glazman, Nayak

- [1] A. Y. Kitaev, Unpaired majorana fermions in quantum wires, *Phys. Usp.* 44, 131 (2001).
- [2] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Majorana Fermions and a Topological Phase Transition in Semiconductor-Superconductor Heterostructures, *Phys. Rev. Lett.* 105, 077001 (2010).
- [3] Y. Oreg, G. Refael, and F. von Oppen, Helical Liquids and Majorana Bound States in Quantum Wires, *Phys. Rev. Lett.* 105, 177002 (2010).
- [4] J. Alicea, New directions in the pursuit of Majorana fermions in solid state systems, *Rep. Prog. Phys.* 75, 076501 (2012).
- [5] J. Alicea, Y. Oreg, G. Refael, F. Von Oppen, and M. Fisher, Non-Abelian statistics and topological quantum information processing in 1d wire networks, *Nat. Phys.* 7, 412 (2011).
- [6] D. Aasen, M. Hell, R. V. Mishmash, A. Higginbotham, J. Danon, M. Leijnse, T. S. Jespersen, J. A. Folk, C. M. Marcus, K. Flensberg, and J. Alicea, Milestones Toward Majorana-Based Quantum Computing, *Phys. Rev. X* 6, 031016 (2016).
- [7] L. Fu and C. L. Kane, Josephson current and noise at a superconductor/quantum-spin-Hall-insulator/superconductor junction, *Phys. Rev. B* 79, 161408(R) (2009).
- [8] M. Cheng and R. M. Lutchyn, Josephson current through a superconductor/semiconductor-nanowire/superconductor junction: Effects of strong spin-orbit coupling and Zeeman splitting, *Phys. Rev. B* 86, 134522 (2012).
- [9] D. I. Pikulin and Y. V. Nazarov, Phenomenology and dynamics of a Majorana Josephson junction, *Phys. Rev. B* 86, 140504(R) (2012).
- [10] P. San-Jose, J. Cayao, E. Prada, and R. Aguado, Multiple Andreev reflection and critical current in topological superconducting nanowire junctions, *New J. Phys.* 15, 075019 (2013).
- [11] J. Cayao, E. Prada, P. San-Jose, and R. Aguado, SNS junctions in nanowires with spin-orbit coupling: Role of confinement and helicity on the subgap spectrum, *Phys. Rev. B* 91, 024514 (2015).
- [12] P. Marra, R. Citro, and A. Braggio, Signatures of topological phase transitions in Josephson current-phase discontinuities, *Phys. Rev. B* 93, 220507(R) (2016).
- [13] Y. Peng, F. Pientka, E. Berg, Y. Oreg, and F. von Oppen, Signatures of topological josephson junctions, *Phys. Rev. B* 94, 085409 (2016).
- [14] K. N. Nesterov, M. Houzet, and J. S. Meyer, Anomalous Josephson effect in semiconducting nanowires as a signature of the topologically nontrivial phase, *Phys. Rev. B* 93, 174502 (2016).
- [15] J. Cayao, P. San-Jose, A. M. Black-Schaffer, R. Aguado, and E. Prada, Majorana splitting from critical currents in Josephson junctions, *Phys. Rev. B* 96, 205425 (2017).
- [16] C. W. J. Beenakker, Universal Limit of Critical-Current Fluctuations in Mesoscopic Josephson Junctions, *Phys. Rev. Lett.* 67, 3836 (1991).
- [17] B. van Heck, J. I. Väyrynen, and L. I. Glazman, Zeeman and spin-orbit effects in the Andreev spectra of nanowire junctions, *Phys. Rev. B* 96, 075404 (2017).
- [18] J. Klinovaja and D. Loss, Composite Majorana fermion wave functions in nanowires, *Phys. Rev. B* 86, 085408 (2012).
- [19] E. Akkermans, A. Auerbach, J. E. Avron, and B. Shapiro, Relation between Persistent Currents and the Scattering Matrix, *Phys. Rev. Lett.* 66, 76 (1991).
- [20] S. Souma and A. Suzuki, Local density of states and scattering matrix in quasi-one-

dimensional systems, *Phys. Rev. B* 65, 115307 (2002).

[21] N. M. Chtchelkatchev and Y. V. Nazarov, Andreev Quantum Dots for Spin Manipulation, *Phys. Rev. Lett.* 90, 226806 (2003).

[22] B. Béri, J. H. Bardarson, and C. W. J. Beenakker, Splitting of Andreev levels in a Josephson junction by spin-orbit coupling, *Phys. Rev. B* 77, 045311 (2008).

[23] L. Tosi, C. Metzger, M. F. Goffman, C. Urbina, H. Pothier, S. Park, A. L. Yeyati, J. Nygård, and P. Krogstrup, Spin-Orbit Splitting of Andreev States Revealed by Microwave Spectroscopy, *Phys. Rev. X* 9, 011010 (2019).

[24] L. P. Rokhinson, X. Liu, and J. K. Furdyna, The fractional a.c. Josephson effect in a semiconductor-superconductor nanowire as a signature of Majorana particles, *Nat. Phys.* 8, 795 (2012).

[25] D. Laroche, D. Bouman, D. J. van Woerkom, A. Proutski, C. Murthy, D. I. Pikulin, C. Nayak, R. J. J. van Gulik, J. Nygård, P. Krogstrup, L. P. Kouwenhoven, and A. Geresdi, Observation of the 4π -periodic Josephson effect in indium arsenide nanowires, *Nat. Commun.* 10, 245 (2019).

[26] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Signatures of Majorana

fermions in hybrid superconductor-semiconductor nanowire devices, *Science* 336, 1003 (2012).

[27] H. Zhang, C.-X. Liu, S. Gazibegovic, D. Xu, J. A. Logan, G. Wang, N. van Loo, J. D. S. Bommer, M. W. A. de Moor, D. Car, R. L. M. Op het Veld, P. J. van Veldhoven, S. Koelling, M. A. Verheijen, M. Pendharkar, D. J. Pennachio, B. Shojaei, J. S. Lee, C. J. Palmstrøm, E. P. A. M. Bakkers, S. D. Sarma, and L. P. Kouwenhoven, Quantized Majorana conductance, *Nature (London)* 556, 74 (2018).

[28] S. M. Albrecht, A. P. Higginbotham, M. Madsen, F. Kuemmeth, T. S. Jespersen, J. Nygård, P. Krogstrup, and C. M. Marcus, Exponential protection of zero modes in Majorana islands, *Nature (London)* 531, 206 (2016).

[29] M. T. Deng, S. Vaitiekenas, E. B. Hansen, J. Danon, M. Leijnse, K. Flensberg, J. Nygård, P. Krogstrup, and C. M. Marcus, Majorana bound state in a coupled quantum-dot hybrid-nanowire system, *Science* 354, 1557 (2016).

[30] L. Bretheau, Ç. Girit, H. Pothier, D. Esteve, and C. Urbina, Exciting Andreev pairs in a superconducting atomic contact, *Nature (London)* 499, 312 (2013).

[31] D. J. van Woerkom, A. Proutski, B. van Heck, D. Bouman, J. I. Väyrynen, L. I. Glazman, P. Krogstrup, J. Nygård, L. P.

Kouwenhoven, and A. Geresdi, Microwave spectroscopy of spinful Andreev bound states in ballistic semiconductor Josephson junctions, *Nat. Phys.* 13, 876 (2017).

[32] M. Hays, G. de Lange, K. Serniak, D. J. van Woerkom, D. Bouman, P. Krogstrup, J. Nygård, A. Geresdi, and M. H. Devoret, Direct Microwave Measurement of Andreev-BoundState Dynamics in A Semiconductor-Nanowire Josephson Junction, *Phys. Rev. Lett.* 121, 047001 (2018).

[33] M. Hays, V. Fatemi, K. Serniak, D. Bouman, S. Diamond, G. de Lange, P. Krogstrup, J. Nygård, A. Geresdi, and M. H. Devoret, Continuous monitoring of a trapped, superconducting spin, *arXiv:1908.02800 [cond-mat.mes-hall]*.

[34] S. Tewari, J. D. Sau, V. W. Scarola, C. Zhang, and S. Das Sarma, Probing a topological quantum critical point in semiconductor-superconductor heterostructures, *Phys. Rev. B* 85, 155302 (2012).

[35] J. I. Väyrynen, G. Rastelli, W. Belzig, and L. I. Glazman, Microwave signatures of Majorana states in a topological Josephson junction, *Phys. Rev. B* 92, 134508 (2015).

[36] L. D. Landau and L. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory*, 3rd ed. (Butterworth-Heinemann, 1981).

From Macroscopic quantization and the proximity effect in SNS junctions by Kulik

- ¹ A. F. Andreev, Zh. Eksp. Teor. Fiz. 46, 1823 (1964) and 49, 655 (1965) [Sov. Phys.-JETP 19, 1228 (1964) and 22, 455 (1966)].
 - ² B. D. Josephson, Adv. Phys. 14, 419 (1965).
 - ³ P. W. Anderson, Ravello Lectures on the Many Body Problem v. 2, Ed. E. R. Gaianello, Acad. Press, 1963, p. 115.
 - ⁴ L. G. Aslamazov, A. I. Larkin, and Yu. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. 55, 323 (1968) [Sov. Phys.-JETP 28, 171 (1969)].
 - ⁵ N. N. Bogolyubov, Usp. Fiz. Nauk 67, 549 (1959) [Sov. Phys.-Usp. 2, 236 (1959)].
 - ⁶ P. de Gennes, Superconductivity of Metals and Alloys, Benjamin, 1967.
 - ⁷ V. P. Galaiko, Zh. Eksp. Teor. Fiz. 57, 941 (1969) [Sov. Phys.-JETP 30, 514 (1970)].
 - ⁸ I. O. Kulik, ZhETF Pis. Red. 6, 652 (1967) [JETP Lett. 6, 143 (1967)].
 - ⁹ G. A. Gogadze and I. O. Kulik, Fiz. Tverd. Tela 11, 2182 (1969) [Sov. Phys.-Solid State 11, 1762 (1970)].
 - ¹⁰ A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskiĭ, Metody kvantovoi teorii polya v statisticheskoi fizike (Quantum Field Theoretical Methods in Statistical Physics), Fizmatgiz, 1962 [Pergamon, 1965].
 - ¹¹ I. O. Kulik and A. E. Gorbonosov, VINITI Preprint No. 239 (1968).
 - ¹² I. O. Kulik and I. K. Yanson, "Effekt Dzhozefsona," Fizika kondensirovannogo sostoyaniya (The Josephson Effect. Physics of the Condensed State). FTINT AN UkrSSR, No. 3, 1968, p. 3.
 - ¹³ V. P. Galaiko, A. V. Svidzinski, and V. A. Slyusarev, Zh. Eksp. Teor. Fiz. 56, 835 (1969) [Sov. Phys.-JETP 29, 454 (1969)].
 - ¹⁴ I. M. Lifshitz and A. M. Kosevich, ibid. 29, 730 (1955) [2, 636 (1956)].
 - ¹⁵ W. J. Tomasch, Phys. Rev. Lett. 15, 672 (1965); 16, 16 (1966).
 - ¹⁶ W. L. McMillan and P. W. Anderson, Phys. Rev. Lett. 16, 85 (1966).
- Translated by J. G. Adashko 207

From Ballistic Josephson-current flow through an asymmetric superconductor-normal-metal-superconductor junction by Chang, Bagwell

- ¹ P.G. deGennes, Superconductivity of Metals and Alloys (Addison-Wesley, New York, 1989).
- ² A.F. Andreev, Zh. Eksp. Teor. Fiz. 46, 1823 (1964) [Sov. Phys. JETP 19, 1228 (1964)]; 49, 655 (1966) [22, 455 (1966)].
- ³ I.O. Kulik, Zh. Eksp. Teor. Fiz. 57, 1745 (1969) [Sov. Phys. JETP 30, 944 (1970)].
- ⁴ C.W.J. Beenakker and H. van Houten, Phys. Rev. Lett. 66, 3056 (1991).
- ⁵ C.W.J. Beenakker, Phys. Rev. Lett. 67, 3836 (1991).
- ⁶ B.J. van Wees, K.M.H. Lenssen, and C.J.P.M. Harmans, Phys. Rev. B 44, 470 (1991).
- ⁷ P.F. Bagwell, Phys. Rev. B 46, 12573 (1992).
- ⁸ G.E. Rittenhouse and J.M. Graybeal, Phys. Rev. B 49, 1182 (1994).
- ⁹ P.F. Bagwell, Phys. Rev. B 49, 6841 (1994).
- ¹⁰ G.E. Blonder, M. Tinkham, and T.M. Klapwijk, Phys. Rev. B 25, 4515 (1982).
- ¹¹ R.A. Riedel and P.F. Bagwell, Phys. Rev. B 48, 15198 (1993).
- ¹² I.O. Kulik and A.N. Omel'yanchuk, Fiz. Nizk. Temp. 4, 296 (1978) [Sov. J. Low Temp. Phys. 4, 142 (1978)].
- ¹³ I.O. Kulik and A.N. Omel'yanchuk, Fiz. Nizk. Temp. 18, 1167 (1992) [Sov. J. Low Temp. Phys. 18, 819 (1992)].
- ¹⁴ S.V. Kuplevakhskii and I.I. Fal'ko, Fiz. Nizk. Temp. 17, 961 (1991) [Sov. J. Low Temp.

Phys. 17, 501 (1991)].

¹⁵ S.V. Kuplevakhskii and I.I. Fal'ko, Fiz. Nizk. Temp. 17, 1208 (1991) [Sov. J. Low Temp. Phys. 17, 636 (1991)].

¹⁶ S.V. Kuplevakhskii and I.I. Fal'ko, Fiz. Nizk. Temp. 18, 623 (1992) [Sov. J. Low Temp. Phys., 18, 440 (1992)].

¹⁷ L.D. Landau and E.M. Lifshitz, Quantum Mechanics: Nonrelativistic Theory (Pergamon, New York, 1958), p. 63.

¹⁸ - about parameters, see main text

¹⁹ P.W. Anderson, in The Many-Body Problem, edited by E.R. Caianiello (Academic, New York, 1964), Vol. 2.

From Multi-terminal Josephson junctions as topological materials by Riwar, Houzet, Meyer, Nazarov

References

- [1] B. D. Josephson, *Physics Letters* **1**, 251–253 (1962).
- [2] A. A. Golubov, M. Y. Kupriyanov, E. Il'ichev, *Rev. Mod. Phys.* **76**, 411 (2004).
- [3] M. H. Devoret, A. Wallraff, J. M. Martinis, *arXiv:0411174* (2004).
- [4] C. W. J. Beenakker, *Phys. Rev. Lett.* **67**, 3836 (1991).
- [5] M. L. Della Rocca, *et al.*, *Phys. Rev. Lett.* **99**, 127005 (2007).
- [6] J.-D. Pillet, *et al.*, *Nature Physics* **6**, 965 (2010).
- [7] L. Bretheau, Ç. O. Girit, H. Pothier, D. Esteve, C. Urbina, *Nature* **499**, 312 (2013).
- [8] A. Y. Kitaev, *Physics-Uspekhi* **44**, 131–136 (2001).
- [9] L. Fu, C. L. Kane, *Phys. Rev. B* **79**, 161408 (2009).
- [10] J. Alicea, *Reports on Progress in Physics* **75**, 076501 (2012).
- [11] D. M. Badiane, L. I. Glazman, M. Houzet, J. S. Meyer, *Comptes Rendus Physique* **14**, 840–856 (2013).
- [12] C. Herring, *Phys. Rev.* **52**, 365 (1937).
- [13] S. Murakami, *New Journal of Physics* **9**, 356–356 (2007).
- [14] X. Wan, A. M. Turner, A. Vishwanath, S. Y. Savrasov, *Phys. Rev. B* **83**, 205101 (2011).
- [15] A. A. Burkov, L. Balents, *Phys. Rev. Lett.* **107**, 127205 (2011).
- [16] P. Hosur, X. Qi, *Comptes Rendus Physique* **14**, 857 (2013).
- [17] A. H. Castro Neto, F. Guinea, N. M. R. Peres, K. S. Novoselov, A. K. Geim, *Rev. Mod. Phys.* **81**, 109 (2009).
- [18] S. Matsuura, P.-Y. Chang, A. P. Schnyder, S. Ryu, *New J. Phys.* **15**, 065001 (2013).
- [19] Y. X. Zhao, Z. D. Wang, *Phys. Rev. Lett.* **110**, 240404 (2013).

- [20] F. Zhang, C. L. Kane, *Phys. Rev. B* **90**, 020501 (2014).
- [21] M. Zgirski, *et al.*, *Phys. Rev. Lett.* **106**, 257003 (2011).
- [22] A. M. Hriscu, Y. V. Nazarov, *Phys. Rev. Lett.* **110**, 097002 (2013).
- [23] R. Leone, L. P. Lévy, P. Lafarge, *Phys. Rev. Lett.* **100**, 117001 (2008).
- [24] H. B. Nielsen, M. Ninomiya, *Nuclear Physics B* **185**, 20 (1981).
- [25] M. L. Mehta, *Random matrices*, vol. 142 (Elsevier/Academic Press, Amsterdam, 2004).
- [26] S. R. Plissard, *et al.*, *Nature Nanotechnology* **8**, 859 (2014).
- [27] B. van Heck, S. Mi, A. R. Akhmerov, *Phys. Rev. B* **90**, 155450 (2014).

References

- [15] C. W. J. Beenakker, *Phys. Rev. Lett.* **67**, 3836 (1991).
- [42] For an odd number of channels, there is an additional eigenvalue 1 corresponding to a state at the gap edge.
- [45] Note that in the main text we restrict ourselves to positive energies while keeping both spin sectors.
- [47] Note that the Chern number may be computed using either the outgoing or the incoming wave functions as they are related through S_N which does not depend on the superconducting phases.
- [48] As pointed out previously, in the case of an odd number of total channels, there is a state at Δ for any value of the phases.
- [S6] M. L. Mehta, *Random matrices*, vol. 142 (Elsevier/Academic Press, Amsterdam, 2004).

From Energy spectrum and current-phase relation of a nanowire Josephson junction close to the topological transition by Murthy, Kurilovich, Kurilovich, van Heck, Glazman, Nayak

References

- [1] A. Y. Kitaev, Unpaired majorana fermions in quantum wires, *Phys. Uspekhi* **44**, 131 (2001).
- [2] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Majorana fermions and a topological phase transition in semiconductor-superconductor heterostructures, *Phys. Rev. Lett.* **105**, 077001 (2010).
- [3] Y. Oreg, G. Refael, and F. von Oppen, Helical liquids and Majorana bound states in quantum wires, *Phys. Rev. Lett.* **105**, 177002 (2010).
- [4] J. Alicea, New directions in the pursuit of Majorana fermions in solid state systems, *Rep. Prog. Phys.* **75**, 076501 (2012).

- [5] J. Alicea, Y. Oreg, G. Refael, F. Von Oppen, and M. Fisher, Non-Abelian statistics and topological quantum information processing in 1d wire networks, [Nature Phys. **7**, 412 \(2011\)](#).
- [6] D. Aasen, M. Hell, R. V. Mishmash, A. Higginbotham, J. Danon, M. Leijnse, T. S. Jespersen, J. A. Folk, C. M. Marcus, K. Flensberg, and J. Alicea, Milestones toward Majorana-based quantum computing, [Phys. Rev. X **6**, 031016 \(2016\)](#).
- [7] L. Fu and C. L. Kane, Josephson current and noise at a superconductor/quantum-spin-Hall-insulator/superconductor junction, [Phys. Rev. B **79**, 161408 \(2009\)](#).
- [8] M. Cheng and R. M. Lutchyn, Josephson current through a superconductor/semiconductor-nanowire/superconductor junction: effects of strong spin-orbit coupling and Zeeman splitting, [Phys. Rev. B **86**, 134522 \(2012\)](#).
- [9] D. I. Pikulin and Y. V. Nazarov, Phenomenology and dynamics of a Majorana Josephson junction, [Phys. Rev. B **86**, 140504 \(2012\)](#).
- [10] P. San-Jose, J. Cayao, E. Prada, and R. Aguado, Multiple Andreev reflection and critical current in topological superconducting nanowire junctions, [New J. Phys. **15**, 075019 \(2013\)](#).
- [11] J. Cayao, E. Prada, P. San-Jose, and R. Aguado, SNS junctions in nanowires with spin-orbit coupling: Role of confinement and helicity on the subgap spectrum, [Phys. Rev. B **91**, 024514 \(2015\)](#).
- [12] P. Marra, R. Citro, and A. Braggio, Signatures of topological phase transitions in Josephson current-phase discontinuities, [Phys. Rev. B **93**, 220507 \(2016\)](#).
- [13] Y. Peng, F. Pientka, E. Berg, Y. Oreg, and F. von Oppen, Signatures of topological josephson junctions, [Phys. Rev. B **94**, 085409 \(2016\)](#).
- [14] K. N. Nesterov, M. Houzet, and J. S. Meyer, Anomalous Josephson effect in semiconducting nanowires as a signature of the topologically nontrivial phase, [Phys. Rev. B **93**, 174502 \(2016\)](#).
- [15] J. Cayao, P. San-Jose, A. Black-Schaffer, R. Aguado, and E. Prada, Majorana splitting from critical currents in Josephson junctions, [Phys. Rev. B **96**, 205425 \(2017\)](#).
- [16] C. W. J. Beenakker, Universal limit of critical-current fluctuations in mesoscopic Josephson junctions, [Phys. Rev. Lett. **67**, 3836 \(1991\)](#).
- [17] B. van Heck, J. I. Väyrynen, and L. I. Glazman, Zeeman and spin-orbit effects in the Andreev spectra of nanowire junctions, [Phys. Rev. B **96**, 075404 \(2017\)](#).
- [18] J. Klinovaja and D. Loss, Composite Majorana fermion wave functions in nanowires, [Phys. Rev. B **86**, 085408 \(2012\)](#).
- [19] E. Akkermans, A. Auerbach, J. E. Avron, and B. Shapiro, Relation between persistent currents and the scattering matrix, [Phys. Rev. Lett. **66**, 76 \(1991\)](#).
- [20] S. Souma and A. Suzuki, Local density of states and scattering matrix in quasi-one-dimensional systems, [Phys. Rev. B **65**, 115307 \(2002\)](#).
- [21] N. M. Chtchelkatchev and Y. V. Nazarov, Andreev quantum dots for spin manipulation, [Phys. Rev. Lett. **90**, 226806 \(2003\)](#).

- [22] B. Béri, J. H. Bardarson, and C. W. J. Beenakker, Splitting of Andreev levels in a Josephson junction by spin-orbit coupling, *Phys. Rev. B* **77**, 045311 (2008).
- [23] L. Tosi, C. Metzger, M. F. Goffman, C. Urbina, H. Pothier, S. Park, A. L. Yeyati, J. Nygård, and P. Krogstrup, Spin-orbit splitting of Andreev states revealed by microwave spectroscopy, *Phys. Rev. X* **9**, 011010 (2019).
- [24] L. P. Rokhinson, X. Liu, and J. K. Furdyna, The fractional a.c. Josephson effect in a semiconductor–superconductor nanowire as a signature of Majorana particles, *Nature Phys.* **8**, 795 (2012).
- [25] D. Laroche, D. Bouman, D. J. van Woerkom, A. Proutski, C. Murthy, D. I. Pikulin, C. Nayak, R. J. J. van Gulik, J. Nygård, P. Krogstrup, L. P. Kouwenhoven, and A. Geresdi, Observation of the 4π -periodic Josephson effect in indium arsenide nanowires, *Nature Commun.* **10**, 245 (2019).
- [26] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Signatures of Majorana fermions in hybrid superconductor-semiconductor nanowire devices, *Science* **336**, 1003 (2012).
- [27] H. Zhang, C.-X. Liu, S. Gazibegovic, D. Xu, J. A. Logan, G. Wang, N. van Loo, J. D. S. Bommer, M. W. A. de Moor, D. Car, R. L. M. Op het Veld, P. J. van Veldhoven, S. Koelling, M. A. Verheijen, M. Pendharkar, D. J. Pennachio, B. Shojaei, J. S. Lee, C. J. Palmstrøm, E. P. A. M. Bakkers, S. D. Sarma, and L. P. Kouwenhoven, Quantized Majorana conductance, *Nature* **556**, 74 (2018).
- [28] S. M. Albrecht, A. P. Higginbotham, M. Madsen, F. Kuemmeth, T. S. Jespersen, J. Nygård, P. Krogstrup, and C. M. Marcus, Exponential protection of zero modes in Majorana islands, *Nature* **531**, 206 (2016).
- [29] M. T. Deng, S. Vaitiekenas, E. B. Hansen, J. Danon, M. Leijnse, K. Flensberg, J. Nygård, P. Krogstrup, and C. M. Marcus, Majorana bound state in a coupled quantum-dot hybrid-nanowire system, *Science* **354**, 1557 (2016).
- [30] L. Bretheau, Ç. Girit, H. Pothier, D. Esteve, and C. Urbina, Exciting Andreev pairs in a superconducting atomic contact, *Nature* **499**, 312 (2013).
- [31] D. J. van Woerkom, A. Proutski, B. van Heck, D. Bouman, J. I. Väyrynen, L. I. Glazman, P. Krogstrup, J. Nygård, L. P. Kouwenhoven, and A. Geresdi, Microwave spectroscopy of spinful Andreev bound states in ballistic semiconductor Josephson junctions, *Nature Phys.* **13**, 876 (2017).
- [32] M. Hays, G. de Lange, K. Serniak, D. J. van Woerkom, D. Bouman, P. Krogstrup, J. Nygård, A. Geresdi, and M. H. Devoret, Direct microwave measurement of Andreev-bound-state dynamics in a semiconductor-nanowire Josephson junction, *Phys. Rev. Lett.* **121**, 047001 (2018).
- [33] M. Hays, V. Fatemi, K. Serniak, D. Bouman, S. Diamond, G. de Lange, P. Krogstrup, J. Nygård, A. Geresdi, and M. H. Devoret, Continuous monitoring of a trapped, superconducting spin, [arXiv:1908.02800 \[cond-mat.mes-hall\]](https://arxiv.org/abs/1908.02800) .
- [34] S. Tewari, J. D. Sau, V. W. Scarola, C. Zhang, and S. Das Sarma, Probing a topological quantum critical point in semiconductor-superconductor heterostructures, *Phys. Rev. B* **85**, 155302 (2012).

- [35] J. I. Väyrynen, G. Rastelli, W. Belzig, and L. I. Glazman, Microwave signatures of Majorana states in a topological Josephson junction, *Phys. Rev. B* **92**, 134508 (2015).
- [36] L. D. Landau and L. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory*, 3rd ed. (Butterworth-Heinemann, 1981).

From Coulomb blockade of Majorana fermion induced transport by Zazunov, Levy Yeyati, Egger

References

- [1] C. Nayak, S.H. Simon, A. Stern, M. Freedman, and S. Das Sarma, *Rev. Mod. Phys.* **80**, 1083 (2008).
- [2] M.Z. Hasan and C.L. Kane, *Rev. Mod. Phys.* **82**, 3045 (2010).
- [3] X.L. Qi and S.C. Zhang, *Rev. Mod. Phys.* (in press); arXiv:1008.2026.
- [4] A.Yu. Kitaev, *Phys. Usp.* **44**, 131 (2001).
- [5] L. Fu and C.L. Kane, *Phys. Rev. Lett.* **100**, 096407 (2008).
- [6] Y. Tanaka, T. Yokoyama, and N. Nagaosa, *Phys. Rev. Lett.* **103**, 107002 (2009).
- [7] K.T. Law, P.A. Lee, and T.K. Ng, *Phys. Rev. Lett.* **103**, 237001 (2009).
- [8] J. Linder, Y. Tanaka, T. Yokoyama, A. Sudbø, and N. Nagaosa, *Phys. Rev. Lett.* **104**, 067001 (2010).
- [9] C.J. Bolech and E. Demler, *Phys. Rev. Lett.* **98**, 237002 (2007).
- [10] J. Nilsson, A.R. Akhmerov, and C.W.J. Beenakker, *Phys. Rev. Lett.* **101**, 120403 (2008).
- [11] K. Flensberg, *Phys. Rev. B* **82**, 180516(R) (2010).
- [12] A. Golub and B. Horovitz, *Phys. Rev. B* **83**, 153415 (2011).
- [13] L. Fu, *Phys. Rev. Lett.* **104**, 056402 (2010).
- [14] M. Cook and M. Franz, arXiv:1105.1787.
- [15] L. Fu and C.L. Kane, *Phys. Rev. B* **79**, 161408(R) (2009).
- [16] M. Sato and S. Fujimoto, *Phys. Rev. B* **79**, 094504 (2009).
- [17] R.M. Lutchyn, J.D. Sau, and S. Das Sarma, *Phys. Rev. Lett.* **105**, 077001 (2010); Y. Oreg, G. Refael, and F.v. Oppen, *ibid.* **105**, 177002 (2010).
- [18] S. Gangadharaiah, B. Braunecker, P. Simon, and D. Loss, *Phys. Rev. Lett.* **107**, 036801 (2011); E.M. Stoudenmire, J. Alicea, O.A. Starykh, and M.P.A. Fisher, *Phys. Rev. B* **84**, 014503 (2011); E. Sela, A. Altland, and A. Rosch, *Phys. Rev. B* **84**, 085114 (2011).
- [19] D.V. Averin and Yu.V. Nazarov, *Phys. Rev. Lett.* **69**, 1993 (1992).
- [20] B. van Heck, F. Hassler, A.R. Akhmerov, and C.W.J. Beenakker, arXiv:1108.1095.

- [21] Yu.V. Nazarov and Ya.M. Blanter, *Quantum transport* (Cambridge University Press, Cambridge, 2009).
- [22] A. Altland and B.D. Simons, *Condensed matter field theory*, 2nd edition (Cambridge University Press, Cambridge, 2010).
- [23] G. Schön and A.D. Zaikin, Phys. Rep. **198**, 237 (1990).
- [24] Yu.V. Nazarov, Phys. Rev. Lett. **82**, 1245 (1999); D.S. Golubev and A.D. Zaikin, Phys. Rev. Lett. **86**, 4887 (2001); Phys. Rev. B **69**, 075318 (2004); P.W. Brouwer, A. Lamacraft, and K. Flensberg, Phys. Rev. B **72**, 075316 (2005).
- [25] A. Altland and R. Egger, Phys. Rev. Lett. **102**, 026805 (2009).
- [26] H. Grabert and M.H. Devoret (eds.), *Single Charge Tunneling*, NATO ASI Series B, Physics v.294 (Plenum Press, New York, 1992).
- [27] The weak CB regime is eventually left for $\Gamma \ll E_c$, where Eq. (14.39) ceases to be valid.

From Quasiclassical description of transport through superconducting contacts by Cuevas, Fogelström

References

- [1] G. Eilenberger, Z. Phys. **214**, 195 (1968); A. I. Larkin & Y. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. **55**, 2262 (1968), [Sov. Phys. JETP **28**, 1200 (1969)]; G. M. Eliashberg, Zh. Eksp. Teor. Fiz. **61**, 1254 (1971), [Sov. Phys. JETP **34**, 668 (1972)].
- [2] J. W. Serene and D. Rainer, Phys. Rep. **101**, 221 (1983).
- [3] A. V. Zaitsev, Sov. Phys. JETP **59**, 1015 (1984).
- [4] A. Millis, D. Rainer & J. A. Sauls, Phys. Rev. B **38**, 4504 (1988)
- [5] M. Eschrig, Phys. Rev. B **61**, 9061 (2000).
- [6] A. Shelankov and M. Ozana, Phys. Rev. B **61**, 7077 (2000).
- [7] M. Fogelström, Phys. Rev. B **62**, 11812 (2000).
- [8] D. J. Van Harlingen, Rev. Mod. Phys. **67**, 515 (1995), and references therein.
- [9] M. Covington, M. Aprili, E. Paraoanu, L. H. Greene, F. Xu, J. Zhu, and C. A. Mirkini, Phys. Rev. Lett. **79**, 277 (1997).
- [10] M. Aprili, M. Covington, E. Paraoanu, B. Niedermayer, and L. H. Greene, Phys. Rev. B **57**, R8139 (1998).
- [11] L. H. Greene, M. Covington, M. Aprili, E. Badica, and D. E. Pugel Physica B **284-288**, 159 (2000).
- [12] M. Covington and L. H. Greene, Phys. Rev. B **62**, 12440 (2000).
- [13] L. Alff, S. Kleefisch, U. Schoop, M. Zittartz, A. Marx, and R. Gross, Eur. Phys. J. B **5**, 423 (1998).

- [14] D. Koelle, R. Kleiner, F. Ludwig, E. Dantsker, and John Clarke, *Rev. Mod. Phys.* **71**, 631 (1999).
- [15] C. C. Tsuei and J. R. Kirtley, *Rev. Mod. Phys.* **72**, 969 (2000).
- [16] E. Polturak, G. Koren, D. Cohen, and E. Aharoni, *Phys. Rev. B* **47**, 5270 (1993).
- [17] A. Engelhardt, R. Dittmann, and A. I. Braginski, *Phys. Rev. B* **59**, 3815 (1999).
- [18] O. Neshor and G. Koren, *Phys. Rev. B* **60**, 9287 (1999).
- [19] M. Hurd, *Phys. Rev. B* **55**, R11993 (1997).
- [20] Yu. S. Barash, H. Burkhardt, and D. Rainer, *Phys. Rev. Lett.* **77**, 4070 (1996)
- [21] A. Poenicke, Yu. S. Barash, C. Bruder, and V. Istyukov, *Phys. Rev. B* **59**, 7102 (1999).
- [22] M. Fogelström and J. A. Sauls, unpublished. M. Fogelström, J. A. Sauls, M. Covington, M. Aprili, and L. H. Greene, unpublished.
- [23] A. Poenicke, M. Fogelström, and J A. Sauls, *Physica B* **284-288**, 589 (2000).
- [24] H. Takayanagi, T. Akazaki and J. Nitta, *Phys. Rev. Lett.* **75**, 3533 (1995).
- [25] E. Scheer, W. Belzig, Y. Naveh, M.H. Devoret, D. Esteve and C. Urbina, to be published in *Phys. Rev. Lett.* (2000).
- [26] M. Fogelström and J. C. Cuevas, in preparation.
- [27] F. Stageberg, R. Cantor, A. M. Goldman, and G. B. Arnold, *Phys. Rev. B* **32**, 3292 (1985).
- [28] R. Meservey and P. M. Tedrow, *Phys. Rep.* **238**, 173 (1994).
- [29] M. Giroud, H. Courtois, K. Hasselbach, D. Mailly and B. Pannetier, *Phys. Rev. B* **58**, R11872 (1998).
- [30] V. T. Petrashov, I. A. Sosnin, I. Cox, A. Parsons, and C. Troadec, *Phys. Rev. Lett.* **83**, 3281 (1999).
- [31] C. L. Chien and D. H. Reich, *J. Magn. Magn. Mater.* **200**, 83 (1999).
- [32] R. J. Soulen Jr., J. M. Byers, M. S. Osofsky, B. Nadgorny, T. Ambrose, S. F. Cheng, P. R. Broussard, C. T. Tanaka, J. Nowak, J. S. Moodera, A. Barry, J. M. D. Coey, *Science* **282**, 85 (1998).
- [33] S. K. Upadhyay, A. Palanisami, R. N. Louie, and R. A. Buhrman, *Phys. Rev. Lett.* **81**, 3274 (1998).
- [34] A. V. Veretennikov, V. V. Ryazanov, V. A. Oboznov, A. Yu. Rusanov, V. A. Larkin, and J. Aarts, *Physica B* **284-288**, 495 (2000).
- [35] P. Gandit, O. Bourgeois, J. Lesueur, R. Melin, A. Sulpice, X. Grison and J. Chaussy, *Physica B* **284-288** 497 (2000); O. Bourgeois, P. Gandit, A. Sulpice, J. Chaussy, J. Lesueur, and X. Grison, *Phys. Rev. B* (in press).
- [36] A. I. Buzdin, L. N. Bulaevskii & S. V. Panyukov, *JETP Lett.* **35**, 178 (1982).

- [37] Z. Radović, M. Ledvij , L. Dobrosavljević-Grujić, A. I. Buzdin, and J. R. Clem, Phys. Rev. B **44**, 759 (1991).
- [38] S. V. Kuplevakhskii and I. I. Fal'ko, JETP Lett. **52**, 340 (1990).
- [39] R. Zikić, L. Dobrosavljević-Grujić, and Z. Radović, Phys. Rev. B **59**, 14644 (1999).
- [40] L. N. Bulaevskii, V. V. Kuzii and A. A. Sobyanin, JETP Lett. **25**, 290 (1977).
- [41] M. J. DeWeert and G. B. Arnold, Phys. Rev. Lett **55**, 1522 (1985); Phys. Rev. B **39**, 11307 (1989).
- [42] M. J. M. de Jong and C. W. J. Beenakker, Phys. Rev. Lett **74**, 1657 (1995).
- [43] Y. Tanaka and S. Kashiwaya, Physica C **274**, 357 (1997).
- [44] S. Kashiwaya, Y. Tanaka, N. Yoshida, and M. R. Beasley, Phys. Rev. B **60**, 3572 (1999).
- [45] I. Žutić and O. T. Valls, Phys. Rev. B **61**, 1555 (2000).
- [46] A. Martín-Rodero, F. J. García-Vidal and A. Levy Yeyati, Phys. Rev. Lett. **72**, 554 (1994). A. Levy Yeyati, A. Martín-Rodero, F. J. García-Vidal, Phys. Rev. B **51**, 3743 (1995).
- [47] J.C. Cuevas, A. Martín-Rodero and A. Levy Yeyati, Phys. Rev. B **54**, 7366 (1996).
- [48] L.V. Keldysh, Sov. Phys. JETP **20**, 1018 (1965).
- [49] J. Bardeen, Phys. Rev. Lett. **6**, 57 (1961).
- [50] B.D. Josephson, Phys. Lett. **1**, 251 (1962).
- [51] V. Ambegaokar and A. Baratoff, Phys. Rev. Lett. **10**, 486 (1963), errata: Phys. Rev. Lett. **11**, 104 (1963).
- [52] J.R. Schrieffer and J.W. Wilkins, Phys. Rev. Lett. **10**, 17 (1963).
- [53] C. Caroli, R. Combescot, P. Nozieres, and D. Saint-James, J. Phys. C **4**, 916 (1971). C. Caroli, R. Combescot, D. Lederer, P. Nozieres, and D. Saint-James, J. Phys. C **4**, 2598 (1971). R. Combescot, J. Phys. C **4**, 2611 (1971). C. Caroli, R. Combescot, P. Nozieres, and D. Saint-James, J. Phys. C **5**, 21 (1972).
- [54] E. Scheer, P. Joyez, D. Esteve, C. Urbina and M.H. Devoret, Phys. Rev. Lett. **78**, 3535 (1997).
- [55] E. Scheer, N. Agraït, J. C. Cuevas, A. Levy Yeyati, B. Ludoph, A. Martín-Rodero, G. Rubio, J. M. van Ruitenbeek and C. Urbina, Nature **394**, 154 (1998).
- [56] R. Landauer, IBM J. Res. Develop. **1**, 223 (1957); Philos. Mag. **21**, 863 (1970).
- [57] M. Büttiker, Phys. Rev. Lett. **57**, 1761 (1986).
- [58] G. E. Blonder, M. Tinkham, and T. M. Klapwijk, Phys. Rev. B **25**, 4515 (1982).
- [59] A. L. Shelankov, Sov. Phys. Solid State **26** 981 (1984)
- [60] C.W.J. Beenakker, Phys. Rev. Lett. **67**, 3836 (1991).

- [61] L. J. Buchholtz and D. Rainer, Z. Phys. B **35**, 151 (1979).
- [62] J. Kurkijärvi, D. Rainer, and J. A. Sauls, Can. J. Phys. **65**, 1440 (1997).
- [63] E. V. Thuneberg, J. Kurkijärvi, and D. Rainer, J. Phys. C **14**, 5615 (1981).
- [64] E. V. Thuneberg, J. Kurkijärvi, and D. Rainer, Phys. Rev. B **29**, 3913 (1984).
- [65] A. Furusaki and M. Tsukada, Physica B **165-166**, 967 (1990).
- [66] C.-R. Hu, Phys. Rev. Lett. **72**, 1526 (1994).
- [67] M. Fogelström, D. Rainer, and J. A. Sauls, Phys. Rev. Lett. **79**, 281 (1997).
- [68] Yu. S. Barash, Phys. Rev. B **61**, 678 (2000).
- [69] L. J. Buchholtz, M. Palumbo, D. Rainer, and J. A. Sauls, J. Low Temp. Phys. **101**, 1079 (1995); **101**, 1099 (1995).
- [70] R. A. Riedel and P. F. Bagwell, Phys. Rev. B **57**, 6084 (1998).
- [71] T. Tokuyasu, J. A. Sauls, and D. Rainer, Phys. Rev. B **38**, 8823 (1988).
- [72] The term $m = 0$ in the series appearing in the integrand should be understood as follows: $(1 - |\gamma_0|^2 |\gamma_1|^2) \{f_{FD}(\epsilon) - f_{FD}(\epsilon + eV)\}$.
- [73] Yu. S. Barash, and A. A. Svidzinsky, Zh. Eksp. Teor. Fiz. **111**, 1120 (1997) [JETP **84**, 619 (1997)].
- [74] M. P. Samanta and S. Datta, Phys. Rev. B **57**, 10972 (1998).
- [75] T. Löfwander, G. Johansson, and G. Wendin, preprint cond-mat/9908261.
- [76] A. Levy Yeyati, J. C. Cuevas, A. Lopez-Davalos, and A. Martín-Rodero., Phys. Rev. B **55**, R6137 (1997).
- [77] G. Johansson, E. N. Bratus, V. S. Shumeiko, G. Wendin, Phys. Rev. B **60**, 1382 (1999).
- [78] For recent review, see Y. Blanter and M. Büttiker, Phys. Rep. **336**, 1 (2000).
- [79] D. V. Averin and H. T. Imam, Phys. Rev. Lett. **76**, 3814 (1996).
- [80] P. Dieleman, H. G. Bukkems, T. M. Klapwijk, M. Schicke, and K. H. Gundlach, Phys. Rev. Lett. **79**, 3486 (1997).
- [81] J. C. Cuevas, A. Martín-Rodero and A. Levy Yeyati, Phys. Rev. Lett. **82**, 4086 (1999).
- [82] Y. Naveh and D. V. Averin, Phys. Rev. Lett. **82**, 4090 (1999).
- [83] J. X. Zhu and C. S. Ting, Phys. Rev. B **59**, R14165 (1999).
- [84] Y. Tanaka, T. Asai, N. Yoshida, J. Inoue, and S. Kashiwaya, Phys. Rev. B **61**, R11902 (2000).
- [85] A. Auerbach and E. Altman, Phys. Rev. Lett. **85**, 3480 (2000).

References

- [1] Affleck, I., J.-S. Caux, and A. Zagoskin, 2000, Phys. Rev. B **62**, 1433.
- [2] Anderson, P., 1961, Phys. Rev. **124**, 41.
- [3] Andreev, A., 1964, Sov. Phys. JETP **19**, 1228.
- [4] Arnold, G., 1987, J. Low Temp. Phys. **68**, 1.
- [5] Averin, D., and A. Bardas, 1995, Phys. Rev. Lett. **75**, 1831.
- [6] Avishai, Y., A. Golub, and A. D. Zaikin, 2003, Phys. Rev. B **67**, 041301 (4pp).
- [7] Bauer, J., A. Oguri, and A. Hewson, 2007, J. Phys.: Cond. Matt. **19**, 486211 (19pp).
- [8] Beenakker, C., 1992, Phys. Rev. B **46**, 12841.
- [9] Beenakker, C., 1997, Rev. Mod. Phys. **69**, 731.
- [10] Benjamin, C., T. Jonckheere, A. Zazunov, and T. Martin, 2007, Eur. Phys. J. B **57**, 279.
- [11] Bergeret, F. S., A. L. Yeyati, and A. Martín-Rodero, 2006, Phys. Rev. B **74**, 132505 (4pp).
- [12] Bergeret, F. S., A. L. Yeyati, and A. Martín-Rodero, 2007, Phys. Rev. B **76**, 174510 (6pp).
- [13] Bickers, N., 1987, Rev. Mod. Phys. **59**, 845.
- [14] Bignon, G., M. Houzet, F. Pistolesi, and F. Hekking, 2004, Europhys. Lett. **67**, 110.
- [15] Bratus, E., V. Shumeiko, and G. Wendin, 1995, Phys. Rev. Lett. **74**, 2110.
- [16] Buitelaar, M. R., W. Belzig, T. Nussbaumer, B. Babic, C. Bruder, and C. Schönenberger, 2003, Phys. Rev. Lett. **91**(5), 057005 (4pp).
- [17] Buitelaar, M. R., T. Nussbaumer, and C. Schönenberger, 2002, Phys. Rev. Lett. **89**(25), 256801 (4pp).
- [18] Buizert, C., A. Oiwa, K. Shibata, K. Hirakawa, and S. Tarucha, 2007, Phys. Rev. Lett. **99**, 136806 (4pp).
- [19] Cho, S. Y., K. Kang, C. M. Ryu, and C. K. Kim, 1999, Super. Latt. Mic. **26**(6), 405.
- [20] Choi, M., M. Lee, K. Kang, and W. Belzig, 2004, Phys. Rev. B **70**, 020502 (4pp).
- [21] Choi, M., M. Lee, K. Kang, and W. Belzig, 2005, Phys. Rev. Lett. **94**, 229701 (1pp).
- [22] Choi, M. S., C. Bruder, and D. Loss, 2000, Phys. Rev. B **62**(20), 13569.
- [23] Choi, M. S., R. López, and R. Aguado, 2005, Phys. Rev. Lett. **95**, 067204 (4pp).
- [24] Clerk, A., and V. Ambegaokar, 2000, Phys. Rev. B **61**, 9109.
- [25] Clerk, A. A., V. Ambegaokar, and S. Hershfield, 2000, Phys. Rev. B **61**(5), 3555.

- [26] Cleuziou, J.-P., W. Wernsdorfer, V. Bouchiat, T. Ondarcuhu, and M. Monthieux, 2006, *Nature* **1**, 53.
- [27] Coleman, P., 1984, *Phys. Rev. B* **29**, 3035.
- [28] Cronenwett, S., T. Oosterkamp, and L. Kouwenhoven, 1998, *Science* **24**, 540.
- [29] Cuevas, J., A. Martín-Rodero, and A. L. Yeyati, 1996, *Phys. Rev. B* **54**, 7366.
- [30] Cuevas, J. C., A. L. Yeyati, and A. Martín-Rodero, 2001, *Phys. Rev. B* **63**, 094515 (5pp).
- [31] Deacon, R., Y. Tanaka, A. Oiwa, K. Yosida, K. Shibata, K. Hirakawa, and S. Tarucha, 2010, *Phys.Rev. B* **81**, 121308 (4pp).
- [32] Deacon, R. S., Y. Tanaka, A. Oiwa, R. Sakano, K. Yoshida, K. Shibata, K. Hirakawa, and S. Tarucha, 2010, *Phys. Rev. Lett.* **104**, 076805 (4pp).
- [33] Dell’Anna, L., A. Zazunov, and R. Egger, 2008, *Phys. Rev. B* **77**, 104525 (6pp).
- [34] Dirks, T., T. Hughes, S. Lal, B. Uchoa, Y. Chen, C. Chialvo, P. Goldbart, and N. Mason, 2011, *Nature Phys.* **7**, 386.
- [35] Domanski, T., and A. Donabidowicz, 2008, *Phys. Rev. B* **78**, 073105 (4pp).
- [36] Eichler, A., R. Deblock, M. Weiss, C. Karrasch, V. Meden, C. Schönenberger, and H. Bouchiat, 2009, *Phys. Rev. B* **79**, 161407 (4pp).
- [37] Eichler, A., M. Weiss, S. Oberholzer, C. Schönenberger, A. L. Yeyati, J. C. Cuevas, and A. Martín-Rodero, 2007, *Phys. Rev. Lett.* **99**, 126602 (4pp).
- [38] Eldridge, J., M. Pala, M. Governale, and J. König, 2010, *Phys. Rev. B* **82**, 184507 (5pp).
- [39] Fazio, R., and R. Raimondi, 1998, *Phys. Rev. Lett.* **80**(13), 2913.
- [40] Fazio, R., and R. Raimondi, 1999, *Phys. Rev. Lett.* **82**, 4950 (1pp).
- [41] Ferrer, J., A. Martín-Rodero, and F. Flores, 1986, *Phys. Rev. B* **36**, 6149.
- [42] Franceschi, S. D., L. Kouwenhoven, C. Schönenberger, and W. Wernsdorfer, 2010, *Nature Nano.* **5**, 703.
- [43] Futterer, D., M. Governale, M. Pala, and J. König, 2009, *Phys. Rev. B* **79**, 054505 (5pp).
- [44] Glazman, L., and K. Matveev, 1989, *JETP Lett.* **49**, 659.
- [45] Glazman, L., and M. Raikh, 1988, *JETP Lett.* **47**, 452.
- [46] Goldhaber-Gordon, D., H. Shtrikman, D. Mahalu, D. Abusch-Magder, U. Meirav, and M. Kastner, 1998, *Nature* **391**, 156.
- [47] Golub, A., I. Kuzmenko, and Y. Avishai, 2011, arXiv:1105.0289 .
- [48] Golubov, A., M. Kupriyanov, and E. Il ichev, 2004, *Rev. Mod. Phys.* **76**, 411.
- [49] Governale, M., M. Pala, and J. König, 2008, *Phys. Rev. B* **77**, 134513 (14pp).
- [50] Graber, M. R., T. Nussbaumer, W. Belzig, and C. Shonenberger, 2004, *Nanotechnology* **15**(7), S479.

- [51] Grove-Rasmussen, K., H. I. Jorgensen, and P. Lindelof, 2007, New J. Phys. **1**, 124 (10pp).
- [52] Herrmann, L., F. Portier, P. Roche, A. L. Yeyati, T. Kontos, and C. Strunk, 2010, Phys. Rev. Lett. **104**, 026801 (4pp).
- [53] Hewson, A., 1993, *The Kondo problem to heavy fermions*, Cambridge Studies in Magnetism (Cambridge University Press, Cambridge, UK).
- [54] Hofstetter, L., S. Csonka, J. Nygard, and C. Schönenberger, 2009, Nature **461**, 960.
- [55] Hur, K. L., P. Recher, E. Dupont, and D. Loss, 2006, Phys. Rev. Lett. **96**, 106803 (4pp).
- [56] Jarillo-Herrero, P., J. Kong, H. van der Zant, C. Dekker, L. Kouwenhoven, and S. D. Franceschi, 2005, Nature **434**, 484.
- [57] Jarillo-Herrero, P., J. Kong, H. van der Zant, C. Dekker, L. Kouwenhoven, and S. D. Franceschi, 2007, Phys. Rev. Lett. **99**, 066801 (4pp).
- [58] Jarillo-Herrero, P., J. van Dam, and L. P. Kouwenhoven, 2006, Nature **439**(23), 953.
- [59] Johansson, G., E. Bratus, V. Shumeiko, and G. Wendin, 1999, Phys. Rev. B **60**, 1382.
- [60] Jonckheere, T., A. Zazunov, K. V. Bayandin, V. Shumeiko, and T. Martin, 2009, Phys. Rev. B **80**, 184510 (11pp).
- [61] Jonckheere, T., A. Zazunov, K. V. Bayandin, V. Shumeiko, and T. Martin, 2009, Phys. Rev. B **80**, 184510 (11pp).
- [62] Jorgensen, H. I., T. Novotný, K. Grove-Rasmussen, K. Flensberg, and P. Lindelof, 2007, Nano Lett. **7**, 2441.
- [63] Kang, K., 1998, Phys. Rev. B **57**(19), 11891.
- [64] Karrasch, C., A. Oguri, and V. Meden, 2008, Phys. Rev. B **77**, 024517 (14pp).
- [65] Kastner, M., 1993, Phys. Today **46**, 24.
- [66] Kasumov, A., K. Tsukagoshi, M. Kawamura, T. Kobayashi, Y. Aoyagi, K. Senba, T. Kodama, H. Nishikawa, I. Ikemoto, K. Kikuchi, V. Volkov, Y. Kasumov, *et al.*, 2005, Phys. Rev. B **72**, 033414 (4pp).
- [67] Klapwijk, T., G. Blonder, and M. Tinkham, 1982, Physica B **109 & 110**, 1657.
- [68] Koerting, V., B. Andersen, K. Flensberg, and J. Paaske, 2010, Phys. Rev. B **82**, 245108 (12pp).
- [69] König, J., J. Schmid, H. Schoeller, and G. Schon, 1996, Phys. Rev. B **54**, 16820.
- [70] Kotliar, G., and A. Ruckenstein, 1986, Phys. Rev. Lett. **57**, 1362.
- [71] Kouwenhoven, L., and L. Glazman, 2001, Physics World **14**, 33.
- [72] Krawiec, M., and K. Wysokinski, 2004, Supercond. Sci. Technol. **17**, 103.
- [73] Krishna-murthy, H., J. Wilkins, and K. Wilson, 1980, Phys. Rev. B **21**, 1003.
- [74] Kuemmeth, F., S. Ilani, D. C. Ralph, and P. McEuen, 2008, Nature **452**, 448.

- [75] Lee, M., T. Jonckheere, and T. Martin, 2010, Phys. Rev. B **81**, 155114 (20pp).
- [76] Lee, P., and T. Ng, 1988, Phys. Phys. Lett. **47**, 1768.
- [77] Lim, J., R. Lopez, M. Choi, and R. Aguado, 2011, arXiv:1104.0513 .
- [78] Lim, J. S., and M. S. Choi, 2008, J. Phys.: Condens. Matter **20**(41), 415225 (6pp).
- [79] Lim, J. S., M. S. Choi, R. López, and R. Aguado, 2006, Phys. Rev. B **74**, 205119 (4pp).
- [80] López, R., M. S. Choi, and R. Aguado, 2007, Phys. Rev. B **75**, 045132 (6pp).
- [81] Luitz, D., and F. Assaad, 2010, Phys. Rev. B **81**, 024509 (16pp).
- [82] Martín-Rodero, A., A. L. Yeyati, and J. Cuevas, 1999, Superlattices Micro. **25**, 925.
- [83] Martín-Rodero, A., A. L. Yeyati, and J. Cuevas, 1999, Superlattices Microstruct. **25**, 925.
- [84] Matsumoto, D., 2001, J. Phys. Soc. Jpn. **70**(2), 492.
- [85] Ng, T. K., 1993, Phys. Rev. Lett. **70**, 3635.
- [86] Nygard, J., D. Cobden, and P. Lindelof, 2000, Nature **408**, 342.
- [87] Octavio, M., M. Tinkham, G. Blonder, and T. Klapwijk, 1983, Phys. Rev. B **27**, 6739.
- [88] Oguri, A., Y. Tanaka, and A. C. Hewson, 2004, J. Phys. Soc. Jpn. **73**(9), 2494.
- [89] Pala, M., M. Governale, and J. König, 2007, New J. Phys. **9**, 278 (10pp).
- [90] Pillet, J.-D., C. Quay, P. Morfin, C. Bena, A. L. Yeyati, and P. Joyez, 2010, Nature Phys. **6**, 965.
- [91] Ralph, D., C. Black, and M. Tinkham, 1995, Phys. Rev. Lett. **74**, 3241.
- [92] Ralph, D., and R. Burhman, 1994, Phys. Phys. Lett. **69**, 2118.
- [93] Recher, P., E. Sukhorukov, and D. Loss, 2001, Phys. Rev. B **63**, 165314 (11pp).
- [94] Roch, N., S. Florens, T. Costi, W. Wernsdorfer, and F. Balestro, 2009, Phys. Rev. Lett. **103**, 197202 (4pp).
- [95] Rozhkov, A., and D. Arovas, 2000, Phys. Rev. B **62**, 6687.
- [96] Rozhkov, A., D. Arovas, and F. Guinea, 2001, Phys. Rev. B **64**, 233301 (4pp).
- [97] Rozhkov, A. V., and D. Arovas, 1999, Phys. Rev. Lett. **82**, 2788.
- [98] Samuelsson, P., and M. Buttiker, 2002, Phys. Rev. Lett. **89**, 046601 (4pp).
- [99] Scheer, E., N. Agraït, J. Cuevas, A. L. Yeyati, B. Ludoph, A. Martín-Rodero, G. R. Bollinger, J. van Ruitenbeek, and C. Urbina, 1998, Nature **394**, 154.
- [100] Scheer, E., P. Joyez, D. Esteve, C. Urbina, and M. H. Devoret, 1998, Phys. Rev. Lett. **78**, 3535.
- [101] Schwab, P., and R. Raimondi, 1999, Phys. Rev. B **59**(3), 1637.
- [102] Schweitzer, H., and G. Czycholl, 1990, Solid State Commun. **74**, 735.

- [103] Sellier, G., T. Kopp, J. Kroha, and Y. S. Barash, 2005, Phys. Rev. B **72**, 174502 (12pp).
- [104] Shiba, H., 1973, Prog. Theor. Phys. **50**, 50.
- [105] Shimizu, Y., H. Horii, Y. Takane, and Y. Isawa, 1998, J. Phys. Soc. Jpn. **67**, 1525.
- [106] Siano, F., and R. Egger, 2004, Phys. Rev. Lett. **93**(4), 047002 (4pp).
- [107] Siano, F., and R. Egger, 2005, Phys. Rev. Lett. **94**, 039902 (1pp).
- [108] Siano, F., and R. Egger, 2005, Phys. Rev. Lett. **94**, 229702 (1pp).
- [109] Soller, H., and A. Komnik, 2011, Eur. Phys. J. D **63**, 3.
- [110] Spivak, B., and S. Kivelson, 1964, Phys. Rev. B **43**, 3740.
- [111] Splettstoesser, J., M. Governale, J. König, F. Taddei, and R. Fazio, 2007, Phys. Rev. B **75**, 235302 (10pp).
- [112] Sun, Q. F., H. Guo, and T. H. Lin, 2001, Phys. Rev. Lett. **87**(17), 176601 (4pp).
- [113] Sun, Q. F., H. Guo, and J. Wang, 2002, Phys. Rev. B **65**, 075315 (11pp).
- [114] T. Meng, S. F., and P. Simon, 2009, Phys. Rev. B **79**, 224521 (10pp).
- [115] Tanaka, Y., N. Kawakami, and A. Oguri, 2007, J. Phys. Soc. Jpn. **76**(7), 074701.
- [116] Tanaka, Y., N. Kawakami, and A. Oguri, 2008, Phys. Rev. B **78**, 035444 (6pp).
- [117] Tanaka, Y., N. Kawakami, and A. Oguri, 2010, Phys. Rev. B **81**, 075404 (11pp).
- [118] Tanaka, Y., A. Oguri, and A. C. Hewson, 2007, New J. Phys. **9**, 115.
- [119] van Dam, J., Y. Nazarov, E. Bakkers, S. D. Franceschi, and L. Kouwenhoven, 2006, Nature **442**(10), 667.
- [120] Varma, C., and Y. Yafet, 1976, Phys. Rev. B **13**, 2950.
- [121] Vecino, E., M. R. Buitelaar, A. Martín-Rodero, C. Schönenberger, and A. L. Yeyati, 2004, SSC **131**, 625.
- [122] Vecino, E., A. Martín-Rodero, and A. L. Yeyati, 2003, Phys. Rev. B **68**, 035105 (9pp).
- [123] Whan, C. B., and T. P. Orlando, 1996, Phys. Rev. B **54**(8), 5255.
- [124] Wiegmann, P., and A. Tsvelick, 1983, J. Phys. C: Solid State Phys. **12**, 2281.
- [125] Wilson, K., 1975, Rev. Mod. Phys. **47**, 773.
- [126] Yamada, K., and K. Yoshida, 1975, Prog. Theor. Phys. **53**, 1286.
- [127] Yamada, Y., Y. Tanaka, and N. Kawakami, 2010, J. Phys. Soc. Jpn. **79**(4), 043705.
- [128] Yamada, Y., Y. Tanaka, and N. Kawakami, 2011, Phys. Rev. B **84**, 075484 (18pp).
- [129] Yeyati, A. L., J. Cuevas, and A. Martín-Rodero, 2005, Phys. Rev. Lett. **95**, 056804 (4pp).
- [130] Yeyati, A. L., J. C. Cuevas, A. López-Dávalos, and A. Martín-Rodero, 1997, Phys. Rev. B **55**(10), 6137.

- [131] Yeyati, A. L., F. Flores, and A. Martín-Rodero, 1999, Phys. Rev. Lett. **83**, 600.
- [132] Yeyati, A. L., A. Martín-Rodero, and F. Flores, 1993, Phys. Rev. Lett. **71**, 2991.
- [133] Yeyati, A. L., A. Martín-Rodero, and E. Vecino, 2003, Phys. Rev. Lett. **91**(26), 266802 (4pp).
- [134] Yoshioka, T., and Y. Ohashi, 2000, J. Phys. Soc. Jpn. **69**(6), 1812.
- [135] Zazunov, A., R. Egger, C. Mora, and T. Martin, 2006, Phys. Rev. B **73**, 214501 (7pp).
- [136] Zazunov, A., A. L. Yeyati, and R. Egger, 2010, Phys. Rev. B **81**, 012502 (4pp).
- [137] Zhao, H. K., 1998, Phys. Rev. B **58**(20), 13660.
- [138] Zhao, H. K., and J. Wang, 2001, Phys. Rev. B **64**, 094505 (10pp).
- [139] Zhu, Y., Q. F. Sun, and T. H. Lin, 2002, Phys. Rev. B **66**, 085306 (5pp).
- [140] Zitko, R., M. Lee, R. López, R. Aguado, and M.-S. Choi, 2010, Phys. Rev. Lett. **105**, 116803 (4pp).

From (????)

References

- [1] Wendin, G. Quantum information processing with superconducting circuits: a review. *Rep. Prog. Phys.* **2017**, *80*, 106001.
- [2] Clarke, J.; Braginski, A. I. *The SQUID Handbook Fundamentals and Technology of SQUIDs and SQUID Systems, Vol. 1*; Wiley-VCH: Weinheim, 2006.
- [3] Benz, S. P.; Hamilton, C. A. Application of the Josephson effect to voltage metrology. *Proc. IEEE* **2004**, *92*, 1617–1629.
- [4] Lamata, L.; Parra-Rodriguez, A.; Sanz, M.; Solano, E. Digital-analog quantum simulations with superconducting circuits. *Adv. Phys.: X* **2018**, *3*, 1457981.
- [5] Likharev, *Dynamics of Josephson Junctions and Circuits*; CRC Press, 1986.
- [6] Hansen, J. B.; Lindelof, P. E. Static and dynamic interactions between Josephson junctions. *Rev. Mod. Phys.* **1984**, *56*, 431–459.
- [7] Deminova, G.; Kovalenko, A. A calculation of interaction between two Josephson junctions. *IEEE Trans. Magn.* **1979**, *15*, 291–294.
- [8] Brinkman, A.; Golubov, A. A. Coherence effects in double-barrier Josephson junctions. *Phys. Rev. B* **2000**, *61*, 11297–11300.
- [9] Golubov, A. A.; Kupriyanov, M. Y.; Il'ichev, E. The current-phase relation in Josephson junctions. *Rev. Mod. Phys.* **2004**, *76*, 411–469.
- [10] Freyn, A.; Douçot, B.; Feinberg, D.; Mélin, R. Production of Nonlocal Quartets and Phase-Sensitive Entanglement in a Superconducting Beam Splitter. *Phys. Rev. Lett.* **2011**, *106*, 257005.

- [11] Jonckheere, T.; Rech, J.; Martin, T.; Douçot, B.; Feinberg, D.; Mélin, R. Multipair de Josephson resonances in a biased all-superconducting bijunction. *Phys. Rev. B* **2013**, *87*, 214501.
- [12] Mélin, R.; Feinberg, D.; Courtois, H.; Padurariu, C.; Pfeffer, A.; Duvauchelle, J. E.; Lefloch, F.; Jonckheere, T.; Rech, J.; Martin, T.; Doucot, B. D.C. Josephson transport by quartets and other Andreev resonances in superconducting bijunctions. *J. Phys.: Conf. Ser.* **2014**, *568*, 052006.
- [13] Feinberg, D.; Jonckheere, T.; Rech, J.; Martin, T.; Douçot, B.; Mélin, R. Quartets and the current-phase structure of a double quantum dot superconducting bijunction at equilibrium. *Eur. Phys. J. B* **2015**, *88*.
- [14] Mélin, R.; Sotto, M.; Feinberg, D.; Caputo, J.-G.; Douçot, B. Gate-tunable zero-frequency current cross correlations of the quartet state in a voltage-biased three-terminal Josephson junction. *Phys. Rev. B* **2016**, *93*, 115436.
- [15] Mélin, R.; Caputo, J.-G.; Yang, K.; Douçot, B. Simple Floquet-Wannier-Stark-Andreev viewpoint and emergence of low-energy scales in a voltage-biased three-terminal Josephson junction. *Phys. Rev. B* **2017**, *95*, 085415.
- [16] Pfeffer, A. H.; Duvauchelle, J. E.; Courtois, H.; Mélin, R.; Feinberg, D.; Lefloch, F. Subgap structure in the conductance of a three-terminal Josephson junction. *Phys. Rev. B* **2014**, *90*, 075401.
- [17] Cohen, Y.; Ronen, Y.; Kang, J.-H.; Heiblum, M.; Feinberg, D.; Mélin, R.; Shtrikman, H. Nonlocal supercurrent of quartets in a three-terminal Josephson junction. *PNAS* **2018**, *115*, 6991–6994.
- [18] Bretheau, L.; Girit, Ç. Ö.; Pothier, H.; Esteve, D.; Urbina, C. Exciting Andreev pairs in a superconducting atomic contact. *Nature* **2013**, *499*, 312–315.
- [19] Janvier, C.; Tosi, L.; Bretheau, L.; Girit, Ç. Ö.; Stern, M.; Bertet, P.; Joyez, P.; Vion, D.; Esteve, D.; Goffman, M. F.; Pothier, H.; Urbina, C. Coherent manipulation of Andreev states in superconducting atomic contacts. *Science* **2015**, *349*, 1199–1202.
- [20] Della Rocca, M. L.; Chauvin, M.; Huard, B.; Pothier, H.; Esteve, D.; Urbina, C. Measurement of the Current-Phase Relation of Superconducting Atomic Contacts. *Phys. Rev. Lett.* **2007**, *99*.
- [21] Krogstrup, P.; Ziino, N. L. B.; Chang, W.; Albrecht, S. M.; Madsen, M. H.; Johnson, E.; Nygård, J.; Marcus, C. M.; Jespersen, T. S. Epitaxy of semiconductor–superconductor nanowires. *Nature Materials* **2015**, *14*, 400–406.
- [22] Albrecht, S. M.; Higginbotham, A. P.; Madsen, M.; Kuemmeth, F.; Jespersen, T. S.; Nygård, J.; Krogstrup, P.; Marcus, C. M. Exponential protection of zero modes in Majorana islands. *Nature* **2016**, *531*, 206–209.
- [23] Deng, M. T.; Vaitiekėnas, S.; Hansen, E. B.; Danon, J.; Leijnse, M.; Flensberg, K.; Nygård, J.; Krogstrup, P.; Marcus, C. M. Majorana bound state in a coupled quantum-dot hybrid-nanowire system. *Science* **2016**, *354*, 1557–1562.
- [24] Goffman, M. F.; Urbina, C.; Pothier, H.; Nygård, J.; Marcus, C. M.; Krogstrup, P. Conduction channels of an InAs-Al nanowire Josephson weak link. *New J. Phys.* **2017**, *19*, 092002.

- [25] Tosi, L.; Metzger, C.; Goffman, M. F.; Urbina, C.; Pothier, H.; Park, S.; Yeyati, A. L.; Nygård, J.; Krogstrup, P. Spin-Orbit Splitting of Andreev States Revealed by Microwave Spectroscopy. *Phys. Rev. X* **2019**, *9*, 011010.
- [26] Bagwell, P. F. Suppression of the Josephson current through a narrow, mesoscopic, semiconductor channel by a single impurity. *Phys. Rev. B* **1992**, *46*, 12573–12586.
- [27] Olivares, D. G.; Yeyati, A. L.; Bretheau, L.; Girit, Ç. Ö.; Pothier, H.; Urbina, C. Dynamics of quasiparticle trapping in Andreev levels. *Phys. Rev. B* **2014**, *89*, 104504.
- [28] Hays, M.; de Lange, G.; Serniak, K.; van Woerkom, D. J.; Bouman, D.; Krogstrup, P.; Nygård, J.; Geresdi, A.; Devoret, M. H. Direct Microwave Measurement of Andreev-Bound-State Dynamics in a Semiconductor-Nanowire Josephson Junction. *Phys. Rev. Lett.* **2018**, *121*, 047001.
- [29] Blonder, G. E.; Tinkham, M.; Klapwijk, T. M. Transition from metallic to tunneling regimes in superconducting microconstrictions: Excess current, charge imbalance, and supercurrent conversion. *Phys. Rev. B* **1982**, *25*, 4515–4532.
- [30] Beenakker, C. W. J.; van Houten, H. Josephson current through a superconducting quantum point contact shorter than the coherence length. *Phys. Rev. Lett.* **1991**, *66*, 3056–3059.
- [31] Bardeen, J.; Kümmel, R.; Jacobs, A. E.; Tewordt, L. Structure of Vortex Lines in Pure Superconductors. *Phys. Rev.* **1969**, *187*, 556–569.
- [32] Buzdin, A.; Koshelev, A. E. Periodic alternating 0- and π -junction structures as realization of φ -Josephson junctions. *Phys. Rev. B* **2003**, *67*, 220504.
- [33] Pillet, J.-D.; Quay, C. H. L.; Morfin, P.; Bena, C.; Yeyati, A. L.; Joyez, P. Andreev bound states in supercurrent-carrying carbon nanotubes revealed. *Nature Physics* **2010**, *6*, 965–969.
- [34] Deng, M.-T.; Vaitiekėnas, S.; Prada, E.; San-Jose, P.; Nygård, J.; Krogstrup, P.; Aguado, R.; Marcus, C. M. Nonlocality of Majorana modes in hybrid nanowires. *Phys. Rev. B* **2018**, *98*, 085125.
- [35] Goffman, M. F.; Cron, R.; Levy Yeyati, A.; Joyez, P.; Devoret, M. H.; Esteve, D.; Urbina, C. Supercurrent in Atomic Point Contacts and Andreev States. *Phys. Rev. Lett.* **2000**, *85*, 170–173.
- [36] English, C. D.; Hamilton, D. R.; Chialvo, C.; Moraru, I. C.; Mason, N.; Van Harlingen, D. J. Observation of non sinusoidal current-phase relation in graphene Josephson junctions. *Phys. Rev. B* **2016**, *94*.
- [37] Delagrè, R.; Weil, R.; Kasumov, A.; Ferrier, M.; Bouchiat, H.; Deblock, R. 0- π quantum transition in a carbon nanotube Josephson junction: Universal phase dependence and orbital degeneracy. *Phys. Rev. B* **2016**, *93*, 195437.

From Hidden symmetry in interacting-quantum-dot-based multi-terminal Josephson junctions

References

- [1] K. Likharev and V. Semenov, [IEEE Transactions on Applied Superconductivity](#) **1**, 3 (1991).

- [2] J. P. Cleuziou, W. Wernsdorfer, V. Bouchiat, T. Ondarcuhu, and M. Monthieux, *Nat. Nanotechnol.* **1**, 53 (2006).
- [3] J. Clarke and F. K. Wilhelm, *Nature* **453**, 1031 (2008).
- [4] A. Martín-Rodero and A. Levy Yeyati, *Adv. Phys.* **60**, 899 (2011).
- [5] V. Meden, *Journal of Physics: Condensed Matter* **31**, 163001 (2019).
- [6] B. van Heck, S. Mi, and A. R. Akhmerov, *Phys. Rev. B* **90**, 155450 (2014).
- [7] R.-P. Riwar, M. Houzet, J. S. Meyer, and Y. V. Nazarov, *Nature Communications* **7**, 11167 (2016).
- [8] E. Eriksson, R.-P. Riwar, M. Houzet, J. S. Meyer, and Y. V. Nazarov, *Phys. Rev. B* **95**, 075417 (2017).
- [9] R. M. Lutchyn, E. P. A. M. Bakkers, L. P. Kouwenhoven, P. Krogstrup, C. M. Marcus, and Y. Oreg, *Nature Reviews Materials* **3**, 52 (2018).
- [10] H.-Y. Xie, M. G. Vavilov, and A. Levchenko, *Phys. Rev. B* **97**, 035443 (2018).
- [11] H.-Y. Xie and A. Levchenko, *Phys. Rev. B* **99**, 094519 (2019).
- [12] E. V. Repin and Y. V. Nazarov, *Phys. Rev. B* **105**, L041405 (2022).
- [13] L. P. Gavensky, G. Usaj, and C. A. Balseiro, *Europhysics Letters* **141**, 36001 (2023).
- [14] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M. P. A. Fisher, *Nature Physics* **7**, 412 (2011).
- [15] A. Zazunov, R. Egger, M. Alvarado, and A. L. Yeyati, *Phys. Rev. B* **96**, 024516 (2017).
- [16] A. Freyn, B. Douçot, D. Feinberg, and R. Mélin, *Phys. Rev. Lett.* **106**, 257005 (2011).
- [17] A. W. Draelos, M.-T. Wei, A. Seredinski, H. Li, Y. Mehta, K. Watanabe, T. Taniguchi, I. V. Borzenets, F. Amet, and G. Finkelstein, *Nano Letters* **19**, 1039 (2019).
- [18] N. Pankratova, H. Lee, R. Kuzmin, K. Wickramasinghe, W. Mayer, J. Yuan, M. G. Vavilov, J. Shabani, and V. E. Manucharyan, *Phys. Rev. X* **10**, 031051 (2020).
- [19] G. V. Graziano, M. Gupta, M. Pendharkar, J. T. Dong, C. P. Dempsey, C. Palmstrøm, and V. S. Pribiag, *Nature Communications* **13**, 5933 (2022).
- [20] M. Gupta, G. V. Graziano, M. Pendharkar, J. T. Dong, C. P. Dempsey, C. Palmstrøm, and V. S. Pribiag, *Nature Communications* **14**, 3078 (2023).
- [21] R. Mélin and D. Feinberg, *Phys. Rev. B* **107**, L161405 (2023).
- [22] F. Zhang, A. S. Rashid, M. T. Ahari, W. Zhang, K. M. Ananthanarayanan, R. Xiao, G. J. de Coster, M. J. Gilbert, N. Samarth, and M. Kayyalha, *Phys. Rev. B* **107**, L140503 (2023).
- [23] M. Coraiola, D. Z. Haxell, D. Sabonis, H. Weisbrich, A. E. Svetogorov, M. Hinderling, S. C. ten Kate, E. Cheah, F. Krizek, R. Schott, W. Wegscheider, J. C. Cuevas, W. Belzig, and F. Nichele, Hybridisation of andreev bound states in three-terminal josephson junctions (2023), [arXiv:2302.14535 \[cond-mat.mes-hall\]](https://arxiv.org/abs/2302.14535) .

- [24] S. Matsuo, T. Imoto, T. Yokoyama, Y. Sato, T. Lindemann, S. Gronin, G. C. Gardner, S. Nakosai, Y. Tanaka, M. J. Manfra, and S. Tarucha, Phase-dependent andreev molecules and superconducting gap closing in coherently coupled josephson junctions (2023), [arXiv:2303.10540 \[cond-mat.supr-con\]](#) .
- [25] T. Meng, S. Florens, and P. Simon, [Phys. Rev. B](#) **79**, 224521 (2009).
- [26] K. Grove-Rasmussen, G. Steffensen, A. Jellinggaard, M. H. Madsen, R. Žitko, J. Paaske, and J. Nygård, [Nature Communications](#) **9**, 2376 (2018).
- [27] M. Žonda, P. Zalom, T. Novotný, G. Loukeris, J. Bätge, and V. Pokorný, [Phys. Rev. B](#) **107**, 115407 (2023).
- [28] J.-D. Pillet, C. H. L. Quay, P. Morfin, C. Bena, A. L. Yeyati, and P. Joyez, [Nat. Phys.](#) **6**, 965 (2010).
- [29] D. J. Luitz, F. F. Assaad, T. Novotný, C. Karrasch, and V. Meden, [Phys. Rev. Lett.](#) **108**, 227001 (2012).
- [30] P. Zalom, Rigorous wilsonian renormalization group for impurity models with a spectral gap (2023), [arXiv:2307.07479 \[cond-mat.str-el\]](#) .
- [31] K. Satori, H. Shiba, O. Sakai, and Y. Shimizu, [J. Phys. Soc. Japan.](#) **61**, 3239 (1992).
- [32] O. Sakai, Y. Shimizu, H. Shiba, and K. Satori, [J. Phys. Soc. Japan.](#) **62**, 3181 (1993).
- [33] T. Hecht, A. Weichselbaum, J. von Delft, and R. Bulla, [J. Phys.: Cond. Mat.](#) **20**, 275213 (2008).
- [34] F. Siano and R. Egger, [Phys. Rev. Lett.](#) **93**, 047002 (2004).
- [35] F. Siano and R. Egger, [Phys. Rev. Lett.](#) **94**, 229702 (2005).
- [36] J. E. Gubernatis, *Quantum Monte Carlo methods : algorithms for lattice models* (Cambridge University Press, New York, NY, 2016).
- [37] C. Karrasch, A. Oguri, and V. Meden, [Phys. Rev. B](#) **77**, 024517 (2008).
- [38] V. Pokorný and M. Žonda, [Phys. Rev. B](#) **107**, 155111 (2023).
- [39] V. V. Baran, E. J. P. Frost, and J. Paaske, [arXiv:2307.11646 \[cond-mat.supr-con\]](#) (2023).
- [40] A. Bargerbos, M. Pita-Vidal, R. Žitko, J. Ávila, L. J. Splitthoff, L. Grünhaupt, J. J. Wesdorp, C. K. Andersen, Y. Liu, L. P. Kouwenhoven, R. Aguado, A. Kou, and B. van Heck, [PRX Quantum](#) **3**, 030311 (2022).
- [41] L. Pavešić, R. Aguado, and R. Žitko, [arXiv:2304.12456 \[cond-mat.mes-hall\]](#) (2023).
- [42] The mechanism behind both effects is novel and relies on the full phase control over the three terminal architecture of the device and the presence of Coulomb interactions in the junction region.
- [1] P. Zalom, V. Pokorný, and T. Novotný, [Phys. Rev. B](#) **103**, 035419 (2021).
- [44] [URL_will_be_inserted_by_publisher.](#)

- [45] The dot Green function actually changes, but it has no physical consequences, for a detailed account of this issue for the two-terminal setup see Ref. [46], the situation for multi-terminal setup is completely analogous.
- [46] A. Kadlecová, M. Žonda, and T. Novotný, *Phys. Rev. B* **95**, 195114 (2017).
- [47] The Hamiltonian is invariant under $d_{\uparrow} \rightarrow d_{\downarrow}$, $d_{\downarrow} \rightarrow d_{\uparrow}$ followed by a complex conjugation. Note that this antiunitary symmetry is different from time-reversal symmetry attained for two-terminal systems at $\varphi = 0$ as noted in Ref. [41].
- [48] The first BZ collapses to 1D in **C** and the green line thus corresponds always to one and the same $\chi = 0$ point.
- [2] *J. Phys.: Cond. Mat.* **19**, 486211 (2007).
- [3] P. Zalom and T. Novotný, *Phys. Rev. B* **104**, 035437 (2021).
- [4] P. Zalom and M. Žonda, *Phys. Rev. B* **105**, 205412 (2022).
- [6] J. C. Estrada Saldaña, A. Vekris, G. Steffensen, R. Žitko, P. Krogstrup, J. Paaske, K. Grove-Rasmussen, and J. Nygård, *Phys. Rev. Lett.* **121**, 257701 (2018).
- [7] R. Žitko, *NRG Ljubljana* (8f90ac4), Zenodo, <https://doi.org/10.5281/zenodo.4841076> (2021).

From Supplementary Material: Hidden symmetry in multi-terminal Josephson junctions by Zalom, Žonda et al.

References

- [1] P. Zalom, V. Pokorný, and T. Novotný, *Phys. Rev. B* **103**, 035419 (2021).
- [2] *J. Phys.: Cond. Mat.* **19**, 486211 (2007).
- [3] P. Zalom and T. Novotný, *Phys. Rev. B* **104**, 035437 (2021).
- [4] P. Zalom and M. Žonda, *Phys. Rev. B* **105**, 205412 (2022).
- [5] V. Meden, *Journal of Physics: Condensed Matter* **31**, 163001 (2019).
- [6] J. C. Estrada Saldaña, A. Vekris, G. Steffensen, R. Žitko, P. Krogstrup, J. Paaske, K. Grove-Rasmussen, and J. Nygård, *Phys. Rev. Lett.* **121**, 257701 (2018).
- [7] R. Žitko, “Nrg Ljubljana,” (2021), *NRG Ljubljana* (8f90ac4), Zenodo, <https://doi.org/10.5281/zenodo.4841076>.

References

- [1] K. K. Likharev, Rev. Mod. Phys. 51 (1979).
- [2] I.O.Kulik, A.N.Omelyanchouk, E. A. Kel'man, Sov.J. Low Temp. Phys. 5 (1979) 55.
- [3] A. Barone and G. Paterno, Physics and applications of the Josephson effect, Wiley, New York, 1982.
- [4] E. D. Vol, A.N.Omelyanchouk, Low Temp. Phys. 20 (1994) 87.
- [5] R. de Bruyn Ouboter, A.N.Omelyanchouk, E. D. Vol, Physica B 205 (1995) 153; 239 (1997) 203.
- [6] L.G. Aslamazov and A.I. Larkin, JETP Lett. **9**, 87 (1969).
- [7] R. de Bruyn Ouboter, A.N.Omelyanchouk, Superlatt. & Microstruct., **23**, N3-4, (1999).
- [8] B. J. Vleeming, Ph. D. thesis, Leiden University (1998).
- [9] I.O. Kulik, Sov. Phys. JETP **30**, 944, 1970.
- [10] I.O.Kulik and A.N.Omelyanchouk, Sov.J. Low Temp. Phys.**4**, 142 (1978); C.W.J. Beenakker and H. van Houten, Physical Review Letters **66**, 3056 (1991).
- [11] Y. Imry, "Introduction to Mesoscopic Physics", Oxford, New York (1997).
- [12] A.F. Andreev, Sov. Phys. JETP **19**, 1228 (1964).
- [13] J. P. Heida, B. J. van Wees, T. M. Klapwijk and G. Borghs, Phys. Rev. B **57**, R5618 (1998).
- [14] V. Barzykin and A.M. Zagoskin, Superlatt. and Microstructures **25**, 797 (1999).
- [15] U. Ledermann, A.L. Fauchère, and G. Blatter, Phys. Rev. B **59**, R9027 (1999).
- [16] Malek Zareyan and A.N.Omelyanchouk, Low Temp. Phys. **25**, 175 (1999).
- [17] A.N.Omelyanchouk and Malek Zareyan, Physica B 291 (2000) 81.
- [18] M.Buttiker, Phys.Rev.Lett. 37, 1761 (1986).
- [19] G. Eilenberger, Zs. Phys., **214**, 195 (1968).
- [20] I.O. Kulik, A.N. Omelyanchuk, and R.I. Shekhter, Sov. J. Low Temp. Phys. **3**,1543 (1977).
- [21] A. Huck, A. van Otterlo, and M. Sigrist, Phys. Rev. B **56**, 14163 (1997); A.M. Zagoskin, J. Phys.: Condensed Matter **9**, L419 (1997); cond-mat/9903170 (1999).
- [22] S.Kashiwaya and Y. Tanaka, Rep. Progr. Phys. **63**, 1641 (2000).

References

- [1] Klitzing K. v., Dorda G. Pepper M. Phys. Rev. Lett.451980494.
- [2] Thouless D. J., Kohmoto M., Nightingale M. P. den Nijs M. Phys. Rev. Lett.491982405.
- [3] Niu Q., Thouless D. J. Wu Y.-S. Phys. Rev. B3119853372.
- [4] Niu Q. Thouless D. J. Phys. Rev. B3519872188.
- [5] Avron J. E. Seiler R. Phys. Rev. Lett.541985259.
- [6] Avron J. E., Osadchy D. Seiler R. Physics Today56200338.
- [7] Avron J. E., Raveh A. Zur B. Rev. Mod. Phys.601988873.
- [8] Avron J. E. Cross M. C. Phys. Rev. B391989756.
- [9] Avron J. E. Sadun L. Annals of Physics2061991440.
- [10] Wigner E. von Neumann J. Phys. Z301929467.
- [11] Riwar R.-P., Houzet M., Meyer J. S. Nazarov Y. V. Nature Communications7201611167.
- [12] Andreev A. F. Soviet Journal of Experimental and Theoretical Physics1919641228.
- [13] Klapwijk T. M. Journal of Superconductivity172004593.
- [14] Nazarov Y. V. Blanter Y. M. Quantum Transport (Cambridge University Press) 2009.
- [15] Beenakker C. W. J. Phys. Rev. Lett.6719913836.
- [16] Nielsen H. Ninomiya M. Nuclear Physics B185198120.
- [17] Yokoyama T. Nazarov Y. V. Phys. Rev. B922015155437.
- [18] Eriksson E., Riwar R.-P., Houzet M., Meyer J. S. Nazarov Y. V. Phys. Rev. B952017075417.
- [19] Yokoyama T., Reutlinger J., Belzig W. Nazarov Y. V. Phys. Rev. B952017045411.
- [20] Xie H.-Y., Vavilov M. G. Levchenko A. Phys. Rev. B972018035443.
- [21] Klees R. L., Cuevas J. C., Belzig W. Rastelli G. Phys. Rev. B1032021014516.
- [22] Meyer J. S. Houzet M. Phys. Rev. Lett.1192017136807.
- [23] Xie H.-Y., Vavilov M. G. Levchenko A. Phys. Rev. B962017161406.
- [24] Peralta Gavensky L., Usaj G., Feinberg D. Balseiro C. A. Phys. Rev. B972018220505.
- [25] Oka T. Kitamura S. Annual Review of Condensed Matter Physics102019387.
- [26] Houzet M. Meyer J. S. Phys. Rev. B1002019014521.
- [27] Peralta Gavensky L., Usaj G. Balseiro C. A. Phys. Rev. B1002019014514.

- [28] Xie H.-Y. Levchenko A. Phys. Rev. B992019094519.
- [29] Kotetes P., Mercaldo M. T. Cuoco M. Phys. Rev. Lett.1232019126802.
- [30] Sakurai K., Mercaldo M. T., Kobayashi S., Yamakage A., Ikegaya S., Habe T., Kotetes P., Cuoco M. Asano Y. Phys. Rev. B1012020174506.
- [31] Meyer J. S. Houzet M. Phys. Rev. B1032021174504.
- [32] Ozawa T. Goldman N. Phys. Rev. B972018201117.
- [33] Klees R. L., Rastelli G., Cuevas J. C. Belzig W. Phys. Rev. Lett.1242020197002.
- [34] Peralta Gavensky L. Quantum transport in hybrid superconducting devices with topological properties. Phd thesis UNCuyo, Instituto Balseiro (2022).
- [35] Volovik G. E. The Universe in a Helium Droplet (Oxford University Press) 2009.
- [36] Repin E. V., Chen Y. Nazarov Y. V. Phys. Rev. B992019165414.
- [37] Wang Z. Zhang S.-C. Phys. Rev. X22012031008.
- [38] Wang Z. Yan B. Journal of Physics: Condensed Matter252013155601.
- [39] Pfeffer A. H., Duvauchelle J. E., Courtois H., Mélin R., Feinberg D. Lefloch F. Phys. Rev. B902014075401.
- [40] Cohen Y., Ronen Y., Kang J.-H., Heiblum M., Feinberg D., Mélin R. Shtrikman H. Proceedings of the National Academy of Sciences11520186991.
- [41] Graziano G. V., Lee J. S., Pendharkar M., Palmstrøm C. J. Pribiag V. S. Phys. Rev. B1012020054510.
- [42] Pankratova N., Lee H., Kuzmin R., Wickramasinghe K., Mayer W., Yuan J., Vavilov M. G., Shabani J. Manucharyan V. E. Phys. Rev. X102020031051.
- [43] Graziano G. V., Gupta M., Pendharkar M., Dong J. T., Dempsey C. P., Palmstrøm C. Pribiag V. S. Nature Communications1320225933.
- [44] Draelos A. W., Wei M.-T., Serebinski A., Li H., Mehta Y., Watanabe K., Taniguchi T., Borzenets I. V., Amet F. Finkelstein G. Nano Letters1920191039.
- [45] Arnault E. G., Larson T. F. Q., Serebinski A., Zhao L., Idris S., McConnell A., Watanabe K., Taniguchi T., Borzenets I., Amet F. Finkelstein G. Nano Letters2120219668.
- [46] Strambini E., D'Ambrosio S., Vischi F., Bergeret F. S., Nazarov Y. V. Giazotto F. Nature Nanotechnology1120161055.
- [47] Vischi F., Carrega M., Strambini E., D'Ambrosio S., Bergeret F. S., Nazarov Y. V. Giazotto F. Phys. Rev. B952017054504.
- [48] Zgirski M., Bretheau L., Le Masne Q., Pothier H., Esteve D. Urbina C. Phys. Rev. Lett.1062011257003.
- [49] Weisbrich H., Klees R., Rastelli G. Belzig W. PRX Quantum22021010310.
- [50] Xie H.-Y., Hasan J. Levchenko A. Phys. Rev. B1052022L241404.

- [51] Park S., Lee W., Jang S., Choi Y.-B., Park J., Jung W., Watanabe K., Taniguchi T., Cho G. Y. Lee G.-H. Nature **603** 2022 421.
- [52] Fatemi V., Akhmerov A. R. Bretheau L. Phys. Rev. Research **3** 2021 013288.
- [53] Leone R., Lévy L. P. Lafarge P. Phys. Rev. Lett. **100** 2008 117001.
- [54] Peyruchat L., Griesmar J., Pillet J.-D. Girit i. m. c. O. Phys. Rev. Research **3** 2021 013289.
- [55] Herrig T. Riwar R.-P. Phys. Rev. Research **4** 2022 013038.
- [56] Septembre I., Meyer J. S., Solnyshkov D. D. Malpuech G. arXiv:2210.11088 2022.

From Tunable pseudogaps due to non-local coherent transport in voltage-biased three-terminal Josephson junctions

References

- [1] T.M. Klapwijk, G.E. Blonder, and M. Tinkham, Physica B & C **109-110**, 1657 (1982); M. Octavio, M. Tinkham, G. E. Blonder, and T. M. Klapwijk, Phys. Rev. B **27**, 6739 (1983).
- [2] E. Scheer, P. Joyez, D. Esteve, C. Urbina, and M. H. Devoret, Phys. Rev. Lett. **78**, 3535 (1997).
- [3] F. Pierre, A. Anthore, H. Pothier, C. Urbina, and D. Esteve, Phys. Rev. Lett. **86**, 1078 (2001).
- [4] D. Averin and A. Bardas, Phys. Rev. Lett. **75**, 1831 (1995); A. Bardas and D. V. Averin, Phys. Rev. B **56**, R8518(R) (1997).
- [5] J. C. Cuevas and H. Pothier, Phys. Rev. B **75**, 174513 (2007).
- [6] M. Houzet and P. Samuelsson, Phys. Rev. B **82**, 060517 (2010).
- [7] A. Freyn, B. Douçot, D. Feinberg, R. Mélin, Phys. Rev. Lett. **106**, 257005 (2011).
- [8] T. Jonckheere, J. Rech, T. Martin, B. Douçot, D. Feinberg, and R. Mélin, Phys. Rev. B **87**, 214501 (2013).
- [9] A. H. Pfeffer, J. E. Duvauchelle, H. Courtois, R. Mélin, D. Feinberg, and F. Lefloch, Phys. Rev. B **90**, 075401 (2014).
- [10] J. E. Duvauchelle, A. H. Pfeffer, H. Courtois, and F. Lefloch, IEEE Trans. Appl. Supercond. **26**, 1801104 (2016).
- [11] Y. Cohen, Y. Ronen, J-H. Kang, M. Heiblum, D. Feinberg, R. Mélin, and H. Shtrikman, arXiv:1606.08436.
- [12] E. Strambini, S. D'Ambrosio, F. Vischi, F. S. Bergeret, and Yu. V. Nazarov, F. Giazotto, Nat. Nanotechnol. **11**, 1055 (2016); F. Vischi, M. Carrega, E. Strambini, S. D'Ambrosio, F. S. Bergeret, Yu. V. Nazarov, and F. Giazotto, Phys. Rev. B **95**, 054504 (2017).
- [13] C. Padurariu, T. Jonckheere, J. Rech, R. Mélin, D. Feinberg, T. Martin, Yu. V. Nazarov, Phys. Rev. B **92**, 205409 (2015).

- [14] B. van Heck, S. Mi, and A. R. Akhmerov, Phys. Rev. B **90**, 155450 (2014).
- [15] R.-P. Riwar, M. Houzet, J. S. Meyer, and Yu. V. Nazarov, Nat. Commun. **7**, 11167 (2016).
- [16] E. Eriksson, R.-P. Riwar, M. Houzet, J. S. Meyer, and Yu. V. Nazarov, Phys. Rev. B **95**, 075417 (2017).
- [17] Yu. V. Nazarov, Superlattices Microstruct. **25**, 1221 (1999).
- [18] Yu. V. Nazarov and Ya. M. Blanter, Quantum transport: Introduction to Nanoscience, (Cambridge University Press, Cambridge, 2009).
- [19] M. Vanević, PhD Thesis, (Basel University, 2008).
- [20] J. T. Edwards and D. J. Thouless, J. Phys. C **5**, 807 (1972).
- [21] P. Dubos, H. Courtois, B. Pannetier, F. K. Wilhelm, A. D. Zaikin, and G. Schön, Phys. Rev. B **63**, 064502 (2001).
- [22] H. Courtois, Ph. Gandit, and B. Pannetier, Phys. Rev. B **52**, 1162 (1995)
- [23] S. Guéron, H. Pothier, N. O. Birge, D. Esteve, and M. H. Devoret, Phys. Rev. Lett. **77**, 3025 (1996).
- [24] H. le Sueur, P. Joyez, H. Pothier, C. Urbina, and D. Esteve, Phys. Rev. Lett. **100**, 197002 (2008).
- [25] A. I. Larkin and Yu. V. Ovchinnikov, Zh. Eksp. Teor. Fiz. **68**, 1915 (1975) [Sov. Phys. JETP **41**, 960 (1976)].
- [26] R. Mélin, J.-G. Caputo, K. Yang, B. Douçot, Phys. Rev. B **95**, 085415 (2017).
- [27] E. V. Bezuglyi, E. N. Bratus, V. S. Shumeiko, Phys. Review B **83**(18), 184517 (2011).
- [28] E. V. Bezuglyi, E. N. Bratus, and V. S. Shumeiko, Physica C **499**, 15 (2014).
- [29] A. Kastalsky, A. W. Kleinsasser, L. H. Greene, R. Bhat, F. P. Milliken, and J. P. Harbison, Phys. Rev. Lett. **67**, 3026 (1991).
- [30] T. Jonckheere, A. Zazunov, K. V. Bayandin, V. Shumeiko, and T. Martin, Phys. Rev. B **80**, 184510 (2009).
- [31] M. Vanević and W. Belzig, Europhys. Lett. **75**, 604 (2006).
- [32] G. B. Lesovik, T. Martin, and G. Blatter, Eur. Phys. J. B **24**, 287 (2001).
- [33] P. Recher, E. V. Sukhorukov, and D. Loss, Phys. Rev. B **63**, 165314 (2001).